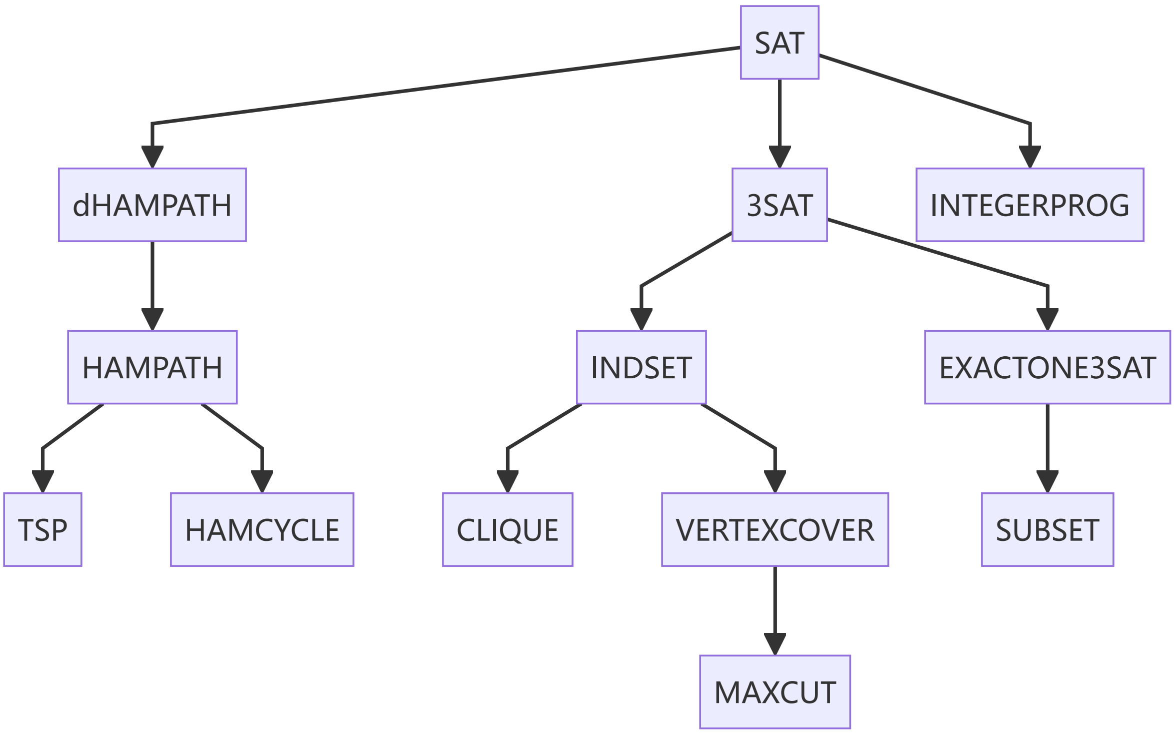
Last class

- **Cook-Levin Theorem**:SAT is $NP\text{-}complete$.
    - $NP\text{-}complete = NP \cap NP\text{-}hard$
    - L is $NP\text{-}hard$ if and only if $\forall K \in NP, K \leq_P L$.
- **Karp Reduction**(Poly-time many-one reduction)
    - $L \leq_P K$ if $\exists$ poly-time TM M s.t. $(\forall x)(x \in L \Leftrightarrow M(x) \in K)$.
- **Cook Reduction**(Poly-time Turing reduction)
    - L is Cook-reducible to K if $\exists$ an oracle TM with oracle access to K that decides L in polynomial time.
- SAT is $NP\text{-}complete$ under Karp reduction. So, SAT is also $NP\text{-}complete$ under Cook reduction.
- L is $NP\text{-}hard$ under Karp reduction $\Leftrightarrow$ L is $NP\text{-}hard$ under Cook reduction.
    - $\Rightarrow$:Obvious
    - $\Leftarrow$:???
- $SAT \leq_P 3SAT$
- $SAT \leq_P INTEGERPROG \leq_P 01PROG$



- $INDSET = \{\langle G, k\rangle | G \textit{ has an independent set of size } k\}$

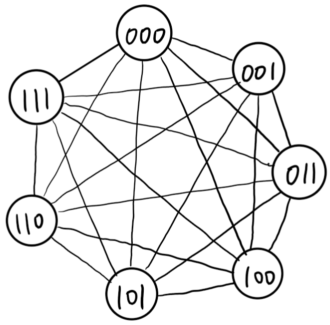
Thm 5.32 $3SAT \leq_P INDSET$

Proof: $3CNF$ φ with m clauses $\rightarrow$ G with $7m$ vertices.

φ is satisfiable $\Leftrightarrow$ $\langle G, m\rangle \in INDSET$.

For each clause, make a cluster, consisting of 7 vertices (satisfying assignments). Connect all.

eg. $C_1 = x_1 \vee \bar{x_2} \vee x_3 \quad (x_1, x_2, x_3) \neq (0, 1, 0)$



For different clauses, connect the assignments if they are inconsistent.

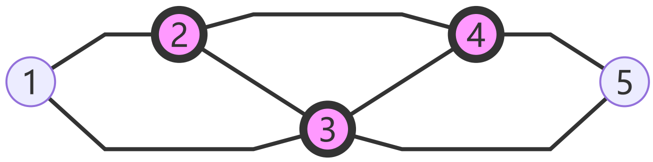
$\Rightarrow$: For each clause, take the vertex in the cluster that corresponds to the satisfying assignment.

$\Leftarrow$: Observe that if there exists an independent set of size m, then the set has exactly one vertex from each cluster. Since there is no edge between any pair, the assignment must be consistent.

Q.E.D.

- $CLIQUE = \{\langle G, k\rangle | G \ has \ a \ K_k \ subgraph\}$

eg. Consider graph G below. $\langle G, 3\rangle \in CLIQUE$, $\langle G, 4\rangle \notin CLIQUE$.



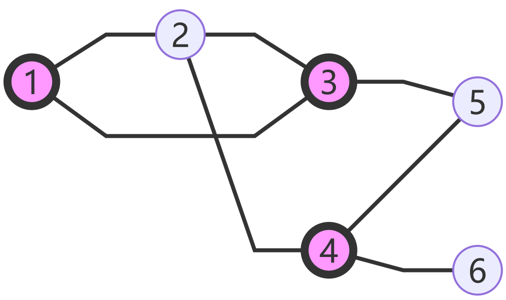
Thm 5.33 $CLIQUE$ **is NP-complete.**

Proof: $INDSET \leq_P CLIQUE$

Given a certificate $w = \{v_1, \ldots, v_k\}$, verifying $\langle G, k\rangle \in CLIQUE$ is in P.

- $VERTEXCOVER = \{\langle G, k\rangle | G \ has \ a \ vertex \ cover \ of \ size \ k\}$

eg. Consider graph G below. $\langle G, 3\rangle \in VERTEXCOVER$.



- A **vertex cover** of a graph is a set of vertices that includes at least one endpoint of every edge of the graph.

Thm 5.34 $VERTEXCOVER$ **is NP-complete.**

Proof:

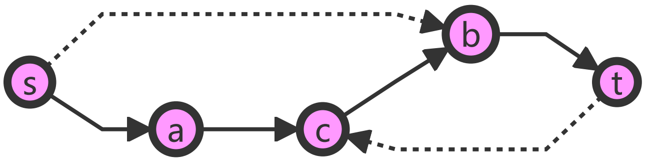
1. We prove $VERTEXCOVER$ is in NP.
2. We prove $INDSET \leq_P VERTEXCOVER$.

$\langle G, k\rangle \in VERTEXCOVER \Leftrightarrow \langle G, |V(G)| - k\rangle \in INDSET$.

Because $C \subseteq V(G)$ is a vertex cover if and only if $V(G) \setminus C$ is an independent set. *Q.E.D.*

- $dHAMPATH = \{\langle G, s, t\rangle | Directed\ graph\ G\ has\ a\ Hamilton\ path\ from\ s\ to\ t\}$

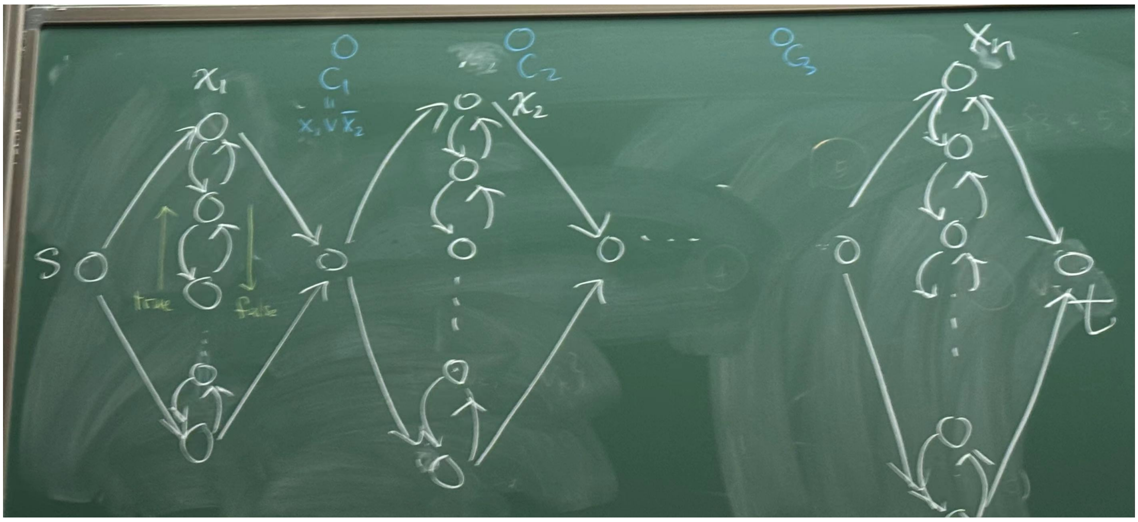
eg. Consider graph G below. $\langle G, s, t\rangle \in dHAMPATH$.



Thm 5.35 $dHAMPATH$ **is NP-complete.**

Proof: We claim $dHAMPATH \in NP$. We show $3SAT \leq_P dHAMPATH$.

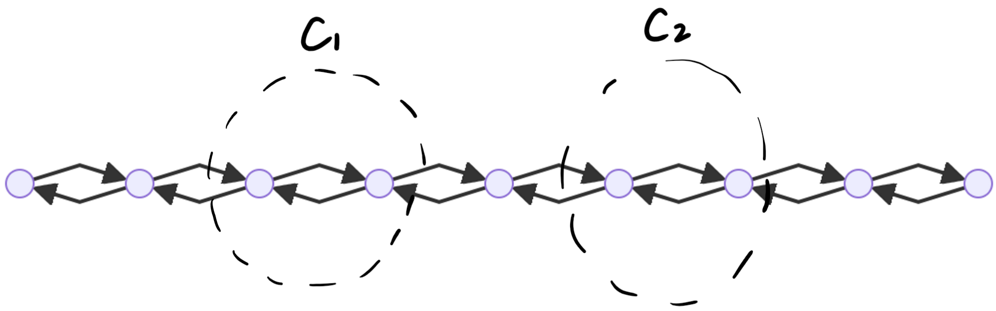
For any $3CNF\ \phi$, construct a directed graph G with two vertices s and t s.t. an s-t Hamilton path exists $\Leftrightarrow$ ϕ is satisfiable.

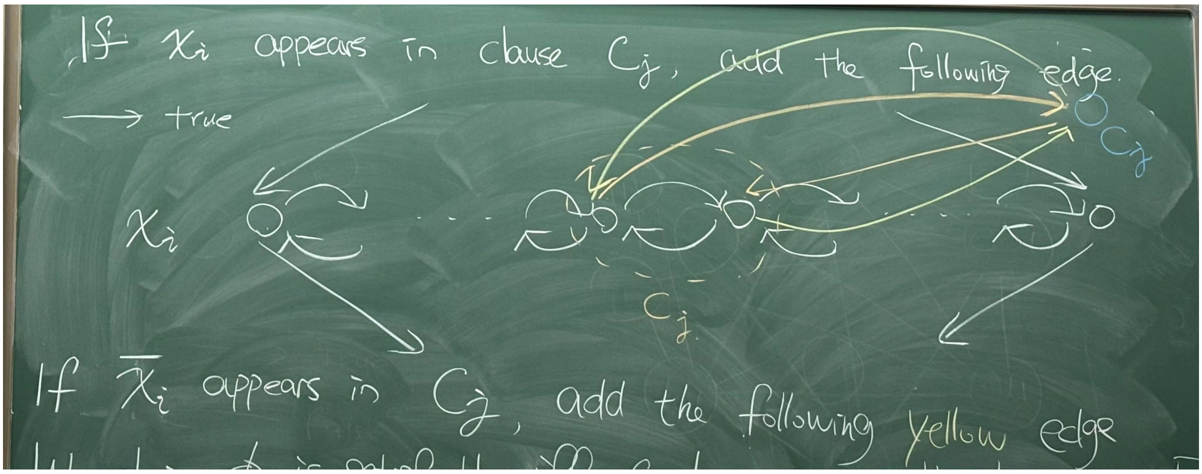


Suppose $\phi = (a_1 \vee b_1 \vee c_1) \wedge \ldots \wedge (a_k \vee b_k \vee c_k)$, where $a_i, b_i, c_i \in \{x_1, \bar{x_1}, \ldots, x_l, \bar{x_l}\}$.

Represent each x_i with a diamond-shaped structure, that can be traversed in either of the two ways, corresponding to two truth settings. Each diamond structure contains a horizontal row, that contains $3k+1$ vertices (in addition to the two nodes on the end).

eg. $k = 2$ case:





We claim ϕ is satisfiable if and only if G has an s-t Hamilton path. *Q.E.D.*

- $HAMPATH = \{\langle G, s, t\rangle | Undirected\ graph\ G\ has\ a\ Hamilton\ path\ from\ s\ to\ t\}$

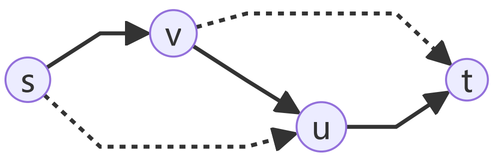
Thm 5.36 $HAMPATH$ **is NP-complete.**

Proof: $HAMPATH \in NP$

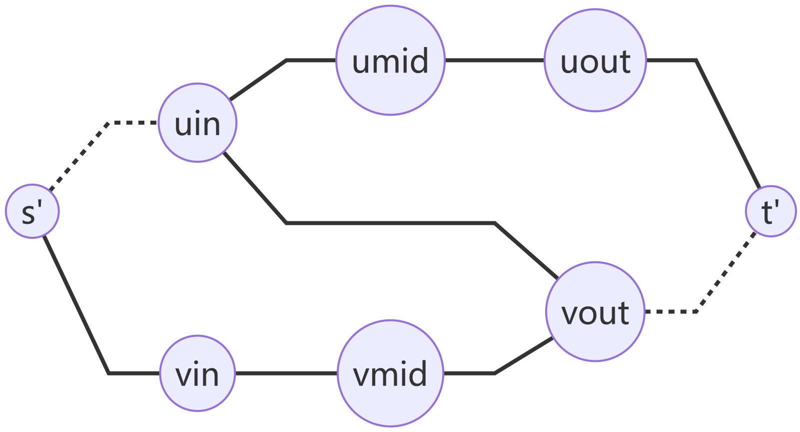
Let us prove $dHAMPATH \leq_P HAMPATH$.

Given a directed graph G, and $s, t \in V(G)$, construct an undirected graph G', $s', t' \in V(G')$ s.t. G has an s-t Hamilton path if and only if G' has an s'-t' Hamilton path.

eg. Consider G:



Construct G':



Each vertex $u \in V(G) \setminus \{s, t\}$ is replaced by a triple u^{in}, u^{mid}, u^{out} in G'. Replace s by s', t by t'. Connect u^{mid} with u^{in} and u^{out}.

If $(u, v) \in E(G)$, then connect u^{out} with v^{in}.

Claim: G has an s-t Hamilton path if and only if G' has an s'-t' Hamilton path. *Q.E.D.*

Time Hierarchy Theorem

- Question: Is $DTIME(n) \subsetneq DTIME(n^2)$? Is $P \subsetneq EXP$?