Turing Machine

eg. Decide $L=\{0^{2^n}|n\geq 0\}$, where $\Sigma=\{0\}$

- Idea:
    1. If there is a single 0, accept
    2. Sweep from left to right, crossing off every other 0
    3. If the number of 0's is odd, reject
    4. Return the head to the left-hand end
    5. Goto step 1
- $\Sigma=\{0\},\Gamma=\{0,\sqcup,\triangleright,x\}$

Def 3.2 **Let $L\subseteq\{0,1\}^*$, let M be a TM. Say M decides L in time $T(n)$ if for every $x\in\{0,1\}^*$,**

1. M halts in $T(n)$ steps
2. If $x\in L$, then M accepts x
3. If $x\notin L$, then M rejects x

Def 3.3 **Let $L\subseteq\{0,1\}^*$. Call L (Turing) decidable if there is a TM that decides it.**

- Note that, on an input x, a TM may *accept, reject or loop forever*.
- In *Def 3.2*, the machine should never loop forever.

Def 3.4 **Let M be a TM. The set of strings that M accepts is the language recognized by M, denoted by $L(M)$.**

Def 3.5 **Let $L\subseteq\{0,1\}^*$. Call L (Turing) recognizable if there is some TM that recognizes it.**

- Obviously, every (Turing) decidable language is (Turing) recognizable.
- The converse is not true. eg. $L=\{\langle M,x\rangle|M\ halts\ on\ x\}$

Def 3.6 **Let $f:\{0,1\}^*\to\{0,1\}^*\cup\{undefined\}$. Say TM M computes f if for every $x\in\{0,1\}^*$ with $f(x)\neq undefined$, M halts with $f(x)$ on its tape in at most $T(|x|)$ steps.**

> An algorithm is a Turing Machine.
>
> ——Alan Turing

- Despite its simplicity, TM is capable of implementing any computer algorithm.

Variants of Turing Machines

Lem 3.7 **If language $L\subseteq\{0,1\}^*$ is decidable in time $T(n)$ by a TM on alphabet Γ, then it is decidable in time $O(\log|\Gamma|\cdot T(n))=O_\Gamma(T(n))$ by a TM on alphabet $\Gamma=\{0,1,\sqcup,\triangleright\}$.**

Proof: Encode any symbol in Γ using $k = \lceil \log_2 |\Gamma| \rceil = O(\log_2 |\Gamma|)$ bits. To simulate one step of M, the new TM M' will

1. Use k steps to read a symbol $a \in \Gamma$
2. Transit to next step q', and get the new symbol b (to overwrite a)
3. Overwrite a by b
4. Go left or right for k steps or stay

Def 3.8 **A k-tape($O(1)$-tape) TM M is a 7-tuple $(Q, \Sigma, \Gamma, \delta, q_0, q_{accept}, q_{reject})$, where**

$$\delta : Q \times \Gamma^k \to Q \times \Gamma^k \times \{L, R, S\}^k$$

Usually,the first tape is the *input tape*, the last tape is the *output tape*, and the rest are *work tapes*.

Lem 3.9 **Let $L \subseteq \{0,1\}^*$. If L is decidable by a k-tape TM in time $T(n)$, then L is decidable by a single-tape TM in time $O(k \cdot T(n)^2)$.**