Last class

1. **Nondeterministic TM**

$$\delta : Q \times \Gamma \to P(Q \times \Gamma \times \{L, R, S\})$$

2. $NP = \cup_{c \geq 1} NTIME(n^c)$

- An equivalent definition of NP
    - $L \in NP \Leftrightarrow \exists$ *polynomial* $p : \mathbb{N} \to \mathbb{N}$ and $\exists$ *poly-time TM* M s.t.
        1. $x \in L$ if $\exists w \in \{0,1\}^{P(|x|)}$ and $M(x, w) = 1$
        2. $x \notin L$ if $\forall w \in \{0,1\}^{P(|x|)}$ and $M(x, w) = 0$

3. **Karp Reduction** $L \leq_P K$: $\exists$ *poly-time TM* M s.t. $\forall x \in \{0,1\}^*, x \in L \Leftrightarrow M(x) \in K$

- If $K \in P$, then $L \in P$.
- If $L \notin P$, then $K \notin P$.

4. **NP-hard**

- $L \in$**NP-hard** $\Leftrightarrow \forall K \in NP, K \leq_p L$

5. **NP-complete**

Cook-Levin Theorem

Thm 5.29 **SAT is NP-complete.**

SAT (Boolean Satisfiability Problem)

- Variable: $x, y, z, \ldots$ can take TRUE or FALSE.
- Literal: A variable or its negation. eg. $x, \neg x, y, \neg y, \ldots$
- Clause: OR(Disjunction) of one or more literals. eg. $\neg x \vee y, \neg y \vee z, \ldots$
- Formula: AND(Conjunction) of one or more clauses.

$SAT = \{\langle \phi \rangle | \phi \ is \ satisfiable\}$

eg. $\phi = (\neg x \vee y) \wedge (x \vee \neg y) \wedge (x \vee y) \wedge (\neg x \vee \neg y)$ is not satisfiable.

Proof: To prove SAT is NP-complete, we need to show $\forall L \in NP, L \leq_P SAT$.

So, it suffices to show, for any poly-time NTM M, and for any input $x \in \{0,1\}^*$, we can algorithmically construct a formula $\phi_{M,x}$ (in polynomial time) s.t. M accepts x ($x \in L$) if and only if $\phi_{M,x} \in SAT$.

$$\overset{x}{\to} efficent \ TM \overset{\phi_{M,x}}{\to} SAT \ solver \to Yes/No$$

- Snapshot

0	1	1	0	1	0	0	...
			$\uparrow Q_1$				

0	1	1	1	1	0	0	...
				$\uparrow Q_2$			

0	1	1	1	1	0	0	...
					$\uparrow Q_3$		

- $T_{i,j,k} : i \in [1, T(n)], j \in \Gamma, k \in [1, T(n)]$: Cell i contains symbol j at step k.

  eg. in the snapshots above, $T_{1,0,3} = true, T_{1,1,3} = false, T_{1,\sqcup,3} = false$
- $H_{i,k} : i, k \in [1, T(n)]$: Head is on cell i at step k.

  eg. in the snapshots above, $H_{4,3} = true, H_{1,3} = H_{2,3} = H_{3,3} = H_{5,3} = \ldots = false$
- $Q_{q,k}$: The head is on state q at step k.
- In total, we have $T(n)^2 \times |\Gamma| + T(n)^2 + |Q| \times T(n) = O_M(T(n)^2)$ variables.

Initialization

x_1	x_2	x_3	...	x_n	$\sqcup$	$\sqcup$	...
$\uparrow$							

Input $x \in \{0, 1\}^*$ is of length n.

$T_{i,x_i,0} = true$ for $i = 1, 2, \ldots, n$, $T_{i,\sqcup,0} = true$ for $i = n + 1, n + 2, \ldots, T(n)$. Otherwise $T_{i,j,0} = false$

$H_{1,0} = true$, $H_{i,0} = false$ for $i = 2, 3, \ldots, T(n)$

At step 0, the head is on the leftmost position.

The initial state is q_0. i.e. $Q_{q_0,0} = true$, $Q_{q,0} = false$ for every $q \in Q \setminus \{q_0\}$.

Restrictions

1. At most one symbol per cell.

$$(\forall i \in [1, T(n)], \forall k \in [1, T(n)], \forall j \neq j' \in \Gamma)(\neg T_{i,j,k} \vee \neg T_{i,j',k})$$

2. At least one symbol per cell.

$$(\forall i, k \in [1, T(n)])(\vee_{j \in \Gamma} T_{i,j,k})$$

3. At most one state at a time.

$$(\forall k \in [1, T(n)], \forall q \neq q' \in Q)(\neg Q_{q,k} \vee \neg Q_{q',k})$$

4. At least one state at a time.

$$(\forall k \in [1, T(n)])(\vee_{q \in Q} Q_{q,k})$$

5. At most one head position at a time.

$$(\forall k \in [1, T(n)], \forall i \neq i' \in [1, T(n)])(\neg H_{i,k} \vee \neg H_{i',k})$$

6. At least one head position at a time.

$$(\forall k \in [1, T(n)])(\vee_{i \in [1,T(n)]} H_{i,k})$$

Transition Rule

- $A \to B$ is equivalent to $\neg A \vee B$.

1. Cell remains unchanged unless written.

$$(\forall i \in [1, T(n)], \forall k \in [1, T(n)-1], \forall j \neq j' \in \Gamma)(T_{i,j,k} \wedge T_{i,j',k+1} \to H_{i,k})$$

2. Transition function δ.

$$(\forall i \in [1, T(n)], \forall k \in [1, T(n)-1], \forall q \in Q, \forall j \in \Gamma)$$
$$(H_{i,k} \wedge Q_{q,k} \wedge T_{i,j,k} \to \vee_{(q',j',d) \in \delta(q,j), d \in \{-1,0,1\}} (H_{i+d,k+1} \wedge Q_{q',k+1} \wedge T_{i,j',k+1}))$$

Halt in an accept state

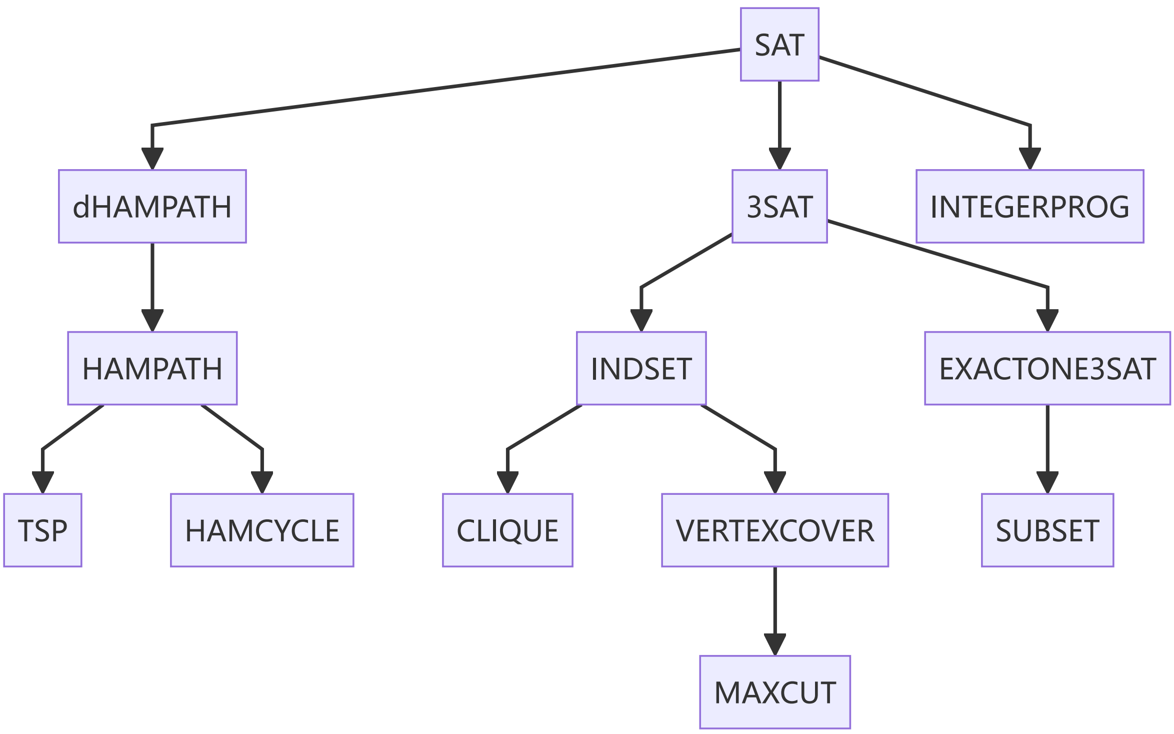
Without loss of generality , there is a self loop in the accept state for $\forall x \in \Gamma$.

$$(\vee_{k \in [1,T(n)]} Q_{q_{accept},k})$$

Take the AND of all the above clauses. The number of clauses is $n^{O(1)}$.

Goal

NTM M accepts $x \Leftrightarrow \phi_{M,x} \in SAT$.



- In **3SAT**, each clause has (at most) 3 literals.
- eg. $\phi = (x \vee y \vee \neg z) \wedge (\neg x \vee \neg y \vee z) \wedge (x \vee \neg z)$

Thm 5.30 $SAT \leq_P 3SAT$.

Proof:

- A **conjunctive normal form (CNF)** is the AND of several clauses. eg. $(x \vee y \vee \neg z) \wedge (\neg x \vee \neg y \vee z) \wedge (x \vee \neg z)$.
- A **3CNF** is a **CNF** where each clause has 3 literals.

Let ϕ be a CNF with m clauses.

If each clause has ≤ 3 literals, then we are done.

If some clause has ≥ 3 literals, we replace it by an equivalent 3CNF.

Let $C = l_1 \vee l_2 \vee \ldots \vee l_k, k \geq 4$, where $l_i \in \{x_1, \bar{x_1}, \ldots, x_n, \bar{x_n}\}$.

Replace C by

$$C' = (l_1 \vee l_2 \vee z_1) \wedge (\bar{z_1} \vee l_3 \vee z_2) \wedge (\bar{z_2} \vee l_4 \vee z_3) \wedge \ldots \wedge (\bar{z_{k-3}} \vee l_{k-1} \vee l_k)$$

($z_1, z_2, \ldots, z_{k-2}$ are new variables)

Claim C is satisfiable if and only if C' is satisfiable. *Q.E.D.*

eg. $x_1 \vee x_2 \vee \bar{x_3} \vee x_5$ is satisfiable if and only if $(x_1 \vee x_2 \vee z) \wedge (\bar{z} \vee \bar{x_3} \vee x_5)$.

Thm 5.31 $SAT \leq_P INTEGERPROG$.

- **INTEGERPROG**: $x_1, x_2, \ldots, x_n \in \{0, 1\}, a_{i,1}x_1 + a_{i,2}x_2 + \ldots + a_{i,n}x_n \geq a_{i,0}$

Proof: Let ϕ be a CNF on n variables $x_1, \ldots, x_n$ with m clauses.

For each variable x_i, introduce a variable y_i in $INTEGERPROG$. Add constraints $o \leq y_i \leq 1$ for all i.

For each clause in ϕ, sums up all literals (replacing x_i by y_i, and replacing $\bar{x}_i$ by $1 - y_i$), and asserts the sum is at least 1.

eg. $x_1 \vee \bar{x_2} \vee \bar{x_3} \rightarrow y_1 + (1 - y_2) + (1 - y_3) \geq 1$. *Q.E.D.*

- $INTEGERPROG \in NPC$
- $01PROG \leq_P INTEGERPROG$
- $01PROG \in NPC$

- $INDSET = \{\langle G, k\rangle | G \; has \; an \; independent \; set \; of \; size \; k\}$.