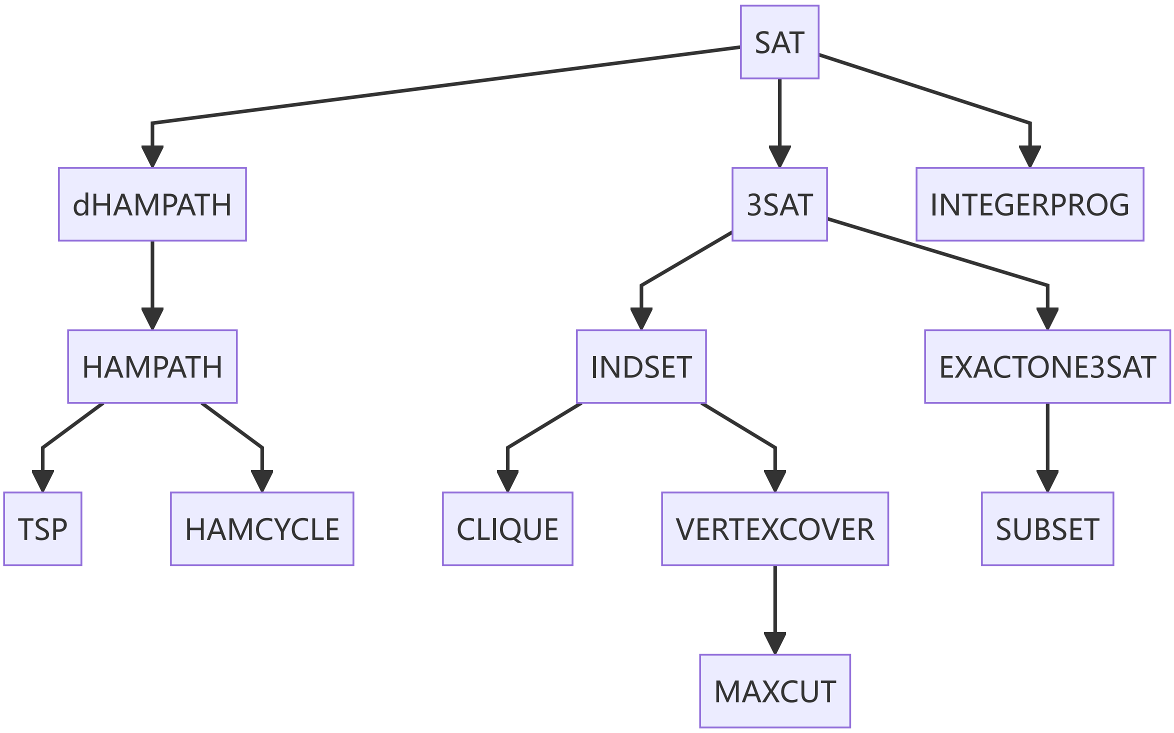
Last class

- NP-complete problems: $NP\text{-}complete = NP \cap NP\text{-}hard$
  - eg. SAT, $3SAT$, $01PROG$, $INTEGERPROG$, $CLIQUE$, $VERTEXCOVER$, $INDSET$, $dHAMPATH$, $HAMPATH$, $MAXCUT$...



Time Hierarchy Theorem

- Question: Is $DTIME(n) \subsetneqq DTIME(n^2)$? Is $P \subsetneqq EXP$?

Def 5.37 **(Time-constructible)** Function $T : \mathbb{N} \to \mathbb{N}$ is time-constructible if $T(n) \geq n$ and there is a TM M that computes the function $1^n \to T(n)$ (in binary) in time $O(T(n))$.

eg. n, n^2, n^3, $n\lfloor \log_2 n \rfloor$, 2^n, 2^{n^2}... are time-constructible.

Def 5.38 **(Time Hierarchy Theorem)** If f, g are time-constructible functions satisfying $f(n) \log f(n) = o(g(n))$, then

$$DTIME(f(n)) \subsetneqq DTIME(g(n))$$

eg. $DTIME(n) \subsetneqq DTIME(n^2) \subsetneqq DTIME(n^3)$, $P \subsetneqq EXP$, $DTIME(n^2) \subsetneqq DTIME(n^{2.1})$, $DTIME(n^2) \subsetneqq DTIME(n^2 \log^2 n)$ (since $n^2 \log(n^2) = o(n^2 \log^2 n)$)

Proof: Diagonalization

Construct L s.t. $L \in DTIME(g(n))$ and $L \notin DTIME(f(n))$.

\	M_0	M_1	M_2	M_3	M_4	...
$\epsilon = 0$	R	A	R			
$0 = 1$	R	A	L			

\	M_0	M_1	M_2	M_3	M_4	...
$1 = 2$	R	R				
$00 = 3$	R					
$01 = 4$	R					
...						

(A: accept R: reject L: loop forever)

Simulate $\frac{g(n)}{\log g(n)}$ steps.

Flip the output:

$$\begin{cases} A \to R \\ R \to A \\ L \to R \end{cases}$$

Claim: $\frac{g(n)}{\log g(n)} = \omega f(n)$

Proof of the claim:

- Case 1: $g(n) \geq f(n)^2$

$$\frac{g(n)}{\log g(n)} \geq \frac{g(n)}{g(n)^{\frac{1}{3}}} = g(n)^{\frac{2}{3}} \geq f(n)^{\frac{4}{3}} = \omega(f(n))$$

- Case 2: $g(n) < f(n)^2$

$$\frac{g(n)}{\log g(n)} \geq \frac{g(n)}{2 \log f(n)} = \omega(f(n))$$

Since $f(n), g(n)$ are time-constructible, we can construct $\lfloor \frac{g(n)}{\lceil \log_2 g(n) \rceil} \rfloor$ in time $O(g(n))$.

Construct the following TM:

On input $x \in \{0, 1\}^n$, simulate M_x on input x for $t(n)$ steps, and flip the output, i.e.

(1) If M_x accepts x in $t(n)$ steps, reject.

(2) If M_x rejects x in $t(n)$ steps, accept.

(3) If M_x does not halt in $t(n)$ steps, reject.

Let $L = L(M)$.

The simulation takes $O(t(n) \cdot \log t(n)) = O(\frac{g(n)}{\log g(n)} \log \frac{g(n)}{\log g(n)}) = O(g(n))$.

So, $L \in DTIME(g(n))$.

Claim: $L \notin DTIME(f(n))$

Assume for contradiction that L is decidable by some TM M_α in time $O(f(n))$.

On input α, M_α accepts or rejects α. By the definition of L, if M_α accepts α, then $\alpha \notin L$, contradiction. If M_α rejects α, then $\alpha \in L$, contradiction. *Q.E.D.*

NP-immediate Problems

Thm 5.39 **(Ladner 1975)** Suppose $P \neq NP$. There exists $L \in NP \setminus P$ that is not NP*-complete*.

Proof idea: **Padding**

$$SAT_H = \{\phi 01^{mH(m)} | \phi \in SAT, m = |\phi|\}$$

Open problem: prove *Graph Isomorphism*, *Factoring* are NP*-immediate*, assuming $P \neq NP$.

- **Computational Complexity Theory** focuses on classifying problems according to their resource usage, and relating these classes to each other.
- Resources: time, space, randomness, parallism, ...

Def 5.40 **(Space complexity)** TM M runs in space $S(n)$ if for every input $x \in \{0,1\}^*$, it uses at most $S(|x|)$ cells on its work tapes (excluding the read-only tapes)

Def 5.41 ($SPACE(S(n))$) Let $S : \mathbb{N} \to \mathbb{N}$. $L \subseteq \{0,1\}^*$ is in $SPACE(S(n))$ if there exists a TM that decides L in space $O(S(n))$.

eg. $SAT = \{\langle\phi\rangle | \phi \text{ } has \text{ } a \text{ } satisfiable \text{ } assignment\}$. $SAT \in NP$*-complete.*

$SAT \in DTIME(2^n), SAT \in SPACE(n)$

Construct the following TM M:

On input ϕ, where ϕ is a boolean formula. For every assignment $\rho : \{x_1, \ldots, x_n\} \to \{0,1\}$, check if $f|_\rho$ is true. Accept if there exists a ρ s.t. $f|_\rho$ is true. Otherwise, rejects.

M decides SAT in time $O(2^n)$ in space $O(n)$.

Def 5.42 Let $S : \mathbb{N} \to \mathbb{N}$. $L \subseteq \{0,1\}^*$ is in $NSPACE(S(n))$ if there exists an NTM that runs in space $O(S(n))$ and decides L.

Thm 5.43

$$DTIME(S(n)) \subseteq SPACE(S(n)) \subseteq NSPACE(S(n)) \subseteq DTIME(2^{O(S(n))})$$

Proof:

(1) $DTIME(S(n)) \subseteq SPACE(S(n))$

Let $L \in DTIME(S(n))$. Then L is decidable by a TM M that runs in time $O(S(n))$. So, M uses at most $O(S(n))$ space. Thus, $L \in SPACE(S(n))$.

(2) $SPACE(S(n)) \subseteq NSPACE(S(n))$

A DTM is also an NTM.

(3) $NSPACE(S(n)) \subseteq DTIME(2^{O(S(n))})$ Idea: Configuration graph

Input tape (read-only) Work tapes: 1~m

- Bits to encode a configuration:

$$\underbrace{\lceil \log_2 n \rceil}_{\text{(head position on the input tape)}} + \underbrace{m \lceil \log_2 O(S(n)) \rceil}_{\text{(heads on m work tapes)}} + \underbrace{O(S(n)) \cdot \lceil \log_2 |\Gamma| \rceil}_{\text{(symbols on all work tapes)}} + \underbrace{\lceil \log_2 |Q| \rceil}_{\text{(states)}}$$
$$= O_M(S(n)) \text{ (Assuming } S(n) \geq \log_2 n)$$

Let $G_{M,x}$ be the configuration graph for NTM M and input x. ($\delta : Q \times \Gamma \to P(Q \times \Gamma \times \{L, R, S\}^m)$)

$|V(G_{M,x})| \leq 2^{O_M(S(n))}$, because each vertex can be encoded by $O_M(S(n))$ bits.

$|E(G_{M,x})| \leq |V(G_{M,x})| \times \underset{(=O_M(1))}{maxdeg} = 2^{O_M(S(n))} \times O_M(1) = 2^{O_M(S(n))}$

$G_{M,x}$ has a start configuration c_{start} and many accept configurations.

Use BFS to check if c_{start} is connected to an accept configuration.

The time complexity of BFS is $O(|V| + |E|) = 2^{O_M(S(n))}$. *Q.E.D.*

Def 5.44 **(Space-constructible functions)** Function $S : \mathbb{N} \to \mathbb{N}$ is space-constructible if there exists a TM s.t. on input 1^n, output the binary representation of $S(n)$ in space $O(S(n))$.

Thm 5.45 **(Space Hierarchy Theorem)** Let f, g be space-constructible functions satisfying $f(n) = o(g(n))$. Then $SPACE(f(n)) \subsetneq SPACE(g(n))$.

Proof idea: Diagonalization