Gödel's Incompleteness Theorem

Thm 4.17 **All consistent axiomatic formulation of number theory which include Peano arithmetic include undecidable proposition.**

Peano Arithmetic

1. Constant 0
2. Successor Operator S (s.t. $1 = S(0), 2 = S(S(0)), \ldots$)
3. Addition($+$), Multiplication($\cdot$)
4. Logical Conjunction: $\wedge, \vee, \neg$
5. Quantifier: $\forall, \exists$
6. Binary: $<, =$
7. Parenthesis: $()$
8. Variables: $x, x^*, x^{**} \ldots$

- Examples:
  1. "x divides y"

  $$DIVIDES(x, y) : (\exists z)(y = x \cdot z)$$

  2. "y is a prime number"

  $$PRIME(y) : (\forall x)(x = S(0) \vee x = y \vee \neg DIVIDES(x, y))$$

  3. **Goldbach's Conjecture**

  $$(\forall x)(x \geq S(S(0)) \Rightarrow (\exists y)(\exists z)(x + x = y + z \ \wedge PRIME(y) \ \wedge PRIME(z)))$$

     - $A \Rightarrow B$ is equivalent to $\neg A \vee B$

Thm 4.17's Proof idea: For any $\alpha, x \in \{0, 1\}^*$, construct a first-order Peano arithmetic formula $\phi_{\alpha,x}$ s.t. $\phi_{\alpha,x}$ is true if and only if M_α halts on x.

Assume for contradiction that the proof system is complete, i.e. every formula can either be proved or disproved. We construct a TM that enumerates all proofs π up to length k, and verify if π is a valid proof of $\phi_{\alpha,x}$ or $\neg\phi_{\alpha,x}$.

Since the proof system is complete, the TM M must halt, which implies that L_{halt} is decidable. Contradiction. *Q.E.D.*

Diophantine Equation

- **A Diophantine Equation is a polynomial equation with integer coefficients and a finite number of unknowns.**
- For example, $x^2 + y^2 + 1 = 0$ has no solution. $3x^2 - 2xy - y^2z - 7 = 0$ has a solution: $x = 1, y = 2, z = -2$.

Hilbert 10^{th} Problem

L_{Dio}

- $L_{Dio} = \{\langle p(x_1, \ldots, x_n)\rangle | Diophantine\ equation\ p(x_1, \ldots, x_n) = 0\ has\ a\ solution\}$

Thm 4.18(MRDP Thm.) L_{Dio} **is undecidable.**

Proof idea: For any $\alpha, w \in \{0,1\}^*$, construct a Diophantine equation $P_{\alpha,w}$ s.t. M_α halts on w if and only if $P_{\alpha,w}$ has an integral solution. Therefore, if L_{Dio} is decidable, L_{halt} is decidable. Contradiction. *Q.E.D.*

Complexity Theory

Def 5.1 Let $T : \mathbb{N} \to \mathbb{N}$. Let $DTIME(T(n))$ be the class of languages that can be decided by a TM in time $O(T(n))$.

$$P = \cup_{c\geq 1} DTIME(n^c)$$
$$EXP = \cup_{c\geq 1} DTIME(2^{n^c})$$