# Gödel's Incompleteness Theorem

*Thm 4.17* **All consistent axiomatic formulation of number theory which include Peano arithmetic include undecidable proposition.**

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## Peano Arithmetic

1. Constant $0$
2. Successor Operator S (s.t. $1 = S(0), 2 = S(S(0)), \ldots$)
3. Addition($+$), Multiplication($\cdot$)
4. Logical Conjunction: $\wedge, \vee, \neg$
5. Quantifier: $\forall, \exists$
6. Binary: $<, =$
7. Parenthesis: $()$
8. Variables: $x, x^*, x^{**}\ldots$

- Examples:
  1. "$x$ divides $y$"

  $$DIVIDES(x, y) : (\exists z)(y = x \cdot z)$$

  2. "$y$ is a prime number"

  $$PRIME(y) : (\forall x)(x = S(0) \vee x = y \vee \neg DIVIDES(x, y))$$

  3. **Goldbach's Conjecture**

  $$(\forall x)(x \geq S(S(0)) \Rightarrow (\exists y)(\exists z)(x + x = y + z \ \wedge PRIME(y) \ \wedge PRIME(z)))$$

     - $A \Rightarrow B$ is equivalent to $\neg A \vee B$

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*Thm 4.17's Proof idea:* For any $\alpha, x \in \{0, 1\}^*$, construct a first-order Peano arithmetic formula $\phi_{\alpha,x}$ s.t. $\phi_{\alpha,x}$ is true if and only if $M_\alpha$ halts on $x$.

Assume for contradiction that the proof system is complete, i.e. every formula can either be proved or disproved. We construct a TM that enumerates all proofs $\pi$ up to length $k$, and verify if $\pi$ is a valid proof of $\phi_{\alpha,x}$ or $\neg\phi_{\alpha,x}$.

Since the proof system is complete, the TM $M$ must halt, which implies that $L_{halt}$ is decidable. Contradiction. *Q.E.D.*

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# Diophantine Equation

- **A Diophantine Equation is a polynomial equation with integer coefficients and a finite number of unknowns.**
- For example, $x^2 + y^2 + 1 = 0$ has no solution. $3x^2 - 2xy - y^2z - 7 = 0$ has a solution: $x = 1, y = 2, z = -2$.

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# Hilbert $10^{th}$ Problem

## $L_{Dio}$

- $L_{Dio} = \{\langle p(x_1, \ldots, x_n)\rangle | Diophantine\ equation\ p(x_1, \ldots, x_n) = 0\ has\ a\ solution\}$

*Thm 4.18(MRDP Thm.)* $L_{Dio}$ **is undecidable.**

*Proof idea:* For any $\alpha, w \in \{0,1\}^*$, construct a Diophantine equation $P_{\alpha,w}$ s.t. $M_\alpha$ halts on $w$ if and only if $P_{\alpha,w}$ has an integral solution. Therefore, if $L_{Dio}$ is decidable, $L_{halt}$ is decidable. Contradiction. *Q.E.D.*

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# Complexity Theory

*Def 5.1* **Let $T : \mathbb{N} \to \mathbb{N}$. Let $DTIME(T(n))$ be the class of languages that can be decided by a TM in time $O(T(n))$.**

$$P = \cup_{c\geq 1} DTIME(n^c)$$
$$EXP = \cup_{c\geq 1} DTIME(2^{n^c})$$