Lem 3.9 **Let $L \subseteq \Sigma^*$. If L is decidable by a k-tape TM in time $T(n)$, then it is decidable by a single-tape TM in time $O(k \cdot T(n)^2)$.**

Proof: Use location $i-1, k+i-1, 2k+i-1, \ldots$ to store the content of the i^{th} tape, where $i = 1, 2, \ldots, k$. For every $a \in \Gamma$, introduce $a, \hat{a} \in \Gamma'$, where $\hat{a}$ denotes the location of the head.

To simulate one step of M, the single-tape TM M' will

1. Sweep the tape from left to right to read k symbols marked by ˆ.
2. Apply M's transition function δ to determine the next state.
3. Sweep back from right to left to update k symbols, if needed, and move ˆ, if needed.

In total, 1.2.3. take $O(k \cdot T(n))$ time.

- **A Bidirectional-tape TM is a TM where tape is infinite in both directions.**

Lem 3.10 **Let $L \subseteq \Sigma^*$. If L is decidable by a Bidirectional-tape TM in time $T(n)$, then it is decidable by a single-tape TM in time $O(T(n))$.**

Proof: Index the Bidirectional tape by $\mathbb{Z}$:

(Assuming that the Bidirectional tape was indexed by: $\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$)

$$Mapping : \{^{i \to 2i}_{-i \to 2i-1}$$

In this way, we create a new (single) tape indexed by: $0, -1, 1, -2, 2, \ldots$

For every step of M, M' will:

1. Read the symbol
2. Transit to the next state
3. Update the symbol
4. Move left or right for 2 steps if needed

It takes $O(1)$ to simulate 1 step. In total, the running time is $O(T(n))$. *Q.E.D.*

RAM TM

Def 3.11 **A RAM TM is a TM with random access memory.**

1. M has an infinite memory tape A indexed by $\mathbb{N}$.
2. One of M's tape is the address tape.
3. Γ contains 2 special symbols R (Read) and W (Write).
4. Q has some special states $Q_{access} \subseteq Q$.

   Whenever M gets into a state $q \in Q_{access}$:

   1. If the address tape contains iR, the value $A[i]$ is written to the cell next to R.
   2. If the address tape contains $iW\sigma$, then set $A[i]$ to symbol σ.

Assume the RAM TM M has k work tapes, and an address tape. Then

$$M = (Q, \Sigma, \Gamma, \delta, q_0, q_{accept}, q_{reject}, Q_{access})$$
$$\delta : Q \times \Gamma^{k+1} \to Q \times \Gamma^{k+1} \times \{L, R, S\}^{k+1}$$

Lem 3.12 **Let $L \subseteq \{0,1\}^*$. If L is decidable by a RAM TM in time $T(n)$, it is decidable by a Multitape TM in time $O(T(n)^3)$.**

- Moreover, if the address length is $O(1)$, then L is decidable by a Multitape TM in time $O(T(n)^2)$.

Proof: Use an extra work tape as memory.

- **Ignoring polynomial factors, all TM variants are equivalent.**

$$\begin{matrix} C++ \\ T(n) \end{matrix} \to \begin{matrix} Assembly \\ O(T(n)) \end{matrix} \to \begin{matrix} RAM\ TM \\ O(T(n)) \end{matrix} \to \begin{matrix} Multitape \\ O(T(n)^3)/O(T(n)^2)^* \end{matrix} \to \begin{matrix} Single-tape \\ O(T(n)^6)/O(T(n)^4)^* \end{matrix}$$