Lem 4.2 **Almost all languages are undecidable.**

Proof: $\#languages = 2^{\aleph_0} = \aleph_1, \#\{L \subseteq \{0,1\}^*\} = TMs = \aleph_0$. *Q.E.D.*

L_{flip}

- $L_{flip} = \{\alpha | M_\alpha \ does \ not \ accept \ \alpha\}$

Lem 4.3 L_{flip} **is undecidable.**

Proof: Assume for contradiction that L_{flip} is decided by a TM M_β, which implies that $L(M_\beta) = L_{flip}$.

- Case 1: $\beta \in L_{flip}$. By definition, M_β does not accept β, i.e. M_β rejects β. So, $\beta \notin L(M_\beta) = L_{flip}$. Contradiction.
- Case 2: $\beta \notin L_{flip}$. By definition, M_β accepts β. So, $\beta \in L(M_\beta) = L_{flip}$. Contradiction. *Q.E.D.*

Turing Halting Problem

- $L_{halt} \overset{def}{=} \{(\alpha, x) | M_\alpha \ halts \ on \ input \ x\}$
- **Fermat's Last Theorem:** $(\forall m \geq 3)(\forall a, b, c \geq 1)(a^m + b^m \neq c^m)$
  - M_α for *FLT*:
    - $T = 2$
    - while true:
      - $T = T + 1$
      - for $d = 3$ to T
        - for $a, b, c \in \{1, 2, \ldots, T\}$
          - if $a^d + b^d = c^d$, exit
  - *FLT* $\Leftrightarrow (M_\alpha, \epsilon) \notin L_{halt}$.

Reduction

Def 4.4 **Let $L_1, L_2 \subseteq \{0,1\}^*$. Write $L_1 \leq L_2$ if there is a reduction from L_1 to L_2. That is, there exists a TM $M : \{0,1\}^* \to \{0,1\}^*$ (On any input x, M always halts and outputs a string $M(x)$) s.t.:**

1. $(\forall x \in L_1)(M(x) \in L_2)$
2. $(\forall x \notin L_1)(M(x) \notin L_2)$

- Let $L_1 \leq L_2$. If L_2 is decidable, then L_1 is decidable. Contrapositively, if L_1 is undecidable, then L_2 is undecidable.

Thm 4.5 L_{halt} **is undecidable.**

Proof: We will prove $L_{flip} \leq L_{halt}$.

Assume that L_{halt} is decidable by a TM M_{halt}, we will prove L_{flip} is decidable, which would be a contradiction.

Create a TM M_{flip} as follows:

- Run M_{halt} on input (α, α).
    1. If M_{halt} rejects (α, α), let M_{flip} accept α.
    2. If M_{halt} accepts (α, α), simulate M_α on input α (using a UTM), and flip the output.

It's easy to verify M_{flip} decides L_{flip}. Contradiction. *Q.E.D.*

L_{accept}

- $L_{accept} = \{(\alpha, x) | M_\alpha \ accepts \ x\}$

Lem 4.6 L_{accept} **is undecidable.**

Proof: We will prove $L_{halt} \leq L_{accept}$. Assuming for contradiction that L_{accept} is decidable, i.e. there exists a TM M_{accept} that decides L_{accept}, we construct a TM M_{halt} that decides L_{halt} as follows:

1. On input (α, x), create a new TM M_β, which simulates M_α on input x, and always accepts whenever M_α halts. (If M_α loops forever, M_β loops forever as well)
2. Run M_{accept} on input (β, x), and forward its output. Clearly, M_{halt} decides L_{halt}. Contradiction. *Q.E.D.*

L_{empty}

- $L_{empty} = \{\langle M \rangle | M \ does \ not \ accept \ any \ input, \ i.e. \ L(M) = \phi\}$

Lem 4.7 L_{empty} **is undecidable.**

Proof: We will prove $L_{halt} \leq L_{empty}$. Assume for contradiction that L_{empty} can be decided by a TM M_{empty}. We construct a TM M_{halt} as follows.

1. On input (α, x), we construct a new TM M_β, whose input is $y \in \{0, 1\}^*$ as follows:

    a. Simulate M_α on input x

    b. If step a halts, always accepy y

    Clearly, $L(M_\beta) = \phi$ if M_α does not halt on x. Otherwise, $L(M_\beta) = \{0, 1\}^*$.
2. Run M_{empty} on input β and flip the output. We can verify that M_{halt} decides L_{halt}.

Contradiction. *Q.E.D.*

$L_{regular}$

- $L_{regular} = \{\langle M \rangle | M \ is \ a \ TM \ s.t. \ L(M) \ is \ regular\}$

Thm 4.8 $L_{regular}$ **is undecidable.**

Proof: Assume for contradiction that $L_{regular}$ is decidable, i.e. $\exists$ a TM $M_{regular}$ that decides $L_{regular}$. We will prove L_{accept} is decidable.

On input (α, x), construct a TM M_{accept} as follows:

1. Construct a TM M_β, where $\beta = \beta(\alpha, x)$, and the input of M_β is denoted by y.

    a. If $y \in \{0^n 1^n | n \geq 0\}$, accept

b. Otherwise, simulate M_α on x, and accept if and only if M_α accepts x

2. Run $M_{regular}$ on β, and forward its output.
   - Case 1: $(\alpha, x) \notin L_{accept}$. i.e. M_α does not accept x. So, $L(M_\beta) = \{0^n 1^n | n \geq 0\}$. Thus, $M_{regular}$ rejects β, which implies that M_{halt} rejects β.
   - Case 2: $(\alpha, x) \in L_{accept}$. So $L(M_\beta) = \{0, 1\}^*$. As such, $M_{regular}$ accepts β, and so does M_{halt}.

Contradiction. *Q.E.D.*

L_{equal}

- $L_{equal} = \{(\langle M_1 \rangle, \langle M_2 \rangle) | M_1 \ and \ M_2 \ are \ TMs \ s.t. \ L(M_1) = L(M_2)\}$

Lem 4.9 L_{equal} **is undecidable.**

Proof: Assume for contradiction that L_{equal} is decidable. i.e. L_{equal} is decided by a TM M_{equal}. We will prove L_{empty} is decidable.

On input $\langle M \rangle$, construct a TM M_{empty} as follows:

1. Run M_{equal} on input $(\langle M \rangle, \langle M_0 \rangle)$, where M_0 rejects immediately.
2. Forward the above output. *Q.E.D.*