

Übungszettel 12

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1 Aufgabe 1.

Sei

$$A = \begin{pmatrix} 6 & 1 & 2 & 3 & 1 & 6 \\ 3 & 6 & 5 & 0 & 4 & 5 \\ 1 & 0 & 0 & 3 & 5 & 4 \\ 2 & 4 & 5 & 2 & 0 & 0 \end{pmatrix} \in M(4 \times 6, \mathbb{F}_7) \quad (1)$$

$$b = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \in \mathbb{F}_7^6 \quad (2)$$

Löse: $Ax = b$ ($x \in \mathbb{F}_7^6$).

$\text{Lös}(A, b)$ ist **affiner Unterraum** von \mathbb{F}_7^6

$$A \mid b \xrightarrow{\text{GauSS}} \left(\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 6 & 5 & 0 \\ 0 & 0 & 1 & 0 & 2 & 2 & 6 \\ 0 & 0 & 0 & 1 & 4 & 6 & 1 \end{array} \right) \quad (3)$$

$$\text{Rang}(A) = \text{Rang}(A \mid b) \implies \text{Lös}(A, b) \neq \emptyset \quad (4)$$

Pivotspalten i und übrige Spalten k :

$$i_1 = 1, i_2 = 2, i_3 = 3, i_4 = 4, k_1 = 5, k_2 = 6 \quad (5)$$

spezielle Lösung:

$$v = \begin{pmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ k_1 \\ k_2 \end{pmatrix} = (0 \quad 0 \quad 6 \quad 1 \quad 0 \quad 0) \quad (6)$$

Lösung des homogenen Gleichungssystems:

$$B = \begin{pmatrix} 0 & 0 \\ 6 & 5 \\ 2 & 2 \\ 4 & 6 \end{pmatrix} \Rightarrow -B = \begin{pmatrix} 0 & 0 \\ 1 & 2 \\ 5 & 5 \\ 3 & 1 \end{pmatrix} \quad (7)$$

$$\text{Lös}(A, 0) = \text{Lin}\left(\begin{pmatrix} 0 \\ 1 \\ 5 \\ 3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 5 \\ 1 \\ 0 \\ 1 \end{pmatrix}\right) \quad (8)$$

$$\text{Lös}(A, b) = v + \text{Lös}(A, 0) \quad (9)$$

Beispiel zur speziellen Lösung

$$\left(\begin{array}{cccccc|c} 1 & 2 & 0 & 3 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right) \quad (10)$$

$$i_1 = 1, i_2 = 3, i_3 = 5, i_4 = 6 \quad (11)$$

$$k_1 = 2, k_2 = 4 \quad (12)$$

Spezielle Lösung:

$$\begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \\ 3 \\ 4 \end{pmatrix} \quad (13)$$

Aufgabe 2.

$$\left(\begin{array}{cccc|cccc} 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \rightarrow \left(\begin{array}{cccc|cccc} 1 & 2 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \rightarrow \left(\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \quad (14)$$

$$\rightarrow \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & -2 & -2 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \rightarrow \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & -2 & -2 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 3 & 2 & -1 & 1 & 1 \end{array}\right) \quad (15)$$

$$\rightarrow \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & -2 & -2 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 3 & 1 & -1 & 1 & 1 \end{array}\right) \rightarrow \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & -2 & -2 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right) \quad (16)$$

$$\rightarrow \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -2 & -2 & 1 & -2 \\ 0 & 0 & 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right) \quad (17)$$

Aufgabe 3.

$$A = (a_{ij}) \in M(2 \times 2, \mathbb{C}) = V \quad (18)$$

(a)

$$\Phi_A : V \rightarrow V, X \mapsto AX - XA = [A, X] \quad (19)$$

zu zeigen: Φ_A ist \mathbb{C} -linear:

Seien $X, Y \in V, \lambda \in \mathbb{C}$:

$$\Phi_A(X + Y) = A(X + Y) - (X + Y)A = \Phi_A(X) + \Phi_A(Y) \quad (20)$$

$$\Phi_A(\lambda X) = A(\lambda X) - (\lambda X)A = \lambda(AX - XA) = \lambda\Phi_A(X) \quad (21)$$

(b)

Sei

$$\mathcal{B}_1 = (b_1, b_2, b_3, b_4) = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right) \quad (22)$$

$$\mathcal{B}_2 = (e_1, e_2, e_3, e_4) \quad \text{die Standard-Basis} \quad = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right) \quad (23)$$

zu zeigen: \mathcal{B}_1 ist eine Basis von V :

$$e_1 = \frac{1}{2}(b_1 + b_3) \quad e_2 = \frac{1}{2}(b_2 + b_4) \quad (24)$$

$$e_3 = \frac{1}{2}(b_2 - b_4) \quad e_4 = \frac{1}{2}(b_1 - b_3)$$

$$e_1, e_2, e_3, e_4 \in \text{Lin}(b_i) \quad (25)$$

$$V = \text{Lin}(e_i) \subset \text{Lin}(b_i) \quad (26)$$

$\implies \mathcal{B}_1$ ist ES von V

$\implies \mathcal{B}_1$ ist Basis von V

(c)

Bestimme $M_{\mathcal{B}_1}^{\mathcal{B}_2}(\Phi_A)$

$$\Phi_A(e_1) = Ae_1 - e_1A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{11} & 0 \\ a_{21} & 0 \end{pmatrix} - \begin{pmatrix} a_{11} & a_{12} \\ 0 & 0 \end{pmatrix} \quad (27)$$

$$= \begin{pmatrix} 0 & -a_{12} \\ a_{21} & 0 \end{pmatrix} = -a_{12} \frac{1}{2}(b_2 + b_4) + a_{21} \frac{1}{2}(b_2 - b_4) \quad (28)$$

$$= \frac{a_{21} - a_{12}}{2} b_2 + \frac{-a_{21} - a_{12}}{2} b_4 \quad (29)$$

$$M_{\mathcal{B}_1}^{\mathcal{B}_2}(\Phi_A) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{a_{21} - a_{12}}{2} & \frac{a_{11} - a_{22}}{2} & \frac{a_{22} - a_{11}}{2} & \frac{a_{12} - a_{21}}{2} \\ 0 & -a_{21} & a_{12} & 0 \\ \frac{-a_{21} - a_{12}}{2} & \frac{a_{11} - a_{22}}{2} & \frac{a_{11} - a_{22}}{2} & \frac{a_{12} + a_{21}}{2} \end{pmatrix} \quad (30)$$

Aufgabe 4.

$$V = \text{Abb}(S_3, \mathbb{C}), W = \text{Abb}(\{1, 2, 3\}, \mathbb{C}) \quad (31)$$

$$S_3 = \{\sigma_1, \dots, \sigma_6\} = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \right\} \quad (32)$$

(a)

$$\varepsilon : V \rightarrow W \quad (33)$$

$$\varepsilon(f)(r) = \sum_{\sigma \in S_3} f(\sigma)\sigma(r) \quad (34)$$

zu zeigen: ε ist \mathbb{C} -linear

Seien $f, g \in V, \lambda \in \mathbb{C}, r \in \{1, 2, 3\}$

$$\varepsilon(f + g)(r) = \sum_{\sigma \in S_3} (f + g)(\sigma)\sigma(r) = \sum_{\sigma \in S_3} f(\sigma)\sigma(r) + \sum_{\sigma \in S_3} g(\sigma)\sigma(r) = \varepsilon(f)(r) + \varepsilon(g)(r) \quad (35)$$

$$= (\varepsilon(f) + \varepsilon(g))(r) \quad (36)$$

$$\varepsilon(\lambda f)(r) = \sum_{\sigma \in S_3} (\lambda f)(\sigma)\sigma(r) = \lambda \sum_{\sigma \in S_3} f(\sigma)\sigma(r) = \lambda \varepsilon(f)(r) = (\lambda \varepsilon(f))(r) \quad (37)$$

(b)

Basis von $V : E = (e_1, \dots, e_6)$ definiert durch

$$e_i(\sigma_j) = \delta_{ij} = \begin{cases} 1 & i = 0j \\ 0 & \text{sonst} \end{cases} \quad (38)$$

Sei $f \in V$. Dann ist

$$f = \sum_{i=1}^6 f(\sigma_i)e_i \quad (39)$$

Sei

$$f = \sum_{i=1}^6 \lambda_i e_i \in V \quad (40)$$

Dann ist

$$f \in \ker(\varepsilon) \iff \forall r \in \{1, 2, 3\} \varepsilon(f)(r) = 0 \iff \left| \begin{array}{l} \varepsilon(f)(1) = \lambda_1 \cdot 1 + \lambda_2 \cdot 2 + \lambda_3 \cdot 1 + \lambda_4 \cdot 3 + \lambda_5 \cdot 2 + \lambda_6 \cdot \\ \varepsilon(f)(2) = \dots = 0 \\ \varepsilon(f)(3) = \dots = 0 \end{array} \right. \quad (41)$$

$$A = \begin{pmatrix} 1 & 2 & 1 & 3 & 2 & 3 \\ 2 & 1 & 3 & 2 & 3 & 1 \\ 3 & 3 & 2 & 1 & 1 & 2 \end{pmatrix} \xrightarrow{\text{GauSS}} \begin{pmatrix} 1 & 0 & 0 & -3 & -2 & -2 \\ 0 & 1 & 0 & 2 & 1 & 2 \\ 0 & 0 & 1 & 2 & 2 & 1 \end{pmatrix} \in M(3 \times 6, \mathbb{C}) \quad (42)$$

$$i_1 = 1, i_2 = 2, i_3 = 3, k_1 = 4, k_2 = 5, k_3 = 6 \quad (43)$$

$$B = \begin{pmatrix} -3 & -2 & -2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix} \rightarrow -B = \begin{pmatrix} 3 & 2 & 2 \\ -2 & -1 & -2 \\ -2 & -2 & -1 \end{pmatrix} \quad (44)$$

$$\text{Lös}(A, 0) = \text{Lin}\left(\underbrace{\begin{pmatrix} 3 \\ -2 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}}_{(x_i)}, \underbrace{\begin{pmatrix} 2 \\ -1 \\ -2 \\ 0 \\ 1 \\ 0 \end{pmatrix}}_{(y_i)}, \underbrace{\begin{pmatrix} 2 \\ -2 \\ -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}}_{(z_i)}\right) \subseteq \mathbb{C}^6 \implies \ker(\varepsilon) = \text{Lin}\left(\sum_{i=1}^6 x_i e_i, \sum_{i=1}^6 y_i e_i, \sum_{i=1}^6 z_i e_i\right) \quad (45)$$