Übungszettel 12

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1 Aufgabe 1.

Sei

$$A = \begin{pmatrix} 6 & 1 & 2 & 3 & 1 & 6 \\ 3 & 6 & 5 & 0 & 4 & 5 \\ 1 & 0 & 0 & 3 & 5 & 4 \\ 2 & 4 & 5 & 2 & 0 & 0 \end{pmatrix} \in M(4 \times 6, \mathbb{F}_7) \tag{1}$$

$$b = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \in \mathbb{F}_7^6 \tag{2}$$

Löse: $Ax = b \quad (x \in \mathbb{F}_7^6)$.
$\text{Lös}(A, b)$ ist **affiner Unterraum** von $\mathbb{F}_7^6$

$$A \mid b \xrightarrow{\text{GauSS}} \left(\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 6 & 5 & 0 \\ 0 & 0 & 1 & 0 & 2 & 2 & 6 \\ 0 & 0 & 0 & 1 & 4 & 6 & 1 \end{array}\right) \tag{3}$$

$$\text{Rang}(A) = \text{Rang}(A \mid b) \implies \text{Lös}(A, b) \neq \emptyset \tag{4}$$

Pivotspalten i und übrige Spalten k:

$$i_1 = 1, i_2 = 2 i_3 = 3, i_4 = 4, k_1 = 5, k_2 = 6 \tag{5}$$

spezielle Lösung:

$$v = \begin{pmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 6 & 1 & 0 & 0 \end{pmatrix} \tag{6}$$

Lösung des homogenen Gleichungssystems:

$$B = \begin{pmatrix} 0 & 0 \\ 6 & 5 \\ 2 & 2 \\ 4 & 6 \end{pmatrix} \implies -B = \begin{pmatrix} 0 & 0 \\ 1 & 2 \\ 5 & 5 \\ 3 & 1 \end{pmatrix} \tag{7}$$

$$\text{Lös}(A, 0) = \text{Lin}(\begin{pmatrix} 0 \\ 1 \\ 5 \\ 3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 5 \\ 1 \\ 0 \\ 1 \end{pmatrix}) \tag{8}$$

$$\text{Lös}(A, b) = v + \text{Lös}(A, 0) \tag{9}$$

Beispiel zur speziellen Lösung

$$\left(\begin{array}{cccccc|c} 1 & 2 & 0 & 3 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right) \tag{10}$$

$$i_1 = 1, i_2 = 3, i_3 = 5, i_4 = 6 \tag{11}$$

$$k_1 = 2, k_2 = 4 \tag{12}$$

Spezielle Lösung:

$$\begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \\ 3 \\ 4 \end{pmatrix} \tag{13}$$

Aufgabe 2.

$$\left(\begin{array}{cccc|cccc} 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \rightarrow \left(\begin{array}{cccc|cccc} 1 & 2 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \rightarrow \left(\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \tag{14}$$

$$\rightarrow \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & -2 & -2 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \rightarrow \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & -2 & -2 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 3 & 2 & -1 & 1 & 1 \end{array}\right) \tag{15}$$

$$\rightarrow \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & -2 & -2 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 3 & 1 & -1 & 1 & 1 \end{array}\right) \rightarrow \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & -2 & -2 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right) \tag{16}$$

$$\rightarrow \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -2 & -2 & 1 & -2 \\ 0 & 0 & 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right) \tag{17}$$

Aufgabe 3.

$$A = (a_{ij}) \in M(2 \times 2, \mathbb{C}) = V \tag{18}$$

(a)

$$\Phi_A : V \to V, X \mapsto AX - XA = [A, X] \tag{19}$$

zu zeigen: Φ_A ist $\mathbb{C}$ -linear:
Seien $X, Y \in V, \lambda \in \mathbb{C}$:

$$\Phi_A(X+Y) = A(X+Y) - (X+Y)A = \Phi_A(X) + \Phi_A(Y) \tag{20}$$

$$\Phi_A(\lambda X) = A(\lambda X) - (\lambda X)A = \lambda(AX - XA) = \lambda \Phi_A(X) \tag{21}$$

(b)

Sei

$$\mathcal{B}_1 = (b_1, b_2, b_3, b_4) = (\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}) \tag{22}$$

$$\mathcal{B}_2 = (e_1, e_2, e_3, e_4) \quad \text{die Standard-Basis} \quad = (\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}) \tag{23}$$

zu zeigen: $\mathcal{B}_1$ ist eine Basis von V:

$$\begin{aligned} e_1 &= \tfrac{1}{2}(b_1 + b_3) \quad e_2 = \tfrac{1}{2}(b_2 + b_4) \\ e_3 &= \tfrac{1}{2}(b_2 - b_4) \quad e_4 = \tfrac{1}{2}(b_1 - b_3) \end{aligned} \tag{24}$$

$$e_1, e_2, e_3, e_4 \in \mathrm{Lin}(b_i) \tag{25}$$

$$V = \mathrm{Lin}(e_i) \subset \mathrm{Lin}(b_i) \tag{26}$$

$\implies \mathcal{B}_1$ ist ES von V
$\implies \mathcal{B}_1$ ist Basis von V

(c)

Bestimme $M_{\mathcal{B}_1}^{\mathcal{B}_2}(\Phi_A)$

$$\Phi_A(e_1) = Ae_1 - e_1 A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{11} & 0 \\ a_{21} & 0 \end{pmatrix} - \begin{pmatrix} a_{11} & a_{12} \\ 0 & 0 \end{pmatrix} \tag{27}$$

$$= \begin{pmatrix} 0 & -a_{12} \\ a_{21} & 0 \end{pmatrix} = -a_{12}\frac{1}{2}(b_2 + b_4) + a_{21}\frac{1}{2}(b_2 - b_4) \tag{28}$$

$$= \frac{a_{21} - a_{12}}{2} b_2 + \frac{-a_{21} - a_{12}}{2} b_4 \tag{29}$$

$$M_{\mathcal{B}_1}^{\mathcal{B}_2}(\Phi_A) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{a_{21}-a_{12}}{2} & \frac{a_{11}-a_{22}}{2} & \frac{a_{22}-a_{11}}{2} & \frac{a_{12}-a_{21}}{2} \\ 0 & -a_{21} & a_{12} & 0 \\ \frac{-a_{21}-a_{12}}{2} & \frac{a_{11}-a_{22}}{2} & \frac{a_{11}-a_{22}}{2} & \frac{a_{12}+a_{21}}{2} \end{pmatrix} \tag{30}$$

Aufgabe 4.

$$V = \mathrm{Abb}(S_3, \mathbb{C}), W = \mathrm{Abb}(\{1,2,3\}, \mathbb{C}) \tag{31}$$

$$S_3 = \{\sigma_1, \dots, \sigma_6\} = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \right\} \tag{32}$$

(a)

$$\varepsilon : V \to W \tag{33}$$

$$\varepsilon(f)(r) = \sum_{\sigma \in S_3} f(\sigma)\sigma(r) \tag{34}$$

zu zeigen: ε ist $\mathbb{C}$ -linear
Seien $f, g \in V, \lambda \in \mathbb{C}, r \in \{1,2,3\}$

$$\varepsilon(f+g)(r) = \sum_{\sigma \in S_3} (f+g)(\sigma)\sigma(r) = \sum_{\sigma \in S_3} f(\sigma)\sigma(r) + \sum_{\sigma \in S_3} g(\sigma)\sigma(r) = \varepsilon(f)(r) + \varepsilon(g)(r) \tag{35}$$

$$= (\varepsilon(f) + \varepsilon(g))(r) \tag{36}$$

$$\varepsilon(\lambda f)(r) = \sum_{\sigma \in S_3} (\lambda f)(\sigma)\sigma(r) = \lambda \sum_{\sigma \in S_3} f(\sigma)\sigma(r) = \lambda\varepsilon(f)(r) = (\lambda\varepsilon(f))(r) \tag{37}$$

(b)

Basis von $V : E = (e_1, \dots, e_6)$ definiert durch

$$e_i(\sigma_j) = \delta_{ij} = \begin{cases} 1 & i = 0j \\ 0 & \text{sonst} \end{cases} \tag{38}$$

Sei $f \in V$. Dann ist

$$f = \sum_{i=1}^{6} f(\sigma_i) e_i \tag{39}$$

Sei

$$f = \sum_{i=1}^{6} \lambda_i e_i \in V \tag{40}$$

Dann ist

$$f \in \ker(\varepsilon) \iff \forall r \in \{1,2,3\} \varepsilon(f)(r) = 0 \iff \left| \begin{array}{r} \varepsilon(f)(1) = \lambda_1 \cdot 1 + \lambda_2 \cdot 2 + \lambda_3 \cdot 1 + \lambda_4 \cdot 3 + \lambda_5 \cdot 2 + \lambda_6 \cdot \\ \varepsilon(f)(2) = \ldots = 0 \\ \varepsilon(f)(3) = \ldots = 0 \end{array} \right. \tag{41}$$

$$A = \begin{pmatrix} 1 & 2 & 1 & 3 & 2 & 3 \\ 2 & 1 & 3 & 2 & 3 & 1 \\ 3 & 3 & 2 & 1 & 1 & 2 \end{pmatrix} \xrightarrow{\text{GauSS}} \begin{pmatrix} 1 & 0 & 0 & -3 & -2 & -2 \\ 0 & 1 & 0 & 2 & 1 & 2 \\ 0 & 0 & 1 & 2 & 2 & 1 \end{pmatrix} \in M(3 \times 6, \mathbb{C}) \tag{42}$$

$$i_1 = 1, i_2 = 2, i_3 = 3, k_1 = 4, k_2 = 5, k_3 = 6 \tag{43}$$

$$B = \begin{pmatrix} -3 & -2 & -2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix} \to -B = \begin{pmatrix} 3 & 2 & 2 \\ -2 & -1 & -2 \\ -2 & -2 & -1 \end{pmatrix} \tag{44}$$

$$\text{Lös}(A,0) = \text{Lin}(\underbrace{\begin{pmatrix} 3 \\ -2 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}}_{(x_i)}, \underbrace{\begin{pmatrix} 2 \\ -1 \\ -2 \\ 0 \\ 1 \\ 0 \end{pmatrix}}_{(y_i)}, \underbrace{\begin{pmatrix} 2 \\ -2 \\ -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}}_{(z_i)}) \subseteq \mathbb{C}^6 \implies \ker(\varepsilon) = \text{Lin}(\sum_{i=1}^{6} x_i e_i, \sum_{i=1}^{6} y_i e_i, \sum_{i=1}^{6} z_i e_i) \tag{45}$$