Enumerating Symmetrical Domino Tilings

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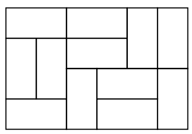
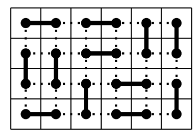
Background

A domino tiling is an arrangement of dominoes (2×1 rectangles) which exactly covers a two-dimensional region as shown in Figure 1a.

Domino tilings are equivalent to perfect matchings (or 1-factors) of the graph with vertices at the center of each grid cell and edges connecting each neighboring cell. A perfect matching is a set of edges such that each vertex is incident to exactly one edge of the set as shown in Figure 1b. Counting perfect matchings is, in general, quite hard, but I hope to simplify the computation using the constraints of the problem.



(a) Domino tiling

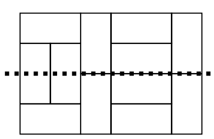


(b) Perfect matching

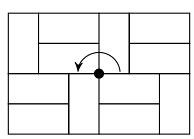
Figure 1: Sample domino tiling of a 4×6 rectangle and corresponding perfect matching

The enumeration of domino tilings was first used in the field of statistical mechanics to quantify the statistical properties of a system of dimer molecules dissolved on a surface. In 1961, Temperley & Fisher [1] and Kasteleyn [2] independently calculated the number of domino tilings for a $n \times m$ rectangle with m even (equal to the number of dimer arrangements on an $n \times m$ lattice) by transforming the problem into a Pfaffian of a skew-symmetric matrix. Kasteleyn [3] then generalized the method into a polynomial-time algorithm for any planar graph in 1963.

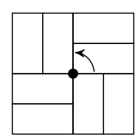
In this project, I will consider domino tilings which are symmetrical as shown in Figure 2. Although there is no known formula for the number of domino tilings which are unique up to symmetry, Mathar [4] calculated the values for many small rectangles in 2013. If one side of the rectangle has length 2, then the number of distinct domino tilings is a variation of the Fibonacci sequence [5].



(a) Axial



(b) 180° Rotation



(c) 90° Rotation

Figure 2: Symmetries of a rectangular region

Research Question

In how many different ways can dominoes (1×2 rectangles) be arranged such that they exactly cover a $m \times n$ rectangle such that each arrangement is unique up to symmetry?

Project Goals and Objectives

The primary goal of this project is to develop a method for calculating the number of domino tilings of a rectangle that are distinct up to symmetry. Specifically, I aim to derive a closed formula, an asymptotic approximation formula, or an algorithm for the enumeration problem and provide proof for the method's correctness (and accuracy in the case of an approximation).

Project Impact

Although domino tilings are not particularly useful, their equivalent perfect matchings are an important concept in graph theory and other fields. In relation to the dimer problem studied by Kasteleyn and others, it will be helpful to count the number of dimer arrangements which are distinct up to symmetry because this ensures symmetrical dimer arrangements are not over-counted. Additionally, my project will provide a way to determine the probability that a random domino tiling (or perfect matching/dimer arrangement) is symmetrical. My original motivation was for solving the recreational math problem of determining the number of ways to arrange dominoes on the faces of a cube.

Project Design

I am currently working on this project with Dr. Szekely as part of an independent research course (Math 499), and I plan to develop the project into a senior thesis for my graduation with distinction in mathematics. So far, I have found a close formula for the number of distinct domino tilings of a $3\times n$ rectangle and the number of horizontally symmetrical tilings of a $4\times n$ rectangle. I have also implemented a brute-force algorithm in Python which counts the number of domino tilings of any rectangle which satisfy each type of symmetry and also counts the number of distinct domino tilings up to symmetry. Although this brute force algorithm is slow, I can use it to numerically verify the first few terms.

To count symmetrical domino tilings in general, I will first explore the method put forth by Kasteleyn [3] for counting tilings on planar graphs. For the two types of rotational symmetry, I can use Kasteleyn's method on a planar graph which represents the symmetry by additional edges. I will then attempt to simplify the resulting matrix problem for a closed form.

For line symmetry, it may not be possible to represent the symmetrical tilings using a planar graph, so I will not be able to use Kasteleyn's algorithm. Instead, I will attempt to find upper and lower bounds which converge asymptotically for an approximation formula. Additionally, I will attempt to prove the computational hardness of the problem.

Project Timeline

Task	pre-grant	Jan.	Feb.	Mar.	Apr.
Formulation of special cases – one side small	X				
Formulation for 180-degree rotation symmetry	X	X			
Formulation for 90-degree rotation symmetry		X	X		
Formulation for line symmetry			X	X	X
Article and presentation preparation				X	X

Anticipated Results

I hope to discover a closed formula, asymptotic formula, or algorithm for the enumeration of domino tilings which can be calculated efficiently. For whatever end result I achieve, I will prove the method's correctness rigorously. The problem should be solvable, but it is possible that no efficient method exists – if this is the case, I will hope for a proof of the problem's computational hardness.

In addition to Discover USC, I hope to present my work at an American Mathematical Society (AMS) student conference, sectional meeting, or national meeting. I also hope to give a presentation in the math department's weekly combinatorics seminar.

Personal Statement

I have been involved in undergraduate research for three years as a member of the Autonomous Field Robotics Lab under Dr. Ioannis Rekleitis. Because of my positive experience in undergraduate research and as a math major, I began working on the project with Dr. Szekely this semester as part of an independent research course (Math 499). I plan to develop the project into my senior thesis for graduation with distinction in mathematics, and I was awarded the math department's Victor W. Laurie Undergraduate Research Scholarship to support my research. To prepare for discrete math research, I have taken Math 574 (discrete math) and Math 575 (graph theory), and I am currently enrolled in the graduate-level discrete math course, Math 774. After graduation, I plan to enroll in a theoretical computer science Ph.D. program with an emphasis on geometric and graph algorithms.

References

[1] H. N. V. Temperley and M. E. Fisher. "Dimer problem in statistical mechanics-an exact result". In: *The Philosophical Magazine: A Journal of Theoretical Experimental and Applied Physics* 6.68 (1961), pp. 1061–1063. DOI: 10.1080/14786436108243366.

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[3] P. W. Kasteleyn. "Dimer Statistics and Phase Transitions". In: *Journal of Mathematical Physics* 4.2 (1963), pp. 287–293. DOI: 10.1063/1.1703953.

[4] R. J. Mathar. *Paving Rectangular Regions with Rectangular Tiles: Tatami and Non-Tatami Tilings*. 2013. arXiv: 1311.6135 [math.CO].

[5] W. E. Patten (proposer) and S. W. Golomb (solver). "Problem E1470: Covering a $2\times n$ rectangle with dominoes". In: *American Mathematical Monthly* 69 (1962), pp. 61–62.