Parallel Algorithms for Trees

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Parallel Computation: Models and Methods
Selim G. Akl — Chapter 6

CISC 879 — Algorithms and Applications
Queen's University

October 29, 2008

Outline

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1. Introduction
   - Graphs
   - Euler Tours
2. Building Euler Tours
3. Applications of Euler Tours
4. Conclusion

Graphs

- A Graph $G = (V, E)$ is a set of vertices V connected by a set of edges E.
- If the edges have orientation, the graph is directed.
- A path is an ordered list of edges in the form $(v_i, v_j), (v_j, v_k), (v_k, v_l)$.
- A cycle is a path that begins and ends at the same vertex.
- The degree of a vertex in an undirected graph is the number of edges adjacent to the vertex.
- The in-degree of a vertex v in a directed graph is the number of edges entering v,
  The out-degree of a vertex v in a directed graph is the number of edges leaving v.

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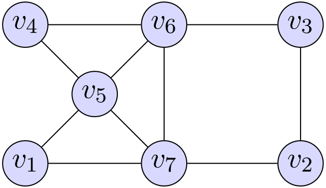
Euler Tours

Euler Tour (ET)

A cycle where every edge of the graph appears in the cycle exactly once.

An undirected graph has an Euler tour if each vertex has an even degree.



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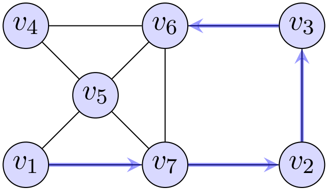
$(v_1, v_7), (v_7, v_2), (v_2, v_3), (v_3, v_6)$



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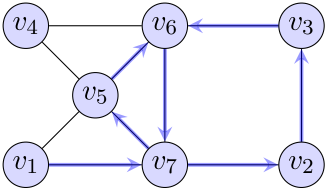
$(v_6, v_7), (v_7, v_5), (v_5, v_6)$



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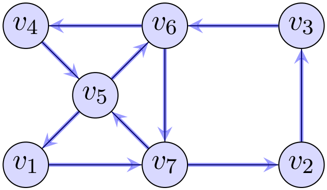
$(v_6, v_7), (v_7, v_5), (v_5, v_6)$

$(v_6, v_4), (v_4, v_5), (v_5, v_1)$



Euler Tours

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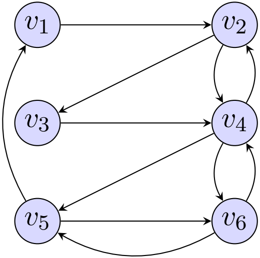

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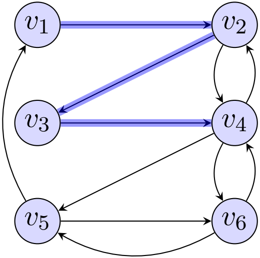
$(v_1, v_2), (v_2, v_3), (v_3, v_4)$



Euler Tours

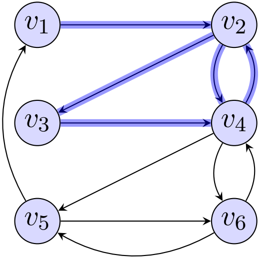
Euler Tour (ET)

A cycle where every edge of the graph appears in the cycle exactly once.

A directed graph has an Euler tour if the in-degree of each vertex is equal to the out-degree of the vertex.

$(v_1, v_2), (v_2, v_3), (v_3, v_4)$

$(v_4, v_2), (v_2, v_4)$



Euler Tours

Euler Tour (ET)

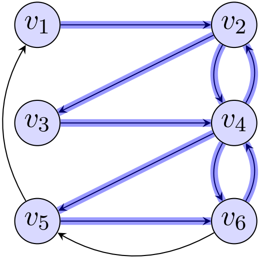
A cycle where every edge of the graph appears in the cycle exactly once.

A directed graph has an Euler tour if the in-degree of each vertex is equal to the out-degree of the vertex.

$(v_1, v_2), (v_2, v_3), (v_3, v_4)$

$(v_4, v_2), (v_2, v_4)$

$(v_4, v_5), (v_5, v_6), (v_6, v_4), (v_4, v_6)$



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$(v_4, v_2), (v_2, v_4)$

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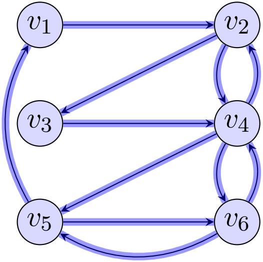
$(v_6, v_5), (v_5, v_1)$

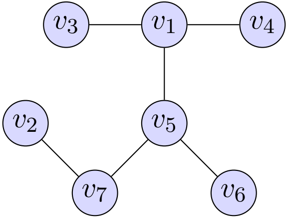


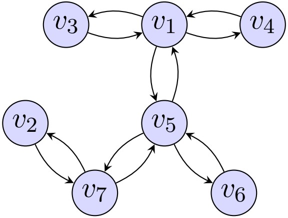
Trees

Tree

An undirected graph that is connected and contains no cycles.

- A tree with n vertices has exactly $n-1$ edges.
- A directed tree (DT) with n vertices has $2n-2$ edges.
    - ➩ Its ET is a sequence of $2n-2$ edges.





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Directed Trees (DT) as Linked Lists

- n linked lists, one for each vertex.
- The nodes of a list are edges leaving that vertex.

head(v_1) $\longrightarrow$ $\boxed{(v_1, v_3)} \rightarrow \boxed{(v_1, v_4)} \rightarrow \boxed{(v_1, v_5)}$

head(v_2) $\longrightarrow$ $\boxed{(v_2, v_7)}$

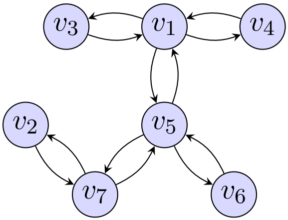
head(v_3) $\longrightarrow$ $\boxed{(v_3, v_1)}$

head(v_4) $\longrightarrow$ $\boxed{(v_4, v_1)}$

head(v_5) $\longrightarrow$ $\boxed{(v_5, v_1)} \rightarrow \boxed{(v_5, v_6)} \rightarrow \boxed{(v_5, v_7)}$

head(v_6) $\longrightarrow$ $\boxed{(v_6, v_5)}$

head(v_7) $\longrightarrow$ $\boxed{(v_7, v_2)} \rightarrow \boxed{(v_7, v_5)}$



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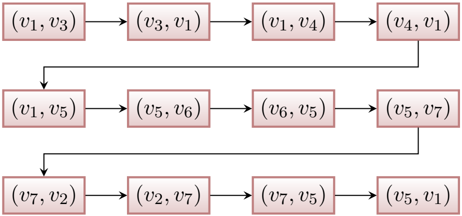
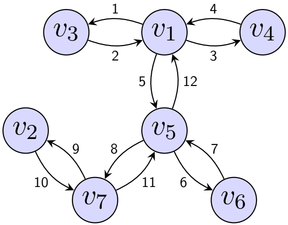
Algorithm Input and Output

- **Input:**
  n linked lists representing a directed tree (DT).
- **Output:**
  Compute the Euler tour (ET) for the given DT:
  - Arrange all the edges of the DT in a single linked list.
  - Each edge (v_i, v_j) is followed by an edge (v_j, v_k).
  - The first edge leaves some vertex v_l, the last edge enters v_l.





Building the Euler Tour in Parallel

- Assuming the shared-memory model, with $n-1$ processors.
- Each processor $P_{ij}, i < j$, is in charge of two edges: (v_i, v_j) and (v_j, v_i).
- P_{ij} determines the position (actually, the successor) of the two nodes holding (v_i, v_j) and (v_j, v_i) (or ij and ji) — in ET.

Successor of (v_i, v_j)

if next$(ji) = jk$ **then**
  succ$(ij) \leftarrow jk$
else
  succ$(ij) \leftarrow$ head(v_j)
end if

Successor of (v_j, v_i)

if next$(ij) = im$ **then**
  succ$(ji) \leftarrow im$
else
  succ$(ji) \leftarrow$ head(v_i)
end if

Example: $P_{ij} = P_{13}$

Successor of (v_1, v_3)

Successor of (v_i, v_j)

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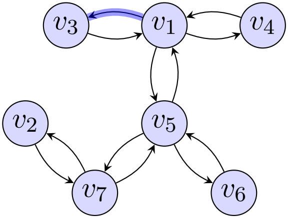


head$(v_1) \longrightarrow \boxed{(v_1, v_3)} \longrightarrow \boxed{(v_1, v_4)} \longrightarrow \boxed{(v_1, v_5)}$

$\vdots$

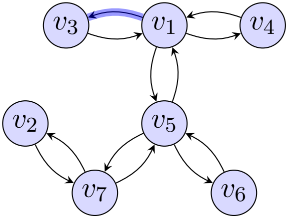
head$(v_3) \longrightarrow \boxed{(v_3, v_1)}$

Example: $P_{ij} = P_{13}$

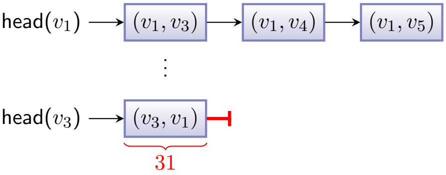
Successor of (v_1, v_3)

Successor of (v_i, v_j)

> **if** next(ji) $= jk$ **then**
> succ$(ij) \leftarrow jk$
> **else**
> succ$(ij) \leftarrow$ head(v_j)
> **end if**



next$(31) =$ `null`



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Successor of (v_1, v_3)

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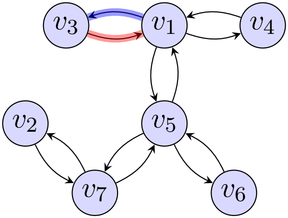

head$(v_1) \longrightarrow \overbrace{(v_1, v_3)}^{13} \longrightarrow (v_1, v_4) \longrightarrow (v_1, v_5)$

$\vdots$

head$(v_3) \longrightarrow (v_3, v_1)$

succ$(13) \leftarrow$ head(v_3)

$(v_1, v_3) \longrightarrow (v_3, v_1)$

Example: $P_{ij} = P_{13}$

Successor of (v_3, v_1)

Successor of (v_j, v_i)

if next$(ij) = im$ **then**
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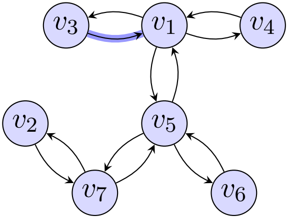
end if



head$(v_1) \longrightarrow$ (v_1, v_3) $\longrightarrow$ (v_1, v_4) $\longrightarrow$ (v_1, v_5)

$\vdots$

head$(v_3) \longrightarrow$ (v_3, v_1)

Example: $P_{ij} = P_{13}$

Successor of (v_3, v_1)

Successor of (v_j, v_i)

if next$(ij) = im$ **then**
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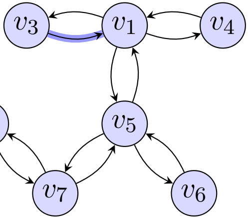
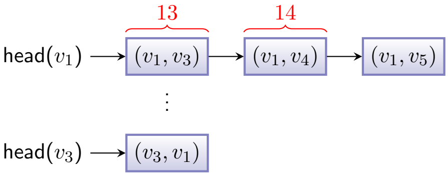
else
  succ$(ji) \leftarrow$ head(v_i)
end if

next$(13) = 14$





Example: $P_{ij} = P_{13}$

Successor of (v_3, v_1)

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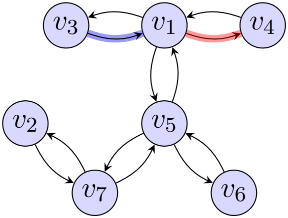

$$\text{succ}(31) \leftarrow 14$$

$(v_1, v_3) \rightarrow (v_3, v_1) \rightarrow (v_1, v_4)$

head$(v_1) \longrightarrow (v_1, v_3) \rightarrow \overbrace{(v_1, v_4)}^{14} \rightarrow (v_1, v_5)$

$\vdots$

head$(v_3) \longrightarrow \underbrace{(v_3, v_1)}_{31}$

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Correctness

- Visits all the edges:
  - Every pair of edges is assigned to a processor.
- Produces a single cycle, not several small cycles:

$\boxed{(v_i, v_j)} \longrightarrow \boxed{(v_j, v_k)}$

head(v_j) $\dashrightarrow \boxed{(v_j, v_i)} \longrightarrow \boxed{(v_j, v_k)} \dashrightarrow$

next(ji) $= jk$

Correctness

- Visits all the edges:
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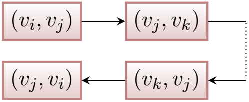
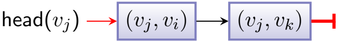




next(ji) $= jk$

next$(jk) = ji$, or head(v_j) $= ji$

Complexity

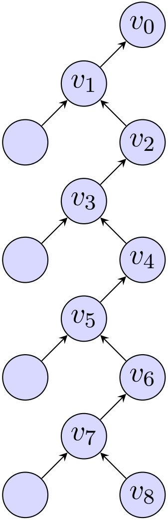
- $t(n) = O(1)$
- $p(n) = n - 1 = O(n)$

$$\Rightarrow \qquad c(n) = O(n)$$

Outline

Pointer Jumping $\text{parent}(v_i) \leftarrow \text{parent}(\text{parent}(v_i))$

Find the root of v_8.



Pointer Jumping

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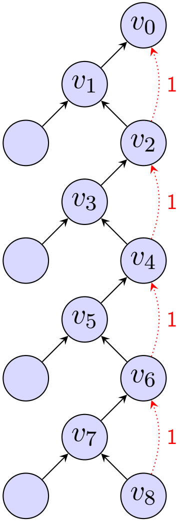
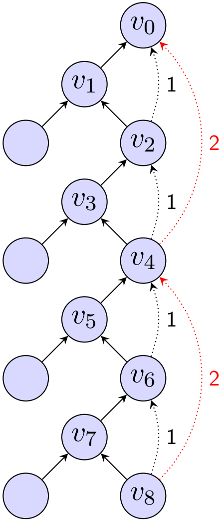


Pointer Jumping

$$\text{parent}(v_i) \leftarrow \text{parent}(\text{parent}(v_i))$$

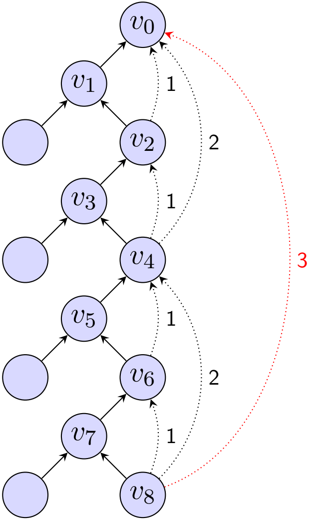
Find the root of v_8.



Pointer Jumping

parent(v_i) $\leftarrow$ parent(parent(v_i))

Find the root of v_8.



Time Complexity

Sequential: $O(n)$

Parallel: $O(\log n)$

Parallel Prefix Computation for Linked Lists

Step 1: **forall** i **do in parallel**
  next(i) $\leftarrow$ succ(i)

Step 2: $\mathit{finished} \leftarrow$ **false**

Step 3: **while not** $\mathit{finished}$ **do**
  (3.1) $\mathit{finished} \leftarrow$ **true**
  (3.2) **forall** i **do in parallel**
    (i) **if** next(i) $\neq \mathit{nil}$ **then**
      (a) val(next(i)) $\leftarrow$ val(i) $\circ$ val(next(i))
      (b) next(i) $\leftarrow$ next(next(i))
    (ii) **if** next(i) $\neq \mathit{nil}$ **then**
      $\mathit{finished} \xleftarrow{\text{common}}$ **false**

Input & Output

Input: A linked list
Output: Prefix computation

$$x_0 \to x_1 \to x_2 \to x_3 \to x_4 \to x_5 \to x_6 \to x_7$$

Parallel Prefix Computation for Linked Lists

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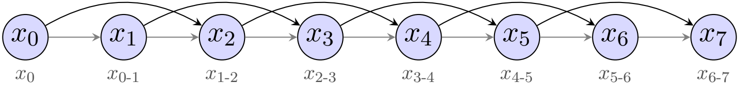
finished $\xleftarrow{\text{common}}$ **false**

Notation

$$x_{a\text{-}b} = x_a \circ x_{a+1} \circ \cdots \circ x_b$$



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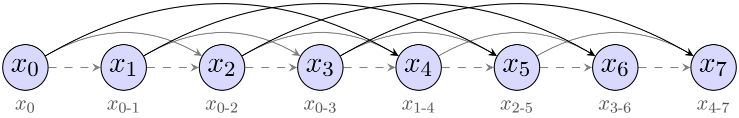

Parallel Prefix Computation for Linked Lists

Step 1: **forall** i **do in parallel**
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Step 2: $finished \leftarrow$ **false**

Step 3: **while not** $finished$ **do**

(3.1) $finished \leftarrow$ **true**

(3.2) **forall** i **do in parallel**

(i) **if** next(i) $\neq nil$ **then**

(a) val(next(i)) $\leftarrow$ val(i) $\circ$ val(next(i))

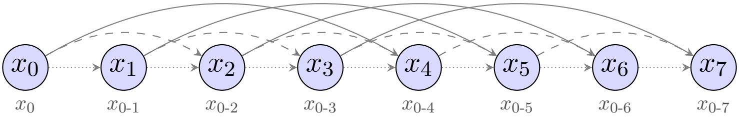
(b) next(i) $\leftarrow$ next(next(i))

(ii) **if** next(i) $\neq nil$ **then**

$finished \xleftarrow{\text{common}}$ **false**

Complexity

$$t(n) = O(\log n)$$
$$p(n) = n$$



Parallel Suffix Computation for Linked Lists

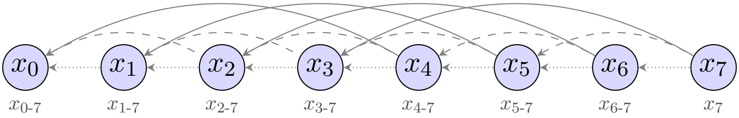
Step 1: **forall** i **do in parallel**
  next$(i) \leftarrow$ succ(i)

Step 2: $\mathit{finished} \leftarrow$ **false**

Step 3: **while not** $\mathit{finished}$ **do**
  (3.1) $\mathit{finished} \leftarrow$ **true**
  (3.2) **forall** i **do in parallel**
    (i) **if** next$(i) \neq \mathit{nil}$ **then**
      (a) val$(i) \leftarrow$ val$(i) \circ$ val(next(i))
      (b) next$(i) \leftarrow$ next(next(i))
    (ii) **if** next$(i) \neq \mathit{nil}$ **then**
      $\mathit{finished} \xleftarrow{\text{common}}$ **false**

Complexity

$$t(n) = O(\log n)$$
$$p(n) = n$$



List Sequencing

List Sequencing

Computing the distance of each node from the *beginning* of the list (node position).

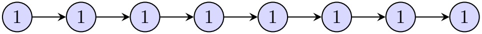
Input: Linked list L

Output: Sequence numbers

forall i **do in parallel**

  val(i) $\leftarrow$ 1

Parallel-Prefix(L, $+$)



List Sequencing

List Sequencing

Computing the distance of each node from the *beginning* of the list (node position).

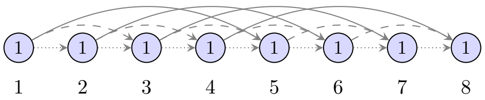
Input: Linked list L

Output: Sequence numbers

forall i **do in parallel**

val(i) $\leftarrow$ 1

Parallel-Prefix(L, $+$)



List Ranking

List Ranking

Computing the distance of each node from the *end* of the list.

Input: Linked list L **Output:** Node ranks

forall i **do in parallel**

(3.1) val(i) $\leftarrow$ 1

(3.2) **if** succ(i) $\neq$ nil **then**

succ(succ(i)) $\leftarrow$ i

Parallel-Prefix(L, $+$)

$$1 \to 1 \to 1 \to 1 \to 1 \to 1 \to 1 \to 1$$

List Ranking

List Ranking

Computing the distance of each node from the *end* of the list.

Input: Linked list L **Output:** Node ranks

forall i **do in parallel**

(3.1) val$(i) \leftarrow 1$

(3.2) **if** succ$(i) \neq nil$ **then**

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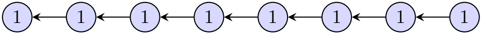
Parallel-Prefix$(L, +)$



List Ranking

List Ranking

Computing the distance of each node from the *end* of the list.

Input: Linked list L

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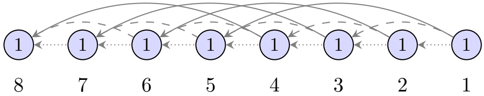
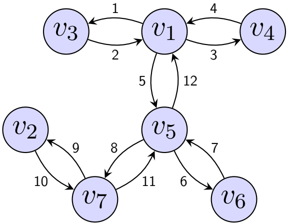


Complexity

$$t(n) = O(\log n), \qquad p(n) = O(n)$$

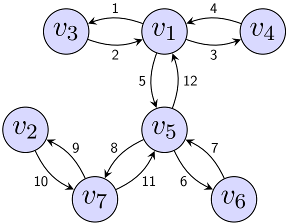
Outline

An Euler Tour is a . . .



An Euler Tour returns to the root of a subtree only after all vertices in the subtree have been visited.

Depth-First Traversal

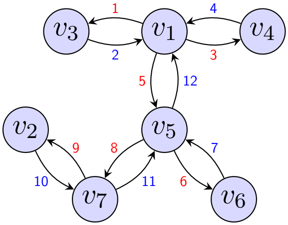


An Euler Tour returns to the root of a subtree only after all vertices in the subtree have been visited.

Finding Parents

- Recall:
  - pos(i) is node i's sequence number.
  - In an Euler Tour, a node looks like (v_i, v_j).
  - If (v_i, v_j) is a node, then (v_j, v_i) is also a node.
- pos(v_i, v_j) $<$ pos(v_j, v_i) $\Rightarrow$ (v_i, v_j) is an advance edge
  otherwise (v_i, v_j) is a retreat edge.

parent(root) $\leftarrow$ *nil*
forall $(v_i, v_j) \in$ ET **do in parallel**
    if pos(v_i, v_j) $<$ pos(v_j, v_i) **then**
        parent(v_j) $\leftarrow v_i$

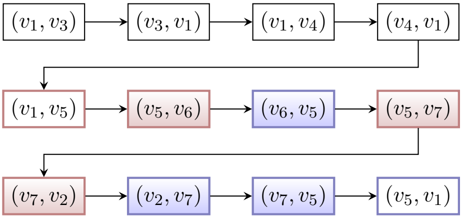


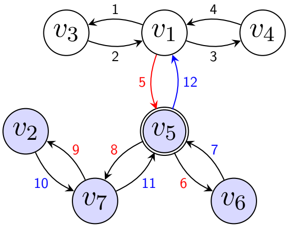
Counting Descendants

$\text{des}(v_i)$

- The subtree of v_i edges appear between the advance edge $(\text{parent}(v_i),\ v_i)$ and the retreat edge $(v_i,\ \text{parent}(v_i))$ in ET.
- Half of these edges are advance edges and half are retreat.
- $\text{des}(v_i)$ is the number of advance edges in the subtree $+\,1$.

$$\text{des}(v_i) = \frac{\text{pos}(v_i,\ \text{parent}(v_i)) - 1 - \text{pos}(\text{parent}(v_i),\ v_i)}{2} + 1$$

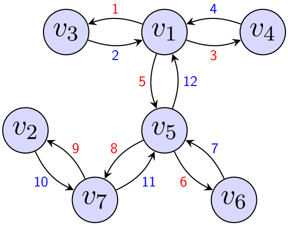




Numbering Vertices Preorder (top-down)

The preorder number of a vertex v_j is the number of advance edges in ET traversed before reaching v_j for the first time $+ 1$.

preorder(root) $\leftarrow 1$
forall $(v_i, v_j) \in$ ET **do in parallel**
  if pos(v_i, v_j) $<$ pos(v_j, v_i) **then**
    val(ij) $\leftarrow$ 1
  else
    val(ij) $\leftarrow$ 0
Parallel-Prefix(ET, $+$)
forall $(v_i, v_j) \in$ ET **do in parallel**
  if pos(v_i, v_j) $<$ pos(v_j, v_i) **then**
    preorder(v_j) $\leftarrow$ val(ij) $+$ 1



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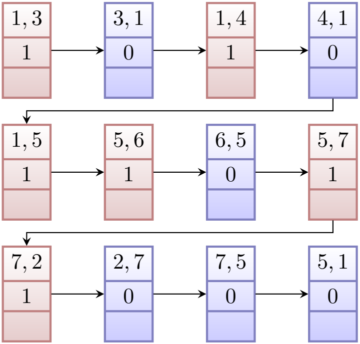
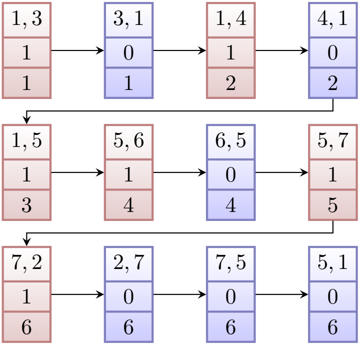


Numbering Vertices — Preorder (top-down)

The preorder number of a vertex v_j is the number of advance edges in ET traversed before reaching v_j for the first time $+ 1$.

preorder(root) $\leftarrow 1$
forall $(v_i, v_j) \in$ ET **do in parallel**
  if pos$(v_i,\, v_j) <$ pos$(v_j,\, v_i)$ **then**
    val$(ij) \leftarrow 1$
  else
    val$(ij) \leftarrow 0$
Parallel-Prefix(ET, $+$)
forall $(v_i, v_j) \in$ ET **do in parallel**
  if pos$(v_i,\, v_j) <$ pos$(v_j,\, v_i)$ **then**
    preorder$(v_j) \leftarrow$ val$(ij) + 1$



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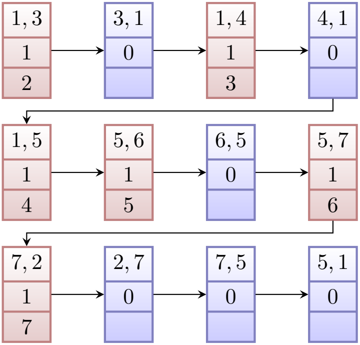
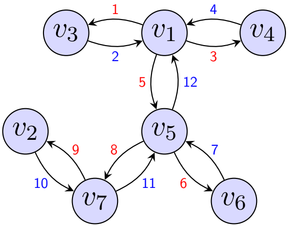


Numbering Vertices — Postorder (bottom-up)

The postorder number of a vertex v_i is the number of retreat edges in ET traversed before reaching v_i for the last time $+1$.

postorder(root) $\leftarrow n$
forall $(v_i, v_j) \in$ ET **do in parallel**
  if pos(v_i, v_j) $<$ pos(v_j, v_i) **then**
    val(ij) $\leftarrow$ 0
  else
    val(ij) $\leftarrow$ 1
Parallel-Prefix(ET, $+$)
forall $(v_i, v_j) \in$ ET **do in parallel**
  if pos(v_i, v_j) $>$ pos(v_j, v_i) **then**
    postorder(v_i) $\leftarrow$ val(ij)



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postorder(root) $\leftarrow n$
forall $(v_i, v_j) \in$ ET **do in parallel**
  if pos(v_i, v_j) $<$ pos(v_j, v_i) **then**
    val(ij) $\leftarrow$ 0
  else
    val(ij) $\leftarrow$ 1
Parallel-Prefix(ET, $+$)
forall $(v_i, v_j) \in$ ET **do in parallel**
  if pos(v_i, v_j) $>$ pos(v_j, v_i) **then**
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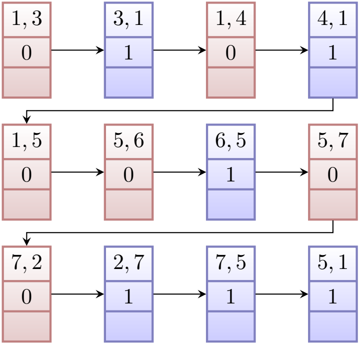

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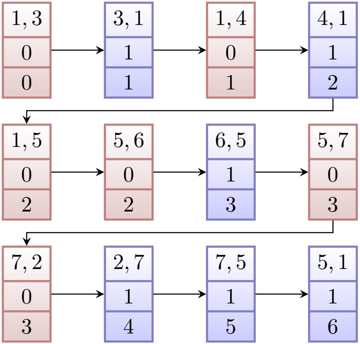

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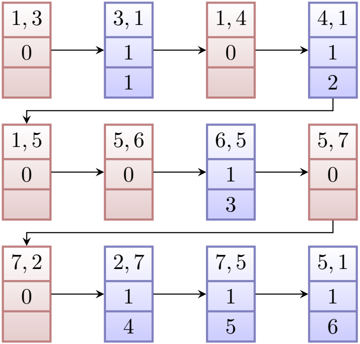


Tree Binary Relations

- "An Euler Tour returns to the root of a subtree only after all vertices in the subtree have been visited."
- v_i is an ancestor of v_j (v_j is a descendant of v_i) iff:

$$\text{preorder}(v_i) \leq \text{preorder}(v_j) < \text{preorder}(v_i) + \text{des}(v_i)$$

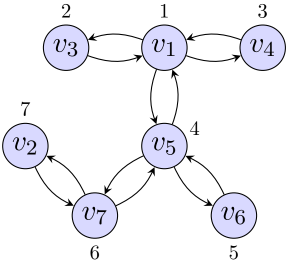
Examples:

v_7 is an ancestor of v_2:

$$6 \leq 7 < 6 + 2 = 8$$

v_6 is *not* an ancestor of v_2:

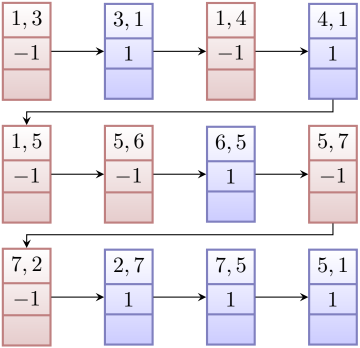
$$5 \leq 7 \not< 5 + 1 = 6$$



Levels of Vertices

level(v_j) $=$ retreat edges $-$ advance edges, following first v_j in ET

level(root) $\leftarrow$ 0
forall $(v_i, v_j) \in$ ET **do in parallel**
  if pos(v_i, v_j) $<$ pos(v_j, v_i) **then**
    val(ij) $\leftarrow -1$
  else
    val(ij) $\leftarrow 1$
Parallel-Suffix(ET, $+$)
forall $(v_i, v_j) \in$ ET **do in parallel**
  if pos(v_i, v_j) $<$ pos(v_j, v_i) **then**
    level(v_j) $\leftarrow$ val(ij) $+ 1$



Levels of Vertices

level(v_j) = retreat edges − advance edges, following first v_j in ET

level(root) $\leftarrow$ 0
forall $(v_i, v_j) \in$ ET **do in parallel**
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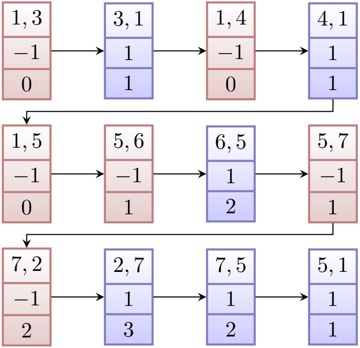

Levels of Vertices

level(v_j) = retreat edges − advance edges, following first v_j in ET

level(root) $\leftarrow$ 0
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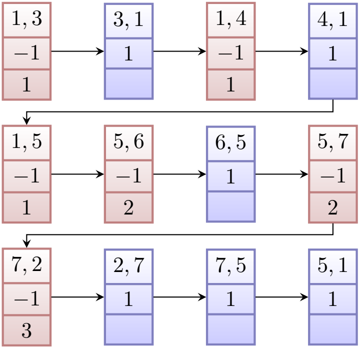

Levels of Vertices

$$\begin{aligned}\text{level}(v_j) &= {\color{blue}\text{retreat}} \text{ edges} - {\color{red}\text{advance}} \text{ edges, following first } v_j \text{ in ET} \\ &= {\color{red}\text{advance}} \text{ edges} - {\color{blue}\text{retreat}} \text{ edges, up to first } v_j \text{ in ET}\end{aligned}$$

level(root) $\leftarrow$ 0
forall $(v_i, v_j) \in$ ET **do in parallel**
  if pos(v_i, v_j) $<$ pos(v_j, v_i) **then**
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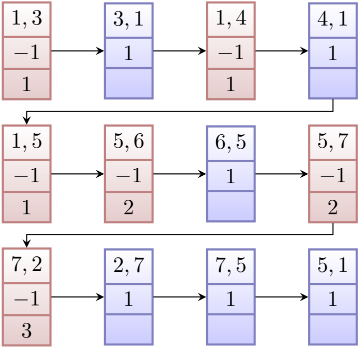


Outline

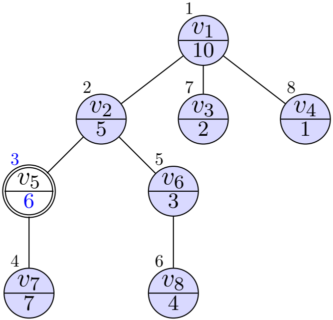
1. Introduction
2. Building Euler Tours
3. Applications of Euler Tours
   - Utility Computations
   - Simple Applications of Euler Tours
   - Computing Minima
4. Conclusion

The Interval Minima Problem

$$S = \{s_k : k = \text{preorder}(v_i) \Rightarrow s_k = \text{val}(v_i)\}$$

Example:

$$3 = \text{preorder}(v_5), s_3 = 6$$
$$S = \{10, 5, 6, 7, 3, 4, 2, 1\}$$



The Interval Minima Problem

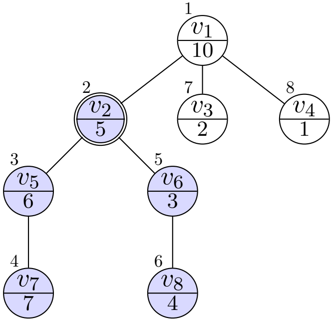
$$S = \{s_k : k = \text{preorder}(v_i) \Rightarrow s_k = \text{val}(v_i)\}$$

$$S_{v_i} = \{s_k, s_{k+1}, \ldots, s_\ell\}, \quad \ell = k + \text{des}(v_i) - 1$$

Example:

$$3 = \text{preorder}(v_5), s_3 = 6$$
$$S = \{10, 5, 6, 7, 3, 4, 2, 1\}$$

$$S_{v_2} : k = 2, \ell = 2 + 5 - 1 = 6$$
$$S_{v_2} = \{5, 6, 7, 3, 4\}$$



The Interval Minima Problem

$$S = \{s_k : k = \text{preorder}(v_i) \Rightarrow s_k = \text{val}(v_i)\}$$

$$S_{v_i} = \{s_k, s_{k+1}, \ldots, s_\ell\}, \quad \ell = k + \text{des}(v_i) - 1$$

$$\boxed{\text{find } \min(S_{v_i}) \quad \forall\, v_i}$$

Example:

$$3 = \text{preorder}(v_5), s_3 = 6$$
$$S = \{10, 5, 6, 7, 3, 4, 2, 1\}$$

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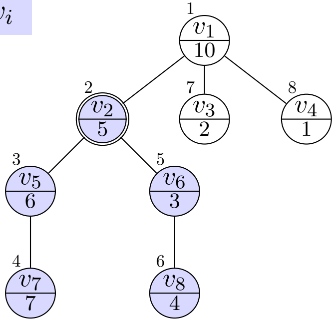

The Interval Minima Tree

Prefix Minima (PM)

The prefix minima of a sequence $\{a_1, a_2, \ldots, a_n\}$ are given by a sequence $\{b_1, b_2, \ldots, b_n\}$ where $b_i = \min(a_1, a_2, \ldots, a_i)$.

Suffix Minima (SM)

The suffix minima of a sequence $\{a_1, a_2, \ldots, a_n\}$ are given by a sequence $\{d_1, d_2, \ldots, d_n\}$ where $d_i = \min(a_i, a_{i+1}, \ldots, a_n)$.

PM	2	1	1	1	1	1
Seq.	2	1	3	4	5	6
SM	1	1	3	4	5	6

The Interval Minima Tree

Prefix Minima (PM)

The prefix minima of a sequence $\{a_1, a_2, \ldots, a_n\}$ are given by a sequence $\{b_1, b_2, \ldots, b_n\}$ where $b_i = \min(a_1, a_2, \ldots, a_i)$.

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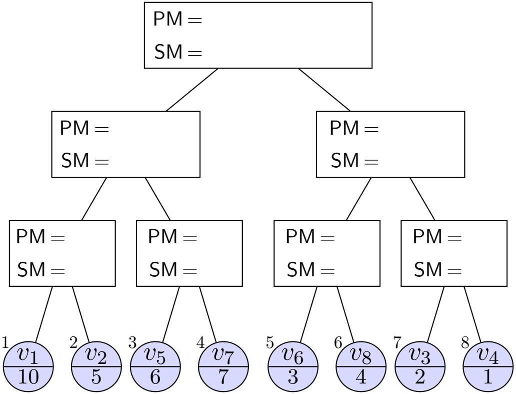
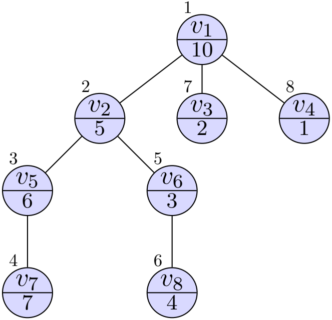
A complete binary tree such that:

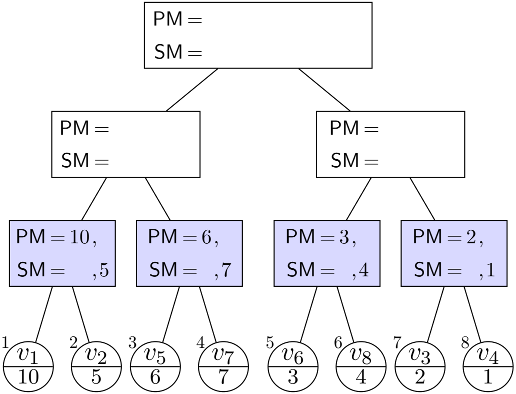
- Leaves are $s_1, s_2, \ldots, s_n$.
- Each internal node x contains two sequences: the prefix minima (PM) and suffix minima (SM) of the values held by leaves in the subtree rooted at x.

Constructing the Tree





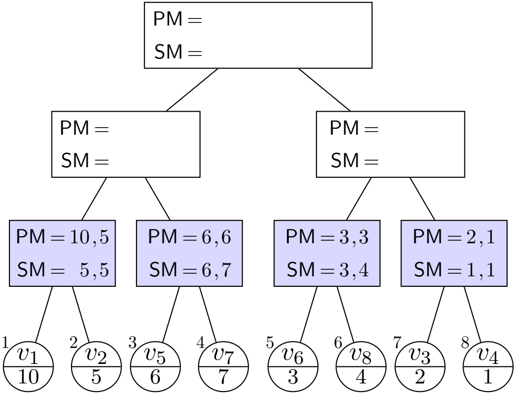
Constructing the Tree



PM:	L	$\{\min(l_p, r_i)\}$
SM:	$\{\min(r_1, l_i)\}$	R

L	R
$l_1, l_2, \ldots, l_p$	$r_1, r_2, \ldots, r_p$

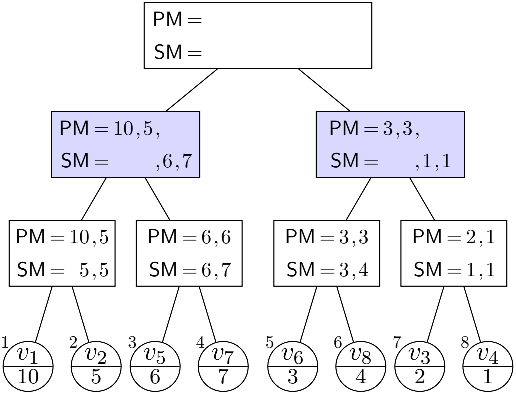
Constructing the Tree

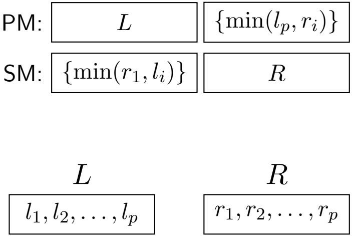


PM:	L	$\{\min(l_p, r_i)\}$
SM:	$\{\min(r_1, l_i)\}$	R

L	R
$l_1, l_2, \ldots, l_p$	$r_1, r_2, \ldots, r_p$

Constructing the Tree





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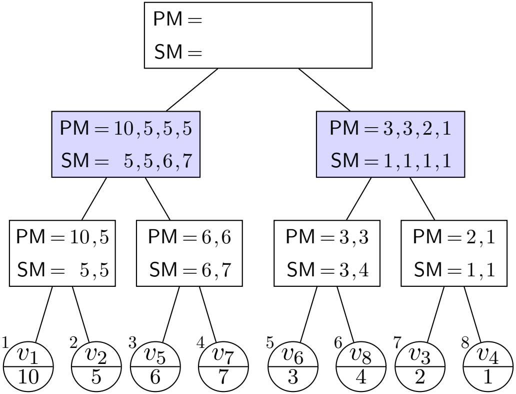
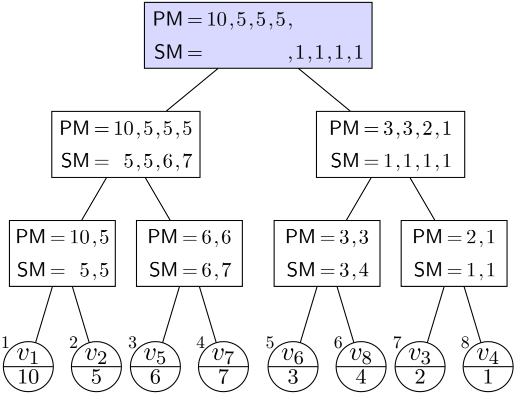


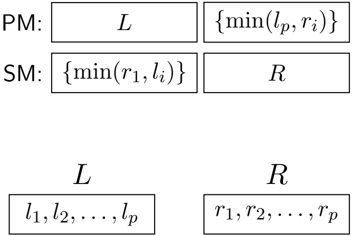
PM:	L	$\{\min(l_p, r_i)\}$
SM:	$\{\min(r_1, l_i)\}$	R

L: $l_1, l_2, \ldots, l_p$

R: $r_1, r_2, \ldots, r_p$

Constructing the Tree





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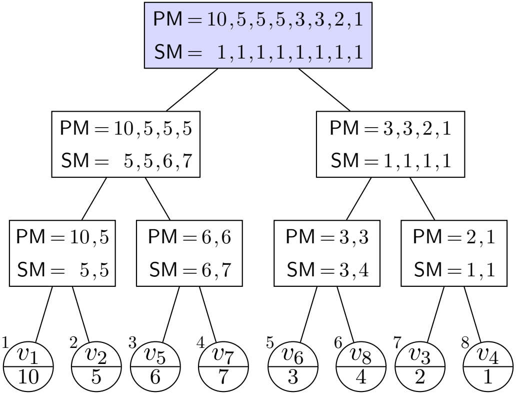
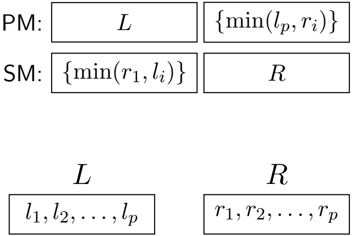
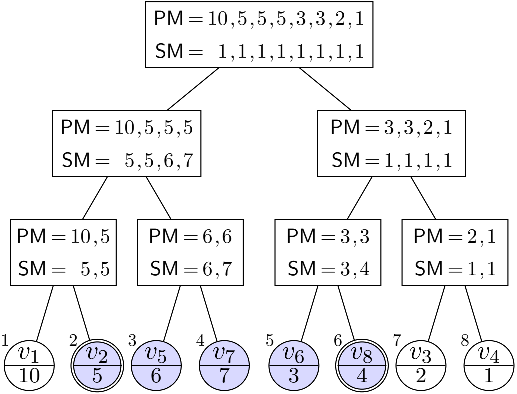


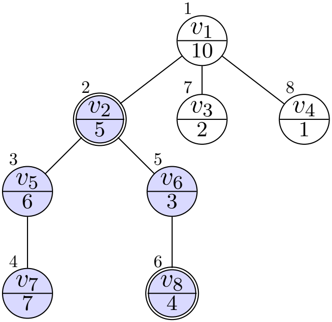


Finding the Minima Using the Tree

An Example: S_{v_2}

$$k = 2, \qquad \ell = k + 5 - 1 = 6 \qquad \Rightarrow \qquad S_{v_2} = \{s_2, \ldots, s_6\}$$





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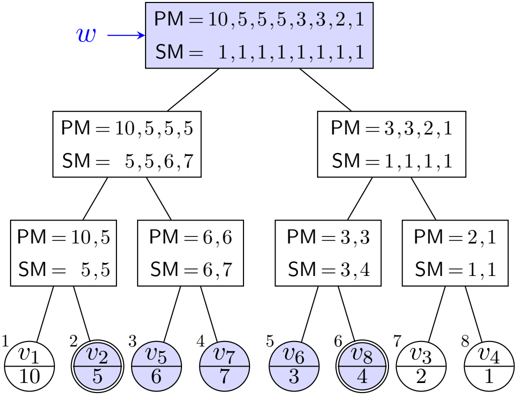

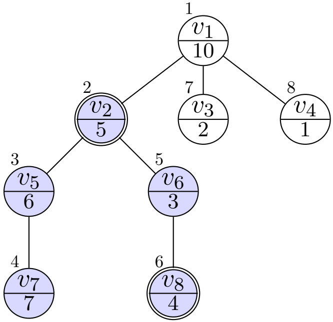

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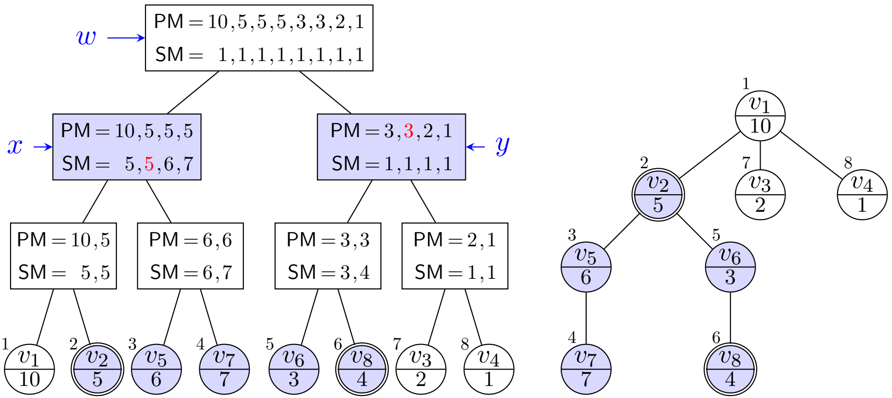


The Interval Minima Algorithm

Input: A tree T **Output:** $\min(S_{v_i})$ for all $v_i \in T$

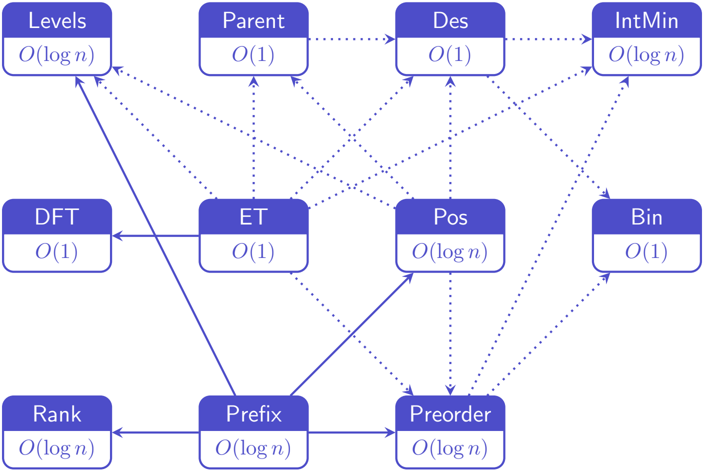
$O(1)$ ET $\leftarrow$ Parallel-Euler-Tour(T)
$O(\log n)$ Parallel-Preorder(ET)
$O(\log n)$ Parallel-Descendants(ET)
$O(\log n)$ $T' \leftarrow$ Interval-Minima-Tree(T)
forall $v_i \in T$ **do in parallel**
$O(1)$ $k \leftarrow$ preorder(v_i), $\ell \leftarrow k +$ des(v_i) $- 1$
$O(\log n)$ $w \leftarrow$ Lowest-Common-Ancestor(s_k, s_ℓ) in T'
$O(1)$ $x \leftarrow$ Left-Child(w), $y \leftarrow$ Right-Child(w);
$O(1)$ $\min(S_{v_i}) \leftarrow \min(\mathsf{SM}_k^{(x)}, \mathsf{PM}_\ell^{(y)})$

Complexity:

$$t(n) = O(\log n) \qquad p(n) = O(n) \qquad \Rightarrow \qquad c(n) = n\log(n)$$

Outline

Summary


Levels
$O(\log n)$
Parent
$O(1)$
Des
$O(1)$
IntMin
$O(\log n)$
DFT
$O(1)$
ET
$O(1)$
Pos
$O(\log n)$
Bin
$O(1)$
Rank
$O(\log n)$
Prefix
$O(\log n)$
Preorder
$O(\log n)$

Thank You

Thank You!

Outline

1. Introduction
2. Building Euler Tours
3. Applications of Euler Tours
4. Conclusion
   - Summary
   - Questions

Questions

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