Algebraic Geometry With $YZ = X^2$

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Projective Space

How do we describe the solutions to $YZ = X^2$ in $\mathbb{C}$?

- $\rightarrow$ if $(X, Y, Z) = (a, b, c)$ is a solution, then $\lambda(a, b, c)$ is also a solution for all $\lambda \in \mathbb{C}$
- $\rightarrow$ it seems each solution corresponds to a solution set...
- $\rightarrow$ we will see these solution sets can be described as equivalence classes

Projective Space 2

Define the equivalence relation $\sim$ on nonzero 3-tuples in $\mathbb{C}$ as $(x_0, x_1, x_2) \sim (y_0, y_1, y_2)$ if and only if there exists $\lambda \in \mathbb{C}^*$ such that $x_i = \lambda y_i$ for all $0 \leq i \leq 2$.

$\rightarrow$ Then $\mathbb{P}^2(\mathbb{C})$, or $\mathbb{P}^2$, is the set of these equivalence classes. Denote the equivalence classes of (x_0, x_1, x_2) by $[x_0 : x_1 : x_2]$

All solutions to $YZ = X^2$ that are not $(0, 0, 0)$ can be described by points in $\mathbb{P}^2$.

Varieties

The notation $V : YZ = X^2$ defines V as the set of points in $\mathbb{P}^2$ satisfying $YZ = X^2$. i.e. $V = \{[X : Y : Z] \in \mathbb{P}^2 \mid YZ = X^2\}$.

- $\rightarrow$ V is what we call an algebraic set. In this case, it is also called a variety.
- $\rightarrow$ $\mathbb{P}^n$, as well, is a variety for any $n \in \mathbb{Z}^+$.

Functions

How do we understand functions on $\mathbb{P}^2$?

- $\rightarrow$ We say a polynomial $f \in \mathbb{C}[X, Y, Z]$ is homogenous of degree d if each term has the same degree d. Equivalently, $f(\lambda(X, Y, Z)) = \lambda^d f(X, Y, Z)$ for any $\lambda \in \mathbb{C}^*$.
- $\rightarrow$ For example, $YZ - X^2$ is homogenous, but $YZ - 1$ and $Y^3 - XZ$ are not.

Functions 2

We define the function field of $\mathbb{P}^2$, denoted $\mathbb{C}(\mathbb{P}^2)$, as the subfield of functions $\frac{f}{g}$ in $\mathbb{C}(X, Y, Z)$ where f and g are homogenous polynomials of the same degree.

$\rightarrow$ These functions are well-defined as functions of $\mathbb{P}^2$

Functions 3

Like with $\mathbb{P}^2$, we wish to define a function field over V. To do this, we essentially set $YZ - X^2$ to 0 in $\mathbb{C}(\mathbb{P}^2)$.

- $\rightarrow$ To do this, first take the elements of $\mathbb{C}(\mathbb{P}^2)$
- $\rightarrow$ Remove the elements $\frac{f}{g}$ where g is a multiple of $YZ - X^2$.
- $\rightarrow$ Set that two functions $\frac{f_1}{g_1}$ and $\frac{f_2}{g_2}$ are equivalent if $\frac{f_1}{g_1} - \frac{f_2}{g_2}$ is a multiple of $YZ - X^2$.

Rational Maps

A rational map is a map between two varieties. We define rational maps between varieties V_1 and V_2 by:

$$\varphi : V_1 \to V_2, \quad \varphi = [f_0 : \ldots : f_n]$$

where the functions $f_0, \ldots, f_n \in \mathbb{C}(V_1)$, $V_2 \in \mathbb{P}^n$, and for any point $P \in V_1$ where $f_0, \ldots, f_n$ are all defined, $\varphi(P) \in V_2$.

Example: Using the V from before, define

$$\varphi : V \to \mathbb{P}^1, \quad \varphi = \left[\frac{Y}{Z} : 1\right].$$

$\rightarrow$ The point $[2 : 4 : 1] \in V$ maps to $[4 : 1]$

Injections of Function Fields

We can use the map φ to define a map between function fields:

$$\varphi^* : \mathbb{C}(\mathbb{P}^1) \to \mathbb{C}(V), \quad \varphi^* f = f \circ \varphi.$$

$$\begin{tikzcd}
V \arrow[r, "\varphi"] \arrow[dr, dashed, "\varphi^* f = f \circ \varphi"'] & \mathbb{P}^1 \arrow[d, "f"] \\
& \mathbb{C}
\end{tikzcd}$$

$\rightarrow$ $\varphi^* \frac{X}{Y} = \frac{Y}{Z}$

$\rightarrow$ $\varphi^* \frac{XY^2 - X^3}{Y^3} = \frac{YZ^2 - Y^3}{Z^3}$

Injections of Function Fields 2

- Our map φ^* takes any function in $\mathbb{C}(\mathbb{P}^1)$ and maps $X \mapsto Y$, $Y \mapsto Z$
- This is an injective homomorphism of fields that fixes $\mathbb{C}$
- No other rational map can induce this injection

Now, we generalize this

Curves

First, we define **curves**.

Essentially, a variety is a curve if its function field can be described in one free variable over $\mathbb{C}$.

The prior examples V and $\mathbb{P}^1$ are curves. We state without proof that

$$\mathbb{C}(\mathbb{P}^1) \cong \mathbb{C}(V) \cong \mathbb{C}(x)$$

General Claim

1. Let $\varphi : V_1 \to V_2$ be a nonconstant rational map where V_1 and V_2 are curves. Then φ induces an injection of function fields $\varphi : \mathbb{C}(V_2) \to \mathbb{C}(V_1)$ that fixes $\mathbb{C}$.
2. For each injection of function fields $\iota : \mathbb{C}(V_2) \to \mathbb{C}(V_1)$ that fixes $\mathbb{C}$, there is a unique nonconstant rational map $\varphi : V_1 \to V_2$ such that $\varphi^* = \iota$.
3. Thus, there is a one-to-one correspondence between nonconstant rational maps $V_1 \to V_2$ and injections $\mathbb{C}(V_2) \to \mathbb{C}(V_1)$ that fix $\mathbb{C}$.