

Algebraic Geometry With $YZ = X^2$

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Theorem:

1. Let $\varphi : V_1 \rightarrow V_2$ be a nonconstant rational map where V_1 and V_2 are curves. Then φ induces an injection of function fields $\varphi^* : \mathbb{C}(V_2) \rightarrow \mathbb{C}(V_1)$ that fixes \mathbb{C} .
2. For each injection of function fields $\iota : \mathbb{C}(V_2) \rightarrow \mathbb{C}(V_1)$ that fixes \mathbb{C} , there is a unique nonconstant rational map $\varphi : V_1 \rightarrow V_2$ such that $\varphi^* = \iota$.

How do we describe the solutions to $YZ = X^2$ in \mathbb{C} ?

- if $(X, Y, Z) = (a, b, c)$ is a solution, then $\lambda(a, b, c)$ is also a solution for all $\lambda \in \mathbb{C}$
- it seems each solution corresponds to a solution set...
- we will describe these solution sets as equivalence classes

Projective Space 2

Define the equivalence relation \sim on nonzero 3-tuples in \mathbb{C} as $(x_0, x_1, x_2) \sim (y_0, y_1, y_2)$ if and only if there exists $\lambda \in \mathbb{C}^\times$ such that $x_i = \lambda y_i$ for all $0 \leq i \leq 2$.

→ Then $\mathbb{P}^2(\mathbb{C})$, or \mathbb{P}^2 , is the set of these equivalence classes.

Denote the equivalence classes of (x_0, x_1, x_2) by $[x_0 : x_1 : x_2]$

All solutions to $YZ = X^2$ that are not $(0, 0, 0)$ can be described by points in \mathbb{P}^2 .

The notation $V : YZ = X^2$ defines V as the set of points in \mathbb{P}^2 satisfying $YZ = X^2$. i.e. $V = \{[X : Y : Z] \in \mathbb{P}^2 \mid YZ = X^2\}$.

→ V is called a variety.

→ \mathbb{P}^n , as well, is a variety for any $n \in \mathbb{Z}^+$.

How do we understand functions on \mathbb{P}^2 ?

- We say a polynomial $f \in \mathbb{C}[X, Y, Z]$ is homogenous of degree d if each term has the same degree d . Equivalently, $f(\lambda(X, Y, Z)) = \lambda^d f(X, Y, Z)$ for any $\lambda \in \mathbb{C}^\times$.
- For example, $YZ - X^2$ is homogenous, but $YZ - 1$ and $Y^3 - XZ$ are not.

Functions 2

We define the function field of \mathbb{P}^2 , denoted $\mathbb{C}(\mathbb{P}^2)$, as the subfield of functions $\frac{f}{g}$ in $\mathbb{C}(X, Y, Z)$ where f and g are homogenous polynomials of the same degree.

→ These functions are well-defined as functions of \mathbb{P}^2

$$\frac{f(\lambda(X, Y, Z))}{g(\lambda(X, Y, Z))} = \frac{\lambda^d f(X, Y, Z)}{\lambda^d g(X, Y, Z)} = \frac{f(X, Y, Z)}{g(X, Y, Z)}$$

Functions 3

Like with \mathbb{P}^2 , we wish to define a function field over V . To do this, we essentially wish to set $YZ - X^2$ to 0 in $\mathbb{C}(\mathbb{P}^2)$.

→ To do this, first form the set of $\frac{f}{g}$ in $\mathbb{C}(\mathbb{P}^2)$ such that $g \notin (YZ - X^2) \subseteq \mathbb{C}[X, Y, Z]$

→ Set that two functions $\frac{f_1}{g_1}$ and $\frac{f_2}{g_2}$ are equivalent if $f_1 g_2 - f_2 g_1 \in (YZ - X^2)$

★ Spiritually we are saying $\frac{f_1}{g_1} - \frac{f_2}{g_2}$ is a “multiple” of $YZ - X^2$

Rational Maps

A rational map is a map between two varieties. We define rational maps between varieties V_1 and V_2 by:

$$\varphi : V_1 \rightarrow V_2, \quad \varphi = [f_0 : \dots : f_n]$$

where the functions $f_0, \dots, f_n \in \mathbb{C}(V_1)$, $V_2 \subseteq \mathbb{P}^n$, and for any point $P \in V_1$ where f_0, \dots, f_n are all defined, $\varphi(P) \in V_2$.

Example: Using the V from before, define

$$\varphi : V \rightarrow \mathbb{P}^1, \quad \varphi = \left[\frac{Y}{Z} : 1 \right].$$

→ The point $[2 : 4 : 1] \in V$ maps to $[4 : 1]$

Injectons of Function Fields

We can use the map φ to define a map between function fields:

$$\varphi^* : \mathbb{C}(\mathbb{P}^1) \rightarrow \mathbb{C}(V), \quad \varphi^* f = f \circ \varphi.$$

$$\begin{array}{ccc} V & \xrightarrow{\varphi} & \mathbb{P}^1 \\ & \searrow \varphi^* f = f \circ \varphi & \downarrow f \\ & & \mathbb{C} \end{array}$$

$$\rightarrow \varphi^* \frac{X}{Y} = \frac{Y}{Z}$$

$$\rightarrow \varphi^* \frac{XY^2 - X^3}{Y^3} = \frac{YZ^2 - Y^3}{Z^3}$$

Injectons of Function Fields 2

- Our map φ^* takes any function in $\mathbb{C}(\mathbb{P}^1)$ and maps $X \mapsto Y$,
 $Y \mapsto Z$
- This is an injective homomorphism of fields that fixes \mathbb{C}

First, we define **curves**.

Essentially, a variety is a curve if its function field can be described in one free variable over \mathbb{C} .

The prior examples V and \mathbb{P}^1 are curves. In fact,

$$\mathbb{C}(\mathbb{P}^1) \cong \mathbb{C}(V) \cong \mathbb{C}(x)$$

Big Claim

1. Let $\varphi : V_1 \rightarrow V_2$ be a nonconstant rational map where V_1 and V_2 are curves. Then φ induces an injection of function fields $\varphi^* : \mathbb{C}(V_2) \rightarrow \mathbb{C}(V_1)$ that fixes \mathbb{C} .
2. For each injection of function fields $\iota : \mathbb{C}(V_2) \rightarrow \mathbb{C}(V_1)$ that fixes \mathbb{C} , there is a unique nonconstant rational map $\varphi : V_1 \rightarrow V_2$ such that $\varphi^* = \iota$.
3. Thus, there is a one-to-one correspondence between nonconstant rational maps $V_1 \rightarrow V_2$ and injections $\mathbb{C}(V_2) \rightarrow \mathbb{C}(V_1)$ that fix \mathbb{C} .

Silverman, Joseph H. *The Arithmetic of Elliptic Curves*. Springer, 2016.