Balanced Partitions of Weighted Graphs: Overview and a Topological Approach

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1 Introduction

1.1 Acknowledgements

Thank you to professor Ali Daemi for being an invaluable thesis-advisor and caring mentor. Also, thank you to the members of the WashU mathematics department as a whole as for providing me a wonderful and seamless experience as a math major.

1.2 Abstract

In this paper, we study the partitioning of k-connected graphs into connected subgraphs, a problem that has significant applications in areas such as network design and power grid islanding. We first prove the Lovasz-Gyori theorem, emulating the combinatorial approach used by Gyori. Additionally, we provide the proof of a weighted analog in the cases of 2-connected and 3-connected graphs. We then transition into Lovasz's method using topology, summarizing his approach and providing the proof of Lovasz-Gyori for 3-connected graphs by constructing a cell-complex out of the spanning trees of the graph. Finally, we employ Lovasz's approach to provide a new proof of the weighted variant for 3-connected graphs, illustrating a technique we believe can be used to solve our conjectured weighted analog of the Lovasz-Gyori theorem.

1.3 Main Theorem/Conjecture

Theorem 1.1 (Lovász/Györi). *Suppose G is an unweighted, k-connected graph with n vertices. Then for any k vertices $\{v_1, \dots, v_k\}$, and any k positive integers $n_1, \dots, n_k$ such that*

$$n_1 + \cdots + n_k = n,$$

there exists a partition of G, $\{V_1, \dots, V_k\}$, such that $v_i \in V_i$ for all i, the induced subgraph of V_i is connected for all i, and the induced subgraph of V_i has n_i elements for each i.

Problem 1.1 (Balanced Connected Integer k-Partition Problem). *Given a weighted graph G with positive weight function p, find a partition of G into induced connected subgraphs $\{G_1, \dots, G_k\}$ such that $\min_{i\in[k]} p(G_i)$ is maximized.*

Conjecture 1.1. *Suppose G is a weighted, k-connected graph with n vertices and a weight function $p\colon V(G) \to \mathbb{R}$, and suppose that $p(G) = 0$. Let $v_1, \dots, v_k$ be k vertices of G such that $p(v_i)$ has the same sign for all i. Then there exists a partition of G, $\{V_1, \dots, V_k\}$, such that $v_i \in V_i$ for all i, the induced subgraph G_i of V_i is connected, and*

$$|p(G_1)| + |p(G_2)| + \cdots + |p(G_k)| \leq (k-1)\Big(\max_{v\in V(G)} w(v)\Big).$$

1.4 Literature Review

Lovász proved theorem 1.1 in 1978. He used homology theory, a branch of algebraic topology, constructing a cell-complex from the spanning arborescences of a directed graph and showing that the homology groups of the cell-complex are trivial [2]. Ervin Györi also proved theorem 1.1 in 1978, using a strictly combinatorial approach [3]. In 2019, Sultan discussed and provided proofs of conjecture 1.1 in the cases $k = 2, 3$ [1]. Hoyer and Thomas provided a self-contained exposition of

Gyori's proof in 2016; we emulate their proof style in this paper in the proof of theorem 1.1, with minor changes [4]. In general, problem 1.1 is frequently studied, with most literature focused on efficient algorithms for finding balanced partitions, and is related to theorem 1.1 and conjecture 1.1.

1.5 Applications

Conjecture 1.1 has numerous potential applications related to networks. For instance, power grid islanding—dividing an interconnected power grid into disconnected regions—requires balancing supply and demand of energy efficiently. Proving this conjecture would provide conditions that guarantee a configuration with a bounded supply-demand difference.

As a hypothetical example, consider a data network with m servers and n messages, where server s_i can hold k_i bytes and message ℓ_i is b_i bytes, with

$$k_1 + \cdots + k_m = b_1 + \cdots + b_n.$$

Modeling the network as a graph G, where servers and messages are nodes and the weight function maps a server to negative its capacity and a message to its size, conjecture 1.1 provide conditions which imply that a valid partition exists that distributes messages with bounded overload.

1.6 A New Proof of Conjecture 1.1 when $k = 3$

The proof of Theorem 1.2 for $k = 3$, as given in [1], utilizes a combinatorial approach. In this paper, we adapt the methods used by Lovász to prove Theorem 1.2 when $k = 3$, and we discuss the potential for generalizing this strategy into a complete proof of Theorem 1.2.

2 The Case $k = 2$

The results for $k = 2$ in both the unweighted and weighted cases follow from the following lemma, proved by Lovász:

Lemma 2.1. *We call two spanning trees of G* neighboring with respect to *a if their greatest common subtree has $n-1$ vertices and includes a. Let G be a 2-connected graph with n vertices and let a be a fixed vertex of G. Then for any two spanning trees of G, T_1, T_2, there exists a chain of neighboring spanning trees of G connecting T_1 and T_2.*

Proof. Let A be the largest common subtree of T_1 and T_2. If $|V(G)| - |V(A)| = 1$, T_1 and T_2 are already neighbors. Suppose $|V(G)| - |V(A)| = k > 1$. Let $e = (u, v)$ and $e' = (u', v')$ be edges in T_1 and T_2 respectively connecting v and v' to A. First, suppose $v \neq v'$. Then $A + e + e'$ is a subtree of G and can be completed into a spanning-tree of G, T'. By induction, there exists chains connecting T_1 to T' and T' to T_2, so the theorem holds.

So assume that $v = v'$. Because $|V(G)| - |V(A)| > 1$ and because G is 2-connected, there exists an edge of G, $e'' = (x, y)$, with $y \in V(G) - V(A) - \{v\}$ and $x \in V(A)$. Then the $A + e + e''$ and $A + e' + e''$ are subtrees and can be completed into spanning trees T' and T'', and, by induction, there exists a chain connecting T_1 to T' to T'' to T_2. ∎

2.1 Unweighted Case

We first give a proof of Theorem 1.1 in the case $k = 2$.

Theorem 2.1. *Theorem 1.1 is true when $k = 2$.*

Proof. Create a new graph G' by adding a new vertex a with edges to v_1 and v_2. Then G' is also 2-connected, so $K = G' - \{v_1\}$ is connected. Take a spanning-tree of K and adjoin v_1 to K through (a, v_1). Repeat this process for v_2, creating two spanning trees of G', T_1 and T_2. By Lemma 2.1, there exists a chain of spanning trees of G neighboring with respect to a and connecting T_1 to T_2, i.e.,

$$(T_1 = A_1, A_2, \ldots, A_m = T_2).$$

Let $\phi(T)$ denote the number of vertices separated from a through v_1. Then $\phi(T_1) = 1$ and $\phi(T_2) = n - 1$. For each transition in the chain, the number of vertices separated from a by v_1 changes by at most 1, and thus ϕ maps the chain onto $\{1, 2, \ldots, n-1\}$ by the intermediate value theorem. Thus, we can take A_j such that $\phi(A_j) = n_1$. The n_1 vertices separated from a through v_1 and the $n_2 = n - n_1$ vertices separated from a through v_2 give the desired V_1, V_2.

□

2.2 Weighted Case

Next, we prove conjecture 1.1 in the case $k = 2$.

Theorem 2.2. *Conjecture 1.1 is true when $k = 2$.*

Proof. Without loss of generality, take v_1, v_2 with positive weights. Create the chain

$$(T_1 = A_1, A_2, \ldots, A_m = T_2),$$

the same way as in the unweighted case, and let $\phi(T)$ denote the total weight of the vertices separated from a through v_1. Then

$$|\phi(T_i) - \phi(T_{i+1})| \leq \max_{v \in V(G)} p(v).$$

Furthermore, $\phi(T_1) = p(v_1) > 0$ and $\phi(T_2) = -p(v_2) < 0$. Thus, we can take A_j, A_{j+1} such that $\phi(A_j) > 0$ and $\phi(A_{j+1}) < 0$. By considering signs,

$$|\phi(A_j) - \phi(A_{j+1})| = |\phi(A_j)| + |\phi(A_{j+1})| \leq \max_{v \in V(G)} p(v),$$

so without loss of generality suppose $|\phi(A_j)| \leq \frac{\max_{v \in V(G)} p(v)}{2}$. Then taking V_1, V_2 as in the unweighted case gives

$$|p(V_1)| + |p(V_2)| \leq 2 \cdot \frac{\max_{v \in V(G)} p(v)}{2} = \max_{v \in V(G)} p(v),$$

and the proof is complete. □

2.3 Discussion

The idea of showing the existence of the desired partition through the intermediate value theorem is crucial to the proofs for higher k. In Sultan, the weighted case for $k = 3$ is shown using a generalization of the idea where $k = 2$, invoking the intermediate value theorem multiple times and over multiple chains. In Lovász, the proof for $k = 3$ invokes the intermediate value theorem on a continuous mapping from a cell-complex into $\mathbb{R}^2$. We will make this idea explicit later in the paper.

3 Gyori's Proof

3.1 Preliminaries

We provide a proof of Theorem 1.1 using the combinatorial approach from Gyori, emulating the exposition provided by Hoyer and Thomas.

We prove the following theorem, which implies Theorem 1.1:

Theorem 3.1. *Let $k \geq 2$ be an integer, let G be a k-connected graph on n vertices, let $v_1, \ldots, v_k$ be distinct vertices of G, and let $n_1, \ldots, n_k$ be positive integers with $n_1 + \cdots + n_k < n$. Let $G_1, \ldots, G_k$ be disjoint connected subgraphs of G such that, for $i = 1, \ldots, k$, the graph G_i has n_i vertices and $v_i \in V(G_i)$. Then G has disjoint connected subgraphs $G'_1, \ldots, G'_k$ such that $v_i \in V(G'_i)$ for $i = 1, \ldots, k$, the graph G'_1 has $n_1 + 1$ vertices, and for $i = 2, \ldots, k$ the graph G'_i has n_i vertices.*

We now introduce some definitions using the terminology of [4].

Definition 3.1. *Reservoir: Let v be a vertex in G_i for some i. The* reservoir *of v, denoted $R(v)$, is the set of all vertices u in G_i such that there exists a path in G_i from u to v_i that is disjoint from v_i.*

Note that $v_i \in R(v)$ for all $v \in G_i - \{v_i\}$ and $R(v_i) = \emptyset$. Moreover, if $u \in G_i$ is not in the reservoir of v, then u is separated from $R(v)$ through v.

Definition 3.2. *Cascade: A* cascade *is a sequence of distinct vertices $w_1, \ldots, w_m$ in $G_i - \{v_i\}$ such that for each j, $w_{j+1} \notin R(w_j)$.*

Definition 3.3. *Configuration: A* configuration *is a choice of subgraphs $G_1, \ldots, G_k$ as defined in the theorem statement, together with a set of k distinct cascades corresponding to each G_i.*

Definition 3.4. *Rank: If v is a vertex in a cascade of G_i, its* rank *is defined recursively: If v has a neighbor in G_1, then its rank is 1. Otherwise, the rank of v is the smallest integer n such that v has a neighbor of rank $n-1$ in some G_j with $j \neq i$. If no such neighbor exists, the rank of v is undefined.*

We will let p_r denote the total number of vertices in the reservoirs of vertices of rank r. A configuration is said to be *valid* if every cascade vertex has a defined rank and the ranks in a cascade are strictly increasing.

Definition 3.5. *r-optimal: A valid configuration is* r-optimal *if it maximizes p_r, subject to the condition that p_{r-1} (and recursively p_{r-2}, …, p_1) is maximized.*

In other words, if a is the maximum value of p_1 over all valid configurations and K is the set of valid configurations with $p_1 = a$, then a 2-optimal configuration is one (not necessarily unique) in K that maximizes p_2. By definition, an r-optimal configuration is k-optimal for all $1 \leq k \leq r$. A configuration is called *optimal* if it is r-optimal for all r.

Definition 3.6. *Bridge: Let S be the set of vertices not contained in any G_i. A* bridge *is an edge from a vertex in S to the reservoir of a cascade vertex. The* rank *of a bridge is defined as the minimum rank among all cascade vertices w such that the bridge has an endpoint in $R(w)$.*

3.2 Proof of Theorem 1.1

We now begin proving the main theorem.

Lemma 3.1. *Suppose there exists an optimal configuration containing a bridge. Then the conclusion of the theorem is true.*

Proof. Assume there is an optimal configuration containing a bridge e. If this bridge has rank k, let $e = (a, b)$ be an edge from S into $R(w)$, where w is a cascade vertex of rank k. Note that this configuration is k-optimal.

First, suppose that w separates G_i; that is, if we remove w from G_i the subgraph becomes disconnected. Consider the subgraph K of G_i with vertex set $B = V(G_i) - R(w)$. For any $u, v \in B$, one may construct a path from u to v by concatenating the paths from u to w and from w to v, so K is connected. Thus, there exists a vertex $z \in V(G_i) - R(w) - \{w\}$ that does not disconnect G_i when removed. Note that any cascade vertex in $V(G_i) - R(w) - \{w\}$ has rank greater than k by definition. Removing all cascade vertices with rank greater than k (which preserves validity) and then moving a to G_i and z to S increases $R(w)$ by 1 (i.e., p_k increases by 1), contradicting k-optimality. Thus, we may assume that w does not separate G_i.

We will prove that there exists a $1 - optimal$ configuration with a rank 1 bridge by induction. As w has rank k, there exists a cascade vertex w' with rank $k - 1$ such that w has a neighbor in $R(w')$. Move w to S and a to G_i as before. Then G_i remains connected as w does not separate G_i. If any cascade vertices in the configuration become undefined, remove them from their cascades. This new configuration now has a bridge of rank $k - 1$ (the edge (w, w')) and no reservoirs of vertices with rank less than k have been affected. Thus, there exists a $k - 1$ optimal configuration with a bridge of rank $k - 1$, and, by induction, there exists a 1-optimal configuration with a bridge of rank 1.

In a 1-optimal configuration of rank 1, we can move the bridge vertex in S to G_1, preserving connectivity and increasing the size of G_1 by 1, as desired. Interestingly, notice that if we can find an optimal configuration, this proof gives a clear algorithm to construct the graph desired in the theorem.

□

Lemma 3.2. *Suppose there is an optimal configuration with an edge (a, b) such that either $a \in V(G_1)$ or a is in the reservoir of a cascade vertex, and $b \in V(G) - V(G_1) - S$ and does not belong to any reservoir. Then b is a cascade vertex and is the last vertex in its cascade.*

Proof. Let $b \in V(G_j)$. First, note that if the cascade of G_j is empty, one may create a new valid configuration by adding a cascade consisting solely of b (since b has a defined rank as it is adjacent to a). However, then $R(b)$ would contain v_j, contradicting optimality. Because b is not in any reservoir, if it is a cascade vertex it must be the last vertex of its cascade. Indeed, if b were not the last, every path from b to v_j would pass through all subsequent cascade vertices, implying the existence of a path from those cascade vertices to v_j disjoint from b. Now suppose b is not a cascade vertex. Suppose a is in the reservoir of a vertex with rank r. If we remove all vertices in the cascade of G_j with rank greater than $r + 1$ and add b at the end of the cascade, p_k remains unchanged for $k \leq r$. Note that this new configuration is valid because b has rank $r + 1$. Moreover, since b is not in any reservoir, the reservoirs of the removed cascade vertices are contained in $R(b)$. Furthermore, the last vertex in the original cascade (which was not in any reservoir) is replaced, thereby increasing p_{r+1} by at least 1, which contradicts optimality.

□

Now we are ready to prove theorem 3.1, thus proving theorem 1.1.

Suppose an optimal configuration does not contain a bridge. Then every edge from S connects either to a non-reservoir vertex not in G_1 or to a vertex that is the last vertex of some cascade. For $G_2, \ldots, G_k$, remove either its last cascade vertex or v_j if the cascade is empty. By Lemma 3.2, non-reservoir vertices are only adjacent to other non-reservoir vertices, so we have removed $k-1$ vertices and disconnected S and the non-reservoir vertices from the rest of the graph, contradicting k-connectedness.

4 The Combinatorial Proof of Conjecture 1.1 when $k = 3$

4.1 Preliminaries

We next provide a proof of the weighted case when $k = 3$, using the combinatorics approach in Sultan. The proof is a generalization of the intermediate value theorem approach when $k = 2$, which is accomplished through the following structure of 3-connected graphs:

Definition 4.1. *Non-Separating Ear Decomposition:*
Let u, v, w be vertices. A nonseparating ear decomposition

$$P_1 \cup P_2 \cup \cdots \cup P_r$$

through the edge (u, v) and avoiding w is an ear decomposition of G such that:

- *P_1 contains the edge (u, v),*
- *the last nontrivial ear has only w as its internal node,*
- *the complements of the induced subgraphs $A_i = P_1 \cup \cdots \cup P_i$ (denoted G_i) are connected, and*
- *each internal vertex of ear P_i has a neighbor in G_i.*

It is known that given a 3-connected graph, an edge (u, v), and a vertex w, we can find a nonseparating ear decomposition through (u, v) and avoiding w in polynomial time.

In Section 2.4 of Sultan, nonseparating induced cycles are discussed. Note that we can find a nonseparating ear decomposition such that P_1 is a nonseparating induced cycle by decomposing the original cycle into components,

$$P_1 = P_1^* \cup P_2^* \cup \cdots \cup P_m^*,$$

where P_1^* is the smallest subcycle (and thus chordless) containing the edge (u, v).

There is a nice property of nonseparating induced ear decompositions with P_1 chordless. By Menger's theorem (since G is 3-connected), any vertex of P_1 has at least 3 vertex-disjoint paths into P_1^c; hence, P_1 being chordless implies that each vertex of P_1 has a neighbor in P_1^c (indeed, this follows from the definition of nonseparating induced ear decompositions).

Definition 4.2. *St-Numbering:*
Given an edge (s, t), an st-numbering is an ordering of the vertices of G such that each vertex (except s and t) is adjacent to vertices with both higher and lower order.

It is known that any 2-connected graph has an st-numbering for any edge (s, t). It is not difficult to see that given an st-numbering $(s, v_1, \ldots, v_n, t)$, the subgraphs of the form $\{s, v_1, \ldots, v_i\}$ and $\{v_j, v_{j+1}, \ldots, t\}$ are connected. In the first case, every vertex v_k (with $k \leq i$) is adjacent to a vertex of lower order, which provides a path from v_k to s; concatenating such paths yields connectivity. A similar argument holds for the other form.

Furthermore, for an ear decomposition $P_1, \ldots, P_n$, any subgraph of the form $P_1 \cup \cdots \cup P_i$ is 2-connected and thus has an st-numbering. Suppose A is 2-connected and B is an ear of A. If we remove a vertex from B in $A \cup B$, any two vertices can still be connected by traversing the side of the ear that remains unblocked. Similarly, if we remove a vertex from A, the remaining vertices in $A \cup B$ remain connected because A without the removed vertex is still connected. Thus, $A \cup B$ is 2-connected, and by induction, $P_1 \cup \cdots \cup P_i$ is 2-connected and has an st-numbering.

4.2 Proof of Conjecture 1.1 when $k = 3$

Now we have the tools to prove conjecture 1.1 in the case $k = 3$.

Theorem 4.1. *Conjecture 1.1 is true when $k = 3$.*

Proof. Let G a weighted graph and $w : V(G) \to \mathbb{R}$ a weight function with $w(G) = 0$, and u, v, w three nodes with weights all greater than 0 (w.l.o.g). We will show that we can split G into connected subgraphs V_1, V_2, V_3 such that u, v, w belong to distinct subgraphs, and

$$|w(V_1)|, |w(V_2)| \leq \frac{\max_{v \in V(G)}\{w(v)\}}{2}.$$

Because $w(G) = 0$, this would imply

$$|w(V_1)| + |w(V_2)| + |w(V_3)| \leq 2 \max_{v \in V(G)}\{w(v)\}.$$

If the weight function takes values plus or minus 1, we see that $w(V_1)$, and $w(V_2)$ are 0, and $w(V_3) = 0$ because $w(G) = 0$.

Consider $G' = G + (u, v)$, which is also 3-connected. Note that none of the subgraphs we'd like to find can contain this edge, so G' has a solution if and only if G has a solution. We know there is a nonseparating induced ear decomposition of G' through the edge (u, v) avoiding w,

$$G' = P_1 \cup \ldots \cup P_r,$$

with P_1 chordless. Let $V_i = P_1 \cup \ldots \cup P_i$. We break the proof into cases.

First consider $w(P_1) \leq 0$. Order the cycle $P_1 = (v_1 \ldots v_n)$ naturally such that the first vertex is u and the last vertex is v (ordered as their position in the path). Let $V_i = (v_1 \ldots v_i)$. Because $w(V_1) = w(u) > 0$, and $w(P_1) \leq 0$, there exists some index k with

$$w(V_k) > 0, \quad w(V_{k+1}) \leq 0,$$

and similarly, there exists some index m with

$$w(P_1 - V_{m+1}) > 0, \quad w(P_1 - V_m) \leq 0.$$

Let k and m be the first indices such that these occur (for m, we go backwards, so that this m is the greatest m such that this property holds. We do this so that $w(P_1 - V_i) > 0$ for all $i \geq m$). Now we have $k \leq m$. Suppose otherwise, that $k > m$. Then $w(P_1 - V_k) > 0$ and $w(V_k) > 0$, which contradicts that $w(P_1) < 0$.

Now if k is strictly less than m we find our connected subgraphs. We have

$$|w(v_{k+1})| = |\, w(V_k) - w(V_{k+1})\,| = |\, w(V_k)| + |\, w(V_{k+1})|.$$

Thus, at least one of the weights of V_k, V_{k+1} has absolute value less than

$$\frac{|\, w(v_{k+1})|}{2}.$$

Similarly, at least one of

$$|\, w(P_1 - V_m)|, \quad |\, w(P_1 - V_{m+1})|$$

is less than or equal to

$$\frac{|\, w(v_{m+1})|}{2}.$$

Then, choosing V_1 and V_2 from these sets respectively gives two connected subgraphs containing u and v. P_1^c is connected by definition of nonseparating ear decomposition, and each vertex in $P_1 - V_1 - V_2$ has a neighbor in P_1^c, so

$$V_3 = (V_1 + V_2)^c$$

is connected, and contains w, as desired (note that we only need one vertex in $P_1 - V_1 - V_2$ to have a neighbor in P_1^c).

Next, suppose $k = m$, so that

$$w(V_k) > 0, \quad w(V_{k+1}) \leq 0, \quad w(P_1 - V_k) \leq 0, \quad w(P_1 - V_{k+1}) > 0.$$

If $w(V_k)$ and $w(P_1 - V_{k+1})$ are both less than or equal to

$$\frac{|\, w(v_{k+1})|}{2},$$

it is clear by the logic in the last paragraph that we can find a solution to the problem. Furthermore, both cannot be greater than

$$\frac{|\, w(v_{k+1})|}{2},$$

otherwise,

$$w(P_1) = |\,w(V_k)| + |\,w(P_1 - V_{k+1})| + w(v_{k+1}) > 0.$$

So suppose only $w(V_k) > \frac{|\,w(v_{k+1})|}{2}$. Then

$$|\,w(V_{k+1})| = \big|\,w(V_k) - |\,w(v_{k+1})|\big| < \frac{|\,w(v_{k+1})|}{2},$$

so we can choose

$$V_1 = V_{k+1}, \quad V_2 = P_1 - V_{k+1}, \quad V_3 = G - V_1 - V_2.$$

If we instead assume $w(P_1 - V_{k+1}) > \frac{|\,w(v_{k+1})|}{2}$, we use the same inequality in the same manner to get a solution. Thus, when $w(P_1) \leq 0$, there is a solution.

Next, consider $w(P_1) > 0$. Because $w(w) > 0$ and $w(G) = 0$, $w(P_1 \cup \ldots \cup P_{r-1}) < 0$. It follows that there exists j such that, where $V_i = P_1 \cup ... \cup P_i$,

$$w(V_j) = w(P_1 \cup \ldots \cup P_j) > 0, \quad w(V_{j+1}) = w(P_1 \cup \ldots \cup P_{j+1}) \leq 0.$$

Define an st numbering of V_j with $u = s$, $v = t$, which we showed exists because V_j is 2-connected. Let

$$V_j^i = \{u, v_1, \ldots, v_i\},$$

and let

$$Q^i = (q_1, \ldots, q_i),$$

the first i internal vertices of the path (ear) P_{j+1} (orientation does not matter w.l.o.g). Now P_{j+1} meets V_j at two vertices, denote these v_x, v_y, where (w.l.o.g) $x < y$. First, suppose there exists $1 \leq i < y - 1$ such that $w(V_j^i) > 0$, $w(V_j^{i+1}) \leq 0$. By the same inequalities of the case $w(P_1) \leq 0$, either

$$|\,w(V_j^i)| \leq \frac{|\,w(v_{i+1})|}{2} \quad \text{or} \quad w(V_j^{i+1}) \leq \frac{|\,w(v_{i+1})|}{2}.$$

Assume that $|\,w(V_j^i)| \leq \frac{|\,w(v_{i+1})|}{2}$, and let

$$H = V_j - V_j^i.$$

Now, we have some cases:

First suppose $w(H) \leq 0$. Because $w(V_j^{i+1}) \leq 0$ and $w(V_j^i) > 0$, $w(H - v_{i+1}) > 0$. So again, either

$$|\,w(H)| \leq \frac{|\,w(v_{i+1})|}{2} \quad \text{or} \quad |\,w(H - v_{i+1})| \leq \frac{|\,w(v_{i+1})|}{2}.$$

Suppose $|\,w(H - v_{i+1})| \leq \frac{|\,w(v_{i+1})|}{2}$. Then

$$w(H - v_{i+1}) = w(V_j - V_j^i - v_{i+1}) = w(V_j) + w(v_{i+1}) - w(V_j^i) \leq \frac{|\,w(v_{i+1})|}{2}$$

because $w(v_{i+1}) < 0, w(V_j) > 0$. But then

$$w(V_j) + \left(\frac{|\,w(v_{i+1})|}{2} - w(V_j^i)\right) \leq 0,$$

which is impossible because $w(V_j^i) \leq \frac{|w(v_{i+1})|}{2}$ by assumption. Thus,

$$|w(H)| \leq \frac{|w(v_{i+1})|}{2}.$$

By the lemma on st numberings, V_j^i and H are connected. Furthermore, V_j^c is connected by definition of an st-numbering, so we have found a solution.

Next, suppose $w(H) > 0$. Because $w(V_j) > 0, w(V_j^i) > 0$, and $w(V_{j+1}) = w(V_j \cup P_{j+1}) \leq 0$, there exists some k with

$$w\big(H \cup (P_{j+1} - Q^h)\big) > 0, \quad w\big(H \cup (P_{j+1} - Q^{h-1})\big) \leq 0,$$

and, again, one of the two has absolute value less than

$$\frac{|w(q_h)|}{2}.$$

Both of these subgraphs are connected because H is connected and the side of an ear is connected. By definition of an st numbering, each internal vertex of P_{j+1} has a neighbor in V_{j+1}^c. Thus, setting V_2 to either $H \cup (P_{j+1} - Q^h)$ or $H \cup (P_{j+1} - Q^h)$, V_2 is connected and contains v, V_j is connected by the lemma on st numberings and contains u, and

$$(V_2 + V_j)^c$$

is connected because by definition of a nonseparating ear decomposition, each internal vertex of P_{j+1} has a neighbor in V_{j+1}^c, and V_{j+1}^c is connected. So we have found a solution.

If instead $w(V_j^{i+1}) \leq \frac{|w(v_{i+1})|}{2}$, letting $H = V_j - V_j^{i+1}$, we must have $w(H) > 0$ because $w(V_j) > 0$, $w(V_j^{i+1}) \leq 0$. Again, because $w(V_{j+1}) = w(V_j \cup P_{j+1}) \leq 0$, there exists some k with

$$w\big(H \cup (P_{j+1} - Q^h)\big) > 0, \quad w\big(H \cup (P_{j+1} - Q^{h-1})\big) \leq 0.$$

It is clear by the last paragraph that we have a solution in this case.

If there is an index $x+1 < i < |V_j|$ such that $w(V_j - V_j^{i-1}) > 0$, $w(V_j - V_j^i) \leq 0$, there is a solution by symmetry to the case above.

Finally, suppose for every $1 \leq i < y$, $w(V_j^i) > 0$, and for every $x < i < |V_j|$, $w(V_j - V_j^i) > 0$. Let

$$V_1 = V_j^{y-1}, \quad V_2 = V_1^c,$$

and note that subgraphs of the form $V_1 \cup Q^i$, $V_2 \cup (P_{i+1} - Q^i)$ are connected by construction of the nonseparating ear decomposition. Because $w(V_{j+1}) \leq 0$, by the same argument in the case $w(P_1) \leq 0$, we can find indices $1 \leq k \leq m < |P_{j+1}|$ such that

$$w(V_1 \cup Q^k) > 0, \quad w(V_1 \cup Q^{k+1}) \leq 0,$$

and

$$w(V_2 \cup (P_{j+1} - Q^{m+1})) > 0, \quad w(V_2 \cup (P_{j+1} - Q^m)) \leq 0.$$

Thus, we have a situation analogous to the case where $w(P_1) \leq 0$, and a solution exists. $\square$

Corollary 4.1. *Suppose G is a weighted, 3-connected graph with n vertices and a weight function $p\colon V(G) \to \{-1, 1\}$, and suppose that $p(G) = 0$. Let v_1, v_2, v_3 be 3 vertices of G such that the weights of these vertices have the same sign. Then there exists a partition of G, $\{V_1, V_2, V_3\}$, such that $v_i \in V_i$ for all i, the induced subgraph G_i of V_i is connected, and*

$$|p(G_1)| = |p(G_2)| = |p(G_3)| = 0.$$

5 Lovasz's Proof of Theorem 1.1 when $k = 3$

5.1 Preliminaries

We now transition into Lovasz's approach in solving theorem 1.1 in the case $k = 3$. Lovasz solves the general problem by creating a cell complex with vertices as spanning-trees of a graph and showing that its homology groups are trivial. Here, we create a similar cell-complex, but with only enough structure necessary to solve theorem 1.1 in the case $k = 3$, which provides a more concise transition into the weighted analog when $k = 3$.

Given a graph G with n vertices and m edges, we can represent a subgraph B as a point in $\mathbb{R}^m$. Order the edges $(e_1, ..., e_m)$, and let $f_i(B) = 1$ if $e_i \in E(B)$ and 0 otherwise. Then $f(B) = (f_1(B), f_2(B), ..., f_m(B))$ is a bijection from the set of subgraphs of G onto the vertices of the $m-$dimensional cube. Fix a vertex a of G. Our cell-complex will consist of the vertices corresponding to spanning trees of G rooted at a. We will now suppose throughout the rest of the paper that G is 3-connected. Spanning trees of G are edge-connected if and only if they are neighboring in the sense of theorem lemma 2.1, and, furthermore, the cell complex is connected by lemma 2.1. Now consider a subtree of G with $n-1$ vertices and consider 3 distinct edges e_1, e_2, e_3 connecting the remaining vertex v to the sub-tree (these exist by 3-connectedness). Then the 3 spanning trees created by adjoining one of these edges are pairwise edge-connected in the 1-skeleton, and thus we append their convex hull as a 2-cell. Next, consider a subtree of G with $n-2$ vertices, and let e_1, e_2 be edges connecting a remaining vertex v to the sub-tree and d_1, d_2 edges connecting the other remaining vertex u to the subtree. The four spanning trees created by adjoining these vertices to the subtree are each edge-connected with two other spanning trees, and thus these spanning trees form a square in the 1-skeleton, and we append their convex hull as a 2-cell. We will call our cell-complex, K, the union of all cells created in this manner. We now show K has trivial fundamental group.

5.2 The Fundamental Group of K is Trivial

Lemma 5.1. *K has trivial fundamental group:*

Proof. We will show that cycles in our cell-complex are contractible. Let $(T_0, T_1, ..., T_k = T_0)$ a cycle in K. Let A the greatest common subtree of the cycle containing a. If $|V(A)| = n$, the cycle has one spanning tree and thus is contractible.

If $|V(A)| = n-1$, let $\vec{x}$ the representation of A in $\mathbb{R}^m$. Then each spanning tree in the chain equals $\vec{x} + e_i$, where e_i is some vector in the canonical basis of $\mathbb{R}^m$. Thus, the union of all triangles formed by the spanning trees in the cycle form the 2-skeleton of a $k-1$ simplex, so the cycle is contractible.

We prove the theorem when $|V(A)| = n-2$ by induction. If the cycle has four distinct spanning trees, it is the boundary of a square and is contractible. Now suppose the cycle has $k > 4$ distinct spanning trees. Let v_1, v_2 the vertices not in A and let (e_{1i}, e_{2i}) represent the edges connecting v_1 and v_2 to A in T_i. If there exists i such that, w.l.o.g, $e_{1i} = e_{1(i+1)} = e_{1(i+1)}$, then T_i, T_{i+1}, and T_{i+2} form a triangle, which is contractible. The remaining cycle has 1 less spanning tree and thus is contractible by induction.

In the other case, there exists i such that $e_{1(i)} = e_{1(i+1)} \neq e_{1(i+2)} = e_{1(i+3)}$ and $e_{2(i)} \neq e_{2(i+1)} = e_{2(i+2)} \neq e_{2(i+3)}$. Consider a spanning tree T_z that completes T_i, T_{i+1}, and T_{i+2} into a square. We have $e_{1z} = e_{1(i+2)}$ and $e_{2z} = e_{2(i)}$. Thus, T_z, T_{i+2}, and T_{i+3} form a triangle, so the cycle $(T_0, T_1, ...T_i, T_z, T_{i+2}, T_{i+3}, ..., T_k = T_0)$ is contractible, and T_i, T_{i+1}, T_{i+2}, and T_z form a square and thus is contractible.

Now, for the general case, let A be the largest common subtree contained in the cycle. There are no edges e contained in each spanning-tree of the cycle such that $e = (x, y) \notin E(A)$. To see this, in a neighboring move from T_i to T_{i+1}, the edges of the path from x to A remain invariant because e is not removed, and thus A would not be the largest common subtree of the cycle. Now assume there is an edge $e = (x, y)$, $x, y \notin V(A)$ which belongs to some spanning trees in the cycle and an edge e' in G connecting x to A. We know e does not belong to each spanning-tree, so there exists T_i, T_{i+1} such that x is a leaf in T_i and e is removed from T_i via a neighboring move. Thus, we can consider a sub-sequence of spanning trees such that $e \notin E(T_i)$, $e \in E(T_{i+1}), E(T_{i+2}), ..., E(T_j)$ and $e \notin E(T_{j+1})$. For each of these spanning trees containing e, replace e with e'. Then the new sequence $(T'_{i+1}, T'_{i+1}, ..., T'_j)$ is a chain in K. First suppose e was in every spanning-tree except for one. Then $e \notin T_i$, and $T_{i-1} = T_{i+1}$, contradicting that we start with a cycle. So suppose otherwise. Adjoining the chains $(T_{i+1}, T_{i+2}, ..., T_j)$ and $(T'_{i+1}, T'_{i+2}, ..., T'_j)$ by concatenating them at their endpoints forms a cycle in K. Because $T_{i+1}, ...T_j$ all contain e, they all contain some edge d connecting $V(A) - V(G)$ to $V(A)$ along a path through e. By construction, each T'_k also contains d, and thus this cycle has a common subtree with one more vertex, and is contractible by induction. The remaining cycle $(T_0, T_1, ..., T_i, T'_{i+1}, ..., T'_j, T_{j+1}, ...T_p = T_0)$ now contains one less edge of the form e. Thus, we may assume that any vertex in $V(G) - V(A)$ such that there exists an edge in G connecting it to A is edge-connected to A in each spanning-tree in the cycle.

By 3-connectedness, there are at least 3 vertices in $V(G) - V(A)$ which have edges in G connecting them to A (this is why we prove the first three cases in the beginning). Let v one of these vertices. For each spanning-tree T_i, form the spanning-tree T'_i by fixing an edge e connecting v to A and replacing the edge in T_i connecting v to A with e. Then $(T'_0, T'_1, ...T'_p = T'_0)$ is a cycle and has a common subtree with one more vertex, and thus is contractible by induction. By 1.2, for each i, there exists a chain in K connecting T_i to T'_i. Notice that in any chain, we only need to move vertices under v. Furthermore, because there are at least 3 vertices edge-connected to A, there exists a chain from T_i to T'_i fixing both A and the other 2 vertices. Letting B, C be these chains corresponding to T_i and T_{i+1}, and letting C^r denote the reversal of C, concatenating B with C^r through T'_i and T'_{i+1} and concatenating C^r to B through T_i and T_{i+1} is a cycle. B and C both fix A and two additional vertices, and thus the full cycle fixes A and at least one additional vertex because the neighboring move from T_i to T_{i+1} and from T'_i to T'_{i+1} affect at most one of the other vertices edge-connected with A. Thus, by induction, this cycle is contractible, and thus the original cycle is contractible as it decomposes as a union of these cycles and $(T'_0, T'_1, ...T'_p = T'_0)$.

□

5.3 Proof of Theorem 1.1 Using Lovasz when k = 3

We can now prove theorem 1.1 when $k = 3$ using Lovasz's methods.

Theorem 5.1. *Theorem 1.1 is true when $k = 3$.*

Proof. First, add to G a new vertex a with edges from a to u, v, and w, and note that this is 3-connected as well. We'll call this new graph B. Now, we turn K into a simplicial-complex by cutting the diagonals of the "square" 2-cells. We define a function on the spanning trees (the vertices of K). Let $f_u(T)$ equal the number of vertices in the branch below and including the vertex u in T. Define identical functions for v and w. Then $f(T) = (f_u(T), f_v(T))$ maps lattice triangles to vertices of (potentially degenerate) lattice triangles. Now extend f affinely over all simplices of K.

Now by 3-connectedness, if we remove v and w the remaining graph is connected, so take a spanning-tree of $B - \{v\} - \{w\}$. Now add back the vertex v and the edge (a, v), and call this spanning-tree of $B - \{w\}$ T_1. Using the same procedure, define a spanning-tree T_2 of the graph $B - \{w\}$ which has all vertices except u in the branch of v. By lemma 2.1, there exists a sequence of spanning trees of $B - \{w\}$ connecting T_1 to T_2 such that adjacent spanning tree in this sequence are neighbors. Letting $|V(G)| = n$, we have

$$f(T_1) = (n - 2, 1), \quad f(T_2) = (1, n - 2)$$

. Now because adjacent spanning trees are neighbors, when we go from a spanning-tree A_i to a spanning-tree A_{i+1} in this sequence, we have either

$$f(A_i) = f(A_{i+1}),$$

or

$$f(A_i) = (f_u(A_i), f_v(A_i)) = (f_u(A_{i+1}) \pm 1,\ f_v(A_{i+1}) \mp 1).$$

Because we extend f affinely over simplices, the line from $(n - 2, 1)$ to $(1, n - 2)$ is in the image of f. Now, if we repeat the same procedure for T_2 (treating it as an element of $B - \{u\}$) and T_3, the spanning-tree which has all vertices except u (respectively v) under the branch of w (which is treated as an element of $B - \{u\}$, respectively $B - \{v\}$), we see that the boundary of the triangle with vertices $(n - 2, 1)$, $(1, n - 2)$, and $(1, 1)$ is contained in the image of f. Then, because we've shown K is simply-connected, we know the interior of this triangle is contained in the image of f. Now all we need to do is show that a point in the image with integer coordinates necessarily has a spanning-tree in its pre-image.

Consider the image of a triangle in K. The vertices are spanning trees T_1, T_2 and T_3 that share a common subtree with at least $n - 2$ points. Let $f(T_1) = (a, b)$. Depending on which branch the at most 2 remaining vertices are underneath (u, v, or w) we have the following possibilities. $f(T_2), f(T_3) = (a, b), (a \pm 1, b \mp 1), (a \pm 1, b), (a, b \pm 1)$. In the case that the image is a degenerate triangle, the image is a single point, a line segment on the horizontal or vertical axis, or the line segment from $(-1, 1)$ to $(1, -1)$. In each case, the integer coordinates in the image correspond to the vertices of the triangle in K. In the case of a non-degenerate triangle, the spanning trees map to vertices of a triangle of the form $(a,b),(a\pm 1,b),(a,b\pm 1)$, or $(a,b),(a\pm 1,b\mp 1),(a,b\pm 1)$ or $(a,b),(a\pm 1,b\mp 1),(a\pm 1,b)$. In each case, the image of the triangle in K is the convex-hull of these vertices in $\mathbb{R}^2$, which does not contain any points with integer coordinates in its interior. Thus, integer coordinates in the image of f are mapped onto by spanning trees in K, and therefore, for each n_1,n_2 with $n_1+n_2<|V(G)|$, there exists a spanning-tree T' with $f(T_1,T_2)=(n_1,n_2)$, and separating T' through its branches under u,v and w gives the desired partition.

□

6 Proof of Conjecture 1.1 Using Lovasz when $k=3$

Now, we create a new proof of theorem 1.2 *in the case* $k=3$, *using Lovasz's techniques as inspiration.*

Theorem 6.1. *Conjecture 1.1 is true when* $k=3$.

Proof. For vertices u,v, and w with weights greater than 0 (w.l.o.g), construct the same new graph as above, and define f the same way, but letting $f_u(T), f_v(T)$, and $f_w(T)$ denote the sum of the weights under the branches of u,v and w. Again, by 3-connectedness, if we remove v and w the remaining graph is connected, so take a spanning-tree of $B-\{v\}-\{w\}$. Now add back the vertex v and the edge (a,v), and call this spanning-tree of $B-\{w\}$ T_1. Using the same procedure, define a spanning-tree T_2 of the graph $B-\{w\}$ which has all vertices except u in the branch of v. By lemma 2.1, there exists a sequence of spanning trees of $B-\{w\}$ connecting T_1 to T_2 such that adjacent spanning trees in this sequence are neighbors. Because $p(G)=0$, we have

$$f(T_1)=(-p(v)-p(w),p(v)),\quad f(T_2)=(p(u),-p(u)-p(w)).$$

Because we extend f affinely over simplices, the line from $(-p(v)-p(w),p(v))$ to $(p(u),-p(u)-p(w))$ is in the image of f. Now, if we repeat the same procedure for T_2 (treating it as an element of $B-\{u\}$) and T_3, the spanning-tree which has all vertices except u, respectively v, under the branch of w (which is treated as an element of $B-\{u\}$, respectively $B-\{v\}$), we see that the boundary of the triangle with vertices

$$(-p(v)-p(w),p(v)),\quad (p(u),-p(u)-p(w)),\quad (p(u),p(w))$$

is contained in the range of f. Then, because we've shown K is simply-connected, we know the interior of this triangle is contained in the image of f. Furthermore, the coordinate $(0,0)$ is contained in the triangle. Let $p=\left(\max_{v\in V(G)}|w(v)|\right)$. We will show that for any point $\vec{x}$ in the image of f, there exists a spanning-tree T' such that the taxicab distance between $f(T')$ and $\vec{x}$ is less than or equal to p.

Consider the image of a triangle in K. Again, the vertices are spanning trees T_1,T_2 and T_3 that share a common subtree with at least $n-2$ points. We will be bounding distances from points to vertices in the image of triangles under f, so we can assume that vertices not in the common subtree have absolute value of weight equal to p. Let $f(T_1)=(0,0)$ by translating each of the coordinates in the image appropriately. Depending on which branches the remaining vertices are underneath we have the following possibilities. $f(T_2), f(T_3) = (0,0), (+-p, -+p), (+-p, 0), (0, +-p)$. In the case that the image is a degenerate triangle, it is clear that for any point $\vec{x}$ in the image of the triangle there exists a vertex such that their taxicab distances are less than or equal to p. In the case of a non-degenerate triangle, the spanning trees map to vertices of a triangle of the form $((0,0), (+-p, -+p), (+-p, 0))$, $((0,0), (+-p, -+p), (0, +-p))$, $((0,0), (+-p, 0), (0, +-p))$, or $((0,0), (0, +-p), (+-p, 0))$. There are two cases to consider: The image of the triangle in K with vertices T_1, T_2, T_3 is congruent to the the triangle with vertices $(0,0), (p,p), (0,p)$ or congruent to the triangle with vertices $(0,0), (-p,p), (p,0)$.

In the first case, the taxicab distance from any point $\vec{x}$ in the triangle to $(p,0)$, $\partial(\vec{x}, (p,0))$ is less than or equal to p. Note that if we move the $\vec{x}$ horizontally to the line $x = y$, $\partial(\vec{x}, (p,0))$ strictly increases. Because the line $x = y$ has slope 1, for any point on this line and contained in the triangle, $\partial(\vec{x}, (p,0)) = p$.

In the second case, for any point $\vec{x}$ in the triangle, let (x_1, x_2) its horizontal and vertical coordinate. Letting $v_1 = (-p,p), v_2 = (0,0)$, and $v_3 = (p,0)$, we have

$$\partial(\vec{x}, v_1) = |-p - x_1| + |p - x_2| = x_1 - x_2 + 2p,$$

$$\partial(\vec{x}, v_2) = |x_1| + x_2,$$
$$\partial(\vec{x}, v_3) = |p - x_1| + x_2 = p - x_1 + x_2.$$

First suppose $x_1 \leq 0$. Then $\partial(\vec{x}, v_2) < \partial(\vec{x}, v_3)$, and $\partial(\vec{x}, v_1) + \partial(\vec{x}, v_2) = 2p$, so one of $\partial(\vec{x}, v_1)$ or $\partial(\vec{x}, v_2)$ is less than or equal to p.

Now suppose that $x_1 > 0$. Note that x_2 is bounded by the line $\frac{p-x_1}{2}$, and thus we have the inequality

$$\partial(\vec{x}, v_2) \leq \frac{x_1 + p}{2} \leq p,$$

as desired.

Now $(0,0)$ is in the image of f and thus is in the image of some triangle in K. It follows that there exists a vertex in the image, $f(T') = (f_u(T'), f_v(T'))$ with $|f_u(T')| + |f_v(T')| \leq p$. As $f_u(T') + f_v(T') + f_w(T') = 0$, it follows that $|f_u(T')| + |f_v(T')| + |f_w(T')| \leq 2p$ as desired.

$\square$

Note that corollary 5.1 also follows from this proof. If the weight function maps into $\{-1, 1\}$, consider the image of the triangle containing $(0,0)$. Then there exists T' such that $|f_u(T')| + |f_v(T')| \leq p = 1$. If $(0,0)$ is not a vertex of the triangle, we must have both $|f_u(T')|, |f_v(T')| < 1$, but then $f(T') = (0,0)$.

7 Conclusion:

While we have not yet generalized the result of section 6 *to a proof of conjecture 1.2, we believe we will be able to make a similar argument in bounding the distance from* $(0,0)$ *to vertices when* $(0,0)$ *is contained in the image of a higher-dimensional cell. Furthermore, finding a generalization of the nonseparating ear decomposition may be beneficial in proving our conjecture through a combinatorial approach.*

8 References

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