

**Definition.** A matrix represent a collection of numbers in rows and cols.

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

**Order of matrix**  $\#rows \times \#cols$

$$A_{3 \times 3} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

**Note:** by matrix we consider the square matrix

### Operations on Matrices

**Addition/ Subtraction:**  $A = (a_{i,j})_{m \times n}$  and  $B = (b_{i,j})_{m \times n} \implies A \pm B = (a_{i,j} \pm b_{i,j})_{m \times n}$

### Properties of Addition:

- Commutative :  $A + B = B + A$
- Associative :  $(A + B) + C = A + (B + C)$

**Multiplication:** iff  $\#cols(A) = \#rows(B) \implies AB_{rows(A) \times cols(B)}$

In general,  $A = [a_{i,j}]_{m \times n}$  and  $B = [b_{i,j}]_{n \times p}$  then  $AB = [c_{i,k}]_{m \times p}$  where,  $c_{i,k} = \sum_{j=1}^n a_{ij}b_{jk}$

### Properties of Multiplication:

- Not Commutative :  $AB \neq BA$
- Associative :  $(AB)C = A(BC)$

**Scalar Multiplication** multiplying each elements with a real number. Let  $A = [a_{i,j}]$  and let  $k \in \mathbb{R}$  then  $kA = [ka_{i,j}]$ .

**Transpose of Matrix:** interchange of rows and cols, denoted as  $A^T$

### Types of Matrices

**Square Matrix:** matrix of same order i.e  $A_{n \times n}$

**Upper triangular Matrix:**  $A_{n \times n}$  s.t all the elements below the main diagonal are 0 i.e

$$A = [a_{ij}] \iff a_{ij} = 0 \forall i > j$$

**Lower triangular Matrix:**  $A_{n \times n}$  s.t all the elements above the main diagonal are 0 i.e

$$A = [a_{ij}] \iff a_{ij} = 0 \forall i < j$$

**Symmetric Matrix:**  $A^T = A$  (above/below diagonal elements are same)

**Skew Symmetric Matrix (Anti-symmetric):**  $A^T = -A$

**Diagonal Matrix:**  $A = [a_{ij}]_{n \times n}$  where,  $a_{ij} = 0 \forall i \neq j$

**Identity or Unit Matrix:**  $I = [a_{ij}]_{n \times n}$  where,

$$a_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{Otherwise} \end{cases}$$

**Orthogonal Matrix:**  $A^T A = A A^T = I$

**Idempotent Matrix:**  $A^2 = A$

**Involuntary Matrix:**  $A^2 = I$

**Singular Matrix:**  $|A| = 0$

**Non-Singular Matrix:**  $|A| \neq 0$

**Minor of  $a_{ij}$**  denoted as  $M_{ij}$  is obtained by deleting  $i^{th}$  rows and  $j^{th}$  cols

$$\delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \implies M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

**Co-factor of Matrix:**  $a_{ij} = (-1)^{i+j} M_{ij}$

**Adjoint of Matrix** matrix obtained by taking the transpose of cofactor matrix of a given matrix.

**Inverse of Matrix:**  $A^{-1} = \frac{adj(A)}{|A|}$  where,  $|A| \neq 0$

**Theorem 1.**  $Aadj(A) = adj(A)A = |A|.I$

**Theorem 2.**  $A$  is said to be invertible iff  $AA^{-1} = A^{-1}A = I$

**Theorem 3.** Invertible matrix has a unique inverse

**Theorem 4.** If  $A$  and  $B$  are invertible then  $AB$  also invertible s.t  $(AB)^{-1} = B^{-1}A^{-1}$

**Theorem 5.** If  $A$  is invertible then  $A^T$  is also invertible.

**Theorem 6.** If  $A$  is invertible symmetric then  $A^{-1}$  also symmetric

**Theorem 7.** If  $A$  and  $B$  are nonsingular then  $adj(AB) = adj(B)adj(A)$

**Theorem 8.** If  $|A| \neq 0 \implies |adj(A)| = |A|^{n-1}$

**Theorem 9.** If  $|A| \neq 0 \implies adj(adj(A)) = |A|^{n-2}.A$

**Definition.** Determinants are scalar quantities that can be calculated from a square matrix. Denoted as  $det(A)$  or  $|A|$ .

**Expansion of determinant:**  $|A| = a_{ij} + Cofactor\ of\ a_{ij}$

**Properties of Determinant**

- Determinant evaluated across any rows/col are same.
- If all elements of row or col are 0 then the  $det(A) = 0$
- $|I_n| = 1$
- $|A^T| = |A|$
- $|AB| = |A||B| = |B||A|$

- $|A^n| = |A|^n$
- The interchange of any two rows or cols changes the sign of a determinant without altering its absolute value.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = - \begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

- If two rows or cols in a det. are same the value of det is 0
- If the elements of row or col of a det is multiplied by scalar, then the value of a new det is equal to some scalar times the value of original det.

$$\begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = k \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

- In a det each elements in any row or col consist of the sum of two terms, then the det can be expressed as sum of two det of same order.

$$\begin{vmatrix} a_{11} + x_1 & a_{12} & a_{13} \\ a_{21} + x_2 & a_{22} & a_{23} \\ a_{31} + x_3 & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} x_1 & a_{12} & a_{13} \\ x_2 & a_{22} & a_{23} \\ x_3 & a_{32} & a_{33} \end{vmatrix}$$

- If  $|A| = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$   $|B| = \begin{vmatrix} b_1 + ca_1 & b_2 + ca_2 & b_3 + ca_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$  then  $|A| = |B|$

- Det. of a diagonal matrix, triangular(Upper or lower) matrix is product of element of principle diagonal.

- $|A^{-1}| = \frac{1}{|A|}$

- If  $A_{n \times n}$  then  $|kA| = k^n |A|$

## Elementary Transformation

- Interchange of any rows or cols:  $R_i \leftrightarrow R_j$  or  $C_i \leftrightarrow C_j$
- Multiplication of  $i^{th}$  row or col by  $k \neq 0$ :  $R_i \leftrightarrow kR_j$  or  $C_i \leftrightarrow kC_j$
- Addition of  $k$  times the  $j^{th}$  row or col to the  $i^{th}$  row or col :  $R_i \leftrightarrow R_i + kR_j$  or  $C_i \leftrightarrow C_i + kC_j$

**Fact:** Trace of matrix is the sum of diagonal element i.e for  $A_{n \times n}$   $\sum_{i=1}^n \text{diag}[a_i]$

## Echelon Form of a Matrix

- Any rows of all zeroes are below any other non-zero rows (*not always the case*)
- Each leading entry of a row is in column to the right of leading entry of the row above it.
- All entries in a col below a leading entry are zeroes.

$$A = \begin{pmatrix} \textcircled{3} & 2 & 0 & 7 & 9 \\ 0 & \textcircled{4} & 5 & 10 & 0 \\ 0 & 0 & 0 & \textcircled{-4} & 1 \\ 0 & 0 & 0 & 0 & \textcircled{6} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Here,  $\textcircled{a_i}$  is the leading entry

## Reduced Echelon Form

- Matrix has to be in echelon form
- The leading entry in each non-zero row is 1
- Each leading 1 is the only non-zero entry in its column

$$A = \begin{pmatrix} \textcircled{1} & 0 & 3 & 0 & 9 \\ 0 & \textcircled{1} & 4 & 0 & -6 \\ 0 & 0 & 0 & \textcircled{1} & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

**Fact:** Reduce Echelon Form (REF) is unique but not the Echelon Form