Definition. A matrix represent a collection of numbers in rows and cols.

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Order of matrix $\#rows \times \#cols$

$$A_{3\times 3} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Note: by matrix we consider the square matrix
Operations on Matrices
Addition/ Subtraction: $A = (a_{i,j})_{m\times n}$ and $B = (b_{i,j})_{m\times n} \implies A \pm B = (a_{i,j} \pm b_{i,j})_{m\times n}$
Properties of Addition:

- Commutative : $A + B = B + A$

- Associative : $(A + B) + C = A + (B + C)$

Multiplication: iff $\#cols(A) = \#rows(B) \implies AB_{rows(A)\times cols(B)}$
In general, $A = [a_{i,j}]_{m\times n}$ and $B = [b_{i,j}]_{n\times p}$ then $AB = [c_{i,k}]_{m\times p}$ where, $c_{i,k} = \sum_{j=1}^{n} a_{ij}b_{jk}$
Properties of Multiplication:

- Not Commutative : $AB \neq BA$

- Associative : $(AB)C = A(BC)$

Scalar Multiplication multiplying each elements with a real number. Let $A = [a_{i,j}]$ and let $k \in \mathbb{R}$ then $kA = [ka_{i,j}]$.
Transpose of Matrix: interchange of rows and cols, denoted as A^T
Types of Matrices
Square Matrix: matrix of same order i.e $A_{n\times n}$
Upper triangular Matrix: $A_{n\times n}$ s.t all the elements below the main diagonal are 0 i.e $A = [a_{ij}] \impliedby a_{ij} = 0 \,\forall i > j$
Lower triangular Matrix: $A_{n\times n}$ s.t all the elements above the main diagonal are 0 i.e $A = [a_{ij}] \impliedby a_{ij} = 0 \,\forall i < j$
Symmetric Matrix: $A^T = A$ (above/below diagonal elements are same)
Skew Symmetric Matrix (Anti-symmetric): $A^T = -A$
Diagonal Matrix: $A = [a_{ij}]_{n\times n}$ where, $a_{ij} = 0 \,\forall i \neq j$
Identity or Unit Matrix: $I = [a_{ij}]_{n\times n}$ where,

$$a_{ij} = \begin{cases} 1 & \text{if i = j} \\ 0 & \text{Otherwise} \end{cases}$$

Orthogonal Matrix: $A^T A = AA^T = I$

Idempotent Matrix: $A^2 = A$
Involutary Matrix: $A^2 = I$
Singular Matrix: $|A| = 0$
Non-Singular Matrix: $|A| \neq 0$

Minor of a_{ij} denoted as M_{ij} is obtained by deleting i^{th} rows and j^{th}cols

$$\delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \implies M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

Co-factor of Matrix: $a_{ij} = (-1)^{i+j} M_{ij}$
Adjoint of Matrix matrix obtained by taking the transpose of cofactor matrix of a given matrix.
Inverse of Matrix: $A^{-1} = \frac{adj(A)}{|A|}$ where, $|A| \neq 0$

Theorem 1. $A \, adj(A) = adj(A)A = |A|.I$

Theorem 2. A is said to be invertible iff $AA^{-1} = A^{-1}A = I$

Theorem 3. Invertible matrix has an unique inverse

Theorem 4. If A and B are invertible then AB also invertible s.t $(AB)^{-1} = B^{-1}A^{-1}$

Theorem 5. If A is invertible then A^T is also invertible.

Theorem 6. If A is invertible symmetric then A^{-1} also symmetric

Theorem 7. If A and B are nonsingular then $adj(AB) = adj(B)adj(A)$

Theorem 8. If $|A| \neq 0 \implies |adj(A)| = |A|^{n-1}$

Theorem 9. If $|A| \neq 0 \implies adj(adj(A)) = |A|^{n-2}.A$

Definition. Determinants are scalar quantities that can be calculated from a square matrix. Denoted as $det(A)$ or $|A|$.

Expansion of determinant: $|A| = a_{ij} + Cofactor \, of \, a_{ij}$
Properties of Determinant

- Determinant evaluated across any rows/col are same.

- If all elements of row or col are 0 then the $det(A) = 0$

- $|I_n| = I$

- $|A^T| = |A|$

- The interchange of any two rows or cols changes the sign of a determinant without altering its absolute value.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = - \begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

- If two rows or cols in a det. are same the value of det is 0

- If the elements of row or col of a det is multiplied by scalar, then the value of a new det is equal to some scalar times the value of original det.

$$\begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = k \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

- In a det each elements in any row or col consist of the sum of two terms, then the det can be expressed as sum of two det of same order.

$$\begin{vmatrix} a_{11}+x_1 & a_{12} & a_{13} \\ a_{21}+x_2 & a_{22} & a_{23} \\ a_{31}+x_3 & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} x_1 & a_{12} & a_{13} \\ x_2 & a_{22} & a_{23} \\ x_3 & a_{32} & a_{33} \end{vmatrix}$$

- If $|A| = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ $|B| = \begin{vmatrix} b_1+ca_1 & b_2+ca_2 & b_3+ca_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ then $|A| = |B|$

- Det. of a diagonal matrix, triangular(Upper or lower) matrix is product of element of principle diagonal.

- $|A^{-1}| = \frac{1}{|A|}$

- If $A_{n \times n}$ then $|kA| = k^n |A|$