Definition. Set F with 2 operation: $\cdot, +$ such that:

- $(F, +)$ is a commutative group.
- $(F, \cdot)$ is a commutative group
- $a(b+c) = ab + ac$
- $(b+c)a = ba + ca$

Note: A field is basically a set that consist of all the 4 operations: $+, /, -, \times$

Definition. A vector space V over a field $\mathbb{R}$ is a set on which two operations: vector addition $(+)$ and scalar multiplication$(\cdot)$ are defined such that -

- $(V, +)$ forms commutative group
- The operation $(\times)$ is define between scalars and vectors such that: $\forall a \in \mathbb{R}$ and $v \in V \implies av \in V$
- $\forall a \in \mathbb{R}$ and $v, w \in V \implies a(v+w) = av + aw$
- $\forall a, b \in \mathbb{R}$ and $v \in V,\ (a+b)v = av + bv$
- $\forall a, b \in \mathbb{R}$ and $v \in V,\ (ab)v = a(bv)$
- Unitary Law: $\exists 1 \in \mathbb{R}$ s.t $1 \cdot v = v\ \forall v \in V$

$(V, +, \cdot)$ is a vector space

Example 1. $V = P(\mathbb{R})$ where, P is set of all polynomials in variable x and coefficient from $\mathbb{R}$

Example 2. $V =$ set of all polynomial of degree n does not form a vector space over $\mathbb{R}$

Definition. The vector in **Euclidean Space** consist of $n-$tuple of $\mathbb{R}$, $(x_1, x_2, \ldots x_n)$, Euclidean Space is used mathemarically represent physical space with notions such as distance, length and angles.

Definition. If $\mathbb{R}^n$ is a vector space, then $S \subseteq \mathbb{R}^n$ is said to be a subspace of $\mathbb{R}^n(\mathbb{R})$ if -

- $0 \in S$
- S is closed under vector addition: $x, y \in S \implies x + y \in S$
- S is closed under scalar multiplication: $x \in S, \alpha \in \mathbb{R} \implies \alpha x \in S$

Definition (Aliter)**.** If $\mathbb{R}^n(R)$ a vector space then $S \subseteq \mathbb{R}^n$ is said to be subspace if $S(\mathbb{R})$

Theorem 1. Let $W \subseteq \mathbb{R}^n$ then W is a subspace of $\mathbb{R}^n$ iff-

- W is non-empty

- For any $a, b \in \mathbb{R}$ and any $\vec{u}, \vec{v} \in W,\ a\vec{u} + b\vec{v} \in W$

Theorem 2. The intersection of any non-empty collection of subspace of $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$.

Theorem 3. The union of two subspace of $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$ iff one of them is contained in the other.

Definition. Let $V(F)$ be a vector space let $S = \{v_1, v_2, \dots v_n\} \ \subseteq V$ then

$$L(S) = \{c_1v_1 + c_2v_2 \cdots + c_nv_n \mid c_i \in \mathbb{R}, i \in [1, n]\}$$

is a linear span of the set S.

Theorem 4. Let $V(\mathbb{R})$ is a vector space and let $S \subseteq V$. Then, $L(S)$ subspace of V.

Definition. $B = \{v_1, v_2 \dots v_n\}$ is a basis of $n-$ dimensional vector space V if-

- $\{v_1, v_2 \dots v_n\}$ is linearly independent.
- For any vector $v \in V$ we have, $V = \alpha_1v_1 + \alpha_2v_2 + \cdots + \alpha_nv_n$ where, $\alpha_1, \alpha_2, \dots \alpha_n$ are scalars from the field $\mathbb{R}$.

Example 3. If $\mathbb{R}^2(\mathbb{R})$ then $S = \{(1, 0), (0, 1)\}$ forms a *basis* because S is linearly independent and if $(a, b) \in \mathbb{R}^2$ then $(a, b) = a(1, 0) + b(0, 1)$

Definition. The no. of vector in a *basis* of $V(F)$ is called *dimension* of vector space $V(F)$

- Dim of $\mathbb{R}^2(\mathbb{R})$ is 2
- Dim of $\mathbb{R}^n(\mathbb{R})$ is n
- Dim of $M_{m\times n}(\mathbb{R})$ is $m \times n$

Theorem 5. Let $V(F)$ be a finite dimensional vector space and let S_1 and S_2 be two subspace of V then

$$dim(S_1) + dim(S_2) = dim(S_1 + S_2) + dim(S_1 \cap S_2)$$

Definition. A matrix represent a collection of numbers in rows and cols.

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Order of matrix $\#rows \times \#cols$

$$A_{3\times 3} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Note: by matrix we consider the square matrix
Operations on Matrices
Addition/ Subtraction: $A = (a_{i,j})_{m\times n}$ and $B = (b_{i,j})_{m\times n} \implies A \pm B = (a_{i,j} \pm b_{i,j})_{m\times n}$
Properties of Addition:

- Commutative : $A + B = B + A$
- Associative : $(A + B) + C = A + (B + C)$

Multiplication: iff $\#cols(A) = \#rows(B) \implies AB_{rows(A)\times cols(B)}$
In general, $A = [a_{i,j}]_{m\times n}$ and $B = [b_{i,j}]_{n\times p}$ then $AB = [c_{i,k}]_{m\times p}$ where, $c_{i,k} = \sum_{j=1}^{n} a_{ij}b_{jk}$
Properties of Multiplication:

- Not Commutative : $AB \neq BA$
- Associative : $(AB)C = A(BC)$

Scalar Multiplication multiplying each elements with a real number. Let $A = [a_{i,j}]$ and let $k \in \mathbb{R}$ then $kA = [ka_{i,j}]$.
Transpose of Matrix: interchange of rows and cols, denoted as A^T
Types of Matrices
Square Matrix: matrix of same order i.e $A_{n\times n}$
Upper triangular Matrix: $A_{n\times n}$ s.t all the elements below the main diagonal are 0 i.e $A = [a_{ij}] \impliedby a_{ij} = 0\, \forall i > j$
Lower triangular Matrix: $A_{n\times n}$ s.t all the elements above the main diagonal are 0 i.e $A = [a_{ij}] \impliedby a_{ij} = 0\, \forall i < j$
Symmetric Matrix: $A^T = A$ (above/below diagonal elements are same)
Skew Symmetric Matrix (Anti-symmetric): $A^T = -A$
Diagonal Matrix: $A = [a_{ij}]_{n\times n}$ where, $a_{ij} = 0\, \forall i \neq j$
Identity or Unit Matrix: $I = [a_{ij}]_{n\times n}$ where,

$$a_{ij} = \begin{cases} 1 & \text{if i = j} \\ 0 & \text{Otherwise} \end{cases}$$

Orthogonal Matrix: $A^T A = AA^T = I$
Idempotent Matrix: $A^2 = A$
Involutary Matrix: $A^2 = I$
Singular Matrix: $|A| = 0$
Non-Singular Matrix: $|A| \neq 0$

Minor of a_{ij} denoted as M_{ij} is obtained by deleting i^{th} rows and j^{th}cols

$$\delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \implies M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

Co-factor of Matrix: $a_{ij} = (-1)^{i+j} M_{ij}$
Adjoint of Matrix matrix obtained by taking the transpose of cofactor matrix of a given matrix.
Inverse of Matrix: $A^{-1} = \frac{adj(A)}{|A|}$ where, $|A| \neq 0$

Theorem 6. $Aadj(A) = adj(A)A = |A|.I$

Theorem 7. A is said to be invertible iff $AA^{-1} = A^{-1}A = I$

Theorem 8. Invertible matrix has an unique inverse

Theorem 9. If A and B are invertible then AB also invertible s.t $(AB)^{-1} = B^{-1}A^{-1}$

Theorem 10. If A is invertible then A^T is also invertible.

Theorem 11. If A is invertible symmetric then A^{-1} also symmetric

Theorem 12. If A and B are nonsingular then $adj(AB) = adj(B)adj(A)$

Theorem 13. If $|A| \neq 0 \implies |adj(A)| = |A|^{n-1}$

Theorem 14. If $|A| \neq 0 \implies adj(adj(A)) = |A|^{n-2}.A$

Definition. Determinants are scalar quantities that can be calculated from a square matrix. Denoted as $det(A)$ or $|A|$.

Expansion of determinant: $|A| = a_{ij} + Cofactor\ of\ a_{ij}$

Properties of Determinant

- Determinant evaluated across any rows/col are same.
- If all elements of row or col are 0 then the $det(A) = 0$
- $|I_n| = I$
- $|A^T| = |A|$
- $|AB| = |A||B| = |B||A|$
- $|A^n| = |A|^n$
- The interchange of any two rows or cols changes the sign of a determinant without altering its absolute value.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = - \begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

- If two rows or cols in a det. are same the value of det is 0
- If the elements of row or col of a det is multiplied by scalar, then the value of a new det is equal to some scalar times the value of original det.

$$\begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = k \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

- In a det each elements in any row or col consist of the sum of two terms, then the det can be expressed as sum of two det of same order.

$$\begin{vmatrix} a_{11}+x_1 & a_{12} & a_{13} \\ a_{21}+x_2 & a_{22} & a_{23} \\ a_{31}+x_3 & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} x_1 & a_{12} & a_{13} \\ x_2 & a_{22} & a_{23} \\ x_3 & a_{32} & a_{33} \end{vmatrix}$$

- If $|A| = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ $|B| = \begin{vmatrix} b_1+ca_1 & b_2+ca_2 & b_3+ca_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ then $|A| = |B|$

- Det. of a diagonal matrix, triangular(Upper or lower) matrix is product of element of principle diagonal.

- $|A^{-1}| = \frac{1}{|A|}$

- If $A_{n\times n}$ then $|kA| = k^n|A|$

Elementary Transformation

- Interchange of any rows or cols: $R_i \leftrightarrow R_j$ or $C_i \leftrightarrow C_j$

- Multiplication of i^{th} row or col by $k \neq 0$: $R_i \leftrightarrow kR_j$or $C_i \leftrightarrow kC_j$

- Addition of k times the j^{th} row or col to the i^{th} row or col : $R_i \leftrightarrow R_i + kR_j$or $C_i \leftrightarrow C_i + kC_j$

Fact: *Trace of matrix is the sum of diagonal element* i.e for $A_{n\times n}$ $\sum_{i=1}^{n} diag[a_i]$

Echelon Form of a Matrix

- Any rows of all zeroes are below any other non-zero rows (*not always the case*)

- Each leading entry of a row is in column to the right of leading entry of the row above it.

- All entries in a col below a leading entry are zeroes.

$$A = \begin{pmatrix} 3 & 2 & 0 & 7 & 9 \\ 0 & 4 & 5 & 10 & 0 \\ 0 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Here, a_i is the leading entry

Reduced Echelon Form

- Matrix has to be in echelon form
- The leading entry in each non-zero row is 1
- Each leading 1 is the only non-zero entry in its column

$$A = \begin{pmatrix} 1 & 0 & 3 & 0 & 9 \\ 0 & 1 & 4 & 0 & -6 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Fact: Reduce Echelon Form(REF) is unique but not the Echelon Form

System of Linear Equation

Consider eqn.

$$a_i x + b_i y + c_i z = d_i,\ i \in \mathbb{N}$$

in matrix form it's represented as $AX = B$,

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

Consistency of Linear Equation

- If $|A| \neq 0$ then the system is consistent and has a unique solution i.e $X = A^{-1}B$
- If $|A| = 0$ then the system has either no solution or have infinite # solutions.

Augmented Matrix Form: $AX = B \implies [A : B]$

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1m} & d_1 \\ a_{21} & a_{22} & \dots & a_{2m} & d_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} & d_m \end{array}\right]$$

Homogeneous System of Equation

Equation of the form: $AX = 0$
For Homog. : $[A : 0]$

Consistency of Homog. System

- $X = 0$ (Trivial Soln)
- If $\rho(A) = \#unknown\,variable$, unique soln.
- If $\rho(A) < \#unknown\,variable$, infinite soln. implies $det(A) = 0$

Non-Homog. System of Equation

Equation of the form: $AX = B,\ B \neq 0$
For Homog. : $[A : B]$
Consistency of Non-Homog. System

- If $\rho[A : B] \neq \rho(A)$, No solution.
- If $\rho[A : B] = \rho(A) = \#unknown\,variable$, unique soln.
- If $\rho[A : B] = \rho(A) \neq \#unknown\,variable$, infinite no. of soln.

Eigen Values and Eigen Vector

Definition. Let $A_{n\times n}$ consider the homog. system of eqn.

$$AX = \lambda X \implies (A - \lambda I)X = 0$$

where, I is the identity matrix, λ is scalar. Then, λ is called an *eigen values* and non-zero vector X is *eigen vector*

Definition. Polynomial obtained from $|A - \lambda I| = 0$ is called the *Characteristics Polynomial* and the roots of polynomial is the *eigen values.*

Guassian Elimination: convert augmented matrix to REF and solve the linear system of equation.

Method of finding eigen values and eigen vector

- Solving the Characteristics polynomial we get the roots as eigen values.
- Now we find the eigen vector for its eigen values by putting λ values in $(A - \lambda I)X = 0$ and using Gaussian Elimination.