

A 0-1 law and cusp excursion for  
geometrically finite actions on  
coarsely hyperbolic metric spaces

Joint with Giulio Tiozzo

Khintchine's Theorem 1926 :  $\psi: \mathbb{N} \rightarrow \mathbb{R}^+$

monotone decr (maybe other hypotheses).

Let  $\mathbb{H}(\psi) := \left\{ x \in \mathbb{R} : \left| x - \frac{p}{q} \right| < \frac{\psi(q)}{q} \text{ for } \text{only many } \frac{p}{q} \in \mathbb{Q} \right\}$

Then

$\sum_{q \in \mathbb{N}} \psi(q) = \infty \Rightarrow \mathbb{H}(\psi) \text{ has full measure}$

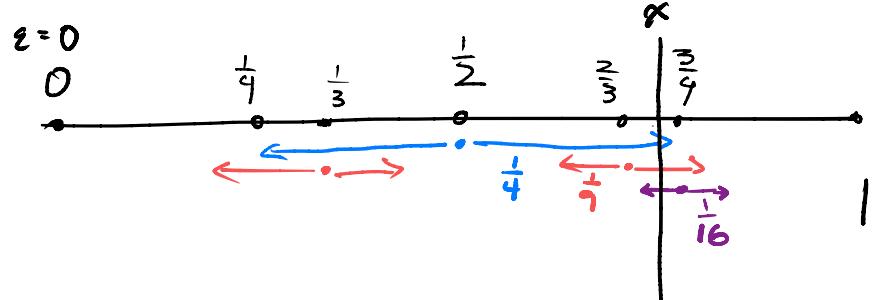
and

$\sum_{q \in \mathbb{N}} \psi(q) < \infty \Rightarrow \mathbb{H}(\psi) \text{ has measure zero}$

restrict to  $[0,1]$  to get a "0-1" law

Application  $\psi_\varepsilon(q) = \frac{1}{q^{1+\varepsilon}}$

$$\mathbb{H}(\psi) = \left\{ x \in \mathbb{R} : \left| x - \frac{p}{q} \right| < \frac{1}{q^{2+\varepsilon}} \text{ for only many } \frac{p}{q} \in \mathbb{Q} \right\}$$



so far this  $x$  is in 3 of these balls. Will it be in only many such balls?  
if  $\varepsilon > 0$  then the balls get even smaller

by Khinchine:

$$\sum_{q \in \mathbb{N}} \psi_\varepsilon(q) = \sum_{q \in \mathbb{N}} \frac{1}{q^{1+\varepsilon}} \begin{cases} = \infty \text{ for } \varepsilon = 0 \\ < \infty \text{ for } \varepsilon > 0 \end{cases}$$

hence

$\mathbb{H}(\psi_\varepsilon)$   $\begin{cases} \text{has full measure for } \varepsilon = 0 \\ \text{has measure zero for } \varepsilon > 0 \end{cases}$

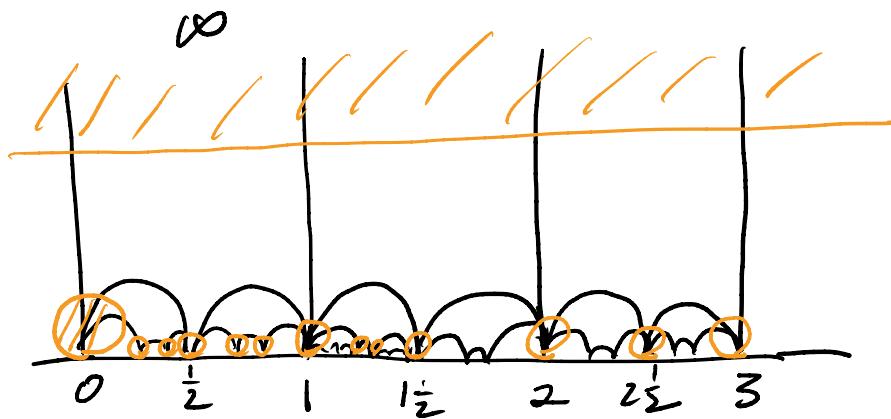
In our picture, with probability 1,  $x$  is in only many balls of radius  $\frac{1}{q^2}$ .

# horoball packings for the hyperbolic plane

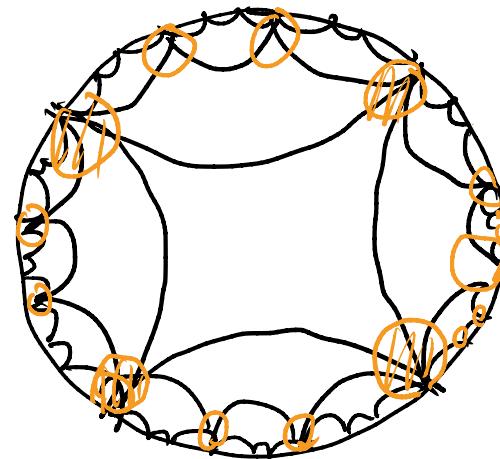
by example

$$\Gamma \curvearrowright \mathbb{H}^2 \text{ finite area}$$

tiling of  $\mathbb{H}^2$   
w/ horoball packing ( $\Gamma$ -equivariant)



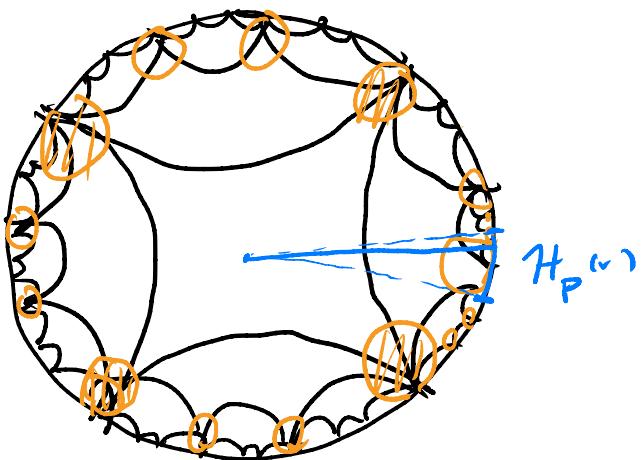
Define horoball packing?



$P = \{\text{centers of horoballs in the packing}\}$

$r_p = \text{Euclidean radius of the horoball centered at } p \text{ from a fixed original packing}$

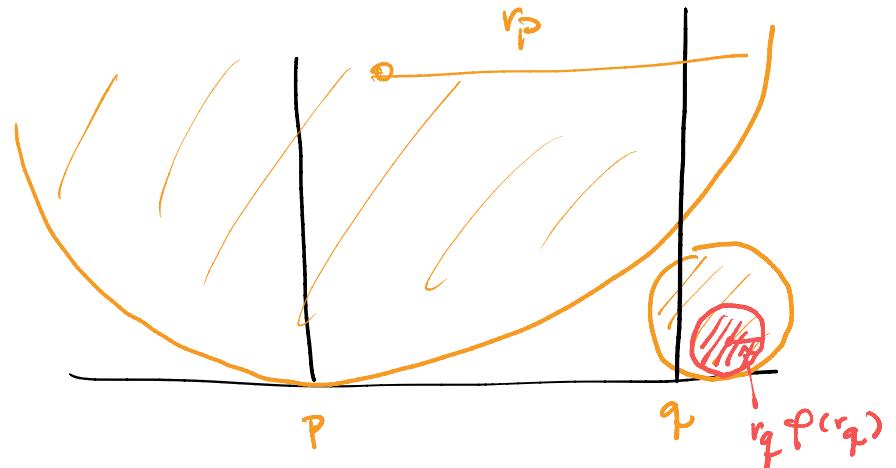
$H_p(r) = \text{shadow of the horoball}$   
centered at  $p$  with radius  $r$ .



radius of shadow  $H_p(r)$  is  $s'$   
is  $\sim r$

- $\varphi: \mathbb{R}^+ \rightarrow \mathbb{R}^+$  increasing  
(so that  $\varphi(r)$  decr. as  $r \rightarrow 0$ )

- $(\text{H})_{(\varphi)} = \{x \in \partial H^2 : x \in H_p(r_p \varphi(r_p))\}$   
for  $\infty$ -ly many  $p \in \mathcal{P}\}$



$\varphi \equiv 1 \Rightarrow \text{no shrinking}$

in this case,  $x$  in  $\infty$ -ly many  
shadows.

$\varphi < 1 \Rightarrow \text{shrinking}, x$  may no longer be  
in  $\infty$ -ly many shadows

Thm: (Stratmann-Velani, Sullivan)

[Khintchin-type Theorem] for small  $\lambda < 1$ ,

$\sum_{n \in \mathbb{N}} \varphi(\lambda^n) < \infty \iff (\text{H})_{(\varphi)}$  has  
Leb measure zero

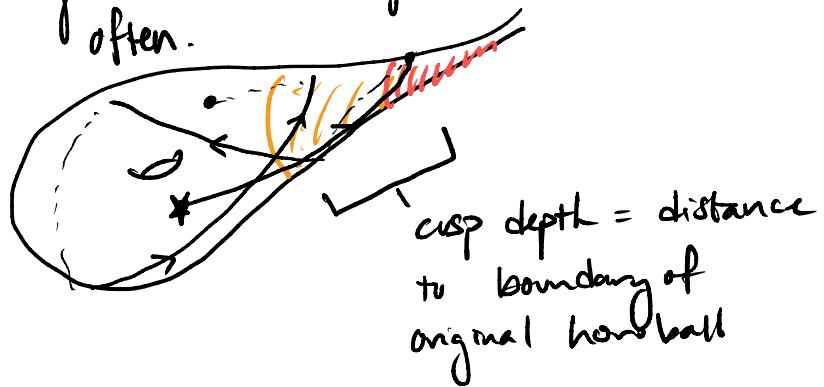
$= \infty \iff (\text{H})_{(\varphi)}$  has  
Leb measure one

## Application to cusp excursion

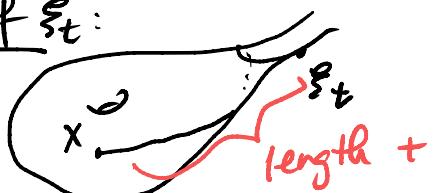
Horoball packing projects to neighborhood of the cusp.  
of shrinks neighborhoods.

if they stay the "same size" ( $\varphi = 1$ ),

a.e. geodesic visits neighborhood  $\infty$ -ly often.



defn of  $\xi_t$ :



Thm (Stratmann-Velani, Sullivan)

[Logarithm Law] For a.e.  $\xi \in S^1$ ,

$$\limsup_{t \rightarrow \infty} \frac{\text{cusp depth } (\xi_t)}{\log(t)} = 1.$$

with Tiozzo, we generalize these to the setting of coarsely hyp. metric spaces

and geom. fin. grp actions

Defn

For functions  $f, g : U \rightarrow \mathbb{R}$

- $f \approx g$  iff  $\exists c$  s.t. on  $U$ ,

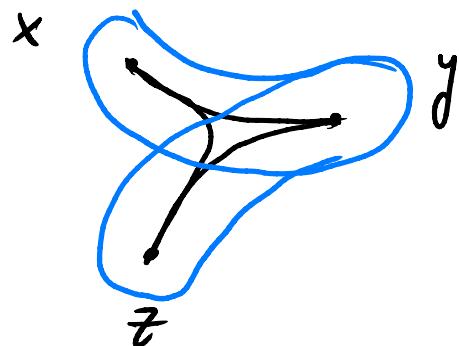
$$f - c \leq g \leq f + c$$

- $f \asymp g$  iff  $\exists c$  s.t. on  $U$ ,

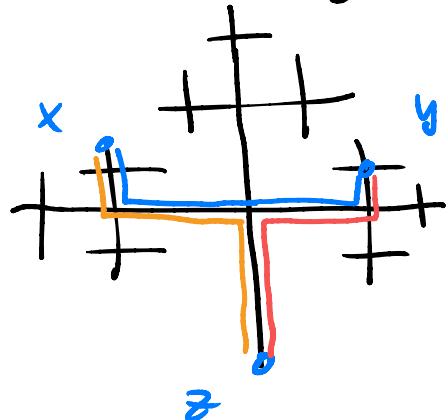
$$\frac{1}{c}g \leq f \leq cg$$

Recall: hyperbolic spaces

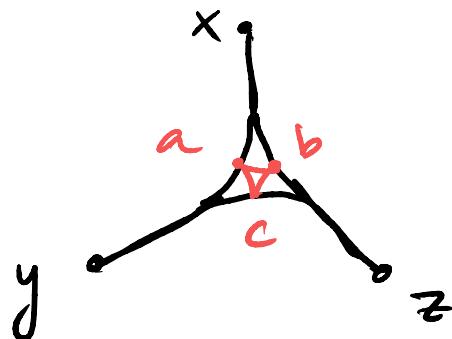
$(X, d)$   $\delta$ -hyperbolic if



$(X, d)$  Grushov hyp or  
coarsely hyp if  $\delta$ -hyp some  $\delta > D$   
e.g. trees are 0-hyperbolic



when  $(X, d)$  coarsely hyp,  
"triangles are tripods"  
Alt defn:  $\text{diam}(\text{inner triangle}) \leq \delta$



Note: for a tree, inner  $\Delta = \text{pt}$

Fact:

"geodesics are Morse" meaning

$$d(x, [y, z]) \approx \frac{1}{2}(d(x, z) + d(x, y) - d(y, z))$$

$=:$  Grushov product  
 $\langle y, z \rangle_x$

$(X, d)$  hyp metric space

fix  $o \in X$   
 $\partial X := \{ \text{geodesic rays based at } o \}$

bndl equiv

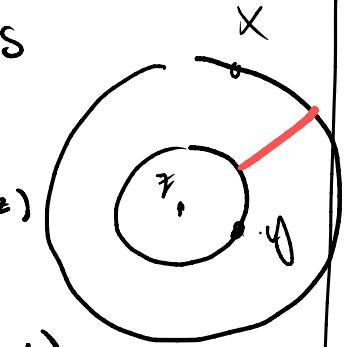
How to get horoball packing?

Defn Busemann functions

$x, y, z \in X$ ,

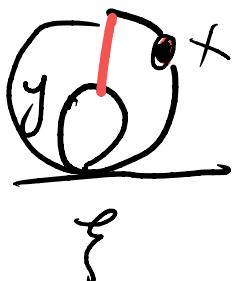
$$\beta_z(x, y) := d(x, z) - d(y, z)$$

Notice  $|\beta_z(x, y)| \leq d(x, y)$ .



for  $\xi \in \partial X$ , define

$$\beta_\xi(x, y) := \liminf_{z \rightarrow \xi} \beta_z(x, y)$$



Defn:

Fix  $\xi \in X$ . horoball of radius  $r$  centered at  $\xi$

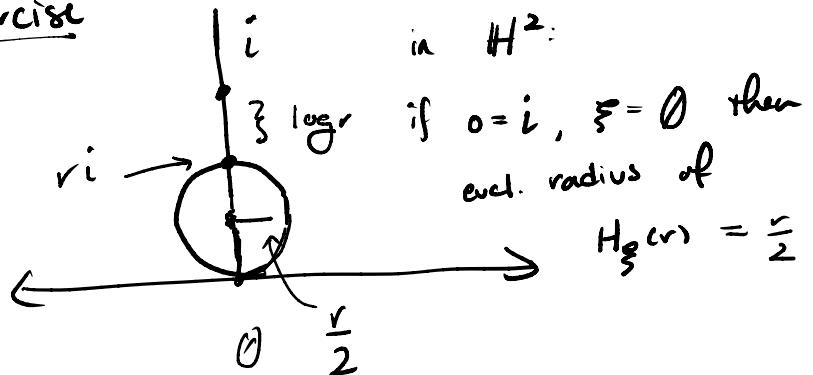
$$H_{\xi^r} := \{x \in X : \beta_\xi(x, o) \leq \log r\}$$

Horosphere =  $\log r$

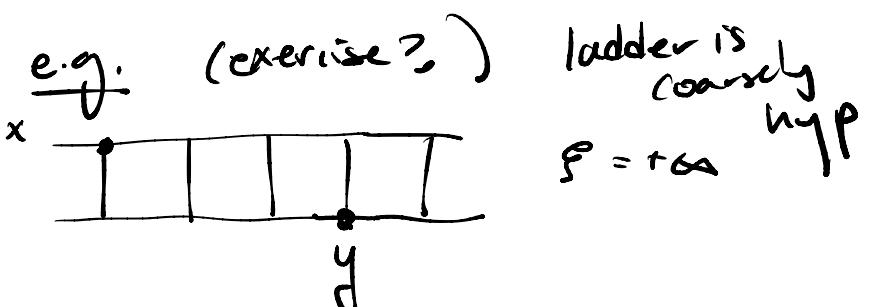
example  $R^2$

example  $H^2$   $\beta_\xi$  is a limit  
recover usual horoballs

Exercise

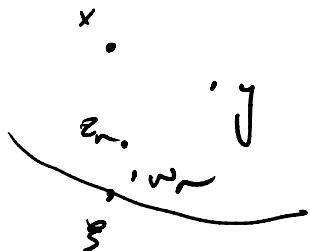


In a hyp metric space  $\beta_\xi(x, y)$  is not a limit



In a hyp metric space  $\beta_{\mathcal{S}}(x, y)$   
is coarsely well-defined

idea:  $z_{tn}, w_{tn} \rightarrow \mathfrak{s}$ .



exercise

choose to lie on

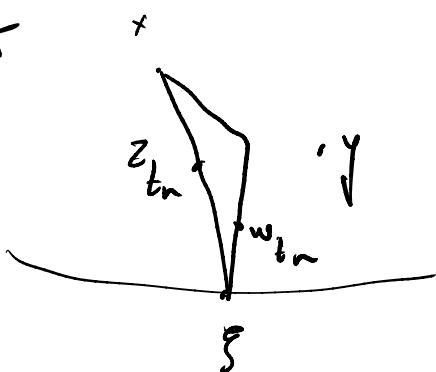
geodesics from  $x$  to  $\mathfrak{s}$

dist.  $tn$  away

want

$$\beta_{z_{tn}}(x, y) \approx \beta_{w_{tn}}(x, y).$$

see  
that

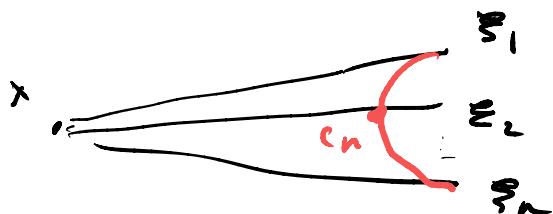


$$\begin{aligned} & |\beta_{z_{tn}}(x, y) - \beta_{w_{tn}}(x, y)| \\ &= |d(y, z_{tn}) - d(y, w_{tn})| \\ &\leq d(z_{tn}, w_{tn}) \leq c \end{aligned}$$

by defn of  $\mathcal{D}X$

$c$  is uniform by hyp of  $X$

Pf:



$$\begin{aligned} & d(s_1(\epsilon), s_n(\epsilon)) \not\equiv n \forall t. \\ & \Rightarrow c_n \rightarrow c \in X. \end{aligned}$$

Properties of Busemann  
functions:

- 1-Lipschitz

$$|\beta_{\mathcal{S}}(x, y)| \leq d(x, y)$$

- $\Gamma$ -inv

$$\beta_{\gamma_{\mathcal{S}}}(x, y) = \beta_{\mathcal{S}}(x, y)$$

- cocycle

$$\begin{aligned} & \beta_{\mathcal{S}}(x, y) + \beta_{\mathcal{S}}(y, z) \\ & \approx \beta_{\mathcal{S}}(x, z) \end{aligned}$$

Def  $H_g(r) = \text{shadow of } H_g(r)$

$$= \{ \gamma \in \partial X \text{ such that}$$

some gd. o to  $\gamma$   
intersects  $H_g(r)$  }

Defn (Recall) Define property discs

$\Gamma$  property discs, non-elm geometrically finite if every pt in  $\Lambda_\Gamma$  is either conical or <sup>bnd</sup> parabolic.

$\Gamma$  convex cocompact if only conical.

From now on:

Assume  $\Gamma \subset I$  s.t.  $X$  is prop. discs, non-elm and acting geom. fin. on  $X$

$\Lambda_\Gamma := \text{limit set} = \overline{\Gamma_0} \setminus \Gamma_0 \subseteq \partial X$ .

$C_\Gamma = \text{convex hull of } \Lambda_\Gamma$ .

Fact (Tukia, Yaman, Bowditch)

- fin. many conjugacy classes  $\Pi_1, \dots, \Pi_d$  parabolic subgroups
- $\Gamma$  hyp relative to  $\{\Pi_1, \dots, \Pi_d\}$

Prop: (Bowditch)

$\Gamma \curvearrowright X$  geom. fin.  $\Rightarrow \exists$  quasi- $\Gamma$ -inv., pairwise disjoint, horoball packing of  $X$   $\{H_p(r_p)\}_{p \in P}$ , and

$\Gamma \curvearrowright C_\Gamma = \cup H_p(r_p)$  cocompact

Defn:  $\Pi \subset \Gamma$  parabolic subgp

has mixed exponential growth if

$\exists a_\Pi, \delta_\Pi > 0$  s.t. for  $t \geq 0$ ,

$B_\Pi(t) :=$

$\#\{g \in \Pi : d(0, g^0) \leq t\} \asymp e^{\delta_\Pi t} (t+1)^{a_\Pi}$

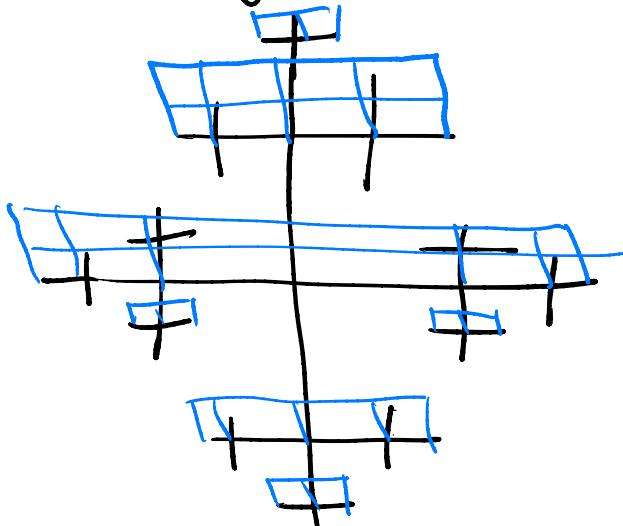
## understanding mixed exp. growth

Exercise:  $\pi \curvearrowright \mathbb{H}^n$

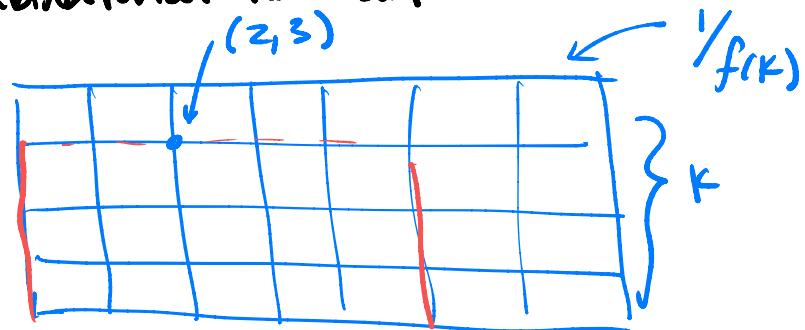
$$a_\pi = 0 \quad \delta_\pi = \frac{\text{rank } \pi}{2}$$

recall Madeline's discussion  
of the Graves-Manning cusp  
space:

$\text{Cay}(F_2)$



combinatorial horoball



$$\text{eg. } f(k) = 2^{-k} \quad 2^{-5} = \frac{1}{32}$$

$$\begin{aligned} d((0,0), (0,32)) &\leq 2 \times 5 + 32 \times 2^{-5} \\ &= 11 < 32 \end{aligned}$$

$X = \text{Cay}(F_2) \cup \text{combinatorial horoballs}$

$d$  = induced metric by  $f$

then  $(G \cdot M, \text{Hruska-Healy}, \dots)$  for certain  $f$   
w/  $e^{at} \leq f(t) \leq e^{bt}$  for some  $a, b > 0$

then  $(X, d)$  a hyperbolic metric space

and  $\Gamma = F_2$  acts geometrically finitely

exercise in that case,  $B_\pi(t) \asymp f(t)$

Can create mixed exp growth or anything.

## Khinchin-type Theorem

Let  $\Pi_1, \dots, \Pi_d$  be the finitely many parabolic subgroups of  $\Gamma$  up to conjugation.

$$P^i = \Gamma \cdot P_i \text{ where } \Pi_i P_i = P_i$$

Defn:  $\varphi: \mathbb{R}^+ \rightarrow \mathbb{R}^+$  Khinchin

if  $\varphi$  incr and  $\exists b_1 < 1, b_2 > 0$  such that

$$\varphi(b_1 x) \geq b_2 \varphi(x)$$

$\forall x \in \mathbb{R}^+$  (important only for small  $x$ )

Defn.

$$\Theta^i(\varphi) = \left\{ x \in \Lambda_{P^i} : x \in H_P(v_P \varphi(v_P)) \text{ for } \begin{array}{c} \text{co-} \\ \text{by} \end{array} \text{ many} \right. \\ \left. p \in P^i \right\}$$

Khinchin series

$$K_\lambda^i(\varphi) = \sum \varphi(\lambda^n)^{2(\delta_p - \delta_{\Pi_i})} (-2 \log \varphi(\lambda^n) + 1)^{a_{\Pi_i}}$$

for  $i = 1, \dots, d$

where

$$\delta_\Gamma = \limsup_{t \rightarrow \infty} \frac{1}{t} \log \#\{x \in \Gamma : d(o, x) \leq t\}.$$

Thm (B.-Tiozzo)

[Khinchin-type theorem]

next time

$\mu$  Patterson-Sullivan measure \*

$\Gamma$  mixed exp growth

for any Khinchine function  $\varphi$ ,

$$(1) \quad \mu(\Theta_\lambda(\varphi)) = 0 \text{ if } K_\lambda(\varphi) < \infty$$

$$(2) \quad \mu(\Theta_\lambda(\varphi)) = 1 \text{ if } K_\lambda(\varphi) = \infty.$$

For  $S = \mathbb{H}^2 / \Gamma$  finite area,

$$\delta_\Gamma = 1, \quad \delta_{\Pi_i} = \frac{1}{2}, \quad a_{\Pi_i} = 0$$

$$\text{hence } K_\lambda^i(\varphi) = \sum \varphi(\lambda^n).$$

(Kleinian (Bishop-Jones, Sullivan))

$$\delta_\Gamma = \text{Hausdorff dim } \Lambda_\Gamma$$

### Thm (B.-Tiozzo)

[Logarithm law] Same  $\mu$ .

$$\delta_{\pi}^* = \max \delta_{\pi_i};$$

For  $\mu$ -a.e.  $\beta \in \Lambda_{\Gamma}$ ,

$$\limsup_{t \rightarrow \infty} \frac{\text{cusp depth}(\beta_t)}{\log t} = \frac{1}{2(\delta_{\pi} - \delta_{\pi}^*)}.$$

end day 1

Note: recover  $H^2$  case, Anteola

Note: the  $2(\delta_{\pi} - \delta_{\pi}^*)$  is coming from the fine scaling properties of  $\mu$ .

### Prior results

- Sullivan / Stratmann-Velani geom.fin  
 $\Gamma \subset H^n$

- Pavlın-Hersonsky X Riem. pinched reg. curvature and  $a_{\pi_i} = 0 \quad \forall i=1,\dots,d$

### Future results

(Horton) filtration by each conjugacy class of parabolic subgroups

### Application

by Benoist-Oh as in Blasco-Zhu,  
 $\pi$  discrete parabolic  
preserves strictly conv  $\Omega$  with  $C^1$  boundary  
 $\Rightarrow$  mixed exponential growth.

Thm applies to  $(\Omega, d_{\Omega})$  Hilbert metric if Gromov hyperbolic

Growth condition is actually more general for the Khinchin-type theorem, but not for log law.

## Day 2

Fix:

$\Gamma \curvearrowright X$  non-elem, geom. fin  
 $(X, d)$  proper geodesic coarsely by  $\Gamma$

Let  $\Pi_1, \dots, \Pi_d$  be the finitely many  
 parabolic subgroups of  $\Gamma$  up to  
 conjugation.

$$P = \{p \in \Lambda_P \text{ parabolic f.p.s}\}$$

For  $\Gamma' < \Gamma$ , define

$$B_{\Gamma'}(t) = \{g \in \Gamma' : d(o, go) \leq t\}.$$

$\Pi < \Gamma$  has mixed exp growth if

$$B_\Pi(t) \asymp e^{\delta_\Pi t} (t+1)^{a_\Pi}$$

$$\text{some } 0 < \delta_\Pi, 0 \leq a_\Pi.$$

$$H_p(r) = \{x \in X : \beta_p(o, x) \leq \log r\}$$

Bowditch  $\Rightarrow$  quasi- $\Gamma$ -inv  
 horoball packing  $\{H_p(r_p)\}_{p \in P}$   
 $\text{stab}_p(p) H_p \approx H_p$

and

$$\Gamma \curvearrowright C(\Lambda_P) \setminus \cup H_p(r_p)$$

cocompactly.

$\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  decr, "Khinchin"

$\Theta(\varphi) = \{x \in \Lambda_P : x \in H_p(r_p \varphi(r_p)) \text{ for}$   
 w.l.o.g. many  
 $p \in P\}$

Khinchin series

$$K_\lambda(\varphi) = \sum_{i=1}^d \sum_{n \in \mathbb{N}} \varphi(\lambda^n)^{2(\delta_p - \delta_{\Pi_i})} (-2 \log \varphi(\lambda^n) + 1)^{a_{\Pi_i}}$$

$$\text{some } 0 < \lambda < 1$$

Thm (B.-Tiozzo)

[Khinchin-type theorem] *next time*

$\mu$  Patterson-Sullivan measure \*

$\Gamma$  mixed exp growth,  $\delta_\pi < \delta_\Gamma$

for any Khinchine function  $\varphi$ ,

(1)  $\mu(\Theta_\lambda(\varphi)) = 0$  if  $K_\lambda(\varphi) < \infty$

(2)  $\mu(\Theta_\lambda(\varphi)) = 1$  if  $K_\lambda(\varphi) = \infty$ .

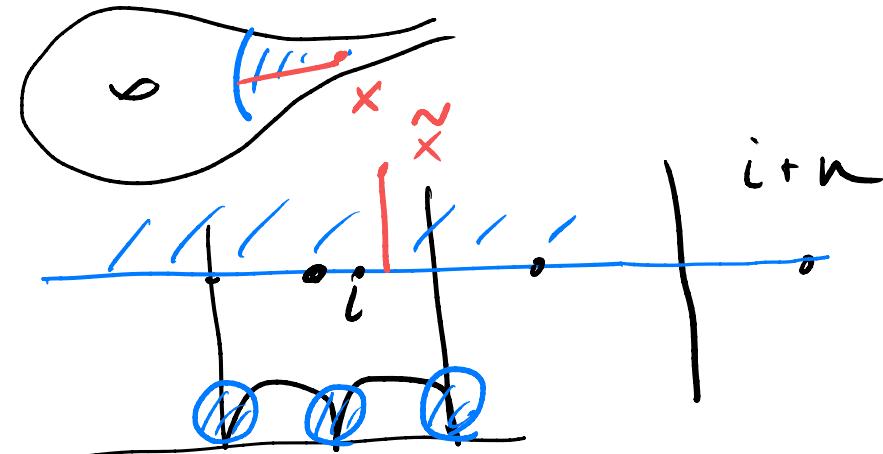
Thm (B.-Tiozzo)

[Logarithm law] Same  $\mu$ .

$$\delta_\pi^* = \max \delta_{\pi_i}$$

For  $\mu$ -a.e.  $\xi \in \Lambda_\Gamma$ ,

$$\limsup_{t \rightarrow \infty} \frac{d(\xi_t, \Gamma_0)}{\log t} = \frac{1}{2(\delta_\pi - \delta_\pi^*)}.$$



$\Gamma_0$  does not enter  
horoballs

$$d(\Gamma_0, C_\Gamma - \bigcup_{P \in P} H_P(c_P)) \text{ bounded}$$

$\Rightarrow \sim$   
cusp depth( $x$ )

$$= d(x, C_\Gamma - \bigcup_{P \in P} H_P(c_P))$$

$$\approx d(x, \Gamma_0).$$

□

Thm (Coornaert)

$$B_p(t) \propto e^{\delta_p t}$$

for some  $\delta_p$  which we call the critical exponent

goal: the  $2(\delta_p - \delta_{\pi}^*)$  is coming from the fine scaling properties of  $\mu$ .

Prior results

- Sullivan / Stratmann-Velani geom.fin  
 $\Gamma \subset \mathbb{H}^n$
- Pavlenko-Hersonsky X Riem. pinched neg. curvature and  $a_{\pi_i} = 0 \quad \forall i=1,\dots,d$

Future results

(Horton) filtration by each conjugacy class of parabolic subgroups

Growth condition is actually more general for the Khinchin-type theorem,

our proof of log law  
Needs mixed exp growth  
but Madeline is generalizing

## Application

certain relatively Anosov rep<sup>ns</sup>

recall  $S = H^2/\Gamma$  means

$\Gamma \hookrightarrow \mathrm{PSL}(2, \mathbb{R})$  <sup>any hyp  $\Gamma'$ .</sup>

can study  $\Gamma \hookrightarrow \mathrm{PSL}(d, \mathbb{R})$

In some cases, natural  $\Gamma$ -inv  
set in  $\mathbb{R}\mathbb{P}^d$  is Gromov-hyp

Benoist-Oh, Blasco-Zhu: has  
mixed-exp growth

many examples  $a_\Gamma \neq 0$  but  
cannot realize any explicitly.

Kim-Oh: others act on GM-cusp space.

Side comment on  $\delta_\Gamma$

Thm: (B-Canary-Kao-Martone)  $\delta_\Gamma T$

$$\# \text{"closed curves"} \sim \frac{e^{\delta_\Gamma T}}{\delta_\Gamma T}$$

Margulis counting for rel Anosov rep<sup>ns</sup>

## Patterson-Sullivan construction

from Didac

recall the critical exponent of  $\Gamma$  is

$$\delta_\Gamma = \limsup_{t \rightarrow \infty} \frac{1}{t} \log \#\{ \gamma \in \Gamma : d(o, \gamma o) \leq t \}.$$

$$\text{e.g. } \#\{ \gamma \in \Gamma : d(o, \gamma o) \leq t \} = e^{2t} \Rightarrow \delta_\Gamma = 2$$

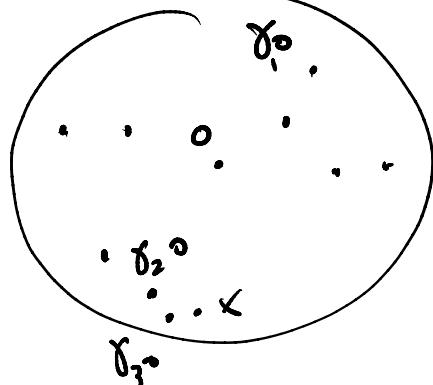
exercise  $\delta_\Gamma$  = abscissa of convergence for

$$P(o, x, s) := \sum_{\gamma \in \Gamma} e^{-s d(x, \gamma o)}$$

(first for  $x=0$ , then triangle inequality)

Defn measure on  $\overline{X}$

$$\mu_{x,s} = \sum_{\gamma \in \Gamma} \frac{e^{-s d(x, \gamma o)}}{P(o, o, s)} \delta_{\gamma o}$$



up to extraction, define Patterson-Sullivan measure

$$\mu_x := \lim^*_{s \searrow \delta_\Gamma} \mu_{x,s}$$

when  $\Gamma$  is divergent type

$$P(o, x, \delta_\Gamma) = \infty$$

$$\text{supp } \mu_x = \Lambda_\Gamma.$$

when  $\Gamma$  is not divergent type, modify w/  $n$  which does not change  $\delta_\Gamma$ .

see that if compact  $K \subseteq X$ ,

$$\mu_{x,s}(K) \rightarrow 0 \quad s \searrow \delta_\Gamma$$

and if open  $O \subseteq \overline{X} \setminus \overline{\Lambda_\Gamma}$ ,

$$\mu_{x,s}(O) = 0.$$

Let  $\mu := \mu_o$ .

Pf ideas

defn  $\{v_x\}_{x \in X}$  is a  
 $\delta$ -quasi-conformal density at  $\infty$ :  $|v_x| < \infty$  and

$$\gamma_x v_x = v_{\gamma x} \quad \Gamma\text{-inv}$$

$$\frac{d v_x}{d v_y}(\xi) \asymp e^{-\delta \beta_\xi(x, y)} \quad \begin{matrix} \text{transf.} \\ \text{rule} \end{matrix}$$

Thm:  $\{\mu_x\}_{x \in X}$  is  $\delta_\Gamma$ -q.c. density

Patterson: Fuchsian

Sullivan: Kleinian

Cornnaert: non-elementary,  
proper, coarsely hyperbolic

Let  $\mu := \mu_0$

for  $s > \delta_\Gamma$

$$g_* \mu_{x,s}(A) = \mu_{x,s}(g^{-1}A)$$

$$= \sum_{\gamma \in \Gamma} e^{-s d(x, \gamma_0)} \frac{\delta_{\gamma_0}(g^{-1}A)}{P(0,0,s)}$$

$$\gamma_0 \in g^{-1}A \Leftrightarrow g\gamma_0 \in A$$

$$= \frac{1}{P(0,0,s)} \sum_{\gamma \in \Gamma} e^{-s d(gx, g\gamma_0)} \frac{\delta_{g\gamma_0}(A)}{g\gamma_0}$$

$$= \frac{1}{P(0,0,s)} \sum_{g\gamma \in \Gamma} e^{-s d(gx, g\gamma_0)} \frac{\delta_{g\gamma_0}(A)}{g\gamma_0}$$

$$\rightarrow \mu_{gx}$$

transf- rule

$$\xi = \lim y_n^0$$

$$\frac{d\mu_x}{d\mu_y}(\xi)$$

$$\frac{d\mu_{x,s}}{d\mu_{y,s}}(y_n^0) = \frac{P(0,0,s)}{P(0,0,s)} \frac{e^{-s d(x, y_n^0)}}{e^{-s d(y, y_n^0)}}$$

$$= e^{-s(d(x, y_n^0) - d(y, y_n^0))}$$

$$\xrightarrow{\text{coarsely!}} e^{-\delta_r \beta_\xi(x, y)}$$

side bar:

Patterson-Sullivan current:

$$d\mu_{PS}(\xi, \eta) =$$

$$e^{-\delta_r \langle \xi, \eta \rangle_0} d\mu(\xi) d\mu(\eta)$$

my fav current

exercis?  $\mathbb{P} \cap \mathbb{H}^2$  cocompact

$$d\mu_{PS}(\xi, \eta) = \frac{1}{(\xi - \eta)^2} d\lambda(\xi, \eta)$$

on  $\mathbb{R}^2 \setminus \{\xi = \eta\}$

Fact = Liouville

Notn.: shadow of a ball radius  $r$

$$\mathcal{O}_r(x, y) = \{\xi \in \partial X \text{ s.t. } \exists \text{ geodesic ray } [x, \xi] \text{ intersecting } B(y, r)\}$$

shadows generate topology on  $\partial X$  for fixed  $r$ .

Sullivan's  
shadow lemma  $\Gamma$  non-elliptic,  
 $x \in X, \{\mu_x\} \text{ S.p.-q.e.}, r \gg 0,$

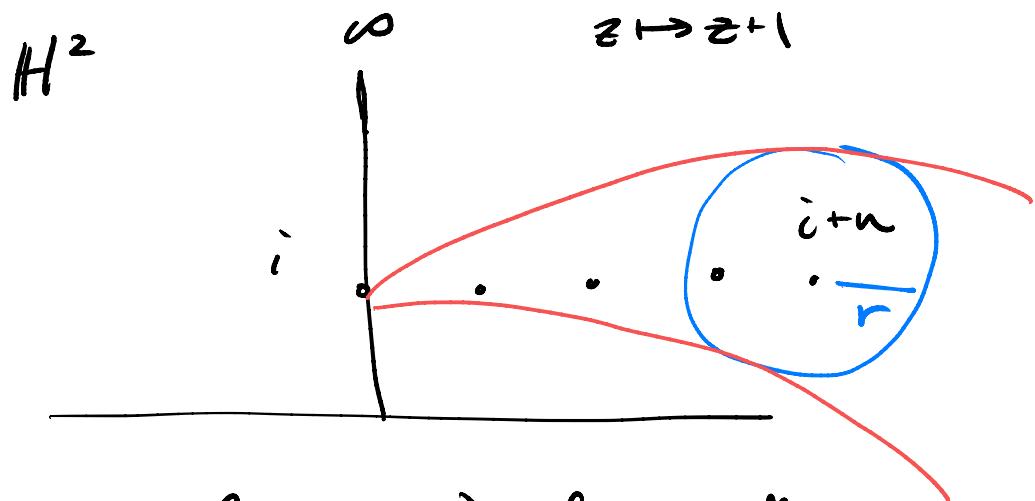
$$\begin{aligned} \frac{1}{c} e^{-\delta_r d(x, \delta x)} \\ \leq \mu_x(\mathcal{O}_r(x, \delta x)) \\ \leq c e^{-\delta_r d(x, \delta x)} \end{aligned}$$

Cor:  $\xi \in \Lambda_p$  conical,  
then  $\mu(\{\xi\}) = 0$

Pf.:  $y_n x \rightarrow \xi$  conically  $\Rightarrow$   
 $d(x, y_n x) \rightarrow \infty$  and

for  $n$  large,  $\xi \in \mathcal{O}_r(x, y_n x)$  hence  
 $\mu(\{\xi\}) \leq C e^{-d(x, y_n x)} \rightarrow 0$

Note does not work for parabolic fixed points



$\infty \notin \mathcal{O}_r(i, i+n)$  for  $n$  suff. large

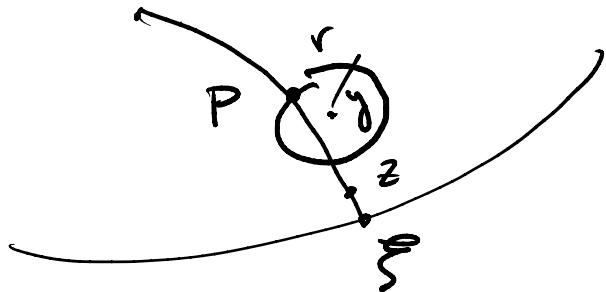
## Pf of Shadow lemma (Roblin)

Pf:

Lemma:  $\forall \xi \in \partial_r(x,y),$

$$d(x,y) - 2r \leq \beta_\xi(x,y) \leq d(x,y)$$

Pf:



$$d(x,y) \leq d(x,P) + r$$

$$d(y,z) \leq d(z,P) + r$$

so

$$d(x,z) = d(x,y) + d(y,z)$$

$$= d(x,P) + d(P,z) - d(y,z)$$

$$\geq d(x,y) - r + d(y,z) - r - d(y,z).$$

By quasi- $\Gamma$ -invariance,

$$\begin{aligned} \mu_x(\partial_r(x,y)) &= \mu_x(\gamma \partial_r(\gamma^{-1}x, y)) \\ &\stackrel{*}{=} \mu_{\gamma^{-1}x}(\partial_r(\gamma^{-1}x, y)) \quad (*) \end{aligned}$$

and the transf. rule,

$$\begin{aligned} \frac{1}{c} \int_{\partial_r(\gamma^{-1}x, y)} e^{-\delta_p \beta_\xi(\gamma^{-1}x, y)} d\mu_x(\xi) \\ \leq * \\ \leq c \int_{\partial_r(\gamma^{-1}x, y)} e^{-\delta_p \beta_\xi(\gamma^{-1}x, y)} d\mu_x(\xi) \end{aligned}$$

Then 1-Lipschitz and Lemma  $\Rightarrow$

$$\begin{aligned} \frac{1}{c} \int_{\partial_r(\gamma^{-1}x, y)} e^{-\delta_p d(\gamma^{-1}x, y)} d\mu_x(\xi) \\ \leq * \\ \leq c \int_{\partial_r(\gamma^{-1}x, y)} e^{-\delta_p(d(\gamma^{-1}x, y) - 2r)} d\mu_x(\xi). \end{aligned}$$

□

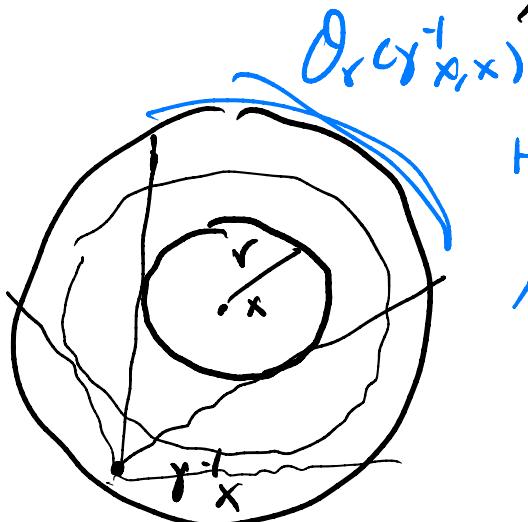
Then

$$* \leq e^{2\delta_p r} e^{-\delta_p d(x, \gamma_x)} \|\mu_x\|$$

completes the upper bound.

For the lower bound, we have

$$* \geq e^{-\delta_p d(x, \gamma_x)} \mu_x(\partial_r(\gamma_{x,x}^{-1}))$$



How small  
could  
 $\mu_x(\partial_r(\gamma_{x,x}^{-1}))$   
be?

if  $r$  large, use ping-pong argument to show  $\forall x, \exists O_1, O_2$  open in  $N_p$  s.t. if  $y \in X$ , one of  $O_i \subseteq \partial_r(\gamma_y)$ .  $\square$

Finish here

Cor:  $\delta_p$ -q.c. densities equiv. for convex cocompact

Pf:  $\{\mu_x\}, \{v_x\}$

$$\frac{1}{c^2} \leq \frac{\mu_x(\partial_r(x, \gamma_x))}{v_x(\partial_r(x, \gamma_x))} \leq c^2$$

$\forall \xi \in N_p$ ,  $\exists$  seq.  $\gamma_n \rightarrow \xi$  conically again, so  $\forall n$  large  $\xi \in \partial_r(x, \gamma_n)$  hence

$$\frac{d\mu_x(\xi)}{dv_x} \in [\frac{1}{c^2}, c^2] \subseteq (0, \infty) \quad \forall \xi.$$

# Talk #3

Last time (Rem: typo)

$$\mathcal{O}_r(x,y) = \{\xi \in \partial X \text{ s.t. } \exists \text{ geodesic ray } [x, \xi] \text{ intersecting } B(y, r)\}$$

$\{\mu_x\}_{x \in X}$  Patterson-Sullivan

Sullivan's  
shadow lemma  $\Gamma$  non-elliptic,  
 $x \in X$ ,  $\{\mu_x\} \subset \mathcal{O}_r$  q.c.,  $r \gg 0$ ,

$$\begin{aligned} \frac{1}{c} e^{-\delta_r d(x, \gamma x)} \\ &= \mu_x(\mathcal{O}_r(x, \gamma x)) \\ &\leq c e^{-\delta_r d(x, \gamma x)} \end{aligned}$$

when  $\Gamma$  convex cocompact,  
 $\mathcal{O}_r = \text{Hausdorff } \Lambda_\rho$  (Patterson)

Rem: Ricks, Gekhtman-Ma

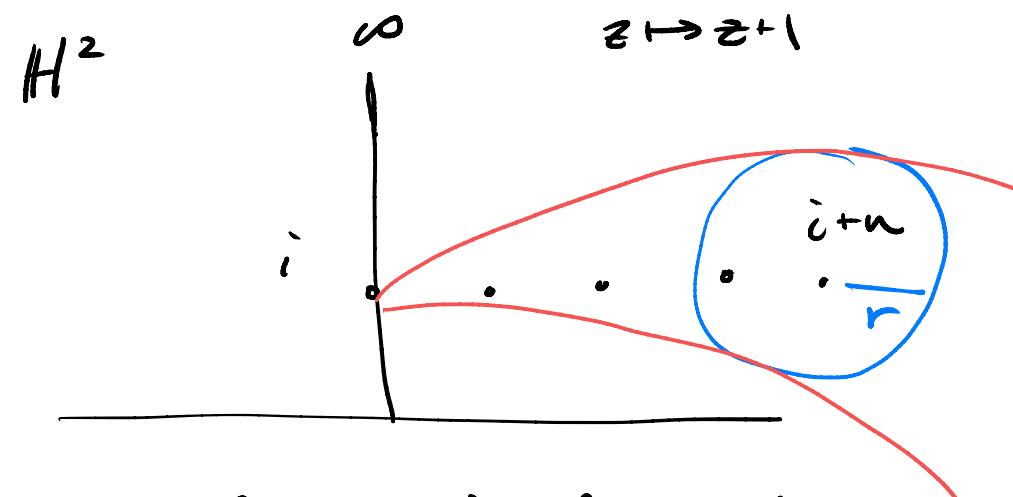
Cor:  $\xi \in \Lambda_\rho$  conical,  
then  $\mu(\{\xi\}) = 0$

Pf:  $\gamma_n x \rightarrow \xi$  conically  $\Rightarrow$   
 $d(x, \gamma_n x) \rightarrow \infty$  and

for  $n$  large,  $\xi \in \mathcal{O}_r(x, \gamma_n x)$  hence

$$\mu(\{\xi\}) \leq C e^{-d(x, \gamma_n x)} \rightarrow 0$$

Note does not work for parabolic fixed points



$\infty \notin \mathcal{O}_r(i, irn)$  for  $n$  suff. large

Cor:  $\delta_p$ -q.c. densities  
equiv. for convex cocompact  $\Gamma$

Pf:  $\{\mu_x\}\{\nu_x\}$

$$\frac{1}{c^2} \leq \frac{\mu_x(\partial_r(x, \gamma_x))}{\nu_x(\partial_r(x, \gamma_x))} \leq c^2$$

$\forall \xi \in \Lambda_\Gamma$ ,  $\exists$  seq.  $\gamma_n x \rightarrow \xi$   
conically again, so  $\forall n$  large  
 $\xi \in \partial_r(x, \gamma_n x)$  hence

$$\frac{d\mu_x(\xi)}{d\nu_x(\xi)} \in [\frac{1}{c^2}, c^2] \subseteq (0, \infty)$$

$\forall \xi.$

Defn:  $A \subseteq \Lambda_\Gamma$  is  $\Gamma$ -inv if  
 $\forall \gamma \in \Gamma^1, \gamma A = A$ .

Defn:  $\mu$  is ergodic for  $\Gamma \curvearrowright X$   
if every  $\Gamma$ -invariant set has  
trivial  $\mu$ -measure, meaning it  
is null or co-null.

e.g.  $\mathcal{O}_\gamma$  where  $\gamma$  closed  
orbit for geodesic flow

Let

Cor:  $\mu$  is ergodic for  
 $\cap_{\Gamma \text{ CVX compact}} \Omega \cap \Lambda_\Gamma$

Pf of cor:

Assume  $A \subseteq \Lambda_\Gamma$  is  $\Gamma$ -inv  
and  $\mu(A) > 0$ . Then

define

$$\mu_A(E) := \mu(E \cap A)$$

see that  $\mu_A$  is also a q.e.  
density:

- quasi- $\Gamma$ -invariant

$$\gamma_* \bar{\mu}_x(B) = \bar{\mu}_x(\gamma^{-1}B)$$

$$= \mu_x(A \cap \gamma^{-1}B)$$

$$= \mu_x(\gamma^{-1}A \cap \gamma^{-1}B)$$

$$= \mu_x(\gamma^{-1}(A \cap B))$$

$$= \mu_{\gamma x}(A \cap B)$$

$$= \bar{\mu}_{\gamma x}(B)$$

- transformation rule

open  $B_n \downarrow \xi$

$$\frac{d\bar{\mu}_y}{d\bar{\mu}_x}(\xi) = \lim_{n \rightarrow \infty} \frac{\mu_y(B_n \cap A)}{\mu_x(B_n \cap A)}$$

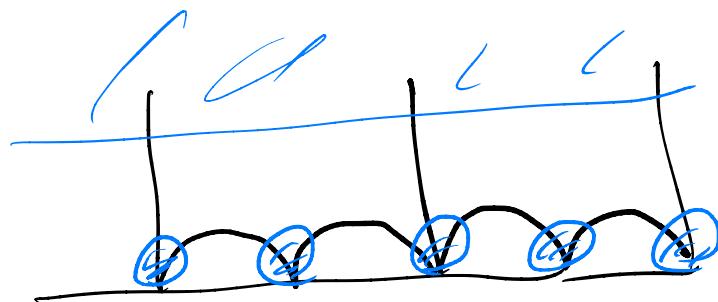
$$= \lim_{n \rightarrow \infty} \frac{\int_{B_n \cap A} e^{-\delta_p \beta_\xi(x, y)} d\mu_x(\xi)}{\mu_x(B_n \cap A)}$$

$$\rightarrow e^{-\delta_p \beta_\xi(x, y)}.$$

Then  $\{\mu_x\}, \{\bar{\mu}_x\}$  equivalent

hence  $\mu_x(A^c) = \bar{\mu}_x(A^c) = 0$

for  $\Gamma$  geom. fin,



## Global Shadow Lemma / fluctuating density theorem

Sullivan, Stratmann-Velani for hyp

Paulin-Hersonsky, Schapira for neg. curvature

B.-Tiozzo

$(X, d)$  hyp,  $\Gamma$  geom. fin.

one  $\pi \subset \Gamma$

$\delta_\pi < \delta_\Gamma$ . Then

$$\mu(O_r(x,y)) \asymp e^{-\delta_\Gamma d(x,y)} e^{(2\delta_\pi - \delta_\Gamma) d(y, \Gamma^0)}$$

Note:  $y \notin U H_p(r_p) \Rightarrow \asymp e^{-\delta_\Gamma d(x,y)}$

Cor:  $\mu$  has no atoms

Pf:  $d(x,y) > d(y, \Gamma^0)$

$$\Rightarrow \mu(O_r(x,y)) \leq e^{2\delta_\pi - 2\delta_\Gamma d(y, \Gamma^0)}$$

$\rightarrow 0$  as  $y \rightarrow \partial X$

since  $\delta_\pi < \delta_\Gamma$ .

Note:  $y \notin U H_p(r_p)$   
recover usual shadow lem.

Matsuaki - Tabuki - Jaerisch:

Patterson - Sullivan measure exists and is ergodic

pictures, how is this different from the original.

Next: say something now about upgrading from standard shadow lemma

Idea one piece of the argument: where does orbit growth come in

Credit for strategy to Schapira

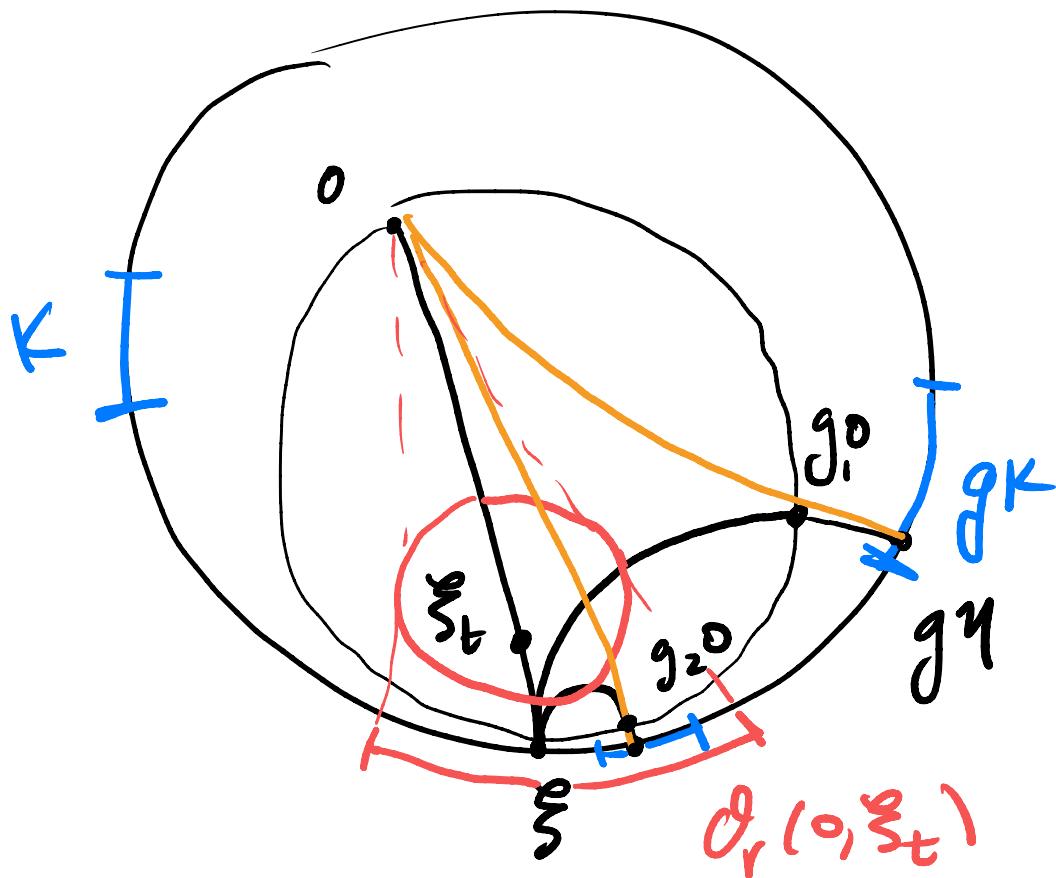
$$\Pi = \text{Stab}_P \xi, \quad \xi_t = \text{pt dist. } t \text{ on geod } [0, \xi]$$

Key Lemma

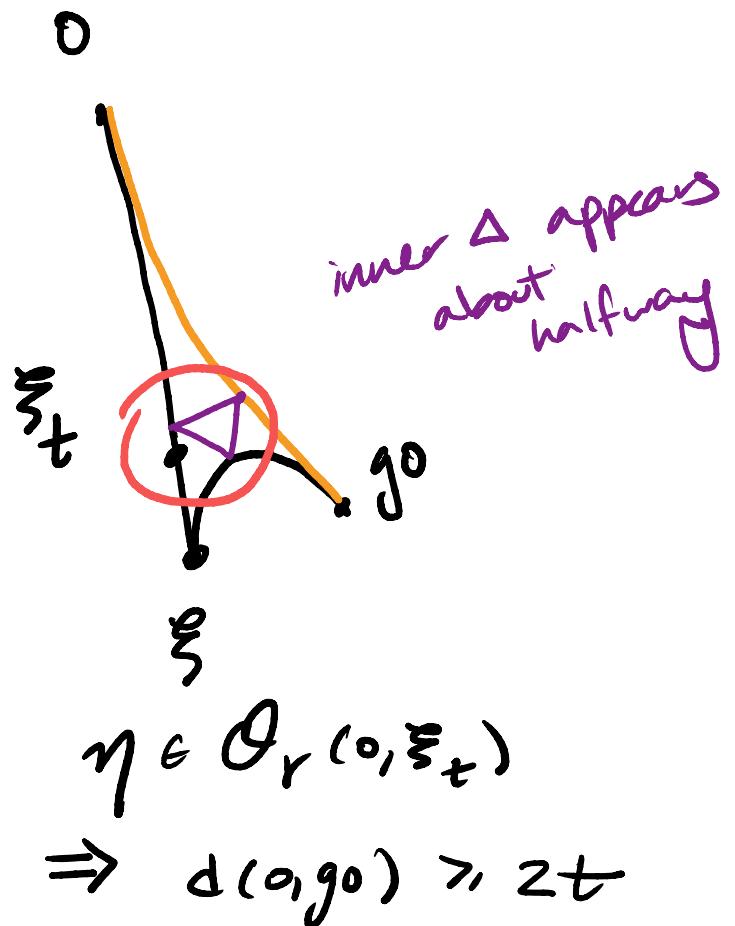
$$\bigcup_{g \in \Pi} gk \sim \mathcal{O}_r(0, \xi_t)$$

$d(0, g_0) \geq 2t$

$\bigcup_{g \in \Gamma} gK \sim O_r(0, \xi_t)$   
 $d(0, g_0) \geq 2t$



it is super technical  
so let's just understand  
& appreciate the statement



density lemma

$$\mu(\mathcal{O}_r(x, \xi_t)) \asymp \sum_{g \in \pi} e^{-d_\Gamma d(o, go)} \cdot \delta_r \approx$$

$$d(o, go) \geq 2t$$

choose  $K$  cpt fundamental domain for  $\pi \cap \Lambda_\Gamma \setminus \{\xi\}$   
 by geom. finiteness

$$\mu(\mathcal{O}_r(x, \xi_t)) \asymp \sum_{g \in \pi} \mu(gK)$$

$$\uparrow \begin{matrix} g \in \pi \\ d(o, go) \geq 2t \end{matrix}$$

by key lemma

note: no atoms!

$$\frac{d\mu_{g^{\xi_t}}(\eta)}{d\mu_{\xi_t}} = \frac{d\mu_{g^{\xi_t}}(\eta)}{d\mu_{\xi_t}} \asymp e^{-\delta_r \beta_\eta(g^{\xi_t}, \xi_t)}$$

coarse hyp lemma: If  $K \subseteq \Lambda_F \setminus \{g_0\}$  cpt,  $\beta_{g^{\xi_t}}(g_0, \cdot)$  large,

$$\beta_{g^{\xi_t}}(\xi_t, g^{\xi_t}) \asymp d(g_0, g_0) + 2t \quad \text{if } g^{\xi_t} = \xi_t \text{ parab.}$$

density lemma follows □

Observe: density lemma is where  $B_T(\epsilon)$  affects the measure in a significant way.

## Talk 4

The Khinchin Theorem

quasi-independence

The logarithm law

Then (B.-Tierra)  $\Gamma_{\text{g.f.}}(\chi_1, \lambda)$

[Khinchin-type theorem] one  $\pi$  mixed growth,  $\delta_\pi < \delta_\tau$

$\mu$  PS measure then

for any Khinchine function  $\varphi$ ,

- (1)  $\mu(\Theta_\lambda(\varphi)) = 0$  if  $K_\lambda(\varphi) < \infty$
- (2)  $\mu(\Theta_\lambda(\varphi)) = 1$  if  $K_\lambda(\varphi) = \infty$ .

Define for  $\lambda < 1$

$$P_m = \{p \in \Lambda_p \text{ parabolic s.t. } r_p \in [\lambda^{n+1}, \lambda^n)\}$$

$$S_m = \bigcup_{p \in P_m(\lambda)} H(r_p \varphi(r_p))$$

PICTURES

$$\Leftrightarrow (\varphi) = \limsup S_m(\lambda)$$

$$= \bigcap_{n=0}^{\infty} \bigcup_{m \geq n} \bigcup_{p \in P_m(\lambda)} H_p(r_p \varphi(r_p))$$

Let  $b(t) = (-2 \log t + 1)^{\frac{1}{1-t}}$ . Then

$$K(\varphi) = \sum_{n \in \mathbb{N}} \varphi(\lambda^n) b(\varphi(\lambda^n))^{2(\delta_p - \delta_\pi)}$$

Defn:

$(Y, \mathcal{P})$  measure space  $A_n \subseteq Y$  measurable are quasi-indep. if  $\exists c$  s.t.  $\forall n \neq m$ ,

$$P(A_n \cap A_m) \leq c P(A_n) P(A_m)$$

e.g. coin-tosses are exactly independent

$A_n = \text{flip heads at step } n$

$A_m = \text{flip heads at step } m$

## Borel - Cantelli Lemma

- ①  $\sum P(A_n) < \infty \Rightarrow P(\limsup A_n) = 0$   
 ②  $\sum P(A_n) = \infty$  and quasi-independence  
 $\Rightarrow P(\limsup A_n) > 0.$

## Quasi-independence lemma

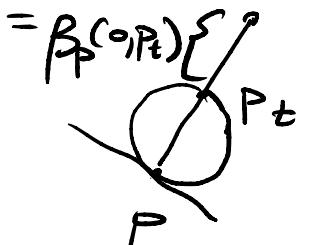
(B-T) The family  $\{S_n(\gamma)\}$   
 is quasi-independent.

Horoball shadow lemma: Fix  $\Theta$ ,

$$\mu(H_p(\theta r_p)) \asymp \theta^{2(\delta - \delta_\pi)} r_p^\delta b(\theta)$$

Pf:  $P_t$  on geod  $(o, p)$

$$\begin{aligned} \mu(O_r(x, P_t)) &\asymp \\ e^{-\delta_p t} e^{(2\delta_\pi - \delta_p)d(P_t, \Gamma_0)} b(d(P_t, \Gamma_0)) & \end{aligned}$$

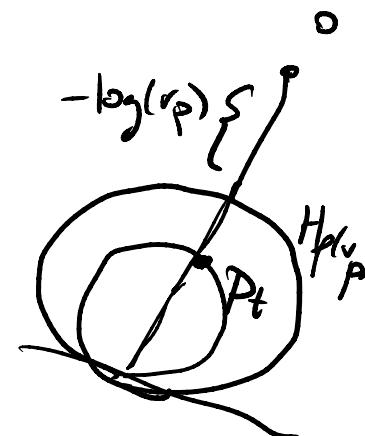


Let  $t = -\log \theta r_p$ .

$$O_r(x, P_t) \asymp H_p(t)$$

$$e^{-\delta_p t} = \theta^{r_p \delta}$$

$$d(P_t, \Gamma_0) \approx -\log \theta$$



Note:  $\theta = 1$

Defn:  $H_p(\varphi) = H_p^{(r_p)} \varphi(r_p)$

$$H_p = H_p^{(1)}$$

Cor 1:

$$P \in O_n(\gamma),$$

$$\begin{aligned} \mu(H_p \varphi) &\asymp \gamma^{n\delta} \varphi(\gamma^n)^{2(\delta_p - \delta_\pi)} \\ &\quad \times b(\varphi \gamma^n) \end{aligned}$$

all about the same size

Lemma 2

\*picture

$$\begin{aligned}\mu(S_n) &\asymp \sum_{P \in \mathcal{P}_n} \mu(H_P \cap \varnothing) \\ &\asymp \#\mathcal{P}_n \mu(H_P \cap \varnothing)\end{aligned}$$

Yang  $\#S_n = \#\mathcal{P}_n \asymp \gamma^{-\delta n}$

Lemma 3 For any  $P \in \mathcal{P}_n$ ,

$$\mu(S_n) \asymp \frac{\mu(H_P \cap \varnothing)}{\mu(H_P)} \xrightarrow{\varnothing=1}$$

Pf: Lem 2, Yang,

Cor 1 for  $\varnothing$  and  $\varnothing = 1$

Now, assume  $S_n \cap S_m \neq \emptyset$ .

Fix  $P^* \in \mathcal{P}_n(\lambda)$

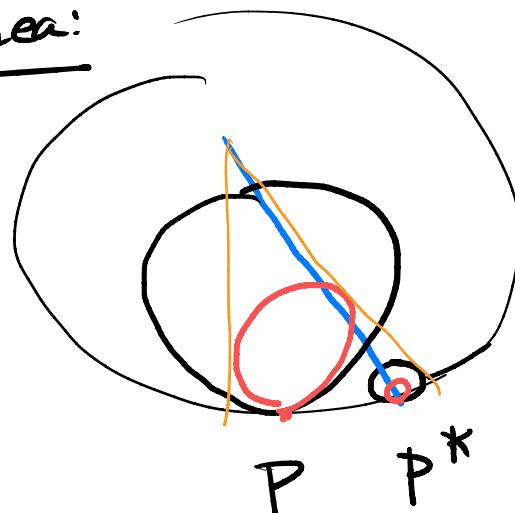
Let  $I(P^*) = \{P \in \mathcal{P}_m(\lambda) : H_P \cap H_{P^*} \neq \emptyset\}$

wLOG  $m > n$

Lemma 40

$$H_P \subset H_{P^*}$$

idea:



Cor 2:

$$\#I(P^*) \leq \frac{\mu(H_{P^*})}{\mu(H_P)}$$

Pf of quasi-independence

$$\mu(S_n \cap S_m)$$

$$\leq \sum_{P^* \in \mathcal{P}_n} \sum_{P \in I(P^*)} \mu(H_p(\varphi))$$

subadditivity

$$\leq \sum_{P^* \in \mathcal{P}_n} \#I(P^*) \mu(H_p(\varphi))$$

Cor 1

$$\lesssim \sum_{P^* \in \mathcal{P}_n} \frac{\mu(H_{P^*}(\varphi))}{\mu(H_P)} \mu(H_p(\varphi))$$

$P \in \mathcal{P}_m$  Cor 2

$$= \left( \sum_{P^* \in \mathcal{P}_n} \mu(H_{P^*}(\varphi)) \right) \underbrace{\frac{\mu(H_p(\varphi))}{\mu(H_p)}}_{\text{Lemma 3}} \gtrsim \mu(S_n) \mu(S_m)$$

Lemma 2

Lemma 3

□

Pf: of Khinchin-type thm

First, Lemma 1  $\Rightarrow$

$$K(\varphi) \asymp \sum_{n \in \mathbb{N}} \mu(S_n)$$

Since  $\Theta(\varphi) = \limsup S_n$

Borel-Cantelli ①  $\Rightarrow$  Khinchin theorem<sup>(1)</sup>

Conversely, Borel-Cantelli ②  
+ quasi-independence lemma

$$\Rightarrow \mu(\Theta_\lambda(\varphi)) > 0.$$

To prove  $\mu(\Theta(\varphi)) = 1$ , by ergodicity of  $\mu$  wrt  $\Gamma$ , it suffices to show  $\forall g \in \Gamma$ ,  $\Theta(\varphi)$  is almost  $\Gamma$ -inv. meaning  $\mu(g \Theta(\varphi) \Delta \Theta(\varphi)) = 0$   
i.e.  $\mu(g \Theta(\varphi)) = \mu(\Theta(\varphi))$

by

Fact:  $\mu$  ergodic  $\Rightarrow$  almost  $\Gamma$ -inv sets have trivial measure.

Note:  $\Leftarrow$  is obvious

$A \text{ } \Gamma\text{-inv} \Rightarrow$

$$\mu(\gamma A) = \mu(A)$$

Black box  $\Theta(\varphi)$  is almost  $\Gamma$ -inv.

# Logarithm Law

final goal: why  
we stopped at  
mixed exp. and  
everyone else stopped  
at exp.

Thm (B.-Tiozzo)

a.a.e.  $\xi \in \Lambda_P$ ,

$$\limsup_{t \rightarrow \infty} \frac{d(\xi_t, P_0)}{\log t} = \frac{1}{2(\delta_P - \delta_\Pi)}$$



"prove" log law, don't  
verify the limsup exactly,  
just identify the two seg  
☒ compute the limit.

understand  
where these come  
from also

" $\rho_\pi^f$ " of logarithm law

Let

$$\varphi_\varepsilon(x) := \log(x^{-1}) - \frac{1+\varepsilon}{2(\delta_p - \delta_\pi)} K(\varepsilon)$$

to cancel out in

$$\# \Pi_i = 1$$

exercise  $\varphi_\varepsilon$  is Khinchine

$$K_\lambda(\varphi_\varepsilon) = \sum \varphi_\varepsilon(\lambda^n) (-2 \log \varphi_\varepsilon(\lambda^n))^{\alpha_\pi}$$

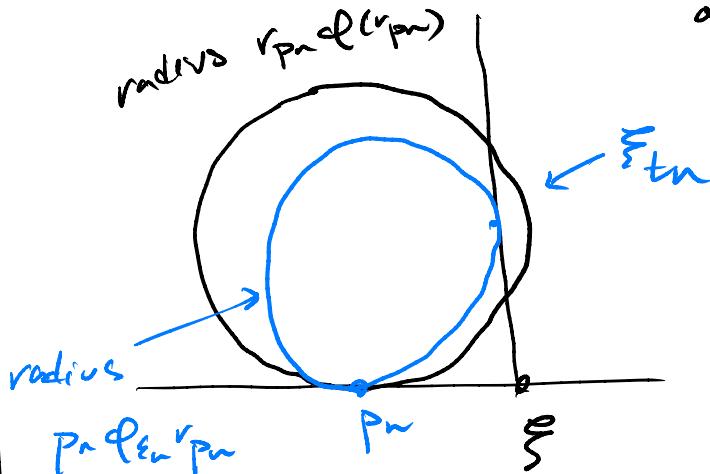
$$= \sum \log(\lambda^{-n})^{-1+\varepsilon} (-2)^{\alpha_\pi} (\log((\log \lambda^{-n})^{1+\varepsilon}/2(\delta_\pi + \varepsilon)))^{\alpha_\pi}$$

$$\approx \sum \frac{1}{n^{1+\varepsilon}} \log(n+1)^{\alpha_\pi} \quad \alpha_\pi > 0$$

calculus exercise  $\uparrow$  diverges if  $\varepsilon = 0$   
and converges if  $\varepsilon > 0$ .

Then w.l.o.g.  $\xi \in \Theta_\lambda(\varphi_0)$ , choose  
maximal seq.  $p_n \in P$  so that  
geodesic  $(0, \xi)$  passes through  $H_{p_n}^{(r_{p_n}, \delta_{p_n})}$   
in order, and  $r_{p_n} \downarrow 0$  monotone decr.

then,  $\exists \varepsilon_n > 0$  s.t.  $[0, \varepsilon]$  tangent to  
 $H_{p_n}^{(r_{p_n}, \delta_{p_n})}$ .  $\xi_{t_n}$  pt of tangency  
on  $[0, \varepsilon]$ .



claim:

$$\limsup_{t \rightarrow \infty} \frac{d(\xi_t, \Gamma_0)}{\log t} = \limsup_{n \rightarrow \infty} \frac{d(\xi_{t_n}, \Gamma_0)}{\log t_n}$$

idea:

$d(\xi_{t_n}, \Gamma_0) \approx$  distance between horospheres  
maximized at  $t = t_n$

## Technical hyperbolicity lemmas



$$d(\beta_{t_n}, \Gamma_0) \approx -\log (\varphi_{t_n}(\gamma_{p_n}))$$

*not really but close enough*

$$\sim \frac{1 + \varepsilon_n}{2(\delta_p - \delta_\pi)} \log t_n$$

If  $\varepsilon_n > \varepsilon > 0$  some  $\varepsilon$ , then

$$\xi \in \mathbb{H}_\lambda(\varphi_\varepsilon). \text{ But } \mu(\mathbb{O}_\lambda(\varphi_\varepsilon)) = 0,$$

so we can choose  $\xi$  s.t.

$$\xi \in \mathbb{O}_\lambda(\varphi_0) \text{ but } \xi \notin \mathbb{O}_\lambda(\varphi_\varepsilon)$$

$\forall \varepsilon > 0$ . Thus,  $\varepsilon_n \rightarrow 0$  and

$$\limsup = \frac{1}{2(\delta_p - \delta_\pi)}$$

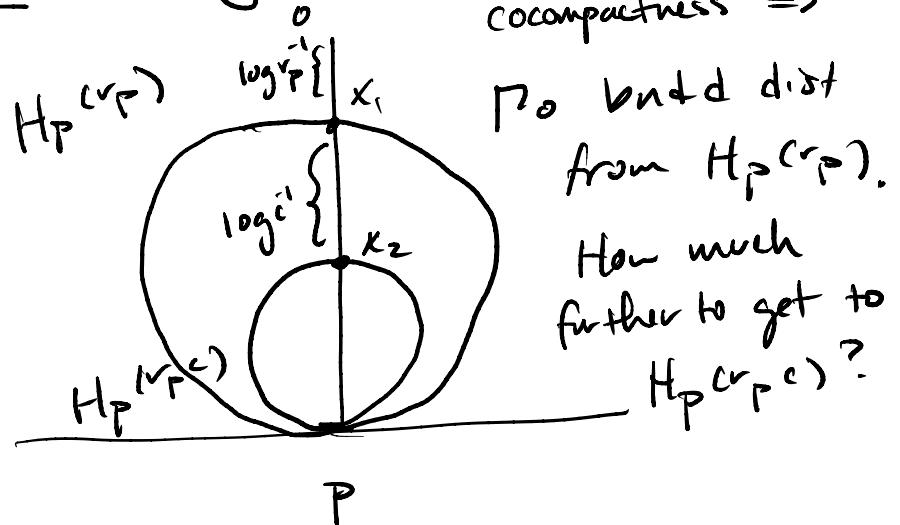


*more on this*

### Lemma 1:

$$d(\xi_{t_n}, \Gamma_0) \approx -\log (\varphi_{\varepsilon_n}(r_{p_n}))$$

idea: for any  $c < 1$



$$\beta_P(0, x_1) = -\log r_P$$

$$\beta_P(0, x_2) = -\log r_P^c$$

$$\beta_P(x_1, 0) + \beta_P(0, x_2) = \beta_P(x_1, x_2) = d(x_1, x_2)$$

"cocycle property"

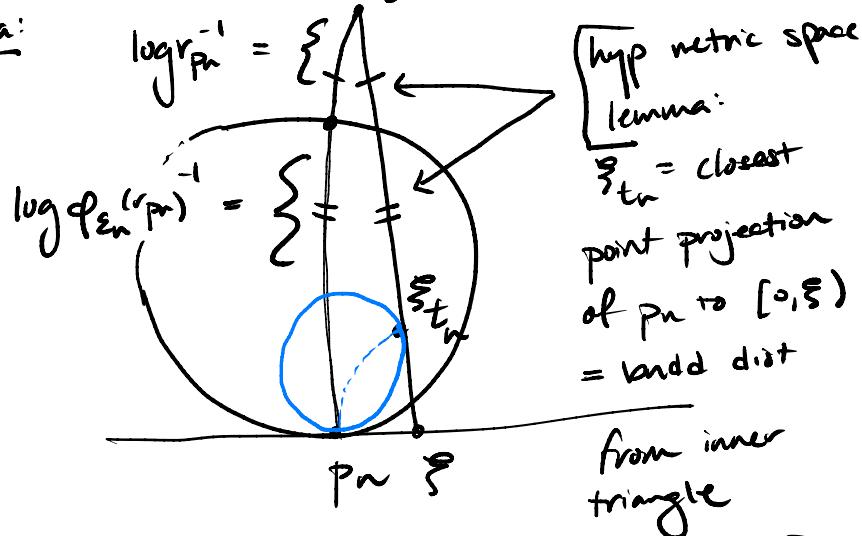
$$\begin{aligned} \log r_P - \log r_P - \log c \\ = -\log c. \end{aligned}$$

Now take  $c = \varphi_{\varepsilon_n}(r_{p_n})$

### Lemma 2:

$$t_n + \log \varphi_{\varepsilon_n}(r_{p_n}) \approx \log r_{p_n}^{-1} \approx t_n$$

idea:



(L1)

$$\Rightarrow d(\xi_{t_n}, \Gamma_0) \approx -\log \varphi_{\varepsilon_n}(r_{p_n})$$

$$= -\log ((\log r_{p_n}^{-1})^{-(1+\varepsilon_n)/2(\delta_P-\delta_\pi)})$$

$$= \frac{1+\varepsilon_n}{2(\delta_P-\delta_\pi)} \log \log r_{p_n}^{-1}$$

$$\leq \frac{1+\varepsilon_n}{2(\delta_P-\delta_\pi)} \log t_n \quad (\text{L2})$$

$\Rightarrow$

$$\limsup \leq \frac{1+\varepsilon_n}{2(\delta_P-\delta_\pi)} \quad \forall n.$$

If  $\varepsilon_n > \varepsilon > 0$  some  $\varepsilon$ , then

$\xi \in \Theta_{\gamma}(\varphi_{\varepsilon})$ . But  $\mu(\Theta_{\gamma}(\varphi_{\varepsilon})) = 0$ ,

so we can choose  $\xi$  s.t.

$\xi \in \Theta_{\gamma}(\varphi_0)$  but  $\xi \notin \Theta_{\gamma}(\varphi_{\varepsilon})$

$\forall \varepsilon > 0$ . Thus,  $\varepsilon_n \rightarrow 0$  and

the upper bound follows -

Lower bound

More details  
if desired

$$d(\xi_{t_n}, \Gamma) \approx -\log \varphi_{\varepsilon_n} r_{p_n}$$

$$= \frac{1 + \varepsilon_n}{2(\delta_r - \delta_{\pi})} \log \log r_{p_n}^{-1}$$

$$(L2)_{\text{lower}} \gtrsim \frac{1 + \varepsilon_n}{2(\delta_r - \delta_{\pi})} \left( \log \left( t_n - \frac{1 + \varepsilon_n}{2(\delta_r - \delta_{\pi})} \log \log (r_{p_n}^{-1}) \right) \right)$$

$$(L2)_{\text{upper}} \gtrsim \frac{1 + \varepsilon_n}{2(\delta_r - \delta_{\pi})} \log \left( t_n - \frac{1 + \varepsilon_n}{2(\delta_r - \delta_{\pi})} \log(t_n) \right)$$

$$\Rightarrow \limsup > \frac{1}{2(\delta_r - \delta_{\pi})} \quad \begin{matrix} \text{since } t_n \rightarrow \infty \\ \varepsilon_n \rightarrow 0 \end{matrix}$$

□