A 0-1 law for circle packings of coarsely hyperbolic metric spaces and applications to cusp excursion

Joint with Giulio Tiozzo

$\boxed{\text{PART I}}$

Khinchin's Theorem 1926 : $\psi : \mathbb{N} \to \mathbb{R}^+$

monotone decr (maybe other hypotheses).

Let $\Theta(\psi) := \{x \in \mathbb{R} : |x - \frac{p}{q}| < \frac{\psi(q)}{q}$ for ∞-ly many $\frac{p}{q} \in \mathbb{Q}\}$

Then

$$\sum_{q \in \mathbb{N}} \psi(q) = \infty \Rightarrow \Theta(\psi) \text{ has full measure}$$

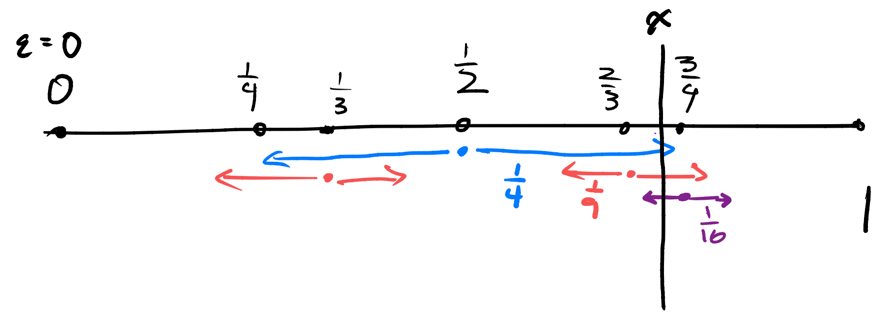
and

$$\sum_{q \in \mathbb{N}} \psi(q) < \infty \Rightarrow \Theta(\psi) \text{ has measure zero}$$

restrict to $[0,1]$ to get a "0-1" law

Application $\psi_\varepsilon(q) = \frac{1}{q^{1+\varepsilon}}$

$\Theta(\psi) = \{x \in \mathbb{R} : |x - \frac{p}{q}| < \frac{1}{q^{2+\varepsilon}}$ for ∞-ly many $\frac{p}{q} \in \mathbb{Q}\}$



so for this x is in 3 of these balls. Will it be in ∞-ly many such balls?
If $\varepsilon > 0$ then the balls get even smaller

by Khinchin:

$$\sum_{q \in \mathbb{N}} \psi_\varepsilon(q) = \sum_{q \in \mathbb{N}} \frac{1}{q^{1+\varepsilon}} \begin{cases} = \infty & \text{for } \varepsilon = 0 \\ < \infty & \text{for } \varepsilon > 0 \end{cases}$$

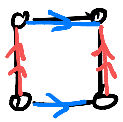
hence

$$\Theta(\psi_\varepsilon) \begin{cases} \text{has full measure for } \varepsilon = 0 \\ \text{has measure zero for } \varepsilon > 0. \end{cases}$$

In our picture, with probability 1, x is in ∞-ly many balls of radius $\frac{1}{q^2}$.

circle packings for the hyperbolic plane

by example

$\Sigma_{1,1}$ = 

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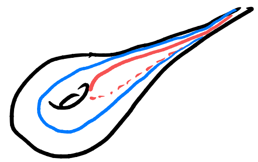
punctured torus

S = 

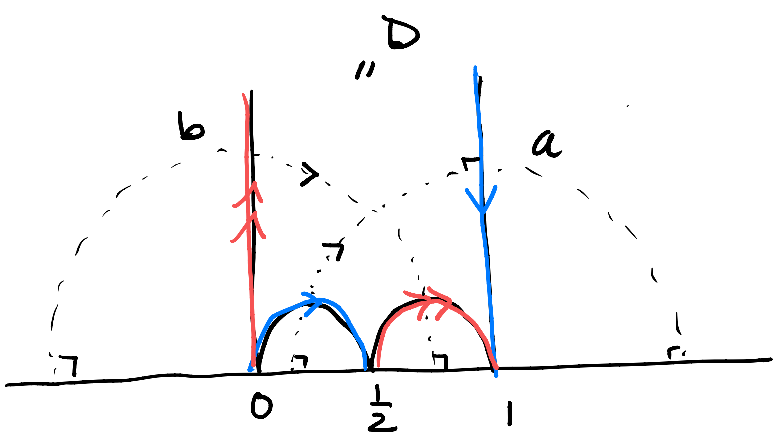
punctured torus with a metric

Let $S = \mathbb{H}^2/\Gamma \cong \Sigma_{1,1}$ finite area

where $\pi_1(\Sigma_{1,1}) \cong F_2 \cong \Gamma < \mathrm{Isom}(\mathbb{H}^2)$

is a hyperbolic structure on $\Sigma_{1,1}$

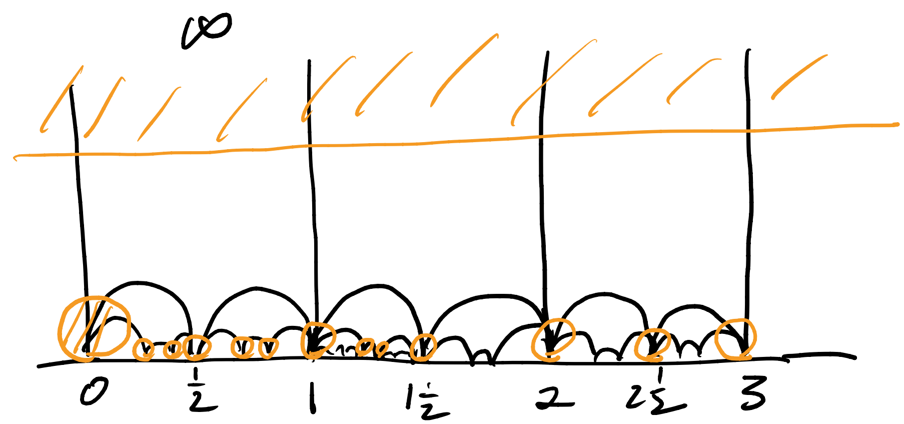
e.g. $\Gamma = \langle a, b \rangle$ where

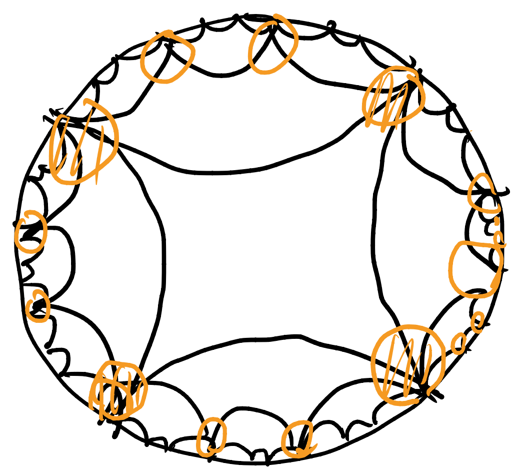


exercise: use that $g \in \mathrm{Isom}\,\mathbb{H}^2 = \{$Möbius transf. with coeffs in $\mathbb{R}\}$ is determined by the image of any 3 points in $\partial\mathbb{H}^2$ to identify the functions a, b.

tiling of $\mathbb{H}^2$ by $\Gamma . D$

$\rightsquigarrow$ circle packing (Γ-equivariant)

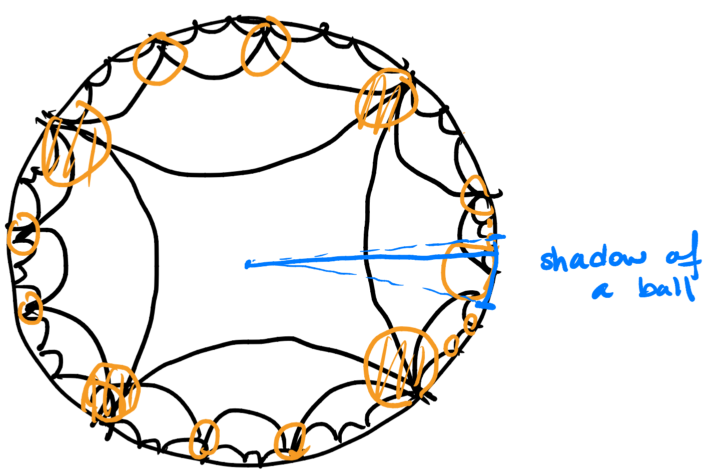




Defn: a circle in $\mathbb{H}^2$ tangent to $\partial\mathbb{H}^2$ is a horosphere. Its interior is a horoball. The point of tangency is the center of the horosphere/ball.

Let $\mathcal{P}$ = {centers of horoballs in the packing} and r_p = Euclidean radius of the horoball centered at p from a fixed original packing

Let $H_p(r)$ be a shadow of the horoball centered at p with radius r.



analogy to classical setting of Khinchin:

$\mathbb{R} \rightsquigarrow S^1$

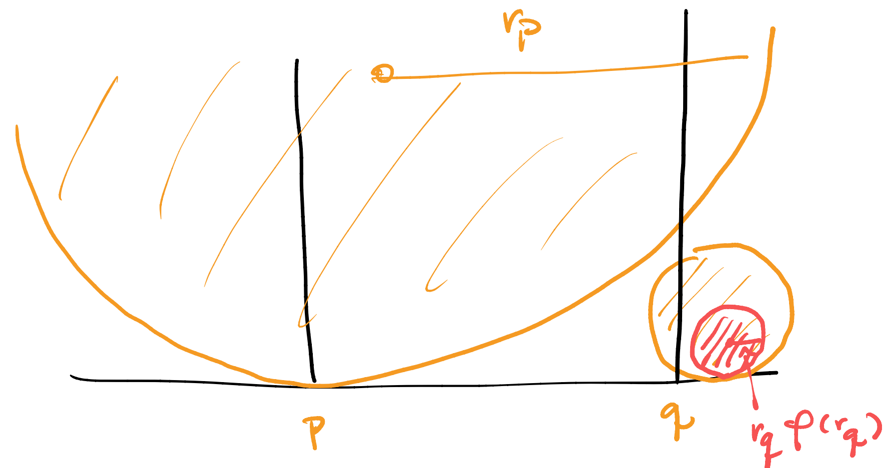
Lebesgue $\rightsquigarrow$ arclength

$\psi : \mathbb{N} \to \mathbb{R}^+$ decr $\rightsquigarrow$ $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ increasing

Rationals $\frac{p}{q}$ $\rightsquigarrow$ horoball centers $p \in \mathcal{P}$

$\{x : |x - \frac{p}{q}| < \frac{\psi(q)}{q} = \frac{1}{q}\psi(q)$ ∞-ly often$\}$ $\rightsquigarrow$ $\{x : x \in H_p(r_p\varphi(r_p))$ ∞-ly often$\}$

$= \Theta(\psi)$ $\qquad =: \Theta(\varphi)$



$\varphi \equiv 1 \Rightarrow$ no shrinking

in this case, a.e. x in ∞-ly many shadows.

$\varphi < 1 \Rightarrow$ shrinking, x may no longer be in ∞-ly many shadows

Thm: (Stratmann-Velani, Sullivan)

[Khinchin-type Theorem] for small $\lambda < 1$,

$$\sum_{n\in\mathbb{N}} \varphi(\lambda^n) < \infty \iff \Theta(\varphi) \text{ has measure zero}$$

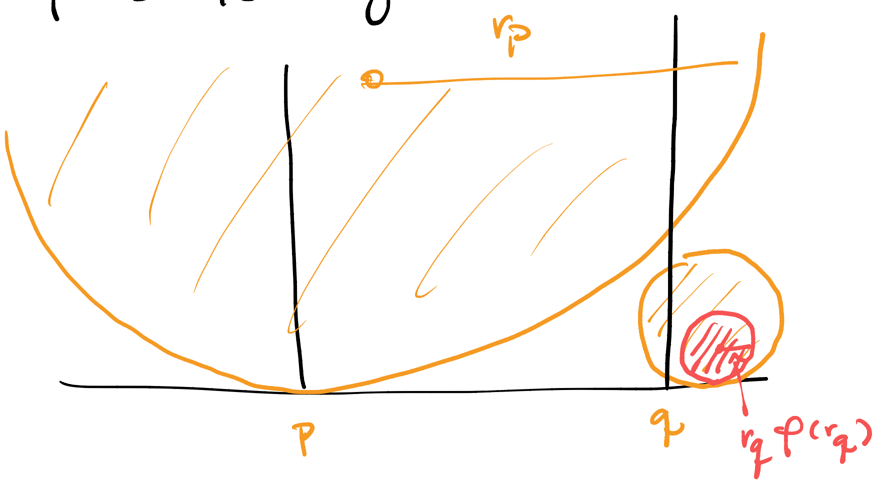
$$= \infty \iff \Theta(\varphi) \text{ has measure one}$$

Note: φ incr $\Rightarrow$ $\varphi(\lambda^n)$ decr. in n

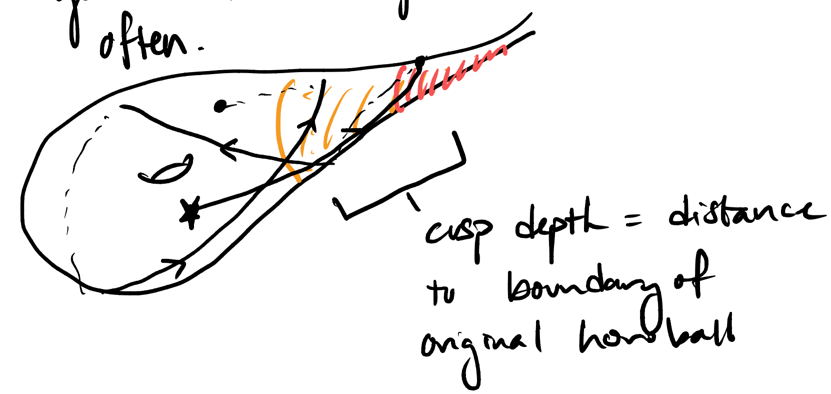
Application to cusp excursion

Horoball packing projects to neighborhood of the cusp.

φ shrinks neighborhoods.

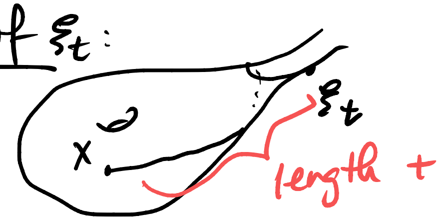


if they stay the "same size" ($\varphi \equiv 1$), a.e. geodesic visits neighborhood ∞-ly often.



Q: What is the optimal amount of shrinking?

defn of ξ_t:



Thm (Stratmann-Velani, Sullivan)

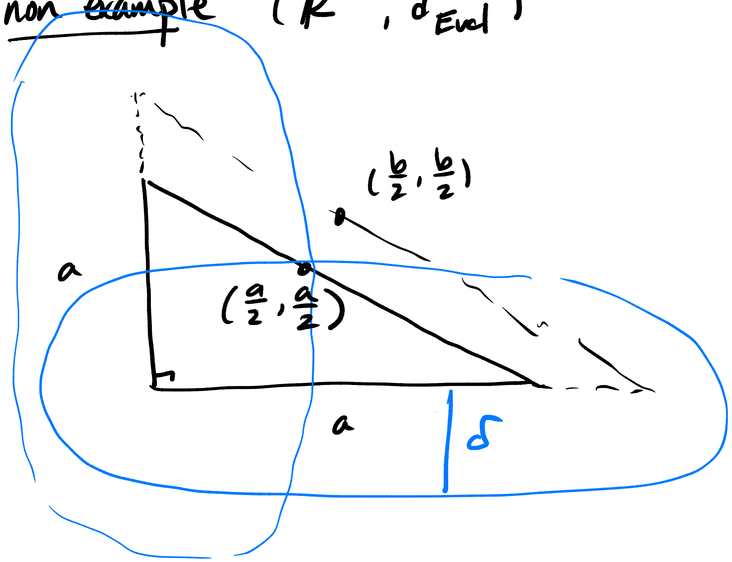
[Logarithm Law] For a.e. $\xi \in S^1$,

$$\limsup_{t\to\infty} \frac{\text{cusp depth}(\xi_t)}{\log(t)} = 1.$$

with Tiozzo, we generalize these to the setting of coarsely hyp. metric spaces which are geometrically finite. Will introduce these concepts now.

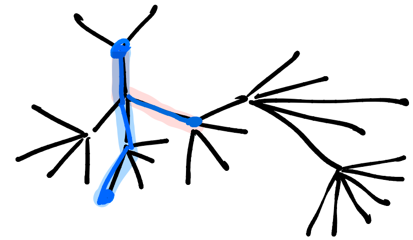
<u>Defn:</u> (X,d) metric space is (Gromov) <u>hyperbolic</u> if $\exists\, \delta > 0$ s.t. for any geodesic triangle, δ-nhd of 2 edges contains the third.

<u>non example</u> $(\mathbb{R}^2, d_{Eucl})$



δ depends on a

<u>example</u> $X = T$ tree, d = path metric



can be ∞ valence!

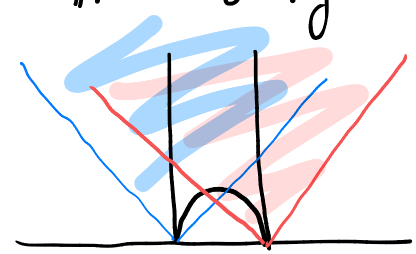
geodesic triangles are degenerate

Any $\delta > 0$ works.

"hyperbolic metric spaces are almost trees"

<u>exercise</u>

$\mathbb{H}^2$ $\quad \delta = \log 2$



δ-nhds in $\mathbb{H}^2$

$\boxed{\text{Hint:}}$ explain, and use again, this fact:

<u>Fact:</u> since $PSL(2,\mathbb{R}) \cong \operatorname{Isom} \mathbb{H}^2$ is triply transitive on $\partial \mathbb{H}^2$, all ideal triangles are isometric

example hyperbolic crochet and MEGL "reverse escher" project

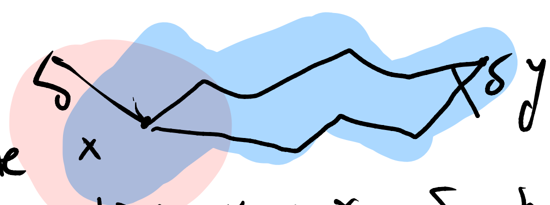
Defn: (X,d) is **proper** if closed metric balls are compact, and **geodesic** if $\forall\, x,y\in X\ \exists$ geodesic x to y.

non-ex. of proper: infinite valence tree

$\boxed{\text{From now on, } (X,d) \text{ always proper geod. hyp metric space}}$

Fact: any 2 geodesics γ_1, γ_2 x to y are unif. bndd distance (dep only on δ)

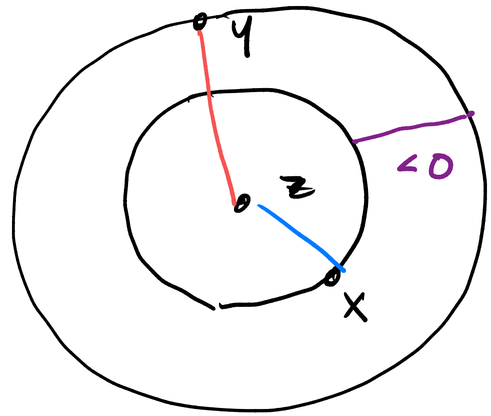
pf:



degenerate Δ vertices x, y, x. δ-nbd of any 2 sides contains the third

Defn: Busemann function centered at z

$$\beta_z(x,y) = d(x,z) - d(y,z)$$



signed relative distance

exercise: $\beta_z(x,y) = -\beta_z(y,x)$ **anti-symmetric**

$\beta_z(x,w) + \beta_z(w,y) = \beta_z(x,y)$ **cocycle**

$g\in \mathrm{Isom}(X)$,

$\beta_{gz}(gx,gy) = \beta_z(x,y)$ **equivariance**

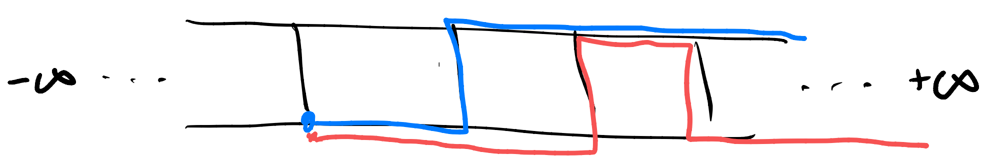
Defn: fix $o \in X$.

$$\partial X := \{ \text{geodesic rays based at } o \} / \sim$$

where $\gamma_1 \sim \gamma_2$ if $\exists\, c$ s.t.

$$d(\gamma_1(t), \gamma_2(t)) \leq c \quad \forall\, t \geq 0.$$

e.g. Ladder graph



$$\partial X = \{ +\infty, -\infty \}$$

e.g. $\partial \mathbb{H}^2 = \mathbb{R} \cup \{\infty\} = S^1$

Defn for $\xi \in \partial X$, define

$$\beta_\xi(x,y) = \liminf_{z \to \xi} \beta_z(x,y)$$

exercise: a) $\liminf = \limsup + O(\delta)$

b) $\beta_\xi(x,y) = -\beta_\xi(y,x) + O(\delta)$

c) $\beta_\xi(x,w) + \beta_\xi(y,w) = \beta_\xi(x,y) + O(\delta)$

quasi-anti-sym & quasi-cocycle

d) equivariance still true for $\xi \in \partial X$.

Defn:

Fix $o \in X$. A horosphere centered at $\xi \in \partial X$ of radius r is

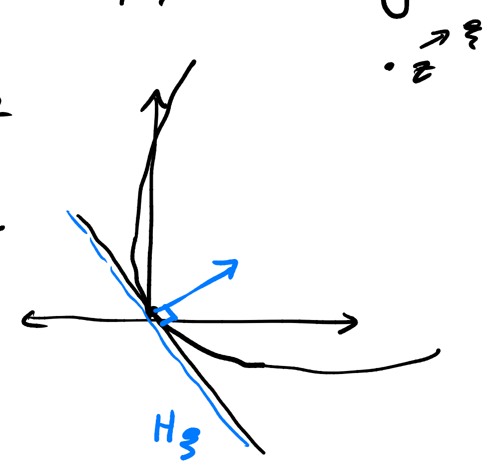
$$S_\xi = \{ x \in X : \beta_\xi(x,o) = \log r \}$$

and horoball is

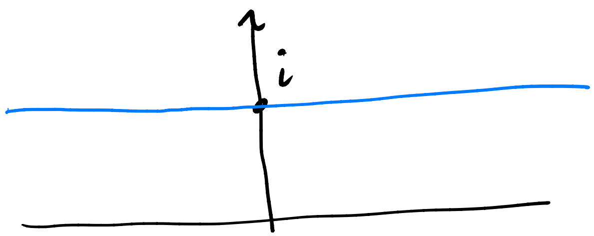
$$H_\xi = \{ x \in X : \beta_\xi(x,o) \leq \log r \}$$

example $\mathbb{R}^2$

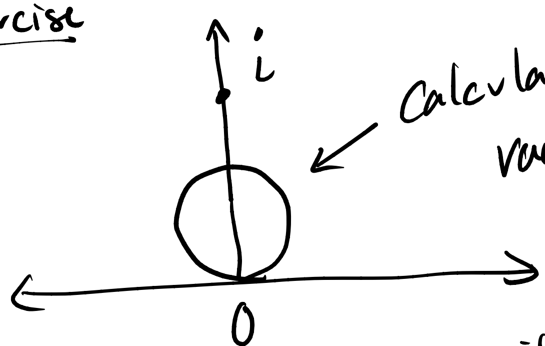
β_ξ is a limit (no $O(\delta)$)



example $\mathbb{H}^2$ β_ξ is a limit no $O(\delta)$



exercise



Calculate Eucl. radius of $H_0(r)$ with the basepoint i

exercise describe horospheres for ladder, $Cay(\mathbb{F}_2)$

Def $H_\xi(r)$ = shadow of $H_\xi(r)$

$= \{ \eta \in \partial X$ such that some geod. o to η intersects $H_\xi(r) \}$

Fact: $\{ H_\xi(r) \mid \xi \in \partial X,\ r > 0 \}$ generates the topology on ∂X.

exercise: the shadow topology on $\partial \mathbb{H}^2$ agrees with the usual topology on S^1.