A GLOBAL SHADOW LEMMA AND LOGARITHM LAW FOR GEOMETRICALLY FINITE HILBERT GEOMETRIES

HARRISON BRAY AND GIULIO TIOZZO

Abstract. We prove a global shadow lemma for Patterson-Sullivan measures in geometrically finite Hilbert manifolds. We also prove a Dirichlet-type theorem for hyperbolic metric spaces which have sufficiently regular Busemann functions. We apply these results to prove the logarithm law for excursion of geodesics into cusps in the setting of Hilbert geometries.

1. Introduction

In this work, we prove a global version of the *shadow lemma* ([Sul79, Sul84, SV95]) for the Patterson-Sullivan measures associated to geometrically finite strictly convex real projective manifolds. We then apply it to obtain a *logarithm law*, as in [Sul82], which provides asymptotics for the maximal cusp excursion for generic geodesics and relates it to the dimension of the limit set. Several of our results also apply in the more general metric context of δ-hyperbolic spaces which satisfy certain regularity and growth conditions.

A convex real projective structure is given by a properly convex domain Ω in real projective space $\mathbb{R}P^n$, with an action by a discrete group Γ of projective transformations preserving Ω. The quotient manifold $M = \Omega/\Gamma$ inherits a natural metric d_Ω called the *Hilbert metric*. If Ω is strictly convex, geodesics for the Hilbert metric are simply straight lines. The moduli space of these geometries is frequently nontrivial and includes the example of hyperbolic space of constant negative curvature.

A Hilbert geometry (Ω, d_Ω) is in general only Finsler, meaning the metric comes from a norm, but this norm does not necessarily come from an inner product. Once Ω is preserved by a noncompact group of projective transformations, the Hilbert geometry (Ω, d_Ω) is Riemannian if and only if Ω is an ellipsoid ([SM02], [Cra14, Theorem 2.2]). Moreover, aside from the special case of the ellipsoid, these Hilbert geometries are not CAT(k) for any k [Mar14]. Nonetheless, as Marquis states in [Mar14], we may think of Hilbert geometries as having "damaged nonpositive curvature." In particular, a strictly convex Hilbert geometry with a large isometry group has many properties resembling negative curvature.

Hyperbolic manifolds are equipped with a natural boundary of their universal cover which carries several measures; in particular, the *Patterson-Sullivan* measure, obtained by taking limits of Dirac measures supported on the group orbit ([Pat76, Sul79, Sul84]). In that context, Sullivan's *shadow lemma* establishes the scaling properties of such measures near boundary points. These properties turn out to depend subtly on the location of the parabolic points, and are related to the fine structure of the limit set.

In this paper, we extend this result to Hilbert geometries. If Ω is strictly convex with C^1 boundary, then the visual boundary of Ω coincides with the geometric boundary $\partial\Omega$ in projective space. An analogue of the Patterson-Sullivan measure has been constructed on this boundary (see [Ben04, Cra11, CM14b, Zhu20, BZ21]).

Date: November 3, 2021.

George Mason University, hbray@gmu.edu.

University of Toronto, tiozzo@math.utoronto.ca.

To state the theorem, let δ_Γ denote the critical exponent for the action of a group Γ of projective transformations on Ω endowed with the Hilbert metric. Define the *shadow* $V(o,\xi,t)$ from a point $o \in \Omega$ to a boundary point ξ of depth $t \geq 0$ to be the set of all boundary points $\eta \in \partial\Omega$ such that the projection of η to the geodesic ray $[o,\xi)$ from o to ξ is distance greater than t from o. Our main result is the following.

Theorem 1.1 (Shadow Lemma). *Let Ω be a properly convex, strictly convex domain in $\mathbb{R}P^n$ with C^1 boundary, and Γ a discrete, nonelementary group of projective transformations acting geometrically finitely on Ω. For each basepoint $o \in \Omega$, let μ_o be the associated Patterson-Sullivan measure. Then there exists a constant C such that for all $\xi \in \partial\Omega$, letting ξ_t be the point on the geodesic ray from o to ξ which is distance t from o, the measure of the shadow satisfies*

$$C^{-1}e^{-\delta_\Gamma t+(2\delta_\Pi-\delta_\Gamma)d(\xi_t,\Gamma o)} \leq \mu_o(V(o,\xi,t)) \leq Ce^{-\delta_\Gamma t+(2\delta_\Pi-\delta_\Gamma)d(\xi_t,\Gamma o)} \tag{1.1}$$

where $\Pi = \{id\}$ if ξ_t is in the thick part, and otherwise Π is the largest parabolic subgroup preserving the horoball containing ξ_t.

In fact, Theorem 1.1 holds in greater generality, for any *conformal density* of dimension δ, as long as δ is larger than the critical exponent of any parabolic subgroup. The most general statement appears as Theorem 7.1. The statement directly generalizes the main theorem of [SV95] for hyperbolic manifolds, and of [Sch04] for Riemannian manifolds with non-constant negative curvature.

Cooper-Long-Tillmann proved that if a properly convex domain Ω admits a quotient of finite volume, then Ω is strictly convex if and only if Ω has C^1-boundary [CLT15]. Thus, we have the following corollary:

Corollary 1.2. *Let Ω be a properly convex domain in $\mathbb{R}P^n$ which is either strictly convex or has C^1-boundary and admits a quotient of finite volume by a discrete, nonelementary group of projective transformations. Then the result of Equation* (1.1) *holds for the Patterson-Sullivan measure associated to any geometrically finite action on Ω.*

1.1. **Dirichlet Theorem.** We define a hyperbolic metric space (X,d) to be *Busemann regular* if X is a proper, uniquely geodesic, and Busemann functions extend continuously to its boundary ∂X. For example, if a properly convex, strictly convex Ω with C^1 boundary admits a geometrically finite group action, then its convex core is a Busemann regular hyperbolic metric space with the Hilbert metric.

Let Γ be a group of isometries of a hyperbolic, Busemann regular metric space (X,d) acting geometrically finitely on X. Consider a base point $o \in X$ and a horoball H. The *radius* of the horoball H is $e^{-d(o,H)}$. If p is a boundary point, then $H_p(r)$ is the unique horoball centered at p with radius r. Then we fix a Γ-invariant horoball packing, where each parabolic point p in the set P of all parabolic points determines a unique horoball H_p centered at p in the packing, and we denote the radius of H_p by r_p. Let $\mathcal{H}_p(s)$ be the shadow of the horoball $H_p(s)$.

Theorem 1.3 (Dirichlet-type Theorem). *Let (X,d) be a Busemann regular hyperbolic metric space and Γ a group of isometries acting geometrically finitely on X. Then there exists a constant c such that for all s sufficiently small, the union*

$$\bigcup_{\substack{p\in P \\ r_p \geq s}} \mathcal{H}_p(c\sqrt{sr_p})$$

covers the limit set Λ_Γ with bounded multiplicity.

We can see this is a Dirichlet-type theorem by considering the classical case of $\mathrm{SL}(2,\mathbb{Z})$ acting on the hyperbolic plane $\mathbb{H}^2$, where the horoballs in the standard horoball packing are centered at rational points $\frac{p}{q}$ with radii $\frac{1}{q^2}$.

1.2. **Applications.** As an application of the shadow lemma (Theorem 1.1) and the Dirichlet theorem (Theorem 1.3), we prove a horoball counting theorem (Proposition 8.4), and a Khinchin-type theorem (Theorem 8.6), culminating in a version of Sullivan's logarithm law for geodesics in the setting of Hilbert geometries:

Theorem 1.4 (Logarithm Law)**.** *Let Ω be a properly convex, strictly convex domain in $\mathbb{R}P^n$ with C^1 boundary, and Γ a discrete, nonelementary group of projective transformations acting geometrically finitely on Ω. Let μ be the associated Patterson-Sullivan measure. Then for μ-almost every ξ in the limit set Λ_Γ, the following holds:*

$$\limsup_{t\to+\infty} \frac{d(\xi_t, \Omega_{\text{thick}})}{\log t} = \frac{1}{2\delta - k_{\max}}$$

where $k_{\max}$ is the maximal rank of any parabolic subgroup and ξ_t is the point on the geodesic ray (o, ξ) that is distance t from o.

In fact, we only use that the space is a Busemann regular hyperbolic metric space and the measure satisfies a shadow lemma (see Theorem 8.7 for the statement in full generality). In particular, our result in Theorem 1.4 also applies to Riemannian manifolds with pinched negative curvature under the growth condition on the cusps given by Equation 1.2. As far as we know, this result appears to be new even in that setting.

In a different vein, we also obtain as a consequence of the shadow lemma:

Corollary 1.5 (Singularity with harmonic measure)**.** *Let Ω be a properly convex, strictly convex domain in $\mathbb{R}P^n$ with C^1 boundary, and Γ a discrete, nonelementary group of projective transformations acting geometrically finitely on Ω. Let μ be a measure on Γ with finite superexponential moment, and let ν be the hitting measure of the random walk driven by μ. If Γ contains at least one parabolic element, then ν is singular with respect to the Patterson-Sullivan measure.*

Proof. Compare the shadow lemma in Theorem 1.1 with the shadow lemma for the hitting measure from [GT20, Proposition 2.3]. □

1.3. **Methods.** The methods we use follow the strategies of Schapira [Sch04] for geometrically finite manifolds of negative curvature. Though the metric is not Riemannian, sufficiently regular Hilbert geometries display a certain degree of hyperbolicity in the metric sense [CM14a]. This is our main tool to adapt the arguments of Schapira to this setting.

For the applications, such as the Dirichlet Theorem (Theorem 1.3), our approach parallels that of Stratmann-Velani [SV95] and Sullivan [Sul82]. The main obstruction is that, unlike Sullivan and Stratmann-Velani, we do not have the Euclidean model. Thus, we intepret the objects in terms of Busemann functions and replace the calculations in the Euclidean model with these intrinsic calculations.

1.3.1. *Projections.* To adapt our arguments to this setting, we develop the notion of closest point projection in Hilbert geometries. In general, Hilbert geometries are not CAT(k) for any k, and need not be δ-hyperbolic. Hence, closest point projection may not be well-defined, even coarsely.

To set up our result, define $\xi \in \partial\Omega$ to be *smooth* if there is a unique supporting hyperplane to Ω at ξ. We prove the following characterization, which may be of independent interest:

Proposition 1.6. *Suppose Ω is a strictly convex domain in real projective space. Let $x \in \Omega$ and let H be a horosphere for the Hilbert metric centered at a smooth boundary point ξ. Then the closest point projection of x onto H is the intersection point of the geodesic containing x and ξ with H.*

We also have a similar projection result for closed convex sets (Lemma 2.4). Note that we do not assume Ω to be δ-hyperbolic, or that Ω admits a group action.

1.3.2. *Cusp subgroups.* Another ingredient of the argument involves a growth estimate for the cusp subgroups. Schapira assumes this condition as a hypothesis [Sch04]. In our case, we are able to directly prove this condition:

Proposition 1.7. *Let Ω be a properly convex, strictly convex domain in $\mathbb{R}P^n$ with C^1 boundary, and Γ a discrete, nonelementary group of projective transformations acting geometrically finitely on Ω. Let Π denote a parabolic subgroup of Γ preserving Ω with critical exponent δ_Π. Fix $o \in \Omega$. Then there exists a constant S such that*

$$\#\{g \in \Pi : d_\Omega(o, go) \in [T, T+S)\} \asymp e^{\delta_\Pi T}. \tag{1.2}$$

If Π is a maximal rank parabolic group, the result of Proposition 1.7 holds without the assumption of a geometrically finite action. See Lemma 5.6.

The two main tools we use in this paper are the smoothness of points in the limit set to define the projection, and hyperbolicity of the convex core. By Crampon-Marquis [CM14a], this is implied by the assumption that the domain is strictly convex with C^1-boundary. It is possible that Crampon-Marquis' hyperbolicity result holds under the weaker assumption of smoothness of the limit set. Thus, we state many of our results on projections in greater generality, so that the proofs of the main results still apply.

1.4. **Historical remarks.** The Patterson-Sullivan measures in the setting of geometrically finite Hilbert geometries with sufficient regularity were first constructed and studied by Crampon in his thesis [Cra11], though the results are only written for surfaces. Zhu has recently expanded and refined these results in our setting [Zhu20], and Blayac-Zhu generalized the results to include certain nonstrictly convex cases [BZ21]. They show that Patterson-Sullivan measures exist for any geometrically finite action on a strictly convex Ω with C^1 boundary, have full support on the limit set of the group, and have no atoms [Zhu20, BZ21]. It follows that the conformal dimension of Patterson-Sullivan measure, which is equal to the critical exponent of the group Γ, is strictly larger than the critical exponent of any parabolic subgroup (see for instance [Zhu20, Lemma 11] for a proof specific to our setting).

The dynamics of the Hilbert geodesic flow was first studied by Benoist in the cocompact setting. Benoist proved that strict convexity of Ω, C^1-regularity of the boundary, and Gromov hyperbolicity of the Hilbert metric are all equivalent [Ben04]. More recently, [CLT15] generalized this result to the non-compact, finite volume case. Finally, [CM14a] introduced and studied a definition of geometrically finite action in Hilbert geometry, which is the one we use. Assuming Ω is both strictly convex and with C^1 boundary, they prove the convex core C_Γ of Ω is hyperbolic in the sense of Gromov [CM14a].

For hyperbolic groups, a version of the shadow lemma for the Patterson-Sullivan measure associated to the word metric is proven by Coornaert [Coo93]. This has been more recently generalized by Yang for relatively hyperbolic groups [Yan13].

1.5. **Structure of the paper.** In Section 2, we discuss some background on Hilbert geometries, and in particular we define and formulate precisely several properties of the closest point projection in Hilbert geometry, which may also be of independent interest. Note that in this part, we do *not* assume the boundary to be C^1. In Section 3, we recall and establish some consequences of the hyperbolicity of the convex hull. In Section 5, we use the local geometry of the cusp to prove Proposition 1.7. In Sections 6 and 7 we prove the main result, Theorem 1.1. The Dirichlet Theorem (Theorem 1.3) and the applications are addressed in Section 8, including the Logarithm Law (Theorem 1.4).

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2. Hilbert geometries

A subset Ω of real projective space $\mathbb{R}P^n$ is *properly convex* if there exists an affine chart in which Ω is bounded and convex, meaning its intersection with any line segment is connected. We say Ω is *strictly convex* if, moreover, the topological boundary $\partial\Omega$ in an affine chart does not contain any open line segments. Any properly convex domain admits a natural, projectively invariant metric called the *Hilbert metric* which is central to this study. The Hilbert metric is defined as follows. Choose an affine chart in which Ω is bounded; then for each $x, y \in \Omega$, any projective line passing through x and y must intersect $\partial\Omega$ at exactly two points, a, b. Then

$$d_\Omega(x, y) := \frac{1}{2}\big|\log[a; x; y; b]\big|$$

where $[a; x; y; b] := \frac{|a-y||b-x|}{|a-x||b-y|}$ is the cross-ratio with respect to the ambient affine metric inherited from the chart. The cross-ratio is a projective invariant, hence the metric does not depend on the chart, and projective transformations which preserve Ω are isometries with respect to d_Ω. Straight lines are geodesics for this metric, and are the only geodesics when Ω is strictly convex. The normalization factor of $\frac{1}{2}$ ensures that if Ω is an ellipsoid, then (Ω, d_Ω) is the Beltrami-Klein model for hyperbolic space of constant curvature -1.

2.1. **Busemann functions and horospheres.** For any properly convex domain Ω endowed with the Hilbert metric, we can define *Busemann functions* in the usual way, and then extend them to the boundary. For $x, y, z \in \Omega$, the *Busemann distance* from x to y relative to z is

$$\beta_z(x, y) := d_\Omega(x, z) - d_\Omega(y, z).$$

Note that level sets of the Busemann function are metric spheres centered at z. Busemann functions are anti-symmetric, 1-Lipschitz, and equivariant for any group of isometries of the Hilbert metric. Moreover, the Busemann function is a cocycle, meaning for $x, y, z, w \in \Omega$,

$$\beta_z(x, y) = \beta_z(x, w) + \beta_z(w, y).$$

Busemann functions extend canonically to the topological boundary of Ω in an affine chart for points in the boundary which have enough regularity. To formulate the exact condition, for a boundary point ξ in $\partial\Omega$, we say H is a *supporting hyperplane* to Ω at ξ if H is a codimension one projective subspace in the complement of Ω which intersects $\partial\Omega$ at ξ. Using terminology from convex geometry, we say ξ is *smooth* if there is a unique supporting hyperplane to Ω at ξ. Let Ω^s denote the set of smooth points in $\partial\Omega$.

Recall that the *cross-ratio of four lines*, $[L_1; L_2; L_3; L_4]$, is well-defined for any four lines $L_1, \ldots, L_4$ which intersect at a single point in projective space, and is equal to the cross-ratio of any 4 distinct collinear points, each on one of the lines, taken in order. For any two distinct points p, q in a shared affine chart in $\mathbb{R}P^n$, let $\overline{pq}$ be the unique projective line which passes through p and q. Let (p, q) be the open projective line segment from p to q inside this affine chart.

Lemma 2.1 ([Bra20, Lemma 3.2]). *Let Ω be a properly convex domain in projective space. If $\xi \in \partial\Omega$ is smooth, then for all $x, y \in \Omega$, the Busemann function*

$$\beta_\xi(x, y) := \lim_{z\to\xi} \beta_z(x, y)$$

exists and does not depend on the path to ξ. Moreover, let x^-, y^- be the intersection points of the lines $\overline{x\xi}, \overline{y\xi}$ with $\partial\Omega \smallsetminus \{\xi\}$, respectively. If $x^- \neq y^-$, then let $q = q(x, y, \xi)$ be the unique intersection point in projective space of the line $\overline{x^-y^-}$ with the unique supporting hyperplane to Ω at ξ. Then

$$\beta_\xi(x, y) = \frac{1}{2}\log[\overline{x^-q}; \overline{xq}; \overline{yq}; \overline{\xi q}]$$

and otherwise (if $x^- = y^-$),

$$\beta_\xi(x, y) = \frac{1}{2}\log[x^-; x; y; \xi].$$

It is straightforward to show with this geometric description that the Busemann functions vary continuously over smooth boundary points [Bra20, Lemma 3.4] and that the anti-symmetric, isometry-equivariance, 1-Lipschitz, and cocycle properties for the Busemann functions centered at points inside Ω extend to $\partial\Omega$. Lastly, a *horosphere* centered at a smooth boundary point ξ is a level set of the Busemann function β_ξ, and a *horoball* centered at ξ is a sublevel set for β_ξ.

Remark 2.2. It is clear to see from Lemma 2.1 and continuity of $\partial\Omega$ that horospheres and horoballs centered at smooth points inherit the same regularity and convexity properties of $\partial\Omega$ and Ω, respectively. If Ω is strictly convex, then Busemann functions are strictly convex along geodesics, and horoballs centered at smooth points of $\partial\Omega$ are strictly convex. If Ω has C^1 boundary, then Busemann functions are C^1 functions, and all horospheres in Ω are C^1.

2.2. **The Gromov product.** For points $x, y, o \in \Omega$, the *Gromov product* of x and y with respect to o is

$$\langle x, y\rangle_o := \frac{1}{2}(d_\Omega(x, o) + d_\Omega(y, o) - d_\Omega(x, y)). \tag{2.1}$$

Clearly the Gromov product is isometry equivariant, symmetric in x and y, and nonnegative by the triangle inequality.

This expression can easily be rewritten in terms of Busemann functions, allowing us to observe quickly that the Gromov product extends continuously to the boundary. In particular, we can notice that for $x, y, o \in \Omega$,

$$\langle x, y\rangle_o = \frac{1}{2}(d_\Omega(x, o) + \beta_y(o, x)) = \frac{1}{2}(d_\Omega(y, o) + \beta_x(o, y)). \tag{2.2}$$

If we take p to be any point on the geodesic segment from x to y, then $\beta_x(o, y) = \beta_x(o, p) + \beta_x(p, y) = \beta_x(o, p) - d_\Omega(p, y)$ by the cocycle property and definition of the Busemann function. Thus, we have the following expression which extends continuously to the boundary, hence applies to all $x, y \in \overline{\Omega}$ and $o \in \Omega$:

$$\langle x, y\rangle_o = \frac{1}{2}(\beta_x(o, p) + \beta_y(o, p)). \tag{2.3}$$

2.3. **Projections.** In this subsection we will see that in our setting, although *orthogonal* projection cannot be defined because there is not a notion of angle, we still have that *closest point projection* onto a convex set is well-defined, continuous, and extends continuously to the boundary. We will also describe geometrically the closest point projection onto a horosphere.

Assume Ω is a strictly convex domain in real projective n-space endowed with the Hilbert metric. Let K be a closed, convex subset of Ω. For $o \in \Omega$, denote by $\mathrm{proj}_K(o)$ the unique point on K which is closest to o. It is straightforward to check that this closest point projection function is well-defined because Ω, and hence any Hilbert metric ball, is strictly convex and the metric topology is locally compact. We wish to extend the domain of projection to infinity.

In order to do that, note that, if $x \in \Omega$ and o is a base point, the closest point projection of x onto K is the point $y \in K$ which minimizes $d_\Omega(x,y) - d_\Omega(x,o) = \beta_x(y,o)$. Since Busemann functions extend to smooth points, this leads to the following definition.

Definition 2.3. Let K be a closed, convex subset of $\Omega \cup \partial\Omega$ and fix a point o in Ω. Define the closest point projection onto K, denoted

$$\mathrm{proj}_K : (\Omega \cup \partial\Omega^s) \smallsetminus K \to K,$$

as follows:

- If $x \in K$, then $\mathrm{proj}_K(x) = x$;
- If $x \notin K$, then $\mathrm{proj}_K(x)$ is the unique point y in K which minimizes $\beta_x(y,o)$.

Note that projection is isometry-equivariant by definition, meaning for any isometry g,

$$g\,\mathrm{proj}_K(z) = \mathrm{proj}_{gK}(gz).$$

Uniqueness of closest point projection follows from strict convexity of horofunctions. More precisely:

Lemma 2.4. *Let Ω be a strictly convex domain in real projective n-space, and let K be a closed, convex subset K of $\Omega \cup \partial\Omega$. Then closest point projection onto K with respect to the Hilbert metric is well-defined for all x in $(\Omega \cup \partial\Omega^s) \smallsetminus K$, it is continuous, and does not depend on the point o. The image of the complement of K under this projection is contained in Ω.*

Proof. Note that if K is a convex subset of $\Omega \cup \partial\Omega$ then by strict convexity of Ω, there is a point in K which is inside of Ω. Then for $x \in \Omega \smallsetminus K$, there exists a closest point projection of x to K which is inside of Ω because Busemann functions are continuous, $K \cap \Omega$ is closed as a subset of Ω with the Hilbert metric, and Ω with the Hilbert metric is a locally compact topological space. If x is in the boundary of Ω and outside of K, then by convexity of K and strict convexity of horospheres, there exists a horoball centered at x which is disjoint from K and a horoball centered at x which intersects K. Then it suffices to show that the annulus of these horoballs intersected with K is a compact subset of Ω. But this is true as long as x is not in K, since the only way for a sequence to be unbounded is by diverging to x.

Now suppose $q_1, q_2 \in K$ are both closest point projections of x to K. Consider any point q on the segment between q_1 and q_2, which must be in K by convexity. Then $\beta_x(q_i,o) \leq \beta_x(q,o)$ by definition of closest point projection. But Busemann functions are strictly convex, so $\beta_x(q,o) \leq \beta_x(q_1,o) = \beta_x(q_2,o)$, with equality if and only if q is equal to q_1 or q_2. Then q_1 and q_2 must be the same point.

Continuity of closest point projection now follows from continuity and well-definedness of Busemann functions. □

2.3.1. *Projection onto a horosphere.* We will often need to project from inside a horoball onto its horosphere boundary in our arguments. The complement of a horoball is not convex, so we cannot immediately expect the previous definition to carry through. Nonetheless, there is a geometric interpretation analogous to that of a Riemannian symmetric space.

Lemma 2.5. *If Ω is a strictly convex Hilbert geometry, $x \in \Omega$, and ξ is a smooth point in $\partial\Omega$, then $|\beta_\xi(x,q)| \leq d_\Omega(x,q)$ with equality if and only if x, q, and ξ are collinear.*

Proof. The inequality is always true, and the converse direction is clear. Let p be the geometric intersection point of the projective ray (x,ξ) with the sphere about x of radius $d_\Omega(x,q)$. plane of the lines $\overline{x^-y^-}$ and $\overline{\xi y^+}$. See that $\beta_\xi(x,p) = d_\Omega(x,p) = d_\Omega(x,q)$. It suffices to show that if q is

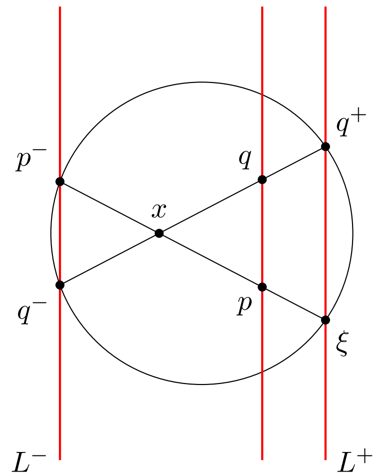

FIGURE 2.1. The configuration in Lemma 2.5.

not collinear with x, p, and ξ, then $\beta_\xi(p, q) \neq 0$. This is sufficient because the inequality in the following expression would necessarily be strict:

$$|d_\Omega(x, q) + \beta_\xi(p, q)| = |\beta_\xi(x, p) + \beta_\xi(p, q)| = |\beta_\xi(x, q)| \leq d_\Omega(x, q)$$

Consider any projective plane containing p, q, and ξ. We can make our arguments in this two dimensional cross-section, since x is collinear with p and ξ and is therefore also in this plane. Thus for simplicity, we abuse notation and assume Ω is a subset of the 2-dimensional projective plane for the remainder of the argument.

Let q^- be the point in $\partial\Omega$ on the line $\overline{xq}$ which is closer to x, and similarly for p, and let L^- be the line $\overline{q^-p^-}$. We can also define q^+, p^+ to be the other intersection points of the lines $\overline{xq}, \overline{xp}$ respectively with $\partial\Omega$ and let $L^+ = \overline{q^+p^+}$. See that the geometric construction of p in fact guarantees $p^+ = \xi$.

The condition that $\beta_x(p, q) = 0$ ensures that the lines L^-, $\overline{pq}$, and L^+ intersect at some point in projective space, and this point must be in the complement of $\overline{\Omega}$ since Ω is strictly convex. Then we can send this intersection point to infinity and ensure Ω remains bounded in this affine chart. Now, the lines $L^-, \overline{pq}$, and L^+ are parallel lines in a copy of $\mathbb{R}^2$.

Choose orthonormal coordinates for $\mathbb{R}^2$ so that the lines are vertical, and without loss of generality, assume that q^+ has larger vertical coordinate than ξ, as pictured in figure 2.1.

In these coordinates, since Ω is strictly convex, the tangent line $T_\xi\Omega$ must have positive slope. Let y be the other intersection point of the line $\overline{q\xi}$ with $\partial\Omega$. Since $\beta_x(p, q) = 0$, in these coordinates, p and q have the same horizontal coordinate, which is between the horizontal coordinates of L^- and L^+. Then for $\beta_\xi(p, q) = 0$, by Lemma 2.1, the lines $\overline{yp^-}$ and $T_\xi\Omega$ must intersect at a point with the same horizontal coordinate as p and q, in between the horizontal coordinates of L^- and L^+. This is only possible when $\overline{yp^-} = L^-$, implying $p = p^-$ and $q = q^-$. Thus q, p, x, and hence ξ, would have to be collinear. $\square$

We now prove Proposition 1.6.

Corollary 2.6. *Suppose Ω is a strictly convex domain in real projective space. Let $x \in \Omega \cup \partial\Omega^s$ and let H be a horosphere for the Hilbert metric centered at a smooth boundary point ξ. Then the closest point projection of x onto H is the intersection point of the geodesic containing x and ξ with H.*

Proof. Let x be in Ω and let p be the intersection of the line $\overline{xp}$ with the horosphere H centered at ξ, and q another point on the same horosphere. Then

$$d_\Omega(x,p) = |\beta_\xi(x,p) + \beta_\xi(p,q)| = |\beta_\xi(x,q)| < d_\Omega(x,q)$$

as desired. This extends to x in $\partial\Omega^s$, by replacing $d_\Omega(x,q)$ with $\beta_x(q,o)$, because horoballs are strictly convex whenever Ω is strictly convex. $\square$

3. Hyperbolic metric spaces

In this section, we discuss properties of a general hyperbolic metric space (X,d), which we will apply to the Hilbert metric in later sections. Most results should be well-known to experts, but we report them here in the precise form we need them.

Recall a geodesic metric space (X,d) is (Gromov) hyperbolic if there exists $\alpha > 0$ such that

$$\langle x,y\rangle_o \geq \min\{\langle x,z\rangle_o, \langle z,y\rangle_o\} - \alpha \tag{3.1}$$

for all $x,y,z,o \in X$. We denote as ∂X the hyperbolic boundary of X, that (if X is proper) is the set of geodesic rays from o, where we identify rays which lie within bounded distance of each other. Finally, we denote as $O(\alpha)$ a quantity which depends only on the hyperbolicity constant α.

3.1. **Shadows.**

Definition 3.1. The *shadow* from o to ξ of depth t is the set $V(o,\xi,t)$ of all points η in the boundary ∂X such that the distance between o and the projection of η onto the geodesic ray $[o,\xi]$ is more than t (this set includes ξ, which is equal to its projection onto $[o,\xi]$ and is infinitely far from o).

Note that by definition, for any isometry $g \in \Gamma$,

$$gV(o,\xi,t) = V(go, g\xi, t).$$

Lemma 3.2. *Let $o \in X$, $\eta,\xi \in X \cup \partial X$, and let p be the projection of η onto $[o,\xi)$. Then*

$$\langle \eta,\xi\rangle_o = d(o,p) + O(\alpha).$$

Proof. Let us first suppose that $\xi,\eta \in X$. Then

$$d(o,\eta) = d(o,p) + d(p,\eta) + O(\alpha)$$
$$d(\xi,\eta) = d(p,\xi) + d(p,\eta) + O(\alpha)$$

hence

$$\begin{aligned} 2\langle \eta',\xi'\rangle_o &= d(o,\eta) + d(o,\xi) - d(\eta,\xi) \\ &= d(o,p) + d(p,\eta') + d(o,p) + d(p,\xi) - d(p,\xi') + d(p,\eta') + O(\alpha) \\ &= 2d(o,p) + O(\alpha). \end{aligned}$$

The claim then follows letting ξ,η go to the boundary, since both the Gromov product and the projection extend continuously. $\square$

From the previous lemma it immediately follows:

Lemma 3.3. *Let us fix $o \in X$. Then there exists C such that, for any $\xi \in \partial X$ and any $t \geq 0$ we have the inclusion*

$$V(o,\xi,t) \subseteq \{\eta \in X \cup \partial X \ : \ \langle \eta,\xi\rangle_o \geq t\} \subseteq V(o,\xi,t+O(\alpha)).$$

Remark 3.4. Shadows of varying depth generate the topology on $\partial\Omega$ because as discussed in Section 2.3 if η is not equal to ξ, then $\mathrm{proj}_{[o,\xi)}(\eta)$ is inside Ω, and is in particular a bounded distance from o.

3.2. **Projections and Busemann functions.** Given three points $x, y, z \in X$, we say that two points $p \in [x, y]$ and $q \in [x, z]$ are *comparable* if $d(p, x) = d(q, x) \leq \langle y, z\rangle_x$. The following is essentially [GdlH90, Theorem 12, page 33].

Lemma 3.5. *Let X be a hyperbolic metric space, and for any $x, y, z \in X \cup \partial X$ and let $p \in [x, y]$, $q \in [x, z]$ be comparable points. Then, there exists C which depends on α, such that $d(p, q) \leq C$.*

From this we obtain:

Lemma 3.6. *If $\eta \in V(o, \xi, t)$, then $d(\eta_t, \xi_t) \leq O(\alpha)$.*

Proof. Let p be the closest point projection of η onto $[o, \xi)$. Then $d(o, p) \geq t$ by definition, and η_s and ξ_s are comparable points as long as $s \leq \langle \eta, p\rangle_o = d(o, p) + O(\alpha)$. □

The next lemma readily follows from, for instance, [MT18, Proposition 3.5].

Lemma 3.7. *Let γ be a (finite or infinite) geodesic, let $\eta \in \Omega \cup (\partial\Omega \setminus \overline{\gamma})$ and let p be the closest point projection of η to γ. Then for any $x, y \in \gamma$ we have*

$$\beta_\eta(x, y) = \beta_p(x, y) + O(\alpha).$$

Proof. Let x_0 be a base point and $\eta \in \Omega$. By the cocycle property

$$\beta_\eta(x, y) = \beta_\eta(x, x_0) - \beta_\eta(y, x_0)$$

and using [MT18, Proposition 3.5]

$$\begin{aligned} &= \beta_\eta(p, x_0) + d(x, p) - \beta_\eta(p, x_0) - d(y, p) + O(\alpha) \\ &= \beta_p(x, y). \end{aligned}$$

The equality then also holds for $\eta \in \partial\Omega \setminus \overline{\gamma}$ as the closest point projection extends continuously. □

Lemma 3.8. *Let $o \in X$, $\xi \in \partial X$ and ξ_t the point on the geodesic ray $[o, \xi)$ at distance t from o. If $\eta \in V(o, \xi, t)$, then*

$$\beta_\eta(o, \xi_t) = t + O(\alpha).$$

On the other hand, if $\eta \notin V(o, \xi, D)$, then

$$-t \leq \beta_\eta(o, \xi_t) \leq -t + 2D + O(\alpha).$$

Proof. Let p be the closest point projection of η onto $[o, \xi)$. Since $\eta \in V(o, \xi, t)$, p lies between ξ_t and ξ. Then by Lemma 3.7

$$\beta_\eta(o, \xi_t) = \beta_p(o, \xi_t) + O(\alpha) = t + O(\alpha).$$

To prove the second part, if $\eta \notin V(o, \xi, D)$, then $d(o, p) \leq D$, so

$$\beta_p(o, \xi_t) = -t + 2d(o, p) \leq -t + 2D$$

so the upper bound follows from Lemma 3.7. The lower bound follows from the triangle inequality. □

We will also use the following result, which is well-known in the literature. We refer the reader to [MT18], where the presentation of the statements most closely resembles our presentation here.

Proposition 3.9 ([MT18], Proposition 2.2)**.** *Let γ be a geodesic in X, $y \in X$ a point, and q a nearest point projection of y to γ. Then for any $z \in \gamma$,*

$$d(y, z) = d(y, q) + d(z, q) + O(\alpha).$$

An immediate corollary is that

Corollary 3.10. *Let γ be a geodesic in X, $y \in X \cup \partial X$ a point, and q a nearest point projection of y to γ. Then for any $z \in \gamma$,*

$$|\beta_y(z, q)| = d(z, q) + O(\alpha).$$

Lemma 3.11. *Let X be a hyperbolic metric space, with $o \in X$ and $p, \xi \in \partial X$. Let $q \in [o, p)$ and $w \in [o, \xi)$ with $\beta_p(o, w) \geq \beta_p(o, q)$. Then there exists $z \in [o, \xi)$ such that*

$$\beta_p(o, z) \geq \frac{d(o, q) + d(o, w)}{2} - O(\alpha).$$

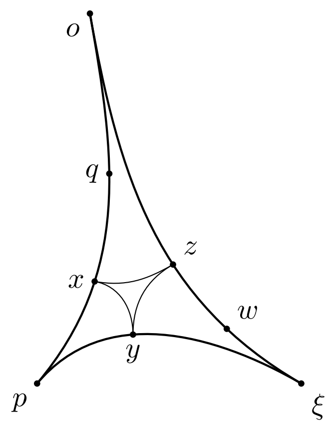

FIGURE 3.1. An approximate tree for the proof of Lemma 3.11.

Proof. By hyperbolicity, the triangle $[o, p) \cup (p, \xi) \cup (\xi, o]$ is thin. Let x, y, z be the vertices of its inner triangle, with $x \in [o, p)$, $y \in (p, \xi)$, $z \in [o, \xi)$.

Let us pick a sequence p_n of points on $[o, p)$, converging to p. By looking at the approximate tree in Figure 3.1.

$$d(o, p_n) = d(o, z) + d(z, p_n) + O(\alpha)$$
$$d(w, p_n) = d(z, w) + d(z, p_n) + O(\alpha)$$

hence by subtraction

$$d(o, p_n) - d(w, p_n) = d(o, z) - d(z, w) + O(\alpha).$$

By taking the limit we obtain

$$\begin{aligned} \beta_p(o, w) &= \lim_{n \to \infty} (d(o, p_n) - d(w, p_n)) \\ &= d(o, z) - d(z, w) + O(\alpha). \end{aligned}$$

Moreover, since p lies on $[o, p)$,

$$\beta_p(o, q) = d(o, q)$$

hence, from $\beta_p(o, w) \geq \beta_p(o, q)$ we obtain

$$d(z, w) \leq d(z, q) + O(\alpha).$$

Moreover,

$$\begin{aligned} d(o, w) &= d(o, q) + d(q, z) + d(z, w) + O(\alpha) \\ &\leq d(o, q) + 2d(q, z) + O(\alpha) \end{aligned}$$

hence

$$\begin{aligned}\frac{d(o,w)+d(o,q)}{2} &\leq d(o,q)+d(q,z)+O(\alpha)\\ &= \beta_p(o,z)+O(\alpha)\end{aligned}$$

which proves the claim. $\square$

3.3. **Shadows in hyperbolic spaces.** We will now need two lemmas on hyperbolic metric spaces and shadows.

Lemma 3.12. *Let X be a hyperbolic metric space, $x, y \in X$, and $\xi, \eta \in \partial X$.*

(1) If $\eta \in V(o, \xi, t)$, then

$$V(o, \eta, t + O(\alpha)) \subseteq V(o, \xi, t) \subseteq V(o, \eta, t - O(\alpha)).$$

(2) For all $M > 0$, there is a constant $A > 0$ such that if $d(x, y) \leq M$, then for all $\xi \in \partial X$ and all $t > 0$,

$$V(x, \xi, t + A) \subset V(y, \xi, t) \subset V(x, \xi, t - A).$$

In an hyperbolic metric space (X, d), there is a metric on ∂X called *the Gromov metric* with the property that

$$c^{-1}e^{-\langle\xi,\eta\rangle_o} \leq d_{\partial X}(\xi, \eta) \leq ce^{-\langle\xi,\eta\rangle_o}$$

for some uniform constant c and any $\eta, \xi \in \partial X$. We refer the reader to Bridson-Haefliger [BH99, Prop. III.H.3.21] for this result.

Lemma 3.13. *There exists a constant C such that any horoball H of radius r has a shadow of radius s in the Gromov metric, where $C^{-1}r \leq s \leq Cr$.*

The proofs of Lemmas 3.12 and 3.13 are well-known in the literature and thus omitted.

4. Shadows in Hilbert geometries

Lemma 4.1 (Asymptotic geodesics in a Hilbert geometry)**.** *Let Ω be any properly convex domain in real projective space. Fix $\xi \in \partial\Omega^s$ and $x, y \in \Omega$ and denote by x_t, y_t the points on the projective rays $(x, \xi), (y, \xi)$ which are Hilbert distance t from x and y, respectively. Then for all $t > 0$,*

$$d_\Omega(x_t, y_t) \leq d_\Omega(x, y),$$

and moreover, this inequality is strict if Ω is strictly convex.

Proof. Observe that since x, y, x_t, y_t and ξ are coplanar, it suffices to consider Ω in $\mathbb{R}P^2$. Now, see that if the projective lines $\overline{xy}$ and $\overline{x_ty_t}$ intersect outside of Ω, then by considering the cross-ratio of four lines taken from ξ, it is straightforward to confirm that $d_\Omega(x_t, y_t) \leq d_\Omega(x, y)$ using the convexity of Ω. If Ω is strictly convex then moreover this inequality must be strict.

Thus, it suffices to show that if x_t and y_t are both distance t from x and y, respectively, then the projective lines $\overline{xy}$ and $\overline{x_ty_t}$ cannot intersect in Ω.

Suppose by contradiction that the intersection point of $\overline{xy}$ with $\overline{x_ty_t}$ is in Ω, and denote this intersection point by p. Let x^-, y^- be the intersection points other than ξ of the projective lines $\overline{x\xi}$ and $\overline{y\xi}$ with $\partial\Omega$.

Consider the cross-ratio of four lines based at p. By definition of p, we have x, y, p are collinear and x_t, y_t, p are collinear; in other words, $\overline{xp} = \overline{yp}$ and $\overline{x_tp} = \overline{y_tp}$. To compare $d_\Omega(x, x_t)$ and $d_\Omega(y, y_t)$, it suffices to compare the cross ratios

$$[\overline{\xi p}; \overline{x_tp}; \overline{xp}; \overline{x^-p}], \;\; [\overline{\xi p}; \overline{y_tp}; \overline{yp}; \overline{y^-p}].$$

See Figure 4.1 for clarification. In particular, the relative position of the lines $\overline{x^- p}$ and $\overline{y^- p}$ completely determine the relationship between these cross-ratios, and the cross-ratios coincide if and only if $\overline{x^- p} = \overline{y^- p}$. But if indeed x^-, y^- and p were collinear, then since $x^-, y^- \in \partial\Omega$ and the intersection of any projective line with a convex set Ω is connected, p would lie on the open line segment (x^-, y^-). But this line segment does not intersect the projective line $\overline{xy}$ by construction. Thus p is not collinear with x^- and y^-, and thus $d_\Omega(x_t, x) \neq d_\Omega(y_t, y)$, completing the contradiction argument. $\square$

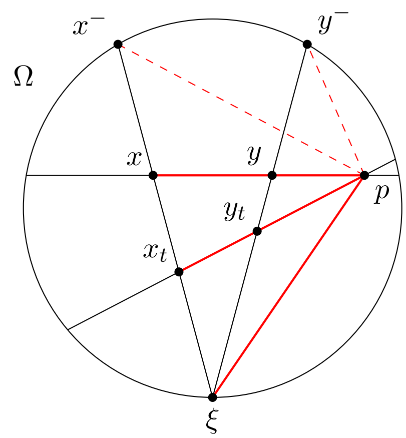

FIGURE 4.1. The setup for Lemma 4.1.

Lemma 4.2 (Fellow traveling)**.** *Let Ω be a strictly convex Hilbert geometry and let Δ be any convex subset of $\Omega \cup \partial\Omega$ which is a hyperbolic metric space when endowed with the Hilbert metric. Let $x, y \in \Delta$ and ξ, η in $\Delta \cap \partial\Omega^s$. Denote by ξ_t, η_t the geodesic rays from x to ξ and from y to η, respectively, parameterized at unit Hilbert speed.*

If $\eta \in V(x, \xi, t)$ then $d_\Omega(\xi_s, \eta_s) \leq d_\Omega(x, y) + O(\alpha)$ for all $s \in [o, t + O(\alpha)]$.

Proof. Since η is in $V(x, \xi, t)$, the projection of η onto (x, ξ) is distance greater than t from x. Let q be the point on the geodesic ray (x, η) which is distance t from x. Then (up to $O(\alpha)$), q and ξ_t are comparable points on the thin triangle with vertices x, ξ, and η, and thus their distance is bounded above by $O(\alpha)$. On the other hand, the distance from q to η_t is bounded above by $d_\Omega(x, y)$ by Lemma 4.1. The conclusion follows from the triangle inequality. $\square$

The next lemma closely resembles an analogous lemma of Schapira written in French [Sch04, Lemme 2.9]. We include a proof in Appendix A for the reader's convenience.

Lemma 4.3. *Let Ω be a strictly convex Hilbert geometry and let C be any convex subset of $\Omega \cup \partial\Omega^s$ which is a hyperbolic metric space when endowed with the Hilbert metric d_Ω. Fixing a point o in C, a parabolic fixed point ξ in $C \cap \partial\Omega$ and a compact subset K of $C \cap \partial\Omega \smallsetminus \{\xi\}$, there exists a constant A such that for every parabolic group element g fixing ξ and all $t > A$, we have*

(a) $d_\Omega(o, go) \geq 2t$ implies that gK lies in the shadow $V(o, \xi, t - A)$ and for all $\eta \in K$, we have

$$|\beta_{g\eta}(\xi_t, g\xi_t) - d_\Omega(o, go) + 2t| \leq 2A;$$

(b) if $d_\Omega(o, go) < 2t$, then gK is disjoint from the shadow $V(o, \xi, t + A)$ and $|\beta_{g\eta}(\xi_t, g\xi_t)| \leq 2A$ for all $\eta \in K$.

5. Cusps and convex projective geometry

We define the *limit set* Λ_Γ of a discrete group Γ of projective transformations acting on a properly convex set Ω in real projective space to be the smallest closed Γ-invariant subset of $\partial\Omega$. The *convex core* C_Γ of Ω is the convex hull of the limit set Λ_Γ, and projects to the convex core of the quotient. When Ω is strictly convex with C^1 boundary, the limit set Λ_Γ is equal to the set of accumulation points of Γo in $\partial\Omega$ for any o in the convex core [CM14a].

5.1. Geometrical finiteness.

An isometry g of a properly convex domain Ω is *parabolic* if $\inf_{x\in\Omega} d_\Omega(x, gx) = 0$ and the infimum is not realized, and a group of projective transformations preserving Ω is *parabolic* if every nontrivial element is parabolic. A boundary point in $\partial\Omega$ is a *parabolic point* if its stabilizer is a parabolic subgroup. A parabolic point ξ in Λ_Γ is a *bounded parabolic point* if the stabilizer acts cocompactly on $\Lambda_\Gamma \smallsetminus \{\xi\}$. A parabolic point in Λ_Γ is moreover a *uniformly bounded parabolic point* if the stabilizer acts cocompactly on the set of all lines from ξ to points in $\Lambda_\Gamma \smallsetminus \{\xi\}$, as a subset of the affine space of all lines through ξ. A *cusp* in a convex real projective manifold M is a neighborhood of an end with parabolic holonomy.

The *rank* of a bounded parabolic point in $\partial\Omega$ is the virtual cohomological dimension of its stabilizer in Γ, and the rank of a cusp is the rank of any of its lifts. A parabolic point is uniformly bounded if and only if its stabilizer is conjugate to a subgroup of $\mathrm{SO}(n,1)$ [CM14a, Proposition 7.21], and this parabolic stabilizer is virtually $\mathbb{Z}^d$ for some $d \leq \dim\Omega - 1$ [CM14a, Théorème 1.7]. The maximal rank of a cusp is $\dim\Omega - 1$.

Finally, $\xi \in \Lambda_\Gamma$ is a *conical limit point* if ξ is the limit point of a sequence $(\gamma_n o)$ which remains uniformly bounded distance from the geodesic ray from o to ξ.

Definition 5.1. The action of Γ on Ω is *geometrically finite* if every point ξ in Λ_Γ is either a uniformly bounded parabolic point or a conical limit point.

Crampon-Marquis show that this definition of geometrical finiteness is equivalent to several other classical criteria under the assumption that Ω is strictly convex with C^1 boundary [CM14a, Théorème 1.3]. In particular, there are finitely many cusps in the quotient, and the convex core of the quotient is hyperbolic and has finite volume for any natural projectively invariant volume form. [1] It also follows that Γ is finitely presented.

Weak geometrical finiteness. Crampon-Marquis also study a weaker notion of geometrical finiteness, which they call "geometrical finiteness of the action of Γ on $\partial\Omega$", rather than "on Ω" [CM14a, Définition 5.14]. This notion is *not* equivalent to hyperbolicity of the convex core: in [CM14a, Théorème 10.4], they show that, assuming Ω is strictly convex with C^1 boundary, geometrical finiteness of the action of Γ on Ω is equivalent to both hyperbolicity of (Ω, d_Ω) and geometrical finiteness of the action of Γ on $\partial\Omega$. In [CM14a, Proposition 10.7], they produce a group Γ in $\mathrm{PSL}(5,\mathbb{R})$ acting on a strictly convex set Ω with C^1 boundary in $\mathbb{R}P^4$ such that the action of Γ on $\partial\Omega$ is geometrically finite, but the action of Γ on Ω is not geometrically finite. Hence the convex core is not hyperbolic. As hyperbolicity of the convex core will be crucial to our arguments, we assume the stronger notion of geometrical finiteness of the action of Γ on Ω.

5.2. Thick-thin decomposition and hyperbolic metric spaces.

A strictly convex Ω with a geometrically finite action admits a *thick-thin decomposition* resembling that of negatively curved spaces:

[1] We remark that all projectively invariant volume forms associated to a properly convex domain are equivalent due to Benzecri [Ben60]. In particular, the notion of an action of finite covolume is well-defined. See the survey of [Mar14] for more elaboration in English.

Theorem 5.2 ([CM14a, Théorème 8.1]). *Assume Ω is a properly convex domain in $\mathbb{R}P^n$ which is strictly convex with C^1 boundary, and Γ is a discrete subgroup of $\mathrm{PSL}(n+1,\mathbb{R})$ which acts geometrically finitely on Ω with quotient $M = \Omega/\Gamma$. Then $M = M_{\text{thick}} \cup M_{\text{thin}}$, where M_{thick} and M_{thin} are smooth submanifolds, and $\overline{M}_{\text{thick}} \cap \overline{M}_{\text{thin}} = \partial M_{\text{thick}} = \partial M_{\text{thin}}$, and M_{thin} is a possibly empty submanifold of cusps. The convex core of M intersected with M_{thick} is compact.*

Letting $\Omega_{\text{thick}}, \Omega_{\text{thin}}$ be the lifts of $M_{\text{thick}}, M_{\text{thin}}$, respectively, in Ω, note that we may express Ω_{thin} as a union of horoballs foliated by horospheres, each centered at a bounded parabolic point and invariant under the parabolic stabilizer. Clearly $\overline{\Omega_{\text{thin}}} \cap \overline{\Omega_{\text{thick}}}$ is a union of horospherical boundaries of the components of Ω_{thin}. Note that $\Omega_{\text{thick}} \cup \Omega_{\text{thin}} = \Omega$. There are many different choices of thick-thin decomposition. We fix one choice and note that the constants in many subsequent lemmas and the main result inevitably depends on this choice.

A crucial feature in this work is hyperbolicity of the Hilbert metric.

Theorem 5.3 ([CM14a, Théorème 1.8]). *If Ω is strictly convex with C^1 boundary and Γ acts geometrically finitely on Ω, then Γ is relatively hyperbolic, relative to the cusp stabilizers, and equivalently, the convex core C_Γ with the Hilbert metric is a hyperbolic metric space.*

5.3. **Growth of orbits of the parabolic subgroup.** In this section we prove Proposition 1.7, a coarse estimate on the growth rate of orbits under a parabolic subgroup.

We say two positive real valued functions f and h are *asymptotically equivalent*, denoted $f \asymp h$, if there exists a uniform constant $k \geq 1$ for which $k^{-1}h \leq f \leq kh$.

First note that for any ellipsoid $\mathcal{E}$ in $\mathbb{R}P^n$ and point $o \in \mathcal{E}$, there is a constant R_0 such that

$$\#\{g \in \Pi : d_{\mathcal{E}}(o, go) \in [T, T + R_0]\} \asymp e^{\delta_\Pi T},$$

since $\mathcal{E}$ with the Hilbert metric is the Beltrami-Klein model of hyperbolic space.

Let $p \in \partial\Omega$ and $Cone(p, C_\Gamma)$ denote the union of all open projective line segments from p to x over all x in the convex hull of the limit set C_Γ. If a properly convex domain Ω is strictly convex with C^1 boundary, then, by [CM14a, Corollaire 7.18], at any uniformly bounded parabolic point ξ with stabilizer Π and rank r, Ω admits Π-invariant inner and outer *almost osculating ellipsoids* to Ω at ξ; that is, there are Π-invariant ellipsoids $\mathcal{E}_1, \mathcal{E}_2$ such that

$$\mathcal{E}_1 \cap Cone(p, C_\Gamma) \subset \Omega \cap Cone(p, C_\Gamma) \subset \mathcal{E}_2 \cap Cone(p, C_\Gamma)$$

and $\partial\mathcal{E}_1 \cap \partial\Omega = \partial\mathcal{E}_2 \cap \Omega = \{\xi\}$. Moreover, $\mathcal{E}_1 \subset \mathcal{E}_2$ is a horoball for $\mathcal{E}_2$ with the Hilbert metric.

When Π has maximal rank, these Π-invariant ellipsoids $\mathcal{E}_1, \mathcal{E}_2$ are *osculating ellipsoids* at ξ; meaning that they are almost osculating and in particular one has the inclusion $\mathcal{E}_1 \subset \Omega \subset \mathcal{E}_2$, without the need to intersect with $Cone(p, C_\Gamma)$ [CM14a, Théorème 7.14].

Remark 5.4. Crampon-Marquis' Corollaire 7.18 follows from Théorème 7.14 and a crucial lemma which states that the parabolic stabilizer of a uniformly bounded parabolic point of rank r preserves a projective subspace H of dimension r and acts on the (properly convex) intersection of Ω with H as a maximal rank parabolic subgroup [CM14a, Lemme 7.17]. We remark that the uniform boundedness of the parabolic subgroup is crucial for Crampon-Marquis' proof of Corollaire 7.18.

In [CLT15, Theorem 0.5], Cooper-Long-Tillman construct the osculating ellipses in the maximal rank setting without the presence of the geometrically finite action assumed by Crampon-Marquis.

Lemma 5.5. *Let Ω be a properly convex, strictly convex domain in $\mathbb{R}P^n$ with C^1 boundary, and Γ a discrete, nonelementary group of projective transformations acting geometrically finitely on Ω. Let Π be a discrete parabolic subgroup of Γ which preserves Ω with fixed point ξ. Let $\mathcal{E}_1 \subset \mathcal{E}_2$ be the almost osculating ellipses to Ω at ξ. Then there exists a point o in $\mathcal{E}_1$ and a uniform constant R such that for all $g \in \Pi$,*

$$d_{\mathcal{E}_1}(o, go) - R \leq d_\Omega(o, go) \leq d_{\mathcal{E}_1}(o, go).$$

By the preceding remarks, to prove Lemma 5.5, it will suffice to prove the following lemma:

Lemma 5.6. *Assume Ω is a proper, strictly convex domain with C^1 boundary in $\mathbb{R}P^n$ and ξ in $\partial\Omega$ is a parabolic point. Let Π be the discrete parabolic subgroup of $\mathrm{PSL}(n+1,\mathbb{R})$ which preserves Ω and stabilizes ξ, and assume Π has maximal rank. Let $\mathcal{E}_1 \subset \mathcal{E}_2$ be the osculating ellipses to Ω at ξ. Then for all $o \in \mathcal{E}_1$, there is a uniform constant R such that for all $g \in \Pi$,*

$$d_{\mathcal{E}_1}(o, go) - R \leq d_\Omega(o, go) \leq d_{\mathcal{E}_1}(o, go).$$

Proof. First, notice that existence of a constant R independent of g such that $d_{\mathcal{E}_1}(o, go) \leq d_{\mathcal{E}_2}(o, go) + R$ implies the result, because the containments $\mathcal{E}_1 \subset \Omega \subset \mathcal{E}_2$ imply $d_{\mathcal{E}_2} \leq d_\Omega \leq d_{\mathcal{E}_1}$.

We will do a calculation in a convenient chart, since the Hilbert metric does not depend on the choice of affine chart. The chart is what Cooper-Long-Tillman define as *parabolic coordinates* [CLT15, pg 203-204, preceding Proposition 3.4]. We will choose the chart for the largest ellipsoid $\mathcal{E}_2$ with respect to the parabolic fixed point.

Let ξ be the parabolic fixed point by the maximal rank parabolic group Π. Let H_ξ be the supporting hyperplane to $\mathcal{E}_2$ at ξ. Choose any other $\eta \in \partial\mathcal{E}_2$ with supporting hyperplane H_η to $\mathcal{E}_2$ at η. Choose a chart with coordinates $(x_1, \ldots, x_n) \in \mathbb{R}^n$ for $\mathcal{E}_2$ in which H_ξ is sent to infinity, H_η is sent to the hyperplane $x_n = 0$, and η is sent to the origin, so that the geodesic line (ξ, η) is sent to the x_n-axis and the parabolic fixed point ξ is sent to the point at infinity in the vertical direction, which is the x_n-direction. In this chart, since $\mathcal{E}_2$ is an ellipsoid, its boundary $\partial\mathcal{E}_2$ is sent to the graph of a paraboloid tangent to the hyperplane $x_n = 0$ at the origin. Since all paraboloids are projectively equivalent, we can choose the paraboloid so that it is the graph of the function $x_1^2 + \cdots + x_{n-1}^2 = x_n$; in particular, in this affine chart, $\partial\mathcal{E}_2$ has rotational symmetry about the x_n-axis.

Moreover, *algebraic horospheres* centered at ξ for the ellipsoid $\mathcal{E}_2$ are vertical translates of $\mathcal{E}_2$, and these algebraic horospheres coincide with horospheres for the Hilbert metric on $\mathcal{E}_2$ since the boundary of $\mathcal{E}_2$ is C^1 at ξ (this claim is effectively consolidated in [CLT15, Proposition 3.4] which is stated without proof, but one can avoid the notion of algebraic horospheres and directly prove this fact with a calculation using the geometric description of the Busemann functions in Lemma 2.1, the definition of parabolic coordinates, and the fact that changing a chart sends lines to lines).

Crampon-Marquis prove that $\mathcal{E}_1$ is an algebraic, hence geometric, horosphere for $\mathcal{E}_2$ [CM14a, Theorem 7.14]. More explicitly, $\mathcal{E}_1$ is the orbit of some point under the Zariski closure $\mathcal{Z}(\Pi)$ of Π. Let $\mathcal{E}_0$ be the orbit of o under $\mathcal{Z}(\Pi)$; thus for all $g \in \Pi$, $go \in \mathcal{E}_0$, and $\mathcal{E}_0 \subset \mathcal{E}_1$ are both horospheres in the exterior ellipsoid $\mathcal{E}_2$. In parabolic coordinates, this implies $\mathcal{E}_0$ and $\mathcal{E}_1$ are vertical translates of $\mathcal{E}_2$. Let K_i be the intersection of $\partial\mathcal{E}_i$ with the vertical axis. Note that we choose coordinates so that $K_2 = 0$.

Now we have the technical set-up prepared for the calculation. We will replace go with an arbitrary point on the curve $\partial\mathcal{E}_0$ in parabolic coordinates. Since the claim follows for any $o \in \mathcal{E}_1$ by the triangle inequality as soon as it is proved for one $o \in \mathcal{E}_1$, we may choose $o = (0, K_0)$ on $\partial\mathcal{E}_0$. Also, $\mathcal{E}_2$ has rotational symmetry and $\mathcal{E}_0$ and $\mathcal{E}_1$ are vertical translates of $\mathcal{E}_2$, so to compute the Hilbert distance, it suffices to take a two dimensional slice of $\mathcal{E}_2$ in these coordinates, and represent any point on $\partial\mathcal{E}_0$ as the pair

$$o_t = (t, t^2 + K_0)$$

in $\mathbb{R}^2$. Let $a_{i,t}, b_{i,t}$ be the intersection points of the line determined by o and o_t with the curves $\partial\mathcal{E}_i$ for $i = 1, 2$, chosen so that

$$d_{\mathcal{E}_i}(o, o_t) = \frac{1}{2} \log \frac{|a_{i,t} - o_t||b_{i,t} - o|}{|a_{i,t} - o||b_{i,t} - o_t|}$$

where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^2$. Since the graphs are symmetric, it suffices to consider

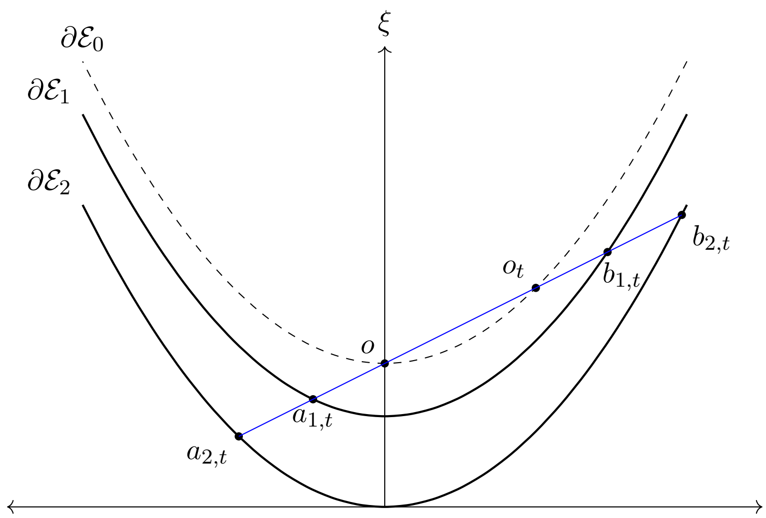

FIGURE 5.1. For the proof of Lemma 5.6.

$t > 0$. Now we compare $d_{\mathcal{E}_1}$ and $d_{\mathcal{E}_2}$. Observe that $|a_{1,t} - o_t| \le |a_{2,t} - o_t|$ and $|b_{1,t} - o| \le |b_{2,t} - o|$, while $|a_{1,t} - o|$ and $|a_{2,t} - o|$ are bounded away from zero when t is large. Hence there is a constant r_0 so that

$$d_{\mathcal{E}_1}(o, o_t) \le d_{\mathcal{E}_2}(o, o_t) + r_0 + \frac{1}{2}\log\frac{|b_{2,t} - o_t|}{|b_{1,t} - o_t|}. \tag{5.1}$$

To determine the coordinates of $b_{i,t}$, see that the line passing through o and o_t has slope t and y-intercept K_0, so we calculate

$$b_{i,t} = \left(\frac{t + \sqrt{t^2 - 4(K_i - K_0)}}{2}, \frac{t^2 + \sqrt{t^4 - 4t^2(K_i - K_0)}}{2} + K_0\right).$$

Then

$$|b_{i,t} - o_t| = \frac{1}{2}\sqrt{\left(t - \sqrt{t^2 - 4(K_i - K_0)}\right)^2 + \left(t^2 - \sqrt{t^4 - 4t^2(K_i - K_0)}\right)^2}.$$

Now we compare $|b_{2,t} - o_t|$ with $|b_{1,t} - o_t|$ and see that these expressions are bounded and away from 0 as $t \to +\infty$, hence $\frac{|b_{2,t} - o_t|}{|b_{1,t} - o_t|}$ is bounded and by equation (5.1), the conclusion follows. $\square$

Proof of Proposition 1.7. First, if the claim holds for some o in Ω, then it also holds for any other point in Ω, so we may prove the claim for o in the interior ellipse $\mathcal{E}_1$ given by Lemma 5.5. Let R_1 be such that

$$\#\{g \in \Pi : d_{\mathcal{E}_1}(o, go) \in [T, T + R_1)\} \asymp e^{\delta_\Pi T}.$$

Take $S = R_1 + R$, which will be the constant needed for the theorem.

Since $\mathcal{E}_1 \subset \Omega$, we have $d_{\mathcal{E}_1} \geq d_\Omega$. Then $d_\Omega(o, go) \geq T$ implies $d_{\mathcal{E}_1}(o, go) \geq T$. By Lemma 5.6, $d_\Omega(o, go) \leq T + R_1$ implies $d_{\mathcal{E}_1}(o, go) \leq T + R_1 + R$. Then there is some constant c_1 such that

$$\begin{aligned}\#\{g : d_\Omega(o, go) \in [T, T + R_1)\} &\leq \#\{g : d_{\mathcal{E}_1}(o, go) \in [T, T + R_1 + R)\} \\ &\leq \sum_{i=0}^{\lceil R/R_1 \rceil} \#\{g : d_{\mathcal{E}_1}(o, go) \in [T + iR_1, T + (i+1)R_1)\} \\ &\leq \sum_{i=0}^{\lceil R/R_1 \rceil} c_1 e^{\delta_\Pi (T + iR_1)} \leq \left(\sum_{i=0}^{\lceil R/R_1 \rceil} c_1 e^{\delta_\Pi i R_1} \right) e^{\delta_\Pi T},\end{aligned}$$

and the upper bound follows.

For the lower bound, let $g \in \Pi$ be such that $d_{\mathcal{E}_1}(o, go) \in [T + R, T + R + R_1)$. Then

$$d_\Omega(o, go) \leq d_{\mathcal{E}_1}(o, go) < T + R + R_1 = T + S$$

and by Lemma 5.5,

$$d_\Omega(o, go) \geq d_{\mathcal{E}_1}(o, go) - R \geq T.$$

Then there is another constant $c_2 > 0$ such that

$$\begin{aligned}c_2 e^{\delta_\Pi R} \cdot e^{\delta_\Pi T} = c_2 e^{\delta_\Pi (T+R)} &\leq \#\{g \in \Pi \ : \ d_{\mathcal{E}_1}(o, go) \in [T + R, T + R + R_1)\} \\ &\leq \#\{g \in \Pi \ : \ d_\Omega(o, go) \in [T, T + S)\},\end{aligned}$$

completing the proof. □

6. Conformal densities and estimates near the cusps

Let Ω be a strictly convex domain in $\mathbb{R}P^n$ with C^1 boundary, and Γ a discrete subgroup of $\mathrm{PSL}(n{+}1, \mathbb{R})$ which preserves Ω. Note that Busemann functions are globally defined and continuous. Then a *conformal density of dimension* $\delta > 0$ is a family $\{\mu_x\}_{x \in \Omega}$ of finite measures supported on $\partial\Omega$ with the following properties:

- (quasi-Γ-invariance) for all $\gamma \in \Gamma$, $x \in \Omega$, we have $\gamma_* \mu_x = \mu_{\gamma x}$;
- (transformation rule) for all $x, y \in \Omega$ and $\eta \in \partial\Omega$, we have

$$\frac{d\mu_x}{d\mu_y}(\eta) = e^{-\delta \beta_\eta (x,y)}.$$

A measure is a δ-conformal measure if it is part of a conformal density of dimension δ. A particularly famous example of a conformal density is the *Patterson-Sullivan density*, first constructed by Patterson for Fuchsian groups and extended by Sullivan to geometrically finite actions on hyperbolic spaces ([Pat76, Sul79, Sul84]). The Patterson-Sullivan density has conformal dimension δ_Γ, which is the *critical exponent* of the Poincaré series, or equivalently, for any $o \in \Omega$,

$$\delta_\Gamma = \limsup_{t \to \infty} \frac{1}{t} \log \#\{\gamma \in \Gamma : d_\Omega(o, \gamma o) \leq t\}.$$

Moreover, a corollary of the construction is that any conformal density for Ω must have conformal dimension at least δ_Γ [Sul79]. Although the Patterson-Sullivan density is the only known conformal density associated to a geometrically finite action on a Hilbert geometry, one appeal of results stated in this generality is that the proof is intrinsic to these defining properties, rather than the Patterson construction.

Crampon showed in his thesis that Patterson's construction can be adapted to the setting of geometrically finite actions of Γ on Ω, when Ω is strictly convex with C^1 boundary. [Cra11, Theorem 4.2.1]. The measures have full support on the limit set [Cra11, Section 4.2.1], which in the finite volume case is all of $\partial\Omega$ [CM14b, Corollaire 1.5]. Crampon then proves in the case of surfaces that the Patterson-Sullivan measures have no atoms [Cra11, Lemma 4.3.3, Proposition 4.3.5], but these arguments generalize to higher dimensions due to [CM14b, Corollaire 7.18], which generalizes [Cra11, Lemma 1.3.4]. In recent work, Zhu confirms that these results extend to higher dimensions (see [Zhu20, Lemma 11, Proposition 12, Corollary 13]). These results hinge on a calculation and proof that any uniformly bounded parabolic group preserving Ω with rank r has critical exponent $\delta_\Pi = \frac{r}{2}$, and if Π is a subgroup of a nonelementary geometrically finite group Γ, then $\delta_\Pi < \delta_\Gamma$ [Zhu20, Lemma 11], [CM14b, Lemme 9.8].

Lemma 6.1. *Let Ω be a properly convex, strictly convex domain in $\mathbb{R}P^n$ with C^1 boundary, and Γ a discrete, nonelementary group of projective transformations acting geometrically finitely on Ω. Let $\{\mu_x\}_{x\in\Omega}$ be a conformal density of dimension δ with full support on Λ_Γ and no atoms. Fix $o \in C_\Gamma$ and $\xi \in \Lambda_\Gamma$, and let α be the hyperbolicity constant of C_Γ with respect to the Hilbert metric. Then for all $\eta \in V(o,\xi,t)$ and $t \geq 0$,*

$$|\beta_\eta(o,\xi_t) - t| \leq 4\alpha$$

and thus for all $t + s \geq 0$,

$$\mu_{\xi_t}(V(o,\xi,t)) \asymp e^{-\delta s}\mu_{\xi_{t+s}}(V(o,\xi,t))$$

with uniform constants.

Proof. To prove the first part, let p be the closest point projection of η onto $[o,\xi)$. By definition, since $\eta \in V(o,\xi,t)$ we have $d(o,p) \geq t$, hence $\beta_p(o,\xi_t) = t$ and by Lemma 3.7 we have

$$\beta_\eta(o,\xi_t) = \beta_p(o,\xi_t) + O(\alpha) = t + O(\alpha).$$

The first part implies the second part because, by the transformation rule of conformal densities,

$$\begin{aligned}
\mu_{\xi_t}(V(o,\xi,t)) &= \int_{V(o,\xi,t)} e^{-\delta\beta_\eta(\xi_t,\xi_{t+s})}\, d\mu_{\xi_{t+s}}(\eta) \\
&= \int_{V(o,\xi,t)} e^{-\delta(-\beta_\eta(o,\xi_t)+\beta_\eta(o,\xi_{t+s}))}\, d\mu_{\xi_{t+s}}(\eta) \\
&\asymp e^{-\delta s}\mu_{\xi_{t+s}}(V(o,\xi,t))
\end{aligned}$$

where the cocycle and antisymmetric properties of the Busemann function are applied in the second equality. ∎

6.1. **The measure of shadows at parabolic fixed points.**

Lemma 6.2. *Let Ω be a properly convex, strictly convex domain in $\mathbb{R}P^n$ with C^1 boundary, and Γ a discrete, nonelementary group of projective transformations acting geometrically finitely on Ω. Let $\{\mu_x\}_{x\in\Omega}$ be a conformal density of dimension δ with full support on Λ_Γ and no atoms.*

Let ξ be a uniformly bounded parabolic point in $\partial\Omega$ with stabilizer the parabolic subgroup Π, and $o \in C_\Gamma$. Let ξ_t be the point on the ray $[o,\xi)$ at distance t from o. Then there exist constants A and C depending on ξ and o such that for all $t > A$,

$$C^{-1}\mu_{\xi_t}(V(o,\xi,t+A)) \leq \sum_{\substack{g\in\Pi \\ d_\Omega(o,go)\geq 2t}} e^{-\delta d_\Omega(o,go)+\delta t} \leq C\mu_{\xi_t}(V(o,\xi,t-A)),$$

and

$$C^{-1}\mu_{\xi_t}(\partial\Omega \smallsetminus V(o,\xi,t-A)) \leq \sum_{\substack{g\in\Pi \\ d_\Omega(o,go)\leq 2t}} e^{-\delta t} \leq C\mu_{\xi_t}(\partial\Omega \smallsetminus V(o,\xi,t+A)).$$

Proof. First let us show the upper bound. By Lemma 4.3(a), there is a constant A such that for all $t > A$,

$$\bigcup_{\substack{g\in\Pi \\ d_\Omega(o,go)\geq 2t}} gK \subset V(o,\xi,t-A)$$

where K is a compact fundamental domain for the action of the parabolic subgroup Π on $\Lambda_\Gamma \smallsetminus \{\xi\}$, given by geometrical finiteness of the action of Γ on Ω. Since the conformal density has full support and no atoms, there is a subset K' of K with positive measure for which the sets $\{gK'\}_{g\in\Pi}$ are pairwise disjoint. Then

$$\sum_{\substack{g\in\Pi \\ d_\Omega(o,go)\geq 2t}} \mu_{\xi_t}(gK') = \mu_{\xi_t}\left(\bigcup_{\substack{g\in\Pi \\ d_\Omega(o,go)\geq 2t}} gK'\right) \leq \mu_{\xi_t}(V(o,\xi,t-A)). \tag{6.1}$$

Moreover, Lemma 4.3(a) gives us control over $\beta_{g\eta}(\xi_t, g\xi_t)$ for all such $g \in \Pi$ and all $\eta \in K$, hence for any subset K'' of K (applying the defining properties of a conformal density),

$$\mu_{\xi_t}(gK'') = \int_{gK''} \frac{d\mu_{\xi_t}}{d\mu_{g\xi_t}}(\lambda)\ d\mu_{g\xi_t}(\lambda) = \int_{gK''} e^{-\delta\beta_\lambda(\xi_t,g\xi_t)}\ d\mu_{g\xi_t}(\lambda) \tag{6.2}$$

and, setting $\lambda = g\eta$ and using Lemma 4.3(a),

$$\asymp \int_{gK''} e^{-\delta(d_\Omega(o,go)-2t)}\ d\mu_{g\xi_t}(\lambda) = e^{-\delta d_\Omega(o,go)+2\delta t}\mu_{g\xi_t}(gK'') \tag{6.3}$$

$$= e^{-\delta d_\Omega(o,go)+2\delta t}\mu_{\xi_t}(K''). \tag{6.4}$$

Since K is compact and disjoint from ξ, there is a constant D such that K is disjoint from $V(o,\xi,D)$, hence by Lemma 3.8, for t sufficiently large and $\eta' \in K$,

$$t - 2D + O(\alpha) = d_\Omega(o,\xi_t) - 2D + O(\alpha) \leq \beta_{\eta'}(\xi_t, o) \leq t.$$

Then another computation using the defining properties of a conformal density gives

$$\mu_{\xi_t}(K'') \asymp e^{-\delta t}\mu_o(K'') \tag{6.5}$$

for any subset K'' of K. Since the fixed compact subset K' of K from equation (6.1) has nonempty interior and the measures have full support, $\mu_o(K')$ is some positive constant, so we apply equations (6.2) and (6.5) to the compact subset K' of K and obtain a constant B independent of t such that

$$\frac{1}{B}\sum_{\substack{g\in\Pi \\ d_\Omega(o,go)\geq 2t}} e^{-\delta d_\Omega(o,go)+\delta t} \leq \mu_{\xi_t}(V(o,\xi,t-A)).$$

The argument for the lower bound is similar. By the contrapositive of Lemma 4.3(b), and using that K is a fundamental domain for the parabolic subgroup, there is a constant A such that

$$V(o,\xi,t+A) \smallsetminus \{\xi\} \subset \bigcup_{\substack{g\in\Pi \\ d_\Omega(o,go)\geq 2t}} gK.$$

Then by subadditivity and the assumption that parabolic fixed points are not atoms for the measures,

$$\mu_{\xi_t}(V(o,\xi,t+A)) \leq \sum_{\substack{g\in\Pi \\ d_\Omega(o,go)\geq 2t}} \mu_{\xi_t}(gK).$$

Now, by applying the estimates from equations (6.2) and (6.5), and adjusting the previous constant B if needed, we have

$$\mu_{\xi_t}(V(o,\xi,t+A)) \leq B \sum_{\substack{g\in\Pi \\ d_\Omega(o,go)\geq 2t}} e^{-\delta d_\Omega(o,go)+\delta t}.$$

The estimate for the complement of the shadow is similar and uses Lemma 4.3 as well, hence the proof is omitted. For more details see [Sch04, Proposition 3.6]. □

6.2. **Uniform control over all parabolic fixed points.**

Lemma 6.3. *Let Ω be a properly convex, strictly convex domain in $\mathbb{R}P^n$ with C^1 boundary, and Γ a discrete, nonelementary group of projective transformations acting geometrically finitely on Ω. Let $\{\mu_x\}_{x\in\Omega}$ be a conformal density of dimension δ with full support on Λ_Γ and no atoms. For all parabolic fixed points ξ with stabilizer Π, and $o \in \Omega$, there exists a constant K (note that it depends on all the above) such that for all ξ_t on the geodesic ray $[o,\xi)$ distance t from o,*

$$\frac{1}{K}e^{(2\delta_\Pi-\delta)t} \leq \mu_{\xi_t}(V(o,\xi,t)) \leq Ke^{(2\delta_\Pi-\delta)t}$$

and

$$\frac{1}{K}e^{(2\delta_\Pi-\delta)t} \leq \mu_{\xi_t}(\partial\Omega \smallsetminus V(o,\xi,t)) \leq Ke^{(2\delta_\Pi-\delta)t}.$$

Proof. Note that we may prove the claim for all t sufficiently large since by adjusting constants, the claim then applies to all $t \geq 0$. First, write Π as the disjoint union

$$\Pi = \bigcup_{n\in\mathbb{N}} \{g \in \Pi : d_\Omega(o,go) \in [Rn-R, Rn)\}$$

and denote

$$a_n := \#\{g \in \Pi \ : \ d_\Omega(o,go) \in [Rn-R,Rn)\}.$$

Then for t sufficiently large, a short calculation gives

$$\begin{aligned}\sum_{d_\Omega(o,go)\geq 2t} e^{-\delta d_\Omega(o,go)+\delta t} &\asymp e^{\delta t} \sum_{Rn-R\geq 2t} \sum_{\substack{d_\Omega(o,go)\in \\ [Rn-R,Rn)}} e^{-\delta Rn} \\ &= e^{\delta t} \sum_{Rn-R\geq 2t} a_n e^{-\delta Rn}\end{aligned}$$

and, by Proposition 1.7, $a_n \asymp e^{\delta_\Pi Rn}$, hence

$$\asymp e^{\delta t} \sum_{n\geq 2t/R+1} e^{(\delta_\Pi-\delta)Rn}$$

and by summing the geometric series

$$\asymp e^{\delta t}e^{(2\delta_\Pi-2\delta)t} = e^{(2\delta_\Pi-\delta)t}.$$

Finally, by Lemma 6.2,

$$\mu_{\xi_t}(V(o,\xi,t-A)) \geq C^{-1}e^{(2\delta_\Pi-\delta)t}.$$

An analogous argument for the upper bound gives

$$\mu_{\xi_t}(V(o,\xi,t+A)) \leq C'e^{(2\delta_\Pi-\delta)t}.$$

To obtain the conclusion, apply the transformation rule and use $|\beta_\eta(\xi_t,\xi_{t\pm A})| \leq \pm A$ to compare $\mu_{\xi_{t\pm A}}(V(o,\xi,t\pm A))$ with $\mu_{\xi_t}(V(o,\xi,t\pm A))$.

To estimate the complement of the shadow, the argument is very similar. The calculation of the geometric sum given by Lemma 6.2 is slightly different but nonetheless produces the estimate

$$\sum_{d_\Omega(o,po)\le 2t} e^{-\delta t} \asymp e^{(2\delta_\Pi-\delta)t} + e^{-\delta t} \asymp e^{(2\delta_\Pi-\delta)t}$$

since $\delta_\Pi = \frac{r}{2} > 0$ [Zhu20, Lemma 11] and hence $2\delta_\Pi - \delta > -\delta$. □

Note that so far, the constants depend on a particular parabolic point ξ.

Lemma 6.4. *Let Ω be a properly convex, strictly convex domain in $\mathbb{R}P^n$ with C^1 boundary, and Γ a discrete, nonelementary group of projective transformations acting geometrically finitely on Ω. Let $\{\mu_x\}_{x\in\Omega}$ be a conformal density of dimension δ with full support on Λ_Γ and no atoms. For all $o \in \Omega$, there exists a constant C such that for all parabolic fixed points ξ, letting ξ_t be the point on the geodesic ray $[o,\xi)$ at distance t from o, if ξ_t is in the thin part of Ω then*

$$C^{-1}e^{(2\delta_\Pi-\delta)d_\Omega(\xi_t,\Gamma o)} \le \mu_{\xi_t}(V(o,\xi,t)) \le Ce^{(2\delta_\Pi-\delta)d_\Omega(\xi_t,\Gamma o)}$$

and

$$C^{-1}e^{(2\delta_\Pi-\delta)d_\Omega(\xi_t,\Gamma o)} \le \mu_{\xi_t}(\partial\Omega \smallsetminus V(o,\xi,t)) \le Ce^{(2\delta_\Pi-\delta)d_\Omega(\xi_t,\Gamma o)}.$$

Proof. Since there are finitely many Γ-orbits of parabolic points, it will suffice to show that the lemma holds for all ξ in the Γ-orbit of the parabolic fixed point η.

Now, let ξ_s be the projection of o (and hence ξ_t, by the horosphere projection Lemma 2.6) to the boundary horosphere H_ξ of the thick part of the convex core C_Γ. Similarly, let $\eta_{s'}$ be the projection of o to the boundary horosphere H_η of the thick part of Ω. Since the thick part is Γ-invariant, any group element γ for which $\gamma\eta = \xi$ also identifies the horospheres H_η with H_ξ. Hence, for any such group element γ, we have $\gamma\eta_{s'}$ and ξ_s both lie in the boundary horosphere H_ξ. Since the parabolic stabilizer of ξ acts cocompactly on the boundary horosphere H_ξ, and the diameter of this fundamental domain is uniformly bounded over all orbits of η, we can choose a particular γ such that $\gamma\eta_{s'}$ and ξ_s are uniformly bounded distance apart. Denote this bound by M, and thus

$$d_\Omega(\xi_s,\gamma o) \le d_\Omega(\xi_s,\gamma\eta_{s'}) + d_\Omega(\gamma\eta_{s'},o) \le M + s' =: M'. \tag{6.6}$$

Then since geodesic rays meeting at the same boundary point ξ are asymptotic in a strictly convex Hilbert geometry (Lemma 4.1),

$$d_\Omega(\xi_t,\gamma\eta_{t-s}) \le d_\Omega(\xi_s,\gamma\eta_0) = d_\Omega(\xi_s,\gamma o) \le M'$$

as well, and by conformality of the Patterson-Sullivan measures and since Busemann functions are 1-Lipschitz,

$$\mu_{\xi_t}(V(o,\xi,t)) \asymp \mu_{\gamma\eta_{t-s}}(V(o,\xi,t)) \tag{6.7}$$

and the same holds for the measures of the complementary shadow. On the other hand, equation (6.6) suffices to apply the hyperbolicity Lemma 3.12(2); for all points such as ξ_s and γo which are bounded distance, there is a constant C depending on this bound such that

$$V(\gamma o,\xi,t-s+C) \subset V(\xi_s,\xi,t-s) \subset V(\gamma o,\xi,t-s-C), \tag{6.8}$$

and the containments apply in reverse to the complementary shadow. It is straightforward to check from the definition of projection that $V(\xi_s,\xi,t-s) = V(o,\xi,t)$. Hence recalling $\xi = \gamma\eta$ and applying equation (6.8), Γ-equivariance of the conformal measures and shadows, and Lemma 6.3 (it does apply because ξ_t is in the thin part, so $t-s$ is positive),

$$\begin{aligned}\mu_{\xi_t}(V(o,\xi,t)) &= \mu_{\xi_t}(V(\xi_s,\xi,t-s)) \asymp \mu_{\gamma\eta_{t-s}}(V(\gamma o,\gamma\eta,t-s))\\ &= \mu_{\eta_{t-s}}(V(o,\eta,t-s)) \asymp e^{(2\delta_\Pi-\delta)(t-s)},\end{aligned}$$

and again, with similar expressions for the complementary shadow. To conclude the proof, see that $t-s$ is the distance of ξ_t to the thick part of Ω, which is equal to $d_\Omega(\xi_t, \Gamma o)$ up to uniform additive constants. $\square$

7. Proof of the global shadow lemma

In this final section, we complete the proof of the following theorem.

Theorem 7.1. *Let Ω be a properly convex, strictly convex domain in $\mathbb{R}P^n$ with C^1 boundary, and Γ a discrete, nonelementary group of projective transformations acting geometrically finitely on Ω. Let $\{\mu_x\}_{x\in\Omega}$ be a conformal density of dimension δ with full support on Λ_Γ and no atoms.*

For each $o \in \Omega$, there exists a constant C such that for all $\xi \in \partial\Omega$, letting ξ_t be the point on the geodesic ray from o to ξ which is distance t from o,

$$C^{-1}e^{-\delta t+(2\delta_\Pi-\delta)d_\Omega(\xi_t,\Gamma o)} \le \mu_o(V(o,\xi,t)) \le Ce^{-\delta t+(2\delta_\Pi-\delta)d_\Omega(\xi_t,\Gamma o)}$$

where $\Pi = \{id\}$ if ξ_t is in the thick part, and otherwise Π is the largest parabolic subgroup preserving the horoball containing ξ_t.

Note that Theorem 7.1 implies Theorem 1.1, since Patterson-Sullivan measures satisfy the assumptions of the theorem [Cra11, Zhu20].

7.1. **Shadows in the thick part.**

Lemma 7.2. *There is a constant B such that for all x in the thick part of Ω, and any $\xi \in \partial\Omega$,*

$$\frac{1}{B} \le \mu_x(V(x,\xi,0)) \le B.$$

Proof. Every point in the thick part is uniformly bounded distance from the Γ-orbit of o for any fixed point o in the thick part. Let γo be some closest point to x which is in the Γ-orbit of o. Then by quasi-Γ-invariance of the measures and shadows,

$$\mu_x(V(x,\xi,0)) \asymp \mu_{\gamma o}(V(\gamma o,\xi,0)) = \mu_o(V(o,\xi',0))$$

where $\xi' = \gamma^{-1}\xi$ varies over $\partial\Omega$.

Now, let us choose some $t > 0$. Then by compactness we can cover $\partial\Omega$ with finitely many shadows of type $V(o,\xi_i,t)$ for $i = 1,\dots,k$. Since μ_o has full support, we have

$$B := \inf_i \mu_o(V(o,\xi_i,t)) > 0.$$

Now, let $\xi \in \partial\Omega$. Then there is a ξ_i such that $\xi \in V(o,\xi_i,t)$, and we claim that $V(o,\xi_i,t) \subseteq V(o,\xi,t/2)$.

Indeed, $\xi \in V(o,\xi_i,t)$ implies $\langle\xi,\xi_i\rangle_o \ge t - O(\alpha)$. Moreover, if $\eta \in V(o,\xi_i,t)$ then $\langle\eta,\xi_i\rangle_o \ge t - O(\alpha)$, hence by (3.1) one gets $\langle\xi,\eta\rangle_o \ge t - O(\alpha) \ge t/2$ for t large enough. Thus,

$$\mu_o(V(o,\xi,0)) \ge \mu_o(V(o,\xi,t/2)) \ge \mu_o(V(o,\xi_i,t)) \ge B.$$

Now the upper bound is clear, since μ_o is a probability measure. $\square$

Proof of Theorem 7.1. First, by Lemma 6.1 comparing $\mu_{\xi_t}(V(o,\xi,t))$ with $\mu_{\xi_0}(V(o,\xi,t)) = \mu_o(V(o,\xi,t))$, it suffices to show that there is a constant C such that

$$C^{-1}e^{(2\delta_\Pi-\delta)d_\Omega(\xi_t,\Gamma o)} \le \mu_{\xi_t}(V(o,\xi,t)) \le Ce^{(2\delta_\Pi-\delta)d_\Omega(\xi_t,\Gamma o)}. \tag{7.1}$$

The case where ξ_t is in the thick part of Ω now follows from Lemma 7.2; to elaborate,

$$V(o,\xi,t) = V(\xi_t,\xi,0)$$

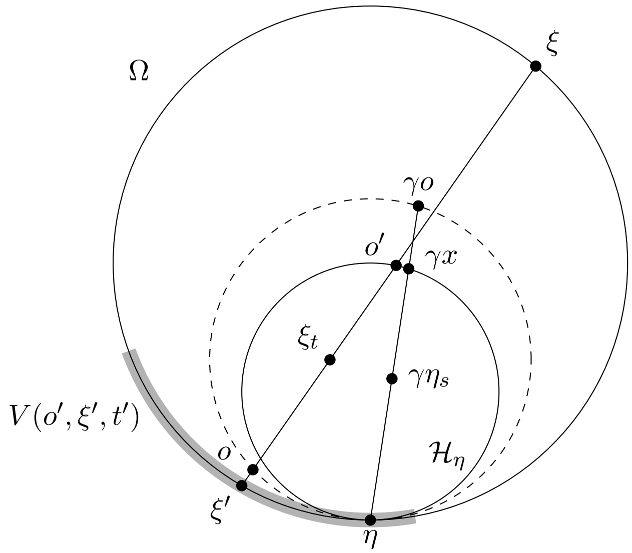

FIGURE 7.1. Case 2 in the proof of Theorem 7.1.

is clear to see from the definition of shadows, so Lemma 7.2 applied with $x = \xi_t$ gives the estimate

$$\mu_{\xi_t}(V(o,\xi,t)) = \mu_{\xi_t}(V(\xi_t,\xi,0)) \asymp 1$$

and the conclusion follows for ξ_t in the thick part, since all such ξ_t are bounded distance from Γo.

The case where ξ is a parabolic fixed point follows directly from Lemma 6.4.

It remains to consider the case where ξ is not a parabolic fixed point, and ξ_t is in the thin part of Ω. Suppose that ξ_t lies in the thin part, and we denote as η the boundary point of the horoball to which ξ_t belongs. We have two cases.

Case 1: If $\eta \in V(o,\xi,t)$, then by Lemma 3.12(1)

$$V(o,\eta,t+O(\alpha)) \subseteq V(o,\xi,t) \subseteq V(o,\eta,t-O(\alpha)).$$

By Lemma 3.6, we have for any $\eta \in \partial\Omega$

$$|\beta_\eta(\eta_t,\xi_t)| \leq d_\Omega(\eta_t,\xi_t) \leq O(\alpha)$$

hence conformal invariance yields

$$C^{-1}\mu_{\eta_t}(V(o,\eta,t+K)) \leq \mu_{\xi_t}(V(o,\xi,t)) \leq C\mu_{\eta_t}(V(o,\eta,t-K))$$

where C, K depend on α, hence the claim follows by Lemma 6.4 and the fact that $d(\eta_t,\Gamma o) = d(\xi(t),\Gamma o) + O(\alpha)$.

Case 2: Suppose that $\eta \notin V(o,\xi,t)$. By taking a small time change of $\pm\epsilon$, it will suffice to assume $\eta \notin \overline{V(o,\xi,t)}$. Let us introduce some notation; see Figure 7.1 for guidance. Let ξ' be the antipodal point to ξ with respect to o. Let o' denote the other intersection point of the geodesic (ξ',ξ) with the horosphere $\mathcal{H}_\eta$ centered at η bounding the thick part. Let $t' = d_\Omega(\xi_t,o')$. Notice that t' is chosen so that

$$V(o,\xi,t) = \partial\Omega \smallsetminus \overline{V(o',\xi',t')}.$$

Hence, by taking a small time change of $\pm\epsilon$, it suffices to estimate $\mu_{\xi_t}(\partial\Omega \smallsetminus V(o',\xi',t'))$. Let η_t be the point on the geodesic ray from o to η which is distance t from o. Let γ be an element of the stabilizer of η such that the geodesic ray from γo to η intersects the same fundamental domain for the action of the stabilizer of η on $\mathcal{H}_\eta$ as o'. Let x be such that γx is the intersection of the geodesic from γo to η with the horosphere $\mathcal{H}_\eta$. In particular, the distance between γx and o' is uniformly bounded, independently of η.

The Case 2 assumption implies η is in $V(o', \xi', t')$, hence by Lemma 3.12(1) we have

$$V(o', \eta, t' + O(\alpha)) \subset V(o', \xi', t') \subset V(o', \eta, t - O(\alpha)).$$

Thus, to estimate $\mu_{\xi_t}(V(o', \xi', t'))$, it suffices to estimate

$$\mu_{\xi_t}(\partial\Omega \smallsetminus V(o', \eta, t')).$$

In order to do so, set $s = t' + d_\Omega(o, x)$. Then η is in $V(o', \xi', t' - \epsilon)$, so by the fellow traveler property of Lemma 4.2 and the triangle inequality, comparablee points on the geodesic rays $[o', \xi')$ and $[\gamma x, \eta)$ are uniformly bounded distance apart up to time $t' - \epsilon + O(\alpha)$. By our choice of parameterizations, the points on these geodesic rays at distance t' from o' and γx are ξ_t and $\xi_{i,s}$, respectively. Hence, $\gamma\xi_{i,s}$ is uniformly close to ξ_t; since also γx is close to o', we have

$$\mu_{\xi_t}(\partial\Omega \smallsetminus V(o', \eta, t')) \asymp \mu_{\gamma\xi_{i,s}}(\partial\Omega \smallsetminus V(\gamma x, \eta, t')) \tag{7.2}$$

and, by shifting perspective along the geodesic, we obtain

$$\asymp \mu_{\gamma\xi_{i,s}}(\partial\Omega \smallsetminus V(\gamma o, \eta, s)) \tag{7.3}$$

hence, since $\gamma\eta = \eta$, and by Γ-equivariance,

$$= \mu_{\xi_{i,s}}(\partial\Omega \smallsetminus V(o, \eta, s)) \tag{7.4}$$

thus we have

$$\asymp e^{(2\delta_\Pi - \delta)d_\Omega(\xi_{i,s}, \Gamma o)} \tag{7.5}$$

by direct application of Lemma 6.4, and finally

$$\asymp e^{(2\delta_\Pi - \delta)d_\Omega(\xi_t, \Gamma o)} \tag{7.6}$$

again because ξ_t is uniformly bounded distance from $\gamma\xi_s$. This yields (7.1), thus completing the proof. □

8. Applications of the shadow lemma

Consider a base point o and a horoball H. Recall that the *radius* of the horoball H is $e^{-d(o,H)}$. If p is a boundary point, then $H_p(r)$ is the unique horoball centered at p with radius r. Let P be the set of all parabolic fixed points in ∂X. Then we fix a Γ-invariant horoball packing where each parabolic point p in the set P determines a unique horoball H_p centered at p in the packing, and we denote the radius of H_p by r_p.

When we consider Hilbert geometries, this fixed horoball packing will be a thick-thin decomposition lifted to the universal cover. Let $\mathcal{H}_p(r)$ be the shadow of the horoball $H_p(r)$.

Recall that a hyperbolic metric space (X, d) is defined to be *Busemann regular* if (X, d) is proper and Busemann functions extend continuously to ∂X and X is uniquely geodesic.

8.1. Disjointness.

Lemma 8.1. *Let (X, d) be a Busemann regular hyperbolic metric space and Γ a group of isometries acting geometrically finitely on X. Let $\xi_1, \xi_2 \in \partial X$ and let H_1, H_2 be horoballs based at ξ_1, ξ_2. Define as q_i the intersection between H_i and $[o, \xi_i)$ for $i = 1, 2$. If $H_1 \cap H_2 = \emptyset$, then*

$$\langle \xi_1, \xi_2 \rangle_o \leq \frac{d(o, q_1) + d(o, q_2)}{2} + O(\alpha).$$

Proof. By symmetry, let us assume that $d(o,q_1) \le d(o,q_2)$. Let z be a point on $[0,\xi_1)$ at distance $\langle \xi_1, \xi_2\rangle_o$. By definition of Gromov product and hyperbolicity, z lies within distance $O(\alpha)$ of $[0,\xi_2)$. If $q_1 \in [z,\xi_1)$, then $d(o,q_2) \ge d_o,q_1) \ge d(o,z)$, hence the claim is trivially true. Suppose $q_1 \in [0,z]$; since H_1 and H_2 are disjoint, then q_2 does not belong to H_1, hence $\beta_{\xi_1}(o,q_2) < \beta_{\xi_1}(o,q_1)$. Hence $d(z,q_2) \ge d(z,q_1) + O(\alpha)$. Then from

$$\begin{aligned} d(q_1,z) &= d(o,z) - d(o,q_1) \\ d(q_2,z) &= d(o,q_2) - d(o,z) + O(\alpha) \end{aligned}$$

we obtain

$$\langle \xi_1,\xi_2\rangle_o = d(o,z) \le \frac{d(o,q_1)+d(o,q_2)}{2} + O(\alpha).$$

□

Corollary 8.2. *There exists a constant $C > 0$ such that, if the horoballs $H_{p_1}(r_1)$ and $H_{p_2}(r_2)$ are disjoint, then*

$$d_{\partial X}(p_1,p_2) \ge C\sqrt{r_1 r_2}.$$

8.2. **Dirichlet Theorem.** We now prove the Dirichlet-type theorem, which does not rely on the shadow lemma.

Theorem 8.3 (Dirichlet-type theorem)**.** *Let (X,d) be a Busemann regular hyperbolic metric space and Γ a group of isometries acting geometrically finitely on X. Then there exist constants c_1, c_2 and c_3 such that for all $s \le c_1$, the set*

$$\bigcup_{\substack{p\in P\\ r_p\ge s}} \mathcal{H}_p(c_2\sqrt{s r_p})$$

covers the limit set with multiplicity at most c_3.

Note that Theorem 8.3 is effectively the same statement as Theorem 1.3.

Proof. First, by compactness of the thick part, note that we can rescale all horoballs in each of the finitely many Γ-orbits of parabolic points by a multiplicative constant c so that the limit set Λ_Γ satisfies

$$\Lambda_\Gamma \subseteq \bigcup_{p\in P} \mathcal{H}_p(c r_p).$$

Let us fix $s > 0$, and let ξ be a point in the limit set. Then by the above equation the line $[o,\xi)$ intersects some horoball $H = H_p(cr_p)$. Let q be the closest point projection of o onto H, so that $cr_p = e^{-d(o,q)}$, and let $w \in H \cap [o,\xi)$ with $s = e^{-d(o,w)}$. By Lemma 3.11, there exists a point z on $[0,\xi)$ with

$$\beta_p(o,z) \ge \frac{d(o,q)+d(o,w)}{2} - O(\alpha)$$

hence also

$$e^{-\beta_p(o,z)} \le e^{-\frac{d(o,q)+d(o,w)}{2}} e^{O(\alpha)} = c_2\sqrt{r_p s}.$$

This shows that z belongs to $H_p(c_2\sqrt{r_p s})$, hence also ξ belongs to $\mathcal{H}_p(c_2\sqrt{r_p s})$.

To prove the second part, note that, since the horoballs are disjoint, we have by Corollary 8.2

$$d_{\partial X}(p_1,p_2) \gtrsim \sqrt{r_1 r_2}.$$

Now, by Lemma 3.13, we have for each $i = 1,2$ that

$$\operatorname{diam} \mathcal{H}_{p_i}(\sqrt{r_i s}) \asymp \sqrt{r_i s}$$

hence, since

$$d_{\partial X}(p_1, p_2) \gtrsim \sqrt{r_1 r_2} \gtrsim \sqrt{sr_1} + \sqrt{sr_2}$$

thus, the shadows $\mathcal{H}_{p_1}(\sqrt{r_1 s})$ and $\mathcal{H}_{p_2}(\sqrt{r_2 s})$ are disjoint. $\square$

Proposition 8.4 (Horoball counting)**.** *Let Γ be a geometrically finite group of isometries of a Busemann regular hyperbolic metric space (X, d) which admits a finite δ-conformal measure that satisfies the global shadow lemma. Let us define*

$$H_n(\lambda) := \{p \in P \ : \ \lambda^{n+1} \leq r_p \leq \lambda^n\}.$$

Then there exist $\lambda < 1$ and constants such that

$$\# H_n(\lambda) \asymp \lambda^{-n\delta}$$

for all $p \in P$ and $n \in \mathbb{N}$.

Proof. It follows exactly as in [SV95, Theorem 3] from the Dirichlet-type Theorem 8.3 and from the fact that there are finitely many orbits of parabolic points in the boundary. $\square$

8.3. **Khinchin functions.** A *Khinchin function* is a positive, increasing function $\varphi : \mathbb{R}^+ \to \mathbb{R}$ such that there exist constants b_1, b_2 for which

$$\varphi(b_1 x) \geq b_2 \varphi(x) \qquad \text{for any } x \in \mathbb{R}^+.$$

Recall that $H_p(r)$ is the unique horoball centered at a boundary point p with radius r.

Note that, as a corollary of the shadow lemma (Theorem 1.1),

$$\mu(\mathcal{H}_p(r_p \varphi(r_p))) \asymp r_p^{\delta} (\varphi(r_p))^{2\delta - k} \tag{8.1}$$

where k is the rank of the parabolic fixed point p.

8.4. **Quasi-independence.** Let $S_n = S(n, \lambda, \varphi)$ be the union of the shadows of $H_p(r_p \varphi(r_p))$ for $\lambda^n \leq r_p \leq \lambda^{n+1}$.

Lemma 8.5 (Quasi-independence)**.** *Let Γ be a geometrically finite group of isometries of a Busemann regular hyperbolic metric space (X, d) which admits a finite δ-conformal measure μ that satisfies the global shadow lemma. There exists a positive constant C such that for all $n, m \in \mathbb{N}$ sufficiently large,*

$$\mu(S_n \cap S_m) \leq C \mu(S_n) \mu(S_m).$$

Proof. Let $H_{p_1}(r_1)$ and $H_{p_2}(r_2)$ be two disjoint horoballs. By Corollary 8.2, we obtain

$$d_{\partial X}(p_1, p_2) \geq C_\alpha \sqrt{r_1 r_2}.$$

where $C_\alpha > 0$ only depends on the hyperbolicity constant. Let $\mathcal{H}_1 = \mathcal{H}_{p_1}(r_1 \varphi(r_1)), \mathcal{H}_2 = \mathcal{H}_{p_2}(r_2 \varphi(r_2))$, with $r_1 > r_2$. Now, suppose $\mathcal{H}_1 \cap \mathcal{H}_2 \neq \emptyset$. Then since φ is increasing,

$$d_{\partial X}(p_1, p_2) \leq C \varphi(r_1) r_1 + C \varphi(r_2) r_2 \leq 2C \varphi(r_1) r_1$$

where C comes from Lemma 3.13, hence

$$C_\alpha \sqrt{r_1 r_2} \leq 2C \varphi(r_1) r_1$$

thus since φ is increasing and r_1 is bounded, φ is bounded by some constant M and

$$\frac{M \varphi(r_1) r_1}{r_2} \geq \frac{\varphi(r_1)^2 r_1}{r_2} \geq \frac{C_\alpha^2}{4C^2}.$$

Hence, if $\xi \in \mathcal{H}_{p_2}(r_2)$, we estimate

$$d_{\partial X}(\xi, p_1) \leq d_{\partial X}(\xi, p_2) + d_{\partial X}(p_2, p_1) \leq C r_2 + 2C \varphi(r_1) r_1 \leq C' r_1 \varphi(r_1)$$

with $C' = \frac{M4C^3}{C_\alpha^2} + 2C$, thus

$$\mathcal{H}_{p_2}(r_2) \subseteq \mathcal{H}_{p_1}(C' r_1 \varphi(r_1)). \tag{8.2}$$

We denote as $S(H)$ the shadow of the horoball H. Given a horoball H of radius r, we denote as φH the horoball with the same boundary point as H and radius $r\varphi(r)$.

Now let $m > n$, and pick an element $H_\star$ of H_n. Let us consider the set

$$I(H_\star) := \{H \in H_m \ : \ S(\varphi H) \cap S(\varphi H_\star) \neq \emptyset\}.$$

By the shadow lemma (Theorem 1.1), for any $H \in H_m$ we have

$$\mu(S(H)) \asymp \lambda^{m\delta}$$

while by the counting lemma (Prop. 8.4)

$$\# H_m \asymp \lambda^{-m\delta}.$$

Hence, if the shadows in H_m are pairwise disjoint,

$$\mu(S_m) \asymp \# H_m \cdot \mu(S(\varphi H)) \asymp \frac{\mu(S(\varphi H))}{\mu(S(H))} \asymp \varphi(\lambda^m)^{2\delta - k} \tag{8.3}$$

Since all shadows in H_m are approximately the same size, we can apply a uniform rescaling so that the shadows are disjoint and by the shadow lemma (Theorem 1.1), Equation 8.3 remains true. Now, note that, if $H \in I(H_\star)$, then by eq. (8.2)

$$S(H) \subseteq S(C' \varphi H_\star)$$

and, by eq. (8.1),

$$\mu(S(C'\varphi H_\star)) \asymp (C')^{2\delta - k} \mu(S(\varphi H_\star)).$$

Moreover, since the elements in H_m are disjoint and have same size, their shadows are also disjoint, so

$$(C')^{k-2\delta} \mu(S(\varphi H_\star)) \asymp \mu(S(C'\varphi H_\star)) \geq \# I(H_\star) \inf_{H \in I(H_\star)} \mu(S(H))$$

hence

$$\begin{aligned}
\mu(S_n \cap S_m) &\leq \sum_{H_\star \in H_n} \sum_{H \in I(H_\star)} \mu(S(\varphi H)) \\
&\leq \sum_{H_\star \in H_n} \# I(H_\star) \sup_H \mu(S(\varphi H)) \\
&\lesssim \sum_{H_\star \in H_n} \frac{\mu(S(\varphi H_\star))}{\inf_H \mu(S(H))} \sup_H \mu(S(\varphi H)) \\
&= \mu(S_n) \frac{\sup_H \mu(S(\varphi H))}{\inf_H \mu(S(H))} \asymp \mu(S_n)\mu(S_m)
\end{aligned}$$

where the last comparison follows the shadow lemma again (Theorem 1.1) with Equation 8.3. This completes the proof. □

8.5. **Khinchin theorem.** Given a Khinchin function φ, we define the set

$$\Omega_\infty^\varphi := \limsup_{n \to \infty} S_n = \bigcap_{n=n_0}^{\infty} \bigcup_{m \geq n} \bigcup_{p \in H_n} \mathcal{H}_p(r_p \varphi(r_p)).$$

Moreover, we have the *Khinchin series*

$$K(\varphi) := \sum_{n=0}^{\infty} (\varphi(\lambda^n))^{2\delta - k_{max}}.$$

Theorem 8.6 (Khinchin-type theorem)**.** *Let Γ be a geometrically finite group of isometries of a Busemann regular hyperbolic metric space (X, d) which admits a finite δ-conformal measure μ that satisfies the global shadow lemma. Let φ be a Khinchin function. Then:*

(1) $\mu(\Omega_\infty^\varphi) = 0$ if $K(\varphi) < \infty$;
(2) $\mu(\Omega_\infty^\varphi) = 1$ if $K(\varphi) = \infty$.

Proof. Note that by Equation 8.1, for any $H \in S_n$,

$$\mu(S_n) \asymp \#H_n \cdot \mu(H) \asymp \lambda^{-n\delta}\lambda^{n\delta}\varphi(\lambda^n)^{2\delta - k_{max}} = \varphi(\lambda^n)^{2\delta-k_{max}}.$$

Now, (1) follows from the standard Borel-Cantelli lemma.

Conversely, (2) follows from the "converse" Borel-Cantelli lemma (e.g. [SV95, Lemma 4.8]), using the quasi-independence from Lemma 8.5. □

8.6. **The logarithm law.** The following result compares to [SV95, Proposition 4.9].

Theorem 8.7 (Logarithm Law)**.** *Let Γ be a geometrically finite group of isometries of a Busemann regular hyperbolic metric space (X, d) which admits a finite δ-conformal measure μ that satisfies the global shadow lemma. For μ-almost every ξ in the limit set Λ_Γ,*

$$\limsup_{t\to+\infty} \frac{d(\xi_t, \Omega_{\text{thick}})}{\log t} = \frac{1}{2\delta - k_{\max}}$$

where $k_{\max}$ is the maximal rank of any parabolic subgroup, and ξ_t is the point on the geodesic ray (o, ξ) that is distance t from o.

Theorem 8.7 immediately implies Theorem 1.4.

Proof. We recall the set-up for the proof provided in Stratmann-Velani [SV95]. For $\epsilon \geq 0$ we define

$$\varphi_\epsilon(x) = (\log x^{-1})^{-\frac{1+\epsilon}{2\delta - k_{\max}}}.$$

Observe that φ_ϵ is a Khinchin function. Then by Theorem 8.6, the limsup set $\Omega_\infty^{\varphi_\epsilon}$ with respect to φ_ϵ is μ-null for all $\epsilon > 0$, and has full measure for $\epsilon = 0$. Fix an arbitrary $\epsilon > 0$ and choose a boundary point ξ in the full measure set $\Omega_{\infty,0} \setminus \Omega_{\infty,\epsilon}$. We adopt the notation $r_{p,\varphi} := r_p\varphi(r_p)$. By definition of the limsup sets, there exists a sequences of parabolic points p_n in P such that the geodesic (o, ξ) passes through horoballs $H_{p_n}(r_{p_n,\varphi_0})$ in order, and passes through no other horoballs; in other words, the radii r_{p_n,φ_0} are monotone decreasing in n, and $(o, \xi) \cap H_p(r_{p,\varphi_0}) \neq \varnothing$ implies $p = p_n$ for some n.

At this point in the proof, our presentation differs from that of Stratmann-Velani.

Since $x \notin \Omega_\infty^{\varphi_\epsilon}$, for all n sufficiently large, (o, ξ) is disjoint from the rescaled horoball $H_{p_n}(r_{p_n,\varphi_\epsilon})$. For each n, choose a sequence $0 < \epsilon_n \leq \epsilon$ so that the geodesic (o, ξ) is tangent to the horoball $H_{p_n}(r_{p_n,\varphi_{\epsilon_n}})$. Denote the point of tangency by ξ_{t_n}, recalling that t_n is the distance from o to ξ_{t_n}. Note that ξ_{t_n} is both the point of maximal penetration of the geodesic (o, ξ) into the larger horoball $H_{p_n}(r_{p_n})$, and the closest point projection of p_n onto the geodesic (o, ξ).

See that $\log r_{p_n}^{-1} \leq t_n$ because $\log r_{p_n}^{-1}$ is the distance from o to the horoball $H_{p_n}(r_{p_n})$, which contains the point ξ_{t_n}. Also, note that by definition of the horoball $H_{p_n}(r_{p_n,\varphi_{\epsilon_n}})$, the distance from o to the horoball $H_{p_n}(r_{p_n,\varphi_{\epsilon_n}})$ is $-\log r_{p_n}\varphi_{\epsilon_n}(r_{p_n})$. Since ξ_{t_n} is on the boundary of $H_{p_n}(r_{p_n,\varphi_{\epsilon_n}})$, for any point a on the boundary of $H_{p_n}(r_{p_n})$, we have $\beta_{p_n}(\xi_{t_n}, a) = -\log\varphi_{\epsilon_n}(r_{p_n})$. Since each r_p is chosen so that the union of all $H_p(r_p)$ is the thin part, there exists a C_1 such that

$$
\begin{aligned}
d(\xi_{t_n}, \Gamma.o) \leq \beta_{p_n}(\xi_{t_n}, a) + C_1 &= -\log \varphi_{\epsilon_n}(r_{p_n}) + C_1 \\
&= \left(\frac{1+\epsilon_n}{2\delta - k_{\max}}\right) \log\left((\log r_{p_n}^{-1})\right) + C_1 \\
&\leq \left(\frac{1+\epsilon_n}{2\delta - k_{\max}}\right) \log(t_n) + C_1. \qquad (8.4)
\end{aligned}
$$

On the other hand, let q be the closest point to o on $[o, p_n) \cap H_{p_n}(r_{p_n})$. Since q lies on the boundary of the horoball,

$$-\log \varphi_{\epsilon_n}(r_{p_n}) = \beta_{p_n(p)}(q, \xi_{t_n}).$$

Since ξ_{t_n} is the closest point projection of $p_n(p)$ onto $[o, \xi)$, Corollary 3.10 gives us that

$$t_n + \log(\varphi_{\epsilon_n}(r_{p_n})) = d(o, \xi_{t_n}) - \beta_{p_n(p)}(q, \xi_{t_n}) + O(\alpha) = d(o, q) + O(\alpha).$$

Thus, by definition of r_{p_n} we obtain

$$t_n + \log(\varphi_{\epsilon_n}(r_{p_n})) \leq \log r_{p_n}^{-1} + C_2 \qquad (8.5)$$

where C_2 is a constant depending only on the hyperbolicity constant. Thus,

$$
\begin{aligned}
d(\xi_{t_n}, \Gamma.o) &\geq -\log(\varphi_{\epsilon_n}(r_{p_n})) - C_1 \\
&= \left(\frac{1+\epsilon_n}{2\delta - k_{\max}}\right) \log(\log r_{p_n}^{-1}) - C_1 \\
&\geq \left(\frac{1+\epsilon_n}{2\delta + k_{\max}}\right) \log(t_n + \log(\varphi_{\epsilon_n}(r_{p_n})) - C_2) - C_1 \\
&= \left(\frac{1+\epsilon_n}{2\delta + k_{\max}}\right) \log\left(t_n - \left(\frac{1+\epsilon_n}{2\delta - k_{\max}}\right) \log(\log(r_{p_n}^{-1})) - C_2\right) - C_1 \\
&\geq \left(\frac{1+\epsilon_n}{2\delta + k_{\max}}\right) \log\left(t_n - \left(\frac{1+\epsilon_n}{2\delta - k_{\max}}\right) \log(t_n) - C_2\right) - C_1.
\end{aligned}
$$

Thus,

$$\frac{1}{2\delta - k_{\max}} \leq \limsup_{t \to +\infty} \frac{d(\xi_t, \Omega_{\text{thick}})}{\log t}.$$

It remains to prove the upper bound on the limsup. For values of t such that $\xi_t \in \Omega_{\text{thick}}$, the result is trivial. Recall that each t_n is chosen so that for all values t so that $\xi_t \in H_{p_n}(r_{p_n})$, the distance $d(\xi_t, \Omega_{\text{thick}})$ is maximized at $t = t_n$. Then for such $t \geq t_n$, $\frac{d(\xi_t, \Omega_{\text{thick}})}{\log(t)} \leq \frac{d(\xi_{t_n}, \Omega_{\text{thick}})}{\log(t_n)}$ as desired. Now consider $t \leq t_n$. Let t' be the time such that $d(\xi_t, \xi_{t_n})$ is maximized over $\xi_t \in H_{p_n}(r_{p_n})$ where $t \leq t_n$; in other words, $\xi_{t'}$ is the point in the boundary of the horoball $H_{p_n}(r_{p_n})$ which is closest to o. Then applying Corollary 3.10, since ξ_{t_n} is the closest point projection of p_n to (o, ξ),

$$
\begin{aligned}
d(\xi_{t_n}, \Omega_{\text{thick}}) = \beta_{p_n}(\xi_{t'}, \xi_{t_n}) &= d(\xi_{t_n}, \xi_{t'}) - O(\alpha) \\
&= |t' - t_n| - O(\alpha) \geq t_n - t - O(\alpha).
\end{aligned}
$$

Thus, $t \geq t_n - d(\xi_{t_n}, \Omega_{\text{thick}}) + O(\alpha)$, and by Equation (8.4),

$$\frac{d(\xi_t, \Omega_{\text{thick}})}{\log(t)} \leq \frac{d(\xi_{t_n}, \Omega_{\text{thick}})}{\log(t_n - d_\Omega(\xi_{t_n}, \Omega_{\text{thick}}) + O(\alpha))} \leq \frac{d(\xi_{t_n}, \Omega_{\text{thick}})}{\log(t_n - C_3 \log(\xi_{t_n}) + O(\alpha))}$$

for some constant $C_3 > 0$. The result follows. □

Appendix A. Proofs of some lemmas

In this section, we include proofs of results that closely resemble Schapira [Sch04] for the readers' convenience.

For the first lemma, we will need the following definition: the *interior triangle* of a given triangle whose vertices are a, b, c in $\Omega \cup \partial\Omega$ is the triangle with vertices x, y, z, with $x \in (a, c), y \in (a, b), z \in (b, c)$ such that $\beta_a(x, y) = \beta_b(y, z) = \beta_c(z, x) = 0$.

Proof of Lemma 4.3. Following the set-up of Schapira; for $\eta \in K$, let y be the point on (ξ, η) which is on the same horosphere at ξ as o. Then y is bounded distance from o for all $\eta \in K$ by compactness of K; let C be an upper bound on $d_\Omega(o, y)$.

Consider the geodesic triangle with endpoints ξ, o, and $g\eta \in gK$. This triangle has a unique interior triangle with vertices $a \in [o, \xi), b \in [o, g\eta)$, and $c \in (\xi, g\eta)$ such that $\beta_o(a, b) = \beta_\xi(a, c) = \beta_{g\eta}(b, c) = 0$.

We will first compare $d_\Omega(o, go)$ with $2d_\Omega(o, a)$ (notice that a of course depends on g). Then we will estimate $2d_\Omega(o, a)$ to prove part (a) of the Lemma, before we complete the proof of part (b).

Note that since c and gy are both on the geodesic $(\xi, g\eta)$ and g preserves horospheres centered at ξ,

$$d_\Omega(gy, c) = |\beta_\xi(gy, c)| = |\beta_\xi(gy, a)| = |\beta_\xi(o, a)| = d_\Omega(o, a). \tag{A.1}$$

By hyperbolicity of the space, there is a uniform constant α which bounds the diameter of this interior triangle. Then by the triangle inequality and (A.1),

$$d_\Omega(o, gy) \leq d_\Omega(o, a) + d_\Omega(a, c) + d_\Omega(c, gy) \leq 2d_\Omega(o, a) + \alpha.$$

Moreover, let q be the comparable point on the line $\overline{g\eta\, o}$ to gy on the line $\overline{g\eta\, \xi}$; this means q is the point in the same horosphere centered at $g\eta$ as the point gy. In particular,

$$d_\Omega(q, b) = |\beta_{g\eta}(q, b)| = |\beta_{g\eta}(gy, c)| = d_\Omega(gy, c) = d_\Omega(o, a) = d_\Omega(o, b)$$

hence $d_\Omega(o, q) = 2d_\Omega(o, a)$, and since q is comparable to gy, their distance is bounded above by α by Lemma 3.5. Then we obtain a lower bound

$$2d_\Omega(o, a) - \alpha \leq d_\Omega(o, q) - d_\Omega(gy, q) \leq d_\Omega(o, gy).$$

Then by the triangle inequality, the fact that g is an isometry, and the upper bound on $d_\Omega(o, y)$,

$$2d_\Omega(o, a) - C - \alpha \leq d_\Omega(o, go) \leq 2d_\Omega(o, a) + C + \alpha. \tag{A.2}$$

In a hyperbolic space with hyperbolicity constant α, the projection of $g\eta$ onto the geodesic ray $[o, \xi)$ is distance at most α from the vertex a of the interior triangle lying on the ray $[o, \xi)$. Let r be this projection, so $d_\Omega(o, r) - \alpha \leq d_\Omega(o, a) \leq d_\Omega(o, r) + \alpha$.

Now, equation (A.2) implies that if $d_\Omega(o, go) \geq 2t$, then

$$2t \leq 2d_\Omega(o, a) + C + \alpha,$$

and hence

$$d_\Omega(o, r) \geq d_\Omega(o, a) - \alpha \geq t - \frac{C}{2} - \frac{3\alpha}{2}.$$

Thus, by Definition 3.1, $g\eta$ lies in the shadow of depth $t - A$ with $A = C/2 + 3\alpha/2$.

On the other hand, if $d_\Omega(o, go) \leq 2t$, then $2d_\Omega(o, a) - C - \alpha \leq 2t$, and $d_\Omega(o, r) \leq d_\Omega(o, a) + \alpha \leq t + A$. Then again by Definition 3.1, $g\eta$ is not in any shadow of depth larger than $t + A$.

Let us now prove the bound on the Busemann functions. By the cocycle property,

$$\beta_{g\eta}(o, go) = \beta_{g\eta}(o, \xi_t) + \beta_{g\eta}(\xi_t, g\xi_t) + \beta_{g\eta}(g\xi_t, go).$$

Now, since $g\eta \in V(o, \xi, t - A)$, we have

$$\beta_{g\eta}(o, \xi_t) = t + O(\alpha).$$

Moreover, since the group acts by isometries,

$$\beta_{g\eta}(g\xi_t, go) = \beta_\eta(\xi_t, o).$$

Further, by compactness we can choose a constant D such that K is disjoint from $V(o, \xi, D)$. Hence, $\eta \in K$ implies

$$t - D \leq \beta_\eta(\xi_t, o) \leq t.$$

Finally, since q lies on $[o, g\eta)$,

$$\beta_{g\eta}(o, q) = d(o, q)$$

and, as discussed before,

$$d(q, go) \leq d(q, gy) + d(gy, go) \leq \alpha + C$$

hence

$$|\beta_{g\eta}(o, go) - d(o, go)| \leq 2\alpha + 2C$$

which yields

$$|\beta_{g\eta}(\xi_t, g\xi_t) - d(o, go) + 2t| \leq B$$

for a suitable choice of B, as required. ∎

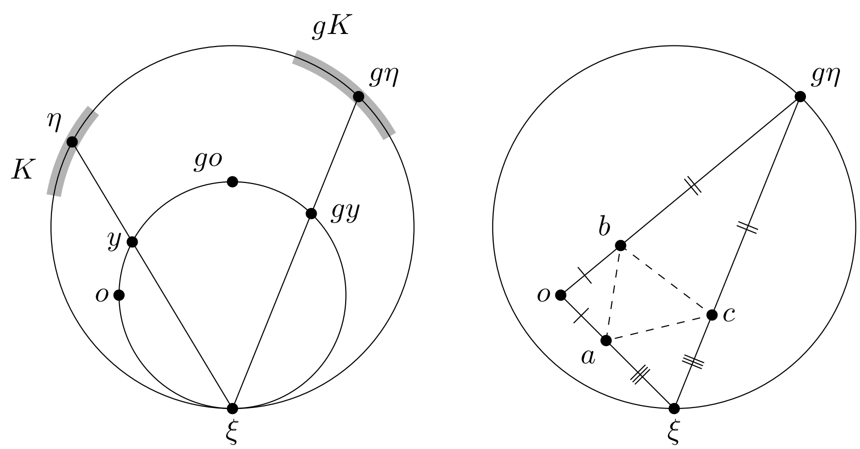

Figure A.1. The setup of Lemma 4.3.

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