A 0-1 LAW AND CUSP EXCURSION FOR GEOMETRICALLY FINITE ACTIONS ON COARSELY HYPERBOLIC METRIC SPACES

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Abstract. Based on joint work with Giulio Tiozzo.

Contents

1. Introduction

Given a function $\psi \colon \mathbb{N} \to \mathbb{R}+$, define

$$\Theta(\psi) = \{x \in [0,1] : |x - \frac{p}{q}| < \frac{\psi(q)}{q} \text{ for infinitely many reduced rationals } \frac{p}{q}\}$$

Theorem 1.1 (Khinchin, 1926)**.** *Let $\psi \colon \mathbb{N} \to \mathbb{R}^+$ be monotone decreasing.* **(1) Harry:** *is this even needed? any other hypotheses? Then*

$$\sum_{q \in \mathbb{R}} \psi(q) = \infty \quad \textit{then} \quad \Theta(\psi) \textit{ has measure 1}$$

and

$$\sum_{q \in \mathbb{R}} \psi(q) = 0 \quad \textit{then} \quad \Theta(\psi) \textit{ has measure zero.}$$

Thus, Theorem 1.1 is a strong 0-1 law for the interval. As an application, we have the following classical example:

Example 1.2. Let $\psi_\epsilon(q) = q^{-(1+\epsilon)}$. Then

$$\Theta(\psi_\epsilon) = \{x \in \mathbb{R} : |x - \frac{p}{q}| < \frac{1}{q^{2+\epsilon}} \text{ for infinitely many } \frac{p}{q} \in \mathbb{Q}\}.$$

Let us illustrate the limsup set $\Theta(\psi_0)$.

See that

$$\sum_{q \in \mathbb{N}} \psi_\epsilon(q) = \sum_{q \in \mathbb{N}} \frac{1}{q^{1+\epsilon}} \begin{cases} = \infty & \text{for } \epsilon = 0 \\ < \infty & \text{for } \epsilon > 0 \end{cases}$$

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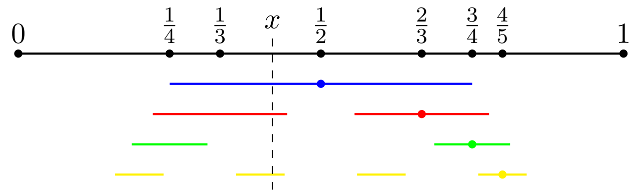

FIGURE 1. An illustration of the first levels of the limsup set $\Theta(\psi_0)$. The intervals around $\frac{1}{2}$ have radius $\frac{1}{4}$, around $\frac{2}{3}$ have radius $\frac{1}{9}$, and so on.

hence Khinchin's theorem implies

$$\Theta(\psi_\epsilon) \text{ has measure } \begin{cases} \text{one for } \epsilon = 0 \\ \text{zero for } \epsilon > 0. \end{cases}$$

Thus in Figure 1, x is in infinitely many balls of radius $\frac{1}{q^2}$ about $\frac{p}{q}$ with probability 1.

We will now discuss an analogy of Khinchin's Theorem (1.1) for the hyperbolic plane, due originally to Sullivan [**?**].

1.1. **Horoball packings for the hyperbolic plane.** The results of Sullivan generalize but we present them for surfaces, or really for a particular surface, to communicate the concept. A statement in full generality appears **(2) Harry:** ref and is the goal of these notes.

Recall a *Fuchsian group* is a discrete subgroup Γ of $\mathsf{PSL}(2, \mathbb{R})$, which acts on the upper half-plane model of hyperbolic 2-space $\mathbb{H}^2$ by isometries via Möbius transformations. Let $\Sigma_{g,n}$ denote a surfaces of genus g with n punctures. Consider a representation $\Gamma < \mathsf{PSL}(2, \mathbb{R})$ of $\pi_1(\Sigma_{0,1})$ which acts cofinitely on $\mathbb{H}^2$; that is, the quotient $\mathbb{H}^2/\Gamma$ has finite area.

[**?**]