A 0-1 LAW AND CUSP EXCURSION FOR GEOMETRICALLY FINITE ACTIONS ON COARSELY HYPERBOLIC METRIC SPACES

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ABSTRACT. Based on joint work with Giulio Tiozzo.

CONTENTS

1. INTRODUCTION

1.1. **History.** Given a function $\psi\colon \mathbb{N} \to \mathbb{R}+$, define

$$\Theta(\psi) = \{x \in [0,1] : |x - \frac{p}{q}| < \frac{\psi(q)}{q} \text{ for infinitely many reduced rationals } \frac{p}{q}\}$$

Note that if $\psi \equiv 1$ then $\Theta(\psi) = [0,1]$. A century ago, Khinchin proved the following celebrated theorem:

Theorem 1.1 (Khinchin, 1926)**.** *Let $\psi\colon \mathbb{N} \to \mathbb{R}^+$ be monotone decreasing.* **(1) Harry:** *is this even needed? any other hypotheses? Then*

$$\sum_{q \in \mathbb{N}} \psi(q) = \infty \quad \textit{then} \quad \Theta(\psi) \textit{ has measure 1}$$

and

$$\sum_{q \in \mathbb{N}} \psi(q) = 0 \quad \textit{then} \quad \Theta(\psi) \textit{ has measure zero.}$$

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Thus, Theorem 1.1 is a strong 0-1 law for the interval. As an application, we have the following classical example:

Example 1.2. Let $\psi_\epsilon(q) = q^{-(1+\epsilon)}$. Then

$$\Theta(\psi_\epsilon) = \{x \in \mathbb{R} : |x - \frac{p}{q}| < \frac{1}{q^{2+\epsilon}} \text{ for infinitely many } \frac{p}{q} \in \mathbb{Q}\}.$$

Let us illustrate the limsup set $\Theta(\psi_0)$.

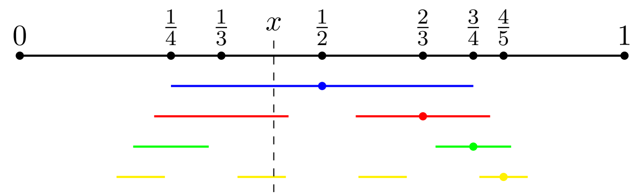

FIGURE 1. An illustration of the first levels of the limsup set $\Theta(\psi_0)$. The intervals around $\frac{1}{2}$ have radius $\frac{1}{4}$, around $\frac{2}{3}$ have radius $\frac{1}{9}$, and so on.

See that

$$\sum_{q\in\mathbb{N}} \psi_\epsilon(q) = \sum_{q\in\mathbb{N}} \frac{1}{q^{1+\epsilon}} \begin{cases} = \infty & \text{for } \epsilon = 0 \\ < \infty & \text{for } \epsilon > 0 \end{cases}$$

hence Khinchin's theorem implies

$$\Theta(\psi_\epsilon) \text{ has measure } \begin{cases} \text{one for } \epsilon = 0 \\ \text{zero for } \epsilon > 0. \end{cases}$$

Thus in Figure 1, x is in infinitely many balls of radius $\frac{1}{q^2}$ about $\frac{p}{q}$ with probability 1.

We will now discuss an analogy of Khinchin's Theorem (1.1) for the hyperbolic plane, due originally to Sullivan [**?**].

1.2. **Horoball packings at infinity for hyperbolic 2-space.** The results of Sullivan generalize but we present them for surfaces, or really for a particular surface, to communicate the concept. A statement in full generality appears **(2) Harry:** ref and is the goal of these notes.

Let $\mathbb{D}$ denote the disk model of hyperbolic 2-space. Recall a *Fuchsian group* is a discrete subgroup Γ of $\mathsf{Isom}(\mathbb{D})$ acting by Möbius transformations. Let $\Sigma_{g,n}$ denote a surfaces of genus g with n punctures. Consider a representation $\Gamma < \mathsf{Isom}(\mathbb{D})$ of $\pi_1(\Sigma_{0,1})$ which acts cofinitely on $\mathbb{D}$; that is, the quotient $\mathbb{D}/\Gamma$ has finite area. In particular, let Γ be a representation for which the ideal quadrilateral with vertices $1, -i, 1, i \in \partial\mathbb{D} = S^1$ is a fundamental domain for the Γ-action.

We now introduce a concept of horoball packing at infinity. We will say a *horoball centered at* $p \in S^1$ is a horoball in $\mathbb{D}$ (which is a Euclidean ball in this model) tangent to S^1 at p. Let $\mathcal{P} = \Gamma.\{\pm 1, \pm i\}$. These *parabolic* *points* will be in analogy with radional points. For $p \in \{\pm 1, \pm i\}$, choose a collection of (sufficiently small) horoballs H_p centered at p. Then for $gp \in \mathcal{P}$, let $H_{gp} = gH_p$. Thus we have a family $\{H_p\}_{p \in \mathcal{P}}$ of horoballs, with each H_p centered at p. Let r_p denote the Euclidean radius of H_p in the disk model $\mathcal{D}$. Now, to allow for rescaling, we denote by $H_p(r)$ the horoball centered at p of Euclidean radius r. Thus, r_p is chosen so that $H_p = H_p(r_p)$. Finally, we define the *horoball shadow* $\mathcal{H}_p(r)$ *centered at* p *with radius* r to be the set of a endpoints ξ of geodesic rays $[0, \xi)$ such that $[0, \xi)$ intersects the horoball $H_p(r)$ in $\mathbb{D}$. See that $\mathcal{H}_p(r)$ is an interval in S^1 centered at p with radius approximately r.

1.3. **0-1 law and cusp excursion in hyperbolic 2-space.** Now, for $\phi \colon \mathbb{R}^+ \to \mathbb{R}^+$ **(3) Harry:** increasing and? we define the limsup set

$$\Theta(\phi) = \{x \in S^1 : x \in \mathcal{H}_p(r_p \phi(r_p)) \text{ for infinitely many } p \in \mathcal{P}\}.$$

When $\phi \equiv 1$, $\Theta(1)$ is the shadows of all the horoballs from the original packing, without any rescaling. These horoballs all project to the same horoball neighborhood of the cusp in the punctured torus quotient. It is a fact due to ergodicity of the geodesic flow of $\Sigma_{0,1}$ that in this case, Lebesgue-almost every geodesic visits each horoball infinitely often. Hence, $\Theta(1)$ has Lebesgue measure 1. We should think of r_p as analogous to the standard radius $\frac{1}{q}$ of $\frac{p}{q}$ for the corresponding limsup set for the real line.

The following 0-1 for hyperbolic 2-space is due to Sullivan:

Theorem 1.3 (Khinchin)**.** *For sufficiently small $\lambda \in (0,1)$, Then*

$$\sum_{n \in \mathbb{N}} \phi(q) = \infty \;\; \textit{then} \;\; \Theta(\phi) \textit{ has Lebesgue measure 1}$$

and

$$\sum_{n \in \mathbb{N}} \phi(q) = 0 \;\; \textit{then} \;\; \Theta(\phi) \textit{ has Lebesgue measure zero.}$$

This theorem provides an application to cusp excursion called the *logarithm law.* First, define the *non-cuspidal* part of $\mathbb{D}$ to be

$$NC = \mathbb{D} \smallsetminus \cup_{p \in \mathcal{P}} H_p.$$

Then for a point $x \in \mathbb{D}$, define the $\mathrm{CuspDepth}(x)$ to be 0 if $x \in NC$, and

$$\mathrm{CuspDepth}(x) = d(x, \partial H_p) \;\; \text{where } x \in H_p.$$

Given a point $\xi \in S^1$, denote by ξ_t the point on the geodesic ray $[0, \xi)$ which is distance t from 0.

Theorem 1.4. *For Lebesgue almost every $\xi \in S^1$,*

$$\limsup_{t \to +\infty} \frac{\mathrm{CuspDepth}(\xi_t)}{\log t} = 1.$$

1.4. **Overview of the notes.** With Tiozzo, we generalize Theorems 1.3 and 1.4 to the setting of certain geometrically finite actions on coarsely hyperbolic metric spaces. In these notes, I present the proof strategy for these results at that level of generality. I introduce the setting with fundamental intuition and examples. Along the way I review some of the foundational Patterson–Sullivan theory needed for these results.

2. Hyperbolic metric spaces

For a more complete treatment of hyperbolic metric spaces, **(4) Harry:** add refs

First for any set U and functions $f, g\colon U \to \mathbb{R}$, we write

- $f \approx g$ if there exists a constant C such that $f - C \le g \le f + C$ on U, and
- $f \asymp g$ if there exists a constant K such that $\frac{1}{K} f \le g \le K f$.

Recall that a metric space (X, d) is *(Gromov) hyperbolic* or *δ-hyperbolic* if there exists a $\delta > 0$ such that all geodesic triangles in X are δ-thin: more specifically, for any geodesic triangle Δ with vertices $x, y, z \in X$, any edge $[x, y]$ of Δ is contained in the δ-neighborhood of the two other edges $[y, z] \cup [x, z]$ of Δ. For example, trees are 0-hyperbolic, and $\mathbb{H}^2$ is $\log 2$-hyperbolic. One should think of a hyperbolic metric space as coarsely a tree.

As an alternate definition, (X, d) is Gromov-hyperbolic if and only if for every geodesic triangle Δ with vertices $x, y, z \in X$, there exist three points $a \in [x, y], b \in [y, z]$, and $c \in [x, z]$ such that $\operatorname{diam}(\{a, b, c\}) \le \delta$. A geodesic triangle with vertices a, b, c is said to be an *inner triangle* of Δ. Note that for a tree, we have $a = b = c$.

Given a triple $x, y, z \in X$, *the Gromov product* is

$$\langle x, y\rangle_z = \frac{1}{2}(d(x, z) + d(y, z) - d(x, z)).$$

Notice that for a tree, $\langle x, y\rangle_z = d(z, [x, y])$. More generally, the Gromov product measures the defect from colinearity of the triple x, y, z, up to additive error:

Fact 2.1. For a δ-hyperbolic metric space (X, d),

$$\langle x, y\rangle_z \approx d(z, [x, y])$$

with constants depending only on δ.

Fixing a perspective point $o \in X$, the *visual boundary* of X is denoted by ∂X, and is the set of geodesic rays based at o up to bounded equivalence. Define the *shadow* $\mathcal{O}_r(x, y)$ of $y \in X$ of radius r by

$$\mathcal{O}_r(x, y) = \{\xi \in \partial X \mid [x, \xi) \cap B_r(y) \ne \varnothing \text{ for some geodesic representation } [x, \xi) \text{ of } \xi\}.$$

The notation $\mathcal{O}$ stands for ombre, which is French for shadow. The visual boundary is a compactification of X when endowed with the shadow topology. Let $\bar{X} = X \cup \partial X$.

2.1. **Busemann functions and horoballs.** The notion of horoball in a hyperbolic metric space is defined using Busemann functions. This generalizes the setting of hyperbolic space and shares many essential properties.

For $x, y, z \in X$, define the *Busemann function* centered at z by

$$\beta_z(x, y) = d(x, z) - d(y, z).$$

This expression measures the signed distance between spheres centered at z passing through x and y. Let $\xi \in \partial X$. Then

$$\beta_\xi(x, y) = \liminf_{z \to \xi} \beta_z(x, y).$$

In hyperbolic space, β_ξ is in fact a limit. In general it is not. For example, one could take the infinite graph given by a ladder with infinitely many rungs and the standard path metric. This space is hyperbolic (it is quasi-isometric to a tree), but the Busemann function is not a limit. It is nonetheless well-defined up to choice of path:

Fact 2.2. In a δ-hyperbolic metric space, for any two sequences $z_, w_n \to \xi \in \partial X$,

$$\beta_{z_n}(x, y) \approx \beta_{w_n}(x, y)$$

with constants depending only on δ.

Some straightforward but useful properties of Busemann functions include:

For all $x, y, w \in X$, $z \in \bar{X}$,

1. (Γ-invariant) $\beta_{\gamma z}(x, y) = \beta_z(x, y)$ for all
2. (asymmetric) $\beta_z(x, y) \approx -\beta_z(y, x)$.
3. (1-Lipshitz) $|\beta_z(x, y)| \leq d(x, y)$.
4. (cocycle property) $\beta_z(x, y) \approx \beta_z(x, w) + \beta_z(w, y)$.

Notice that the 1-Lipshitz property implies finiteness of β_ξ for $\xi \in \partial X$.

Definition 2.3. Fixing $o \in X$, the *horoball of radius r centered at ξ* is

$$H_\xi(r) = \{x \in X \mid \beta_\xi(x, o) \leq \log r\}.$$

A *horosphere of radius r centered at ξ* $\{x \in X \mid \beta_\xi(x, o) = \log r\}$.

We think of horospheres as a limit of metric spheres S_n which contain a fixed point x, with each sphere S_n centered at z_n which are going to ∂X with n. It is straightforward to verify that in $\mathbb{R}^2$, horospheres are lines, and horoballs are halfspaces. In $\mathbb{D}$, horospheres are Euclidean circles tangent to the boundary S^1, and horoballs are the disks contained in these circles. In the upper-half space model $\mathbb{H}^2$ of hyperbolic 2-space, horospheres centered at the vertical point at infinity are horizontal lines, and all other horospheres are Euclidean circles tangent to the real line. In particular, a straightforward exercise is to confirm in $\mathbb{H}^2$ that for $o = i$ and $\xi = 0$, we have $H_\xi(2r)$ is a Euclidean ball of radius r about ir. This motivates the choice of $\log r$ in the definition of the horoball.

Definition 2.4. The *shadow of a horoball of radius* r centered at ξ is

$$\mathcal{H}_\xi(r) := \{\eta \in \partial X \mid [o,\eta) \cap H_\xi(r) \neq \varnothing \text{ for some geodesic representative of } \eta\}.$$

2.2. **Geometrical finiteness.** Fix a group Γ of isometries of a hyperbolic metric space (X,d) acting properly discontinuously. Assume (X,d) is proper. The *limit set* of Γ is the set of accumulation points of a Γ-orbit:

$$\overline{\Gamma.o}/\Gamma.o$$

for any fixed $o \in X$. By proper discontinuity, $\Lambda_\Gamma \subset \partial X$. Since X is hyperbolic, Λ_Γ does not depend on the basepoint **(5) Harry:** Coornaert Théorème 5.1. The group Γ is *non-elementary* if $|\Lambda_\Gamma| \geq 3$. Equivalently, Γ does not fix any point. **(6) Harry:** add refs It follows that Λ_Γ is uncountable. When Γ is non-elementary, Λ_Γ is the smallest closed Γ-invariant subset of ∂X. Let C_Γ denote the *convex hull* of Λ_Γ; this is the intersection with X of the smallest convex set in $\bar{X}$ containing Λ_Γ.

Isometries of X are classified by their translation distance. The *translation distance* of $\gamma \in \mathsf{Isom}(X)$ is $\tau(\gamma) = \inf_{o\in X} d(o,\gamma o)$. Then γ is *elliptic* if $\tau(\gamma) = 0$ is realized in X, *parabolic* if $\tau(\gamma) = 0$ is not realized, and *hyperbolic* or *loxodromic* if $\tau(\gamma) > 0$, which implies in this setting that it is realized in X. If γ is parabolic then it is infinite order with a single fixed point in ∂X, and if γ is loxodromic then γ is infinite order with each two fixed points in ∂X, one attracting and one repelling. **(7) Harry:** Bowditch Lemma 2.1 A subgroup Π of $\mathsf{Isom}(X)$ is a *parabolic group* if Π has infinite order, fixes a single point in ∂X, and contains no loxodromic elements.

A point $x \in \Lambda_\Gamma$ is *conical* if for any geodesic ray ξ in the class of x, there exists a constant D and a sequence $x_n \in \Gamma.o$ converging to x such that $\{x_n\}_{n\in\mathbb{N}}$ is contained in a D neighborhood of ξ. Equivalently, x is conical in the sense of a convergence group action; in particular, there exists a sequence $\gamma_n \in \Gamma$ and $a \neq b \in \Lambda_\Gamma$ such that $\gamma_n x \to a$ and $\gamma_n y \to b$ for all $y \in \Lambda_\Gamma \smallsetminus \{x\}$.

Exercise 2.5.
1. If $\gamma \in \mathsf{Isom}(\mathbb{H}^2)$ is loxodromic, then the attracting and repelling fixed points of γ are conical limit points for any subgroup of isometries containing γ.
2. Verify that every fixed point of a parabolic isometry is in Λ_Γ.
3. Assume g is loxodromic, h is parabolic and $g,h \in \Gamma$. If Γ is properly discontinuous then g and h cannot fix a common point in Λ_Γ.

Then a point $x \in \Lambda_\Gamma$ is *parabolic* if its stabilizer $\mathrm{stab}_\Gamma(x)$ is a parabolic subgroup, and is moreover *bounded parabolic* if Γ acts on $\Lambda_\Gamma \smallsetminus \{x\}$ cocompactly.

Example 2.6. A first example for a parabolic subgroup is $\langle z \mapsto z+1\rangle$ acting on the upper-half plane, which fixes ∞. Every parabolic isometry in this model is conjugate to a horizontal translation, which is upper triangular as a matrix. Every loxodromic isometry is conjugate to a scaling $z \mapsto \lambda z$ for some $\lambda > 0$, which is diagonal as a matrix.

In $\mathbb{H}^3$, **(8) Harry:** discuss a $\mathbb{Z}^2$ subgroup. Remark that parabolic subgroups in $\mathbb{H}^n$ are virtually abelian with rank $\leq n-1$.

Then Γ is *geometrically finite* if every point of Λ_Γ is either conical or bounded parabolic, and Γ is more specifically *convex cocompact* if every point in Λ_Γ is conical. Classically, for Kleinian groups $\Gamma < \mathbb{H}^3$, Beardon–Maskit proved that Γ is geometrically finite if and only if Γ admits a polygonal fundamental domain with finitely many sides. For higher dimensions this is false, but the framework of geometrical finiteness is fairly robust.

We discuss some examples below, but first here are some crucial facts that will be enlightened via examples.

Fact 2.7 (Tukia, Yaman, Bowditch). finitely many cusps
horoball packing
cocompact action on noncuspidal part

Here are some examples. First, every finite covolume action on $\mathbb{H}^n$ is geometrically finite. A good example to have in mind is the punctured torus, with genus 1 and 1 puncture, denoted $\Sigma_{1,1}$. This surface admits a hyperbolic metric of finite area by choosing opposite edge identifications for an ideal quadtrilateral, for instance as pictured in **(9) Harry:** figure. The corresponding representation of $\pi_1(S)$ is computable; the generators are loxodromic with axes intersecting the identifying edges orthogonally. The one can verify that the commutators of this representation are parabolic, and in particular there is one commutator which is equal to a horizontal translation. In $\mathbb{H}^2$, when Γ has parabolics, by definition this introduces a cuspidal region to the quotient, which lifts to a Γ-invariant collection of horoballs in $\mathbb{H}^2$, each preserved by a parabolic subgroup. Since the property of being a parabolic fixed point is Γ-invariant, by computating the representation we confirm that the rationals are parabolic, hence $\Lambda_\Gamma = \partial H^2 = S^1$. Now it is straightfoward to verify that the parabolic points are bounded parabolic. The fact that all other points in S^1 are conical follows from **(10) Harry:** the fact

Exercise 2.8. Verify that when Γ is cocompact, $\Lambda_\Gamma = \partial X$ and hence $C_\Gamma = X$. By the translation distance definition, Γ has no parabolic isometries.

It follows from the previous exercise that cocompact actions are the first examples of convex cocompact actions, and that moreover Γ is convex cocompact if and only if the Cayley graph of Γ is quasi-isometrically embedded in X via the orbit map $\gamma \mapsto \gamma.o$. We can think of geometrical finiteness as a natural extension of convex cocompactness where orbits do not quasi-isometrically embed but the action retains some finiteness properties.

Exercise 2.9. Orbits of the parabolic subgroup $G = \langle z \mapsto z+1 \rangle$ acting on $\mathbb{H}^2$ grow logarithmically in n, and hence the orbit map does not quasi-isometrically embed.

2.3. **Growth conditions.** To state the results, we will need to assume some growth conditions on the parabolic subgroups. For a parabolic subgroup $\Pi < \Gamma$, define the growth function by

$$B_\Pi(t) = \#\{g \in \Pi \mid d(o, go) \leq t\}.$$

Then Π has *mixed exponential growth* if there exist a_Π, δ_Π such that

$$B_\Pi(t) \asymp e^{\delta_\Pi t}(t+1)^{a_\Pi}.$$

Exercise 2.10. If Π is a parabolic subgroup acting on $\mathbb{H}^2$ then $\delta_\Pi = \frac{1}{2}$ and $a_\Pi = 0$. More generally, if Π acts on $\mathbb{H}^n$ then $\delta_\Pi = \frac{r}{n}$ where r is the abelian rank of Π.

Example 2.11 (Groves–Manning cusp space)**.** Here is a another example of a geometrically finite group action which allows for many possible growth functions.

Though our theorems are more general, we will present the notes under these growth assumptions for ease of readability, and refer the reader to **(11) Harry:** b-t for details in greater generality.

(12) Harry: resume on page 8 with the facts of Tukia, Yaman, Bowditch