Project 10: Eigenvalue Problem

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1 Introduction

Generally, we can solve PDEs and ODEs through assuming the form of solutions and substitutions. However, such method is time intensive and difficult to code. Therefore, we can approach the problem with linear algebra through eigenvalues and eigenvectors. With this approach, we simplify the problem by decomposing them to lower order system and performing simple linear algebra; furthermore, this approach is better for programming.

2 Methods

2.1 Buckling of A Strut

A strut will buckle when the applied load surpasses the strut critical load. Hence, we want to compute the critical load to assess structural failure. We have a governing equation for buckling in term of second order ordinary differential equation.

$$EI\frac{d^2u}{dy^2} + Pu = 0$$
$$\frac{d^2u}{dy^2} + \frac{P}{EI}u = 0$$

Let us solves the equation analytically. First we will write the characteristic equation.

$$r^2 + \frac{P}{EI} = 0$$
$$\implies r = \pm\frac{\sqrt{\frac{-4P}{EI}}}{2}$$
$$= \pm i\frac{\sqrt{4\alpha}}{2}, \text{ where } \alpha = \frac{P}{EI}$$
$$\implies r = \pm i\sqrt{\alpha}$$

Assume the solution is in the form of $u = e^{rt}$:

$$u = c_1 e^{iy\sqrt{\alpha}} + c_2 e^{-iy\sqrt{\alpha}}$$

$$= c_1(cos(y\sqrt{\alpha} + isin(y\sqrt{\alpha}) + c_2(cos(y\sqrt{\alpha} - isin(y\sqrt{\alpha})$$

$$= (c_1 + c_2)cos(y\sqrt{\alpha}) + i(c_1 - c_1)sin(y\sqrt{\alpha}$$

$$\implies u = \bar{c_1}cos(y\sqrt{\alpha}) + \bar{c_2}sin(y\sqrt{\alpha})$$

Boundary Conditions:

$$u(0) = 0 \implies \bar{c_1} = 0 \implies u = \bar{c_2}sin(y\sqrt{\alpha})$$

$$u(2) = 0 \implies sin(2\sqrt{\alpha}) = 0 \implies 2\sqrt{\alpha} = \pi$$

$$\implies \alpha = \tfrac{P}{EI} = \tfrac{\pi^2}{4}$$

$$\implies P = \tfrac{EI\pi^2}{4} = 2.4674$$

Let us compute the eigenvalue of the ODE, where $\alpha = \frac{-P}{EI}$.

$$\begin{bmatrix} 0 & 1 \\ \alpha & 0 \end{bmatrix} \vec{u} = \vec{u}'$$

$$\det(A - \lambda I) = 0 \implies \begin{vmatrix} -\lambda & 1 \\ \alpha & -\lambda \end{vmatrix} = 0$$

$$\implies |A - \lambda I| = \lambda^2 - \alpha = 0$$

$$\implies \lambda = \alpha = \frac{-P}{EI} = -2.4674$$

Now, let us solve the same equation numerically. The grid:

$$y = [0, L] = [0, 2]$$

$$\Delta y = \frac{L}{n} = \frac{2}{n}, \text{ where n is number of elements}$$

The scheme:

$$\frac{d^2u}{dy^2} + \frac{P}{EI}u = 0$$

$$\frac{(u_{i+1} - 2u_i + ui - 1)}{\Delta y^2} = \frac{-P}{EI}u$$

The solver: We can break the left side into a tridigonal matrix multiplying $\vec{u}$. Since the left side is a matrix multiplying an array of the interest variable, and the right side is a scalar multiplying the same array of the interest variable, we have an eigenvalue function. The tridiagonal matrix composes of a lower diagonal with element $\frac{1}{\Delta y^2}$; a main diagonal element with $\frac{-2}{\Delta y^2}$; an upper diagonal with element $\frac{1}{\Delta y^2}$. For the largest eigenvalue, we will perform basic matrix multiplication until the residual value of the eigenvalue is within a chosen tolerance.

$$A\vec{x_0} = \bar{\vec{x_1}}, \text{ where } \bar{\vec{x_1}} = \lambda_1 \vec{x_1}$$

$$A\vec{x_1} = \bar{\vec{x_3}}, \text{ where } \bar{\vec{x_3}} = \lambda_1 \vec{x_3}$$

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$$A\vec{x_{n-1}} = \bar{\vec{x}}, \text{ where } \bar{\vec{x}} = \lambda_1 \vec{x}$$

We can decompose any $\bar{\vec{x}}$ into $\lambda_1 \vec{x}$ by normalizing the $\bar{\vec{x}}$. This method allows us to solve for the largest eigenvalue. For the smallest eigenvalue, we simply replace A with A^{-1} and λ with $\frac{1}{\lambda}$. However, instead of multiplying matrices, we solve the tridiagonal system.

3 Programming

- Inputs: number of elements

- Outputs: grid, eigenvector, eigenvalue, critical load

Set up the grid and tridiagonal matrix

while residuals > tolerance

    solve the tridiagonal matrix (starting with a guess)

    compute new eigenvalue

    compute new eigenvector

    compute residuals by comparing new eigenvalue with old eigenvalue

end

Compute critical load from eigenvalue and given information

4 Results

5 Discussion