Project 1: Steady, Inviscid, Adiabatic, Incompressible, Irrotational 2-D Flow Over a Rotating Cylinder.

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Introduction:

Lift makes it possible for us to fly. In this report, we examine in detail the specific mechanics of how lift is made possible through simplified Navier-Stokes Equations, 1st Law of Thermodynamics, Conservation of Mass, and elementary flows. We construct a 2D rotating circular body and forward movement with uniform, source, sink, and vortex flows. Uniform flow mimics the relative air speed. Aerodynamics forces are the result of relative motion between the object and the fluid. Source and sink give us a doublet, which is a circular body. Lastly, the vortex core allows us to imitate the circulation phenomenon in the boundary layer of a wing, producing a velocity differential. Ultimately, the results would show us how exactly an object obtain lift through vorticity and circulation.

Methods:

<u>Problem 1:</u>

Navier-Stokes Equation, which is a governing equation for fluid motion, describes the behavior of Newtonian fluid (shear stress rate does not affect viscosity). We can find a fluid parameters like velocity or pressure among other things by solving the governing equation. However, the complete form of the equation poses a great difficulty to solve. Furthermore, it is impossible to solve analytically at the presence. The complete equation describes a fluid flow with the foundation of Newton 2nd Law of Motion in unit volume.

$$\text{Conservation of momentum: } \rho\left(\frac{D\vec{V}}{Dt}\right) = -\vec{\nabla}\mathbf{p} + \mu\nabla^2\vec{V} + \rho\vec{g}$$

The left-hand side of the equation represents the Newton Law of Motion $F = ma$. The terms on the right-hand side of the equation accounts for changes in pressure, viscous effect, and external force (gravity in this case), respectively. It is worth noting that pressure drives flow, and viscous effect introduces shear stress. Once we decompose the equation with respect to space and time, the decomposition contains 3 partial differential equations matches 3 dimensions with the same parameters. Moreover, we also have 2 additional equations to account for conservation of mass and energy.

$$\text{Conservation of mass: } \frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{V}) = 0$$

$$\text{Conservation of energy: } \rho\left(\frac{\partial\varepsilon}{\partial t} + \mathbf{u}\cdot\nabla\varepsilon\right) - \nabla\cdot(K_H\nabla T) + p\nabla\cdot\mathbf{u} = 0$$

Therefore, we impose assumptions on the fluid of interest to simplify the governing equation, reducing the number of variables and associated equations. The first assumption is steady flow, which states the properties of the fluid does not change with time. Effectively, the assumption sets the derivative with respect to time equal 0. The second assumption makes the fluid inviscid, making the viscous effect disappear. Without viscous effect, the fluid does not shear and have no slip condition. Adiabatic assumption eliminates the need for conservation of energy equation. Incompressibility allows us to set density as a constant, reducing the conservation of mass to 1 equation and 1 unknown. Irrotational suggest the fluid element does not rotate about itself; therefore, vorticity does not exist. At first the assumption appears counterproductive because we wish to investigate circulation about a cylinder or circular body. However, we could add a vortex to introduce circulation. At this point, the irrotational assumption simplifies the preservation of circulation, unifying the circulation of streamlines beyond the vortex source. Lastly, we assume flow in 2D to reduce the number of equations, which depends on space, down to 2. This assumption is particularly useful when variation of properties in one dimension is negligible. In our case, we are only interested in vertical (lift) and axial (thrust) directions.

Problem 2:

The 2D rotating cylinder could be model using elementary flow and superposition. First, we need to construct a circular body using a combination of source and sink. Such combination is called the Rankine body. As the distance between the two flows approaches zero, the Rankine body becomes circular or doublet. The parameter we want to find is the velocity components. They allow us to compute other parameters through conservation of energy statement (Bernoulli). The velocity of a sink or source comes from conservation of mass or flux.

$$2\pi r \mathrm{V_r} \rho = Q$$

$$\rightarrow V_{r(source)} = \frac{Q}{2\pi r\rho}$$

The source and sink flows outward or inward at a point. Therefore, they do not have angular velocity. Q is the strength of the source and sink: Q is positive for source and negative for sink. To simplify the equation, we will call the quantity $\bar{Q} = \frac{Q}{2\pi\rho}$.

$$\Rightarrow V_{r(source)} = \frac{\bar{Q}}{r}$$

Mathematically, we use differentiation or taking the limit as the distance (d) between the source and sink approaches zero to transform the Rankine body into a doublet (circular body). In this case we take the limit of function $V_{r(source)}$.

$$V_{r(doublet)} = \lim_{d\to 0} \frac{\left(\frac{1}{r}^{+} - \frac{1}{r}^{-}\right)}{2d} \bar{Q}2d$$

Note: 2d is added for simplification and $D = \bar{Q}d$

$$\rightarrow V_{r(doublet)} = \frac{\partial\left(\frac{1}{r}\right)}{\partial x} 2D$$

$$\rightarrow V_{r(doublet)} = \frac{\partial\left(\frac{1}{r}\right)}{\partial r}\left(\frac{\partial r}{\partial x}\right) 2D$$

x is the axis in which the source and sink moving towards each other. Another implicit assumption is that $\bar{Q}$ goes to infinity as d goes to 0 to keep D finite.

$$r^2 = x^2 + y^2$$

$$2r\left(\frac{dr}{dx}\right) = 2x$$

$$\rightarrow \frac{dr}{dx} = \frac{x}{r} = \frac{rcos\theta}{r} = cos\theta$$

Substitute and differentiate to get:

$$V_{r(doublet)} = \frac{Dcos\theta}{r^2}$$

The convention we use makes counterclockwise angular rotation positive. However, we want to model the flow as clockwise, so we will make velocity expression negative. Lastly, sink and source flow do not have angular velocity by nature as they flow radially.

$$V_{r(doublet)} = -\frac{Dcos\theta}{r^2}$$

Next, we find the velocity components of a vortex flow to model the rotational aspect of the problem. The vortex flow is a simple circular motion problem. The vortex strength is circulation (Γ), which is a function of annular area and angular rotation (Ω). Furthermore, vortex is purely rotational, so it does not have radial component.

$$\Gamma = A\Omega = 2\pi R\Omega$$

The velocity at the vortex edge is the rate of angular change multiply by the radius of the vortex. Moreover, circulation is conserved, which stays constant as r changes. Therefore, the velocity must go down as radius increases.

$$V_{\theta(vortex)} = \Omega R$$

$$\Rightarrow V_{\theta(vortex)} = \frac{\Gamma}{2\pi R}$$

We superimpose the vortex solution onto the solution of the doublet to get the spinning circular body. Similar differentiation approach is carried out. However, the two rotating vortices are moving along the y direction rather than x.

$$V_{\theta(doublet)} = \lim_{d\to 0} \frac{\left(\frac{1}{r}^{+} - \frac{1}{r}^{-}\right)}{2d} \frac{\Gamma}{2\pi} 2d$$

$$V_{\theta(doublet)} = D\frac{d\left(\frac{1}{r}\right)}{dr} \cdot \frac{dr}{dy} = -\frac{D\sin\ \theta}{r^2}$$

Finally, we add the uniform flow into the mix. The uniform flow velocity components in the polar coordinate are:

$$V_r = U_\infty \cos\ \theta$$

$$V_\theta = -U_\infty \sin\ \theta$$

The final superposition equation for uniform, source, sink, and vortex flow is:

$$V_r = U_\infty \cos\ \theta - \frac{D}{r^2}\cos\ \theta$$

$$V_\theta = -U_\infty \sin\ \theta - \frac{D}{r^2}\sin\ \theta - \frac{\Gamma}{2\pi}\frac{1}{r}$$

$$V_{total} = \sqrt{V_r^2 + V_\theta^2}$$

The complete velocity components of superposition flow inform us about the critical circulation and bifurcation point(s) as well. Critical circulation occurs when the bifurcation points meet or intersect. We could derive the expression for critical circulation knowing the quality of stagnation point. At stagnation points, velocity is zero and along the surface. Thus, we set V_r and V_θ equal to zero and solve.

$$V_r = 0 = U_\infty \cos\theta - \frac{D}{r^2}\cos\theta$$

$$\rightarrow D = U_\infty R_{cylinder}^2$$

$$V_\theta = 0 = -U_\infty \sin\ \theta_{stag} - \frac{D}{R_{cylinder}^2}\sin\theta_{stag} - \frac{\Gamma_c}{2\pi}\frac{1}{r}$$

$$\rightarrow \frac{\Gamma_c}{2\pi}\left(\frac{1}{R_{cylinder}}\right) = -sin\theta_{stag}\left(U_\infty + \frac{U_\infty R_{cylinder}^2}{R_{cylinder}^2}\right) = -sin\theta_{stag}(2U_\infty)$$

$$\rightarrow \Gamma_c = -4\pi R_{cylinder} sin\theta_{stag} U_\infty$$

We could find information about pressure with velocity through the Bernoulli equation. The Bernoulli's equation is a special case of energy conservation statement.

$$\frac{dE}{dt} = \frac{P}{\rho} + \frac{V_r^2 + V_\theta^2}{2} + gz + H$$

The change of energy accounts for the static pressure, dynamic pressure, external forces (often gravity), and head loss. However, the change in energy (left side), external forces, and head loss terms get eliminated due to assumptions we made initially. Therefore, the equation we work with is:

$$\frac{P}{\rho} + \frac{V_r^2 + V_\theta^2}{2} = \frac{P}{\rho} + \frac{U_\infty^2}{2}$$

The definition of coefficient of pressure is: $C_p = \frac{2\Delta P}{\rho_\infty U_\infty^2}$

Rearrange the Bernoulli's equation to substitute into C_p equation and we obtain an equation for C_p in terms of velocity components.

$$C_p = 1 - \left(\frac{V_r^2 + V_\theta^2}{U_\infty^2}\right)$$

Since the solid body we work with is in 2D not 3D, we integrate C_p over the circumference (unit of length) rather than area to obtain lift and drag coefficient; take the vertical projection for lift and horizontal for drag. However, we also need to divide the integral with nondimensionalize term c ($2R_{cylinder}$).

$$C_l = -\frac{1}{c}\int_0^{2\pi} C_{p_{surf}} R_{cylinder} sin\theta d\theta = -\frac{1}{2}\int_0^{2\pi} C_{p_{surf}} sin\theta d\theta$$

$$C_d = -\frac{1}{c}\int_0^{2\pi} C_{p_{surf}} R_{cylinder} cos\theta d\theta = -\frac{1}{2}\int_0^{2\pi} C_{p_{surf}} cos\theta d\theta$$

For lift and drag, we perform similar computation; however, we do not have to apply critical chord for non-dimensionalizing.

$$L = -\int_0^{2\pi} C_{p_{surf}} R_{cylinder} sin\theta d\theta$$

$$D = -\int_0^{2\pi} C_{p_{surf}} R_{cylinder} cos\theta d\theta$$

Problem 3:

Problem 3 is an extension of problem 2. All the computing procedure for aerodynamics parameters remain the same; however, we discretize (control) the circulation. The circulation becomes a vector with fine incremental steps rather than discrete values.

Programming:

Problem 2:

We established a coordinate as the first step. Since the derivations used were done in polar coordinates, we made a grid with respect to radial (r) and angle (theta). We also create variables to hold the cartesian values for plotting conveniency called x_polar and y_polar. The function linspace was used to create an evenly space vector. Meshgrid combines 2 vectors into corresponding multipliable matrix (every element in one vector has another corresponding element in another).

```
    % Polar grid
    r_range = linspace(cylinder_radius, 3, 5);
    theta_range = linspace(0, 2*pi, 31);
    [r, theta] = meshgrid(r_range, theta_range);
    x_polar = r.*cos(theta);
    y_polar = r.*sin(theta);
```

The control parameters we change in problem 2 is the circulation Γ. Therefore, we initialize and compute critical circulation and any independent parameters with respect to circulation. In this case, radial velocity is independent of Γ.

```
    critical_circulation = 4*pi*farField_velocity*cylinder_radius;
    radial_velocity = cos(theta).* (1-((cylinder_radius.^2)./(r.^2)));
```

Since we want to change Γ, an if-else statement is used to toggle between the choices of circulation. Once the main function is called with an integer parameter, we set the value of Γ. Default value of Γ is 0; $\Gamma < \Gamma_c$ for option 1; $\Gamma = \Gamma_c$ for option 2; $\Gamma > \Gamma_c$ for option 3. The angular velocity component depends on Γ, so we compute them accordingly based on the derivation. A value of .4 or 40% was chosen to set the lower and higher Γ value.

```
    if gamma == 0 % no circulation
        angular_velocity = -
farField_velocity*sin(theta).*(1+((cylinder_radius.^2)./(r.^2)));
    elseif gamma == 1 % circulation less than critical
        angular_velocity = -
farField_velocity*sin(theta).*(1+((cylinder_radius.^2)./(r.^2)))-
(critical_circulation*.4)./(2*pi*r);
    elseif gamma == 2 % circulation equal critical
        angular_velocity = -
farField_velocity*sin(theta).*(1+((cylinder_radius.^2)./(r.^2)))-
critical_circulation./(2*pi*r);
    elseif gamma == 3 % circulation greater than critical
        angular_velocity = -
farField_velocity*sin(theta).*(1+((cylinder_radius.^2)./(r.^2)))-
(critical_circulation*1.4)./(2*pi*r);
    end
```

With the velocities computed, we move on to compute the total velocity and aerodynamics coefficients. Since the values correspond to a 2D space, we must specify the first column as the surface values. Ergo, the first column of the pressure coefficient represents the pressure coefficients along the surface of the body. We use built-in 'trapz' to approximate the integral based on trapezoidal rule.

```
    x_velocity = -sin(theta).*angular_velocity + cos(theta).*radial_velocity;
    y_velocity = cos(theta).*angular_velocity + sin(theta).*radial_velocity;
    total_velocity = sqrt(y_velocity.^2+x_velocity.^2);
```

```
    % aerodynamic coefficients
    pressure_coefficient = 1 -
((radial_velocity.^2+angular_velocity.^2)./farField_velocity);
    surface_pressure_coefficient = pressure_coefficient(:,1)';
    lift_coefficient = -.5.*trapz(theta_range,
(surface_pressure_coefficient.*sin(theta_range)));
    drag_coefficient = -.5.*trapz(theta_range,
(surface_pressure_coefficient.*cos(theta_range)));
```

Lastly, we plot all the relevant values using subplots. The title of the figure is also based on the Γ values chosen via if-else statement. Function 'quiver' was used to plot vector field; 'plot' graphs 1D vectors; 'countourf' shows the heat map in 2D.

```
    figure('units','normalized','outerposition',[0 0 1 1]);
    subplot(2,3,1)
    contourf(x_polar,y_polar,radial_velocity,20);
    colorbar;
    title('Component of Radial Velocity Distribution');
    subplot(2,3,2)
    contourf(x_polar,y_polar,angular_velocity,20);
    colorbar;
    title('Component of Tangential Velocity Distribution');
    subplot(2,3,3)
    contourf(x_polar,y_polar,total_velocity,20);
    colorbar;
    title('Total Velocity Distribution');
    subplot(2,3,4)
    quiver(x_polar,y_polar,x_velocity,y_velocity);
    axis equal;
    title('Velocity Vector Field')
    subplot(2,3,5)
    hold on
    contourf(x_polar,y_polar,pressure_coefficient,20);
    contour(x_polar, y_polar, pressure_coefficient,
[0,0],'r','LineWidth',2,'ShowText','on');
    colorbar;
    title('Pressure Coefficient Distribution');
    subplot(2,3,6)
    plot(theta_range,surface_pressure_coefficient);
    title('Pressure Coefficient at Surface')
    xlabel('\theta (radians)',Interpreter='tex');
    ylabel('Cp');
    if gamma == 0
            sgtitle('Problem 2: Circulation = 0')
        elseif gamma == 1
            sgtitle('Problem 2: Circulation Less Than Critical')
        elseif gamma == 2
            sgtitle('Problem 2: Circulation Equal Critical')
        else
            sgtitle('Problem 2: Circulation Greater Than Critical')
    end
```

Problem 3:

All the mathematical computations stay the same for problem 3. However, we no longer use if-else statement to choose Γ because Γ would be discretized into a vector of 20x1 size. However, we do not want to run the computation every time the main function is called (computation codes for problem 2 and 3 is within a main function). Therefore, we add a second parameter to turn on the computation code for problem 3.

```
    if p3==1
        circulation_vector = linspace(0,critical_circulation*1.4,20); % circulation
gamma as a vector
```

In order to go through each circulation value, we use for-loop. The final results of each parameter would be presented as a 1D vector.

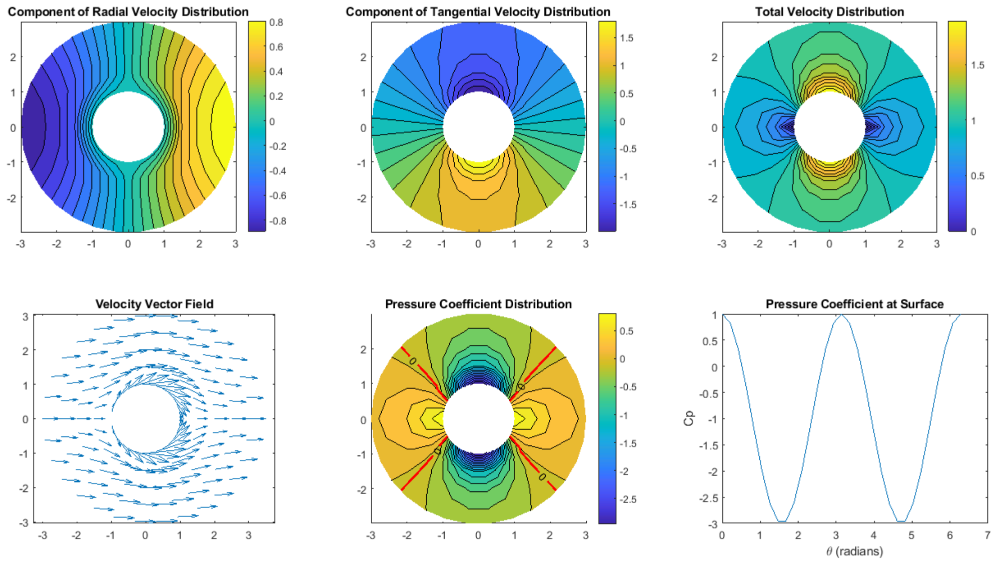
```
        for i = 1:length(circulation_vector)
            tangential_velocity = -
farField_velocity*sin(theta).*(1+((cylinder_radius.^2)./(r.^2)))-
circulation_vector(i)./(2*pi*r);
            Cp = 1 -
((radial_velocity.^2+tangential_velocity.^2)./farField_velocity);
            Cp_surf = Cp(:,1)';
            lift_integration_formula = Cp_surf.*cylinder_radius.*sin(theta_range);
            drag_integration_formula = Cp_surf.*cylinder_radius.*cos(theta_range);
            Cl(1,i) = -.5.*trapz(theta_range, (Cp_surf.*sin(theta_range)));
            Cl_analytical(1,i) =
circulation_vector(i)/(cylinder_radius*farField_velocity);
            Cd(1,i) = -.5.*trapz(theta_range, (Cp_surf.*cos(theta_range)));
            lift(1,i) = -trapz(theta_range, lift_integration_formula);
            drag(1,i) = -trapz(theta_range, drag_integration_formula);
            surface_velocity =
sqrt(tangential_velocity(:,1).^2+radial_velocity(:,1).^2);
            [~,X] = min(surface_velocity);
            theta_stagnation_numerical(1,i) = abs(wrapToPi(theta(X,1)));
            theta_stagnation_analytical(1,i) =
asin(circulation_vector(i)/(4*pi*farField_velocity*cylinder_radius));
        end
```

Lastly, we plot the results per requirements.

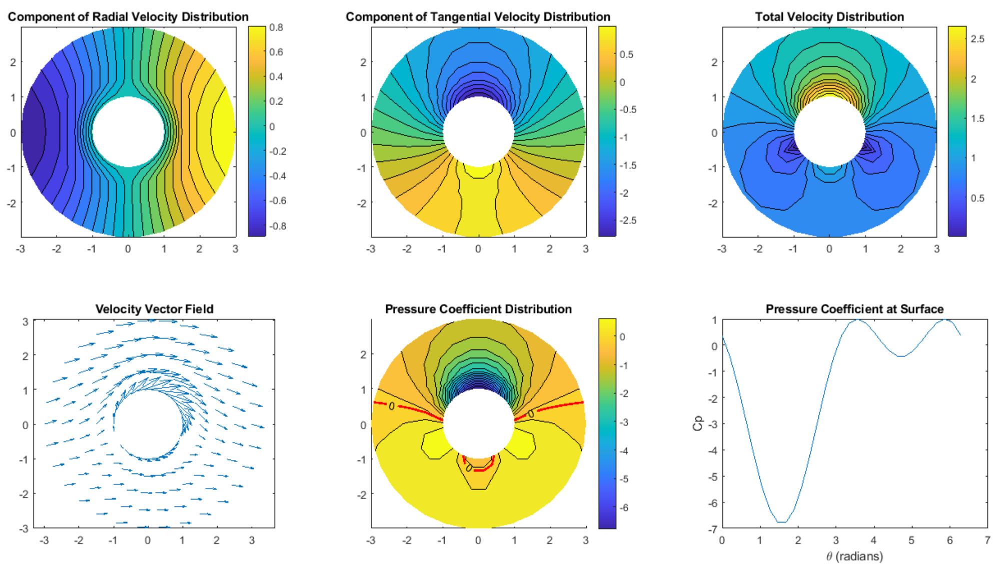
Results:

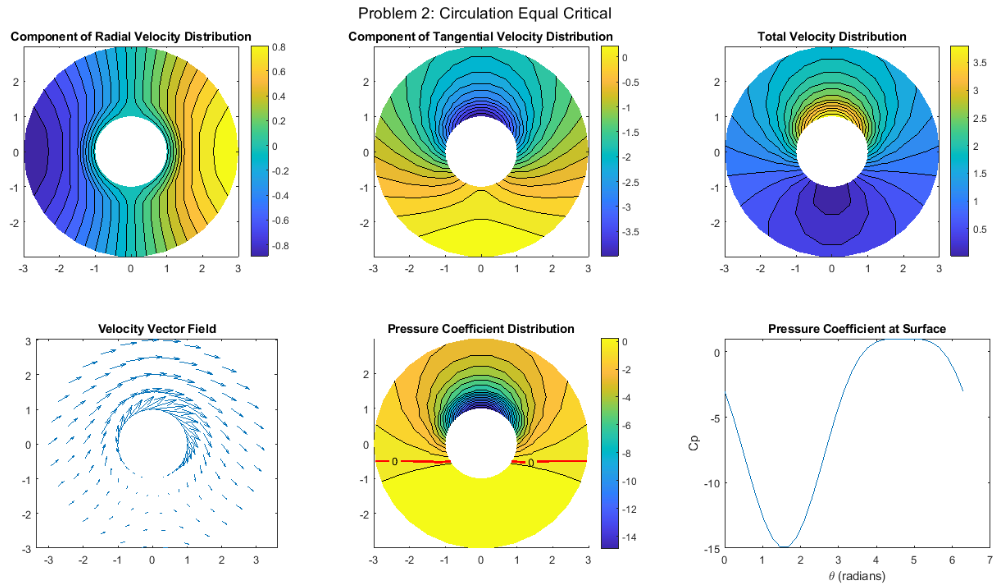
Γ	C_l	C_d
0	*1.9429e-16*	*1.1102e-16*
$< \Gamma_c$	*5.0265*	*1.3878e-16*
$= \Gamma_c$	*12.5664*	*-1.2212e-15*
$> \Gamma_c$	*17.5929*	*-6.6613e-16*

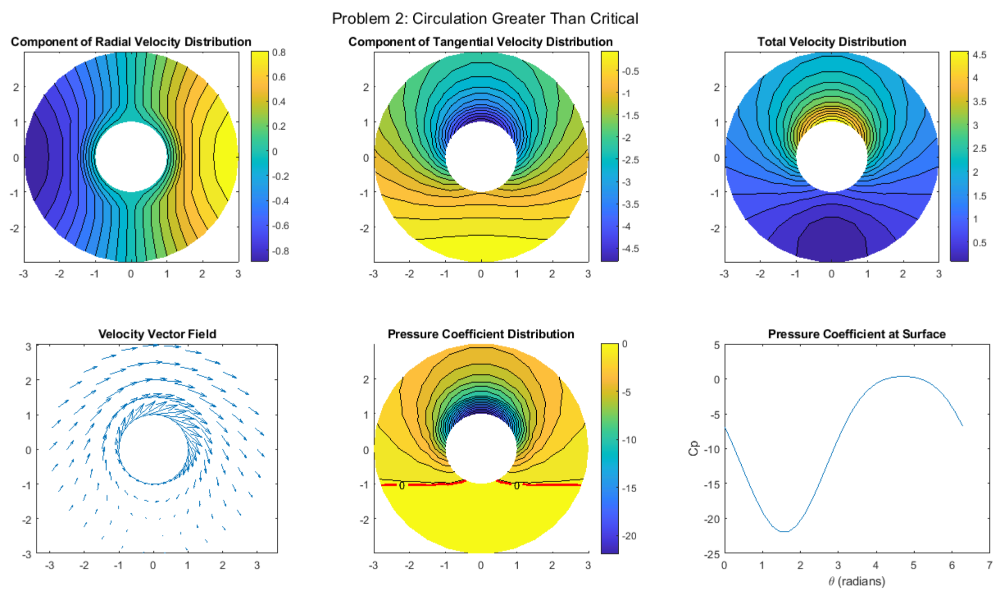
Problem 2: Circulation = 0

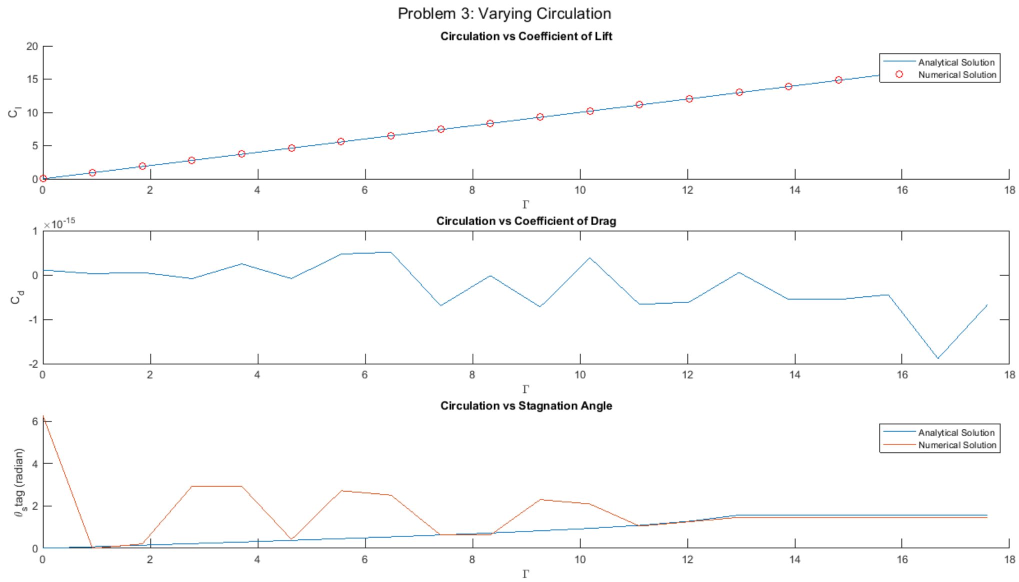


Problem 2: Circulation Less Than Critical




Problem 2: Circulation Equal Critical
Component of Radial Velocity Distribution
Component of Tangential Velocity Distribution
Total Velocity Distribution
Velocity Vector Field
Pressure Coefficient Distribution
Pressure Coefficient at Surface
Cp
θ (radians)


Problem 2: Circulation Greater Than Critical
Component of Radial Velocity Distribution
Component of Tangential Velocity Distribution
Total Velocity Distribution
Velocity Vector Field
Pressure Coefficient Distribution
Pressure Coefficient at Surface
Cp
θ (radians)



Discussion:

From the contour plots, we could see that as circulation increases, the number of 0 levels for coefficient of pressure decreases. It makes sense as the stagnation points merge once passed the critical threshold. The coefficient of lift also increases with circulation. One way to justify such trend is that we obtain lift through coefficient of pressure. However, we only take the vertical projection of θ for lift calculations. Circulation increases the value of theta as it relocates the stagnation point from the horizontal axis. Therefore, the coefficient of lift increases linearly with circulation. We need circulation and vorticity for lift! On the other hand, the coefficient of drag hovers around zero; this is due to the inviscid assumption we made. Overall, the trends for all the parameters follow each other nicely.