

Traces & Kernels

We describe the framework to extend from finite to locally compact case.

(Traces).

Let $(H, \langle \cdot, \cdot \rangle)$ be a separable Hilbert space. For $v \in H$, let $\|v\| = \sqrt{\langle v, v \rangle}$ be the Norm. We want to describe properties of bounded linear maps:

$$T: H \rightarrow H. \quad (\text{continuous})$$

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(Compact) T is compact if the closure of the image of the unit ball is compact. $\rightarrow \{v: \|v\| \leq 1\}$

Prop T is self-adjoint compact operator, then there is an orthonormal basis for H consisting of eigenvectors for T . The eigenspaces of non-zero eigenvalues are finite dimensional.

e.g. \bullet $H = L^2(\mathbb{R})$, $T_a: H \rightarrow H$ is compact
 \bullet (non) $H = L^2(\mathbb{R})$, $T_a: H \rightarrow H$ is not compact

\bullet $f \mapsto xf$ is self-adjoint, but doesn't have eigenvectors.

(Polar decomposition) First assume that T^*T is a positive (i.e. $\langle T^*v, v \rangle \geq 0, \forall v$) self-adjoint bounded operator. Using "spectral calculus" we can define:

$$|T| = \sqrt{T^*T}$$

$$\|V(x)\| = \|x\| \text{ if } x \in \text{range } V$$

Moreover, $T = V|T|$ and $|T| = V^*T$; where V is a partial isometry.

Existence of V :

- * $\||T|(x)\| = \|T(x)\|$
- * $\ker |T| = \ker (T^*T) = \ker T$

(Trace class) We say T is trace-class if there exists a orthonormal basis $\{e_i\}$ such that $\sum \langle T e_i, e_i \rangle < \infty$.

Proof Let $\{x_n\}$ be another orthonormal basis of H .

$$\begin{aligned}
 \sum_n \langle T x_n, x_n \rangle &= \sum_n \sum_i \langle \langle x_n, e_i \rangle T e_i, x_n \rangle \quad (x_n = \sum_i \langle x_n, e_i \rangle e_i) \\
 &= \sum_i \sum_n \langle T e_i, \langle e_i, x_n \rangle x_n \rangle \quad (\text{positive sum}) \\
 &= \sum_i \langle T e_i, e_i \rangle .
 \end{aligned}$$

Prop. Suppose that T is trace class.

- i) $\forall S: H \rightarrow H$ bounded, ST and TS are trace-class
(*-Dual.)
- ii) T is compact
- iii) For all orthonormal basis $\{x_n\}$, $\sum \langle T x_n, x_n \rangle$ converges absolutely
and it is independent of $\{x_n\}$. We denote this value $\text{Tr}(T)$.

Question: How can we get these type of operator?