Commutative Algebra—USTC 2020

1. Let A be a ring and let $A[x]$ be the ring of polynomials in an indeterminate x with coefficients in A. Let $f = a_0 + a_1x + \cdots + a_nx^n \in A[x]$ and f is said to be primitive if $(a_0, a_1, \cdots, a_n) = (1)$. Prove that if $f, g \in A[x]$, then

$$fg \text{ is primitive if and only if } f \text{ and } g \text{ are primitive.}$$

证明: $f = a_0 + a_1x + \cdots + a_nx^n$, $g = b_0 + b_1x + \cdots + b_mx^m$.

$fg = a_0b_0 + (a_0b_1 + a_1b_0)x + \cdots + a_nb_mx^{n+m}$

($\Rightarrow$) 由 $(1) = (a_0b_0, a_0b_1 + a_1b_0, \cdots, a_nb_m) \subseteq (a_0, a_1, \cdots, a_n)$ 或 $(b_0, b_1, \cdots, b_m)$

则 由 fg 本原, 可知 f 或 g 本原,

($\Leftarrow$) 反证假设 fg 不本原, 存在极大理想 m s.t. $(a_0b_0, a_0b_1 + a_1b_0, \cdots, a_nb_m) \subseteq m \neq (1)$

在 $A[x]/m[x] \cong (A/m)[x]$ 整环中, 由 $\overline{fg} = \bar{f} \cdot \bar{g} = 0$ 可得, 记 $\bar{f} = f(m$

$\bar{f} = 0$ 或 $\bar{g} = 0$

i.e. f 或 g 不本原, 矛盾. 因此, fg 本原.

2. Let $\mathbb{Z}[x]$ be the ring of polynomials in an indeterminate x with coefficients in $\mathbb{Z}$. Determine all the prime ideals and maximal ideals of $\mathbb{Z}[x]$.

$\varphi: \mathbb{Z} \longrightarrow \mathbb{Z}[x]$ 令 $\mathfrak{p}$ 是 $\mathbb{Z}[x]$ 的素理想

$\mathfrak{p}$. 则 $\varphi^{-1}(\mathfrak{p}) = \mathfrak{p} \cap \mathbb{Z}$ 为 $\mathfrak{p}$ 在 $\mathbb{Z}$ 中的局限

为 $\mathbb{Z}$ 的素理想.

case 1: $\mathfrak{p} \cap \mathbb{Z} = (0)$

则 $\mathfrak{p} = (0)$ 或 $\mathfrak{p} = (f)$ f 为 $\mathbb{Z}[x]$ 中的不可约多项式 且 容度 $c(f) = 1$.

case 2: $\mathfrak{p} \cap \mathbb{Z} = p\mathbb{Z}$, p 为 $\mathbb{Z}$中 素数

由 $\mathfrak{p} = p\mathbb{Z}[x] = (p)$. 或 $\mathfrak{p} = (p, f)$ f在 $\mathbb{F}_p$上不可约.

极大理想: $\mathbb{Z}[x]/(p, f) \cong \mathbb{F}_p[x]/(f) \cong \mathbb{F}_{p^{\deg(f)}}$ 域

(p, f) 为 $\mathbb{Z}[x]$ 的极大理想.

3. Let A, B be rings , let M be an A-module, P a B-module and N an (A, B)-bimodule (that is, N is simultaneously an A-module and a B-module and the two structures are compatible in the sense that $a(xb) = (ax)b$ for all $a \in A, b \in B, x \in N$). Then $M \otimes_A N$ is naturally a B-module, $N \otimes_B P$ an A-module, and we have

$$(M \otimes_A N) \otimes_B P \cong M \otimes_A (N \otimes_B P).$$

证明： · $x \in M$, $y \in N$; $a \in A$, $b \in B$, $z \in P$.

$a(x \otimes_A y) = ax \otimes_A y$, $(x \otimes_A y) b = x \otimes_A by$

$a(y \otimes_B z) = ay \otimes_B z$, $(y \otimes_B z) b = y \otimes_B bz$.

可知 $M \otimes_A N$ 为 A, B 双模， $N \otimes_B P$ 为 A、B 双模.

· 双线性映射 $(M \otimes_A N) \times P \longrightarrow M \otimes_A (N \otimes_B P)$

诱导 $\varphi : (M \otimes_A N) \otimes_B P \to M \otimes_A (N \otimes_B P)$

[mod m[x]) $(m \otimes_A n) \otimes_B p \mapsto m \otimes_A (n \otimes_B p)$.

同理可得 $\psi : M \otimes_A (N \otimes_B P) \longrightarrow (M \otimes_A N) \otimes_B P$.

$m \otimes_A (n \otimes_B p) \mapsto (m \otimes_A n) \otimes_B p$.

可知 φ, ψ 互逆，则. $(M \otimes_A N) \otimes_B P \cong M \otimes_A (N \otimes_B P)$.

4. Let A be a ring, $\mathbf{a}$ an ideal, M an A-module. Show that $(A/\mathbf{a}) \otimes_A M$ is isomorphic to $M/\mathbf{a}M$.

证明： 由 $0 \to \mathfrak{a} \to A \to A/\mathfrak{a} \to 0$ 正合，可知

$$\begin{tikzcd}
 & 0 \arrow[d] & \ker\varphi \arrow[d] & \\
\mathfrak{a}\otimes M \arrow[r] \arrow[d, "\psi"] & A\otimes M \arrow[r] \arrow[d, phantom, "\cong"] & (A/\mathfrak{a})\otimes M \arrow[r] \arrow[d, "\varphi"] & 0 \\
0 \arrow[r] & \mathfrak{a}M \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & M/\mathfrak{a}M \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & \operatorname{Coker}\varphi &
\end{tikzcd}$$

正合.

正合.

$\forall \sum a_i m_i \in \mathfrak{a}M$. $\psi(\sum a_i \otimes m_i) = \sum a_i m_i$ 满射

由 $A \otimes M \cong M$. 可 诱导 ~~$\mathfrak{a} \otimes M \cong \mathfrak{a}M$~~. 满

由蛇形引理，可知 $0 \to \ker\varphi \to 0 \to 0 \to \operatorname{Coker}\varphi \to 0$ 正合.

即 $\ker\varphi = \operatorname{Coker}\varphi = 0$

因此 $(A/\mathfrak{a}) \otimes M \cong M/\mathfrak{a}M$.

法二： $A/\mathfrak{a} \times M \to M/\mathfrak{a}M$.

$(a+\mathfrak{a}, m) \mapsto am + \mathfrak{a}M$. 双线性映射.

诱导 $\varphi : (A/\mathfrak{a}) \otimes M \to M/\mathfrak{a}M$.

$(a+\mathfrak{a}) \otimes m \mapsto am + \mathfrak{a}M$

$\psi : M/\mathfrak{a}M \to (A/\mathfrak{a}) \otimes M$

$m + \mathfrak{a}M \mapsto (1+\mathfrak{a}) \otimes m$.

ψ, φ 互逆. 同构.

5. If $f: A \to B$ is a ring homomorphism and M is a flat A-module, then $M_B = B \otimes_A M$ is a flat B-module.

$0 \to N' \xrightarrow{\varphi} N$ 正合 B-模列, i.e. φ 单.

$\quad\;\; \cong \qquad\; \cong$

$0 \to N' \otimes_B B \to N \otimes_B B$. 正合 B-模序列，具有 A-模结构；作为 A-模亦正合.

由 M 是平坦 A-模，可知

$0 \to (N' \otimes_B B) \otimes_A M \to (N \otimes_B B) \otimes_A M$ 正合.

$\qquad\qquad \cong \qquad\qquad\qquad \cong$ 由练习题 3 可知

$0 \to N' \otimes_B (B \otimes_A M) \to N \otimes_B (B \otimes_A M)$

证明

即. $0 \to N' \otimes_B M_B \to N \otimes_B M_B$ 正合，即 M_B 平坦 B-模.

6. Let $\mathbb{Z}[x]$ be the ring of polynomials in an indeterminate x with coefficients in $\mathbb{Z}$. Prove that $q = (p^n, x)$ for $1 < n \in \mathbb{N}$ is a primary ideal, but is not a power of prime ideal.

证明： · $r(q) = r(p^n, x) = (p, x)$.

$\mathbb{Z}[x]/(p,x) \cong \mathbb{F}_p$ 域 因此 (p,x) 为 $\mathbb{Z}[x]$ 的极大理想

$\dfrac{2A = I^2}{\text{准素分解}}$.

· 由命题 4.2，可知 $q = (p^n, x)$ 为准素理想

· 假如 $q = \mathfrak{p}^n$ 素理想 $\mathfrak{p}$.

则 $r(q) = r(\mathfrak{p}^n) = \mathfrak{p} = (p, x)$.

即 $(p,x)^n = (p^n, p^{n-1}x, \dots, x^n)$

易见 $(p^n, x) \neq (p^n, p^{n-1}x, \dots, x^n)$ 对 $n > 1$.

因此. q 不是素理想之幂.

7. Let $A = \mathbb{Z}[\sqrt{-5}]$ be the ring of integers in the quadratic field $\mathbb{Q}(\sqrt{-5})$ and let I be the prime ideal $(2, 1+\sqrt{-5})$ of A generated by 2 and $1+\sqrt{-5}$. Note that every nonzero prime ideal P of A contains a prime $p \in \mathbb{Z}$.

(1) If P is a prime ideal of A not containing 2, then prove that $I_P = A_P$;

(2) If P is a prime ideal of A containing 2, then prove that $P = I$ and $I_P = (1+\sqrt{-5})A_P$;

(3) Prove that $I_P \cong A_P$ as A_P-modules for every prime ideal P of A, but that I and A are not isomorphic as $\cancel{\mathbb{Z}}$ A-modules.

证明：(1). $2 \notin P$，则 $2 \in (A-P) \cap I \neq \phi$，i.e. $1 = \frac{2}{2} \in I_P$ 为 A_P 中单位.

因此 $I_P = A_P$.

(2). 由 $2 \in P$，且知 $(1+\sqrt{-5})\cdot(1-\sqrt{-5}) = 6 = 2\times 3 \in P$. 素理想.

则 $1+\sqrt{-5} \in P$ 或 $1-\sqrt{-5} \in P$.

如果 $1+\sqrt{-5} \in P$，$2 \in P \Rightarrow I = (2, 1+\sqrt{-5}) \in P$

即 $\mathbb{Z}[\sqrt{-5}]/(2, 1+\sqrt{-5}) \cong \mathbb{F}_2$ 域，I 极大理想

则 $P = I$.

如果 $1-\sqrt{-5} \in P$，则 $2-(1-\sqrt{-5}) = 1+\sqrt{-5} \in P$，则 $P = I$.

$= I^2$

素分解.

· $I_P = (2, 1+\sqrt{-5})A_P$

即 $2 = \frac{(1+\sqrt{-5})(1-\sqrt{-5})}{3} = (1+\sqrt{-5})\cdot \underbrace{\frac{1-\sqrt{-5}}{3}}_{\in A_P} \in (1+\sqrt{-5})A_P$. $3 \notin (2, 1+\sqrt{5})$

(3). · $2 \notin P$，$I_P = A_P$

· $2 \in P$，$I_P = (1+\sqrt{-5})A_P \cong A_P$.

· $I = (2, 1+\sqrt{-5}) \ncong A$.

i.e. I 不能由一个元素生成，即证明 I 不是主理想 $(x) = (a+b\sqrt{-5})$

反证，假设 $I = (x) = (2, 1+\sqrt{-5})$

$$\begin{cases} 2 = xy \\ 1+\sqrt{-5} = xz \end{cases} \quad y, z \in \mathbb{Z}[\sqrt{-5}]. \Rightarrow \begin{cases} Nm(x) \mid Nm(2) = 4 \\ Nm(x) \mid Nm(1+\sqrt{-5}) = (1+\sqrt{-5})(1-\sqrt{-5}) = 6. \end{cases}$$

$x\bar{x} = Nm(x) \in \mathbb{Z}$，则 $\underset{a^2+5b^2}{\underset{\parallel}{Nm(x)}} \mid (4,6) = 2$.

而. $a^2+5b^2 = 2$ 无解（模5，同余0,1,4），则 $(x) = (\pm 1)$ 矛盾.

10. Let $\mathbf{p}_i$ be prime ideals of the integral domain A for $1 \le i \le n$ and let $S = A - \cup_{i=1}^n \mathbf{p}_i$. Prove that:

(1) $S^{-1}A$ is a semi-local ring, i.e., has only finitely many maximal ideals.

(2) $A_{\mathbf{p}_i} \cong (S^{-1}A)_{S^{-1}\mathbf{p}_i}$.

(3) $S^{-1}A = \cap_{i=1}^n A_{\mathbf{p}_i}$.

证明：(1) $S^{-1}A$ 中的素理想形式为 $S^{-1}p$，p 为 A 的素理想且 $p \cap S = \phi$

$$\Downarrow$$

$$p \subseteq A \setminus S = \bigcup_{i=1}^{n} p_i$$

由命题1.11 可知 $p \subseteq p_i$ 对某个 $1 \le i \le n$.

因此 $S^{-1}A$ 的极大理想恰为 $\{S^{-1}p_i : 1 \le i \le n\}$ 中的极大元，有限

$\Rightarrow S^{-1}A$ 为半局部环

(2). $\varphi: A_{p_i} \to (S^{-1}A)_{S^{-1}p_i}$　　$x \in A$

$\frac{x}{s} \mapsto \frac{x/1}{s/1}$　　$s \in A \setminus p_i$

单：$\varphi(\frac{x}{s}) = \frac{x/1}{s/1} = 0$　i.e. $\exists \frac{a}{t} \in S^{-1}A \setminus S^{-1}p_i$ s.t. $\frac{a}{t} \cdot \frac{x}{1} = 0$.

$\Rightarrow \exists t' \in A \setminus p_i$ s.t. $\underline{a} x \underline{t'} = 0$，$at' \in A \setminus p_i$

i.e. $\frac{x}{s} = 0$

满：$\forall \frac{a/t_1}{b/t_2} \in (S^{-1}A)_{S^{-1}p_i}$　　$\underline{t_1, t_2 \in S}$，$\underline{b \notin A \setminus p_i}$

$$\downarrow$$

$$t_1 b \in A \setminus p_i$$

$\varphi(\frac{at_2}{t_1 b}) = \frac{at_2/1}{t_1 b/1} = \frac{a/t_1}{b/t_2}$

因此 $A_{p_i} \cong (S^{-1}A)_{S^{-1}p_i}$

(3). • $A - p_i \supseteq A - \bigcup_{i=1}^{n} p_i = S$ $\Rightarrow$ $S^{-1}A \subseteq A_{p_i}$ $\forall 1 \le i \le n$.

i.e. $S^{-1}A \subseteq \bigcap_{i=1}^{n} A_{p_i}$

• 反之，若 $\frac{a}{s} \in \bigcap_{i=1}^{n} A_{p_i}$

即 $\frac{a}{s} = \frac{a_1}{s_1} = \frac{a_2}{s_2} = \cdots = \frac{a_n}{s_n}$　　$s_i \in A \setminus p_i^{\subseteq S}$，$a_i \in A$.

$$\Downarrow$$

$= \frac{a_i \prod_{j \ne i} s_j}{\prod_{j=1}^{n} s_j} \in S^{-1}A$.　　$\prod_{i=1}^{n} s_i \in S$.

8. Determining the ring of integers in $\mathbb{Q}(\sqrt{5})$, i.e., the integral closure of $\mathbb{Z}$ in $\mathbb{Q}(\sqrt{5})$.

证明： $A+B\sqrt{5}$ 在 $\mathbb{Z}$ 上整

$\Leftrightarrow$ $\min(A+B\sqrt{5}, \mathbb{Q}) = x^2 - 2Ax + A^2 - 5B^2 \in \mathbb{Z}[x]$.

$\Leftrightarrow$ $2A \in \mathbb{Z}$, $A^2 - 5B^2 \in \mathbb{Z}$.

$\Leftrightarrow$ $2A \in \mathbb{Z}$, $(2A)^2 - 5\cdot(2B)^2 \in 4\mathbb{Z}$.

$\Leftrightarrow$ $2A \in \mathbb{Z}$, $(2A)^2 \equiv 5\cdot(2B)^2 \equiv (2B)^2 \pmod 4$

$\Leftrightarrow$ $2A \in \mathbb{Z}$, $2A$ 与 $2B$ 同奇偶.

$K=\mathbb{Q}(\sqrt{5})$ 的整闭包 $\mathcal{O}_K = \{ \underbrace{A+B\sqrt{5}}_{= A-B+\frac{2B(1+\sqrt{5})}{2}} \mid 2A \in \mathbb{Z}, 2A$ 与 $2B$ 同奇偶$\}$

$= \mathbb{Z} \oplus \mathbb{Z}\frac{1+\sqrt{5}}{2} = \mathbb{Z}[\frac{1+\sqrt{5}}{2}]$.

9. Let $\mathbf{a}$ be a decomposable ideal in a ring A and let $\mathbf{p}$ be a maximal element of the set of ideals of $(\mathbf{a} : x)$, where $x \in A$ and $x \notin \mathbf{a}$. Show that $\mathbf{p}$ is a prime ideal belonging to $\mathbf{a}$.

证明： • 由 ~~奇~~ 第一唯一性定理 4.5，可知

只~~须~~需证明 $\mathfrak{p}$ 是素理想

• 由 $yz \in \mathfrak{p} = (\mathfrak{a} : x)$，可知 $xyz \in \mathfrak{a}$

如果 $z \notin (\mathfrak{a}:x)$，即为 $xz \notin \mathfrak{a} = \bigcap_{j=1}^{n} \mathfrak{q}_j$ 31习题4.4.

因此， $y \in (\mathfrak{a} : xz) = (\bigcap_{j=1}^{n} \mathfrak{q}_j : xz) = \bigcap_{j=1}^{n} (\mathfrak{q}_j : xz) \neq (1)$

由 $\mathfrak{p}$ 为 $\{(\mathfrak{a}:x) : x \in A, x \notin \mathfrak{a}\}$ 中的极大元.

~~由~~知 . 而 $(\mathfrak{a} : xz) \supseteq (\mathfrak{a} : x) = \mathfrak{p}$.

则有 $(\mathfrak{a} : xz) = \mathfrak{p}$. i.e. $y \in \mathfrak{p}$

法二． 假设 $\mathfrak{a} = (0)$. 否则考虑 $A/\mathfrak{a}$， $\pi : A \to A/\mathfrak{a}$.

$\mathfrak{p} = (0 : x) = \mathrm{Ann}(x)$， $a \notin \mathrm{Ann}(x) \Rightarrow \mathrm{Ann}(x) = \mathrm{Ann}(ax)$

由 $\mathrm{Ann}(x) \subseteq \mathrm{Ann}(ax)$ 且 $\mathrm{Ann}(x)$ 极大

$yz \in \mathrm{Ann}(x)$. $y \notin \mathrm{Ann}(x) \Rightarrow$ ~~$y \notin \mathrm{Ann}(x) = \mathrm{Ann}(xy)$~~

$xyz = 0$

$z \in \mathrm{Ann}(xy) = \mathrm{Ann}(x)$.

即 $\mathrm{Ann}(x)$ 素理想 .