ALGEBRAIC GEOMETRY: EXAM (USTC 2020-2021)

Let k be an algebraically closed field.

1. (10 points)
(1) State the definition of dimension of varieties.
(2) Let $f : X \to Y$ be a **surjective** morphism of varieties. Prove that $\dim X \geq \dim Y$.

2. (15 points)
(1) State the definition of affine variety.
(2) Is $\mathbb{A}^2_{(x,y)} \setminus \{x = 0\}$ affine? Compute $\Gamma(\mathbb{A}^2_{(x,y)} \setminus \{x = 0\})$.
(3) Compute $\Gamma(\mathbb{A}^2_{(x,y)} \setminus \{(0, 0)\})$. Is $\mathbb{A}^2_{(x,y)} \setminus \{(0, 0)\}$ affine?

3. (15 points)
(1) Describe the construction of the blowup Z of $\mathbb{A}^2$ at $(0, 0)$, and decompose Z into two open subsets with both isomorphic to $\mathbb{A}^2$.
(2) Let $C = V(y^n - x^{n+1})$ and denote by $\tilde{C}$ the strict transform under the blow up $\pi : Z \to \mathbb{A}^2$ at $(0, 0)$. Show that $\tilde{C}$ is nonsingular and is isomorphic to $\mathbb{A}^1$.

4. (15 points)
(1) Embed $\mathbb{P}^1 \times \mathbb{P}^2$ into a projective space as a closed subvariety and present the defining equations.
(2) Construct a birational map from $\mathbb{P}^2$ to $\mathbb{P}^1 \times \mathbb{P}^1$.
(3) Prove there does not exist a birational morphism from $\mathbb{P}^2$ to $\mathbb{P}^1 \times \mathbb{P}^1$.

5. (15 points)
Let $X \subset \mathbb{A}^3$ be a curve defined by $V(f(x, y, z), g(x, y, z))$ where $f, g \in k[x, y, z]$.
(1) Prove that if char $k \neq 2$ and $f = x^2 + y^2 + z^2, g = x + y + z^3$, then X is isomorphic to a plane curve.
(2) State the definition of simple point of a curve.
(3) A closed point $P \in X$ is simple if and only if

$$\operatorname{rank} \begin{pmatrix} f_x & f_y & f_z \\ g_x & g_y & g_z \end{pmatrix} |_P = 2.$$

6. (30 points)
Assume char $k = 0$. Let X be defined by $X^5 + Y^5 + Z^5 = 0$ in $\mathbb{P}^2$. Let $D_0 = div(X + Y)$ and denote by K the canonical divisor.
(1) Show that X is nonsingular.
(2) Compute $g(X)$, $\deg(K)$.
(3) State the Riemann-Roch Theorem and prove that $l(K) = g(X)$.
(4) Prove that $D_0 = 5P_0$ where $P_0 = [1, -1, 0]$, and for $n \geq 3$, compute $l(nD_0)$.
(5) Compute $l(D_0)$, find a basis of $L(D_0)$.

(6) Compute $l(D_0 - P_0)$ and $l(D_0 - 2P_0)$.
(7) Find the divisor $div(\mathrm{d}x)$ where $x = \frac{X}{Z}$, and find a basis of the space of all the differential forms such that $div(\omega) \geq 0$.