Coxeter functors and Auslander-Reiten translations

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Let Q be a finite connected acyclic quiver, and let $A = KQ$ be the path algebra of Q with coefficients in a field K. $\mathrm{rep}(Q) := \mathrm{rep}_K(Q)$.

Theorem 1 (Gabriel)**.** *There are functorial isomorphisms*

$$TC^{\pm}(M) \cong \tau^{\pm}(M)$$

for all $M \in \mathrm{rep}(Q)$, where $\tau(-)$ is the Auslander-Reiten translation, and T is a twist functor, for $M \in \mathrm{rep}(Q)$,

$$(TM)(i) = M(i), (TM)(\alpha) = -M(\alpha), i \in Q_0, \alpha \in Q_1$$

A vertex i of Q is called *sink*(resp. *source*) if there is no arrow in Q starting (resp. ending) at i.

Given any vertex i, the quiver $\sigma_i(Q)$ is obtained from Q by reversing all arrows which start or end at i.

Let $i \in Q_0$ be a sink of Q. We define a $reflection\ functors$

$$\Sigma_i^+ : \mathrm{rep}(Q) \longrightarrow \mathrm{rep}(\sigma_i(Q))$$

For $M \in \mathrm{rep}(Q)$,set

$$(\Sigma_i^+(M))(j) = \begin{cases} M(j) & ,j \neq i \\ \ker(M_{i,in}) & ,j = i. \end{cases}$$

where

$$0 \longrightarrow \ker(M_{i,in}) \xrightarrow{\mathrm{inc}} \bigoplus_{\substack{\alpha \in Q_1 \\ t(\alpha)=i}} M(s(\alpha)) \xrightarrow{M_{i,in}} M(i)$$

and $M_{i,in} := \bigoplus_{\substack{\alpha \in Q_1 \\ t(\alpha)=i}} M(\alpha)$.

$$(\Sigma_i^+(M))(\alpha) = \begin{cases} M(\alpha) & ,t(\alpha) \neq i \\ \mathrm{proj}_{s(\alpha)} \circ \mathrm{inc} & ,t(\alpha) = i. \end{cases}$$

where $\mathrm{proj}_{s(\alpha)} : \bigoplus_{\substack{\alpha \in Q_1 \\ t(\alpha)=i}} M(s(\alpha)) \longrightarrow M(s(\alpha))$ is canonical projection.

Similarly, we can define Σ_i^- for $i \in Q_0$ is a source of Q.

Assume that $Q_0 = \{1, 2, \cdots, n\}$,and $(1, 2, \cdots, n)$ is a $+-$admissible sequence for Q,that is 1 is a sink in Q and k is a sink in $\sigma_{k-1} \cdots \sigma_1(Q)$.

The *Coxeter functor* is the functor

$$C^+ := \Sigma_n^+ \circ \Sigma_{n-1}^+ \circ \cdots \circ \Sigma_1^+ : \mathrm{rep}(Q) \longrightarrow \mathrm{rep}(Q)$$

$$C^- := \Sigma_1^- \circ \Sigma_2^- \circ \cdots \circ \Sigma_n^- : \mathrm{rep}(Q) \longrightarrow \mathrm{rep}(Q)$$

We define a new quiver $\widetilde{Q}$ as follow:

$$\widetilde{Q}_0 = \{(i, a) | i \in Q_0, a = 0, 1\}$$

$$\widetilde{Q}_1 = \{\alpha^{(a)} : (i, a) \to (j, a) | \alpha : i \to j \in Q_1, a = 0, 1\} \cup \{\alpha^* : (j, 1) \to (i, 0) | \alpha : i \to j \in Q_1\}$$

Let $\tilde{A} = K\tilde{Q}/I$,where I is the ideal of $K\widetilde{Q}$ defined by the *mesh relation*

$$\rho_i = \sum_{\substack{\alpha \in Q_1 \\ s(\alpha)=i}} \alpha^* \alpha^{(1)} + \sum_{\substack{\alpha \in Q_1 \\ t(\alpha)=i}} \alpha^{(0)} \alpha^* = 0, \quad \forall i \in Q_0$$

Example 1. Let Q be

$$\begin{tikzcd}
 & 1 \\
 & 2 \arrow[u, "\alpha"'] \\
3 \arrow[ur, "\beta"'] \arrow[uur, "\gamma"] &
\end{tikzcd}$$

Then $\widetilde{Q}$ is

$$\begin{tikzcd}
 & (1,1) \arrow[rrr, dashed, "\alpha^*", start anchor=east, end anchor=north west, to path={-- (\tikztotarget)}] \arrow[rdd, dashed, "\gamma^*"] & & & (1,0) \\
 & (2,1) \arrow[u, "\alpha^{(1)}"'] \arrow[rd, dashed, "\beta^*"] & & & (2,0) \arrow[u, "\alpha^{(0)}"'] \\
(3,1) \arrow[uur, "\gamma^{(1)}"] \arrow[ur, "\beta^{(1)}"'] & & (3,0) \arrow[uurr, "\gamma^{(0)}"] \arrow[urr, "\beta^{(0)}"'] & &
\end{tikzcd}$$

The mesh relation is

$$\alpha^{(0)} \alpha^* + \gamma^{(0)} \gamma^* = 0, \ i = 1$$

$$\beta^{(0)} \beta^* + \alpha^* \alpha^{(1)} = 0, \ i = 2$$

$$\gamma^* \gamma^{(1)} + \beta^* \beta^{(1)} = 0, \ i = 3$$

For $a = 0, 1$, let $Q_{(a)}$ be a full subquiver of $\widetilde{Q}$ with $(Q_{(a)})_0 = \{(i,a)|i \in Q_0\}$.Set

$$\mathbb{1}_a := \sum_{i\in Q_0} e_{(i,a)}, \quad A_a := \mathbb{1}_a \widetilde{A} \mathbb{1}_a (\cong KQ_{(a)}).$$

Then we have natural algebra isomorphisms

$$\eta_a : A \to A_a,\ e_i \longmapsto e_{(i,a)};\ \alpha \longmapsto \alpha^{(a)}$$

We obtain for $a \in \{0, 1\}$ restriction functors

$$\mathrm{Res}_a : \mathrm{rep}(\widetilde{Q}, I) \to \mathrm{rep}(Q_{(a)}),\ M \longmapsto \mathbb{1}_a \widetilde{A} \otimes_{\widetilde{A}} M = \mathbb{1}_a M.$$

Then

$$\mathrm{Res}_0^* : \mathrm{rep}(Q_{(0)}) \to \mathrm{rep}(\widetilde{Q}, I),\ N \longmapsto \mathrm{Hom}_{A_0}(\mathbb{1}_0 \widetilde{A}, N)$$

is right adjoint to Res_0, and $\mathrm{Res}_0 \circ \mathrm{Res}_0^* \cong \mathrm{id}_{\mathrm{rep}(Q^{(0)})}$.

For $l \in Q_0 \cup \{0\}$, let $Q^{(l)}$ be the full subquiver of $\widetilde{Q}$ with

$$Q_0^{(l)} = \{(i,0), (j,1)|i > l, j \le l\}$$

and $\widetilde{Q}^{(l)}$ be a full subquiver of $\widetilde{Q}$ with

$$\widetilde{Q}_0^{(l)} = \{(i,0), (j,1)|i \in Q_0, j \le l\}$$

and

$$I^{(l)} = \langle \rho_j | j \le l \rangle$$

Then $Q^{(k)} = \sigma_k \sigma_{l-1} \cdots \sigma_1(Q^{(0)}),\ k \in Q_0$ and $Q^{(k)} \subset \widetilde{Q}^{(k)} \supset \widetilde{Q}^{(k-1)}$.

Define idempotents in $\widetilde{A}$:

$$\mathbb{1}^{(l)} = \sum_{i>l} e_{(i,0)} + \sum_{i\le l} e_{(i,1)}$$

$$\mathbb{1}_0^{(l)} = \mathbb{1}_0 + \sum_{i\le l} e_{(i,1)}$$

for $l \in Q_0 \cup \{0\}$.We have

$$A^{(l)} = KQ^{(l)} \cong \mathbb{1}^{(l)} \widetilde{A} \mathbb{1}^{(l)}$$
$$\widetilde{A}^{(l)} = K\widetilde{Q}^{(l)} \cong \mathbb{1}_0^{(l)} \widetilde{A} \mathbb{1}_0^{(l)}$$

Then $A^{(0)} = \widetilde{A}^{(0)} = A_0, A^{(n)} = A_1, \widetilde{A}^{(n)} = \widetilde{A}$.

Consider the restriction functors

$$\mathrm{Res}^{(l)} : \mathrm{rep}(\widetilde{A}) \to \mathrm{rep}(A^{(l)}),\ M \longmapsto \mathbb{1}^{(l)} \widetilde{A} \otimes_{\widetilde{A}} M = \mathbb{1}^{(l)} M$$

$$\mathrm{Res}_{(l,m)} : \mathrm{rep}(\widetilde{A}^{(m)}) \to \mathrm{rep}(\widetilde{A}^{(l)}),\ M \longmapsto \mathbb{1}_0^{(l)} \widetilde{A}^{(m)} \otimes_{\widetilde{A}^{(m)}} M = \mathbb{1}_0^{(l)} M \quad (l < m)$$

Then $\mathrm{Res}_{(l,m)}$ admits a right adjoint

$$\mathrm{Res}_{(l,m)}^*(-) = \mathrm{Hom}_{\widetilde{A}^{(l)}}(\mathbb{1}_0^{(l)} \widetilde{A}^{(m)}, -)$$

and

$$\mathrm{Res}_{(l,m)} \circ \mathrm{Res}^*_{(l,m)} = \mathrm{id}$$

Note that

$$\mathrm{Res}_0 = \mathrm{Res}_{(0,1)} \circ \mathrm{Res}_{(1,2)} \circ \cdots \circ \mathrm{Res}_{(n-1,n)}$$

Thus

$$\mathrm{Res}^*_0 \cong \mathrm{Res}^*_{(n-1,n)} \circ \cdots \circ \mathrm{Res}^*_{(1,2)} \circ \mathrm{Res}^*_{(0,1)}$$

For $M \in \mathrm{rep}(\widetilde{A}^{(i-1)})$,we define $R^*_{(i-1,i)}(M)$ as follow:

$$\mathrm{Res}_{(i-1,i)} R^*_{(i-1,i)}(M) = M$$

$$(R^*_{(i-1,i)}M)(i,1) = \ker M_{i,in}$$

where

$$M_{i,in} = (M(\alpha))_\alpha : \bigoplus_{\substack{\alpha \in Q_1^{(i-1)} \\ t(\alpha)=(i,0)}} M(s(\alpha)) \longrightarrow M(i,0)$$

Note that $\{\alpha \in \widetilde{Q}_1 | t(\alpha) = (i,0)\} = \{\alpha \in Q_1^{(i-1)} | t(\alpha) = (i,0)\}$.It is easy to see that we have

$$R^*_{(i-1,i)}(M) \in \mathrm{rep}(\widetilde{A}^{(i)}), \quad \mathrm{Res}^{(i)} \circ R^*_{(i-1,i)}(M) \cong \Sigma_i^+ \circ \mathrm{Res}^{(i-1)}(M)$$

for all $M \in \mathrm{rep}(\widetilde{A}^{(i-1)}), i \in Q_0$.

Lemma 1. *With the above notations,we have functorial isomorphisms*

$$\mathrm{Res}^{(i)} \circ \mathrm{Res}^*_{(i-1,i)}(M) \cong \Sigma_i^+ \circ \mathrm{Res}^{(i-1)}(M)$$

for all $M \in \mathrm{rep}(\widetilde{A}^{(i-1)}), i \in Q_0$.

Proof. Only need to prove that $R^*_{(i-1,i)}$ is right adjoint to $\mathrm{Res}_{(i-1,i)}$.

Let $N \in \mathrm{rep}(\widetilde{A}^{(i)}), M \in \mathrm{rep}(\widetilde{A}^{(i-1)})$,we consider the natural map

$$\mathrm{Hom}_{\widetilde{A}^{(i)}}(N, R^*_{(i-1,i)}(M)) \to \mathrm{Hom}_{\widetilde{A}^{(i-1)}}(\mathrm{Res}_{(i-1,i)}(N), M), \; f \longmapsto \mathrm{Res}_{(i-1,i)}(f)$$

For $\forall g \in \mathrm{Hom}_{\widetilde{A}^{(i-1)}}(\mathrm{Res}_{(i-1,i)}(N), M)$,we have a commutative diagram

$$\begin{tikzcd}
 & N(i,1) \arrow[r, "N_{i,out}"] \arrow[d, dashed, "g(i,1)"] & \bigoplus_{\substack{\alpha\in\widetilde{Q}_1 \\ t(\alpha)=(i,0)}} N(s(\alpha)) \arrow[r, "N_{i,in}"] \arrow[d, "\oplus g(s(\alpha))"] & N(i,0) \arrow[d, "g(i,0)"] \\
0 \arrow[r] & \ker(M_{i,in}) \arrow[r] & \bigoplus_{\substack{\alpha\in\widetilde{Q}_1 \\ t(\alpha)=(i,0)}} M(s(\alpha)) \arrow[r, "M_{i,in}"] & M(i,0)
\end{tikzcd}$$

since $N \in \mathrm{rep}(\widetilde{A}^{(i)})$,by mesh relation ,we have $N_{i,in} \circ N_{i,out} = 0$,hence $\exists! g(i,1) : N(i,1) \to \ker M_{i,in} = \mathrm{R}^*_{(i-1,i)}(M)(i,1)$ which makes the left-hand square commutative.Therefore exists a unique $\widetilde{g} \in \mathrm{Hom}_{\widetilde{A}^{(i)}}(N, R^*_{(i-1,i)}(M))$,such that $\mathrm{Res}_{(i-1,i)}(\widetilde{g}) = g$.

Thus,$R^*_{(i-1,i)} \cong \mathrm{Res}^*_{(i-1,i)}$.So

$$\mathrm{Res}^{(i)} \circ \mathrm{Res}^*_{(i-1,i)}(M) \cong \Sigma_i^+ \circ \mathrm{Res}^{(i-1)}(M)$$

for all $M \in \mathrm{rep}(\widetilde{A}^{(i-1)}), i \in Q_0$. □

Lemma 2. *With the above notations we have functorial isomorphism*

$$\mathrm{Res}_1 \circ \mathrm{Res}_0^*(M) \cong C^+(M)$$

for $M \in \mathrm{rep}(Q)(= \mathrm{rep}(Q_{(0)}))$.

Proof.

$$\begin{aligned} \mathrm{Res}_1 \circ \mathrm{Res}_0^*(M) &= \mathrm{Res}^{(n)} \circ \mathrm{Res}^*_{(n-1,n)} \cdots \circ \mathrm{Res}^*_{(0,1)}(M) \\ &\cong \Sigma_n^+ \circ \mathrm{Res}^{(n-1)} \circ \mathrm{Res}^*_{(n-2,n-2)} \circ \cdots \circ \mathrm{Res}^*_{(0,1)}(M) \\ &\cong \Sigma_n^+ \circ \Sigma_{n-1}^+ \circ \cdots \circ \Sigma_1^+ \circ \mathrm{Res}^{(0)}(M) \\ &= C^+(M) \end{aligned}$$

□

For $M \in \mathrm{rep}(Q)$,we have a projective resolution

$$0 \longrightarrow \bigoplus_{\alpha \in Q_1} Ae_{t(\alpha)} \otimes M(s(\alpha)) \xrightarrow{d} \bigoplus_{i \in Q_0} Ae_i \otimes M(i) \xrightarrow{mult} M \longrightarrow 0$$

where

$$d(a \otimes m) = a\alpha \otimes m - a \otimes \alpha m, \quad \alpha \in Q_1, a \in Ae_{t(\alpha)}, m \in M(s(\alpha))$$

$$mult(a \otimes m) = am, \quad a \in Ae_i, m \in M(i), i \in Q_0.$$

Hence we have a projective resolution for $T(M)$

$$0 \longrightarrow \bigoplus_{\alpha \in Q_1} Ae_{t(\alpha)} \otimes M(s(\alpha)) \xrightarrow{d} \bigoplus_{i \in Q_0} Ae_i \otimes M(i) \xrightarrow{mult} T(M) \longrightarrow 0 \qquad (1)$$

where

$$d(a \otimes m) = a\alpha \otimes m + a \otimes \alpha m, \quad \alpha \in Q_1, a \in Ae_{t(\alpha)}, m \in M(s(\alpha))$$

$$mult(a \otimes m) = a.m \text{ (action in } T(M)), \quad a \in Ae_i, m \in M(i), i \in Q_0.$$

The projective resolution (1) of $T(M)$ is not minimal in general. However, it is the direct sum of a minimal projection resolution and an exact sequence of the form

$$0 \longrightarrow P = P \longrightarrow 0 \longrightarrow 0$$

Applying the Nakayama functor $v = D\mathrm{Hom}_A(-, A)$ to (1),we get an exact sequence

$$0 \longrightarrow \tau(TM) \longrightarrow \bigoplus_{\alpha \in Q_1} D(e_{t(\alpha)}A) \otimes M(s(\alpha)) \xrightarrow{v(d)} D(e_iA) \otimes M(i)$$

where

$$v(d)(f\otimes m)=f(\alpha\cdot -)\otimes m+f\otimes \alpha m,\quad \alpha\in Q_1, f\in D(e_{t(\alpha)}A), m\in M(s(\alpha))$$

Since we have natural isomorphism for $\alpha\in Q_1$:

$$D(e_{t(\alpha)}A)\otimes M(s(\alpha))\cong \mathrm{Hom}_K(e_{t(\alpha)}A, M(s(\alpha))),\ f\otimes m\longmapsto (x\longmapsto f(x)m)$$

Then we have a commutative diagram

$$\begin{array}{ccc}
\bigoplus\limits_{\alpha\in Q_1} D(e_{t(\alpha)}A)\otimes M(s(\alpha)) & \xrightarrow{v(d)} & \bigoplus\limits_{i\in Q_0} D(e_iA)\otimes M(i) \\
\downarrow & & \downarrow \\
\bigoplus\limits_{\alpha\in Q_1} \mathrm{Hom}_K(\alpha^*A_1, M(s(\alpha))) & \xrightarrow{\Phi} & \bigoplus\limits_{i\in Q_0} \mathrm{Hom}_K(e_{(i,1)}A_1, M(i))
\end{array}$$

where

$$\Phi:(\phi_\alpha)_\alpha\longmapsto\left(\sum_{s(\alpha)=i}\phi_\alpha(\alpha^*\alpha^{(1)}\cdot -)+\sum_{t(\alpha)=i}M(\alpha)(\phi_\alpha(\alpha^*\cdot -))\right)_i$$

For $M\in \mathrm{rep}(A_0)$. We first define $\widetilde{M}\in\mathrm{rep}(\widetilde{Q})$ by requiring that

$$\mathrm{Res}_0(\widetilde{M})=M$$

$$\mathrm{Res}_1(\widetilde{M})=\bigoplus_{\alpha\in Q_1}\mathrm{Hom}_K(\alpha^*A_1, M(s(\alpha),0))$$

It remains to define the structure map for $\beta: i\to j\in Q_1$:

$$\widetilde{M}(\beta^*):\widetilde{M}(j,1)\longrightarrow \widetilde{M}(i,0)=M(i,0)$$

This is given by the following composition:

$$\bigoplus_{\alpha\in Q_1}\mathrm{Hom}_K(\alpha^*A_1e_{(j,1)}, M(s(\alpha),0))\xrightarrow{proj}\mathrm{Hom}_K(\beta^*A_1e_{(j,1)}, M(i,0))=\mathrm{Hom}_K(K\beta^*, M(i,0))\xrightarrow{\text{eval}} M(i,0)$$

That is

$$\widetilde{M}(\beta^*)((\phi_\alpha)_\alpha)=\phi_\beta(\beta^*).$$

Secondly,we define a subrepresentation $R_0^*(M)$ of $\widetilde{M}$ as follow:

$$(R_0^*(M))(i,0)=\widetilde{M}(i,0)=M(i,0),\ i\in Q_0$$

$$\begin{aligned}&(R_0^*(M))(i,1)\\
=&\left\{(\phi_\alpha)_\alpha\in\widetilde{M}(i,1)\ \middle|\ \sum_{s(\alpha)=j}\phi_\alpha(\alpha^*\alpha^{(1)}x)+\sum_{t(\alpha)=j}M(\alpha^{(0)})(\phi_\alpha(\alpha^*x))=0,\forall j\in Q_0, x\in e_{(j,1)}A_1e_{(i,1)}\right\}\end{aligned}$$

Check:$R_0^*(M) \in \mathrm{rep}(\widetilde{Q}, I)$.

For $\forall(\phi_\beta)_\beta \in (R_0^*M)(i,1)$, we have

$$\begin{aligned}
&\sum_{s(\alpha)=i} (R_0^*M)(\alpha^*)(R_0^*M)(\alpha^{(1)})((\phi_\beta)_\beta) + \sum_{s(\alpha)=i} (R_0^*M)(\alpha^{(0)})(R_0^*M)(\alpha^*)((\phi_\beta)_\beta) \\
=&\sum_{s(\alpha)=i} (R_0^*M)(\alpha^*)((\phi_\beta(- \cdot \alpha^{(1)}))_\beta) + \sum_{s(\alpha)=i} (R_0^*M)(\alpha^{(0)})(\phi_\alpha(\alpha^*)) \\
=&\sum_{s(\alpha)=i} \phi_\alpha(\alpha^*\alpha^{(1)}) + \sum_{t(\alpha)=i} M(\alpha^{(0)})(\phi_\alpha(\alpha^*)) \\
=&0
\end{aligned}$$

Hence $R_0^*(M) \in \mathrm{rep}(\widetilde{Q}, I)$.

Lemma 3. *R_0^* is right adjoint to* Res_0.

Proof. Let $N \in \mathrm{rep}(\widetilde{Q}, I), M \in \mathrm{rep}(Q^{(0)})$.Consider $\varphi \in \mathrm{Hom}_{\widetilde{A}}(N, R_0^*(M))$.Thus,$\varphi$ is given by a family of $K-$linear maps

$$\varphi^{(i,a)} \in \mathrm{Hom}_K(N(i,a), (R_0^*M)(i,a))$$

subject to the usual commutativity relations.Then

$$\varphi_1 = (\varphi^{(i,0)})_{i \in Q_0} \in \mathrm{Hom}_{A_0}(\mathrm{Res}_0(N), M)$$

Hence we have a restriction

$$r_{N,M} : \mathrm{Hom}_{\widetilde{A}}(N, R_0^*(M)) \longrightarrow \mathrm{Hom}_{A_0}(\mathrm{Res}_0(N), M),\ \varphi \longmapsto \varphi_1$$

Let us denote by $\varphi_\beta^{(i,1)}(-, n)$ the β-component of $\varphi^{(i,1)}(n), n \in N(i,1)$.

For each arrow $\alpha : i \to j$ in Q_1.We have the commutative diagram

$$\begin{tikzcd}
N(i,1) \arrow[r, "N(\alpha^{(1)})"] \arrow[d, "\varphi^{(i,1)}"] & N(j,1) \arrow[d, "\varphi^{(j,1)}"] \\
(R_0^*M)(i,1) \arrow[r, "(R_0^*M)(\alpha^{(1)})"] & (R_0^*M)(j,1)
\end{tikzcd}$$

So

$$\varphi_\beta^{(i,1)}(- \cdot \alpha^{(1)}, n) = \varphi_\beta^{(j,1)}(-, \alpha^{(1)} n),\ \forall \beta \in Q_1, n \in N(i,1) \tag{2}$$

For each arrow $\gamma : k \to i$, we have the commutative diagram

$$\begin{tikzcd}
N(i,1) \arrow[r, "N(\gamma^*)"] \arrow[d, "\varphi^{(i,1)}"] & N(k,0) \arrow[d, "\varphi^{(k,0)}"] \\
(R_0^*M)(i,1) \arrow[r, "(R_0^*M)(\gamma^*)"] & (R_0^*M)(k,0) = M(k,0)
\end{tikzcd}$$

So

$$\varphi_{\gamma}^{(i,1)}(\gamma^*, n) = \varphi^{(k,0)}(N(\gamma^*)(n)), \ \forall n \in N(i,1). \tag{3}$$

Combining (2) and (3) we can see that the map $\varphi_{\beta}^{(j,1)}$ for $\beta \in Q_1$ and $j \in Q_0$ are determined by the map $\varphi^{(i,0)}$ with $i \in Q_0$,in other words $r_{N,M}$ is injective.

For $\psi \in \mathrm{Hom}_{A_0}(\mathrm{Res}_0(N), M)$.

Define

$$\varphi^{(i,0)} = \psi^{(i,0)}, \ i \in Q_0$$

$$\begin{array}{rccc} \varphi^{(i,1)}: & N(i,1) & \longrightarrow & (R_0^*M)(i,1) \\ & n & \longmapsto & (\varphi_{\alpha}^{(i,1)}(-,n) : a \in \alpha^* A_1 e_{(i,1)} \longmapsto \psi^{(s(\alpha),0)}(an))_{\alpha} \end{array}$$

Since for $\forall j \in Q_0, x \in e_{(j,1)} A_1 e_{(i,1)}, n \in N(i,1)$,we have

$$\begin{aligned} & \sum_{s(\alpha)=j} \varphi_{\alpha}^{(i,1)}(\alpha^* \alpha^{(1)} x, n) + \sum_{t(\alpha)=j} M(\alpha^{(0)})(\varphi_{\alpha}^{(i,1)}(\alpha^* x, n)) \\ = & \sum_{s(\alpha)=j} \psi^{(j,0)}(\alpha^* \alpha^{(1)} x n) + \sum_{t(\alpha)=j} M(\alpha^{(0)}) \psi^{(s(\alpha),0)}(\alpha^* x n) \\ = & \sum_{s(\alpha)=j} \psi^{(j,0)}(\alpha^* \alpha^{(1)} x n) + \sum_{t(\alpha)=j} \psi^{(j,0)} N(\alpha^{(0)})(\alpha^* x n) \\ = & \psi^{(j,0)} \left[\left(\sum_{s(\alpha)=j} \alpha^* \alpha^{(1)} + \sum_{t(\alpha)=j} \alpha^{(0)} \alpha^* \right) x n \right] \\ = & 0 \ \ (\text{since } N \in \mathrm{rep}(\widetilde{Q}, I)) \end{aligned}$$

It is easy to check that $\varphi \in \mathrm{Hom}_{\widetilde{A}}(N, R_0^* M)$, and $r_{N,M}(\varphi) = \psi$. □

Lemma 4. *For $M \in \mathrm{rep}(Q)$,we have*

$$\tau(TM) \cong \mathrm{Res}_1 \circ R_0^*(M)$$

where in the right-hand side A_0 and A_1 are identified with A by means of the isomorphisms η_0, η_1.

Proof. By the construction of $R_0^*(M)$. □

Finally,for $M \in \mathrm{rep}(Q)$,there is

$$\tau(M) = \tau(T^2 M) \cong \mathrm{Res}_1 \circ R_0^*(TM) \cong \mathrm{Res}_1 \circ \mathrm{Res}_0^*(TM) \cong C^+ T(M) = TC^+(M).$$

This proves theorem1.

Since C^- is left adjoint to C^+, and gl.dim$A \leq 1$,we have

$$\mathrm{Hom}_A(\tau^-(M), N) \cong \mathrm{Hom}_A(M, \tau(N)), \quad M, N \in A-mod$$

that is τ^- is left adjoint to τ,hence

$$\tau^-(M) \cong TC^-(M), \quad \forall M \in \mathrm{rep}(Q).$$

References

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