Let k be an algebraically closed field. Please answer the following questions independently.

1. Let $P_1, P_2, \cdots, P_n \in \mathbb{P}_k^2$ and $[X, Y, Z]$ be the homogeneous coordinate of $\mathbb{P}_k^2$. Prove that

(1) there exists a projective transform T such that $P_1^T, P_2^T, \cdots, P_n^T \subset \{Z \neq 0\}$;

(2) for a given curve C there exists a projective transform T such that none of $P_1^T, P_2^T, \cdots, P_n^T$ lies on C.

2. Let $P = (0,0) \in \mathbb{A}^2$ and $F = 3x^2 + y^3, G = x + xy + y^4, H = xy + y^2$.

(1) Compute $I_P(F \cap G)$;

(2) P is a simple point of G and write out a uniformizer;

(3) whether H satisfies Noether's condition for F and G at P.

3. Let $P = (0,0) \in \mathbb{A}^2$ and $I = (x, y)_P$. Let $f, g \in k[x, y]$ be two elements prime to each other. Write that $f = f_{m_1} + f_{>m_1}$ and $g = g_{m_2} + g_{>m_2}$ where f_{m_1}, g_{m_2} are the homogeneous part of minimal degree of f, g respectively. Show that

(1) there exists a number n such that $I^n \subseteq (f, g)_P$;

(2) if $I^n \subseteq (f_{m_1}, g_{m_2})_P$ then $I^n \subseteq (f, g)_P$.

4. Let $C \subset \mathbb{P}^2$ be the cubic curve defined by $X^3 + Y^3 = Z^3$ ($char\ k \neq 3$).

(1) Show that C is nonsingular;

(2) for a fixed point $P \in C$, write out the equation of the tangent line L_P;

(3) find all possible point $P \in C$ such that $I_P(L_P \cap C) = 3$.

5. Assume $char\ k = 0$. Show that

(1) a projective plane curve $F(X, Y, Z) = 0$ is non-singular if and only if, for sufficiently large N the ideal (F, F_X, F_Y, F_Z) contains X^N, Y^N, Z^N;

(2) for any $d > 0$, there exists a nonsingular projective plane curve of degree d;

(3) the set of singular plane curves of degree $d = 2$ is a proper closed subset of the linear system $V(d) \cong \mathbb{P}^5$.