A Mathematica symbolic code to evaluate Weingarten functions

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Abstract

We provide a simple Mathematica code to quickly evaluate Weingarten functions and related integrals for arbitrary d.

1 Introduction

There is a multiplicity of applications requiring the computation of averages of various functions over the unitary groups in dimension d or the behaviour of such functions in the limit of large d [1, 2, 3, 4]. The computation of this type of integral is considerably simplified using the so-called *Weingarten functions*, a nomenclature introduced in Ref. [5].

Weingarten functions depend only on a class of symmetric group S_p and on the dimension d of the unitaries that are averaged. A convenient closed form expression has been given in Ref. [5]:

$$\int U_{i_1 j_1} \dots U_{i_p j_p} U^*_{i'_1 j'_1} \dots U^*_{i'_p, j'_p}\, dU = \sum_{\sigma,\tau \in S_p} \mathrm{Wg}([\sigma\tau^{-1}]; d)\,, \tag{1}$$

$$\mathrm{Wg}(\sigma\tau^{-1}; d) = \frac{1}{(p!)^2} \sum_{\lambda} \frac{\chi^\lambda(\mathbf{1})^2 \chi^\lambda([\sigma\tau^{-1}])}{s_{\lambda,d}} \tag{2}$$

where U is a Haar-random $d \times d$ unitary matrix, dU is the Haar measure over $U(d)$, and $[\sigma]$ is the class of element σ. The sum in Eq.(1) is a sum over all $\sigma \in S_p$ and all the $\tau \in S_p$ so that

$$(i'_{\sigma(1)}, \dots, i'_{\sigma(p)}) = (i_1, \dots, i_p)\,, \tag{3}$$

$$(j'_{\tau(1)}, \dots, j'_{\tau(p)}) = (j_1, \dots, j_p)\,, \tag{4}$$

with the integral 0 is the i', i, j' or j strings have different lengths. In other words, the integral on the left of Eq. (1) is a sum of Weingarten functions. In the expression for Wg, $\chi^\lambda(\mathbf{1})^2$ is the dimension of irrep λ of S_p, $\chi^\lambda(\sigma)$ is the character of element σ in the irrep λ, $s_{\lambda,d}$ is the dimension of irrep λ of $U(d)$, and the sum over λ is a sum over all partitions of p. Alternate derivations and properties, including generalizations to the orthogonal and the symplectic groups, can be found in Refs. [6, 7, 8, 9].

In this note I describe the workings of a Mathematica code to quickly evaluate Weingarten functions as given in Eq. (2). The code is built on an implementation of the Murnaghan-Nakayama rule for the characters irreducible representations of the symmetric group S_p, provided to me by Dr. Justin Kulp [10]. The dimensionality factor $s_{\lambda,d}$ can be computed in the usual way using the hook-rule [11]. To speed up calculations it was found useful to write a dedicated function to evaluate the dimension of the irrep λ of S_p, also using the hook-rule.

2 The main functions: eWg and cWg

The code contains two main functions: eWg and cWg. The functions differ in their arguments, as explained in details a little later. The LHS of Eq.(2) is seen to depend on the product $\sigma\tau^{-1}$ of elements in S_p but in fact the RHS depends only on the class of $\sigma\tau^{-1}$. Furthermore, the sum on the RHS of Eq.(1) is a sum over products of elements in S_p so it is convenient to define eWg when group elements are used as inputs, and cWg for use when classes of elements are used as inputs.

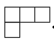
The cWg functions takes as input a class and a dimension parameter d:

```
cWg[class_,d_]
```

and returns the RHS of Eq. (2). Classes are partitions inside curly brackets. Thus for instance:

```
In[1]:=   cWg[{3,1},d]
```

$$\texttt{Out[1]=}\quad \frac{-3+2\ d^2}{(-3+d)\ (-2+d)\ (-1+d)\ d^2\ \ (1+d)\ (2+d)\ (3+d)}$$

is the Weingarten function for the integration of $U(d)$ functions, for the class $\{3,1\}$ corresponding to the Young diagram .

The function eWg has slightly different inputs:

```
eWg[sigma_,p_,d_]
```

Mathematica constructs elements of S_p in the form of cycles, such as Cycles[{{2,3,4}}]. However, it is not possible to full determine from the information in Cycles if this is an element of S_4, S_5 *etc* hence the need to supply the additional parameter p to indicate this is an element in S_p. The output is of course the same as cWg if the order p and an element in the class are given as inputs:

```
In[2]:=   cWg[Cycles[{{2,3,4}}],4,d]
```

$$\texttt{Out[2]=}\quad \frac{-3+2\ d^2}{(-3+d)\ (-2+d)\ (-1+d)\ d^2\ \ (1+d)\ (2+d)\ (3+d)}$$

The dimension d of the unitaries can be passed directly to either functions:

```
In[3]:=   cWg[{3,1},6]
```

$$\texttt{Out[3]=}\quad \frac{23}{362880}$$

3 Auxiliary functions

3.1 murnNaka

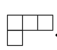
This function comes from the code of [10]. The inputs are a partition and a class:

```
murnNaka[partition_,class_]
```

The output is the character of the elements in `class` of the irrep of the symmetric group labelled by `partition`. For example:

```
In[4]:=   murnNaka[{3,1},{1,1,1,1}]

Out[4]=   3
```

which is also the dimension of irrep .

3.2 getClass

This function inputs an element in S_p and returns the class of this element:

```
getClass[p_,cycles_]
```

For example:

```
In[5]:=   getClass[4,Cycles[{{1,2,3}}]]

Out[5]=   {3,1}
```

3.3 snDimension

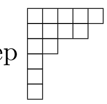
This function inputs a `partition`

```
snDimension[partition_]
```

and returns the dimension of the irrep of S_p labelled by a `partition`. It uses the hook-length formula:
For instance

```
In[6]:=   snDimension[{5,4,2,1,1,1}]

Out[6]=   63063
```

snDimension gives the same result as murnNaka when the class is $(1, \ldots, 1)$ but is considerably faster than murnNaka when p is large and the partition contains many parts. For instance, the dimension of irrep  of S_{14} evaluated using snDimension returns 63063 in $\sim 10^{-4}$ second on a MacBook Air whereas murnNaka returns the same result in roughly $\lesssim 90$ seconds.

3.4 udDimension

This functions outputs the dimension of the $U(d)$ labelled by `ppartition`:

```
udDimension[ppartition_, d_].
```

Thus, for the partition $(5, 4, 2, 1, 1, 1)$, we have

```
In[7]:=   udDimension[{5,4,2,1,1,1},d]
```

$$\text{Out[7]=}\quad \frac{(-5+d)\ (-4+d)\ (-3+d)\ (-2+d)\ (-1+d)^2\ d^2\ (1+d)^2\ (2+d)^2\ (3+d)\ (4+d)}{1382400}$$

```
In[8]:=   udDimension[{5,4,2,1,1,1},8]

Out[8]=   873180
```

The dimension d of the unitary must be greater than or equal to the number of parts in the partition, or alternatively the d must be greater than or equal to the number of rows in the Young diagram associated with the partition, else there is no irrep for this partition and the function returns 0:

```
In[9]:=   udDimension[{5,4,2,1,1,1},4]

Out[9]=   0
```

3.5 gClass

This functions outputs the number of elements in the class of S_p labelled by `partition`:

```
gClass[partition_].
```

Thus, for the partition $(5, 1, 1)$ of S_7, we have

```
In[10]:=   gClass[{5,1,1}]

Out[10]=   504
```

4 Tables of Weingarten functions for $n \leq 6$

S_2: integrals of the type $\int dU U_{i\alpha} U_{j\beta} U^*_{k\eta} U^*_{\ell\nu}$.

Class	Wg
$\{2\}$	$-\frac{1}{(d-1)d(d+1)}$
$\{1,1\}$	$\frac{1}{(d-1)(d+1)}$

(5)

S_3 integrals of the type $\int dU U_{i\alpha} U_{j\beta} U_{k\eta} U^*_{m\mu} U^*_{\ell\nu} U^*_{p\kappa}$.

Class	Wg
$\{3\}$	$\frac{2}{(d-2)(d-1)d(d+1)(d+2)}$
$\{2,1\}$	$-\frac{1}{(d-2)(d-1)(d+1)(d+2)}$
$\{1,1,1\}$	$\frac{d^2-2}{(d-2)(d-1)d(d+1)(d+2)}$

(6)

S_4:

Class	Wg
$\{4\}$	$-\frac{5}{(d-3)(d-2)(d-1)d(d+1)(d+2)(d+3)}$
$\{3,1\}$	$\frac{2d^2-3}{(d-3)(d-2)(d-1)d^2(d+1)(d+2)(d+3)}$
$\{2,2\}$	$\frac{d^2+6}{(d-3)(d-2)(d-1)d^2(d+1)(d+2)(d+3)}$
$\{2,1,1\}$	$-\frac{1}{(d-3)(d-1)d(d+1)(d+3)}$
$\{1,1,1,1\}$	$\frac{d^4-8d^2+6}{(d-3)(d-2)(d-1)d^2(d+1)(d+2)(d+3)}$

(7)

S_5

Class	Wg
$\{5\}$	$\frac{14}{(d-4)(d-3)(d-2)(d-1)d(d+1)(d+2)(d+3)(d+4)}$
$\{4,1\}$	$\frac{24-5d^2}{(d-4)(d-3)(d-2)(d-1)d^2(d+1)(d+2)(d+3)(d+4)}$
$\{3,2\}$	$-\frac{2\left(d^2+12\right)}{(d-4)(d-3)(d-2)(d-1)d^2(d+1)(d+2)(d+3)(d+4)}$
$\{3,1,1\}$	$\frac{2}{(d-4)(d-2)(d-1)d(d+1)(d+2)(d+4)}$
$\{2,2,1\}$	$\frac{d^2-2}{(d-4)(d-3)(d-2)(d-1)d(d+1)(d+2)(d+3)(d+4)}$
$\{2,1,1,1\}$	$\frac{-d^4+14d^2-24}{(d-4)(d-3)(d-2)(d-1)d^2(d+1)(d+2)(d+3)(d+4)}$
$\{1,1,1,1,1\}$	$\frac{d^4-20d^2+78}{(d-4)(d-3)(d-2)(d-1)d(d+1)(d+2)(d+3)(d+4)}$

(8)

S_6:

Class	Wg
$\{6\}$	$-\frac{42}{(d-5)(d-4)(d-3)(d-2)(d-1)d(d+1)(d+2)(d+3)(d+4)(d+5)}$
$\{5,1\}$	$\frac{14\left(d^2-10\right)}{(d-5)(d-4)(d-3)(d-2)(d-1)d^2(d+1)(d+2)(d+3)(d+4)(d+5)}$
$\{4,2\}$	$\frac{5\left(d^4+15d^2+8\right)}{(d-5)(d-4)(d-3)(d-2)(d-1)^2d^2(d+1)^2(d+2)(d+3)(d+4)(d+5)}$
$\{4,1,1\}$	$\frac{13-5d^2}{(d-5)(d-3)(d-2)(d-1)^2d(d+1)^2(d+2)(d+3)(d+5)}$
$\{3,3\}$	$\frac{4\left(d^4+29d^2-90\right)}{(d-5)(d-4)(d-3)(d-2)(d-1)^2d^2(d+1)^2(d+2)(d+3)(d+4)(d+5)}$
$\{3,2,1\}$	$\frac{-2d^2-13}{(d-5)(d-4)(d-2)(d-1)^2d(d+1)^2(d+2)(d+4)(d+5)}$
$\{3,1,1,1\}$	$\frac{2d^6-51d^4+229d^2-60}{(d-5)(d-4)(d-3)(d-2)(d-1)^2d^2(d+1)^2(d+2)(d+3)(d+4)(d+5)}$
$\{2,2,2\}$	$\frac{-d^4-d^2-358}{(d-5)(d-4)(d-3)(d-2)(d-1)^2d(d+1)^2(d+2)(d+3)(d+4)(d+5)}$
$\{2,2,1,1\}$	$\frac{d^4-3d^2+10}{(d-5)(d-3)(d-2)(d-1)^2d^2(d+1)^2(d+2)(d+3)(d+5)}$
$\{2,1,1,1,1\}$	$\frac{-d^4+24d^2-38}{(d-5)(d-4)(d-2)(d-1)^2d(d+1)^2(d+2)(d+4)(d+5)}$
$\{1,1,1,1,1,1\}$	$\frac{d^8-41d^6+458d^4-1258d^2+240}{(d-5)(d-4)(d-3)(d-2)(d-1)^2d^2(d+1)^2(d+2)(d+3)(d+4)(d+5)}$

(9)

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A Appendix: Some examples

A.1 First example

Suppose we wish to evaluate the sum of products

$$\sum_{j,k=1}^{d} \sum_{\alpha,\beta} \int dU \, U_{j\alpha} U_{k\alpha} U^*_{k\beta} U^*_{j\beta} \tag{10}$$

using Weingarten functions.

For convenience, we make two arrays. The first contains the Latin indices of the unstarred product $U_{j\alpha}U_{k\alpha}$ on the first row, and the Latin indices of the starred product $U^*_{k\beta}U^*_{j\beta}$ on the second row. The second array contains the Greek indices of these products:

$$\begin{pmatrix} j & k \\ j & k \end{pmatrix}, \qquad \begin{pmatrix} \alpha & \alpha \\ \beta & \beta \end{pmatrix} \tag{11}$$

An integral of the type

$$\int dU \, U_{j\alpha} U_{k\alpha} U^*_{k\beta} U^*_{j\beta} \tag{12}$$

is 0 unless there is a permutation $\sigma \in S_2$ that takes the bottom row (j,k) of the first array to the first row (j,k) of this array, and unless there is a permutation $\tau \in S_2$ that takes the bottom row (β,β) of the second array to the first row (α,α) of this array.

The two elements of S_2 are $P(e)$ and $P(12)$ and they act by $P(e)(a,b) = (a,b)$ and $P(12)(a,b) = (b,a)$. Thus

$$P(e)(\beta,\beta) = (\beta,\beta)\,, \qquad P(12)(\beta,\beta) = (\beta,\beta)\,. \tag{13}$$

Hence, the Weingarten integrals are all 0 unless $\alpha = \beta$, and Eq.(10) simplifies to

$$\sum_{j,k=1}^{d} \sum_{\alpha} \int dU \, U_{j\alpha} U_{k\alpha} U^*_{k\alpha} U^*_{j\alpha} \tag{14}$$

Next, we need to consider separately the cases $j = k$ and $j \neq k$.

Fix some α, suppose $j = k$, and consider

$$\int dU \, U_{j\alpha} U_{j\alpha} U^*_{j\alpha} U^*_{j\alpha} \tag{15}$$

Then the integral is a sum of Weingarten functions over permutations $\sigma \in S_2$ and τ such that

$$\sigma(j,j) = (j,j)\,, \qquad \tau(\alpha,\alpha) = (\alpha,\alpha). \tag{16}$$

This is true for every $\sigma \in S_2$ and every $\tau \in S_2$ so

$$\int dU \, U_{j\alpha} U_{j\alpha} U^*_{j\alpha} U^*_{j\alpha} = \sum_{\sigma=e,(12)} \sum_{\tau=e,(12)} \mathrm{Wg}(\tau^{-1} \circ \sigma, d)\,, \tag{17}$$

$$= 2\mathrm{Wg}(e,d) + 2\mathrm{Wg}((12),d)\,, \tag{18}$$

$$= \frac{2}{d^2-1} - \frac{2}{d(d^2-1)} = \frac{2}{d(d+1)} \tag{19}$$

There are obviously d terms of this type (one for each j) so we get

$$\sum_{j} \int dU \, U_{j\alpha} U_{j\alpha} U^*_{j\alpha} U^*_{j\alpha} = \frac{2}{(d+1)} \tag{20}$$

Next, suppose $j \neq k$, and consider

$$\int dU \, U_{j\alpha} U_{k\alpha} U^*_{k\alpha} U^*_{j\alpha} \tag{21}$$

Then the integral is a sum of Weingarten functions over permutations $\sigma \in S_2$ and τ such that

$$\sigma(j,k) = (j,k)\,, \qquad \tau(\alpha,\alpha) = (\alpha,\alpha). \tag{22}$$

This is true for $\sigma = \mathbf{1} \in S_2$ and every $\tau \in S_2$ so

$$\int dU \, U_{j\alpha} U_{k\alpha} U^*_{k\alpha} U^*_{j\alpha} = \sum_{\sigma=e} \sum_{\tau=e,(12)} \mathrm{Wg}(\tau^{-1} \circ \sigma, d)\,, \tag{23}$$

$$= \mathrm{Wg}(e,d) + \mathrm{Wg}((12),d)\,, \tag{24}$$

$$= \frac{1}{d^2-1} - \frac{1}{d(d^2-1)} = \frac{1}{d(d+1)}\,. \tag{25}$$

There are $d(d-1)$ such terms so we have

$$\sum_{j\neq k=1}^{d} \int dU \, U_{j\alpha} U_{k\alpha} U^*_{k\alpha} U^*_{j\alpha} = \frac{d(d-1)}{d(d+1)}\,. \tag{26}$$

Thus, for each α we have

$$\sum_{j,k=1}^{d} \int dU \, U_{j\alpha} U_{k\alpha} U^*_{k\alpha} U^*_{j\alpha} = \frac{2}{d+1} + \frac{d-1}{d+1} = \frac{d+1}{d+1} = 1\,, \tag{27}$$

and since there are d values of α we have finally

$$\sum_{j,k=1}^{d} \sum_{\alpha=1}^{d} \int dU \, U_{j\alpha} U_{k\alpha} U^*_{k\alpha} U^*_{j\alpha} = d\,. \tag{28}$$

A.2 Circular ensembles and Weingarten functions

Let

$$V_{ij} = \sum_{k=1^d} U_{ki} U_{kj}\,. \tag{29}$$

A.2.1 $\int dU |V_{ii}|^4$

Matsumoto in Ref. [3] considers the evaluation of certain polynomials in the entries of symmetric unitary random matrices V from a circular orthogonal example.

Here, we verify the expression given in Theorem 1.1 of Ref. [3] for $n = 2$:

$$\int dU |V_{ii}|^4 = \frac{8}{(d+1)(d+3)}\,, \tag{30}$$

$$= \sum_{kmnp} \int dU U_{ki} U_{ki} U_{mi} U_{mi} U^*_{ni} U^*_{ni} U^*_{pi} U^*_{pi} \tag{31}$$

using Weingarten calculus.

Again we construct two arrays using the indices in the products of starred and unstarred U's:

$$\begin{pmatrix} k & k & m & m \\ n & n & p & p \end{pmatrix}, \qquad \begin{pmatrix} i & i & i & i \\ i & i & i & i \end{pmatrix}. \tag{32}$$

There are two types of terms. First, $k = m$ which in turn implies $k = n = p$, so we have

$$\int dU U_{ki} U_{ki} U_{ki} U_{ki} U^*_{ki} U^*_{ki} U^*_{ki} U^*_{ki} \tag{33}$$

Here, for fixed k, the set of S_4 transformations mapping the indices $(kkkk)$ of the bottom row of the first array to $(kkkk)$ is S_4, and the set of S_4 transformations mapping the indices $(iiii)$ to $(iiii)$ is also S_4. Now, for fixed σ, the sum over all $\tau \in S_4$ of $\tau^{-1}\sigma$ simply gives every element in S_4 again. As there are $4! = 24$ possible σ, we find

$$\int dU U_{ki} U_{ki} U_{ki} U_{ki} U^*_{ki} U^*_{ki} U^*_{ki} U^*_{ki} = 24 \sum_{\tau \in S_4} \mathrm{Wg}(\tau; d)\,, \tag{34}$$

$$= 24 \sum_{\lambda \vdash 4} g_\lambda \mathrm{Wg}(\lambda; d) \tag{35}$$

where λ is a partition of 4 and g_λ is the number of elements in the class λ of S_4. For this case we have

$$g_{(4)} = 6\,, \quad g_{(3,1)} = 8, \quad g_{(2,2)} = 3, \quad g_{(2,1,1)} = 6\,, \quad g_{(1,1,1,1)} = 1 \tag{36}$$

and thus

$$\int dU U_{ki} U_{ki} U_{ki} U_{ki} U^*_{ki} U^*_{ki} U^*_{ki} U^*_{ki} = 24\,(6\mathrm{Wg}((4); d) + 8\mathrm{Wg}((3,1); d) + 3\mathrm{Wg}((2,2); d)$$
$$+6\mathrm{Wg}((2,1,1); d) + 6\mathrm{Wg}((1,1,1,1); d))\;. \tag{37}$$

We can use the results of Eq.(7) to obtain the result

$$\int dU U_{ki} U_{ki} U_{ki} U_{ki} U^*_{ki} U^*_{ki} U^*_{ki} U^*_{ki} = \sum_{\sigma \in S_4} \sum_{\tau \in S_4} \mathrm{Wg}(\tau^{-1}\sigma; d)\,, \tag{38}$$

$$= \frac{24}{d(d+1)(d+2)(d+3)}\,. \tag{39}$$

There are d possible values of k.

Next, there are terms of the form $k \neq m$, $n \neq p$. For instance:

$$\int dU U_{1i} U_{1i} U_{2i} U_{2i} U^*_{ni} U^*_{ni} U^*_{pi} U^*_{pi}\,. \tag{40}$$

Here, the indices $(nnpp)$ must be a permutation of (1122) so that $n = 1, p = 2$ or $n = 2, p = 1$. In the first case, the $\sigma \in S_4$ that permute (1122) into (1122) are

$$\sigma \in \{e, P(34), P(12), P((12)(34))\}\,. \tag{41}$$

The τ's are again all the elements in S_4 so

$$\int dU U_{1i} U_{1i} U_{2i} U_{2i} U^*_{ni} U^*_{ni} U^*_{pi} U^*_{pi} = \sum_{\sigma \in \{e, P(34), P(12), P((12)(34))\}} \sum_{\tau \in S_4} \mathrm{Wg}(\tau^{-1}\sigma; d)\,, \tag{42}$$

$$= \frac{4}{d(d+1)(d+2)(d+3)}\,. \tag{43}$$

There are $2d(d-1)$ such terms.

Thus summing all the pieces:

$$\int dU |V_{ii}|^4 = \frac{24d}{d(d+1)(d+2)(d+3)} + \frac{4 \times 2d(d-1)}{d(d+1)(d+2)(d+3)} = \frac{8}{(d+1)(d+3)} \tag{44}$$

again in agreement with Ref. [3].

A.2.2 $\int dU |V_{ij}|^4$

The expanded form of this integral is given by

$$\sum_{kmnp} \int dU U_{ki} U_{kj} U_{mi} U_{mj} U^*_{ni} U^*_{nj} U^*_{pi} U^*_{pj} \,, \qquad i \neq j \,. \tag{45}$$

There are two types of terms. First, $k = m$ which in turn implies $k = n = p$, so we have

$$\int dU U_{ki} U_{kj} U_{ki} U_{kj} U^*_{ki} U^*_{kj} U^*_{ki} U^*_{kj} \tag{46}$$

Here, for fixed k, the set of S_4 transformations mapping the indices $(kkkk)$ to $(kkkk)$ is S_4, and the set of S_4 transformations mapping the indices $(ijij)$ to $(ijij)$ is $\{\mathbb{1}, P_{24}, P_{13}, P_{13}P_{24}\}$. Thus, we have

$$\int dU U_{ki} U_{kj} U_{ki} U_{kj} U^*_{ki} U^*_{kj} U^*_{ki} U^*_{kj} = \sum_{\sigma \in S_4} \sum_{\tau = \mathbb{1}, P_{24}, P_{13}, P_{13}P_{24}} \mathrm{Wg}(\tau^{-1}\sigma; d) \,, \tag{47}$$

$$= \frac{4}{d(d+1)(d+2)(d+3)} \,. \tag{48}$$

There are d possible values of k.

Next, terms with $k \neq m, n \neq p$. This time:

$$\int dU U_{ki} U_{kj} U_{ki} U_{kj} U^*_{ki} U^*_{kj} U^*_{ki} U^*_{kj} = \sum_{\sigma, \tau = \mathbb{1}, P_{24}, P_{13}, P_{13}P_{24}} \mathrm{Wg}(\tau^{-1}\sigma; d) \,, \tag{49}$$

$$= \frac{d^2 + d + 2}{(d-1)d^2(d+1)(d+2)(d+3)} \,. \tag{50}$$

Again there are $2d(d-1)$ such terms. The total is then

$$\int dU |V_{ij}|^4 = \frac{2}{d(d+3)} \,, \qquad i \neq j \,. \tag{51}$$

For $d = 4$ this yields $1/14$, which agrees with numerical integration based on the parametrization and measure given in Ref. [12].

A.2.3 $\int dU |V_{ij}|^4$

We want to show that

$$\int dU |V_{ii}|^8 = \int dU \sum_{abcd} \sum_{mnpq} U_{ai} U_{ai} U_{bi} U_{bi} U_{ci} U_{ci} U_{di} U_{di} U^*_{mi} U^*_{mi} U^*_{ni} U^*_{ni} U^*_{pi} U^*_{pi} U^*_{qi} U^*_{qi} \,, \tag{52}$$

$$= \frac{384}{(d+1)(d+3)(d+5)(d+7)} \tag{53}$$

in agreement with Theorem 1.1 of Ref. [3].

There is only one type of indices of the form

$$\begin{pmatrix} i\,i\,i\,i\,i\,i\,i\,i \\ i\,i\,i\,i\,i\,i\,i\,i \end{pmatrix} \tag{54}$$

All $\tau \in S_8$ of the first row yield the second row.

For the indices inside the sums, there are 5 types of terms:

$$\begin{pmatrix} aa & bb & cc & dd \\ mm & nn & pp & qq \end{pmatrix} = \begin{pmatrix} aa & aa & aa & aa \\ aa & aa & aa & aa \end{pmatrix} \tag{55}$$

$$\begin{pmatrix} aa & bb & cc & dd \\ mm & nn & pp & qq \end{pmatrix} = \begin{pmatrix} aa & aa & aa & dd \\ aa & aa & aa & dd \end{pmatrix} \quad \text{and permutations of the upper and lower rows,} \tag{56}$$

$$\begin{pmatrix} aa & bb & cc & dd \\ mm & nn & pp & qq \end{pmatrix} = \begin{pmatrix} aa & aa & cc & dd \\ aa & da & cc & dd \end{pmatrix} \quad \text{and permutations of the upper and lower rows,} \tag{57}$$

$$\begin{pmatrix} aa & bb & cc & dd \\ mm & nn & pp & qq \end{pmatrix} = \begin{pmatrix} aa & bb & cc & dd \\ aa & bb & cc & dd \end{pmatrix} \quad \text{and permutations of the upper and lower rows,} \tag{58}$$

$$\begin{pmatrix} aa & bb & cc & dd \\ mm & nn & pp & qq \end{pmatrix} = \begin{pmatrix} aa & aa & cc & cc \\ aa & aa & cc & cc \end{pmatrix} \quad \text{and permutations of the upper and lower rows.} \tag{59}$$

1. For terms of the type

$$\begin{pmatrix} aa & bb & cc & dd \\ mm & nn & pp & qq \end{pmatrix} = \begin{pmatrix} aa & aa & aa & aa \\ aa & aa & aa & aa \end{pmatrix} \tag{60}$$

all $\sigma \in S_8$ permute the first row into the second row. There are d possible ways of choosing a so the contributions of this type of terms is

$$d \times \sum_{\sigma,\tau \in S_8} \mathrm{Wg}(\tau^{-1}\sigma; d) \tag{61}$$

The double sum can be simplified by noting that, for every *fixed* σ, the sum over all τ's simply yields all elements in S_8, with their enumeration shifted by σ. As there are 8! elements in S_8, we have

$$d \times \sum_{\sigma,\tau \in S_8} \mathrm{Wg}(\tau^{-1}\sigma; d) = d \times 8! \sum_{\tau} \mathrm{Wg}(\tau; d) \tag{62}$$

since summing over τ^{-1} is the same as summing over τ.

Next, we recall that the Weingarten functions are actually functions of classes so we can rewrite the sum as

$$d \times 8! \sum_{\tau} \mathrm{Wg}(\tau; d) = d \times 8! \times \sum_{\lambda \vdash 8} g_\lambda \mathrm{Wg}(\lambda; d)\,, \tag{63}$$

$$= \frac{40320\, d}{d(d+1)(d+2)(d+3)(d+4)(d+5)(d+6)(d+7)}\,. \tag{64}$$

where λ is a partion of 8 and g_λ is the number of elements in the class λ.

2. Next, we have terms of the type

$$\begin{pmatrix} aa & bb & cc & dd \\ mm & nn & pp & qq \end{pmatrix} = \begin{pmatrix} aa & aa & aa & dd \\ aa & aa & aa & dd \end{pmatrix} \tag{65}$$

The permutations that permute the first row into the second row are in $S_6 \times S_2$. Note that if instead we consider

$$\begin{pmatrix} aa & bb & cc & dd \\ mm & nn & pp & qq \end{pmatrix} = \begin{pmatrix} aa & aa & aa & dd \\ aa & aa & dd & aa \end{pmatrix} \tag{66}$$

the permutations are still in $S_6 \times S_2$ although this subgroup is not the same as the one in the previous case.

Fixed the second row; for each fixed $\sigma \in S_6 \times S_2$, the sum over all τ's in S_8 simply yields all elements in S_8, with their enumeration shifted by σ. Thus, for every fixed second row, there are now $6! \times 2!$ sums over all τ's.

Next, there are $d(d-1)$ ways of choosing $a \neq d$. For a fixed bottom row, there are 4 possible permutations of a's and d's in the first row, and there are 4 possible permutations of a and d's in the bottom row, so the contributions from all terms of this type are

$$\begin{aligned} d(d-1) \times 4 \times 4 \times \times 6! \times 2! \sum_{\tau \in S_8} \mathrm{Wg}(\tau; d) \\ = \frac{23040\, d(d-1)}{d(d+1)(d+2)(d+3)(d+4)(d+5)(d+6)(d+7)} . \end{aligned} \tag{67}$$

3. Now, terms of the type

$$\begin{pmatrix} aa & bb & cc\, dd \\ mm\, nn\, pp\, qq \end{pmatrix} = \begin{pmatrix} aa\, aa\, cc\, dd \\ aa\, aa\, cc\, dd \end{pmatrix} \tag{68}$$

Fix the second row. The elements that permute the first row into the second row are in $S_4 \times S_2 \times S_2$. Again, for each fixed $\sigma \in S_4 \times S_2 \times S_2$, the sum over all τ's simply yields all elements in S_8, with their enumeration shifted by σ. Thus, there are now $4! \times 2! \times 2!$ sums over all τ's.

There are $6 \times d(d-1)(d-2)$ possible choices (with permutations) of (aa, aa, cc, dd) on the first row, and for each of these there are 12 permutations of (aa, aa, cc, dd) on the bottom row.

Thus, we have for terms of this type

$$\begin{aligned} 6 \times d(d-1)(d-2) \times 12 \times 4! \times 2! \times 2! \sum_{\tau \in S_8} \mathrm{Wg}(\tau; d) \\ = \frac{6912\, d(d-1)(d-2)}{d(d+1)(d+2)(d+3)(d+4)(d+5)(d+6)(d+7)} \end{aligned} \tag{69}$$

4. Now, terms of the type

$$\begin{pmatrix} aa & bb & cc\, dd \\ mm\, nn\, pp\, qq \end{pmatrix} = \begin{pmatrix} aa\, bb\, cc\, dd \\ aa\, bb\, cc\, dd \end{pmatrix} \tag{70}$$

Fix the second row. The elements that permute the first row into the second row are in $S_2 \times S_2 \times S_2 \times S_2$. Again, for each fixed $\sigma \in S_2 \times S_2 \times S_2 \times S_2$, the sum over all τ's simply yields all elements in S_8, with their enumeration shifted by σ. Thus, there are now $2! \times 2! \times 2! \times 2!$ sums over all τ's.

There are $d(d-1)(d-2)(d-3)$ possible choices (with permutations) of (aa, bb, cc, dd) on the first row, and for each of these there are 24 permutations of (aa, bb, cc, dd) on the bottom row.

Thus, we have for terms of this type

$$\begin{aligned} d(d-1)(d-2)(d-3) \times 24 \times 2! \times 2! \times 2! \times 2! \sum_{\tau \in S_8} \mathrm{Wg}(\tau; d), \\ = \frac{384\, d(d-1)(d-2)(d-3)}{d(d+1)(d+2)(d+3)(d+4)(d+5)(d+6)(d+7)} \end{aligned} \tag{71}$$

5. Finally, terms of the type

$$\begin{pmatrix} aa & bb & cc\, dd \\ mm\, nn\, pp\, qq \end{pmatrix} = \begin{pmatrix} aa\, aa\, cc\, cc \\ aa\, aa\, cc\, cc \end{pmatrix} \tag{72}$$

Fix the second row. The elements that permute the first row into the second row are in $S_4 \times S_4$. Again, for each fixed $\sigma \in S_4 \times S_4$, the sum over all τ's simply yields all elements in S_8, with their enumeration shifted by σ. Thus, there are now $4! \times 4!$ sums over all τ's.

There are $3d(d-1)$ possible choices (with permutations) of (aa, aa, cc, cc) on the first row, and for each of these there are 6 permutations of (aa, aa, cc, cc) on the bottom row.

Thus, we have for terms of this type

$$
\begin{aligned}
&3d(d-1) \times 6 \times 4! \times 4! \sum_{\tau \in S_8} \mathrm{Wg}(\tau; d), \\
&= \frac{10368\, d(d-1)}{d(d+1)(d+2)(d+3)(d+4)(d+5)(d+6)(d+7)}
\end{aligned} \tag{73}
$$

Summing all these contribution yields

$$
\begin{aligned}
&\frac{40320\, d + 23040\, d(d-1) + 6912\, d(d-1)(d-2) + 384\, d(d-1)(d-2)(d-3) + 10368\, d(d-1)}{d(d+1)(d+2)(d+3)(d+4)(d+5)(d+6)(d+7)} \\
&= \frac{384 d(d+2)(d+4)(d+6)}{d(d+1)(d+2)(d+3)(d+4)(d+5)(d+6)(d+7)} = \frac{384}{(d+1)(d+3)(d+5)(d+7)}\,,
\end{aligned} \tag{74}
$$

as expected