The Bowl's Sequence

Deping Huang

2020-04-02

Contents

1 The Power Series of $\arctan\left(1-\sqrt{1-x^2}\right)$ and Bowl's Sequence.

I have discussed about a function and its power series at $x = 0$. Here is the expression of the function. Obviously, $f(x)$ is an even function, only the even powers of x is contained in the series.

$$\begin{aligned}
f(x) &= \arctan\left(1-\sqrt{1-x^2}\right) \\
&= \sum_{n=1}^{\infty}\frac{a_n}{(2n)!}x^{2n} \\
a_n &= 1, 3, 15, 315, 36855, 4833675, \\
&\quad 711485775, 133449190875, \\
&\quad 33399969978375, 10845524928112875, \\
&\quad 4368604540935009375, \\
&\quad 2121018409773134746875, \cdots
\end{aligned}$$

I name the a_n Bowl's sequence, for the curve of $f(x)$ looks like a bowl. Now I try to get the recurrence formula. Firstly, note that

$$\sqrt{1-x^2} = 1-\sum_{n=0}^{\infty}\frac{\prod_{k=1}^{n-1}\left(k-\frac{1}{2}\right)}{2n!}x^{2n}$$

$$1-\sum_{n=1}^{\infty}\frac{(2n-2)!}{(n!)\,(n-1)!2^{2n-1}}x^{2n}$$

$$\left(\arctan x\right)^{'} = \frac{1}{1+x^2}$$

Then one can write

$$\begin{aligned}
f^{'}(x) &= \sum_{n=1}^{\infty}\frac{a_n}{(2n-1)!}x^{2n-1} \\
&= \frac{x}{\sqrt{1-x^2}\,(3-x^2)-2\,(1-x^2)} \\
&\quad \sum_{n=0}^{\infty}\frac{a_{n+1}}{(2n+1)!}x^{2n} \\
&= \frac{1}{\sqrt{1-x^2}\,(3-x^2)-2\,(1-x^2)} \\
&= \frac{1}{(x^2-3)\sum_{n=1}^{\infty}k_nx^{2n}+1+x^2} \\
&= \frac{1}{1-\frac{1}{2}x^2+\sum_{n=2}^{\infty}(k_{n-1}-3k_n)\,x^{2n}} \\
&= \frac{1}{\sum_{m=0}^{\infty}b_mx^{2m}} \\
k_n &= \frac{1}{2},\frac{1}{8},\frac{1}{16},\frac{5}{128},n\geq 1 \\
k_n &= \frac{(2n-2)!}{(n!)\,(n-1)!2^{2n-1}} \\
&\quad k_{n-1}-3k_n \\
&= \frac{(2n-4)!}{(n-1)!\,(n-2)!2^{2n-3}} \\
&\quad -\frac{3\,(2n-2)!}{(n!)\,(n-1)!2^{2n-1}} \\
&= \frac{(2n-4)!}{(n!)\,(n-1)!2^{2n-1}}\times \\
&\quad (4n\,(n-1)-3\,(2n-2)\,(2n-3)) \\
&= \frac{-(2n-4)!2\,(n-1)\,(4n-9)}{(n!)\,(n-1)!2^{2n-1}} \\
&= -\frac{(2n-4)!\,(4n-9)}{(n!)\,(n-2)!2^{2n-2}} \\
b_{0,1,2\cdots} &= 1,-\frac{1}{2},k_{n-1}-3k_n
\end{aligned}$$

$$
\begin{aligned}
1 &= \sum_{n=0}^{\infty} \frac{a_{n+1}}{(2n+1)!} x^{2n} \sum_{m=0}^{\infty} b_m x^{2m} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{a_{k+1} b_{n-k}}{(2k+1)!} x^{2n} \\
0 &= \frac{a_2}{3!} + a_1 b_1 \\
0 &= \sum_{k=0}^{n} \frac{a_{k+1} b_{n-k}}{(2k+1)!}
\end{aligned}
$$

Then we get the recurrent formula

$$
\begin{aligned}
a_1 &= 1 \\
b_{0,1,2} &= 1, -\frac{1}{2}, \frac{1}{8} \\
b_n &= -\frac{(2n-4)!\,(4n-9)}{(n!)\,(n-2)!2^{2n-2}}, n \geq 2, \frac{10! * 19}{7! * 5! *} \\
a_{n+1} &= -\,(2n+1)! \sum_{k=0}^{n-1} \frac{a_{k+1} b_{n-k}}{(2k+1)!} \\
&= \sum_{k=1}^{n-1} \frac{a_k b_{n+1-k}}{(2k-1)!} + n\,(2n+1)\,a_n \\
&= \frac{1}{2^{2n}} \sum_{k=1}^{n-1} A_{n,k} a_k + n\,(2n+1)\,a_n \\
A_{n,k} &= \frac{2^{2k}\,(2n-2k-2)!\,(4n-4k-5)\,(2n+1)!}{(2k-1)!\,(n-k+1)!\,(n-k-1)!} \\
A_{n,n-1} &= \frac{-2^{2n-2}\,(2n+1)!}{(2n-3)!2!} \\
&= -3 \times 2^{2n} C_{2n+1}^{4} \\
A_{n,k+1} &= \frac{2^{2k+2}\,(2n-2k-4)!\,(4n-4k-9)\,(2n+1)!}{(2k+1)!\,(n-k)!\,(n-k-2)!} \\
\frac{A_{n,k}}{A_{n,k+1}} &= \frac{C_{2n-2k-2}^{2}\,(4n-4k-5)}{(4n-4k-9)} \frac{C_{2k+1}^{2}}{(n-k+1)\,(n-k-1)} \\
&= \frac{C_{2k+1}^{2} C_{2n-2k-2}^{2}\,(4n-4k-5)}{(4n-4k-9)\left((n-k)^2-1\right)} \\
A_{n,k} &= \frac{C_{2k+1}^{2} C_{2n-2k-2}^{2}\,(4n-4k-5)}{(4n-4k-9)\left((n-k)^2-1\right)} A_{n,k+1}
\end{aligned}
$$

2 Some Properties of a_n

There are some properties about a_n, especially the numerator and denominator of $\frac{a_n}{(2n)!}$

$$\begin{aligned}
\frac{a_n}{(2n)!} &= \frac{p_n}{q_n} \\
q_n &= 2^{n_2} m, 2 \nmid m \\
n &= 2^{[\log_2^n]} + k \\
f_k &= 2n - n_2 \\
&= 1, 2, 2, 3, 2, 3, 3, 4, \\
&\quad 2, 3, 3, 4, 3, 4, 4, 5, \\
&\quad 2, 3, 3, 4, 3, 4, 4, 5, \cdots
\end{aligned}$$

There is an interesting property with the sequence f_n, one can determine f_n with a folding algorithm

$$\begin{aligned}
f_0 &= 1 \\
f_n &= f_{n-2^{[\log_2^n]}} \\
f_{2^n+k} &= f_k, \geq 0, \\
0 &\leq k \leq 2^n - 1
\end{aligned}$$

Or more directly, f_n is the number of 1s in the binary form of n. For instance

$$\begin{aligned}
2^m &= 1\underbrace{00\cdots0}_{m} \\
f_{2^m} &= 1 \\
2^m - 1 &= \underbrace{111\cdots1}_{m} \\
f_{2^m-1} &= m
\end{aligned}$$

Finally, one can write

$$\begin{aligned}
k &= n - 2^{[\log_2^n]} \\
q_n &\equiv 0 \left(mod\ 2^{2n-f_k}\right)
\end{aligned}$$

Especially, when n is the power of 2, one can write

$$q_{2^m} = 2^{2^{m-1}}$$

As for the numerator p_n, there is a conjecture that there are infinitely n so that p_n is prime, some prime p_n as bellow

$$
\begin{aligned}
p_5 &= 13\\
p_6 &= 31\\
p_{10} &= 5843\\
p_{15} &= 2458109\\
p_{18} &= 287091419\\
p_{24} &= 254342741399\\
p_{35} &= 35295012453058848 67\\
p_{39} &= 427860028793103252967\\
p_{83} &= 906377099957202739168\\
&\ 439729625276641710281\\
&\ 62149\\
p_{104} &= 890041453097372994863\\
&\ 389952648819701669139\\
&\ 68061129264810447\\
p_{109} &= 178041567983613612685\\
&\ 170776564228683871018\\
&\ 804351665817679734203\\
p_{120} &= 593034744586068109565\\
&\ 075377561604220776300\\
&\ 623247248105927795763\\
&\ 23897\\
p_{437} &= 132826198763401313280\\
&\ 585119934525107382711\\
&\ 284452214529798920088\\
&\ 069952738221413540575\\
&\ 646323455756363232682\\
&\ 095220830661203976318\\
&\ 730041110236955737761\\
&\ 420708939239504355127\\
&\ 804643551416330663663\\
&\ 997210832053536500045\\
&\ 423635910869675730412\\
&\ 885022471452421547266\\
&\ 30814189\\
p_{490} &= 508818332708978663103
\end{aligned}
$$

$$
\begin{array}{lcl}
 & & 107911527620184502586 \\
 & & 073511194666269794065 \\
 & & 763105253803021305545 \\
 & & 331613391179050179918 \\
 & & 750722619885974464239 \\
 & & 453380533927033735587 \\
 & & 851768367857263396369 \\
 & & 992989008221344478305 \\
 & & 633246748311053178344 \\
 & & 493170345298176422170 \\
 & & 127717655694908284303 \\
 & & 180577462897540480172 \\
 & & 0266422118919186 49 \\
p_{552} & = & 20390526841203180 62 \\
 & & 90325548891069832 31 \\
 & & 81564583960828066 51 \\
 & & 08404767794487691 04 \\
 & & 40588848615526986 20 \\
 & & 89393129452028971 \\
 & & 4756847971993219664 \\
 & & 2272381333662993054 \\
 & & 3935108519650164989 \\
 & & 0540453125429789130 \\
 & & 04252955375193693 \\
 & & 6274967897166378869 \\
 & & 3431710245426744510 \\
 & & 5743490756545708646 \\
 & & 119920490625 \\
 & & 5146902790276966220 \\
 & & 8837000889825196913 \\
 & & 98449971025842337 \\
\cdots & \cdots & \cdots
\end{array}
$$

3 Summary

The Bowl's sequence was given in this note, and the recurrence formulas was given too.

Take-home message is here
(a.) The power series of $\arctan\left(1-\sqrt{1-x^2}\right)$ is

$$\begin{aligned} & \arctan\left(1-\sqrt{1-x^2}\right) \\ = & \sum_{n=1}^{\infty} \frac{a_n}{(2n)!} x^{2n} \\ a_1 = & 1 \\ a_{n+1} = & \frac{1}{2^{2n}} \sum_{k=1}^{n-1} A_{n,k} a_k + n\,(2n+1)\,a_n \end{aligned}$$

(b.) A property of a_n

$$\begin{aligned} \frac{p_n}{q_n} &= \frac{a_n}{(2n)!} \\ q_{2^m} &= 2^{2^{m-1}} \end{aligned}$$