The Bowl's Sequence

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1 The Power Series of $\arctan\left(1 - \sqrt{1 - x^2}\right)$ and Bowl's Sequence.

I have discussed about a function and its power series at $x = 0$. Here is the expression of the function. Obviously, $f(x)$ is an even function, only the even powers of x is contained in the series.

$$
\begin{aligned}
f(x) &= \arctan\left(1 - \sqrt{1 - x^2}\right) \\
&= \sum_{n=1}^{\infty} \frac{a_n}{(2n)!} x^{2n} \\
a_n &= 1, 3, 15, 315, 36855, 4833675, \\
&\quad 711485775, 133449190875, \\
&\quad 33399969978375, 10845524928112875, \\
&\quad 4368604540935009375, \\
&\quad 212101840977313474 6875, \cdots
\end{aligned}
$$

I name the a_n Bowl's sequence, for the curve of $f(x)$ looks like a bowl. Now I try to get the recurrence formula. Firstly, note that

$$
\begin{aligned}
\sqrt{1 - x^2} &= 1 - \sum_{n=0}^{\infty} \frac{\prod_{k=1}^{n-1}\left(k - \frac{1}{2}\right)}{2n!} x^{2n} \\
&\quad 1 - \sum_{n=1}^{\infty} \frac{(2n-2)!}{(n!)(n-1)! 2^{2n-1}} x^{2n} \\
(\arctan x)' &= \frac{1}{1 + x^2}
\end{aligned}
$$

Then one can write

$$
\begin{aligned}
f'(x) &= \sum_{n=1}^{\infty} \frac{a_n}{(2n-1)!} x^{2n-1} \\
&= \frac{x}{\sqrt{1 - x^2}\left(3 - x^2\right) - 2\left(1 - x^2\right)} \\
&\quad \sum_{n=0}^{\infty} \frac{a_{n+1}}{(2n+1)!} x^{2n} \\
&= \frac{1}{\sqrt{1 - x^2}\left(3 - x^2\right) - 2\left(1 - x^2\right)} \\
&= \frac{1}{\left(x^2 - 3\right)\sum_{n=1}^{\infty} k_n x^{2n} + 1 + x^2} \\
&= \frac{1}{1 - \frac{1}{2}x^2 + \sum_{n=2}^{\infty}\left(k_{n-1} - 3k_n\right)x^{2n}} \\
&= \frac{1}{\sum_{m=0}^{\infty} b_m x^{2m}} \\
k_n &= \frac{1}{2}, \frac{1}{8}, \frac{1}{16}, \frac{5}{128}, n \geq 1 \\
k_n &= \frac{(2n-2)!}{(n!)(n-1)! 2^{2n-1}} \\
k_{n-1} &- 3k_n \\
&= \frac{(2n-4)!}{(n-1)!(n-2)! 2^{2n-3}} \\
&\quad - \frac{3(2n-2)!}{(n!)(n-1)! 2^{2n-1}} \\
&= \frac{(2n-4)!}{(n!)(n-1)! 2^{2n-1}} \times \\
&\quad \left(4n(n-1) - 3(2n-2)(2n-3)\right) \\
&= \frac{-(2n-4)! 2(n-1)(4n-9)}{(n!)(n-1)! 2^{2n-1}} \\
&= -\frac{(2n-4)!(4n-9)}{(n!)(n-2)! 2^{2n-2}}
\end{aligned}
$$

$$b_{0,1,2\cdots} = 1, -\frac{1}{2}, k_{n-1} - 3k_n$$

$$1 = \sum_{n=0}^{\infty} \frac{a_{n+1}}{(2n+1)!} x^{2n} \sum_{m=0}^{\infty} b_m x^{2m}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{a_{k+1} b_{n-k}}{(2k+1)!} x^{2n}$$

$$0 = \frac{a_2}{3!} + a_1 b_1$$

$$0 = \sum_{k=0}^{n} \frac{a_{k+1} b_{n-k}}{(2k+1)!}$$

Then we get the recurrent formula

$$a_1 = 1$$

$$b_{0,1,2} = 1, -\frac{1}{2}, \frac{1}{8}$$

$$b_n = -\frac{(2n-4)!\,(4n-9)}{(n!)\,(n-2)!2^{2n-2}}, n \geq 2, \frac{10!*19}{7!*5!*}$$

$$a_{n+1} = -(2n+1)! \sum_{k=0}^{n-1} \frac{a_{k+1} b_{n-k}}{(2k+1)!}$$

$$= \sum_{k=1}^{n-1} \frac{a_k b_{n+1-k}}{(2k-1)!} + n\,(2n+1)\,a_n$$

$$= \frac{1}{2^{2n}} \sum_{k=1}^{n-1} A_{n,k} a_k + n\,(2n+1)\,a_n$$

$$A_{n,k} = \frac{2^{2k}\,(2n-2k-2)!\,(4n-4k-5)\,(2n+1)!}{(2k-1)!\,(n-k+1)!\,(n-k-1)!}$$

$$A_{n,n-1} = \frac{-2^{2n-2}\,(2n+1)!}{(2n-3)!2!}$$

$$= -3 \times 2^{2n} C_{2n+1}^4$$

$$A_{n,k+1} = \frac{2^{2k+2}\,(2n-2k-4)!\,(4n-4k-9)\,(2n+1)!}{(2k+1)!\,(n-k)!\,(n-k-2)!}$$

$$\frac{A_{n,k}}{A_{n,k+1}} = \frac{C_{2n-2k-2}^2\,(4n-4k-5)}{(4n-4k-9)} \frac{C_{2k+1}^2}{(n-k+1)\,(n-k-1)}$$

$$= \frac{C_{2k+1}^2 C_{2n-2k-2}^2\,(4n-4k-5)}{(4n-4k-9)\left((n-k)^2-1\right)}$$

$$A_{n,k} = \frac{C_{2k+1}^2 C_{2n-2k-2}^2\,(4n-4k-5)}{(4n-4k-9)\left((n-k)^2-1\right)} A_{n,k+1}$$

2 Some Properties of a_n

2.1 Factors of a_n

There are some properties about a_n. It is observed that

$$n \equiv 0\,(mod\,3), n \geq 2$$

$$n \equiv 0\,(mod\,5), n \geq 3$$

$$a_n \equiv 0\,(mod\,3^{f_{n,3}})$$

$$a_n \equiv 0\,(mod\,5^{f_{n,5}})$$

$$f_{n,3} = 0,1,1,2,4,4,7,6,6,8,$$
$$9,9,10,13,13,14,17,$$
$$15,17,18,19,19,21,$$
$$21,24,23$$

$$f_{n,5} = 0,0,1,1,1,2,2,3,3,3,$$
$$5,5,6,6,6,7,7,8,8,8\cdots$$

2.2 Denominator of $\frac{a_n}{(2n)!}$

It is more interesting about the numerator and denominator of $\frac{a_n}{(2n)!}$

$$\frac{a_n}{(2n)!} = \frac{p_n}{q_n}$$

$$q_n = 2^{n_2} m, 2 \nmid m$$

$$n = 2^{[\log_2^n]} + k$$

$$f_k = 2n - n_2$$

$$= 1,2,2,3,2,3,3,4,$$
$$2,3,3,4,3,4,4,5,$$
$$2,3,3,4,3,4,4,5,\cdots$$

There is an interesting property with the sequence f_n, one can determine f_n with a folding algorithm

$$f_0 = 1$$

$$f_n = f_{n-2^{[\log_2^n]}}$$

$$f_{2^n+k} = f_k, \geq 0,$$

$$0 \leq k \leq 2^n - 1$$

Or more directly, f_n is the number of 1s in the binary form of n. For instance

$$
\begin{aligned}
2^m &= 1\underbrace{00\cdots0}_{m} \\
f_{2^m} &= 1 \\
2^m - 1 &= \underbrace{111\cdots1}_{m} \\
f_{2^m-1} &= m \\
f_0 &= 0 \\
f_{2n} &= f_n \\
f_{2n+1} &= f_n + 1 \\
f_n &= f_{\left[\frac{n}{2}\right]} + n \bmod 2
\end{aligned}
$$

See https://oeis.org/A000120, which is called Haming weight of n.

Finally, one can write

$$
\begin{aligned}
k &= n - 2^{[\log_2^n]} \\
q_n &\equiv 0 \left(mod\ 2^{2n-f_k}\right)
\end{aligned}
$$

Especially, when n is the power of 2, one can write

$$
q_{2^m} = 2^{2^{m+1}-1} \tag{1}
$$

Furthermore, when n is the multiplication of a prime and a power of 2, one can write

$$
\begin{aligned}
n &= 2^m p \\
q_n &= p \times 2^{2n-f_k}
\end{aligned}
$$

And, when n is a Mersenne prime, one can get q_n via

$$
q_{2^m-1} = 2^{2^{m+1}-2(m+1)}\left(2^m - 1\right)
$$

where m is prime and $2^m - 1$ is prime.

2.3 Numerator of $\frac{a_n}{(2n)!}$

As for the numerators p_n, one can prove that all of p_n is odd since the denominators are always even. Also, there is a conjecture that there are infinitely n so that p_n is prime, some prime p_n are listed in the Appendix.

3 Summary

The Bowl's sequence was given in this note, and the recurrence formulas was given too.

Take-home message is here

(a.) The power series of $\arctan\left(1 - \sqrt{1 - x^2}\right)$ is

$$
\begin{aligned}
\arctan&\left(1 - \sqrt{1 - x^2}\right) \\
&= \sum_{n=1}^{\infty} \frac{a_n}{(2n)!} x^{2n} \\
a_1 &= 1 \\
a_{n+1} &= \frac{1}{2^{2n}} \sum_{k=1}^{n-1} A_{n,k} a_k + n\left(2n + 1\right) a_n
\end{aligned}
$$

(b.) A property of a_n

$$
\begin{aligned}
\frac{p_n}{q_n} &= \frac{a_n}{(2n)!} \\
q_{2^m} &= 2^{2^{m-1}}
\end{aligned}
$$

Appendix

a_n **with** $n \leq 20$

$$
\begin{aligned}
a_1 &= 1 \\
a_2 &= 3 \\
a_3 &= 15 \\
a_4 &= 315 \\
a_5 &= 36855 \\
a_6 &= 4833675 \\
a_7 &= 711485775 \\
a_8 &= 133449190875 \\
a_9 &= 33399969978375 \\
a_{10} &= 10845524928112875 \\
a_{11} &= 4368604540935009375 \\
a_{12} &= 2121018409773134 \\
&\quad\ 746875 \\
a_{13} &= 1222083076784378
\end{aligned}
$$

918484375

$$a_{14} = 8260130176741322\,44878796875$$

$$a_{15} = 6477241138419361\,42199672859375$$

$$a_{16} = 5831696435249193\,52829283528046875$$

$$a_{17} = 59735917746314449\,1308077497692734375$$

$$a_{18} = 69070563474827507\,73890069987317042\,96875$$

$$a_{19} = 89530877093508634\,76640969618485120\,87109375$$

$$a_{20} = 12930147565151909\,18978281765759779\,961360546875$$

p_n **with** $n \leq 20$

$$p_1 = 1$$
$$p_2 = 1$$
$$p_3 = 1$$
$$p_4 = 1$$
$$p_5 = 13$$
$$p_6 = 31$$
$$p_7 = 117$$
$$p_8 = 209$$
$$p_9 = 3077$$
$$p_{10} = 5843$$
$$p_{11} = 22415$$
$$p_{12} = 43015$$
$$p_{13} = 330457$$
$$p_{14} = 636347$$
$$p_{15} = 2458109$$
$$p_{16} = 4759409$$
$$p_{17} = 147733749$$
$$p_{18} = 287091419$$
$$p_{19} = 1117521323$$
$$p_{20} = 2178052043$$
$$p_{21} = 5667208289$$
$$p_{22} = 33216221057$$

q_n **with** $n \leq 20$

$$q_1 = 2 = 2^1$$
$$q_2 = 8 = 2^3$$
$$q_3 = 48 = 2^4 \times 3^1$$
$$q_4 = 128 = 2^7$$
$$q_5 = 1280 = 2^8 \times 5^1$$
$$q_6 = 3072 = 2^{10} \times 3^1$$
$$q_7 = 14336 = 2^{11} \times 7^1$$
$$q_8 = 32768 = 2^{15}$$
$$q_9 = 589824 = 2^{16} \times 3^2$$
$$q_{10} = 1310720 = 2^{18} \times 5^1$$
$$q_{11} = 5767168 = 2^{19} \times 11^1$$
$$q_{12} = 12582912 = 2^{22} \times 3^1$$
$$q_{13} = 109051904 = 2^{23} \times 13^1$$
$$q_{14} = 234881024 = 2^{25} \times 7^1$$

$$q_{15} = 1006632960$$
$$= 2^{26} \times 3^1 \times 5^1$$
$$q_{16} = 2147483648$$
$$= 2^{31}$$
$$q_{17} = 73014444032$$
$$= 2^{32} \times 17^1$$
$$q_{18} = 154618822656$$
$$= 2^{34} \times 3^2$$
$$q_{19} = 652835028992$$
$$= 2^{35} \times 19^1$$
$$q_{20} = 1374389534720$$
$$= 2^{38} \times 5^1$$
$$q_{21} = 3848290697216$$
$$= 2^{39} \times 7^1$$
$$q_{22} = 24189255811072$$
$$= 2^{41} \times 11^1$$

Prime p_n within 1000

$$p_5 = 13$$
$$p_6 = 31$$
$$p_{10} = 5843$$
$$p_{15} = 2458109$$
$$p_{18} = 287091419$$
$$p_{24} = 254342741399$$
$$p_{35} = 3529501245305884867$$
$$p_{39} = 427860028793103252967$$
$$p_{83} = 90637709995720273916843972962527664171028162149$$
$$p_{104} = 890041453097372994863889952648819701669139680611\\29264810447$$
$$p_{109} = 178041567983613612685170776564228683871018804351665817679734203$$
$$p_{120} = 593034744586068109565$$

$$p_{437} = 13282619876340131328058511993452510738271128445221452979892008806995273822141354057564632345575636323268209522083066120397631873004110236955737761420708939239504355127804643551416330663663997210832053536500045423635910869675730412885022471452421547266$$
$$30814189$$

$$p_{490} = 50881833270897866310310791152762018450258607351119466626979406576310525380302130554533161339117905017991875072261988597446423945338053392703373558785176836785726339636999298900822134447830563324674831105317834449317034529817642217012771765569490828430318057746289754048017202664221189191864$$

$$p_{552} = 2039052684120318062903255488910698323181564583960828066510840476779448769104405888486155269862089393129452028971475684797199321966$$

$$
\begin{aligned}
2272381333662993054 & \\
3935108519650164989 & \\
0540453125429789130 & \\
04252955375193693 & \\
6274967897166378869 & \\
3431710245426744510 & \\
5743490756545708646 & \\
119920490625 & \\
5146902790276966220 & \\
8837000889825196913 & \\
98449971025842337 & \\
\cdots \quad \cdots \quad \cdots &
\end{aligned}
\qquad
\begin{aligned}
&= \sum_{k=0}^{n}\left(C_{2n+1}^{2k} I_{2k+1} - C_{2n+1}^{2k+1} I_{2k+2}\right)\\[4pt]
&= \sum_{k=0}^{n}\left(\frac{C_{2n+1}^{2k} 2^{2k}}{(2k+1)\,C_{2k}^{k}} - \frac{C_{2n+1}^{2k+1} C_{2k+2}^{k+1} \pi}{2^{2k+3}}\right)\\[4pt]
x &= \sin\theta\\[4pt]
\tan y &= 1-\sqrt{1-x^2}\\[4pt]
&= 1-\cos\theta = 2\sin^2\frac{\theta}{2}\\[4pt]
\int_0^{\frac{\pi}{2}} d\theta\,\cos^{2n}\theta &= 2^{-2n-1} C_{2n}^{n}\pi = I_{2n}\\[4pt]
\int_0^{\frac{\pi}{2}} d\theta\,\cos^{2n+1}\theta &= \frac{\Gamma(n+1)\,\Gamma\!\left(\frac{1}{2}\right)}{2\Gamma\!\left(n+\frac{3}{2}\right)\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)}\\[4pt]
&= \frac{n!\,2^n}{(2n+1)!!}\\[4pt]
&= \frac{(n!)^2\,2^{2n}}{(2n+1)!} = \frac{2^{2n}}{(2n+1)\,C_{2n}^{n}}\\[4pt]
&= I_{2n+1}\\[4pt]
I_{2n} I_{2n+1} &= \frac{\pi}{2(2n+1)}\\[4pt]
I_0 &= \frac{\pi}{2},\ I_1 = 1\\[4pt]
I_{m-1} I_m &= \frac{\pi}{2m}\\[4pt]
I_m I_{m+1} &= \frac{\pi}{2m+2}
\end{aligned}
$$

Integral of $f(x)$ in the region $[0,1]$

$$
\begin{aligned}
\int_0^1 f(x)\,dx &= \int_0^1 dx\,\arctan\left(1-\sqrt{1-x^2}\right)\\[6pt]
&= \int_0^{\frac{\pi}{2}} d\theta\,\cos\theta\,\arctan(1-\cos\theta)\\[6pt]
&= \int_0^{\frac{\pi}{2}} d\theta \sum_{n=0}^{\infty} \frac{(-1)^n \cos\theta\,(1-\cos\theta)^{2n+1}}{(2n+1)!}\\[6pt]
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(-1)^n \left(C_{2n+1}^{2k} I_{2k+1} - C_{2n+1}^{2k+1} I_{2k+2}\right)}{(2n+1)!}\\[6pt]
&= \sum_{n=0}^{\infty}\sum_{k=0}^{2n+1} \frac{(-1)^{n+k} C_{2n+1}^{k} I_{k+1}}{(2n+1)!}\\[6pt]
&= \sum_{n=1}^{\infty} \frac{a_n}{(2n+1)!}\\[6pt]
&\ \int_0^{\frac{\pi}{2}} d\theta\,\cos\theta\,(1-\cos\theta)^{2n+1}\\[6pt]
&= \int_0^{\frac{\pi}{2}} d\theta\,\cos\theta \sum_{k=0}^{2n+1} C_{2n+1}^{k}(-1)^k \cos^k\theta\\[6pt]
&= \int_0^{\frac{\pi}{2}} d\theta \sum_{k=0}^{2n+1} (-1)^k C_{2n+1}^{k} \cos^{k+1}\theta\\[6pt]
&= \sum_{k=0}^{2n+1} (-1)^k C_{2n+1}^{k} I_{k+1}
\end{aligned}
$$