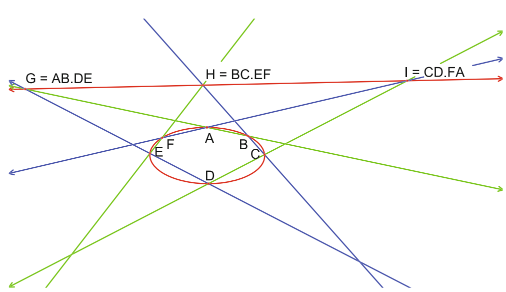
The Pascal Mysticum Demystified

JOHN CONWAY AND ALEX RYBA

The object we call the Pascal Mysticum is a remarkable configuration of 95 points and 95 lines derived from 6 points on a conic. In what may be the simplest account yet published, we explain all the incidences of the Mysticum in a readily memorable way and provide complete proofs in just a few pages.

In 1639, Blaise Pascal discovered, at age 16, the famous hexagon theorem that bears his name. His "Essai pour les Coniques," printed in 1640, is a densely written page; an English translation of this essay by Frances Marguerite Clarke appears in David Eugene Smith's sourcebook in mathematics [11]. In the essay, Pascal declared his intention of writing a treatise on conics in which he would derive the major theorems of Apollonius from his new theorem. The treatise was lost, but a sketch had been read by Leibnitz, who left a summary describing its major sections.

Pascal's Hexagon Theorem states: If a hexagon *ABCDEF* is inscribed in a conic, the three meeting points $G = AB.DE$, $H = BC.EF$, $I = CD.FA$ of pairs of opposite sides are collinear. The line *GHI* is called the Pascal line of *ABCDEF*.



The Pascal Figure.

Pascal's essay does not include a proof of his theorem. Many synthetic proofs have been published (e.g., [4]), but we find the following analytic proof (sketched in Wikipedia) most memorable.

PROOF. The triples of lines *AB*, *CD*, *EF* and *BC*, *DE*, *FA* can be regarded as cubic curves $c_1 = 0$ and $c_2 = 0$ that intersect in the nine points *A*, *B*, *C*, *D*, *E*, *F*, *G*, *H*, *I*. By a well-known theorem, proved by Cayley and Bacharach, all cubics through *A*, *B*, *C*, *D*, *E*, *F*, *G*, *H* automatically pass through *I*. Pascal's theorem follows by considering the cubic formed by the conic and the line *GH*.

This particular case of the Cayley-Bacharach theorem is easier than the general one. The reason that all cubics through *A*, *B*, *C*, *D*, *E*, *F*, *G*, *H* pass through *I* is that they form the one-parameter family $\lambda c_1 + \mu c_2 = 0$. For it is easy to see that for any of the points $A, \ldots, H$, there is a cubic that omits that point but passes through all its predecessors. The dimension of the (projective) space of cubics is therefore reduced from its original value of 9 to 1 by the successive conditions that they pass through *A*, *B*, *C*, *D*, *E*, *F*, *G*, *H*.

In the next section, we discuss the remarkable configuration we call the Mysticum Hexagrammaticum. The "permutation" notation we use later makes it easy to understand the incidences, and we have found uniform proofs for all the propositions. This permutation notation was used in a proposed book on triangle geometry by Conway and Steve Sigur. After Sigur's death,[1] we were surprised to find that essentially the same notation was already developed in Christine Ladd's 1879 paper [7]! As she stated, it greatly simplifies the description of incidence. It is a pity that more recent treatments have obscured her beautifully simple description of the configuration by reverting to the more cumbersome older notations. It seems that later discussions in English derive from Salmon's book [10], which rather surprisingly does not contain a proof for the Salmon Prop., see below, and is burdened by an awkward notation.

Ladd herself pointed out the need for another notation that would further simplify the more subtle incidences. Ladd's

[1] Steve Sigur died of a brain tumor on 5 July 2008.

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desideratum is supplied by our "dual notation" (to be described in a companion paper [15]), which is obtained by applying the outer automorphism of S_6. Ladd would surely have discovered this too if only she had consulted her mentor, Sylvester, who had found the outer automorphism in 1844 [13, 14].

Developments from Pascal's Theorem

The 19th century saw a series of grand developments from Pascal's theorem. By taking the same six points in different orders, we obtain just 60 Pascal lines, since there are 12 permutations preserving a hexagon. In 1828, Jakob Steiner asserted[2] [12] that these concur in threes in 20 points we call "Steiner nodes." In 1829, his great rival Julius Plücker showed [9] that these Steiner nodes lie in fours on 15 "Plücker lines." The sequence was continued on 27 June 1849 when Thomas Kirkman announced [5][3] (in a newspaper advertisement!) that the Pascal lines also concur in threes at 60 "Kirkman nodes," and a month later, when Arthur Cayley showed that these lie in threes on 20 "Cayley lines" [1] (which also each contain one Steiner node). After one further month, George Salmon closed the system[4] by showing that the Cayley lines meet in fours at 15 "Salmon nodes."

The figure formed by these 95 lines and 95 nodes has in recent years often been called the *hexagrammum mysticum*, a name that properly only applies to the hexagon of Pascal's theorem. We prefer to interchange the roles of adjective and noun to give the new name *mysticum hexagrammaticum* ("mysterious six-lettered construction"), abbreviated to *mysticum*.

Occasionally and somewhat inconsistently, a concept has been named after two people. For instance, our *Plücker lines* are often called *Steiner-Plücker lines*. The shorter name is more appropriate, because in fact Plücker discovered these lines while correcting a mistaken assertion of Steiner. The term "Cayley-Salmon lines" has similarly been used for what we just call "Cayley lines." However, Cayley's publication preceded Salmon's, although he magnanimously acknowledged Salmon's independent discovery [2].

Which Nodes Lie on Which Lines?

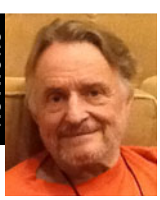
The capital letters in our "Mystic H" provide a handy way to recall which objects are points and which lines. We call the named points nodes, because the investigators whose names contain the letter "n" discovered nodes, whereas those with a nonsilent "l" discovered lines (the silent "l" in "Salmon" quietly reminds us of his independent discovery of the Cayley lines).

The Pascal line of the hexagon[5] abcdef will be called L(abcdef). We can regard the portion (abcdef) of this symbol as representing a permutation—in this case a 6-cycle—in the usual way. It turns out that Kirkman nodes N(abcdef) are also conveniently indexed by 6-cycles. In a similar way Steiner nodes N(ace)(bdf) and Cayley lines L(ace)(bdf) are indexed by products of two 3-cycles, whereas Plücker lines L(ad)(be)(cf) and Salmon nodes N(ad)(be)(cf) are indexed by products of three 2-cycles. The permutations in these names can be replaced by their inverses, which we alternatively call opposites so that we can say that permutations index the same object just if they are equal or opposite.

The Mystic H

The "Props" of our "Mystic H" display all the incidences of the Mysticum Hexagrammaticum. We use a black Prop when the incidence condition is the usual one that the permutations indexing a point and line commute without being equal or opposite. The white Prop, however, indicates that a Pascal line contains a Kirkman node just when their indexing hexagons are disjoint (have no common edge).

AUTHORS



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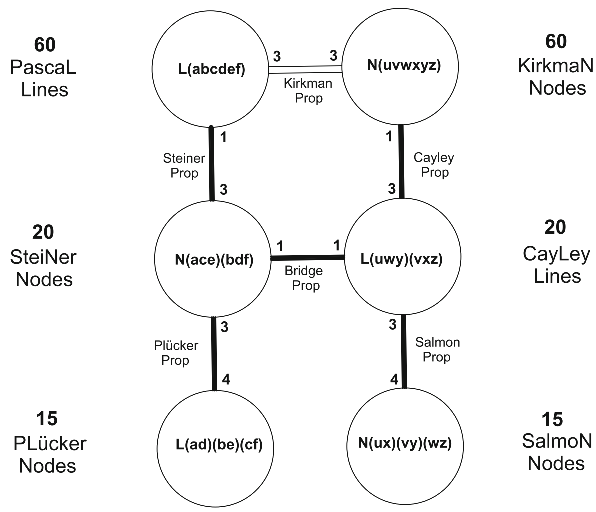
[2]Beware! His one-page paper contains as many false assertions as true ones.

[3]and slightly later in [6].

[4]Remarkably this is the only part of the mysticum not proved in [10], the first book to describe the mysticum, or any of his papers. His proof was seen by Kirkman and Cayley [6, 2].

[5]We have switched to lower case letters since these are more usual for permutations.



The Mystic H for the Mysticum Hexagrammaticum.

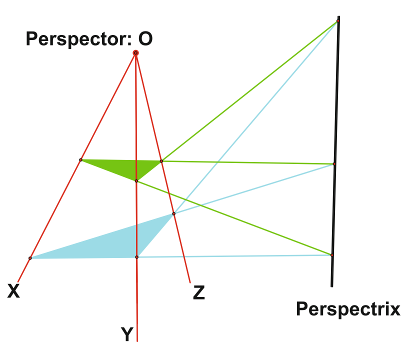
These Props also name incidence propositions due to the indicated mathematicians. The numbers near them count the corresponding incidences. For example, the labels 4 and 3 on the Plücker Prop tell us that 4 Steiner nodes lie on each Plücker line and that 3 Plücker lines meet at each Steiner node.

The mystic H is a remarkably good mnemonic. One should read it by columns, because the left column contains the three earliest discoveries, which were made in chronological order by French and German speakers, whereas the right column contains later discoveries by English-speaking writers, again in chronological order.

Desargues

Our proofs of all these propositions involve Desargues's theorem. Already in 1636, it seems that Pascal's teacher Girard Desargues opened up the new subject of projective geometry by proving this celebrated theorem to Mersenne's circle of friends, including Pascal's father and perhaps Pascal himself.

Desargues's Theorem If two triangles Δ_1 and Δ_2 have a perspector O they have a perspectrix, and conversely.



The Desargues Figure.

The *perspector*[6] of two triangles, if it exists, is the meeting point of the *joins* of corresponding vertices. Their *perspectrix*[7] is the joining line of the *meets* of corresponding edges, if that exists. (The traditional terms are *center* and *axis* of perspective.)

The proof is to observe (by lifting Y out of the plane OXZ) that we can regard this figure as a picture (projection) of a 3-dimensional figure in which the triangles are in two different planes. Despite this, corresponding sides intersect because they are coplanar. The intersection of the two planes becomes the perspectrix in the picture. We need not prove the converse because it is also the dual.

We remark that the same kind of lifting argument proves what we call

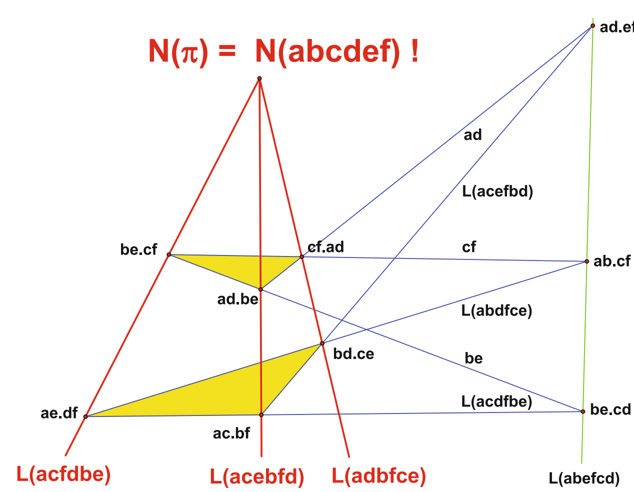
The Double Desargues Theorem If a third triangle Δ_3 is in perspective from O with Δ_2 (and so also with Δ_1), then the perspectrices of the three pairs of triangles chosen from Δ_1, Δ_2, and Δ_3 are concurrent.

The proof is that the lifted perspectrices are the intersections of pairs of planes of the three triangles, and these three planes meet in a point.

Proving the Propositions

Let us call ab, bc, . . ., fa the *edges* of π = (abcdef) and call 6-cycles disjoint if they have no common edge. Then we have:

The Kirkman Prop. The Pascal lines $L(\pi_1)$, $L(\pi_2)$, $L(\pi_3)$ indexed by 6-cycles disjoint from a 6-cycle π meet in a Kirkman node we can call $N(\pi)$.



The Kirkman Figure.

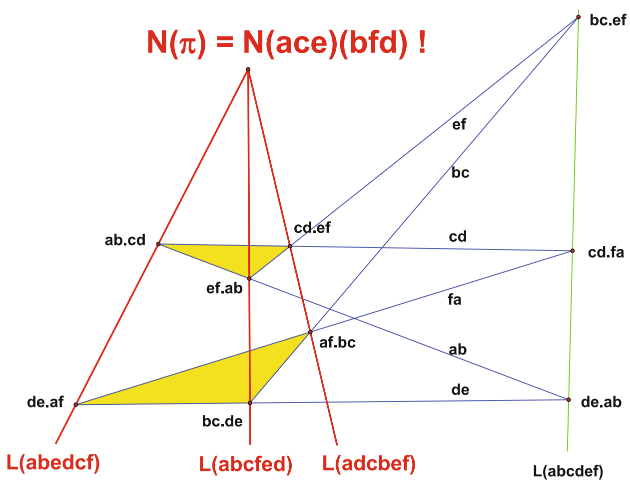
Proof. The Kirkman Figure shows two triangles that are already known to have a perspectrix, so they must have a perspector $N(\pi)$ = N(abcdef), at which the three lines $L(\pi_1)$ = L(acfdbe), $L(\pi_2)$ = L(acebfd), $L(\pi_3)$ = L (adbfce) concur. We could label the perspector by the set $\{\pi_1, \pi_2, \pi_3\}$, but this set determines and is determined by π, the unique 6-cycle disjoint from π_1, π_2, π_3.

[6]Abbreviating perspective–center.
[7]Analogous to directrix.

To verify such proofs, it suffices to check that the incidences of all lines and points other than the new one are already known. For instance, that the Pascal line L(acdfbe) at the bottom of this diagram does indeed join the three meeting points ae.df, ac.bf, be.cd. The checking is dramatically reduced by symmetries of the diagram—here, (ace)(bdf) and (af)(be)(cd).

The Steiner Prop. If three 6-cycles π_1, π_2, π_3 have a common square $\pi = \pi_1^2 = \pi_2^2 = \pi_3^2$, then the three Pascal lines L(π_1), L(π_2) and L(π_3) concur in a Steiner node that we can call $N(\pi)$.

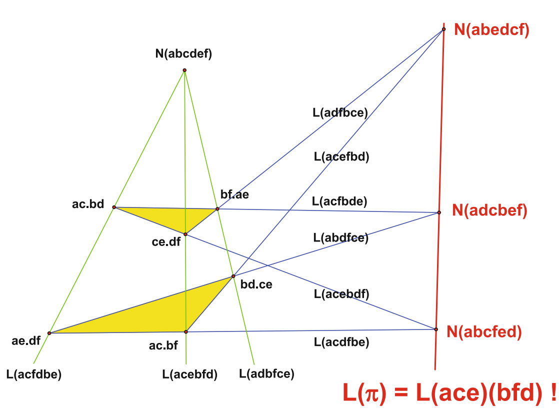
Proof. This is proved by a similar argument using the Steiner Figure. We could label the Steiner node by the set $\{\pi_1, \pi_2, \pi_3\}$ but again π conveys the same information.



The Steiner Figure.

The Cayley Prop. The three Kirkman nodes N(π_1), N(π_2), and N(π_3) are collinear if $\pi = \pi_1^2 = \pi_2^2 = \pi_3^2$. This defines the Cayley line we shall call L(π).

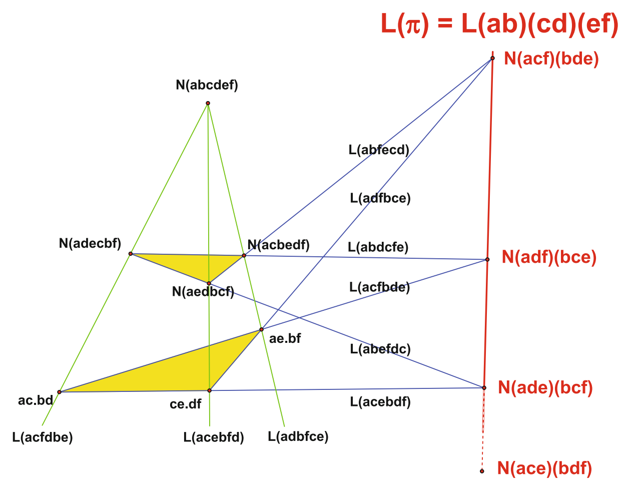
Proof. This time the two triangles of the Cayley Figure are known to have a perspector and therefore must have a perspectrix, which is the Cayley line.



The Cayley Figure.

The Plücker Prop. The Steiner nodes N(π_1), N(π_2), N(π_3), and N (π_4) of the Plücker Figure lie on a line called L(π), where π is the unique product of three 2-cycles that commutes with $\{\pi_1, \pi_2, \pi_3, \pi_4\}$, or indeed with any pair of them.

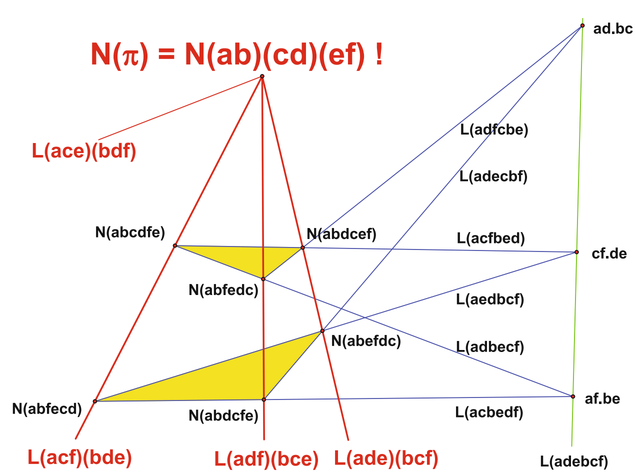
Proof. The triangles of the Plücker Figure have a perspector N(abcdef), and therefore must have a perspectrix containing the points N(π_1), N(π_2), N(π_3). The same argument shows that N(π_1), N(π_2), and N(π_4) are also collinear, and so all four Steiner nodes lie on one line, which we can call L(π).



The Plücker Figure.

The Salmon Prop. Reciprocally, and for exactly the same permutations, the Cayley lines L(π_1), L(π_2), L(π_3), and L(π_4) meet in a Salmon node $N(\pi)$.

Proof. Not quite reciprocally, the triangles of the Salmon Figure have a perspectrix and therefore have a perspector. As in the Plücker Prop., this shows that the lines L(π_1), L(π_2), L(π_3), and L(π_4) meet at a node that can be named N(π).



The Salmon Figure.

Our closing proposition, proved by Salmon in 1849 [2, 6], uses the remark we made just before our opening one.

THE BRIDGE PROP. The Steiner node N(ace)(bdf) lies on the Cayley line L(ace)(bfd).

PROOF. The two pairs of triangles in the Kirkman and Cayley figures share one triangle and have a common perspector. The Double Desargues theorem now says that the three perspectrices of the three pairs of these triangles concur. The perspectrices are L(ace)(bfd), L(abefcd), and L(adebcf). The intersection of the last two perspectrices is N(ace)(bdf), so this must also lie on the first.

Closure

Many people have now produced diagrams of the whole mysticum by machine, and we have used a virtual diagram of this kind to verify that the classical mysticum contains every point or line that is incident with three or more lines or points named after the same mathematician. The first published diagram[8] of the mysticum was hand drawn by Anne and Elizabeth Linton in 1921 [8], presumably to verify this closure property.

However, there are weaker notions of closure that allow for extensions of other kinds, in particular the "multiMysticum" found by Veronese in 1879 and our own recently discovered "polar Mysticum," which we shall describe in a future article [15]. Veronese drops the requirement that the three points or lines be named after the same mathematician and finds "higher" Pascal lines and Kirkman nodes that form what we call his multiMysticum. It starts from 90 lines already appearing in Kirkman's advertisement that contain two of his nodes and a meeting point.

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