Extending the Pascal Mysticum

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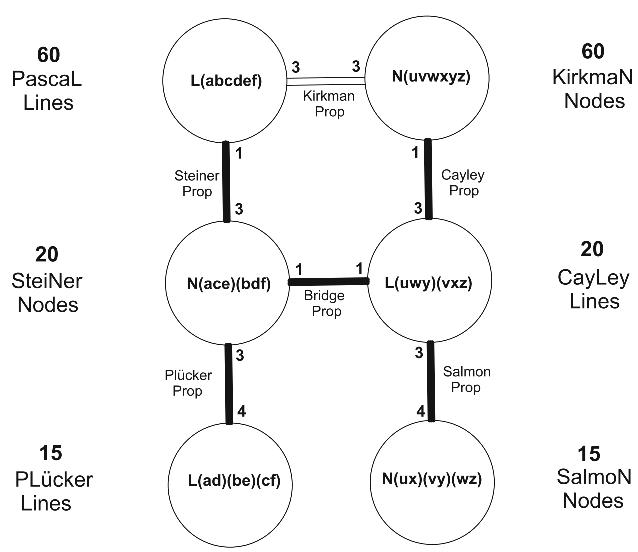
Our recent article in this journal [2] discusses the remarkable configuration of 95 points and 95 lines that was derived from 6 points on a conic by various mathematicians in the 19th century. The "mystic H," given there and copied here, compactly displays both the history of these objects and their incidences (which yield definitions). Its first column describes the earlier 19th-century discoveries that happened around 1829 and its second the later ones of 1849.



The Mystic H.

Of course, the first discovery of all had been Pascal's 1639 theorem[1] that the three meeting points of pairs of opposite edges of a hexagon inscribed in a conic are collinear. In 1828, Steiner started the study of the 60 such Pascal lines obtained from different hexagons with the same six vertices. He found that they pass in threes through 20 Steiner nodes. Plücker completed these early discoveries in 1829, by showing that the 20 Steiner nodes lie in fours on 15 Plücker lines.

The later discoveries began with Kirkman's 1849 observation that the 60 Pascal lines also meet in threes at 60 Kirkman nodes. This was followed in short order by Cayley's observation that these nodes lie in threes on 20 Cayley lines, and Salmon's that those lines pass in fours through 15 Salmon nodes. Finally, Salmon noted (in a letter to Cayley and Kirkman) that each Cayley line passes through a Steiner node.

For the "black props" that form the mystic H, the incidence condition is that the two corresponding objects are incident just when the permutations that index them commute without being equal or inverse. The remaining white or "Kirkman prop" indicates a different incidence condition described in the following text.

The present article describes two extensions of this "classical mysticum," namely the "multimysticum" discovered by Veronese [6] and Kirkman [3], and our own "polar mysticum." Several interesting theorems arise in the description and are proved in the text, but the detailed verification of the incidences involves repeated applications of Desargues's theorem that are like those of [2], but are relegated to an appendix so as not to burden the reader. The appendix is short, and it provides in two pages what was spread over more than 50 pages of [6].

The Six Decagrams and Sylvester's Duality of S_6

Our discussion involves Sylvester's remarkable duality of S_6, which we shall deduce from:

VERONESE'S DECAGRAM THEOREM *The figure formed by the 60 Pascal lines and 60 Kirkman nodes has six connected components that are Desargues configurations (or "decagrams").*

[1] The theorem was published as Pascal's "Essay pour les Coniques" in 1640.

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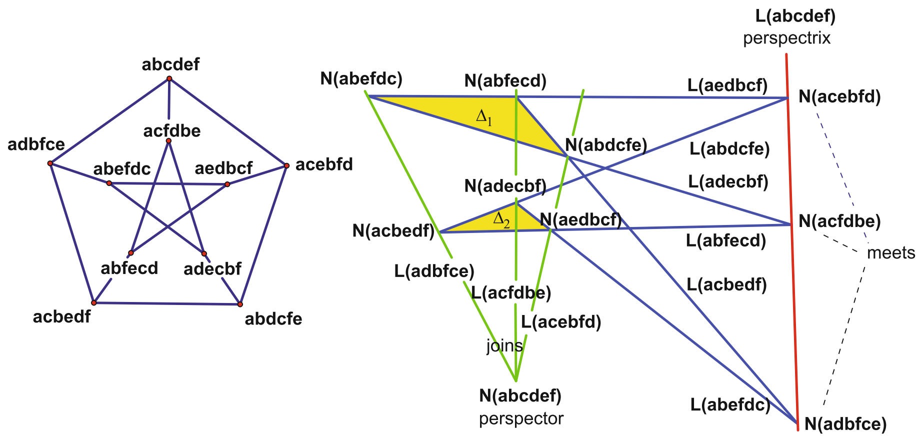


The Petersen Graph halves the Desargues Decagram.

PROOF We showed in [2] that a hexagon (abcdef) indexes a Pascal line L(abcdef) that passes through the three Kirkman nodes N(acebfd), N(aecfbd), and N(acfdbe) indexed by the hexagons disjoint from it.[2] The other hexagons disjoint from these three will in turn index the remaining Pascal lines through those nodes, and so on.

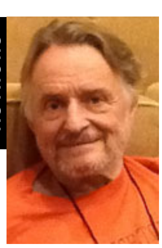
The theorem is proved by the observation that in the graph formed by joining hexagons when they are disjoint, the component of (abcdef) has only 10 vertices. This component is the well known Petersen graph whose 10 vertices each index both a Pascal line and a Kirkman node in the Desargues configuration (or "decagram") of the figure. Since we could have started with any of the 60 Pascal lines, we obtain exactly six decagrams that we call 0,1,2,3,4,5.

The following table shows the 6-cycles that label the Pascal lines and Kirkman nodes of these six decagrams. Any of these gives a *hexal name* for its decagram.

0	1	2	3	4	5
abcfde	abcdfe	abcdef	abcedf	abcfed	abcefd
abdefc	abdcef	abdcfe	abdfec	abdecf	abdfce
abedcf	abefcd	abefdc	abecfd	abedfc	abecdf
abfced	abfedc	abfecd	abfdce	abfcde	abfdec
acbdfe	acbfde	acbedf	acbdef	acbefd	acbfed
acdbef	acedbf	acebfd	acdbfe	acdfbe	acdebf
acefbd	acfbed	acfdbe	acfebd	acebdf	acfbde
adcebf	adbecf	adbfce	adcfbe	adbcef	adbcfe
adfbce	adfcbe	adecbf	adebcf	adcbfe	adcbef
aebcdf	aecbdf	aedbcf	aedcbf	aecdbf	aebdcf

Hexal names for the six decagrams.

AUTHORS



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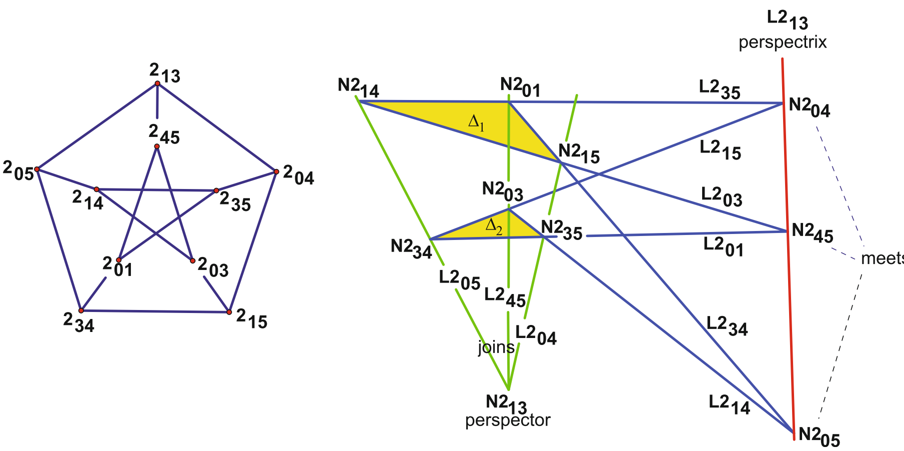
The authors previous paper on the classical mysticum [2] contains biographical information. While working on that paper they discovered the results of this one, and were later surprised to find that a good number of them were already in an 1879 paper by Christine Ladd which in turn attributed them to Veronese. They hope this paper will encourage other explorers.

[2] Two hexagons are *disjoint* if they have no common edge.

The Dual Actions of the Symmetric Group S_6

Any permutation of the six points a, b, c, d, e, f also permutes the six decagrams. However, this *duality* does not preserve cycle shape, for instance the permutation (abcdef) dualizes to (2)(13)(045) (because it takes the names from column 0 to those from column 4 and these to column 5, etc.). We shall call a, b, c, d, e, f the primal population, and 0, 1, 2, 3, 4, 5 the dual population, whose members we call *totals*.[3] Any element of S_6 can be specified by its action on either population, which completely determines its action on the other.

The dual actions of S_6 were discovered by Sylvester in 1844 [4], as he makes clear in a plaintive note of 1861 [5]. If we were to fix a bijection that identifies the elements of the two populations, we would obtain an outer automorphism of S_6. However, for our purposes it is better not to do so.



Dual labelling of the Petersen Graph for Desargues Decagram number 2.

Dual Notation

In the primal notation, the elements of the mysticum are labelled by 6-cycles such as (abcdef) and their powers. In dual notation[4] they are labelled by elements such as $(u)(vw)(xyz)$ and their component 3-cycles (xyz) and 2-cycles (vw). Because $(u)(vw)(xyz)$ and its inverse index the same objects we can abbreviate it to u_{vw} without ambiguity, and this leads to some very brief notation:

Pascal Lines and Kirkman Nodes: Lu_{vw} and Nu_{vw}.
Steiner Nodes and Cayley Lines: Nxyz and Lxyz.
Plücker Lines and Salmon Nodes: Luv and Nuv.

This also neatly redescribes the incidences between these objects. Most of these incidences (the "black props" of the mystic H) are between nodes and lines whose labels are disjoint. The exception (the "white prop") is that a Kirkman node Nu_{vw} lies on a Pascal line Lu_{xy} just if their names involve disjoint subscripts on the same total u. We have dualized the labels of the Petersen graph and Desargues decagram to illustrate this simple incidence rule. Our brief notation amply fulfills the desideratum of Christine Ladd that we mentioned in [2].

Dual notation is usually shorter than primal. For edges of the original hexagons this is not the case, for example the edge line ab becomes 04.12.35 in dual notation.

The Multimysticum Picture that Extends the Mystic H

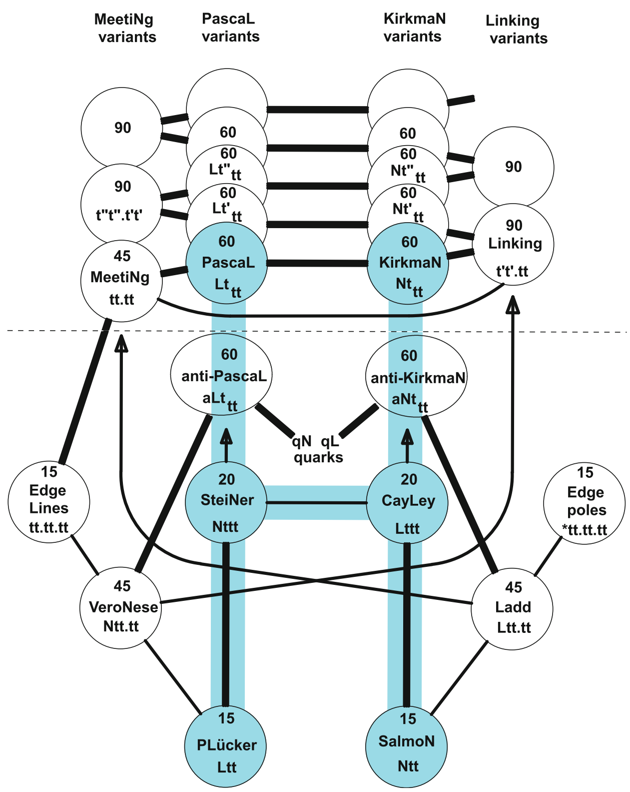
We shall now extend our mystic H to show the incidences for the multimysticum (in which the original mystic H appears in blue). Like the original mystic H, the multimysticum picture presents a remarkably compact summary of the incidences it describes. Many pages of [6] are summarized in this wonderful diagram.

We continue the convention of [2] in which a capitalized L or N indicates a line or node. In the typical symbol, which is in dual notation, the letters t represent distinct totals. The four arrowheads indicate incidences with all the objects above them. So, for instance, Steiner nodes are incident with anti-Pascal lines, Pascal lines, and all the higher Pascal lines.

Sevenfold Symmetry: The Septum

The following remarkable fact, which seems not to have been noticed previously, becomes trivial using dual notation.

[3]Abbreviating Sylvester's "synthematic totals."
[4]We use italic letters u, v, w, x, y, z for the six totals.



The Multimysticum Picture extends the Mystic H.

The Septal Theorem *The incidences of the Steiner and Salmon nodes and the Plücker and Cayley lines have sevenfold symmetry.*

We call these 35 points and 35 lines the *septum* for this reason, and because in many ways they naturally separate from the rest of the mysticum.

Proof In dual notation, the permutations that index objects of the septum are 2-cycles and 3-cycles on the six digits 0, 1, 2, 3, 4, 5. If we adjoin a new digit $\diamondsuit$ and replace the typical 2-cycle (12) by the set $\{\diamondsuit, 1, 2\}$ and the typical 3-cycle (345) by the set {3,4,5}, then the incidence condition becomes very simple: a point and line are incident just when the corresponding 3-element sets are disjoint. Because this condition is invariant under all permutations of the seven digits, it proves the theorem.

This proof is typical of the simplifications that are achieved by passing from primal to dual notation.

Reciprocity

In 1806 Brianchon made a striking application of the principle of projective duality:

Brianchon's Theorem *If a hexagon is circumscribed to a conic, the three lines joining pairs of opposite vertices meet at a point we may call the Brianchon node.*

The six tangents at a, b, c, d, e, f can be taken in 60 different orders to produce 60 Brianchon nodes, which are the basis of a dual mysticum. The fact that this has exactly the same incidences as the Pascal mysticum led many people to suppose that they are essentially the same. It is indeed true, as Hesse noted, that the incidences of the classical mysticum display what he called "a certain reciprocity," namely that if we interchange Pascal lines with Kirkman nodes, Cayley lines with Steiner nodes, and Plücker lines with Salmon nodes, all the incidences between these concepts are preserved. Hesse seems to have thought that his "reciprocity" was reciprocation in the conic (in which case the Kirkman nodes of the original hexagon would be the Brianchon nodes of its hexagon of tangents). This would explain why the incidences of the mysticum display a left-right symmetry. Unfortunately, as Veronese noted, Hesse's reciprocity cannot extend to a completely incidence-preserving one. The meeting point ab.de lies on four Pascal lines L(abcdef), L(abcedf), L(bacdef), L(bacedf), but the would-be Hessian reciprocal is false: no four Kirkman nodes are collinear, in general.

The Objects of the Multimysticum

In his 1877 paper [6], Veronese defined, among other things, various points and lines (a few already known to Kirkman [3]) that form what we call the multimysticum. It essentially consists of infinitely many copies of the mysticum that all share the same Steiner and Salmon nodes and Plücker and Cayley lines; in our terms, the same septum. However, there are "higher" versions Lu'_{vw}, Lu''_{vw}, ... of the Pascal lines and Nu'_{vw}, Nu''_{vw}, ... of the Kirkman nodes, which we have represented by the rungs of the "ladder" that forms the upper central part of the multimysticum picture.

In general, we use a numerical superscript h to indicate a string of h primes so that the Pascal and Kirkman variants at height h become Lu^h_{vw} and Nu^h_{vw}. The incidence condition is that Nu^h_{vw} lies on Lu^h_{xy} where $\{x, y\}$ is one of the three pairs that do not mention u, v, or w. In other words, a higher Pascal line and Kirkman node are incident just if they have the same height h and total u, but disjoint subscripts.

The classical mysticum starts from the 45 meeting points that are used to define the Pascal lines. The typical one ab.de lies on four Pascal lines L(abcdef), L(abcedf), L(bacdef), L(bacedf). Because elements of cycle shape 2.2 have the same shape in dual notation, ab.cd becoming 03.45 for example, the meeting point of the edges ab and cd also becomes 03.45 in dual notation; similarly other meeting points take the form *tt.tt*, each t representing a distinct total. The Pascal lines through $uv.wx$ are Lu_{wx}, Lv_{wx}, Lw_{uv}, and Lx_{uv}. However, at larger heights, there are higher meeting points that lie on Pascal lines of two adjacent heights, the larger of which is even.

The picture represents these by the handles on the left side of the ladder. For instance, the four higher Pascal lines Lu''_{wx}, Lv''_{wx}, Lw'_{uv}, Lx'_{uv}, two of height 2 and two of height 1, meet at the higher meeting point we call $u''v''.w'x'$. The rule is that the four Pascal lines Lu^h_{wx}, Lv^h_{wx}, Lw^{h-}_{uv}, Lx^{h-}_{uv} pass through the higher meeting point $u^hv^h.w^{h-}x^{h-}$, where h is even. Here $h-$ means $h-1$, except that to include the height zero case we declare that $0-$ is 0. So much for meeting points.

There is a reciprocal system of lines that we call *linking lines*, represented by the handles on the right side of the ladder. A linking line passes through two pairs of Kirkman nodes of adjacent heights, the larger of which is now odd. The typical linking line is called $u^h v^h.w^{h-}x^{h-}$, where h is odd. It passes through the higher Kirkman nodes $\mathrm{N}u^h_{wx}$, $\mathrm{N}v^h_{wx}$, $\mathrm{N}w^{h-}_{uv}$, $\mathrm{N}x^{h-}_{uv}$.

Veronese Nodes and Ladd Lines

Veronese's other discoveries remain to be explained. For each of the 45 permutations $(uv)(wx)$ and each odd height, there are two linking lines $u^h v^h.w^{h-}x^{h-}$ and $w^h x^h.u^{h-}v^{h-}$. Veronese discovered the remarkable fact that the intersection of these two lines is independent of h; and it is the "Veronese[5] node," which we may write in dual notation as N*uv.wx*. There are 45 of these Veronese nodes, which lie in threes on a Plücker line, the rule being that the three Veronese nodes N*uv.ux*, N*uw.xv*, N*ux.vw* all lie on the Plücker line L*yz*, where *y* and *z* are the two remaining totals.

Hessian reciprocity works in this case. Namely, the join of the two meeting points $u^h v^h.w^{h-}x^{h-}$ and $w^h x^h.u^{h-}v^{h-}$ of the same height $h > 0$ (which in this case must be even) is again independent of height and is the "Ladd[6] line" L*uv.wx*.

Although we have described the Ladd line L*uv.wx* as the join of infinitely many meeting variants $u^h v^h.w^{h-}x^{h-}$, it is not yet clear that all these points are collinear, so this cannot serve as a definition. We define it formally to be the join of the Salmon node N*yz* and the meeting point *uv.wx*. Almost reciprocally, the Veronese node N*uv.wx* is the intersection of L*yz* and the edge line named in dual notation by the permutation *uv.wx.yz*. (This definition is based on Kirkman's observation [3] that a Veronese node lies on an edge line.)

The Anti-Pascal Lines and Anti-Kirkman Nodes: Quarks

Like the 45 meeting points, the 45 Veronese nodes are indexed by pairs of commuting duads. Pascal lines arise from collinear triples of meeting points, and it turns out that a similarly indexed set of 60 anti-Pascal lines is formed by collinear triples of Veronese nodes. The "anti-Pascal line" $\mathrm{aL}x_{yz}$ passes through the Steiner node N*uvw* and the three Veronese nodes N*uv.yz*, N*uw.yz*, and N*vw.yz*, where *u*, *v*, *w* are the totals other than *x*, *y*, *z*. To explain the name, we use the more cumbersome primal notation and note that the Pascal line L(abcdef) joins the three meeting points ab.de, bc.ef, cd.fa. By interchanging pairs of antipodal vertices in these, we obtain ae.bd, bf.ec, ca.fd, which name the three Veronese nodes that lie on the corresponding anti-Pascal line (abbreviating "antipodal Pascal line") aL(abcdef). Reciprocally, the three Ladd lines L*uv.yz*, L*uw.yz*, L*vw.yz* and the Cayley line L*uvw* meet at an "anti-Kirkman node" $\mathrm{aN}x_{yz}$.

Let us explain the broken rung at the foot of the ladder. The Kirkman node $\mathrm{N}u_{vw}$ is the intersection of the three Pascal lines $\mathrm{L}u_{xy}$, $\mathrm{L}u_{xz}$, $\mathrm{L}u_{yz}$ whose subscripts are disjoint from *v* and *w*. However, the three corresponding anti-Pascal lines $\mathrm{aL}u_{xy}$, $\mathrm{aL}u_{xz}$, $\mathrm{aL}u_{yz}$ are not concurrent. Instead, they intersect in pairs in three distinct points $\mathrm{qN}u_{vw}(x)$, $\mathrm{qN}u_{vw}(y)$, $\mathrm{qN}u_{vw}(z)$, where $\mathrm{qN}u_{vw}(x)$ is the intersection of $\mathrm{aL}u_{xy}$ and $\mathrm{aL}u_{xz}$. We call these three points *quark nodes* and regard them as parts of a broken copy of the anti-Kirkman node $\mathrm{aN}u_{vw}$.

Altogether there are 180 quark nodes and a reciprocal set of 180 quark lines, the typical one $\mathrm{qL}u_{vw}(x)$ being the join of the two anti-Kirkman nodes $\mathrm{aN}u_{xy}$ and $\mathrm{aN}u_{xz}$. Two mysterious (because nonreciprocal) facts are that, although the quark node $\mathrm{qN}u_{vw}(x)$ lies on the Pascal line $\mathrm{L}x_{yz}$, the quark line $\mathrm{qL}u_{vw}(x)$ passes instead through the meeting point *ux.vw*.

Elevation

We put these new objects (the Veronese and anti-Kirkman nodes, Ladd and anti-Pascal lines, and the quarks) into the lower part of the multimysticum picture (below the dotted line) because they share with the septum the property that they do not depend on the parameter *h*. There are no problems with reciprocity in the lower part of the picture. However, for the ladder and its handles, there is a kind of reciprocity that not only interchanges Pascals and Kirkmans, and meeting points and linking lines, but also increases heights by 1. We call this *elevation*. It is still not perfect, as can be seen from the differences between the left and right handles in the picture. Each right handle joins Kirkman nodes of two adjacent heights by a linking line, but the lowest left handle is a single line between meeting points and Pascal lines (because $0-$ is 0). This is why there are only 45 meeting points of height 0 but 90 of all other heights.

We have now finished describing[7] the multimysticum picture. It amounts to a whole host of incidence assertions, which we relegate to an appendix. The appendix also includes a proof of a further theorem of Kirkman [3]. This is that the meeting point *wx.yz* lies on the linking lines $w'x'.uv$ and $y'z'.tu$, a fact that does not elevate.

The Polar Multimysticum, a New Discovery

So far, everything[8] we've mentioned already appeared in Veronese's 1877 paper. We now present a new discovery, that associates to each node N of the multimysticum a line $*N$ we call its *polar*, and to each line L a node $*L$ called its *pole*. It will always be understood that pole and polar are indexed by the same numbers in the same way. The polar multimysticum is formed by these objects and overlaps Veronese's multimysticum in its lower part as shown in our new picture. The polar mystic H, shown in green, is different from the original H because the Pascal lines and Kirkman nodes have been replaced by the Kirkman polars and Pascal poles, respectively.

We call this "the Full Polarity" because it is defined for all objects of the multimysticum. It interchanges already known objects in the lower part of the picture, namely Steiner, Salmon, Veronese, and anti-Kirkman nodes, with Cayley, Plücker, Ladd, and anti-Pascal lines. However for the upper part of the picture it defines new objects called the Pascal poles

[5]These were actually discovered by Kirkman, but most of their properties are due to Veronese.

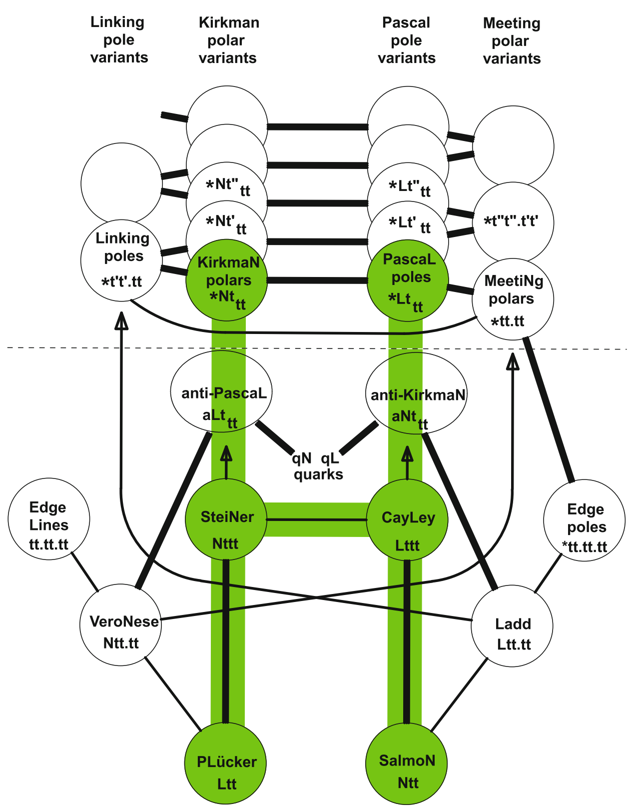
[6]These were actually discovered by Veronese but are described in some detail by Ladd (whose name contains an L!).

[7]The papers [3] and [6] include a few other points and lines that we omit. These are of lesser interest and do not contribute to the construction of the multimysticum.

[8]Except the quarks, which are mere scaffolding.

and linking poles and Kirkman polars and meeting polars (and their elevations).

If at least two objects of type A are incident with any one of another type B, we shall say "A locates B," because the locations of all the objects of type A determine those of all objects of type B. For instance, the Veronese nodes locate the edge lines, because there are three Veronese nodes on each edge line. In the picture, a heavy connection between two types means that they locate each other. Because the upper part of the picture is the body of a heavily connected "snake" whose head consists of the edge lines, the locations of the objects of any one type in the upper part of the picture determine the edge lines, and conversely. Also, if we know where our desired full polarity takes all the objects of any one of these types, we know where it takes all the others.



The Polar (multi)Mysticum and the Polar Mystic H.

We define the full polarity by saying that it interchanges each Veronese node with the Ladd line indexed in the same way. The heavy connections that almost make an "M" in the lower part of the picture show that it interchanges anti-Pascal lines with anti-Kirkman nodes and quark nodes with quark lines. Because the edge line $uv.wx.yz$ is located by the three Veronese nodes N$uv.wx$, N$uv.yz$, N$wx.yz$ it contains, we must define the edge pole ∗$uv.wx.yz$ to be the intersection of the corresponding Ladd lines L$uv.wx$, L$uv.yz$, L$wx.yz$. Because two edge lines intersect in a meeting point, we define the meeting polars as joins of edge poles. Then, because a Pascal line is the join of three meeting points, its pole can only be defined as the intersection of three meeting polars, and so on all the way back along the snake. This argument is in fact reversible, and proves:

The Polarity Theorem *The full polarity is completely determined by its action on any one of the types of object in the multimysticum except those in the septum.*

Proof Since the snake is heavily connected, if a polarity agrees with full polarity on any one of the upper types, it does so for all of them, and in particular for the linking lines. This agreement then continues to the Veronese nodes because N$uv.wx$ is located as the intersection of two linking lines $w'x'.uv$ and $u'v'.wx$. Moreover the heavy connections of the "M" in the lower part of the figure extend the argument to the antis and the quarks.

In fact we can also deduce that the poles of the Plücker and Cayley lines must be the Salmon and Steiner nodes, respectively. This is because the Veronese nodes locate the Plücker lines, which are heavily connected to the Steiner nodes. We have not yet proved that there really is a polarity interchanging Veronese nodes with Ladd lines. This requires, for instance, the concurrence of three Ladd lines that supposedly define an edge pole and the three meeting polars that define a Pascal pole, and so on. These proofs are provided in the appendix.

Like Hesse's reciprocity, our full polarity is not completely incidence-preserving. It does not extend to the six original vertices. The poles ∗af, ∗bf, ∗cf, ∗df, ∗ef of the five edges through the vertex f are not collinear. In fact no three of the fifteen edge poles are collinear, in general. However, in a sense polarity works for everything except the six vertices, because all the incidences of the (multi)mysticum were deduced by repeated use of Desargues's theorem, starting with the collinearities of Pascal's theorem, namely that of three points such as ab.de, bc.ef, cd.fa. So to prove that full polarity works (to the extent that it does) we need only two new theorems.

The Edge Pole Theorem *A triple of Ladd lines* L(cd.ef), L(ce.df), L(cf.de) *concur at a new point called the edge pole* ∗ab.

This theorem defines the edge poles that start our new polarity. The reciprocal fact that the corresponding Veronese nodes lie on the edge line ab was already discovered by Kirkman. The join of two edge poles ∗ab and ∗cd defines the meeting polar ∗ab.cd.

The Polar Pascal Theorem *Three meeting-point polars* ∗ab.de, ∗bc.ef *and* ∗cd.fa *concur at a Pascal Pole* ∗L(abcdef).

We found the Polar Pascal theorem very hard to prove, because the standard method of applying Desargues's theorem to earlier points and lines does not work. Perhaps this is why there is no hint of the new polarity in Veronese's paper. Our proof, given in the Appendix, involves the introduction of the quarks, whose only role is to facilitate this proof.

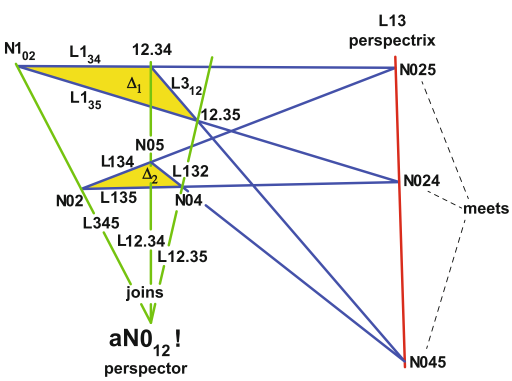
Reciprocity remains a tantalizing subject. Full polarity avoids some of the defects of Hesse's reciprocity, but it has its own problems. Can we somehow define a perfect reciprocity? Can we enlarge the mysticum so that it includes the Brianchon nodes? In a paper of 1881 [1] that we have not understood, Caporali defines a set of six lines permuted like the totals. This raises a number of questions. Do these six lines touch a conic? Can they be defined analogously to points and lines of the multimysticum and polar mysticum?

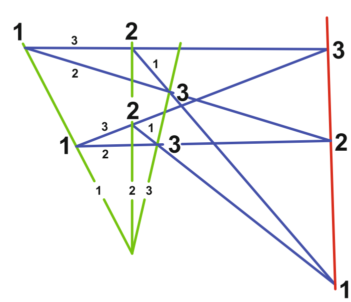
Appendix: Proofs

The proofs, like those of [2] (which we do not repeat here), are applications of Desargues' theorem. We present them in dual notation in a telegraphic style. As usual, each proof will be that two triangles have either a perspectrix or a perspector and therefore have the other. Each proof is a table, with rows for the vertices and edges of both triangles, a row for meets and joins, and a row with a perspectrix and perspector, one of which is new.

Where possible, a symmetry is noted to shorten the tables and their verifications. In the tables, "&c" after an entry means that the next entry is obtained by interchanging totals 4 and 5. This significantly reduces the work of verification.

To clarify these conventions, we present the first case in the longer style of [2]. It is the left-hand Desargues configuration below, which shows that a Cayley line and two Ladd lines pass through a point. The right-hand diagram shows how the vertices and edges of the two triangles Δ_1 and Δ_2 are numbered, for use in all later tables.





In the abbreviated format this proof reads:

Δ_1 vertices	edges	$N1_{02}$	12.34 &c	$L3_{12}$	$L1_{35}$ &c
Δ_2 vertices	edges	N02	N05 &c	L132	L135 &c
joins	meets	L345	L12.34 &c	N045	N024 &c
perspector	perspectrix	$aN0_{12}$ (new)		L13 (old)	

The permutation (345) transforms the three joins to a set containing the same Cayley line and one of the same Ladd lines. Hence the other transformed Ladd line passes through the same perspector. This perspector, being the intersection of a Cayley line and three Ladd lines, satisfies the incidence requirements of an anti-Kirkman node.

An almost reciprocal argument and its image under (345) applies for anti-Pascal lines:

Δ_1 vertices	edges	03.45	01.24 &c	$L2_{01}$	05.12.34 &c
Δ_2 vertices	edges	N123	N235 &c	L01	L05 &c
joins	meets	$L0_{45}$	$L4_{01}$ &c	N345	N12.34 &c
perspector	perspectrix	01.45 (old)		$aL0_{12}$ (new)	

After completing the lower incidences, we now proceed to the ladder. Here, the first incidences outside the classical mysticum concern the linking lines. The following Desargues configuration and its image under the permutation (01) show that each linking line passes through two Kirkman nodes, a Veronese node, and one meeting point:

Δ_1 vertices	edges	N234	02.15	N01.34	01.25.34	$aL5_{01}$	$L0_{15}$
Δ_2 vertices	edges	N235	02.14	N01.35	01.24.35	$aL4_{01}$	$L0_{14}$
joins	meets	L01	$L1_{02}$	$aL2_{01}$	23.45	N01.23	$N0_{23}$
perspector	perspectrix	N345 (old)			$2'3'.01$ (new)		

The remaining incidences (the rungs and handles of the ladder) follow in sequence from three Desargues decagrams. For the Kirkman nodes of height 1, the triangles are:

Δ_1 vertices	edges	N01	$N1_{05}$ &c	$L1_{23}$	L235 &c
Δ_2 vertices	edges	N02	$N2_{05}$ &c	$L2_{13}$	L135 &c
joins	meets	L345	$0'5'.12$ &c	N045	N04 &c
perspector	perspectrix	$N0'_{12}$ (new)		L123 (old)	

and their images under (345). For Pascal lines of height 1, we use the following:

Δ_1 vertices	edges	$N3_{01}$	N01.35 &c	$aL2_{01}$	$0'1'.34$ &c
Δ_2 vertices	edges	$N3_{02}$	N02.35 &c	$aL1_{02}$	$0'2'.34$ &c
joins	meets	$L3_{45}$	$aL4_{35}$ &c	N345	$N0'_{34}$ &c
perspector	perspectrix	N012 (old)		$L0'_{12}$ (new)	

and their images under (345). For the meeting points between heights 1 and 2, the decagram is:

Δ_1 vertices	edges	23.45	N014 &c	L23	$L4_{23}$ &c
Δ_2 vertices	edges	N01	$N5'_{01}$ &c	$4'5'.01$	L235 &c
joins	meets	L23.45	$L5'_{23}$ &c	N01.45	$N4_{01}$ &c
perspector	perspectrix	$2''3''.4'5'$ (new)		$0'1'.45$ (old)	

By elevating these three arguments, we obtain corresponding ones for all greater heights. In the third argument, the first elevation of the meeting point 23.45 should be $4'5'.23$ rather than $2'3'.45$. (Because $0-$ is 0, both of these could be considered.)

Two more Desargues decagrams form our justification of the polar mysticum. Our first decagram proves the Edge Pole Theorem, in dual notation.

Δ_1 vertices	edges	N01	$aN2_{01}$	$aN5_{01}$	L01.34	L234	L345
Δ_2 vertices	edges	$aN0_{23}$	N23	$aN5_{23}$	L014	L23.14	L145
joins	meets	L23.45	L01.45	L01.23	N25	N05	N02
perspector	perspectrix	$*01.23.45$ (new)			L134 (old)		

Number, Name and	Symbol	Numbers and Symbols of Incident Objects				
45 MeetiNg Points	$uv.wx$	4 $\mathrm{L}u_{wx}$	2 $u'v'.yz$	1 $\mathrm{L}uv.wx$	2 $uw.vx.yz$	4 $\mathrm{qL}u_{wx}(v)$
90 Linking Lines	$w'x'.uv$	2 $\mathrm{N}u_{wx}$	2 $\mathrm{N}w'_{uv}$	1 $\mathrm{N}uv.wx$	1 $wx.yz$	
90 Higher MeetiNgs[0]	$u^hv^h.w^{h-}x^{h-}$	2 $\mathrm{L}u^h_{wx}$	2 $\mathrm{L}w^{h-}_{uv}$	1 $\mathrm{L}uv.wx$		
90 Higher Linkings[1]	$u^hv^h.w^{h-}x^{h-}$	2 $\mathrm{N}u^h_{wx}$	2 $\mathrm{N}w^{h-}_{uv}$	1 $\mathrm{N}uv.wx$		
60 PascaL Lines	$\mathrm{L}x_{yz}$	3 $ux.yz$	3 $\mathrm{N}x_{uv}$	1 $\mathrm{N}uvw$	3 $\mathrm{qN}w_{uv}(x)$	
60 KirkmaN Nodes	$\mathrm{N}x_{yz}$	3 $y'z'.ux$	3 $\mathrm{L}x_{uv}$	1 $\mathrm{L}uvw$		
60 Higher PascaLs[0,2]	$\mathrm{L}x^h_{yz}$	3 $\mathrm{N}x^h_{uv}$	1 $\mathrm{N}uvw$	3 $u^hx^h.y^{h\pm}z^{h\pm}$		
60 Higher KirkmaNs[1,2]	$\mathrm{N}x^h_{yz}$	3 $\mathrm{L}x^h_{uv}$	1 $\mathrm{L}uvw$	3 $u^hx^h.y^{h\pm}z^{h\pm}$		
60 anti-PascaLs	$\mathrm{aL}x_{yz}$	1 $\mathrm{N}uvw$	3 $\mathrm{N}yz.uv$	6 $\mathrm{qN}x_{uv}(y)$		
60 anti-KirkmaNs	$\mathrm{aN}x_{yz}$	1 $\mathrm{L}uvw$	3 $\mathrm{L}yz.uv$	6 $\mathrm{qL}x_{uv}(y)$		
20 SteiNer Nodes	$\mathrm{N}xyz$	3 $\mathrm{L}u^h_{vw}$	3 $\mathrm{aL}u_{vw}$	3 $\mathrm{L}uv$	1 $\mathrm{L}uvw$	
20 CayLey Lines	$\mathrm{L}xyz$	3 $\mathrm{N}u^h_{vw}$	3 $\mathrm{aN}u_{vw}$	3 $\mathrm{N}uv$	1 $\mathrm{N}uvw$	
15 PLücker Lines	$\mathrm{L}uv$	4 $\mathrm{N}xyz$	3 $\mathrm{N}wx.yz$			
15 SalmoN Nodes	$\mathrm{N}uv$	4 $\mathrm{L}xyz$	3 $\mathrm{L}wx.yz$			
15 Edge Lines	$uv.wx.yz$	6 $uw.vx$	3 $\mathrm{N}uv.wx$			
15 Edge Poles	$*uv.wx.yz$	6 $*uw.vx$	3 $\mathrm{L}uv.wx$			
180 Quark Nodes	$\mathrm{qN}x_{yz}(w)$	2 $\mathrm{aL}x_{wu}$	1 $\mathrm{L}w_{uv}$	1 $*xw.yz$		
180 Quark Lines	$\mathrm{qL}x_{yz}(w)$	2 $\mathrm{aN}x_{wu}$	1 $*\mathrm{L}w_{uv}$	1 $xw.yz$		
45 VeroNese Nodes[1]	$\mathrm{N}wx.yz$	4 $\mathrm{aL}u_{yz}$	1 $\mathrm{L}uv$	1 $uv.wx.yz$	2 $w^hx^h.y^{h-}z^{h-}$	
45 Ladd Lines[0]	$\mathrm{L}wx.yz$	4 $\mathrm{aN}u_{yz}$	1 $\mathrm{N}uv$	1 $*uv.wx.yz$	2 $w^hx^h.y^{h-}z^{h-}$	1 $wx.yz$

[0] For the meeting points mentioned here, the larger superscript must be even.
[1] For the linking lines mentioned here, the larger superscript must be odd.
[2] $h\pm$ means $h \pm 1$ ($u^hx^h.y^{h+1}z^{h+1}$ is an alternative name for $y^{h+1}z^{h+1}.u^hx^h$).

Because a meeting point is defined as the intersection of (just) two edges, we can define a meeting polar to be the join of the corresponding edge poles. Our second decagram proves that the three lines L045, $\mathrm{qL}5_{04}(2)$, $*02.13$ are concurrent. The first and third are invariant under interchanging 4 and 5, whereas the first and second are invariant under interchanging 0 and 4. The perspector is therefore the intersection of the three meeting polars $*02.13, *42.13, *52.13$, proving the Polar Pascal Theorem.

Δ_1 vertices	edges	N12	$\mathrm{aN}5_{12}$	$*03.12.45$	L03.12	L03.45	L034
Δ_2 vertices	edges	N23	$\mathrm{aN}5_{23}$	$*01.23.45$	L01.23	L01.45	L014
joins	meets	L045	$\mathrm{qL}5_{04}(2)$	$*02.13$	N45	$\mathrm{aN}2_{45}$	N25
perspector	perspectrix		$*\mathrm{L}2_{13}$ (new)			L013 (old)	

This also proves that a Pascal pole lies on a quark line, the polar version of the first mysterious fact we mentioned. The second is proved by yet another decagram.

Δ_1 vertices	edges	N01	24.35 &c	01.23.45	L25.34 &c
Δ_2 vertices	edges	N02	14.35 &c	02.13.45	L15.34 &c
joins	meets	L345	$\mathrm{L}4_{35}$ &c	03.12	$\mathrm{aN}0_{34}$ &c
perspector	perspectrix	N012 (old)		$\mathrm{qL}0_{12}(3)$ (new)	

We end with a table of all the incidences in the multimysticum.

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