Quadrature Histograms in Maximum Likelihood Quantum State Tomography

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Quantum state tomography (QST) aims to determine the quantum state of a system from measured data and is an essential tool for quantum information. When dealing with quantum states of light, QST is done by measuring quantum noise statistics of the field amplitudes at different optical phases using homodyne detection. The quadrature-phase homodyne measurement outputs a continuous variable, but we can histogram the continuous measurements and make the statistical estimation faster without losing too much information. This paper investigate different ways to determine the quadrature histograms for optical homodyne QST.

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I. INTRODUCTION

Quantum information science and engineering is now at the point where rudimentary quantum computers are available in the laboratory and commercially [1–4]. Consequently, precise reconstruction and diagnostic tools used to estimate quantum states [5–12], processes [13–20], and measurements [21–24] are fundamental.

Quantum tomography techniques for quantum states of light became a subject of major interest in recent years, since quantum light sources are essential for implementations of continuous-variable (CV) quantum computation [25–29]. These source are also extensively exploited in quantum criptography [30–34], quantum metrology [35, 36], state teleportation [37–39], dense coding [40, 41] and cloning [42, 43].

In quantum state tomography, we perform a large number of experimental measurements on a collection of quantum systems, all prepared in a same unknown state. The goal is to estimate this unknown state from the experimental measurements results. This estimation can be done from the experimental statistical data by different methods. In here we will be dealing with Maximum Likelihood estimation, that finds among all possible states, the one which maximizes the probability of obtaining the experimental data set in hand.

Quantum homodyne tomography is one of the most popular optical tomography techniques available. It rapidly became a versatile tool and has been applied in many different quantum optics experimental settings since it was proposed by Vogel and Risken in 1989 [5] and first implemented by Smithey *et al.* in 1993 [6]. This technique permits to characterize a light quantum state by means of the electric field quantum noise statistics collected through multiple phase-sensitive measurements.

A homodyne measurement generates a continuous value. While no data binning is necessarily needed, we believe that the loss due to binning may be insignificant. On the other hand, discretization of the data by binning it reduces considerably the number of data, expediting the reconstruction algorithm. But how can we estimate the quadrature bin width, such that the bins are not too small nor too big? Bins should not be too small in order to avoid significant statistical fluctuation due to scarcity of samples in each bin. And bins should not be too large in order to avoid a lack of resolution, making the histogram a poor representation of the underlying distribution shape.

In this paper, we use numerical experiments to simulate optical homodyne tomography of quantum optical states and perform maximum likelihood tomography on the data with and without binning. When choosing a quadrature bin width, we use and compare two different ways: Scott's rule [44] and equation (5.136) from Leonhardt's book [45]. The paper is divide as follow: in Section II we review maximum likelihood in homodyne tomography. In Section III we describe our numerical experiments and present our results. In Section IV we discuss the interpretation of our results and make some concluding remarks.

II. MAXIMUM LIKELIHOOD IN HOMODYNE TOMOGRAPHY

Let us consider N quantum systems, each of them prepared in an optical state described by a density matrix ρ_{true}. In each experimental run, we measure the field quadrature at different phases θ of a local oscillator, i.e. a reference system prepared in a high amplitude coherent state. Each measurement is associated with an observable $\hat{X}_\theta = \hat{X}\cos\theta + \hat{P}\sin\theta$, where $\hat{X}$ and $\hat{P}$ are the position and momentum operators, respectively. For a given phase θ, we measure a quadrature value x, resulting on a data set $\{(\theta_i, x_i)\}$.

The outcome of the i-th measurement is described by a positive-operator-valued measure (POVM) element

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$\Pi(x_i|\theta_i) = \Pi_i$. Given the data set $\{(\theta_i, x_i) : i = 1, ..., N\}$, we can write the likelihood of a candidate density matrix ρ as

$$\mathcal{L}(\rho) = \prod_{i=1}^{N} \mathrm{Tr}(\Pi_i \rho), \quad (1)$$

where $\mathrm{Tr}(\rho\Pi_i)$ is the probability, when measuring with phase θ_i, to obtain outcome x_i, according to the candidate density matrix ρ.

MLE searches within the density matrix space the one that maximizes the likelihood in (**??**). Equivalently, it usually is more convenient to maximize the logarithm of the likelihood (the "log-likelihood"):

$$L(\rho) = \ln \mathcal{L}(\rho) = \sum_{i=1}^{N} \ln[\mathrm{Tr}(\Pi_i \rho)], \quad (2)$$

which is maximized by the same density matrix as the likelihood. The MLE is essentially a function optimization problem, and since the log-likelihood function is concave, the convergence to an unique solution will be achieved by most iterative optimization methods.

In our numerical simulations, we use an algorithm for likelihood maximization that begins with interactions of the $R\rho R$ algorithm [**?**] followed by iterations of a regularized gradient ascent algorithm (RGA). The main reason to switch from one algorithm to another is the fact that an expressive slow-down is observed in the $R\rho R$ algorithm after about $(n+1)^2/4$ iterations. In the RGA, $\rho^{(k+1)}$ is parametrized as

$$\rho^{(k+1)} = \frac{\left(\sqrt{\rho^{(k)}} + A\right)\left(\sqrt{\rho^{(k)}} + A^\dagger\right)}{\mathrm{Tr}\left[\left(\sqrt{\rho^{(k)}} + A\right)\left(\sqrt{\rho^{(k)}} + A^\dagger\right)\right]}, \quad (3)$$

where $\rho^{(k)}$ is the density found by the last interaction of $R\rho R$, and A may be any complex matrix of the same dimensions as ρ. Eq. (**??**) ensures that $\rho^{(k+1)}$ is a physical density matrix for any choosed A. The matrix A should maximize the quadratic approximation of the log-likelihood subject to $\mathrm{Tr}(AA^\dagger) \leq u$, where u is a positive number adjusted by the algorithm to guarantee that the log-likelihood increases with each iteration. To halt the interactions, we use the stopping criterion of [**?**], $L(\rho_{\mathrm{ML}}) - L(\rho^{(k)}) \leq 0.2$, where $L(\rho_{\mathrm{ML}})$ is the maximum of the log-likelihood.

III. NUMERICAL EXPERIMENTS

IV. CONCLUSION

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