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# A Computational Toolbox for Teaching Quantum Mechanics and Quantum Optics

**Uma Ferramenta Computational para o Ensino da Mecânica Quântica e da Óptica Quântica**

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We describe a collection of code~~s~~ that ~~may be~~are useful in teaching of quantum mechanics and quantum optics courses. The toolbox has been developed using Matlab programming language, but we expect that a translation to an open source mathematics software could be easily done [SG: Want to ask Italo to do this?]. These codes can be used to help visualize quantum states of light using Wigner function, how these states may be affected by loss, among other things.
**Keywords:** Quantum states of light, Wigner function, purity, fidelity.

Descrevemos um conjunto de códigos computacionais que podem ser úteis no ensino da mecânica quântica e da óptica quântica. Esse ferramental foi desenvolvido usando a linguagem de programação Matlab, mas esperamos que uma tradução desses códigos para uma linguagem de software aberto possa ser feito facilmente. Os códigos aqui descritos ajudam a visualizar estados quânticos da luz usando a função de Wigner, como estes estados podem ser afetados por perdas, entre outras coisas.
**Palavras-chave:** Estados quânticos da luz, função de Wigner, pureza, fidelidade.

## 1. Introduction

Quantum mechanics is one of the most crucial, and yet, challenging, topics of modern physics [1, 2]. With its application to many and important technological problems, the challenges in teaching quantum mechanics are now part not only of the physical science undergraduate curricula, but also of engineering education, especially in the area of communications [3, 4]. Some of the challenges are also faced by professors when teaching quantum optics in a graduate (or undergraduate) level. Any visualization tools in these areas of knowledge can be of great assistance towards an appropriated grasp of concepts by students [5, 6, 7, 8, 9].

We ~~are going to~~ describe here a set of codes developed using Matlab programming language and their use to give insight into the nature of quantum mechanics and quantum optics. In the context of our work, we use a formulation of quantum mechanics that is equivalent to the standard pedagogical approach: the Wigner function [10].

If the wave function $\psi(x)$ of a system is known, we may easily determine the probability density $|\psi(x)|^2$ in position space $x$. On the other hand, the momentum distribution, $|\phi(p)|^2$, is difficult to visualize if we have only $\psi(x)$. That is the standard formulation of quantum mechanics. We may alternatively approach quantum mechanics through a function that displays the probability distribution simultaneously in the $x$ and $p$ variables. This function was introduced in 1932 by Eugene Wigner [11] and is defined as:

$$W(x,p) = \frac{1}{h}\int e^{-ipy/\hbar}\psi(x+y/2)\psi^*(x-y/2)\,dy. \quad (1)$$

Quantum mechanical probability densities in position space $x$ and momentum space $p$ can be obtained from the marginals of the Wigner function, since

$$\int W(x,p)\,dp = \psi^*(x)\psi(x) = |\psi(x)|^2,$$
$$\int W(x,p)\,dx = \phi^*(p)\phi(p) = |\phi(p)|^2. \quad (2)$$

The Wigner function $W(x,p)$ is known as a quasi-probability ~~function~~distribution because, even though it is real

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and normalized to one, ~~it is not necessarily non-negativefo~~ some states it is negative over some of $(x, p)$ space. The amount of negativity is a measure of the "nonclassicality" of a state.[citation needed]

This paper is organized as follows: in Section 2 we describe the use of the computational toolbox to study the harmonic oscillator. In Section 3 we show the use of the visualization toolbox for other states of the light, and in Section 4 we make some concluding remarks.

## 2. Example: simple harmonic oscillator Fock states

We can use the wave function for each energy state of the harmonic oscillator and Eq. (1) to find the corresponding Wigner function for each of these states. Alternatively, we may use the Algebraic Method described in [12], [SG: Something is wrong with this sentence, and I'm not sure what it is trying to say.] continue the discussion to introduce the number operator, $\hat{n} = \hat{a}^\dagger \hat{a}$, and the ket notation for the harmonic oscillator's eigenstates $|n\rangle$, as done in [13]. [SG: I would explain that the code represents the states in the photon number basis here. Also why we need to specify the maximum number of photons and how to choose the maximum number.]

In the toolbox, there is a routine for generating the pure state vector for photon number eigenstate $n$ in the photon number basis, as show bellow. When calling this routine, the students need to specify two variables: the number of photons, $n$, and the photon number at which the Hilbert space is truncated ~~(since the routines where developed to study Quantum State Tomography, we need to specify where we truncate the Hilbert space)~~.

```
function psi = generate_fock_vector(n, maxPhoton)
% Creates photon number state
%    psi = generate_fock_vector(n, maxPhoton) returns the pure state vector
%    for photon number eigenstate n in the photon number basis. maxPhoton is
%    the photon number at which the Hilbert space is truncated.

psi = zeros(maxPhoton+1,1);
psi(n+1,1) = 1;
```

[SG: LaTeX has nice tools for inserting code into text. Check out `https://en.wikibooks.org/wiki/LaTeX/Source_Code_Listings`. I think that will be easer to mannage than inserting the code as images.]

The following lines of code plot the Wigner function of a vacuum state ($n = 0$). Notice that to plot the Wigner function of any $n$ photon state, all that we need to do is change the number of photons $n$ in the routine. The resulting plot for the vacuum state can be seen in Figure 1. [SG: I would remove the lines for `etaState = 1` and `rho = apply_loss` because they are not used here and they add clutter. Also, maybe we should write a graphing function that encapsulates the last four lines, gives a default step size, and draws the graph.]

```
% The infinite dimensional state space for the harmonic oscillator will be
% represented in the photon number basis.
% We will truncate the Hilbert space at maxPhotonNumber photons.
maxPhotonNumber = 10;

% First, pre-compute a lot of numbers, such as coefficients for Hermite
% polynomials, factorials, binomial coefficients.
S = init_tables(maxPhotonNumber);

% Make state vector for Fock state.
n = 0;  % number of photons
psi = generate_fock_vector(n, maxPhotonNumber)
% The Fock state may suffer from some loss by passing through a
% medium with etaState efficiency.
etaState = 1;
rho = apply_loss(psi,etaState,S);
% Now it must be represented by a density matrix, rho.

wignerStepSize = 0.1;
[x,p] = meshgrid(-4:wignerStepSize:4,-4:wignerStepSize:4);
wigner2 = wigner(rho1, x,p);
mesh(x,p,wigner2); xlabel('x'); ylabel('p'); zlabel('W(x,p)');
```

[SG: After introducing each figure, I recommend writing a few sentences about what features of the figure students should notice to increase their understanding of the state's properties.]

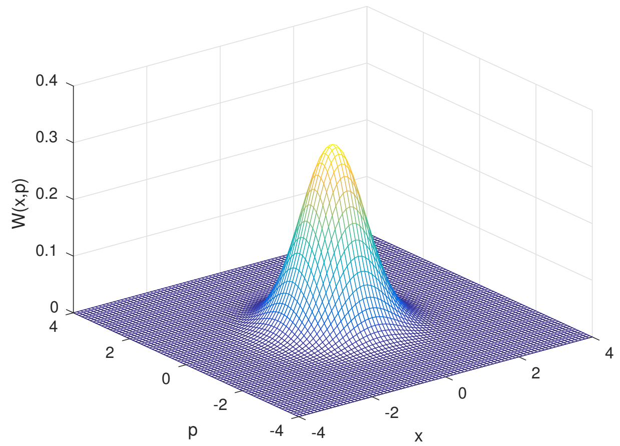


**Figura 1:** Wigner function of a vacuum state $n = 0$.

Figure 2 shows the plot of $W(x, p)$ of a $n = 4$ number state. ~~In here it can be seen~~One can see the negative part of the Wigner function, usually related to the nonclassicality of the state.

## 3. Example: other states of light

In the context of a quantum optics course, it is natural to study other states of light, such as coherent and squeezed states. Moreover, these states may also be introduced in an upperlevel quantum mechanics course. A coherent state is an eigenstate of the annihilation

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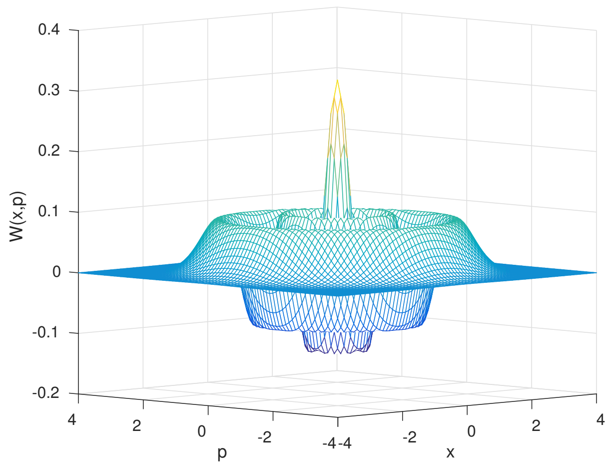


**Figura 2:** Wigner function of a number state $n = 4$.

operator, $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$, where $\alpha$ is a complex number. Coherent states are minimum uncertainty states.

A squeezed state is a state of light where fluctuations are reduced below the standard quantum limit in one quadrature at the expense of increased fluctuations in the conjugate quadrature. The toolbox provides routines to generate a coherent and a squeezed vacuum state, as shown bellow.

```
function psi = generate_coherent_vector(alpha, maxPhoton)
% Creates coherent state
%   psi = generate_coherent_vector(alpha, maxPhoton) returns the pure state
%   vector in the photon number basis of complex amplitude alpha. maxPhoton
%   is the photon number at which the Hilbert space is truncated.

n = (0:maxPhoton).';
psi = alpha.^n./sqrt(factorial(n));
normalization = exp(-0.5 .* abs(alpha).^2);
psi = normalization * psi;

psi = normalize(psi,'check');
```

[SG: I have always hated the `ratioSwitch` in `generate_squeezed_vacuum_vector`. It adds a lot of clutter here, so maybe we should remove it for this paper.]

To plot the Wigner function of a coherent state and a squeezed vacuum state, we use the lines of code showed. We may plot the Wigner function of coherent states of different amplitude by changing the value for $\alpha$ in the code. When plotting the Wigner function of a squeezed vacuum state, we need to specify if we are using the state's variance (as in the example showed) or the ratio of the state's variance with respect to vacuum's variance (that is 1/2). Plotting different squeezed vacuum states is done by changing the variance or the ratio.

In Figure 3 we have the Wigner function of a coherent state with $\alpha = 1$. When students compare Figures 1

```
function psi = generate_squeezed_vacuum_vector(varianceOrRatio, maxPhotons, ratioSwitch)
% state vector for squeezed vacuum state in Fock basis
%    generate_squeezed_vacuum_vector(varianceOrRatio, maxPhotons, ...
%        ratioSwitch)
%    returns the state vector in the Fock basis for a squeezed vacuum
%    state.  The level of squeezing is given by varianceOrRatio, which may
%    be the state's x-quadrature variance or the ratio of this variance to
%    the vacuum variance.  ratioSwitch = 'true variance' indicates that
%    varianceOrRatio is the state's variance. ratioSwotch = 'ratio'
%    indicates that varianceOrRatio is ratio of the state's variance with
%    vacuum.  Vacuum variance = 1/2.  The state will be expressed in a
%    Hilbert space with at most maxPhotons.  maxPhotons may also be the
%    struct S generated by init_tables.

vacuumVariance = 0.5;

if exist('ratioSwitch', 'var') && strcmp(ratioSwitch,'true variance')
    variance = varianceOrRatio;
elseif exist('ratioSwitch', 'var') && strcmp(ratioSwitch,'ratio')
    variance = varianceOrRatio*vacuumVariance;
else  % if no ratioSwitch is present, assume the ratio is given
    variance = varianceOrRatio*vacuumVariance;
end

lambda = (variance-vacuumVariance)/(variance+vacuumVariance);

if isstruct(maxPhotons)
    maxPhotons = maxPhotons.photons;
end

% psi will have occupation of only even numbered photons.
% n is the number of nonzero elements of psi
n = floor(maxPhotons / 2);

a=(0:n).';
b=realsqrt(factorial(2*a));
c=factorial(a);
d=(lambda/2).^(a);
psi=b./c.*d;
psi = psi .* ((1 - lambda^2)^(1/4));

psi = normalize(psi, 'check');

% here we add zeros into the odd numbered photons places
psi=[psi.';zeros(1,n+1)];
psi=psi(1:(maxPhotons+1));
psi=psi(:);

end
```

```
% The infinite dimensional state space for the harmonic oscillator will be
% represented in the photon number basis.
% We will truncate the Hilbert space at maxPhotonNumber photons.
maxPhotonNumber = 10;

% First, pre-compute a lot of numbers, such as coefficients for Hermite
% polynomials, factorials, binomial coefficients.
S = init_tables(maxPhotonNumber);

% Make state vector for coherent state.
alpha = 1;  % amplitude of coherent state
psi = generate_coherent_vector(alpha, maxPhotonNumber);
% The coherent state may suffer from some loss by passing through a
% medium with etaState efficiency.
etaState = 1;
rho = apply_loss(psi,etaState,S);
% Now it must be represented by a density matrix, rho.

wignerStepSize = 0.1;
[x,p] = meshgrid(-4:wignerStepSize:4,-4:wignerStepSize:4);
wigner2 = wigner(rho, x,p);
mesh(x,p,wigner2); xlabel('x'); ylabel('p'); zlabel('W(x,p)');
```

```
% The infinite dimensional state space for the harmonic oscillator will be
% represented in the photon number basis.
% We will truncate the Hilbert space at maxPhotonNumber photons.
maxPhotonNumber = 10;

% First, pre-compute a lot of numbers, such as coefficients for Hermite
% polynomials, factorials, binomial coefficients.
S = init_tables(maxPhotonNumber);

% Make state vector for squeezed vacuum state.
v = 1/6;  %variance
ratioSwitch = 'true variance';
psi = generate_squeezed_vacuum_vector(v, maxPhotonNumber, ratioSwitch);
% The squeezed vacuum state may suffer from some loss by passing through a
% medium with etaState efficiency.
etaState = 1;
rho = apply_loss(psi,etaState,S);
% Now it must be represented by a density matrix, rho.

wignerStepSize = 0.1;
[x,p] = meshgrid(-4:wignerStepSize:4,-4:wignerStepSize:4);
wigner2 = wigner(rho, x,p);
mesh(x,p,wigner2); xlabel('x'); ylabel('p'); zlabel('W(x,p)');
```

and 3, they see that a coherent state is a displaced vacuum state. Figure 4 shows the Wigner function of a squeezed vacuum state with variance equals to 1/6 of vacuum variance.

[SG: I would not repeat the maxPhotonNumber and S assignments every time.]

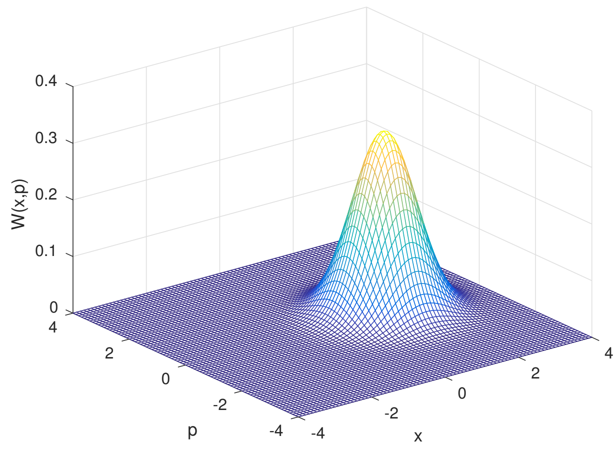


**Figura 3:** Wigner function of a coherent state with $\alpha = 1$.

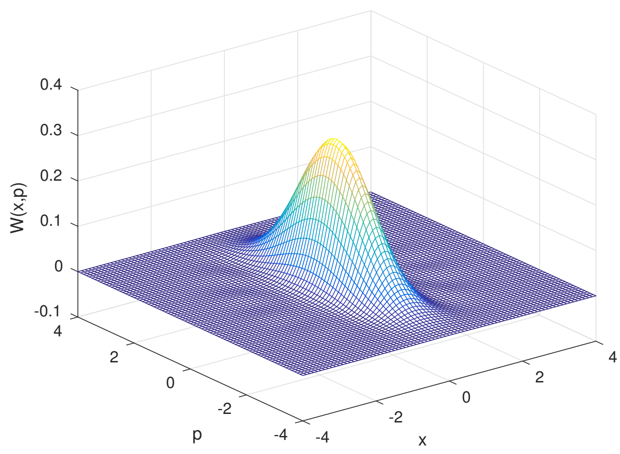


**Figura 4:** Wigner function of a squeezed vacuum state with variance equals to 1/6 of vacuum variance.

Students from both courses [SG: What are the two courses?] can explore the fundamental superposition principle of quantum mechanics using superpositions of coherent states, also known as Schrödinger cat states, such as:

$$|\text{cat}\rangle = \frac{1}{\sqrt{2\left(1+e^{-2|\alpha|^2}\cos\theta\right)}}\left(e^{i\theta}|-\alpha\rangle + |\alpha\rangle\right). \qquad (3)$$

The routines to generate a cat state with amplitude $\alpha$ and to plot the corresponding Wigner function for such states are shown bellow.

Figure 5 shows the Wigner function of a cat state with amplitude of coherent states in the superposition of $\alpha = 2$ and superposition phase $\theta = 0$.

The effect of loss in any of the states shown here can be considered and observed by changing the `etaS-`

```
function catPsi = generate_cat_vector(alpha, theta, maxPhotons)
% Creates Schrodinger cat state
%    catPsi = generate_cat_vector(alpha, theta, maxPhoton) returns the pure
%    state vector for a Schrodinger cat state (namely,
%    e^{i*theta}|-alpha>+|alpha>) in the photon number basis. maxPhoton is
%    the photon number at which the Hilbert space is truncated or the table
%    made by init_tables.
%

if isstruct(maxPhotons)
    maxPhotons = maxPhotons.photons;
end

catPsi = (exp(1i .* theta) .* (-alpha).^(0:maxPhotons) + alpha.^(0:maxPhotons)) ./
sqrt(factorial(0:maxPhotons));
catPsi = catPsi.';

normalization = exp(-abs(alpha).^2./2)./sqrt(2.*(1+exp(-2*abs(alpha)^2).*cos(theta)));
catPsi = normalization .* catPsi;

catPsi = normalize(catPsi,'check');
```

```
% The infinite dimensional state space for the harmonic oscillator will be
% represented in the photon number basis.
% We will truncate the Hilbert space at maxPhotonNumber photons.
maxPhotonNumber = 10;

% First, pre-compute a lot of numbers, such as coefficients for Hermite
% polynomials, factorials, binomial coefficients.
S = init_tables(maxPhotonNumber);

% Make state vector for Schrodinger cat state.
alpha = 2;  % amplitude of coherent states in the superposition
phase = 0;  % phase between superposition
psi = generate_cat_vector(alpha, phase, S);
% The Schrodinger cat state suffers from some loss by passing through a
% medium with etaState efficiency.
etaState = 1;
rho = apply_loss(psi,etaState,S);
% Now it must be represented by a density matrix, rho.

wignerStepSize = 0.1;
[x,p] = meshgrid(-5:wignerStepSize:5,-5:wignerStepSize:5);
wigner2 = wigner(rho, x,p);
mesh(x,p,wigner2); xlabel('x'); ylabel('p'); zlabel('W(x,p)');
```

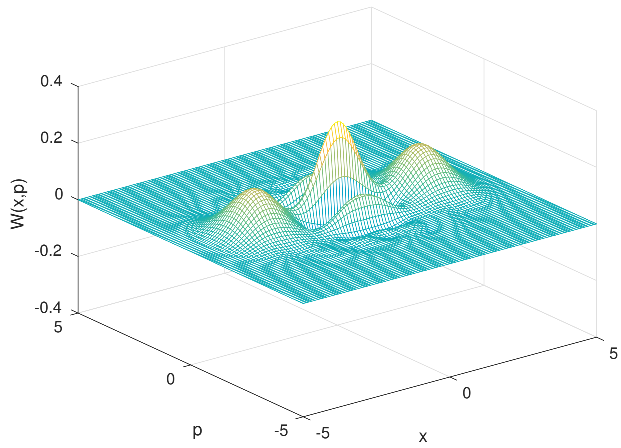


**Figura 5:** Wigner function of a cat state with amplitude of coherent states in the superposition $\alpha = 2$ and superposition phase $\theta = 0$.

`tate.` parameter when using the lines of codes. Let us consider, for example, the effect of loss in the cat state of Figure 5. For that, consider that the state passed through a medium with 90 % efficiency (`etaState = 0.9`). The Wigner function for the state after the loss is shown in figure 6. [SG: Since we are now talking about mixed states, maybe we should explain how to calculate the Wigner function given a density matrix.]

We then may use the following lines of code to calculate how "close"the state with loss is of the original

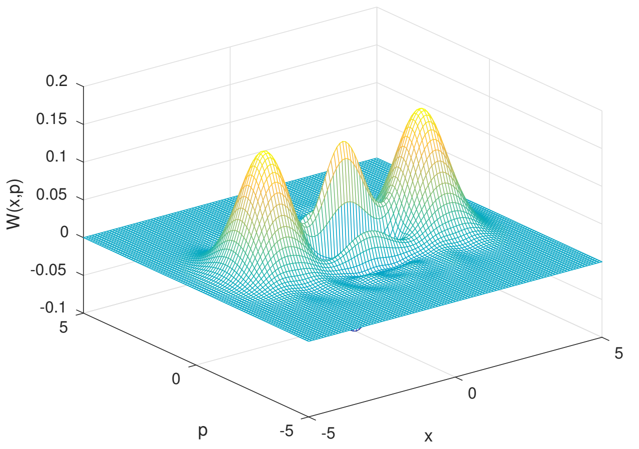


**Figura 6:** Wigner function of a cat state with amplitude of coherent states in the superposition $\alpha = 2$ and superposition phase $\theta = 0$ after going through a medium with 90% efficiency.

state. That is done using the concept of fidelity, defined, for any two density matrices, $\rho$ and $\sigma$, as

$$F(\rho,\sigma) = \left( Tr\left( \sqrt{\sqrt{\rho}\sigma\sqrt{\rho}} \right) \right)^2, \tag{4}$$

where the fidelity is equal to 1 if and only if ~~the two states do coincide~~$\rho = \sigma$.

```
% The infinite dimensional state space for the harmonic oscillator will be
% represented in the photon number basis.
% We will truncate the Hilbert space at maxPhotonNumber photons.
maxPhotonNumber = 10;

% First, pre-compute a lot of numbers, such as coefficients for Hermite
% polynomials, factorials, binomial coefficients.
S = init_tables(maxPhotonNumber);

% Make state vector for Schrodinger cat state.
alpha = 2;  % amplitude of coherent states in the superposition
phase = 0;  % phase between superposition
psi1 = generate_cat_vector(alpha, phase, S);
% The Schrodinger cat state suffers from some loss by passing through a
% medium with etaState efficiency.
etaState1 = 1;
rho1 = apply_loss(psi1,etaState1,S);
% Now it must be represented by a density matrix, rho.
psi2 = generate_cat_vector(alpha, phase, S);
etaState2 = 0.9;
rho2 = apply_loss(psi2,etaState2,S);

F = fidelity(rho1, rho2)
```

In the example given, the fideliy is $F = 0.7171$.

[SG: Some other tools that you might want to add: Phase evolution using `phase_evolution.m` can help students visualize the harmonic oscillator dynamics. Graphing quadrature probability distribution functions with `quadrature_pdf.m`. Calculating quadrature variances and comparing states' product of variances to the Heisenburg limit. Calculating photon probability distributions and mean photon number.]

## 4. Conclusion

The toolbox of computational routines described in this paper has been proved useful for teaching quantum mechanics and quantum optics courses. The lines of code developed for this toolbox facilitates the visualization of quantum states using the Wigner quasi-probability distibution, also called the Wigner function. Our code is available at ....

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