# Basic Game Theory for competitive programing

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# Contents

# Chapter 1

# Game theory

Not to be confused with Game studies.
This article is about the mathematical study of optimizing agents. For other uses, see Game theory (disambiguation).

**Game theory** is "the study of mathematical models of conflict and cooperation between intelligent rational decision-makers."[1] Game theory is mainly used in economics, political science, and psychology, as well as logic, computer science, biology and poker.[2] Originally, it addressed zero-sum games, in which one person's gains result in losses for the other participants. Today, game theory applies to a wide range of behavioral relations, and is now an umbrella term for the science of logical decision making in humans, animals, and computers.

Modern game theory began with the idea regarding the existence of mixed-strategy equilibria in two-person zero-sum games and its proof by John von Neumann. Von Neumann's original proof used Brouwer fixed-point theorem on continuous mappings into compact convex sets, which became a standard method in game theory and mathematical economics. His paper was followed by the 1944 book *Theory of Games and Economic Behavior*, co-written with Oskar Morgenstern, which considered cooperative games of several players. The second edition of this book provided an axiomatic theory of expected utility, which allowed mathematical statisticians and economists to treat decision-making under uncertainty.

This theory was developed extensively in the 1950s by many scholars. Game theory was later explicitly applied to biology in the 1970s, although similar developments go back at least as far as the 1930s. Game theory has been widely recognized as an important tool in many fields. With the Nobel Memorial Prize in Economic Sciences going to game theorist Jean Tirole in 2014, eleven game-theorists have now won the economics Nobel Prize. John Maynard Smith was awarded the Crafoord Prize for his application of game theory to biology.

## 1.1 History

Early discussions of examples of two-person games occurred long before the rise of modern, mathematical game theory. The first known discussion of game theory occurred in a letter written by Charles Waldegrave, an active Jacobite, and uncle to James Waldegrave, a British diplomat, in 1713.[3] In this letter, Waldegrave provides a minimax mixed strategy solution to a two-person version of the card game le Her, and the problem is now known as Waldegrave problem. James Madison made what we now recognize as a game-theoretic analysis of the ways states can be expected to behave under different systems of taxation.[4][5] In his 1838 *Recherches sur les principes mathématiques de la théorie des richesses* (*Researches into the Mathematical Principles of the Theory of Wealth*), Antoine Augustin Cournot considered a duopoly and presents a solution that is a restricted version of the Nash equilibrium.

In 1913 Ernst Zermelo published *Über eine Anwendung der Mengenlehre auf die Theorie des Schachspiels*. It proved that the optimal chess strategy is strictly determined. This paved the way for more general theorems.[6]:429

The Danish mathematician Zeuthen proved that the mathematical model had a winning strategy by using Brouwer's fixed point theorem. In his 1938 book *Applications aux Jeux de Hasard* and earlier notes, Émile Borel proved a minimax theorem for two-person zero-sum matrix games only when the pay-off matrix was symmetric. Borel conjectured that non-existence of mixed-strategy equilibria in two-person zero-sum games would occur, a conjecture that was proved false.

*John von Neumann*

Game theory did not really exist as a unique field until John von Neumann published a paper in 1928.[7] Von Neumann's original proof used Brouwer's fixed-point theorem on continuous mappings into compact convex sets, which became a standard method in game theory and mathematical economics. His paper was followed by his 1944 book *Theory of Games and Economic Behavior* co-authored with Oskar Morgenstern.[8] The second edition of this book provided an axiomatic theory of utility, which reincarnated Daniel Bernoulli's old theory of utility (of the money) as an independent discipline. Von Neumann's work in game theory culminated in this 1944 book. This foundational work contains the method for finding mutually consistent solutions for two-person zero-sum games. During the following time period, work on game theory was primarily focused on cooperative game theory, which analyzes optimal strategies for groups of individuals, presuming that they can enforce agreements between them about proper strategies.[9]

In 1950, the first mathematical discussion of the prisoner's dilemma appeared, and an experiment was undertaken by notable mathematicians Merrill M. Flood and Melvin Dresher, as part of the RAND Corporation's investigations into game theory. RAND pursued the studies because of possible applications to global nuclear strategy.[10] Around this same time, John Nash developed a criterion for mutual consistency of players' strategies, known as Nash equilibrium, applicable to a wider variety of games than the criterion proposed by von Neumann and Morgenstern. This equilibrium is sufficiently general to allow for the analysis of non-cooperative games in addition to cooperative ones.

Game theory experienced a flurry of activity in the 1950s, during which time the concepts of the core, the extensive form game, fictitious play, repeated games, and the Shapley value were developed. In addition, the first applications of game theory to philosophy and political science occurred during this time.

### 1.1.1 Prize-winning achievements

In 1965, Reinhard Selten introduced his solution concept of subgame perfect equilibria, which further refined the Nash equilibrium (later he would introduce trembling hand perfection as well). In 1967, John Harsanyi developed the concepts of complete information and Bayesian games. Nash, Selten and Harsanyi became Economics Nobel Laureates in 1994 for their contributions to economic game theory.

In the 1970s, game theory was extensively applied in biology, largely as a result of the work of John Maynard Smith and his evolutionarily stable strategy. In addition, the concepts of correlated equilibrium, trembling hand perfection, and common knowledge[11] were introduced and analyzed.

In 2005, game theorists Thomas Schelling and Robert Aumann followed Nash, Selten and Harsanyi as Nobel Laureates. Schelling worked on dynamic models, early examples of evolutionary game theory. Aumann contributed more to the equilibrium school, introducing an equilibrium coarsening, correlated equilibrium, and developing an extensive formal analysis of the assumption of common knowledge and of its consequences.

In 2007, Leonid Hurwicz, together with Eric Maskin and Roger Myerson, was awarded the Nobel Prize in Economics "for having laid the foundations of mechanism design theory." Myerson's contributions include the notion of proper equilibrium, and an important graduate text: *Game Theory, Analysis of Conflict*.[1] Hurwicz introduced and formalized the concept of incentive compatibility.

In 2012, Alvin E. Roth and Lloyd S. Shapley were awarded the Nobel Prize in Economics "for the theory of stable allocations and the practice of market design" and, in 2014, the Nobel went to game theorist Jean Tirole.

## 1.2 Game types

### 1.2.1 Cooperative / Non-cooperative

Main articles: Cooperative game and Non-cooperative game

A game is *cooperative* if the players are able to form binding commitments externally enforced (e.g. through contract law). A game is *non-cooperative* if players cannot form alliances or if all agreements need to be self-enforcing (e.g. through credible threats).[1]

Cooperative games are often analysed through the framework of *cooperative game theory*, which focuses on predicting which coalitions will form, the joint actions that groups take and the resulting collective payoffs. It is opposed to the traditional *non-cooperative game theory* which focuses on predicting individual players' actions and payoffs and analyzing Nash equilibriums.[2][3]

Cooperative game theory provides a high-level approach as it only describes the structure, strategies and payoffs of coalitions, whereas non-cooperative game theory also looks at how bargaining procedures will affect the distribution of payoffs within each coalition. As non-cooperative game theory is more general, cooperative games can be analyzed through the approach of non-cooperative game theory (the converse does not hold) provided that sufficient assumptions are made to encompass all the possible strategies available to players due to the possibility of external enforcement of cooperation. While it would thus be optimal to have all games expressed under a non-cooperative framework, in many instances insufficient information is available to accurately model the formal procedures available to the players during the strategic bargaining process, or the resulting model would be of too high complexity to offer a practical tool in the real world. In such cases, cooperative game theory provides a simplified approach that allows to analyze the game at large without having to make any assumption about bargaining powers.

### 1.2.2 Symmetric / Asymmetric

Main article: Symmetric game

A symmetric game is a game where the payoffs for playing a particular strategy depend only on the other strategies employed, not on who is playing them. If the identities of the players can be changed without changing the payoff to the strategies, then a game is symmetric. Many of the commonly studied 2×2 games are symmetric. The standard representations of chicken, the prisoner's dilemma, and the stag hunt are all symmetric games. Some scholars would consider certain asymmetric games as examples of these games as well. However, the most common payoffs for each of these games are symmetric.

Most commonly studied asymmetric games are games where there are not identical strategy sets for both players. For instance, the ultimatum game and similarly the dictator game have different strategies for each player. It is possible, however, for a game to have identical strategies for both players, yet be asymmetric. For example, the game pictured to the right is asymmetric despite having identical strategy sets for both players.

### 1.2.3 Zero-sum / Non-zero-sum

Main article: Zero–sum game

Zero-sum games are a special case of constant-sum games, in which choices by players can neither increase nor decrease the available resources. In zero-sum games the total benefit to all players in the game, for every combination of strategies, always adds to zero (more informally, a player benefits only at the equal expense of others).[12] Poker exemplifies a zero-sum game (ignoring the possibility of the house's cut), because one wins exactly the amount one's opponents lose. Other zero-sum games include matching pennies and most classical board games including Go and chess.

Many games studied by game theorists (including the famed prisoner's dilemma) are non-zero-sum games, because the outcome has net results greater or less than zero. Informally, in non-zero-sum games, a gain by one player does not necessarily correspond with a loss by another.

Constant-sum games correspond to activities like theft and gambling, but not to the fundamental economic situation in which there are potential gains from trade. It is possible to transform any game into a (possibly asymmetric) zero-sum game by adding a dummy player (often called "the board") whose losses compensate the players' net winnings.

### 1.2.4 Simultaneous / Sequential

Main articles: Simultaneous game and Sequential game

Simultaneous games are games where both players move simultaneously, or if they do not move simultaneously, the later players are unaware of the earlier players' actions (making them *effectively* simultaneous). Sequential games (or dynamic games) are games where later players have some knowledge about earlier actions. This need not be perfect information about every action of earlier players; it might be very little knowledge. For instance, a player may know that an earlier player did not perform one particular action, while he does not know which of the other available actions the first player actually performed.

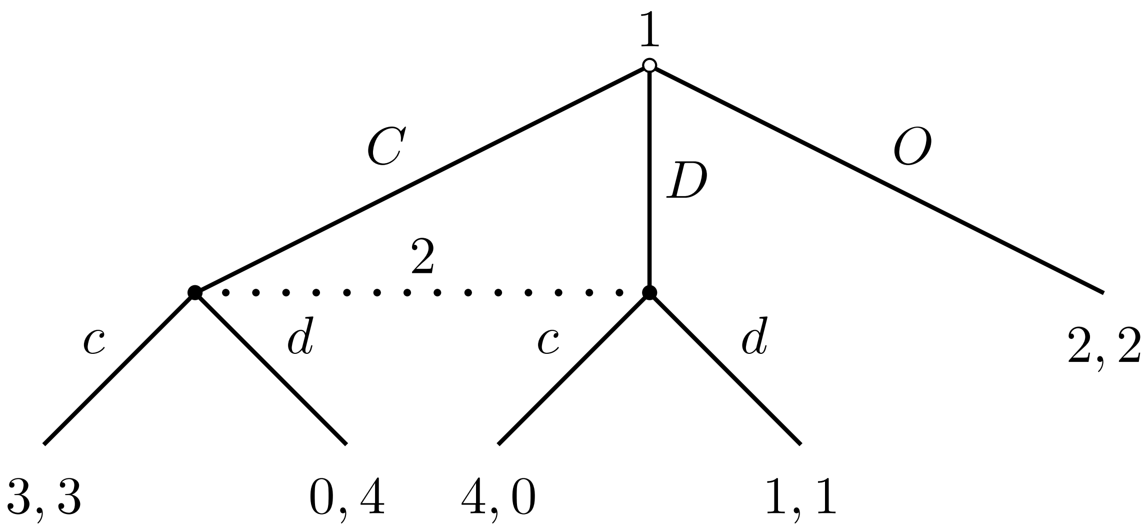
The difference between simultaneous and sequential games is captured in the different representations discussed above. Often, normal form is used to represent simultaneous games, while extensive form is used to represent sequential ones. The transformation of extensive to normal form is one way, meaning that multiple extensive form games correspond to the same normal form. Consequently, notions of equilibrium for simultaneous games are insufficient for reasoning about sequential games; see subgame perfection.

In short, the differences between sequential and simultaneous games are as follows:

### 1.2.5 Perfect information and imperfect information

Main article: Perfect information

An important subset of sequential games consists of games of perfect information. A game is one of perfect informa-




*A game of imperfect information (the dotted line represents ignorance on the part of player 2, formally called an information set)*

tion if, in extensive form, all players know the moves previously made by all other players. Simultaneous games can not be games of perfect information, because the conversion to extensive form converts simultaneous moves into a sequence of moves with earlier moves being unknown. Most games studied in game theory are imperfect-information games. Interesting examples of perfect-information games include the ultimatum game and centipede game. Recreational games of perfect information games include chess and checkers. Many card games are games of imperfect information, such as poker or contract bridge.[13]

Perfect information is often confused with complete information, which is a similar concept. Complete information requires that every player know the strategies and payoffs available to the other players but not necessarily the actions taken. Games of incomplete information can be reduced, however, to games of imperfect information by introducing "moves by nature".[14]

### 1.2.6 Combinatorial games

Games in which the difficulty of finding an optimal strategy stems from the multiplicity of possible moves are called combinatorial games. Examples include chess and go. Games that involve imperfect or incomplete information may also have a strong combinatorial character, for instance backgammon. There is no unified theory addressing combinatorial elements in games. There are, however, mathematical tools that can solve particular problems and answer general questions.[15]

Games of perfect information have been studied in combinatorial game theory, which has developed novel representations, e.g. surreal numbers, as well as combinatorial and algebraic (and sometimes non-constructive) proof methods to solve games of certain types, including “loopy” games that may result in infinitely long sequences of moves. These methods address games with higher combinatorial complexity than those usually considered in traditional (or “economic”) game theory.[16][17] A typical game that has been solved this way is hex. A related field of study, drawing from computational complexity theory, is game complexity, which is concerned with estimating the computational difficulty of finding optimal strategies.[18]

Research in artificial intelligence has addressed both perfect and imperfect (or incomplete) information games that have very complex combinatorial structures (like chess, go, or backgammon) for which no provable optimal strategies have been found. The practical solutions involve computational heuristics, like alpha-beta pruning or use of artificial neural networks trained by reinforcement learning, which make games more tractable in computing practice.[15][19]

### 1.2.7 Infinitely long games

Main article: Determinacy

Games, as studied by economists and real-world game players, are generally finished in finitely many moves. Pure mathematicians are not so constrained, and set theorists in particular study games that last for infinitely many moves, with the winner (or other payoff) not known until *after* all those moves are completed.

The focus of attention is usually not so much on the best way to play such a game, but whether one player has a winning strategy. (It can be proven, using the axiom of choice, that there are games – even with perfect information and where the only outcomes are "win" or "lose" – for which *neither* player has a winning strategy.) The existence of such strategies, for cleverly designed games, has important consequences in descriptive set theory.

### 1.2.8 Discrete and continuous games

Much of game theory is concerned with finite, discrete games, that have a finite number of players, moves, events, outcomes, etc. Many concepts can be extended, however. Continuous games allow players to choose a strategy from a continuous strategy set. For instance, Cournot competition is typically modeled with players' strategies being any non-negative quantities, including fractional quantities.

### 1.2.9 Differential games

Differential games such as the continuous pursuit and evasion game are continuous games where the evolution of the players' state variables is governed by differential equations. The problem of finding an optimal strategy in a differential game is closely related to the optimal control theory. In particular, there are two types of strategies: the open-loop strategies are found using the Pontryagin maximum principle while the closed-loop strategies are found using Bellman's Dynamic Programming method.

A particular case of differential games are the games with a random time horizon.[20] In such games, the terminal time is a random variable with a given probability distribution function. Therefore, the players maximize the mathematical expectation of the cost function. It was shown that the modified optimization problem can be reformulated as a discounted differential game over an infinite time interval.

### 1.2.10 Many-player and population games

Games with an arbitrary, but finite, number of players are often called n-person games.[21] Evolutionary game theory considers games involving a population of decision makers, where the frequency with which a particular decision is made can change over time in response to the decisions made by all individuals in the population. In biology, this is intended to model (biological) evolution, where genetically programmed organisms pass along some of their strategy programming to their offspring. In economics, the same theory is intended to capture population changes because people play the game many times within their lifetime, and consciously (and perhaps rationally) switch strategies.[22]

### 1.2.11 Stochastic outcomes (and relation to other fields)

Individual decision problems with stochastic outcomes are sometimes considered "one-player games". These situations are not considered game theoretical by some authors. They may be modeled using similar tools within the related disciplines of decision theory, operations research, and areas of artificial intelligence, particularly AI planning (with uncertainty) and multi-agent system. Although these fields may have different motivators, the mathematics involved are substantially the same, e.g. using Markov decision processes (MDP).

Stochastic outcomes can also be modeled in terms of game theory by adding a randomly acting player who makes "chance moves" ("moves by nature").[23] This player is not typically considered a third player in what is otherwise a two-player game, but merely serves to provide a roll of the dice where required by the game.

For some problems, different approaches to modeling stochastic outcomes may lead to different solutions. For example, the difference in approach between MDPs and the minimax solution is that the latter considers the worst-case over a set of adversarial moves, rather than reasoning in expectation about these moves given a fixed probability distribution. The minimax approach may be advantageous where stochastic models of uncertainty are not available, but may also be overestimating extremely unlikely (but costly) events, dramatically swaying the strategy in such scenarios if it is assumed that an adversary can force such an event to happen.[24] (See Black swan theory for more discussion on this kind of modeling issue, particularly as it relates to predicting and limiting losses in investment banking.)

General models that include all elements of stochastic outcomes, adversaries, and partial or noisy observability (of moves by other players) have also been studied. The "gold standard" is considered to be partially observable stochastic game (POSG), but few realistic problems are computationally feasible in POSG representation.[24]

### 1.2.12 Metagames

These are games the play of which is the development of the rules for another game, the target or subject game. Metagames seek to maximize the utility value of the rule set developed. The theory of metagames is related to mechanism design theory.

The term metagame analysis is also used to refer to a practical approach developed by Nigel Howard.[25] whereby a situation is framed as a strategic game in which stakeholders try to realise their objectives by means of the options available to them. Subsequent developments have led to the formulation of confrontation analysis.

### 1.2.13 Pooling games

These are games prevailing over all forms of society. Pooling games are repeated plays with changing payoff table in general over an experienced path and their equilibrium strategies usually take a form of evolutionary social convention and economic convention. Pooling game theory emerges to formally recognize the interaction between optimal choice in one play and the emergence of forthcoming payoff table update path, identify the invariance existence and robustness, and predict variance over time. The theory is based upon topological transformation classification of payoff table update over time to predict variance and invariance, and is also within the jurisdiction of the computational law of reachable optimality for ordered system.[26]

## 1.3 Representation of games

See also: List of games in game theory

The games studied in game theory are well-defined mathematical objects. To be fully defined, a game must specify the following elements: the *players* of the game, the *information* and *actions* available to each player at each decision point, and the *payoffs* for each outcome. (Eric Rasmusen refers to these four "essential elements" by the acronym "PAPI".)[27] A game theorist typically uses these elements, along with a solution concept of their choosing, to deduce a set of equilibrium strategies for each player such that, when these strategies are employed, no player can profit by unilaterally deviating from their strategy. These equilibrium strategies determine an equilibrium to the game—a stable state in which either one outcome occurs or a set of outcomes occur with known probability.

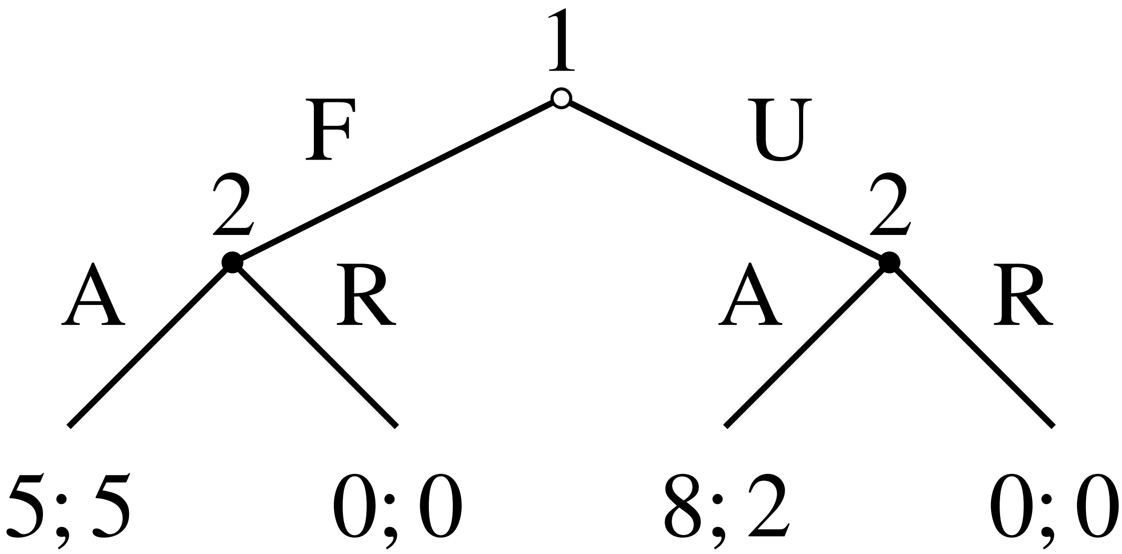
Most cooperative games are presented in the characteristic function form, while the extensive and the normal forms are used to define noncooperative games.

### 1.3.1 Extensive form

Main article: Extensive form game

The extensive form can be used to formalize games with a time sequencing of moves. Games here are played on trees (as pictured here). Here each vertex (or node) represents a point of choice for a player. The player is specified by a number listed by the vertex. The lines out of the vertex represent a possible action for that player. The payoffs are specified at the bottom of the tree. The extensive form can be viewed as a multi-player generalization of a decision tree.[28]

The game pictured consists of two players. The way this particular game is structured (i.e., with sequential decision making and perfect information), *Player 1* "moves" first by choosing either *F* or *U* (letters are assigned arbitrarily for




*An extensive form game*

mathematical purposes). Next in the sequence, *Player 2*, who has now seen *Player 1*'s move, chooses to play either *A* or *R*. Once *Player 2* has made his/ her choice, the game is considered finished and each player gets their respective payoff. Suppose that *Player 1* chooses *U* and then *Player 2* chooses *A*: *Player 1* then gets a payoff of "eight" (which in real-world terms can be interpreted in many ways, the simplest of which is in terms of money but could mean things such as eight days of vacation or eight countries conquered or even eight more opportunities to play the same game against other players) and *Player 2* gets a payoff of "two".

The extensive form can also capture simultaneous-move games and games with imperfect information. To represent it, either a dotted line connects different vertices to represent them as being part of the same information set (i.e. the players do not know at which point they are), or a closed line is drawn around them. (See example in the imperfect information section.)

### 1.3.2 Normal form

Main article: Normal-form game

The normal (or strategic form) game is usually represented by a matrix which shows the players, strategies, and payoffs (see the example to the right). More generally it can be represented by any function that associates a payoff for each player with every possible combination of actions. In the accompanying example there are two players; one chooses the row and the other chooses the column. Each player has two strategies, which are specified by the number of rows and the number of columns. The payoffs are provided in the interior. The first number is the payoff received by the row player (Player 1 in our example); the second is the payoff for the column player (Player 2 in our example). Suppose that Player 1 plays *Up* and that Player 2 plays *Left.* Then Player 1 gets a payoff of 4, and Player 2 gets 3.

When a game is presented in normal form, it is presumed that each player acts simultaneously or, at least, without knowing the actions of the other. If players have some information about the choices of other players, the game is usually presented in extensive form.

Every extensive-form game has an equivalent normal-form game, however the transformation to normal form may result in an exponential blowup in the size of the representation, making it computationally impractical.[29]

### 1.3.3 Characteristic function form

Main article: Cooperative game

In games that possess removable utility, separate rewards are not given; rather, the characteristic function decides the payoff of each unity. The idea is that the unity that is 'empty', so to speak, does not receive a reward at all.

The origin of this form is to be found in John von Neumann and Oskar Morgenstern's book; when looking at these instances, they guessed that when a union **C** appears, it works against the fraction $\left(\frac{\mathbf{N}}{\mathbf{C}}\right)$ as if two individuals were playing a normal game. The balanced payoff of C is a basic function. Although there are differing examples that help determine coalitional amounts from normal games, not all appear that in their function form can be derived from such.

Formally, a characteristic function is seen as: (N,v), where N represents the group of people and $v : 2^N \to \mathbf{R}$ is a normal utility.

Such characteristic functions have expanded to describe games where there is no removable utility.

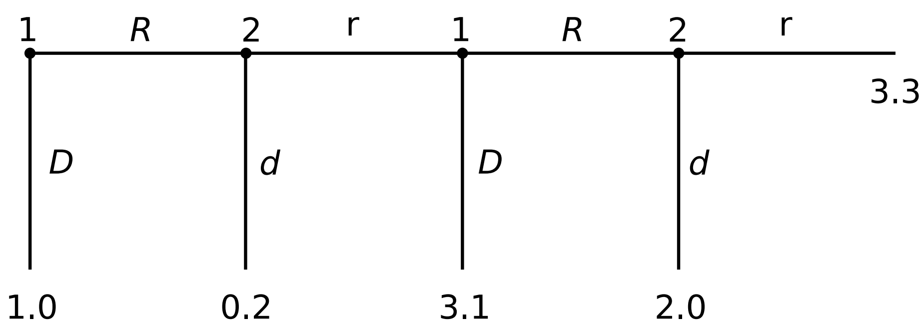
## 1.4 General and applied uses

As a method of applied mathematics, game theory has been used to study a wide variety of human and animal behaviors. It was initially developed in economics to understand a large collection of economic behaviors, including behaviors of firms, markets, and consumers. The first use of game-theoretic analysis was by Antoine Augustin Cournot in 1838 with his solution of the Cournot duopoly. The use of game theory in the social sciences has expanded, and game theory has been applied to political, sociological, and psychological behaviors as well.

Although pre-twentieth century naturalists such as Charles Darwin made game-theoretic kinds of statements, the use of game-theoretic analysis in biology began with Ronald Fisher's studies of animal behavior during the 1930s. This work predates the name "game theory", but it shares many important features with this field. The developments in economics were later applied to biology largely by John Maynard Smith in his book *Evolution and the Theory of Games*.

In addition to being used to describe, predict, and explain behavior, game theory has also been used to develop theories of ethical or normative behavior and to prescribe such behavior.[30] In economics and philosophy, scholars have applied game theory to help in the understanding of good or proper behavior. Game-theoretic arguments of this type can be found as far back as Plato.[31]

### 1.4.1 Description and modeling




*A four-stage centipede game*

The primary use of game theory is to describe and model how human populations behave. Some scholars believe that by finding the equilibria of games they can predict how actual human populations will behave when confronted with situations analogous to the game being studied. This particular view of game theory has been criticized. It is argued that the assumptions made by game theorists are often violated when applied to real world situations. Game theorists usually assume players act rationally, but in practice, human behavior often deviates from this model. Game theorists respond by comparing their assumptions to those used in physics. Thus while their assumptions do not always hold, they can treat game theory as a reasonable scientific ideal akin to the models used by physicists. However, empirical work has shown that in some classic games, such as the centipede game, guess 2/3 of the average game, and the dictator game, people regularly do not play Nash equilibria. There is an ongoing debate regarding the importance of these experiments and whether the analysis of the experiments fully captures all aspects of the relevant situation.[32]

Some game theorists, following the work of John Maynard Smith and George R. Price, have turned to evolutionary game theory in order to resolve these issues. These models presume either no rationality or bounded rationality on the part of players. Despite the name, evolutionary game theory does not necessarily presume natural selection in the biological sense. Evolutionary game theory includes both biological as well as cultural evolution and also models of individual learning (for example, fictitious play dynamics).

### 1.4.2 Prescriptive or normative analysis

Some scholars, like Leonard Savage, see game theory not as a predictive tool for the behavior of human beings, but as a suggestion for how people ought to behave. Since a strategy, corresponding to a Nash equilibrium of a game constitutes one's best response to the actions of the other players – provided they are in (the same) Nash equilibrium – playing a strategy that is part of a Nash equilibrium seems appropriate. This normative use of game theory has also come under criticism.

### 1.4.3 Economics and business

Game theory is a major method used in mathematical economics and business for modeling competing behaviors of interacting agents.[33] Applications include a wide array of economic phenomena and approaches, such as auctions, bargaining, mergers & acquisitions pricing,[34] fair division, duopolies, oligopolies, social network formation, agent-based computational economics,[35] general equilibrium, mechanism design,[36] and voting systems;[37] and across such broad areas as experimental economics,[38] behavioral economics,[39] information economics,[27] industrial organization,[40] and political economy.[41][42]

This research usually focuses on particular sets of strategies known as "solution concepts" or "equilibria". A common assumption is that players act rationally. In non-cooperative games, the most famous of these is the Nash equilibrium. A set of strategies is a Nash equilibrium if each represents a best response to the other strategies. If all the players are playing the strategies in a Nash equilibrium, they have no unilateral incentive to deviate, since their strategy is the best they can do given what others are doing.[43][44]

The payoffs of the game are generally taken to represent the utility of individual players.

A prototypical paper on game theory in economics begins by presenting a game that is an abstraction of a particular economic situation. One or more solution concepts are chosen, and the author demonstrates which strategy sets in the presented game are equilibria of the appropriate type. Naturally one might wonder to what use this information should be put. Economists and business professors suggest two primary uses (noted above): *descriptive* and *prescriptive*.[30]

### 1.4.4 Political science

The application of game theory to political science is focused in the overlapping areas of fair division, political economy, public choice, war bargaining, positive political theory, and social choice theory. In each of these areas, researchers have developed game-theoretic models in which the players are often voters, states, special interest groups, and politicians.

Early examples of game theory applied to political science are provided by Anthony Downs. In his book *An Economic Theory of Democracy*,[45] he applies the Hotelling firm location model to the political process. In the Downsian model, political candidates commit to ideologies on a one-dimensional policy space. Downs first shows how the political candidates will converge to the ideology preferred by the median voter if voters are fully informed, but then argues that voters choose to remain rationally ignorant which allows for candidate divergence. Game Theory was applied in 1962 to the Cuban missile crisis during the presidency of John F. Kennedy.[46]

It has also been proposed that game theory explains the stability of any form of political government. Taking the simplest case of a monarchy, for example, the king, being only one person, does not and cannot maintain his authority by personally exercising physical control over all or even any significant number of his subjects. Sovereign control is instead explained by the recognition by each citizen that all other citizens expect each other to view the king (or other established government) as the person whose orders will be followed. Coordinating communication among citizens to replace the sovereign is effectively barred, since conspiracy to replace the sovereign is generally punishable as a crime. Thus, in a process that can be modeled by variants of the prisoner's dilemma, during periods of stability no citizen will find it rational to move to replace the sovereign, even if all the citizens know they would be better off if they were all to act collectively.[47]

A game-theoretic explanation for democratic peace is that public and open debate in democracies send clear and reliable information regarding their intentions to other states. In contrast, it is difficult to know the intentions of nondemocratic leaders, what effect concessions will have, and if promises will be kept. Thus there will be mistrust and unwillingness to make concessions if at least one of the parties in a dispute is a non-democracy.[48]

Game theory could also help predict a nation's responses when there is a new rule or law to be applied to that nation. One example would be Peter John Wood's (2013) research when he looked into what nations could do to help reduce climate change. Wood thought this could be accomplished by making treaties with other nations to reduce green house gas emissions. However, he concluded that this idea could not work because it would create a prisoner's dilemma to the nations.[49]

### 1.4.5 Biology

Unlike those in economics, the payoffs for games in biology are often interpreted as corresponding to fitness. In addition, the focus has been less on equilibria that correspond to a notion of rationality and more on ones that would be maintained by evolutionary forces. The best known equilibrium in biology is known as the *evolutionarily stable strategy* (ESS), first introduced in (Smith & Price 1973). Although its initial motivation did not involve any of the mental requirements of the Nash equilibrium, every ESS is a Nash equilibrium.

In biology, game theory has been used as a model to understand many different phenomena. It was first used to explain the evolution (and stability) of the approximate 1:1 sex ratios. (Fisher 1930) suggested that the 1:1 sex ratios are a result of evolutionary forces acting on individuals who could be seen as trying to maximize their number of grandchildren.

Additionally, biologists have used evolutionary game theory and the ESS to explain the emergence of animal communication.[50] The analysis of signaling games and other communication games has provided insight into the evolution of communication among animals. For example, the mobbing behavior of many species, in which a large number of prey animals attack a larger predator, seems to be an example of spontaneous emergent organization. Ants have also been shown to exhibit feed-forward behavior akin to fashion (see Paul Ormerod's *Butterfly Economics*).

Biologists have used the game of chicken to analyze fighting behavior and territoriality.[51]

According to Maynard Smith, in the preface to *Evolution and the Theory of Games*, "paradoxically, it has turned out that game theory is more readily applied to biology than to the field of economic behaviour for which it was originally designed". Evolutionary game theory has been used to explain many seemingly incongruous phenomena in nature.[52]

One such phenomenon is known as biological altruism. This is a situation in which an organism appears to act in a way that benefits other organisms and is detrimental to itself. This is distinct from traditional notions of altruism because such actions are not conscious, but appear to be evolutionary adaptations to increase overall fitness. Examples can be found in species ranging from vampire bats that regurgitate blood they have obtained from a night's hunting and give it to group members who have failed to feed, to worker bees that care for the queen bee for their entire lives and never mate, to vervet monkeys that warn group members of a predator's approach, even when it endangers that individual's chance of survival.[53] All of these actions increase the overall fitness of a group, but occur at a cost to the individual.

Evolutionary game theory explains this altruism with the idea of kin selection. Altruists discriminate between the individuals they help and favor relatives. Hamilton's rule explains the evolutionary rationale behind this selection with the equation $c<b*r$ where the cost (c) to the altruist must be less than the benefit (b) to the recipient multiplied by the coefficient of relatedness (r). The more closely related two organisms are causes the incidences of altruism to increase because they share many of the same alleles. This means that the altruistic individual, by ensuring that the alleles of its close relative are passed on, (through survival of its offspring) can forgo the option of having offspring itself because the same number of alleles are passed on. Helping a sibling for example (in diploid animals), has a coefficient of ½, because (on average) an individual shares ½ of the alleles in its sibling's offspring. Ensuring that enough of a sibling's offspring survive to adulthood precludes the necessity of the altruistic individual producing offspring.[53] The coefficient values depend heavily on the scope of the playing field; for example if the choice of whom to favor includes all genetic living things, not just all relatives, we assume the discrepancy between all humans only accounts for approximately 1% of the diversity in the playing field, a co-efficient that was ½ in the smaller field becomes 0.995. Similarly if it is considered that information other than that of a genetic nature (e.g. epigenetics, religion, science, etc.) persisted through time the playing field becomes larger still, and the discrepancies smaller.

### 1.4.6 Computer science and logic

Game theory has come to play an increasingly important role in logic and in computer science. Several logical theories have a basis in game semantics. In addition, computer scientists have used games to model interactive computations. Also, game theory provides a theoretical basis to the field of multi-agent systems.

Separately, game theory has played a role in online algorithms; in particular, the k-server problem, which has in the past been referred to as *games with moving costs* and *request-answer games*.[54] Yao's principle is a game-theoretic technique for proving lower bounds on the computational complexity of randomized algorithms, especially online algorithms.

The emergence of the internet has motivated the development of algorithms for finding equilibria in games, markets, computational auctions, peer-to-peer systems, and security and information markets. Algorithmic game theory[55] and within it algorithmic mechanism design[56] combine computational algorithm design and analysis of complex systems with economic theory.[57]

### 1.4.7 Philosophy

Game theory has been put to several uses in philosophy. Responding to two papers by W.V.O. Quine (1960, 1967), Lewis (1969) used game theory to develop a philosophical account of convention. In so doing, he provided the first analysis of common knowledge and employed it in analyzing play in coordination games. In addition, he first suggested that one can understand meaning in terms of signaling games. This later suggestion has been pursued by several philosophers since Lewis.[58] Following Lewis (1969) game-theoretic account of conventions, Edna Ullmann-Margalit (1977) and Bicchieri (2006) have developed theories of social norms that define them as Nash equilibria that result from transforming a mixed-motive game into a coordination game.[59][60]

Game theory has also challenged philosophers to think in terms of interactive epistemology: what it means for a collective to have common beliefs or knowledge, and what are the consequences of this knowledge for the social outcomes resulting from agents' interactions. Philosophers who have worked in this area include Bicchieri (1989, 1993),[61][62] Skyrms (1990),[63] and Stalnaker (1999).[64]

In ethics, some authors have attempted to pursue Thomas Hobbes' project of deriving morality from self-interest. Since games like the prisoner's dilemma present an apparent conflict between morality and self-interest, explaining why cooperation is required by self-interest is an important component of this project. This general strategy is a component of the general social contract view in political philosophy (for examples, see Gauthier (1986) and Kavka (1986)).[65]

Other authors have attempted to use evolutionary game theory in order to explain the emergence of human attitudes about morality and corresponding animal behaviors. These authors look at several games including the prisoner's dilemma, stag hunt, and the Nash bargaining game as providing an explanation for the emergence of attitudes about morality (see, e.g., Skyrms (1996, 2004) and Sober and Wilson (1999)).

## 1.5 In popular culture

Based on the book by Sylvia Nasar,[66] the life story of game theorist and mathematician John Nash was turned into the biopic *A Beautiful Mind* starring Russell Crowe.[67]

"Games theory" and "theory of games" are mentioned in the military science fiction novel *Starship Troopers* by Robert A. Heinlein.[68] In the 1997 film of the same name, the character Carl Jenkins refers to his assignment to military intelligence as to "games and theory."

The film *Dr. Strangelove* satirizes game theoretic ideas about deterrence theory. For example, nuclear deterrence depends on the threat to retaliate catastrophically if a nuclear attack is detected. A game theorist might argue that such threats can fail to be *credible*, in the sense that they can lead to subgame imperfect equilibria. The movie takes this idea one step further, with the Russians irrevocably committing to a catastrophic nuclear response without making the threat public.

*Liar Game* is a popular Japanese Manga, television program and movie, where each episode presents the main characters with a Game Theory type game. The show's supporting characters reflect and explore game theory's predictions around self-preservation strategies used in each challenge. The main character however, who is portrayed as an innocent, naive and good hearted young lady Kansaki Nao, always attempts to convince the other players to follow a mutually beneficial strategy where everybody wins. Kansaki Nao's seemingly simple strategies that appear to be the product of her innocent good nature actually represent optimal equilibrium solutions which Game Theory attempts to solve. Other players however, usually use her naivety against her to follow strategies that serve self-preservation. The show improvises heavily on Game Theory predictions and strategies to provide each episode's script, the players decisions. In a sense, each episode exhibits a Game Theory game and the strategies/ equilibria/ solutions provide the script which is coloured in by the actors.

## 1.6 See also

- Chainstore paradox
- Collective Intentionality
- Combinatorial game theory
- Confrontation analysis
- Glossary of game theory
- Intra-household bargaining
- Parrondo's paradox
- Quantum game theory
- Quantum refereed game
- Rationality
- Reverse game theory
- Self-confirming equilibrium
- Zermelo's theorem (game theory)
- Tragedy of the commons
- Law and economics

**Lists**

- List of emerging technologies
- List of games in game theory
- Outline of artificial intelligence

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# Chapter 2

# Zero-sum game

Not to be confused with Empty sum or Zero game.
For other uses, see Zero sum (disambiguation).

In game theory and economic theory, a **zero-sum game** is a mathematical representation of a situation in which each participant's gain (or loss) of utility is exactly balanced by the losses (or gains) of the utility of the other participant(s). If the total gains of the participants are added up and the total losses are subtracted, they will sum to zero. Thus cutting a cake, where taking a larger piece reduces the amount of cake available for others, is a zero-sum game if all participants value each unit of cake equally (see marginal utility).

In contrast, **non-zero-sum** describes a situation in which the interacting parties' aggregate gains and losses can be less than or more than zero. A zero-sum game is also called a *strictly competitive* game while non-zero-sum games can be either competitive or non-competitive. Zero-sum games are most often solved with the minimax theorem which is closely related to linear programming duality,[1] or with Nash equilibrium.

## 2.1 Definition

The zero-sum property (if one gains, another loses) means that any result of a zero-sum situation is Pareto optimal (generally, any game where all strategies are Pareto optimal is called a conflict game).[2]

Zero-sum games are a specific example of constant sum games where the sum of each outcome is always zero. Such games are distributive, not integrative; the pie cannot be enlarged by good negotiation.

Situations where participants can all gain or suffer together are referred to as non-zero-sum. Thus, a country with an excess of bananas trading with another country for their excess of apples, where both benefit from the transaction, is in a non-zero-sum situation. Other non-zero-sum games are games in which the sum of gains and losses by the players are sometimes more or less than what they began with.

The idea of Pareto optimal payoff in a zero-sum game gives rise to a generalized relative selfish rationality standard, the punishing-the-opponent standard, where both players always seek to minimize the opponent's payoff at a favorable cost to himself rather to prefer more than less. The punishing-the-opponent standard can be used in both zero-sum games (i.e. warfare game, Chess) and non-zero-sum games (i.e. pooling selection games).[3]

## 2.2 Solution

For two-player finite zero-sum games, the different game theoretic solution concepts of Nash equilibrium, minimax, and maximin all give the same solution. If the players are allowed to play a mixed strategy, the game always has an equilibrium.

### 2.2.1 Example

A game's payoff matrix is a convenient representation. Consider for example the two-player zero-sum game pictured at right or above.

The order of play proceeds as follows: The first player (red) chooses in secret one of the two actions 1 or 2; the second player (blue), unaware of the first player's choice, chooses in secret one of the three actions A, B or C. Then, the choices are revealed and each player's points total is affected according to the payoff for those choices.

*Example: Red chooses action 2 and Blue chooses action B. When the payoff is allocated, Red gains 20 points and Blue loses 20 points.*

Now, in this example game both players know the payoff matrix and attempt to maximize the number of their points. What should they do?

Red could reason as follows: "With action 2, I could lose up to 20 points and can win only 20, while with action 1 I can lose only 10 but can win up to 30, so action 1 looks a lot better." With similar reasoning, Blue would choose action C. If both players take these actions, Red will win 20 points. But what happens if Blue anticipates Red's reasoning and choice of action 1, and goes for action B, so as to win 10 points? Or if Red in turn anticipates this devious trick and goes for action 2, so as to win 20 points after all?

Émile Borel and John von Neumann had the fundamental and surprising insight that probability provides a way out of this conundrum. Instead of deciding on a definite action to take, the two players assign probabilities to their respective actions, and then use a random device which, according to these probabilities, chooses an action for them. Each player computes the probabilities so as to minimize the maximum expected point-loss independent of the opponent's strategy. This leads to a linear programming problem with the optimal strategies for each player. This minimax method can compute probably optimal strategies for all two-player zero-sum games.

For the example given above, it turns out that Red should choose action 1 with probability 4/5 and action 2 with probability 1/5, while Blue should assign the probabilities 0, 4/7, and 3/7 to the three actions A, B, and C. Red will then win 20/7 points on average per game.

### 2.2.2 Solving

The Nash equilibrium for a two-player, zero-sum game can be found by solving a linear programming problem. Suppose a zero-sum game has a payoff matrix $M$ where element $M_{i,j}$ is the payoff obtained when the minimizing player chooses pure strategy $i$ and the maximizing player chooses pure strategy $j$ (i.e. the player trying to minimize the payoff chooses the row and the player trying to maximize the payoff chooses the column). Assume every element of $M$ is positive. The game will have at least one Nash equilibrium. The Nash equilibrium can be found (see ref. [2], page 740) by solving the following linear program to find a vector $u$ :

Minimize:

$$\sum_i u_i$$

Subject to the constraints:

$u$

$Mu$

The first constraint says each element of the $u$ vector must be nonnegative, and the second constraint says each element of the $Mu$ vector must be at least 1. For the resulting $u$ vector, the inverse of the sum of its elements is the value of the game. Multiplying $u$ by that value gives a probability vector, giving the probability that the maximizing player will choose each of the possible pure strategies.

If the game matrix does not have all positive elements, simply add a constant to every element that is large enough to make them all positive. That will increase the value of the game by that constant, and will have no effect on the equilibrium mixed strategies for the equilibrium.

The equilibrium mixed strategy for the minimizing player can be found by solving the dual of the given linear program. Or, it can be found by using the above procedure to solve a modified payoff matrix which is the transpose and negation of $M$ (adding a constant so it's positive), then solving the resulting game.

If all the solutions to the linear program are found, they will constitute all the Nash equilibria for the game. Conversely, any linear program can be converted into a two-player, zero-sum game by using a change of variables that puts it in the form of the above equations. So such games are equivalent to linear programs, in general.

### 2.2.3 Universal Solution

If avoiding a zero-sum game is an action choice with some probability for players, avoiding is always an equilibrium strategy for at least one player at a zero-sum game. For any two players zero-sum game where a zero-zero draw is impossible or incredible after the play is started, such as Poker, there is no Nash equilibrium strategy other than avoiding the play. Even if there is a credible zero-zero draw after a zero-sum game is started, it is not better than the avoiding strategy. In this sense, it's interesting to find reward-as-you-go in optimal choice computation shall prevail over all two players zero-sum games with regard to starting the game or not.[4]

## 2.3 Non-zero-sum

### 2.3.1 Economics

Many economic situations are not zero-sum, since valuable goods and services can be created, destroyed, or badly allocated in a number of ways, and any of these will create a net gain or loss of utility to numerous stakeholders. Specifically, all trade is by definition positive sum, because when two parties agree to an exchange each party must consider the goods it is receiving to be more valuable than the goods it is delivering. In fact, all economic exchanges must benefit both parties to the point that each party can overcome its transaction costs, or the transaction would simply not take place.

There is some semantic confusion in addressing exchanges under coercion. If we assume that “Trade X”, in which Adam trades Good A to Brian for Good B, does not benefit Adam sufficiently, Adam will ignore Trade X (and trade his Good A for something else in a different positive-sum transaction, or keep it). However, if Brian uses force to ensure that Adam will exchange Good A for Good B, then this says nothing about the original Trade X. Trade X was not, and still is not, positive-sum (in fact, this non-occurring transaction may be zero-sum, if Brian's net gain of utility coincidentally offsets Adam's net loss of utility). What has in fact happened is that a new trade has been proposed, “Trade Y”, where Adam exchanges Good A for two things: Good B and escaping the punishment imposed by Brian for refusing the trade. Trade Y is positive-sum, because if Adam wanted to refuse the trade, he theoretically has that option (although it is likely now a much worse option), but he has determined that his position is better served in at least temporarily putting up with the coercion. Under coercion, the coerced party is still doing the best they can under their unfortunate circumstances, and any exchanges they make are positive-sum.

There is additional confusion under asymmetric information. Although many economic theories assume perfect information, economic participants with imperfect or even no information can always avoid making trades that they feel are not in their best interest. Considering transaction costs, then, no zero-sum exchange would ever take place, although asymmetric information can reduce the number of positive-sum exchanges, as occurs in "The Market for Lemons".

### 2.3.2 Psychology

The most common or simple example from the subfield of social psychology is the concept of "social traps". In some cases pursuing our personal interests can enhance our collective well-being, but in others personal interest results in mutually destructive behavior.

See also: The 7 Habits of Highly Effective People § Abundance mentality

### 2.3.3 Complexity

It has been theorized by Robert Wright in his book *Nonzero: The Logic of Human Destiny*, that society becomes increasingly non-zero-sum as it becomes more complex, specialized, and interdependent.

## 2.4 Extensions

In 1944 John von Neumann and Oskar Morgenstern proved that any non-zero-sum game for $n$ players is equivalent to a zero-sum game with $n + 1$ players; the $(n + 1)$th player representing the global profit or loss.[5]

## 2.5 Misunderstandings

Zero-sum games and particularly their solutions are commonly misunderstood by critics of game theory, usually with respect to the independence and rationality of the players, as well as to the interpretation of utility functions. Furthermore, the word "game" does not imply the model is valid only for recreational games.[1]

Elections are often improperly cited as examples of zero-sum games. Unless the election enforces mandatory voting and utilizes a plurality voting system, it is non-zero-sum.

## 2.6 Zero-sum mentality

In community psychology, the "Zero-sum mentality" is a way of thinking that assumes all games are zero-sum: that for every winner there is a loser; for every gain there is a loss.

## 2.7 See also

- Comparative advantage
- Lump of labour fallacy
- Gains from trade
- Free trade
- Bimatrix game

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[3] Wenliang Wang (2015). Pooling Game Theory and Public Pension Plan. ISBN 978-1507658246. Chapter 1 and Chapter 4.

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[5] "Theory of Games and Economic Behavior". Princeton University Press (1953). June 25, 2005. Retrieved 2010-11-11.

## 2.9 Further reading

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- *Handbook of Game Theory - volume 2*, chapter *Zero-sum two-person games*, (1994) Elsevier Amsterdam, by Raghavan, T. E. S., Edited by Aumann and Hart, pp. 735–759, ISBN 0-444-89427-6
- *Power:Its Forms, Bases and Uses* (1997) Transaction Publishers, by Dennis Wrong

## 2.10 External links

- Play zero-sum games online by Elmer G. Wiens.
- Game Theory & its Applications - comprehensive text on psychology and game theory. (Contents and Preface to Second Edition.)
- A playable zero-sum game and its mixed strategy Nash equilibrium.

# Chapter 3

# Nim

For other uses, see Nim (disambiguation).

**Nim** is a mathematical game of strategy in which two players take turns removing objects from distinct heaps. On

*Mine A rack used to play nim with heaps of size three, five, and seven.*

each turn, a player must remove at least one object, and may remove any number of objects provided they all come from the same heap. The goal of the game is to be the player to remove the last object.

Variants of Nim have been played since ancient times.[1] The game is said to have originated in China—it closely resembles the Chinese game of "Tsyan-shizi", or "picking stones"[2]—but the origin is uncertain; the earliest European references to Nim are from the beginning of the 16th century. Its current name was coined by Charles L. Bouton of Harvard University, who also developed the complete theory of the game in 1901,[3] but the origins of the name were never fully explained. The name is probably derived from German *nimm* meaning "take [imperative]", or the obsolete English verb *nim* of the same meaning.[4]

Nim can be played as a *misère* game, in which the player to take the last object loses. Nim can also be played as a *normal play* game, which means that the person who makes the last move (i.e., who takes the last object) wins. This is called normal play because most games follow this convention, even though Nim usually does not.

Normal play Nim (or more precisely the system of nimbers) is fundamental to the Sprague–Grundy theorem, which essentially says that in normal play every impartial game is equivalent to a Nim heap that yields the same outcome when played in parallel with other normal play impartial games (see disjunctive sum).

While all normal play impartial games can be assigned a Nim value, that is not the case under the misère convention. Only tame games can be played using the same strategy as misère nim.

Nim is a special case of a poset game where the poset consists of disjoint chains (the heaps).

At the 1940 New York World's Fair Westinghouse displayed a machine, the Nimatron, that played Nim.[5] It was also one of the first ever electronic computerized games. Ferranti built a Nim playing computer which was displayed at the Festival of Britain in 1951. In 1952 Herbert Koppel, Eugene Grant and Howard Bailer, engineers from the W. L. Maxon Corporation, developed a machine weighing 50 pounds which played Nim against a human opponent and regularly won.[6] A Nim Playing Machine has been described made from TinkerToy.[7]

The game of Nim was the subject of Martin Gardner's February 1958 Mathematical Games column in Scientific American. A version of Nim is played—and has symbolic importance—in the French New Wave film *Last Year at Marienbad* (1961).[8]

## 3.1 Game play and illustration

The normal game is between two players and played with three heaps of any number of objects. The two players alternate taking any number of objects from any single one of the heaps. The goal is to be the last to take an object. In misère play, the goal is instead to ensure that the opponent is forced to take the last remaining object.

The following example game is played between fictional players Bob and Alice who start with heaps of three, four and five objects.

Sizes of heaps Moves A B C 3 4 5 Bob takes 2 from A 1 4 5 Alice takes 3 from C 1 4 2 Bob takes 1 from B 1 3 2 Alice takes 1 from B 1 2 2 Bob takes entire A heap, leaving two 2s. 0 2 2 Alice takes 1 from B 0 1 2 Bob takes 1 from C leaving two 1s. (*In misère play he would take 2 from C leaving (0, 1, 0).*) 0 1 1 Alice takes 1 from B 0 0 1 Bob takes entire C heap and wins.

## 3.2 Winning positions

The practical strategy to win at the game of *Nim* is for a player to get the other into one of the following positions, and every successive turn afterwards they should be able to make one of the lower positions. Only the last move changes between misere and normal play.

* Only valid for normal play, ** Only valid for misere. For the generalisations, $n$ and $m$ can be any value > 0, and they may be the same.

## 3.3 Mathematical theory

Nim has been mathematically solved for any number of initial heaps and objects, and there is an easily calculated way to determine which player will win and what winning moves are open to that player. In a game that starts with heaps of three, four, and five, the first player will win with optimal play, whether the misère or normal play convention is followed.

The key to the theory of the game is the binary digital sum of the heap sizes, that is, the sum (in binary) neglecting all carries from one digit to another. This operation is also known as "exclusive or" (xor) or "vector addition over **GF**(2)". Within combinatorial game theory it is usually called the **nim-sum**, as it will be called here. The nim-sum of $x$ and $y$ is written $x \oplus y$ to distinguish it from the ordinary sum, $x + y$. An example of the calculation with heaps of size 3, 4, and 5 is as follows:

Binary Decimal $011_2$ $3_{10}$ Heap A $100_2$ $4_{10}$ Heap B $101_2$ $5_{10}$ Heap C --- $010_2$ $2_{10}$ The nim-sum of heaps A, B, and C, $3 \oplus 4 \oplus 5 = 2$

An equivalent procedure, which is often easier to perform mentally, is to express the heap sizes as sums of distinct powers of 2, cancel pairs of equal powers, and then add what's left:

3 = 0 + 2 + 1 = 2 1 Heap A 4 = 4 + 0 + 0 = 4 Heap B 5 = 4 + 0 + 1 = 4 1 Heap C --- 2 = 2 What's left after canceling 1s and 4s

In normal play, the winning strategy is to finish every move with a nim-sum of 0. This is always possible if the nim-sum is not zero before the move. If the nim-sum is zero, then the next player will lose if the other player does not make a mistake. To find out which move to make, let X be the nim-sum of all the heap sizes. Find a heap where the nim-sum of X and heap-size is less than the heap-size - the winning strategy is to play in such a heap, reducing that heap to the nim-sum of its original size with X. In the example above, taking the nim-sum of the sizes is X = 3 ⊕ 4 ⊕ 5 = 2. The nim-sums of the heap sizes A=3, B=4, and C=5 with X=2 are

A ⊕ X = 3 ⊕ 2 = 1 [Since (011) ⊕ (010) = 001 ]

B ⊕ X = 4 ⊕ 2 = 6

C ⊕ X = 5 ⊕ 2 = 7

The only heap that is reduced is heap A, so the winning move is to reduce the size of heap A to 1 (by removing two objects).

As a particular simple case, if there are only two heaps left, the strategy is to reduce the number of objects in the bigger heap to make the heaps equal. After that, no matter what move your opponent makes, you can make the same move on the other heap, guaranteeing that you take the last object.

When played as a misère game, Nim strategy is different only when the normal play move would leave no heap of size two or larger. In that case, the correct move is to leave an odd number of heaps of size one (in normal play, the correct move would be to leave an even number of such heaps).

In a misère game with heaps of sizes three, four and five, the strategy would be applied like this:

A B C nim-sum 3 4 5 $010_2$=$2_{10}$ I take 2 from A, leaving a sum of 000, so I will win. 1 4 5 $000_2$=$0_{10}$ You take 2 from C 1 4 3 $110_2$=$6_{10}$ I take 2 from B 1 2 3 $000_2$=$0_{10}$ You take 1 from C 1 2 2 $001_2$=$1_{10}$ I take 1 from A 0 2 2 $000_2$=$0_{10}$ You take 1 from C 0 2 1 $011_2$=$3_{10}$ The normal play strategy would be to take 1 from B, leaving an even number (2) heaps of size 1. For misère play, I take the entire B heap, to leave an odd number (1) of heaps of size 1. 0 0 1 $001_2$=$1_{10}$ You take 1 from C, and lose.

The previous strategy for a misère game can be easily implemented (for example in Python, below).

def nim(heaps, misere=True): """Computes next move for Nim in a normal or misère (default) game, returns tuple (chosen_heap, nb_remove)""" X = reduce(lambda x,y: x^y, heaps) if X == 0: # Will lose unless all non-empty heaps have size one if max(heaps) > 1: print "You will lose :(" for i, heap in enumerate(heaps): if heap > 0: # Empty any (non-empty) heap chosen_heap, nb_remove = i, heap break else: sums = [t^X < t for t in heaps] chosen_heap = sums.index(True) nb_remove = heaps[chosen_heap] - (heaps[chosen_heap]^X) heaps_twomore = 0 for i, heap in enumerate(heaps): n = heap-nb_remove if chosen_heap == i else heap if n>1: heaps_twomore += 1 # If move leaves no heap of size 2 or larger, leave an odd (misère) or even (normal) number of heaps of size 1 if heaps_twomore == 0: chosen_heap = heaps.index(max(heaps)) heaps_one = sum(t==1 for t in heaps) # misère (resp. normal) strategy: if it is even (resp. odd) make it odd (resp. even), else do not change nb_remove = heaps[chosen_heap]−1 if heaps_one%2!=misere else heaps[chosen_heap] return chosen_heap, nb_remove

## 3.4 Proof of the winning formula

The soundness of the optimal strategy described above was demonstrated by C. Bouton.

**Theorem**. In a normal Nim game, the player making the first move has a winning strategy if and only if the nim-sum of the sizes of the heaps is nonzero. Otherwise, the second player has a winning strategy.

*Proof:* Notice that the nim-sum (⊕) obeys the usual associative and commutative laws of addition (+) and also satisfies an additional property, $x \oplus x = 0$ (technically speaking, that the nonnegative integers under ⊕ form an Abelian group of exponent 2).

Let $x_1$, ..., $xn$ be the sizes of the heaps before a move, and $y_1$, ..., $yn$ the corresponding sizes after a move. Let $s = x_1 \oplus ... \oplus xn$ and $t = y_1 \oplus ... \oplus yn$. If the move was in heap $k$, we have $xi = yi$ for all $i \neq k$, and $xk > yk$. By the properties of ⊕ mentioned above, we have

$t = 0 \oplus t = s \oplus s \oplus t = s \oplus (x_1 \oplus ... \oplus xn) \oplus (y_1 \oplus ... \oplus yn) = s \oplus (x_1 \oplus y_1) \oplus ... \oplus (xn \oplus yn) = s \oplus 0 \oplus ... \oplus 0 \oplus (xk \oplus yk) \oplus 0 \oplus ... \oplus 0 = s \oplus xk \oplus yk$ (*) $t = s \oplus xk \oplus yk$.

The theorem follows by induction on the length of the game from these two lemmas.

**Lemma 1**. If $s = 0$, then $t \neq 0$ no matter what move is made.

*Proof:* If there is no possible move, then the lemma is vacuously true (and the first player loses the normal play game by definition). Otherwise, any move in heap $k$ will produce $t = xk \oplus yk$ from (*). This number is nonzero, since $xk \neq yk$.

**Lemma 2**. If $s \neq 0$, it is possible to make a move so that $t = 0$.

*Proof:* Let $d$ be the position of the leftmost (most significant) nonzero bit in the binary representation of $s$, and choose $k$ such that the $d$th bit of $xk$ is also nonzero. (Such a $k$ must exist, since otherwise the $d$th bit of $s$ would be 0.) Then letting $yk = s \oplus xk$, we claim that $yk < xk$: all bits to the left of $d$ are the same in $xk$ and $yk$, bit $d$ decreases from 1 to 0 (decreasing the value by $2^d$), and any change in the remaining bits will amount to at most $2^d-1$. The first player can thus make a move by taking $xk - yk$ objects from heap $k$, then

$t = s \oplus xk \oplus yk$ (by (*)) $= s \oplus xk \oplus (s \oplus xk) = 0$.

The modification for misère play is demonstrated by noting that the modification first arises in a position that has only one heap of size 2 or more. Notice that in such a position $s \neq 0$, therefore this situation has to arise when it is the turn of the player following the winning strategy. The normal play strategy is for the player to reduce this to size 0 or 1, leaving an even number of heaps with size 1, and the misère strategy is to do the opposite. From that point on, all moves are forced.

## 3.5 Variations

### 3.5.1 Dividing natural number

Give any natural number $n$, the two people can divide $n$ by a prime power (OEIS A000961) which is a factor of $n$ (except 1), the person who gets 1 wins (or loses).

If $n = 2^{a_1}3^{a_2}5^{a_3}7^{a_4}\cdots p_k^{a_k}$ , where $p_k$ is the $k$-th prime, then it is a Nim game with $k$ groups of stones, and the $r$-th groups has $a_r$ stones.

If the divisor changes to “a power of squarefree numbers” (OEIS A072774) except 1, it is Wythoff’s game.

The divisor can also change to “a divisor of $m$" for fixed $m$, where $m$ is a divisor of $n$. ($m$ should be divisible by all of the prime factors of $n$ and should be less than $n$)

Of course, you can choose a set of allowed divisors. For example, {2, 3, 4, 12, 15, 20, 24, 25, 30, 32, 36}.

### 3.5.2 The subtraction game *S*(1, 2, . . ., *k*)

Interactive subtraction game: Players take turns removing 1, 2 or 3 objects from an initial pool of 21 objects. The player taking the last object wins.

In another game which is commonly known as Nim (but is better called the subtraction game *S* (1,2,...,*k*)), an upper bound is imposed on the number of objects that can be removed in a turn. Instead of removing arbitrarily many objects, a player can only remove 1 or 2 or ... or $k$ at a time. This game is commonly played in practice with only one heap (for instance with $k = 3$ in the game *Thai 21* on Survivor: Thailand, where it appeared as an Immunity Challenge).

Bouton’s analysis carries over easily to the general multiple-heap version of this game. The only difference is that as a first step, before computing the Nim-sums, we must reduce the sizes of the heaps modulo $k + 1$. If this makes all the heaps of size zero (in misère play), the winning move is to take $k$ objects from one of the heaps. In particular, in ideal play from a single heap of $n$ objects, the second player can win if and only if

$n \equiv 0 \pmod{k + 1}$ (in normal play), or

$n \equiv 1 \pmod{k + 1}$ (in misère play).

This follows from calculating the nim-sequence of $S(1,2,...,k)$,

$$0.123\ldots k0123\ldots k0123\cdots = \dot{0}.123\ldots\dot{k},$$

from which the strategy above follows by the Sprague–Grundy theorem.

### 3.5.3 The 21 game

The game "21" is played as a misère game with any number of players who take turns saying a number. The first player says "1" and each player in turn increases the number by 1, 2, or 3, but may not exceed 21; the player forced to say "21" loses. This can be modeled as a subtraction game with a heap of 21–*n* objects. The winning strategy for the two-player version of this game is to always say a multiple of 4; it is then guaranteed that the other player will ultimately have to say 21 – so in the standard version where the first player opens with "1", they start with a losing move.

The 21 game can also be played with different numbers, like "Add at most 5; lose on 34".

A sample game of 21 in which the second player follows the winning strategy:

Player Number 1 1 2 4 1 5, 6 or 7 2 8 1 9, 10 or 11 2 12 1 13, 14 or 15 2 16 1 17, 18 or 19 2 20 1 21

### 3.5.4 The 100 game

A similar version is the "100 game": two players start from 0 and alternatively add a number from 1 to 10 to the sum. The player who reaches 100 wins. The winning strategy is to reach a number in which the digits are subsequent (e.g. 01, 12, 23, 34,...) and control the game by jumping through all the numbers of this sequence. Once reached 89, the opponent has lost (he can only tell numbers from 90 to 99, and the next answer can in any case be 100).

### 3.5.5 A multiple-heap rule

See also: Wythoff's game

In another variation of Nim, besides removing any number of objects from a single heap, one is permitted to remove the same number of objects from each heap.

### 3.5.6 Circular Nim

See also: Kayles

Yet another variation of Nim is 'Circular Nim', where any number of objects are placed in a circle, and two players alternately remove one, two or three adjacent objects. For example, starting with a circle of ten objects,

. . . . . . . . . .

three objects are taken in the first move

_ . . . . . . . . _ _

then another three

_ . _ _ _ . . . _ _

then one

_ . _ _ _ . . _ _ _

but then three objects cannot be taken out in one move.

### 3.5.7 Grundy's game

In Grundy's game, another variation of Nim, a number of objects are placed in an initial heap, and two players alternately divide a heap into two nonempty heaps of different sizes. Thus, six objects may be divided into piles of 5+1 or 4+2, but not 3+3. Grundy's game can be played as either misère or normal play.

### 3.5.8 Greedy Nim

*Greedy Nim* is a variation where the players are restricted to choosing stones from only the largest pile.[9] It is a finite impartial game. *Greedy Nim Misère* has the same rules as Greedy Nim, but here the last player able to make a move loses.

Let the largest number of stones in a pile be $m$, the second largest number of stones in a pile be $n$. Let $pm$ be the number of piles having $m$ stones, $pn$ be the number of piles having $n$ stones. Then there is a theorem that game positions with $pm$ even are $P$ positions. [10] This theorem can be shown by considering the positions where $pm$ is odd. If $pm$ is larger than 1, all stones may be removed from this pile to reduce $pm$ by 1 and the new $pm$ will be even. If $pm = 1$ (i.e. the largest heap is unique), there are two cases:

- If $pn$ is odd, the size of the largest heap is reduced to $n$ (so now the new $pm$ is even).
- If $pn$ is even, the largest heap is removed entirely, leaving an even number of largest heaps.

Thus there exists a move to a state where $pm$ is even. Conversely, if $pm$ is even, if any move is possible ($pm \neq 0$) then it must take the game to a state where $pm$ is odd. The final position of the game is even ($pm = 0$). Hence each position of the game with $pm$ even must be a $P$ position.

### 3.5.9 Index-*k* Nim

A generalization of multi-heap Nim was called "Nim $_k$ " or "index-$k$ Nim by E. H. Moore,[11] who analyzed it in 1910. In index-$k$ Nim, instead of removing objects from only one heap, players can remove objects from at least one but up to $k$ different heaps. The number of elements that may be removed from each heap may be either arbitrary, or limited to at most $r$ elements, like in the "subtraction game" above.

The winning strategy is as follows: Like in ordinary multi-heap Nim, one considers the binary representation of the heap sizes (or heap sizes modulo $r + 1$). In ordinary Nim one forms the XOR-sum (or sum modulo 2) of each binary digit, and the winning strategy is to make each XOR sum zero. In the generalization to index-$k$ Nim, one forms the sum of each binary digit modulo $k + 1$.

Again the winning strategy is to move such that this sum is zero for every digit. Indeed, the value thus computed is zero for the final position, and given a configuration of heaps for which this value is zero, any change of at most $k$ heaps will make the value non-zero. Conversely, given a configuration with non-zero value, one can always take from at most $k$ heaps, carefully chosen, so that the value will become zero.

### 3.5.10 Building Nim

**Building Nim** is a variant of Nim where the two players first construct the game of Nim. Given $n$ stones and $s$ empty piles, the players alternate turns placing exactly one stone into a pile of their choice.[12] Once all the stones are placed, a game of Nim begins, starting with the next player that would move. This game is denoted *BN(n,s)*.

## 3.6 See also

- Dr. NIM
- Fuzzy game
- Nimber

- Nimrod (computing)
- Octal games
- Star (game theory)
- Subtract a square
- Zero game
- Android Nim
- Raymond Redheffer
- Last Year at Marienbad

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[12] Larsson, Urban; Heubach, Silvia; Dufour, Matthieu; Duchêne, Eric. "Building Nim". arXiv:1502.04068.


## 3.8 Additional reading


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- M. Kaitchik: *Mathematical Recreations*, W. W. Norton, 1942.
- Donal D. Spencer: *Game Playing with Computers*, Hayden Book Company, Inc., 1968.


## 3.9 External links

- 50-pound computer plays Nim- The New Yorker magazine “Talk of the Town” August, 1952(subscription required)
- The hot game of Nim – Nim theory and connections with other games at cut-the-knot
- Nim and 2-dimensional SuperNim at cut-the-knot
- Nim on NCTM’s Illuminations
- NIM in Excel Excel spreadsheet using basic formulas to determine best moves for a 3,4,5 start point.
- Mind Nimmer - Implementation of Nim for iOS.
- Nim game: a good illustration of the necessity of knowledge and mastery when failure must not be risked, however remote the probability of total loss may seem.
- Subtraction Game: a Subtraction Game illustration on Appstore.

## Chapter 4

# Sprague–Grundy theorem

*The Sprague-Grundy theorem says that many simple games are mathematically equivalent to the stone-taking game of Nim (*pictured*).*

In combinatorial game theory, the **Sprague–Grundy theorem** states that every impartial game under the normal play convention is equivalent to a nimber. The **Grundy value** or **nim-value** of an impartial game is then defined as the unique nimber that the game is equivalent to. In the case of a game whose positions (or summands of positions) are indexed by the natural numbers (for example the possible heap sizes in nim-like games), the sequence of nimbers for successive heap sizes is called the **nim-sequence** of the game.

The theorem and its proof encapsulate the main results of a theory discovered independently by R. P. Sprague (1935) and P. M. Grundy (1939).[1]

## 4.1 Definitions

For the purposes of the Sprague–Grundy theorem, a *game* is a two-player sequential game of perfect information satisfying the *ending condition* (all games come to an end: there are no infinite lines of play) and the *normal play condition* (a player who cannot move loses).

An *impartial game* is one such as nim, in which each player has exactly the same available moves as the other player in any position. Note that games such as tic-tac-toe, checkers, and chess are *not* impartial games. In the case of checkers and chess, for example, players can only move their own pieces, not their opponent's pieces. And in tic-tac-toe, one player puts down X's, while the other puts down O's. Positions in impartial games fall into two *outcome classes*: either the next player (the one whose turn it is) wins (an *N-position*) or the previous player wins (a *P-position*).

In this proof, a position is identified with the set of positions that can be reached in one move (these positions are called *options*). For example, the position $\{\}$ is a P-position under normal play, because the current player has no moves and therefore loses. The position $\{\{\}\}$ , in contrast, is an N-position; the current player has only one option, and that option is a losing position for their opponent.

A *nimber* is a special position denoted $*n$ for some ordinal $n$ . $*0$ is $\{\}$ , the P-position given as an example earlier. The other nimbers are defined inductively by $*(n+1) = *n \cup \{*n\}$ ; in particular, $*1 = \{*0\}$ (the example N-position from above), $*2 = \{*0, *1\}$ (a choice between the two example positions), etc. $*n$ therefore corresponds to a heap of $n$ counters in the game of nim, hence the name.

Two positions $G$ and $H$ can be *added* to make a new position $G + H$ in a combined game where the current player can choose either to move in $G$ or in $H$ . Explicit computation of the set $G + H$ is by repeated application of the rule $G + H = \{G + h \mid h \in H\} \cup \{g + H \mid g \in G\}$ , which incidentally indicates that position addition is both commutative and associative as expected.

Two positions $G$ and $G'$ are defined to be *equivalent* if for every position $H$ , the position $G + H$ is in the same outcome class as $G' + H$ . Such an equivalence is written $G \approx G'$ .

## 4.2 First Lemma

As an intermediate step to proving the main theorem, we show that, for every position $G$ and every P-position $A$ , the equivalence $G \approx A + G$ holds. By the above definition of equivalence, this amounts to showing that $G + H$ and $A + G + H$ share an outcome class for all $H$ .

Suppose that $G + H$ is a P-position. Then the previous player has a winning strategy for $A + G + H$ : respond to moves in $A$ according to their winning strategy for $A$ (which exists by virtue of $A$ being a P-position), and respond to moves in $G + H$ according to their winning strategy for $G + H$ (which exists for analogous reason). So $A + G + H$ must also be a P-position.

On the other hand, if $G + H$ is an N-position, then the next player has a winning strategy: choose a P-position from among the $G + H$ options, putting their opponent in the case above. Thus, in this case, $A + G + H$ must be a N-position, just like $G + H$ .

As these are the only two cases, the lemma holds.

## 4.3 Second Lemma

As a further step, we show that $G \approx G'$ if and only if $G + G'$ is a P-position.

In the forward direction, suppose that $G \approx G'$ . Applying the definition of equivalence with $H = G$ , we find that $G' + G$ (which is equal to $G + G'$ by commutativity of addition) is in the same outcome class as $G + G$ . But $G + G$ must be a P-position: for every move made in one copy of $G$ , the previous player can respond with the same move in the other copy, and so always make the last move.

In the reverse direction, since $A = G + G'$ is a P-position by hypothesis, it follows from the first lemma, $G \approx G + A$ , that $G \approx G + (G + G')$ . Similarly, since $A = G + G$ is also a P-position, it follows from the first lemma in the form $G' \approx G' + A$ that $G' \approx G' + (G + G)$ . By associativity and commutativity, the right-hand sides of these results are equal. Furthermore, $\approx$ is an equivalence relation because equality is an equivalence relation on outcome classes. Via the transitivity of $\approx$ , we can conclude that $G \approx G'$ .

## 4.4 Proof

We prove that all positions are equivalent to a nimber by structural induction. The more specific result, that the given game's initial position must be equivalent to a nimber, shows that the game is itself equivalent to a nimber.

Consider a position $G = \{G_1, G_2, \ldots, G_k\}$ . By the induction hypothesis, all of the options are equivalent to nimbers, say $G_i \approx *n_i$ . So let $G' = \{*n_1, *n_2, \ldots, *n_k\}$ . We will show that $G \approx *m$ , where $m$ is the mex (minimum exclusion) of the numbers $n_1, n_2, \ldots, n_k$ , that is, the smallest non-negative integer not equal to some $n_i$ .

The first thing we need to note is that $G \approx G'$ , by way of the second lemma. If $k$ is zero, the claim is trivially true. Otherwise, consider $G + G'$ . If the next player makes a move to $G_i$ in $G$ , then the previous player can move to $*n_i$ in $G'$ , and conversely if the next player makes a move in $G'$ . After this, the position is a P-position by the lemma's forward implication. Therefore, $G + G'$ is a P-position, and, citing the lemma's reverse implication, $G \approx G'$ .

Now let us show that $G' + *m$ is a P-position, which, using the second lemma once again, means that $G' \approx *m$ . We do so by giving an explicit strategy for the previous player.

Suppose that $G'$ and $*m$ are empty. Then $G' + *m$ is the null set, clearly a P-position.

Or consider the case that the next player moves in the component $*m$ to the option $*m'$ where $m' < m$ . Because $m$ was the *minimum* excluded number, the previous player can move in $G'$ to $*m'$ . And, as shown before, any position plus itself is a P-position.

Finally, suppose instead that the next player moves in the component $G'$ to the option $*n_i$ . If $n_i < m$ then the previous player moves in $*m$ to $*n_i$ ; otherwise, if $n_i > m$ , the previous player moves in $*n_i$ to $*m$ ; in either case the result is a position plus itself. (It is not possible that $n_i = m$ because $m$ was defined to be different from all the $n_i$ .)

In summary, we have $G \approx G'$ and $G' \approx *m$ . By transitivity, we conclude that $G \approx *m$ , as desired.

## 4.5 Development

If $G$ is a position of an impartial game, the unique integer $m$ such that $G \approx *m$ is called its Grundy value, or Grundy number, and the function which assigns this value to each such position is called the Sprague–Grundy function. R.L.Sprague and P.M.Grundy independently gave an explicit definition of this function, not based on any concept of equivalence to nim positions, and showed that it had the following properties:

- The Grundy value of a single nim pile of size $m$ (i.e. of the position $*m$ ) is $m$ ;
- A position is a loss for the next player to move (i.e. a P-position) if and only if its Grundy value is zero; and
- The Grundy value of the sum of a finite set of positions is just the nim-sum of the Grundy values of its summands.

It follows straightforwardly from these results that if a position $G$ has a Grundy value of $m$ , then $G + H$ has the same Grundy value as $*m + H$ , and therefore belongs to the same outcome class, for any position $H$ . Thus, although Sprague and Grundy never explicitly stated the theorem described in this article, it is nevertheless an almost trivial consequence of their results. These results have subsequently been developed into the field of combinatorial game theory, notably by Richard Guy, Elwyn Berlekamp, John Horton Conway and others, where they are now encapsulated in the Sprague–Grundy theorem and its proof in the form described here. The field is presented in the books *Winning Ways for your Mathematical Plays* and *On Numbers and Games*.

## 4.6 See also

- Genus theory
- Indistinguishability quotient

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## 4.8 External links

- Grundy's game at cut-the-knot
- Easily readable, introductory account from the UCLA Math Department
- The Game of Nim at sputsoft.com

## 4.9 Text and image sources, contributors, and licenses

### 4.9.1 Text

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