Set Theory

CS 5012

UNIVERSITY OF VIRGINIA
DATA SCIENCE INSTITUTE

Why Study Set Theory

- Set theory provides a clear understanding of the fact that the things we deal with in our lives are in some way members of bigger groups. This sharpens and enhances our ability to analyze

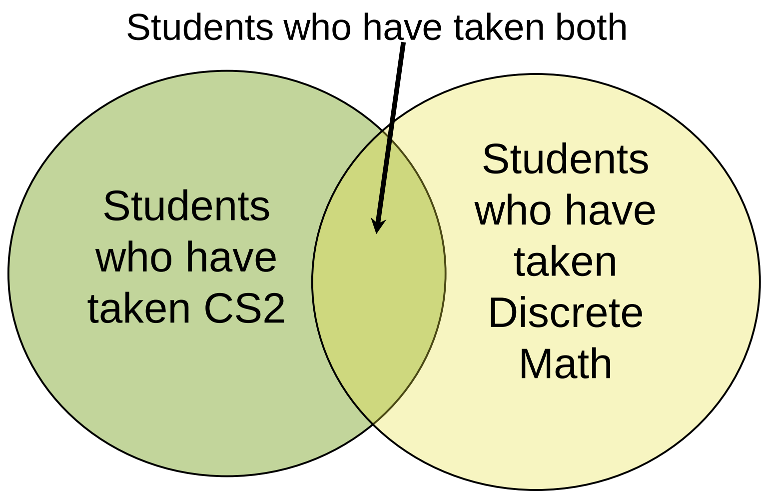
- Much of our understanding of knowledge is based on intuitive notions about sets of objects. Every human being, object, idea or property belongs to a broader group of things

Why Study Set Theory

- Many mathematical concepts can be defined precisely using only set theoretic concepts
- Set theory is a foundation for
  - Mathematical analysis
  - Topology
  - Abstract Algebra, and
  - Discrete mathematics
- Mathematicians accept that (in principle) theorems in these areas can be derived from the relevant definitions and axioms of set theory

Why Study Set Theory

- Set theory is helpful in formulating **query requests for databases**
- From formulating logical foundations for geometry, calculus and topology to creating algebra revolving around fields, rings and groups, **applications of set theory extend to** mathematics, biology, chemistry, physics, as well as computer science and electrical engineering



CS2 and Discrete Math are two required courses in CS. Let's assume the CS dept. wishes to remind students who have taken CS2 to take Discrete Math.
Required: Students who have taken CS2 but have not yet taken Discrete Math (dark green portion of the Venn diagram)

Some Special Sets

- **Universal set (U):** a ***universe of reference*** **(universal set) i**s a set of all the possible elements related to the subject and relevant for various other sets under discussion
- **Empty set:** { } or Ø i.e. set with no elements
- **Natural numbers (N):** {0, 1, 2, …} *(some exclude 0 from this set)*
- **Integers (Z):** {… -5, -4, 3, -2, -1, 0, 1, 2, 3, 4, 5 …}
- **Real numbers (R):** points on an infinitely long line called the **number line or real line**
- *See the Supplemental Material for additional special sets*

Some Basic Terms and Concepts

- **Set Theory**: Set theory is the field of mathematics that deals with the properties of sets that are independent of the things that make up the set
- **Set**: An **unordered** and *unique* homogeneous collection of objects (called *elements*)
- **Subset**: A set T in which every member of T is also a member of some other set S
- **Superset**: Set A is a superset of set B, such that each of the elements of set B is also an element of the set A

Some Basic Terms and Concepts

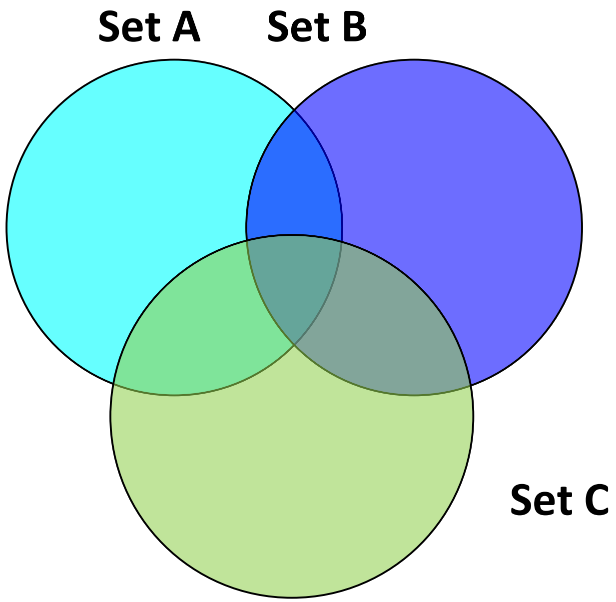
- **Union of sets**
- **Intersection of sets**
- **Difference of sets**
- **Equality of sets**: Two sets are said to be equal if every element in one set is also an element of the second set. In other words, two sets are equal if, and only if, they both contain exactly the same type and number of elements
- **Complement of a set**

Some Basic Terms and Concepts

- **Cartesian Product of sets**
- **Disjoint sets**
- **Cardinality of a set**: The cardinality (*size*) of a set is a measure of the "*number of elements of the set*." It is denoted by |A| or #A. Also as n(A), or card(A)
- **Empty Set**: An empty set is the set with no elements. Its size or cardinality is **zero**. If A is a set, then the empty set is one of its subsets

Some Basic Terms and Concepts

- **Tuple** : A finite sequence of elements; a finite ordered set
- **Relation**: A set of ordered tuples
- **Venn Diagram**: A diagram representing sets by circles or ellipses



- *See the "Supplemental Material" for additional information on basic terms/concepts*

Some Common Symbols

- **Set**: {2, 4, 6, 8 }   **An empty set** { } or ∅
- **Subset**: $A \subseteq B$
- **Proper subset** / strict subset: $A \subset B$ (Fewer elements than set)
- **Not a subset**: $A \not\subset B$
- **Superset**: $A \supseteq B$
- **Proper superset** / strict superset: $A \supset B$ (More elements than set)
- **Not Superset**: $A \not\supset B$

Some Common Symbols

- **Union**: A U B   (All the elements of A + All the elements of B)
- **Intersection**: $A \cap B$ (only the elements that are in both)
- **Equality** of sets: A = B   (A= { 4,7,9}; B = {4,7,9})
- **Element of**: "a" is element of A: $a \in A$
- **Not an element of**: "a" not element of A: $a \notin A$
- **"Set Builder" notation**:
  $\{ x \mid P(x)\}$ *all elements in Universe that satisfy predicate P*

Set-Builder Notation

- One way of declaring a set is to write down **all** the elements within curly brackets:
  - A = { 10, 20, 30 }
  - B = { Jon, Tom, Ann, May, Sara, Jim}
- If we have to define a set U giving all the students of UVA, we have to *list lots of names!* There is a simple way to do this job: use the **set-builder notation**
- Set builder is particularly useful if we have to build a set from very large or infinite elements
- Set builders notation state the *type of the elements that can be included*

Set-Builder Notation

- **Set-builder**: is a notation for describing a Set by stating the properties, restriction or conditions that its members must satisfy
  - *Set-builder notation is a shorthand way of writing sets using formulas, notation and restrictions*
- M= { x| x is a movie being shown in Richmond today }
- *The set of all x such that x is a movie being shown in Richmond today*
- The set {x | x < 100} is read aloud as, "*the set of all x such that x is less than 100*"
- S = { x| x is a student at UVA }
- *The set of all x such that x is a student at UVA. This set contains every student of UVA as element of S*

Set-Builder Notation

More Examples

- $\{x \mid x \neq 7\}$ *the set of all real numbers except 7*
- $\{x \mid x > 5000\}$ *the set of all real numbers greater than 5000*
- $\{2n + 1 : n \text{ is an integer}\}$ *the set of all odd integers (e.g. ..., -3, -1, 1, 3, 5,...)*

- Note that every set builder notation has three parts:
  1. *a variable*
  2. *a colon or vertical bar separator { **|** }*
  3. *a logical **predicate**, condition, property, restriction or requirement for adding the variable to the set*

- $S = \{x : \text{every } x \text{ is } \beta\}$
- Read as: "*Every x is a member of S, if it meets the condition β*"

Set-Builder Notation

More Examples

- Let N be the set of natural numbers, then

  {n | n ∈ N AND n is even}

  is the set of even natural numbers

  (The set of all n s.t. n ∈ N and n is even)

Set Operations: Equality

- **Definition** (Equality of sets): Two sets are equal if and only if they have the same elements. Elements found in B are exactly the same as the elements found in A
- More formally, for any sets A and B, $A = B$ if and only if $\forall x \, [\, x \in A \leftrightarrow x \in B \,]$
- Note here the application of quantifier symbols from the **predicate logic**
  *For all values of x, if x is a member of A then it's a member of B and **vice versa***

Set Operations: Subsets

- **Definition**: If A and B are sets, and if every element of B belongs to A, then B is a *subset* of A, denoted by $B \subseteq A$
- **Definition**: If A and B are sets, and if B is a subset of A but $B \neq A$, then B is a *proper subset* of A, denoted by $B \subset A$

Given a set A such that: $A = \{2, 4, 6, 8\}$

- $B = \{2, 4, 6, 8\}$ is a subset of A: $B \subseteq A$
- $C = \{2, 4, 6\}$ is a *proper* subset of A: $B \subset A$
- $D = \{2, 4\}$ is a *proper* subset of A: $B \subset A$
- $K = \{2\}$ is a *proper* subset of A: $B \subset A$

Set Operations: Complement of a Set

- Let's say that we have a set B that is a *subset* of set A. The **complement** of set B is the set of elements of set A that are <u>not</u> elements of set B

- Let there be a set A and a set B such that set B is a subset of set A:
  - A = {2, 4, 6, 8, 11}
  - B= {2, 4, 6}
  - $B \subset A$ Set B is a (proper) subset of set A

  Then the complement of B = B' = B^C = {8, 11}

Complement of B is denoted by: B' (also written as B^C)

Cartesian Product

Definition: A Cartesian product for sets A and B, A X B, is the set of all ordered pairs (a, b) where a ∈ A and b ∈ B.

- Products can be specified using *Set builder Notation*:

  A X B = { (a,b) | a ∈ A ∩ b ∈ B}

- Let set **A** = {Jim, Jon}; Let set **B** = {Fay, May}

  A X B = { (Jim, Fay), (Jim, May), (Jon, Fay), (Jon, May) }

  B X A = { (Fay, Jim), (Fay, Jon), (May, Jim), (May, Jon) }

Please note:

1. A X B is *not* same the same as B X A: A X B ≠ B X A
2. In the Cartesian sets above, 2 numbers together within parentheses make one element such as: (Jim, Fay)

Cartesian Product

More Examples

- $A = \{1,2\}$; $B = \{3,4\}$
  - $A \times B = \{1,2\} \times \{3,4\} = \{(1,3), (1,4), (2,3), (2,4)\}$
  - $B \times A = \{3,4\} \times \{1,2\} = \{(3,1), (3,2), (4,1), (4,2)\}$
- $A = B = \{1,2\}$
  - $A \times B = B \times A = \{1,2\} \times \{1,2\} = \{(1,1), (1,2), (2,1), (2,2)\}$

What happens when dealing with empty sets?

- $A = \{1,2\}$; $B = \emptyset$
  - $A \times B = \{1,2\} \times \emptyset = \emptyset$
  - $B \times A = \emptyset \times \{1,2\} = \emptyset$

Advanced Set Operations: Example 1

- Let there be a universal set U defined with the set builder as: $U = \{x \mid 0 < x < 16\}$

(U contains every x such that x is between 0 and 16, both ***ex***clusive. Thus U = all whole numbers 1-15)

$U = \{1,2,3,4,5,6,7,8,9,10,11,12,13,14,15\}$

- Let there be:
  - $K = \{5, 6, 7, 8, 11, 13, 15\}$
  - $P = \{7, 13, 15\}$

- Find the set S that is represented by: $\mathbf{(K \cap P')'}$

Example 1 (Solution)

U = {1,2,3,4,5,6,7,8,9,10,11,12,13,14,15}
K = {5, 6, 7, 8, 11, 13, 15}
P = { 7, 13, 15}
Find the set S that is represented by: **(K ∩ P')'**

- Step 1: Find the elements of P' (complement of P). This will be the list of elements not found in P.
  - U = {1,2,3,4,5,6,**7**,8,9,10,11,12,**13**,14,**15**}
  - Remove the underlined numbers because they are members of P
  - P' = {1,2,3,4,5,6, 8,9,10,11,12,14}
- Step 2: Find (K ∩ P') (Elements found both is K and P')
  - (K ∩ P') = { 5, 6, 8, 11 }
- Step 3: If (K ∩ P') = { 5, 6, 8, 11 } then (K ∩ P')' will include all the elements of the universe that are not in (K ∩ P')
  - **(K ∩ P)' = {1, 2, 3, 4, 7, 9, 10, 12, 13, 14, 15}**

Advanced Set Operations: Example 2

U = {1,2,3,4,5,6,7,8,9,10,11,12,13,14,15}

K = {5, 6, 7, 8, 11} ; P = { 7, 13, 15 }

Find the set S that is represented by: **(K U P')'**

- Step 1: Find the elements of P' (complement of P). This will be the list of elements not found in P.
  - U = {1,2,3,4,5,6,**7**,8,9,10,11,12,**13**,14,**15**}
  - Remove the underlined numbers because they are members of P
  - P' = {1,2,3,4,5,6, 8,9,10,11,12,14}
- Step 2: Find (K U P') (Elements found either in K or in P')
  - Add everything in the list that is found in the two sets: K, P'
  - (K U P') = {1,2,3,4,5,6,7,8,9,10,11,12,14}
- Step 3: If (K U P') = {1,2,3,4,5,6,7,8,9,10,11,12,14} then (K U P')' will include all the elements of the universe that are not in (K U P')
  - **(K U P')' = {13, 15}**

Venn Diagrams in Set Theory

- John Venn, (4 August 1834 – 4 April 1923) was an English Logician and Philosopher. He introduced the Venn Diagram, *a graphical way of presenting concepts*. Venn diagram are being used in the fields of set theory, logic, Statistics, and Computer Science
- When dry formulae and rules of set theory are presented in graphical form it enhances the clarity and understanding of the concepts
- *See the "Supplemental Material" for information about the use of Venn Diagrams in set theory*
- *Note: De Morgan's Law illustrated using Venn Diagrams!*