Set Theory
{*Supplemental Material*}
CS 5012

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Why Study Set Theory

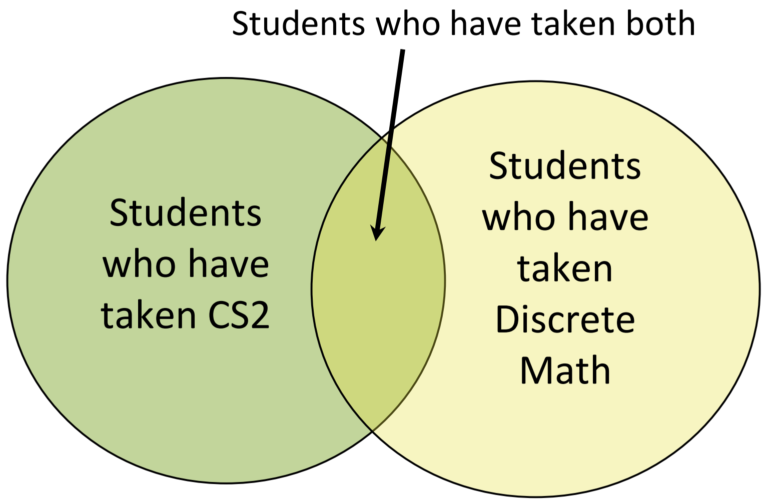
- Set theory provides a clear understanding of the fact that the things we deal with in our lives are in some way members of bigger groups. This sharpens and enhances our ability to analyze
- Set theory provides a context for mathematicians to define numbers and manipulation of numbers
- Studying set theory may provide insights into the structure of our number system and higher-level mathematics
- Much of our understanding of knowledge is based on intuitive notions about sets of objects. Every human being, object, idea or property belongs to a broader group of things

Why Study Set Theory

- Many mathematical concepts can be defined precisely using only set theoretic concepts
- Set theory is a foundation for
  - Mathematical analysis
  - Topology
  - Abstract Algebra, and
  - Discrete mathematics
- Mathematicians accept that (in principle) theorems in these areas can be derived from the relevant definitions and axioms of set theory

Why Study Set Theory

- Set theory is helpful in formulating **query requests for databases**
- From formulating logical foundations for geometry, calculus and topology to creating algebra revolving around fields, rings and groups, **applications of set theory extend to** mathematics, biology, chemistry, physics, as well as computer science and electrical engineering



CS2 and Discrete Math are two required courses in CS. Let's assume the CS dept. wishes to remind students who have taken CS2 to take Discrete Math.
Required: Students who have taken CS2 but have not yet taken Discrete Math (dark green portion of the Venn diagram)

Some Special Sets

- **Universal set (U):** a ***universe of reference*** **(universal set) is** a set of all the possible elements related to the subject and relevant for various other sets under discussion
- **Empty set:** { } or Ø i.e. set with no elements
- **Whole numbers (W):** {0, 1, 2, …}
- **Natural numbers (N):** {0, 1, 2, …} *(some exclude 0 from this set)*
- **Integers (Z):** {… -5, -4, 3, -2, -1, 0, 1, 2, 3, 4, 5 …}
- **Real numbers (R):** points on an infinitely long line called the **number line or real line**
- **Prime numbers:** a natural number greater than 1 that has no positive divisors other than 1 and itself
- **Rational numbers (Q):** a rational number is a real number that can be written as a simple fraction (***ratio***). 1.25, 1.5 are examples of rational numbers. (1.25 = 5/4,   1.5 = 3/2)

Some Basic Terms and Concepts

- **Set Theory**: Set theory is the field of mathematics that deals with the properties of sets that are independent of the things that make up the set
- **Set**: An **unordered** and *unique* homogeneous collection of objects (called *elements*)
- **Subset**: A set T in which every member of T is also a member of some other set S
- **Superset**: Set A is a superset of set B, such that each of the elements of set B is also an element of the set A
- **Elements**: The contents of a set are called its members or elements

Some Basic Terms and Concepts

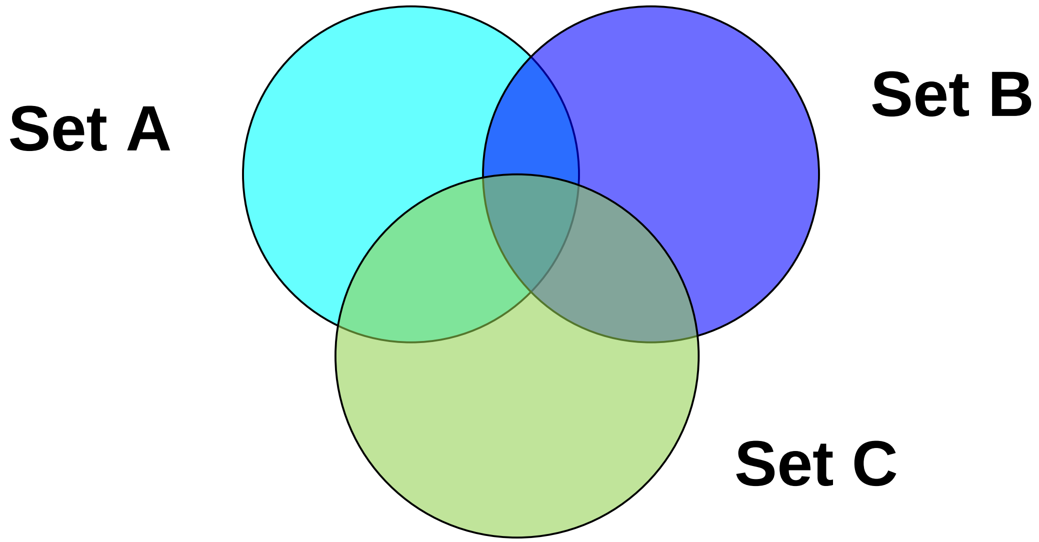
- **Union of two sets**: The set that contains all the elements found in either of both of two sets. **Intersection of sets**: The intersection of two sets is defined as the collection of elements that belong to both of the two sets
- **Difference of two sets**: The difference between two sets is defined as the collection of elements that belong in one set but not in the other
- **Equality of two sets**: Two sets are said to be equal if every element in one set is also an element of the second set. In other words, two sets are equal if, and only if, they both contain exactly the same type and number of elements
- **Complement**: That part of a set S that is not contained in a particular subset T

Some Basic Terms and Concepts

- **Cartesian Product**: The set of all possible pairs of elements whose components are members of two sets
- **Disjoint**: Of two or more sets, having no members in common; having an intersection equal to the empty set
- **Member** of a set is another way of saying “element of a set”
- **Partition**: When a set is divided in such a way that all elements of the set are contained in exactly one of the subsets)
- **Cardinality**: The cardinality (size) of a set is a measure of the "*number of elements of the set*." It is denoted by |A| or #A. Also as n(A), or card(A)
- **Empty Set**: An empty set is the set with no elements. Its size or cardinality (count of elements in a set) is zero. If A is a set, then the empty set is one of its subsets

Some Basic Terms and Concepts

- **Tuple** : A finite sequence of elements; a finite ordered set
- **Relation**: A set of ordered tuples
- **Venn Diagram**: A diagram representing sets by circles or ellipses



Some Basic Definitions

- A **set** is a collection of objects
- If every element in a set A is also a member of set B then A is a **subset** of B, i.e., $A \subset B$
- Two sets, A and B, are **equal**, denoted A=B, if and only if all elements in A belongs to the set B and every element in B belongs to set A, i.e., $A \subseteq B$ **and** $A \supseteq B$
- B is a **proper subset** of A if B is a subset of A, but B *does not equal* A
- The **empty set**, or null set, is a set which contains no elements, and is denoted by the symbol $\emptyset$
- Suppose that $A \subset S$ (subset). The **complement** of set A, denoted as A' or A^c , is the set containing all elements in S that are not in A. i.e., $A^c = \{\gamma : \gamma \in S \text{ and } \gamma \notin A\}$

Some Basic Definitions

- The **intersection** of sets A and B, denoted $A \cap B$, is the set containing all elements in both A and B. i.e., $A \cap B = \{\gamma : \gamma \in A \text{ and } \gamma \in B\}$
- Two sets, A and B, are called **disjoint** or *mutually exclusive* if they contain no common element. i.e., if $A \cap B = \emptyset$
- The *set of all possible outcomes of a random experiment* is called the **sample space** (or **universal set**) and is denoted by U

Set Operations: Equality

Let there be four sets A, B, C and D:

A = {2, 4, 6, 8}
B = {8, 6, 2, 4}
C = {4, 8, 6, 2}
D = {2, 4, 100, 9}

- A = B
- A = C
- B = C
- A ≠ D
- B ≠ D
- C ≠ D
- It is important to emphasize that in a set, **repetition of elements is not permitted** and that the **order of the elements is not important**
- (*A set is an unordered list of elements without repetition*)

Subsets - Exercises

EXERCISE

- Given a set **C** of all the letters of the word "*demographics*"

a) How many elements are there in set C?
b) Is set J={d, e, m, o} a *proper subset* of C? $J \subset C$?
c) If set K = {g, r, a, p, h, i, c, s}
   - i) Is $K \subset C$ True or False?
   - ii) Is $K \subseteq C$ True or False?

   If set L = {p, h, i}
   - iii) Is $L \subset K$ True or False?
   - iv) Is $L \subseteq K$ True or False?

Complement of a set

- Let U = {x| x is a UVA student}

(Here we define a universal set with all the students of UVA)

  - Let E be a subset of U containing all the students of UVA who are Engineering majors
  - Let M be a subset of U containing all the students of UVA who are Math majors
  - Let S be a subset of U containing all the students of UVA who are Statistics majors

1. What are the elements of E’
2. What are the elements of M’
3. What are the elements of S’
4. What are the elements of (E U M U S)’

Complement of a set

1. What are the elements of E’
2. What are the elements of M’
3. What are the elements of S’
4. What are the elements of (E U M U S)’

- The elements of the sets that are complements of the sets 1, 2, 3 and 4 above, respectively, are:
  - All the students of UVA who are not Engineering majors
  - All the students of UVA who are not Math major
  - All the students of UVA who are not Statistics major
  - All the students of UVA who are not majoring in any of the following: Engineering or Math or Statistics

Set Operations: Intersection

- **Definition**: Form a new set K whose elements are those that are common for sets A and B. An element will be included in the new set K, if and only if it is found in both A and B

  K = {Ann, Bob}
- This new set is called the **intersection** of A and B and denoted as: A ∩B

2013 TV shoppers: A = {Jon, Ann, Sam, Bob, Fay}
2013 cable shoppers: B = {Ann, Bob, Dan, Zoe, Jeb}
2013 computer shoppers: C = {Leo, Zoe, Iva, Ray, Bob}

Intersection of set A and set B is: A∩B

Intersection

- Let U be the universe of discourse, a universal set, set of all the possible values related to the subject
- Let Estore be a super store selling electronics
- Let U = {x| x is a shopper at Estore}
- In this example, U contains all the shoppers of Estore from January 1 to December 31, 2013

  – On the next slide:
    - Set A contains all the customers who bought LED televisions
    - Set B contains all the customers who bought HDMI cables
    - Set C contains all the customers who bought computers

Intersection (cont'd)

A = {Jon, Tim, Sam, Bob, Fay}
B = {Ann, Bob, Dan, Zoe, Jeb}
C = {Leo, Zoe, Iva, Ray, Bob}

Exercises:

1. What is the intersection of B and C?
2. What is the intersection of A, B and C?
3. What is the What is the intersection of A, B and (NOT C)? (Hint: include the name if it's found in A and B, but *not* in C)

Intersection (cont'd)

A = {Jon, Tim, Sam, Bob, Fay}
B = {Ann, Bob, Dan, Zoe, Jeb}
C = {Leo, Zoe, Iva, Ray, Bob}

Questions:

1. What is the intersection of B and C?
2. What is the intersection of A, B and C?
3. What is the What is the intersection of A, B and (NOT C)?
   (include the name if it's found in A and B, but not in C)

Symbolic Presentation:

1. Intersection of B and C: $B \cap C$
2. Intersection of A, B and C: $A \cap B \cap C$
3. Intersection of A, B and (not C): $A \cap B \cap C'$

Elements of the sets:

1. $B \cap C$ = {Bob}
2. $A \cap B \cap C$ = {Bob}
3. $A \cap B \cap C'$ = { } This is an empty set, also denoted as $\emptyset$

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Set Operations: Union

- **Definition**: The union of a collection of sets is the set of all distinct elements in the collection (*repeated entries will be dropped*). Union of two sets A and B is denoted by A U B

- Let V be a set of fruits grown in Virginia
  V = {peaches, pears, apples, plums}
- Let F be a set of fruits grown in Florida
- F = {oranges, tangerines, grapefruits, lemon, lime}
- Let C be a set of fruits grown in California
  C = {pears, apples, blueberries, lime, strawberries}

What are the elements of: a) V U F, b) V U C, c) F U C ?

Union

Let us have three sets A, B, C

A = {Jon, Ann, Sam, Bob, Fay}

B = {Ann, Bob, Dan, Zoe, Jeb}

C = {Leo, Zoe, Iva, Ray, Bob}

EXERCISE

a) Form a new set K whose elements are those that are found either in sets A or in set B or in set C
   – K is a union of 3 sets: A U B U C
b) How many elements are in set K?

A Union of set A and set B is: A U B

Union (cont'd)

EXERCISE

- Let U be a universal set of all the popular names with three letters
- U = {Jon, Ann, Sam, Bob, Fay, Dan, Zoe, Jeb Leo, Iva, Ray, Mae, Nat, Ron, Tim, Tod, Ada, Ama, Amy, Ana, Aya Deb Eva, Eve, Fae, Fay, Ida, Ima, Iza, Jan, Joy, Kay, Kia, Kya, Lee, Liz }

  » A = {Jon, Ann, Sam, Bob, Fay}
  » B = {Ann, Bob, Dan, Zoe, Jeb}
  » C = {Leo, Zoe, Iva, Ray, Bob}

Exercises (Describe the elements)

a) What is the Union of B and C?: B U C
b) What is the Union of A, B and C?: A U B U C
c) What is the Union of A, B and (not C)?: A U B U C'

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Cartesian Product

EXERCISE

- Let there be three sets A, B, C:

  $A = \{1,2\}$

  $B = \{3,4\}$

  $C = \emptyset$

- Let there be two Cartesian Products:

  $K = A \times B$, $P = A \times C$

a) What are the elements of A X B ?

- What is the cardinality (*size*) of A X B ?

b) What are the elements A X C ?

- What is the cardinality (*size*) of A X C ?

Set Operations: Disjoint (Mutually Exclusive) sets

Definition: Two sets A and B are disjoint if their intersection $A \cap B = \emptyset$. There are no common elements between the two sets

- Let set C = {x| a person with Canadian citizenship}
- Let set U = {x| a person with USA citizenship}

- If there is no one having a dual citizenship then sets A and B are disjoint: $C \cap U = \emptyset$
- If some people have dual nationality then $C \cap U \neq \emptyset$, the sets C and U are *not* disjoint

Disjoint (Mutually Exclusive) sets

- Example 1

  B = {x | x is a blue car in Virginia}

  R = {x | x is a red car in Virginia}

  $B \cap R = \emptyset$

  A car can be either all blue or all red, *not both*

- Example 2 [*not disjoint*]

  C = {x | x is a Computer Science major

  M = {x | x is Math major}

  $C \cap N \neq \emptyset$

  There is a possibility of 2 majors

Disjoint (Mutually Exclusive) sets

- Example 3

  H = {x | x was a hot day with temperature >99}

  S = {x | x was a snowy day}

  H ∩ S = Ø

  It has never snowed when temperature is >99

- Example 4
  - A = {x | x > 10}
  - B = {x | x < 10}
  - B ∩ R = Ø

Mutually Exclusive Sets

- Example 5

  A = {x | x is an even number}

  B = {x | x is an odd number}

  $A \cap B = \emptyset$

- Example 6 [*not disjoint*]
  - A = {x | x is an integer}
  - B = {x | x is a natural number}
  - $H \cap S \neq \emptyset$

EXERCISE

Can you think of <u>two</u> examples of Disjoint sets?

Use of Venn Diagrams in Set Theory

Venn Diagrams

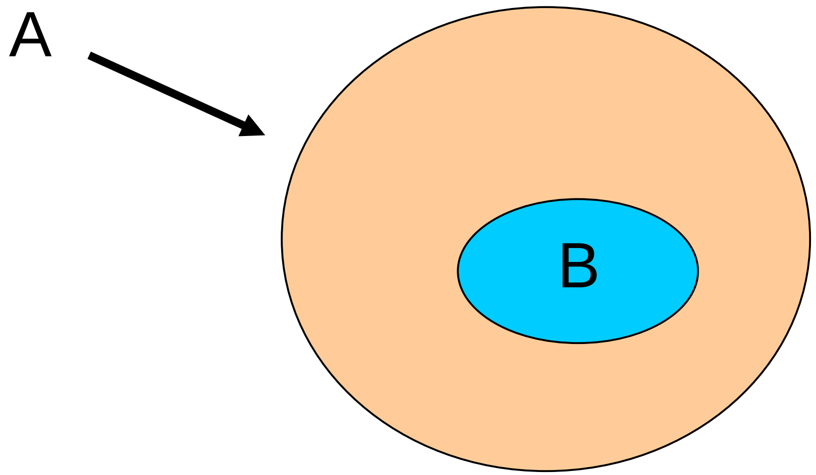
- John Venn, (4 August 1834 – 4 April 1923) was an English Logician and Philosopher. He introduced the Venn Diagram, *a graphical way of presenting concepts*. Venn diagram are being used in the fields of set theory, logic, Statistics, and Computer Science

- When dry formulae and rules of set theory are presented in graphical form it enhances the clarity and understanding of the concepts

Venn Diagrams

Subsets

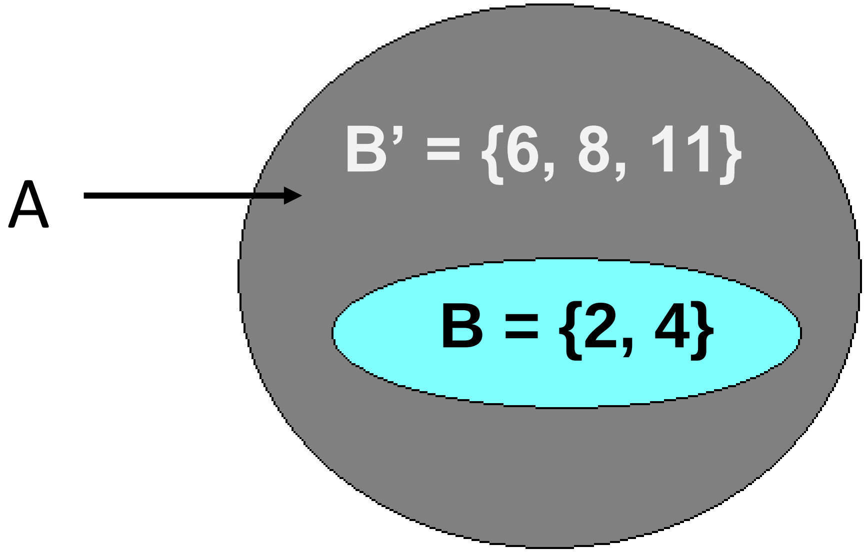
- Set A and set B are two proper subsets of set C
  - Set A has some of the elements of set C as well as set B had some elements of set C.
  - Below: Set A = all the contents of the outer circle
  - $B \subset A$ Set B is a (proper) subset of set A



Venn Diagrams

Complement of a set

- Let's say that we have a set B that is a subset of set A. The complement of set B (denoted B’) be the set of elements of set A that are not elements of set B
  - Below: A = {2, 4, 6, 8, 11}, B U B’ = A
  - B = {2, 4}, Complement of B = B’ = {6, 8, 11}



Venn Diagrams

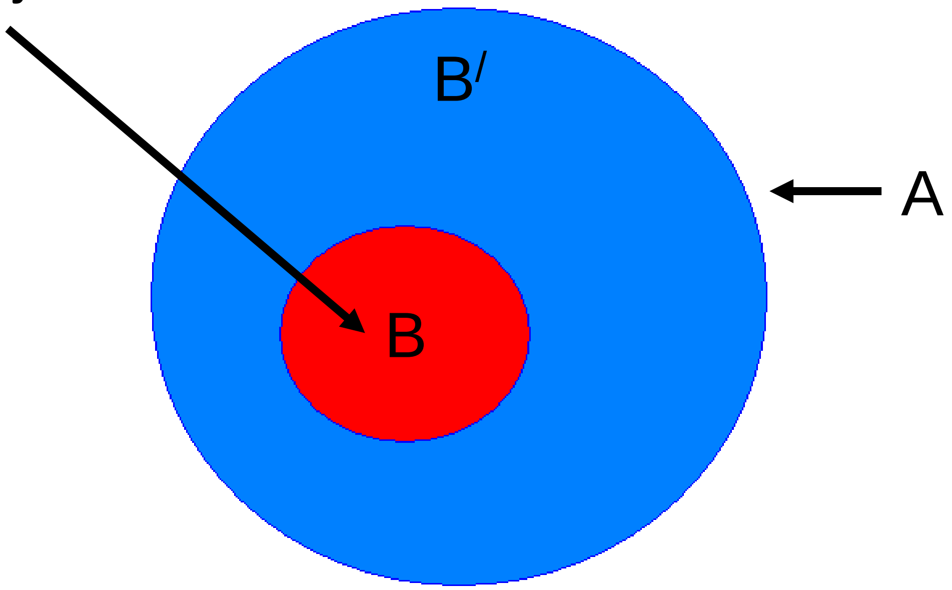
Another example: compliment of a set

- A = {x | x is a relative of John}
- Read as "A is a set of all the values of x such that x is a relative of John"
- B = {x | A relative of John living in Virginia}

The bigger (blue) circle contains both B and B'.

Thus the bigger circle is B U B' = A =

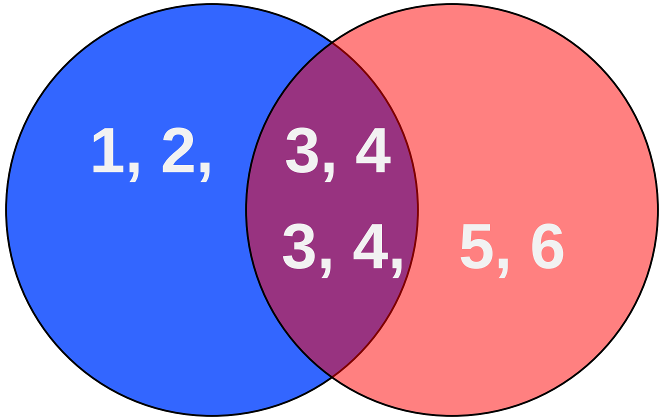
All the relatives of John



Venn Diagrams

Intersection of Sets

- Definition: An intersection of two sets A and B is the set of all the element found in set A AND set B, It denoted as A ∩ B
- Let A = {1, 2, 3, 4}; Let B = {3, 4, 5, 6}
- A ∩ B = {3, 4}
- The overlapping area of the two circles is A ∩ B



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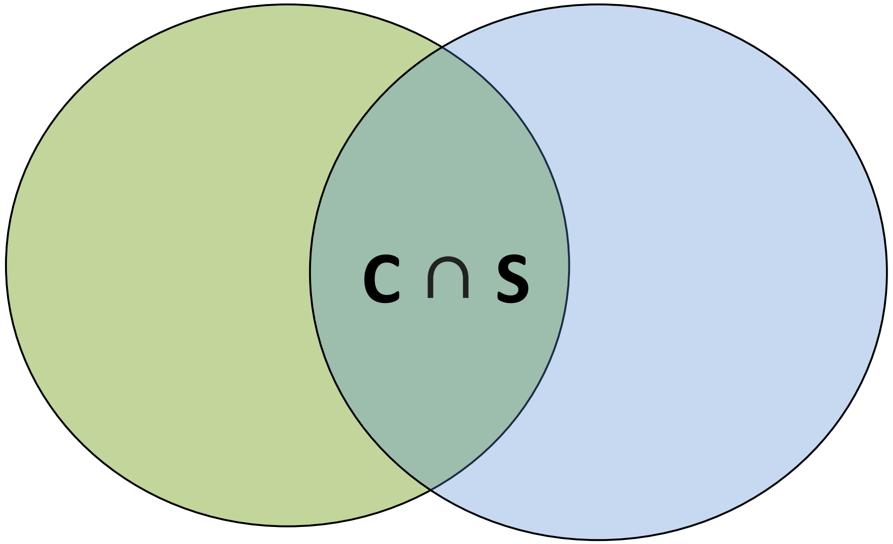
Venn Diagrams

Intersection of Sets

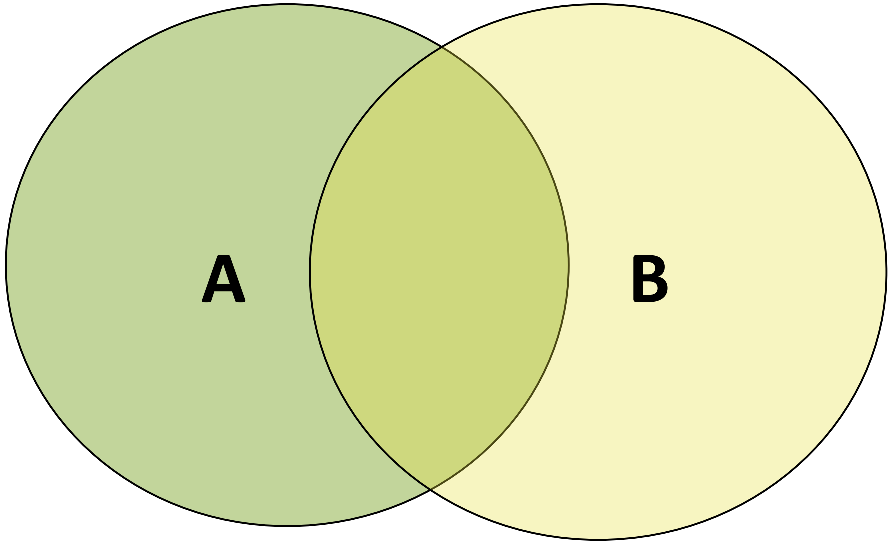
- Let C = {x | x is student taking computer courses}
- Let S = {x | x is student taking Statistics courses}
- C ∩ S = {x | x is a student taking computer Science and Math courses}



Venn Diagrams

Union of Sets

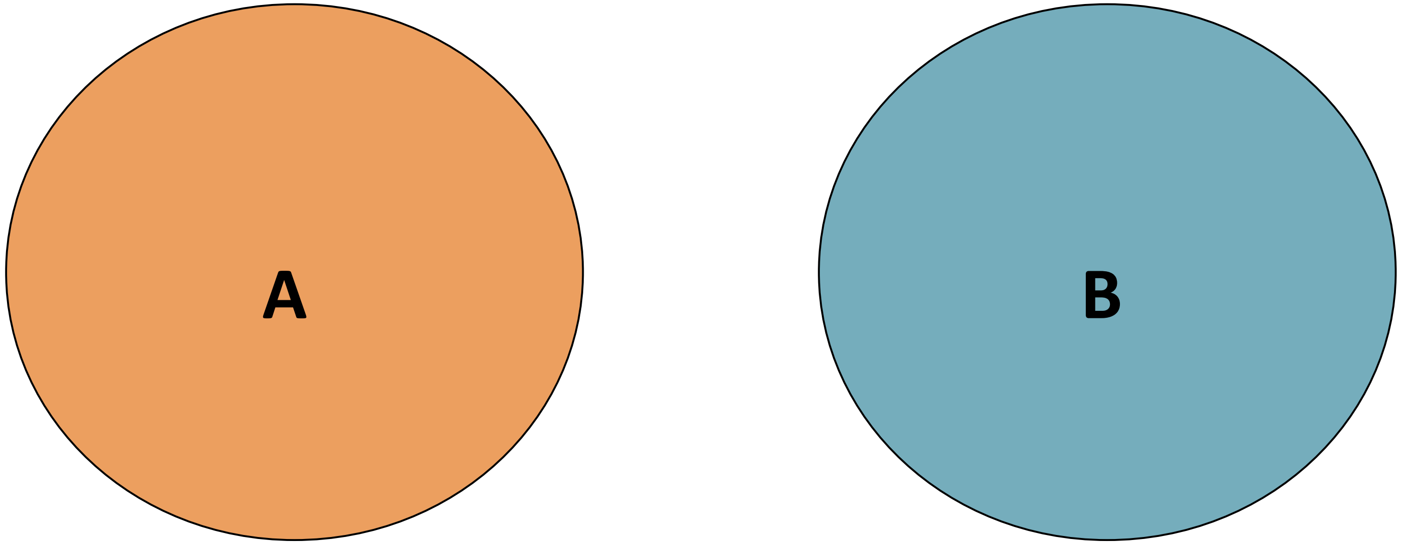
- A union of two sets A and B is a collection of all the elements of A and B (All elements of A and B included with duplications being removed). Union is denoted as: A U B
- Below: A U B will be the collection of all the items of A and B



Venn Diagrams

Disjoint of two Sets

- Two sets A and B are disjoint if they have no elements in common
- Intersection of disjoint sets is empty set: $A \cap B = \emptyset$



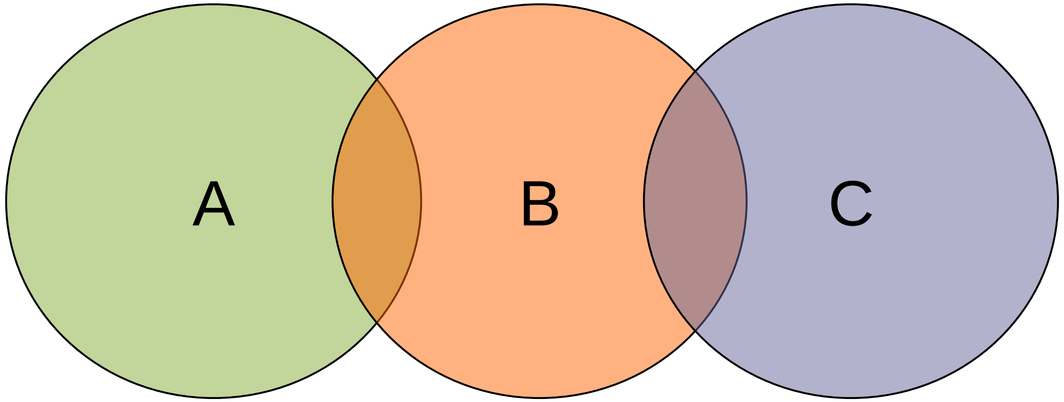
Set Operations on Three Sets

- In the previous examples we have talked of operations involving **two sets** i.e. intersection of two sets , union of two sets etc. This was done to simplify the examples for better understanding
- Most of the set operations can be done on **any number of sets**, however the explanation and understanding of the concepts becomes more complex
- In the following few slides, we will discuss examples involving **three sets**. This illustrates that the *operations on sets can be done for any number of sets*

Venn Diagrams

Union of Three Sets

- Let there be three sets A, B, C and D such that
- **A** = { 1, 2, 3, 4, 5}; **B** = { 5, 6, 7, 8, 9}; **C** = {11, 8, 5, 2, 29}
- **D** = A U B U C = {1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 29}
- Total number elements of A, B and C = 5 + 5 + 5 = 15
- How come D has only 11 elements?



Venn Diagrams

Example 1: Intersection of Two or Three Sets

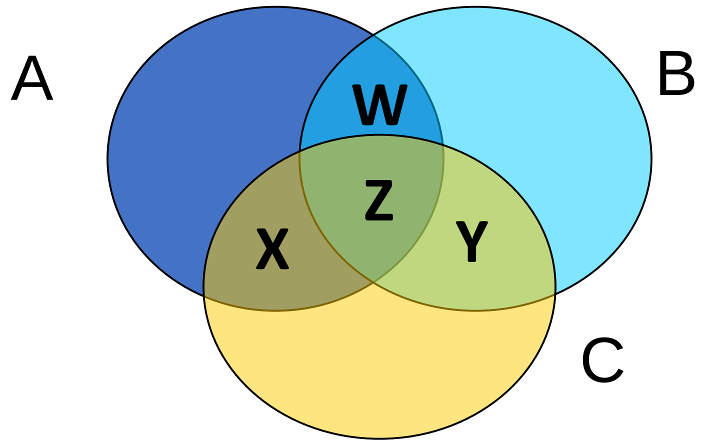
- Let there be three sets A, B, C and D such that
- A = { 1, 2, 3, 4, 5};   B = { 5, 6, 7, 8, 9}; C = {11, 8, 5, 2, 29};
- D = A ∩ B ∩ C = {5}
- '5' is the only element found in all three sets
- Remember that the sequence (*order*) of elements in a set is not important: {1,2,3} ≡ {3,2,1} ≡ {2,1,3}

Set W = A ∩ B = {5}

Set X = A ∩ C = {2, 5}

Set Y = B ∩ C = {5, 8}

Set Z = A ∩ B ∩ C = {5}



Venn Diagrams

Example 2: Intersection of Two or Three Sets

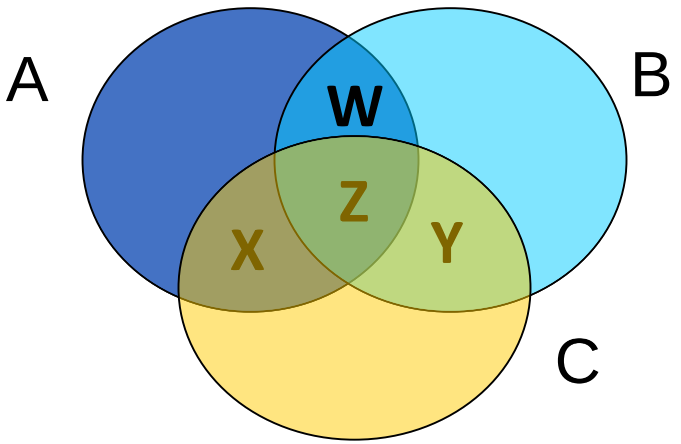
- Here are three customers who bought some items at an electronics store:
- A = {TV, iPad, iPod, HD}; B = {TV, HD}; C = {iPad, iPod, HD}
- D = A ∩ B ∩ C = {} = Ø (no items were bought by all three customers)

Set W = A ∩ B = {TV, HD}

Set X = A ∩ C = {iPad, iPod, HD}

Set Y = B ∩ C = {HD}

Set Z = A ∩ B ∩ C = Ø



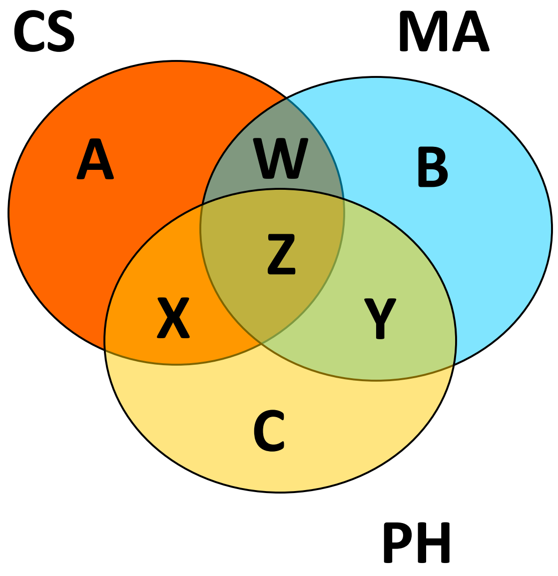
Exercise 1

Let there be three sets:

- CS: {x | x is taking Computer Science courses}
- MA: {x | x is taking Math courses}
- PH: {x | x is taking Physics courses}

Below are three circles, each circle representing one of the three sets

Can you figure out what are the elements of the sets A, B, C, W, X, Y, Z?



Exercise 1 (Solution)

Let there be three sets:

- CS: {x | x is taking Computer Science courses}
- MA: {x | x is taking Math courses}
- PH: {x | x is taking Physics courses}

Solution:

A = {x | x = CS ∩ (MA U PH)’}

B = {x | x = MA ∩ (CS U PH)’}

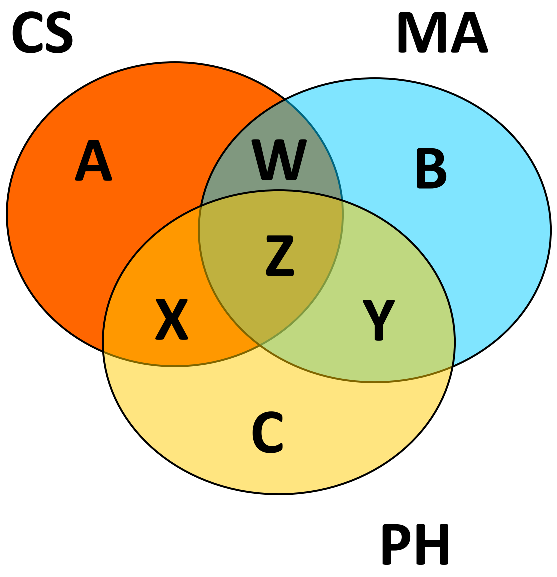
C = {x | x = = PH ∩ (CS U MA)’}

W = {x | x = CS ∩ MA ∩ PH’}

X = {x | x = CS ∩ PH ∩ MA’}

Y = {x | x = MA ∩ PH ∩ CS’}

Z = {x | x = CS ∩ MA ∩ PH}



Exercise 2 (with Solution)

Let there be three sets:

- U= {S1, S2, S3, S4, S5, S6, S7, S8, S9 } (Nine students at UVA)
- C = { S1, S2, S3, S4, S7, S9 } (Five Computer Science majors)
- S = {S3, S4, S7 } (Three Statistics majors)

Find : (C’ U S’)’

- Step 1: Find all the students who are NOT Computer Science majors
  - C’ = {S5, S6, S8} (i)
- Step 2: Find all the students who are NOT Statistics majors
  - S’ = {S1, S2, S5, S6, S8, S9 } (ii)
- Step 3: Find: C’ U S’ = all the elements of (i) and (ii) added together
  - C’ U S’ = {S1, S2, S5, S6, S8, S9 } (iii)
- Step 4: Find **(C’ U S’)’** = Universe – (iii) = **{ S3, S4, S7 }**

Note that **(C’ U S’)’** = C ∩ S (this is why it’s good to know/understand equivalences!)

Exercise 3 (with Solution)

Let there be five sets:

- U= {1, 2, 3, 4, 5, 6, 7, 8, 9 }
- A= { 4, 5, 7, 9 }, B = {1, 3, 4, 7, 8, 9}
- X = {2, 4, 5, 8, 9}, Y = {1, 2, 3, 4, 5}

Find: (A’ ∩ B’) ∩ (X U Y)

- Step 1: Find A’ = {1, 2, 3, 6, 8 }
- Step 2: Find B’ = {2, 5, 6}
- Step 3:Find (A’ ∩ B’) = { 2, 6 } (i)
- Step 4: Find (X U Y) = { 1, 2, 3, 4, 5, 8, 9 } (ii)
- Step 5: Write all the elements common to (i) and (ii)
  - Only ‘**2**’ is found in both sets
  - **(AC ∩ BC) ∩ (X U Y) = { 2 }**

De Morgan's Rules

De Morgan's Rules in Logic and Set Theory

- De Morgan's **Rule # 1**: In our study of logic we learnt that the *negation of a the conjunction* of two statements is the *disjunction of the negation of these statements*:

  ¬(P ∧ Q) is logically equivalent to ¬P ∨ ¬Q

- De Morgan's **Rule # 2**: We also learnt that the *negation of a the disjunction of two statements* is the *conjunction of the negation of these statements*:

  ¬(P ∨ Q) is logically equivalent to ¬P ∧ ¬Q

- You will notice in the next 2 slides that the De Morgan's Laws of Propositions are similar to the De Morgan's Laws of sets. Only the *symbols* are different

De Morgan’s law #1 for set theory

Proof of De Morgan’s law # 1

- De Morgan’s Law # 1: **¬(P ∧ Q) is logically equivalent to ¬P ∨ ¬Q**

- U = { 1, 2, 3, 4, 5, 6, 7, 8, 9}
- A = {2, 3, 5, 6, 7}; B = (1, 2, 4, 6, 7, 9}
- Step 1: (A U B) = {1, 2, 3, 4, 5, 6, 7, 9}
- Step 2: (A U B)’ = { 8 }
- Step 3: A’ = { 1, 4, 8, 9}
- Step 4: B’ = {3, 5, 8}
- Step 5: A’ ∩ B’ = { 8 }
- Steps 1 through 5 lead to the conclusion that:
    - (A U B)’= A’ ∩ B’

De Morgan’s law #2 for set theory

Proof of De Morgan’s law # 2

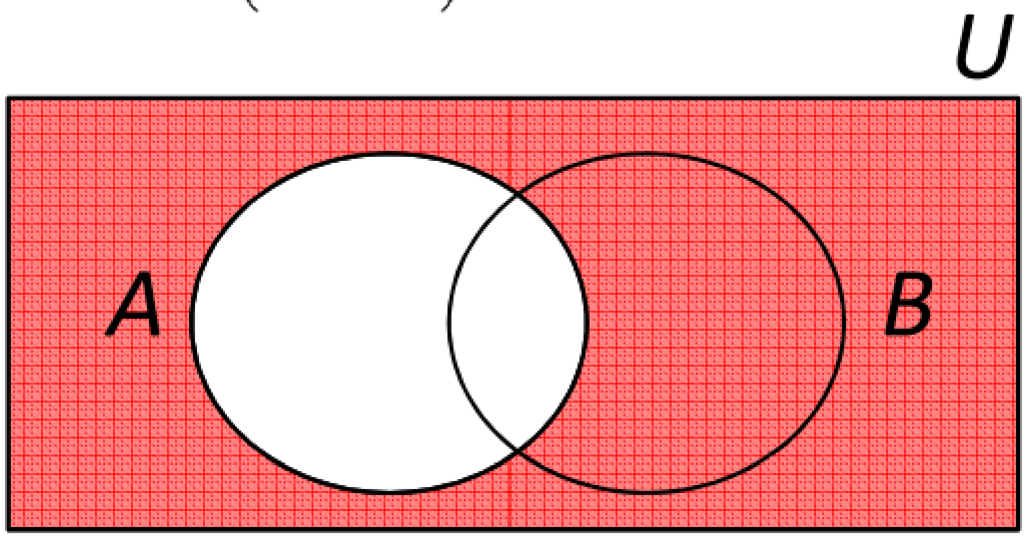
- De Morgan’s Law # 2: **¬(P ∨ Q) is logically equivalent to ¬P ∧ ¬Q**

- U = { 1, 2, 3, 4, 5, 6, 7, 8, 9}
- A = { 2, 3, 5, 6, 7}; B = (1, 2, 4, 6, 7, 9}
- Step 1: (A ∩ B) = {2, 6, 7}
- Step 2: (A ∩ B)’ = { 1, 3, 4, 5, 8, 9}
- Step 3: A’ = { 1, 4, 8, 9}
- Step 4: B’ = { 3, 5, 8 }
- Step 5: A’ U B’ = { 1, 3, 4, 5, 8, 9}
- Steps 1 through 5 lead to the conclusion that:
    - (A ∩ B)’ = A’ U B’

De Morgan's Law & Venn Diagrams

Look at the area in red!

$$(A \cup B)' = A' \cap B'$$

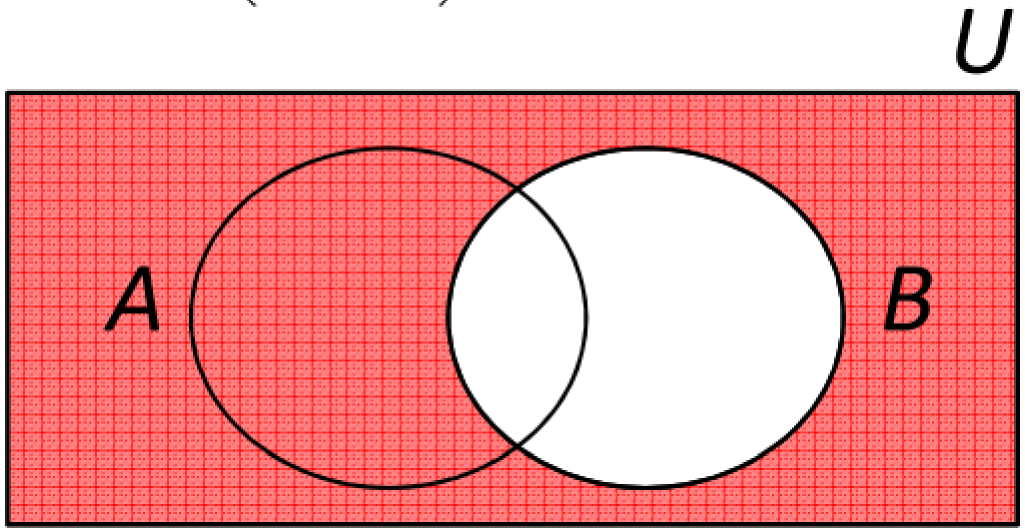


This is A'.

De Morgan's Law & Venn Diagrams

Look at the area in red!

$$(A \cup B)' = A' \cap B'$$

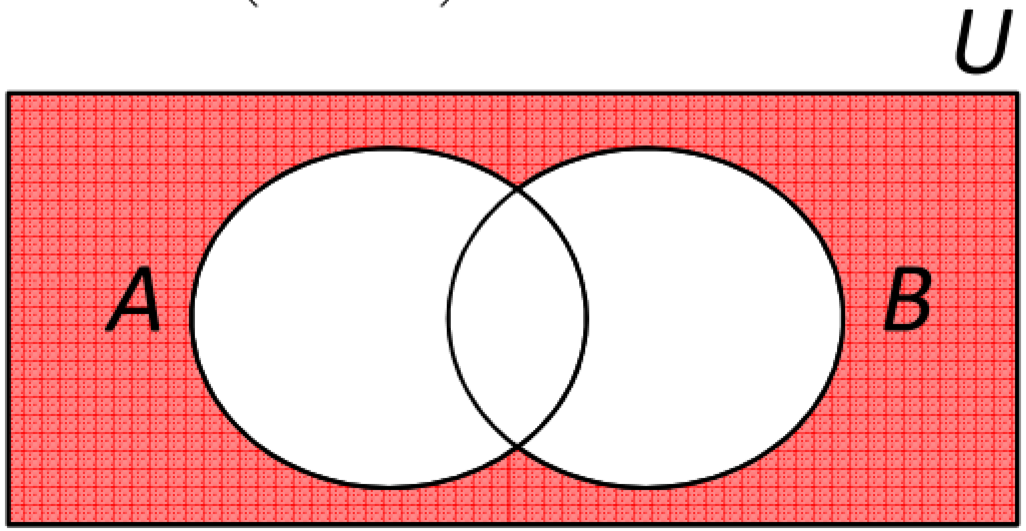


This is B'.

De Morgan’s Law & Venn Diagrams

Look at the area in red!

$$(A \cup B)' = A' \cap B'$$



This is their intersection.