Relations & Functions

CS 5012

UNIVERSITY OF VIRGINIA
DATA SCIENCE INSTITUTE

Broader Goal of these Lessons

- The broader goal of the following lessons is to sharpen your thinking and analytical skills by discussing the concepts of Logic, Sets, Relations And Functions and their connections with each other

- Our discussion will evolve from sets, to relations to Functions
- *You will recognize that these subjects are very closely interconnected*

Objectives

- Understanding of the sets and their connection with Relations and Functions
- Developing the understanding from sets to Relations, Relations to Functions
- Applications of these concepts to solve real life problems
- Seeing some examples and solving exercises to understand a very wide variety of applications of functions in real life situations

Main Topics to be Covered

- Evolution of discussion from Sets to Relations
- What distinguishes Functions from Relations
- Problems involving Relations and Functions

Quantifiers (*Recall*)

Universal Quantifier (∀)

- ∀ x| P(x) means "for all x, the case P(x) holds"
- Instead of "for all x", the words "for each x" and "for every x" are also used

Existential Quantifier (∃)

- ∃ x | P(x) means "there exists an x such that P(x) holds"
- For each x (for at least one x), predicate P(x) holds

Part I:

RELATIONS

Informal Definitions of Relation

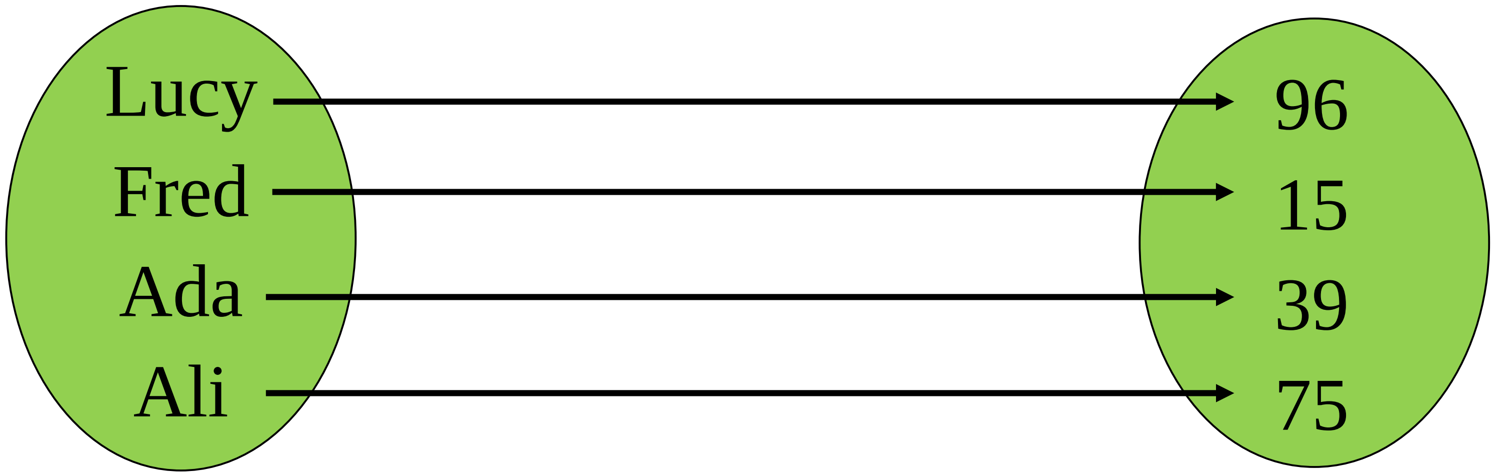
- An **association** between the elements of an ordered pair of objects (humans, animals, plants or things)
  - (mother, daughter), (Orange, Yellow)
- A **relation** is a connection between two variables x and y such as price per pound and total cost
- An association between inputs and outputs
  - (1,2), (2,4), (3,6)
- An **ordered pair** consisting of a x and y-coordinate

Formal Definitions of Relation

- A **relation** is an association between inputs and outputs
- A relation is any association between elements of one set, called the **domain** , and another set, called the **range**
- A **relation (a,b)** is a subset of ordered pairs drawn from the Cartesian product of set A and set B, written as
  - $R \subseteq A \times B = \{(a, b) \mid a \in X, b \in Y\}$

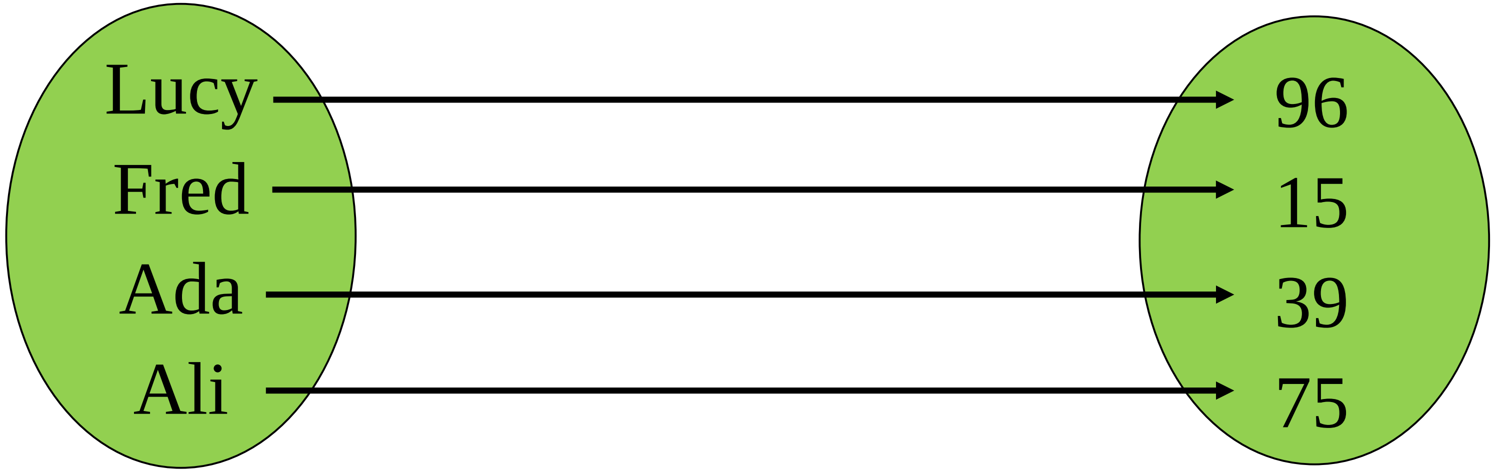
Domain, Codomain and Range

- The longest documented lifespan of a human being is that of Jeanne Calment of France who lived 122 years
- Lucy, Fred, Ada, Ali and their ages at death



Domain, Codomain and Range

- Domain = {Lucy, Fred, Ada, Ali}
- Range = {96, 15, 39, 75} (*actual*)
- Codomain: {x| 0 < x < 123, x is a positive integer (Natural number) (*possible*)
- R = {(Lucy,96), (Fred,15), (Ada,39), (Ali,75)}



Relation expressed as a Set

- Here, we explicitly express **Relation** as a set of ordered pairs such that:
- $R \subseteq A \times B = \{(a, b) \mid a \in A, b \in B\}$
- Set **A** = {Lucy, Fred, Ada, Ali}
- Set **B** = {96, 15, 39, 75}
- R = {(Lucy,96), (Fred,15), (Ada,39), (Ali,75)}

Relation within one Set

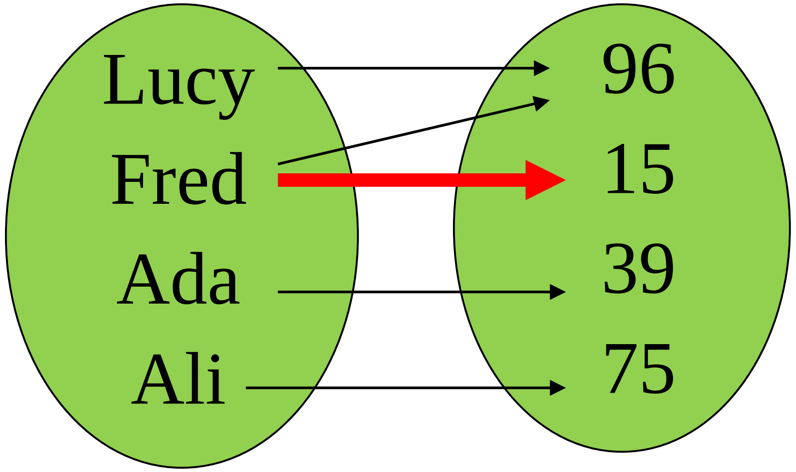
- We saw relation between two sets A and B. Set A provided the domain and set B provided the range of the ordered pairs of the relation
- A relation can also be between the elements of the *same set*:
- Set A = {1,2,3,4}
- R = {(1,2), (1,3), (1,4), (2,3), (2,4), (3,4)}
- Set F = {Father, Mother, Daughter, Son}
- R = {(Father,Mother), (Father,Daughter), (Father,Son), (Mother,Daughter), (Mother,Son), (Daughter, Son)}

Multiple links between Domain & Range

- Previously we saw that each item (person) in the domain is connected to **exactly one element** of the range (age at death) and vice versa. Elements of domain and range can, also, have **multiple connections**

In our example, a relation shown by red line cannot exist. Fred could not have died twice.

However, two people can die at the same age

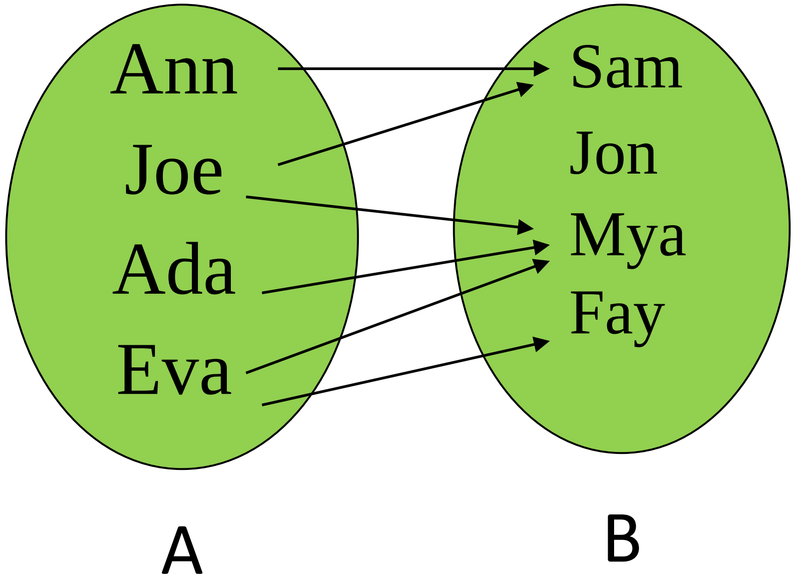


Multiple links between Domain & Range

- Here is an example where **multiple relations can exist, both ways, between domain and range**
- Left side set is set A (*students*)
- Right side set is set B (*professors*)
- Relation shows the courses being taken

Set of Relations:

R = {(Ann,Sam), (Joe,Sam), (Joe,Mya), (Ada,Mya), (Eva,Mya), (Eva,Fay)}



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Types of Relations

- Some important types of relations:
  - Symmetric
  - Transitive
  - Reflexive
  - Equivalence
  - Identity
  - Asymmetric

Symmetric Relation

- Let there be a set A = {1, 2, 3 }
- Let R be a binary relation on set A
- R is **symmetric** if for all a, b ∈ A, (a,b) ∈ R implies (b,a) ∈ R
- A relation R is symmetric if whenever (a, b) ∈ R then (b, a) ∈ R
- R is symmetric if **aRb** implies **bRa** for all a and b in A
- The relation R on {1,2,3} given by
- R = {(1,1), (1,2), (2,1), (2,3), (3,2),(1,3), (3,1)} is symmetric
- (All paths are 2-way)

Transitive Relation

- Let R be a binary relation on A
- R is **transitive** if for all a,b,c ∈ A, (a,b) ∈ R and (b,c) ∈ R implies (a,c) ∈ R
- A relation is transitive if whenever (a, b) ∈ R and (b, c) ∈ R then (a, c) ∈ R
- R is transitive if **aRb** and **bRc** together imply that **aRc** holds for all a, b, and c in X
- The relation R on {1,2,3} given by
- R = {(1,1), (1,2), (2,1), (2,2), (2,3), (1,3)} is transitive (If I can get from one point to another in 2 steps, then I can get there in 1 step)

Reflexive Relation

- Let R be a binary relation on A
- R is reflexive if for all a ∈ A, (a,a) ∈ R
- A relation on a set A is **reflexive** if (a, a) ∈ R for all a∈A
- The relation R on {1,2,3} given by R = {(1,1), (2,2), (2,3), (3,3)} is reflexive. (All loops are present)

Equivalence Relation

- A relation that is reflexive, symmetric ***and*** transitive is termed an **equivalence** relation

- Example:

  Let the set {a, b, c} have the equivalence relation

  { (a,a), (b,b), (c,c), (b,c), (c,b) }

Identity Relation

- An **identity relation** is a relation that always returns the same value that was used as its argument. In terms of equations, the relation is given by f(x) = x
- Example 1:
  - Let A = {x, y, z}
  - I = {(x,x), (y,y), (z,z)}
- Example 2:
  - If price of sugar is $1 per pound then here is the quantity-cost relationship
  - R = {(1,1), (2,2), (3,3), (4,4)}
  - If we buy one lb, we pay 1 Dollar, for 2 lbs we pay 2 Dollars and so on

Asymmetric Relation

- Let there be set A= {1,2,3}
- Let R be a relation on A
- Relation R is called **asymmetric** if $(x,y) \in R$ always implies $(y,x) \notin R$
- R = { (1,2), (2,3), (1,3) } is Asymmetric, because $(x,y) \in R$ and $(y,x) \notin R$

Operations on Relations

- Some common operations on relations:
  - Union
  - Intersection
  - Difference
  - Inverse
  - Composition

Union Of Relations

- **Union** of two relations includes elements that are in either relations R or in relation S *(remove duplicates!)*

- R = {(2,3), (3,5), (3,9), (4,8), (6,9), (7,8)}
- S = {(3,5), (6,4), (7,5), (7,8)}
- **Union of R and S**:
  - R U S = {(x,y) | xRy V xSy}
  - R U S = { (2,3),(3,5),(3,9),(4,8),(6,9),(7,8),(6,4),(7,5) }

Intersection Of Relations

- An **intersection** of two relations R and S includes elements that are common to both relations: R and S. An ordered pair must be **in both R and S** to be included in the result

- $R = \{(2,3), (4,5), (5,9), (5,6), (5,7), (6,7)\}$
- $S = \{(2,3), (6,4), (5,6), (7,8)\}$
- **Intersection of R and S:**
  - $R \cap S = \{(x,y) \mid xRy \land xSy\}$
  - $R \cap S = \{(2,3),(5,6)\}$

Difference of Relations

- The **difference** between two relations R and S, denoted R-S, includes the elements found in relation R but not in relation S

- R = {(2,3), (3,5), (3,9), (5,9), (6,8), (7,8)}
- S = {(3,5), (6,4), (6,8), (7,5), (7,8)}
- **Difference between R and S:**
  - $R - S = \{(x,y) \mid xRy \land \neg(xSy)\}$
  - $R - S = \{(2,3),(3,9),(5,9)\}$

Inverse of a Relation

- The **inverse** relation of a binary relation is the relation that occurs when the order of the elements is switched in the relation
- The inverse of a relation is the relation where the two components of an ordered pair exchange their location (position)
- Let us say that tomatoes are two Dollars a pound. Here is the relation between the price, P, and the cost, C:
  - { (P1,C1), (P2,C2), (P3,C3)}
  - { (1,2), (2,4), (3,6) }
- **Inverse of the function:**
  - { (C1,P1), (C2,P2), (C3,P3)}
  - { (2,1), (4,2), (6,3) }

Composition of a Relation

- **Composition** of relations
- Let X, Y and Z be three sets, R be a relation from X to Y, S be a relation from Y to Z
- A composition of R and S is a relation from X to Z
- xRySz, i.e. $(x,y) \in R$ and $(y,z) \in S$
- In everyday life, the relation from grandparents to grandchildren is a composition of relations
- For the construction of a house, Price per bag for cement (X), affects the cost of cement (Y) that affects the cost of building the house (Z)

Presenting the Relations

- A set of ordered pairs
- Table of values
- Mapping
- Equation
- Graph

Presenting the Relations

- **Ordered pairs format**
- *(One of the ways we're most familiar with)*
- For simplification, let us say that tomatoes are 2 Dollars per pound. With this price, following is the relation given by ordered pairs of amount purchased and the total cost paid by the customer:
- {(1,2), (2,4), (3,6), (4,8), (5,10), (6,12)}

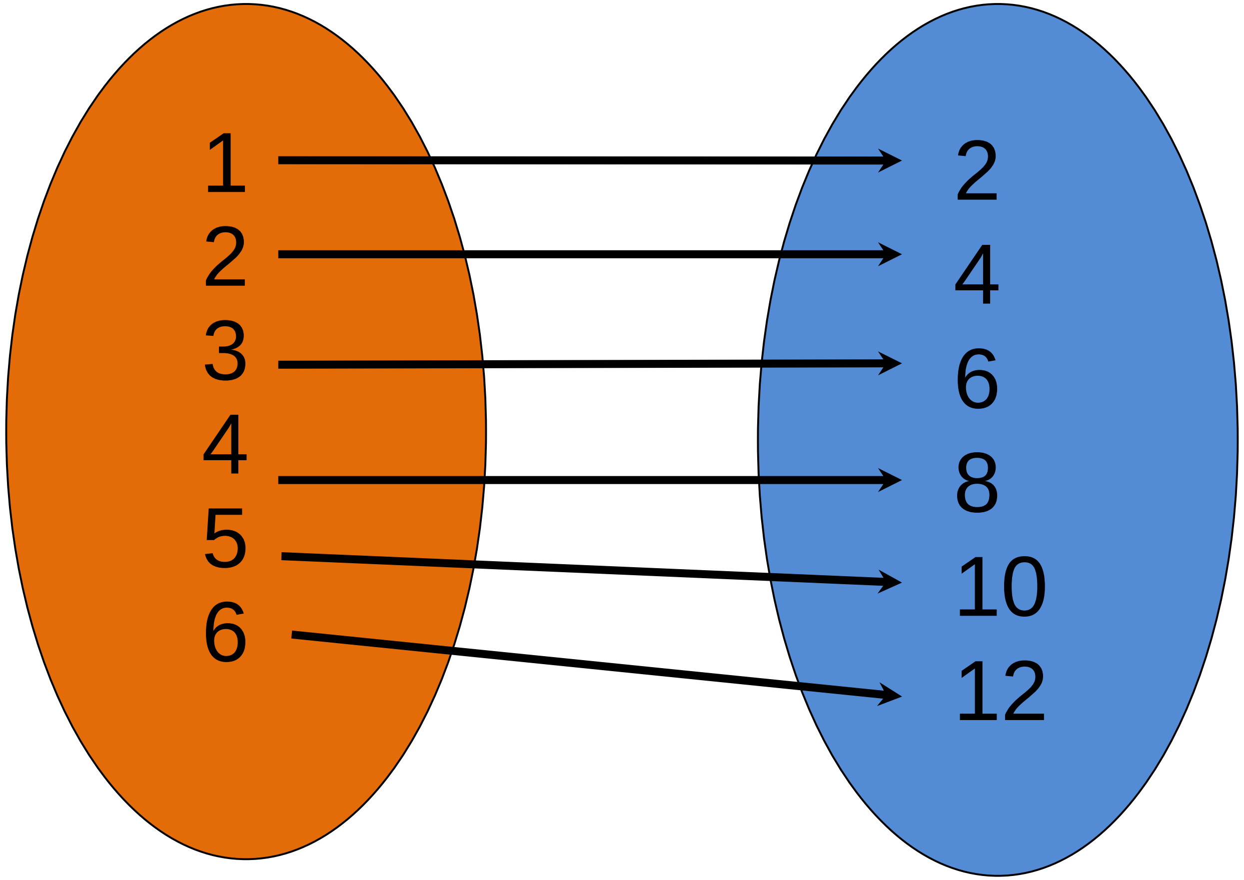
Presenting the Relations

- **Table Format**

Quantity	Cost
1	2
2	4
3	6
4	8
5	10
6	12

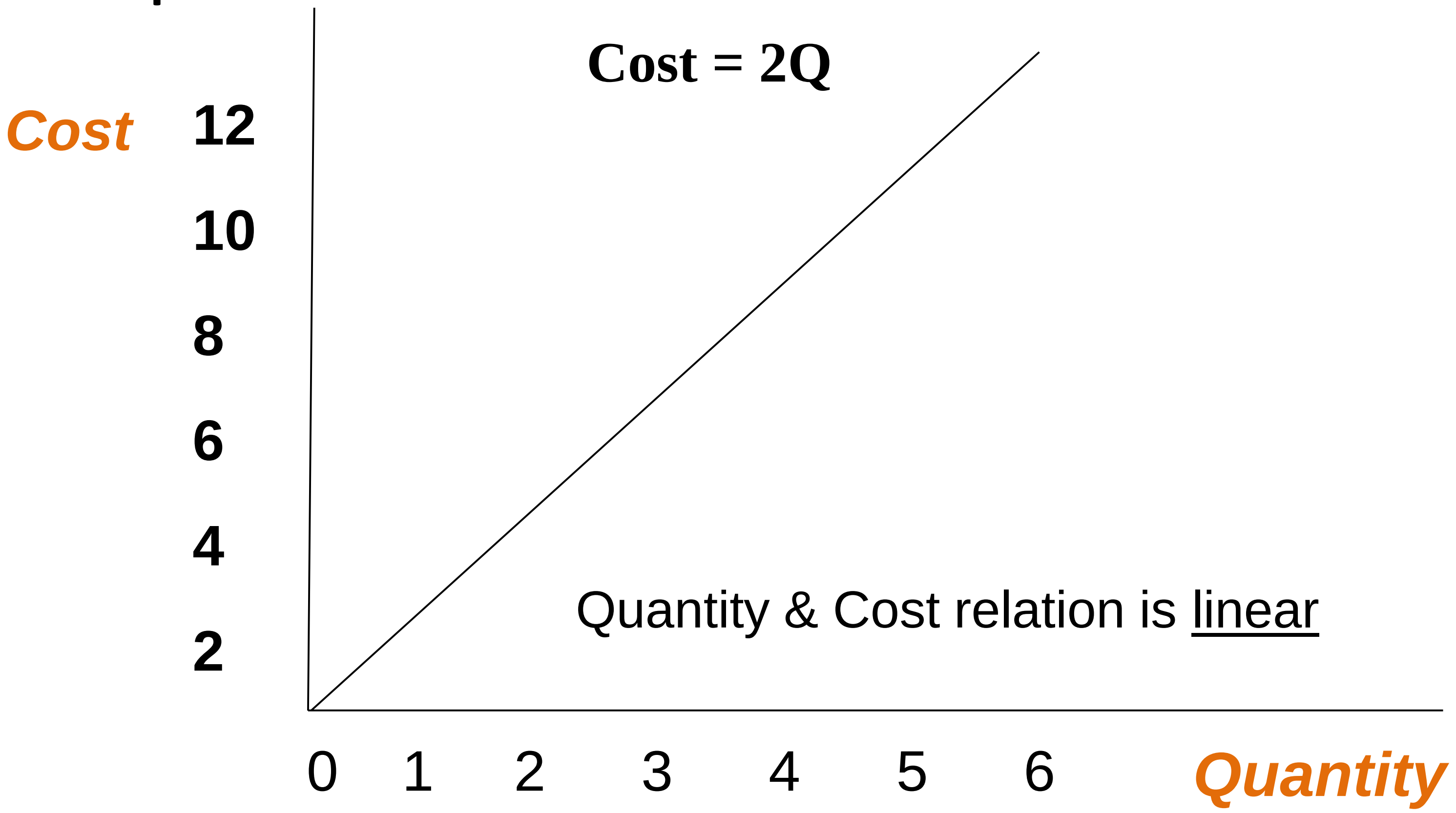
Presenting the Relations

- **Mapping Format**



Presenting the Relations

- **Equation and Graph**



Part II:

FUNCTIONS

Definition of a Function

- A **function** is a special relationship between values: Each of its input values gives back exactly one output value
- A function is a set of ordered pairs in which each x-element has only one y-element associated with it
- Functions are **relations** with the following special property:
  - A function from a set A (called **domain**) to a set B (called **range**) is a relation between A and B such that for each $a \in A$ there is *one and only one* associated $b \in B$

Function

- As we know from the definition, Functions are special type of relations. Every function is also a relation. However every relation is not a functions. Some relations are also functions
- Many of the relations we just discussed are also functions
- When we discuss some examples, we will discover that almost all aspects of our business activities and scientific endeavors can be ***explained, analyzed and improved by understanding and applying appropriate functions***

Some Most Common Functions

- Linear Function
- Quadratic Function
- Power Function
- Exponential Function
- Logarithmic Function
- Sine Function
- Cosine Function

Linear Functions

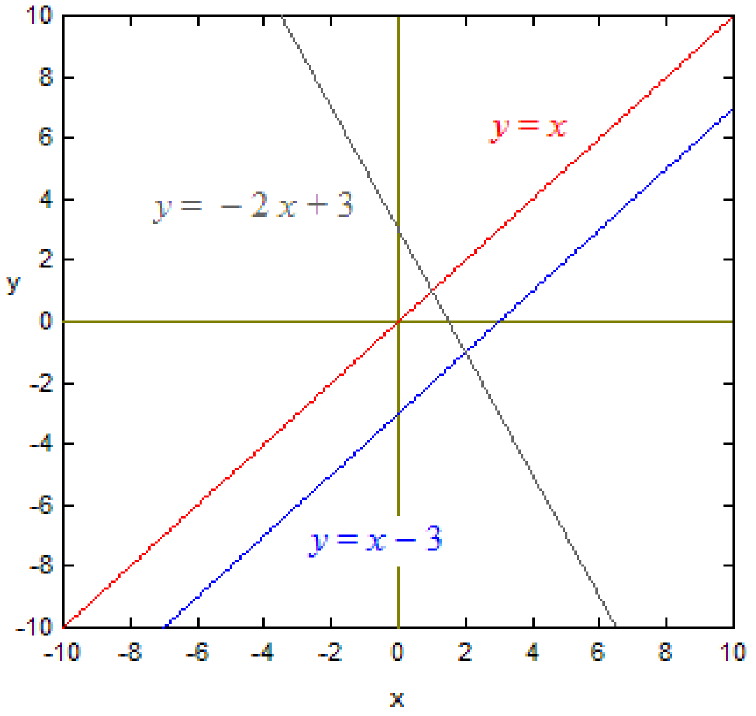
- f(x) = a + bx *(a is 'y' intercept and b is the slope of the line)*
- In spite of being so simple, **Linear function** has many examples in real life…

Here are some examples:

- Quantity bought at a store (x) and money paid f(x)

- Dough used and loafs of bread produced

- Distance traveled and gasoline used



Quadratic Function

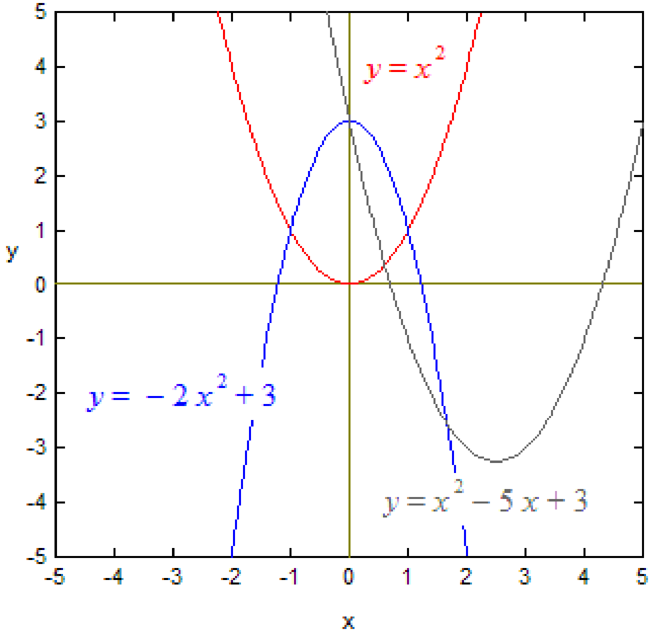
- The **Quadratic** equation has the form: $Ax^2 + bx + c = 0$
- The name Quadratic comes from "*quad*" meaning **square**, because the variable (x) gets squared. It is also called an "Equation of degree 2"

There are many examples where y increases in the beginning then starts declining and vice versa

Applying fertilizer to a crop, taking medicine for a disease, or using yeast in the dough; we see positive results to some point and then we start seeing harm of adding more of the input

(Too much of a good thing can be harmful)

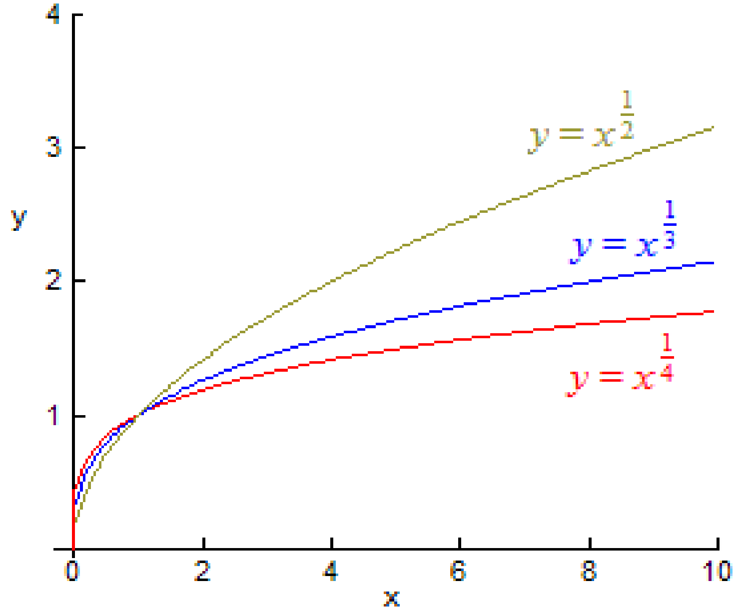
Throwing a ball in the air or firing a rocket are two other examples



Power Function

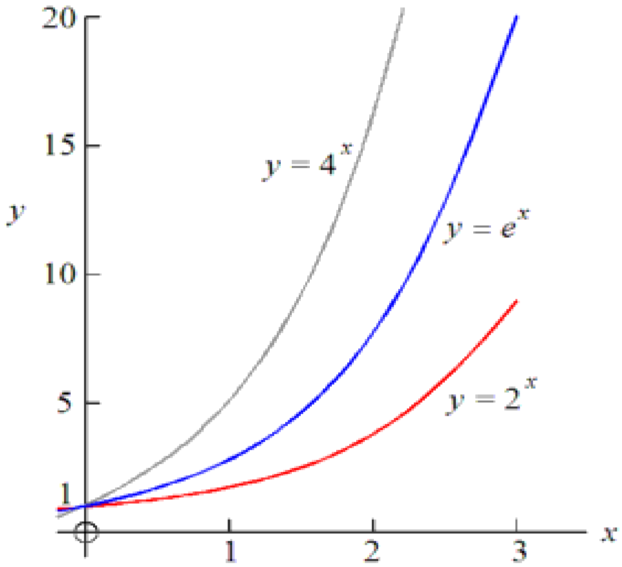
- **Power** function general form: $f(x) = x^p$
- In a number of real life situations, the marginal (additional) benefit of our efforts (or inputs) starts declining…

  - Years of use and minerals extracted from a coal mine
  - Fertilizer used and growth of a plant
  - Continuous hours of study and intensity of retention



Exponential Function

- **Exponential** functions have the form: $f(x) = b^x$
- There are many quantities in real life that grow **exponentially**. Two examples are population growth and compound interest

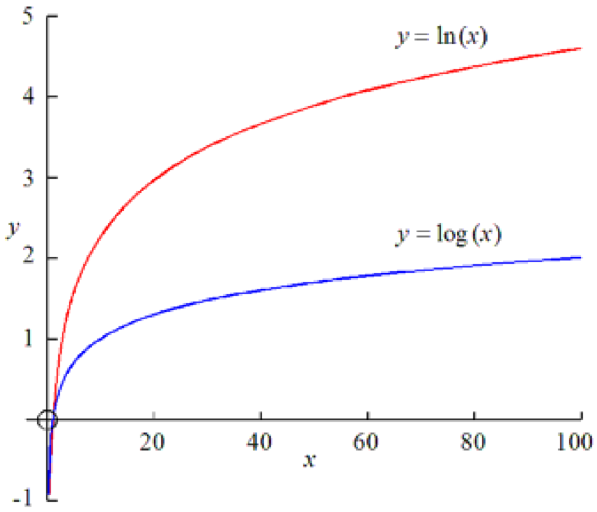


Logarithmic Function

- The **logarithmic** function is defined as: $f(x) = \log_b x$
- A **logarithm** is simply an exponent that is written in a special way. $10^3 = 1000$ is the same thing as $\log_{10} 1000 = 3$ (log of 1000 to the base 10 = 3). *This graph has some resemblance to power function*

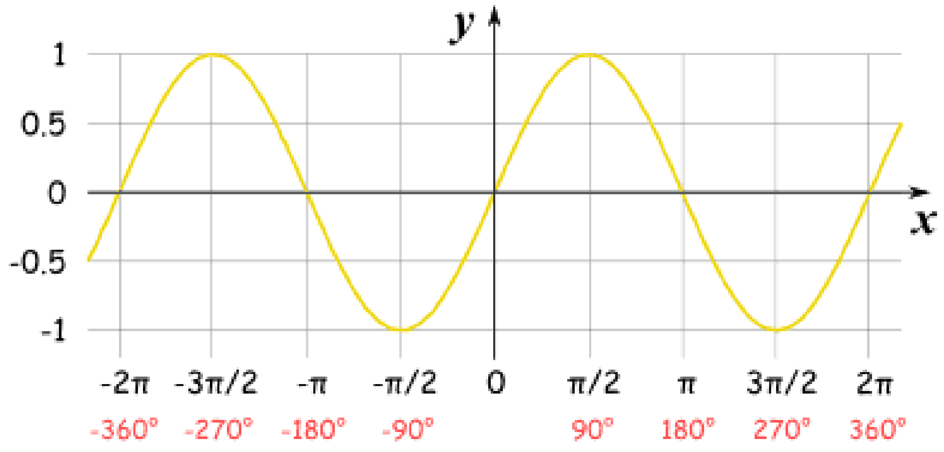
In a number of real life situations, the marginal (additional) benefit of our efforts (or inputs) starts declining…

- Years of use and minerals extracted from a coal mine
- Fertilizer used and growth of a plant
- Continuous hours of study and intensity of retention



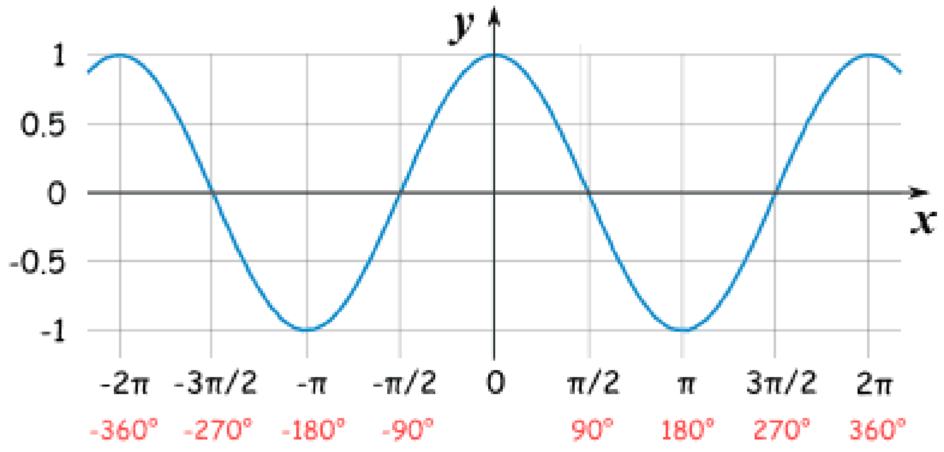
Sine Function

- **Sine** of θ = Opposite / Hypotenuse. F(x) = sine(x)
- Drawing the graph with *angle* being x and *sine* (θ) being y creates a graph of **wave form**
- This wave pattern occurs often in nature, including ocean waves, sound waves, and light waves



Cosine Function

- General form of the **Cosine** function: F(x) = cosine(x)
- Graph of Cosine function is similar to sine function and has many similar applications as the sine function



Operations on Function

Operation	Definition
Addition :	$(f + g)(x) = f(x) + g(x)$
Subtraction:	$(f - g)(x) = f(x) - g(x)$
Multiplication:	$(f \, . \, g)(x) = f(x).g(x)$
Division:	$(f/g)(x) = f(x)/g(x)$
Composition:	$(g \circ f)(x) = g(f(x))$

Addition of Functions

- Let us take two functions

  f(x) = 4x + 2 and **g(x)** = 3x + 1

  The sum of these functions is:

  $f(x) + g(x) = 7x + 3$

- Example:
- Let's say x = 6, then the combined effect of 2 functions will give a result of:

  $f(x) = 7(6) + 3 = 45$

Subtraction of Functions

- Let us take two functions

  $\mathbf{f(x)} = 4x + 2$ and $\mathbf{g(x)} = 3x + 1$

  The difference of these functions is:

  $f(x) – g(x) = (4x + 2) – (3x + 1)$

  $= 4x + 2 – 3x – 1 = x - 1$

  $f(x) – g(x) = x – 1$

- Example:
- Let's say x = 6, then the effect of subtraction will give a result of:

  $f(x) = 6 – 1 = 5$

Multiplication of Functions

- To illustrate the point, let us take the same 2 functions

  f(x) = 4x + 2 and **g(x)** = 3x + 1

The product of these functions is:

$$f(x).g(x) = 12x^2 + 4x + 6x + 2 = 12x^2 + 10x + 2$$

$$f(x).g(x) = 12x^2 + 10x + 2$$

- Example:
- Let's say x = 2, then the effect of multiplication will give a result of:

$$12(4) + 10(2) + 2 = 60$$

Division of a Functions

- Let us have two functions:

  f(x) = $x^4 + 4x^3 + x - 10$ and **g(x)** = $x^2 + 3x - 5$

- Here is the long division: f(x) / g(x)

$$
\begin{array}{r|l l}
 & \qquad\qquad 1x^2 + 1x + 2 & \\
\cline{2-2}
1x^2 + 3x - 5 & 1x^4 + 4x^3 + 0x^2 + 1x - 10 & \\
 & 1x^4 + 3x^3 - 5x^2 & \text{subtract} \\
\cline{2-2}
 & 0 \quad 1x^3 + 5x^2 + 1x - 10 & \\
 & \quad\;\; 1x^3 + 3x^2 - 5x & \text{subtract} \\
\cline{2-2}
 & \qquad 0 \quad 2x^2 + 6x - 10 & \\
 & \qquad\quad\;\; 2x^2 + 6x - 10 & \text{subtract} \\
\cline{2-2}
 & \qquad\qquad\qquad\qquad 0 &
\end{array}
$$

Composition of Functions

- $(g \circ f)(x) = g(f(x))$
- This is a two stage function where f(x) becomes input for g(x)
- Let us take an example of composition of the following 2 simple functions:

  f(x) $= x + 2$ and **g(x)** $= x^2 + 10$
- Example: find g(f(x)) of the functions above

  $g(x) = x^2 + 10$ [*a*]

  In [*a*] above replace 'x' by x + 2

  $g(x) = (x+2)^2 + 10 = (x^2 + 4x + 4) + 10$

  $g(f(x)) = x^2 + 4x + 14$