Stat 6021: Homework Set 8

1. In this question, you will revisit the `swiss` data set that you worked on in Homeworks 4 and 5. The data set contains information regarding a standardized fertility measure and socio-economic indictors for each of the 47 French-speaking provinces of Switzerland around the year 1888. In Homework 5, you found that the model with just three predictors: *Education*, *Catholic*, and *Infant Mortality* was preferred to a model with all the predictors. Fit the model with the three predictors, and answer the following questions.

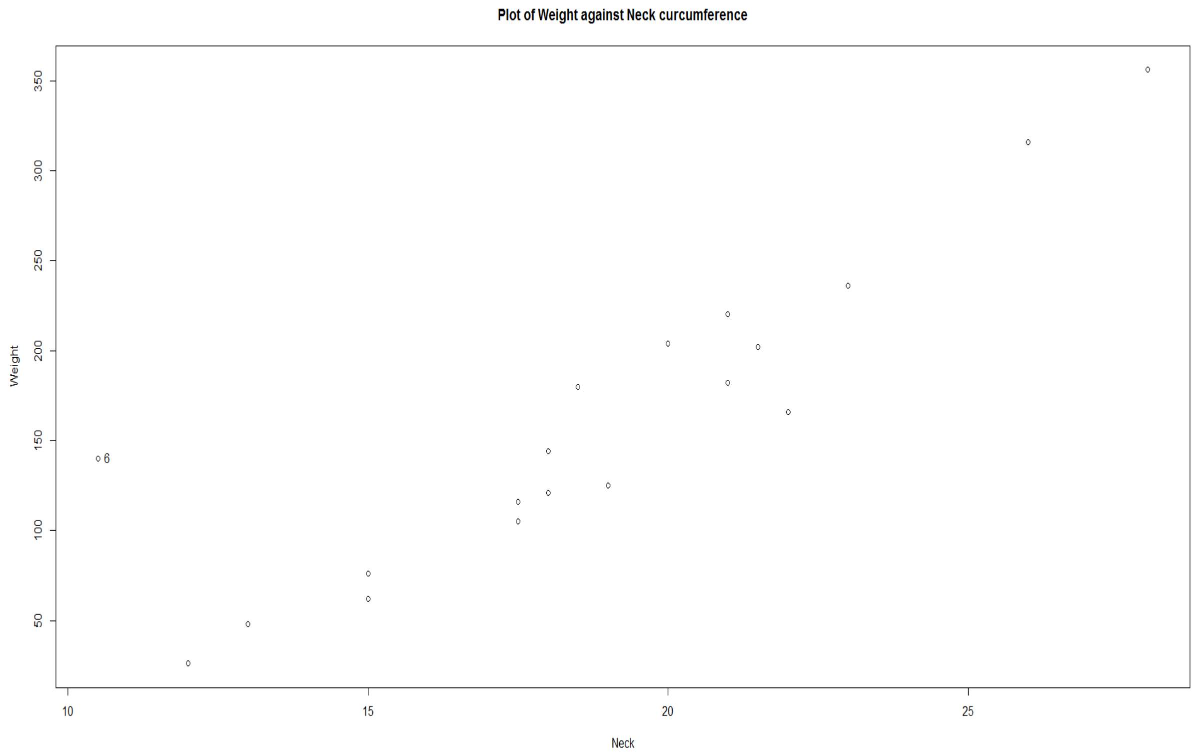
   (a) Are there any observations that are outlying in the response variable? Be sure to show your work and explain how you arrived at your answer.

   (b) Are there any observations that have high leverage? Be sure to show your work and explain how you arrived at your answer.

   (c) Are there any influential observations based on DFFITs and Cook's Distance?

   (d) Briefly describe the difference in what DFFITS and Cook's distance are measuring.

2. (No R Required) Data from $n = 19$ bears of varying ages are used to develop an equation for estimating *Weight* from *Neck* circumference. From a visual inspection of the scatterplot, it appears observation 6 may be an outlier.



The output below comes from fitting the linear regression model on the data.

```
##with all 19 bears
Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  -158.78      40.46  -3.924  0.00109 **
Neck           16.95       2.10   8.071 3.24e-07 ***

Residual standard error: 40.13 on 17 degrees of freedom
Multiple R-squared:  0.793,     Adjusted R-squared:  0.7809
F-statistic: 65.14 on 1 and 17 DF,  p-value: 3.235e-07
```

The output below comes from fitting the linear regression model on the data, with the outlier removed.

```
##with outlier removed, so 18 bears
Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  -234.60      25.93  -9.049 1.08e-07 ***
Neck           20.54       1.32  15.562 4.39e-11 ***
```

```
Residual standard error: 22.6 on 16 degrees of freedom
Multiple R-squared:  0.938,     Adjusted R-squared:  0.9342
F-statistic: 242.2 on 1 and 16 DF,  p-value: 4.394e-11
```

The output below displays the values of the predictor and response for the 6th observation.

```
> data[6,]
   Neck Weight
6  10.5    140
```

Some additional information from R, regarding ordinary residuals, e_i, and leverages, h_{ii} shown below, from the full data.

```
> result$residuals ##residuals
         1          2          3          4          5          6          7
-25.276933 -48.066801  22.880666  23.828133  -2.276933 120.829070 -32.803200
         8          9         10         11         12         13         14
-18.592131 -38.224400  25.249333 -21.803200 -15.119334  40.248397  34.143331
        15         16         17         18         19
 -3.593068 -33.434532   4.985732 -19.434532 -13.539598
```

```
> tmp$hat ##leverages
         1          2          3          4          5          6          7
0.05422642 0.08132161 0.06633278 0.05682064 0.05422642 0.23960510 0.05700079
         8          9         10         11         12         13         14
0.17788427 0.05278518 0.05282121 0.05700079 0.06633278 0.28626504 0.19604381
        15         16         17         18         19
0.07314261 0.09141025 0.10178713 0.09141025 0.14358291
```

(a) Calculate the externally studentized residual, t_i, for observation 6. Will this be considered outlying in the response?

(b) What is the leverage for observation 6? Based on the criterion that leverages greater than $\frac{2p}{n}$ are considered outlying in the predictor(s), is this observation high leverage?

(c) Calculate the DFFITS for observation 6. Briefly describe the role of leverages in DFFITS.

(d) Calculate Cook's distance for observation 6.

3. (No R Required) Cook's distance has the equivalent formulae

$$D_i = \frac{\left(\hat{\boldsymbol{\beta}} - \hat{\boldsymbol{\beta}}_{(i)}\right)' (\boldsymbol{X'X}) \left(\hat{\boldsymbol{\beta}} - \hat{\boldsymbol{\beta}}_{(i)}\right)}{p\text{MSres}} \tag{1}$$

$$= \frac{r_i^2}{p} \frac{h_{ii}}{1 - h_{ii}}. \tag{2}$$

where r_i denotes studentized residuals. Show that (1) and (2) are equivalent. You may use the following without proof:

$$\hat{\boldsymbol{\beta}} - \hat{\boldsymbol{\beta}}_{(i)} = (1 - h_{ii})^{-1} (\boldsymbol{X'X})^{-1} \boldsymbol{X_i} e_i. \tag{3}$$

4. **This question is optional** (No R Required) Recall that leverage, h_{ii}, is defined by

$$h_{ii} = \boldsymbol{X_i}' (\boldsymbol{X'X})^{-1} \boldsymbol{X_i},$$

where

$$\boldsymbol{X} = \begin{bmatrix} 1 & x_{1,1} & \cdots & x_{1,k} \\ 1 & x_{2,1} & \cdots & x_{2,k} \\ \vdots & & & \\ 1 & x_{n,1} & \cdots & x_{n,k} \end{bmatrix}$$

and

$$\boldsymbol{X_i} = \begin{bmatrix} 1 \\ x_{i,1} \\ \vdots \\ x_{i,k} \end{bmatrix}.$$

$\boldsymbol{X}$ is called the design matrix, containing the values of the k predictors for all observations, with 1s appended in the first column for the intercept, and $\boldsymbol{X_i}$ is a column vector that contains the values of the k predictors for observation i, with 1 appended in the first entry. It turns out that the sum of the leverages is

$$\sum_{i=1}^{n} h_{ii} = p, \tag{4}$$

where p denotes the number of parameters in the regression model.

(a) Show that (4) is true in simple linear regression, i.e. when $p = 2$. In case you forgot, the inverse of a 2×2 matrix is

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Hint: In your intermediate step, you will need to show that

$$(\boldsymbol{X'X})^{-1} = \begin{bmatrix} \frac{1}{n} + \frac{\bar{x}^2}{\sum_{i=1}^n (x_i - \bar{x})^2} & \frac{-\bar{x}}{\sum_{i=1}^n (x_i - \bar{x})^2} \\ \frac{-\bar{x}}{\sum_{i=1}^n (x_i - \bar{x})^2} & \frac{1}{\sum_{i=1}^n (x_i - \bar{x})^2} \end{bmatrix}$$

(b) Show that

$$\sum_{i=1}^{n} \sigma^2\{\hat{y}_i\} = p\sigma^2. \tag{5}$$

In other words, that the sum of the variances of $\hat{y}_i$ is $p\sigma^2$. Hint: how are $\hat{y}_i$ and y_i related? You may also use (4).