3 ☐ DIFFERENTIATION RULES

3.1 Derivatives of Polynomials and Exponential Functions

1. (a) e is the number such that $\lim_{h\to 0}\dfrac{e^h-1}{h}=1$.

(b)

x	$\dfrac{2.7^x-1}{x}$
-0.001	0.9928
-0.0001	0.9932
0.001	0.9937
0.0001	0.9933

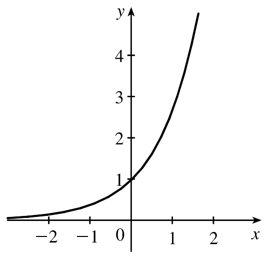
x	$\dfrac{2.8^x-1}{x}$
-0.001	1.0291
-0.0001	1.0296
0.001	1.0301
0.0001	1.0297

From the tables (to two decimal places),

$\lim_{h\to 0}\dfrac{2.7^h-1}{h}=0.99$ and $\lim_{h\to 0}\dfrac{2.8^h-1}{h}=1.03$.

Since $0.99<1<1.03$, $2.7<e<2.8$.

2. (a)



The function value at $x=0$ is 1 and the slope at $x=0$ is 1.

(b) $f(x)=e^x$ is an exponential function and $g(x)=x^e$ is a power function. $\dfrac{d}{dx}\left(e^x\right)=e^x$ and $\dfrac{d}{dx}\left(x^e\right)=ex^{e-1}$.

(c) $f(x)=e^x$ grows more rapidly than $g(x)=x^e$ when x is large.

3. $f(x)=2^{40}$ is a constant function, so its derivative is 0, that is, $f'(x)=0$.

4. $f(x)=e^5$ is a constant function, so its derivative is 0, that is, $f'(x)=0$.

5. $f(x)=5.2x+2.3 \quad\Rightarrow\quad f'(x)=5.2(1)+0=5.2$

6. $g(x)=\frac{7}{4}x^2-3x+12 \quad\Rightarrow\quad g'(x)=\frac{7}{4}(2x)-3(1)+0=\frac{7}{2}x-3$

7. $f(t)=2t^3-3t^2-4t \quad\Rightarrow\quad f'(t)=2(3t^2)-3(2t)-4(1)=6t^2-6t-4$

8. $f(t)=1.4t^5-2.5t^2+6.7 \quad\Rightarrow\quad f'(t)=1.4(5t^4)-2.5(2t)+0=7t^4-5t$

9. $g(x)=x^2(1-2x)=x^2-2x^3 \quad\Rightarrow\quad g'(x)=2x-2(3x^2)=2x-6x^2$

10. $H(u)=(3u-1)(u+2)=3u^2+5u-2 \quad\Rightarrow\quad H'(u)=3(2u)+5(1)-0=6u+5$

11. $g(t)=2t^{-3/4} \quad\Rightarrow\quad g'(t)=2\left(-\frac{3}{4}t^{-7/4}\right)=-\frac{3}{2}t^{-7/4}$

12. $B(y)=cy^{-6} \quad\Rightarrow\quad B'(y)=c(-6y^{-7})=-6cy^{-7}$

13. $F(r)=\dfrac{5}{r^3}=5r^{-3} \quad\Rightarrow\quad F'(r)=5(-3r^{-4})=-15r^{-4}=-\dfrac{15}{r^4}$

14. $y=x^{5/3}-x^{2/3} \quad\Rightarrow\quad y'=\frac{5}{3}x^{2/3}-\frac{2}{3}x^{-1/3}$

15. $R(a) = (3a+1)^2 = 9a^2 + 6a + 1 \quad \Rightarrow \quad R'(a) = 9(2a) + 6(1) + 0 = 18a + 6$

16. $h(t) = \sqrt[4]{t} - 4e^t = t^{1/4} - 4e^t \quad \Rightarrow \quad h'(t) = \frac{1}{4}t^{-3/4} - 4(e^t) = \frac{1}{4}t^{-3/4} - 4e^t$

17. $S(p) = \sqrt{p} - p = p^{1/2} - p \quad \Rightarrow \quad S'(p) = \frac{1}{2}p^{-1/2} - 1$ or $\dfrac{1}{2\sqrt{p}} - 1$

18. $y = \sqrt[3]{x}\,(2+x) = 2x^{1/3} + x^{4/3} \quad \Rightarrow \quad y' = 2\left(\frac{1}{3}x^{-2/3}\right) + \frac{4}{3}x^{1/3} = \frac{2}{3}x^{-2/3} + \frac{4}{3}x^{1/3}$ or $\dfrac{2}{3\sqrt[3]{x^2}} + \dfrac{4}{3}\sqrt[3]{x}$

19. $y = 3e^x + \dfrac{4}{\sqrt[3]{x}} = 3e^x + 4x^{-1/3} \quad \Rightarrow \quad y' = 3(e^x) + 4(-\frac{1}{3})x^{-4/3} = 3e^x - \frac{4}{3}x^{-4/3}$

20. $S(R) = 4\pi R^2 \quad \Rightarrow \quad S'(R) = 4\pi(2R) = 8\pi R$

21. $h(u) = Au^3 + Bu^2 + Cu \quad \Rightarrow \quad h'(u) = A(3u^2) + B(2u) + C(1) = 3Au^2 + 2Bu + C$

22. $y = \dfrac{\sqrt{x} + x}{x^2} = \dfrac{\sqrt{x}}{x^2} + \dfrac{x}{x^2} = x^{1/2-2} + x^{1-2} = x^{-3/2} + x^{-1} \quad \Rightarrow \quad y' = -\frac{3}{2}x^{-5/2} + (-1x^{-2}) = -\frac{3}{2}x^{-5/2} - x^{-2}$

23. $y = \dfrac{x^2 + 4x + 3}{\sqrt{x}} = x^{3/2} + 4x^{1/2} + 3x^{-1/2} \quad \Rightarrow$

$y' = \frac{3}{2}x^{1/2} + 4\left(\frac{1}{2}\right)x^{-1/2} + 3\left(-\frac{1}{2}\right)x^{-3/2} = \frac{3}{2}\sqrt{x} + \dfrac{2}{\sqrt{x}} - \dfrac{3}{2x\sqrt{x}}$ $\left[\text{note that } x^{3/2} = x^{2/2} \cdot x^{1/2} = x\sqrt{x}\right]$

The last expression can be written as $\dfrac{3x^2}{2x\sqrt{x}} + \dfrac{4x}{2x\sqrt{x}} - \dfrac{3}{2x\sqrt{x}} = \dfrac{3x^2 + 4x - 3}{2x\sqrt{x}}$.

24. $G(t) = \sqrt{5t} + \dfrac{\sqrt{7}}{t} = \sqrt{5}\,t^{1/2} + \sqrt{7}\,t^{-1} \quad \Rightarrow \quad G'(t) = \sqrt{5}\left(\frac{1}{2}t^{-1/2}\right) + \sqrt{7}\left(-1t^{-2}\right) = \dfrac{\sqrt{5}}{2\sqrt{t}} - \dfrac{\sqrt{7}}{t^2}$

25. $j(x) = x^{2.4} + e^{2.4} \quad \Rightarrow \quad j'(x) = 2.4x^{1.4} + 0 = 2.4x^{1.4}$

26. $k(r) = e^r + r^e \quad \Rightarrow \quad k'(r) = e^r + er^{e-1}$

27. $G(q) = \left(1 + q^{-1}\right)^2 = 1 + 2q^{-1} + q^{-2} \quad \Rightarrow \quad G'(q) = 0 + 2(-1q^{-2}) + (-2q^{-3}) = -2q^{-2} - 2q^{-3}$

28. $F(z) = \dfrac{A + Bz + Cz^2}{z^2} = \dfrac{A}{z^2} + \dfrac{Bz}{z^2} + \dfrac{Cz^2}{z^2} = Az^{-2} + Bz^{-1} + C \quad \Rightarrow$

$F'(z) = A(-2z^{-3}) + B(-1z^{-2}) + 0 = -2Az^{-3} - Bz^{-2} = -\dfrac{2A}{z^3} - \dfrac{B}{z^2}$ or $-\dfrac{2A + Bz}{z^3}$

29. $f(v) = \dfrac{\sqrt[3]{v} - 2ve^v}{v} = \dfrac{\sqrt[3]{v}}{v} - \dfrac{2ve^v}{v} = v^{-2/3} - 2e^v \quad \Rightarrow \quad f'(v) = -\frac{2}{3}v^{-5/3} - 2e^v$

30. $D(t) = \dfrac{1 + 16t^2}{(4t)^3} = \dfrac{1 + 16t^2}{64t^3} = \frac{1}{64}t^{-3} + \frac{1}{4}t^{-1} \quad \Rightarrow$

$D'(t) = \frac{1}{64}(-3t^{-4}) + \frac{1}{4}(-1t^{-2}) = -\frac{3}{64}t^{-4} - \frac{1}{4}t^{-2}$ or $-\dfrac{3}{64t^4} - \dfrac{1}{4t^2}$

31. $z = \dfrac{A}{y^{10}} + Be^y = Ay^{-10} + Be^y \quad \Rightarrow \quad z' = -10Ay^{-11} + Be^y = -\dfrac{10A}{y^{11}} + Be^y$

32. $y = e^{x+1} + 1 = e^x e^1 + 1 = e \cdot e^x + 1 \quad \Rightarrow \quad y' = e \cdot e^x = e^{x+1}$

33. $y = 2x^3 - x^2 + 2 \quad \Rightarrow \quad y' = 6x^2 - 2x$. At $(1, 3)$, $y' = 6(1)^2 - 2(1) = 4$ and an equation of the tangent line is $y - 3 = 4(x - 1)$ or $y = 4x - 1$.

34. $y = 2e^x + x \quad \Rightarrow \quad y' = 2e^x + 1$. At $(0, 2)$, $y' = 2e^0 + 1 = 3$ and an equation of the tangent line is $y - 2 = 3(x - 0)$ or $y = 3x + 2$.

35. $y = x + \dfrac{2}{x} = x + 2x^{-1} \quad \Rightarrow \quad y' = 1 - 2x^{-2}$. At $(2, 3)$, $y' = 1 - 2(2)^{-2} = \frac{1}{2}$ and an equation of the tangent line is $y - 3 = \frac{1}{2}(x - 2)$ or $y = \frac{1}{2}x + 2$.

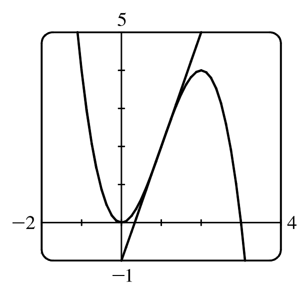
36. $y = \sqrt[4]{x} - x = x^{1/4} - x \quad \Rightarrow \quad y' = \frac{1}{4}x^{-3/4} - 1 = \dfrac{1}{4\sqrt[4]{x^3}} - 1$. At $(1, 0)$, $y' = \frac{1}{4} - 1 = -\frac{3}{4}$ and an equation of the tangent line is $y - 0 = -\frac{3}{4}(x - 1)$ or $y = -\frac{3}{4}x + \frac{3}{4}$.

37. $y = x^4 + 2e^x \quad \Rightarrow \quad y' = 4x^3 + 2e^x$. At $(0, 2)$, $y' = 2$ and an equation of the tangent line is $y - 2 = 2(x - 0)$ or $y = 2x + 2$. The slope of the normal line is $-\frac{1}{2}$ (the negative reciprocal of 2) and an equation of the normal line is $y - 2 = -\frac{1}{2}(x - 0)$ or $y = -\frac{1}{2}x + 2$.

38. $y^2 = x^3 \quad \Rightarrow \quad y = x^{3/2}$ [since x and y are positive at $(1, 1)$] $\Rightarrow \quad y' = \frac{3}{2}x^{1/2}$. At $(1, 1)$, $y' = \frac{3}{2}$ and an equation of the tangent line is $y - 1 = \frac{3}{2}(x - 1)$ or $y = \frac{3}{2}x - \frac{1}{2}$. The slope of the normal line is $-\frac{2}{3}$ $\left(\text{the negative reciprocal of } \frac{3}{2}\right)$ and an equation of the normal line is $y - 1 = -\frac{2}{3}(x - 1)$ or $y = -\frac{2}{3}x + \frac{5}{3}$.

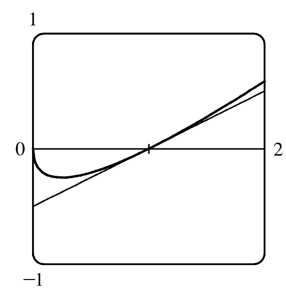
39. $y = 3x^2 - x^3 \quad \Rightarrow \quad y' = 6x - 3x^2$.

At $(1, 2)$, $y' = 6 - 3 = 3$, so an equation of the tangent line is $y - 2 = 3(x - 1)$ or $y = 3x - 1$.



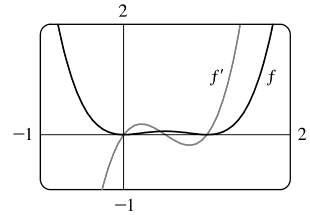
40. $y = x - \sqrt{x} \quad \Rightarrow \quad y' = 1 - \frac{1}{2}x^{-1/2} = 1 - \dfrac{1}{2\sqrt{x}}$.

At $(1, 0)$, $y' = \frac{1}{2}$, so an equation of the tangent line is $y - 0 = \frac{1}{2}(x - 1)$ or $y = \frac{1}{2}x - \frac{1}{2}$.



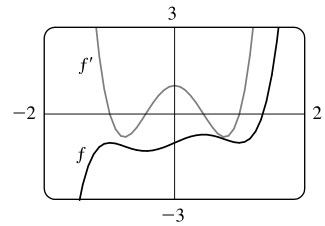
41. $f(x) = x^4 - 2x^3 + x^2 \quad \Rightarrow \quad f'(x) = 4x^3 - 6x^2 + 2x$

Note that $f'(x) = 0$ when f has a horizontal tangent, f' is positive when f is increasing, and f' is negative when f is decreasing.

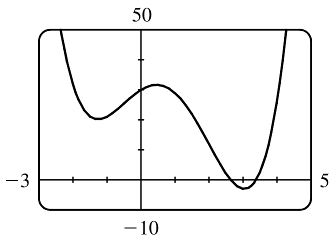


42. $f(x) = x^5 - 2x^3 + x - 1 \quad\Rightarrow\quad f'(x) = 5x^4 - 6x^2 + 1$

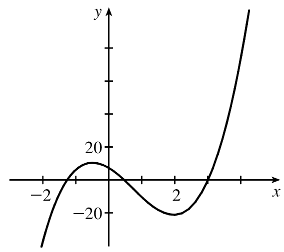
Note that $f'(x) = 0$ when f has a horizontal tangent, f' is positive when f is increasing, and f' is negative when f is decreasing.



43. (a)

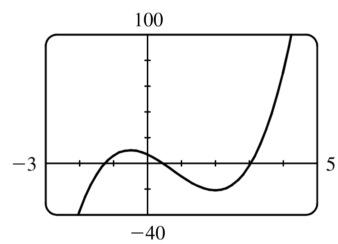


(b) From the graph in part (a), it appears that f' is zero at $x_1 \approx -1.25$, $x_2 \approx 0.5$, and $x_3 \approx 3$. The slopes are negative (so f' is negative) on $(-\infty, x_1)$ and (x_2, x_3). The slopes are positive (so f' is positive) on (x_1, x_2) and (x_3, ∞).

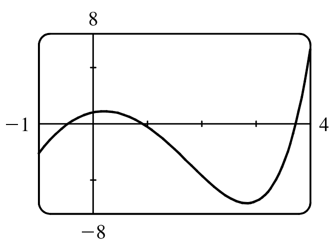


(c) $f(x) = x^4 - 3x^3 - 6x^2 + 7x + 30 \quad\Rightarrow$

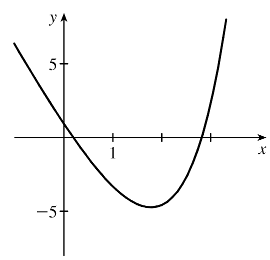
$f'(x) = 4x^3 - 9x^2 - 12x + 7$



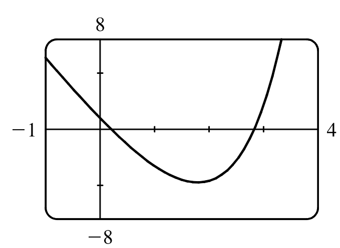
44. (a)



(b) From the graph in part (a), it appears that f' is zero at $x_1 \approx 0.2$ and $x_2 \approx 2.8$. The slopes are positive (so f' is positive) on $(-\infty, x_1)$ and (x_2, ∞). The slopes are negative (so f' is negative) on (x_1, x_2).



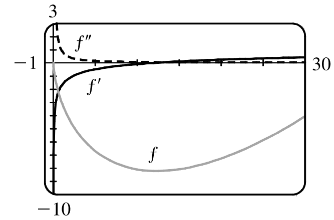
(c) $g(x) = e^x - 3x^2 \quad\Rightarrow\quad g'(x) = e^x - 6x$



45. $f(x) = 0.001x^5 - 0.02x^3 \quad\Rightarrow\quad f'(x) = 0.005x^4 - 0.06x^2 \quad\Rightarrow\quad f''(x) = 0.02x^3 - 0.12x$

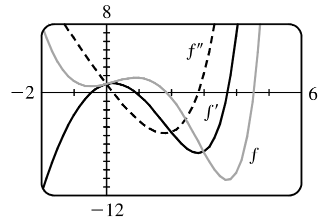
46. $G(r) = \sqrt{r} + \sqrt[3]{r} \quad\Rightarrow\quad G'(r) = \frac{1}{2}r^{-1/2} + \frac{1}{3}r^{-2/3} \quad\Rightarrow\quad G''(r) = -\frac{1}{4}r^{-3/2} - \frac{2}{9}r^{-5/3}$

47. $f(x) = 2x - 5x^{3/4} \quad\Rightarrow\quad f'(x) = 2 - \frac{15}{4}x^{-1/4} \quad\Rightarrow\quad f''(x) = \frac{15}{16}x^{-5/4}$



Note that f' is negative when f is decreasing and positive when f is increasing. f'' is always positive since f' is always increasing.

48. $f(x) = e^x - x^3 \quad\Rightarrow\quad f'(x) = e^x - 3x^2 \quad\Rightarrow\quad f''(x) = e^x - 6x$



Note that $f'(x) = 0$ when f has a horizontal tangent and that $f''(x) = 0$ when f' has a horizontal tangent.

49. (a) $s = t^3 - 3t \quad\Rightarrow\quad v(t) = s'(t) = 3t^2 - 3 \quad\Rightarrow\quad a(t) = v'(t) = 6t$

(b) $a(2) = 6(2) = 12\text{ m/s}^2$

(c) $v(t) = 3t^2 - 3 = 0$ when $t^2 = 1$, that is, $t = 1\ [t \geq 0]$ and $a(1) = 6\text{ m/s}^2$.

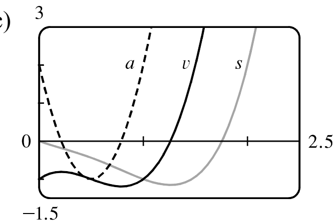
50. (a) $s = t^4 - 2t^3 + t^2 - t \quad\Rightarrow$

$v(t) = s'(t) = 4t^3 - 6t^2 + 2t - 1 \quad\Rightarrow$

$a(t) = v'(t) = 12t^2 - 12t + 2$

(b) $a(1) = 12(1)^2 - 12(1) + 2 = 2\text{ m/ s}^2$

(c)



51. $L = 0.0155A^3 - 0.372A^2 + 3.95A + 1.21 \quad\Rightarrow\quad \dfrac{dL}{dA} = 0.0465A^2 - 0.744A + 3.95$, so

$\left.\dfrac{dL}{dA}\right|_{A=12} = 0.0465(12)^2 - 0.744(12) + 3.95 = 1.718$. The derivative is the instantaneous rate of change of the length of an Alaskan rockfish with respect to its age when its age is 12 years.

52. $S(A) = 0.882A^{0.842} \quad\Rightarrow\quad S'(A) = 0.882(0.842A^{-0.158}) = 0.742644A^{-0.158}$, so

$S'(100) = 0.742644(100)^{-0.158} \approx 0.36$. The derivative is the instantaneous rate of change of the number of tree species with respect to area. Its units are number of species per square meter.

53. (a) $P = \dfrac{k}{V}$ and $P = 50$ when $V = 0.106$, so $k = PV = 50(0.106) = 5.3$. Thus, $P = \dfrac{5.3}{V}$ and $V = \dfrac{5.3}{P}$.

(b) $V = 5.3P^{-1} \quad\Rightarrow\quad \dfrac{dV}{dP} = 5.3(-1P^{-2}) = -\dfrac{5.3}{P^2}$. When $P = 50$, $\dfrac{dV}{dP} = -\dfrac{5.3}{50^2} = -0.00212$. The derivative is the instantaneous rate of change of the volume with respect to the pressure at $25\,^\circ\text{C}$. Its units are m^3/kPa.

54. (a) $L = aP^2 + bP + c$, where $a \approx -0.275428$, $b \approx 19.74853$, and $c \approx -273.55234$.

(b) $\dfrac{dL}{dP} = 2aP + b$. When $P = 30$, $\dfrac{dL}{dP} \approx 3.2$, and when $P = 40$, $\dfrac{dL}{dP} \approx -2.3$. The derivative is the instantaneous rate of change of tire life with respect to pressure. Its units are (thousands of miles)$/(\mathrm{lb/in}^2)$. When $\dfrac{dL}{dP}$ is positive, tire life is increasing, and when $\dfrac{dL}{dP} < 0$, tire life is decreasing.

55. The curve $y = 2x^3 + 3x^2 - 12x + 1$ has a horizontal tangent when $y' = 6x^2 + 6x - 12 = 0 \iff 6(x^2 + x - 2) = 0 \iff 6(x + 2)(x - 1) = 0 \iff x = -2$ or $x = 1$. The points on the curve are $(-2, 21)$ and $(1, -6)$.

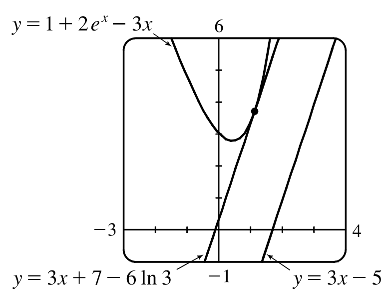
56. $f(x) = e^x - 2x \Rightarrow f'(x) = e^x - 2$. $f'(x) = 0 \Rightarrow e^x = 2 \Rightarrow x = \ln 2$, so f has a horizontal tangent when $x = \ln 2$.

57. $y = 2e^x + 3x + 5x^3 \Rightarrow y' = 2e^x + 3 + 15x^2$. Since $2e^x > 0$ and $15x^2 \geq 0$, we must have $y' > 0 + 3 + 0 = 3$, so no tangent line can have slope 2.

58. $y = x^4 + 1 \Rightarrow y' = 4x^3$. The slope of the line $32x - y = 15$ (or $y = 32x - 15$) is 32, so the slope of any line parallel to it is also 32. Thus, $y' = 32 \iff 4x^3 = 32 \iff x^3 = 8 \iff x = 2$, which is the x-coordinate of the point on the curve at which the slope is 32. The y-coordinate is $2^4 + 1 = 17$, so an equation of the tangent line is $y - 17 = 32(x - 2)$ or $y = 32x - 47$.

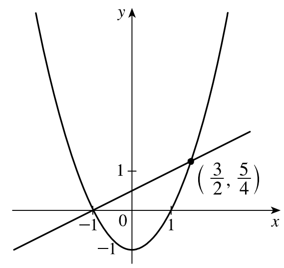
59. The slope of the line $3x - y = 15$ (or $y = 3x - 15$) is 3, so the slope of both tangent lines to the curve is 3.
$y = x^3 - 3x^2 + 3x - 3 \Rightarrow y' = 3x^2 - 6x + 3 = 3(x^2 - 2x + 1) = 3(x - 1)^2$. Thus, $3(x - 1)^2 = 3 \Rightarrow (x - 1)^2 = 1 \Rightarrow x - 1 = \pm 1 \Rightarrow x = 0$ or 2, which are the x-coordinates at which the tangent lines have slope 3. The points on the curve are $(0, -3)$ and $(2, -1)$, so the tangent line equations are $y - (-3) = 3(x - 0)$ or $y = 3x - 3$ and $y - (-1) = 3(x - 2)$ or $y = 3x - 7$.

60. The slope of $y = 1 + 2e^x - 3x$ is given by $m = y' = 2e^x - 3$.
The slope of $3x - y = 5 \iff y = 3x - 5$ is 3.
$m = 3 \Rightarrow 2e^x - 3 = 3 \Rightarrow e^x = 3 \Rightarrow x = \ln 3$.
This occurs at the point $(\ln 3, 7 - 3\ln 3) \approx (1.1, 3.7)$.

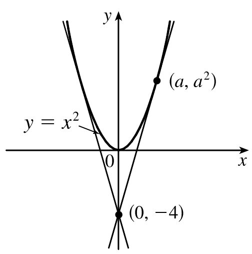


61. The slope of $y = \sqrt{x}$ is given by $y = \frac{1}{2}x^{-1/2} = \dfrac{1}{2\sqrt{x}}$. The slope of $2x + y = 1$ (or $y = -2x + 1$) is -2, so the desired normal line must have slope -2, and hence, the tangent line to the curve must have slope $\frac{1}{2}$. This occurs if $\dfrac{1}{2\sqrt{x}} = \dfrac{1}{2} \Rightarrow \sqrt{x} = 1 \Rightarrow x = 1$. When $x = 1$, $y = \sqrt{1} = 1$, and an equation of the normal line is $y - 1 = -2(x - 1)$ or $y = -2x + 3$.

62. $y = f(x) = x^2 - 1 \quad \Rightarrow \quad f'(x) = 2x$. So $f'(-1) = -2$, and the slope of the normal line is $\frac{1}{2}$. The equation of the normal line at $(-1, 0)$ is $y - 0 = \frac{1}{2}[x - (-1)]$ or $y = \frac{1}{2}x + \frac{1}{2}$. Substituting this into the equation of the parabola, we obtain $\frac{1}{2}x + \frac{1}{2} = x^2 - 1 \quad \Leftrightarrow \quad x + 1 = 2x^2 - 2 \quad \Leftrightarrow$ $2x^2 - x - 3 = 0 \quad \Leftrightarrow \quad (2x - 3)(x + 1) = 0 \quad \Leftrightarrow \quad x = \frac{3}{2}$ or -1. Substituting $\frac{3}{2}$ into the equation of the normal line gives us $y = \frac{5}{4}$. Thus, the second point of intersection is $\left(\frac{3}{2}, \frac{5}{4}\right)$, as shown in the sketch.



63.



Let (a, a^2) be a point on the parabola at which the tangent line passes through the point $(0, -4)$. The tangent line has slope $2a$ and equation $y - (-4) = 2a(x - 0) \quad \Leftrightarrow \quad y = 2ax - 4$. Since (a, a^2) also lies on the line, $a^2 = 2a(a) - 4$, or $a^2 = 4$. So $a = \pm 2$ and the points are $(2, 4)$ and $(-2, 4)$.

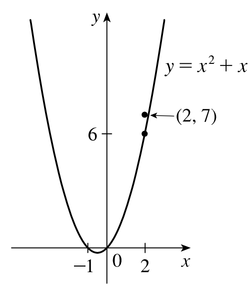
64. (a) If $y = x^2 + x$, then $y' = 2x + 1$. If the point at which a tangent meets the parabola is $(a, a^2 + a)$, then the slope of the tangent is $2a + 1$. But since it passes through $(2, -3)$, the slope must also be $\dfrac{\Delta y}{\Delta x} = \dfrac{a^2 + a + 3}{a - 2}$.

Therefore, $2a + 1 = \dfrac{a^2 + a + 3}{a - 2}$. Solving this equation for a we get $a^2 + a + 3 = 2a^2 - 3a - 2 \quad \Leftrightarrow$ $a^2 - 4a - 5 = (a - 5)(a + 1) = 0 \quad \Leftrightarrow \quad a = 5$ or -1. If $a = -1$, the point is $(-1, 0)$ and the slope is -1, so the equation is $y - 0 = (-1)(x + 1)$ or $y = -x - 1$. If $a = 5$, the point is $(5, 30)$ and the slope is 11, so the equation is $y - 30 = 11(x - 5)$ or $y = 11x - 25$.

(b) As in part (a), but using the point $(2, 7)$, we get the equation

$$2a + 1 = \frac{a^2 + a - 7}{a - 2} \quad \Rightarrow \quad 2a^2 - 3a - 2 = a^2 + a - 7 \quad \Leftrightarrow \quad a^2 - 4a + 5 = 0.$$

The last equation has no real solution (discriminant $= -16 < 0$), so there is no line through the point $(2, 7)$ that is tangent to the parabola. The diagram shows that the point $(2, 7)$ is "inside" the parabola, but tangent lines to the parabola do not pass through points inside the parabola.



65. $$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{\dfrac{1}{x + h} - \dfrac{1}{x}}{h} = \lim_{h \to 0} \frac{x - (x + h)}{hx(x + h)} = \lim_{h \to 0} \frac{-h}{hx(x + h)} = \lim_{h \to 0} \frac{-1}{x(x + h)} = -\frac{1}{x^2}$$

66. (a) $f(x) = x^n \quad \Rightarrow \quad f'(x) = nx^{n-1} \quad \Rightarrow \quad f''(x) = n(n - 1)x^{n-2} \quad \Rightarrow \quad \cdots \quad \Rightarrow$

$f^{(n)}(x) = n(n - 1)(n - 2) \cdots 2 \cdot 1x^{n-n} = n!$

(b) $f(x) = x^{-1} \quad \Rightarrow \quad f'(x) = (-1)x^{-2} \quad \Rightarrow \quad f''(x) = (-1)(-2)x^{-3} \quad \Rightarrow \quad \cdots \quad \Rightarrow$

$f^{(n)}(x) = (-1)(-2)(-3) \cdots (-n)x^{-(n+1)} = (-1)^n n! x^{-(n+1)}$ or $\dfrac{(-1)^n\, n!}{x^{n+1}}$

67. Let $P(x) = ax^2 + bx + c$. Then $P'(x) = 2ax + b$ and $P''(x) = 2a$. $P''(2) = 2 \;\Rightarrow\; 2a = 2 \;\Rightarrow\; a = 1$.

$P'(2) = 3 \;\Rightarrow\; 2(1)(2) + b = 3 \;\Rightarrow\; 4 + b = 3 \;\Rightarrow\; b = -1$.

$P(2) = 5 \;\Rightarrow\; 1(2)^2 + (-1)(2) + c = 5 \;\Rightarrow\; 2 + c = 5 \;\Rightarrow\; c = 3$. So $P(x) = x^2 - x + 3$.

68. $y = Ax^2 + Bx + C \;\Rightarrow\; y' = 2Ax + B \;\Rightarrow\; y'' = 2A$. We substitute these expressions into the equation $y'' + y' - 2y = x^2$ to get

$$\begin{aligned}(2A) + (2Ax + B) - 2(Ax^2 + Bx + C) &= x^2\\ 2A + 2Ax + B - 2Ax^2 - 2Bx - 2C &= x^2\\ (-2A)x^2 + (2A - 2B)x + (2A + B - 2C) &= (1)x^2 + (0)x + (0)\end{aligned}$$

The coefficients of x^2 on each side must be equal, so $-2A = 1 \;\Rightarrow\; A = -\frac{1}{2}$. Similarly, $2A - 2B = 0 \;\Rightarrow\; A = B = -\frac{1}{2}$ and $2A + B - 2C = 0 \;\Rightarrow\; -1 - \frac{1}{2} - 2C = 0 \;\Rightarrow\; C = -\frac{3}{4}$.

69. $y = f(x) = ax^3 + bx^2 + cx + d \;\Rightarrow\; f'(x) = 3ax^2 + 2bx + c$. The point $(-2, 6)$ is on f, so $f(-2) = 6 \;\Rightarrow\; -8a + 4b - 2c + d = 6$ **(1)**. The point $(2, 0)$ is on f, so $f(2) = 0 \;\Rightarrow\; 8a + 4b + 2c + d = 0$ **(2)**. Since there are horizontal tangents at $(-2, 6)$ and $(2, 0)$, $f'(\pm 2) = 0$. $f'(-2) = 0 \;\Rightarrow\; 12a - 4b + c = 0$ **(3)** and $f'(2) = 0 \;\Rightarrow\; 12a + 4b + c = 0$ **(4)**. Subtracting equation **(3)** from **(4)** gives $8b = 0 \;\Rightarrow\; b = 0$. Adding **(1)** and **(2)** gives $8b + 2d = 6$, so $d = 3$ since $b = 0$. From **(3)** we have $c = -12a$, so **(2)** becomes $8a + 4(0) + 2(-12a) + 3 = 0 \;\Rightarrow\; 3 = 16a \;\Rightarrow\; a = \frac{3}{16}$. Now $c = -12a = -12\left(\frac{3}{16}\right) = -\frac{9}{4}$ and the desired cubic function is $y = \frac{3}{16}x^3 - \frac{9}{4}x + 3$.

70. $y = ax^2 + bx + c \;\Rightarrow\; y'(x) = 2ax + b$. The parabola has slope 4 at $x = 1$ and slope -8 at $x = -1$, so $y'(1) = 4 \;\Rightarrow\; 2a + b = 4$ **(1)** and $y'(-1) = -8 \;\Rightarrow\; -2a + b = -8$ **(2)**. Adding **(1)** and **(2)** gives us $2b = -4 \;\Leftrightarrow\; b = -2$. From **(1)**, $2a - 2 = 4 \;\Leftrightarrow\; a = 3$. Thus, the equation of the parabola is $y = 3x^2 - 2x + c$. Since it passes through the point $(2, 15)$, we have $15 = 3(2)^2 - 2(2) + c \;\Rightarrow\; c = 7$, so the equation is $y = 3x^2 - 2x + 7$.

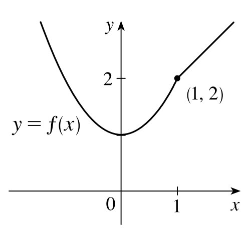
71. $f(x) = \begin{cases} x^2 + 1 & \text{if } x < 1 \\ x + 1 & \text{if } x \geq 1 \end{cases}$

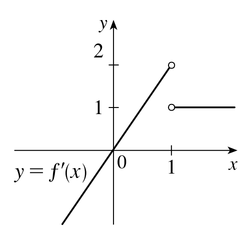
Calculate the left- and right-hand derivatives as defined in Exercise 2.8.64:

$$f'_-(1) = \lim_{h \to 0^-} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^-} \frac{[(1+h)^2 + 1] - (1+1)}{h} = \lim_{h \to 0^-} \frac{h^2 + 2h}{h} = \lim_{h \to 0^-} (h + 2) = 2 \text{ and}$$

$$f'_+(1) = \lim_{h \to 0^+} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^+} \frac{[(1+h) + 1] - (1+1)}{h} = \lim_{h \to 0^+} \frac{h}{h} = \lim_{h \to 0^+} 1 = 1.$$

Since the left and right limits are different, $\displaystyle\lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$ does not exist, that is, $f'(1)$ does not exist. Therefore, f is not differentiable at 1.





72. $g(x) = \begin{cases} 2x & \text{if } x \le 0 \\ 2x - x^2 & \text{if } 0 < x < 2 \\ 2 - x & \text{if } x \ge 2 \end{cases}$

Investigate the left- and right-hand derivatives at $x = 0$ and $x = 2$:

$g'_-(0) = \lim\limits_{h \to 0^-} \dfrac{g(0+h) - g(0)}{h} = \lim\limits_{h \to 0^-} \dfrac{2h - 2(0)}{h} = 2$ and

$g'_+(0) = \lim\limits_{h \to 0^+} \dfrac{g(0+h) - g(0)}{h} = \lim\limits_{h \to 0^+} \dfrac{(2h - h^2) - 2(0)}{h} = \lim\limits_{h \to 0^+} (2 - h) = 2$, so g is differentiable at $x = 0$.

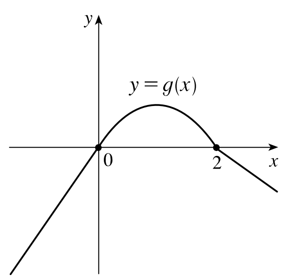
$g'_-(2) = \lim\limits_{h \to 0^-} \dfrac{g(2+h) - g(2)}{h} = \lim\limits_{h \to 0^-} \dfrac{2(2+h) - (2+h)^2 - (2-2)}{h} = \lim\limits_{h \to 0^-} \dfrac{-2h - h^2}{h} = \lim\limits_{h \to 0^-} (-2 - h) = -2$

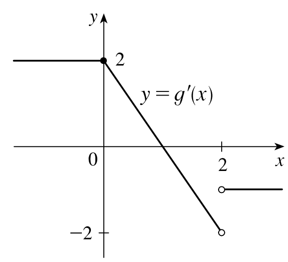
and

$g'_+(2) = \lim\limits_{h \to 0^+} \dfrac{g(2+h) - g(2)}{h} = \lim\limits_{h \to 0^+} \dfrac{[2 - (2+h)] - (2-2)}{h} = \lim\limits_{h \to 0^+} \dfrac{-h}{h} = \lim\limits_{h \to 0^+} (-1) = -1,$

so g is not differentiable at $x = 2$. Thus, a formula for g' is

$g'(x) = \begin{cases} 2 & \text{if } x \le 0 \\ 2 - 2x & \text{if } 0 < x < 2 \\ -1 & \text{if } x > 2 \end{cases}$





73. (a) Note that $x^2 - 9 < 0$ for $x^2 < 9 \quad \Leftrightarrow \quad |x| < 3 \quad \Leftrightarrow \quad -3 < x < 3$. So

$$f(x) = \begin{cases} x^2 - 9 & \text{if } x \le -3 \\ -x^2 + 9 & \text{if } -3 < x < 3 \\ x^2 - 9 & \text{if } x \ge 3 \end{cases} \quad \Rightarrow \quad f'(x) = \begin{cases} 2x & \text{if } x < -3 \\ -2x & \text{if } -3 < x < 3 \\ 2x & \text{if } x > 3 \end{cases} \quad = \begin{cases} 2x & \text{if } |x| > 3 \\ -2x & \text{if } |x| < 3 \end{cases}$$

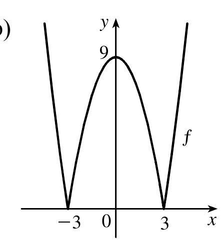
To show that $f'(3)$ does not exist we investigate $\lim\limits_{h \to 0} \dfrac{f(3+h) - f(3)}{h}$ by computing the left- and right-hand derivatives defined in Exercise 2.8.64.

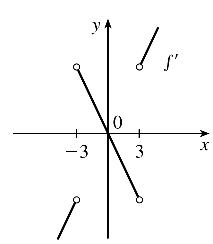
$f'_-(3) = \lim\limits_{h \to 0^-} \dfrac{f(3+h) - f(3)}{h} = \lim\limits_{h \to 0^-} \dfrac{[-(3+h)^2 + 9] - 0}{h} = \lim\limits_{h \to 0^-} (-6 - h) = -6$ and

$f'_+(3) = \lim\limits_{h \to 0^+} \dfrac{f(3+h) - f(3)}{h} = \lim\limits_{h \to 0^+} \dfrac{\left[(3+h)^2 - 9\right] - 0}{h} = \lim\limits_{h \to 0^+} \dfrac{6h + h^2}{h} = \lim\limits_{h \to 0^+} (6 + h) = 6.$

Since the left and right limits are different, $\lim\limits_{h \to 0} \dfrac{f(3+h) - f(3)}{h}$ does not exist, that is, $f'(3)$ does not exist. Similarly, $f'(-3)$ does not exist. Therefore, f is not differentiable at 3 or at -3.

(b)





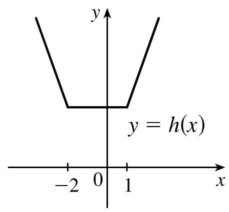
74. If $x \geq 1$, then $h(x) = |x-1| + |x+2| = x - 1 + x + 2 = 2x + 1$.

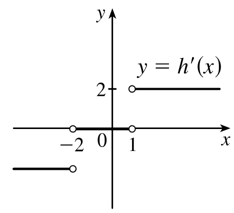
If $-2 < x < 1$, then $h(x) = -(x-1) + x + 2 = 3$.

If $x \leq -2$, then $h(x) = -(x-1) - (x+2) = -2x - 1$. Therefore,

$$h(x) = \begin{cases} -2x - 1 & \text{if } x \leq -2 \\ 3 & \text{if } -2 < x < 1 \\ 2x + 1 & \text{if } x \geq 1 \end{cases} \quad \Rightarrow \quad h'(x) = \begin{cases} -2 & \text{if } x < -2 \\ 0 & \text{if } -2 < x < 1 \\ 2 & \text{if } x > 1 \end{cases}$$

To see that $h'(1) = \lim\limits_{x \to 1} \dfrac{h(x) - h(1)}{x - 1}$ does not exist, observe that $\lim\limits_{x \to 1^-} \dfrac{h(x) - h(1)}{x - 1} = \lim\limits_{x \to 1^-} \dfrac{3 - 3}{3 - 1} = 0$ but $\lim\limits_{x \to 1^+} \dfrac{h(x) - h(1)}{x - 1} = \lim\limits_{x \to 1^+} \dfrac{2x - 2}{x - 1} = 2$. Similarly, $h'(-2)$ does not exist.





75. Substituting $x = 1$ and $y = 1$ into $y = ax^2 + bx$ gives us $a + b = 1$ **(1)**. The slope of the tangent line $y = 3x - 2$ is 3 and the slope of the tangent to the parabola at (x, y) is $y' = 2ax + b$. At $x = 1$, $y' = 3 \Rightarrow 3 = 2a + b$ **(2)**. Subtracting **(1)** from **(2)** gives us $2 = a$ and it follows that $b = -1$. The parabola has equation $y = 2x^2 - x$.

76. $y = x^4 + ax^3 + bx^2 + cx + d \Rightarrow y(0) = d$. Since the tangent line $y = 2x + 1$ is equal to 1 at $x = 0$, we must have $d = 1$. $y' = 4x^3 + 3ax^2 + 2bx + c \Rightarrow y'(0) = c$. Since the slope of the tangent line $y = 2x + 1$ at $x = 0$ is 2, we must have $c = 2$. Now $y(1) = 1 + a + b + c + d = a + b + 4$ and the tangent line $y = 2 - 3x$ at $x = 1$ has y-coordinate -1, so $a + b + 4 = -1$ or $a + b = -5$ **(1)**. Also, $y'(1) = 4 + 3a + 2b + c = 3a + 2b + 6$ and the slope of the tangent line $y = 2 - 3x$ at $x = 1$ is -3, so $3a + 2b + 6 = -3$ or $3a + 2b = -9$ **(2)**. Adding -2 times **(1)** to **(2)** gives us $a = 1$ and hence, $b = -6$. The curve has equation $y = x^4 + x^3 - 6x^2 + 2x + 1$.

77. $y = f(x) = ax^2 \Rightarrow f'(x) = 2ax$. So the slope of the tangent to the parabola at $x = 2$ is $m = 2a(2) = 4a$. The slope of the given line, $2x + y = b \Leftrightarrow y = -2x + b$, is seen to be -2, so we must have $4a = -2 \Leftrightarrow a = -\frac{1}{2}$. So when $x = 2$, the point in question has y-coordinate $-\frac{1}{2} \cdot 2^2 = -2$. Now we simply require that the given line, whose equation is $2x + y = b$, pass through the point $(2, -2)$: $2(2) + (-2) = b \Leftrightarrow b = 2$. So we must have $a = -\frac{1}{2}$ and $b = 2$.

78. The slope of the curve $y = c\sqrt{x}$ is $y' = \dfrac{c}{2\sqrt{x}}$ and the slope of the tangent line $y = \frac{3}{2}x + 6$ is $\frac{3}{2}$. These must be equal at the point of tangency $\left(a, c\sqrt{a}\right)$, so $\dfrac{c}{2\sqrt{a}} = \dfrac{3}{2} \Rightarrow c = 3\sqrt{a}$. The y-coordinates must be equal at $x = a$, so $c\sqrt{a} = \frac{3}{2}a + 6 \Rightarrow \left(3\sqrt{a}\right)\sqrt{a} = \frac{3}{2}a + 6 \Rightarrow 3a = \frac{3}{2}a + 6 \Rightarrow \frac{3}{2}a = 6 \Rightarrow a = 4$. Since $c = 3\sqrt{a}$, we have $c = 3\sqrt{4} = 6$.

79. The line $y = 2x + 3$ has slope 2. The parabola $y = cx^2 \Rightarrow y' = 2cx$ has slope $2ca$ at $x = a$. Equating slopes gives us $2ca = 2$, or $ca = 1$. Equating y-coordinates at $x = a$ gives us $ca^2 = 2a + 3 \Leftrightarrow (ca)a = 2a + 3 \Leftrightarrow 1a = 2a + 3 \Leftrightarrow a = -3$. Thus, $c = \dfrac{1}{a} = -\dfrac{1}{3}$.

80. $f(x) = ax^2 + bx + c \quad \Rightarrow \quad f'(x) = 2ax + b$. The slope of the tangent line at $x = p$ is $2ap + b$, the slope of the tangent line at $x = q$ is $2aq + b$, and the average of those slopes is $\dfrac{(2ap + b) + (2aq + b)}{2} = ap + aq + b$. The midpoint of the interval $[p, q]$ is $\dfrac{p+q}{2}$ and the slope of the tangent line at the midpoint is $2a\left(\dfrac{p+q}{2}\right) + b = a(p+q) + b$. This is equal to $ap + aq + b$, as required.

81. f is clearly differentiable for $x < 2$ and for $x > 2$. For $x < 2$, $f'(x) = 2x$, so $f'_-(2) = 4$. For $x > 2$, $f'(x) = m$, so $f'_+(2) = m$. For f to be differentiable at $x = 2$, we need $4 = f'_-(2) = f'_+(2) = m$. So $f(x) = 4x + b$. We must also have continuity at $x = 2$, so $4 = f(2) = \lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^+} (4x + b) = 8 + b$. Hence, $b = -4$.

82. (a) $xy = c \quad \Rightarrow \quad y = \dfrac{c}{x}$. Let $P = \left(a, \dfrac{c}{a}\right)$. The slope of the tangent line at $x = a$ is $y'(a) = -\dfrac{c}{a^2}$. Its equation is $y - \dfrac{c}{a} = -\dfrac{c}{a^2}(x - a)$ or $y = -\dfrac{c}{a^2}x + \dfrac{2c}{a}$, so its y-intercept is $\dfrac{2c}{a}$. Setting $y = 0$ gives $x = 2a$, so the x-intercept is $2a$. The midpoint of the line segment joining $\left(0, \dfrac{2c}{a}\right)$ and $(2a, 0)$ is $\left(a, \dfrac{c}{a}\right) = P$.

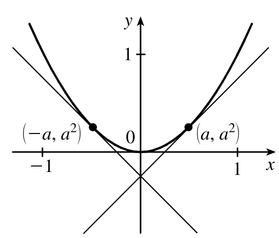
(b) We know the x- and y-intercepts of the tangent line from part (a), so the area of the triangle bounded by the axes and the tangent is $\frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}xy = \frac{1}{2}(2a)(2c/a) = 2c$, a constant.

83. *Solution 1:* Let $f(x) = x^{1000}$. Then, by the definition of a derivative, $f'(1) = \lim\limits_{x \to 1} \dfrac{f(x) - f(1)}{x - 1} = \lim\limits_{x \to 1} \dfrac{x^{1000} - 1}{x - 1}$. But this is just the limit we want to find, and we know (from the Power Rule) that $f'(x) = 1000x^{999}$, so $f'(1) = 1000(1)^{999} = 1000$. So $\lim\limits_{x \to 1} \dfrac{x^{1000} - 1}{x - 1} = 1000$.

Solution 2: Note that $(x^{1000} - 1) = (x - 1)(x^{999} + x^{998} + x^{997} + \cdots + x^2 + x + 1)$. So

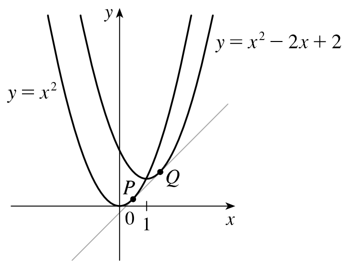
$$\begin{aligned} \lim_{x \to 1} \frac{x^{1000} - 1}{x - 1} &= \lim_{x \to 1} \frac{(x-1)(x^{999} + x^{998} + x^{997} + \cdots + x^2 + x + 1)}{x - 1} = \lim_{x \to 1} (x^{999} + x^{998} + x^{997} + \cdots + x^2 + x + 1) \\ &= \underbrace{1 + 1 + 1 + \cdots + 1 + 1 + 1}_{\text{1000 ones}} = 1000, \text{ as above.} \end{aligned}$$

84. In order for the two tangents to intersect on the y-axis, the points of tangency must be at equal distances from the y-axis, since the parabola $y = x^2$ is symmetric about the y-axis. Say the points of tangency are (a, a^2) and $(-a, a^2)$, for some $a > 0$. Then since the derivative of $y = x^2$ is $dy/dx = 2x$, the left-hand tangent has slope $-2a$ and equation $y - a^2 = -2a(x + a)$, or $y = -2ax - a^2$, and similarly the right-hand tangent line has equation $y - a^2 = 2a(x - a)$, or $y = 2ax - a^2$. So the two lines intersect at $(0, -a^2)$. Now if the lines are perpendicular, then the product of their slopes is -1, so $(-2a)(2a) = -1 \quad \Leftrightarrow \quad a^2 = \frac{1}{4} \quad \Leftrightarrow \quad a = \frac{1}{2}$. So the lines intersect at $\left(0, -\frac{1}{4}\right)$.



85. $y = x^2 \Rightarrow y' = 2x$, so the slope of a tangent line at the point (a, a^2) is $y' = 2a$ and the slope of a normal line is $-1/(2a)$, for $a \neq 0$. The slope of the normal line through the points (a, a^2) and $(0, c)$ is $\dfrac{a^2 - c}{a - 0}$, so $\dfrac{a^2 - c}{a} = -\dfrac{1}{2a} \Rightarrow$ $a^2 - c = -\frac{1}{2} \Rightarrow a^2 = c - \frac{1}{2}$. The last equation has two solutions if $c > \frac{1}{2}$, one solution if $c = \frac{1}{2}$, and no solution if $c < \frac{1}{2}$. Since the y-axis is normal to $y = x^2$ regardless of the value of c (this is the case for $a = 0$), we have three normal lines if $c > \frac{1}{2}$ and one normal line if $c \leq \frac{1}{2}$.

86.



From the sketch, it appears that there may be a line that is tangent to both curves. The slope of the line through the points $P(a, a^2)$ and $Q(b, b^2 - 2b + 2)$ is $\dfrac{b^2 - 2b + 2 - a^2}{b - a}$. The slope of the tangent line at P is $2a$ $[y' = 2x]$ and at Q is $2b - 2$ $[y' = 2x - 2]$. All three slopes are equal, so $2a = 2b - 2 \Leftrightarrow a = b - 1$.

Also, $2b - 2 = \dfrac{b^2 - 2b + 2 - a^2}{b - a} \Rightarrow 2b - 2 = \dfrac{b^2 - 2b + 2 - (b-1)^2}{b - (b-1)} \Rightarrow 2b - 2 = b^2 - 2b + 2 - b^2 + 2b - 1 \Rightarrow$ $2b = 3 \Rightarrow b = \frac{3}{2}$ and $a = \frac{3}{2} - 1 = \frac{1}{2}$. Thus, an equation of the tangent line at P is $y - \left(\frac{1}{2}\right)^2 = 2\left(\frac{1}{2}\right)\left(x - \frac{1}{2}\right)$ or $y = x - \frac{1}{4}$.

APPLIED PROJECT Building a Better Roller Coaster

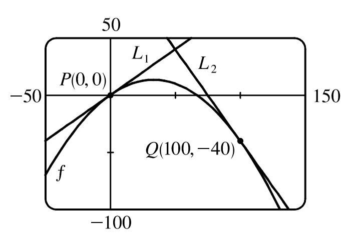
1. (a) $f(x) = ax^2 + bx + c \Rightarrow f'(x) = 2ax + b$.

The origin is at P:	$f(0) = 0$	$\Rightarrow$	$c = 0$
The slope of the ascent is 0.8:	$f'(0) = 0.8$	$\Rightarrow$	$b = 0.8$
The slope of the drop is -1.6:	$f'(100) = -1.6$	$\Rightarrow$	$200a + b = -1.6$

(b) $b = 0.8$, so $200a + b = -1.6 \Rightarrow 200a + 0.8 = -1.6 \Rightarrow 200a = -2.4 \Rightarrow a = -\dfrac{2.4}{200} = -0.012$.

Thus, $f(x) = -0.012x^2 + 0.8x$.

(c) Since L_1 passes through the origin with slope 0.8, it has equation $y = 0.8x$. The horizontal distance between P and Q is 100, so the y-coordinate at Q is $f(100) = -0.012(100)^2 + 0.8(100) = -40$. Since L_2 passes through the point $(100, -40)$ and has slope -1.6, it has equation $y + 40 = -1.6(x - 100)$ or $y = -1.6x + 120$.



(d) The difference in elevation between $P(0, 0)$ and $Q(100, -40)$ is $0 - (-40) = 40$ feet.

2. (a)

Interval	Function	First Derivative	Second Derivative
$(-\infty, 0)$	$L_1(x) = 0.8x$	$L_1'(x) = 0.8$	$L_1''(x) = 0$
$[0, 10)$	$g(x) = kx^3 + lx^2 + mx + n$	$g'(x) = 3kx^2 + 2lx + m$	$g''(x) = 6kx + 2l$
$[10, 90]$	$q(x) = ax^2 + bx + c$	$q'(x) = 2ax + b$	$q''(x) = 2a$
$(90, 100]$	$h(x) = px^3 + qx^2 + rx + s$	$h'(x) = 3px^2 + 2qx + r$	$h''(x) = 6px + 2q$
$(100, \infty)$	$L_2(x) = -1.6x + 120$	$L_2'(x) = -1.6$	$L_2''(x) = 0$

There are 4 values of x (0, 10, 90, and 100) for which we must make sure the function values are equal, the first derivative values are equal, and the second derivative values are equal. The third column in the following table contains the value of each side of the condition — these are found after solving the system in part (b).

At $x =$	Condition	Value	Resulting Equation
0	$g(0) = L_1(0)$	0	$n = 0$
	$g'(0) = L_1'(0)$	$\frac{4}{5}$	$m = 0.8$
	$g''(0) = L_1''(0)$	0	$2l = 0$
10	$g(10) = q(10)$	$\frac{68}{9}$	$1000k + 100l + 10m + n = 100a + 10b + c$
	$g'(10) = q'(10)$	$\frac{2}{3}$	$300k + 20l + m = 20a + b$
	$g''(10) = q''(10)$	$-\frac{2}{75}$	$60k + 2l = 2a$
90	$h(90) = q(90)$	$-\frac{220}{9}$	$729{,}000p + 8100q + 90r + s = 8100a + 90b + c$
	$h'(90) = q'(90)$	$-\frac{22}{15}$	$24{,}300p + 180q + r = 180a + b$
	$h''(90) = q''(90)$	$-\frac{2}{75}$	$540p + 2q = 2a$
100	$h(100) = L_2(100)$	-40	$1{,}000{,}000p + 10{,}000q + 100r + s = -40$
	$h'(100) = L_2'(100)$	$-\frac{8}{5}$	$30{,}000p + 200q + r = -1.6$
	$h''(100) = L_2''(100)$	0	$600p + 2q = 0$

(b) We can arrange our work in a 12×12 matrix as follows.

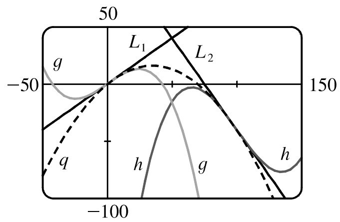
a	b	c	k	l	m	n	p	q	r	s	constant
0	0	0	0	0	0	1	0	0	0	0	0
0	0	0	0	0	1	0	0	0	0	0	0.8
0	0	0	0	2	0	0	0	0	0	0	0
−100	−10	−1	1000	100	10	1	0	0	0	0	0
−20	−1	0	300	20	1	0	0	0	0	0	0
−2	0	0	60	2	0	0	0	0	0	0	0
−8100	−90	−1	0	0	0	0	729,000	8100	90	1	0
−180	−1	0	0	0	0	0	24,300	180	1	0	0
−2	0	0	0	0	0	0	540	2	0	0	0
0	0	0	0	0	0	0	1,000,000	10,000	100	1	−40
0	0	0	0	0	0	0	30,000	200	1	0	−1.6
0	0	0	0	0	0	0	600	2	0	0	0

Solving the system gives us the formulas for q, g, and h.

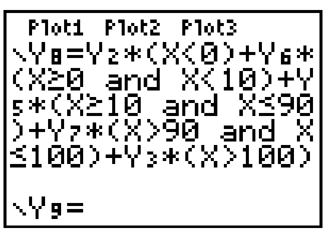
$$\left.\begin{array}{l} a = -0.01\overline{3} = -\frac{1}{75} \\ b = 0.9\overline{3} = \frac{14}{15} \\ c = -0.\overline{4} = -\frac{4}{9} \end{array}\right\} q(x) = -\tfrac{1}{75}x^2 + \tfrac{14}{15}x - \tfrac{4}{9} \qquad \left.\begin{array}{l} k = -0.000\overline{4} = -\frac{1}{2250} \\ l = 0 \\ m = 0.8 = \frac{4}{5} \\ n = 0 \end{array}\right\} g(x) = -\tfrac{1}{2250}x^3 + \tfrac{4}{5}x$$

$$\left.\begin{array}{l} p = 0.000\overline{4} = \frac{1}{2250} \\ q = -0.1\overline{3} = -\frac{2}{15} \\ r = 11.7\overline{3} = \frac{176}{15} \\ s = -324.\overline{4} = -\frac{2920}{9} \end{array}\right\} h(x) = \tfrac{1}{2250}x^3 - \tfrac{2}{15}x^2 + \tfrac{176}{15}x - \tfrac{2920}{9}$$

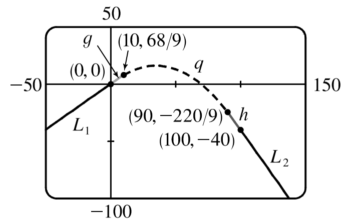
(c) Graph of L_1, q, g, h, and L_2:



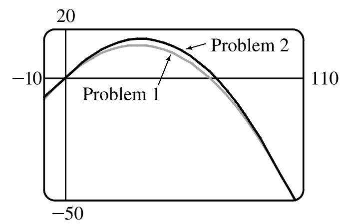
This is the piecewise-defined function assignment on a TI-83/4 Plus calculator, where $\mathrm{Y}_2 = L_1$, $\mathrm{Y}_6 = g$, $\mathrm{Y}_5 = q$, $\mathrm{Y}_7 = h$, and $\mathrm{Y}_3 = L_2$.



The graph of the five functions as a piecewise-defined function:



A comparison of the graphs in part 1(c) and part 2(c):



3.2 The Product and Quotient Rules

1. Product Rule: $f(x) = (1 + 2x^2)(x - x^2) \quad \Rightarrow$

$$f'(x) = (1 + 2x^2)(1 - 2x) + (x - x^2)(4x) = 1 - 2x + 2x^2 - 4x^3 + 4x^2 - 4x^3 = 1 - 2x + 6x^2 - 8x^3.$$

Multiplying first: $f(x) = (1 + 2x^2)(x - x^2) = x - x^2 + 2x^3 - 2x^4 \quad \Rightarrow \quad f'(x) = 1 - 2x + 6x^2 - 8x^3$ (equivalent).

2. Quotient Rule: $F(x) = \dfrac{x^4 - 5x^3 + \sqrt{x}}{x^2} = \dfrac{x^4 - 5x^3 + x^{1/2}}{x^2} \quad \Rightarrow$

$$F'(x) = \frac{x^2(4x^3 - 15x^2 + \frac{1}{2}x^{-1/2}) - (x^4 - 5x^3 + x^{1/2})(2x)}{(x^2)^2} = \frac{4x^5 - 15x^4 + \frac{1}{2}x^{3/2} - 2x^5 + 10x^4 - 2x^{3/2}}{x^4}$$

$$= \frac{2x^5 - 5x^4 - \frac{3}{2}x^{3/2}}{x^4} = 2x - 5 - \tfrac{3}{2}x^{-5/2}$$

Simplifying first: $F(x) = \dfrac{x^4 - 5x^3 + \sqrt{x}}{x^2} = x^2 - 5x + x^{-3/2} \quad \Rightarrow \quad F'(x) = 2x - 5 - \frac{3}{2}x^{-5/2}$ (equivalent).

For this problem, simplifying first seems to be the better method.

3. By the Product Rule, $f(x) = (3x^2 - 5x)e^x \quad \Rightarrow$

$$f'(x) = (3x^2 - 5x)(e^x)' + e^x(3x^2 - 5x)' = (3x^2 - 5x)e^x + e^x(6x - 5)$$
$$= e^x[(3x^2 - 5x) + (6x - 5)] = e^x(3x^2 + x - 5)$$

4. By the Product Rule, $g(x) = (x + 2\sqrt{x}\,)\,e^x \Rightarrow$

$$g'(x) = (x + 2\sqrt{x}\,)(e^x)' + e^x(x + 2\sqrt{x}\,)' = (x + 2\sqrt{x}\,)e^x + e^x\left(1 + 2 \cdot \tfrac{1}{2}x^{-1/2}\right)$$
$$= e^x\left[(x + 2\sqrt{x}\,) + \left(1 + 1/\sqrt{x}\,\right)\right] = e^x\left(x + 2\sqrt{x} + 1 + 1/\sqrt{x}\,\right)$$

5. By the Quotient Rule, $y = \dfrac{x}{e^x} \quad \Rightarrow \quad y' = \dfrac{e^x(1) - x(e^x)}{(e^x)^2} = \dfrac{e^x(1 - x)}{(e^x)^2} = \dfrac{1 - x}{e^x}$.

6. By the Quotient Rule, $y = \dfrac{e^x}{1 - e^x} \quad \Rightarrow \quad y' = \dfrac{(1 - e^x)e^x - e^x(-e^x)}{(1 - e^x)^2} = \dfrac{e^x - e^{2x} + e^{2x}}{(1 - e^x)^2} = \dfrac{e^x}{(1 - e^x)^2}$.

The notations $\overset{\text{PR}}{\Rightarrow}$ and $\overset{\text{QR}}{\Rightarrow}$ indicate the use of the Product and Quotient Rules, respectively.

7. $g(x) = \dfrac{1 + 2x}{3 - 4x} \quad \overset{\text{QR}}{\Rightarrow} \quad g'(x) = \dfrac{(3 - 4x)(2) - (1 + 2x)(-4)}{(3 - 4x)^2} = \dfrac{6 - 8x + 4 + 8x}{(3 - 4x)^2} = \dfrac{10}{(3 - 4x)^2}$

8. $G(x) = \dfrac{x^2 - 2}{2x + 1} \quad \overset{\text{QR}}{\Rightarrow} \quad G'(x) = \dfrac{(2x + 1)(2x) - (x^2 - 2)(2)}{(2x + 1)^2} = \dfrac{4x^2 + 2x - 2x^2 + 4}{(2x + 1)^2} = \dfrac{2x^2 + 2x + 4}{(2x + 1)^2}$

9. $H(u) = (u - \sqrt{u}\,)(u + \sqrt{u}\,) \quad \overset{\text{PR}}{\Rightarrow}$

$$H'(u) = (u - \sqrt{u}\,)\left(1 + \frac{1}{2\sqrt{u}}\right) + (u + \sqrt{u}\,)\left(1 - \frac{1}{2\sqrt{u}}\right) = u + \tfrac{1}{2}\sqrt{u} - \sqrt{u} - \tfrac{1}{2} + u - \tfrac{1}{2}\sqrt{u} + \sqrt{u} - \tfrac{1}{2} = 2u - 1.$$

An easier method is to simplify first and then differentiate as follows:

$$H(u) = (u - \sqrt{u}\,)(u + \sqrt{u}\,) = u^2 - (\sqrt{u}\,)^2 = u^2 - u \quad \Rightarrow \quad H'(u) = 2u - 1$$

10. $J(v) = (v^3 - 2v)(v^{-4} + v^{-2}) \quad \overset{\text{PR}}{\Rightarrow}$

$$J'(v) = (v^3 - 2v)(-4v^{-5} - 2v^{-3}) + (v^{-4} + v^{-2})(3v^2 - 2)$$
$$= -4v^{-2} - 2v^0 + 8v^{-4} + 4v^{-2} + 3v^{-2} - 2v^{-4} + 3v^0 - 2v^{-2} = 1 + v^{-2} + 6v^{-4}$$

11. $F(y) = \left(\dfrac{1}{y^2} - \dfrac{3}{y^4}\right)(y + 5y^3) = (y^{-2} - 3y^{-4})(y + 5y^3) \quad \overset{\text{PR}}{\Rightarrow}$

$$F'(y) = (y^{-2} - 3y^{-4})(1 + 15y^2) + (y + 5y^3)(-2y^{-3} + 12y^{-5})$$
$$= (y^{-2} + 15 - 3y^{-4} - 45y^{-2}) + (-2y^{-2} + 12y^{-4} - 10 + 60y^{-2})$$
$$= 5 + 14y^{-2} + 9y^{-4} \text{ or } 5 + 14/y^2 + 9/y^4$$

12. $f(z) = (1 - e^z)(z + e^z) \quad \overset{\text{PR}}{\Rightarrow}$

$$f'(z) = (1 - e^z)(1 + e^z) + (z + e^z)(-e^z) = 1^2 - (e^z)^2 - ze^z - (e^z)^2 = 1 - ze^z - 2e^{2z}$$

13. $y = \dfrac{x^2+1}{x^3-1} \overset{\text{QR}}{\Rightarrow}$

$$y' = \frac{(x^3-1)(2x)-(x^2+1)(3x^2)}{(x^3-1)^2} = \frac{x\left[(x^3-1)(2)-(x^2+1)(3x)\right]}{(x^3-1)^2} = \frac{x(2x^3-2-3x^3-3x)}{(x^3-1)^2} = \frac{x(-x^3-3x-2)}{(x^3-1)^2}$$

14. $y = \dfrac{\sqrt{x}}{2+x} \overset{\text{QR}}{\Rightarrow}$

$$y' = \frac{(2+x)\left(\dfrac{1}{2\sqrt{x}}\right)-\sqrt{x}\,(1)}{(2+x)^2} = \frac{\dfrac{1}{\sqrt{x}}+\dfrac{\sqrt{x}}{2}-\sqrt{x}}{(2+x)^2} = \frac{\dfrac{2+x-2x}{2\sqrt{x}}}{(2+x)^2} = \frac{2-x}{2\sqrt{x}(2+x)^2}$$

15. $y = \dfrac{t^3+3t}{t^2-4t+3} \overset{\text{QR}}{\Rightarrow}$

$$y' = \frac{(t^2-4t+3)(3t^2+3)-(t^3+3t)(2t-4)}{(t^2-4t+3)^2}$$

$$= \frac{3t^4+3t^2-12t^3-12t+9t^2+9-(2t^4-4t^3+6t^2-12t)}{(t^2-4t+3)^2} = \frac{t^4-8t^3+6t^2+9}{(t^2-4t+3)^2}$$

16. $y = \dfrac{1}{t^3+2t^2-1} \overset{\text{QR}}{\Rightarrow} y' = \dfrac{(t^3+2t^2-1)(0)-1(3t^2+4t)}{(t^3+2t^2-1)^2} = -\dfrac{3t^2+4t}{(t^3+2t^2-1)^2}$

17. $y = e^p(p+p\sqrt{p}\,) = e^p(p+p^{3/2}) \overset{\text{PR}}{\Rightarrow} y' = e^p\left(1+\frac{3}{2}p^{1/2}\right)+(p+p^{3/2})e^p = e^p\left(1+\frac{3}{2}\sqrt{p}+p+p\sqrt{p}\right)$

18. $h(r) = \dfrac{ae^r}{b+e^r} \overset{\text{QR}}{\Rightarrow} h'(r) = \dfrac{(b+e^r)(ae^r)-(ae^r)(e^r)}{(b+e^r)^2} = \dfrac{abe^r+ae^{2r}-ae^{2r}}{(b+e^r)^2} = \dfrac{abe^r}{(b+e^r)^2}$

19. $y = \dfrac{s-\sqrt{s}}{s^2} = \dfrac{s}{s^2}-\dfrac{\sqrt{s}}{s^2} = s^{-1}-s^{-3/2} \Rightarrow y' = -s^{-2}+\frac{3}{2}s^{-5/2} = \dfrac{-1}{s^2}+\dfrac{3}{2s^{5/2}} = \dfrac{3-2\sqrt{s}}{2s^{5/2}}$

20. $y = (z^2+e^z)\sqrt{z} \overset{\text{PR}}{\Rightarrow}$

$$y' = (z^2+e^z)\left(\frac{1}{2\sqrt{z}}\right)+\sqrt{z}\,(2z+e^z) = \frac{z^2}{2\sqrt{z}}+\frac{e^z}{2\sqrt{z}}+2z\sqrt{z}+\sqrt{z}\,e^z$$

$$= \frac{z^2+e^z+4z^2+2ze^z}{2\sqrt{z}} = \frac{5z^2+e^z+2ze^z}{2\sqrt{z}}$$

21. $f(t) = \dfrac{\sqrt[3]{t}}{t-3} \overset{\text{QR}}{\Rightarrow}$

$$f'(t) = \frac{(t-3)\left(\frac{1}{3}t^{-2/3}\right)-t^{1/3}(1)}{(t-3)^2} = \frac{\frac{1}{3}t^{1/3}-t^{-2/3}-t^{1/3}}{(t-3)^2} = \frac{-\frac{2}{3}t^{1/3}-t^{-2/3}}{(t-3)^2} = \frac{\dfrac{-2t}{3t^{2/3}}-\dfrac{3}{3t^{2/3}}}{(t-3)^2} = \frac{-2t-3}{3t^{2/3}(t-3)^2}$$

22. $V(t) = \dfrac{4+t}{te^t} \overset{\text{QR}}{\Rightarrow}$

$$V'(t) = \frac{te^t(1)-(4+t)(te^t+e^t(1))}{(te^t)^2} = \frac{te^t-4te^t-4e^t-t^2e^t-te^t}{t^2e^{2t}}$$

$$= \frac{-4te^t-4e^t-t^2e^t}{t^2e^{2t}} = \frac{-e^t(t^2+4t+4)}{t^2e^{2t}} = -\frac{(t+2)^2}{t^2e^t}$$

23. $f(x) = \dfrac{x^2 e^x}{x^2 + e^x} \quad \overset{\text{QR}}{\Rightarrow}$

$$f'(x) = \frac{(x^2 + e^x)\left[x^2 e^x + e^x(2x)\right] - x^2 e^x (2x + e^x)}{(x^2 + e^x)^2} = \frac{x^4 e^x + 2x^3 e^x + x^2 e^{2x} + 2xe^{2x} - 2x^3 e^x - x^2 e^{2x}}{(x^2 + e^x)^2}$$

$$= \frac{x^4 e^x + 2xe^{2x}}{(x^2 + e^x)^2} = \frac{xe^x(x^3 + 2e^x)}{(x^2 + e^x)^2}$$

24. $F(t) = \dfrac{At}{Bt^2 + Ct^3} = \dfrac{A}{Bt + Ct^2} \quad \overset{\text{QR}}{\Rightarrow}$

$$F'(t) = \frac{(Bt + Ct^2)(0) - A(B + 2Ct)}{(Bt + Ct^2)^2} = \frac{-A(B + 2Ct)}{(t)^2(B + Ct)^2} = -\frac{A(B + 2Ct)}{t^2(B + Ct)^2}$$

25. $f(x) = \dfrac{x}{x + c/x} \quad \Rightarrow \quad f'(x) = \dfrac{(x + c/x)(1) - x(1 - c/x^2)}{\left(x + \dfrac{c}{x}\right)^2} = \dfrac{x + c/x - x + c/x}{\left(\dfrac{x^2 + c}{x}\right)^2} = \dfrac{2c/x}{\dfrac{(x^2 + c)^2}{x^2}} \cdot \dfrac{x^2}{x^2} = \dfrac{2cx}{(x^2 + c)^2}$

26. $f(x) = \dfrac{ax + b}{cx + d} \quad \Rightarrow \quad f'(x) = \dfrac{(cx + d)(a) - (ax + b)(c)}{(cx + d)^2} = \dfrac{acx + ad - acx - bc}{(cx + d)^2} = \dfrac{ad - bc}{(cx + d)^2}$

27. $f(x) = (x^3 + 1)e^x \quad \overset{\text{PR}}{\Rightarrow}$

$$f'(x) = (x^3 + 1)e^x + e^x(3x^2) = e^x\left[(x^3 + 1) + 3x^2\right] = e^x(x^3 + 3x^2 + 1) \quad \overset{\text{PR}}{\Rightarrow}$$

$$f''(x) = e^x(3x^2 + 6x) + (x^3 + 3x^2 + 1)e^x = e^x\left[(3x^2 + 6x) + (x^3 + 3x^2 + 1)\right] = e^x(x^3 + 6x^2 + 6x + 1)$$

28. $f(x) = \sqrt{x}\, e^x \quad \overset{\text{PR}}{\Rightarrow} \quad f'(x) = \sqrt{x}\, e^x + e^x\left(\dfrac{1}{2\sqrt{x}}\right) = \left(\sqrt{x} + \dfrac{1}{2\sqrt{x}}\right)e^x = \dfrac{2x + 1}{2\sqrt{x}}\, e^x.$

Using the Product Rule and $f'(x) = \left(x^{1/2} + \frac{1}{2}x^{-1/2}\right)e^x$, we get

$$f''(x) = \left(x^{1/2} + \tfrac{1}{2}x^{-1/2}\right)e^x + e^x\left(\tfrac{1}{2}x^{-1/2} - \tfrac{1}{4}x^{-3/2}\right) = \left(x^{1/2} + x^{-1/2} - \tfrac{1}{4}x^{-3/2}\right)e^x = \frac{4x^2 + 4x - 1}{4x^{3/2}}\, e^x$$

29. $f(x) = \dfrac{x^2}{1 + e^x} \quad \overset{\text{QR}}{\Rightarrow} \quad f'(x) = \dfrac{(1 + e^x)(2x) - x^2(e^x)}{(1 + e^x)^2} = \dfrac{x[(1 + e^x)2 - xe^x]}{(1 + e^x)^2} = \dfrac{x(2 + 2e^x - xe^x)}{(1 + e^x)^2}.$

Using the Quotient and Product Rules and $f'(x) = \dfrac{2x + 2xe^x - x^2 e^x}{(1 + e^x)^2}$, we get

$$f''(x) = \frac{(1 + e^x)^2\left[2 + 2(xe^x + e^x) - (x^2 e^x + 2xe^x)\right] - (2x + 2xe^x - x^2 e^x)\left[(1 + e^x)e^x + (1 + e^x)e^x\right]}{[(1 + e^x)^2]^2}$$

$$= \frac{(1 + e^x)\left\{\left[(1 + e^x)(2 + 2xe^x + 2e^x - x^2 e^x - 2xe^x)\right] - (2x + 2xe^x - x^2 e^x)(2e^x)\right\}}{(1 + e^x)^4}$$

$$= \frac{(1 + e^x)(2 + 2e^x - x^2 e^x) - 4xe^x - 4xe^{2x} + 2x^2 e^{2x}}{(1 + e^x)^3}$$

$$= \frac{2 + 2e^x - x^2 e^x + 2e^x + 2e^{2x} - x^2 e^{2x} - 4xe^x - 4xe^{2x} + 2x^2 e^{2x}}{(1 + e^x)^3}$$

$$= \frac{2 + 4e^x - x^2 e^x - 4xe^x + 2e^{2x} + x^2 e^{2x} - 4xe^{2x}}{(1 + e^x)^3}$$

30. $f(x) = \dfrac{x}{x^2-1} \quad \Rightarrow \quad f'(x) = \dfrac{(x^2-1)(1) - x(2x)}{(x^2-1)^2} = \dfrac{x^2-1-2x^2}{(x^2-1)^2} = \dfrac{-x^2-1}{(x^2-1)^2} \quad \Rightarrow$

$$f''(x) = \frac{(x^2-1)^2(-2x) - (-x^2-1)(x^4-2x^2+1)'}{[(x^2-1)^2]^2} = \frac{(x^2-1)^2(-2x) + (x^2+1)(4x^3-4x)}{(x^2-1)^4}$$

$$= \frac{(x^2-1)^2(-2x) + (x^2+1)(4x)(x^2-1)}{(x^2-1)^4} = \frac{(x^2-1)[(x^2-1)(-2x) + (x^2+1)(4x)]}{(x^2-1)^4}$$

$$= \frac{-2x^3+2x+4x^3+4x}{(x^2-1)^3} = \frac{2x^3+6x}{(x^2-1)^3}$$

31. $y = \dfrac{x^2-1}{x^2+x+1} \quad \Rightarrow$

$$y' = \frac{(x^2+x+1)(2x) - (x^2-1)(2x+1)}{(x^2+x+1)^2} = \frac{2x^3+2x^2+2x-2x^3-x^2+2x+1}{(x^2+x+1)^2} = \frac{x^2+4x+1}{(x^2+x+1)^2}.$$

At $(1, 0)$, $y' = \dfrac{6}{3^2} = \dfrac{2}{3}$, and an equation of the tangent line is $y - 0 = \frac{2}{3}(x-1)$, or $y = \frac{2}{3}x - \frac{2}{3}$.

32. $y = \dfrac{1+x}{1+e^x} \quad \Rightarrow \quad y' = \dfrac{(1+e^x)(1) - (1+x)e^x}{(1+e^x)^2} = \dfrac{1+e^x-e^x-xe^x}{(1+e^x)^2} = \dfrac{1-xe^x}{(1+e^x)^2}$.

At $\left(0, \frac{1}{2}\right)$, $y' = \dfrac{1}{(1+1)^2} = \dfrac{1}{4}$, and an equation of the tangent line is $y - \frac{1}{2} = \frac{1}{4}(x-0)$ or $y = \frac{1}{4}x + \frac{1}{2}$.

33. $y = 2xe^x \quad \Rightarrow \quad y' = 2(x \cdot e^x + e^x \cdot 1) = 2e^x(x+1)$.

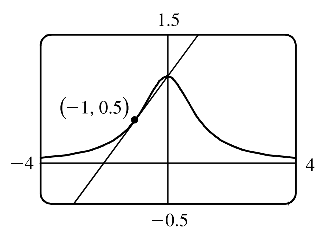
At $(0, 0)$, $y' = 2e^0(0+1) = 2 \cdot 1 \cdot 1 = 2$, and an equation of the tangent line is $y - 0 = 2(x-0)$, or $y = 2x$. The slope of the normal line is $-\frac{1}{2}$, so an equation of the normal line is $y - 0 = -\frac{1}{2}(x-0)$, or $y = -\frac{1}{2}x$.

34. $y = \dfrac{2x}{x^2+1} \quad \Rightarrow \quad y' = \dfrac{(x^2+1)(2) - 2x(2x)}{(x^2+1)^2} = \dfrac{2-2x^2}{(x^2+1)^2}$. At $(1, 1)$, $y' = 0$, and an equation of the tangent line is $y - 1 = 0(x-1)$, or $y = 1$. The slope of the normal line is undefined, so an equation of the normal line is $x = 1$.

35. (a) $y = f(x) = \dfrac{1}{1+x^2} \quad \Rightarrow$

$f'(x) = \dfrac{(1+x^2)(0) - 1(2x)}{(1+x^2)^2} = \dfrac{-2x}{(1+x^2)^2}$. So the slope of the tangent line at the point $\left(-1, \frac{1}{2}\right)$ is $f'(-1) = \dfrac{2}{2^2} = \frac{1}{2}$ and its equation is $y - \frac{1}{2} = \frac{1}{2}(x+1)$ or $y = \frac{1}{2}x + 1$.

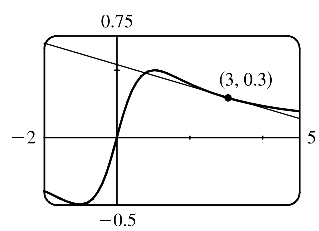
(b)



36. (a) $y = f(x) = \dfrac{x}{1+x^2} \quad \Rightarrow$

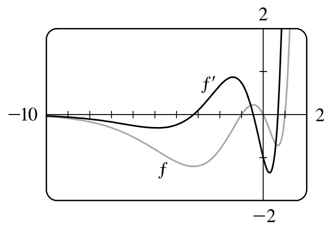
$f'(x) = \dfrac{(1+x^2)1 - x(2x)}{(1+x^2)^2} = \dfrac{1-x^2}{(1+x^2)^2}$. So the slope of the tangent line at the point $(3, 0.3)$ is $f'(3) = \frac{-8}{100}$ and its equation is $y - 0.3 = -0.08(x-3)$ or $y = -0.08x + 0.54$.

(b)



37. (a) $f(x) = (x^3 - x)e^x \quad \Rightarrow \quad f'(x) = (x^3 - x)e^x + e^x(3x^2 - 1) = e^x(x^3 + 3x^2 - x - 1)$

(b)

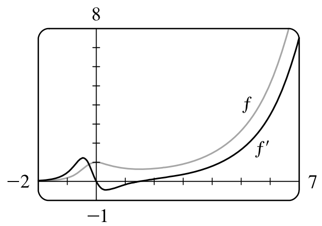


$f' = 0$ when f has a horizontal tangent line, f' is negative when f is decreasing, and f' is positive when f is increasing.

38. (a) $f(x) = \dfrac{e^x}{2x^2 + x + 1} \quad \Rightarrow$

$$f'(x) = \frac{(2x^2 + x + 1)e^x - e^x(4x + 1)}{(2x^2 + x + 1)^2} = \frac{e^x(2x^2 + x + 1 - 4x - 1)}{(2x^2 + x + 1)^2} = \frac{e^x(2x^2 - 3x)}{(2x^2 + x + 1)^2}$$

(b)



$f' = 0$ when f has a horizontal tangent line, f' is negative when f is decreasing, and f' is positive when f is increasing.

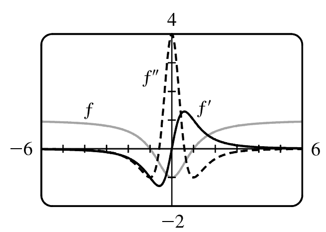
39. (a) $f(x) = \dfrac{x^2 - 1}{x^2 + 1} \quad \Rightarrow$

$$f'(x) = \frac{(x^2 + 1)(2x) - (x^2 - 1)(2x)}{(x^2 + 1)^2} = \frac{(2x)[(x^2 + 1) - (x^2 - 1)]}{(x^2 + 1)^2} = \frac{(2x)(2)}{(x^2 + 1)^2} = \frac{4x}{(x^2 + 1)^2} \quad \Rightarrow$$

$$f''(x) = \frac{(x^2 + 1)^2(4) - 4x(x^4 + 2x^2 + 1)'}{[(x^2 + 1)^2]^2} = \frac{4(x^2 + 1)^2 - 4x(4x^3 + 4x)}{(x^2 + 1)^4}$$

$$= \frac{4(x^2 + 1)^2 - 16x^2(x^2 + 1)}{(x^2 + 1)^4} = \frac{4(x^2 + 1)[(x^2 + 1) - 4x^2]}{(x^2 + 1)^4} = \frac{4(1 - 3x^2)}{(x^2 + 1)^3}$$

(b)

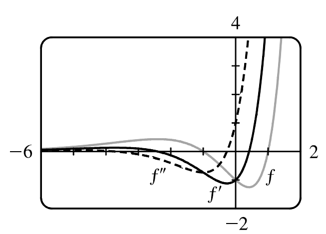


$f' = 0$ when f has a horizontal tangent and $f'' = 0$ when f' has a horizontal tangent. f' is negative when f is decreasing and positive when f is increasing. f'' is negative when f' is decreasing and positive when f' is increasing. f'' is negative when f is concave down and positive when f is concave up.

40. (a) $f(x) = (x^2 - 1)e^x \quad \Rightarrow \quad f'(x) = (x^2 - 1)e^x + e^x(2x) = e^x(x^2 + 2x - 1) \quad \Rightarrow$

$f''(x) = e^x(2x + 2) + (x^2 + 2x - 1)e^x = e^x(x^2 + 4x + 1)$

(b)



We can see that our answers are plausible, since f has horizontal tangents where $f'(x) = 0$, and f' has horizontal tangents where $f''(x) = 0$.

41. $f(x) = \dfrac{x^2}{1+x} \quad \Rightarrow \quad f'(x) = \dfrac{(1+x)(2x) - x^2(1)}{(1+x)^2} = \dfrac{2x + 2x^2 - x^2}{(1+x)^2} = \dfrac{x^2 + 2x}{x^2 + 2x + 1} \quad \Rightarrow$

$$f''(x) = \frac{(x^2+2x+1)(2x+2) - (x^2+2x)(2x+2)}{(x^2+2x+1)^2} = \frac{(2x+2)(x^2+2x+1-x^2-2x)}{[(x+1)^2]^2}$$

$$= \frac{2(x+1)(1)}{(x+1)^4} = \frac{2}{(x+1)^3},$$

so $f''(1) = \dfrac{2}{(1+1)^3} = \dfrac{2}{8} = \dfrac{1}{4}$.

42. $g(x) = \dfrac{x}{e^x} \quad \Rightarrow \quad g'(x) = \dfrac{e^x \cdot 1 - x \cdot e^x}{(e^x)^2} = \dfrac{e^x(1-x)}{(e^x)^2} = \dfrac{1-x}{e^x} \quad \Rightarrow$

$$g''(x) = \frac{e^x \cdot (-1) - (1-x)e^x}{(e^x)^2} = \frac{e^x[-1-(1-x)]}{(e^x)^2} = \frac{x-2}{e^x} \quad \Rightarrow$$

$$g'''(x) = \frac{e^x \cdot 1 - (x-2)e^x}{(e^x)^2} = \frac{e^x[1-(x-2)]}{(e^x)^2} = \frac{3-x}{e^x} \quad \Rightarrow$$

$$g^{(4)}(x) = \frac{e^x \cdot (-1) - (3-x)e^x}{(e^x)^2} = \frac{e^x[-1-(3-x)]}{(e^x)^2} = \frac{x-4}{e^x}.$$

The pattern suggests that $g^{(n)}(x) = \dfrac{(x-n)(-1)^n}{e^x}$. (We could use mathematical induction to prove this formula.)

43. We are given that $f(5) = 1$, $f'(5) = 6$, $g(5) = -3$, and $g'(5) = 2$.

(a) $(fg)'(5) = f(5)g'(5) + g(5)f'(5) = (1)(2) + (-3)(6) = 2 - 18 = -16$

(b) $\left(\dfrac{f}{g}\right)'(5) = \dfrac{g(5)f'(5) - f(5)g'(5)}{[g(5)]^2} = \dfrac{(-3)(6) - (1)(2)}{(-3)^2} = -\dfrac{20}{9}$

(c) $\left(\dfrac{g}{f}\right)'(5) = \dfrac{f(5)g'(5) - g(5)f'(5)}{[f(5)]^2} = \dfrac{(1)(2) - (-3)(6)}{(1)^2} = 20$

44. We are given that $f(4) = 2$, $g(4) = 5$, $f'(4) = 6$, and $g'(4) = -3$.

(a) $h(x) = 3f(x) + 8g(x) \quad \Rightarrow \quad h'(x) = 3f'(x) + 8g'(x)$, so

$h'(4) = 3f'(4) + 8g'(4) = 3(6) + 8(-3) = 18 - 24 = -6$.

(b) $h(x) = f(x)\,g(x) \quad \Rightarrow \quad h'(x) = f(x)\,g'(x) + g(x)\,f'(x)$, so

$h'(4) = f(4)\,g'(4) + g(4)\,f'(4) = 2(-3) + 5(6) = -6 + 30 = 24$.

(c) $h(x) = \dfrac{f(x)}{g(x)} \quad \Rightarrow \quad h'(x) = \dfrac{g(x)\,f'(x) - f(x)\,g'(x)}{[g(x)]^2}$, so

$h'(4) = \dfrac{g(4)\,f'(4) - f(4)\,g'(4)}{[g(4)]^2} = \dfrac{5(6) - 2(-3)}{5^2} = \dfrac{30+6}{25} = \dfrac{36}{25}$.

(d) $h(x) = \dfrac{g(x)}{f(x) + g(x)} \quad \Rightarrow$

$h'(4) = \dfrac{[f(4) + g(4)]\,g'(4) - g(4)\,[f'(4) + g'(4)]}{[f(4) + g(4)]^2} = \dfrac{(2+5)\,(-3) - 5\,[6 + (-3)]}{(2+5)^2} = \dfrac{-21-15}{7^2} = -\dfrac{36}{49}$

45. $f(x) = e^x g(x) \quad \Rightarrow \quad f'(x) = e^x g'(x) + g(x)e^x = e^x[g'(x) + g(x)]$. $f'(0) = e^0[g'(0) + g(0)] = 1(5 + 2) = 7$

46. $\dfrac{d}{dx}\left[\dfrac{h(x)}{x}\right] = \dfrac{xh'(x) - h(x) \cdot 1}{x^2} \quad \Rightarrow \quad \dfrac{d}{dx}\left[\dfrac{h(x)}{x}\right]_{x=2} = \dfrac{2h'(2) - h(2)}{2^2} = \dfrac{2(-3) - (4)}{4} = \dfrac{-10}{4} = -2.5$

47. $g(x) = xf(x) \quad \Rightarrow \quad g'(x) = xf'(x) + f(x) \cdot 1$. Now $g(3) = 3f(3) = 3 \cdot 4 = 12$ and $g'(3) = 3f'(3) + f(3) = 3(-2) + 4 = -2$. Thus, an equation of the tangent line to the graph of g at the point where $x = 3$ is $y - 12 = -2(x - 3)$, or $y = -2x + 18$.

48. $f'(x) = x^2 f(x) \quad \Rightarrow \quad f''(x) = x^2 f'(x) + f(x) \cdot 2x$. Now $f'(2) = 2^2 f(2) = 4(10) = 40$, so $f''(2) = 2^2(40) + 10(4) = 200$.

49. (a) From the graphs of f and g, we obtain the following values: $f(1) = 2$ since the point $(1, 2)$ is on the graph of f; $g(1) = 1$ since the point $(1, 1)$ is on the graph of g; $f'(1) = 2$ since the slope of the line segment between $(0, 0)$ and $(2, 4)$ is $\dfrac{4 - 0}{2 - 0} = 2$; $g'(1) = -1$ since the slope of the line segment between $(-2, 4)$ and $(2, 0)$ is $\dfrac{0 - 4}{2 - (-2)} = -1$.

Now $u(x) = f(x)g(x)$, so $u'(1) = f(1)g'(1) + g(1)\,f'(1) = 2 \cdot (-1) + 1 \cdot 2 = 0$.

(b) $v(x) = f(x)/g(x)$, so $v'(5) = \dfrac{g(5)f'(5) - f(5)g'(5)}{[g(5)]^2} = \dfrac{2\left(-\frac{1}{3}\right) - 3 \cdot \frac{2}{3}}{2^2} = \dfrac{-\frac{8}{3}}{4} = -\dfrac{2}{3}$

50. (a) $P(x) = F(x)\,G(x)$, so $P'(2) = F(2)\,G'(2) + G(2)\,F'(2) = 3 \cdot \frac{2}{4} + 2 \cdot 0 = \frac{3}{2}$.

(b) $Q(x) = F(x)/G(x)$, so $Q'(7) = \dfrac{G(7)\,F'(7) - F(7)\,G'(7)}{[G(7)]^2} = \dfrac{1 \cdot \frac{1}{4} - 5 \cdot \left(-\frac{2}{3}\right)}{1^2} = \dfrac{1}{4} + \dfrac{10}{3} = \dfrac{43}{12}$

51. (a) $y = xg(x) \quad \Rightarrow \quad y' = xg'(x) + g(x) \cdot 1 = xg'(x) + g(x)$

(b) $y = \dfrac{x}{g(x)} \quad \Rightarrow \quad y' = \dfrac{g(x) \cdot 1 - xg'(x)}{[g(x)]^2} = \dfrac{g(x) - xg'(x)}{[g(x)]^2}$

(c) $y = \dfrac{g(x)}{x} \quad \Rightarrow \quad y' = \dfrac{xg'(x) - g(x) \cdot 1}{(x)^2} = \dfrac{xg'(x) - g(x)}{x^2}$

52. (a) $y = x^2 f(x) \quad \Rightarrow \quad y' = x^2 f'(x) + f(x)(2x)$

(b) $y = \dfrac{f(x)}{x^2} \quad \Rightarrow \quad y' = \dfrac{x^2 f'(x) - f(x)(2x)}{(x^2)^2} = \dfrac{xf'(x) - 2f(x)}{x^3}$

(c) $y = \dfrac{x^2}{f(x)} \quad \Rightarrow \quad y' = \dfrac{f(x)(2x) - x^2 f'(x)}{[f(x)]^2}$

(d) $y = \dfrac{1 + xf(x)}{\sqrt{x}} \quad \Rightarrow$

$$y' = \frac{\sqrt{x}\,[xf'(x) + f(x)] - [1 + xf(x)]\,\dfrac{1}{2\sqrt{x}}}{\left(\sqrt{x}\,\right)^2}$$

$$= \frac{x^{3/2} f'(x) + x^{1/2} f(x) - \frac{1}{2}x^{-1/2} - \frac{1}{2}x^{1/2} f(x)}{x} \cdot \frac{2x^{1/2}}{2x^{1/2}} = \frac{xf(x) + 2x^2 f'(x) - 1}{2x^{3/2}}$$

53. If $y = f(x) = \dfrac{x}{x+1}$, then $f'(x) = \dfrac{(x+1)(1) - x(1)}{(x+1)^2} = \dfrac{1}{(x+1)^2}$. When $x = a$, the equation of the tangent line is

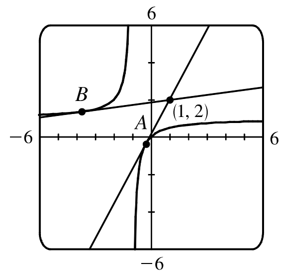
$y - \dfrac{a}{a+1} = \dfrac{1}{(a+1)^2}(x - a)$. This line passes through $(1, 2)$ when $2 - \dfrac{a}{a+1} = \dfrac{1}{(a+1)^2}(1-a) \quad \Leftrightarrow$

$2(a+1)^2 - a(a+1) = 1 - a \quad \Leftrightarrow \quad 2a^2 + 4a + 2 - a^2 - a - 1 + a = 0 \quad \Leftrightarrow \quad a^2 + 4a + 1 = 0.$

The quadratic formula gives the roots of this equation as $a = \dfrac{-4 \pm \sqrt{4^2 - 4(1)(1)}}{2(1)} = \dfrac{-4 \pm \sqrt{12}}{2} = -2 \pm \sqrt{3}$,

so there are two such tangent lines. Since

$$\begin{aligned} f\left(-2 \pm \sqrt{3}\right) &= \frac{-2 \pm \sqrt{3}}{-2 \pm \sqrt{3} + 1} = \frac{-2 \pm \sqrt{3}}{-1 \pm \sqrt{3}} \cdot \frac{-1 \mp \sqrt{3}}{-1 \mp \sqrt{3}} \\ &= \frac{2 \pm 2\sqrt{3} \mp \sqrt{3} - 3}{1 - 3} = \frac{-1 \pm \sqrt{3}}{-2} = \frac{1 \mp \sqrt{3}}{2}, \end{aligned}$$



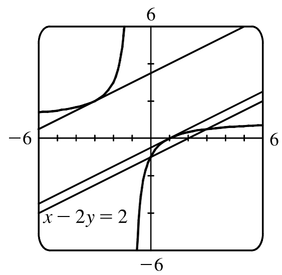
the lines touch the curve at $A\left(-2 + \sqrt{3}, \frac{1 - \sqrt{3}}{2}\right) \approx (-0.27, -0.37)$

and $B\left(-2 - \sqrt{3}, \frac{1 + \sqrt{3}}{2}\right) \approx (-3.73, 1.37)$.

54. $y = \dfrac{x-1}{x+1} \quad \Rightarrow \quad y' = \dfrac{(x+1)(1) - (x-1)(1)}{(x+1)^2} = \dfrac{2}{(x+1)^2}$. If the tangent intersects

the curve when $x = a$, then its slope is $2/(a+1)^2$. But if the tangent is parallel to

$x - 2y = 2$, that is, $y = \frac{1}{2}x - 1$, then its slope is $\frac{1}{2}$. Thus, $\dfrac{2}{(a+1)^2} = \dfrac{1}{2} \quad \Rightarrow$



$(a+1)^2 = 4 \quad \Rightarrow \quad a + 1 = \pm 2 \quad \Rightarrow \quad a = 1$ or -3. When $a = 1$, $y = 0$ and the

equation of the tangent is $y - 0 = \frac{1}{2}(x - 1)$ or $y = \frac{1}{2}x - \frac{1}{2}$. When $a = -3$, $y = 2$ and

the equation of the tangent is $y - 2 = \frac{1}{2}(x + 3)$ or $y = \frac{1}{2}x + \frac{7}{2}$.

55. $R = \dfrac{f}{g} \quad \Rightarrow \quad R' = \dfrac{gf' - fg'}{g^2}$. For $f(x) = x - 3x^3 + 5x^5$, $f'(x) = 1 - 9x^2 + 25x^4$,

and for $g(x) = 1 + 3x^3 + 6x^6 + 9x^9$, $g'(x) = 9x^2 + 36x^5 + 81x^8$.

Thus, $R'(0) = \dfrac{g(0)f'(0) - f(0)g'(0)}{[g(0)]^2} = \dfrac{1 \cdot 1 - 0 \cdot 0}{1^2} = \dfrac{1}{1} = 1$.

56. $Q = \dfrac{f}{g} \quad \Rightarrow \quad Q' = \dfrac{gf' - fg'}{g^2}$. For $f(x) = 1 + x + x^2 + xe^x$, $f'(x) = 1 + 2x + xe^x + e^x$,

and for $g(x) = 1 - x + x^2 - xe^x$, $g'(x) = -1 + 2x - xe^x - e^x$.

Thus, $Q'(0) = \dfrac{g(0)f'(0) - f(0)g'(0)}{[g(0)]^2} = \dfrac{1 \cdot 2 - 1 \cdot (-2)}{1^2} = \dfrac{4}{1} = 4$.

57. If $P(t)$ denotes the population at time t and $A(t)$ the average annual income, then $T(t) = P(t)A(t)$ is the total personal income. The rate at which $T(t)$ is rising is given by $T'(t) = P(t)A'(t) + A(t)P'(t) \quad \Rightarrow$

$$\begin{aligned} T'(1999) &= P(1999)A'(1999) + A(1999)P'(1999) = (961{,}400)(\$1400/\text{yr}) + (\$30{,}593)(9200/\text{yr}) \\ &= \$1{,}345{,}960{,}000/\text{yr} + \$281{,}455{,}600/\text{yr} = \$1{,}627{,}415{,}600/\text{yr} \end{aligned}$$

So the total personal income was rising by about \$1.627 billion per year in 1999.

The term $P(t)A'(t) \approx \$1.346$ billion represents the portion of the rate of change of total income due to the existing population's increasing income. The term $A(t)P'(t) \approx \$281$ million represents the portion of the rate of change of total income due to increasing population.

58. (a) $f(20) = 10{,}000$ means that when the price of the fabric is \$20/yard, 10,000 yards will be sold.

$f'(20) = -350$ means that as the price of the fabric increases past \$20/yard, the amount of fabric which will be sold is decreasing at a rate of 350 yards per (dollar per yard).

(b) $R(p) = pf(p) \quad \Rightarrow \quad R'(p) = pf'(p) + f(p) \cdot 1 \quad \Rightarrow \quad R'(20) = 20f'(20) + f(20) \cdot 1 = 20(-350) + 10{,}000 = 3000.$

This means that as the price of the fabric increases past \$20/yard, the total revenue is increasing at \$3000/(\$/yard). Note that the Product Rule indicates that we will lose \$7000/(\$/yard) due to selling less fabric, but this loss is more than made up for by the additional revenue due to the increase in price.

59. $v = \dfrac{0.14[\mathrm{S}]}{0.015 + [\mathrm{S}]} \quad \Rightarrow \quad \dfrac{dv}{d[\mathrm{S}]} = \dfrac{(0.015 + [\mathrm{S}])(0.14) - (0.14[\mathrm{S}])(1)}{(0.015 + [\mathrm{S}])^2} = \dfrac{0.0021}{(0.015 + [\mathrm{S}])^2}.$

$dv/d[\mathrm{S}]$ represents the rate of change of the rate of an enzymatic reaction with respect to the concentration of a substrate S.

60. $B(t) = N(t)\,M(t) \quad \Rightarrow \quad B'(t) = N(t)\,M'(t) + M(t)\,N'(t)$, so

$B'(4) = N(4)\,M'(4) + M(4)\,N'(4) = 820(0.14) + 1.2(50) = 174.8$ g/week.

61. (a) $(fgh)' = [(fg)h]' = (fg)'h + (fg)h' = (f'g + fg')h + (fg)h' = f'gh + fg'h + fgh'$

(b) Putting $f = g = h$ in part (a), we have $\dfrac{d}{dx}[f(x)]^3 = (fff)' = f'ff + ff'f + fff' = 3fff' = 3[f(x)]^2 f'(x)$.

(c) $\dfrac{d}{dx}\left(e^{3x}\right) = \dfrac{d}{dx}\left(e^x\right)^3 = 3(e^x)^2 e^x = 3e^{2x}e^x = 3e^{3x}$

62. (a) We use the Product Rule repeatedly: $F = fg \quad \Rightarrow \quad F' = f'g + fg' \quad \Rightarrow$

$F'' = (f''g + f'g') + (f'g' + fg'') = f''g + 2f'g' + fg''.$

(b) $F''' = f'''g + f''g' + 2\left(f''g' + f'g''\right) + f'g'' + fg''' = f'''g + 3f''g' + 3f'g'' + fg''' \quad \Rightarrow$

$$\begin{aligned} F^{(4)} &= f^{(4)}g + f'''g' + 3\left(f'''g' + f''g''\right) + 3\left(f''g'' + f'g'''\right) + f'g''' + fg^{(4)} \\ &= f^{(4)}g + 4f'''g' + 6f''g'' + 4f'g''' + fg^{(4)} \end{aligned}$$

(c) By analogy with the Binomial Theorem, we make the guess:

$$F^{(n)} = f^{(n)}g + nf^{(n-1)}g' + \binom{n}{2} f^{(n-2)}g'' + \cdots + \binom{n}{k} f^{(n-k)}g^{(k)} + \cdots + nf'g^{(n-1)} + fg^{(n)},$$

where $\dbinom{n}{k} = \dfrac{n!}{k!\,(n-k)!} = \dfrac{n(n-1)(n-2)\cdots(n-k+1)}{k!}$.

63. For $f(x) = x^2e^x$, $f'(x) = x^2e^x + e^x(2x) = e^x(x^2 + 2x)$. Similarly, we have

$$f''(x) = e^x(x^2 + 4x + 2)$$
$$f'''(x) = e^x(x^2 + 6x + 6)$$
$$f^{(4)}(x) = e^x(x^2 + 8x + 12)$$
$$f^{(5)}(x) = e^x(x^2 + 10x + 20)$$

It appears that the coefficient of x in the quadratic term increases by 2 with each differentiation. The pattern for the constant terms seems to be $0 = 1 \cdot 0$, $2 = 2 \cdot 1$, $6 = 3 \cdot 2$, $12 = 4 \cdot 3$, $20 = 5 \cdot 4$. So a reasonable guess is that $f^{(n)}(x) = e^x[x^2 + 2nx + n(n-1)]$.

Proof: Let S_n be the statement that $f^{(n)}(x) = e^x[x^2 + 2nx + n(n-1)]$.

1. S_1 is true because $f'(x) = e^x(x^2 + 2x)$.
2. Assume that S_k is true; that is, $f^{(k)}(x) = e^x[x^2 + 2kx + k(k-1)]$. Then

$$\begin{aligned} f^{(k+1)}(x) &= \frac{d}{dx}\left[f^{(k)}(x)\right] = e^x(2x + 2k) + [x^2 + 2kx + k(k-1)]e^x \\ &= e^x[x^2 + (2k+2)x + (k^2 + k)] = e^x[x^2 + 2(k+1)x + (k+1)k] \end{aligned}$$

   This shows that S_{k+1} is true.
3. Therefore, by mathematical induction, S_n is true for all n; that is, $f^{(n)}(x) = e^x[x^2 + 2nx + n(n-1)]$ for every positive integer n.

64. (a) $\dfrac{d}{dx}\left(\dfrac{1}{g(x)}\right) = \dfrac{g(x) \cdot \frac{d}{dx}(1) - 1 \cdot \frac{d}{dx}[g(x)]}{[g(x)]^2}$ [Quotient Rule] $= \dfrac{g(x) \cdot 0 - 1 \cdot g'(x)}{[g(x)]^2} = \dfrac{0 - g'(x)}{[g(x)]^2} = -\dfrac{g'(x)}{[g(x)]^2}$

(b) $\dfrac{d}{dt}\left(\dfrac{1}{t^3 + 2t^2 - 1}\right) = -\dfrac{(t^3 + 2t^2 - 1)'}{(t^3 + 2t^2 - 1)^2} = -\dfrac{3t^2 + 4t}{(t^3 + 2t^2 - 1)^2}$

(c) $\dfrac{d}{dx}(x^{-n}) = \dfrac{d}{dx}\left(\dfrac{1}{x^n}\right) = -\dfrac{(x^n)'}{(x^n)^2}$ [by the Reciprocal Rule] $= -\dfrac{nx^{n-1}}{x^{2n}} = -nx^{n-1-2n} = -nx^{-n-1}$

3.3 Derivatives of Trigonometric Functions

1. $f(x) = x^2 \sin x \overset{\text{PR}}{\Rightarrow} f'(x) = x^2 \cos x + (\sin x)(2x) = x^2 \cos x + 2x \sin x$

2. $f(x) = x \cos x + 2 \tan x \Rightarrow f'(x) = x(-\sin x) + (\cos x)(1) + 2\sec^2 x = \cos x - x \sin x + 2\sec^2 x$

3. $f(x) = e^x \cos x \Rightarrow f'(x) = e^x(-\sin x) + (\cos x)e^x = e^x(\cos x - \sin x)$

4. $y = 2\sec x - \csc x \Rightarrow y' = 2(\sec x\ \tan x) - (-\csc x\ \cot x) = 2\sec x\ \tan x + \csc x\ \cot x$

5. $y = \sec\theta\ \tan\theta \Rightarrow y' = \sec\theta\,(\sec^2\theta) + \tan\theta\,(\sec\theta\tan\theta) = \sec\theta\,(\sec^2\theta + \tan^2\theta)$. Using the identity $1 + \tan^2\theta = \sec^2\theta$, we can write alternative forms of the answer as $\sec\theta\,(1 + 2\tan^2\theta)$ or $\sec\theta\,(2\sec^2\theta - 1)$.

6. $g(\theta) = e^\theta(\tan\theta - \theta) \Rightarrow g'(\theta) = e^\theta(\sec^2\theta - 1) + (\tan\theta - \theta)e^\theta = e^\theta(\sec^2\theta - 1 + \tan\theta - \theta)$

7. $y = c\cos t + t^2 \sin t \quad \Rightarrow \quad y' = c(-\sin t) + t^2(\cos t) + \sin t\,(2t) = -c\sin t + t(t\cos t + 2\sin t)$

8. $f(t) = \dfrac{\cot t}{e^t} \quad \Rightarrow \quad f'(t) = \dfrac{e^t(-\csc^2 t) - (\cot t)e^t}{(e^t)^2} = \dfrac{e^t(-\csc^2 t - \cot t)}{(e^t)^2} = -\dfrac{\csc^2 t + \cot t}{e^t}$

9. $y = \dfrac{x}{2 - \tan x} \quad \Rightarrow \quad y' = \dfrac{(2 - \tan x)(1) - x(-\sec^2 x)}{(2 - \tan x)^2} = \dfrac{2 - \tan x + x\sec^2 x}{(2 - \tan x)^2}$

10. $y = \sin\theta\,\cos\theta \quad \Rightarrow \quad y' = \sin\theta(-\sin\theta) + \cos\theta(\cos\theta) = \cos^2\theta - \sin^2\theta \quad$ [or $\cos 2\theta$]

11. $f(\theta) = \dfrac{\sin\theta}{1 + \cos\theta} \quad \Rightarrow$

$$f'(\theta) = \frac{(1+\cos\theta)\cos\theta - (\sin\theta)(-\sin\theta)}{(1+\cos\theta)^2} = \frac{\cos\theta + \cos^2\theta + \sin^2\theta}{(1+\cos\theta)^2} = \frac{\cos\theta + 1}{(1+\cos\theta)^2} = \frac{1}{1+\cos\theta}$$

12. $y = \dfrac{\cos x}{1 - \sin x} \quad \Rightarrow$

$$y' = \frac{(1-\sin x)(-\sin x) - \cos x(-\cos x)}{(1-\sin x)^2} = \frac{-\sin x + \sin^2 x + \cos^2 x}{(1-\sin x)^2} = \frac{-\sin x + 1}{(1-\sin x)^2} = \frac{1}{1-\sin x}$$

13. $y = \dfrac{t\sin t}{1+t} \quad \Rightarrow$

$$y' = \frac{(1+t)(t\cos t + \sin t) - t\sin t(1)}{(1+t)^2} = \frac{t\cos t + \sin t + t^2\cos t + t\sin t - t\sin t}{(1+t)^2} = \frac{(t^2+t)\cos t + \sin t}{(1+t)^2}$$

14. $y = \dfrac{\sin t}{1 + \tan t} \quad \Rightarrow$

$$y' = \frac{(1+\tan t)\cos t - (\sin t)\sec^2 t}{(1+\tan t)^2} = \frac{\cos t + \sin t - \dfrac{\sin t}{\cos^2 t}}{(1+\tan t)^2} = \frac{\cos t + \sin t - \tan t\sec t}{(1+\tan t)^2}$$

15. Using Exercise 3.2.61(a) , $f(\theta) = \theta\cos\theta\,\sin\theta \quad \Rightarrow$

$$f'(\theta) = 1\cos\theta\,\sin\theta + \theta(-\sin\theta)\sin\theta + \theta\cos\theta(\cos\theta) = \cos\theta\,\sin\theta - \theta\sin^2\theta + \theta\cos^2\theta$$
$$= \sin\theta\,\cos\theta + \theta(\cos^2\theta - \sin^2\theta) = \tfrac{1}{2}\sin 2\theta + \theta\cos 2\theta \quad \text{[using double-angle formulas]}$$

16. Using Exercise 3.2.61(a), $f(t) = te^t\cot t \quad \Rightarrow$

$$f'(t) = 1e^t\cot t + te^t\cot t + te^t(-\csc^2 t) = e^t(\cot t + t\cot t - t\csc^2 t)$$

17. $\dfrac{d}{dx}(\csc x) = \dfrac{d}{dx}\left(\dfrac{1}{\sin x}\right) = \dfrac{(\sin x)(0) - 1(\cos x)}{\sin^2 x} = \dfrac{-\cos x}{\sin^2 x} = -\dfrac{1}{\sin x}\cdot\dfrac{\cos x}{\sin x} = -\csc x\,\cot x$

18. $\dfrac{d}{dx}(\sec x) = \dfrac{d}{dx}\left(\dfrac{1}{\cos x}\right) = \dfrac{(\cos x)(0) - 1(-\sin x)}{\cos^2 x} = \dfrac{\sin x}{\cos^2 x} = \dfrac{1}{\cos x}\cdot\dfrac{\sin x}{\cos x} = \sec x\,\tan x$

19. $\dfrac{d}{dx}(\cot x) = \dfrac{d}{dx}\left(\dfrac{\cos x}{\sin x}\right) = \dfrac{(\sin x)(-\sin x) - (\cos x)(\cos x)}{\sin^2 x} = -\dfrac{\sin^2 x + \cos^2 x}{\sin^2 x} = -\dfrac{1}{\sin^2 x} = -\csc^2 x$

20. $f(x) = \cos x \quad \Rightarrow$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h} = \lim_{h\to 0} \frac{\cos x\,\cos h - \sin x\,\sin h - \cos x}{h}$$

$$= \lim_{h\to 0}\left(\cos x\,\frac{\cos h - 1}{h} - \sin x\,\frac{\sin h}{h}\right) = \cos x\,\lim_{h\to 0}\frac{\cos h - 1}{h} - \sin x\,\lim_{h\to 0}\frac{\sin h}{h}$$

$$= (\cos x)(0) - (\sin x)(1) = -\sin x$$

21. $y = \sin x + \cos x \quad \Rightarrow \quad y' = \cos x - \sin x$, so $y'(0) = \cos 0 - \sin 0 = 1 - 0 = 1$. An equation of the tangent line to the curve $y = \sin x + \cos x$ at the point $(0, 1)$ is $y - 1 = 1(x - 0)$ or $y = x + 1$.

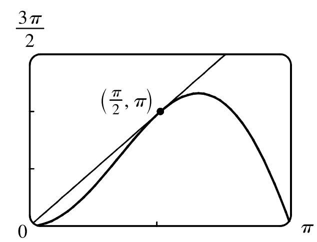
22. $y = e^x \cos x \quad \Rightarrow \quad y' = e^x(-\sin x) + (\cos x)e^x = e^x(\cos x - \sin x) \quad \Rightarrow$ the slope of the tangent line at $(0, 1)$ is $e^0(\cos 0 - \sin 0) = 1(1 - 0) = 1$ and an equation is $y - 1 = 1(x - 0)$ or $y = x + 1$.

23. $y = \cos x - \sin x \quad \Rightarrow \quad y' = -\sin x - \cos x$, so $y'(\pi) = -\sin\pi - \cos\pi = 0 - (-1) = 1$. An equation of the tangent line to the curve $y = \cos x - \sin x$ at the point $(\pi, -1)$ is $y - (-1) = 1(x - \pi)$ or $y = x - \pi - 1$.

24. $y = x + \tan x \quad \Rightarrow \quad y' = 1 + \sec^2 x$, so $y'(\pi) = 1 + (-1)^2 = 2$. An equation of the tangent line to the curve $y = x + \tan x$ at the point (π, π) is $y - \pi = 2(x - \pi)$ or $y = 2x - \pi$.

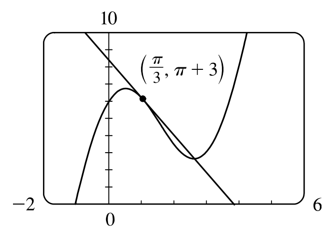
25. (a) $y = 2x\sin x \quad \Rightarrow \quad y' = 2(x\cos x + \sin x \cdot 1)$. At $\left(\frac{\pi}{2}, \pi\right)$, $y' = 2\left(\frac{\pi}{2}\cos\frac{\pi}{2} + \sin\frac{\pi}{2}\right) = 2(0 + 1) = 2$, and an equation of the tangent line is $y - \pi = 2\left(x - \frac{\pi}{2}\right)$, or $y = 2x$.

(b)



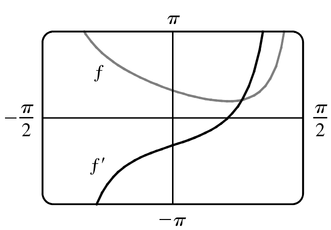
26. (a) $y = 3x + 6\cos x \quad \Rightarrow \quad y' = 3 - 6\sin x$. At $\left(\frac{\pi}{3}, \pi + 3\right)$, $y' = 3 - 6\sin\frac{\pi}{3} = 3 - 6\,\frac{\sqrt{3}}{2} = 3 - 3\sqrt{3}$, and an equation of the tangent line is $y - (\pi + 3) = \left(3 - 3\sqrt{3}\right)\left(x - \frac{\pi}{3}\right)$, or $y = \left(3 - 3\sqrt{3}\right)x + 3 + \pi\sqrt{3}$.

(b)



27. (a) $f(x) = \sec x - x \quad \Rightarrow \quad f'(x) = \sec x\,\tan x - 1$

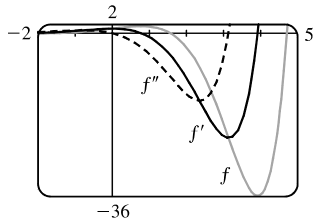
(b)



Note that $f' = 0$ where f has a minimum. Also note that f' is negative when f is decreasing and f' is positive when f is increasing.

28. (a) $f(x) = e^x\cos x \quad \Rightarrow \quad f'(x) = e^x(-\sin x) + (\cos x)\,e^x = e^x(\cos x - \sin x) \quad \Rightarrow$

$$f''(x) = e^x(-\sin x - \cos x) + (\cos x - \sin x)\,e^x = e^x(-\sin x - \cos x + \cos x - \sin x) = -2e^x\sin x$$

(b) 

Note that $f' = 0$ where f has a minimum and $f'' = 0$ where f' has a minimum. Also note that f' is negative when f is decreasing and f'' is negative when f' is decreasing.

29. $H(\theta) = \theta\sin\theta \quad\Rightarrow\quad H'(\theta) = \theta\,(\cos\theta) + (\sin\theta)\cdot 1 = \theta\cos\theta + \sin\theta \quad\Rightarrow$

$H''(\theta) = \theta\,(-\sin\theta) + (\cos\theta)\cdot 1 + \cos\theta = -\theta\sin\theta + 2\cos\theta$

30. $f(t) = \sec t \quad\Rightarrow\quad f'(t) = \sec t\,\tan t \quad\Rightarrow\quad f''(t) = (\sec t)\sec^2 t + (\tan t)\sec t\,\tan t = \sec^3 t + \sec t\,\tan^2 t$, so

$f''\left(\frac{\pi}{4}\right) = \left(\sqrt{2}\right)^3 + \sqrt{2}(1)^2 = 2\sqrt{2} + \sqrt{2} = 3\sqrt{2}.$

31. (a) $f(x) = \dfrac{\tan x - 1}{\sec x} \quad\Rightarrow$

$$f'(x) = \frac{\sec x(\sec^2 x) - (\tan x - 1)(\sec x\tan x)}{(\sec x)^2} = \frac{\sec x(\sec^2 x - \tan^2 x + \tan x)}{\sec^2 x} = \frac{1 + \tan x}{\sec x}$$

(b) $f(x) = \dfrac{\tan x - 1}{\sec x} = \dfrac{\dfrac{\sin x}{\cos x} - 1}{\dfrac{1}{\cos x}} = \dfrac{\dfrac{\sin x - \cos x}{\cos x}}{\dfrac{1}{\cos x}} = \sin x - \cos x \quad\Rightarrow\quad f'(x) = \cos x - (-\sin x) = \cos x + \sin x$

(c) From part (a), $f'(x) = \dfrac{1 + \tan x}{\sec x} = \dfrac{1}{\sec x} + \dfrac{\tan x}{\sec x} = \cos x + \sin x$, which is the expression for $f'(x)$ in part (b).

32. (a) $g(x) = f(x)\sin x \quad\Rightarrow\quad g'(x) = f(x)\cos x + \sin x\cdot f'(x)$, so

$g'\left(\frac{\pi}{3}\right) = f\left(\frac{\pi}{3}\right)\cos\frac{\pi}{3} + \sin\frac{\pi}{3}\cdot f'\left(\frac{\pi}{3}\right) = 4\cdot\frac{1}{2} + \frac{\sqrt{3}}{2}\cdot(-2) = 2 - \sqrt{3}$

(b) $h(x) = \dfrac{\cos x}{f(x)} \quad\Rightarrow\quad h'(x) = \dfrac{f(x)\cdot(-\sin x) - \cos x\cdot f'(x)}{[f(x)]^2}$, so

$$h'\left(\tfrac{\pi}{3}\right) = \frac{f\left(\frac{\pi}{3}\right)\cdot\left(-\sin\frac{\pi}{3}\right) - \cos\frac{\pi}{3}\cdot f'\left(\frac{\pi}{3}\right)}{\left[f\left(\frac{\pi}{3}\right)\right]^2} = \frac{4\left(-\frac{\sqrt{3}}{2}\right) - \left(\frac{1}{2}\right)(-2)}{4^2} = \frac{-2\sqrt{3} + 1}{16} = \frac{1 - 2\sqrt{3}}{16}$$

33. $f(x) = x + 2\sin x$ has a horizontal tangent when $f'(x) = 0 \quad\Leftrightarrow\quad 1 + 2\cos x = 0 \quad\Leftrightarrow\quad \cos x = -\frac{1}{2} \quad\Leftrightarrow$

$x = \frac{2\pi}{3} + 2\pi n$ or $\frac{4\pi}{3} + 2\pi n$, where n is an integer. Note that $\frac{4\pi}{3}$ and $\frac{2\pi}{3}$ are $\pm\frac{\pi}{3}$ units from π. This allows us to write the solutions in the more compact equivalent form $(2n+1)\pi \pm \frac{\pi}{3}$, n an integer.

34. $f(x) = e^x\cos x$ has a horizontal tangent when $f'(x) = 0$. $f'(x) = e^x(-\sin x) + (\cos x)e^x = e^x(\cos x - \sin x)$.

$f'(x) = 0 \quad\Leftrightarrow\quad \cos x - \sin x = 0 \quad\Leftrightarrow\quad \cos x = \sin x \quad\Leftrightarrow\quad \tan x = 1 \quad\Leftrightarrow\quad x = \frac{\pi}{4} + n\pi$, n an integer.

35. (a) $x(t) = 8\sin t \quad\Rightarrow\quad v(t) = x'(t) = 8\cos t \quad\Rightarrow\quad a(t) = x''(t) = -8\sin t$

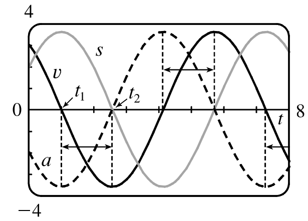
(b) The mass at time $t = \frac{2\pi}{3}$ has position $x\left(\frac{2\pi}{3}\right) = 8\sin\frac{2\pi}{3} = 8\left(\frac{\sqrt{3}}{2}\right) = 4\sqrt{3}$, velocity $v\left(\frac{2\pi}{3}\right) = 8\cos\frac{2\pi}{3} = 8\left(-\frac{1}{2}\right) = -4$,

and acceleration $a\left(\frac{2\pi}{3}\right) = -8\sin\frac{2\pi}{3} = -8\left(\frac{\sqrt{3}}{2}\right) = -4\sqrt{3}$. Since $v\left(\frac{2\pi}{3}\right) < 0$, the particle is moving to the left.

36. (a) $s(t) = 2\cos t + 3\sin t \quad\Rightarrow\quad v(t) = -2\sin t + 3\cos t \quad\Rightarrow$

$a(t) = -2\cos t - 3\sin t$

(b)

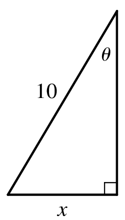


(c) $s = 0 \quad\Rightarrow\quad t_2 \approx 2.55$. So the mass passes through the equilibrium position for the first time when $t \approx 2.55$ s.

(d) $v = 0 \quad\Rightarrow\quad t_1 \approx 0.98$, $s(t_1) \approx 3.61$ cm. So the mass travels a maximum of about 3.6 cm (upward and downward) from its equilibrium position.

(e) The speed $|v|$ is greatest when $s = 0$, that is, when $t = t_2 + n\pi$, n a positive integer.

37.

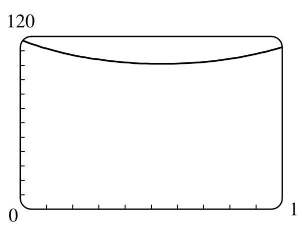


From the diagram we can see that $\sin\theta = x/10 \quad\Leftrightarrow\quad x = 10\sin\theta$. We want to find the rate of change of x with respect to θ, that is, $dx/d\theta$. Taking the derivative of $x = 10\sin\theta$, we get $dx/d\theta = 10(\cos\theta)$. So when $\theta = \frac{\pi}{3}$, $\frac{dx}{d\theta} = 10\cos\frac{\pi}{3} = 10\left(\frac{1}{2}\right) = 5$ ft/rad.

38. (a) $F = \dfrac{\mu W}{\mu\sin\theta + \cos\theta} \quad\Rightarrow\quad \dfrac{dF}{d\theta} = \dfrac{(\mu\sin\theta + \cos\theta)(0) - \mu W(\mu\cos\theta - \sin\theta)}{(\mu\sin\theta + \cos\theta)^2} = \dfrac{\mu W(\sin\theta - \mu\cos\theta)}{(\mu\sin\theta + \cos\theta)^2}$

(b) $\dfrac{dF}{d\theta} = 0 \quad\Leftrightarrow\quad \mu W(\sin\theta - \mu\cos\theta) = 0 \quad\Leftrightarrow\quad \sin\theta = \mu\cos\theta \quad\Leftrightarrow\quad \tan\theta = \mu \quad\Leftrightarrow\quad \theta = \tan^{-1}\mu$

(c)



From the graph of $F = \dfrac{0.6(50)}{0.6\sin\theta + \cos\theta}$ for $0 \le \theta \le 1$, we see that $\dfrac{dF}{d\theta} = 0 \quad\Rightarrow\quad \theta \approx 0.54$. Checking this with part (b) and $\mu = 0.6$, we calculate $\theta = \tan^{-1} 0.6 \approx 0.54$. So the value from the graph is consistent with the value in part (b).

39. $\displaystyle\lim_{x\to 0}\frac{\sin 5x}{3x} = \lim_{x\to 0}\frac{5}{3}\left(\frac{\sin 5x}{5x}\right) = \frac{5}{3}\lim_{x\to 0}\frac{\sin 5x}{5x} = \frac{5}{3}\lim_{\theta\to 0}\frac{\sin\theta}{\theta} \quad [\theta = 5x] \quad = \frac{5}{3}\cdot 1 = \frac{5}{3}$

40. $\displaystyle\lim_{x\to 0}\frac{\sin x}{\sin\pi x} = \lim_{x\to 0}\frac{\sin x}{x}\cdot\frac{\pi x}{\sin\pi x}\cdot\frac{1}{\pi} = \lim_{x\to 0}\frac{\sin x}{x}\cdot\lim_{\theta\to 0}\frac{\theta}{\sin\theta}\cdot\frac{1}{\pi} \quad [\theta = \pi x]$

$\displaystyle = 1\cdot\lim_{\theta\to 0}\frac{1}{\dfrac{\sin\theta}{\theta}}\cdot\frac{1}{\pi} = 1\cdot 1\cdot\frac{1}{\pi} = \frac{1}{\pi}$

41. $\displaystyle\lim_{t\to 0}\frac{\tan 6t}{\sin 2t} = \lim_{t\to 0}\left(\frac{\sin 6t}{t}\cdot\frac{1}{\cos 6t}\cdot\frac{t}{\sin 2t}\right) = \lim_{t\to 0}\frac{6\sin 6t}{6t}\cdot\lim_{t\to 0}\frac{1}{\cos 6t}\cdot\lim_{t\to 0}\frac{2t}{2\sin 2t}$

$\displaystyle = 6\lim_{t\to 0}\frac{\sin 6t}{6t}\cdot\lim_{t\to 0}\frac{1}{\cos 6t}\cdot\frac{1}{2}\lim_{t\to 0}\frac{2t}{\sin 2t} = 6(1)\cdot\frac{1}{1}\cdot\frac{1}{2}(1) = 3$

42. $\displaystyle\lim_{\theta\to 0}\frac{\cos\theta - 1}{\sin\theta} = \lim_{\theta\to 0}\frac{\dfrac{\cos\theta - 1}{\theta}}{\dfrac{\sin\theta}{\theta}} = \frac{\displaystyle\lim_{\theta\to 0}\frac{\cos\theta - 1}{\theta}}{\displaystyle\lim_{\theta\to 0}\frac{\sin\theta}{\theta}} = \frac{0}{1} = 0$

43. $\lim\limits_{x\to 0} \dfrac{\sin 3x}{5x^3 - 4x} = \lim\limits_{x\to 0} \left(\dfrac{\sin 3x}{3x} \cdot \dfrac{3}{5x^2 - 4} \right) = \lim\limits_{x\to 0} \dfrac{\sin 3x}{3x} \cdot \lim\limits_{x\to 0} \dfrac{3}{5x^2 - 4} = 1 \cdot \left(\dfrac{3}{-4} \right) = -\dfrac{3}{4}$

44. $\lim\limits_{x\to 0} \dfrac{\sin 3x \sin 5x}{x^2} = \lim\limits_{x\to 0} \left(\dfrac{3 \sin 3x}{3x} \cdot \dfrac{5 \sin 5x}{5x} \right) = \lim\limits_{x\to 0} \dfrac{3 \sin 3x}{3x} \cdot \lim\limits_{x\to 0} \dfrac{5 \sin 5x}{5x}$

$$= 3 \lim_{x\to 0} \frac{\sin 3x}{3x} \cdot 5 \lim_{x\to 0} \frac{\sin 5x}{5x} = 3(1) \cdot 5(1) = 15$$

45. Divide numerator and denominator by θ. ($\sin\theta$ also works.)

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta + \tan\theta} = \lim_{\theta\to 0} \frac{\dfrac{\sin\theta}{\theta}}{1 + \dfrac{\sin\theta}{\theta} \cdot \dfrac{1}{\cos\theta}} = \frac{\lim\limits_{\theta\to 0} \dfrac{\sin\theta}{\theta}}{1 + \lim\limits_{\theta\to 0} \dfrac{\sin\theta}{\theta} \lim\limits_{\theta\to 0} \dfrac{1}{\cos\theta}} = \frac{1}{1 + 1 \cdot 1} = \frac{1}{2}$$

46. $\lim\limits_{x\to 0} \csc x \, \sin(\sin x) = \lim\limits_{x\to 0} \dfrac{\sin(\sin x)}{\sin x} = \lim\limits_{\theta\to 0} \dfrac{\sin\theta}{\theta}$ [As $x \to 0$, $\theta = \sin x \to 0$.] $= 1$

47. $\lim\limits_{\theta\to 0} \dfrac{\cos\theta - 1}{2\theta^2} = \lim\limits_{\theta\to 0} \dfrac{\cos\theta - 1}{2\theta^2} \cdot \dfrac{\cos\theta + 1}{\cos\theta + 1} = \lim\limits_{\theta\to 0} \dfrac{\cos^2\theta - 1}{2\theta^2(\cos\theta + 1)} = \lim\limits_{\theta\to 0} \dfrac{-\sin^2\theta}{2\theta^2(\cos\theta + 1)}$

$$= -\frac{1}{2} \lim_{\theta\to 0} \frac{\sin\theta}{\theta} \cdot \frac{\sin\theta}{\theta} \cdot \frac{1}{\cos\theta + 1} = -\frac{1}{2} \lim_{\theta\to 0} \frac{\sin\theta}{\theta} \cdot \lim_{\theta\to 0} \frac{\sin\theta}{\theta} \cdot \lim_{\theta\to 0} \frac{1}{\cos\theta + 1}$$

$$= -\frac{1}{2} \cdot 1 \cdot 1 \cdot \frac{1}{1+1} = -\frac{1}{4}$$

48. $\lim\limits_{x\to 0} \dfrac{\sin(x^2)}{x} = \lim\limits_{x\to 0} \left[x \cdot \dfrac{\sin(x^2)}{x \cdot x} \right] = \lim\limits_{x\to 0} x \cdot \lim\limits_{x\to 0} \dfrac{\sin(x^2)}{x^2} = 0 \cdot \lim\limits_{y\to 0^+} \dfrac{\sin y}{y}$ [where $y = x^2$] $= 0 \cdot 1 = 0$

49. $\lim\limits_{x\to\pi/4} \dfrac{1 - \tan x}{\sin x - \cos x} = \lim\limits_{x\to\pi/4} \dfrac{\left(1 - \dfrac{\sin x}{\cos x}\right) \cdot \cos x}{(\sin x - \cos x) \cdot \cos x} = \lim\limits_{x\to\pi/4} \dfrac{\cos x - \sin x}{(\sin x - \cos x)\cos x} = \lim\limits_{x\to\pi/4} \dfrac{-1}{\cos x} = \dfrac{-1}{1/\sqrt{2}} = -\sqrt{2}$

50. $\lim\limits_{x\to 1} \dfrac{\sin(x-1)}{x^2 + x - 2} = \lim\limits_{x\to 1} \dfrac{\sin(x-1)}{(x+2)(x-1)} = \lim\limits_{x\to 1} \dfrac{1}{x+2} \lim\limits_{x\to 1} \dfrac{\sin(x-1)}{x-1} = \frac{1}{3} \cdot 1 = \frac{1}{3}$

51. $\dfrac{d}{dx}(\sin x) = \cos x \;\Rightarrow\; \dfrac{d^2}{dx^2}(\sin x) = -\sin x \;\Rightarrow\; \dfrac{d^3}{dx^3}(\sin x) = -\cos x \;\Rightarrow\; \dfrac{d^4}{dx^4}(\sin x) = \sin x.$

The derivatives of $\sin x$ occur in a cycle of four. Since $99 = 4(24) + 3$, we have $\dfrac{d^{99}}{dx^{99}}(\sin x) = \dfrac{d^3}{dx^3}(\sin x) = -\cos x.$

52. Let $f(x) = x \sin x$ and $h(x) = \sin x$, so $f(x) = xh(x)$. Then $f'(x) = h(x) + xh'(x)$,

$f''(x) = h'(x) + h'(x) + xh''(x) = 2h'(x) + xh''(x)$,

$f'''(x) = 2h''(x) + h''(x) + xh'''(x) = 3h''(x) + xh'''(x), \cdots, f^{(n)}(x) = nh^{(n-1)}(x) + xh^{(n)}(x).$

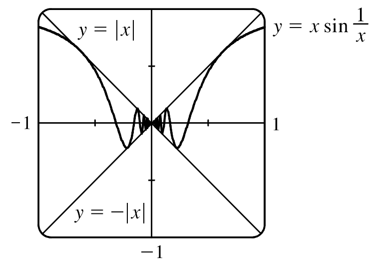
Since $34 = 4(8) + 2$, we have $h^{(34)}(x) = h^{(2)}(x) = \dfrac{d^2}{dx^2}(\sin x) = -\sin x$ and $h^{(35)}(x) = -\cos x$.

Thus, $\dfrac{d^{35}}{dx^{35}}(x \sin x) = 35h^{(34)}(x) + xh^{(35)}(x) = -35 \sin x - x \cos x$.

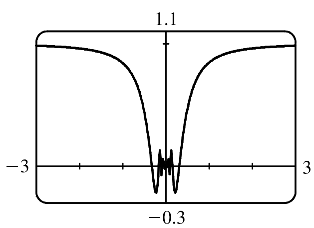
53. $y = A\sin x + B\cos x \quad\Rightarrow\quad y' = A\cos x - B\sin x \quad\Rightarrow\quad y'' = -A\sin x - B\cos x$. Substituting these expressions for y, y', and y'' into the given differential equation $y'' + y' - 2y = \sin x$ gives us $(-A\sin x - B\cos x) + (A\cos x - B\sin x) - 2(A\sin x + B\cos x) = \sin x \quad\Leftrightarrow$ $-3A\sin x - B\sin x + A\cos x - 3B\cos x = \sin x \quad\Leftrightarrow\quad (-3A - B)\sin x + (A - 3B)\cos x = 1\sin x$, so we must have $-3A - B = 1$ and $A - 3B = 0$ (since 0 is the coefficient of $\cos x$ on the right side). Solving for A and B, we add the first equation to three times the second to get $B = -\frac{1}{10}$ and $A = -\frac{3}{10}$.

54. (a) Let $\theta = \dfrac{1}{x}$. Then as $x \to \infty$, $\theta \to 0^+$, and $\displaystyle\lim_{x\to\infty} x\sin\frac{1}{x} = \lim_{\theta\to 0^+}\frac{1}{\theta}\sin\theta = \lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$.

(b) Since $-1 \le \sin(1/x) \le 1$, we have (as illustrated in the figure) $-|x| \le x\sin(1/x) \le |x|$. We know that $\displaystyle\lim_{x\to 0}(|x|) = 0$ and $\displaystyle\lim_{x\to 0}(-|x|) = 0$; so by the Squeeze Theorem, $\displaystyle\lim_{x\to 0} x\sin(1/x) = 0$.



(c)



55. (a) $\dfrac{d}{dx}\tan x = \dfrac{d}{dx}\dfrac{\sin x}{\cos x} \quad\Rightarrow\quad \sec^2 x = \dfrac{\cos x\,\cos x - \sin x\,(-\sin x)}{\cos^2 x} = \dfrac{\cos^2 x + \sin^2 x}{\cos^2 x}$. So $\sec^2 x = \dfrac{1}{\cos^2 x}$.

(b) $\dfrac{d}{dx}\sec x = \dfrac{d}{dx}\dfrac{1}{\cos x} \quad\Rightarrow\quad \sec x\,\tan x = \dfrac{(\cos x)(0) - 1(-\sin x)}{\cos^2 x}$. So $\sec x\,\tan x = \dfrac{\sin x}{\cos^2 x}$.

(c) $\dfrac{d}{dx}(\sin x + \cos x) = \dfrac{d}{dx}\dfrac{1 + \cot x}{\csc x} \quad\Rightarrow$

$$\cos x - \sin x = \frac{\csc x\,(-\csc^2 x) - (1 + \cot x)(-\csc x\,\cot x)}{\csc^2 x} = \frac{\csc x\,[-\csc^2 x + (1 + \cot x)\,\cot x]}{\csc^2 x}$$
$$= \frac{-\csc^2 x + \cot^2 x + \cot x}{\csc x} = \frac{-1 + \cot x}{\csc x}$$

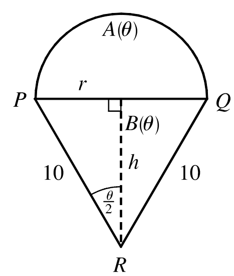
So $\cos x - \sin x = \dfrac{\cot x - 1}{\csc x}$.

56. We get the following formulas for r and h in terms of θ:

$$\sin\frac{\theta}{2} = \frac{r}{10} \quad\Rightarrow\quad r = 10\sin\frac{\theta}{2} \quad\text{and}\quad \cos\frac{\theta}{2} = \frac{h}{10} \quad\Rightarrow\quad h = 10\cos\frac{\theta}{2}$$

Now $A(\theta) = \frac{1}{2}\pi r^2$ and $B(\theta) = \frac{1}{2}(2r)h = rh$. So

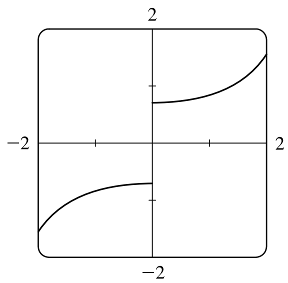
$$\lim_{\theta\to 0^+}\frac{A(\theta)}{B(\theta)} = \lim_{\theta\to 0^+}\frac{\frac{1}{2}\pi r^2}{rh} = \tfrac{1}{2}\pi\lim_{\theta\to 0^+}\frac{r}{h} = \tfrac{1}{2}\pi\lim_{\theta\to 0^+}\frac{10\sin(\theta/2)}{10\cos(\theta/2)}$$
$$= \tfrac{1}{2}\pi\lim_{\theta\to 0^+}\tan(\theta/2) = 0$$



57. By the definition of radian measure, $s = r\theta$, where r is the radius of the circle. By drawing the bisector of the angle θ, we can see that $\sin\dfrac{\theta}{2} = \dfrac{d/2}{r} \quad\Rightarrow\quad d = 2r\sin\dfrac{\theta}{2}$. So $\displaystyle\lim_{\theta\to 0^+}\frac{s}{d} = \lim_{\theta\to 0^+}\frac{r\theta}{2r\sin(\theta/2)} = \lim_{\theta\to 0^+}\frac{2\cdot(\theta/2)}{2\sin(\theta/2)} = \lim_{\theta\to 0}\frac{\theta/2}{\sin(\theta/2)} = 1$.

[This is just the reciprocal of the limit $\lim\limits_{x\to 0}\frac{\sin x}{x} = 1$ combined with the fact that as $\theta\to 0$, $\frac{\theta}{2}\to 0$ also.]

58. (a)



It appears that $f(x) = \dfrac{x}{\sqrt{1-\cos 2x}}$ has a jump discontinuity at $x = 0$.

(b) Using the identity $\cos 2x = 1 - \sin^2 x$, we have $\dfrac{x}{\sqrt{1-\cos 2x}} = \dfrac{x}{\sqrt{1-(1-2\sin^2 x)}} = \dfrac{x}{\sqrt{2\sin^2 x}} = \dfrac{x}{\sqrt{2}\,|\sin x|}$.

Thus,
$$\lim_{x\to 0^-}\frac{x}{\sqrt{1-\cos 2x}} = \lim_{x\to 0^-}\frac{x}{\sqrt{2}\,|\sin x|} = \frac{1}{\sqrt{2}}\lim_{x\to 0^-}\frac{x}{-(\sin x)}$$
$$= -\frac{1}{\sqrt{2}}\lim_{x\to 0^-}\frac{1}{\sin x/x} = -\frac{1}{\sqrt{2}}\cdot\frac{1}{1} = -\frac{\sqrt{2}}{2}$$

Evaluating $\lim\limits_{x\to 0^+} f(x)$ is similar, but $|\sin x| = +\sin x$, so we get $\frac{1}{2}\sqrt{2}$. These values appear to be reasonable values for the graph, so they confirm our answer to part (a).

Another method: Multiply numerator and denominator by $\sqrt{1+\cos 2x}$.

3.4 The Chain Rule

1. Let $u = g(x) = 1 + 4x$ and $y = f(u) = \sqrt[3]{u}$. Then $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = (\frac{1}{3}u^{-2/3})(4) = \dfrac{4}{3\sqrt[3]{(1+4x)^2}}$.

2. Let $u = g(x) = 2x^3 + 5$ and $y = f(u) = u^4$. Then $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = (4u^3)(6x^2) = 24x^2(2x^3+5)^3$.

3. Let $u = g(x) = \pi x$ and $y = f(u) = \tan u$. Then $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = (\sec^2 u)(\pi) = \pi\sec^2\pi x$.

4. Let $u = g(x) = \cot x$ and $y = f(u) = \sin u$. Then $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = (\cos u)(-\csc^2 x) = -\cos(\cot x)\csc^2 x$.

5. Let $u = g(x) = \sqrt{x}$ and $y = f(u) = e^u$. Then $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = (e^u)\left(\frac{1}{2}x^{-1/2}\right) = e^{\sqrt{x}}\cdot\dfrac{1}{2\sqrt{x}} = \dfrac{e^{\sqrt{x}}}{2\sqrt{x}}$.

6. Let $u = g(x) = 2 - e^x$ and $y = f(u) = \sqrt{u}$. Then $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = (\frac{1}{2}u^{-1/2})(-e^x) = -\dfrac{e^x}{2\sqrt{2-e^x}}$.

7. $F(x) = (5x^6 + 2x^3)^4 \quad\Rightarrow\quad F'(x) = 4(5x^6+2x^3)^3\cdot\dfrac{d}{dx}(5x^6+2x^3) = 4(5x^6+2x^3)^3(30x^5+6x^2)$.

We can factor as follows: $4(x^3)^3(5x^3+2)^3 6x^2(5x^3+1) = 24x^{11}(5x^3+2)^3(5x^3+1)$

8. $F(x) = (1 + x + x^2)^{99} \quad \Rightarrow \quad F'(x) = 99(1 + x + x^2)^{98} \cdot \dfrac{d}{dx}\left(1 + x + x^2\right) = 99(1 + x + x^2)^{98}(1 + 2x)$

9. $f(x) = \sqrt{5x+1} = (5x + 1)^{1/2} \quad \Rightarrow \quad f'(x) = \frac{1}{2}(5x + 1)^{-1/2}(5) = \dfrac{5}{2\sqrt{5x+1}}$

10. $f(x) = \dfrac{1}{\sqrt[3]{x^2 - 1}} = (x^2 - 1)^{-1/3} \quad \Rightarrow \quad f'(x) = -\dfrac{1}{3}(x^2 - 1)^{-4/3}(2x) = \dfrac{-2x}{3(x^2 - 1)^{4/3}}$

11. $f(\theta) = \cos(\theta^2) \quad \Rightarrow \quad f'(\theta) = -\sin(\theta^2) \cdot \dfrac{d}{d\theta}(\theta^2) = -\sin(\theta^2) \cdot (2\theta) = -2\theta \sin(\theta^2)$

12. $g(\theta) = \cos^2\theta = (\cos\theta)^2 \quad \Rightarrow \quad g'(\theta) = 2(\cos\theta)^1(-\sin\theta) = -2\sin\theta\,\cos\theta = -\sin 2\theta$

13. $y = x^2 e^{-3x} \quad \Rightarrow \quad y' = x^2 e^{-3x}(-3) + e^{-3x}(2x) = e^{-3x}(-3x^2 + 2x) = xe^{-3x}(2 - 3x)$

14. $f(t) = t\sin \pi t \quad \Rightarrow \quad f'(t) = t(\cos \pi t) \cdot \pi + (\sin \pi t) \cdot 1 = \pi t \cos \pi t + \sin \pi t$

15. $f(t) = e^{at}\sin bt \quad \Rightarrow \quad f'(t) = e^{at}(\cos bt) \cdot b + (\sin bt)e^{at} \cdot a = e^{at}(b\cos bt + a\sin bt)$

16. $g(x) = e^{x^2 - x} \quad \Rightarrow \quad g'(x) = e^{x^2 - x}(2x - 1)$

17. $f(x) = (2x - 3)^4(x^2 + x + 1)^5 \quad \Rightarrow$

$$\begin{aligned}
f'(x) &= (2x - 3)^4 \cdot 5(x^2 + x + 1)^4(2x + 1) + (x^2 + x + 1)^5 \cdot 4(2x - 3)^3 \cdot 2 \\
&= (2x - 3)^3(x^2 + x + 1)^4[(2x - 3) \cdot 5(2x + 1) + (x^2 + x + 1) \cdot 8] \\
&= (2x - 3)^3(x^2 + x + 1)^4(20x^2 - 20x - 15 + 8x^2 + 8x + 8) = (2x - 3)^3(x^2 + x + 1)^4(28x^2 - 12x - 7)
\end{aligned}$$

18. $g(x) = (x^2 + 1)^3(x^2 + 2)^6 \quad \Rightarrow$

$$\begin{aligned}
g'(x) &= (x^2 + 1)^3 \cdot 6(x^2 + 2)^5 \cdot 2x + (x^2 + 2)^6 \cdot 3(x^2 + 1)^2 \cdot 2x \\
&= 6x(x^2 + 1)^2(x^2 + 2)^5[2(x^2 + 1) + (x^2 + 2)] = 6x(x^2 + 1)^2(x^2 + 2)^5(3x^2 + 4)
\end{aligned}$$

19. $h(t) = (t + 1)^{2/3}(2t^2 - 1)^3 \quad \Rightarrow$

$$\begin{aligned}
h'(t) &= (t + 1)^{2/3} \cdot 3(2t^2 - 1)^2 \cdot 4t + (2t^2 - 1)^3 \cdot \tfrac{2}{3}(t + 1)^{-1/3} = \tfrac{2}{3}(t + 1)^{-1/3}(2t^2 - 1)^2[18t(t + 1) + (2t^2 - 1)] \\
&= \tfrac{2}{3}(t + 1)^{-1/3}(2t^2 - 1)^2(20t^2 + 18t - 1)
\end{aligned}$$

20. $F(t) = (3t - 1)^4(2t + 1)^{-3} \quad \Rightarrow$

$$\begin{aligned}
F'(t) &= (3t - 1)^4(-3)(2t + 1)^{-4}(2) + (2t + 1)^{-3} \cdot 4(3t - 1)^3(3) \\
&= 6(3t - 1)^3(2t + 1)^{-4}[-(3t - 1) + 2(2t + 1)] = 6(3t - 1)^3(2t + 1)^{-4}(t + 3)
\end{aligned}$$

21. $y = \sqrt{\dfrac{x}{x+1}} = \left(\dfrac{x}{x+1}\right)^{1/2} \quad \Rightarrow$

$$\begin{aligned}
y' &= \frac{1}{2}\left(\frac{x}{x+1}\right)^{-1/2}\frac{d}{dx}\left(\frac{x}{x+1}\right) = \frac{1}{2}\,\frac{x^{-1/2}}{(x+1)^{-1/2}}\,\frac{(x+1)(1) - x(1)}{(x+1)^2} \\
&= \frac{1}{2}\,\frac{(x+1)^{1/2}}{x^{1/2}}\,\frac{1}{(x+1)^2} = \frac{1}{2\sqrt{x}\,(x+1)^{3/2}}
\end{aligned}$$

22. $y = \left(x + \dfrac{1}{x}\right)^5 \quad \Rightarrow \quad y' = 5\left(x + \dfrac{1}{x}\right)^4 \dfrac{d}{dx}\left(x + \dfrac{1}{x}\right) = 5\left(x + \dfrac{1}{x}\right)^4 \left(1 - \dfrac{1}{x^2}\right).$

Another form of the answer is $\dfrac{5(x^2+1)^4(x^2-1)}{x^6}$.

23. $y = e^{\tan\theta} \quad \Rightarrow \quad y' = e^{\tan\theta}\dfrac{d}{d\theta}(\tan\theta) = (\sec^2\theta)e^{\tan\theta}$

24. Using Formula 5 and the Chain Rule, $f(t) = 2^{t^3} \quad \Rightarrow \quad f'(t) = 2^{t^3}\ln 2\,\dfrac{d}{dt}\,(t^3) = 3(\ln 2)t^2 2^{t^3}$.

25. $g(u) = \left(\dfrac{u^3-1}{u^3+1}\right)^8 \quad \Rightarrow$

$$g'(u) = 8\left(\frac{u^3-1}{u^3+1}\right)^7 \frac{d}{du}\frac{u^3-1}{u^3+1} = 8\,\frac{\left(u^3-1\right)^7}{(u^3+1)^7}\,\frac{(u^3+1)(3u^2)-(u^3-1)(3u^2)}{(u^3+1)^2}$$

$$= 8\,\frac{(u^3-1)^7}{(u^3+1)^7}\,\frac{3u^2[(u^3+1)-(u^3-1)]}{(u^3+1)^2} = 8\frac{(u^3-1)^7}{(u^3+1)^7}\,\frac{3u^2(2)}{(u^3+1)^2} = \frac{48u^2(u^3-1)^7}{(u^3+1)^9}$$

26. $s(t) = \sqrt{\dfrac{1+\sin t}{1+\cos t}} = \left(\dfrac{1+\sin t}{1+\cos t}\right)^{1/2} \quad \Rightarrow$

$$s'(t) = \frac{1}{2}\left(\frac{1+\sin t}{1+\cos t}\right)^{-1/2} \frac{(1+\cos t)\cos t - (1+\sin t)(-\sin t)}{(1+\cos t)^2}$$

$$= \frac{1}{2}\,\frac{(1+\sin t)^{-1/2}}{(1+\cos t)^{-1/2}}\,\frac{\cos t + \cos^2 t + \sin t + \sin^2 t}{(1+\cos t)^2} = \frac{\cos t + \sin t + 1}{2\sqrt{1+\sin t}\,(1+\cos t)^{3/2}}$$

27. Using Formula 5 and the Chain Rule, $r(t) = 10^{2\sqrt{t}} \quad \Rightarrow$

$$r'(t) = 10^{2\sqrt{t}}\ln 10\,\frac{d}{dt}\,(2\sqrt{t}\,) = 10^{2\sqrt{t}}\ln 10\left(2\cdot\tfrac{1}{2}t^{-1/2}\right) = \frac{(\ln 10)\,10^{2\sqrt{t}}}{\sqrt{t}}$$

28. $f(z) = e^{z/(z-1)} \quad \Rightarrow \quad f'(z) = e^{z/(z-1)}\dfrac{d}{dz}\dfrac{z}{z-1} = e^{z/(z-1)}\,\dfrac{(z-1)(1)-z(1)}{(z-1)^2} = -\dfrac{e^{z/(z-1)}}{(z-1)^2}$

29. $H(r) = \dfrac{(r^2-1)^3}{(2r+1)^5} \quad \Rightarrow$

$$H'(r) = \frac{(2r+1)^5\cdot 3(r^2-1)^2(2r)-(r^2-1)^3\cdot 5(2r+1)^4(2)}{[(2r+1)^5]^2} = \frac{2(2r+1)^4(r^2-1)^2[3r(2r+1)-5(r^2-1)]}{(2r+1)^{10}}$$

$$= \frac{2(r^2-1)^2(6r^2+3r-5r^2+5)}{(2r+1)^6} = \frac{2(r^2-1)^2(r^2+3r+5)}{(2r+1)^6}$$

30. $J(\theta) = \tan^2(n\theta) = [\tan(n\theta)]^2 \quad \Rightarrow$

$$J'(\theta) = 2\,[\tan(n\theta)]^1\,\frac{d}{d\theta}\tan(n\theta) = 2\tan(n\theta)\,\sec^2(n\theta)\cdot n = 2n\tan(n\theta)\,\sec^2(n\theta)$$

31. By (9), $F(t) = e^{t\sin 2t} \quad \Rightarrow$

$$F'(t) = e^{t\sin 2t}(t\sin 2t)' = e^{t\sin 2t}(t\cdot 2\cos 2t + \sin 2t\cdot 1) = e^{t\sin 2t}(2t\cos 2t + \sin 2t)$$

32. $F(t)=\dfrac{t^2}{\sqrt{t^3+1}} \quad\Rightarrow$

$$F'(t)=\frac{(t^3+1)^{1/2}(2t)-t^2\cdot\frac{1}{2}(t^3+1)^{-1/2}(3t^2)}{\left(\sqrt{t^3+1}\right)^2}=\frac{t(t^3+1)^{-1/2}\left[2(t^3+1)-\frac{3}{2}t^3\right]}{(t^3+1)^1}$$

$$=\frac{t\left(\frac{1}{2}t^3+2\right)}{(t^3+1)^{3/2}}=\frac{t(t^3+4)}{2(t^3+1)^{3/2}}$$

33. Using Formula 5 and the Chain Rule, $G(x)=4^{C/x} \quad\Rightarrow$

$$G'(x)=4^{C/x}(\ln 4)\,\frac{d}{dx}\frac{C}{x}\quad\left[\frac{C}{x}=Cx^{-1}\right]\quad=4^{C/x}(\ln 4)\left(-Cx^{-2}\right)=-C(\ln 4)\,\frac{4^{C/x}}{x^2}$$

34. $U(y)=\left(\dfrac{y^4+1}{y^2+1}\right)^5 \quad\Rightarrow$

$$U'(y)=5\left(\frac{y^4+1}{y^2+1}\right)^4\frac{(y^2+1)(4y^3)-(y^4+1)(2y)}{(y^2+1)^2}=\frac{5(y^4+1)^4 2y[2y^2(y^2+1)-(y^4+1)]}{(y^2+1)^4(y^2+1)^2}$$

$$=\frac{10y(y^4+1)^4(y^4+2y^2-1)}{(y^2+1)^6}$$

35. $y=\cos\left(\dfrac{1-e^{2x}}{1+e^{2x}}\right) \quad\Rightarrow$

$$y'=-\sin\left(\frac{1-e^{2x}}{1+e^{2x}}\right)\cdot\frac{d}{dx}\left(\frac{1-e^{2x}}{1+e^{2x}}\right)=-\sin\left(\frac{1-e^{2x}}{1+e^{2x}}\right)\cdot\frac{(1+e^{2x})(-2e^{2x})-(1-e^{2x})(2e^{2x})}{(1+e^{2x})^2}$$

$$=-\sin\left(\frac{1-e^{2x}}{1+e^{2x}}\right)\cdot\frac{-2e^{2x}\left[(1+e^{2x})+(1-e^{2x})\right]}{(1+e^{2x})^2}=-\sin\left(\frac{1-e^{2x}}{1+e^{2x}}\right)\cdot\frac{-2e^{2x}(2)}{(1+e^{2x})^2}=\frac{4e^{2x}}{(1+e^{2x})^2}\cdot\sin\left(\frac{1-e^{2x}}{1+e^{2x}}\right)$$

36. $y=x^2e^{-1/x} \quad\Rightarrow\quad y'=x^2e^{-1/x}\left(\dfrac{1}{x^2}\right)+e^{-1/x}(2x)=e^{-1/x}+2xe^{-1/x}=e^{-1/x}(1+2x)$

37. $y=\cot^2(\sin\theta)=[\cot(\sin\theta)]^2 \quad\Rightarrow$

$$y'=2[\cot(\sin\theta)]\cdot\frac{d}{d\theta}[\cot(\sin\theta)]=2\cot(\sin\theta)\cdot[-\csc^2(\sin\theta)\cdot\cos\theta]=-2\cos\theta\,\cot(\sin\theta)\,\csc^2(\sin\theta)$$

38. $y=\sqrt{1+xe^{-2x}} \quad\Rightarrow\quad y'=\frac{1}{2}\left(1+xe^{-2x}\right)^{-1/2}\left[x\left(-2e^{-2x}\right)+e^{-2x}\right]=\dfrac{e^{-2x}(-2x+1)}{2\sqrt{1+xe^{-2x}}}$

39. $f(t)=\tan(\sec(\cos t)) \quad\Rightarrow$

$$f'(t)=\sec^2(\sec(\cos t))\,\frac{d}{dt}\,\sec(\cos t)=\sec^2(\sec(\cos t))[\sec(\cos t)\,\tan(\cos t)]\,\frac{d}{dt}\,\cos t$$

$$=-\sec^2(\sec(\cos t))\,\sec(\cos t)\,\tan(\cos t)\,\sin t$$

40. $y=e^{\sin 2x}+\sin(e^{2x}) \quad\Rightarrow$

$$y'=e^{\sin 2x}\frac{d}{dx}\sin 2x+\cos(e^{2x})\frac{d}{dx}e^{2x}=e^{\sin 2x}(\cos 2x)\cdot 2+\cos(e^{2x})\,e^{2x}\cdot 2$$

$$=2\cos 2x\,e^{\sin 2x}+2e^{2x}\cos(e^{2x})$$

41. $f(t) = \sin^2\left(e^{\sin^2 t}\right) = \left[\sin\left(e^{\sin^2 t}\right)\right]^2 \quad \Rightarrow$

$$f'(t) = 2\left[\sin\left(e^{\sin^2 t}\right)\right] \cdot \frac{d}{dt} \sin\left(e^{\sin^2 t}\right) = 2\sin\left(e^{\sin^2 t}\right) \cdot \cos\left(e^{\sin^2 t}\right) \cdot \frac{d}{dt}\, e^{\sin^2 t}$$

$$= 2\sin\left(e^{\sin^2 t}\right)\cos\left(e^{\sin^2 t}\right) \cdot e^{\sin^2 t} \cdot \frac{d}{dt}\, \sin^2 t = 2\sin\left(e^{\sin^2 t}\right)\cos\left(e^{\sin^2 t}\right) e^{\sin^2 t} \cdot 2\sin t\, \cos t$$

$$= 4\sin\left(e^{\sin^2 t}\right)\cos\left(e^{\sin^2 t}\right) e^{\sin^2 t} \sin t\, \cos t$$

42. $y = \sqrt{x + \sqrt{x + \sqrt{x}}} \quad \Rightarrow \quad y' = \frac{1}{2}\left(x + \sqrt{x + \sqrt{x}}\right)^{-1/2}\left[1 + \frac{1}{2}\left(x + \sqrt{x}\right)^{-1/2}\left(1 + \frac{1}{2}x^{-1/2}\right)\right]$

43. $g(x) = (2ra^{rx} + n)^p \quad \Rightarrow$

$$g'(x) = p(2ra^{rx} + n)^{p-1} \cdot \frac{d}{dx}(2ra^{rx} + n) = p(2ra^{rx} + n)^{p-1} \cdot 2ra^{rx}(\ln a) \cdot r = 2r^2 p(\ln a)(2ra^{rx} + n)^{p-1} a^{rx}$$

44. $y = 2^{3^{4^x}} \quad \Rightarrow$

$$y' = 2^{3^{4^x}} (\ln 2)\, \frac{d}{dx} 3^{4^x} = 2^{3^{4^x}} (\ln 2)\, 3^{4^x} (\ln 3)\, \frac{d}{dx} 4^x = 2^{3^{4^x}} (\ln 2)\, 3^{4^x} (\ln 3)\, 4^x (\ln 4) = (\ln 2)(\ln 3)(\ln 4) 4^x 3^{4^x} 2^{3^{4^x}}$$

45. $y = \cos\sqrt{\sin(\tan \pi x)} = \cos(\sin(\tan \pi x))^{1/2} \quad \Rightarrow$

$$y' = -\sin(\sin(\tan \pi x))^{1/2} \cdot \frac{d}{dx}\, (\sin(\tan \pi x))^{1/2} = -\sin(\sin(\tan \pi x))^{1/2} \cdot \tfrac{1}{2}(\sin(\tan \pi x))^{-1/2} \cdot \frac{d}{dx}\, (\sin(\tan \pi x))$$

$$= \frac{-\sin\sqrt{\sin(\tan \pi x)}}{2\sqrt{\sin(\tan \pi x)}} \cdot \cos(\tan \pi x) \cdot \frac{d}{dx}\, \tan \pi x = \frac{-\sin\sqrt{\sin(\tan \pi x)}}{2\sqrt{\sin(\tan \pi x)}} \cdot \cos(\tan \pi x) \cdot \sec^2(\pi x) \cdot \pi$$

$$= \frac{-\pi \cos(\tan \pi x) \sec^2(\pi x) \sin\sqrt{\sin(\tan \pi x)}}{2\sqrt{\sin(\tan \pi x)}}$$

46. $y = \left[x + (x + \sin^2 x)^3\right]^4 \quad \Rightarrow \quad y' = 4\left[x + (x + \sin^2 x)^3\right]^3 \cdot \left[1 + 3(x + \sin^2 x)^2 \cdot (1 + 2\sin x\, \cos x)\right]$

47. $y = \cos(\sin 3\theta) \quad \Rightarrow \quad y' = -\sin(\sin 3\theta) \cdot (\cos 3\theta) \cdot 3 = -3\cos 3\theta\, \sin(\sin 3\theta) \quad \Rightarrow$

$$y'' = -3\left[(\cos 3\theta)\cos(\sin 3\theta)(\cos 3\theta) \cdot 3 + \sin(\sin 3\theta)(-\sin 3\theta) \cdot 3\right] = -9\cos^2(3\theta)\cos(\sin 3\theta) + 9(\sin 3\theta)\sin(\sin 3\theta)$$

48. $y = \dfrac{1}{(1 + \tan x)^2} = (1 + \tan x)^{-2} \quad \Rightarrow \quad y' = -2(1 + \tan x)^{-3}\sec^2 x = \dfrac{-2\sec^2 x}{(1 + \tan x)^3}$.

Using the Product Rule with $y' = \left[-2(1 + \tan x)^{-3}\right](\sec x)^2$, we get

$$y'' = -2(1 + \tan x)^{-3} \cdot 2(\sec x)(\sec x\, \tan x) + (\sec x)^2 \cdot 6(1 + \tan x)^{-4}\sec^2 x$$

$$= 2\sec^2 x\, (1 + \tan x)^{-4}\left[-2(1 + \tan x)\tan x + 3\sec^2 x\right] \qquad \left[\begin{array}{c}\text{2 is the lesser exponent for } \sec x \\ \text{and } -4 \text{ for } (1 + \tan x)\end{array}\right]$$

$$= 2\sec^2 x\, (1 + \tan x)^{-4}\left[-2\tan x - 2\tan^2 x + 3(\tan^2 x + 1)\right]$$

$$= \frac{2\sec^2 x\left(\tan^2 x - 2\tan x + 3\right)}{(1 + \tan x)^4}$$

49. $y = \sqrt{1 - \sec t} \quad \Rightarrow \quad y' = \frac{1}{2}(1 - \sec t)^{-1/2}(-\sec t\, \tan t) = \dfrac{-\sec t\, \tan t}{2\sqrt{1 - \sec t}}$.

Using the Product Rule with $y' = \left(-\frac{1}{2}\sec t\, \tan t\right)(1 - \sec t)^{-1/2}$, we get

$y'' = \left(-\frac{1}{2}\sec t\,\tan t\right)\left[-\frac{1}{2}(1-\sec t)^{-3/2}(-\sec t\,\tan t)\right] + (1-\sec t)^{-1/2}\left(-\frac{1}{2}\right)[\sec t\,\sec^2 t + \tan t\,\sec t\,\tan t]$.

Now factor out $-\frac{1}{2}\sec t(1-\sec t)^{-3/2}$. Note that $-\frac{3}{2}$ is the lesser exponent on $(1-\sec t)$. Continuing,

$$\begin{aligned} y'' &= -\tfrac{1}{2}\sec t\,(1-\sec t)^{-3/2}\left[\tfrac{1}{2}\sec t\,\tan^2 t + (1-\sec t)(\sec^2 t + \tan^2 t)\right] \\ &= -\tfrac{1}{2}\sec t\,(1-\sec t)^{-3/2}\left(\tfrac{1}{2}\sec t\,\tan^2 t + \sec^2 t + \tan^2 t - \sec^3 t - \sec t\,\tan^2 t\right) \\ &= -\tfrac{1}{2}\sec t\,(1-\sec t)^{-3/2}\left[-\tfrac{1}{2}\sec t\,(\sec^2 t - 1) + \sec^2 t + (\sec^2 t - 1) - \sec^3 t\right] \\ &= -\tfrac{1}{2}\sec t\,(1-\sec t)^{-3/2}\left(-\tfrac{3}{2}\sec^3 t + 2\sec^2 t + \tfrac{1}{2}\sec t - 1\right) \\ &= \sec t\,(1-\sec t)^{-3/2}\left(\tfrac{3}{4}\sec^3 t - \sec^2 t - \tfrac{1}{4}\sec t + \tfrac{1}{2}\right) \\ &= \frac{\sec t\,(3\sec^3 t - 4\sec^2 t - \sec t + 2)}{4(1-\sec t)^{3/2}} \end{aligned}$$

There are many other correct forms of y'', such as $y'' = \dfrac{\sec t\,(3\sec t + 2)\sqrt{1-\sec t}}{4}$. We chose to find a factored form with only secants in the final form.

50. $y = e^{e^x} \quad\Rightarrow\quad y' = e^{e^x}\cdot(e^x)' = e^{e^x}\cdot e^x \quad\Rightarrow$

$y'' = e^{e^x}\cdot(e^x)' + e^x\cdot\left(e^{e^x}\right)' = e^{e^x}\cdot e^x + e^x\cdot e^{e^x}\cdot e^x = e^{e^x}\cdot e^x(1+e^x) \quad\text{or}\quad e^{e^x+x}(1+e^x)$

51. $y = 2^x \quad\Rightarrow\quad y' = 2^x\ln 2$. At $(0,1)$, $y' = 2^0\ln 2 = \ln 2$, and an equation of the tangent line is $y - 1 = (\ln 2)(x-0)$ or $y = (\ln 2)x + 1$.

52. $y = \sqrt{1+x^3} = (1+x^3)^{1/2} \quad\Rightarrow\quad y' = \frac{1}{2}(1+x^3)^{-1/2}\cdot 3x^2 = \dfrac{3x^2}{2\sqrt{1+x^3}}$. At $(2,3)$, $y' = \dfrac{3\cdot 4}{2\sqrt{9}} = 2$, and an equation of the tangent line is $y - 3 = 2(x-2)$, or $y = 2x - 1$.

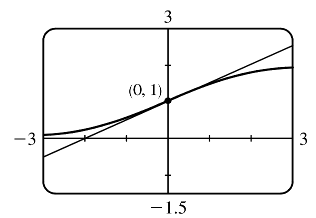
53. $y = \sin(\sin x) \quad\Rightarrow\quad y' = \cos(\sin x)\cdot\cos x$. At $(\pi,0)$, $y' = \cos(\sin\pi)\cdot\cos\pi = \cos(0)\cdot(-1) = 1(-1) = -1$, and an equation of the tangent line is $y - 0 = -1(x-\pi)$, or $y = -x + \pi$.

54. $y = xe^{-x^2} \quad\Rightarrow\quad y' = xe^{-x^2}(-2x) + e^{-x^2}(1) = e^{-x^2}(-2x^2+1)$. At $(0,0)$, $y' = e^0(1) = 1$, and an equation of the tangent line is $y - 0 = 1(x-0)$ or $y = x$.

55. (a) $y = \dfrac{2}{1+e^{-x}} \quad\Rightarrow\quad y' = \dfrac{(1+e^{-x})(0) - 2(-e^{-x})}{(1+e^{-x})^2} = \dfrac{2e^{-x}}{(1+e^{-x})^2}$.

At $(0,1)$, $y' = \dfrac{2e^0}{(1+e^0)^2} = \dfrac{2(1)}{(1+1)^2} = \dfrac{2}{2^2} = \dfrac{1}{2}$. So an equation of the tangent line is $y - 1 = \frac{1}{2}(x-0)$ or $y = \frac{1}{2}x + 1$.

(b)

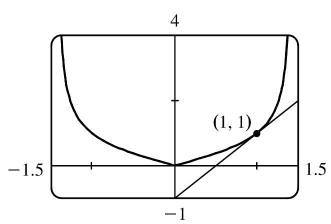


56. (a) For $x > 0$, $|x| = x$, and $y = f(x) = \dfrac{x}{\sqrt{2-x^2}} \quad\Rightarrow$

$$\begin{aligned} f'(x) &= \frac{\sqrt{2-x^2}\,(1) - x\left(\frac{1}{2}\right)(2-x^2)^{-1/2}(-2x)}{\left(\sqrt{2-x^2}\right)^2}\cdot\frac{(2-x^2)^{1/2}}{(2-x^2)^{1/2}} \\ &= \frac{(2-x^2)+x^2}{(2-x^2)^{3/2}} = \frac{2}{(2-x^2)^{3/2}} \end{aligned}$$

So at $(1,1)$, the slope of the tangent line is $f'(1) = 2$ and its equation is $y - 1 = 2(x-1)$ or $y = 2x - 1$.

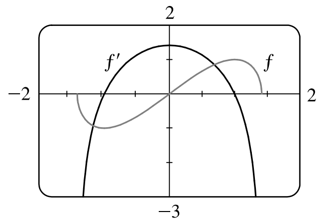
(b)



57. (a) $f(x) = x\sqrt{2-x^2} = x(2-x^2)^{1/2} \quad \Rightarrow$

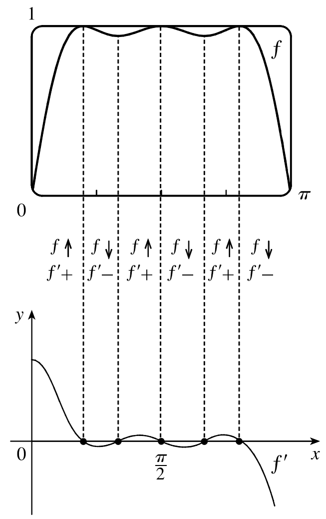
$$f'(x) = x \cdot \tfrac{1}{2}(2-x^2)^{-1/2}(-2x) + (2-x^2)^{1/2} \cdot 1 = (2-x^2)^{-1/2}\left[-x^2 + (2-x^2)\right] = \frac{2-2x^2}{\sqrt{2-x^2}}$$

(b)



$f' = 0$ when f has a horizontal tangent line, f' is negative when f is decreasing, and f' is positive when f is increasing.

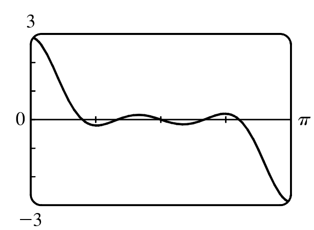
58. (a)



From the graph of f, we see that there are 5 horizontal tangents, so there must be 5 zeros on the graph of f'. From the symmetry of the graph of f, we must have the graph of f' as high at $x = 0$ as it is low at $x = \pi$. The intervals of increase and decrease as well as the signs of f' are indicated in the figure.

(b) $f(x) = \sin(x + \sin 2x) \quad \Rightarrow$

$$f'(x) = \cos(x + \sin 2x) \cdot \frac{d}{dx}\,(x + \sin 2x) = \cos(x + \sin 2x)(1 + 2\cos 2x)$$



59. For the tangent line to be horizontal, $f'(x) = 0$. $f(x) = 2\sin x + \sin^2 x \quad \Rightarrow \quad f'(x) = 2\cos x + 2\sin x\,\cos x = 0 \quad \Leftrightarrow$ $2\cos x(1 + \sin x) = 0 \quad \Leftrightarrow \quad \cos x = 0$ or $\sin x = -1$, so $x = \frac{\pi}{2} + 2n\pi$ or $\frac{3\pi}{2} + 2n\pi$, where n is any integer. Now $f\left(\frac{\pi}{2}\right) = 3$ and $f\left(\frac{3\pi}{2}\right) = -1$, so the points on the curve with a horizontal tangent are $\left(\frac{\pi}{2} + 2n\pi, 3\right)$ and $\left(\frac{3\pi}{2} + 2n\pi, -1\right)$, where n is any integer.

60. $y = \sqrt{1+2x} \quad \Rightarrow \quad y' = \frac{1}{2}(1+2x)^{-1/2} \cdot 2 = \dfrac{1}{\sqrt{1+2x}}$. The line $6x + 2y = 1$ $\left(\text{or } y = -3x + \frac{1}{2}\right)$ has slope -3, so the tangent line perpendicular to it must have slope $\frac{1}{3}$. Thus, $\dfrac{1}{3} = \dfrac{1}{\sqrt{1+2x}} \quad \Leftrightarrow \quad \sqrt{1+2x} = 3 \quad \Rightarrow \quad 1 + 2x = 9 \quad \Leftrightarrow$ $2x = 8 \quad \Leftrightarrow \quad x = 4$. When $x = 4$, $y = \sqrt{1 + 2(4)} = 3$, so the point is $(4, 3)$.

61. $F(x) = f(g(x)) \quad \Rightarrow \quad F'(x) = f'(g(x)) \cdot g'(x)$, so $F'(5) = f'(g(5)) \cdot g'(5) = f'(-2) \cdot 6 = 4 \cdot 6 = 24$.

62. $h(x) = \sqrt{4+3f(x)} \quad \Rightarrow \quad h'(x) = \frac{1}{2}(4+3f(x))^{-1/2} \cdot 3f'(x)$, so

$h'(1) = \frac{1}{2}(4+3f(1))^{-1/2} \cdot 3f'(1) = \frac{1}{2}(4+3\cdot 7)^{-1/2} \cdot 3 \cdot 4 = \frac{6}{\sqrt{25}} = \frac{6}{5}$.

63. (a) $h(x) = f(g(x)) \quad \Rightarrow \quad h'(x) = f'(g(x)) \cdot g'(x)$, so $h'(1) = f'(g(1)) \cdot g'(1) = f'(2) \cdot 6 = 5 \cdot 6 = 30$.

(b) $H(x) = g(f(x)) \quad \Rightarrow \quad H'(x) = g'(f(x)) \cdot f'(x)$, so $H'(1) = g'(f(1)) \cdot f'(1) = g'(3) \cdot 4 = 9 \cdot 4 = 36$.

64. (a) $F(x) = f(f(x)) \quad \Rightarrow \quad F'(x) = f'(f(x)) \cdot f'(x)$, so $F'(2) = f'(f(2)) \cdot f'(2) = f'(1) \cdot 5 = 4 \cdot 5 = 20$.

(b) $G(x) = g(g(x)) \quad \Rightarrow \quad G'(x) = g'(g(x)) \cdot g'(x)$, so $G'(3) = g'(g(3)) \cdot g'(3) = g'(2) \cdot 9 = 7 \cdot 9 = 63$.

65. (a) $u(x) = f(g(x)) \quad \Rightarrow \quad u'(x) = f'(g(x))g'(x)$. So $u'(1) = f'(g(1))g'(1) = f'(3)g'(1)$. To find $f'(3)$, note that f is linear from $(2,4)$ to $(6,3)$, so its slope is $\dfrac{3-4}{6-2} = -\dfrac{1}{4}$. To find $g'(1)$, note that g is linear from $(0,6)$ to $(2,0)$, so its slope is $\dfrac{0-6}{2-0} = -3$. Thus, $f'(3)g'(1) = \left(-\frac{1}{4}\right)(-3) = \frac{3}{4}$.

(b) $v(x) = g(f(x)) \quad \Rightarrow \quad v'(x) = g'(f(x))f'(x)$. So $v'(1) = g'(f(1))f'(1) = g'(2)f'(1)$, which does not exist since $g'(2)$ does not exist.

(c) $w(x) = g(g(x)) \quad \Rightarrow \quad w'(x) = g'(g(x))g'(x)$. So $w'(1) = g'(g(1))g'(1) = g'(3)g'(1)$. To find $g'(3)$, note that g is linear from $(2,0)$ to $(5,2)$, so its slope is $\dfrac{2-0}{5-2} = \dfrac{2}{3}$. Thus, $g'(3)g'(1) = \left(\frac{2}{3}\right)(-3) = -2$.

66. (a) $h(x) = f(f(x)) \quad \Rightarrow \quad h'(x) = f'(f(x))f'(x)$. So $h'(2) = f'(f(2))f'(2) = f'(1)f'(2) \approx (-1)(-1) = 1$.

(b) $g(x) = f(x^2) \quad \Rightarrow \quad g'(x) = f'(x^2) \cdot \dfrac{d}{dx}(x^2) = f'(x^2)(2x)$. So $g'(2) = f'(2^2)(2 \cdot 2) = 4f'(4) \approx 4(2) = 8$.

67. The point $(3,2)$ is on the graph of f, so $f(3) = 2$. The tangent line at $(3,2)$ has slope $\dfrac{\Delta y}{\Delta x} = \dfrac{-4}{6} = -\dfrac{2}{3}$.

$g(x) = \sqrt{f(x)} \quad \Rightarrow \quad g'(x) = \frac{1}{2}[f(x)]^{-1/2} \cdot f'(x) \quad \Rightarrow$

$g'(3) = \frac{1}{2}[f(3)]^{-1/2} \cdot f'(3) = \frac{1}{2}(2)^{-1/2}(-\frac{2}{3}) = -\dfrac{1}{3\sqrt{2}}$ or $-\frac{1}{6}\sqrt{2}$.

68. (a) $F(x) = f(x^\alpha) \quad \Rightarrow \quad F'(x) = f'(x^\alpha)\dfrac{d}{dx}(x^\alpha) = f'(x^\alpha)\alpha x^{\alpha-1}$

(b) $G(x) = [f(x)]^\alpha \quad \Rightarrow \quad G'(x) = \alpha\,[f(x)]^{\alpha-1}\,f'(x)$

69. (a) $F(x) = f(e^x) \quad \Rightarrow \quad F'(x) = f'(e^x)\dfrac{d}{dx}(e^x) = f'(e^x)e^x$

(b) $G(x) = e^{f(x)} \quad \Rightarrow \quad G'(x) = e^{f(x)}\dfrac{d}{dx}f(x) = e^{f(x)}f'(x)$

70. (a) $g(x) = e^{cx} + f(x) \quad \Rightarrow \quad g'(x) = e^{cx} \cdot c + f'(x) \quad \Rightarrow \quad g'(0) = e^0 \cdot c + f'(0) = c + 5$.

$g'(x) = ce^{cx} + f'(x) \quad \Rightarrow \quad g''(x) = ce^{cx} \cdot c + f''(x) \quad \Rightarrow \quad g''(0) = c^2e^0 + f''(0) = c^2 - 2$.

(b) $h(x) = e^{kx}f(x) \quad \Rightarrow \quad h'(x) = e^{kx}f'(x) + f(x) \cdot ke^{kx} \quad \Rightarrow \quad h'(0) = e^0f'(0) + f(0) \cdot ke^0 = 5 + 3k$.

An equation of the tangent line to the graph of h at the point $(0, h(0)) = (0, f(0)) = (0,3)$ is

$y - 3 = (5+3k)(x-0)$ or $y = (5+3k)x + 3$.

71. $r(x) = f(g(h(x))) \quad \Rightarrow \quad r'(x) = f'(g(h(x))) \cdot g'(h(x)) \cdot h'(x)$, so

$$r'(1) = f'(g(h(1))) \cdot g'(h(1)) \cdot h'(1) = f'(g(2)) \cdot g'(2) \cdot 4 = f'(3) \cdot 5 \cdot 4 = 6 \cdot 5 \cdot 4 = 120$$

72. $f(x) = xg(x^2) \quad \Rightarrow \quad f'(x) = xg'(x^2)\,2x + g(x^2) \cdot 1 = 2x^2 g'(x^2) + g(x^2) \quad \Rightarrow$

$$f''(x) = 2x^2 g''(x^2)\,2x + g'(x^2)\,4x + g'(x^2)\,2x = 4x^3 g''(x^2) + 4xg'(x^2) + 2xg'(x^2) = 6xg'(x^2) + 4x^3 g''(x^2)$$

73. $F(x) = f(3f(4f(x))) \quad \Rightarrow$

$$\begin{aligned} F'(x) &= f'(3f(4f(x))) \cdot \frac{d}{dx}(3f(4f(x))) = f'(3f(4f(x))) \cdot 3f'(4f(x)) \cdot \frac{d}{dx}(4f(x)) \\ &= f'(3f(4f(x))) \cdot 3f'(4f(x)) \cdot 4f'(x), \quad \text{so} \end{aligned}$$

$$F'(0) = f'(3f(4f(0))) \cdot 3f'(4f(0)) \cdot 4f'(0) = f'(3f(4 \cdot 0)) \cdot 3f'(4 \cdot 0) \cdot 4 \cdot 2 = f'(3 \cdot 0) \cdot 3 \cdot 2 \cdot 4 \cdot 2 = 2 \cdot 3 \cdot 2 \cdot 4 \cdot 2 = 96.$$

74. $F(x) = f(xf(xf(x))) \quad \Rightarrow$

$$\begin{aligned} F'(x) &= f'(xf(xf(x))) \cdot \frac{d}{dx}(xf(xf(x))) = f'(xf(xf(x))) \cdot \left[x \cdot f'(xf(x)) \cdot \frac{d}{dx}(xf(x)) + f(xf(x)) \cdot 1\right] \\ &= f'(xf(xf(x))) \cdot [xf'(xf(x)) \cdot (xf'(x) + f(x) \cdot 1) + f(xf(x))], \quad \text{so} \end{aligned}$$

$$\begin{aligned} F'(1) &= f'(f(f(1))) \cdot [f'(f(1)) \cdot (f'(1) + f(1)) + f(f(1))] = f'(f(2)) \cdot [f'(2) \cdot (4 + 2) + f(2)] \\ &= f'(3) \cdot [5 \cdot 6 + 3] = 6 \cdot 33 = 198. \end{aligned}$$

75. $y = e^{2x}(A\cos 3x + B\sin 3x) \quad \Rightarrow$

$$\begin{aligned} y' &= e^{2x}(-3A\sin 3x + 3B\cos 3x) + (A\cos 3x + B\sin 3x) \cdot 2e^{2x} \\ &= e^{2x}(-3A\sin 3x + 3B\cos 3x + 2A\cos 3x + 2B\sin 3x) \\ &= e^{2x}[(2A + 3B)\cos 3x + (2B - 3A)\sin 3x] \quad \Rightarrow \end{aligned}$$

$$\begin{aligned} y'' &= e^{2x}[-3(2A + 3B)\sin 3x + 3(2B - 3A)\cos 3x] + [(2A + 3B)\cos 3x + (2B - 3A)\sin 3x] \cdot 2e^{2x} \\ &= e^{2x}\{[-3(2A + 3B) + 2(2B - 3A)]\sin 3x + [3(2B - 3A) + 2(2A + 3B)]\cos 3x\} \\ &= e^{2x}[(-12A - 5B)\sin 3x + (-5A + 12B)\cos 3x] \end{aligned}$$

Substitute the expressions for y, y', and y'' in $y'' - 4y' + 13y$ to get

$$\begin{aligned} y'' - 4y' + 13y &= e^{2x}[(-12A - 5B)\sin 3x + (-5A + 12B)\cos 3x] \\ &\qquad - 4e^{2x}[(2A + 3B)\cos 3x + (2B - 3A)\sin 3x] + 13e^{2x}(A\cos 3x + B\sin 3x) \\ &= e^{2x}[(-12A - 5B - 8B + 12A + 13B)\sin 3x + (-5A + 12B - 8A - 12B + 13A)\cos 3x] \\ &= e^{2x}[(0)\sin 3x + (0)\cos 3x] = 0 \end{aligned}$$

Thus, the function y satisfies the differential equation $y'' - 4y' + 13y = 0$.

76. $y = e^{rx} \quad \Rightarrow \quad y' = re^{rx} \quad \Rightarrow \quad y'' = r^2 e^{rx}$. Substituting y, y', and y'' into $y'' - 4y' + y = 0$ gives us $r^2 e^{rx} - 4re^{rx} + e^{rx} = 0 \quad \Rightarrow \quad e^{rx}(r^2 - 4r + 1) = 0$. Since $e^{rx} \neq 0$, we must have

$$r^2 - 4r + 1 = 0 \quad \Rightarrow \quad r = \frac{4 \pm \sqrt{16 - 4}}{2} = 2 \pm \sqrt{3}.$$

77. The use of $D, D^2, \ldots, D^n$ is just a derivative notation (see text page 159). In general, $Df(2x) = 2f'(2x)$, $D^2 f(2x) = 4f''(2x), \ldots, D^n f(2x) = 2^n f^{(n)}(2x)$. Since $f(x) = \cos x$ and $50 = 4(12) + 2$, we have $f^{(50)}(x) = f^{(2)}(x) = -\cos x$, so $D^{50} \cos 2x = -2^{50} \cos 2x$.

78. $f(x) = xe^{-x}$, $f'(x) = e^{-x} - xe^{-x} = (1-x)e^{-x}$, $f''(x) = -e^{-x} + (1-x)(-e^{-x}) = (x-2)e^{-x}$. Similarly, $f'''(x) = (3-x)e^{-x}$, $f^{(4)}(x) = (x-4)e^{-x}, \ldots, f^{(1000)}(x) = (x-1000)e^{-x}$.

79. $s(t) = 10 + \frac{1}{4}\sin(10\pi t) \quad\Rightarrow\quad$ the velocity after t seconds is $v(t) = s'(t) = \frac{1}{4}\cos(10\pi t)(10\pi) = \frac{5\pi}{2}\cos(10\pi t)$ cm/s.

80. (a) $s = A\cos(\omega t + \delta) \quad\Rightarrow\quad$ velocity $= s' = -\omega A \sin(\omega t + \delta)$.

(b) If $A \neq 0$ and $\omega \neq 0$, then $s' = 0 \quad\Leftrightarrow\quad \sin(\omega t + \delta) = 0 \quad\Leftrightarrow\quad \omega t + \delta = n\pi \quad\Leftrightarrow\quad t = \dfrac{n\pi - \delta}{\omega}$, n an integer.

81. (a) $B(t) = 4.0 + 0.35\sin\dfrac{2\pi t}{5.4} \quad\Rightarrow\quad \dfrac{dB}{dt} = \left(0.35\cos\dfrac{2\pi t}{5.4}\right)\left(\dfrac{2\pi}{5.4}\right) = \dfrac{0.7\pi}{5.4}\cos\dfrac{2\pi t}{5.4} = \dfrac{7\pi}{54}\cos\dfrac{2\pi t}{5.4}$

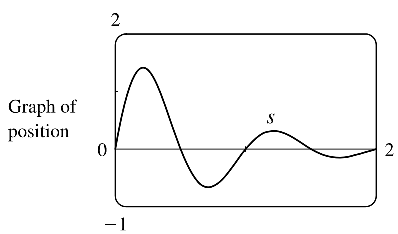
(b) At $t = 1$, $\dfrac{dB}{dt} = \dfrac{7\pi}{54}\cos\dfrac{2\pi}{5.4} \approx 0.16$.

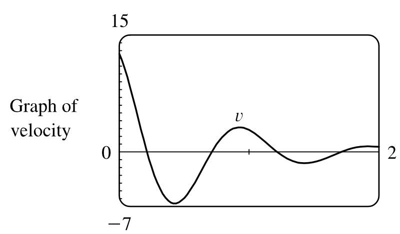
82. $L(t) = 12 + 2.8\sin\left(\frac{2\pi}{365}(t-80)\right) \quad\Rightarrow\quad L'(t) = 2.8\cos\left(\frac{2\pi}{365}(t-80)\right)\left(\frac{2\pi}{365}\right)$.

On March 21, $t = 80$, and $L'(80) \approx 0.0482$ hours per day. On May 21, $t = 141$, and $L'(141) \approx 0.02398$, which is approximately one-half of $L'(80)$.

83. $s(t) = 2e^{-1.5t}\sin 2\pi t \quad\Rightarrow$

$v(t) = s'(t) = 2[e^{-1.5t}(\cos 2\pi t)(2\pi) + (\sin 2\pi t)e^{-1.5t}(-1.5)] = 2e^{-1.5t}(2\pi\cos 2\pi t - 1.5\sin 2\pi t)$

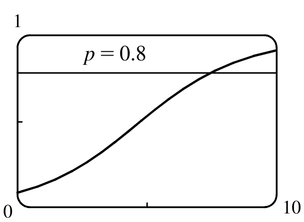




84. (a) $\displaystyle\lim_{t\to\infty} p(t) = \lim_{t\to\infty} \frac{1}{1 + ae^{-kt}} = \frac{1}{1 + a\cdot 0} = 1$, since $k > 0 \quad\Rightarrow\quad -kt \to -\infty \quad\Rightarrow\quad e^{-kt} \to 0$.

(b) $p(t) = (1 + ae^{-kt})^{-1} \quad\Rightarrow\quad \dfrac{dp}{dt} = -(1 + ae^{-kt})^{-2}(-kae^{-kt}) = \dfrac{kae^{-kt}}{(1 + ae^{-kt})^2}$

(c)



From the graph of $p(t) = (1 + 10e^{-0.5t})^{-1}$, it seems that $p(t) = 0.8$ (indicating that 80% of the population has heard the rumor) when $t \approx 7.4$ hours.

85. (a) Use $C(t) = ate^{bt}$ with $a = 0.0225$ and $b = -0.0467$ to get $C'(t) = a(te^{bt}\cdot b + e^{bt}\cdot 1) = a(bt+1)e^{bt}$. $C'(10) = 0.0225(0.533)e^{-0.467} \approx 0.0075$, so the BAC was increasing at approximately 0.0075 (mg/mL)/min after 10 minutes.

(b) A half an hour later gives us $t = 10 + 30 = 40$. $C'(40) = 0.0225(-0.868)e^{-18.68} \approx -0.0030$, so the BAC was decreasing at approximately $0.0030\ (\text{mg/mL})/\text{min}$ after 40 minutes.

86. $P(t) = (1436.53) \cdot (1.01395)^t \quad \Rightarrow \quad P'(t) = (1436.53) \cdot (1.01395)^t(\ln 1.01395)$. The units for $P'(t)$ are millions of people per year. The rates of increase for 1920, 1950, and 2000 are $P'(20) \approx 26.25$, $P'(50) \approx 39.78$, and $P'(100) \approx 79.53$, respectively.

87. By the Chain Rule, $a(t) = \dfrac{dv}{dt} = \dfrac{dv}{ds}\dfrac{ds}{dt} = \dfrac{dv}{ds}\,v(t) = v(t)\,\dfrac{dv}{ds}$. The derivative dv/dt is the rate of change of the velocity with respect to time (in other words, the acceleration) whereas the derivative dv/ds is the rate of change of the velocity with respect to the displacement.

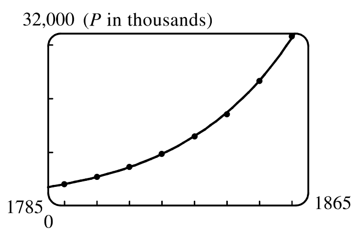
88. (a) The derivative dV/dr represents the rate of change of the volume with respect to the radius and the derivative dV/dt represents the rate of change of the volume with respect to time.

(b) Since $V = \frac{4}{3}\pi r^3$, $\dfrac{dV}{dt} = \dfrac{dV}{dr}\dfrac{dr}{dt} = 4\pi r^2\,\dfrac{dr}{dt}$.

89. (a) Using a calculator or CAS, we obtain the model $Q = ab^t$ with $a \approx 100.0124369$ and $b \approx 0.000045145933$.

(b) Use $Q'(t) = ab^t \ln b$ (from Formula 5) with the values of a and b from part (a) to get $Q'(0.04) \approx -670.63\ \mu\text{A}$. The result of Example 2.1.2 was $-670\ \ \mu\text{A}$.

90. (a) $P = ab^t$ with $a = 4.502714 \times 10^{-20}$ and $b = 1.029953851$, where P is measured in thousands of people. The fit appears to be very good.



(b) **For 1800:** $m_1 = \dfrac{5308 - 3929}{1800 - 1790} = 137.9$, $m_2 = \dfrac{7240 - 5308}{1810 - 1800} = 193.2$.

So $P'(1800) \approx (m_1 + m_2)/2 = 165.55$ thousand people/year.

For 1850: $m_1 = \dfrac{23{,}192 - 17{,}063}{1850 - 1840} = 612.9$, $m_2 = \dfrac{31{,}443 - 23{,}192}{1860 - 1850} = 825.1$.

So $P'(1850) \approx (m_1 + m_2)/2 = 719$ thousand people/year.

(c) Using $P'(t) = ab^t \ln b$ (from Formula 7) with the values of a and b from part (a), we get $P'(1800) \approx 156.85$ and $P'(1850) \approx 686.07$. These estimates are somewhat less than the ones in part (b).

(d) $P(1870) \approx 41{,}946.56$. The difference of 3.4 million people is most likely due to the Civil War (1861–1865).

91. (a) Derive gives $g'(t) = \dfrac{45(t-2)^8}{(2t+1)^{10}}$ without simplifying. With either Maple or Mathematica, we first get

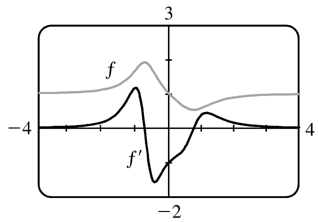
$g'(t) = 9\dfrac{(t-2)^8}{(2t+1)^9} - 18\dfrac{(t-2)^9}{(2t+1)^{10}}$, and the simplification command results in the expression given by Derive.

(b) Derive gives $y' = 2(x^3 - x + 1)^3(2x+1)^4(17x^3 + 6x^2 - 9x + 3)$ without simplifying. With either Maple or Mathematica, we first get $y' = 10(2x+1)^4(x^3 - x + 1)^4 + 4(2x+1)^5(x^3 - x + 1)^3(3x^2 - 1)$. If we use

Mathematica's `Factor` or `Simplify`, or Maple's `factor`, we get the above expression, but Maple's `simplify` gives the polynomial expansion instead. For locating horizontal tangents, the factored form is the most helpful.

92. (a) $f(x) = \left(\dfrac{x^4 - x + 1}{x^4 + x + 1}\right)^{1/2}$. Derive gives $f'(x) = \dfrac{(3x^4 - 1)\sqrt{\dfrac{x^4 - x + 1}{x^4 + x + 1}}}{(x^4 + x + 1)(x^4 - x + 1)}$ whereas either Maple or Mathematica give $f'(x) = \dfrac{3x^4 - 1}{\sqrt{\dfrac{x^4 - x + 1}{x^4 + x + 1}}\,(x^4 + x + 1)^2}$ after simplification.

(b) $f'(x) = 0 \quad \Leftrightarrow \quad 3x^4 - 1 = 0 \quad \Leftrightarrow \quad x = \pm\sqrt[4]{\frac{1}{3}} \approx \pm 0.7598.$



(c) Yes. $f'(x) = 0$ where f has horizontal tangents. f' has two maxima and one minimum where f has inflection points.

93. (a) If f is even, then $f(x) = f(-x)$. Using the Chain Rule to differentiate this equation, we get

$f'(x) = f'(-x)\dfrac{d}{dx}(-x) = -f'(-x)$. Thus, $f'(-x) = -f'(x)$, so f' is odd.

(b) If f is odd, then $f(x) = -f(-x)$. Differentiating this equation, we get $f'(x) = -f'(-x)(-1) = f'(-x)$, so f' is even.

94.
$$\left[\frac{f(x)}{g(x)}\right]' = \left\{f(x)\,[g(x)]^{-1}\right\}' = f'(x)\,[g(x)]^{-1} + (-1)\,[g(x)]^{-2}\,g'(x)f(x)$$
$$= \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{[g(x)]^2} = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

This is an alternative derivation of the *formula* in the Quotient Rule. But part of the purpose of the Quotient Rule is to show that if f and g are differentiable, so is f/g. The proof in Section 3.2 does that; this one doesn't.

95. (a)
$$\begin{aligned}
\frac{d}{dx}(\sin^n x\,\cos nx) &= n\sin^{n-1}x\,\cos x\,\cos nx + \sin^n x\,(-n\sin nx) && \text{[Product Rule]}\\
&= n\sin^{n-1}x\,(\cos nx\,\cos x - \sin nx\,\sin x) && [\text{factor out } n\sin^{n-1}x]\\
&= n\sin^{n-1}x\cos(nx + x) && \text{[Addition Formula for cosine]}\\
&= n\sin^{n-1}x\cos[(n+1)x] && [\text{factor out } x]
\end{aligned}$$

(b)
$$\begin{aligned}
\frac{d}{dx}(\cos^n x\,\cos nx) &= n\cos^{n-1}x\,(-\sin x)\cos nx + \cos^n x\,(-n\sin nx) && \text{[Product Rule]}\\
&= -n\cos^{n-1}x\,(\cos nx\,\sin x + \sin nx\,\cos x) && [\text{factor out } -n\cos^{n-1}x]\\
&= -n\cos^{n-1}x\sin(nx + x) && \text{[Addition Formula for sine]}\\
&= -n\cos^{n-1}x\sin[(n+1)x] && [\text{factor out } x]
\end{aligned}$$

96. "The rate of change of y^5 with respect to x is eighty times the rate of change of y with respect to x" $\Leftrightarrow$
$\dfrac{d}{dx}y^5 = 80\dfrac{dy}{dx} \quad \Leftrightarrow \quad 5y^4\dfrac{dy}{dx} = 80\dfrac{dy}{dx} \quad \Leftrightarrow \quad 5y^4 = 80$ (Note that $dy/dx \neq 0$ since the curve never has a horizontal tangent) $\Leftrightarrow \quad y^4 = 16 \quad \Leftrightarrow \quad y = 2$ (since $y > 0$ for all x)

97. Since $\theta^\circ = \left(\frac{\pi}{180}\right)\theta$ rad, we have $\dfrac{d}{d\theta}\left(\sin\theta^\circ\right) = \dfrac{d}{d\theta}\left(\sin\frac{\pi}{180}\theta\right) = \frac{\pi}{180}\cos\frac{\pi}{180}\theta = \frac{\pi}{180}\cos\theta^\circ$.

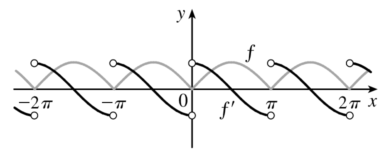
98. (a) $f(x) = |x| = \sqrt{x^2} = (x^2)^{1/2} \quad\Rightarrow\quad f'(x) = \frac{1}{2}(x^2)^{-1/2}(2x) = x/\sqrt{x^2} = x/|x|$ for $x \neq 0$.

f is not differentiable at $x = 0$.

(b) $f(x) = |\sin x| = \sqrt{\sin^2 x} \quad\Rightarrow$

$$f'(x) = \tfrac{1}{2}(\sin^2 x)^{-1/2} 2\sin x\,\cos x = \frac{\sin x}{|\sin x|}\cos x$$

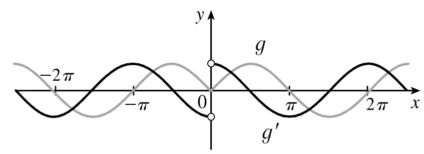
$$= \begin{cases} \cos x & \text{if } \sin x > 0 \\ -\cos x & \text{if } \sin x < 0 \end{cases}$$



f is not differentiable when $x = n\pi$, n an integer.

(c) $g(x) = \sin|x| = \sin\sqrt{x^2} \quad\Rightarrow$

$$g'(x) = \cos|x| \cdot \frac{x}{|x|} = \frac{x}{|x|}\cos x = \begin{cases} \cos x & \text{if } x > 0 \\ -\cos x & \text{if } x < 0 \end{cases}$$



g is not differentiable at 0.

99. The Chain Rule says that $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$, so

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{dy}{du}\frac{du}{dx}\right) = \left[\frac{d}{dx}\left(\frac{dy}{du}\right)\right]\frac{du}{dx} + \frac{dy}{du}\frac{d}{dx}\left(\frac{du}{dx}\right) \quad \text{[Product Rule]}$$

$$= \left[\frac{d}{du}\left(\frac{dy}{du}\right)\frac{du}{dx}\right]\frac{du}{dx} + \frac{dy}{du}\frac{d^2u}{dx^2} = \frac{d^2y}{du^2}\left(\frac{du}{dx}\right)^2 + \frac{dy}{du}\frac{d^2u}{dx^2}$$

100. From Exercise 99, $\dfrac{d^2y}{dx^2} = \dfrac{d^2y}{du^2}\left(\dfrac{du}{dx}\right)^2 + \dfrac{dy}{du}\dfrac{d^2u}{dx^2} \quad\Rightarrow$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{d^2y}{du^2}\left(\frac{du}{dx}\right)^2\right] + \frac{d}{dx}\left[\frac{dy}{du}\frac{d^2u}{dx^2}\right]$$

$$= \left[\frac{d}{dx}\left(\frac{d^2y}{du^2}\right)\right]\left(\frac{du}{dx}\right)^2 + \left[\frac{d}{dx}\left(\frac{du}{dx}\right)^2\right]\frac{d^2y}{du^2} + \left[\frac{d}{dx}\left(\frac{dy}{du}\right)\right]\frac{d^2u}{dx^2} + \left[\frac{d}{dx}\left(\frac{d^2u}{dx^2}\right)\right]\frac{dy}{du}$$

$$= \left[\frac{d}{du}\left(\frac{d^2y}{du^2}\right)\frac{du}{dx}\right]\left(\frac{du}{dx}\right)^2 + 2\frac{du}{dx}\frac{d^2u}{dx^2}\frac{d^2y}{du^2} + \left[\frac{d}{du}\left(\frac{dy}{du}\right)\frac{du}{dx}\right]\left(\frac{d^2u}{dx^2}\right) + \frac{d^3u}{dx^3}\frac{dy}{du}$$

$$= \frac{d^3y}{du^3}\left(\frac{du}{dx}\right)^3 + 3\frac{du}{dx}\frac{d^2u}{dx^2}\frac{d^2y}{du^2} + \frac{dy}{du}\frac{d^3u}{dx^3}$$

APPLIED PROJECT Where Should a Pilot Start Descent?

1. Condition (i) will hold if and only if all of the following four conditions hold:

(α) $P(0) = 0$

(β) $P'(0) = 0$ (for a smooth landing)

(γ) $P'(\ell) = 0$ (since the plane is cruising horizontally when it begins its descent)

(δ) $P(\ell) = h$.

First of all, condition α implies that $P(0) = d = 0$, so $P(x) = ax^3 + bx^2 + cx \Rightarrow P'(x) = 3ax^2 + 2bx + c$. But $P'(0) = c = 0$ by condition β. So $P'(\ell) = 3a\ell^2 + 2b\ell = \ell\,(3a\ell + 2b)$. Now by condition γ, $3a\ell + 2b = 0 \Rightarrow a = -\dfrac{2b}{3\ell}$.

Therefore, $P(x) = -\dfrac{2b}{3\ell}x^3 + bx^2$. Setting $P(\ell) = h$ for condition δ, we get $P(\ell) = -\dfrac{2b}{3\ell}\ell^3 + b\ell^2 = h \Rightarrow$ $-\dfrac{2}{3}b\ell^2 + b\ell^2 = h \Rightarrow \dfrac{1}{3}b\ell^2 = h \Rightarrow b = \dfrac{3h}{\ell^2} \Rightarrow a = -\dfrac{2h}{\ell^3}$. So $y = P(x) = -\dfrac{2h}{\ell^3}x^3 + \dfrac{3h}{\ell^2}x^2$.

2. By condition (ii), $\dfrac{dx}{dt} = -v$ for all t, so $x(t) = \ell - vt$. Condition (iii) states that $\left|\dfrac{d^2y}{dt^2}\right| \le k$. By the Chain Rule, we have $\dfrac{dy}{dt} = \dfrac{dy}{dx}\dfrac{dx}{dt} = -\dfrac{2h}{\ell^3}\left(3x^2\right)\dfrac{dx}{dt} + \dfrac{3h}{\ell^2}(2x)\dfrac{dx}{dt} = \dfrac{6hx^2v}{\ell^3} - \dfrac{6hxv}{\ell^2}$ (for $x \le \ell$) $\Rightarrow$

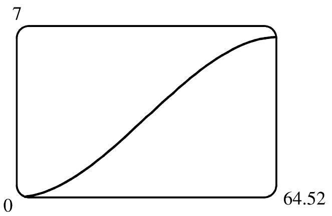
$\dfrac{d^2y}{dt^2} = \dfrac{6hv}{\ell^3}(2x)\dfrac{dx}{dt} - \dfrac{6hv}{\ell^2}\dfrac{dx}{dt} = -\dfrac{12hv^2}{\ell^3}x + \dfrac{6hv^2}{\ell^2}$. In particular, when $t = 0$, $x = \ell$ and so

$\left.\dfrac{d^2y}{dt^2}\right|_{t=0} = -\dfrac{12hv^2}{\ell^3}\ell + \dfrac{6hv^2}{\ell^2} = -\dfrac{6hv^2}{\ell^2}$. Thus, $\left|\dfrac{d^2y}{dt^2}\right|_{t=0} = \dfrac{6hv^2}{\ell^2} \le k$. (This condition also follows from taking $x = 0$.)

3. We substitute $k = 860\ \text{mi/h}^2$, $h = 35{,}000\ \text{ft} \times \dfrac{1\ \text{mi}}{5280\ \text{ft}}$, and $v = 300\ \text{mi/h}$ into the result of part (b):

$\dfrac{6\left(35{,}000 \cdot \frac{1}{5280}\right)(300)^2}{\ell^2} \le 860 \Rightarrow \ell \ge 300\sqrt{6 \cdot \dfrac{35{,}000}{5280 \cdot 860}} \approx 64.5$ miles.

4. Substituting the values of h and ℓ in Problem 3 into $P(x) = -\dfrac{2h}{\ell^3}x^3 + \dfrac{3h}{\ell^2}x^2$ gives us $P(x) = ax^3 + bx^2$, where $a \approx -4.937 \times 10^{-5}$ and $b \approx 4.78 \times 10^{-3}$.



3.5 Implicit Differentiation

1. (a) $\dfrac{d}{dx}(9x^2 - y^2) = \dfrac{d}{dx}(1) \Rightarrow 18x - 2y\,y' = 0 \Rightarrow 2y\,y' = 18x \Rightarrow y' = \dfrac{9x}{y}$

(b) $9x^2 - y^2 = 1 \Rightarrow y^2 = 9x^2 - 1 \Rightarrow y = \pm\sqrt{9x^2 - 1}$, so $y' = \pm\frac{1}{2}(9x^2 - 1)^{-1/2}(18x) = \pm\dfrac{9x}{\sqrt{9x^2 - 1}}$.

(c) From part (a), $y' = \dfrac{9x}{y} = \dfrac{9x}{\pm\sqrt{9x^2 - 1}}$, which agrees with part (b).

2. (a) $\dfrac{d}{dx}(2x^2 + x + xy) = \dfrac{d}{dx}(1) \Rightarrow 4x + 1 + xy' + y \cdot 1 = 0 \Rightarrow xy' = -4x - y - 1 \Rightarrow y' = -\dfrac{4x + y + 1}{x}$

(b) $2x^2 + x + xy = 1 \Rightarrow xy = 1 - 2x^2 - x \Rightarrow y = \dfrac{1}{x} - 2x - 1$, so $y' = -\dfrac{1}{x^2} - 2$

(c) From part (a),

$y' = -\dfrac{4x + y + 1}{x} = -4 - \dfrac{1}{x}y - \dfrac{1}{x} = -4 - \dfrac{1}{x}\left(\dfrac{1}{x} - 2x - 1 - \dfrac{1}{x}\right) = -4 - \dfrac{1}{x^2} + 2 + \dfrac{1}{x} - \dfrac{1}{x} = -\dfrac{1}{x^2} - 2$, which agrees with part (b).

3. (a) $\dfrac{d}{dx}\left(\sqrt{x}+\sqrt{y}\right)=\dfrac{d}{dx}(1) \;\Rightarrow\; \dfrac{1}{2}x^{-1/2}+\dfrac{1}{2}y^{-1/2}y'=0 \;\Rightarrow\; \dfrac{1}{2\sqrt{y}}\,y'=-\dfrac{1}{2\sqrt{x}} \;\Rightarrow\; y'=-\dfrac{\sqrt{y}}{\sqrt{x}}$

(b) $\sqrt{x}+\sqrt{y}=1 \;\Rightarrow\; \sqrt{y}=1-\sqrt{x} \;\Rightarrow\; y=\left(1-\sqrt{x}\right)^2 \;\Rightarrow\; y=1-2\sqrt{x}+x$, so

$y'=-2\cdot\dfrac{1}{2}x^{-1/2}+1=1-\dfrac{1}{\sqrt{x}}$.

(c) From part (a), $y'=-\dfrac{\sqrt{y}}{\sqrt{x}}=-\dfrac{1-\sqrt{x}}{\sqrt{x}}$ [from part (b)] $=-\dfrac{1}{\sqrt{x}}+1$, which agrees with part (b).

4. (a) $\dfrac{d}{dx}\left(\dfrac{2}{x}-\dfrac{1}{y}\right)=\dfrac{d}{dx}(4) \;\Rightarrow\; -2x^{-2}+y^{-2}y'=0 \;\Rightarrow\; \dfrac{1}{y^2}\,y'=\dfrac{2}{x^2} \;\Rightarrow\; y'=\dfrac{2y^2}{x^2}$

(b) $\dfrac{2}{x}-\dfrac{1}{y}=4 \;\Rightarrow\; \dfrac{1}{y}=\dfrac{2}{x}-4 \;\Rightarrow\; \dfrac{1}{y}=\dfrac{2-4x}{x} \;\Rightarrow\; y=\dfrac{x}{2-4x}$, so

$y'=\dfrac{(2-4x)(1)-x(-4)}{(2-4x)^2}=\dfrac{2}{(2-4x)^2}\;\left[\text{or }\dfrac{1}{2(1-2x)^2}\right]$.

(c) From part (a), $y'=\dfrac{2y^2}{x^2}=\dfrac{2\left(\dfrac{x}{2-4x}\right)^2}{x^2}$ [from part (b)] $=\dfrac{2x^2}{x^2(2-4x)^2}=\dfrac{2}{(2-4x)^2}$, which agrees with part (b).

5. $\dfrac{d}{dx}(x^2-4xy+y^2)=\dfrac{d}{dx}(4) \;\Rightarrow\; 2x-4[xy'+y(1)]+2y\,y'=0 \;\Rightarrow\; 2y\,y'-4xy'=4y-2x \;\Rightarrow$

$y'(y-2x)=2y-x \;\Rightarrow\; y'=\dfrac{2y-x}{y-2x}$

6. $\dfrac{d}{dx}(2x^2+xy-y^2)=\dfrac{d}{dx}(2) \;\Rightarrow\; 4x+xy'+y(1)-2y\,y'=0 \;\Rightarrow\; xy'-2y\,y'=-4x-y \;\Rightarrow$

$(x-2y)y'=-4x-y \;\Rightarrow\; y'=\dfrac{-4x-y}{x-2y}$

7. $\dfrac{d}{dx}(x^4+x^2y^2+y^3)=\dfrac{d}{dx}(5) \;\Rightarrow\; 4x^3+x^2\cdot 2y\,y'+y^2\cdot 2x+3y^2y'=0 \;\Rightarrow\; 2x^2y\,y'+3y^2y'=-4x^3-2xy^2 \;\Rightarrow$

$(2x^2y+3y^2)y'=-4x^3-2xy^2 \;\Rightarrow\; y'=\dfrac{-4x^3-2xy^2}{2x^2y+3y^2}=-\dfrac{2x(2x^2+y^2)}{y(2x^2+3y)}$

8. $\dfrac{d}{dx}(x^3-xy^2+y^3)=\dfrac{d}{dx}(1) \;\Rightarrow\; 3x^2-x\cdot 2y\,y'-y^2\cdot 1+3y^2y'=0 \;\Rightarrow\; 3y^2y'-2x\,y\,y'=y^2-3x^2 \;\Rightarrow$

$(3y^2-2xy)\,y'=y^2-3x^2 \;\Rightarrow\; y'=\dfrac{y^2-3x^2}{3y^2-2xy}=\dfrac{y^2-3x^2}{y(3y-2x)}$

9. $\dfrac{d}{dx}\left(\dfrac{x^2}{x+y}\right)=\dfrac{d}{dx}(y^2+1) \;\Rightarrow\; \dfrac{(x+y)(2x)-x^2(1+y')}{(x+y)^2}=2y\,y' \;\Rightarrow$

$2x^2+2xy-x^2-x^2\,y'=2y(x+y)^2\,y' \;\Rightarrow\; x^2+2xy=2y(x+y)^2\,y'+x^2\,y' \;\Rightarrow$

$x(x+2y)=[2y(x^2+2xy+y^2)+x^2]\,y' \;\Rightarrow\; y'=\dfrac{x(x+2y)}{2x^2y+4xy^2+2y^3+x^2}$

Or: Start by clearing fractions and then differentiate implicitly.

10. $\dfrac{d}{dx}(xe^y)=\dfrac{d}{dx}(x-y) \;\Rightarrow\; xe^y\,y'+e^y\cdot 1=1-y' \;\Rightarrow\; xe^y\,y'+y'=1-e^y \;\Rightarrow\; y'(xe^y+1)=1-e^y \;\Rightarrow$

$y'=\dfrac{1-e^y}{xe^y+1}$

11. $\dfrac{d}{dx}(y\cos x)=\dfrac{d}{dx}(x^2+y^2) \;\Rightarrow\; y(-\sin x)+\cos x\cdot y'=2x+2y\,y' \;\Rightarrow\; \cos x\cdot y'-2y\,y'=2x+y\sin x \;\Rightarrow$

$y'(\cos x-2y)=2x+y\sin x \;\Rightarrow\; y'=\dfrac{2x+y\sin x}{\cos x-2y}$

12. $\dfrac{d}{dx}\cos(xy)=\dfrac{d}{dx}(1+\sin y) \;\Rightarrow\; -\sin(xy)(xy'+y\cdot 1)=\cos y\cdot y' \;\Rightarrow\; -xy'\sin(xy)-\cos y\cdot y'=y\sin(xy) \;\Rightarrow$

$y'[-x\sin(xy)-\cos y]=y\sin(xy) \;\Rightarrow\; y'=\dfrac{y\sin(xy)}{-x\sin(xy)-\cos y}=-\dfrac{y\sin(xy)}{x\sin(xy)+\cos y}$

13. $\dfrac{d}{dx}\sqrt{x+y}=\dfrac{d}{dx}\left(x^4+y^4\right) \;\Rightarrow\; \frac{1}{2}\left(x+y\right)^{-1/2}\left(1+y'\right)=4x^3+4y^3y' \;\Rightarrow$

$\dfrac{1}{2\sqrt{x+y}}+\dfrac{1}{2\sqrt{x+y}}\,y'=4x^3+4y^3y' \;\Rightarrow\; \dfrac{1}{2\sqrt{x+y}}-4x^3=4y^3y'-\dfrac{1}{2\sqrt{x+y}}y' \;\Rightarrow$

$\dfrac{1-8x^3\sqrt{x+y}}{2\sqrt{x+y}}=\dfrac{8y^3\sqrt{x+y}-1}{2\sqrt{x+y}}\,y' \;\Rightarrow\; y'=\dfrac{1-8x^3\sqrt{x+y}}{8y^3\sqrt{x+y}-1}$

14. $\dfrac{d}{dx}(e^y\sin x)=\dfrac{d}{dx}(x+xy) \;\Rightarrow\; e^y\cos x+\sin x\cdot e^y y'=1+xy'+y\cdot 1 \;\Rightarrow$

$e^y\sin x\cdot y'-xy'=1+y-e^y\cos x \;\Rightarrow\; y'(e^y\sin x-x)=1+y-e^y\cos x \;\Rightarrow\; y'=\dfrac{1+y-e^y\cos x}{e^y\sin x-x}$

15. $\dfrac{d}{dx}(e^{x/y})=\dfrac{d}{dx}(x-y) \;\Rightarrow\; e^{x/y}\cdot\dfrac{d}{dx}\left(\dfrac{x}{y}\right)=1-y' \;\Rightarrow$

$e^{x/y}\cdot\dfrac{y\cdot 1-x\cdot y'}{y^2}=1-y' \;\Rightarrow\; e^{x/y}\cdot\dfrac{1}{y}-\dfrac{xe^{x/y}}{y^2}\cdot y'=1-y' \;\Rightarrow\; y'-\dfrac{xe^{x/y}}{y^2}\cdot y'=1-\dfrac{e^{x/y}}{y} \;\Rightarrow$

$y'\left(1-\dfrac{xe^{x/y}}{y^2}\right)=\dfrac{y-e^{x/y}}{y} \;\Rightarrow\; y'=\dfrac{\dfrac{y-e^{x/y}}{y}}{\dfrac{y^2-xe^{x/y}}{y^2}}=\dfrac{y(y-e^{x/y})}{y^2-xe^{x/y}}$

16. $\dfrac{d}{dx}(xy)=\dfrac{d}{dx}\sqrt{x^2+y^2} \;\Rightarrow\; xy'+y(1)=\frac{1}{2}\left(x^2+y^2\right)^{-1/2}(2x+2y\,y') \;\Rightarrow$

$xy'+y=\dfrac{x}{\sqrt{x^2+y^2}}+\dfrac{y}{\sqrt{x^2+y^2}}\,y' \;\Rightarrow\; xy'-\dfrac{y}{\sqrt{x^2+y^2}}\,y'=\dfrac{x}{\sqrt{x^2+y^2}}-y \;\Rightarrow$

$\dfrac{x\sqrt{x^2+y^2}-y}{\sqrt{x^2+y^2}}\,y'=\dfrac{x-y\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}} \;\Rightarrow\; y'=\dfrac{x-y\sqrt{x^2+y^2}}{x\sqrt{x^2+y^2}-y}$

17. $\dfrac{d}{dx}\tan^{-1}(x^2y)=\dfrac{d}{dx}(x+xy^2) \;\Rightarrow\; \dfrac{1}{1+(x^2y)^2}(x^2y'+y\cdot 2x)=1+x\cdot 2y\,y'+y^2\cdot 1 \;\Rightarrow$

$\dfrac{x^2}{1+x^4y^2}\,y'-2xy\,y'=1+y^2-\dfrac{2xy}{1+x^4y^2} \;\Rightarrow\; y'\left(\dfrac{x^2}{1+x^4y^2}-2xy\right)=1+y^2-\dfrac{2xy}{1+x^4y^2} \;\Rightarrow$

$y'=\dfrac{1+y^2-\dfrac{2xy}{1+x^4y^2}}{\dfrac{x^2}{1+x^4y^2}-2xy}$ or $y'=\dfrac{1+x^4y^2+y^2+x^4y^4-2xy}{x^2-2xy-2x^5y^3}$

18. $\dfrac{d}{dx}(x\sin y + y\sin x) = \dfrac{d}{dx}(1) \quad\Rightarrow\quad x\cos y\cdot y' + \sin y\cdot 1 + y\cos x + \sin x\cdot y' = 0 \quad\Rightarrow$

$x\cos y\cdot y' + \sin x\cdot y' = -\sin y - y\cos x \quad\Rightarrow\quad y'(x\cos y + \sin x) = -\sin y - y\cos x \quad\Rightarrow\quad y' = \dfrac{-\sin y - y\cos x}{x\cos y + \sin x}$

19. $\dfrac{d}{dx}\sin(xy) = \dfrac{d}{dx}\cos(x+y) \quad\Rightarrow\quad \cos(xy)\cdot(xy' + y\cdot 1) = -\sin(x+y)\cdot(1+y') \quad\Rightarrow$

$x\cos(xy)\,y' + y\cos(xy) = -\sin(x+y) - y'\sin(x+y) \quad\Rightarrow$

$x\cos(xy)\,y' + y'\sin(x+y) = -y\cos(xy) - \sin(x+y) \quad\Rightarrow$

$[x\cos(xy) + \sin(x+y)]\,y' = -1\,[y\cos(xy) + \sin(x+y)] \quad\Rightarrow\quad y' = -\dfrac{y\cos(xy) + \sin(x+y)}{x\cos(xy) + \sin(x+y)}$

20. $\tan(x-y) = \dfrac{y}{1+x^2} \quad\Rightarrow\quad (1+x^2)\tan(x-y) = y \quad\Rightarrow\quad (1+x^2)\sec^2(x-y)\cdot(1-y') + \tan(x-y)\cdot 2x = y' \quad\Rightarrow$

$(1+x^2)\sec^2(x-y) - (1+x^2)\sec^2(x-y)\cdot y' + 2x\tan(x-y) = y' \quad\Rightarrow$

$(1+x^2)\sec^2(x-y) + 2x\tan(x-y) = \left[1 + (1+x^2)\sec^2(x-y)\right]\cdot y' \quad\Rightarrow$

$y' = \dfrac{(1+x^2)\sec^2(x-y) + 2x\tan(x-y)}{1 + (1+x^2)\sec^2(x-y)}$

21. $\dfrac{d}{dx}\left\{f(x) + x^2[f(x)]^3\right\} = \dfrac{d}{dx}(10) \quad\Rightarrow\quad f'(x) + x^2\cdot 3[f(x)]^2\cdot f'(x) + [f(x)]^3\cdot 2x = 0.$ If $x = 1$, we have

$f'(1) + 1^2\cdot 3[f(1)]^2\cdot f'(1) + [f(1)]^3\cdot 2(1) = 0 \quad\Rightarrow\quad f'(1) + 1\cdot 3\cdot 2^2\cdot f'(1) + 2^3\cdot 2 = 0 \quad\Rightarrow$

$f'(1) + 12f'(1) = -16 \quad\Rightarrow\quad 13f'(1) = -16 \quad\Rightarrow\quad f'(1) = -\frac{16}{13}.$

22. $\dfrac{d}{dx}[g(x) + x\sin g(x)] = \dfrac{d}{dx}(x^2) \quad\Rightarrow\quad g'(x) + x\cos g(x)\cdot g'(x) + \sin g(x)\cdot 1 = 2x.$ If $x = 0$, we have

$g'(0) + 0 + \sin g(0) = 2(0) \quad\Rightarrow\quad g'(0) + \sin 0 = 0 \quad\Rightarrow\quad g'(0) + 0 = 0 \quad\Rightarrow\quad g'(0) = 0.$

23. $\dfrac{d}{dy}(x^4y^2 - x^3y + 2xy^3) = \dfrac{d}{dy}(0) \quad\Rightarrow\quad x^4\cdot 2y + y^2\cdot 4x^3\,x' - (x^3\cdot 1 + y\cdot 3x^2\,x') + 2(x\cdot 3y^2 + y^3\cdot x') = 0 \quad\Rightarrow$

$4x^3y^2\,x' - 3x^2y\,x' + 2y^3\,x' = -2x^4y + x^3 - 6xy^2 \quad\Rightarrow\quad (4x^3y^2 - 3x^2y + 2y^3)\,x' = -2x^4y + x^3 - 6xy^2 \quad\Rightarrow$

$x' = \dfrac{dx}{dy} = \dfrac{-2x^4y + x^3 - 6xy^2}{4x^3y^2 - 3x^2y + 2y^3}$

24. $\dfrac{d}{dy}(y\sec x) = \dfrac{d}{dy}(x\tan y) \quad\Rightarrow\quad y\cdot\sec x\,\tan x\cdot x' + \sec x\cdot 1 = x\cdot\sec^2 y + \tan y\cdot x' \quad\Rightarrow$

$y\sec x\,\tan x\cdot x' - \tan y\cdot x' = x\sec^2 y - \sec x \quad\Rightarrow\quad (y\sec x\,\tan x - \tan y)\,x' = x\sec^2 y - \sec x \quad\Rightarrow$

$x' = \dfrac{dx}{dy} = \dfrac{x\sec^2 y - \sec x}{y\sec x\,\tan x - \tan y}$

25. $y\sin 2x = x\cos 2y \quad\Rightarrow\quad y\cdot\cos 2x\cdot 2 + \sin 2x\cdot y' = x(-\sin 2y\cdot 2y') + \cos(2y)\cdot 1 \quad\Rightarrow$

$\sin 2x\cdot y' + 2x\sin 2y\cdot y' = -2y\cos 2x + \cos 2y \quad\Rightarrow\quad y'(\sin 2x + 2x\sin 2y) = -2y\cos 2x + \cos 2y \quad\Rightarrow$

$y' = \dfrac{-2y\cos 2x + \cos 2y}{\sin 2x + 2x\sin 2y}$. When $x = \frac{\pi}{2}$ and $y = \frac{\pi}{4}$, we have $y' = \dfrac{(-\pi/2)(-1) + 0}{0 + \pi \cdot 1} = \dfrac{\pi/2}{\pi} = \dfrac{1}{2}$, so an equation of the tangent line is $y - \frac{\pi}{4} = \frac{1}{2}(x - \frac{\pi}{2})$, or $y = \frac{1}{2}x$.

26. $\sin(x + y) = 2x - 2y \quad\Rightarrow\quad \cos(x + y) \cdot (1 + y') = 2 - 2y' \quad\Rightarrow\quad \cos(x + y) \cdot y' + 2y' = 2 - \cos(x + y) \quad\Rightarrow$

$y'[\cos(x + y) + 2] = 2 - \cos(x + y) \quad\Rightarrow\quad y' = \dfrac{2 - \cos(x + y)}{\cos(x + y) + 2}$. When $x = \pi$ and $y = \pi$, we have $y' = \dfrac{2 - 1}{1 + 2} = \dfrac{1}{3}$, so an equation of the tangent line is $y - \pi = \frac{1}{3}(x - \pi)$, or $y = \frac{1}{3}x + \frac{2\pi}{3}$.

27. $x^2 - xy - y^2 = 1 \quad\Rightarrow\quad 2x - (xy' + y \cdot 1) - 2y\,y' = 0 \quad\Rightarrow\quad 2x - xy' - y - 2y\,y' = 0 \quad\Rightarrow\quad 2x - y = xy' + 2y\,y' \quad\Rightarrow$

$2x - y = (x + 2y)\,y' \quad\Rightarrow\quad y' = \dfrac{2x - y}{x + 2y}$. When $x = 2$ and $y = 1$, we have $y' = \dfrac{4 - 1}{2 + 2} = \dfrac{3}{4}$, so an equation of the tangent line is $y - 1 = \frac{3}{4}(x - 2)$, or $y = \frac{3}{4}x - \frac{1}{2}$.

28. $x^2 + 2xy + 4y^2 = 12 \quad\Rightarrow\quad 2x + 2x\,y' + 2y + 8y\,y' = 0 \quad\Rightarrow\quad 2x\,y' + 8y\,y' = -2x - 2y \quad\Rightarrow$

$(x + 4y)\,y' = -x - y \quad\Rightarrow\quad y' = -\dfrac{x + y}{x + 4y}$. When $x = 2$ and $y = 1$, we have $y' = -\dfrac{2 + 1}{2 + 4} = -\dfrac{1}{2}$, so an equation of the tangent line is $y - 1 = -\frac{1}{2}(x - 2)$ or $y = -\frac{1}{2}x + 2$.

29. $x^2 + y^2 = (2x^2 + 2y^2 - x)^2 \quad\Rightarrow\quad 2x + 2y\,y' = 2(2x^2 + 2y^2 - x)(4x + 4y\,y' - 1)$. When $x = 0$ and $y = \frac{1}{2}$, we have $0 + y' = 2(\frac{1}{2})(2y' - 1) \quad\Rightarrow\quad y' = 2y' - 1 \quad\Rightarrow\quad y' = 1$, so an equation of the tangent line is $y - \frac{1}{2} = 1(x - 0)$ or $y = x + \frac{1}{2}$.

30. $x^{2/3} + y^{2/3} = 4 \quad\Rightarrow\quad \frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}y' = 0 \quad\Rightarrow\quad \dfrac{1}{\sqrt[3]{x}} + \dfrac{y'}{\sqrt[3]{y}} = 0 \quad\Rightarrow\quad y' = -\dfrac{\sqrt[3]{y}}{\sqrt[3]{x}}$. When $x = -3\sqrt{3}$ and $y = 1$, we have $y' = -\dfrac{1}{\left(-3\sqrt{3}\right)^{1/3}} = -\dfrac{\left(-3\sqrt{3}\right)^{2/3}}{-3\sqrt{3}} = \dfrac{3}{3\sqrt{3}} = \dfrac{1}{\sqrt{3}}$, so an equation of the tangent line is $y - 1 = \frac{1}{\sqrt{3}}\left(x + 3\sqrt{3}\right)$ or $y = \frac{1}{\sqrt{3}}x + 4$.

31. $2(x^2 + y^2)^2 = 25(x^2 - y^2) \quad\Rightarrow\quad 4(x^2 + y^2)(2x + 2y\,y') = 25(2x - 2y\,y') \quad\Rightarrow$

$4(x + y\,y')(x^2 + y^2) = 25(x - y\,y') \quad\Rightarrow\quad 4y\,y'(x^2 + y^2) + 25yy' = 25x - 4x(x^2 + y^2) \quad\Rightarrow$

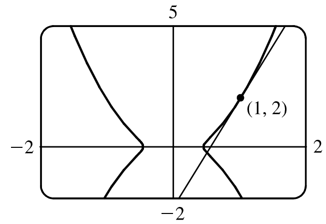
$y' = \dfrac{25x - 4x(x^2 + y^2)}{25y + 4y(x^2 + y^2)}$. When $x = 3$ and $y = 1$, we have $y' = \frac{75 - 120}{25 + 40} = -\frac{45}{65} = -\frac{9}{13}$, so an equation of the tangent line is $y - 1 = -\frac{9}{13}(x - 3)$ or $y = -\frac{9}{13}x + \frac{40}{13}$.

32. $y^2(y^2 - 4) = x^2(x^2 - 5) \quad\Rightarrow\quad y^4 - 4y^2 = x^4 - 5x^2 \quad\Rightarrow\quad 4y^3\,y' - 8y\,y' = 4x^3 - 10x$. When $x = 0$ and $y = -2$, we have $-32y' + 16y' = 0 \quad\Rightarrow\quad -16y' = 0 \quad\Rightarrow\quad y' = 0$, so an equation of the tangent line is $y + 2 = 0(x - 0)$ or $y = -2$.

33. (a) $y^2 = 5x^4 - x^2 \quad\Rightarrow\quad 2y\,y' = 5(4x^3) - 2x \quad\Rightarrow\quad y' = \dfrac{10x^3 - x}{y}$.

So at the point $(1, 2)$ we have $y' = \dfrac{10(1)^3 - 1}{2} = \dfrac{9}{2}$, and an equation of the tangent line is $y - 2 = \frac{9}{2}(x - 1)$ or $y = \frac{9}{2}x - \frac{5}{2}$.

(b)



34. (a) $y^2 = x^3 + 3x^2 \quad\Rightarrow\quad 2y\,y' = 3x^2 + 3(2x) \quad\Rightarrow\quad y' = \dfrac{3x^2 + 6x}{2y}$. So at the point $(1, -2)$ we have

$y' = \dfrac{3(1)^2 + 6(1)}{2(-2)} = -\dfrac{9}{4}$, and an equation of the tangent line is $y + 2 = -\frac{9}{4}(x - 1)$ or $y = -\frac{9}{4}x + \frac{1}{4}$.

(b) The curve has a horizontal tangent where $y' = 0 \quad\Leftrightarrow$

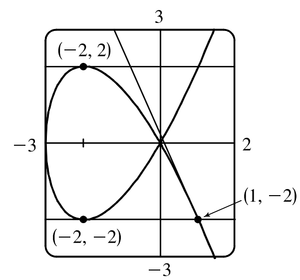
$3x^2 + 6x = 0 \quad\Leftrightarrow\quad 3x(x + 2) = 0 \quad\Leftrightarrow\quad x = 0$ or $x = -2$.

But note that at $x = 0$, $y = 0$ also, so the derivative does not exist.

At $x = -2$, $y^2 = (-2)^3 + 3(-2)^2 = -8 + 12 = 4$, so $y = \pm 2$.

So the two points at which the curve has a horizontal tangent are $(-2, -2)$ and $(-2, 2)$.

(c)



35. $x^2 + 4y^2 = 4 \quad\Rightarrow\quad 2x + 8y\,y' = 0 \quad\Rightarrow\quad y' = -x/(4y) \quad\Rightarrow$

$$y'' = -\frac{1}{4}\,\frac{y \cdot 1 - x \cdot y'}{y^2} = -\frac{1}{4}\,\frac{y - x[-x/(4y)]}{y^2} = -\frac{1}{4}\,\frac{4y^2 + x^2}{4y^3} = -\frac{1}{4}\,\frac{4}{4y^3} \qquad \left[\begin{array}{c}\text{since } x \text{ and } y \text{ must satisfy the}\\ \text{original equation } x^2 + 4y^2 = 4\end{array}\right]$$

Thus, $y'' = -\dfrac{1}{4y^3}$.

36. $x^2 + xy + y^2 = 3 \quad\Rightarrow\quad 2x + xy' + y + 2y\,y' = 0 \quad\Rightarrow\quad (x + 2y)y' = -2x - y \quad\Rightarrow\quad y' = \dfrac{-2x - y}{x + 2y}$.

Differentiating $2x + xy' + y + 2y\,y' = 0$ to find y'' gives $2 + xy'' + y' + y' + 2y\,y'' + 2y'y' = 0 \quad\Rightarrow$

$$(x + 2y)\,y'' = -2 - 2y' - 2(y')^2 = -2\left[1 - \frac{2x + y}{x + 2y} + \left(\frac{2x + y}{x + 2y}\right)^2\right] \quad\Rightarrow$$

$$\begin{aligned} y'' &= -\frac{2}{x + 2y}\left[\frac{(x + 2y)^2 - (2x + y)(x + 2y) + (2x + y)^2}{(x + 2y)^2}\right] \\ &= -\frac{2}{(x + 2y)^3}\,(x^2 + 4xy + 4y^2 - 2x^2 - 4xy - xy - 2y^2 + 4x^2 + 4xy + y^2) \\ &= -\frac{2}{(x + 2y)^3}\,(3x^2 + 3xy + 3y^2) = -\frac{2}{(x + 2y)^3}\,(9) \qquad \left[\begin{array}{c}\text{since } x \text{ and } y \text{ must satisfy the}\\ \text{original equation } x^2 + xy + y^2 = 3\end{array}\right] \end{aligned}$$

Thus, $y'' = -\dfrac{18}{(x + 2y)^3}$.

37. $\sin y + \cos x = 1 \quad \Rightarrow \quad \cos y \cdot y' - \sin x = 0 \quad \Rightarrow \quad y' = \dfrac{\sin x}{\cos y} \quad \Rightarrow$

$$y'' = \frac{\cos y\,\cos x - \sin x(-\sin y)\,y'}{(\cos y)^2} = \frac{\cos y\,\cos x + \sin x\,\sin y(\sin x/\cos y)}{\cos^2 y}$$

$$= \frac{\cos^2 y\,\cos x + \sin^2 x\,\sin y}{\cos^2 y\,\cos y} = \frac{\cos^2 y\,\cos x + \sin^2 x\,\sin y}{\cos^3 y}$$

Using $\sin y + \cos x = 1$, the expression for y'' can be simplified to $y'' = (\cos^2 x + \sin y)/\cos^3 y$.

38. $x^3 - y^3 = 7 \quad \Rightarrow \quad 3x^2 - 3y^2 y' = 0 \quad \Rightarrow \quad y' = \dfrac{x^2}{y^2} \quad \Rightarrow$

$$y'' = \frac{y^2(2x) - x^2(2y\,y')}{(y^2)^2} = \frac{2xy[y - x(x^2/y^2)]}{y^4} = \frac{2x(y - x^3/y^2)}{y^3} = \frac{2x(y^3 - x^3)}{y^3 y^2} = \frac{2x(-7)}{y^5} = \frac{-14x}{y^5}$$

39. If $x = 0$ in $xy + e^y = e$, then we get $0 + e^y = e$, so $y = 1$ and the point where $x = 0$ is $(0, 1)$. Differentiating implicitly with respect to x gives us $xy' + y \cdot 1 + e^y y' = 0$. Substituting 0 for x and 1 for y gives us $0 + 1 + ey' = 0 \quad \Rightarrow \quad ey' = -1 \quad \Rightarrow \quad y' = -1/e$. Differentiating $xy' + y + e^y y' = 0$ implicitly with respect to x gives us $xy'' + y' \cdot 1 + y' + e^y y'' + y' \cdot e^y y' = 0$. Now substitute 0 for x, 1 for y, and $-1/e$ for y'.

$$0 + \left(-\frac{1}{e}\right) + \left(-\frac{1}{e}\right) + ey'' + \left(-\frac{1}{e}\right)(e)\left(-\frac{1}{e}\right) = 0 \quad \Rightarrow \quad -\frac{2}{e} + ey'' + \frac{1}{e} = 0 \quad \Rightarrow \quad ey'' = \frac{1}{e} \quad \Rightarrow \quad y'' = \frac{1}{e^2}.$$

40. If $x = 1$ in $x^2 + xy + y^3 = 1$, then we get $1 + y + y^3 = 1 \quad \Rightarrow \quad y^3 + y = 0 \quad \Rightarrow \quad y(y^2 + 1) \quad \Rightarrow \quad y = 0$, so the point where $x = 1$ is $(1, 0)$. Differentiating implicitly with respect to x gives us $2x + xy' + y \cdot 1 + 3y^2 \cdot y' = 0$. Substituting 1 for x and 0 for y gives us $2 + y' + 0 + 0 = 0 \quad \Rightarrow \quad y' = -2$. Differentiating $2x + xy' + y + 3y^2 y' = 0$ implicitly with respect to x gives us $2 + xy'' + y' \cdot 1 + y' + 3(y^2 y'' + y' \cdot 2yy') = 0$. Now substitute 1 for x, 0 for y, and -2 for y'.
$2 + y'' + (-2) + (-2) + 3(0 + 0) = 0 \quad \Rightarrow \quad y'' = 2$. Differentiating $2 + xy'' + 2y' + 3y^2 y'' + 6y(y')^2 = 0$ implicitly with respect to x gives us $xy''' + y'' \cdot 1 + 2y'' + 3(y^2 y''' + y'' \cdot 2yy') + 6[y \cdot 2y'y'' + (y')^2 y'] = 0$. Now substitute 1 for x, 0 for y, -2 for y', and 2 for y''. $y''' + 2 + 4 + 3(0 + 0) + 6[0 + (-8)] = 0 \quad \Rightarrow \quad y''' = -2 - 4 + 48 = 42$.

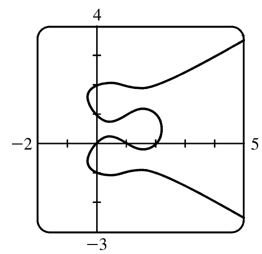
41. (a) There are eight points with horizontal tangents: four at $x \approx 1.57735$ and four at $x \approx 0.42265$.

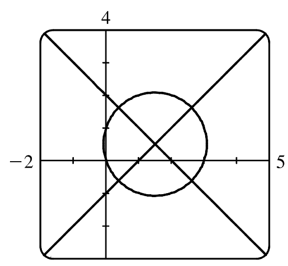
(b) $y' = \dfrac{3x^2 - 6x + 2}{2(2y^3 - 3y^2 - y + 1)} \quad \Rightarrow \quad y' = -1$ at $(0, 1)$ and $y' = \frac{1}{3}$ at $(0, 2)$.

Equations of the tangent lines are $y = -x + 1$ and $y = \frac{1}{3}x + 2$.

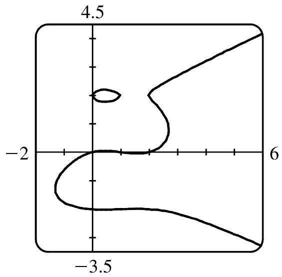
(c) $y' = 0 \quad \Rightarrow \quad 3x^2 - 6x + 2 = 0 \quad \Rightarrow \quad x = 1 \pm \frac{1}{3}\sqrt{3}$

(d) By multiplying the right side of the equation by $x - 3$, we obtain the first graph. By modifying the equation in other ways, we can generate the other graphs.

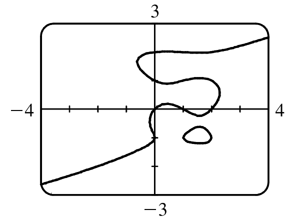




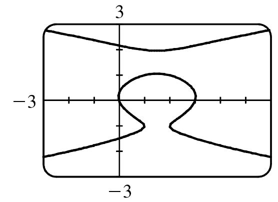
$y(y^2 - 1)(y - 2)$
$= x(x - 1)(x - 2)(x - 3)$



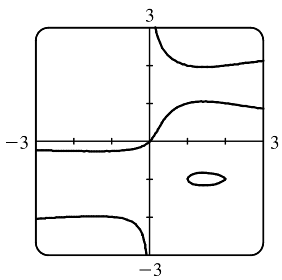
$y(y^2-4)(y-2)$
$= x(x-1)(x-2)$



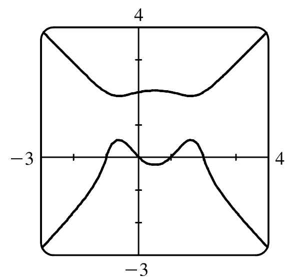
$y(y+1)(y^2-1)(y-2)$
$= x(x-1)(x-2)$



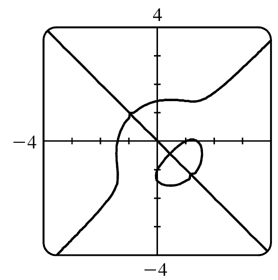
$(y+1)(y^2-1)(y-2)$
$= (x-1)(x-2)$



$x(y+1)(y^2-1)(y-2)$
$= y(x-1)(x-2)$

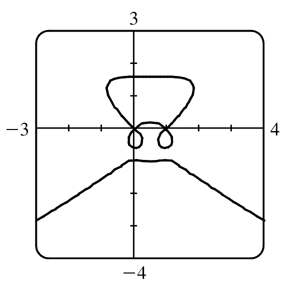


$y(y^2+1)(y-2)$
$= x(x^2-1)(x-2)$



$y(y+1)(y^2-2)$
$= x(x-1)(x^2-2)$

42. (a)



(b) $\dfrac{d}{dx}(2y^3+y^2-y^5) = \dfrac{d}{dx}(x^4-2x^3+x^2) \quad\Rightarrow$

$6y^2y' + 2y\,y' - 5y^4y' = 4x^3 - 6x^2 + 2x \quad\Rightarrow$

$y' = \dfrac{2x(2x^2-3x+1)}{6y^2+2y-5y^4} = \dfrac{2x(2x-1)(x-1)}{y(6y+2-5y^3)}$. From the graph and the values for which $y'=0$, we speculate that there are 9 points with horizontal tangents: 3 at $x=0$, 3 at $x=\frac{1}{2}$, and 3 at $x=1$. The three horizontal tangents along the top of the wagon are hard to find, but by limiting the y-range of the graph (to $[1.6, 1.7]$, for example) they are distinguishable.

43. From Exercise 31, a tangent to the lemniscate will be horizontal if $y'=0 \;\Rightarrow\; 25x - 4x(x^2+y^2) = 0 \;\Rightarrow\; x[25-4(x^2+y^2)] = 0 \;\Rightarrow\; x^2+y^2 = \frac{25}{4}$ (**1**). (Note that when x is 0, y is also 0, and there is no horizontal tangent at the origin.) Substituting $\frac{25}{4}$ for x^2+y^2 in the equation of the lemniscate, $2(x^2+y^2)^2 = 25(x^2-y^2)$, we get $x^2-y^2 = \frac{25}{8}$ (**2**). Solving (**1**) and (**2**), we have $x^2 = \frac{75}{16}$ and $y^2 = \frac{25}{16}$, so the four points are $\left(\pm\frac{5\sqrt{3}}{4}, \pm\frac{5}{4}\right)$.

44. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \;\Rightarrow\; \dfrac{2x}{a^2} + \dfrac{2yy'}{b^2} = 0 \;\Rightarrow\; y' = -\dfrac{b^2x}{a^2y} \;\Rightarrow$ an equation of the tangent line at (x_0, y_0) is $y - y_0 = \dfrac{-b^2x_0}{a^2y_0}(x-x_0)$. Multiplying both sides by $\dfrac{y_0}{b^2}$ gives $\dfrac{y_0y}{b^2} - \dfrac{y_0^2}{b^2} = -\dfrac{x_0x}{a^2} + \dfrac{x_0^2}{a^2}$. Since (x_0, y_0) lies on the ellipse, we have $\dfrac{x_0x}{a^2} + \dfrac{y_0y}{b^2} = \dfrac{x_0^2}{a^2} + \dfrac{y_0^2}{b^2} = 1$.

45. $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1 \Rightarrow \dfrac{2x}{a^2} - \dfrac{2yy'}{b^2} = 0 \Rightarrow y' = \dfrac{b^2x}{a^2y} \Rightarrow$ an equation of the tangent line at (x_0, y_0) is

$y - y_0 = \dfrac{b^2x_0}{a^2y_0}(x - x_0)$. Multiplying both sides by $\dfrac{y_0}{b^2}$ gives $\dfrac{y_0y}{b^2} - \dfrac{y_0^2}{b^2} = \dfrac{x_0x}{a^2} - \dfrac{x_0^2}{a^2}$. Since (x_0, y_0) lies on the hyperbola,

we have $\dfrac{x_0x}{a^2} - \dfrac{y_0y}{b^2} = \dfrac{x_0^2}{a^2} - \dfrac{y_0^2}{b^2} = 1$.

46. $\sqrt{x} + \sqrt{y} = \sqrt{c} \Rightarrow \dfrac{1}{2\sqrt{x}} + \dfrac{y'}{2\sqrt{y}} = 0 \Rightarrow y' = -\dfrac{\sqrt{y}}{\sqrt{x}} \Rightarrow$ an equation of the tangent line at (x_0, y_0)

is $y - y_0 = -\dfrac{\sqrt{y_0}}{\sqrt{x_0}}(x - x_0)$. Now $x = 0 \Rightarrow y = y_0 - \dfrac{\sqrt{y_0}}{\sqrt{x_0}}(-x_0) = y_0 + \sqrt{x_0}\sqrt{y_0}$, so the y-intercept is

$y_0 + \sqrt{x_0}\sqrt{y_0}$. And $y = 0 \Rightarrow -y_0 = -\dfrac{\sqrt{y_0}}{\sqrt{x_0}}(x - x_0) \Rightarrow x - x_0 = \dfrac{y_0\sqrt{x_0}}{\sqrt{y_0}} \Rightarrow$

$x = x_0 + \sqrt{x_0}\sqrt{y_0}$, so the x-intercept is $x_0 + \sqrt{x_0}\sqrt{y_0}$. The sum of the intercepts is

$\left(y_0 + \sqrt{x_0}\sqrt{y_0}\right) + \left(x_0 + \sqrt{x_0}\sqrt{y_0}\right) = x_0 + 2\sqrt{x_0}\sqrt{y_0} + y_0 = \left(\sqrt{x_0} + \sqrt{y_0}\right)^2 = \left(\sqrt{c}\right)^2 = c$.

47. If the circle has radius r, its equation is $x^2 + y^2 = r^2 \Rightarrow 2x + 2yy' = 0 \Rightarrow y' = -\dfrac{x}{y}$, so the slope of the tangent line

at $P(x_0, y_0)$ is $-\dfrac{x_0}{y_0}$. The negative reciprocal of that slope is $\dfrac{-1}{-x_0/y_0} = \dfrac{y_0}{x_0}$, which is the slope of OP, so the tangent line at

P is perpendicular to the radius OP.

48. $y^q = x^p \Rightarrow qy^{q-1}y' = px^{p-1} \Rightarrow y' = \dfrac{px^{p-1}}{qy^{q-1}} = \dfrac{px^{p-1}y}{qy^q} = \dfrac{px^{p-1}x^{p/q}}{qx^p} = \dfrac{p}{q}x^{(p/q)-1}$

49. $y = (\tan^{-1}x)^2 \Rightarrow y' = 2(\tan^{-1}x)^1 \cdot \dfrac{d}{dx}(\tan^{-1}x) = 2\tan^{-1}x \cdot \dfrac{1}{1+x^2} = \dfrac{2\tan^{-1}x}{1+x^2}$

50. $y = \tan^{-1}(x^2) \Rightarrow y' = \dfrac{1}{1+(x^2)^2} \cdot \dfrac{d}{dx}(x^2) = \dfrac{1}{1+x^4} \cdot 2x = \dfrac{2x}{1+x^4}$

51. $y = \sin^{-1}(2x+1) \Rightarrow$

$$y' = \frac{1}{\sqrt{1-(2x+1)^2}} \cdot \frac{d}{dx}(2x+1) = \frac{1}{\sqrt{1-(4x^2+4x+1)}} \cdot 2 = \frac{2}{\sqrt{-4x^2-4x}} = \frac{1}{\sqrt{-x^2-x}}$$

52. $g(x) = \arccos\sqrt{x} \Rightarrow g'(x) = -\dfrac{1}{\sqrt{1-(\sqrt{x})^2}}\dfrac{d}{dx}\sqrt{x} = -\dfrac{1}{\sqrt{1-x}}\left(\dfrac{1}{2}x^{-1/2}\right) = -\dfrac{1}{2\sqrt{x}\sqrt{1-x}}$

53. $F(x) = x\sec^{-1}(x^3) \overset{\text{PR}}{\Rightarrow}$

$$F'(x) = x \cdot \frac{1}{x^3\sqrt{(x^3)^2-1}}\frac{d}{dx}(x^3) + \sec^{-1}(x^3) \cdot 1 = \frac{x(3x^2)}{x^3\sqrt{x^6-1}} + \sec^{-1}(x^3) = \frac{3}{\sqrt{x^6-1}} + \sec^{-1}(x^3)$$

54. $y = \tan^{-1}\left(x - \sqrt{x^2+1}\right) \quad \Rightarrow$

$$y' = \frac{1}{1 + \left(x - \sqrt{x^2+1}\right)^2}\left(1 - \frac{x}{\sqrt{x^2+1}}\right) = \frac{1}{1 + x^2 - 2x\sqrt{x^2+1} + x^2 + 1}\left(\frac{\sqrt{x^2+1} - x}{\sqrt{x^2+1}}\right)$$

$$= \frac{\sqrt{x^2+1} - x}{2\left(1 + x^2 - x\sqrt{x^2+1}\right)\sqrt{x^2+1}} = \frac{\sqrt{x^2+1} - x}{2\left[\sqrt{x^2+1}\,(1+x^2) - x(x^2+1)\right]} = \frac{\sqrt{x^2+1} - x}{2\left[(1+x^2)\left(\sqrt{x^2+1} - x\right)\right]}$$

$$= \frac{1}{2(1+x^2)}$$

55. $h(t) = \cot^{-1}(t) + \cot^{-1}(1/t) \quad \Rightarrow$

$$h'(t) = -\frac{1}{1+t^2} - \frac{1}{1+(1/t)^2}\cdot\frac{d}{dt}\frac{1}{t} = -\frac{1}{1+t^2} - \frac{t^2}{t^2+1}\cdot\left(-\frac{1}{t^2}\right) = -\frac{1}{1+t^2} + \frac{1}{t^2+1} = 0.$$

Note that this makes sense because $h(t) = \dfrac{\pi}{2}$ for $t > 0$ and $h(t) = \dfrac{3\pi}{2}$ for $t < 0$.

56. $R(t) = \arcsin(1/t) \quad \Rightarrow$

$$R'(t) = \frac{1}{\sqrt{1-(1/t)^2}}\,\frac{d}{dt}\,\frac{1}{t} = \frac{1}{\sqrt{1-1/t^2}}\left(-\frac{1}{t^2}\right) = -\frac{1}{\sqrt{1-1/t^2}}\,\frac{1}{\sqrt{t^4}}$$

$$= -\frac{1}{\sqrt{t^4-t^2}} = -\frac{1}{\sqrt{t^2(t^2-1)}} = -\frac{1}{|\,t\,|\sqrt{t^2-1}}$$

57. $y = x\sin^{-1}x + \sqrt{1-x^2} \quad \Rightarrow$

$$y' = x\cdot\frac{1}{\sqrt{1-x^2}} + (\sin^{-1}x)(1) + \frac{1}{2}(1-x^2)^{-1/2}(-2x) = \frac{x}{\sqrt{1-x^2}} + \sin^{-1}x - \frac{x}{\sqrt{1-x^2}} = \sin^{-1}x$$

58. $y = \cos^{-1}(\sin^{-1}t) \quad \Rightarrow \quad y' = -\dfrac{1}{\sqrt{1-(\sin^{-1}t)^2}}\cdot\dfrac{d}{dt}\sin^{-1}t = -\dfrac{1}{\sqrt{1-(\sin^{-1}t)^2}}\cdot\dfrac{1}{\sqrt{1-t^2}}$

59. $y = \arccos\left(\dfrac{b + a\cos x}{a + b\cos x}\right) \quad \Rightarrow$

$$y' = -\frac{1}{\sqrt{1 - \left(\dfrac{b+a\cos x}{a+b\cos x}\right)^2}}\,\frac{(a+b\cos x)(-a\sin x) - (b+a\cos x)(-b\sin x)}{(a+b\cos x)^2}$$

$$= \frac{1}{\sqrt{a^2 + b^2\cos^2 x - b^2 - a^2\cos^2 x}}\,\frac{(a^2-b^2)\,\sin x}{|a+b\cos x|}$$

$$= \frac{1}{\sqrt{a^2-b^2}\,\sqrt{1-\cos^2 x}}\,\frac{(a^2-b^2)\,\sin x}{|a+b\cos x|} = \frac{\sqrt{a^2-b^2}}{|a+b\cos x|}\,\frac{\sin x}{|\sin x|}$$

But $0 \le x \le \pi$, so $|\sin x| = \sin x$. Also $a > b > 0 \quad \Rightarrow \quad b\cos x \ge -b > -a$, so $a + b\cos x > 0$.

Thus $y' = \dfrac{\sqrt{a^2-b^2}}{a+b\cos x}$.

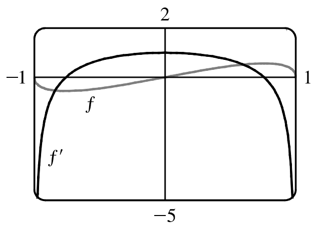
60. $y = \arctan\sqrt{\dfrac{1-x}{1+x}} = \arctan\left(\dfrac{1-x}{1+x}\right)^{1/2} \quad\Rightarrow$

$$y' = \frac{1}{1+\left(\sqrt{\dfrac{1-x}{1+x}}\right)^2} \cdot \frac{d}{dx}\left(\frac{1-x}{1+x}\right)^{1/2} = \frac{1}{1+\dfrac{1-x}{1+x}} \cdot \frac{1}{2}\left(\frac{1-x}{1+x}\right)^{-1/2} \cdot \frac{(1+x)(-1)-(1-x)(1)}{(1+x)^2}$$

$$= \frac{1}{\dfrac{1+x}{1+x}+\dfrac{1-x}{1+x}} \cdot \frac{1}{2}\left(\frac{1+x}{1-x}\right)^{1/2} \cdot \frac{-2}{(1+x)^2} = \frac{1+x}{2} \cdot \frac{1}{2} \cdot \frac{(1+x)^{1/2}}{(1-x)^{1/2}} \cdot \frac{-2}{(1+x)^2}$$

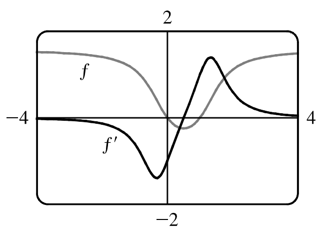
$$= \frac{-1}{2(1-x)^{1/2}(1+x)^{1/2}} = \frac{-1}{2\sqrt{1-x^2}}$$

61. $f(x) = \sqrt{1-x^2}\arcsin x \quad\Rightarrow\quad f'(x) = \sqrt{1-x^2} \cdot \dfrac{1}{\sqrt{1-x^2}} + \arcsin x \cdot \dfrac{1}{2}\left(1-x^2\right)^{-1/2}(-2x) = 1 - \dfrac{x\arcsin x}{\sqrt{1-x^2}}$



Note that $f' = 0$ where the graph of f has a horizontal tangent. Also note that f' is negative when f is decreasing and f' is positive when f is increasing.

62. $f(x) = \arctan(x^2-x) \quad\Rightarrow\quad f'(x) = \dfrac{1}{1+(x^2-x)^2} \cdot \dfrac{d}{dx}(x^2-x) = \dfrac{2x-1}{1+(x^2-x)^2}$



Note that $f' = 0$ where the graph of f has a horizontal tangent. Also note that f' is negative when f is decreasing and f' is positive when f is increasing.

63. Let $y = \cos^{-1}x$. Then $\cos y = x$ and $0 \le y \le \pi \quad\Rightarrow\quad -\sin y\,\dfrac{dy}{dx} = 1 \quad\Rightarrow$

$\dfrac{dy}{dx} = -\dfrac{1}{\sin y} = -\dfrac{1}{\sqrt{1-\cos^2 y}} = -\dfrac{1}{\sqrt{1-x^2}}$. [Note that $\sin y \ge 0$ for $0 \le y \le \pi$.]

64. (a) Let $y = \sec^{-1}x$. Then $\sec y = x$ and $y \in \left(0, \frac{\pi}{2}\right] \cup \left(\pi, \frac{3\pi}{2}\right]$. Differentiate with respect to x: $\sec y\,\tan y\left(\dfrac{dy}{dx}\right) = 1 \quad\Rightarrow$

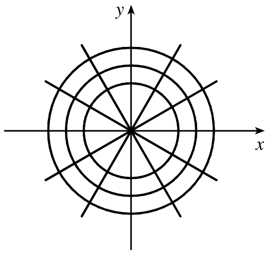
$\dfrac{dy}{dx} = \dfrac{1}{\sec y\,\tan y} = \dfrac{1}{\sec y\sqrt{\sec^2 y - 1}} = \dfrac{1}{x\sqrt{x^2-1}}$. Note that $\tan^2 y = \sec^2 y - 1 \quad\Rightarrow\quad \tan y = \sqrt{\sec^2 y - 1}$ since $\tan y > 0$ when $0 < y < \frac{\pi}{2}$ or $\pi < y < \frac{3\pi}{2}$.

(b) $y = \sec^{-1}x \quad\Rightarrow\quad \sec y = x \quad\Rightarrow\quad \sec y\,\tan y\,\dfrac{dy}{dx} = 1 \quad\Rightarrow\quad \dfrac{dy}{dx} = \dfrac{1}{\sec y\,\tan y}$. Now $\tan^2 y = \sec^2 y - 1 = x^2 - 1$, so $\tan y = \pm\sqrt{x^2-1}$. For $y \in \left[0, \frac{\pi}{2}\right)$, $x \ge 1$, so $\sec y = x = |x|$ and $\tan y \ge 0 \quad\Rightarrow$

$\dfrac{dy}{dx} = \dfrac{1}{x\sqrt{x^2-1}} = \dfrac{1}{|x|\sqrt{x^2-1}}$. For $y \in \left(\frac{\pi}{2}, \pi\right]$, $x \le -1$, so $|x| = -x$ and $\tan y = -\sqrt{x^2-1} \quad\Rightarrow$

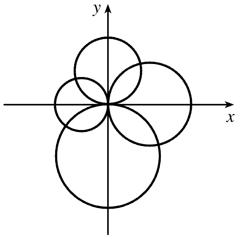
$\dfrac{dy}{dx} = \dfrac{1}{\sec y\,\tan y} = \dfrac{1}{x\left(-\sqrt{x^2-1}\right)} = \dfrac{1}{(-x)\sqrt{x^2-1}} = \dfrac{1}{|x|\sqrt{x^2-1}}$.

65. $x^2 + y^2 = r^2$ is a circle with center O and $ax + by = 0$ is a line through O [assume a and b are not both zero]. $x^2 + y^2 = r^2 \;\Rightarrow\; 2x + 2yy' = 0 \;\Rightarrow\; y' = -x/y$, so the slope of the tangent line at $P_0\,(x_0, y_0)$ is $-x_0/y_0$. The slope of the line OP_0 is y_0/x_0, which is the negative reciprocal of $-x_0/y_0$. Hence, the curves are orthogonal, and the families of curves are orthogonal trajectories of each other.

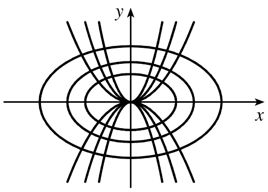


66. The circles $x^2 + y^2 = ax$ and $x^2 + y^2 = by$ intersect at the origin where the tangents are vertical and horizontal [assume a and b are both nonzero]. If (x_0, y_0) is the other point of intersection, then $x_0^2 + y_0^2 = ax_0$ (**1**) and $x_0^2 + y_0^2 = by_0$ (**2**).

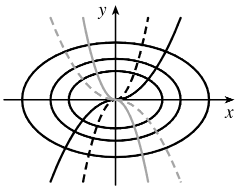
Now $x^2 + y^2 = ax \;\Rightarrow\; 2x + 2yy' = a \;\Rightarrow\; y' = \dfrac{a - 2x}{2y}$ and $x^2 + y^2 = by \;\Rightarrow$ $2x + 2yy' = by' \;\Rightarrow\; y' = \dfrac{2x}{b - 2y}$. Thus, the curves are orthogonal at $(x_0, y_0) \;\Leftrightarrow$ $\dfrac{a - 2x_0}{2y_0} = -\dfrac{b - 2y_0}{2x_0} \;\Leftrightarrow\; 2ax_0 - 4x_0^2 = 4y_0^2 - 2by_0 \;\Leftrightarrow\; ax_0 + by_0 = 2(x_0^2 + y_0^2)$, which is true by (**1**) and (**2**).



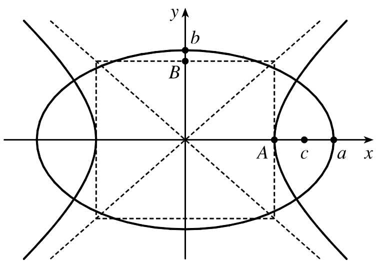
67. $y = cx^2 \;\Rightarrow\; y' = 2cx$ and $x^2 + 2y^2 = k$ [assume $k > 0$] $\Rightarrow\; 2x + 4yy' = 0 \;\Rightarrow$ $2yy' = -x \;\Rightarrow\; y' = -\dfrac{x}{2(y)} = -\dfrac{x}{2(cx^2)} = -\dfrac{1}{2cx}$, so the curves are orthogonal if $c \neq 0$. If $c = 0$, then the horizontal line $y = cx^2 = 0$ intersects $x^2 + 2y^2 = k$ orthogonally at $\left(\pm\sqrt{k}, 0\right)$, since the ellipse $x^2 + 2y^2 = k$ has vertical tangents at those two points.



68. $y = ax^3 \;\Rightarrow\; y' = 3ax^2$ and $x^2 + 3y^2 = b$ [assume $b > 0$] $\Rightarrow\; 2x + 6yy' = 0 \;\Rightarrow$ $3yy' = -x \;\Rightarrow\; y' = -\dfrac{x}{3(y)} = -\dfrac{x}{3(ax^3)} = -\dfrac{1}{3ax^2}$, so the curves are orthogonal if $a \neq 0$. If $a = 0$, then the horizontal line $y = ax^3 = 0$ intesects $x^2 + 3y^2 = b$ orthogonally at $\left(\pm\sqrt{b}, 0\right)$, since the ellipse $x^2 + 3y^2 = b$ has vertical tangents at those two points.



69. Since $A^2 < a^2$, we are assured that there are four points of intersection.



(1) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \;\Rightarrow\; \dfrac{2x}{a^2} + \dfrac{2yy'}{b^2} = 0 \;\Rightarrow\; \dfrac{yy'}{b^2} = -\dfrac{x}{a^2} \;\Rightarrow$

$y' = m_1 = -\dfrac{xb^2}{ya^2}$.

(2) $\dfrac{x^2}{A^2} - \dfrac{y^2}{B^2} = 1 \;\Rightarrow\; \dfrac{2x}{A^2} - \dfrac{2yy'}{B^2} = 0 \;\Rightarrow\; \dfrac{yy'}{B^2} = \dfrac{x}{A^2} \;\Rightarrow$

$y' = m_2 = \dfrac{xB^2}{yA^2}$.

Now $m_1 m_2 = -\dfrac{xb^2}{ya^2} \cdot \dfrac{xB^2}{yA^2} = -\dfrac{b^2B^2}{a^2A^2} \cdot \dfrac{x^2}{y^2}$ (**3**). Subtracting equations, (**1**) − (**2**), gives us $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{x^2}{A^2} + \dfrac{y^2}{B^2} = 0 \;\Rightarrow$

$\dfrac{y^2}{b^2} + \dfrac{y^2}{B^2} = \dfrac{x^2}{A^2} - \dfrac{x^2}{a^2} \quad\Rightarrow\quad \dfrac{y^2B^2 + y^2b^2}{b^2B^2} = \dfrac{x^2a^2 - x^2A^2}{A^2a^2} \quad\Rightarrow\quad \dfrac{y^2(b^2+B^2)}{b^2B^2} = \dfrac{x^2(a^2-A^2)}{a^2A^2}$ **(4)**. Since $a^2 - b^2 = A^2 + B^2$, we have $a^2 - A^2 = b^2 + B^2$. Thus, equation **(4)** becomes $\dfrac{y^2}{b^2B^2} = \dfrac{x^2}{A^2a^2} \quad\Rightarrow\quad \dfrac{x^2}{y^2} = \dfrac{A^2a^2}{b^2B^2}$, and substituting for $\dfrac{x^2}{y^2}$ in equation **(3)** gives us $m_1m_2 = -\dfrac{b^2B^2}{a^2A^2} \cdot \dfrac{a^2A^2}{b^2B^2} = -1$. Hence, the ellipse and hyperbola are orthogonal trajectories.

70. $y = (x+c)^{-1} \quad\Rightarrow\quad y' = -(x+c)^{-2}$ and $y = a(x+k)^{1/3} \quad\Rightarrow\quad y' = \frac{1}{3}a(x+k)^{-2/3}$, so the curves are othogonal if the product of the slopes is -1, that is, $\dfrac{-1}{(x+c)^2} \cdot \dfrac{a}{3(x+k)^{2/3}} = -1 \quad\Rightarrow\quad a = 3(x+c)^2(x+k)^{2/3} \quad\Rightarrow$

$a = 3\left(\dfrac{1}{y}\right)^2 \left(\dfrac{y}{a}\right)^2$ [since $y^2 = (x+c)^{-2}$ and $y^2 = a^2(x+k)^{2/3}$] $\quad\Rightarrow\quad a = 3\left(\dfrac{1}{a^2}\right) \quad\Rightarrow\quad a^3 = 3 \quad\Rightarrow\quad a = \sqrt[3]{3}$.

71. (a) $\left(P + \dfrac{n^2a}{V^2}\right)(V - nb) = nRT \quad\Rightarrow\quad PV - Pnb + \dfrac{n^2a}{V} - \dfrac{n^3ab}{V^2} = nRT \quad\Rightarrow$

$$\frac{d}{dP}(PV - Pnb + n^2aV^{-1} - n^3abV^{-2}) = \frac{d}{dP}(nRT) \quad\Rightarrow$$

$$PV' + V \cdot 1 - nb - n^2aV^{-2} \cdot V' + 2n^3abV^{-3} \cdot V' = 0 \quad\Rightarrow\quad V'(P - n^2aV^{-2} + 2n^3abV^{-3}) = nb - V \quad\Rightarrow$$

$$V' = \frac{nb - V}{P - n^2aV^{-2} + 2n^3abV^{-3}} \text{ or } \frac{dV}{dP} = \frac{V^3(nb - V)}{PV^3 - n^2aV + 2n^3ab}$$

(b) Using the last expression for dV/dP from part (a), we get

$$\frac{dV}{dP} = \frac{(10\text{ L})^3[(1\text{ mole})(0.04267\text{ L/mole}) - 10\text{ L}]}{\left[\begin{array}{l}(2.5\text{ atm})(10\text{ L})^3 - (1\text{ mole})^2(3.592\text{ L}^2\text{-atm/ mole}^2)(10\text{ L}) \\ \qquad + 2(1\text{ mole})^3(3.592\text{ L}^2\text{-atm/ mole}^2)(0.04267\text{ L/ mole})\end{array}\right]}$$

$$= \frac{-9957.33\text{ L}^4}{2464.386541\text{ L}^3\text{-atm}} \approx -4.04\text{ L/ atm}.$$

72. (a) $x^2 + xy + y^2 + 1 = 0 \quad\Rightarrow\quad 2x + xy' + y \cdot 1 + 2yy' + 0 = 0 \quad\Rightarrow\quad y'(x+2y) = -2x - y \quad\Rightarrow\quad y' = \dfrac{-2x - y}{x + 2y}$

(b) Plotting the curve in part (a) gives us an empty graph, that is, there are no points that satisfy the equation. If there were any points that satisfied the equation, then x and y would have opposite signs; otherwise, all the terms are positive and their sum can not equal 0. $\quad x^2 + xy + y^2 + 1 = 0 \quad\Rightarrow\quad x^2 + 2xy + y^2 - xy + 1 = 0 \quad\Rightarrow\quad (x+y)^2 = xy - 1$. The left side of the last equation is nonnegative, but the right side is at most -1, so that proves there are no points that satisfy the equation.

Another solution: $x^2 + xy + y^2 + 1 = \frac{1}{2}x^2 + xy + \frac{1}{2}y^2 + \frac{1}{2}x^2 + \frac{1}{2}y^2 + 1 = \frac{1}{2}(x^2 + 2xy + y^2) + \frac{1}{2}(x^2 + y^2) + 1$

$$= \tfrac{1}{2}(x+y)^2 + \tfrac{1}{2}(x^2 + y^2) + 1 \geq 1$$

Another solution: Regarding $x^2 + xy + y^2 + 1 = 0$ as a quadratic in x, the discriminant is $y^2 - 4(y^2 + 1) = -3y^2 - 4$. This is negative, so there are no real solutions.

(c) The expression for y' in part (a) is meaningless; that is, since the equation in part (a) has no solution, it does not implicitly define a function y of x, and therefore it is meaningless to consider y'.

73. To find the points at which the ellipse $x^2 - xy + y^2 = 3$ crosses the x-axis, let $y = 0$ and solve for x.

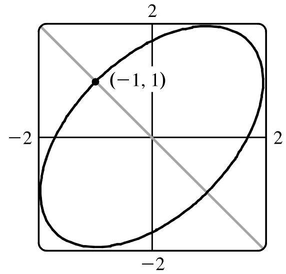
$y = 0 \quad \Rightarrow \quad x^2 - x(0) + 0^2 = 3 \quad \Leftrightarrow \quad x = \pm\sqrt{3}$. So the graph of the ellipse crosses the x-axis at the points $\left(\pm\sqrt{3}, 0\right)$.

Using implicit differentiation to find y', we get $2x - xy' - y + 2yy' = 0 \quad \Rightarrow \quad y'(2y - x) = y - 2x \quad \Leftrightarrow \quad y' = \dfrac{y - 2x}{2y - x}$.

So y' at $\left(\sqrt{3}, 0\right)$ is $\dfrac{0 - 2\sqrt{3}}{2(0) - \sqrt{3}} = 2$ and y' at $\left(-\sqrt{3}, 0\right)$ is $\dfrac{0 + 2\sqrt{3}}{2(0) + \sqrt{3}} = 2$. Thus, the tangent lines at these points are parallel.

74. (a) We use implicit differentiation to find $y' = \dfrac{y - 2x}{2y - x}$ as in Exercise 73. The slope of the tangent line at $(-1, 1)$ is $m = \dfrac{1 - 2(-1)}{2(1) - (-1)} = \dfrac{3}{3} = 1$, so the slope of the normal line is $-\dfrac{1}{m} = -1$, and its equation is $y - 1 = -1(x + 1) \quad \Leftrightarrow$ $y = -x$. Substituting $-x$ for y in the equation of the ellipse, we get $x^2 - x(-x) + (-x)^2 = 3 \quad \Rightarrow \quad 3x^2 = 3 \quad \Leftrightarrow \quad x = \pm 1$. So the normal line must intersect the ellipse again at $x = 1$, and since the equation of the line is $y = -x$, the other point of intersection must be $(1, -1)$.

(b)



75. $x^2y^2 + xy = 2 \quad \Rightarrow \quad x^2 \cdot 2yy' + y^2 \cdot 2x + x \cdot y' + y \cdot 1 = 0 \quad \Leftrightarrow \quad y'(2x^2y + x) = -2xy^2 - y \quad \Leftrightarrow$

$y' = -\dfrac{2xy^2 + y}{2x^2y + x}$. So $-\dfrac{2xy^2 + y}{2x^2y + x} = -1 \quad \Leftrightarrow \quad 2xy^2 + y = 2x^2y + x \quad \Leftrightarrow \quad y(2xy + 1) = x(2xy + 1) \quad \Leftrightarrow$

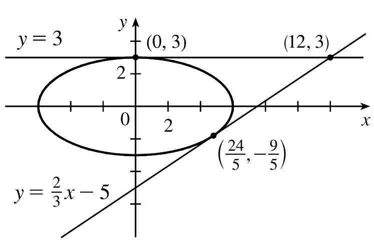
$y(2xy + 1) - x(2xy + 1) = 0 \quad \Leftrightarrow \quad (2xy + 1)(y - x) = 0 \quad \Leftrightarrow \quad xy = -\frac{1}{2}$ or $y = x$. But $xy = -\frac{1}{2} \quad \Rightarrow$

$x^2y^2 + xy = \frac{1}{4} - \frac{1}{2} \neq 2$, so we must have $x = y$. Then $x^2y^2 + xy = 2 \quad \Rightarrow \quad x^4 + x^2 = 2 \quad \Leftrightarrow \quad x^4 + x^2 - 2 = 0 \quad \Leftrightarrow$

$(x^2 + 2)(x^2 - 1) = 0$. So $x^2 = -2$, which is impossible, or $x^2 = 1 \quad \Leftrightarrow \quad x = \pm 1$. Since $x = y$, the points on the curve where the tangent line has a slope of -1 are $(-1, -1)$ and $(1, 1)$.

76. $x^2 + 4y^2 = 36 \quad \Rightarrow \quad 2x + 8yy' = 0 \quad \Rightarrow \quad y' = -\dfrac{x}{4y}$. Let (a, b) be a point on $x^2 + 4y^2 = 36$ whose tangent line passes through $(12, 3)$. The tangent line is then $y - 3 = -\dfrac{a}{4b}(x - 12)$, so $b - 3 = -\dfrac{a}{4b}(a - 12)$. Multiplying both sides by $4b$ gives $4b^2 - 12b = -a^2 + 12a$, so $4b^2 + a^2 = 12(a + b)$. But $4b^2 + a^2 = 36$, so $36 = 12(a + b) \quad \Rightarrow \quad a + b = 3 \quad \Rightarrow$ $b = 3 - a$. Substituting $3 - a$ for b into $a^2 + 4b^2 = 36$ gives $a^2 + 4(3 - a)^2 = 36 \quad \Leftrightarrow \quad a^2 + 36 - 24a + 4a^2 = 36 \quad \Leftrightarrow$ $5a^2 - 24a = 0 \quad \Leftrightarrow \quad a(5a - 24) = 0$, so $a = 0$ or $a = \frac{24}{5}$. If $a = 0$, $b = 3 - 0 = 3$, and if $a = \frac{24}{5}$, $b = 3 - \frac{24}{5} = -\frac{9}{5}$.

So the two points on the ellipse are $(0, 3)$ and $\left(\frac{24}{5}, -\frac{9}{5}\right)$. Using $y - 3 = -\dfrac{a}{4b}(x - 12)$ with $(a, b) = (0, 3)$ gives us the tangent line $y - 3 = 0$ or $y = 3$. With $(a, b) = \left(\frac{24}{5}, -\frac{9}{5}\right)$, we have $y - 3 = -\frac{24/5}{4(-9/5)}(x - 12) \quad \Leftrightarrow \quad y - 3 = \frac{2}{3}(x - 12) \quad \Leftrightarrow \quad y = \frac{2}{3}x - 5$. A graph of the ellipse and the tangent lines confirms our results.



77. (a) If $y = f^{-1}(x)$, then $f(y) = x$. Differentiating implicitly with respect to x and remembering that y is a function of x, we get $f'(y)\,\dfrac{dy}{dx} = 1$, so $\dfrac{dy}{dx} = \dfrac{1}{f'(y)} \quad\Rightarrow\quad (f^{-1})'(x) = \dfrac{1}{f'(f^{-1}(x))}$.

(b) $f(4) = 5 \quad\Rightarrow\quad f^{-1}(5) = 4$. By part (a), $(f^{-1})'(5) = \dfrac{1}{f'(f^{-1}(5))} = \dfrac{1}{f'(4)} = 1\big/\left(\frac{2}{3}\right) = \frac{3}{2}$.

78. (a) Assume $a < b$. Since e^x is an increasing function, $e^a < e^b$, and hence, $a + e^a < b + e^b$; that is, $f(a) < f(b)$. So $f(x) = x + e^x$ is an increasing function and therefore one-to-one.

(b) $f^{-1}(1) = a \quad\Leftrightarrow\quad f(a) = 1$, so we need to find a such that $f(a) = 1$. By inspection, we see that $f(0) = 0 + e^0 = 1$, so $a = 0$, and hence, $f^{-1}(1) = 0$.

(c) $(f^{-1})'(1) = \dfrac{1}{f'(f^{-1}(1))} = \dfrac{1}{f'(0)}$ [by part (b)]. Now $f(x) = x + e^x \quad\Rightarrow\quad f'(x) = 1 + e^x$, so $f'(0) = 1 + e^0 = 2$. Thus, $(f^{-1})'(1) = \frac{1}{2}$.

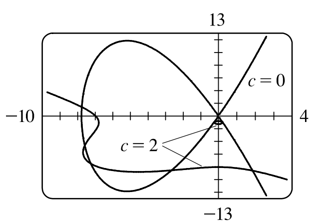
79. (a) $y = J(x)$ and $xy'' + y' + xy = 0 \quad\Rightarrow\quad xJ''(x) + J'(x) + xJ(x) = 0$. If $x = 0$, we have $0 + J'(0) + 0 = 0$, so $J'(0) = 0$.

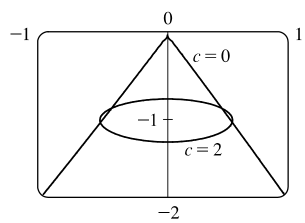
(b) Differentiating $xy'' + y' + xy = 0$ implicitly, we get $xy''' + y'' \cdot 1 + y'' + xy' + y \cdot 1 = 0 \quad\Rightarrow$ $xy''' + 2y'' + xy' + y = 0$, so $xJ'''(x) + 2J''(x) + xJ'(x) + J(x) = 0$. If $x = 0$, we have $0 + 2J''(0) + 0 + 1 \quad [J(0) = 1 \text{ is given}] \quad = 0 \quad\Rightarrow\quad 2J''(0) = -1 \quad\Rightarrow\quad J''(0) = -\frac{1}{2}$.

80. $x^2 + 4y^2 = 5 \quad\Rightarrow\quad 2x + 4(2yy') = 0 \quad\Rightarrow\quad y' = -\dfrac{x}{4y}$. Now let h be the height of the lamp, and let (a, b) be the point of tangency of the line passing through the points $(3, h)$ and $(-5, 0)$. This line has slope $(h - 0)/[3 - (-5)] = \frac{1}{8}h$. But the slope of the tangent line through the point (a, b) can be expressed as $y' = -\dfrac{a}{4b}$, or as $\dfrac{b - 0}{a - (-5)} = \dfrac{b}{a + 5}$ [since the line passes through $(-5, 0)$ and (a, b)], so $-\dfrac{a}{4b} = \dfrac{b}{a + 5} \quad\Leftrightarrow\quad 4b^2 = -a^2 - 5a \quad\Leftrightarrow\quad a^2 + 4b^2 = -5a$. But $a^2 + 4b^2 = 5$ [since (a, b) is on the ellipse], so $5 = -5a \quad\Leftrightarrow\quad a = -1$. Then $4b^2 = -a^2 - 5a = -1 - 5(-1) = 4 \quad\Rightarrow\quad b = 1$, since the point is on the top half of the ellipse. So $\dfrac{h}{8} = \dfrac{b}{a + 5} = \dfrac{1}{-1 + 5} = \dfrac{1}{4} \quad\Rightarrow\quad h = 2$. So the lamp is located 2 units above the x-axis.

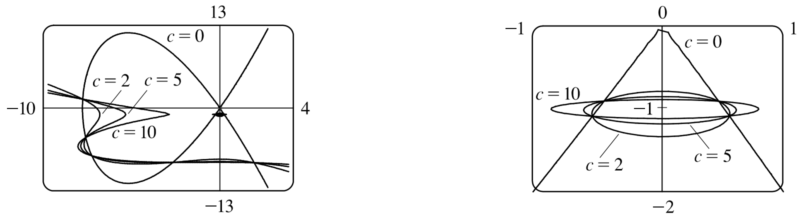
LABORATORY PROJECT Families of Implicit Curves

1. (a) There appear to be nine points of intersection. The "inner four" near the origin are about $(\pm 0.2, -0.9)$ and $(\pm 0.3, -1.1)$. The "outer five" are about $(2.0, -8.9)$, $(-2.8, -8.8)$, $(-7.5, -7.7)$, $(-7.8, -4.7)$, and $(-8.0, 1.5)$.





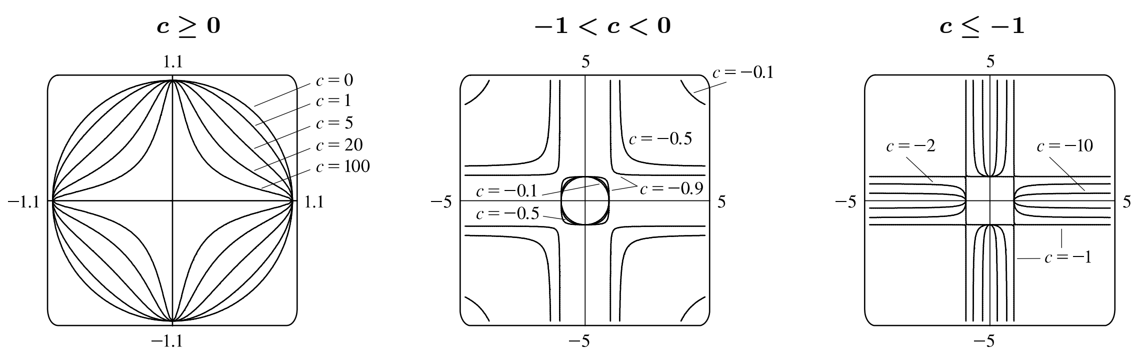
(b) We see from the graphs with $c = 5$ and $c = 10$, and for other values of c, that the curves change shape but the nine points of intersection are the same.



2. (a) If $c = 0$, the graph is the unit circle. As c increases, the graph looks more diamondlike and then more crosslike (see the graph for $c \geq 0$).

For $-1 < c < 0$ (see the graph), there are four hyperboliclike branches as well as an ellipticlike curve bounded by $|x| \leq 1$ and $|y| \leq 1$ for values of c close to 0. As c gets closer to -1, the branches and the curve become more rectangular, approaching the lines $|x| = 1$ and $|y| = 1$.

For $c = -1$, we get the lines $x = \pm 1$ and $y = \pm 1$. As c decreases, we get four test-tubelike curves (see the graph) that are bounded by $|x| = 1$ and $|y| = 1$, and get thinner as $|c|$ gets larger.



(b) The curve for $c = -1$ is described in part (a). When $c = -1$, we get $x^2 + y^2 - x^2y^2 = 1 \quad \Leftrightarrow$ $0 = x^2y^2 - x^2 - y^2 + 1 \quad \Leftrightarrow \quad 0 = (x^2 - 1)(y^2 - 1) \quad \Leftrightarrow \quad x = \pm 1$ or $y = \pm 1$, which algebraically proves that the graph consists of the stated lines.

(c) $\dfrac{d}{dx}(x^2 + y^2 + cx^2y^2) = \dfrac{d}{dx}(1) \quad \Rightarrow \quad 2x + 2y\,y' + c(x^2 \cdot 2y\,y' + y^2 \cdot 2x) = 0 \quad \Rightarrow$

$2y\,y' + 2cx^2y\,y' = -2x - 2cxy^2 \quad \Rightarrow \quad 2y(1 + cx^2)y' = -2x(1 + cy^2) \quad \Rightarrow \quad y' = -\dfrac{x(1 + cy^2)}{y(1 + cx^2)}$.

For $c = -1$, $y' = -\dfrac{x(1 - y^2)}{y(1 - x^2)} = -\dfrac{x(1 + y)(1 - y)}{y(1 + x)(1 - x)}$, so $y' = 0$ when $y = \pm 1$ or $x = 0$ (which leads to $y = \pm 1$) and y' is undefined when $x = \pm 1$ or $y = 0$ (which leads to $x = \pm 1$). Since the graph consists of the lines $x = \pm 1$ and $y = \pm 1$, the slope at any point on the graph is undefined or 0, which is consistent with the expression found for y'.

3.6 Derivatives of Logarithmic Functions

1. The differentiation formula for logarithmic functions, $\dfrac{d}{dx}(\log_a x) = \dfrac{1}{x \ln a}$, is simplest when $a = e$ because $\ln e = 1$.

2. $f(x) = x \ln x - x \quad \Rightarrow \quad f'(x) = x \cdot \dfrac{1}{x} + (\ln x) \cdot 1 - 1 = 1 + \ln x - 1 = \ln x$

3. $f(x) = \sin(\ln x) \quad \Rightarrow \quad f'(x) = \cos(\ln x) \cdot \dfrac{d}{dx} \ln x = \cos(\ln x) \cdot \dfrac{1}{x} = \dfrac{\cos(\ln x)}{x}$

4. $f(x) = \ln(\sin^2 x) = \ln(\sin x)^2 = 2 \ln|\sin x| \quad \Rightarrow \quad f'(x) = 2 \cdot \dfrac{1}{\sin x} \cdot \cos x = 2 \cot x$

5. $f(x) = \ln \dfrac{1}{x} \quad \Rightarrow \quad f'(x) = \dfrac{1}{1/x} \dfrac{d}{dx}\left(\dfrac{1}{x}\right) = x\left(-\dfrac{1}{x^2}\right) = -\dfrac{1}{x}$.

Another solution: $f(x) = \ln \dfrac{1}{x} = \ln 1 - \ln x = -\ln x \quad \Rightarrow \quad f'(x) = -\dfrac{1}{x}$.

6. $y = \dfrac{1}{\ln x} = (\ln x)^{-1} \quad \Rightarrow \quad y' = -1(\ln x)^{-2} \cdot \dfrac{1}{x} = \dfrac{-1}{x(\ln x)^2}$

7. $f(x) = \log_{10}(1 + \cos x) \quad \Rightarrow \quad f'(x) = \dfrac{1}{(1 + \cos x) \ln 10} \dfrac{d}{dx}(1 + \cos x) = \dfrac{-\sin x}{(1 + \cos x) \ln 10}$

8. $f(x) = \log_{10} \sqrt{x} \quad \Rightarrow \quad f'(x) = \dfrac{1}{\sqrt{x} \ln 10} \dfrac{d}{dx} \sqrt{x} = \dfrac{1}{\sqrt{x} \ln 10} \dfrac{1}{2\sqrt{x}} = \dfrac{1}{2(\ln 10)x}$

Or: $f(x) = \log_{10} \sqrt{x} = \log_{10} x^{1/2} = \frac{1}{2} \log_{10} x \quad \Rightarrow \quad f'(x) = \dfrac{1}{2} \dfrac{1}{x \ln 10} = \dfrac{1}{2(\ln 10) x}$

9. $g(x) = \ln(xe^{-2x}) = \ln x + \ln e^{-2x} = \ln x - 2x \quad \Rightarrow \quad g'(x) = \dfrac{1}{x} - 2$

10. $g(t) = \sqrt{1 + \ln t} \quad \Rightarrow \quad g'(t) = \frac{1}{2}(1 + \ln t)^{-1/2} \dfrac{d}{dt}(1 + \ln t) = \dfrac{1}{2\sqrt{1 + \ln t}} \cdot \dfrac{1}{t} = \dfrac{1}{2t\sqrt{1 + \ln t}}$

11. $F(t) = (\ln t)^2 \sin t \quad \Rightarrow \quad F'(t) = (\ln t)^2 \cos t + \sin t \cdot 2 \ln t \cdot \dfrac{1}{t} = \ln t \left(\ln t \, \cos t + \dfrac{2 \sin t}{t}\right)$

12. $h(x) = \ln\left(x + \sqrt{x^2 - 1}\right) \quad \Rightarrow \quad h'(x) = \dfrac{1}{x + \sqrt{x^2 - 1}} \left(1 + \dfrac{x}{\sqrt{x^2 - 1}}\right) = \dfrac{1}{x + \sqrt{x^2 - 1}} \cdot \dfrac{\sqrt{x^2 - 1} + x}{\sqrt{x^2 - 1}} = \dfrac{1}{\sqrt{x^2 - 1}}$

13. $G(y) = \ln \dfrac{(2y + 1)^5}{\sqrt{y^2 + 1}} = \ln(2y + 1)^5 - \ln(y^2 + 1)^{1/2} = 5 \ln(2y + 1) - \frac{1}{2} \ln(y^2 + 1) \quad \Rightarrow$

$G'(y) = 5 \cdot \dfrac{1}{2y + 1} \cdot 2 - \dfrac{1}{2} \cdot \dfrac{1}{y^2 + 1} \cdot 2y = \dfrac{10}{2y + 1} - \dfrac{y}{y^2 + 1} \quad \left[\text{or } \dfrac{8y^2 - y + 10}{(2y + 1)(y^2 + 1)}\right]$

14. $P(v) = \dfrac{\ln v}{1 - v} \quad \Rightarrow \quad P'(v) = \dfrac{(1 - v)(1/v) - (\ln v)(-1)}{(1 - v)^2} \cdot \dfrac{v}{v} = \dfrac{1 - v + v \ln v}{v(1 - v)^2}$

15. $F(s) = \ln \ln s \quad \Rightarrow \quad F'(s) = \dfrac{1}{\ln s} \dfrac{d}{ds} \ln s = \dfrac{1}{\ln s} \cdot \dfrac{1}{s} = \dfrac{1}{s \ln s}$

16. $y = \ln\left|1+t-t^3\right| \quad\Rightarrow\quad y' = \dfrac{1}{1+t-t^3}\,\dfrac{d}{dt}\,(1+t-t^3) = \dfrac{1-3t^2}{1+t-t^3}$

17. $T(z) = 2^z \log_2 z \quad\Rightarrow\quad T'(z) = 2^z\dfrac{1}{z\ln 2} + \log_2 z\cdot 2^z\ln 2 = 2^z\left(\dfrac{1}{z\ln 2} + \log_2 z\,(\ln 2)\right).$

Note that $\log_2 z\,(\ln 2) = \dfrac{\ln z}{\ln 2}\,(\ln 2) = \ln z$ by the change of base theorem. Thus, $T'(z) = 2^z\left(\dfrac{1}{z\ln 2} + \ln z\right).$

18. $y = \ln(\csc x - \cot x) \quad\Rightarrow$

$$y' = \frac{1}{\csc x - \cot x}\,\frac{d}{dx}\,(\csc x - \cot x) = \frac{1}{\csc x - \cot x}(-\csc x\,\cot x + \csc^2 x) = \frac{\csc x(\csc x - \cot x)}{\csc x - \cot x} = \csc x$$

19. $y = \ln(e^{-x} + xe^{-x}) = \ln(e^{-x}(1+x)) = \ln(e^{-x}) + \ln(1+x) = -x + \ln(1+x) \quad\Rightarrow$

$$y' = -1 + \frac{1}{1+x} = \frac{-1-x+1}{1+x} = -\frac{x}{1+x}$$

20. $H(z) = \ln\sqrt{\dfrac{a^2-z^2}{a^2+z^2}} = \ln\left(\dfrac{a^2-z^2}{a^2+z^2}\right)^{1/2} = \dfrac{1}{2}\ln\left(\dfrac{a^2-z^2}{a^2+z^2}\right) = \tfrac{1}{2}\ln(a^2-z^2) - \tfrac{1}{2}\ln(a^2+z^2) \quad\Rightarrow$

$$H'(z) = \frac{1}{2}\cdot\frac{1}{a^2-z^2}\cdot(-2z) - \frac{1}{2}\cdot\frac{1}{a^2+z^2}\cdot(2z) = \frac{z}{z^2-a^2} - \frac{z}{z^2+a^2} = \frac{z(z^2+a^2) - z(z^2-a^2)}{(z^2-a^2)(z^2+a^2)}$$
$$= \frac{z^3+za^2-z^3+za^2}{(z^2-a^2)(z^2+a^2)} = \frac{2a^2z}{z^4-a^4}$$

21. $y = \tan\left[\ln(ax+b)\right] \quad\Rightarrow\quad y' = \sec^2[\ln(ax+b)]\cdot\dfrac{1}{ax+b}\cdot a = \sec^2[\ln(ax+b)]\,\dfrac{a}{ax+b}$

22. $y = \log_2(x\log_5 x) \quad\Rightarrow$

$$y' = \frac{1}{(x\log_5 x)(\ln 2)}\,\frac{d}{dx}\,(x\log_5 x) = \frac{1}{(x\log_5 x)(\ln 2)}\left(x\cdot\frac{1}{x\ln 5} + \log_5 x\right) = \frac{1}{(x\log_5 x)(\ln 5)(\ln 2)} + \frac{1}{x(\ln 2)}.$$

Note that $\log_5 x(\ln 5) = \dfrac{\ln x}{\ln 5}(\ln 5) = \ln x$ by the change of base theorem. Thus, $y' = \dfrac{1}{x\ln x\ln 2} + \dfrac{1}{x\ln 2} = \dfrac{1+\ln x}{x\ln x\ln 2}.$

23. $y = \sqrt{x}\,\ln x \quad\Rightarrow\quad y' = \sqrt{x}\cdot\dfrac{1}{x} + (\ln x)\,\dfrac{1}{2\sqrt{x}} = \dfrac{2+\ln x}{2\sqrt{x}} \quad\Rightarrow$

$$y'' = \frac{2\sqrt{x}\,(1/x) - (2+\ln x)(1/\sqrt{x}\,)}{(2\sqrt{x}\,)^2} = \frac{2/\sqrt{x} - (2+\ln x)(1/\sqrt{x}\,)}{4x} = \frac{2-(2+\ln x)}{\sqrt{x}(4x)} = -\frac{\ln x}{4x\sqrt{x}}$$

24. $y = \dfrac{\ln x}{1+\ln x} \quad\Rightarrow\quad y' = \dfrac{(1+\ln x)(1/x) - (\ln x)(1/x)}{(1+\ln x)^2} = \dfrac{1}{x(1+\ln x)^2} \quad\Rightarrow$

$$y'' = -\frac{\dfrac{d}{dx}[x(1+\ln x)^2]}{[x(1+\ln x)^2]^2} \quad \text{[Reciprocal Rule]} \quad = -\frac{x\cdot 2(1+\ln x)\cdot(1/x) + (1+\ln x)^2}{x^2(1+\ln x)^4}$$
$$= -\frac{(1+\ln x)[2+(1+\ln x)]}{x^2(1+\ln x)^4} = -\frac{3+\ln x}{x^2(1+\ln x)^3}$$

25. $y = \ln|\sec x| \quad\Rightarrow\quad y' = \dfrac{1}{\sec x}\,\dfrac{d}{dx}\sec x = \dfrac{1}{\sec x}\sec x\,\tan x = \tan x \quad\Rightarrow\quad y'' = \sec^2 x$

26. $y = \ln(1 + \ln x) \quad \Rightarrow \quad y' = \dfrac{1}{1 + \ln x} \cdot \dfrac{1}{x} = \dfrac{1}{x(1 + \ln x)} \quad \Rightarrow$

$$y'' = -\frac{\frac{d}{dx}[x(1+\ln x)]}{[x(1+\ln x)]^2} \quad \text{[Reciprocal Rule]} \quad = -\frac{x(1/x) + (1+\ln x)(1)}{x^2(1+\ln x)^2} = -\frac{1+1+\ln x}{x^2(1+\ln x)^2} = -\frac{2+\ln x}{x^2(1+\ln x)^2}$$

27. $f(x) = \dfrac{x}{1 - \ln(x-1)} \quad \Rightarrow$

$$f'(x) = \frac{[1-\ln(x-1)] \cdot 1 - x \cdot \dfrac{-1}{x-1}}{[1-\ln(x-1)]^2} = \frac{\dfrac{(x-1)[1-\ln(x-1)] + x}{x-1}}{[1-\ln(x-1)]^2} = \frac{x - 1 - (x-1)\ln(x-1) + x}{(x-1)[1-\ln(x-1)]^2}$$
$$= \frac{2x - 1 - (x-1)\ln(x-1)}{(x-1)[1-\ln(x-1)]^2}$$

$$\operatorname{Dom}(f) = \{x \mid x - 1 > 0 \ \text{ and } \ 1 - \ln(x-1) \neq 0\} = \{x \mid x > 1 \ \text{ and } \ \ln(x-1) \neq 1\}$$
$$= \{x \mid x > 1 \ \text{ and } \ x - 1 \neq e^1\} = \{x \mid x > 1 \ \text{ and } \ x \neq 1 + e\} = (1, 1+e) \cup (1+e, \infty)$$

28. $f(x) = \sqrt{2 + \ln x} = (2 + \ln x)^{1/2} \quad \Rightarrow \quad f'(x) = \dfrac{1}{2}(2+\ln x)^{-1/2} \cdot \dfrac{1}{x} = \dfrac{1}{2x\sqrt{2+\ln x}}$

$\operatorname{Dom}(f) = \{x \mid 2 + \ln x \geq 0\} = \{x \mid \ln x \geq -2\} = \{x \mid x \geq e^{-2}\} = [e^{-2}, \infty)$.

29. $f(x) = \ln(x^2 - 2x) \quad \Rightarrow \quad f'(x) = \dfrac{1}{x^2 - 2x}(2x - 2) = \dfrac{2(x-1)}{x(x-2)}$.

$\operatorname{Dom}(f) = \{x \mid x(x-2) > 0\} = (-\infty, 0) \cup (2, \infty)$.

30. $f(x) = \ln \ln \ln x \quad \Rightarrow \quad f'(x) = \dfrac{1}{\ln \ln x} \cdot \dfrac{1}{\ln x} \cdot \dfrac{1}{x}$.

$\operatorname{Dom}(f) = \{x \mid \ln \ln x > 0\} = \{x \mid \ln x > 1\} = \{x \mid x > e\} = (e, \infty)$.

31. $f(x) = \ln(x + \ln x) \quad \Rightarrow \quad f'(x) = \dfrac{1}{x + \ln x} \dfrac{d}{dx}(x + \ln x) = \dfrac{1}{x + \ln x}\left(1 + \dfrac{1}{x}\right)$.

Substitute 1 for x to get $f'(1) = \dfrac{1}{1 + \ln 1}\left(1 + \dfrac{1}{1}\right) = \dfrac{1}{1+0}(1+1) = 1 \cdot 2 = 2$.

32. $f(x) = \cos(\ln x^2) \quad \Rightarrow \quad f'(x) = -\sin(\ln x^2)\dfrac{d}{dx} \ln x^2 = -\sin(\ln x^2)\dfrac{1}{x^2}(2x) = -\dfrac{2\sin(\ln x^2)}{x}$.

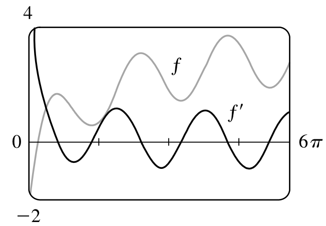
Substitute 1 for x to get $f'(1) = -\dfrac{2\sin(\ln 1^2)}{1} = -2\sin 0 = 0$.

33. $y = \ln(x^2 - 3x + 1) \quad \Rightarrow \quad y' = \dfrac{1}{x^2 - 3x + 1} \cdot (2x - 3) \quad \Rightarrow \quad y'(3) = \frac{1}{1} \cdot 3 = 3$, so an equation of a tangent line at $(3, 0)$ is $y - 0 = 3(x - 3)$, or $y = 3x - 9$.

34. $y = x^2 \ln x \quad \Rightarrow \quad y' = x^2 \cdot \dfrac{1}{x} + (\ln x)(2x) \quad \Rightarrow \quad y'(1) = 1 + 0 = 1$, so an equation of a tangent line at $(1, 0)$ is $y - 0 = 1(x - 1)$, or $y = x - 1$.

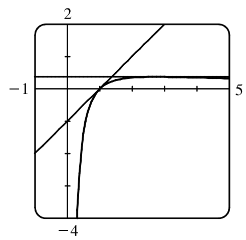
35. $f(x) = \sin x + \ln x \quad \Rightarrow \quad f'(x) = \cos x + 1/x.$

This is reasonable, because the graph shows that f increases when f' is positive, and $f'(x) = 0$ when f has a horizontal tangent.



36. $y = \dfrac{\ln x}{x} \quad \Rightarrow \quad y' = \dfrac{x(1/x) - \ln x}{x^2} = \dfrac{1 - \ln x}{x^2}.$

$y'(1) = \dfrac{1-0}{1^2} = 1$ and $y'(e) = \dfrac{1-1}{e^2} = 0 \quad \Rightarrow \quad$ equations of tangent lines are $y - 0 = 1(x-1)$ or $y = x - 1$ and $y - 1/e = 0(x - e)$ or $y = 1/e$.



37. $f(x) = cx + \ln(\cos x) \quad \Rightarrow \quad f'(x) = c + \dfrac{1}{\cos x} \cdot (-\sin x) = c - \tan x.$

$f'(\frac{\pi}{4}) = 6 \quad \Rightarrow \quad c - \tan\frac{\pi}{4} = 6 \quad \Rightarrow \quad c - 1 = 6 \quad \Rightarrow \quad c = 7.$

38. $f(x) = \log_a(3x^2 - 2) \quad \Rightarrow \quad f'(x) = \dfrac{1}{(3x^2-2)\ln a} \cdot 6x.$

$f'(1) = 3 \quad \Rightarrow \quad \dfrac{1}{\ln a} \cdot 6 = 3 \quad \Rightarrow \quad 2 = \ln a \quad \Rightarrow \quad a = e^2.$

39. $y = (x^2+2)^2(x^4+4)^4 \quad \Rightarrow \quad \ln y = \ln[(x^2+2)^2(x^4+4)^4] \quad \Rightarrow \quad \ln y = 2\ln(x^2+2) + 4\ln(x^4+4) \quad \Rightarrow$

$\dfrac{1}{y}\,y' = 2 \cdot \dfrac{1}{x^2+2} \cdot 2x + 4 \cdot \dfrac{1}{x^4+4} \cdot 4x^3 \quad \Rightarrow \quad y' = y\left(\dfrac{4x}{x^2+2} + \dfrac{16x^3}{x^4+4}\right) \quad \Rightarrow$

$y' = (x^2+2)^2(x^4+4)^4\left(\dfrac{4x}{x^2+2} + \dfrac{16x^3}{x^4+4}\right)$

40. $y = \dfrac{e^{-x}\cos^2 x}{x^2+x+1} \quad \Rightarrow \quad \ln y = \ln\dfrac{e^{-x}\cos^2 x}{x^2+x+1} \quad \Rightarrow$

$\ln y = \ln e^{-x} + \ln|\cos x|^2 - \ln(x^2+x+1) = -x + 2\ln|\cos x| - \ln(x^2+x+1) \quad \Rightarrow$

$\dfrac{1}{y}\,y' = -1 + 2 \cdot \dfrac{1}{\cos x}(-\sin x) - \dfrac{1}{x^2+x+1}(2x+1) \quad \Rightarrow \quad y' = y\left(-1 - 2\tan x - \dfrac{2x+1}{x^2+x+1}\right) \quad \Rightarrow$

$y' = -\dfrac{e^{-x}\cos^2 x}{x^2+x+1}\left(1 + 2\tan x + \dfrac{2x+1}{x^2+x+1}\right)$

41. $y = \sqrt{\dfrac{x-1}{x^4+1}} \quad \Rightarrow \quad \ln y = \ln\left(\dfrac{x-1}{x^4+1}\right)^{1/2} \quad \Rightarrow \quad \ln y = \dfrac{1}{2}\ln(x-1) - \dfrac{1}{2}\ln(x^4+1) \quad \Rightarrow$

$\dfrac{1}{y}\,y' = \dfrac{1}{2}\dfrac{1}{x-1} - \dfrac{1}{2}\dfrac{1}{x^4+1} \cdot 4x^3 \quad \Rightarrow \quad y' = y\left(\dfrac{1}{2(x-1)} - \dfrac{2x^3}{x^4+1}\right) \quad \Rightarrow \quad y' = \sqrt{\dfrac{x-1}{x^4+1}}\left(\dfrac{1}{2x-2} - \dfrac{2x^3}{x^4+1}\right)$

42. $y = \sqrt{x}\,e^{x^2-x}(x+1)^{2/3} \quad \Rightarrow \quad \ln y = \ln\left[x^{1/2}e^{x^2-x}(x+1)^{2/3}\right] \quad \Rightarrow$

$\ln y = \frac{1}{2}\ln x + (x^2 - x) + \frac{2}{3}\ln(x+1) \quad \Rightarrow \quad \dfrac{1}{y}\,y' = \dfrac{1}{2} \cdot \dfrac{1}{x} + 2x - 1 + \dfrac{2}{3} \cdot \dfrac{1}{x+1} \quad \Rightarrow$

$y' = y\left(\dfrac{1}{2x} + 2x - 1 + \dfrac{2}{3x+3}\right) \quad \Rightarrow \quad y' = \sqrt{x}\,e^{x^2-x}(x+1)^{2/3}\left(\dfrac{1}{2x} + 2x - 1 + \dfrac{2}{3x+3}\right)$

43. $y = x^x \Rightarrow \ln y = \ln x^x \Rightarrow \ln y = x \ln x \Rightarrow y'/y = x(1/x) + (\ln x) \cdot 1 \Rightarrow y' = y(1 + \ln x) \Rightarrow$
$y' = x^x(1 + \ln x)$

44. $y = x^{\cos x} \Rightarrow \ln y = \ln x^{\cos x} \Rightarrow \ln y = \cos x \ln x \Rightarrow \dfrac{1}{y}\, y' = \cos x \cdot \dfrac{1}{x} + \ln x \cdot (-\sin x) \Rightarrow$

$y' = y\left(\dfrac{\cos x}{x} - \ln x \sin x\right) \Rightarrow y' = x^{\cos x}\left(\dfrac{\cos x}{x} - \ln x \sin x\right)$

45. $y = x^{\sin x} \Rightarrow \ln y = \ln x^{\sin x} \Rightarrow \ln y = \sin x \ln x \Rightarrow \dfrac{y'}{y} = (\sin x) \cdot \dfrac{1}{x} + (\ln x)(\cos x) \Rightarrow$

$y' = y\left(\dfrac{\sin x}{x} + \ln x \cos x\right) \Rightarrow y' = x^{\sin x}\left(\dfrac{\sin x}{x} + \ln x\, \cos x\right)$

46. $y = \sqrt{x}^{\,x} \Rightarrow \ln y = \ln \sqrt{x}^{\,x} \Rightarrow \ln y = x \ln x^{1/2} \Rightarrow \ln y = \frac{1}{2} x \ln x \Rightarrow \dfrac{1}{y}\, y' = \dfrac{1}{2} x \cdot \dfrac{1}{x} + \ln x \cdot \dfrac{1}{2} \Rightarrow$

$y' = y\left(\frac{1}{2} + \frac{1}{2} \ln x\right) \Rightarrow y' = \frac{1}{2}\sqrt{x}^{\,x}(1 + \ln x)$

47. $y = (\cos x)^x \Rightarrow \ln y = \ln(\cos x)^x \Rightarrow \ln y = x \ln \cos x \Rightarrow \dfrac{1}{y}\, y' = x \cdot \dfrac{1}{\cos x} \cdot (-\sin x) + \ln \cos x \cdot 1 \Rightarrow$

$y' = y\left(\ln \cos x - \dfrac{x \sin x}{\cos x}\right) \Rightarrow y' = (\cos x)^x(\ln \cos x - x \tan x)$

48. $y = (\sin x)^{\ln x} \Rightarrow \ln y = \ln(\sin x)^{\ln x} \Rightarrow \ln y = \ln x \cdot \ln \sin x \Rightarrow \dfrac{1}{y}\, y' = \ln x \cdot \dfrac{1}{\sin x} \cdot \cos x + \ln \sin x \cdot \dfrac{1}{x} \Rightarrow$

$y' = y\left(\ln x \cdot \dfrac{\cos x}{\sin x} + \dfrac{\ln \sin x}{x}\right) \Rightarrow y' = (\sin x)^{\ln x}\left(\ln x \cot x + \dfrac{\ln \sin x}{x}\right)$

49. $y = (\tan x)^{1/x} \Rightarrow \ln y = \ln(\tan x)^{1/x} \Rightarrow \ln y = \dfrac{1}{x} \ln \tan x \Rightarrow$

$\dfrac{1}{y}\, y' = \dfrac{1}{x} \cdot \dfrac{1}{\tan x} \cdot \sec^2 x + \ln \tan x \cdot \left(-\dfrac{1}{x^2}\right) \Rightarrow y' = y\left(\dfrac{\sec^2 x}{x \tan x} - \dfrac{\ln \tan x}{x^2}\right) \Rightarrow$

$y' = (\tan x)^{1/x}\left(\dfrac{\sec^2 x}{x \tan x} - \dfrac{\ln \tan x}{x^2}\right)$ or $y' = (\tan x)^{1/x} \cdot \dfrac{1}{x}\left(\csc x \sec x - \dfrac{\ln \tan x}{x}\right)$

50. $y = (\ln x)^{\cos x} \Rightarrow \ln y = \cos x \ln(\ln x) \Rightarrow \dfrac{y'}{y} = \cos x \cdot \dfrac{1}{\ln x} \cdot \dfrac{1}{x} + (\ln \ln x)(-\sin x) \Rightarrow$

$y' = (\ln x)^{\cos x}\left(\dfrac{\cos x}{x \ln x} - \sin x \ln \ln x\right)$

51. $y = \ln(x^2 + y^2) \Rightarrow y' = \dfrac{1}{x^2 + y^2} \dfrac{d}{dx}(x^2 + y^2) \Rightarrow y' = \dfrac{2x + 2yy'}{x^2 + y^2} \Rightarrow x^2 y' + y^2 y' = 2x + 2yy' \Rightarrow$

$x^2 y' + y^2 y' - 2yy' = 2x \Rightarrow (x^2 + y^2 - 2y)y' = 2x \Rightarrow y' = \dfrac{2x}{x^2 + y^2 - 2y}$

52. $x^y = y^x \Rightarrow y \ln x = x \ln y \Rightarrow y \cdot \dfrac{1}{x} + (\ln x) \cdot y' = x \cdot \dfrac{1}{y} \cdot y' + \ln y \Rightarrow y' \ln x - \dfrac{x}{y}\, y' = \ln y - \dfrac{y}{x} \Rightarrow$

$y' = \dfrac{\ln y - y/x}{\ln x - x/y}$

53. $f(x) = \ln(x-1) \quad \Rightarrow \quad f'(x) = \dfrac{1}{(x-1)} = (x-1)^{-1} \quad \Rightarrow \quad f''(x) = -(x-1)^{-2} \quad \Rightarrow \quad f'''(x) = 2(x-1)^{-3} \quad \Rightarrow$

$f^{(4)}(x) = -2 \cdot 3(x-1)^{-4} \quad \Rightarrow \quad \cdots \quad \Rightarrow \quad f^{(n)}(x) = (-1)^{n-1} \cdot 2 \cdot 3 \cdot 4 \cdot \cdots \cdot (n-1)(x-1)^{-n} = (-1)^{n-1} \dfrac{(n-1)!}{(x-1)^n}$

54. $y = x^8 \ln x$, so $D^9 y = D^8 y' = D^8(8x^7 \ln x + x^7)$. But the eighth derivative of x^7 is 0, so we now have

$D^8(8x^7 \ln x) = D^7(8 \cdot 7x^6 \ln x + 8x^6) = D^7(8 \cdot 7x^6 \ln x) = D^6(8 \cdot 7 \cdot 6x^5 \ln x) = \cdots = D(8!\, x^0 \ln x) = 8!/x.$

55. If $f(x) = \ln(1+x)$, then $f'(x) = \dfrac{1}{1+x}$, so $f'(0) = 1$.

Thus, $\displaystyle\lim_{x \to 0} \frac{\ln(1+x)}{x} = \lim_{x \to 0} \frac{f(x)}{x} = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = f'(0) = 1.$

56. Let $m = n/x$. Then $n = xm$, and as $n \to \infty$, $m \to \infty$.

Therefore, $\displaystyle\lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n = \lim_{m \to \infty} \left(1 + \frac{1}{m}\right)^{mx} = \left[\lim_{m \to \infty} \left(1 + \frac{1}{m}\right)^m\right]^x = e^x$ by Equation 6.

3.7 Rates of Change in the Natural and Social Sciences

1. (a) $s = f(t) = t^3 - 8t^2 + 24t$ (in feet) $\quad \Rightarrow \quad v(t) = f'(t) = 3t^2 - 16t + 24$ (in ft/s)

(b) $v(1) = 3(1)^2 - 16(1) + 24 = 11$ ft/s

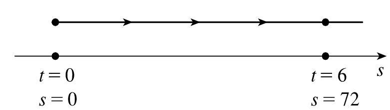
(c) The particle is at rest when $v(t) = 0$. $3t^2 - 16t + 24 = 0 \quad \Rightarrow \quad \dfrac{-(-16) \pm \sqrt{(-16)^2 - 4(3)(24)}}{2(3)} = \dfrac{16 \pm \sqrt{-32}}{6}$.

The negative discriminant indicates that v is never 0 and that the particle never rests.

(d) From parts (b) and (c), we see that $v(t) > 0$ for all t, so the particle is always moving in the positive direction.

(e) The total distance traveled during the first 6 seconds (since the particle doesn't change direction) is $f(6) - f(0) = 72 - 0 = 72$ ft.

(f)

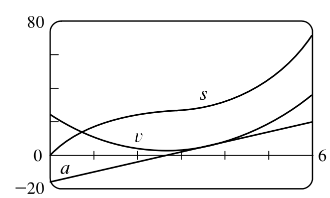


(g) $v(t) = 3t^2 - 16t + 24 \quad \Rightarrow$

$a(t) = v'(t) = 6t - 16$ (in (ft/s)/s or ft/s^2).

$a(1) = 6(1) - 16 = -10$ ft/s^2

(h)



(i) The particle is speeding up when v and a have the same sign. v is always positive and a is positive when $6t - 16 > 0 \quad \Rightarrow$ $t > \frac{8}{3}$, so the particle is speeding up when $t > \frac{8}{3}$. It is slowing down when v and a have opposite signs; that is, when $0 \le t < \frac{8}{3}$.

2. (a) $s = f(t) = \dfrac{9t}{t^2 + 9}$ (in feet) $\quad \Rightarrow \quad v(t) = f'(t) = \dfrac{(t^2+9)(9) - 9t(2t)}{(t^2+9)^2} = \dfrac{-9t^2 + 81}{(t^2+9)^2} = \dfrac{-9(t^2 - 9)}{(t^2+9)^2}$ (in ft/s)

(b) $v(1) = \dfrac{-9(1-9)}{(1+9)^2} = \dfrac{72}{100} = 0.72$ ft/s

(c) The particle is at rest when $v(t) = 0$. $\dfrac{-9(t^2-9)}{(t^2+9)^2} = 0 \quad \Leftrightarrow \quad t^2 - 9 = 0 \quad \Rightarrow \quad t = 3$ s [since $t \geq 0$].

(d) The particle is moving in the positive direction when $v(t) > 0$.

$$\frac{-9(t^2-9)}{(t^2+9)^2} > 0 \quad \Rightarrow \quad -9(t^2-9) > 0 \quad \Rightarrow \quad t^2 - 9 < 0 \quad \Rightarrow \quad t^2 < 9 \quad \Rightarrow \quad 0 \leq t < 3.$$

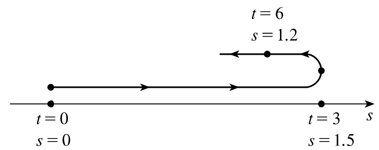
(e) Since the particle is moving in the positve direction and in the negative direction, we need to calculate the distance traveled in the intervals $[0, 3]$ and$[3, 6]$, respectively.

$|f(3) - f(0)| = \left|\frac{27}{18} - 0\right| = \frac{3}{2}$

$|f(6) - f(3)| = \left|\frac{54}{45} - \frac{27}{18}\right| = \frac{3}{10}$

The total distance is $\frac{3}{2} + \frac{3}{10} = \frac{9}{5}$ or 1.8 ft.

(f)

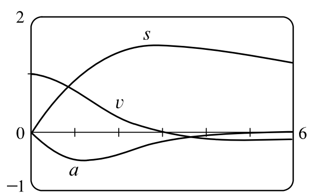


(g) $v(t) = -9\,\dfrac{t^2-9}{(t^2+9)^2} \quad \Rightarrow$

$$a(t) = v'(t) = -9\,\frac{(t^2+9)^2(2t) - (t^2-9)2(t^2+9)(2t)}{[(t^2+9)^2]^2} = -9\,\frac{2t(t^2+9)[(t^2+9) - 2(t^2-9)]}{(t^2+9)^4} = \frac{18t(t^2-27)}{(t^2+9)^3}.$$

$$a(1) = \frac{18(-26)}{10^3} = -0.468 \text{ ft/s}^2$$

(h)



(i) The particle is speeding up when v and a have the same sign. a is negative for $0 < t < \sqrt{27}$ [≈ 5.2], so from the figure in part (h), we see that v and a are both negative for $3 < t < 3\sqrt{3}$. The particle is slowing down when v and a have opposite signs. This occurs when $0 < t < 3$ and when $t > 3\sqrt{3}$.

3. (a) $s = f(t) = \sin(\pi t/2)$ (in feet) $\quad \Rightarrow \quad v(t) = f'(t) = \cos(\pi t/2) \cdot (\pi/2) = \frac{\pi}{2}\cos(\pi t/2)$ (in ft/s)

(b) $v(1) = \frac{\pi}{2}\cos\frac{\pi}{2} = \frac{\pi}{2}(0) = 0$ ft/s

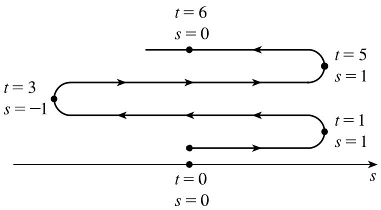
(c) The particle is at rest when $v(t) = 0$. $\frac{\pi}{2}\cos\frac{\pi}{2}t = 0 \quad \Leftrightarrow \quad \cos\frac{\pi}{2}t = 0 \quad \Leftrightarrow \quad \frac{\pi}{2}t = \frac{\pi}{2} + n\pi \quad \Leftrightarrow \quad t = 1 + 2n$, where n is a nonnegative integer since $t \geq 0$.

(d) The particle is moving in the positive direction when $v(t) > 0$. From part (c), we see that v changes sign at every positive odd integer. v is positive when $0 < t < 1$, $3 < t < 5$, $7 < t < 9$, and so on.

(e) v changes sign at $t = 1$, 3, and 5 in the interval $[0, 6]$. The total distance traveled during the first 6 seconds is

$$\begin{aligned} |f(1) - f(0)| + |f(3) - f(1)| + |f(5) - f(3)| + |f(6) - f(5)| &= |1 - 0| + |-1 - 1| + |1 - (-1)| + |0 - 1| \\ &= 1 + 2 + 2 + 1 = 6 \text{ ft} \end{aligned}$$

(f)

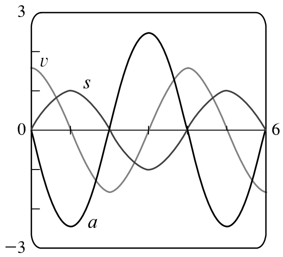


(g) $v(t) = \frac{\pi}{2}\cos(\pi t/2) \quad \Rightarrow$

$$a(t) = v'(t) = \tfrac{\pi}{2}\left[-\sin(\pi t/2)\cdot(\pi/2)\right]$$
$$= (-\pi^2/4)\sin(\pi t/2) \text{ ft/s}^2$$

$$a(1) = (-\pi^2/4)\sin(\pi/2) = -\pi^2/4 \text{ ft/s}^2$$

(h)



(i) The particle is speeding up when v and a have the same sign. From the figure in part (h), we see that v and a are both positive when $3 < t < 4$ and both negative when $1 < t < 2$ and $5 < t < 6$. Thus, the particle is speeding up when $1 < t < 2$, $3 < t < 4$, and $5 < t < 6$. The particle is slowing down when v and a have opposite signs; that is, when $0 < t < 1$, $2 < t < 3$, and $4 < t < 5$.

4. (a) $s = f(t) = t^2 e^{-t}$ (in feet) $\Rightarrow$ $v(t) = f'(t) = t^2(-e^{-t}) + e^{-t}(2t) = te^{-t}(-t+2)$ (in ft/s)

(b) $v(1) = (1)e^{-1}(-1+2) = 1/e$ ft/s

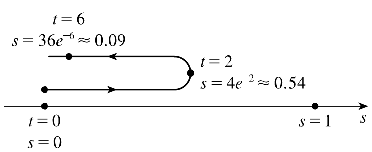
(c) The particle is at rest when $v(t) = 0$. $v(t) = 0 \quad \Leftrightarrow \quad t = 0$ or 2 s.

(d) The particle is moving in the positive direction when $v(t) > 0 \quad \Leftrightarrow \quad te^{-t}(-t+2) > 0 \quad \Leftrightarrow \quad t(-t+2) > 0 \quad \Leftrightarrow$ $0 < t < 2$.

(e) v changes sign at $t = 2$ in the interval $[0, 6]$. The total distance traveled during the first 6 seconds is

$$|f(2) - f(0)| + |f(6) - f(2)| = \left|4e^{-2} - 0\right| + \left|36e^{-6} - 4e^{-2}\right| = 4e^{-2} + 4e^{-2} - 36e^{-6}$$
$$= 8e^{-2} - 36e^{-6} \approx 0.99 \text{ ft}$$

(f)

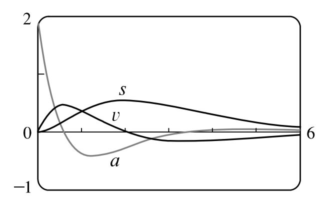


(g) $v(t) = (2t - t^2)e^{-t} \quad \Rightarrow$

$$a(t) = v'(t) = (2t - t^2)(-e^{-t}) + e^{-t}(2 - 2t)$$
$$= e^{-t}\left[-(2t - t^2) + (2 - 2t)\right]$$
$$= e^{-t}(t^2 - 4t + 2) \text{ ft/s}^2$$

$$a(1) = e^{-1}(1 - 4 + 2) = -1/e \text{ ft/s}^2$$

(h)



(i) $a(t) = 0 \quad \Leftrightarrow \quad t^2 - 4t + 2 = 0 \quad [e^{-t} \neq 0] \quad \Leftrightarrow \quad t = \dfrac{4 \pm \sqrt{8}}{2} = 2 \pm \sqrt{2} \quad [\approx 0.6$ and $3.4]$. The particle is speeding up when v and a have the same sign. Using the previous information and the figure in part (h), we see that v and a are both positive when $0 < t < 2 - \sqrt{2}$ and both negative when $2 < t < 2 + \sqrt{2}$. The particle is slowing down when v and a have opposite signs. This occurs when $2 - \sqrt{2} < t < 2$ and $t > 2 + \sqrt{2}$.

5. (a) From the figure, the velocity v is positive on the interval $(0, 2)$ and negative on the interval $(2, 3)$. The acceleration a is positive (negative) when the slope of the tangent line is positive (negative), so the acceleration is positive on the interval $(0, 1)$, and negative on the interval $(1, 3)$. The particle is speeding up when v and a have the same sign, that is, on the interval $(0, 1)$ when $v > 0$ and $a > 0$, and on the interval $(2, 3)$ when $v < 0$ and $a < 0$. The particle is slowing down when v and a have opposite signs, that is, on the interval $(1, 2)$ when $v > 0$ and $a < 0$.

(b) $v > 0$ on $(0, 3)$ and $v < 0$ on $(3, 4)$. $a > 0$ on $(1, 2)$ and $a < 0$ on $(0, 1)$ and $(2, 4)$. The particle is speeding up on $(1, 2)$ $[v > 0, a > 0]$ and on $(3, 4)$ $[v < 0, a < 0]$. The particle is slowing down on $(0, 1)$ and $(2, 3)$ $[v > 0, a < 0]$.

6. (a) The velocity v is positive when s is increasing, that is, on the intervals $(0, 1)$ and $(3, 4)$; and it is negative when s is decreasing, that is, on the interval $(1, 3)$. The acceleration a is positive when the graph of s is concave upward (v is increasing), that is, on the interval $(2, 4)$; and it is negative when the graph of s is concave downward (v is decreasing), that is, on the interval $(0, 2)$. The particle is speeding up on the interval $(1, 2)$ $[v < 0, a < 0]$ and on $(3, 4)$ $[v > 0, a > 0]$. The particle is slowing down on the interval $(0, 1)$ $[v > 0, a < 0]$ and on $(2, 3)$ $[v < 0, a > 0]$.

(b) The velocity v is positive on $(3, 4)$ and negative on $(0, 3)$. The acceleration a is positive on $(0, 1)$ and $(2, 4)$ and negative on $(1, 2)$. The particle is speeding up on the interval $(1, 2)$ $[v < 0, a < 0]$ and on $(3, 4)$ $[v > 0, a > 0]$. The particle is slowing down on the interval $(0, 1)$ $[v < 0, a > 0]$ and on $(2, 3)$ $[v < 0, a > 0]$.

7. (a) $h(t) = 2 + 24.5t - 4.9t^2 \quad \Rightarrow \quad v(t) = h'(t) = 24.5 - 9.8t$. The velocity after 2 s is $v(2) = 24.5 - 9.8(2) = 4.9$ m/s and after 4 s is $v(4) = 24.5 - 9.8(4) = -14.7$ m/s.

(b) The projectile reaches its maximum height when the velocity is zero. $v(t) = 0 \quad \Leftrightarrow \quad 24.5 - 9.8t = 0 \quad \Leftrightarrow$ $t = \dfrac{24.5}{9.8} = 2.5$ s.

(c) The maximum height occurs when $t = 2.5$. $h(2.5) = 2 + 24.5(2.5) - 4.9(2.5)^2 = 32.625$ m $\left[\text{or } 32\frac{5}{8}\text{ m}\right]$.

(d) The projectile hits the ground when $h = 0 \quad \Leftrightarrow \quad 2 + 24.5t - 4.9t^2 = 0 \quad \Leftrightarrow$

$$t = \frac{-24.5 \pm \sqrt{24.5^2 - 4(-4.9)(2)}}{2(-4.9)} \quad \Rightarrow \quad t = t_f \approx 5.08 \text{ s} \ [\text{since } t \geq 0].$$

(e) The projectile hits the ground when $t = t_f$. Its velocity is $v(t_f) = 24.5 - 9.8t_f \approx -25.3$ m/s [downward].

8. (a) At maximum height the velocity of the ball is 0 ft/s. $v(t) = s'(t) = 80 - 32t = 0 \quad \Leftrightarrow \quad 32t = 80 \quad \Leftrightarrow \quad t = \frac{5}{2}$.

So the maximum height is $s\left(\frac{5}{2}\right) = 80\left(\frac{5}{2}\right) - 16\left(\frac{5}{2}\right)^2 = 200 - 100 = 100$ ft.

(b) $s(t) = 80t - 16t^2 = 96 \quad \Leftrightarrow \quad 16t^2 - 80t + 96 = 0 \quad \Leftrightarrow \quad 16(t^2 - 5t + 6) = 0 \quad \Leftrightarrow \quad 16(t - 3)(t - 2) = 0$.
So the ball has a height of 96 ft on the way up at $t = 2$ and on the way down at $t = 3$. At these times the velocities are $v(2) = 80 - 32(2) = 16$ ft/s and $v(3) = 80 - 32(3) = -16$ ft/s, respectively.

9. (a) $h(t) = 15t - 1.86t^2 \quad \Rightarrow \quad v(t) = h'(t) = 15 - 3.72t$. The velocity after 2 s is $v(2) = 15 - 3.72(2) = 7.56$ m/s.

(b) $25 = h \quad \Leftrightarrow \quad 1.86t^2 - 15t + 25 = 0 \quad \Leftrightarrow \quad t = \dfrac{15 \pm \sqrt{15^2 - 4(1.86)(25)}}{2(1.86)} \quad \Leftrightarrow \quad t = t_1 \approx 2.35$ or $t = t_2 \approx 5.71$.

The velocities are $v(t_1) = 15 - 3.72t_1 \approx 6.24$ m/s [upward] and $v(t_2) = 15 - 3.72t_2 \approx -6.24$ m/s [downward].

10. (a) $s(t) = t^4 - 4t^3 - 20t^2 + 20t \quad \Rightarrow \quad v(t) = s'(t) = 4t^3 - 12t^2 - 40t + 20. \quad v = 20 \quad \Leftrightarrow$

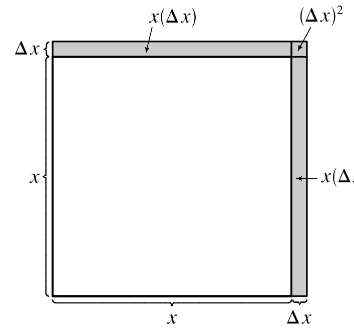
$4t^3 - 12t^2 - 40t + 20 = 20 \quad \Leftrightarrow \quad 4t^3 - 12t^2 - 40t = 0 \quad \Leftrightarrow \quad 4t(t^2 - 3t - 10) = 0 \quad \Leftrightarrow$

$4t(t-5)(t+2) = 0 \quad \Leftrightarrow \quad t = 0$ s or 5 s [for $t \geq 0$].

(b) $a(t) = v'(t) = 12t^2 - 24t - 40. \quad a = 0 \quad \Leftrightarrow \quad 12t^2 - 24t - 40 = 0 \quad \Leftrightarrow \quad 4(3t^2 - 6t - 10) = 0 \quad \Leftrightarrow$

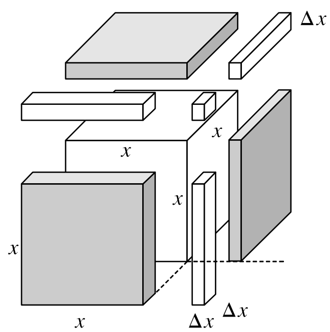
$t = \dfrac{6 \pm \sqrt{6^2 - 4(3)(-10)}}{2(3)} = 1 \pm \dfrac{1}{3}\sqrt{39} \approx 3.08$ s [for $t \geq 0$]. At this time, the acceleration changes from negative to positive and the velocity attains its minimum value.

11. (a) $A(x) = x^2 \quad \Rightarrow \quad A'(x) = 2x$. $A'(15) = 30\ \text{mm}^2/\text{mm}$ is the rate at which the area is increasing with respect to the side length as x reaches 15 mm.



(b) The perimeter is $P(x) = 4x$, so $A'(x) = 2x = \frac{1}{2}(4x) = \frac{1}{2}P(x)$. The figure suggests that if Δx is small, then the change in the area of the square is approximately half of its perimeter (2 of the 4 sides) times Δx. From the figure, $\Delta A = 2x\,(\Delta x) + (\Delta x)^2$. If Δx is small, then $\Delta A \approx 2x\,(\Delta x)$ and so $\Delta A/\Delta x \approx 2x$.

12. (a) $V(x) = x^3 \quad \Rightarrow \quad \dfrac{dV}{dx} = 3x^2$. $\left.\dfrac{dV}{dx}\right|_{x=3} = 3(3)^2 = 27\ \text{mm}^3/\text{mm}$ is the rate at which the volume is increasing as x increases past 3 mm.



(b) The surface area is $S(x) = 6x^2$, so $V'(x) = 3x^2 = \frac{1}{2}(6x^2) = \frac{1}{2}S(x)$. The figure suggests that if Δx is small, then the change in the volume of the cube is approximately half of its surface area (the area of 3 of the 6 faces) times Δx. From the figure, $\Delta V = 3x^2(\Delta x) + 3x(\Delta x)^2 + (\Delta x)^3$. If Δx is small, then $\Delta V \approx 3x^2(\Delta x)$ and so $\Delta V/\Delta x \approx 3x^2$.

13. (a) Using $A(r) = \pi r^2$, we find that the average rate of change is:

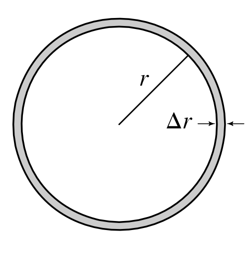
(i) $\dfrac{A(3) - A(2)}{3 - 2} = \dfrac{9\pi - 4\pi}{1} = 5\pi$

(ii) $\dfrac{A(2.5) - A(2)}{2.5 - 2} = \dfrac{6.25\pi - 4\pi}{0.5} = 4.5\pi$

(iii) $\dfrac{A(2.1) - A(2)}{2.1 - 2} = \dfrac{4.41\pi - 4\pi}{0.1} = 4.1\pi$

(b) $A(r) = \pi r^2 \quad \Rightarrow \quad A'(r) = 2\pi r$, so $A'(2) = 4\pi$.

(c) The circumference is $C(r) = 2\pi r = A'(r)$. The figure suggests that if Δr is small, then the change in the area of the circle (a ring around the outside) is approximately equal to its circumference times Δr. Straightening out this ring gives us a shape that is approximately rectangular with length $2\pi r$ and width Δr, so $\Delta A \approx 2\pi r(\Delta r)$. Algebraically, $\Delta A = A(r + \Delta r) - A(r) = \pi(r + \Delta r)^2 - \pi r^2 = 2\pi r(\Delta r) + \pi(\Delta r)^2$. So we see that if Δr is small, then $\Delta A \approx 2\pi r(\Delta r)$ and therefore, $\Delta A/\Delta r \approx 2\pi r$.



14. After t seconds the radius is $r = 60t$, so the area is $A(t) = \pi(60t)^2 = 3600\pi t^2 \quad \Rightarrow \quad A'(t) = 7200\pi t \quad \Rightarrow$

(a) $A'(1) = 7200\pi\ \text{cm}^2/\text{s}$ (b) $A'(3) = 21,600\pi\ \text{cm}^2/\text{s}$ (c) $A'(5) = 36{,}000\pi\ \text{cm}^2/\text{s}$

As time goes by, the area grows at an increasing rate. In fact, the rate of change is linear with respect to time.

15. $S(r) = 4\pi r^2 \quad \Rightarrow \quad S'(r) = 8\pi r \quad \Rightarrow$

(a) $S'(1) = 8\pi\ \text{ft}^2/\text{ft}$ (b) $S'(2) = 16\pi\ \text{ft}^2/\text{ft}$ (c) $S'(3) = 24\pi\ \text{ft}^2/\text{ft}$

As the radius increases, the surface area grows at an increasing rate. In fact, the rate of change is linear with respect to the radius.

16. (a) Using $V(r) = \frac{4}{3}\pi r^3$, we find that the average rate of change is:

(i) $\dfrac{V(8) - V(5)}{8 - 5} = \dfrac{\frac{4}{3}\pi(512) - \frac{4}{3}\pi(125)}{3} = 172\pi\ \mu\text{m}^3/\mu\text{m}$

(ii) $\dfrac{V(6) - V(5)}{6 - 5} = \dfrac{\frac{4}{3}\pi(216) - \frac{4}{3}\pi(125)}{1} = 121.\overline{3}\pi\ \mu\text{m}^3/\mu\text{m}$

(iii) $\dfrac{V(5.1) - V(5)}{5.1 - 5} = \dfrac{\frac{4}{3}\pi(5.1)^3 - \frac{4}{3}\pi(5)^3}{0.1} = 102.01\overline{3}\pi\ \mu\text{m}^3/\mu\text{m}$

(b) $V'(r) = 4\pi r^2$, so $V'(5) = 100\pi\ \mu\text{m}^3/\mu\text{m}$.

(c) $V(r) = \frac{4}{3}\pi r^3 \quad \Rightarrow \quad V'(r) = 4\pi r^2 = S(r)$. By analogy with Exercise 13(c), we can say that the change in the volume of the spherical shell, ΔV, is approximately equal to its thickness, Δr, times the surface area of the inner sphere. Thus, $\Delta V \approx 4\pi r^2(\Delta r)$ and so $\Delta V/\Delta r \approx 4\pi r^2$.

17. The mass is $f(x) = 3x^2$, so the linear density at x is $\rho(x) = f'(x) = 6x$.

(a) $\rho(1) = 6\ \text{kg/m}$ (b) $\rho(2) = 12\ \text{kg/m}$ (c) $\rho(3) = 18\ \text{kg/m}$

Since ρ is an increasing function, the density will be the highest at the right end of the rod and lowest at the left end.

18. $V(t) = 5000\left(1 - \frac{1}{40}t\right)^2 \quad \Rightarrow \quad V'(t) = 5000 \cdot 2\left(1 - \frac{1}{40}t\right)\left(-\frac{1}{40}\right) = -250\left(1 - \frac{1}{40}t\right)$

(a) $V'(5) = -250\left(1 - \frac{5}{40}\right) = -218.75\ \text{gal/min}$ (b) $V'(10) = -250\left(1 - \frac{10}{40}\right) = -187.5\ \text{gal/min}$

(c) $V'(20) = -250\left(1 - \frac{20}{40}\right) = -125\ \text{gal/min}$ (d) $V'(40) = -250\left(1 - \frac{40}{40}\right) = 0\ \text{gal/min}$

The water is flowing out the fastest at the beginning—when $t = 0$, $V'(t) = -250\ \text{gal/min}$. The water is flowing out the slowest at the end—when $t = 40$, $V'(t) = 0$. As the tank empties, the water flows out more slowly.

19. The quantity of charge is $Q(t) = t^3 - 2t^2 + 6t + 2$, so the current is $Q'(t) = 3t^2 - 4t + 6$.

(a) $Q'(0.5) = 3(0.5)^2 - 4(0.5) + 6 = 4.75\ \text{A}$ (b) $Q'(1) = 3(1)^2 - 4(1) + 6 = 5\ \text{A}$

The current is lowest when Q' has a minimum. $Q''(t) = 6t - 4 < 0$ when $t < \frac{2}{3}$. So the current decreases when $t < \frac{2}{3}$ and increases when $t > \frac{2}{3}$. Thus, the current is lowest at $t = \frac{2}{3}$ s.

20. (a) $F = \dfrac{GmM}{r^2} = (GmM)r^{-2} \quad \Rightarrow \quad \dfrac{dF}{dr} = -2(GmM)r^{-3} = -\dfrac{2GmM}{r^3}$, which is the rate of change of the force with respect to the distance between the bodies. The minus sign indicates that as the distance r between the bodies increases, the magnitude of the force F exerted by the body of mass m on the body of mass M is decreasing.

(b) Given $F'(20{,}000) = -2$, find $F'(10{,}000)$. $-2 = -\dfrac{2GmM}{20{,}000^3} \;\Rightarrow\; GmM = 20{,}000^3$.

$$F'(10,000) = -\frac{2(20{,}000^3)}{10{,}000^3} = -2 \cdot 2^3 = -16 \text{ N/km}$$

21. With $m = m_0\left(1 - \dfrac{v^2}{c^2}\right)^{-1/2}$,

$$F = \frac{d}{dt}(mv) = m\frac{d}{dt}(v) + v\frac{d}{dt}(m) = m_0\left(1 - \frac{v^2}{c^2}\right)^{-1/2} \cdot a + v \cdot m_0\left[-\frac{1}{2}\left(1 - \frac{v^2}{c^2}\right)^{-3/2}\right]\left(-\frac{2v}{c^2}\right)\frac{d}{dt}(v)$$

$$= m_0\left(1 - \frac{v^2}{c^2}\right)^{-3/2} \cdot a\left[\left(1 - \frac{v^2}{c^2}\right) + \frac{v^2}{c^2}\right] = \frac{m_0 a}{(1 - v^2/c^2)^{3/2}}$$

Note that we factored out $(1 - v^2/c^2)^{-3/2}$ since $-3/2$ was the lesser exponent. Also note that $\dfrac{d}{dt}(v) = a$.

22. (a) $D(t) = 7 + 5\cos[0.503(t - 6.75)] \;\Rightarrow\; D'(t) = -5\sin[0.503(t - 6.75)](0.503) = -2.515\sin[0.503(t - 6.75)]$.

At 3:00 AM, $t = 3$, and $D'(3) = -2.515\sin[0.503(-3.75)] \approx 2.39$ m/h (rising).

(b) At 6:00 AM, $t = 6$, and $D'(6) = -2.515\sin[0.503(-0.75)] \approx 0.93$ m/h (rising).

(c) At 9:00 AM, $t = 9$, and $D'(9) = -2.515\sin[0.503(2.25)] \approx -2.28$ m/h (falling).

(d) At noon, $t = 12$, and $D'(12) = -2.515\sin[0.503(5.25)] \approx -1.21$ m/h (falling).

23. (a) To find the rate of change of volume with respect to pressure, we first solve for V in terms of P.

$$PV = C \;\Rightarrow\; V = \frac{C}{P} \;\Rightarrow\; \frac{dV}{dP} = -\frac{C}{P^2}.$$

(b) From the formula for dV/dP in part (a), we see that as P increases, the absolute value of dV/dP decreases.

Thus, the volume is decreasing more rapidly at the beginning.

(c) $\beta = -\dfrac{1}{V}\dfrac{dV}{dP} = -\dfrac{1}{V}\left(-\dfrac{C}{P^2}\right) = \dfrac{C}{(PV)P} = \dfrac{C}{CP} = \dfrac{1}{P}$

24. (a) $[\mathrm{C}] = \dfrac{a^2kt}{akt + 1} \;\Rightarrow\;$ rate of reaction $= \dfrac{d[\mathrm{C}]}{dt} = \dfrac{(akt + 1)(a^2k) - (a^2kt)(ak)}{(akt + 1)^2} = \dfrac{a^2k(akt + 1 - akt)}{(akt + 1)^2} = \dfrac{a^2k}{(akt + 1)^2}$

(b) If $x = [\mathrm{C}]$, then $a - x = a - \dfrac{a^2kt}{akt + 1} = \dfrac{a^2kt + a - a^2kt}{akt + 1} = \dfrac{a}{akt + 1}$.

So $k(a - x)^2 = k\left(\dfrac{a}{akt + 1}\right)^2 = \dfrac{a^2k}{(akt + 1)^2} = \dfrac{d[\mathrm{C}]}{dt}$ [from part (a)] $= \dfrac{dx}{dt}$.

(c) As $t \to \infty$, $[\mathrm{C}] = \dfrac{a^2kt}{akt + 1} = \dfrac{(a^2kt)/t}{(akt + 1)/t} = \dfrac{a^2k}{ak + (1/t)} \to \dfrac{a^2k}{ak} = a$ moles/L.

(d) As $t \to \infty$, $\dfrac{d[\mathrm{C}]}{dt} = \dfrac{a^2k}{(akt + 1)^2} \to 0$.

(e) As t increases, nearly all of the reactants A and B are converted into product C. In practical terms, the reaction virtually stops.

25. In Example 6, the population function was $n = 2^t n_0$. Since we are tripling instead of doubling and the initial population is 400, the population function is $n(t) = 400 \cdot 3^t$. The rate of growth is $n'(t) = 400 \cdot 3^t \cdot \ln 3$, so the rate of growth after 2.5 hours is $n'(2.5) = 400 \cdot 3^{2.5} \cdot \ln 3 \approx 6850$ bacteria/hour.

26. $n = f(t) = \dfrac{a}{1 + be^{-0.7t}} \quad \Rightarrow \quad n' = -\dfrac{a \cdot be^{-0.7t}(-0.7)}{(1 + be^{-0.7t})^2}$ [Reciprocal Rule]. When $t = 0$, $n = 20$ and $n' = 12$.

$f(0) = 20 \quad \Rightarrow \quad 20 = \dfrac{a}{1+b} \quad \Rightarrow \quad a = 20(1+b)$. $f'(0) = 12 \quad \Rightarrow \quad 12 = \dfrac{0.7ab}{(1+b)^2} \quad \Rightarrow \quad 12 = \dfrac{0.7(20)(1+b)b}{(1+b)^2} \quad \Rightarrow$

$\dfrac{12}{14} = \dfrac{b}{1+b} \quad \Rightarrow \quad 6(1+b) = 7b \quad \Rightarrow \quad 6 + 6b = 7b \quad \Rightarrow \quad b = 6$ and $a = 20(1+6) = 140$. For the long run, we let t increase without bound. $\lim\limits_{t\to\infty} f(t) = \lim\limits_{t\to\infty} \dfrac{140}{1 + 6e^{-0.7t}} = \dfrac{140}{1 + 6 \cdot 0} = 140$, indicating that the yeast population stabilizes at 140 cells.

27. (a) **1920:** $m_1 = \dfrac{1860 - 1750}{1920 - 1910} = \dfrac{110}{10} = 11$, $m_2 = \dfrac{2070 - 1860}{1930 - 1920} = \dfrac{210}{10} = 21$,

$(m_1 + m_2)/2 = (11 + 21)/2 = 16$ million/year

1980: $m_1 = \dfrac{4450 - 3710}{1980 - 1970} = \dfrac{740}{10} = 74$, $m_2 = \dfrac{5280 - 4450}{1990 - 1980} = \dfrac{830}{10} = 83$,

$(m_1 + m_2)/2 = (74 + 83)/2 = 78.5$ million/year

(b) $P(t) = at^3 + bt^2 + ct + d$ (in millions of people), where $a \approx -0.000\,284\,900\,3$, $b \approx 0.522\,433\,122\,43$, $c \approx -6.395\,641\,396$, and $d \approx 1720.586\,081$.

(c) $P(t) = at^3 + bt^2 + ct + d \quad \Rightarrow \quad P'(t) = 3at^2 + 2bt + c$ (in millions of people per year)

(d) 1920 corresponds to $t = 20$ and $P'(20) \approx 14.16$ million/year. 1980 corresponds to $t = 80$ and $P'(80) \approx 71.72$ million/year. These estimates are smaller than the estimates in part (a).

(e) $f(t) = pq^t$ (where $p = 1.43653 \times 10^9$ and $q = 1.01395$) $\quad \Rightarrow \quad f'(t) = pq^t \ln q$ (in millions of people per year)

(f) $f'(20) \approx 26.25$ million/year [much larger than the estimates in part (a) and (d)].

$f'(80) \approx 60.28$ million/year [much smaller than the estimates in parts (a) and (d)].

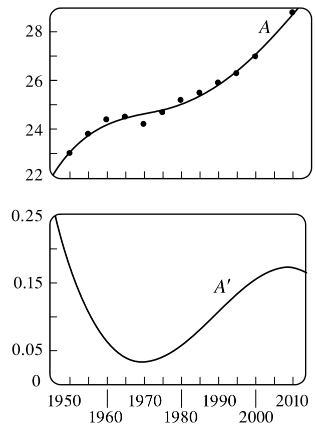
(g) $P'(85) \approx 76.24$ million/year and $f'(85) \approx 64.61$ million/year. The first estimate is probably more accurate.

28. (a) $A(t) = at^4 + bt^3 + ct^2 + dt + e$, where $a \approx -1.199\,781 \times 10^{-6}$, $b \approx 9.545\,853 \times 10^3$, $c \approx -28.478\,550$, $d \approx 37{,}757.105\,467$, and $e \approx -1.877\,031 \times 10^7$.

(b) $A(t) = at^4 + bt^3 + ct^2 + dt + e \quad \Rightarrow \quad A'(t) = 4at^3 + 3bt^2 + 2ct + d$.

(c) Part (b) gives $A'(1990) \approx 0.106$ years of age per year.

(d)



29. (a) Using $v = \dfrac{P}{4\eta l}(R^2 - r^2)$ with $R = 0.01$, $l = 3$, $P = 3000$, and $\eta = 0.027$, we have v as a function of r:

$v(r) = \dfrac{3000}{4(0.027)3}(0.01^2 - r^2)$. $v(0) = 0.\overline{925}$ cm/s, $v(0.005) = 0.69\overline{4}$ cm/s, $v(0.01) = 0$.

(b) $v(r) = \dfrac{P}{4\eta l}(R^2 - r^2) \quad \Rightarrow \quad v'(r) = \dfrac{P}{4\eta l}(-2r) = -\dfrac{Pr}{2\eta l}$. When $l = 3$, $P = 3000$, and $\eta = 0.027$, we have

$v'(r) = -\dfrac{3000r}{2(0.027)3}$. $v'(0) = 0$, $v'(0.005) = -92.\overline{592}$ (cm/s)/cm, and $v'(0.01) = -185.\overline{185}$ (cm/s)/cm.

(c) The velocity is greatest where $r = 0$ (at the center) and the velocity is changing most where $r = R = 0.01$ cm (at the edge).

30. (a) (i) $f = \dfrac{1}{2L}\sqrt{\dfrac{T}{\rho}} = \left(\dfrac{1}{2}\sqrt{\dfrac{T}{\rho}}\right)L^{-1} \quad \Rightarrow \quad \dfrac{df}{dL} = -\left(\dfrac{1}{2}\sqrt{\dfrac{T}{\rho}}\right)L^{-2} = -\dfrac{1}{2L^2}\sqrt{\dfrac{T}{\rho}}$

(ii) $f = \dfrac{1}{2L}\sqrt{\dfrac{T}{\rho}} = \left(\dfrac{1}{2L\sqrt{\rho}}\right)T^{1/2} \quad \Rightarrow \quad \dfrac{df}{dT} = \dfrac{1}{2}\left(\dfrac{1}{2L\sqrt{\rho}}\right)T^{-1/2} = \dfrac{1}{4L\sqrt{T\rho}}$

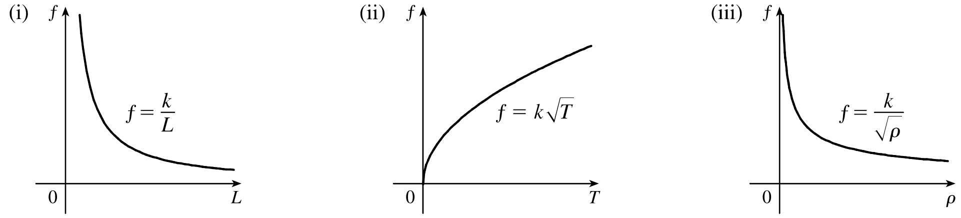
(iii) $f = \dfrac{1}{2L}\sqrt{\dfrac{T}{\rho}} = \left(\dfrac{\sqrt{T}}{2L}\right)\rho^{-1/2} \quad \Rightarrow \quad \dfrac{df}{d\rho} = -\dfrac{1}{2}\left(\dfrac{\sqrt{T}}{2L}\right)\rho^{-3/2} = -\dfrac{\sqrt{T}}{4L\rho^{3/2}}$

(b) *Note:* Illustrating tangent lines on the generic figures may help to explain the results.

(i) $\dfrac{df}{dL} < 0$ and L is decreasing $\Rightarrow$ f is increasing $\Rightarrow$ higher note

(ii) $\dfrac{df}{dT} > 0$ and T is increasing $\Rightarrow$ f is increasing $\Rightarrow$ higher note

(iii) $\dfrac{df}{d\rho} < 0$ and ρ is increasing $\Rightarrow$ f is decreasing $\Rightarrow$ lower note



31. (a) $C(x) = 2000 + 3x + 0.01x^2 + 0.0002x^3 \quad \Rightarrow \quad C'(x) = 0 + 3(1) + 0.01(2x) + 0.0002(3x^2) = 3 + 0.02x + 0.0006x^2$

(b) $C'(100) = 3 + 0.02(100) + 0.0006(100)^2 = 3 + 2 + 6 = \$11/\text{pair}$. $C'(100)$ is the rate at which the cost is increasing as the 100th pair of jeans is produced. It predicts the (approximate) cost of the 101st pair.

(c) The cost of manufacturing the 101st pair of jeans is
$C(101) - C(100) = 2611.0702 - 2600 = 11.0702 \approx \$11.07$. This is close to the marginal cost from part (b).

32. (a) $C(q) = 84 + 0.16q - 0.0006q^2 + 0.000003q^3 \quad \Rightarrow \quad C'(q) = 0.16 - 0.0012q + 0.000009q^2$, and
$C'(100) = 0.16 - 0.0012(100) + 0.000009(100)^2 = 0.13$. This is the rate at which the cost is increasing as the 100th item is produced.

(b) The actual cost of producing the 101st item is $C(101) - C(100) = 97.13030299 - 97 \approx \$0.13$

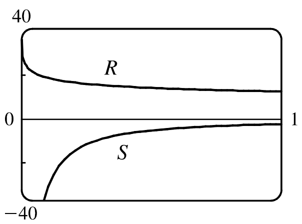
33. (a) $A(x) = \dfrac{p(x)}{x} \quad \Rightarrow \quad A'(x) = \dfrac{xp'(x) - p(x) \cdot 1}{x^2} = \dfrac{xp'(x) - p(x)}{x^2}$.
$A'(x) > 0 \quad \Rightarrow \quad A(x)$ is increasing; that is, the average productivity increases as the size of the workforce increases.

(b) $p'(x)$ is greater than the average productivity $\quad \Rightarrow \quad p'(x) > A(x) \quad \Rightarrow \quad p'(x) > \dfrac{p(x)}{x} \quad \Rightarrow \quad xp'(x) > p(x) \quad \Rightarrow$

$xp'(x) - p(x) > 0 \quad \Rightarrow \quad \dfrac{xp'(x) - p(x)}{x^2} > 0 \quad \Rightarrow \quad A'(x) > 0$.

34. (a) $$S = \frac{dR}{dx} = \frac{(1 + 4x^{0.4})(9.6x^{-0.6}) - (40 + 24x^{0.4})(1.6x^{-0.6})}{(1 + 4x^{0.4})^2}$$
$$= \frac{9.6x^{-0.6} + 38.4x^{-0.2} - 64x^{-0.6} - 38.4x^{-0.2}}{(1 + 4x^{0.4})^2} = -\frac{54.4x^{-0.6}}{(1 + 4x^{0.4})^2}$$

(b)



At low levels of brightness, R is quite large $[R(0) = 40]$ and is quickly decreasing, that is, S is negative with large absolute value. This is to be expected: at low levels of brightness, the eye is more sensitive to slight changes than it is at higher levels of brightness.

35. $t = \ln\left(\dfrac{3c + \sqrt{9c^2 - 8c}}{2}\right) = \ln\left(3c + \sqrt{9c^2 - 8c}\right) - \ln 2 \quad \Rightarrow$

$$\frac{dt}{dc} = \frac{1}{3c + \sqrt{9c^2 - 8c}} \frac{d}{dc} \left(3c + \sqrt{9c^2 - 8c}\right) - 0 = \frac{3 + \frac{1}{2}(9c^2 - 8c)^{-1/2}(18c - 8)}{3c + \sqrt{9c^2 - 8c}}$$
$$= \frac{3 + \dfrac{9c - 4}{\sqrt{9c^2 - 8c}}}{3c + \sqrt{9c^2 - 8c}} = \frac{3\sqrt{9c^2 - 8c} + 9c - 4}{\sqrt{9c^2 - 8c}\left(3c + \sqrt{9c^2 - 8c}\right)}.$$

This derivative represents the rate of change of duration of dialysis required with respect to the initial urea concentration.

36. $f(r) = 2\sqrt{Dr} \quad \Rightarrow \quad f'(r) = 2 \cdot \frac{1}{2}(Dr)^{-1/2} \cdot D = \dfrac{D}{\sqrt{Dr}} = \sqrt{\dfrac{D}{r}}$. $f'(r)$ is the rate of change of the wave speed with respect to the reproductive rate.

37. $PV = nRT \Rightarrow T = \dfrac{PV}{nR} = \dfrac{PV}{(10)(0.0821)} = \dfrac{1}{0.821}(PV)$. Using the Product Rule, we have

$$\frac{dT}{dt} = \frac{1}{0.821}\left[P(t)V'(t) + V(t)P'(t)\right] = \frac{1}{0.821}\left[(8)(-0.15) + (10)(0.10)\right] \approx -0.2436 \text{ K/min.}$$

38. (a) If $dP/dt = 0$, the population is stable (it is constant).

(b) $\dfrac{dP}{dt} = 0 \Rightarrow \beta P = r_0\left(1 - \dfrac{P}{P_c}\right)P \Rightarrow \dfrac{\beta}{r_0} = 1 - \dfrac{P}{P_c} \Rightarrow \dfrac{P}{P_c} = 1 - \dfrac{\beta}{r_0} \Rightarrow P = P_c\left(1 - \dfrac{\beta}{r_0}\right)$.

If $P_c = 10{,}000$, $r_0 = 5\% = 0.05$, and $\beta = 4\% = 0.04$, then $P = 10{,}000\left(1 - \frac{4}{5}\right) = 2000$.

(c) If $\beta = 0.05$, then $P = 10{,}000\left(1 - \frac{5}{5}\right) = 0$. There is no stable population.

39. (a) If the populations are stable, then the growth rates are neither positive nor negative; that is, $\dfrac{dC}{dt} = 0$ and $\dfrac{dW}{dt} = 0$.

(b) "The caribou go extinct" means that the population is zero, or mathematically, $C = 0$.

(c) We have the equations $\dfrac{dC}{dt} = aC - bCW$ and $\dfrac{dW}{dt} = -cW + dCW$. Let $dC/dt = dW/dt = 0$, $a = 0.05$, $b = 0.001$, $c = 0.05$, and $d = 0.0001$ to obtain $0.05C - 0.001CW = 0$ **(1)** and $-0.05W + 0.0001CW = 0$ **(2)**. Adding 10 times **(2)** to **(1)** eliminates the CW-terms and gives us $0.05C - 0.5W = 0 \Rightarrow C = 10W$. Substituting $C = 10W$ into **(1)** results in $0.05(10W) - 0.001(10W)W = 0 \Leftrightarrow 0.5W - 0.01W^2 = 0 \Leftrightarrow 50W - W^2 = 0 \Leftrightarrow W(50 - W) = 0 \Leftrightarrow W = 0$ or 50. Since $C = 10W$, $C = 0$ or 500. Thus, the population pairs (C, W) that lead to stable populations are $(0, 0)$ and $(500, 50)$. So it is possible for the two species to live in harmony.

3.8 Exponential Growth and Decay

1. The relative growth rate is $\dfrac{1}{P}\dfrac{dP}{dt} = 0.7944$, so $\dfrac{dP}{dt} = 0.7944P$ and, by Theorem 2, $P(t) = P(0)e^{0.7944t} = 2e^{0.7944t}$.

Thus, $P(6) = 2e^{0.7944(6)} \approx 234.99$ or about 235 members.

2. (a) By Theorem 2, $P(t) = P(0)e^{kt} = 50e^{kt}$. In 20 minutes $\left(\frac{1}{3}\text{ hour}\right)$, there are 100 cells, so $P\left(\frac{1}{3}\right) = 50e^{k/3} = 100 \Rightarrow e^{k/3} = 2 \Rightarrow k/3 = \ln 2 \Rightarrow k = 3\ln 2 = \ln(2^3) = \ln 8$.

(b) $P(t) = 50e^{(\ln 8)t} = 50 \cdot 8^t$

(c) $P(6) = 50 \cdot 8^6 = 50 \cdot 2^{18} = 13{,}107{,}200$ cells

(d) $\dfrac{dP}{dt} = kP \Rightarrow P'(6) = kP(6) = (\ln 8)P(6) \approx 27{,}255{,}656$ cells/h

(e) $P(t) = 10^6 \Leftrightarrow 50 \cdot 8^t = 1{,}000{,}000 \Leftrightarrow 8^t = 20{,}000 \Leftrightarrow t\ln 8 = \ln 20{,}000 \Leftrightarrow t = \dfrac{\ln 20{,}000}{\ln 8} \approx 4.76$ h

3. (a) By Theorem 2, $P(t) = P(0)e^{kt} = 100e^{kt}$. Now $P(1) = 100e^{k(1)} = 420 \Rightarrow e^k = \frac{420}{100} \Rightarrow k = \ln 4.2$.

So $P(t) = 100e^{(\ln 4.2)t} = 100(4.2)^t$.

(b) $P(3) = 100(4.2)^3 = 7408.8 \approx 7409$ bacteria

(c) $dP/dt = kP \quad \Rightarrow \quad P'(3) = k \cdot P(3) = (\ln 4.2)\big(100(4.2)^3\big)$ [from part (a)] $\approx 10{,}632$ bacteria/h

(d) $P(t) = 100(4.2)^t = 10{,}000 \quad \Rightarrow \quad (4.2)^t = 100 \quad \Rightarrow \quad t = (\ln 100)/(\ln 4.2) \approx 3.2$ hours

4. (a) $y(t) = y(0)e^{kt} \quad \Rightarrow \quad y(2) = y(0)e^{2k} = 400$ and $y(6) = y(0)e^{6k} = 25{,}600$. Dividing these equations, we get $e^{6k}/e^{2k} = 25{,}600/400 \quad \Rightarrow \quad e^{4k} = 64 \quad \Rightarrow \quad 4k = \ln 2^6 = 6\ln 2 \quad \Rightarrow \quad k = \frac{3}{2}\ln 2 \approx 1.0397$, about 104% per hour.

(b) $400 = y(0)e^{2k} \quad \Rightarrow \quad y(0) = 400/e^{2k} \quad \Rightarrow \quad y(0) = 400/e^{3\ln 2} = 400/\left(e^{\ln 2}\right)^3 = 400/2^3 = 50.$

(c) $y(t) = y(0)e^{kt} = 50e^{(3/2)(\ln 2)t} = 50(e^{\ln 2})^{(3/2)t} \quad \Rightarrow \quad y(t) = 50(2)^{1.5t}$

(d) $y(4.5) = 50(2)^{1.5(4.5)} = 50(2)^{6.75} \approx 5382$ bacteria

(e) $\dfrac{dy}{dt} = ky = \left(\dfrac{3}{2}\ln 2\right)(50(2)^{6.75}) \approx 5596$ bacteria/h

(f) $y(t) = 50{,}000 \quad \Rightarrow \quad 50{,}000 = 50(2)^{1.5t} \quad \Rightarrow \quad 1000 = (2)^{1.5t} \quad \Rightarrow \quad \ln 1000 = 1.5t\ln 2 \quad \Rightarrow$

$t = \dfrac{\ln 1000}{1.5\ln 2} \approx 6.64$ h

5. (a) Let the population (in millions) in the year t be $P(t)$. Since the initial time is the year 1750, we substitute $t - 1750$ for t in Theorem 2, so the exponential model gives $P(t) = P(1750)e^{k(t-1750)}$. Then $P(1800) = 980 = 790e^{k(1800-1750)} \quad \Rightarrow$ $\frac{980}{790} = e^{k(50)} \quad \Rightarrow \quad \ln\frac{980}{790} = 50k \quad \Rightarrow \quad k = \frac{1}{50}\ln\frac{980}{790} \approx 0.0043104$. So with this model, we have $P(1900) = 790e^{k(1900-1750)} \approx 1508$ million, and $P(1950) = 790e^{k(1950-1750)} \approx 1871$ million. Both of these estimates are much too low.

(b) In this case, the exponential model gives $P(t) = P(1850)e^{k(t-1850)} \quad \Rightarrow \quad P(1900) = 1650 = 1260e^{k(1900-1850)} \quad \Rightarrow$ $\ln\frac{1650}{1260} = k(50) \quad \Rightarrow \quad k = \frac{1}{50}\ln\frac{1650}{1260} \approx 0.005393$. So with this model, we estimate $P(1950) = 1260e^{k(1950-1850)} \approx 2161$ million. This is still too low, but closer than the estimate of $P(1950)$ in part (a).

(c) The exponential model gives $P(t) = P(1900)e^{k(t-1900)} \quad \Rightarrow \quad P(1950) = 2560 = 1650e^{k(1950-1900)} \quad \Rightarrow$ $\ln\frac{2560}{1650} = k(50) \quad \Rightarrow \quad k = \frac{1}{50}\ln\frac{2560}{1650} \approx 0.008785$. With this model, we estimate $P(2000) = 1650e^{k(2000-1900)} \approx 3972$ million. This is much too low. The discrepancy is explained by the fact that the world birth rate (average yearly number of births per person) is about the same as always, whereas the mortality rate (especially the infant mortality rate) is much lower, owing mostly to advances in medical science and to the wars in the first part of the twentieth century. The exponential model assumes, among other things, that the birth and mortality rates will remain constant.

6. (a) Let $P(t)$ be the population (in millions) in the year t. Since the initial time is the year 1950, we substitute $t - 1950$ for t in Theorem 2, and find that the exponential model gives $P(t) = P(1950)e^{k(t-1950)} \quad \Rightarrow$ $P(1960) = 100 = 83e^{k(1960-1950)} \quad \Rightarrow \quad \frac{100}{83} = e^{10k} \quad \Rightarrow \quad k = \frac{1}{10}\ln\frac{100}{83} \approx 0.0186$. With this model, we estimate $P(1980) = 83e^{k(1980-1950)} = 83e^{30k} \approx 145$ million, which is an underestimate of the actual population of 150 million.

(b) As in part (a), $P(t) = P(1960)e^{k(t-1960)} \Rightarrow P(1980) = 150 = 100e^{20k} \Rightarrow 20k = \ln\frac{150}{100} \Rightarrow$ $k = \frac{1}{20}\ln\frac{3}{2} \approx 0.0203$. Thus, $P(2000) = 100e^{40k} = 225$ million, which is an overestimate of the actual population of 214 million.

(c) As in part (a), $P(t) = P(1980)e^{k(t-1980)} \Rightarrow P(2000) = 214 = 150e^{20k} \Rightarrow 20k = \ln\frac{214}{150} \Rightarrow$ $k = \frac{1}{20}\ln\frac{214}{150} \approx 0.0178$. Thus, $P(2010) = 150e^{30k} \approx 256$, which is an overestimate of the actual population of 243 million.

(d) $P(2020) = 150e^{k(2020-1980)} \approx 305$ million. This estimate will probably be an overestimate since this model gave us an overestimate in part (c) — indicating that k is too large. Creating a model with more recent data would likely result in an improved estimate.

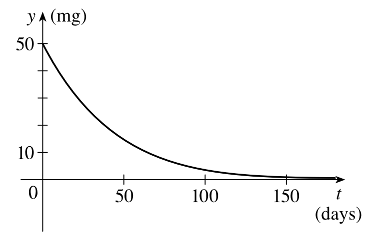
7. (a) If $y = [N_2O_5]$ then by Theorem 2, $\dfrac{dy}{dt} = -0.0005y \Rightarrow y(t) = y(0)e^{-0.0005t} = Ce^{-0.0005t}$.

(b) $y(t) = Ce^{-0.0005t} = 0.9C \Rightarrow e^{-0.0005t} = 0.9 \Rightarrow -0.0005t = \ln 0.9 \Rightarrow t = -2000\ln 0.9 \approx 211$ s

8. (a) The mass remaining after t days is $y(t) = y(0)\,e^{kt} = 50e^{kt}$. Since the half-life is 28 days, $y(28) = 50e^{28k} = 25 \Rightarrow$ $e^{28k} = \frac{1}{2} \Rightarrow 28k = \ln\frac{1}{2} \Rightarrow k = -(\ln 2)/28$, so $y(t) = 50e^{-(\ln 2)t/28} = 50 \cdot 2^{-t/28}$.

(b) $y(40) = 50 \cdot 2^{-40/28} \approx 18.6$ mg

(c) $y(t) = 2 \Rightarrow 2 = 50 \cdot 2^{-t/28} \Rightarrow \frac{2}{50} = 2^{-t/28} \Rightarrow$ $(-t/28)\ln 2 = \ln\frac{1}{25} \Rightarrow t = \left(-28\ln\frac{1}{25}\right)/\ln 2 \approx 130$ days

(d) 

9. (a) If $y(t)$ is the mass (in mg) remaining after t years, then $y(t) = y(0)e^{kt} = 100e^{kt}$.

$y(30) = 100e^{30k} = \frac{1}{2}(100) \Rightarrow e^{30k} = \frac{1}{2} \Rightarrow k = -(\ln 2)/30 \Rightarrow y(t) = 100e^{-(\ln 2)t/30} = 100 \cdot 2^{-t/30}$

(b) $y(100) = 100 \cdot 2^{-100/30} \approx 9.92$ mg

(c) $100e^{-(\ln 2)t/30} = 1 \Rightarrow -(\ln 2)t/30 = \ln\frac{1}{100} \Rightarrow t = -30\,\frac{\ln 0.01}{\ln 2} \approx 199.3$ years

10. (a) If $y(t)$ is the mass after t days and $y(0) = A$, then $y(t) = Ae^{kt}$.

$y(1) = Ae^{k} = 0.945A \Rightarrow e^{k} = 0.945 \Rightarrow k = \ln 0.945$.

Then $Ae^{(\ln 0.945)t} = \frac{1}{2}A \Leftrightarrow \ln e^{(\ln 0.945)t} = \ln\frac{1}{2} \Leftrightarrow (\ln 0.945)t = \ln\frac{1}{2} \Leftrightarrow t = -\frac{\ln 2}{\ln 0.945} \approx 12.25$ years.

(b) $Ae^{(\ln 0.945)t} = 0.20A \Leftrightarrow (\ln 0.945)t = \ln\frac{1}{5} \Leftrightarrow t = -\frac{\ln 5}{\ln 0.945} \approx 28.45$ years

11. Let $y(t)$ be the level of radioactivity. Thus, $y(t) = y(0)e^{-kt}$ and k is determined by using the half-life:

$y(5730) = \frac{1}{2}y(0) \Rightarrow y(0)e^{-k(5730)} = \frac{1}{2}y(0) \Rightarrow e^{-5730k} = \frac{1}{2} \Rightarrow -5730k = \ln\frac{1}{2} \Rightarrow k = -\dfrac{\ln\frac{1}{2}}{5730} = \dfrac{\ln 2}{5730}$.

If 74% of the ^{14}C remains, then we know that $y(t) = 0.74y(0) \Rightarrow 0.74 = e^{-t(\ln 2)/5730} \Rightarrow \ln 0.74 = -\dfrac{t\ln 2}{5730} \Rightarrow$

$t = -\dfrac{5730(\ln 0.74)}{\ln 2} \approx 2489 \approx 2500$ years.

12. From Exercise 11, we have the model $y(t) = y(0)e^{-kt}$ with $k = (\ln 2)/5730$. Thus, $y(68{,}000{,}000) = y(0)e^{-68{,}000{,}000k} \approx y(0) \cdot 0 = 0$. There would be an undetectable amount of ^{14}C remaining for a 68-million-year-old dinosaur.

Now let $y(t) = 0.1\%\, y(0)$, so $0.001y(0) = y(0)e^{-kt} \;\Rightarrow\; 0.001 = e^{-kt} \;\Rightarrow\; \ln 0.001 = -kt \;\Rightarrow$ $t = \dfrac{\ln 0.001}{-k} = \dfrac{\ln 0.001}{-(\ln 2)/5730} \approx 57{,}104$, which is the maximum age of a fossil that we could date using ^{14}C.

13. Let t measure time since a dinosaur died in millions of years, and let $y(t)$ be the amount of ^{40}K in the dinosaur's bones at time t. Then $y(t) = y(0)e^{-kt}$ and k is determined by the half-life: $y(1250) = \frac{1}{2}y(0) \;\Rightarrow\; y(0)e^{-k(1250)} = \frac{1}{2}y(0) \;\Rightarrow$ $e^{-1250k} = \frac{1}{2} \;\Rightarrow\; -1250k = \ln \frac{1}{2} \;\Rightarrow\; k = -\dfrac{\ln \frac{1}{2}}{1250} = \dfrac{\ln 2}{1250}$. To determine if a dinosaur dating of 68 million years is possible, we find that $y(68) = y(0)e^{-k(68)} \approx 0.963y(0)$, indicating that about 96% of the ^{40}K is remaining, which is clearly detectable. To determine the maximum age of a fossil by using ^{40}K, we solve $y(t) = 0.1\%y(0)$ for t.
$y(0)e^{-kt} = 0.001y(0) \;\Leftrightarrow\; e^{-kt} = 0.001 \;\Leftrightarrow\; -kt = \ln 0.001 \;\Leftrightarrow\; t = \dfrac{\ln 0.001}{-(\ln 2)/1250} \approx 12{,}457$ million, or 12.457 billion years.

14. From the information given, we know that $\dfrac{dy}{dx} = 2y \;\Rightarrow\; y = Ce^{2x}$ by Theorem 2. To calculate C we use the point $(0, 5)$: $5 = Ce^{2(0)} \;\Rightarrow\; C = 5$. Thus, the equation of the curve is $y = 5e^{2x}$.

15. (a) Using Newton's Law of Cooling, $\dfrac{dT}{dt} = k(T - T_s)$, we have $\dfrac{dT}{dt} = k(T - 75)$. Now let $y = T - 75$, so $y(0) = T(0) - 75 = 185 - 75 = 110$, so y is a solution of the initial-value problem $dy/dt = ky$ with $y(0) = 110$ and by Theorem 2 we have $y(t) = y(0)e^{kt} = 110e^{kt}$.

$y(30) = 110e^{30k} = 150 - 75 \;\Rightarrow\; e^{30k} = \frac{75}{110} = \frac{15}{22} \;\Rightarrow\; k = \frac{1}{30}\ln\frac{15}{22}$, so $y(t) = 110e^{\frac{1}{30}t\ln\left(\frac{15}{22}\right)}$ and $y(45) = 110e^{\frac{45}{30}\ln\left(\frac{15}{22}\right)} \approx 62^\circ\text{F}$. Thus, $T(45) \approx 62 + 75 = 137^\circ\text{F}$.

(b) $T(t) = 100 \;\Rightarrow\; y(t) = 25$. $y(t) = 110e^{\frac{1}{30}t\ln\left(\frac{15}{22}\right)} = 25 \;\Rightarrow\; e^{\frac{1}{30}t\ln\left(\frac{15}{22}\right)} = \frac{25}{110} \;\Rightarrow\; \frac{1}{30}t\ln\frac{15}{22} = \ln\frac{25}{110} \;\Rightarrow$
$t = \dfrac{30\ln\frac{25}{110}}{\ln\frac{15}{22}} \approx 116$ min.

16. Let $T(t)$ be the temperature of the body t hours after 1:30 PM. Then $T(0) = 32.5$ and $T(1) = 30.3$. Using Newton's Law of Cooling, $\dfrac{dT}{dt} = k(T - T_s)$, we have $\dfrac{dT}{dt} = k(T - 20)$. Now let $y = T - 20$, so $y(0) = T(0) - 20 = 32.5 - 20 = 12.5$, so y is a solution to the initial value problem $dy/dt = ky$ with $y(0) = 12.5$ and by Theorem 2 we have $y(t) = y(0)e^{kt} = 12.5e^{kt}$.

$y(1) = 30.3 - 20 \;\Rightarrow\; 10.3 = 12.5e^{k(1)} \;\Rightarrow\; e^k = \frac{10.3}{12.5} \;\Rightarrow\; k = \ln\frac{10.3}{12.5}$. The murder occurred when

$y(t) = 37 - 20 \quad\Rightarrow\quad 12.5e^{kt} = 17 \quad\Rightarrow\quad e^{kt} = \frac{17}{12.5} \quad\Rightarrow\quad kt = \ln\frac{17}{12.5} \quad\Rightarrow\quad t = \left(\ln\frac{17}{12.5}\right)/\ln\frac{10.3}{12.5} \approx -1.588\text{ h}$

≈ -95 minutes. Thus, the murder took place about 95 minutes before 1:30 PM, or 11:55 AM.

17. $\dfrac{dT}{dt} = k(T - 20)$. Letting $y = T - 20$, we get $\dfrac{dy}{dt} = ky$, so $y(t) = y(0)e^{kt}$. $y(0) = T(0) - 20 = 5 - 20 = -15$, so $y(25) = y(0)e^{25k} = -15e^{25k}$, and $y(25) = T(25) - 20 = 10 - 20 = -10$, so $-15e^{25k} = -10 \quad\Rightarrow\quad e^{25k} = \frac{2}{3}$. Thus, $25k = \ln\left(\frac{2}{3}\right)$ and $k = \frac{1}{25}\ln\left(\frac{2}{3}\right)$, so $y(t) = y(0)e^{kt} = -15e^{(1/25)\ln(2/3)t}$. More simply, $e^{25k} = \frac{2}{3} \quad\Rightarrow\quad e^k = \left(\frac{2}{3}\right)^{1/25} \quad\Rightarrow$ $e^{kt} = \left(\frac{2}{3}\right)^{t/25} \quad\Rightarrow\quad y(t) = -15\cdot\left(\frac{2}{3}\right)^{t/25}$.

(a) $T(50) = 20 + y(50) = 20 - 15\cdot\left(\frac{2}{3}\right)^{50/25} = 20 - 15\cdot\left(\frac{2}{3}\right)^2 = 20 - \frac{20}{3} = 13.\bar{3}\,^\circ\text{C}$

(b) $15 = T(t) = 20 + y(t) = 20 - 15\cdot\left(\frac{2}{3}\right)^{t/25} \quad\Rightarrow\quad 15\cdot\left(\frac{2}{3}\right)^{t/25} = 5 \quad\Rightarrow\quad \left(\frac{2}{3}\right)^{t/25} = \frac{1}{3} \quad\Rightarrow$

$(t/25)\,\ln\left(\frac{2}{3}\right) = \ln\left(\frac{1}{3}\right) \quad\Rightarrow\quad t = 25\ln\left(\frac{1}{3}\right)/\ln\left(\frac{2}{3}\right) \approx 67.74$ min.

18. $\dfrac{dT}{dt} = k(T - 20)$. Let $y = T - 20$. Then $\dfrac{dy}{dt} = ky$, so $y(t) = y(0)e^{kt}$. $y(0) = T(0) - 20 = 95 - 20 = 75$, so $y(t) = 75e^{kt}$. When $T(t) = 70$, $\dfrac{dT}{dt} = -1^\circ\text{C/min}$. Equivalently, $\dfrac{dy}{dt} = -1$ when $y(t) = 50$. Thus, $-1 = \dfrac{dy}{dt} = ky(t) = 50k$ and $50 = y(t) = 75e^{kt}$. The first relation implies $k = -1/50$, so the second relation says $50 = 75e^{-t/50}$. Thus, $e^{-t/50} = \frac{2}{3} \quad\Rightarrow\quad -t/50 = \ln\left(\frac{2}{3}\right) \quad\Rightarrow\quad t = -50\ln\left(\frac{2}{3}\right) \approx 20.27$ min.

19. (a) Let $P(h)$ be the pressure at altitude h. Then $dP/dh = kP \quad\Rightarrow\quad P(h) = P(0)e^{kh} = 101.3e^{kh}$.

$P(1000) = 101.3e^{1000k} = 87.14 \quad\Rightarrow\quad 1000k = \ln\left(\frac{87.14}{101.3}\right) \quad\Rightarrow\quad k = \frac{1}{1000}\ln\left(\frac{87.14}{101.3}\right) \quad\Rightarrow$

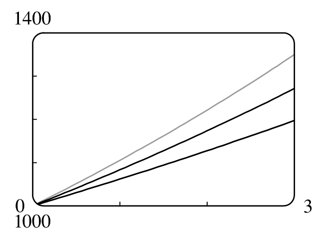
$P(h) = 101.3\,e^{\frac{1}{1000}h\ln\left(\frac{87.14}{101.3}\right)}$, so $P(3000) = 101.3e^{3\ln\left(\frac{87.14}{101.3}\right)} \approx 64.5$ kPa.

(b) $P(6187) = 101.3\,e^{\frac{6187}{1000}\ln\left(\frac{87.14}{101.3}\right)} \approx 39.9$ kPa

20. (a) Using $A = A_0\left(1 + \dfrac{r}{n}\right)^{nt}$ with $A_0 = 1000$, $r = 0.08$, and $t = 3$, we have:

(i) Annually: $n = 1$;	$A = 1000\left(1 + \frac{0.08}{1}\right)^{1\cdot3} = \$1259.71$
(ii) Quarterly: $n = 4$;	$A = 1000\left(1 + \frac{0.08}{4}\right)^{4\cdot3} = \$1268.24$
(iii) Monthly: $n = 12$;	$A = 1000\left(1 + \frac{0.08}{12}\right)^{12\cdot3} = \$1270.24$
(iv) Weekly: $n = 52$	$A = 1000\left(1 + \frac{0.08}{52}\right)^{52\cdot3} = \$1271.01$
(v) Daily: $n = 365$;	$A = 1000\left(1 + \frac{0.08}{365}\right)^{365\cdot3} = \$1271.22$
(vi) Hourly: $n = 365\cdot24$;	$A = 1000\left(1 + \frac{0.08}{365\cdot24}\right)^{365\cdot24\cdot3} = \$1271.25$
(vii) Continuously:	$A = 1000e^{(0.08)3} = \$1271.25$

(b)



$A_{0.10}(3) = \$1349.86$,

$A_{0.08}(3) = \$1271.25$, and

$A_{0.06}(3) = \$1197.22$.

21. (a) Using $A = A_0\left(1 + \dfrac{r}{n}\right)^{nt}$ with $A_0 = 3000$, $r = 0.05$, and $t = 5$, we have:

(i) Annually: $n = 1$; $A = 3000\left(1 + \frac{0.05}{1}\right)^{1\cdot 5} = \$3828.84$

(ii) Semiannually: $n = 2$; $A = 3000\left(1 + \frac{0.05}{2}\right)^{2\cdot 5} = \$3840.25$

(iii) Monthly: $n = 12$; $A = 3000\left(1 + \frac{0.05}{12}\right)^{12\cdot 5} = \$3850.08$

(iv) Weekly: $n = 52$; $A = 3000\left(1 + \frac{0.05}{52}\right)^{52\cdot 5} = \$3851.61$

(v) Daily: $n = 365$; $A = 3000\left(1 + \frac{0.05}{365}\right)^{365\cdot 5} = \$3852.01$

(vi) Continuously: $A = 3000e^{(0.05)5} = \$3852.08$

(b) $dA/dt = 0.05A$ and $A(0) = 3000$.

22. (a) $A_0 e^{0.06t} = 2A_0 \quad \Leftrightarrow \quad e^{0.06t} = 2 \quad \Leftrightarrow \quad 0.06t = \ln 2 \quad \Leftrightarrow \quad t = \frac{50}{3}\ln 2 \approx 11.55$, so the investment will double in about 11.55 years.

(b) The annual interest rate in $A = A_0(1 + r)^t$ is r. From part (a), we have $A = A_0 e^{0.06t}$. These amounts must be equal, so $(1 + r)^t = e^{0.06t} \quad \Rightarrow \quad 1 + r = e^{0.06} \quad \Rightarrow \quad r = e^{0.06} - 1 \approx 0.0618 = 6.18\%$, which is the equivalent annual interest rate.

APPLIED PROJECT Controlling Red Blood Cell Loss During Surgery

1. Let $R(t)$ be the volume of RBCs (in liters) at time t (in hours). Since the total volume of blood is 5 L, the concentration of RBCs is $R/5$. The patient bleeds 2 L of blood in 4 hours, so

$$\frac{dR}{dt} = -\frac{2L}{4h} \cdot \frac{R}{5} = -\frac{1}{10}R$$

From Section 3.8, we know that $dR/dt = kR$ has solution $R(t) = R(0)e^{kt}$. In this case, $R(0) = 45\%$ of $5 = \frac{9}{4}$ and $k = -\frac{1}{10}$, so $R(t) = \frac{9}{4}e^{-t/10}$. At the end of the operation, the volume of RBCs is $R(4) = \frac{9}{4}e^{-0.4} \approx 1.51$ L.

2. Let V be the volume of blood that is extracted and replaced with saline solution. Let $R_A(t)$ be the volume of RBCs with the ANH procedure. Then $R_A(0)$ is 45% of $(5 - V)$, or $\frac{9}{20}(5 - V)$, and hence $R_A(t) = \frac{9}{20}(5 - V)e^{-t/10}$. We want $R_A(4) \geq 25\%$ of $5 \quad \Leftrightarrow \quad \frac{9}{20}(5 - V)e^{-0.4} \geq \frac{5}{4} \quad \Leftrightarrow \quad 5 - V \geq \frac{25}{9}e^{0.4} \quad \Leftrightarrow \quad V \leq 5 - \frac{25}{9}e^{0.4} \approx 0.86$ L. To maximize the effect of the ANH procedure, the surgeon should remove 0.86 L of blood and replace it with saline solution.

3. The RBC loss *without* the ANH procedure is $R(0) - R(4) = \frac{9}{4} - \frac{9}{4}e^{-0.4} \approx 0.74$ L. The RBC loss *with* the ANH procedure is $R_A(0) - R_A(4) = \frac{9}{20}(5 - V) - \frac{9}{20}(5 - V)e^{-0.4} = \frac{9}{20}(5 - V)(1 - e^{-0.4})$. Now let $V = 5 - \frac{25}{9}e^{0.4}$ [from Problem 2] to get $R_A(0) - R_A(4) = \frac{9}{20}\left[5 - \left(5 - \frac{25}{9}e^{0.4}\right)\right](1 - e^{0.4}) = \frac{9}{20} \cdot \frac{25}{9}e^{0.4}(1 - e^{0.4}) = \frac{5}{4}(e^{0.4} - 1) \approx 0.61$ L. Thus, the ANH procedure reduces the RBC loss by about $0.74 - 0.61 = 0.13$ L (about 4.4 fluid ounces).

3.9 Related Rates

1. $V = x^3 \quad\Rightarrow\quad \dfrac{dV}{dt} = \dfrac{dV}{dx}\dfrac{dx}{dt} = 3x^2\,\dfrac{dx}{dt}$

2. (a) $A = \pi r^2 \quad\Rightarrow\quad \dfrac{dA}{dt} = \dfrac{dA}{dr}\dfrac{dr}{dt} = 2\pi r\,\dfrac{dr}{dt}$ (b) $\dfrac{dA}{dt} = 2\pi r\,\dfrac{dr}{dt} = 2\pi(30\text{ m})(1\text{ m/s}) = 60\pi\text{ m}^2/\text{s}$

3. Let s denote the side of a square. The square's area A is given by $A = s^2$. Differentiating with respect to t gives us $\dfrac{dA}{dt} = 2s\,\dfrac{ds}{dt}$. When $A = 16$, $s = 4$. Substitution 4 for s and 6 for $\dfrac{ds}{dt}$ gives us $\dfrac{dA}{dt} = 2(4)(6) = 48\text{ cm}^2\text{/s}$.

4. $A = \ell w \quad\Rightarrow\quad \dfrac{dA}{dt} = \ell\cdot\dfrac{dw}{dt} + w\cdot\dfrac{d\ell}{dt} = 20(3) + 10(8) = 140\text{ cm}^2\text{/s}$.

5. $V = \pi r^2 h = \pi(5)^2 h = 25\pi h \quad\Rightarrow\quad \dfrac{dV}{dt} = 25\pi\,\dfrac{dh}{dt} \quad\Rightarrow\quad 3 = 25\pi\,\dfrac{dh}{dt} \quad\Rightarrow\quad \dfrac{dh}{dt} = \dfrac{3}{25\pi}\text{ m/min.}$

6. $V = \frac{4}{3}\pi r^3 \quad\Rightarrow\quad \dfrac{dV}{dt} = \frac{4}{3}\pi\cdot 3r^2\,\dfrac{dr}{dt} \quad\Rightarrow\quad \dfrac{dV}{dt} = 4\pi\left(\frac{1}{2}\cdot 80\right)^2(4) = 25{,}600\pi\text{ mm}^3\text{/s}$.

7. $S = 4\pi r^2 \quad\Rightarrow\quad \dfrac{dS}{dt} = 4\pi\cdot 2r\,\dfrac{dr}{dt} \quad\Rightarrow\quad \dfrac{dS}{dt} = 4\pi\cdot 2\cdot 8\cdot 2 = 128\pi\text{ cm}^2\text{/min.}$

8. (a) $A = \frac{1}{2}ab\sin\theta \quad\Rightarrow\quad \dfrac{dA}{dt} = \frac{1}{2}ab\cos\theta\,\dfrac{d\theta}{dt} = \frac{1}{2}(2)(3)\left(\cos\frac{\pi}{3}\right)(0.2) = 3\left(\frac{1}{2}\right)(0.2) = 0.3\text{ cm}^2\text{/min.}$

(b) $A = \frac{1}{2}ab\sin\theta \quad\Rightarrow$

$$\begin{aligned}\frac{dA}{dt} &= \frac{1}{2}a\left(b\cos\theta\,\frac{d\theta}{dt} + \sin\theta\,\frac{db}{dt}\right) = \tfrac{1}{2}(2)\left[3\left(\cos\tfrac{\pi}{3}\right)(0.2) + \left(\sin\tfrac{\pi}{3}\right)(1.5)\right] \\ &= 3\left(\tfrac{1}{2}\right)(0.2) + \tfrac{1}{2}\sqrt{3}\left(\tfrac{3}{2}\right) = 0.3 + \tfrac{3}{4}\sqrt{3}\text{ cm}^2\text{/min} \quad [\approx 1.6]\end{aligned}$$

(c) $A = \frac{1}{2}ab\sin\theta \quad\Rightarrow$

$$\begin{aligned}\frac{dA}{dt} &= \frac{1}{2}\left(\frac{da}{dt}\,b\sin\theta + a\,\frac{db}{dt}\sin\theta + ab\cos\theta\,\frac{d\theta}{dt}\right) \qquad \text{[by Exercise 3.2.61(a)]} \\ &= \tfrac{1}{2}\left[(2.5)(3)\left(\tfrac{1}{2}\sqrt{3}\right) + (2)(1.5)\left(\tfrac{1}{2}\sqrt{3}\right) + (2)(3)\left(\tfrac{1}{2}\right)(0.2)\right] \\ &= \left(\tfrac{15}{8}\sqrt{3} + \tfrac{3}{4}\sqrt{3} + 0.3\right) = \left(\tfrac{21}{8}\sqrt{3} + 0.3\right)\ \text{cm}^2\text{/min} \quad [\approx 4.85]\end{aligned}$$

Note how this answer relates to the answer in part (a) [θ changing] and part (b) [b and θ changing].

9. (a) $y = \sqrt{2x+1}$ and $\dfrac{dx}{dt} = 3 \quad\Rightarrow\quad \dfrac{dy}{dt} = \dfrac{dy}{dx}\dfrac{dx}{dt} = \dfrac{1}{2}(2x+1)^{-1/2}\cdot 2\cdot 3 = \dfrac{3}{\sqrt{2x+1}}$. When $x = 4$, $\dfrac{dy}{dt} = \dfrac{3}{\sqrt{9}} = 1$.

(b) $y = \sqrt{2x+1} \quad\Rightarrow\quad y^2 = 2x+1 \quad\Rightarrow\quad 2x = y^2 - 1 \quad\Rightarrow\quad x = \frac{1}{2}y^2 - \frac{1}{2}$ and $\dfrac{dy}{dt} = 5 \quad\Rightarrow$

$\dfrac{dx}{dt} = \dfrac{dx}{dy}\dfrac{dy}{dt} = y\cdot 5 = 5y$. When $x = 12$, $y = \sqrt{25} = 5$, so $\dfrac{dx}{dt} = 5(5) = 25$.

10. (a) $\dfrac{d}{dt}(4x^2 + 9y^2) = \dfrac{d}{dt}(36) \quad\Rightarrow\quad 8x\,\dfrac{dx}{dt} + 18y\,\dfrac{dy}{dt} = 0 \quad\Rightarrow\quad 4x\,\dfrac{dx}{dt} + 9y\,\dfrac{dy}{dt} = 0 \quad\Rightarrow$

$4(2)\,\dfrac{dx}{dt} + 9\left(\dfrac{2}{3}\sqrt{5}\right)\left(\dfrac{1}{3}\right) = 0 \quad\Rightarrow\quad 8\,\dfrac{dx}{dt} = -2\sqrt{5} \quad\Rightarrow\quad \dfrac{dx}{dt} = -\dfrac{1}{4}\sqrt{5}$

(b) $4x\,\dfrac{dx}{dt} + 9y\,\dfrac{dy}{dt} = 0 \;\Rightarrow\; 4(-2)(3) + 9\left(\dfrac{2}{3}\sqrt{5}\right)\dfrac{dy}{dt} = 0 \;\Rightarrow\; 6\sqrt{5}\,\dfrac{dy}{dt} = 24 \;\Rightarrow\; \dfrac{dy}{dt} = \dfrac{4}{\sqrt{5}}$

11. $\dfrac{d}{dt}\left(x^2 + y^2 + z^2\right) = \dfrac{d}{dt}(9) \;\Rightarrow\; 2x\,\dfrac{dx}{dt} + 2y\,\dfrac{dy}{dt} + 2z\,\dfrac{dz}{dt} = 0 \;\Rightarrow\; x\,\dfrac{dx}{dt} + y\,\dfrac{dy}{dt} + z\,\dfrac{dz}{dt} = 0.$

If $\dfrac{dx}{dt} = 5$, $\dfrac{dy}{dt} = 4$ and $(x, y, z) = (2, 2, 1)$, then $2(5) + 2(4) + 1\,\dfrac{dz}{dt} = 0 \;\Rightarrow\; \dfrac{dz}{dt} = -18.$

12. $\dfrac{d}{dt}(xy) = \dfrac{d}{dt}(8) \;\Rightarrow\; x\,\dfrac{dy}{dt} + y\,\dfrac{dx}{dt} = 0.$ If $\dfrac{dy}{dt} = -3$ cm/s and $(x, y) = (4, 2)$, then $4(-3) + 2\,\dfrac{dx}{dt} = 0 \;\Rightarrow$

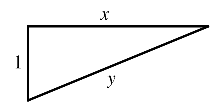
$\dfrac{dx}{dt} = 6$. Thus, the x-coordinate is increasing at a rate of 6 cm/s.

13. (a) Given: a plane flying horizontally at an altitude of 1 mi and a speed of 500 mi/h passes directly over a radar station.

If we let t be time (in hours) and x be the horizontal distance traveled by the plane (in mi), then we are given that $dx/dt = 500$ mi/h.

(b) Unknown: the rate at which the distance from the plane to the station is increasing when it is 2 mi from the station. If we let y be the distance from the plane to the station, then we want to find dy/dt when $y = 2$ mi.

(c)



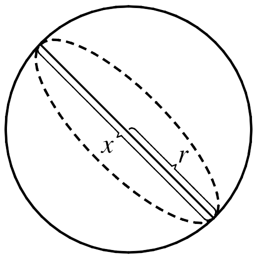
(d) By the Pythagorean Theorem, $y^2 = x^2 + 1 \;\Rightarrow\; 2y\,(dy/dt) = 2x\,(dx/dt)$.

(e) $\dfrac{dy}{dt} = \dfrac{x}{y}\dfrac{dx}{dt} = \dfrac{x}{y}(500)$. Since $y^2 = x^2 + 1$, when $y = 2$, $x = \sqrt{3}$, so $\dfrac{dy}{dt} = \dfrac{\sqrt{3}}{2}(500) = 250\sqrt{3} \approx 433$ mi/h.

14. (a) Given: the rate of decrease of the surface area is 1 cm^2/min. If we let t be time (in minutes) and S be the surface area (in cm^2), then we are given that $dS/dt = -1$ cm^2/s.

(b) Unknown: the rate of decrease of the diameter when the diameter is 10 cm. If we let x be the diameter, then we want to find dx/dt when $x = 10$ cm.

(c)



(d) If the radius is r and the diameter $x = 2r$, then $r = \frac{1}{2}x$ and

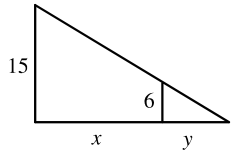
$S = 4\pi r^2 = 4\pi\left(\frac{1}{2}x\right)^2 = \pi x^2 \;\Rightarrow\; \dfrac{dS}{dt} = \dfrac{dS}{dx}\dfrac{dx}{dt} = 2\pi x\,\dfrac{dx}{dt}.$

(e) $-1 = \dfrac{dS}{dt} = 2\pi x\,\dfrac{dx}{dt} \;\Rightarrow\; \dfrac{dx}{dt} = -\dfrac{1}{2\pi x}$. When $x = 10$, $\dfrac{dx}{dt} = -\dfrac{1}{20\pi}$. So the rate of decrease is $\dfrac{1}{20\pi}$ cm/min.

15. (a) Given: a man 6 ft tall walks away from a street light mounted on a 15-ft-tall pole at a rate of 5 ft/s. If we let t be time (in s) and x be the distance from the pole to the man (in ft), then we are given that $dx/dt = 5$ ft/s.

(b) Unknown: the rate at which the tip of his shadow is moving when he is 40 ft from the pole. If we let y be the distance from the man to the tip of his shadow (in ft), then we want to find $\dfrac{d}{dt}(x + y)$ when $x = 40$ ft.

(c)



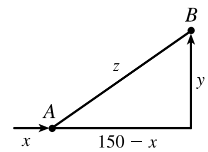
(d) By similar triangles, $\dfrac{15}{6} = \dfrac{x+y}{y} \quad\Rightarrow\quad 15y = 6x + 6y \quad\Rightarrow\quad 9y = 6x \quad\Rightarrow\quad y = \frac{2}{3}x$.

(e) The tip of the shadow moves at a rate of $\dfrac{d}{dt}(x+y) = \dfrac{d}{dt}\left(x + \dfrac{2}{3}x\right) = \dfrac{5}{3}\dfrac{dx}{dt} = \frac{5}{3}(5) = \frac{25}{3}$ ft/s.

16. (a) Given: at noon, ship A is 150 km west of ship B; ship A is sailing east at 35 km/h, and ship B is sailing north at 25 km/h. If we let t be time (in hours), x be the distance traveled by ship A (in km), and y be the distance traveled by ship B (in km), then we are given that $dx/dt = 35$ km/h and $dy/dt = 25$ km/h.

(b) Unknown: the rate at which the distance between the ships is changing at 4:00 PM. If we let z be the distance between the ships, then we want to find dz/dt when $t = 4$ h.

(c)

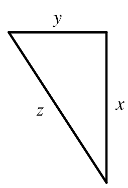


(d) $z^2 = (150 - x)^2 + y^2 \quad\Rightarrow\quad 2z\,\dfrac{dz}{dt} = 2(150 - x)\left(-\dfrac{dx}{dt}\right) + 2y\,\dfrac{dy}{dt}$

(e) At 4:00 PM, $x = 4(35) = 140$ and $y = 4(25) = 100 \quad\Rightarrow\quad z = \sqrt{(150 - 140)^2 + 100^2} = \sqrt{10{,}100}$.

So $\dfrac{dz}{dt} = \dfrac{1}{z}\left[(x - 150)\,\dfrac{dx}{dt} + y\,\dfrac{dy}{dt}\right] = \dfrac{-10(35) + 100(25)}{\sqrt{10{,}100}} = \dfrac{215}{\sqrt{101}} \approx 21.4$ km/h.

17.



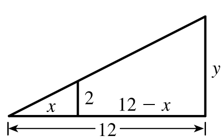
We are given that $\dfrac{dx}{dt} = 60$ mi/h and $\dfrac{dy}{dt} = 25$ mi/h. $z^2 = x^2 + y^2 \quad\Rightarrow$

$2z\,\dfrac{dz}{dt} = 2x\,\dfrac{dx}{dt} + 2y\,\dfrac{dy}{dt} \quad\Rightarrow\quad z\,\dfrac{dz}{dt} = x\,\dfrac{dx}{dt} + y\,\dfrac{dy}{dt} \quad\Rightarrow\quad \dfrac{dz}{dt} = \dfrac{1}{z}\left(x\,\dfrac{dx}{dt} + y\,\dfrac{dy}{dt}\right)$.

After 2 hours, $x = 2\,(60) = 120$ and $y = 2\,(25) = 50 \quad\Rightarrow\quad z = \sqrt{120^2 + 50^2} = 130$,

so $\dfrac{dz}{dt} = \dfrac{1}{z}\left(x\,\dfrac{dx}{dt} + y\,\dfrac{dy}{dt}\right) = \dfrac{120(60) + 50(25)}{130} = 65$ mi/h.

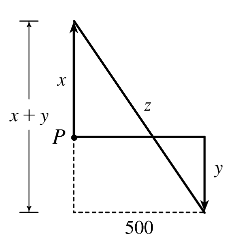
18.



We are given that $\dfrac{dx}{dt} = 1.6$ m/s. By similar triangles, $\dfrac{y}{12} = \dfrac{2}{x} \quad\Rightarrow\quad y = \dfrac{24}{x} \quad\Rightarrow$

$\dfrac{dy}{dt} = -\dfrac{24}{x^2}\dfrac{dx}{dt} = -\dfrac{24}{x^2}(1.6)$. When $x = 8$, $\dfrac{dy}{dt} = -\dfrac{24(1.6)}{64} = -0.6$ m/s, so the shadow is decreasing at a rate of 0.6 m/s.

19.



We are given that $\dfrac{dx}{dt} = 4$ ft/s and $\dfrac{dy}{dt} = 5$ ft/s. $z^2 = (x+y)^2 + 500^2 \quad\Rightarrow$

$2z\,\dfrac{dz}{dt} = 2(x+y)\left(\dfrac{dx}{dt} + \dfrac{dy}{dt}\right)$. 15 minutes after the woman starts, we have

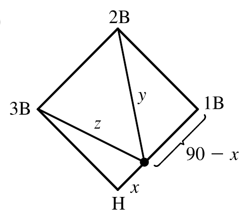
$x = (4 \text{ ft/s})(20 \text{ min})(60 \text{ s/min}) = 4800$ ft and $y = 5 \cdot 15 \cdot 60 = 4500 \quad\Rightarrow$

$z = \sqrt{(4800 + 4500)^2 + 500^2} = \sqrt{86{,}740{,}000}$, so

$\dfrac{dz}{dt} = \dfrac{x+y}{z}\left(\dfrac{dx}{dt} + \dfrac{dy}{dt}\right) = \dfrac{4800 + 4500}{\sqrt{86{,}740{,}000}}(4+5) = \dfrac{837}{\sqrt{8674}} \approx 8.99$ ft/s.

20. We are given that $\dfrac{dx}{dt} = 24$ ft/s.

(a)



$y^2 = (90 - x)^2 + 90^2 \quad\Rightarrow\quad 2y\,\dfrac{dy}{dt} = 2(90 - x)\left(-\dfrac{dx}{dt}\right)$. When $x = 45$,

$y = \sqrt{45^2 + 90^2} = 45\sqrt{5}$, so $\dfrac{dy}{dt} = \dfrac{90 - x}{y}\left(-\dfrac{dx}{dt}\right) = \dfrac{45}{45\sqrt{5}}\,(-24) = -\dfrac{24}{\sqrt{5}}$,

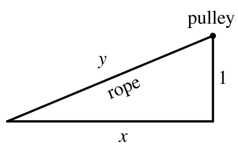
so the distance from second base is decreasing at a rate of $\frac{24}{\sqrt{5}} \approx 10.7$ ft/s.

(b) Due to the symmetric nature of the problem in part (a), we expect to get the same answer—and we do.

$z^2 = x^2 + 90^2 \quad\Rightarrow\quad 2z\,\dfrac{dz}{dt} = 2x\,\dfrac{dx}{dt}$. When $x = 45$, $z = 45\sqrt{5}$, so $\dfrac{dz}{dt} = \dfrac{45}{45\sqrt{5}}(24) = \dfrac{24}{\sqrt{5}} \approx 10.7$ ft/s.

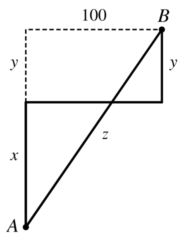
21. $A = \frac{1}{2}bh$, where b is the base and h is the altitude. We are given that $\dfrac{dh}{dt} = 1$ cm/min and $\dfrac{dA}{dt} = 2\ \text{cm}^2/\text{min}$. Using the Product Rule, we have $\dfrac{dA}{dt} = \dfrac{1}{2}\left(b\,\dfrac{dh}{dt} + h\,\dfrac{db}{dt}\right)$. When $h = 10$ and $A = 100$, we have $100 = \frac{1}{2}b(10) \quad\Rightarrow\quad \frac{1}{2}b = 10 \quad\Rightarrow$ $b = 20$, so $2 = \dfrac{1}{2}\left(20 \cdot 1 + 10\,\dfrac{db}{dt}\right) \quad\Rightarrow\quad 4 = 20 + 10\,\dfrac{db}{dt} \quad\Rightarrow\quad \dfrac{db}{dt} = \dfrac{4 - 20}{10} = -1.6$ cm/min.

22.



Given $\dfrac{dy}{dt} = -1$ m/s, find $\dfrac{dx}{dt}$ when $x = 8$ m. $y^2 = x^2 + 1 \quad\Rightarrow\quad 2y\,\dfrac{dy}{dt} = 2x\,\dfrac{dx}{dt} \quad\Rightarrow$ $\dfrac{dx}{dt} = \dfrac{y}{x}\dfrac{dy}{dt} = -\dfrac{y}{x}$. When $x = 8$, $y = \sqrt{65}$, so $\dfrac{dx}{dt} = -\dfrac{\sqrt{65}}{8}$. Thus, the boat approaches the dock at $\dfrac{\sqrt{65}}{8} \approx 1.01$ m/s.

23.



We are given that $\dfrac{dx}{dt} = 35$ km/h and $\dfrac{dy}{dt} = 25$ km/h. $z^2 = (x + y)^2 + 100^2 \quad\Rightarrow$

$2z\,\dfrac{dz}{dt} = 2(x + y)\left(\dfrac{dx}{dt} + \dfrac{dy}{dt}\right)$. At 4:00 PM, $x = 4(35) = 140$ and $y = 4(25) = 100 \quad\Rightarrow$

$z = \sqrt{(140 + 100)^2 + 100^2} = \sqrt{67{,}600} = 260$, so

$\dfrac{dz}{dt} = \dfrac{x + y}{z}\left(\dfrac{dx}{dt} + \dfrac{dy}{dt}\right) = \dfrac{140 + 100}{260}(35 + 25) = \dfrac{720}{13} \approx 55.4$ km/h.

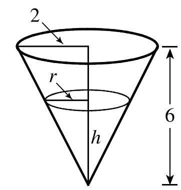
24. The distance z of the particle to the origin is given by $z = \sqrt{x^2 + y^2}$, so $z^2 = x^2 + [2\sin(\pi x/2)]^2 \quad\Rightarrow$

$2z\,\dfrac{dz}{dt} = 2x\,\dfrac{dx}{dt} + 4 \cdot 2\sin\left(\dfrac{\pi}{2}x\right)\cos\left(\dfrac{\pi}{2}x\right) \cdot \dfrac{\pi}{2}\,\dfrac{dx}{dt} \quad\Rightarrow\quad z\,\dfrac{dz}{dt} = x\,\dfrac{dx}{dt} + 2\pi\sin\left(\dfrac{\pi}{2}x\right)\cos\left(\dfrac{\pi}{2}x\right)\dfrac{dx}{dt}$. When

$(x, y) = \left(\dfrac{1}{3}, 1\right)$, $z = \sqrt{\left(\dfrac{1}{3}\right)^2 + 1^2} = \sqrt{\dfrac{10}{9}} = \dfrac{1}{3}\sqrt{10}$, so $\dfrac{1}{3}\sqrt{10}\,\dfrac{dz}{dt} = \dfrac{1}{3}\sqrt{10} + 2\pi\,\sin\,\dfrac{\pi}{6}\,\cos\,\dfrac{\pi}{6} \cdot \sqrt{10} \quad\Rightarrow$

$\dfrac{1}{3}\,\dfrac{dz}{dt} = \dfrac{1}{3} + 2\pi\left(\dfrac{1}{2}\right)\left(\dfrac{1}{2}\sqrt{3}\right) \quad\Rightarrow\quad \dfrac{dz}{dt} = 1 + \dfrac{3\sqrt{3}\,\pi}{2}$ cm/s.

25. If $C =$ the rate at which water is pumped in, then $\dfrac{dV}{dt} = C - 10{,}000$, where

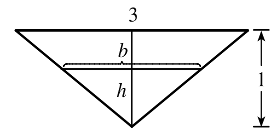


$V = \frac{1}{3}\pi r^2 h$ is the volume at time t. By similar triangles, $\dfrac{r}{2} = \dfrac{h}{6} \quad\Rightarrow\quad r = \dfrac{1}{3}h \quad\Rightarrow$

$V = \frac{1}{3}\pi\left(\frac{1}{3}h\right)^2 h = \frac{\pi}{27}h^3 \quad\Rightarrow\quad \dfrac{dV}{dt} = \dfrac{\pi}{9}h^2\,\dfrac{dh}{dt}$. When $h = 200$ cm,

$\dfrac{dh}{dt} = 20$ cm/min, so $C - 10{,}000 = \dfrac{\pi}{9}(200)^2(20) \quad\Rightarrow\quad C = 10{,}000 + \dfrac{800{,}000}{9}\pi \approx 289{,}253$ cm^3/min.

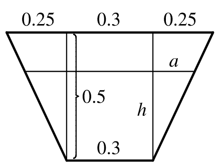
26. By similar triangles, $\dfrac{3}{1} = \dfrac{b}{h}$, so $b = 3h$. The trough has volume



$V = \frac{1}{2}bh(10) = 5(3h)h = 15h^2 \quad\Rightarrow\quad 12 = \dfrac{dV}{dt} = 30h\,\dfrac{dh}{dt} \quad\Rightarrow\quad \dfrac{dh}{dt} = \dfrac{2}{5h}$.

When $h = \frac{1}{2}$, $\dfrac{dh}{dt} = \dfrac{2}{5 \cdot \frac{1}{2}} = \dfrac{4}{5}$ ft/min.

27.



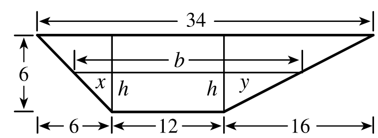
The figure is labeled in meters. The area A of a trapezoid is $\frac{1}{2}(\text{base}_1 + \text{base}_2)(\text{height})$, and the volume V of the 10-meter-long trough is $10A$. Thus, the volume of the trapezoid with height h is $V = (10)\frac{1}{2}[0.3 + (0.3 + 2a)]h$.

By similar triangles, $\dfrac{a}{h} = \dfrac{0.25}{0.5} = \dfrac{1}{2}$, so $2a = h \quad\Rightarrow\quad V = 5(0.6 + h)h = 3h + 5h^2$.

Now $\dfrac{dV}{dt} = \dfrac{dV}{dh}\dfrac{dh}{dt} \quad\Rightarrow\quad 0.2 = (3 + 10h)\dfrac{dh}{dt} \quad\Rightarrow\quad \dfrac{dh}{dt} = \dfrac{0.2}{3 + 10h}$. When $h = 0.3$,

$\dfrac{dh}{dt} = \dfrac{0.2}{3 + 10(0.3)} = \dfrac{0.2}{6}$ m/min $= \dfrac{1}{30}$ m/min or $\dfrac{10}{3}$ cm/min.

28. The figure is drawn without the top 3 feet.



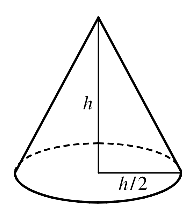
$V = \frac{1}{2}(b + 12)h(20) = 10(b + 12)h$ and, from similar triangles,

$\dfrac{x}{h} = \dfrac{6}{6}$ and $\dfrac{y}{h} = \dfrac{16}{6} = \dfrac{8}{3}$, so $b = x + 12 + y = h + 12 + \dfrac{8h}{3} = 12 + \dfrac{11h}{3}$.

Thus, $V = 10\left(24 + \dfrac{11h}{3}\right)h = 240h + \dfrac{110h^2}{3}$ and so $0.8 = \dfrac{dV}{dt} = \left(240 + \dfrac{220}{3}h\right)\dfrac{dh}{dt}$.

When $h = 5$, $\dfrac{dh}{dt} = \dfrac{0.8}{240 + 5(220/3)} = \dfrac{3}{2275} \approx 0.00132$ ft/min.

29. We are given that $\dfrac{dV}{dt} = 30$ ft^3/min. $V = \dfrac{1}{3}\pi r^2 h = \dfrac{1}{3}\pi\left(\dfrac{h}{2}\right)^2 h = \dfrac{\pi h^3}{12} \quad\Rightarrow$

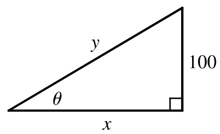


$\dfrac{dV}{dt} = \dfrac{dV}{dh}\dfrac{dh}{dt} \quad\Rightarrow\quad 30 = \dfrac{\pi h^2}{4}\dfrac{dh}{dt} \quad\Rightarrow\quad \dfrac{dh}{dt} = \dfrac{120}{\pi h^2}$.

When $h = 10$ ft, $\dfrac{dh}{dt} = \dfrac{120}{10^2\pi} = \dfrac{6}{5\pi} \approx 0.38$ ft/min.

30. We are given $dx/dt = 8$ ft/s. $\cot\theta = \dfrac{x}{100} \quad\Rightarrow\quad x = 100\cot\theta \quad\Rightarrow$

$\dfrac{dx}{dt} = -100\csc^2\theta\,\dfrac{d\theta}{dt} \quad\Rightarrow\quad \dfrac{d\theta}{dt} = -\dfrac{\sin^2\theta}{100}\cdot 8$. When $y = 200$, $\sin\theta = \dfrac{100}{200} = \dfrac{1}{2} \quad\Rightarrow$



$\dfrac{d\theta}{dt} = -\dfrac{(1/2)^2}{100}\cdot 8 = -\dfrac{1}{50}$ rad/s. The angle is decreasing at a rate of $\frac{1}{50}$ rad/s.

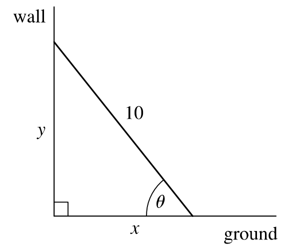
31. The area A of an equilateral triangle with side s is given by $A = \frac{1}{4}\sqrt{3}\,s^2$.

$\dfrac{dA}{dt} = \frac{1}{4}\sqrt{3}\cdot 2s\,\dfrac{ds}{dt} = \frac{1}{4}\sqrt{3}\cdot 2(30)(10) = 150\sqrt{3}\ \text{cm}^2/\text{min}.$

32. $\cos\theta = \dfrac{x}{10} \quad\Rightarrow\quad -\sin\theta\,\dfrac{d\theta}{dt} = \dfrac{1}{10}\,\dfrac{dx}{dt}$. From Example 2, $\dfrac{dx}{dt} = 1$ and

when $x = 6$, $y = 8$, so $\sin\theta = \dfrac{8}{10}$.

Thus, $-\dfrac{8}{10}\,\dfrac{d\theta}{dt} = \dfrac{1}{10}(1) \quad\Rightarrow\quad \dfrac{d\theta}{dt} = -\dfrac{1}{8}$ rad/s.



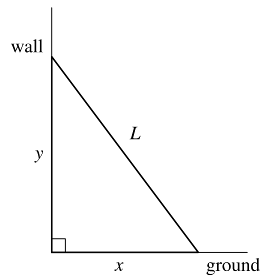
33. From the figure and given information, we have $x^2 + y^2 = L^2$, $\dfrac{dy}{dt} = -0.15$ m/s, and

$\dfrac{dx}{dt} = 0.2$ m/s when $x = 3$ m. Differentiating implicitly with respect to t, we get

$x^2 + y^2 = L^2 \quad\Rightarrow\quad 2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt} = 0 \quad\Rightarrow\quad y\dfrac{dy}{dt} = -x\dfrac{dx}{dt}$. Substituting the given

information gives us $y(-0.15) = -3(0.2) \quad\Rightarrow\quad y = 4$ m. Thus, $3^2 + 4^2 = L^2 \quad\Rightarrow$

$L^2 = 25 \quad\Rightarrow\quad L = 5$ m.



34. According to the model in Example 2, $\dfrac{dy}{dt} = -\dfrac{x}{y}\dfrac{dx}{dt} \to -\infty$ as $y \to 0$, which doesn't make physical sense. For example, the model predicts that for sufficiently small y, the tip of the ladder moves at a speed greater than the speed of light. Therefore the model is not appropriate for small values of y. What actually happens is that the tip of the ladder leaves the wall at some point in its descent. For a discussion of the true situation see the article "The Falling Ladder Paradox" by Paul Scholten and Andrew Simoson in *The College Mathematics Journal,* 27, (1), January 1996, pages 49–54. Also see "On Mathematical and Physical Ladders" by M. Freeman and P. Palffy-Muhoray in the *American Journal of Physics,* 53 (3), March 1985, pages 276–277.

35. The area A of a sector of a circle with radius r and angle θ is given by $A = \frac{1}{2}r^2\theta$. Here r is constant and θ varies, so

$\dfrac{dA}{dt} = \dfrac{1}{2}r^2\,\dfrac{d\theta}{dt}$. The minute hand rotates through $360^\circ = 2\pi$ radians each hour, so $\dfrac{dA}{dt} = \dfrac{1}{2}r^2(2\pi) = \pi r^2\ \text{cm}^2/\text{h}$. This answer makes sense because the minute hand sweeps through the full area of a circle, πr^2, each hour.

36. The volume of a hemisphere is $\frac{2}{3}\pi r^3$, so the volume of a hemispherical basin of radius 30 cm is $\frac{2}{3}\pi(30)^3 = 18{,}000\pi \text{ cm}^3$. If the basin is half full, then $V = \pi\left(rh^2 - \frac{1}{3}h^3\right) \quad\Rightarrow\quad 9000\pi = \pi\left(30h^2 - \frac{1}{3}h^3\right) \quad\Rightarrow\quad \frac{1}{3}h^3 - 30h^2 + 9000 = 0 \quad\Rightarrow$ $h = H \approx 19.58$ [from a graph or numerical rootfinder; the other two solutions are less than 0 and greater than 30].

$$V = \pi\left(30h^2 - \tfrac{1}{3}h^3\right) \quad\Rightarrow\quad \frac{dV}{dt} = \pi\left(60h\,\frac{dh}{dt} - h^2\,\frac{dh}{dt}\right) \quad\Rightarrow\quad \left(2\,\frac{\text{L}}{\text{min}}\right)\left(1000\,\frac{\text{cm}^3}{\text{L}}\right) = \pi(60h - h^2)\,\frac{dh}{dt} \quad\Rightarrow$$

$$\frac{dh}{dt} = \frac{2000}{\pi(60H - H^2)} \approx 0.804 \text{ cm/min}.$$

37. Differentiating both sides of $PV = C$ with respect to t and using the Product Rule gives us $P\,\dfrac{dV}{dt} + V\,\dfrac{dP}{dt} = 0 \quad\Rightarrow$ $\dfrac{dV}{dt} = -\dfrac{V}{P}\,\dfrac{dP}{dt}$. When $V = 600$, $P = 150$ and $\dfrac{dP}{dt} = 20$, so we have $\dfrac{dV}{dt} = -\dfrac{600}{150}(20) = -80$. Thus, the volume is decreasing at a rate of 80 cm^3/min.

38. $PV^{1.4} = C \quad\Rightarrow\quad P \cdot 1.4V^{0.4}\,\dfrac{dV}{dt} + V^{1.4}\,\dfrac{dP}{dt} = 0 \quad\Rightarrow\quad \dfrac{dV}{dt} = -\dfrac{V^{1.4}}{P \cdot 1.4V^{0.4}}\,\dfrac{dP}{dt} = -\dfrac{V}{1.4P}\,\dfrac{dP}{dt}$.

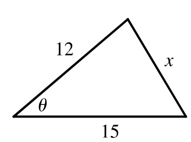
When $V = 400$, $P = 80$ and $\dfrac{dP}{dt} = -10$, so we have $\dfrac{dV}{dt} = -\dfrac{400}{1.4(80)}(-10) = \dfrac{250}{7}$. Thus, the volume is increasing at a rate of $\frac{250}{7} \approx 36$ cm^3/min.

39. With $R_1 = 80$ and $R_2 = 100$, $\dfrac{1}{R} = \dfrac{1}{R_1} + \dfrac{1}{R_2} = \dfrac{1}{80} + \dfrac{1}{100} = \dfrac{180}{8000} = \dfrac{9}{400}$, so $R = \dfrac{400}{9}$. Differentiating $\dfrac{1}{R} = \dfrac{1}{R_1} + \dfrac{1}{R_2}$ with respect to t, we have $-\dfrac{1}{R^2}\,\dfrac{dR}{dt} = -\dfrac{1}{R_1^2}\,\dfrac{dR_1}{dt} - \dfrac{1}{R_2^2}\,\dfrac{dR_2}{dt} \quad\Rightarrow\quad \dfrac{dR}{dt} = R^2\left(\dfrac{1}{R_1^2}\,\dfrac{dR_1}{dt} + \dfrac{1}{R_2^2}\,\dfrac{dR_2}{dt}\right)$. When $R_1 = 80$ and $R_2 = 100$, $\dfrac{dR}{dt} = \dfrac{400^2}{9^2}\left[\dfrac{1}{80^2}(0.3) + \dfrac{1}{100^2}(0.2)\right] = \dfrac{107}{810} \approx 0.132\ \Omega/\text{s}$.

40. We want to find $\dfrac{dB}{dt}$ when $L = 18$ using $B = 0.007W^{2/3}$ and $W = 0.12L^{2.53}$.

$$\begin{aligned}\frac{dB}{dt} &= \frac{dB}{dW}\,\frac{dW}{dL}\,\frac{dL}{dt} = \left(0.007 \cdot \tfrac{2}{3}W^{-1/3}\right)\left(0.12 \cdot 2.53 \cdot L^{1.53}\right)\left(\frac{20-15}{10{,}000{,}000}\right)\\ &= \left[0.007 \cdot \tfrac{2}{3}(0.12 \cdot 18^{2.53})^{-1/3}\right]\left(0.12 \cdot 2.53 \cdot 18^{1.53}\right)\left(\frac{5}{10^7}\right) \approx 1.045 \times 10^{-8} \text{ g/yr}\end{aligned}$$

41. We are given $d\theta/dt = 2°/\text{min} = \frac{\pi}{90}$ rad/min. By the Law of Cosines,

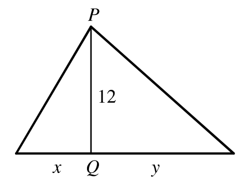


$x^2 = 12^2 + 15^2 - 2(12)(15)\cos\theta = 369 - 360\cos\theta \quad\Rightarrow$

$2x\,\dfrac{dx}{dt} = 360\sin\theta\,\dfrac{d\theta}{dt} \quad\Rightarrow\quad \dfrac{dx}{dt} = \dfrac{180\sin\theta}{x}\,\dfrac{d\theta}{dt}$. When $\theta = 60°$,

$x = \sqrt{369 - 360\cos 60°} = \sqrt{189} = 3\sqrt{21}$, so $\dfrac{dx}{dt} = \dfrac{180\sin 60°}{3\sqrt{21}}\,\dfrac{\pi}{90} = \dfrac{\pi\sqrt{3}}{3\sqrt{21}} = \dfrac{\sqrt{7}\,\pi}{21} \approx 0.396$ m/min.

42. Using Q for the origin, we are given $\frac{dx}{dt} = -2$ ft/s and need to find $\frac{dy}{dt}$ when $x = -5$.



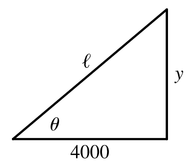
Using the Pythagorean Theorem twice, we have $\sqrt{x^2 + 12^2} + \sqrt{y^2 + 12^2} = 39$, the total length of the rope. Differentiating with respect to t, we get

$$\frac{x}{\sqrt{x^2+12^2}}\frac{dx}{dt} + \frac{y}{\sqrt{y^2+12^2}}\frac{dy}{dt} = 0, \text{ so } \frac{dy}{dt} = -\frac{x\sqrt{y^2+12^2}}{y\sqrt{x^2+12^2}}\frac{dx}{dt}.$$

Now when $x = -5$, $39 = \sqrt{(-5)^2 + 12^2} + \sqrt{y^2+12^2} = 13 + \sqrt{y^2+12^2} \quad \Leftrightarrow \quad \sqrt{y^2+12^2} = 26$, and $y = \sqrt{26^2 - 12^2} = \sqrt{532}$. So when $x = -5$, $\frac{dy}{dt} = -\frac{(-5)(26)}{\sqrt{532}\,(13)}(-2) = -\frac{10}{\sqrt{133}} \approx -0.87$ ft/s.

So cart B is moving towards Q at about 0.87 ft/s.

43. (a) By the Pythagorean Theorem, $4000^2 + y^2 = \ell^2$. Differentiating with respect to t, we obtain $2y\,\frac{dy}{dt} = 2\ell\,\frac{d\ell}{dt}$. We know that $\frac{dy}{dt} = 600$ ft/s, so when $y = 3000$ ft,

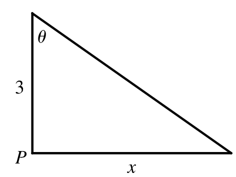


$\ell = \sqrt{4000^2 + 3000^2} = \sqrt{25{,}000{,}000} = 5000$ ft

and $\frac{d\ell}{dt} = \frac{y}{\ell}\frac{dy}{dt} = \frac{3000}{5000}(600) = \frac{1800}{5} = 360$ ft/s.

(b) Here $\tan\theta = \frac{y}{4000} \quad \Rightarrow \quad \frac{d}{dt}(\tan\theta) = \frac{d}{dt}\left(\frac{y}{4000}\right) \quad \Rightarrow \quad \sec^2\theta\,\frac{d\theta}{dt} = \frac{1}{4000}\frac{dy}{dt} \quad \Rightarrow \quad \frac{d\theta}{dt} = \frac{\cos^2\theta}{4000}\frac{dy}{dt}$. When $y = 3000$ ft, $\frac{dy}{dt} = 600$ ft/s, $\ell = 5000$ and $\cos\theta = \frac{4000}{\ell} = \frac{4000}{5000} = \frac{4}{5}$, so $\frac{d\theta}{dt} = \frac{(4/5)^2}{4000}(600) = 0.096$ rad/s.

44. We are given that $\frac{d\theta}{dt} = 4(2\pi) = 8\pi$ rad/min. $x = 3\tan\theta \quad \Rightarrow$

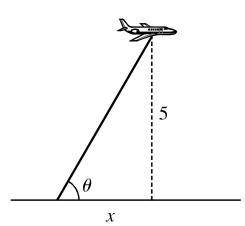


$\frac{dx}{dt} = 3\sec^2\theta\,\frac{d\theta}{dt}$. When $x = 1$, $\tan\theta = \frac{1}{3}$, so $\sec^2\theta = 1 + \left(\frac{1}{3}\right)^2 = \frac{10}{9}$

and $\frac{dx}{dt} = 3\left(\frac{10}{9}\right)(8\pi) = \frac{80}{3}\pi \approx 83.8$ km/min.

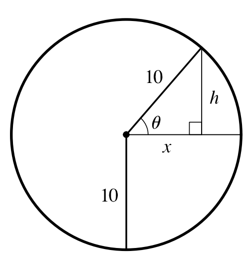
45. $\cot\theta = \frac{x}{5} \quad \Rightarrow \quad -\csc^2\theta\,\frac{d\theta}{dt} = \frac{1}{5}\frac{dx}{dt} \quad \Rightarrow \quad -\left(\csc\frac{\pi}{3}\right)^2\left(-\frac{\pi}{6}\right) = \frac{1}{5}\frac{dx}{dt} \quad \Rightarrow$

$\frac{dx}{dt} = \frac{5\pi}{6}\left(\frac{2}{\sqrt{3}}\right)^2 = \frac{10}{9}\pi$ km/min $[\approx 130$ mi/h$]$

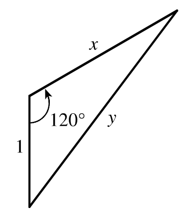


46. We are given that $\frac{d\theta}{dt} = \frac{2\pi \text{ rad}}{2 \text{ min}} = \pi$ rad/min. By the Pythagorean Theorem, when $h = 6$, $x = 8$, so $\sin\theta = \frac{6}{10}$ and $\cos\theta = \frac{8}{10}$. From the figure, $\sin\theta = \frac{h}{10} \quad \Rightarrow$

$h = 10\sin\theta$, so $\frac{dh}{dt} = 10\cos\theta\,\frac{d\theta}{dt} = 10\left(\frac{8}{10}\right)\pi = 8\pi$ m/min.



47. We are given that $\frac{dx}{dt} = 300$ km/h. By the Law of Cosines,

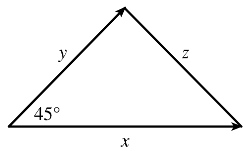


$y^2 = x^2 + 1^2 - 2(1)(x)\cos 120° = x^2 + 1 - 2x\left(-\frac{1}{2}\right) = x^2 + x + 1$, so

$2y\,\frac{dy}{dt} = 2x\,\frac{dx}{dt} + \frac{dx}{dt} \quad\Rightarrow\quad \frac{dy}{dt} = \frac{2x+1}{2y}\,\frac{dx}{dt}$. After 1 minute, $x = \frac{300}{60} = 5$ km $\Rightarrow$

$y = \sqrt{5^2 + 5 + 1} = \sqrt{31}$ km $\quad\Rightarrow\quad \frac{dy}{dt} = \frac{2(5)+1}{2\sqrt{31}}(300) = \frac{1650}{\sqrt{31}} \approx 296$ km/h.

48. We are given that $\frac{dx}{dt} = 3$ mi/h and $\frac{dy}{dt} = 2$ mi/h. By the Law of Cosines,



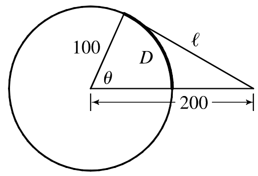
$z^2 = x^2 + y^2 - 2xy\cos 45° = x^2 + y^2 - \sqrt{2}\,xy \quad\Rightarrow$

$2z\frac{dz}{dt} = 2x\,\frac{dx}{dt} + 2y\,\frac{dy}{dt} - \sqrt{2}\,x\,\frac{dy}{dt} - \sqrt{2}\,y\,\frac{dx}{dt}$. After 15 minutes $\left[= \frac{1}{4}\text{ h}\right]$,

we have $x = \frac{3}{4}$ and $y = \frac{2}{4} = \frac{1}{2} \quad\Rightarrow\quad z^2 = \left(\frac{3}{4}\right)^2 + \left(\frac{2}{4}\right)^2 - \sqrt{2}\left(\frac{3}{4}\right)\left(\frac{2}{4}\right) \quad\Rightarrow\quad z = \frac{\sqrt{13 - 6\sqrt{2}}}{4}$ and

$$\frac{dz}{dt} = \frac{2}{\sqrt{13-6\sqrt{2}}}\left[2\left(\tfrac{3}{4}\right)3 + 2\left(\tfrac{1}{2}\right)2 - \sqrt{2}\left(\tfrac{3}{4}\right)2 - \sqrt{2}\left(\tfrac{1}{2}\right)3\right] = \frac{2}{\sqrt{13-6\sqrt{2}}}\,\frac{13-6\sqrt{2}}{2} = \sqrt{13-6\sqrt{2}} \approx 2.125 \text{ mi/h}.$$

49. Let the distance between the runner and the friend be ℓ. Then by the Law of Cosines,

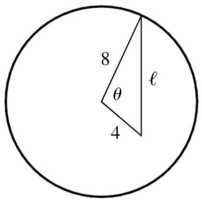


$\ell^2 = 200^2 + 100^2 - 2\cdot 200\cdot 100\cdot\cos\theta = 50{,}000 - 40{,}000\cos\theta$ $(\star)$. Differentiating implicitly with respect to t, we obtain $2\ell\,\frac{d\ell}{dt} = -40{,}000(-\sin\theta)\,\frac{d\theta}{dt}$. Now if D is the distance run when the angle is θ radians, then by the formula for the length of an arc on a circle, $s = r\theta$, we have $D = 100\theta$, so $\theta = \frac{1}{100}D \quad\Rightarrow\quad \frac{d\theta}{dt} = \frac{1}{100}\frac{dD}{dt} = \frac{7}{100}$. To substitute into the expression for $\frac{d\ell}{dt}$, we must know $\sin\theta$ at the time when $\ell = 200$, which we find from $(\star)$: $200^2 = 50{,}000 - 40{,}000\cos\theta \quad\Leftrightarrow$

$\cos\theta = \frac{1}{4} \quad\Rightarrow\quad \sin\theta = \sqrt{1 - \left(\frac{1}{4}\right)^2} = \frac{\sqrt{15}}{4}$. Substituting, we get $2(200)\,\frac{d\ell}{dt} = 40{,}000\frac{\sqrt{15}}{4}\left(\frac{7}{100}\right) \quad\Rightarrow$

$d\ell/dt = \frac{7\sqrt{15}}{4} \approx 6.78$ m/s. Whether the distance between them is increasing or decreasing depends on the direction in which the runner is running.

50. The hour hand of a clock goes around once every 12 hours or, in radians per hour, $\frac{2\pi}{12} = \frac{\pi}{6}$ rad/h. The minute hand goes around once an hour, or at the rate of 2π rad/h. So the angle θ between them (measuring clockwise from the minute hand to the hour hand) is changing at the rate of $d\theta/dt = \frac{\pi}{6} - 2\pi = -\frac{11\pi}{6}$ rad/h. Now, to relate θ to ℓ, we use the Law of Cosines: $\ell^2 = 4^2 + 8^2 - 2\cdot 4\cdot 8\cdot\cos\theta = 80 - 64\cos\theta$ $(\star)$.



Differentiating implicitly with respect to t, we get $2\ell\,\frac{d\ell}{dt} = -64(-\sin\theta)\frac{d\theta}{dt}$. At 1:00, the angle between the two hands is one-twelfth of the circle, that is, $\frac{2\pi}{12} = \frac{\pi}{6}$ radians. We use ($\star$) to find ℓ at 1:00: $\ell = \sqrt{80 - 64\cos\frac{\pi}{6}} = \sqrt{80 - 32\sqrt{3}}$.

Substituting, we get $2\ell\,\dfrac{d\ell}{dt} = 64\sin\dfrac{\pi}{6}\left(-\dfrac{11\pi}{6}\right) \;\Rightarrow\; \dfrac{d\ell}{dt} = \dfrac{64\left(\frac{1}{2}\right)\left(-\frac{11\pi}{6}\right)}{2\sqrt{80 - 32\sqrt{3}}} = -\dfrac{88\pi}{3\sqrt{80 - 32\sqrt{3}}} \approx -18.6$.

So at 1:00, the distance between the tips of the hands is decreasing at a rate of 18.6 mm/h ≈ 0.005 mm/s.

3.10 Linear Approximations and Differentials

1. $f(x) = x^3 - x^2 + 3 \;\Rightarrow\; f'(x) = 3x^2 - 2x$, so $f(-2) = -9$ and $f'(-2) = 16$. Thus,
$L(x) = f(-2) + f'(-2)(x - (-2)) = -9 + 16(x + 2) = 16x + 23$.

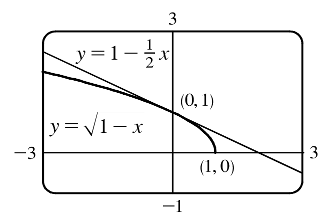
2. $f(x) = \sin x \;\Rightarrow\; f'(x) = \cos x$, so $f\left(\frac{\pi}{6}\right) = \frac{1}{2}$ and $f'\left(\frac{\pi}{6}\right) = \frac{1}{2}\sqrt{3}$. Thus,
$L(x) = f\left(\frac{\pi}{6}\right) + f'\left(\frac{\pi}{6}\right)\left(x - \frac{\pi}{6}\right) = \frac{1}{2} + \frac{1}{2}\sqrt{3}\left(x - \frac{\pi}{6}\right) = \frac{1}{2}\sqrt{3}\,x + \frac{1}{2} - \frac{1}{12}\sqrt{3}\,\pi$.

3. $f(x) = \sqrt{x} \;\Rightarrow\; f'(x) = \frac{1}{2}x^{-1/2} = 1/(2\sqrt{x})$, so $f(4) = 2$ and $f'(4) = \frac{1}{4}$. Thus,
$L(x) = f(4) + f'(4)(x - 4) = 2 + \frac{1}{4}(x - 4) = 2 + \frac{1}{4}x - 1 = \frac{1}{4}x + 1$.

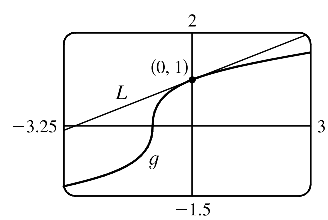
4. $f(x) = 2^x \;\Rightarrow\; f'(x) = 2^x \ln 2$, so $f(0) = 1$ and $f'(0) = \ln 2$. Thus, $L(x) = f(0) + f'(0)(x - 0) = 1 + (\ln 2)x$.

5. $f(x) = \sqrt{1 - x} \;\Rightarrow\; f'(x) = \dfrac{-1}{2\sqrt{1 - x}}$, so $f(0) = 1$ and $f'(0) = -\frac{1}{2}$.

Therefore,
$\sqrt{1 - x} = f(x) \approx f(0) + f'(0)(x - 0) = 1 + \left(-\frac{1}{2}\right)(x - 0) = 1 - \frac{1}{2}x$.
So $\sqrt{0.9} = \sqrt{1 - 0.1} \approx 1 - \frac{1}{2}(0.1) = 0.95$
and $\sqrt{0.99} = \sqrt{1 - 0.01} \approx 1 - \frac{1}{2}(0.01) = 0.995$.

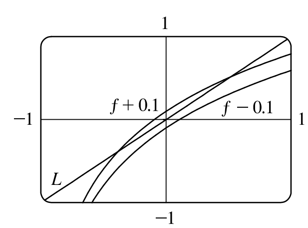


6. $g(x) = \sqrt[3]{1 + x} = (1 + x)^{1/3} \;\Rightarrow\; g'(x) = \frac{1}{3}(1 + x)^{-2/3}$, so $g(0) = 1$ and $g'(0) = \frac{1}{3}$. Therefore, $\sqrt[3]{1 + x} = g(x) \approx g(0) + g'(0)(x - 0) = 1 + \frac{1}{3}x$.
So $\sqrt[3]{0.95} = \sqrt[3]{1 + (-0.05)} \approx 1 + \frac{1}{3}(-0.05) = 0.98\overline{3}$,
and $\sqrt[3]{1.1} = \sqrt[3]{1 + 0.1} \approx 1 + \frac{1}{3}(0.1) = 1.0\overline{3}$.

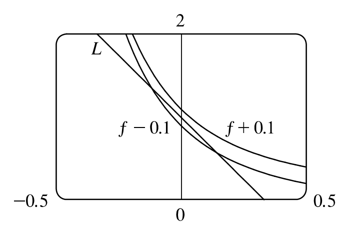


7. $f(x) = \ln(1 + x) \;\Rightarrow\; f'(x) = \dfrac{1}{1 + x}$, so $f(0) = 0$ and $f'(0) = 1$.

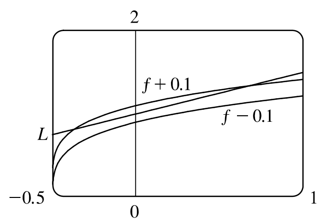
Thus, $f(x) \approx f(0) + f'(0)(x - 0) = 0 + 1(x) = x$. We need $\ln(1 + x) - 0.1 < x < \ln(1 + x) + 0.1$, which is true when $-0.383 < x < 0.516$.



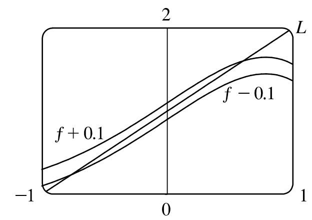
8. $f(x) = (1 + x)^{-3} \;\Rightarrow\; f'(x) = -3(1 + x)^{-4}$, so $f(0) = 1$ and $f'(0) = -3$. Thus, $f(x) \approx f(0) + f'(0)(x - 0) = 1 - 3x$. We need $(1 + x)^{-3} - 0.1 < 1 - 3x < (1 + x)^{-3} + 0.1$, which is true when $-0.116 < x < 0.144$.



9. $f(x) = \sqrt[4]{1+2x} \quad \Rightarrow \quad f'(x) = \frac{1}{4}(1+2x)^{-3/4}(2) = \frac{1}{2}(1+2x)^{-3/4}$, so $f(0) = 1$ and $f'(0) = \frac{1}{2}$. Thus, $f(x) \approx f(0) + f'(0)(x-0) = 1 + \frac{1}{2}x$. We need $\sqrt[4]{1+2x} - 0.1 < 1 + \frac{1}{2}x < \sqrt[4]{1+2x} + 0.1$, which is true when $-0.368 < x < 0.677$.



10. $f(x) = e^x \cos x \quad \Rightarrow \quad f'(x) = e^x(-\sin x) + (\cos x)e^x = e^x(\cos x - \sin x)$, so $f(0) = 1$ and $f'(0) = 1$. Thus, $f(x) \approx f(0) + f'(0)(x-0) = 1 + x$. We need $e^x \cos x - 0.1 < 1 + x < e^x \cos x + 0.1$, which is true when $-0.762 < x < 0.607$.



11. (a) The differential dy is defined in terms of dx by the equation $dy = f'(x)\,dx$. For $y = f(x) = xe^{-4x}$, $f'(x) = xe^{-4x}(-4) + e^{-4x} \cdot 1 = e^{-4x}(-4x+1)$, so $dy = (1-4x)e^{-4x}dx$.

(b) For $y = f(t) = \sqrt{1-t^4}$, $f'(t) = \frac{1}{2}(1-t^4)^{-1/2}(-4t^3) = -\dfrac{2t^3}{\sqrt{1-t^4}}$, so $dy = -\dfrac{2t^3}{\sqrt{1-t^4}}\,dt$.

12. (a) For $y = f(u) = \dfrac{1+2u}{1+3u}$, $f'(u) = \dfrac{(1+3u)(2) - (1+2u)(3)}{(1+3u)^2} = \dfrac{-1}{(1+3u)^2}$, so $dy = \dfrac{-1}{(1+3u)^2}\,du$.

(b) For $y = f(\theta) = \theta^2 \sin 2\theta$, $f'(\theta) = \theta^2(\cos 2\theta)(2) + (\sin 2\theta)(2\theta)$, so $dy = 2\theta(\theta\cos 2\theta + \sin 2\theta)\,d\theta$.

13. (a) For $y = f(t) = \tan\sqrt{t}$, $f'(t) = \sec^2\sqrt{t} \cdot \dfrac{1}{2}t^{-1/2} = \dfrac{\sec^2\sqrt{t}}{2\sqrt{t}}$, so $dy = \dfrac{\sec^2\sqrt{t}}{2\sqrt{t}}\,dt$.

(b) For $y = f(v) = \dfrac{1-v^2}{1+v^2}$,

$$f'(v) = \frac{(1+v^2)(-2v) - (1-v^2)(2v)}{(1+v^2)^2} = \frac{-2v[(1+v^2) + (1-v^2)]}{(1+v^2)^2} = \frac{-2v(2)}{(1+v^2)^2} = \frac{-4v}{(1+v^2)^2},$$

so $dy = \dfrac{-4v}{(1+v^2)^2}\,dv$.

14. (a) For $y = f(\theta) = \ln(\sin\theta)$, $f'(\theta) = \dfrac{1}{\sin\theta}\cos\theta = \cot\theta$, so $dy = \cot\theta\,d\theta$.

(b) For $y = f(x) = \dfrac{e^x}{1-e^x}$, $f'(x) = \dfrac{(1-e^x)e^x - e^x(-e^x)}{(1-e^x)^2} = \dfrac{e^x[(1-e^x) - (-e^x)]}{(1-e^x)^2} = \dfrac{e^x}{(1-e^x)^2}$, so

$dy = \dfrac{e^x}{(1-e^x)^2}\,dx$.

15. (a) $y = e^{x/10} \quad \Rightarrow \quad dy = e^{x/10} \cdot \frac{1}{10}\,dx = \frac{1}{10}e^{x/10}dx$

(b) $x = 0$ and $dx = 0.1 \quad \Rightarrow \quad dy = \frac{1}{10}e^{0/10}(0.1) = 0.01$.

16. (a) $y = \cos\pi x \quad \Rightarrow \quad dy = -\sin\pi x \cdot \pi\,dx = -\pi\,\sin\pi x\,dx$

(b) $x = \frac{1}{3}$ and $dx = -0.02 \quad \Rightarrow \quad dy = -\pi\,\sin\frac{\pi}{3}(-0.02) = \pi\left(\sqrt{3}/2\right)(0.02) = 0.01\pi\sqrt{3} \approx 0.054$.

17. (a) $y = \sqrt{3 + x^2} \quad \Rightarrow \quad dy = \dfrac{1}{2}(3 + x^2)^{-1/2}(2x)\,dx = \dfrac{x}{\sqrt{3 + x^2}}\,dx$

(b) $x = 1$ and $dx = -0.1 \quad \Rightarrow \quad dy = \dfrac{1}{\sqrt{3 + 1^2}}(-0.1) = \dfrac{1}{2}(-0.1) = -0.05.$

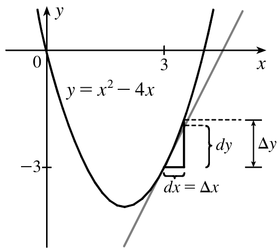
18. (a) $y = \dfrac{x + 1}{x - 1} \quad \Rightarrow \quad dy = \dfrac{(x - 1)(1) - (x + 1)(1)}{(x - 1)^2}\,dx = \dfrac{-2}{(x - 1)^2}\,dx$

(b) $x = 2$ and $dx = 0.05 \quad \Rightarrow \quad dy = \dfrac{-2}{(2 - 1)^2}(0.05) = -2(0.05) = -0.1.$

19. $y = f(x) = x^2 - 4x,\ x = 3,\ \Delta x = 0.5 \quad \Rightarrow$

$\Delta y = f(3.5) - f(3) = -1.75 - (-3) = 1.25$

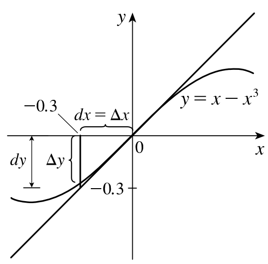
$dy = f'(x)\,dx = (2x - 4)\,dx = (6 - 4)(0.5) = 1$



20. $y = f(x) = x - x^3,\ x = 0,\ \Delta x = -0.3 \quad \Rightarrow$

$\Delta y = f(-0.3) - f(0) = -0.273 - 0 = -0.273$

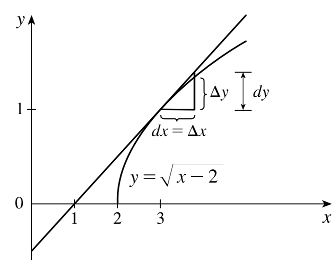
$dy = f'(x)\,dx = (1 - 3x^2)\,dx = (1 - 0)(-0.3) = -0.3$



21. $y = f(x) = \sqrt{x - 2},\ x = 3,\ \Delta x = 0.8 \quad \Rightarrow$

$\Delta y = f(3.8) - f(3) = \sqrt{1.8} - 1 \approx 0.34$

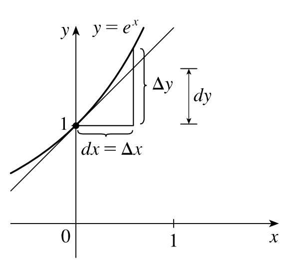
$dy = f'(x)\,dx = \dfrac{1}{2\sqrt{x - 2}}\,dx = \dfrac{1}{2(1)}(0.8) = 0.4$



22. $y = f(x) = e^x,\ x = 0,\ \Delta x = 0.5 \quad \Rightarrow$

$\Delta y = f(0.5) - f(0) = \sqrt{e} - 1\ \ [\approx 0.65]$

$dy = e^x\,dx = e^0(0.5) = 0.5$



23. To estimate $(1.999)^4$, we'll find the linearization of $f(x) = x^4$ at $a = 2$. Since $f'(x) = 4x^3$, $f(2) = 16$, and $f'(2) = 32$, we have $L(x) = 16 + 32(x - 2)$. Thus, $x^4 \approx 16 + 32(x - 2)$ when x is near 2, so $(1.999)^4 \approx 16 + 32(1.999 - 2) = 16 - 0.032 = 15.968.$

24. $y = f(x) = 1/x \quad \Rightarrow \quad dy = -1/x^2\,dx$. When $x = 4$ and $dx = 0.002$, $dy = -\frac{1}{16}(0.002) = -\frac{1}{8000}$, so $\frac{1}{4.002} \approx f(4) + dy = \frac{1}{4} - \frac{1}{8000} = \frac{1999}{8000} = 0.249875$.

25. $y = f(x) = \sqrt[3]{x} \quad \Rightarrow \quad dy = \frac{1}{3}x^{-2/3}\,dx$. When $x = 1000$ and $dx = 1$, $dy = \frac{1}{3}(1000)^{-2/3}(1) = \frac{1}{300}$, so $\sqrt[3]{1001} = f(1001) \approx f(1000) + dy = 10 + \frac{1}{300} = 10.00\overline{3} \approx 10.003$.

26. $y = f(x) = \sqrt{x} \quad \Rightarrow \quad dy = \frac{1}{2}x^{-1/2}\,dx$. When $x = 100$ and $dx = 0.5$, $dy = \frac{1}{2}(100)^{-1/2}\left(\frac{1}{2}\right) = \frac{1}{40}$, so $\sqrt{100.5} = f(100.5) \approx f(100) + dy = 10 + \frac{1}{40} = 10.025$.

27. $y = f(x) = e^x \quad \Rightarrow \quad dy = e^x\,dx$. When $x = 0$ and $dx = 0.1$, $dy = e^0(0.1) = 0.1$, so $e^{0.1} = f(0.1) \approx f(0) + dy = 1 + 0.1 = 1.1$.

28. $y = f(x) = \cos x \quad \Rightarrow \quad dy = -\sin x\,dx$. When $x = 30^\circ$ $[\pi/6]$ and $dx = -1^\circ$ $[-\pi/180]$, $dy = \left(-\sin\frac{\pi}{6}\right)\left(-\frac{\pi}{180}\right) = -\frac{1}{2}\left(-\frac{\pi}{180}\right) = \frac{\pi}{360}$, so $\cos 29^\circ = f(29^\circ) \approx f(30^\circ) + dy = \frac{1}{2}\sqrt{3} + \frac{\pi}{360} \approx 0.875$.

29. $y = f(x) = \sec x \quad \Rightarrow \quad f'(x) = \sec x\,\tan x$, so $f(0) = 1$ and $f'(0) = 1 \cdot 0 = 0$. The linear approximation of f at 0 is $f(0) + f'(0)(x - 0) = 1 + 0(x) = 1$. Since 0.08 is close to 0, approximating $\sec 0.08$ with 1 is reasonable.

30. $y = f(x) = \sqrt{x} \quad \Rightarrow \quad f'(x) = 1/(2\sqrt{x}\,)$, so $f(4) = 2$ and $f'(4) = \frac{1}{4}$. The linear approximation of f at 4 is $f(4) + f'(4)(x - 4) = 2 + \frac{1}{4}(x - 4)$. Now $f(4.02) = \sqrt{4.02} \approx 2 + \frac{1}{4}(0.02) = 2 + 0.005 = 2.005$, so the approximation is reasonable.

31. $y = f(x) = 1/x \quad \Rightarrow \quad f'(x) = -1/x^2$, so $f(10) = 0.1$ and $f'(10) = -0.01$. The linear approximation of f at 10 is $f(10) + f'(10)(x - 10) = 0.1 - 0.01(x - 10)$. Now $f(9.98) = 1/9.98 \approx 0.1 - 0.01(-0.02) = 0.1 + 0.0002 = 0.1002$, so the approximation is reasonable.

32. (a) $f(x) = (x - 1)^2 \quad \Rightarrow \quad f'(x) = 2(x - 1)$, so $f(0) = 1$ and $f'(0) = -2$.

Thus, $f(x) \approx L_f(x) = f(0) + f'(0)(x - 0) = 1 - 2x$.

$g(x) = e^{-2x} \quad \Rightarrow \quad g'(x) = -2e^{-2x}$, so $g(0) = 1$ and $g'(0) = -2$.

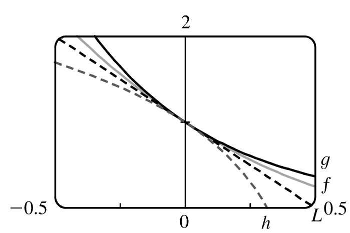
Thus, $g(x) \approx L_g(x) = g(0) + g'(0)(x - 0) = 1 - 2x$.

$h(x) = 1 + \ln(1 - 2x) \quad \Rightarrow \quad h'(x) = \dfrac{-2}{1 - 2x}$, so $h(0) = 1$ and $h'(0) = -2$.

Thus, $h(x) \approx L_h(x) = h(0) + h'(0)(x - 0) = 1 - 2x$.

Notice that $L_f = L_g = L_h$. This happens because f, g, and h have the same function values and the same derivative values at $a = 0$.

(b) The linear approximation appears to be the best for the function f since it is closer to f for a larger domain than it is to g and h. The approximation looks worst for h since h moves away from L faster than f and g do.



33. (a) If x is the edge length, then $V = x^3 \Rightarrow dV = 3x^2\,dx$. When $x = 30$ and $dx = 0.1$, $dV = 3(30)^2(0.1) = 270$, so the maximum possible error in computing the volume of the cube is about 270 cm^3. The relative error is calculated by dividing the change in V, ΔV, by V. We approximate ΔV with dV.

Relative error $= \dfrac{\Delta V}{V} \approx \dfrac{dV}{V} = \dfrac{3x^2\,dx}{x^3} = 3\,\dfrac{dx}{x} = 3\left(\dfrac{0.1}{30}\right) = 0.01$.

Percentage error $=$ relative error $\times 100\% = 0.01 \times 100\% = 1\%$.

(b) $S = 6x^2 \Rightarrow dS = 12x\,dx$. When $x = 30$ and $dx = 0.1$, $dS = 12(30)(0.1) = 36$, so the maximum possible error in computing the surface area of the cube is about 36 cm^2.

Relative error $= \dfrac{\Delta S}{S} \approx \dfrac{dS}{S} = \dfrac{12x\,dx}{6x^2} = 2\,\dfrac{dx}{x} = 2\left(\dfrac{0.1}{30}\right) = 0.00\overline{6}$.

Percentage error $=$ relative error $\times 100\% = 0.00\overline{6} \times 100\% = 0.\overline{6}\%$.

34. (a) $A = \pi r^2 \Rightarrow dA = 2\pi r\,dr$. When $r = 24$ and $dr = 0.2$, $dA = 2\pi(24)(0.2) = 9.6\pi$, so the maximum possible error in the calculated area of the disk is about $9.6\pi \approx 30$ cm^2.

(b) Relative error $= \dfrac{\Delta A}{A} \approx \dfrac{dA}{A} = \dfrac{2\pi r\,dr}{\pi r^2} = \dfrac{2\,dr}{r} = \dfrac{2(0.2)}{24} = \dfrac{0.2}{12} = \dfrac{1}{60} = 0.01\overline{6}$.

Percentage error $=$ relative error $\times 100\% = 0.01\overline{6} \times 100\% = 1.\overline{6}\%$.

35. (a) For a sphere of radius r, the circumference is $C = 2\pi r$ and the surface area is $S = 4\pi r^2$, so

$r = \dfrac{C}{2\pi} \Rightarrow S = 4\pi\left(\dfrac{C}{2\pi}\right)^2 = \dfrac{C^2}{\pi} \Rightarrow dS = \dfrac{2}{\pi}C\,dC$. When $C = 84$ and $dC = 0.5$, $dS = \dfrac{2}{\pi}(84)(0.5) = \dfrac{84}{\pi}$,

so the maximum error is about $\dfrac{84}{\pi} \approx 27$ cm^2. Relative error $\approx \dfrac{dS}{S} = \dfrac{84/\pi}{84^2/\pi} = \dfrac{1}{84} \approx 0.012 = 1.2\%$

(b) $V = \dfrac{4}{3}\pi r^3 = \dfrac{4}{3}\pi\left(\dfrac{C}{2\pi}\right)^3 = \dfrac{C^3}{6\pi^2} \Rightarrow dV = \dfrac{1}{2\pi^2}C^2\,dC$. When $C = 84$ and $dC = 0.5$,

$dV = \dfrac{1}{2\pi^2}(84)^2(0.5) = \dfrac{1764}{\pi^2}$, so the maximum error is about $\dfrac{1764}{\pi^2} \approx 179$ cm^3.

The relative error is approximately $\dfrac{dV}{V} = \dfrac{1764/\pi^2}{(84)^3/(6\pi^2)} = \dfrac{1}{56} \approx 0.018 = 1.8\%$.

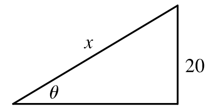
36. For a hemispherical dome, $V = \frac{2}{3}\pi r^3 \Rightarrow dV = 2\pi r^2\,dr$. When $r = \frac{1}{2}(50) = 25$ m and $dr = 0.05$ cm $= 0.0005$ m, $dV = 2\pi(25)^2(0.0005) = \frac{5\pi}{8}$, so the amount of paint needed is about $\frac{5\pi}{8} \approx 2$ m^3.

37. (a) $V = \pi r^2 h \Rightarrow \Delta V \approx dV = 2\pi rh\,dr = 2\pi rh\,\Delta r$

(b) The error is

$\Delta V - dV = [\pi(r+\Delta r)^2 h - \pi r^2 h] - 2\pi rh\,\Delta r = \pi r^2 h + 2\pi rh\,\Delta r + \pi(\Delta r)^2 h - \pi r^2 h - 2\pi rh\,\Delta r = \pi(\Delta r)^2 h$.

38. (a) $\sin\theta = \dfrac{20}{x} \quad\Rightarrow\quad x = 20\csc\theta \quad\Rightarrow$



$$dx = 20(-\csc\theta\,\cot\theta)\,d\theta = -20\csc 30^\circ\ \cot 30^\circ\ (\pm 1^\circ)$$

$$= -20(2)\left(\sqrt{3}\right)\left(\pm\frac{\pi}{180}\right) = \pm\frac{2\sqrt{3}}{9}\pi$$

So the maximum error is about $\pm\frac{2}{9}\sqrt{3}\,\pi \approx \pm 1.21$ cm.

(b) The relative error is $\dfrac{\Delta x}{x} \approx \dfrac{dx}{x} = \dfrac{\pm\frac{2}{9}\sqrt{3}\,\pi}{20(2)} = \pm\dfrac{\sqrt{3}}{180}\pi \approx \pm 0.03$, so the percentage error is approximately $\pm 3\%$.

39. $V = RI \quad\Rightarrow\quad I = \dfrac{V}{R} \quad\Rightarrow\quad dI = -\dfrac{V}{R^2}\,dR$. The relative error in calculating I is $\dfrac{\Delta I}{I} \approx \dfrac{dI}{I} = \dfrac{-(V/R^2)\,dR}{V/R} = -\dfrac{dR}{R}$.

Hence, the relative error in calculating I is approximately the same (in magnitude) as the relative error in R.

40. $F = kR^4 \quad\Rightarrow\quad dF = 4kR^3\,dR \quad\Rightarrow\quad \dfrac{dF}{F} = \dfrac{4kR^3\,dR}{kR^4} = 4\left(\dfrac{dR}{R}\right)$. Thus, the relative change in F is about 4 times the relative change in R. So a 5% increase in the radius corresponds to a 20% increase in blood flow.

41. (a) $dc = \dfrac{dc}{dx}\,dx = 0\,dx = 0$

(b) $d(cu) = \dfrac{d}{dx}(cu)\,dx = c\,\dfrac{du}{dx}\,dx = c\,du$

(c) $d(u+v) = \dfrac{d}{dx}(u+v)\,dx = \left(\dfrac{du}{dx} + \dfrac{dv}{dx}\right)dx = \dfrac{du}{dx}\,dx + \dfrac{dv}{dx}\,dx = du + dv$

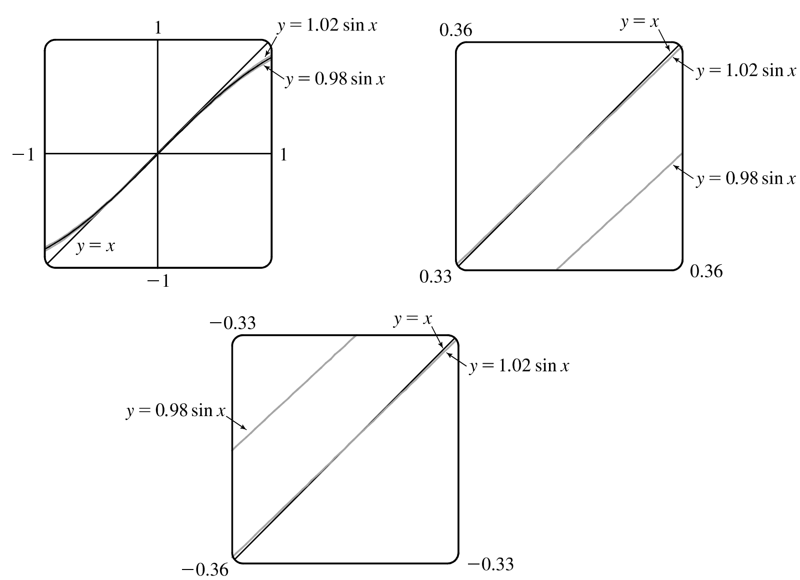
(d) $d(uv) = \dfrac{d}{dx}(uv)\,dx = \left(u\,\dfrac{dv}{dx} + v\,\dfrac{du}{dx}\right)dx = u\,\dfrac{dv}{dx}\,dx + v\,\dfrac{du}{dx}\,dx = u\,dv + v\,du$

(e) $d\left(\dfrac{u}{v}\right) = \dfrac{d}{dx}\left(\dfrac{u}{v}\right)\,dx = \dfrac{v\,\dfrac{du}{dx} - u\,\dfrac{dv}{dx}}{v^2}\,dx = \dfrac{v\,\dfrac{du}{dx}\,dx - u\,\dfrac{dv}{dx}\,dx}{v^2} = \dfrac{v\,du - u\,dv}{v^2}$

(f) $d\,(x^n) = \dfrac{d}{dx}\,(x^n)\,dx = nx^{n-1}\,dx$

42. (a) $f(x) = \sin x \quad\Rightarrow\quad f'(x) = \cos x$, so $f(0) = 0$ and $f'(0) = 1$. Thus, $f(x) \approx f(0) + f'(0)(x-0) = 0 + 1(x-0) = x$.

(b)



[continued]

We want to know the values of x for which $y = x$ approximates $y = \sin x$ with less than a 2% difference; that is, the values of x for which

$$\left|\frac{x - \sin x}{\sin x}\right| < 0.02 \quad \Leftrightarrow \quad -0.02 < \frac{x - \sin x}{\sin x} < 0.02 \quad \Leftrightarrow$$

$$\begin{cases} -0.02 \sin x < x - \sin x < 0.02 \sin x & \text{if } \sin x > 0 \\ -0.02 \sin x > x - \sin x > 0.02 \sin x & \text{if } \sin x < 0 \end{cases} \quad \Leftrightarrow \quad \begin{cases} 0.98 \sin x < x < 1.02 \sin x & \text{if } \sin x > 0 \\ 1.02 \sin x < x < 0.98 \sin x & \text{if } \sin x < 0 \end{cases}$$

In the first figure, we see that the graphs are very close to each other near $x = 0$. Changing the viewing rectangle and using an intersect feature (see the second figure) we find that $y = x$ intersects $y = 1.02 \sin x$ at $x \approx 0.344$. By symmetry, they also intersect at $x \approx -0.344$ (see the third figure). Converting 0.344 radians to degrees, we get $0.344\left(\frac{180^\circ}{\pi}\right) \approx 19.7^\circ \approx 20^\circ$, which verifies the statement.

43. (a) The graph shows that $f'(1) = 2$, so $L(x) = f(1) + f'(1)(x - 1) = 5 + 2(x - 1) = 2x + 3$.

$f(0.9) \approx L(0.9) = 4.8$ and $f(1.1) \approx L(1.1) = 5.2$.

(b) From the graph, we see that $f'(x)$ is positive and decreasing. This means that the slopes of the tangent lines are positive, but the tangents are becoming less steep. So the tangent lines lie *above* the curve. Thus, the estimates in part (a) are too large.

44. (a) $g'(x) = \sqrt{x^2 + 5} \quad \Rightarrow \quad g'(2) = \sqrt{9} = 3$. $g(1.95) \approx g(2) + g'(2)(1.95 - 2) = -4 + 3(-0.05) = -4.15$.

$g(2.05) \approx g(2) + g'(2)(2.05 - 2) = -4 + 3(0.05) = -3.85$.

(b) The formula $g'(x) = \sqrt{x^2 + 5}$ shows that $g'(x)$ is positive and increasing. This means that the slopes of the tangent lines are positive and the tangents are getting steeper. So the tangent lines lie *below* the graph of g. Hence, the estimates in part (a) are too small.

LABORATORY PROJECT Taylor Polynomials

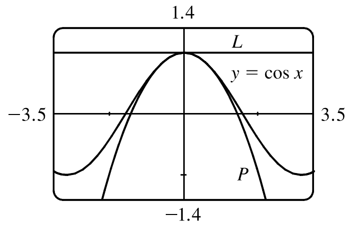
1. We first write the functions described in conditions (i), (ii), and (iii):

$$\begin{aligned} P(x) &= A + Bx + Cx^2 & \qquad f(x) &= \cos x \\ P'(x) &= B + 2Cx & f'(x) &= -\sin x \\ P''(x) &= 2C & f''(x) &= -\cos x \end{aligned}$$

So, taking $a = 0$, our three conditions become

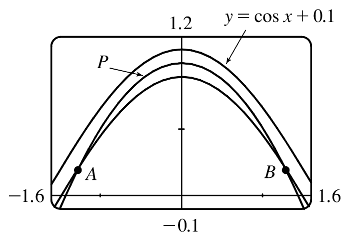
$$\begin{aligned} P(0) &= f(0): & A &= \cos 0 = 1 \\ P'(0) &= f'(0): & B &= -\sin 0 = 0 \\ P''(0) &= f''(0): & 2C &= -\cos 0 = -1 \quad \Rightarrow \quad C = -\tfrac{1}{2} \end{aligned}$$

The desired quadratic function is $P(x) = 1 - \frac{1}{2}x^2$, so the quadratic approximation is $\cos x \approx 1 - \frac{1}{2}x^2$.



The figure shows a graph of the cosine function together with its linear approximation $L(x) = 1$ and quadratic approximation $P(x) = 1 - \frac{1}{2}x^2$ near 0. You can see that the quadratic approximation is much better than the linear one.

2. Accuracy to within 0.1 means that $\left|\cos x - \left(1 - \frac{1}{2}x^2\right)\right| < 0.1 \quad \Leftrightarrow \quad -0.1 < \cos x - \left(1 - \frac{1}{2}x^2\right) < 0.1 \quad \Leftrightarrow$
$0.1 > \left(1 - \frac{1}{2}x^2\right) - \cos x > -0.1 \quad \Leftrightarrow \quad \cos x + 0.1 > 1 - \frac{1}{2}x^2 > \cos x - 0.1 \quad \Leftrightarrow \quad \cos x - 0.1 < 1 - \frac{1}{2}x^2 < \cos x + 0.1.$



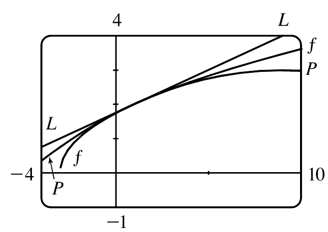
From the figure we see that this is true between A and B. Zooming in or using an intersect feature, we find that the x-coordinates of B and A are about ± 1.26. Thus, the approximation $\cos x \approx 1 - \frac{1}{2}x^2$ is accurate to within 0.1 when $-1.26 < x < 1.26$.

3. If $P(x) = A + B(x-a) + C(x-a)^2$, then $P'(x) = B + 2C(x-a)$ and $P''(x) = 2C$. Applying the conditions (i), (ii), and (iii), we get

$$\begin{aligned} P(a) &= f(a): &\quad A &= f(a) \\ P'(a) &= f'(a): &\quad B &= f'(a) \\ P''(a) &= f''(a): &\quad 2C &= f''(a) \quad \Rightarrow \quad C = \tfrac{1}{2}f''(a) \end{aligned}$$

Thus, $P(x) = A + B(x-a) + C(x-a)^2$ can be written in the form $P(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2$.

4. From Example 3.10.1, we have $f(1) = 2$, $f'(1) = \frac{1}{4}$, and $f'(x) = \frac{1}{2}(x+3)^{-1/2}$.
So $f''(x) = -\frac{1}{4}(x+3)^{-3/2} \quad \Rightarrow \quad f''(1) = -\frac{1}{32}$.
From Problem 3, the quadratic approximation $P(x)$ is
$\sqrt{x+3} \approx f(1) + f'(1)(x-1) + \frac{1}{2}f''(1)(x-1)^2 = 2 + \frac{1}{4}(x-1) - \frac{1}{64}(x-1)^2$.



The figure shows the function $f(x) = \sqrt{x+3}$ together with its linear approximation $L(x) = \frac{1}{4}x + \frac{7}{4}$ and its quadratic approximation $P(x)$. You can see that $P(x)$ is a better approximation than $L(x)$ and this is borne out by the numerical values in the following chart.

	from $L(x)$	actual value	from $P(x)$
$\sqrt{3.98}$	1.9950	1.99499373...	1.99499375
$\sqrt{4.05}$	2.0125	2.01246118...	2.01246094
$\sqrt{4.2}$	2.0500	2.04939015...	2.04937500

5. $T_n(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots + c_n(x-a)^n$. If we put $x = a$ in this equation, then all terms after the first are 0 and we get $T_n(a) = c_0$. Now we differentiate $T_n(x)$ and obtain
$T_n'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots + nc_n(x-a)^{n-1}$. Substituting $x = a$ gives $T_n'(a) = c_1$.
Differentiating again, we have $T_n''(x) = 2c_2 + 2 \cdot 3c_3(x-a) + 3 \cdot 4c_4(x-a^2) + \cdots + (n-1)nc_n(x-a)^{n-2}$ and so

$T_n''(a) = 2c_2$. Continuing in this manner, we get $T_n'''(x) = 2 \cdot 3c_3 + 2 \cdot 3 \cdot 4c_4(x-a) + \cdots + (n-2)(n-1)nc_n(x-a)^{n-3}$ and $T_n'''(a) = 2 \cdot 3c_3$. By now we see the pattern. If we continue to differentiate and substitute $x = a$, we obtain $T_n^{(4)}(a) = 2 \cdot 3 \cdot 4c_4$ and in general, for any integer k between 1 and n, $T_n^{(k)}(a) = 2 \cdot 3 \cdot 4 \cdot 5 \cdot \cdots \cdot kc_k = k!\,c_k \quad \Rightarrow$ $c_k = \dfrac{T_n^{(k)}(a)}{k!}$. Because we want T_n and f to have the same derivatives at a, we require that $c_k = \dfrac{f^{(k)}(a)}{k!}$ for $k = 1, 2, \ldots, n$.

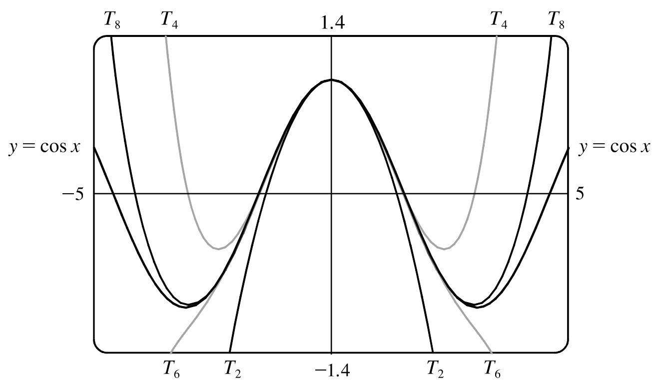
6. $T_n(x) = f(a) + f'(a)(x-a) + \dfrac{f''(a)}{2!}(x-a)^2 + \cdots + \dfrac{f^{(n)}(a)}{n!}(x-a)^n$. To compute the coefficients in this equation we need to calculate the derivatives of f at 0:

$$\begin{aligned} f(x) &= \cos x & f(0) &= \cos 0 = 1 \\ f'(x) &= -\sin x & f'(0) &= -\sin 0 = 0 \\ f''(x) &= -\cos x & f''(0) &= -1 \\ f'''(x) &= \sin x & f'''(0) &= 0 \\ f^{(4)}(x) &= \cos x & f^{(4)}(0) &= 1 \end{aligned}$$

We see that the derivatives repeat in a cycle of length 4, so $f^{(5)}(0) = 0$, $f^{(6)}(0) = -1$, $f^{(7)}(0) = 0$, and $f^{(8)}(0) = 1$. From the original expression for $T_n(x)$, with $n = 8$ and $a = 0$, we have

$$\begin{aligned} T_8(x) &= f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 + \cdots + \frac{f^{(8)}(0)}{8!}(x-0)^8 \\ &= 1 + 0 \cdot x + \frac{-1}{2!}x^2 + 0 \cdot x^3 + \frac{1}{4!}x^4 + 0 \cdot x^5 + \frac{-1}{6!}x^6 + 0 \cdot x^7 + \frac{1}{8!}x^8 = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} \end{aligned}$$

and the desired approximation is $\cos x \approx 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \dfrac{x^8}{8!}$. The Taylor polynomials T_2, T_4, and T_6 consist of the initial terms of T_8 up through degree 2, 4, and 6, respectively. Therefore, $T_2(x) = 1 - \dfrac{x^2}{2!}$, $T_4(x) = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!}$, and $T_6(x) = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!}$. We graph T_2, T_4, T_6, T_8, and f:



Notice that $T_2(x)$ is a good approximation to $\cos x$ near 0, $T_4(x)$ is a good approximation on a larger interval, $T_6(x)$ is a better approximation, and $T_8(x)$ is better still. Each successive Taylor polynomial is a good approximation on a larger interval than the previous one.

3.11 Hyperbolic Functions

1. (a) $\sinh 0 = \frac{1}{2}(e^0 - e^{-0}) = 0$ (b) $\cosh 0 = \frac{1}{2}(e^0 + e^{-0}) = \frac{1}{2}(1+1) = 1$

2. (a) $\tanh 0 = \dfrac{(e^0 - e^{-0})/2}{(e^0 + e^{-0})/2} = 0$ (b) $\tanh 1 = \dfrac{e^1 - e^{-1}}{e^1 + e^{-1}} = \dfrac{e^2 - 1}{e^2 + 1} \approx 0.76159$

3. (a) $\cosh(\ln 5) = \frac{1}{2}(e^{\ln 5} + e^{-\ln 5}) = \frac{1}{2}\left(5 + (e^{\ln 5})^{-1}\right) = \frac{1}{2}(5 + 5^{-1}) = \frac{1}{2}\left(5 + \frac{1}{5}\right) = \frac{13}{5}$

(b) $\cosh 5 = \frac{1}{2}(e^5 + e^{-5}) \approx 74.20995$

4. (a) $\sinh 4 = \frac{1}{2}(e^4 - e^{-4}) \approx 27.28992$

(b) $\sinh(\ln 4) = \frac{1}{2}(e^{\ln 4} - e^{-\ln 4}) = \frac{1}{2}\left(4 - (e^{\ln 4})^{-1}\right) = \frac{1}{2}(4 - 4^{-1}) = \frac{1}{2}\left(4 - \frac{1}{4}\right) = \frac{15}{8}$

5. (a) $\operatorname{sech} 0 = \dfrac{1}{\cosh 0} = \dfrac{1}{1} = 1$ (b) $\cosh^{-1} 1 = 0$ because $\cosh 0 = 1$.

6. (a) $\sinh 1 = \frac{1}{2}(e^1 - e^{-1}) \approx 1.17520$

(b) Using Equation 3, we have $\sinh^{-1} 1 = \ln\left(1 + \sqrt{1^2 + 1}\right) = \ln\left(1 + \sqrt{2}\right) \approx 0.88137$.

7. $\sinh(-x) = \frac{1}{2}[e^{-x} - e^{-(-x)}] = \frac{1}{2}(e^{-x} - e^x) = -\frac{1}{2}(e^{-x} - e^x) = -\sinh x$

8. $\cosh(-x) = \frac{1}{2}[e^{-x} + e^{-(-x)}] = \frac{1}{2}(e^{-x} + e^x) = \frac{1}{2}(e^x + e^{-x}) = \cosh x$

9. $\cosh x + \sinh x = \frac{1}{2}(e^x + e^{-x}) + \frac{1}{2}(e^x - e^{-x}) = \frac{1}{2}(2e^x) = e^x$

10. $\cosh x - \sinh x = \frac{1}{2}(e^x + e^{-x}) - \frac{1}{2}(e^x - e^{-x}) = \frac{1}{2}(2e^{-x}) = e^{-x}$

11.
$$\begin{aligned}\sinh x\,\cosh y + \cosh x\,\sinh y &= \left[\tfrac{1}{2}(e^x - e^{-x})\right]\left[\tfrac{1}{2}(e^y + e^{-y})\right] + \left[\tfrac{1}{2}(e^x + e^{-x})\right]\left[\tfrac{1}{2}(e^y - e^{-y})\right] \\ &= \tfrac{1}{4}[(e^{x+y} + e^{x-y} - e^{-x+y} - e^{-x-y}) + (e^{x+y} - e^{x-y} + e^{-x+y} - e^{-x-y})] \\ &= \tfrac{1}{4}(2e^{x+y} - 2e^{-x-y}) = \tfrac{1}{2}[e^{x+y} - e^{-(x+y)}] = \sinh(x+y)\end{aligned}$$

12.
$$\begin{aligned}\cosh x\,\cosh y + \sinh x\,\sinh y &= \left[\tfrac{1}{2}(e^x + e^{-x})\right]\left[\tfrac{1}{2}(e^y + e^{-y})\right] + \left[\tfrac{1}{2}(e^x - e^{-x})\right]\left[\tfrac{1}{2}(e^y - e^{-y})\right] \\ &= \tfrac{1}{4}\left[(e^{x+y} + e^{x-y} + e^{-x+y} + e^{-x-y}) + (e^{x+y} - e^{x-y} - e^{-x+y} + e^{-x-y})\right] \\ &= \tfrac{1}{4}(2e^{x+y} + 2e^{-x-y}) = \tfrac{1}{2}\left[e^{x+y} + e^{-(x+y)}\right] = \cosh(x+y)\end{aligned}$$

13. Divide both sides of the identity $\cosh^2 x - \sinh^2 x = 1$ by $\sinh^2 x$:

$$\frac{\cosh^2 x}{\sinh^2 x} - \frac{\sinh^2 x}{\sinh^2 x} = \frac{1}{\sinh^2 x} \quad \Leftrightarrow \quad \coth^2 x - 1 = \operatorname{csch}^2 x.$$

14.
$$\tanh(x+y) = \frac{\sinh(x+y)}{\cosh(x+y)} = \frac{\sinh x\,\cosh y + \cosh x\,\sinh y}{\cosh x\,\cosh y + \sinh x\,\sinh y} = \frac{\dfrac{\sinh x\,\cosh y}{\cosh x\,\cosh y} + \dfrac{\cosh x\,\sinh y}{\cosh x\,\cosh y}}{\dfrac{\cosh x\,\cosh y}{\cosh x\,\cosh y} + \dfrac{\sinh x\,\sinh y}{\cosh x\,\cosh y}}$$
$$= \frac{\tanh x + \tanh y}{1 + \tanh x\,\tanh y}$$

15. Putting $y = x$ in the result from Exercise 11, we have

$\sinh 2x = \sinh(x+x) = \sinh x\,\cosh x + \cosh x\,\sinh x = 2\sinh x\,\cosh x$.

16. Putting $y = x$ in the result from Exercise 12, we have

$\cosh 2x = \cosh(x + x) = \cosh x \cosh x + \sinh x \sinh x = \cosh^2 x + \sinh^2 x$.

17. $\tanh(\ln x) = \dfrac{\sinh(\ln x)}{\cosh(\ln x)} = \dfrac{(e^{\ln x} - e^{-\ln x})/2}{(e^{\ln x} + e^{-\ln x})/2} = \dfrac{x - (e^{\ln x})^{-1}}{x + (e^{\ln x})^{-1}} = \dfrac{x - x^{-1}}{x + x^{-1}} = \dfrac{x - 1/x}{x + 1/x} = \dfrac{(x^2-1)/x}{(x^2+1)/x} = \dfrac{x^2-1}{x^2+1}$

18. $\dfrac{1 + \tanh x}{1 - \tanh x} = \dfrac{1 + (\sinh x)/\cosh x}{1 - (\sinh x)/\cosh x} = \dfrac{\cosh x + \sinh x}{\cosh x - \sinh x} = \dfrac{\frac{1}{2}(e^x + e^{-x}) + \frac{1}{2}(e^x - e^{-x})}{\frac{1}{2}(e^x + e^{-x}) - \frac{1}{2}(e^x - e^{-x})} = \dfrac{e^x}{e^{-x}} = e^{2x}$

Or: Using the results of Exercises 9 and 10, $\dfrac{\cosh x + \sinh x}{\cosh x - \sinh x} = \dfrac{e^x}{e^{-x}} = e^{2x}$

19. By Exercise 9, $(\cosh x + \sinh x)^n = (e^x)^n = e^{nx} = \cosh nx + \sinh nx$.

20. $\coth x = \dfrac{1}{\tanh x} \quad\Rightarrow\quad \coth x = \dfrac{1}{\tanh x} = \dfrac{1}{12/13} = \dfrac{13}{12}$.

$\operatorname{sech}^2 x = 1 - \tanh^2 x = 1 - \left(\frac{12}{13}\right)^2 = \frac{25}{169} \quad\Rightarrow\quad \operatorname{sech} x = \frac{5}{13}$ [sech, like cosh, is positive].

$\cosh x = \dfrac{1}{\operatorname{sech} x} \quad\Rightarrow\quad \cosh x = \dfrac{1}{5/13} = \dfrac{13}{5}$.

$\tanh x = \dfrac{\sinh x}{\cosh x} \quad\Rightarrow\quad \sinh x = \tanh x \cosh x \quad\Rightarrow\quad \sinh x = \dfrac{12}{13} \cdot \dfrac{13}{5} = \dfrac{12}{5}$.

$\operatorname{csch} x = \dfrac{1}{\sinh x} \quad\Rightarrow\quad \operatorname{csch} x = \dfrac{1}{12/5} = \dfrac{5}{12}$.

21. $\operatorname{sech} x = \dfrac{1}{\cosh x} \quad\Rightarrow\quad \operatorname{sech} x = \dfrac{1}{5/3} = \dfrac{3}{5}$.

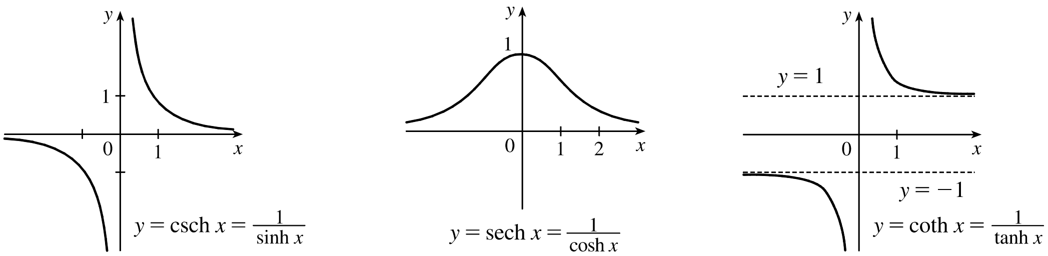
$\cosh^2 x - \sinh^2 x = 1 \quad\Rightarrow\quad \sinh^2 x = \cosh^2 x - 1 = \left(\frac{5}{3}\right)^2 - 1 = \frac{16}{9} \quad\Rightarrow\quad \sinh x = \frac{4}{3}$ [because $x > 0$].

$\operatorname{csch} x = \dfrac{1}{\sinh x} \quad\Rightarrow\quad \operatorname{csch} x = \dfrac{1}{4/3} = \dfrac{3}{4}$.

$\tanh x = \dfrac{\sinh x}{\cosh x} \quad\Rightarrow\quad \tanh x = \dfrac{4/3}{5/3} = \dfrac{4}{5}$.

$\coth x = \dfrac{1}{\tanh x} \quad\Rightarrow\quad \coth x = \dfrac{1}{4/5} = \dfrac{5}{4}$.

22. (a)



23. (a) $\displaystyle\lim_{x\to\infty} \tanh x = \lim_{x\to\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} \cdot \frac{e^{-x}}{e^{-x}} = \lim_{x\to\infty} \frac{1 - e^{-2x}}{1 + e^{-2x}} = \frac{1-0}{1+0} = 1$

(b) $\displaystyle\lim_{x\to-\infty} \tanh x = \lim_{x\to-\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} \cdot \frac{e^x}{e^x} = \lim_{x\to-\infty} \frac{e^{2x} - 1}{e^{2x} + 1} = \frac{0-1}{0+1} = -1$

(c) $\lim\limits_{x\to\infty} \sinh x = \lim\limits_{x\to\infty} \dfrac{e^x - e^{-x}}{2} = \infty$

(d) $\lim\limits_{x\to-\infty} \sinh x = \lim\limits_{x\to-\infty} \dfrac{e^x - e^{-x}}{2} = -\infty$

(e) $\lim\limits_{x\to\infty} \operatorname{sech} x = \lim\limits_{x\to\infty} \dfrac{2}{e^x + e^{-x}} = 0$

(f) $\lim\limits_{x\to\infty} \coth x = \lim\limits_{x\to\infty} \dfrac{e^x + e^{-x}}{e^x - e^{-x}} \cdot \dfrac{e^{-x}}{e^{-x}} = \lim\limits_{x\to\infty} \dfrac{1 + e^{-2x}}{1 - e^{-2x}} = \dfrac{1+0}{1-0} = 1$ [*Or:* Use part (a)]

(g) $\lim\limits_{x\to0^+} \coth x = \lim\limits_{x\to0^+} \dfrac{\cosh x}{\sinh x} = \infty$, since $\sinh x \to 0$ through positive values and $\cosh x \to 1$.

(h) $\lim\limits_{x\to0^-} \coth x = \lim\limits_{x\to0^-} \dfrac{\cosh x}{\sinh x} = -\infty$, since $\sinh x \to 0$ through negative values and $\cosh x \to 1$.

(i) $\lim\limits_{x\to-\infty} \operatorname{csch} x = \lim\limits_{x\to-\infty} \dfrac{2}{e^x - e^{-x}} = 0$

(j) $\lim\limits_{x\to\infty} \dfrac{\sinh x}{e^x} = \lim\limits_{x\to\infty} \dfrac{e^x - e^{-x}}{2e^x} = \lim\limits_{x\to\infty} \dfrac{1 - e^{-2x}}{2} = \dfrac{1-0}{2} = \dfrac{1}{2}$

24. (a) $\dfrac{d}{dx}(\cosh x) = \dfrac{d}{dx}\left[\frac{1}{2}(e^x + e^{-x})\right] = \frac{1}{2}(e^x - e^{-x}) = \sinh x$

(b) $\dfrac{d}{dx}(\tanh x) = \dfrac{d}{dx}\left(\dfrac{\sinh x}{\cosh x}\right) = \dfrac{\cosh x\,\cosh x - \sinh x\,\sinh x}{\cosh^2 x} = \dfrac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \dfrac{1}{\cosh^2 x} = \operatorname{sech}^2 x$

(c) $\dfrac{d}{dx}(\operatorname{csch} x) = \dfrac{d}{dx}\left(\dfrac{1}{\sinh x}\right) = -\dfrac{\cosh x}{\sinh^2 x} = -\dfrac{1}{\sinh x} \cdot \dfrac{\cosh x}{\sinh x} = -\operatorname{csch} x\,\coth x$

(d) $\dfrac{d}{dx}(\operatorname{sech} x) = \dfrac{d}{dx}\left(\dfrac{1}{\cosh x}\right) = -\dfrac{\sinh x}{\cosh^2 x} = -\dfrac{1}{\cosh x} \cdot \dfrac{\sinh x}{\cosh x} = -\operatorname{sech} x\,\tanh x$

(e) $\dfrac{d}{dx}(\coth x) = \dfrac{d}{dx}\left(\dfrac{\cosh x}{\sinh x}\right) = \dfrac{\sinh x\,\sinh x - \cosh x\,\cosh x}{\sinh^2 x} = \dfrac{\sinh^2 x - \cosh^2 x}{\sinh^2 x} = -\dfrac{1}{\sinh^2 x} = -\operatorname{csch}^2 x$

25. Let $y = \sinh^{-1} x$. Then $\sinh y = x$ and, by Example 1(a), $\cosh^2 y - \sinh^2 y = 1 \;\Rightarrow\;$ [with $\cosh y > 0$]
$\cosh y = \sqrt{1 + \sinh^2 y} = \sqrt{1 + x^2}$. So by Exercise 9, $e^y = \sinh y + \cosh y = x + \sqrt{1 + x^2} \;\Rightarrow\; y = \ln\left(x + \sqrt{1 + x^2}\right)$.

26. Let $y = \cosh^{-1} x$. Then $\cosh y = x$ and $y \ge 0$, so $\sinh y = \sqrt{\cosh^2 y - 1} = \sqrt{x^2 - 1}$. So, by Exercise 9,
$e^y = \cosh y + \sinh y = x + \sqrt{x^2 - 1} \;\Rightarrow\; y = \ln\left(x + \sqrt{x^2 - 1}\right)$.

Another method: Write $x = \cosh y = \frac{1}{2}\left(e^y + e^{-y}\right)$ and solve a quadratic, as in Example 3.

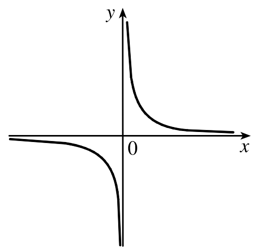
27. (a) Let $y = \tanh^{-1} x$. Then $x = \tanh y = \dfrac{\sinh y}{\cosh y} = \dfrac{(e^y - e^{-y})/2}{(e^y + e^{-y})/2} \cdot \dfrac{e^y}{e^y} = \dfrac{e^{2y} - 1}{e^{2y} + 1} \;\Rightarrow\; xe^{2y} + x = e^{2y} - 1 \;\Rightarrow$

$1 + x = e^{2y} - xe^{2y} \;\Rightarrow\; 1 + x = e^{2y}(1 - x) \;\Rightarrow\; e^{2y} = \dfrac{1+x}{1-x} \;\Rightarrow\; 2y = \ln\left(\dfrac{1+x}{1-x}\right) \;\Rightarrow\; y = \frac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right)$.

(b) Let $y = \tanh^{-1} x$. Then $x = \tanh y$, so from Exercise 18 we have

$$e^{2y} = \frac{1 + \tanh y}{1 - \tanh y} = \frac{1 + x}{1 - x} \quad \Rightarrow \quad 2y = \ln\left(\frac{1 + x}{1 - x}\right) \quad \Rightarrow \quad y = \tfrac{1}{2} \ln\left(\frac{1 + x}{1 - x}\right).$$

28. (a) (i) $y = \operatorname{csch}^{-1} x \quad \Leftrightarrow \quad \operatorname{csch} y = x \quad (x \neq 0)$



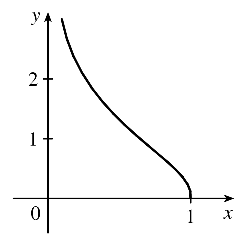
(ii) We sketch the graph of csch^{-1} by reflecting the graph of csch (see Exercise 22) about the line $y = x$.

(iii) Let $y = \operatorname{csch}^{-1} x$. Then $x = \operatorname{csch} y = \dfrac{2}{e^y - e^{-y}} \quad \Rightarrow \quad xe^y - xe^{-y} = 2 \quad \Rightarrow$

$x(e^y)^2 - 2e^y - x = 0 \quad \Rightarrow \quad e^y = \dfrac{1 \pm \sqrt{x^2 + 1}}{x}$. But $e^y > 0$, so for $x > 0$,

$e^y = \dfrac{1 + \sqrt{x^2 + 1}}{x}$ and for $x < 0$, $e^y = \dfrac{1 - \sqrt{x^2 + 1}}{x}$. Thus, $\operatorname{csch}^{-1} x = \ln\left(\dfrac{1}{x} + \dfrac{\sqrt{x^2 + 1}}{|x|}\right)$.

(b) (i) $y = \operatorname{sech}^{-1} x \quad \Leftrightarrow \quad \operatorname{sech} y = x$ and $y > 0$.



(ii) We sketch the graph of sech^{-1} by reflecting the graph of sech (see Exercise 22) about the line $y = x$.

(iii) Let $y = \operatorname{sech}^{-1} x$, so $x = \operatorname{sech} y = \dfrac{2}{e^y + e^{-y}} \quad \Rightarrow \quad xe^y + xe^{-y} = 2 \quad \Rightarrow$

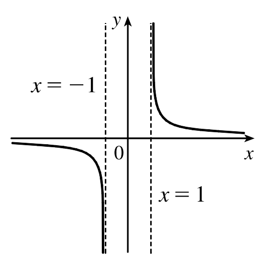
$x\,(e^y)^2 - 2e^y + x = 0 \quad \Leftrightarrow \quad e^y = \dfrac{1 \pm \sqrt{1 - x^2}}{x}$. But $y > 0 \quad \Rightarrow \quad e^y > 1$.

This rules out the minus sign because $\dfrac{1 - \sqrt{1 - x^2}}{x} > 1 \quad \Leftrightarrow \quad 1 - \sqrt{1 - x^2} > x \quad \Leftrightarrow \quad 1 - x > \sqrt{1 - x^2} \quad \Leftrightarrow$

$1 - 2x + x^2 > 1 - x^2 \quad \Leftrightarrow \quad x^2 > x \quad \Leftrightarrow \quad x > 1$, but $x = \operatorname{sech} y \leq 1$.

Thus, $e^y = \dfrac{1 + \sqrt{1 - x^2}}{x} \quad \Rightarrow \quad \operatorname{sech}^{-1} x = \ln\left(\dfrac{1 + \sqrt{1 - x^2}}{x}\right)$.

(c) (i) $y = \coth^{-1} x \quad \Leftrightarrow \quad \coth y = x$



(ii) We sketch the graph of $\coth^{-1}$ by reflecting the graph of coth (see Exercise 22) about the line $y = x$.

(iii) Let $y = \coth^{-1} x$. Then $x = \coth y = \dfrac{e^y + e^{-y}}{e^y - e^{-y}} \quad \Rightarrow$

$xe^y - xe^{-y} = e^y + e^{-y} \quad \Rightarrow \quad (x - 1)e^y = (x + 1)e^{-y} \quad \Rightarrow \quad e^{2y} = \dfrac{x + 1}{x - 1} \quad \Rightarrow$

$2y = \ln \dfrac{x + 1}{x - 1} \quad \Rightarrow \quad \coth^{-1} x = \dfrac{1}{2} \ln \dfrac{x + 1}{x - 1}$

29. (a) Let $y = \cosh^{-1} x$. Then $\cosh y = x$ and $y \geq 0 \quad \Rightarrow \quad \sinh y \dfrac{dy}{dx} = 1 \quad \Rightarrow$

$\dfrac{dy}{dx} = \dfrac{1}{\sinh y} = \dfrac{1}{\sqrt{\cosh^2 y - 1}} = \dfrac{1}{\sqrt{x^2 - 1}}$ [since $\sinh y \geq 0$ for $y \geq 0$]. *Or:* Use Formula 4.

(b) Let $y = \tanh^{-1} x$. Then $\tanh y = x \quad \Rightarrow \quad \operatorname{sech}^2 y \dfrac{dy}{dx} = 1 \quad \Rightarrow \quad \dfrac{dy}{dx} = \dfrac{1}{\operatorname{sech}^2 y} = \dfrac{1}{1 - \tanh^2 y} = \dfrac{1}{1 - x^2}$.

Or: Use Formula 5.

(c) Let $y = \operatorname{csch}^{-1} x$. Then $\operatorname{csch} y = x \Rightarrow -\operatorname{csch} y \coth y \dfrac{dy}{dx} = 1 \Rightarrow \dfrac{dy}{dx} = -\dfrac{1}{\operatorname{csch} y \coth y}$. By Exercise 13, $\coth y = \pm\sqrt{\operatorname{csch}^2 y + 1} = \pm\sqrt{x^2+1}$. If $x > 0$, then $\coth y > 0$, so $\coth y = \sqrt{x^2+1}$. If $x < 0$, then $\coth y < 0$, so $\coth y = -\sqrt{x^2+1}$. In either case we have $\dfrac{dy}{dx} = -\dfrac{1}{\operatorname{csch} y \coth y} = -\dfrac{1}{|x|\sqrt{x^2+1}}$.

(d) Let $y = \operatorname{sech}^{-1} x$. Then $\operatorname{sech} y = x \Rightarrow -\operatorname{sech} y \tanh y \dfrac{dy}{dx} = 1 \Rightarrow$

$\dfrac{dy}{dx} = -\dfrac{1}{\operatorname{sech} y \tanh y} = -\dfrac{1}{\operatorname{sech} y\sqrt{1-\operatorname{sech}^2 y}} = -\dfrac{1}{x\sqrt{1-x^2}}$. [Note that $y > 0$ and so $\tanh y > 0$.]

(e) Let $y = \coth^{-1} x$. Then $\coth y = x \Rightarrow -\operatorname{csch}^2 y \dfrac{dy}{dx} = 1 \Rightarrow \dfrac{dy}{dx} = -\dfrac{1}{\operatorname{csch}^2 y} = \dfrac{1}{1-\coth^2 y} = \dfrac{1}{1-x^2}$ by Exercise 13.

30. $f(x) = e^x \cosh x \overset{\text{PR}}{\Rightarrow} f'(x) = e^x \sinh x + (\cosh x)e^x = e^x(\sinh x + \cosh x)$, or, using Exercise 9, $e^x(e^x) = e^{2x}$.

31. $f(x) = \tanh\sqrt{x} \Rightarrow f'(x) = \operatorname{sech}^2\sqrt{x}\,\dfrac{d}{dx}\sqrt{x} = \operatorname{sech}^2\sqrt{x}\left(\dfrac{1}{2\sqrt{x}}\right) = \dfrac{\operatorname{sech}^2\sqrt{x}}{2\sqrt{x}}$

32. $g(x) = \sinh^2 x = (\sinh x)^2 \Rightarrow g'(x) = 2(\sinh x)^1 \dfrac{d}{dx}(\sinh x) = 2\sinh x \cosh x$, or, using Exercise 15, $\sinh 2x$.

33. $h(x) = \sinh(x^2) \Rightarrow h'(x) = \cosh(x^2)\dfrac{d}{dx}(x^2) = 2x\cosh(x^2)$

34. $F(t) = \ln(\sinh t) \Rightarrow F'(t) = \dfrac{1}{\sinh t}\dfrac{d}{dt}\sinh t = \dfrac{1}{\sinh t}\cosh t = \coth t$

35. $G(t) = \sinh(\ln t) \Rightarrow G'(t) = \cosh(\ln t)\dfrac{d}{dt}\ln t = \dfrac{1}{2}\left(e^{\ln t} + e^{-\ln t}\right)\left(\dfrac{1}{t}\right) = \dfrac{1}{2t}\left(t + \dfrac{1}{t}\right) = \dfrac{1}{2t}\left(\dfrac{t^2+1}{t}\right) = \dfrac{t^2+1}{2t^2}$

Or: $G(t) = \sinh(\ln t) = \dfrac{1}{2}(e^{\ln t} - e^{-\ln t}) = \dfrac{1}{2}\left(t - \dfrac{1}{t}\right) \Rightarrow G'(t) = \dfrac{1}{2}\left(1 + \dfrac{1}{t^2}\right) = \dfrac{t^2+1}{2t^2}$

36. $y = \operatorname{sech} x(1 + \ln \operatorname{sech} x) \overset{\text{PR}}{\Rightarrow}$

$$\begin{aligned} y' &= \operatorname{sech} x\,\frac{d}{dx}(1+\ln\operatorname{sech} x) + (1+\ln\operatorname{sech} x)\frac{d}{dx}\operatorname{sech} x \\ &= \operatorname{sech} x\left(\frac{-\operatorname{sech} x \tanh x}{\operatorname{sech} x}\right) + (1+\ln\operatorname{sech} x)(-\operatorname{sech} x \tanh x) \\ &= -\operatorname{sech} x \tanh x\,[1 + (1+\ln\operatorname{sech} x)] = -\operatorname{sech} x \tanh x\,(2+\ln\operatorname{sech} x) \end{aligned}$$

37. $y = e^{\cosh 3x} \Rightarrow y' = e^{\cosh 3x}\cdot\sinh 3x\cdot 3 = 3e^{\cosh 3x}\sinh 3x$

38. $f(t) = \dfrac{1+\sinh t}{1-\sinh t} \overset{\text{QR}}{\Rightarrow}$

$$\begin{aligned} f'(t) &= \frac{(1-\sinh t)\cosh t - (1+\sinh t)(-\cosh t)}{(1-\sinh t)^2} = \frac{\cosh t - \sinh t\cosh t + \cosh t + \sinh t\cosh t}{(1-\sinh t)^2} \\ &= \frac{2\cosh t}{(1-\sinh t)^2} \end{aligned}$$

39. $g(t) = t\coth\sqrt{t^2+1} \overset{\text{PR}}{\Rightarrow}$

$$g'(t) = t\left[-\operatorname{csch}^2\sqrt{t^2+1}\left(\tfrac{1}{2}(t^2+1)^{-1/2}\cdot 2t\right)\right] + \left(\coth\sqrt{t^2+1}\right)(1) = \coth\sqrt{t^2+1} - \frac{t^2}{\sqrt{t^2+1}}\operatorname{csch}^2\sqrt{t^2+1}$$

40. $y = \sinh^{-1}(\tan x) \quad\Rightarrow\quad y' = \dfrac{1}{\sqrt{1+(\tan x)^2}}\dfrac{d}{dx}(\tan x) = \dfrac{\sec^2 x}{\sqrt{\sec^2 x}} = \dfrac{|\sec^2 x|}{|\sec x|} = |\sec x|$

41. $y = \cosh^{-1}\sqrt{x} \quad\Rightarrow\quad y' = \dfrac{1}{\sqrt{(\sqrt{x})^2-1}}\dfrac{d}{dx}(\sqrt{x}) = \dfrac{1}{\sqrt{x-1}}\dfrac{1}{2\sqrt{x}} = \dfrac{1}{2\sqrt{x(x-1)}}$

42. $y = x\tanh^{-1}x + \ln\sqrt{1-x^2} = x\tanh^{-1}x + \frac{1}{2}\ln(1-x^2) \quad\Rightarrow$

$$y' = \tanh^{-1}x + \frac{x}{1-x^2} + \frac{1}{2}\left(\frac{1}{1-x^2}\right)(-2x) = \tanh^{-1}x$$

43. $y = x\sinh^{-1}(x/3) - \sqrt{9+x^2} \quad\Rightarrow$

$$y' = \sinh^{-1}\left(\frac{x}{3}\right) + x\frac{1/3}{\sqrt{1+(x/3)^2}} - \frac{2x}{2\sqrt{9+x^2}} = \sinh^{-1}\left(\frac{x}{3}\right) + \frac{x}{\sqrt{9+x^2}} - \frac{x}{\sqrt{9+x^2}} = \sinh^{-1}\left(\frac{x}{3}\right)$$

44. $y = \operatorname{sech}^{-1}(e^{-x}) \quad\Rightarrow\quad y' = -\dfrac{1}{e^{-x}\sqrt{1-(e^{-x})^2}}\dfrac{d}{dx}(e^{-x}) = -\dfrac{1}{e^{-x}\sqrt{1-e^{-2x}}}(-e^{-x}) = \dfrac{1}{\sqrt{1-e^{-2x}}}$

45. $y = \coth^{-1}(\sec x) \quad\Rightarrow$

$$y' = \frac{1}{1-(\sec x)^2}\frac{d}{dx}(\sec x) = \frac{\sec x\,\tan x}{1-\sec^2 x} = \frac{\sec x\,\tan x}{1-(\tan^2 x+1)} = \frac{\sec x\,\tan x}{-\tan^2 x}$$

$$= -\frac{\sec x}{\tan x} = -\frac{1/\cos x}{\sin x/\cos x} = -\frac{1}{\sin x} = -\csc x$$

46. $\dfrac{1+\tanh x}{1-\tanh x} = \dfrac{1+(\sinh x)/\cosh x}{1-(\sinh x)/\cosh x} = \dfrac{\cosh x+\sinh x}{\cosh x-\sinh x} = \dfrac{e^x}{e^{-x}}$ [by Exercises 9 and 10] $= e^{2x}$, so

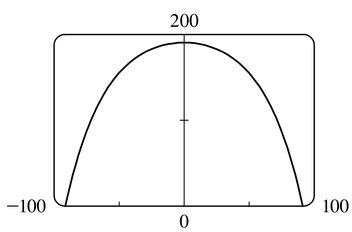
$\sqrt[4]{\dfrac{1+\tanh x}{1-\tanh x}} = \sqrt[4]{e^{2x}} = e^{x/2}$. Thus, $\dfrac{d}{dx}\sqrt[4]{\dfrac{1+\tanh x}{1-\tanh x}} = \dfrac{d}{dx}(e^{x/2}) = \dfrac{1}{2}e^{x/2}$.

47. $\dfrac{d}{dx}\arctan(\tanh x) = \dfrac{1}{1+(\tanh x)^2}\dfrac{d}{dx}(\tanh x) = \dfrac{\operatorname{sech}^2 x}{1+\tanh^2 x} = \dfrac{1/\cosh^2 x}{1+(\sinh^2 x)/\cosh^2 x}$

$$= \frac{1}{\cosh^2 x+\sinh^2 x} = \frac{1}{\cosh 2x} \text{ [by Exercise 16] } = \operatorname{sech} 2x$$

48. (a) Let $a = 0.03291765$. A graph of the central curve,

$y = f(x) = 211.49 - 20.96\cosh ax$, is shown.



(b) $f(0) = 211.49 - 20.96\cosh 0 = 211.49 - 20.96(1) = 190.53$ m.

(c) $y = 100 \quad\Rightarrow\quad 100 = 211.49 - 20.96\cosh ax \quad\Rightarrow$

$20.96\cosh ax = 111.49 \quad\Rightarrow\quad \cosh ax = \dfrac{111.49}{20.96} \quad\Rightarrow$

$ax = \pm\cosh^{-1}\dfrac{111.49}{20.96} \quad\Rightarrow\quad x = \pm\dfrac{1}{a}\cosh^{-1}\dfrac{111.49}{20.96} \approx \pm 71.56$ m. The points are approximately $(\pm 71.56, 100)$.

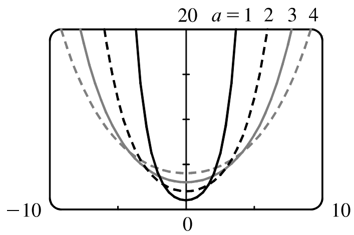
(d) $f(x) = 211.49 - 20.96\cosh ax \quad \Rightarrow \quad f'(x) = -20.96\sinh ax \cdot a$.

$$f'\left(\pm\frac{1}{a}\cosh^{-1}\frac{111.49}{20.96}\right) = -20.96a\sinh\left[a\left(\pm\frac{1}{a}\cosh^{-1}\frac{111.49}{20.96}\right)\right] = -20.96a\sinh\left(\pm\cosh^{-1}\frac{111.49}{20.96}\right) \approx \mp 3.6.$$

So the slope at $(71.56, 100)$ is about -3.6 and the slope at $(-71.56, 100)$ is about 3.6.

49. As the depth d of the water gets large, the fraction $\dfrac{2\pi d}{L}$ gets large, and from Figure 3 or Exercise 23(a), $\tanh\left(\dfrac{2\pi d}{L}\right)$ approaches 1. Thus, $v = \sqrt{\dfrac{gL}{2\pi}\tanh\left(\dfrac{2\pi d}{L}\right)} \approx \sqrt{\dfrac{gL}{2\pi}(1)} = \sqrt{\dfrac{gL}{2\pi}}$.

50.



For $y = a\cosh(x/a)$ with $a > 0$, we have the y-intercept equal to a. As a increases, the graph flattens.

51. (a) $y = 20\cosh(x/20) - 15 \quad \Rightarrow \quad y' = 20\sinh(x/20) \cdot \frac{1}{20} = \sinh(x/20)$. Since the right pole is positioned at $x = 7$, we have $y'(7) = \sinh\frac{7}{20} \approx 0.3572$.

(b) If α is the angle between the tangent line and the x-axis, then $\tan\alpha =$ slope of the line $= \sinh\frac{7}{20}$, so $\alpha = \tan^{-1}\left(\sinh\frac{7}{20}\right) \approx 0.343$ rad $\approx 19.66^\circ$. Thus, the angle between the line and the pole is $\theta = 90^\circ - \alpha \approx 70.34^\circ$.

52. We differentiate the function twice, then substitute into the differential equation: $y = \dfrac{T}{\rho g}\cosh\dfrac{\rho g x}{T} \quad \Rightarrow$

$\dfrac{dy}{dx} = \dfrac{T}{\rho g}\sinh\left(\dfrac{\rho g x}{T}\right)\dfrac{\rho g}{T} = \sinh\dfrac{\rho g x}{T} \quad \Rightarrow \quad \dfrac{d^2y}{dx^2} = \cosh\left(\dfrac{\rho g x}{T}\right)\dfrac{\rho g}{T} = \dfrac{\rho g}{T}\cosh\dfrac{\rho g x}{T}$. We evaluate the two sides separately: LHS $= \dfrac{d^2y}{dx^2} = \dfrac{\rho g}{T}\cosh\dfrac{\rho g x}{T}$ and RHS $= \dfrac{\rho g}{T}\sqrt{1 + \left(\dfrac{dy}{dx}\right)^2} = \dfrac{\rho g}{T}\sqrt{1 + \sinh^2\dfrac{\rho g x}{T}} = \dfrac{\rho g}{T}\cosh\dfrac{\rho g x}{T}$, by the identity proved in Example 1(a).

53. (a) From Exercise 52, the shape of the cable is given by $y = f(x) = \dfrac{T}{\rho g}\cosh\left(\dfrac{\rho g x}{T}\right)$. The shape is symmetric about the y-axis, so the lowest point is $(0, f(0)) = \left(0, \dfrac{T}{\rho g}\right)$ and the poles are at $x = \pm 100$. We want to find T when the lowest point is 60 m, so $\dfrac{T}{\rho g} = 60 \quad \Rightarrow \quad T = 60\rho g = (60 \text{ m})(2 \text{ kg/m})(9.8 \text{ m/s}^2) = 1176\ \dfrac{\text{kg-m}}{\text{s}^2}$, or 1176 N (newtons).

The height of each pole is $f(100) = \dfrac{T}{\rho g}\cosh\left(\dfrac{\rho g \cdot 100}{T}\right) = 60\cosh\left(\dfrac{100}{60}\right) \approx 164.50$ m.

(b) If the tension is doubled from T to $2T$, then the low point is doubled since $\dfrac{T}{\rho g} = 60 \quad \Rightarrow \quad \dfrac{2T}{\rho g} = 120$. The height of the poles is now $f(100) = \dfrac{2T}{\rho g}\cosh\left(\dfrac{\rho g \cdot 100}{2T}\right) = 120\cosh\left(\dfrac{100}{120}\right) \approx 164.13$ m, just a slight decrease.

54. (a) $\lim\limits_{t\to\infty} v(t) = \lim\limits_{t\to\infty} \sqrt{\dfrac{mg}{k}} \tanh\left(t\sqrt{\dfrac{gk}{m}}\right) = \sqrt{\dfrac{mg}{k}} \lim\limits_{t\to\infty} \tanh\left(t\sqrt{\dfrac{gk}{m}}\right) = \sqrt{\dfrac{mg}{k}} \cdot 1 \quad \left[\begin{array}{c}\text{as } t\to\infty, \\ t\sqrt{gk/m}\to\infty\end{array}\right] \quad = \sqrt{\dfrac{mg}{k}}$

(b) Belly-to-earth: $g = 9.8$, $k = 0.515$, $m = 60$, so the terminal velocity is $\sqrt{\dfrac{60(9.8)}{0.515}} \approx 33.79$ m/s.

Feet-first: $g = 9.8$, $k = 0.067$, $m = 60$, so the terminal velocity is $\sqrt{\dfrac{60(9.8)}{0.067}} \approx 93.68$ m/s.

55. (a) $y = A\sinh mx + B\cosh mx \quad \Rightarrow \quad y' = mA\cosh mx + mB\sinh mx \quad \Rightarrow$

$y'' = m^2 A\sinh mx + m^2 B\cosh mx = m^2(A\sinh mx + B\cosh mx) = m^2 y$

(b) From part (a), a solution of $y'' = 9y$ is $y(x) = A\sinh 3x + B\cosh 3x$. So $-4 = y(0) = A\sinh 0 + B\cosh 0 = B$, so $B = -4$. Now $y'(x) = 3A\cosh 3x - 12\sinh 3x \quad \Rightarrow \quad 6 = y'(0) = 3A \quad \Rightarrow \quad A = 2$, so $y = 2\sinh 3x - 4\cosh 3x$.

56. $\cosh x = \cosh[\ln(\sec\theta + \tan\theta)] = \dfrac{1}{2}\left[e^{\ln(\sec\theta+\tan\theta)} + e^{-\ln(\sec\theta+\tan\theta)}\right] = \dfrac{1}{2}\left[\sec\theta + \tan\theta + \dfrac{1}{\sec\theta+\tan\theta}\right]$

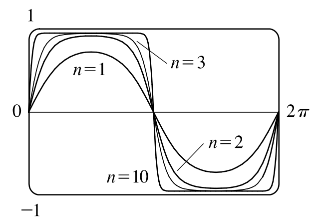
$= \dfrac{1}{2}\left[\sec\theta + \tan\theta + \dfrac{\sec\theta - \tan\theta}{(\sec\theta+\tan\theta)(\sec\theta-\tan\theta)}\right] = \dfrac{1}{2}\left[\sec\theta + \tan\theta + \dfrac{\sec\theta-\tan\theta}{\sec^2\theta - \tan^2\theta}\right]$

$= \frac{1}{2}(\sec\theta + \tan\theta + \sec\theta - \tan\theta) = \sec\theta$

57. The tangent to $y = \cosh x$ has slope 1 when $y' = \sinh x = 1 \quad \Rightarrow \quad x = \sinh^{-1} 1 = \ln\left(1+\sqrt{2}\right)$, by Equation 3.

Since $\sinh x = 1$ and $y = \cosh x = \sqrt{1+\sinh^2 x}$, we have $\cosh x = \sqrt{2}$. The point is $\left(\ln\left(1+\sqrt{2}\right), \sqrt{2}\right)$.

58. $f_n(x) = \tanh(n\sin x)$, where n is a positive integer. Note that $f_n(x+2\pi) = f_n(x)$; that is, f_n is periodic with period 2π. Also, from Figure 3, $-1 < \tanh x < 1$, so we can choose a viewing rectangle of $[0, 2\pi] \times [-1, 1]$. From the graph, we see that $f_n(x)$ becomes more rectangular looking as n increases. As n becomes large, the graph of f_n approaches the graph of $y = 1$ on the intervals $(2k\pi, (2k+1)\pi)$ and $y = -1$ on the intervals $((2k-1)\pi, 2k\pi)$.



59. If $ae^x + be^{-x} = \alpha\cosh(x+\beta)$ [or $\alpha\sinh(x+\beta)$], then

$ae^x + be^{-x} = \frac{\alpha}{2}\left(e^{x+\beta} \pm e^{-x-\beta}\right) = \frac{\alpha}{2}\left(e^x e^\beta \pm e^{-x}e^{-\beta}\right) = \left(\frac{\alpha}{2}e^\beta\right)e^x \pm \left(\frac{\alpha}{2}e^{-\beta}\right)e^{-x}$. Comparing coefficients of e^x and e^{-x}, we have $a = \frac{\alpha}{2}e^\beta$ **(1)** and $b = \pm\frac{\alpha}{2}e^{-\beta}$ **(2)**. We need to find α and β. Dividing equation **(1)** by equation **(2)** gives us $\frac{a}{b} = \pm e^{2\beta} \quad \Rightarrow \quad (\star) \quad 2\beta = \ln\left(\pm\frac{a}{b}\right) \quad \Rightarrow \quad \beta = \frac{1}{2}\ln\left(\pm\frac{a}{b}\right)$. Solving equations **(1)** and **(2)** for e^β gives us

$e^\beta = \dfrac{2a}{\alpha}$ and $e^\beta = \pm\dfrac{\alpha}{2b}$, so $\dfrac{2a}{\alpha} = \pm\dfrac{\alpha}{2b} \quad \Rightarrow \quad \alpha^2 = \pm 4ab \quad \Rightarrow \quad \alpha = 2\sqrt{\pm ab}$.

$(\star)$ If $\frac{a}{b} > 0$, we use the $+$ sign and obtain a cosh function, whereas if $\frac{a}{b} < 0$, we use the $-$ sign and obtain a sinh function.

In summary, if a and b have the same sign, we have $ae^x + be^{-x} = 2\sqrt{ab}\cosh\left(x + \frac{1}{2}\ln\frac{a}{b}\right)$, whereas, if a and b have the opposite sign, then $ae^x + be^{-x} = 2\sqrt{-ab}\sinh\left(x + \frac{1}{2}\ln\left(-\frac{a}{b}\right)\right)$.

3 Review

TRUE-FALSE QUIZ

1. True. This is the Sum Rule.

2. False. See the warning before the Product Rule.

3. True. This is the Chain Rule.

4. True. $\dfrac{d}{dx}\sqrt{f(x)} = \dfrac{d}{dx}[f(x)]^{1/2} = \dfrac{1}{2}[f(x)]^{-1/2}\,f'(x) = \dfrac{f'(x)}{2\sqrt{f(x)}}$

5. False. $\dfrac{d}{dx}\,f(\sqrt{x}\,) = f'(\sqrt{x}\,)\cdot\frac{1}{2}x^{-1/2} = \dfrac{f'(\sqrt{x}\,)}{2\sqrt{x}}$, which is not $\dfrac{f'(x)}{2\sqrt{x}}$.

6. False. $y = e^2$ is a constant, so $y' = 0$, not $2e$.

7. False. $\dfrac{d}{dx}\,(10^x) = 10^x \ln 10$, which is not equal to $x10^{x-1}$.

8. False. $\ln 10$ is a constant, so its derivative, $\dfrac{d}{dx}\,(\ln 10)$, is 0, not $\frac{1}{10}$.

9. True. $\dfrac{d}{dx}\,(\tan^2 x) = 2\,\tan x\,\sec^2 x$, and $\dfrac{d}{dx}\,(\sec^2 x) = 2\,\sec x\,(\sec x\,\tan x) = 2\,\tan x\,\sec^2 x$.

Or: $\dfrac{d}{dx}\,(\sec^2 x) = \dfrac{d}{dx}\,(1+\tan^2 x) = \dfrac{d}{dx}\,(\tan^2 x)$.

10. False. $f(x) = \left|x^2+x\right| = x^2+x$ for $x \geq 0$ or $x \leq -1$ and $\left|x^2+x\right| = -(x^2+x)$ for $-1 < x < 0$.
So $f'(x) = 2x+1$ for $x > 0$ or $x < -1$ and $f'(x) = -(2x+1)$ for $-1 < x < 0$. But $|2x+1| = 2x+1$ for $x \geq -\frac{1}{2}$ and $|2x+1| = -2x-1$ for $x < -\frac{1}{2}$.

11. True. If $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, then $p'(x) = na_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + a_1$, which is a polynomial.

12. True. $f(x) = (x^6 - x^4)^5$ is a polynomial of degree 30, so its 31st derivative, $f^{(31)}(x)$, is 0.

13. True. If $r(x) = \dfrac{p(x)}{q(x)}$, then $r'(x) = \dfrac{q(x)p'(x) - p(x)q'(x)}{[q(x)]^2}$, which is a quotient of polynomials, that is, a rational function.

14. False. A tangent line to the parabola $y = x^2$ has slope $dy/dx = 2x$, so at $(-2, 4)$ the slope of the tangent is $2(-2) = -4$ and an equation of the tangent line is $y - 4 = -4(x+2)$. [The given equation, $y - 4 = 2x(x+2)$, is not even linear!]

15. True. $g(x) = x^5 \quad\Rightarrow\quad g'(x) = 5x^4 \quad\Rightarrow\quad g'(2) = 5(2)^4 = 80$, and by the definition of the derivative,

$$\lim_{x\to 2}\frac{g(x)-g(2)}{x-2} = g'(2) = 5(2)^4 = 80.$$

EXERCISES

1. $y = (x^2 + x^3)^4 \quad \Rightarrow \quad y' = 4(x^2 + x^3)^3(2x + 3x^2) = 4(x^2)^3(1 + x)^3 x(2 + 3x) = 4x^7(x + 1)^3(3x + 2)$

2. $y = \dfrac{1}{\sqrt{x}} - \dfrac{1}{\sqrt[5]{x^3}} = x^{-1/2} - x^{-3/5} \quad \Rightarrow \quad y' = -\dfrac{1}{2}x^{-3/2} + \dfrac{3}{5}x^{-8/5}$ or $\dfrac{3}{5x\sqrt[5]{x^3}} - \dfrac{1}{2x\sqrt{x}}$ or $\dfrac{1}{10}x^{-8/5}(-5x^{1/10} + 6)$

3. $y = \dfrac{x^2 - x + 2}{\sqrt{x}} = x^{3/2} - x^{1/2} + 2x^{-1/2} \quad \Rightarrow \quad y' = \dfrac{3}{2}x^{1/2} - \dfrac{1}{2}x^{-1/2} - x^{-3/2} = \dfrac{3}{2}\sqrt{x} - \dfrac{1}{2\sqrt{x}} - \dfrac{1}{\sqrt{x^3}}$

4. $y = \dfrac{\tan x}{1 + \cos x} \quad \Rightarrow \quad y' = \dfrac{(1 + \cos x)\sec^2 x - \tan x(-\sin x)}{(1 + \cos x)^2} = \dfrac{(1 + \cos x)\sec^2 x + \tan x \, \sin x}{(1 + \cos x)^2}$

5. $y = x^2 \sin \pi x \quad \Rightarrow \quad y' = x^2(\cos \pi x)\pi + (\sin \pi x)(2x) = x(\pi x \cos \pi x + 2 \sin \pi x)$

6. $y = x \cos^{-1} x \quad \Rightarrow \quad y' = x\left(-\dfrac{1}{\sqrt{1 - x^2}}\right) + (\cos^{-1} x)(1) = \cos^{-1} x - \dfrac{x}{\sqrt{1 - x^2}}$

7. $y = \dfrac{t^4 - 1}{t^4 + 1} \quad \Rightarrow \quad y' = \dfrac{(t^4 + 1)4t^3 - (t^4 - 1)4t^3}{(t^4 + 1)^2} = \dfrac{4t^3[(t^4 + 1) - (t^4 - 1)]}{(t^4 + 1)^2} = \dfrac{8t^3}{(t^4 + 1)^2}$

8. $\dfrac{d}{dx}(xe^y) = \dfrac{d}{dx}(y \sin x) \quad \Rightarrow \quad xe^y y' + e^y \cdot 1 = y \cos x + \sin x \cdot y' \quad \Rightarrow \quad xe^y y' - \sin x \cdot y' = y \cos x - e^y \quad \Rightarrow$

$(xe^y - \sin x)y' = y \cos x - e^y \quad \Rightarrow \quad y' = \dfrac{y \cos x - e^y}{xe^y - \sin x}$

9. $y = \ln(x \ln x) \quad \Rightarrow \quad y' = \dfrac{1}{x \ln x}(x \ln x)' = \dfrac{1}{x \ln x}\left(x \cdot \dfrac{1}{x} + \ln x \cdot 1\right) = \dfrac{1 + \ln x}{x \ln x}$

Another method: $y = \ln(x \ln x) = \ln x + \ln \ln x \quad \Rightarrow \quad y' = \dfrac{1}{x} + \dfrac{1}{\ln x} \cdot \dfrac{1}{x} = \dfrac{\ln x + 1}{x \ln x}$

10. $y = e^{mx} \cos nx \quad \Rightarrow$

$y' = e^{mx}(\cos nx)' + \cos nx\,(e^{mx})' = e^{mx}(-\sin nx \cdot n) + \cos nx\,(e^{mx} \cdot m) = e^{mx}(m \cos nx - n \sin nx)$

11. $y = \sqrt{x} \cos \sqrt{x} \quad \Rightarrow$

$y' = \sqrt{x}\,\left(\cos \sqrt{x}\right)' + \cos \sqrt{x}\,\left(\sqrt{x}\right)' = \sqrt{x}\left[-\sin \sqrt{x}\left(\tfrac{1}{2}x^{-1/2}\right)\right] + \cos \sqrt{x}\left(\tfrac{1}{2}x^{-1/2}\right)$

$= \tfrac{1}{2}x^{-1/2}\left(-\sqrt{x}\,\sin \sqrt{x} + \cos \sqrt{x}\right) = \dfrac{\cos \sqrt{x} - \sqrt{x}\,\sin \sqrt{x}}{2\sqrt{x}}$

12. $y = (\arcsin 2x)^2 \quad \Rightarrow \quad y' = 2(\arcsin 2x) \cdot (\arcsin 2x)' = 2 \arcsin 2x \cdot \dfrac{1}{\sqrt{1 - (2x)^2}} \cdot 2 = \dfrac{4 \arcsin 2x}{\sqrt{1 - 4x^2}}$

13. $y = \dfrac{e^{1/x}}{x^2} \quad \Rightarrow \quad y' = \dfrac{x^2(e^{1/x})' - e^{1/x}\,(x^2)'}{(x^2)^2} = \dfrac{x^2(e^{1/x})(-1/x^2) - e^{1/x}(2x)}{x^4} = \dfrac{-e^{1/x}(1 + 2x)}{x^4}$

14. $y = \ln \sec x \quad \Rightarrow \quad y' = \dfrac{1}{\sec x}\,\dfrac{d}{dx}\,(\sec x) = \dfrac{1}{\sec x}\,(\sec x \, \tan x) = \tan x$

15. $\dfrac{d}{dx}(y + x\cos y) = \dfrac{d}{dx}(x^2y) \quad\Rightarrow\quad y' + x(-\sin y \cdot y') + \cos y \cdot 1 = x^2y' + y \cdot 2x \quad\Rightarrow$

$y' - x\sin y \cdot y' - x^2y' = 2xy - \cos y \quad\Rightarrow\quad (1 - x\sin y - x^2)y' = 2xy - \cos y \quad\Rightarrow\quad y' = \dfrac{2xy - \cos y}{1 - x\sin y - x^2}$

16. $y = \left(\dfrac{u-1}{u^2+u+1}\right)^4 \quad\Rightarrow$

$$y' = 4\left(\frac{u-1}{u^2+u+1}\right)^3 \frac{d}{du}\left(\frac{u-1}{u^2+u+1}\right) = 4\left(\frac{u-1}{u^2+u+1}\right)^3 \frac{(u^2+u+1)(1)-(u-1)(2u+1)}{(u^2+u+1)^2}$$
$$= \frac{4(u-1)^3}{(u^2+u+1)^3}\,\frac{u^2+u+1-2u^2+u+1}{(u^2+u+1)^2} = \frac{4(u-1)^3(-u^2+2u+2)}{(u^2+u+1)^5}$$

17. $y = \sqrt{\arctan x} \quad\Rightarrow\quad y' = \dfrac{1}{2}(\arctan x)^{-1/2}\,\dfrac{d}{dx}(\arctan x) = \dfrac{1}{2\sqrt{\arctan x}\,(1+x^2)}$

18. $y = \cot(\csc x) \quad\Rightarrow\quad y' = -\csc^2(\csc x)\,\dfrac{d}{dx}(\csc x) = -\csc^2(\csc x)\cdot(-\csc x\,\cot x) = \csc^2(\csc x)\,\csc x\,\cot x$

19. $y = \tan\left(\dfrac{t}{1+t^2}\right) \quad\Rightarrow$

$$y' = \sec^2\left(\frac{t}{1+t^2}\right)\frac{d}{dt}\left(\frac{t}{1+t^2}\right) = \sec^2\left(\frac{t}{1+t^2}\right)\cdot\frac{(1+t^2)(1)-t(2t)}{(1+t^2)^2} = \frac{1-t^2}{(1+t^2)^2}\sec^2\left(\frac{t}{1+t^2}\right)$$

20. $y = e^{x\sec x} \quad\Rightarrow\quad y' = e^{x\sec x}\,\dfrac{d}{dx}(x\sec x) = e^{x\sec x}(x\sec x\,\tan x + \sec x\cdot 1) = \sec x\, e^{x\sec x}(x\tan x + 1)$

21. $y = 3^{x\ln x} \quad\Rightarrow\quad y' = 3^{x\ln x}(\ln 3)\,\dfrac{d}{dx}(x\ln x) = 3^{x\ln x}(\ln 3)\left(x\cdot\dfrac{1}{x} + \ln x\cdot 1\right) = 3^{x\ln x}(\ln 3)(1+\ln x)$

22. $y = \sec(1+x^2) \quad\Rightarrow\quad y' = 2x\sec(1+x^2)\,\tan(1+x^2)$

23. $y = (1-x^{-1})^{-1} \quad\Rightarrow$

$y' = -1(1-x^{-1})^{-2}[-(-1x^{-2})] = -(1-1/x)^{-2}x^{-2} = -((x-1)/x)^{-2}x^{-2} = -(x-1)^{-2}$

24. $y = \dfrac{1}{\sqrt[3]{x+\sqrt{x}}} = \left(x+\sqrt{x}\right)^{-1/3} \quad\Rightarrow\quad y' = -\frac{1}{3}\left(x+\sqrt{x}\right)^{-4/3}\left(1+\dfrac{1}{2\sqrt{x}}\right)$

25. $\sin(xy) = x^2 - y \quad\Rightarrow\quad \cos(xy)(xy' + y\cdot 1) = 2x - y' \quad\Rightarrow\quad x\cos(xy)y' + y' = 2x - y\cos(xy) \quad\Rightarrow$

$y'[x\cos(xy)+1] = 2x - y\cos(xy) \quad\Rightarrow\quad y' = \dfrac{2x - y\cos(xy)}{x\cos(xy)+1}$

26. $y = \sqrt{\sin\sqrt{x}} \quad\Rightarrow\quad y' = \frac{1}{2}\left(\sin\sqrt{x}\right)^{-1/2}\left(\cos\sqrt{x}\right)\left(\dfrac{1}{2\sqrt{x}}\right) = \dfrac{\cos\sqrt{x}}{4\sqrt{x\sin\sqrt{x}}}$

27. $y = \log_5(1+2x) \quad\Rightarrow\quad y' = \dfrac{1}{(1+2x)\ln 5}\,\dfrac{d}{dx}(1+2x) = \dfrac{2}{(1+2x)\ln 5}$

28. $y = (\cos x)^x \quad\Rightarrow\quad \ln y = \ln(\cos x)^x = x\ln\cos x \quad\Rightarrow\quad \dfrac{y'}{y} = x\cdot\dfrac{1}{\cos x}\cdot(-\sin x) + \ln\cos x\cdot 1 \quad\Rightarrow$

$y' = (\cos x)^x(\ln\cos x - x\tan x)$

29. $y = \ln \sin x - \frac{1}{2}\sin^2 x \quad \Rightarrow \quad y' = \dfrac{1}{\sin x} \cdot \cos x - \frac{1}{2} \cdot 2\sin x \cdot \cos x = \cot x - \sin x \, \cos x$

30. $y = \dfrac{(x^2+1)^4}{(2x+1)^3(3x-1)^5} \quad \Rightarrow$

$$\ln y = \ln \frac{(x^2+1)^4}{(2x+1)^3(3x-1)^5} = \ln(x^2+1)^4 - \ln[(2x+1)^3(3x-1)^5] = 4\ln(x^2+1) - [\ln(2x+1)^3 + \ln(3x-1)^5]$$

$$= 4\ln(x^2+1) - 3\ln(2x+1) - 5\ln(3x-1) \quad \Rightarrow$$

$$\frac{y'}{y} = 4 \cdot \frac{1}{x^2+1} \cdot 2x - 3 \cdot \frac{1}{2x+1} \cdot 2 - 5 \cdot \frac{1}{3x-1} \cdot 3 \quad \Rightarrow \quad y' = \frac{(x^2+1)^4}{(2x+1)^3(3x-1)^5}\left(\frac{8x}{x^2+1} - \frac{6}{2x+1} - \frac{15}{3x-1}\right).$$

[The answer could be simplified to $y' = -\dfrac{(x^2+56x+9)(x^2+1)^3}{(2x+1)^4(3x-1)^6}$, but this is unnecessary.]

31. $y = x\tan^{-1}(4x) \quad \Rightarrow \quad y' = x \cdot \dfrac{1}{1+(4x)^2} \cdot 4 + \tan^{-1}(4x) \cdot 1 = \dfrac{4x}{1+16x^2} + \tan^{-1}(4x)$

32. $y = e^{\cos x} + \cos(e^x) \quad \Rightarrow \quad y' = e^{\cos x}(-\sin x) + [-\sin(e^x) \cdot e^x] = -\sin x \, e^{\cos x} - e^x \sin(e^x)$

33. $y = \ln|\sec 5x + \tan 5x| \quad \Rightarrow$

$$y' = \frac{1}{\sec 5x + \tan 5x}(\sec 5x \, \tan 5x \cdot 5 + \sec^2 5x \cdot 5) = \frac{5\sec 5x\,(\tan 5x + \sec 5x)}{\sec 5x + \tan 5x} = 5\sec 5x$$

34. $y = 10^{\tan \pi\theta} \quad \Rightarrow \quad y' = 10^{\tan \pi\theta} \cdot \ln 10 \cdot \sec^2 \pi\theta \cdot \pi = \pi(\ln 10)10^{\tan \pi\theta} \sec^2 \pi\theta$

35. $y = \cot(3x^2+5) \quad \Rightarrow \quad y' = -\csc^2(3x^2+5)(6x) = -6x\csc^2(3x^2+5)$

36. $y = \sqrt{t\ln(t^4)} \quad \Rightarrow$

$$y' = \frac{1}{2}[t\ln(t^4)]^{-1/2}\,\frac{d}{dt}\,[t\ln(t^4)] = \frac{1}{2\sqrt{t\ln(t^4)}} \cdot \left[1 \cdot \ln(t^4) + t \cdot \frac{1}{t^4} \cdot 4t^3\right] = \frac{1}{2\sqrt{t\ln(t^4)}} \cdot [\ln(t^4) + 4] = \frac{\ln(t^4)+4}{2\sqrt{t\ln(t^4)}}$$

Or: Since y is only defined for $t > 0$, we can write $y = \sqrt{t \cdot 4\ln t} = 2\sqrt{t\ln t}$. Then

$y' = 2 \cdot \dfrac{1}{2\sqrt{t\ln t}} \cdot \left(1 \cdot \ln t + t \cdot \dfrac{1}{t}\right) = \dfrac{\ln t + 1}{\sqrt{t\ln t}}$. This agrees with our first answer since

$$\frac{\ln(t^4)+4}{2\sqrt{t\ln(t^4)}} = \frac{4\ln t + 4}{2\sqrt{t \cdot 4\ln t}} = \frac{4(\ln t + 1)}{2 \cdot 2\sqrt{t\ln t}} = \frac{\ln t + 1}{\sqrt{t\ln t}}.$$

37. $y = \sin\!\left(\tan\sqrt{1+x^3}\right) \quad \Rightarrow \quad y' = \cos\!\left(\tan\sqrt{1+x^3}\right)\left(\sec^2\sqrt{1+x^3}\right)\left[3x^2/\left(2\sqrt{1+x^3}\right)\right]$

38. $y = \arctan\!\left(\arcsin\sqrt{x}\right) \quad \Rightarrow \quad y' = \dfrac{1}{1+\left(\arcsin\sqrt{x}\right)^2} \cdot \dfrac{1}{\sqrt{1-x}} \cdot \dfrac{1}{2\sqrt{x}}$

39. $y = \tan^2(\sin\theta) = [\tan(\sin\theta)]^2 \quad \Rightarrow \quad y' = 2[\tan(\sin\theta)] \cdot \sec^2(\sin\theta) \cdot \cos\theta$

40. $xe^y = y - 1 \quad \Rightarrow \quad xe^y y' + e^y = y' \quad \Rightarrow \quad e^y = y' - xe^y y' \quad \Rightarrow \quad y' = e^y/(1-xe^y)$

41. $y = \dfrac{\sqrt{x+1}\,(2-x)^5}{(x+3)^7} \quad\Rightarrow\quad \ln y = \frac{1}{2}\ln(x+1) + 5\ln(2-x) - 7\ln(x+3) \quad\Rightarrow\quad \dfrac{y'}{y} = \dfrac{1}{2(x+1)} + \dfrac{-5}{2-x} - \dfrac{7}{x+3} \quad\Rightarrow$

$y' = \dfrac{\sqrt{x+1}\,(2-x)^5}{(x+3)^7}\left[\dfrac{1}{2(x+1)} - \dfrac{5}{2-x} - \dfrac{7}{x+3}\right]$ or $y' = \dfrac{(2-x)^4(3x^2-55x-52)}{2\sqrt{x+1}\,(x+3)^8}$.

42. $y = \dfrac{(x+\lambda)^4}{x^4+\lambda^4} \quad\Rightarrow\quad y' = \dfrac{(x^4+\lambda^4)(4)(x+\lambda)^3 - (x+\lambda)^4(4x^3)}{(x^4+\lambda^4)^2} = \dfrac{4(x+\lambda)^3(\lambda^4-\lambda x^3)}{(x^4+\lambda^4)^2}$

43. $y = x\sinh(x^2) \quad\Rightarrow\quad y' = x\cosh(x^2)\cdot 2x + \sinh(x^2)\cdot 1 = 2x^2\cosh(x^2) + \sinh(x^2)$

44. $y = (\sin mx)/x \quad\Rightarrow\quad y' = (mx\cos mx - \sin mx)/x^2$

45. $y = \ln(\cosh 3x) \quad\Rightarrow\quad y' = (1/\cosh 3x)(\sinh 3x)(3) = 3\tanh 3x$

46. $y = \ln\left|\dfrac{x^2-4}{2x+5}\right| = \ln\left|x^2-4\right| - \ln|2x+5| \quad\Rightarrow\quad y' = \dfrac{2x}{x^2-4} - \dfrac{2}{2x+5}$ or $\dfrac{2(x+1)(x+4)}{(x+2)(x-2)(2x+5)}$

47. $y = \cosh^{-1}(\sinh x) \quad\Rightarrow\quad y' = \dfrac{1}{\sqrt{(\sinh x)^2-1}}\cdot\cosh x = \dfrac{\cosh x}{\sqrt{\sinh^2 x - 1}}$

48. $y = x\tanh^{-1}\sqrt{x} \quad\Rightarrow\quad y' = \tanh^{-1}\sqrt{x} + x\,\dfrac{1}{1-\left(\sqrt{x}\right)^2}\,\dfrac{1}{2\sqrt{x}} = \tanh^{-1}\sqrt{x} + \dfrac{\sqrt{x}}{2(1-x)}$

49. $y = \cos\left(e^{\sqrt{\tan 3x}}\right) \quad\Rightarrow$

$$y' = -\sin\left(e^{\sqrt{\tan 3x}}\right)\cdot\left(e^{\sqrt{\tan 3x}}\right)' = -\sin\left(e^{\sqrt{\tan 3x}}\right)\, e^{\sqrt{\tan 3x}}\cdot\tfrac{1}{2}(\tan 3x)^{-1/2}\cdot\sec^2(3x)\cdot 3$$

$$= \frac{-3\sin\left(e^{\sqrt{\tan 3x}}\right) e^{\sqrt{\tan 3x}}\,\sec^2(3x)}{2\sqrt{\tan 3x}}$$

50. $y = \sin^2\left(\cos\sqrt{\sin\pi x}\right) = \left[\sin\left(\cos\sqrt{\sin\pi x}\right)\right]^2 \quad\Rightarrow$

$$y' = 2\left[\sin\left(\cos\sqrt{\sin\pi x}\right)\right]\left[\sin\left(\cos\sqrt{\sin\pi x}\right)\right]' = 2\sin\left(\cos\sqrt{\sin\pi x}\right)\cos\left(\cos\sqrt{\sin\pi x}\right)\left(\cos\sqrt{\sin\pi x}\right)'$$

$$= 2\sin\left(\cos\sqrt{\sin\pi x}\right)\cos\left(\cos\sqrt{\sin\pi x}\right)\left(-\sin\sqrt{\sin\pi x}\right)\left(\sqrt{\sin\pi x}\right)'$$

$$= -2\sin\left(\cos\sqrt{\sin\pi x}\right)\cos\left(\cos\sqrt{\sin\pi x}\right)\sin\sqrt{\sin\pi x}\cdot\tfrac{1}{2}(\sin\pi x)^{-1/2}(\sin\pi x)'$$

$$= \frac{-\sin\left(\cos\sqrt{\sin\pi x}\right)\cos\left(\cos\sqrt{\sin\pi x}\right)\sin\sqrt{\sin\pi x}}{\sqrt{\sin\pi x}}\cdot\cos\pi x\cdot\pi$$

$$= \frac{-\pi\sin\left(\cos\sqrt{\sin\pi x}\right)\cos\left(\cos\sqrt{\sin\pi x}\right)\sin\sqrt{\sin\pi x}\,\cos\pi x}{\sqrt{\sin\pi x}}$$

51. $f(t) = \sqrt{4t+1} \quad\Rightarrow\quad f'(t) = \frac{1}{2}(4t+1)^{-1/2}\cdot 4 = 2(4t+1)^{-1/2} \quad\Rightarrow$

$f''(t) = 2(-\frac{1}{2})(4t+1)^{-3/2}\cdot 4 = -4/(4t+1)^{3/2}$, so $f''(2) = -4/9^{3/2} = -\frac{4}{27}$.

52. $g(\theta) = \theta \sin\theta \quad\Rightarrow\quad g'(\theta) = \theta\cos\theta + \sin\theta \cdot 1 \quad\Rightarrow\quad g''(\theta) = \theta(-\sin\theta) + \cos\theta \cdot 1 + \cos\theta = 2\cos\theta - \theta\sin\theta$,

so $g''(\pi/6) = 2\cos(\pi/6) - (\pi/6)\sin(\pi/6) = 2\left(\sqrt{3}/2\right) - (\pi/6)(1/2) = \sqrt{3} - \pi/12$.

53. $x^6 + y^6 = 1 \quad\Rightarrow\quad 6x^5 + 6y^5 y' = 0 \quad\Rightarrow\quad y' = -x^5/y^5 \quad\Rightarrow$

$$y'' = -\frac{y^5(5x^4) - x^5(5y^4 y')}{(y^5)^2} = -\frac{5x^4y^4\left[y - x(-x^5/y^5)\right]}{y^{10}} = -\frac{5x^4\left[(y^6+x^6)/y^5\right]}{y^6} = -\frac{5x^4}{y^{11}}$$

54. $f(x) = (2-x)^{-1} \quad\Rightarrow\quad f'(x) = (2-x)^{-2} \quad\Rightarrow\quad f''(x) = 2(2-x)^{-3} \quad\Rightarrow\quad f'''(x) = 2\cdot 3(2-x)^{-4} \quad\Rightarrow$

$f^{(4)}(x) = 2\cdot 3\cdot 4(2-x)^{-5}$. In general, $f^{(n)}(x) = 2\cdot 3\cdot 4\cdot\cdots\cdot n(2-x)^{-(n+1)} = \dfrac{n!}{(2-x)^{(n+1)}}$.

55. We first show it is true for $n = 1$: $f(x) = xe^x \quad\Rightarrow\quad f'(x) = xe^x + e^x = (x+1)e^x$. We now assume it is true for $n = k$: $f^{(k)}(x) = (x+k)e^x$. With this assumption, we must show it is true for $n = k+1$:

$$f^{(k+1)}(x) = \frac{d}{dx}\left[f^{(k)}(x)\right] = \frac{d}{dx}\left[(x+k)e^x\right] = (x+k)e^x + e^x = [(x+k)+1]e^x = [x+(k+1)]e^x.$$

Therefore, $f^{(n)}(x) = (x+n)e^x$ by mathematical induction.

56. $$\lim_{t\to 0}\frac{t^3}{\tan^3 2t} = \lim_{t\to 0}\frac{t^3\cos^3 2t}{\sin^3 2t} = \lim_{t\to 0}\cos^3 2t\cdot\frac{1}{8\dfrac{\sin^3 2t}{(2t)^3}} = \lim_{t\to 0}\frac{\cos^3 2t}{8\left(\lim\limits_{t\to 0}\dfrac{\sin 2t}{2t}\right)^3} = \frac{1}{8\cdot 1^3} = \frac{1}{8}$$

57. $y = 4\sin^2 x \quad\Rightarrow\quad y' = 4\cdot 2\sin x\cos x$. At $\left(\frac{\pi}{6}, 1\right)$, $y' = 8\cdot\frac{1}{2}\cdot\frac{\sqrt{3}}{2} = 2\sqrt{3}$, so an equation of the tangent line is $y - 1 = 2\sqrt{3}\left(x - \frac{\pi}{6}\right)$, or $y = 2\sqrt{3}\,x + 1 - \pi\sqrt{3}/3$.

58. $y = \dfrac{x^2-1}{x^2+1} \quad\Rightarrow\quad y' = \dfrac{(x^2+1)(2x) - (x^2-1)(2x)}{(x^2+1)^2} = \dfrac{4x}{(x^2+1)^2}$.

At $(0,-1)$, $y' = 0$, so an equation of the tangent line is $y + 1 = 0(x-0)$, or $y = -1$.

59. $y = \sqrt{1+4\sin x} \quad\Rightarrow\quad y' = \frac{1}{2}(1+4\sin x)^{-1/2}\cdot 4\cos x = \dfrac{2\cos x}{\sqrt{1+4\sin x}}$.

At $(0,1)$, $y' = \dfrac{2}{\sqrt{1}} = 2$, so an equation of the tangent line is $y - 1 = 2(x-0)$, or $y = 2x+1$.

60. $x^2 + 4xy + y^2 = 13 \quad\Rightarrow\quad 2x + 4(xy' + y\cdot 1) + 2yy' = 0 \quad\Rightarrow\quad x + 2xy' + 2y + yy' = 0 \quad\Rightarrow$

$2xy' + yy' = -x - 2y \quad\Rightarrow\quad y'(2x+y) = -x-2y \quad\Rightarrow\quad y' = \dfrac{-x-2y}{2x+y}$.

At $(2,1)$, $y' = \dfrac{-2-2}{4+1} = -\dfrac{4}{5}$, so an equation of the tangent line is $y - 1 = -\frac{4}{5}(x-2)$, or $y = -\frac{4}{5}x + \frac{13}{5}$.

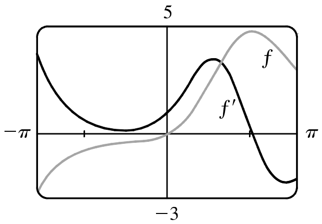
The slope of the normal line is $\frac{5}{4}$, so an equation of the normal line is $y - 1 = \frac{5}{4}(x-2)$, or $y = \frac{5}{4}x - \frac{3}{2}$.

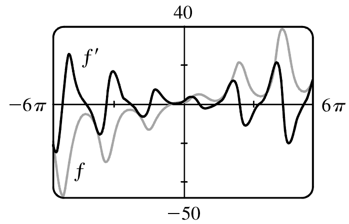
61. $y = (2+x)e^{-x} \quad\Rightarrow\quad y' = (2+x)(-e^{-x}) + e^{-x}\cdot 1 = e^{-x}[-(2+x)+1] = e^{-x}(-x-1)$.

At $(0,2)$, $y' = 1(-1) = -1$, so an equation of the tangent line is $y - 2 = -1(x-0)$, or $y = -x+2$.

The slope of the normal line is 1, so an equation of the normal line is $y - 2 = 1(x-0)$, or $y = x+2$.

62. $f(x) = xe^{\sin x} \quad \Rightarrow \quad f'(x) = x[e^{\sin x}(\cos x)] + e^{\sin x}(1) = e^{\sin x}(x\cos x + 1)$. As a check on our work, we notice from the graphs that $f'(x) > 0$ when f is increasing. Also, we see in the larger viewing rectangle a certain similarity in the graphs of f and f': the sizes of the oscillations of f and f' are linked.





63. (a) $f(x) = x\sqrt{5-x} \quad \Rightarrow$

$$f'(x) = x\left[\frac{1}{2}(5-x)^{-1/2}(-1)\right] + \sqrt{5-x} = \frac{-x}{2\sqrt{5-x}} + \sqrt{5-x}\cdot\frac{2\sqrt{5-x}}{2\sqrt{5-x}} = \frac{-x}{2\sqrt{5-x}} + \frac{2(5-x)}{2\sqrt{5-x}}$$

$$= \frac{-x+10-2x}{2\sqrt{5-x}} = \frac{10-3x}{2\sqrt{5-x}}$$

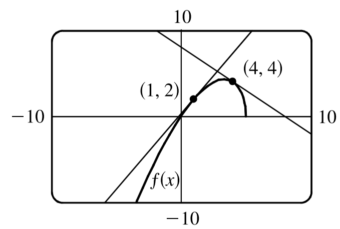
(b) At $(1,2)$: $f'(1) = \frac{7}{4}$.

So an equation of the tangent line is $y - 2 = \frac{7}{4}(x-1)$ or $y = \frac{7}{4}x + \frac{1}{4}$.

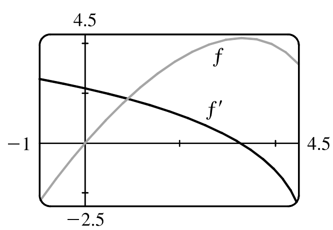
At $(4,4)$: $f'(4) = -\frac{2}{2} = -1$.

So an equation of the tangent line is $y - 4 = -1(x-4)$ or $y = -x + 8$.

(c)



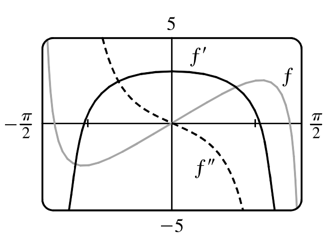
(d)



The graphs look reasonable, since f' is positive where f has tangents with positive slope, and f' is negative where f has tangents with negative slope.

64. (a) $f(x) = 4x - \tan x \quad \Rightarrow \quad f'(x) = 4 - \sec^2 x \quad \Rightarrow \quad f''(x) = -2\sec x\,(\sec x\,\tan x) = -2\sec^2 x\,\tan x$.

(b)



We can see that our answers are reasonable, since the graph of f' is 0 where f has a horizontal tangent, and the graph of f' is positive where f has tangents with positive slope and negative where f has tangents with negative slope. The same correspondence holds between the graphs of f' and f''.

65. $y = \sin x + \cos x \quad \Rightarrow \quad y' = \cos x - \sin x = 0 \quad \Leftrightarrow \quad \cos x = \sin x$ and $0 \le x \le 2\pi \quad \Leftrightarrow \quad x = \frac{\pi}{4}$ or $\frac{5\pi}{4}$, so the points are $\left(\frac{\pi}{4}, \sqrt{2}\right)$ and $\left(\frac{5\pi}{4}, -\sqrt{2}\right)$.

66. $x^2 + 2y^2 = 1 \quad \Rightarrow \quad 2x + 4yy' = 0 \quad \Rightarrow \quad y' = -x/(2y) = 1 \quad \Leftrightarrow \quad x = -2y$. Since the points lie on the ellipse, we have $(-2y)^2 + 2y^2 = 1 \quad \Rightarrow \quad 6y^2 = 1 \quad \Rightarrow \quad y = \pm\frac{1}{\sqrt{6}}$. The points are $\left(-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$ and $\left(\frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right)$.

67. $f(x) = (x-a)(x-b)(x-c) \quad \Rightarrow \quad f'(x) = (x-b)(x-c) + (x-a)(x-c) + (x-a)(x-b)$.

So $\dfrac{f'(x)}{f(x)} = \dfrac{(x-b)(x-c) + (x-a)(x-c) + (x-a)(x-b)}{(x-a)(x-b)(x-c)} = \dfrac{1}{x-a} + \dfrac{1}{x-b} + \dfrac{1}{x-c}$.

Or: $f(x) = (x-a)(x-b)(x-c) \quad \Rightarrow \quad \ln|f(x)| = \ln|x-a| + \ln|x-b| + \ln|x-c| \quad \Rightarrow$

$\dfrac{f'(x)}{f(x)} = \dfrac{1}{x-a} + \dfrac{1}{x-b} + \dfrac{1}{x-c}$

68. (a) $\cos 2x = \cos^2 x - \sin^2 x \quad \Rightarrow \quad -2\sin 2x = -2\cos x \sin x - 2\sin x \cos x \quad \Leftrightarrow \quad \sin 2x = 2\sin x \cos x$

(b) $\sin(x+a) = \sin x \, \cos a + \cos x \, \sin a \quad \Rightarrow \quad \cos(x+a) = \cos x \, \cos a - \sin x \, \sin a$.

69. (a) $S(x) = f(x) + g(x) \quad \Rightarrow \quad S'(x) = f'(x) + g'(x) \quad \Rightarrow \quad S'(1) = f'(1) + g'(1) = 3 + 1 = 4$

(b) $P(x) = f(x)\,g(x) \quad \Rightarrow \quad P'(x) = f(x)\,g'(x) + g(x)\,f'(x) \quad \Rightarrow$

$P'(2) = f(2)\,g'(2) + g(2)f'(2) = 1(4) + 1(2) = 4 + 2 = 6$

(c) $Q(x) = \dfrac{f(x)}{g(x)} \quad \Rightarrow \quad Q'(x) = \dfrac{g(x)\,f'(x) - f(x)\,g'(x)}{[g(x)]^2} \quad \Rightarrow$

$Q'(1) = \dfrac{g(1)\,f'(1) - f(1)\,g'(1)}{[g(1)]^2} = \dfrac{3(3) - 2(1)}{3^2} = \dfrac{9-2}{9} = \dfrac{7}{9}$

(d) $C(x) = f(g(x)) \quad \Rightarrow \quad C'(x) = f'(g(x))\,g'(x) \quad \Rightarrow \quad C'(2) = f'(g(2))\,g'(2) = f'(1) \cdot 4 = 3 \cdot 4 = 12$

70. (a) $P(x) = f(x)\,g(x) \quad \Rightarrow \quad P'(x) = f(x)\,g'(x) + g(x)\,f'(x) \quad \Rightarrow$

$P'(2) = f(2)\,g'(2) + g(2)\,f'(2) = (1)\left(\frac{6-0}{3-0}\right) + (4)\left(\frac{0-3}{3-0}\right) = (1)(2) + (4)(-1) = 2 - 4 = -2$

(b) $Q(x) = \dfrac{f(x)}{g(x)} \quad \Rightarrow \quad Q'(x) = \dfrac{g(x)\,f'(x) - f(x)\,g'(x)}{[g(x)]^2} \quad \Rightarrow$

$Q'(2) = \dfrac{g(2)\,f'(2) - f(2)\,g'(2)}{[g(2)]^2} = \dfrac{(4)(-1) - (1)(2)}{4^2} = \dfrac{-6}{16} = -\dfrac{3}{8}$

(c) $C(x) = f(g(x)) \quad \Rightarrow \quad C'(x) = f'(g(x))g'(x) \quad \Rightarrow$

$C'(2) = f'(g(2))g'(2) = f'(4)g'(2) = \left(\frac{6-0}{5-3}\right)(2) = (3)(2) = 6$

71. $f(x) = x^2 g(x) \quad \Rightarrow \quad f'(x) = x^2 g'(x) + g(x)(2x) = x[xg'(x) + 2g(x)]$

72. $f(x) = g(x^2) \quad \Rightarrow \quad f'(x) = g'(x^2)(2x) = 2xg'(x^2)$

73. $f(x) = [\,g(x)]^2 \quad \Rightarrow \quad f'(x) = 2[\,g(x)] \cdot g'(x) = 2g(x)\,g'(x)$

74. $f(x) = g(g(x)) \quad \Rightarrow \quad f'(x) = g'(g(x))\,g'(x)$

75. $f(x) = g(e^x) \quad \Rightarrow \quad f'(x) = g'(e^x)\,e^x$

76. $f(x) = e^{g(x)} \quad \Rightarrow \quad f'(x) = e^{g(x)} g'(x)$

77. $f(x) = \ln|g(x)| \quad \Rightarrow \quad f'(x) = \dfrac{1}{g(x)} g'(x) = \dfrac{g'(x)}{g(x)}$

78. $f(x) = g(\ln x) \quad \Rightarrow \quad f'(x) = g'(\ln x) \cdot \dfrac{1}{x} = \dfrac{g'(\ln x)}{x}$

79. $h(x) = \dfrac{f(x)\,g(x)}{f(x) + g(x)} \quad \Rightarrow$

$$h'(x) = \frac{[f(x) + g(x)]\,[f(x)\,g'(x) + g(x)\,f'(x)] - f(x)\,g(x)\,[f'(x) + g'(x)]}{[f(x) + g(x)]^2}$$

$$= \frac{[f(x)]^2\,g'(x) + f(x)\,g(x)\,f'(x) + f(x)\,g(x)\,g'(x) + [g(x)]^2\,f'(x) - f(x)\,g(x)\,f'(x) - f(x)\,g(x)\,g'(x)}{[f(x) + g(x)]^2}$$

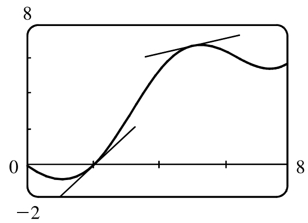
$$= \frac{f'(x)\,[g(x)]^2 + g'(x)\,[f(x)]^2}{[f(x) + g(x)]^2}$$

80. $h(x) = \sqrt{\dfrac{f(x)}{g(x)}} \quad \Rightarrow \quad h'(x) = \dfrac{f'(x)\,g(x) - f(x)\,g'(x)}{2\sqrt{f(x)/g(x)}\,[g(x)]^2} = \dfrac{f'(x)\,g(x) - f(x)\,g'(x)}{2[g(x)]^{3/2}\sqrt{f(x)}}$

81. Using the Chain Rule repeatedly, $h(x) = f(g(\sin 4x)) \quad \Rightarrow$

$$h'(x) = f'(g(\sin 4x)) \cdot \frac{d}{dx}\,(g(\sin 4x)) = f'(g(\sin 4x)) \cdot g'(\sin 4x) \cdot \frac{d}{dx}\,(\sin 4x) = f'(g(\sin 4x))g'(\sin 4x)(\cos 4x)(4).$$

82. (a)



(b) The average rate of change is larger on $[2, 3]$.

(c) The instantaneous rate of change (the slope of the tangent) is larger at $x = 2$.

(d) $f(x) = x - 2\sin x \quad \Rightarrow \quad f'(x) = 1 - 2\cos x$,

so $f'(2) = 1 - 2\cos 2 \approx 1.8323$ and $f'(5) = 1 - 2\cos 5 \approx 0.4327$.

So $f'(2) > f'(5)$, as predicted in part (c).

83. $y = [\ln(x+4)]^2 \quad \Rightarrow \quad y' = 2[\ln(x+4)]^1 \cdot \dfrac{1}{x+4} \cdot 1 = 2\,\dfrac{\ln(x+4)}{x+4}$ and $y' = 0 \quad \Leftrightarrow \quad \ln(x+4) = 0 \quad \Leftrightarrow$

$x + 4 = e^0 \quad \Rightarrow \quad x + 4 = 1 \quad \Leftrightarrow \quad x = -3$, so the tangent is horizontal at the point $(-3, 0)$.

84. (a) The line $x - 4y = 1$ has slope $\frac{1}{4}$. A tangent to $y = e^x$ has slope $\frac{1}{4}$ when $y' = e^x = \frac{1}{4} \quad \Rightarrow \quad x = \ln\frac{1}{4} = -\ln 4$.

Since $y = e^x$, the y-coordinate is $\frac{1}{4}$ and the point of tangency is $\left(-\ln 4, \frac{1}{4}\right)$. Thus, an equation of the tangent line is $y - \frac{1}{4} = \frac{1}{4}(x + \ln 4)$ or $y = \frac{1}{4}x + \frac{1}{4}(\ln 4 + 1)$.

(b) The slope of the tangent at the point (a, e^a) is $\dfrac{d}{dx}\,e^x\Big|_{x=a} = e^a$. Thus, an equation of the tangent line is

$y - e^a = e^a(x - a)$. We substitute $x = 0$, $y = 0$ into this equation, since we want the line to pass through the origin: $0 - e^a = e^a(0 - a) \quad \Leftrightarrow \quad -e^a = e^a(-a) \quad \Leftrightarrow \quad a = 1$. So an equation of the tangent line at the point $(a, e^a) = (1, e)$ is $y - e = e(x - 1)$ or $y = ex$.

85. $y = f(x) = ax^2 + bx + c \quad \Rightarrow \quad f'(x) = 2ax + b$. We know that $f'(-1) = 6$ and $f'(5) = -2$, so $-2a + b = 6$ and $10a + b = -2$. Subtracting the first equation from the second gives $12a = -8 \quad \Rightarrow \quad a = -\frac{2}{3}$. Substituting $-\frac{2}{3}$ for a in the first equation gives $b = \frac{14}{3}$. Now $f(1) = 4 \quad \Rightarrow \quad 4 = a + b + c$, so $c = 4 + \frac{2}{3} - \frac{14}{3} = 0$ and hence, $f(x) = -\frac{2}{3}x^2 + \frac{14}{3}x$.

86. (a) $\lim\limits_{t\to\infty} C(t) = \lim\limits_{t\to\infty}[K(e^{-at} - e^{-bt})] = K\lim\limits_{t\to\infty}(e^{-at} - e^{-bt}) = K(0-0) = 0$ because $-at \to -\infty$ and $-bt \to -\infty$ as $t \to \infty$.

(b) $C(t) = K(e^{-at} - e^{-bt}) \quad\Rightarrow\quad C'(t) = K(e^{-at}(-a) - e^{-bt}(-b)) = K(-ae^{-at} + be^{-bt})$

(c) $C'(t) = 0 \quad\Leftrightarrow\quad be^{-bt} = ae^{-at} \quad\Leftrightarrow\quad \dfrac{b}{a} = e^{(-a+b)t} \quad\Leftrightarrow\quad \ln\dfrac{b}{a} = (b-a)t \quad\Leftrightarrow\quad t = \dfrac{\ln(b/a)}{b-a}$

87. $s(t) = Ae^{-ct}\cos(\omega t + \delta) \quad\Rightarrow$

$v(t) = s'(t) = A\{e^{-ct}\,[-\omega\sin(\omega t+\delta)] + \cos(\omega t+\delta)(-ce^{-ct})\} = -Ae^{-ct}\,[\omega\sin(\omega t+\delta) + c\cos(\omega t+\delta)] \quad\Rightarrow$

$$\begin{aligned}
a(t) = v'(t) &= -A\{e^{-ct}[\omega^2\cos(\omega t+\delta) - c\omega\sin(\omega t+\delta)] + [\omega\sin(\omega t+\delta) + c\cos(\omega t+\delta)](-ce^{-ct})\} \\
&= -Ae^{-ct}[\omega^2\cos(\omega t+\delta) - c\omega\sin(\omega t+\delta) - c\omega\sin(\omega t+\delta) - c^2\cos(\omega t+\delta)] \\
&= -Ae^{-ct}[(\omega^2 - c^2)\cos(\omega t+\delta) - 2c\omega\sin(\omega t+\delta)] = Ae^{-ct}[(c^2-\omega^2)\cos(\omega t+\delta) + 2c\omega\sin(\omega t+\delta)]
\end{aligned}$$

88. (a) $x = \sqrt{b^2 + c^2t^2} \quad\Rightarrow\quad v(t) = x' = \left[1/\left(2\sqrt{b^2+c^2t^2}\right)\right]2c^2t = c^2t/\sqrt{b^2+c^2t^2} \quad\Rightarrow$

$$a(t) = v'(t) = \frac{c^2\sqrt{b^2+c^2t^2} - c^2t\left(c^2t/\sqrt{b^2+c^2t^2}\right)}{b^2+c^2t^2} = \frac{b^2c^2}{(b^2+c^2t^2)^{3/2}}$$

(b) $v(t) > 0$ for $t > 0$, so the particle always moves in the positive direction.

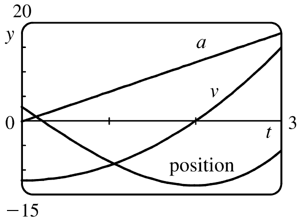
89. (a) $y = t^3 - 12t + 3 \quad\Rightarrow\quad v(t) = y' = 3t^2 - 12 \quad\Rightarrow\quad a(t) = v'(t) = 6t$

(b) $v(t) = 3(t^2-4) > 0$ when $t > 2$, so it moves upward when $t > 2$ and downward when $0 \le t < 2$.

(c) Distance upward $= y(3) - y(2) = -6 - (-13) = 7$,

Distance downward $= y(0) - y(2) = 3 - (-13) = 16$. Total distance $= 7 + 16 = 23$.

(d)



(e) The particle is speeding up when v and a have the same sign, that is, when $t > 2$. The particle is slowing down when v and a have opposite signs; that is, when $0 < t < 2$.

90. (a) $V = \frac{1}{3}\pi r^2 h \quad\Rightarrow\quad dV/dh = \frac{1}{3}\pi r^2 \quad [r \text{ constant}]$

(b) $V = \frac{1}{3}\pi r^2 h \quad\Rightarrow\quad dV/dr = \frac{2}{3}\pi rh \quad [h \text{ constant}]$

91. The linear density ρ is the rate of change of mass m with respect to length x.

$m = x\left(1+\sqrt{x}\right) = x + x^{3/2} \quad\Rightarrow\quad \rho = dm/dx = 1 + \frac{3}{2}\sqrt{x}$, so the linear density when $x = 4$ is $1 + \frac{3}{2}\sqrt{4} = 4$ kg/m.

92. (a) $C(x) = 920 + 2x - 0.02x^2 + 0.00007x^3 \quad\Rightarrow\quad C'(x) = 2 - 0.04x + 0.00021x^2$

(b) $C'(100) = 2 - 4 + 2.1 = \$0.10/\text{unit}$. This value represents the rate at which costs are increasing as the hundredth unit is produced, and is the approximate cost of producing the 101st unit.

(c) The cost of producing the 101st item is $C(101) - C(100) = 990.10107 - 990 = \$0.10107$, slightly larger than $C'(100)$.

93. (a) $y(t) = y(0)e^{kt} = 200e^{kt} \Rightarrow y(0.5) = 200e^{0.5k} = 360 \Rightarrow e^{0.5k} = 1.8 \Rightarrow 0.5k = \ln 1.8 \Rightarrow$
$k = 2\ln 1.8 = \ln(1.8)^2 = \ln 3.24 \Rightarrow y(t) = 200e^{(\ln 3.24)t} = 200(3.24)^t$

(b) $y(4) = 200(3.24)^4 \approx 22{,}040$ bacteria

(c) $y'(t) = 200(3.24)^t \cdot \ln 3.24$, so $y'(4) = 200(3.24)^4 \cdot \ln 3.24 \approx 25{,}910$ bacteria per hour

(d) $200(3.24)^t = 10{,}000 \Rightarrow (3.24)^t = 50 \Rightarrow t\ln 3.24 = \ln 50 \Rightarrow t = \ln 50/\ln 3.24 \approx 3.33$ hours

94. (a) If $y(t)$ is the mass remaining after t years, then $y(t) = y(0)e^{kt} = 100e^{kt}$. $y(5.24) = 100e^{5.24k} = \frac{1}{2} \cdot 100 \Rightarrow$
$e^{5.24k} = \frac{1}{2} \Rightarrow 5.24k = -\ln 2 \Rightarrow k = -\frac{1}{5.24}\ln 2 \Rightarrow y(t) = 100e^{-(\ln 2)t/5.24} = 100 \cdot 2^{-t/5.24}$. Thus,
$y(20) = 100 \cdot 2^{-20/5.24} \approx 7.1$ mg.

(b) $100 \cdot 2^{-t/5.24} = 1 \Rightarrow 2^{-t/5.24} = \dfrac{1}{100} \Rightarrow -\dfrac{t}{5.24}\ln 2 = \ln\dfrac{1}{100} \Rightarrow t = 5.24\dfrac{\ln 100}{\ln 2} \approx 34.8$ years

95. (a) $C'(t) = -kC(t) \Rightarrow C(t) = C(0)e^{-kt}$ by Theorem 3.8.2. But $C(0) = C_0$, so $C(t) = C_0e^{-kt}$.

(b) $C(30) = \frac{1}{2}C_0$ since the concentration is reduced by half. Thus, $\frac{1}{2}C_0 = C_0e^{-30k} \Rightarrow \ln\frac{1}{2} = -30k \Rightarrow$
$k = -\frac{1}{30}\ln\frac{1}{2} = \frac{1}{30}\ln 2$. Since 10% of the original concentration remains if 90% is eliminated, we want the value of t such that $C(t) = \frac{1}{10}C_0$. Therefore, $\frac{1}{10}C_0 = C_0e^{-t(\ln 2)/30} \Rightarrow \ln 0.1 = -t(\ln 2)/30 \Rightarrow t = -\frac{30}{\ln 2}\ln 0.1 \approx 100$ h.

96. (a) If $y = u - 20$, $u(0) = 80 \Rightarrow y(0) = 80 - 20 = 60$, and the initial-value problem is $dy/dt = ky$ with $y(0) = 60$.
So the solution is $y(t) = 60e^{kt}$. Now $y(0.5) = 60e^{k(0.5)} = 60 - 20 \Rightarrow e^{0.5k} = \frac{40}{60} = \frac{2}{3} \Rightarrow k = 2\ln\frac{2}{3} = \ln\frac{4}{9}$,
so $y(t) = 60e^{(\ln 4/9)t} = 60(\frac{4}{9})^t$. Thus, $y(1) = 60(\frac{4}{9})^1 = \frac{80}{3} = 26\frac{2}{3}$ °C and $u(1) = 46\frac{2}{3}$ °C.

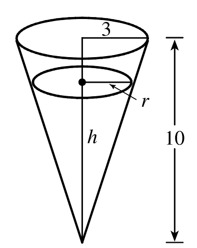
(b) $u(t) = 40 \Rightarrow y(t) = 20$. $y(t) = 60\left(\dfrac{4}{9}\right)^t = 20 \Rightarrow \left(\dfrac{4}{9}\right)^t = \dfrac{1}{3} \Rightarrow t\ln\dfrac{4}{9} = \ln\dfrac{1}{3} \Rightarrow t = \dfrac{\ln\frac{1}{3}}{\ln\frac{4}{9}} \approx 1.35$ h
or 81.3 min.

97. If x = edge length, then $V = x^3 \Rightarrow dV/dt = 3x^2\,dx/dt = 10 \Rightarrow dx/dt = 10/(3x^2)$ and $S = 6x^2 \Rightarrow$
$dS/dt = (12x)\,dx/dt = 12x[10/(3x^2)] = 40/x$. When $x = 30$, $dS/dt = \frac{40}{30} = \frac{4}{3}$ cm²/min.

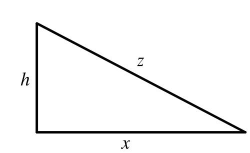
98. Given $dV/dt = 2$, find dh/dt when $h = 5$. $V = \frac{1}{3}\pi r^2 h$ and, from similar triangles, $\dfrac{r}{h} = \dfrac{3}{10} \Rightarrow V = \dfrac{\pi}{3}\left(\dfrac{3h}{10}\right)^2 h = \dfrac{3\pi}{100}h^3$, so

$$2 = \frac{dV}{dt} = \frac{9\pi}{100}h^2\,\frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{200}{9\pi h^2} = \frac{200}{9\pi\,(5)^2} = \frac{8}{9\pi}\text{ cm/s}$$

when $h = 5$.

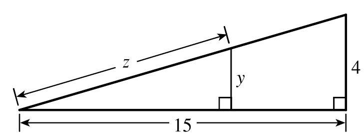


99. Given $dh/dt = 5$ and $dx/dt = 15$, find dz/dt. $z^2 = x^2 + h^2 \Rightarrow$
$2z\dfrac{dz}{dt} = 2x\dfrac{dx}{dt} + 2h\dfrac{dh}{dt} \Rightarrow \dfrac{dz}{dt} = \dfrac{1}{z}(15x + 5h)$. When $t = 3$,
$h = 45 + 3(5) = 60$ and $x = 15(3) = 45 \Rightarrow z = \sqrt{45^2 + 60^2} = 75$,
so $\dfrac{dz}{dt} = \dfrac{1}{75}[15(45) + 5(60)] = 13$ ft/s.



100. We are given $dz/dt = 30$ ft/s. By similar triangles, $\dfrac{y}{z} = \dfrac{4}{\sqrt{241}} \quad \Rightarrow$

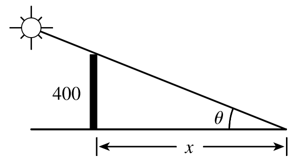
$y = \dfrac{4}{\sqrt{241}} z$, so $\dfrac{dy}{dt} = \dfrac{4}{\sqrt{241}} \dfrac{dz}{dt} = \dfrac{120}{\sqrt{241}} \approx 7.7$ ft/s.



101. We are given $d\theta/dt = -0.25$ rad/h. $\tan\theta = 400/x \quad \Rightarrow$

$x = 400 \cot\theta \quad \Rightarrow \quad \dfrac{dx}{dt} = -400 \csc^2\theta \, \dfrac{d\theta}{dt}$. When $\theta = \frac{\pi}{6}$,

$\dfrac{dx}{dt} = -400(2)^2(-0.25) = 400$ ft/h.

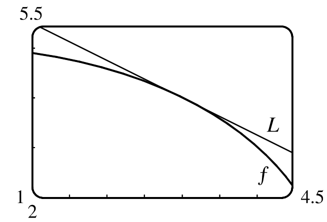


102. (a) $f(x) = \sqrt{25 - x^2} \quad \Rightarrow \quad f'(x) = \dfrac{-2x}{2\sqrt{25 - x^2}} = -x(25 - x^2)^{-1/2}$.

So the linear approximation to $f(x)$ near 3

is $f(x) \approx f(3) + f'(3)(x - 3) = 4 - \frac{3}{4}(x - 3)$.

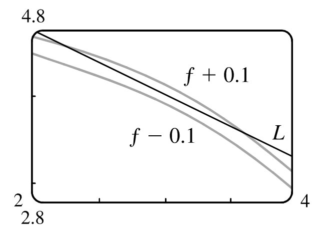
(b)



(c) For the required accuracy, we want $\sqrt{25 - x^2} - 0.1 < 4 - \frac{3}{4}(x - 3)$ and

$4 - \frac{3}{4}(x - 3) < \sqrt{25 - x^2} + 0.1$. From the graph, it appears that these both

hold for $2.24 < x < 3.66$.



103. (a) $f(x) = \sqrt[3]{1 + 3x} = (1 + 3x)^{1/3} \quad \Rightarrow \quad f'(x) = (1 + 3x)^{-2/3}$, so the linearization of f at $a = 0$ is

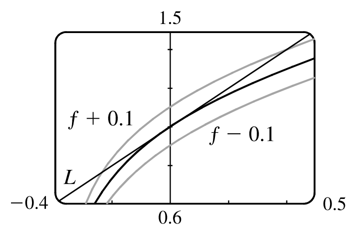
$L(x) = f(0) + f'(0)(x - 0) = 1^{1/3} + 1^{-2/3}x = 1 + x$. Thus, $\sqrt[3]{1 + 3x} \approx 1 + x \quad \Rightarrow$

$\sqrt[3]{1.03} = \sqrt[3]{1 + 3(0.01)} \approx 1 + (0.01) = 1.01$.

(b) The linear approximation is $\sqrt[3]{1 + 3x} \approx 1 + x$, so for the required accuracy

we want $\sqrt[3]{1 + 3x} - 0.1 < 1 + x < \sqrt[3]{1 + 3x} + 0.1$. From the graph,

it appears that this is true when $-0.235 < x < 0.401$.

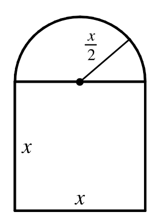


104. $y = x^3 - 2x^2 + 1 \quad \Rightarrow \quad dy = (3x^2 - 4x)\,dx$. When $x = 2$ and $dx = 0.2$, $dy = \left[3(2)^2 - 4(2)\right](0.2) = 0.8$.

105. $A = x^2 + \frac{1}{2}\pi\left(\frac{1}{2}x\right)^2 = \left(1 + \frac{\pi}{8}\right)x^2 \quad \Rightarrow \quad dA = \left(2 + \frac{\pi}{4}\right)x\,dx$. When $x = 60$

and $dx = 0.1$, $dA = \left(2 + \frac{\pi}{4}\right)60(0.1) = 12 + \frac{3\pi}{2}$, so the maximum error is

approximately $12 + \frac{3\pi}{2} \approx 16.7 \text{ cm}^2$.



106. $\lim\limits_{x\to 1}\dfrac{x^{17}-1}{x-1}=\left[\dfrac{d}{dx}\,x^{17}\right]_{x\,=\,1}=17(1)^{16}=17$

107. $\lim\limits_{h\to 0}\dfrac{\sqrt[4]{16+h}-2}{h}=\left[\dfrac{d}{dx}\,\sqrt[4]{x}\right]_{x\,=\,16}=\dfrac{1}{4}x^{-3/4}\Big|_{x\,=\,16}=\dfrac{1}{4\left(\sqrt[4]{16}\right)^3}=\dfrac{1}{32}$

108. $\lim\limits_{\theta\to\pi/3}\dfrac{\cos\theta-0.5}{\theta-\pi/3}=\left[\dfrac{d}{d\theta}\cos\theta\right]_{\theta\,=\,\pi/3}=-\sin\dfrac{\pi}{3}=-\dfrac{\sqrt{3}}{2}$

109.
$$\begin{aligned}
\lim_{x\to 0}\frac{\sqrt{1+\tan x}-\sqrt{1+\sin x}}{x^3}&=\lim_{x\to 0}\frac{\left(\sqrt{1+\tan x}-\sqrt{1+\sin x}\right)\left(\sqrt{1+\tan x}+\sqrt{1+\sin x}\right)}{x^3\left(\sqrt{1+\tan x}+\sqrt{1+\sin x}\right)}\\
&=\lim_{x\to 0}\frac{(1+\tan x)-(1+\sin x)}{x^3\left(\sqrt{1+\tan x}+\sqrt{1+\sin x}\right)}=\lim_{x\to 0}\frac{\sin x\,(1/\cos x-1)}{x^3\left(\sqrt{1+\tan x}+\sqrt{1+\sin x}\right)}\cdot\frac{\cos x}{\cos x}\\
&=\lim_{x\to 0}\frac{\sin x\,(1-\cos x)}{x^3\left(\sqrt{1+\tan x}+\sqrt{1+\sin x}\right)\cos x}\cdot\frac{1+\cos x}{1+\cos x}\\
&=\lim_{x\to 0}\frac{\sin x\cdot\sin^2 x}{x^3\left(\sqrt{1+\tan x}+\sqrt{1+\sin x}\right)\cos x\,(1+\cos x)}\\
&=\left(\lim_{x\to 0}\frac{\sin x}{x}\right)^3\lim_{x\to 0}\frac{1}{\left(\sqrt{1+\tan x}+\sqrt{1+\sin x}\right)\cos x\,(1+\cos x)}\\
&=1^3\cdot\frac{1}{\left(\sqrt{1}+\sqrt{1}\right)\cdot 1\cdot(1+1)}=\frac{1}{4}
\end{aligned}$$

110. Differentiating the first given equation implicitly with respect to x and using the Chain Rule, we obtain $f(g(x))=x \quad\Rightarrow$ $f'(g(x))\,g'(x)=1 \quad\Rightarrow\quad g'(x)=\dfrac{1}{f'(g(x))}$. Using the second given equation to expand the denominator of this expression gives $g'(x)=\dfrac{1}{1+[f(g(x))]^2}$. But the first given equation states that $f(g(x))=x$, so $g'(x)=\dfrac{1}{1+x^2}$.

111. $\dfrac{d}{dx}\,[f(2x)]=x^2 \quad\Rightarrow\quad f'(2x)\cdot 2=x^2 \quad\Rightarrow\quad f'(2x)=\frac{1}{2}x^2$. Let $t=2x$. Then $f'(t)=\frac{1}{2}\left(\frac{1}{2}t\right)^2=\frac{1}{8}t^2$, so $f'(x)=\frac{1}{8}x^2$.

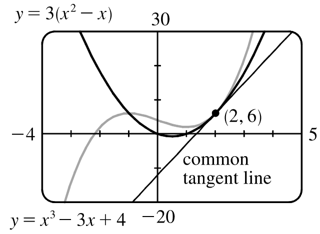
112. Let (b,c) be on the curve, that is, $b^{2/3}+c^{2/3}=a^{2/3}$. Now $x^{2/3}+y^{2/3}=a^{2/3} \quad\Rightarrow\quad \frac{2}{3}x^{-1/3}+\frac{2}{3}y^{-1/3}\dfrac{dy}{dx}=0$, so $\dfrac{dy}{dx}=-\dfrac{y^{1/3}}{x^{1/3}}=-\left(\dfrac{y}{x}\right)^{1/3}$, so at (b,c) the slope of the tangent line is $-(c/b)^{1/3}$ and an equation of the tangent line is $y-c=-(c/b)^{1/3}(x-b)$ or $y=-(c/b)^{1/3}x+(c+b^{2/3}c^{1/3})$. Setting $y=0$, we find that the x-intercept is $b^{1/3}c^{2/3}+b=b^{1/3}(c^{2/3}+b^{2/3})=b^{1/3}a^{2/3}$ and setting $x=0$ we find that the y-intercept is $c+b^{2/3}c^{1/3}=c^{1/3}(c^{2/3}+b^{2/3})=c^{1/3}a^{2/3}$. So the length of the tangent line between these two points is

$$\begin{aligned}
\sqrt{(b^{1/3}a^{2/3})^2+(c^{1/3}a^{2/3})^2}&=\sqrt{b^{2/3}a^{4/3}+c^{2/3}a^{4/3}}=\sqrt{(b^{2/3}+c^{2/3})a^{4/3}}\\
&=\sqrt{a^{2/3}a^{4/3}}=\sqrt{a^2}=a=\text{constant}
\end{aligned}$$

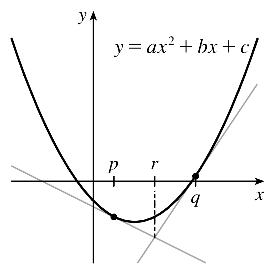
PROBLEMS PLUS

1. Let a be the x-coordinate of Q. Since the derivative of $y = 1 - x^2$ is $y' = -2x$, the slope at Q is $-2a$. But since the triangle is equilateral, $\overline{AO}/\overline{OC} = \sqrt{3}/1$, so the slope at Q is $-\sqrt{3}$. Therefore, we must have that $-2a = -\sqrt{3} \quad\Rightarrow\quad a = \frac{\sqrt{3}}{2}$. Thus, the point Q has coordinates $\left(\frac{\sqrt{3}}{2}, 1 - \left(\frac{\sqrt{3}}{2}\right)^2\right) = \left(\frac{\sqrt{3}}{2}, \frac{1}{4}\right)$ and by symmetry, P has coordinates $\left(-\frac{\sqrt{3}}{2}, \frac{1}{4}\right)$.

2. $y = x^3 - 3x + 4 \quad\Rightarrow\quad y' = 3x^2 - 3$, and $y = 3(x^2 - x) \quad\Rightarrow\quad y' = 6x - 3$. The slopes of the tangents of the two curves are equal when $3x^2 - 3 = 6x - 3$; that is, when $x = 0$ or 2. At $x = 0$, both tangents have slope -3, but the curves do not intersect. At $x = 2$, both tangents have slope 9 and the curves intersect at $(2, 6)$. So there is a common tangent line at $(2, 6)$, $y = 9x - 12$.



3.



We must show that r (in the figure) is halfway between p and q, that is, $r = (p + q)/2$. For the parabola $y = ax^2 + bx + c$, the slope of the tangent line is given by $y' = 2ax + b$. An equation of the tangent line at $x = p$ is $y - (ap^2 + bp + c) = (2ap + b)(x - p)$. Solving for y gives us

$$y = (2ap + b)x - 2ap^2 - bp + (ap^2 + bp + c)$$

or

$$y = (2ap + b)x + c - ap^2 \qquad \textbf{(1)}$$

Similarly, an equation of the tangent line at $x = q$ is

$$y = (2aq + b)x + c - aq^2 \qquad \textbf{(2)}$$

We can eliminate y and solve for x by subtracting equation **(1)** from equation **(2)**.

$$\begin{aligned} [(2aq + b) - (2ap + b)]x - aq^2 + ap^2 &= 0 \\ (2aq - 2ap)x &= aq^2 - ap^2 \\ 2a(q - p)x &= a(q^2 - p^2) \\ x &= \frac{a(q + p)(q - p)}{2a(q - p)} = \frac{p + q}{2} \end{aligned}$$

Thus, the x-coordinate of the point of intersection of the two tangent lines, namely r, is $(p + q)/2$.

4. We could differentiate and then simplify or we can simplify and then differentiate. The latter seems to be the simpler method.

$$\begin{aligned} \frac{\sin^2 x}{1 + \cot x} + \frac{\cos^2 x}{1 + \tan x} &= \frac{\sin^2 x}{1 + \dfrac{\cos x}{\sin x}} \cdot \frac{\sin x}{\sin x} + \frac{\cos^2 x}{1 + \dfrac{\sin x}{\cos x}} \cdot \frac{\cos x}{\cos x} = \frac{\sin^3 x}{\sin x + \cos x} + \frac{\cos^3 x}{\cos x + \sin x} \\ &= \frac{\sin^3 x + \cos^3 x}{\sin x + \cos x} \quad [\text{factor sum of cubes}] \quad = \frac{(\sin x + \cos x)(\sin^2 x - \sin x\,\cos x + \cos^2 x)}{\sin x + \cos x} \\ &= \sin^2 x - \sin x\,\cos x + \cos^2 x = 1 - \sin x\,\cos x = 1 - \tfrac{1}{2}(2 \sin x\,\cos x) = 1 - \tfrac{1}{2}\sin 2x \end{aligned}$$

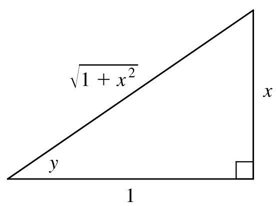
Thus, $\dfrac{d}{dx}\left(\dfrac{\sin^2 x}{1 + \cot x} + \dfrac{\cos^2 x}{1 + \tan x}\right) = \dfrac{d}{dx}\left(1 - \tfrac{1}{2}\sin 2x\right) = -\tfrac{1}{2}\cos 2x \cdot 2 = -\cos 2x$.

5. Using $f'(a) = \lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a}$, we recognize the given expression, $f(x) = \lim\limits_{t \to x} \dfrac{\sec t - \sec x}{t - x}$, as $g'(x)$ with $g(x) = \sec x$. Now $f'(\frac{\pi}{4}) = g''(\frac{\pi}{4})$, so we will find $g''(x)$. $g'(x) = \sec x \tan x \quad \Rightarrow$ $g''(x) = \sec x \sec^2 x + \tan x \sec x \tan x = \sec x(\sec^2 x + \tan^2 x)$, so $g''(\frac{\pi}{4}) = \sqrt{2}(\sqrt{2}^2 + 1^2) = \sqrt{2}(2 + 1) = 3\sqrt{2}$.

6. Using $f'(0) = \lim\limits_{x \to 0} \dfrac{f(x) - f(0)}{x - 0}$, we see that for the given equation, $\lim\limits_{x \to 0} \dfrac{\sqrt[3]{ax + b} - 2}{x} = \dfrac{5}{12}$, we have $f(x) = \sqrt[3]{ax + b}$, $f(0) = 2$, and $f'(0) = \frac{5}{12}$. Now $f(0) = 2 \quad \Leftrightarrow \quad \sqrt[3]{b} = 2 \quad \Leftrightarrow \quad b = 8$. Also $f'(x) = \frac{1}{3}(ax + b)^{-2/3} \cdot a$, so $f'(0) = \frac{5}{12}$ $\Leftrightarrow \quad \frac{1}{3}(8)^{-2/3} \cdot a = \frac{5}{12} \quad \Leftrightarrow \quad \frac{1}{3}(\frac{1}{4})a = \frac{5}{12} \quad \Leftrightarrow \quad a = 5$.

7. Let $y = \tan^{-1} x$. Then $\tan y = x$, so from the triangle we see that $\sin(\tan^{-1} x) = \sin y = \dfrac{x}{\sqrt{1 + x^2}}$. Using this fact we have that

$$\sin(\tan^{-1}(\sinh x)) = \frac{\sinh x}{\sqrt{1 + \sinh^2 x}} = \frac{\sinh x}{\cosh x} = \tanh x.$$



Hence, $\sin^{-1}(\tanh x) = \sin^{-1}(\sin(\tan^{-1}(\sinh x))) = \tan^{-1}(\sinh x)$.

8. We find the equation of the parabola by substituting the point $(-100, 100)$, at which the car is situated, into the general equation $y = ax^2$: $100 = a(-100)^2 \quad \Rightarrow \quad a = \frac{1}{100}$. Now we find the equation of a tangent to the parabola at the point (x_0, y_0). We can show that $y' = a(2x) = \frac{1}{100}(2x) = \frac{1}{50}x$, so an equation of the tangent is $y - y_0 = \frac{1}{50}x_0(x - x_0)$. Since the point (x_0, y_0) is on the parabola, we must have $y_0 = \frac{1}{100}x_0^2$, so our equation of the tangent can be simplified to $y = \frac{1}{100}x_0^2 + \frac{1}{50}x_0(x - x_0)$. We want the statue to be located on the tangent line, so we substitute its coordinates $(100, 50)$ into this equation: $50 = \frac{1}{100}x_0^2 + \frac{1}{50}x_0(100 - x_0) \quad \Rightarrow \quad x_0^2 - 200x_0 + 5000 = 0 \quad \Rightarrow$ $x_0 = \frac{1}{2}\left[200 \pm \sqrt{200^2 - 4(5000)}\right] \quad \Rightarrow \quad x_0 = 100 \pm 50\sqrt{2}$. But $x_0 < 100$, so the car's headlights illuminate the statue when it is located at the point $\left(100 - 50\sqrt{2}, 150 - 100\sqrt{2}\right) \approx (29.3, 8.6)$, that is, about 29.3 m east and 8.6 m north of the origin.

9. We use mathematical induction. Let S_n be the statement that $\dfrac{d^n}{dx^n}(\sin^4 x + \cos^4 x) = 4^{n-1}\cos(4x + n\pi/2)$.

S_1 is true because

$$\begin{aligned}
\frac{d}{dx}(\sin^4 x + \cos^4 x) &= 4\sin^3 x\,\cos x - 4\cos^3 x\,\sin x = 4\sin x\,\cos x\left(\sin^2 x - \cos^2 x\right)x \\
&= -4\sin x\,\cos x\,\cos 2x = -2\sin 2x\,\cos 2 = -\sin 4x = \sin(-4x) \\
&= \cos\left(\tfrac{\pi}{2} - (-4x)\right) = \cos\left(\tfrac{\pi}{2} + 4x\right) = 4^{n-1}\cos\left(4x + n\tfrac{\pi}{2}\right) \text{ when } n = 1
\end{aligned}$$

[continued]

Now assume S_k is true, that is, $\dfrac{d^k}{dx^k}\left(\sin^4 x + \cos^4 x\right) = 4^{k-1}\cos\!\left(4x + k\frac{\pi}{2}\right)$. Then

$$\begin{aligned}\frac{d^{k+1}}{dx^{k+1}}\left(\sin^4 x + \cos^4 x\right) &= \frac{d}{dx}\left[\frac{d^k}{dx^k}\left(\sin^4 x + \cos^4 x\right)\right] = \frac{d}{dx}\left[4^{k-1}\cos\!\left(4x + k\tfrac{\pi}{2}\right)\right] \\ &= -4^{k-1}\sin\!\left(4x + k\tfrac{\pi}{2}\right)\cdot\frac{d}{dx}\left(4x + k\tfrac{\pi}{2}\right) = -4^k\sin\!\left(4x + k\tfrac{\pi}{2}\right) \\ &= 4^k\sin\!\left(-4x - k\tfrac{\pi}{2}\right) = 4^k\cos\!\left(\tfrac{\pi}{2} - \left(-4x - k\tfrac{\pi}{2}\right)\right) = 4^k\cos\!\left(4x + (k+1)\,\tfrac{\pi}{2}\right)\end{aligned}$$

which shows that S_{k+1} is true.

Therefore, $\dfrac{d^n}{dx^n}\left(\sin^4 x + \cos^4 x\right) = 4^{n-1}\cos\!\left(4x + n\frac{\pi}{2}\right)$ for every positive integer n, by mathematical induction.

Another proof: First write

$$\sin^4 x + \cos^4 x = (\sin^2 x + \cos^2 x)^2 - 2\sin^2 x\,\cos^2 x = 1 - \tfrac{1}{2}\sin^2 2x = 1 - \tfrac{1}{4}(1 - \cos 4x) = \tfrac{3}{4} + \tfrac{1}{4}\cos 4x$$

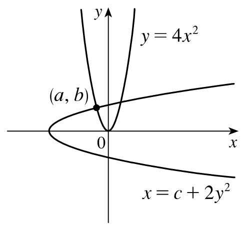
Then we have $\dfrac{d^n}{dx^n}\left(\sin^4 x + \cos^4 x\right) = \dfrac{d^n}{dx^n}\left(\dfrac{3}{4} + \dfrac{1}{4}\cos 4x\right) = \dfrac{1}{4}\cdot 4^n\cos\!\left(4x + n\dfrac{\pi}{2}\right) = 4^{n-1}\cos\!\left(4x + n\dfrac{\pi}{2}\right)$.

10. $$\begin{aligned}\lim_{x\to a}\frac{f(x)-f(a)}{\sqrt{x}-\sqrt{a}} &= \lim_{x\to a}\left[\frac{f(x)-f(a)}{\sqrt{x}-\sqrt{a}}\cdot\frac{\sqrt{x}+\sqrt{a}}{\sqrt{x}+\sqrt{a}}\right] = \lim_{x\to a}\left[\frac{f(x)-f(a)}{x-a}\cdot\left(\sqrt{x}+\sqrt{a}\right)\right] \\ &= \lim_{x\to a}\frac{f(x)-f(a)}{x-a}\cdot\lim_{x\to a}\left(\sqrt{x}+\sqrt{a}\right) = f'(a)\cdot\left(\sqrt{a}+\sqrt{a}\right) = 2\sqrt{a}\,f'(a)\end{aligned}$$

11. We must find a value x_0 such that the normal lines to the parabola $y = x^2$ at $x = \pm x_0$ intersect at a point one unit from the points $\left(\pm x_0, x_0^2\right)$. The normals to $y = x^2$ at $x = \pm x_0$ have slopes $-\dfrac{1}{\pm 2x_0}$ and pass through $\left(\pm x_0, x_0^2\right)$ respectively, so the normals have the equations $y - x_0^2 = -\dfrac{1}{2x_0}(x - x_0)$ and $y - x_0^2 = \dfrac{1}{2x_0}(x + x_0)$. The common y-intercept is $x_0^2 + \dfrac{1}{2}$. We want to find the value of x_0 for which the distance from $\left(0, x_0^2 + \frac{1}{2}\right)$ to $\left(x_0, x_0^2\right)$ equals 1. The square of the distance is $(x_0 - 0)^2 + \left[x_0^2 - \left(x_0^2 + \frac{1}{2}\right)\right]^2 = x_0^2 + \frac{1}{4} = 1 \quad\Leftrightarrow\quad x_0 = \pm\frac{\sqrt{3}}{2}$. For these values of x_0, the y-intercept is $x_0^2 + \frac{1}{2} = \frac{5}{4}$, so the center of the circle is at $\left(0, \frac{5}{4}\right)$.

Another solution: Let the center of the circle be $(0, a)$. Then the equation of the circle is $x^2 + (y - a)^2 = 1$. Solving with the equation of the parabola, $y = x^2$, we get $x^2 + (x^2 - a)^2 = 1 \quad\Leftrightarrow\quad x^2 + x^4 - 2ax^2 + a^2 = 1 \quad\Leftrightarrow\quad x^4 + (1 - 2a)x^2 + a^2 - 1 = 0$. The parabola and the circle will be tangent to each other when this quadratic equation in x^2 has equal roots; that is, when the discriminant is 0. Thus, $(1 - 2a)^2 - 4(a^2 - 1) = 0 \quad\Leftrightarrow\quad 1 - 4a + 4a^2 - 4a^2 + 4 = 0 \quad\Leftrightarrow\quad 4a = 5$, so $a = \frac{5}{4}$. The center of the circle is $\left(0, \frac{5}{4}\right)$.

12. See the figure. The parabolas $y = 4x^2$ and $x = c + 2y^2$ intersect each other at right angles at the point (a, b) if and only if (a, b) satisfies both equations and the tangent lines at (a, b) are perpendicular. $y = 4x^2 \;\Rightarrow\; y' = 8x$ and $x = c + 2y^2 \;\Rightarrow\; 1 = 4y\,y' \;\Rightarrow\; y' = \dfrac{1}{4y}$, so at (a, b) we must

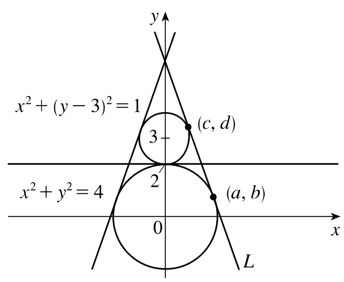


have $8a = -\dfrac{1}{1/(4b)} \;\Rightarrow\; 8a = -4b \;\Rightarrow\; b = -2a$. Since (a, b) is on both parabolas, we have **(1)** $b = 4a^2$ and **(2)** $a = c + 2b^2$. Substituting $-2a$ for b in **(1)** gives us $-2a = 4a^2 \;\Rightarrow\; 4a^2 + 2a = 0 \;\Rightarrow\; 2a(2a+1) = 0 \;\Rightarrow\; a = 0$ or $a = -\frac{1}{2}$.

If $a = 0$, then $b = 0$ and $c = 0$, and the tangent lines at $(0, 0)$ are $y = 0$ and $x = 0$.

If $a = -\frac{1}{2}$, then $b = -2\left(-\frac{1}{2}\right) = 1$ and $-\frac{1}{2} = c + 2(1)^2 \;\Rightarrow\; c = -\frac{5}{2}$, and the tangent lines at $\left(-\frac{1}{2}, 1\right)$ are $y - 1 = -4\left(x + \frac{1}{2}\right)$ [or $y = -4x - 1$] and $y - 1 = \frac{1}{4}\left(x + \frac{1}{2}\right)$ $\left[\text{or } y = \frac{1}{4}x + \frac{9}{8}\right]$.

13. See the figure. Clearly, the line $y = 2$ is tangent to both circles at the point $(0, 2)$. We'll look for a tangent line L through the points (a, b) and (c, d), and if such a line exists, then its reflection through the y-axis is another such line. The slope of L is the same at (a, b) and (c, d). Find those slopes: $x^2 + y^2 = 4 \;\Rightarrow$ $2x + 2y\,y' = 0 \;\Rightarrow\; y' = -\dfrac{x}{y}$ $\left[= -\dfrac{a}{b}\right]$ and $x^2 + (y-3)^2 = 1 \;\Rightarrow$ $2x + 2(y-3)y' = 0 \;\Rightarrow\; y' = -\dfrac{x}{y-3}$ $\left[= -\dfrac{c}{d-3}\right]$.



Now an equation for L can be written using either point-slope pair, so we get $y - b = -\dfrac{a}{b}(x - a)$ $\left[\text{or } y = -\dfrac{a}{b}x + \dfrac{a^2}{b} + b\right]$ and $y - d = -\dfrac{c}{d-3}(x - c)$ $\left[\text{or } y = -\dfrac{c}{d-3}\,x + \dfrac{c^2}{d-3} + d\right]$. The slopes are equal, so $-\dfrac{a}{b} = -\dfrac{c}{d-3} \;\Leftrightarrow$ $d - 3 = \dfrac{bc}{a}$. Since (c, d) is a solution of $x^2 + (y-3)^2 = 1$, we have $c^2 + (d-3)^2 = 1$, so $c^2 + \left(\dfrac{bc}{a}\right)^2 = 1 \;\Rightarrow$ $a^2c^2 + b^2c^2 = a^2 \;\Rightarrow\; c^2(a^2 + b^2) = a^2 \;\Rightarrow\; 4c^2 = a^2$ [since (a, b) is a solution of $x^2 + y^2 = 4$] $\;\Rightarrow\; a = 2c$.

Now $d - 3 = \dfrac{bc}{a} \;\Rightarrow\; d = 3 + \dfrac{bc}{2c}$, so $d = 3 + \dfrac{b}{2}$. The y-intercepts are equal, so $\dfrac{a^2}{b} + b = \dfrac{c^2}{d-3} + d \;\Leftrightarrow$

$$\frac{a^2}{b} + b = \frac{(a/2)^2}{b/2} + \left(3 + \frac{b}{2}\right) \;\Leftrightarrow\; \left[\frac{a^2}{b} + b = \frac{a^2}{2b} + 3 + \frac{b}{2}\right](2b) \;\Leftrightarrow\; 2a^2 + 2b^2 = a^2 + 6b + b^2 \;\Leftrightarrow$$

$a^2 + b^2 = 6b \;\Leftrightarrow\; 4 = 6b \;\Leftrightarrow\; b = \frac{2}{3}$. It follows that $d = 3 + \dfrac{b}{2} = \dfrac{10}{3}$, $a^2 = 4 - b^2 = 4 - \frac{4}{9} = \frac{32}{9} \;\Rightarrow\; a = \frac{4}{3}\sqrt{2}$, and $c^2 = 1 - (d-3)^2 = 1 - \left(\frac{1}{3}\right)^2 = \frac{8}{9} \;\Rightarrow\; c = \frac{2}{3}\sqrt{2}$. Thus, L has equation $y - \dfrac{2}{3} = -\dfrac{(4/3)\sqrt{2}}{2/3}\left(x - \dfrac{4}{3}\sqrt{2}\right) \;\Leftrightarrow$ $y - \frac{2}{3} = -2\sqrt{2}\left(x - \frac{4}{3}\sqrt{2}\right) \;\Leftrightarrow\; y = -2\sqrt{2}\,x + 6$. Its reflection has equation $y = 2\sqrt{2}\,x + 6$.

[continued]

In summary, there are three lines tangent to both circles: $y = 2$ touches at $(0, 2)$, L touches at $\left(\frac{4}{3}\sqrt{2}, \frac{2}{3}\right)$ and $\left(\frac{2}{3}\sqrt{2}, \frac{10}{3}\right)$, and its reflection through the y-axis touches at $\left(-\frac{4}{3}\sqrt{2}, \frac{2}{3}\right)$ and $\left(-\frac{2}{3}\sqrt{2}, \frac{10}{3}\right)$.

14. $f(x) = \dfrac{x^{46} + x^{45} + 2}{1 + x} = \dfrac{x^{45}(x + 1) + 2}{x + 1} = \dfrac{x^{45}(x + 1)}{x + 1} + \dfrac{2}{x + 1} = x^{45} + 2(x + 1)^{-1}$, so

$f^{(46)}(x) = (x^{45})^{(46)} + 2\left[(x + 1)^{-1}\right]^{(46)}$. The forty-sixth derivative of any forty-fifth degree polynomial is 0, so $(x^{45})^{46} = 0$. Thus, $f^{(46)}(x) = 2\left[(-1)(-2)(-3)\cdots(-46)(x + 1)^{-47}\right] = 2(46!)(x + 1)^{-47}$ and $f^{(46)}(3) = 2(46!)(4)^{-47}$ or $(46!)2^{-93}$.

15. We can assume without loss of generality that $\theta = 0$ at time $t = 0$, so that $\theta = 12\pi t$ rad. [The angular velocity of the wheel is 360 rpm $= 360 \cdot (2\pi \text{ rad})/(60 \text{ s}) = 12\pi$ rad/s.] Then the position of A as a function of time is

$A = (40\cos\theta, 40\sin\theta) = (40\cos 12\pi t, 40\sin 12\pi t)$, so $\sin\alpha = \dfrac{y}{1.2 \text{ m}} = \dfrac{40\sin\theta}{120} = \dfrac{\sin\theta}{3} = \dfrac{1}{3}\sin 12\pi t$.

(a) Differentiating the expression for $\sin\alpha$, we get $\cos\alpha \cdot \dfrac{d\alpha}{dt} = \dfrac{1}{3} \cdot 12\pi \cdot \cos 12\pi t = 4\pi\cos\theta$. When $\theta = \dfrac{\pi}{3}$, we have

$\sin\alpha = \dfrac{1}{3}\sin\theta = \dfrac{\sqrt{3}}{6}$, so $\cos\alpha = \sqrt{1 - \left(\dfrac{\sqrt{3}}{6}\right)^2} = \sqrt{\dfrac{11}{12}}$ and $\dfrac{d\alpha}{dt} = \dfrac{4\pi\cos\frac{\pi}{3}}{\cos\alpha} = \dfrac{2\pi}{\sqrt{11/12}} = \dfrac{4\pi\sqrt{3}}{\sqrt{11}} \approx 6.56$ rad/s.

(b) By the Law of Cosines, $|AP|^2 = |OA|^2 + |OP|^2 - 2|OA|\,|OP|\cos\theta \quad\Rightarrow$

$120^2 = 40^2 + |OP|^2 - 2 \cdot 40\,|OP|\cos\theta \quad\Rightarrow\quad |OP|^2 - (80\cos\theta)\,|OP| - 12{,}800 = 0 \quad\Rightarrow$

$|OP| = \frac{1}{2}\left(80\cos\theta \pm \sqrt{6400\cos^2\theta + 51{,}200}\right) = 40\cos\theta \pm 40\sqrt{\cos^2\theta + 8} = 40\left(\cos\theta + \sqrt{8 + \cos^2\theta}\right)$ cm

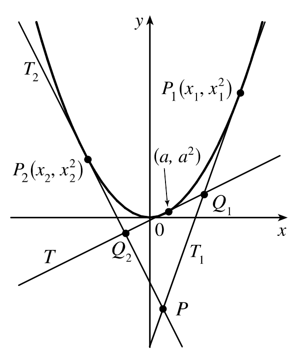
[since $|OP| > 0$]. As a check, note that $|OP| = 160$ cm when $\theta = 0$ and $|OP| = 80\sqrt{2}$ cm when $\theta = \frac{\pi}{2}$.

(c) By part (b), the x-coordinate of P is given by $x = 40\left(\cos\theta + \sqrt{8 + \cos^2\theta}\right)$, so

$$\frac{dx}{dt} = \frac{dx}{d\theta}\frac{d\theta}{dt} = 40\left(-\sin\theta - \frac{2\cos\theta\sin\theta}{2\sqrt{8 + \cos^2\theta}}\right) \cdot 12\pi = -480\pi\sin\theta\left(1 + \frac{\cos\theta}{\sqrt{8 + \cos^2\theta}}\right) \text{ cm/s}.$$

In particular, $dx/dt = 0$ cm/s when $\theta = 0$ and $dx/dt = -480\pi$ cm/s when $\theta = \frac{\pi}{2}$.

16. The equation of T_1 is $y - x_1^2 = 2x_1(x - x_1) = 2x_1 x - 2x_1^2$ or $y = 2x_1 x - x_1^2$.



The equation of T_2 is $y = 2x_2 x - x_2^2$. Solving for the point of intersection, we get $2x(x_1 - x_2) = x_1^2 - x_2^2 \quad\Rightarrow\quad x = \frac{1}{2}(x_1 + x_2)$. Therefore, the coordinates of P are $\left(\frac{1}{2}(x_1 + x_2), x_1 x_2\right)$. So if the point of contact of T is (a, a^2), then Q_1 is $\left(\frac{1}{2}(a + x_1), a x_1\right)$ and Q_2 is $\left(\frac{1}{2}(a + x_2), a x_2\right)$. Therefore,

$|PQ_1|^2 = \frac{1}{4}(a - x_2)^2 + x_1^2(a - x_2)^2 = (a - x_2)^2\left(\frac{1}{4} + x_1^2\right)$ and

$|PP_1|^2 = \frac{1}{4}(x_1 - x_2)^2 + x_1^2(x_1 - x_2)^2 = (x_1 - x_2)^2\left(\frac{1}{4} + x_1^2\right)$.

So $\dfrac{|PQ_1|^2}{|PP_1|^2} = \dfrac{(a - x_2)^2}{(x_1 - x_2)^2}$, and similarly $\dfrac{|PQ_2|^2}{|PP_2|^2} = \dfrac{(x_1 - a)^2}{(x_1 - x_2)^2}$. Finally, $\dfrac{|PQ_1|}{|PP_1|} + \dfrac{|PQ_2|}{|PP_2|} = \dfrac{a - x_2}{x_1 - x_2} + \dfrac{x_1 - a}{x_1 - x_2} = 1$.

17. Consider the statement that $\dfrac{d^n}{dx^n}(e^{ax}\sin bx) = r^n e^{ax}\sin(bx+n\theta)$. For $n=1$,

$\dfrac{d}{dx}(e^{ax}\sin bx) = ae^{ax}\sin bx + be^{ax}\cos bx$, and

$re^{ax}\sin(bx+\theta) = re^{ax}[\sin bx\cos\theta + \cos bx\sin\theta] = re^{ax}\left(\dfrac{a}{r}\sin bx + \dfrac{b}{r}\cos bx\right) = ae^{ax}\sin bx + be^{ax}\cos bx$

since $\tan\theta = \dfrac{b}{a} \quad\Rightarrow\quad \sin\theta = \dfrac{b}{r}$ and $\cos\theta = \dfrac{a}{r}$. So the statement is true for $n=1$.

Assume it is true for $n=k$. Then

$$\begin{aligned}\frac{d^{k+1}}{dx^{k+1}}(e^{ax}\sin bx) &= \frac{d}{dx}\left[r^k e^{ax}\sin(bx+k\theta)\right] = r^k ae^{ax}\sin(bx+k\theta) + r^k e^{ax} b\cos(bx+k\theta)\\ &= r^k e^{ax}[a\sin(bx+k\theta) + b\cos(bx+k\theta)]\end{aligned}$$

But

$\sin[bx+(k+1)\theta] = \sin[(bx+k\theta)+\theta] = \sin(bx+k\theta)\cos\theta + \sin\theta\cos(bx+k\theta) = \frac{a}{r}\sin(bx+k\theta) + \frac{b}{r}\cos(bx+k\theta)$.

Hence, $a\sin(bx+k\theta) + b\cos(bx+k\theta) = r\sin[bx+(k+1)\theta]$. So

$\dfrac{d^{k+1}}{dx^{k+1}}(e^{ax}\sin bx) = r^k e^{ax}[a\sin(bx+k\theta)+b\cos(bx+k\theta)] = r^k e^{ax}[r\sin(bx+(k+1)\theta)] = r^{k+1}e^{ax}[\sin(bx+(k+1)\theta)]$.

Therefore, the statement is true for all n by mathematical induction.

18. We recognize this limit as the definition of the derivative of the function $f(x) = e^{\sin x}$ at $x = \pi$, since it is of the form

$\displaystyle\lim_{x\to\pi}\frac{f(x)-f(\pi)}{x-\pi}$. Therefore, the limit is equal to $f'(\pi) = (\cos\pi)e^{\sin\pi} = -1\cdot e^0 = -1$.

19. It seems from the figure that as P approaches the point $(0,2)$ from the right, $x_T\to\infty$ and $y_T\to 2^+$. As P approaches the point $(3,0)$ from the left, it appears that $x_T\to 3^+$ and $y_T\to\infty$. So we guess that $x_T\in(3,\infty)$ and $y_T\in(2,\infty)$. It is more difficult to estimate the range of values for x_N and y_N. We might perhaps guess that $x_N\in(0,3)$, and $y_N\in(-\infty,0)$ or $(-2,0)$.

In order to actually solve the problem, we implicitly differentiate the equation of the ellipse to find the equation of the tangent line: $\dfrac{x^2}{9}+\dfrac{y^2}{4}=1 \quad\Rightarrow\quad \dfrac{2x}{9}+\dfrac{2y}{4}y'=0$, so $y' = -\dfrac{4}{9}\dfrac{x}{y}$. So at the point (x_0,y_0) on the ellipse, an equation of the tangent line is $y-y_0 = -\dfrac{4}{9}\dfrac{x_0}{y_0}(x-x_0)$ or $4x_0x+9y_0y = 4x_0^2+9y_0^2$. This can be written as $\dfrac{x_0x}{9}+\dfrac{y_0y}{4} = \dfrac{x_0^2}{9}+\dfrac{y_0^2}{4} = 1$, because (x_0,y_0) lies on the ellipse. So an equation of the tangent line is $\dfrac{x_0x}{9}+\dfrac{y_0y}{4}=1$.

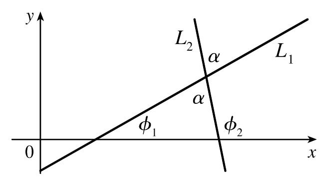
Therefore, the x-intercept x_T for the tangent line is given by $\dfrac{x_0x_T}{9}=1 \quad\Leftrightarrow\quad x_T=\dfrac{9}{x_0}$, and the y-intercept y_T is given by $\dfrac{y_0y_T}{4}=1 \quad\Leftrightarrow\quad y_T=\dfrac{4}{y_0}$.

So as x_0 takes on all values in $(0,3)$, x_T takes on all values in $(3,\infty)$, and as y_0 takes on all values in $(0,2)$, y_T takes on all values in $(2,\infty)$. At the point (x_0,y_0) on the ellipse, the slope of the normal line is $-\dfrac{1}{y'(x_0,y_0)}=\dfrac{9}{4}\dfrac{y_0}{x_0}$, and its equation is $y-y_0=\dfrac{9}{4}\dfrac{y_0}{x_0}(x-x_0)$. So the x-intercept x_N for the normal line is given by $0-y_0=\dfrac{9}{4}\dfrac{y_0}{x_0}(x_N-x_0)\;\Rightarrow\; x_N=-\dfrac{4x_0}{9}+x_0=\dfrac{5x_0}{9}$, and the y-intercept y_N is given by $y_N-y_0=\dfrac{9}{4}\dfrac{y_0}{x_0}(0-x_0)\;\Rightarrow\; y_N=-\dfrac{9y_0}{4}+y_0=-\dfrac{5y_0}{4}$.

So as x_0 takes on all values in $(0,3)$, x_N takes on all values in $\left(0,\frac{5}{3}\right)$, and as y_0 takes on all values in $(0,2)$, y_N takes on all values in $\left(-\frac{5}{2},0\right)$.

20. $\displaystyle\lim_{x\to 0}\frac{\sin(3+x)^2-\sin 9}{x}=f'(3)$ where $f(x)=\sin x^2$. Now $f'(x)=(\cos x^2)(2x)$, so $f'(3)=6\cos 9$.

21. (a) If the two lines L_1 and L_2 have slopes m_1 and m_2 and angles of inclination ϕ_1 and ϕ_2, then $m_1=\tan\,\phi_1$ and $m_2=\tan\,\phi_2$. The triangle in the figure shows that $\phi_1+\alpha+(180^\circ-\phi_2)=180^\circ$ and so $\alpha=\phi_2-\phi_1$. Therefore, using the identity for $\tan(x-y)$, we have $\tan\alpha=\tan(\phi_2-\phi_1)=\dfrac{\tan\phi_2-\tan\phi_1}{1+\tan\phi_2\tan\phi_1}$ and so $\tan\,\alpha=\dfrac{m_2-m_1}{1+m_1m_2}$.



(b) (i) The parabolas intersect when $x^2=(x-2)^2\;\Rightarrow\; x=1$. If $y=x^2$, then $y'=2x$, so the slope of the tangent to $y=x^2$ at $(1,1)$ is $m_1=2(1)=2$. If $y=(x-2)^2$, then $y'=2(x-2)$, so the slope of the tangent to $y=(x-2)^2$ at $(1,1)$ is $m_2=2(1-2)=-2$. Therefore, $\tan\alpha=\dfrac{m_2-m_1}{1+m_1m_2}=\dfrac{-2-2}{1+2(-2)}=\dfrac{4}{3}$ and so $\alpha=\tan^{-1}\left(\frac{4}{3}\right)\approx 53^\circ$ [or 127°].

(ii) $x^2-y^2=3$ and $x^2-4x+y^2+3=0$ intersect when $x^2-4x+(x^2-3)+3=0\;\Leftrightarrow\; 2x(x-2)=0\;\Rightarrow\; x=0$ or 2, but 0 is extraneous. If $x=2$, then $y=\pm 1$. If $x^2-y^2=3$ then $2x-2yy'=0\;\Rightarrow\; y'=x/y$ and $x^2-4x+y^2+3=0\;\Rightarrow\; 2x-4+2yy'=0\;\Rightarrow\; y'=\dfrac{2-x}{y}$. At $(2,1)$ the slopes are $m_1=2$ and $m_2=0$, so $\tan\alpha=\frac{0-2}{1+2\cdot 0}=-2\;\Rightarrow\;\alpha\approx 117^\circ$. At $(2,-1)$ the slopes are $m_1=-2$ and $m_2=0$, so $\tan\alpha=\dfrac{0-(-2)}{1+(-2)(0)}=2\;\Rightarrow\;\alpha\approx 63^\circ$ [or 117°].

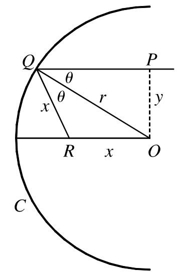
22. $y^2=4px\;\Rightarrow\; 2yy'=4p\;\Rightarrow\; y'=2p/y\;\Rightarrow$ slope of tangent at $P(x_1,y_1)$ is $m_1=2p/y_1$. The slope of FP is $m_2=\dfrac{y_1}{x_1-p}$, so by the formula from Problem 19(a),

$$\tan\alpha=\frac{\dfrac{y_1}{x_1-p}-\dfrac{2p}{y_1}}{1+\left(\dfrac{2p}{y_1}\right)\left(\dfrac{y_1}{x_1-p}\right)}\cdot\frac{y_1\,(x_1-p)}{y_1\,(x_1-p)}=\frac{y_1^2-2p(x_1-p)}{y_1(x_1-p)+2py_1}=\frac{4px_1-2px_1+2p^2}{x_1y_1-py_1+2py_1}$$

$$=\frac{2p(p+x_1)}{y_1(p+x_1)}=\frac{2p}{y_1}=\text{slope of tangent at }P=\tan\beta$$

Since $0\leq\alpha,\beta\leq\frac{\pi}{2}$, this proves that $\alpha=\beta$.

23. Since $\angle ROQ = \angle OQP = \theta$, the triangle QOR is isosceles, so $|QR| = |RO| = x$. By the Law of Cosines, $x^2 = x^2 + r^2 - 2rx\cos\theta$. Hence, $2rx\cos\theta = r^2$, so $x = \dfrac{r^2}{2r\cos\theta} = \dfrac{r}{2\cos\theta}$. Note that as $y \to 0^+$, $\theta \to 0^+$ (since $\sin\theta = y/r$), and hence $x \to \dfrac{r}{2\cos 0} = \dfrac{r}{2}$. Thus, as P is taken closer and closer to the x-axis, the point R approaches the midpoint of the radius AO.



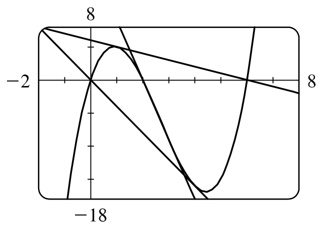
24. $\displaystyle \lim_{x\to 0}\frac{f(x)}{g(x)} = \lim_{x\to 0}\frac{f(x)-0}{g(x)-0} = \lim_{x\to 0}\frac{f(x)-f(0)}{g(x)-g(0)} = \lim_{x\to 0}\frac{\dfrac{f(x)-f(0)}{x-0}}{\dfrac{g(x)-g(0)}{x-0}} = \frac{\displaystyle\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}}{\displaystyle\lim_{x\to 0}\frac{g(x)-g(0)}{x-0}} = \frac{f'(0)}{g'(0)}$

25. $\displaystyle \lim_{x\to 0}\frac{\sin(a+2x)-2\sin(a+x)+\sin a}{x^2}$

$$\begin{aligned}
&= \lim_{x\to 0}\frac{\sin a\,\cos 2x + \cos a\,\sin 2x - 2\sin a\,\cos x - 2\cos a\,\sin x + \sin a}{x^2}\\
&= \lim_{x\to 0}\frac{\sin a\,(\cos 2x - 2\cos x + 1) + \cos a\,(\sin 2x - 2\sin x)}{x^2}\\
&= \lim_{x\to 0}\frac{\sin a\,(2\cos^2 x - 1 - 2\cos x + 1) + \cos a\,(2\sin x\,\cos x - 2\sin x)}{x^2}\\
&= \lim_{x\to 0}\frac{\sin a\,(2\cos x)(\cos x - 1) + \cos a\,(2\sin x)(\cos x - 1)}{x^2}\\
&= \lim_{x\to 0}\frac{2(\cos x - 1)[\sin a\,\cos x + \cos a\,\sin x](\cos x + 1)}{x^2(\cos x + 1)}\\
&= \lim_{x\to 0}\frac{-2\sin^2 x\,[\sin(a+x)]}{x^2(\cos x+1)} = -2\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2\cdot\frac{\sin(a+x)}{\cos x + 1} = -2(1)^2\,\frac{\sin(a+0)}{\cos 0 + 1} = -\sin a
\end{aligned}$$

26. (a) $f(x) = x(x-2)(x-6) = x^3 - 8x^2 + 12x \quad\Rightarrow$

$f'(x) = 3x^2 - 16x + 12$. The average of the first pair of zeros is $(0+2)/2 = 1$. At $x = 1$, the slope of the tangent line is $f'(1) = -1$, so an equation of the tangent line has the form $y = -1x + b$. Since $f(1) = 5$, we have $5 = -1 + b \quad\Rightarrow\quad b = 6$ and the tangent has equation $y = -x + 6$.



Similarly, at $x = \dfrac{0+6}{2} = 3$, $y = -9x + 18$; at $x = \dfrac{2+6}{2} = 4$, $y = -4x$. From the graph, we see that each tangent line drawn at the average of two zeros intersects the graph of f at the third zero.

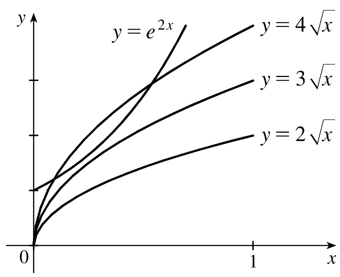
(b) A CAS gives $f'(x) = (x-b)(x-c) + (x-a)(x-c) + (x-a)(x-b)$ or $f'(x) = 3x^2 - 2(a+b+c)x + ab + ac + bc$. Using the `Simplify` command, we get

$f'\left(\dfrac{a+b}{2}\right) = -\dfrac{(a-b)^2}{4}$ and $f\left(\dfrac{a+b}{2}\right) = -\dfrac{(a-b)^2}{8}(a+b-2c)$, so an equation of the tangent line at $x = \dfrac{a+b}{2}$

is $y = -\dfrac{(a-b)^2}{4}\left(x - \dfrac{a+b}{2}\right) - \dfrac{(a-b)^2}{8}(a+b-2c)$. To find the x-intercept, let $y = 0$ and use the `Solve` command. The result is $x = c$.

Using Derive, we can begin by authoring the expression $(x-a)(x-b)(x-c)$. Now load the utility file `DifferentiationApplications`. Next we author tangent $(\#1, x, (a+b)/2)$—this is the command to find an equation of the tangent line of the function in #1 whose independent variable is x at the x-value $(a+b)/2$. We then simplify that expression and obtain the equation $y = \#4$. The form in expression #4 makes it easy to see that the x-intercept is the third zero, namely c. In a similar fashion we see that b is the x-intercept for the tangent line at $(a+c)/2$ and a is the x-intercept for the tangent line at $(b+c)/2$.

```
#1:  (x - a)·(x - b)·(x - c)
#2:  LOAD(C:\Program Files\TI Education\Derive 6\Math\DifferentiationApplications.mth
#3:  TANGENT((x - a)·(x - b)·(x - c), x, (a + b)/2)
#4:  (a^2 - 2·a·b + b^2)·(c-x)/4
```

27. 

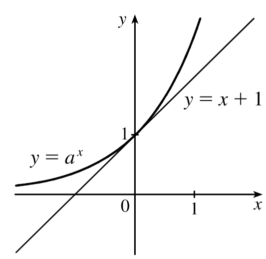
Let $f(x) = e^{2x}$ and $g(x) = k\sqrt{x}$ $[k > 0]$. From the graphs of f and g, we see that f will intersect g exactly once when f and g share a tangent line. Thus, we must have $f = g$ and $f' = g'$ at $x = a$.

$$f(a) = g(a) \;\Rightarrow\; e^{2a} = k\sqrt{a} \quad (\star)$$

and $$f'(a) = g'(a) \;\Rightarrow\; 2e^{2a} = \frac{k}{2\sqrt{a}} \;\Rightarrow\; e^{2a} = \frac{k}{4\sqrt{a}}.$$

So we must have $k\sqrt{a} = \dfrac{k}{4\sqrt{a}} \;\Rightarrow\; \left(\sqrt{a}\right)^2 = \dfrac{k}{4k} \;\Rightarrow\; a = \frac{1}{4}$. From $(\star)$, $e^{2(1/4)} = k\sqrt{1/4} \;\Rightarrow$ $k = 2e^{1/2} = 2\sqrt{e} \approx 3.297$.

28. We see that at $x = 0$, $f(x) = a^x = 1 + x = 1$, so if $y = a^x$ is to lie above $y = 1 + x$, the two curves must just touch at $(0, 1)$, that is, we must have $f'(0) = 1$. [To see this analytically, note that $a^x \geq 1 + x \;\Rightarrow\; a^x - 1 \geq x \;\Rightarrow\; \dfrac{a^x - 1}{x} \geq 1$ for $x > 0$, so $f'(0) = \lim\limits_{x \to 0^+} \dfrac{a^x - 1}{x} \geq 1$. Similarly, for $x < 0$, $a^x - 1 \geq x \;\Rightarrow\; \dfrac{a^x - 1}{x} \leq 1$, so $f'(0) = \lim\limits_{x \to 0^-} \dfrac{a^x - 1}{x} \leq 1$.



[continued]

Since $1 \le f'(0) \le 1$, we must have $f'(0) = 1$.] But $f'(x) = a^x \ln a \;\Rightarrow\; f'(0) = \ln a$, so we have $\ln a = 1 \;\Leftrightarrow\; a = e$.

Another method: The inequality certainly holds for $x \le -1$, so consider $x > -1$, $x \ne 0$. Then $a^x \ge 1 + x \;\Rightarrow$ $a \ge (1+x)^{1/x}$ for $x > 0 \;\Rightarrow\; a \ge \lim\limits_{x \to 0^+} (1+x)^{1/x} = e$, by Equation 3.6.5. Also, $a^x \ge 1 + x \;\Rightarrow\; a \le (1+x)^{1/x}$ for $x < 0 \;\Rightarrow\; a \le \lim\limits_{x \to 0^-} (1+x)^{1/x} = e$. So since $e \le a \le e$, we must have $a = e$.

29. $y = \dfrac{x}{\sqrt{a^2-1}} - \dfrac{2}{\sqrt{a^2-1}} \arctan \dfrac{\sin x}{a + \sqrt{a^2-1} + \cos x}$. Let $k = a + \sqrt{a^2-1}$. Then

$$\begin{aligned}
y' &= \frac{1}{\sqrt{a^2-1}} - \frac{2}{\sqrt{a^2-1}} \cdot \frac{1}{1 + \sin^2 x/(k+\cos x)^2} \cdot \frac{\cos x(k+\cos x) + \sin^2 x}{(k+\cos x)^2} \\
&= \frac{1}{\sqrt{a^2-1}} - \frac{2}{\sqrt{a^2-1}} \cdot \frac{k\cos x + \cos^2 x + \sin^2 x}{(k+\cos x)^2 + \sin^2 x} = \frac{1}{\sqrt{a^2-1}} - \frac{2}{\sqrt{a^2-1}} \cdot \frac{k\cos x + 1}{k^2 + 2k\cos x + 1} \\
&= \frac{k^2 + 2k\cos x + 1 - 2k\cos x - 2}{\sqrt{a^2-1}\,(k^2 + 2k\cos x + 1)} = \frac{k^2 - 1}{\sqrt{a^2-1}\,(k^2 + 2k\cos x + 1)}
\end{aligned}$$

But $k^2 = 2a^2 + 2a\sqrt{a^2-1} - 1 = 2a\left(a + \sqrt{a^2-1}\right) - 1 = 2ak - 1$, so $k^2 + 1 = 2ak$, and $k^2 - 1 = 2(ak - 1)$.

So $y' = \dfrac{2(ak-1)}{\sqrt{a^2-1}\,(2ak + 2k\cos x)} = \dfrac{ak-1}{\sqrt{a^2-1}k\,(a + \cos x)}$. But $ak - 1 = a^2 + a\sqrt{a^2-1} - 1 = k\sqrt{a^2-1}$,

so $y' = 1/(a + \cos x)$.

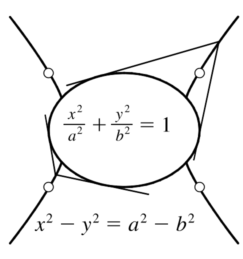
30. Suppose that $y = mx + c$ is a tangent line to the ellipse. Then it intersects the ellipse at only one point, so the discriminant of the equation $\dfrac{x^2}{a^2} + \dfrac{(mx+c)^2}{b^2} = 1 \;\Leftrightarrow\; (b^2 + a^2m^2)x^2 + 2mca^2x + a^2c^2 - a^2b^2 = 0$ must be 0; that is,

$$\begin{aligned}
0 &= (2mca^2)^2 - 4(b^2 + a^2m^2)(a^2c^2 - a^2b^2) = 4a^4c^2m^2 - 4a^2b^2c^2 + 4a^2b^4 - 4a^4c^2m^2 + 4a^4b^2m^2 \\
&= 4a^2b^2(a^2m^2 + b^2 - c^2)
\end{aligned}$$

Therefore, $a^2m^2 + b^2 - c^2 = 0$.

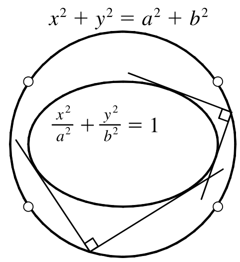
Now if a point (α, β) lies on the line $y = mx + c$, then $c = \beta - m\alpha$, so from above,

$$0 = a^2m^2 + b^2 - (\beta - m\alpha)^2 = (a^2 - \alpha^2)m^2 + 2\alpha\beta m + b^2 - \beta^2 \;\Leftrightarrow\; m^2 + \frac{2\alpha\beta}{a^2-\alpha^2}m + \frac{b^2-\beta^2}{a^2-\alpha^2} = 0.$$



(a) Suppose that the two tangent lines from the point (α, β) to the ellipse have slopes m and $\dfrac{1}{m}$. Then m and $\dfrac{1}{m}$ are roots of the equation $z^2 + \dfrac{2\alpha\beta}{a^2-\alpha^2}z + \dfrac{b^2-\beta^2}{a^2-\alpha^2} = 0$. This implies that $(z - m)\left(z - \dfrac{1}{m}\right) = 0 \;\Leftrightarrow$ $z^2 - \left(m + \dfrac{1}{m}\right)z + m\left(\dfrac{1}{m}\right) = 0$, so equating the constant terms in the two quadratic equations, we get $\dfrac{b^2-\beta^2}{a^2-\alpha^2} = m\left(\dfrac{1}{m}\right) = 1$, and hence $b^2 - \beta^2 = a^2 - \alpha^2$. So (α, β) lies on the hyperbola $x^2 - y^2 = a^2 - b^2$.

(b) If the two tangent lines from the point (α, β) to the ellipse have slopes m and $-\dfrac{1}{m}$, then m and $-\dfrac{1}{m}$ are roots of the quadratic equation, and so $(z - m)\left(z + \dfrac{1}{m}\right) = 0$, and equating the constant terms as in part (a), we get $\dfrac{b^2 - \beta^2}{a^2 - \alpha^2} = -1$, and hence $b^2 - \beta^2 = \alpha^2 - a^2$. So the point (α, β) lies on the circle $x^2 + y^2 = a^2 + b^2$.



31. $y = x^4 - 2x^2 - x \quad \Rightarrow \quad y' = 4x^3 - 4x - 1$. The equation of the tangent line at $x = a$ is $y - (a^4 - 2a^2 - a) = (4a^3 - 4a - 1)(x - a)$ or $y = (4a^3 - 4a - 1)x + (-3a^4 + 2a^2)$ and similarly for $x = b$. So if at $x = a$ and $x = b$ we have the same tangent line, then $4a^3 - 4a - 1 = 4b^3 - 4b - 1$ and $-3a^4 + 2a^2 = -3b^4 + 2b^2$. The first equation gives $a^3 - b^3 = a - b \quad \Rightarrow \quad (a - b)(a^2 + ab + b^2) = (a - b)$. Assuming $a \neq b$, we have $1 = a^2 + ab + b^2$. The second equation gives $3(a^4 - b^4) = 2(a^2 - b^2) \quad \Rightarrow \quad 3(a^2 - b^2)(a^2 + b^2) = 2(a^2 - b^2)$ which is true if $a = -b$. Substituting into $1 = a^2 + ab + b^2$ gives $1 = a^2 - a^2 + a^2 \quad \Rightarrow \quad a = \pm 1$ so that $a = 1$ and $b = -1$ or vice versa. Thus, the points $(1, -2)$ and $(-1, 0)$ have a common tangent line.

As long as there are only two such points, we are done. So we show that these are in fact the only two such points. Suppose that $a^2 - b^2 \neq 0$. Then $3(a^2 - b^2)(a^2 + b^2) = 2(a^2 - b^2)$ gives $3(a^2 + b^2) = 2$ or $a^2 + b^2 = \frac{2}{3}$. Thus, $ab = (a^2 + ab + b^2) - (a^2 + b^2) = 1 - \dfrac{2}{3} = \dfrac{1}{3}$, so $b = \dfrac{1}{3a}$. Hence, $a^2 + \dfrac{1}{9a^2} = \dfrac{2}{3}$, so $9a^4 + 1 = 6a^2 \quad \Rightarrow$ $0 = 9a^4 - 6a^2 + 1 = (3a^2 - 1)^2$. So $3a^2 - 1 = 0 \quad \Rightarrow \quad a^2 = \dfrac{1}{3} \quad \Rightarrow \quad b^2 = \dfrac{1}{9a^2} = \dfrac{1}{3} = a^2$, contradicting our assumption that $a^2 \neq b^2$.

32. Suppose that the normal lines at the three points (a_1, a_1^2), (a_2, a_2^2), and (a_3, a_3^2) intersect at a common point. Now if one of the a_i is 0 (suppose $a_1 = 0$) then by symmetry $a_2 = -a_3$, so $a_1 + a_2 + a_3 = 0$. So we can assume that none of the a_i is 0.

The slope of the tangent line at (a_i, a_i^2) is $2a_i$, so the slope of the normal line is $-\dfrac{1}{2a_i}$ and its equation is $y - a_i^2 = -\dfrac{1}{2a_i}(x - a_i)$. We solve for the x-coordinate of the intersection of the normal lines from (a_1, a_1^2) and (a_2, a_2^2):

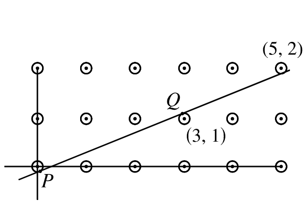
$$y = a_1^2 - \frac{1}{2a_1}(x - a_1) = a_2^2 - \frac{1}{2a_2}(x - a_2) \quad \Rightarrow \quad x\left(\frac{1}{2a_2} - \frac{1}{2a_1}\right) = a_2^2 - a_1^2 \quad \Rightarrow$$

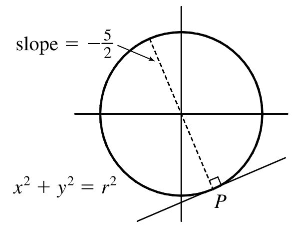
$$x\left(\frac{a_1 - a_2}{2a_1 a_2}\right) = (-a_1 - a_2)(a_1 + a_2) \quad \Leftrightarrow \quad x = -2a_1 a_2(a_1 + a_2) \quad \textbf{(1)}.$$

Similarly, solving for the x-coordinate of the intersections of the normal lines from (a_1, a_1^2) and (a_3, a_3^2) gives $x = -2a_1 a_3(a_1 + a_3)$ **(2)**.

Equating **(1)** and **(2)** gives $a_2(a_1 + a_2) = a_3(a_1 + a_3) \quad \Leftrightarrow \quad a_1(a_2 - a_3) = a_3^2 - a_2^2 = -(a_2 + a_3)(a_2 - a_3) \quad \Leftrightarrow$ $a_1 = -(a_2 + a_3) \quad \Leftrightarrow \quad a_1 + a_2 + a_3 = 0$.

33. Because of the periodic nature of the lattice points, it suffices to consider the points in the 5×2 grid shown. We can see that the minimum value of r occurs when there is a line with slope $\frac{2}{5}$ which touches the circle centered at $(3, 1)$ and the circles centered at $(0, 0)$ and $(5, 2)$.

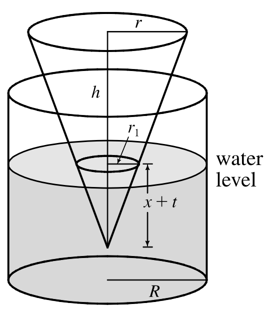




To find P, the point at which the line is tangent to the circle at $(0, 0)$, we simultaneously solve $x^2 + y^2 = r^2$ and $y = -\frac{5}{2}x \quad \Rightarrow \quad x^2 + \frac{25}{4}x^2 = r^2 \quad \Rightarrow \quad x^2 = \frac{4}{29}r^2 \quad \Rightarrow \quad x = \frac{2}{\sqrt{29}}r,\, y = -\frac{5}{\sqrt{29}}r$. To find Q, we either use symmetry or solve $(x-3)^2 + (y-1)^2 = r^2$ and $y - 1 = -\frac{5}{2}(x-3)$. As above, we get $x = 3 - \frac{2}{\sqrt{29}}r,\, y = 1 + \frac{5}{\sqrt{29}}r$. Now the slope of the line PQ is $\frac{2}{5}$, so $m_{PQ} = \dfrac{1 + \frac{5}{\sqrt{29}}r - \left(-\frac{5}{\sqrt{29}}r\right)}{3 - \frac{2}{\sqrt{29}}r - \frac{2}{\sqrt{29}}r} = \dfrac{1 + \frac{10}{\sqrt{29}}r}{3 - \frac{4}{\sqrt{29}}r} = \dfrac{\sqrt{29} + 10r}{3\sqrt{29} - 4r} = \dfrac{2}{5} \quad \Rightarrow$

$5\sqrt{29} + 50r = 6\sqrt{29} - 8r \quad \Leftrightarrow \quad 58r = \sqrt{29} \quad \Leftrightarrow \quad r = \frac{\sqrt{29}}{58}$. So the minimum value of r for which any line with slope $\frac{2}{5}$ intersects circles with radius r centered at the lattice points on the plane is $r = \frac{\sqrt{29}}{58} \approx 0.093$.

34.



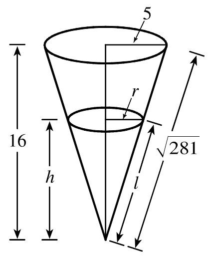
Assume the axes of the cone and the cylinder are parallel. Let H denote the initial height of the water. When the cone has been dropping for t seconds, the water level has risen x centimeters, so the tip of the cone is $x + 1t$ centimeters below the water line. We want to find dx/dt when $x + t = h$ (when the cone is completely submerged). Using similar triangles, $\dfrac{r_1}{x+t} = \dfrac{r}{h} \quad \Rightarrow \quad r_1 = \dfrac{r}{h}(x+t)$.

$$
\begin{array}{ccccc}
\text{volume of water and cone at time } t & = & \text{original volume of water} & + & \text{volume of submerged part of cone} \\
\pi R^2(H+x) & = & \pi R^2 H & + & \frac{1}{3}\pi r_1^2(x+t) \\
\pi R^2 H + \pi R^2 x & = & \pi R^2 H & + & \frac{1}{3}\pi \dfrac{r^2}{h^2}(x+t)^3 \\
3h^2R^2x & = & r^2(x+t)^3 & &
\end{array}
$$

Differentiating implicitly with respect to t gives us $3h^2R^2\dfrac{dx}{dt} = r^2\left[3(x+t)^2\dfrac{dx}{dt} + 3(x+t)^2\dfrac{dt}{dt}\right] \quad \Rightarrow$

$\dfrac{dx}{dt} = \dfrac{r^2(x+t)^2}{h^2R^2 - r^2(x+t)^2} \quad \Rightarrow \quad \left.\dfrac{dx}{dt}\right|_{x+t=h} = \dfrac{r^2h^2}{h^2R^2 - r^2h^2} = \dfrac{r^2}{R^2 - r^2}$. Thus, the water level is rising at a rate of $\dfrac{r^2}{R^2 - r^2}$ cm/s at the instant the cone is completely submerged.

35.



By similar triangles, $\dfrac{r}{5} = \dfrac{h}{16} \quad \Rightarrow \quad r = \dfrac{5h}{16}$. The volume of the cone is $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \left(\dfrac{5h}{16}\right)^2 h = \dfrac{25\pi}{768}h^3$, so $\dfrac{dV}{dt} = \dfrac{25\pi}{256}h^2\,\dfrac{dh}{dt}$. Now the rate of change of the volume is also equal to the difference of what is being added ($2\text{ cm}^3/\text{min}$) and what is oozing out ($k\pi rl$, where πrl is the area of the cone and k is a proportionality constant). Thus, $\dfrac{dV}{dt} = 2 - k\pi rl$.

Equating the two expressions for $\dfrac{dV}{dt}$ and substituting $h = 10$, $\dfrac{dh}{dt} = -0.3$, $r = \dfrac{5(10)}{16} = \dfrac{25}{8}$, and $\dfrac{l}{\sqrt{281}} = \dfrac{10}{16} \quad \Leftrightarrow$ $l = \dfrac{5}{8}\sqrt{281}$, we get $\dfrac{25\pi}{256}(10)^2(-0.3) = 2 - k\pi\dfrac{25}{8}\cdot\dfrac{5}{8}\sqrt{281} \quad \Leftrightarrow \quad \dfrac{125k\pi\sqrt{281}}{64} = 2 + \dfrac{750\pi}{256}$. Solving for k gives us $k = \dfrac{256 + 375\pi}{250\pi\sqrt{281}}$. To maintain a certain height, the rate of oozing, $k\pi rl$, must equal the rate of the liquid being poured in; that is, $\dfrac{dV}{dt} = 0$. Thus, the rate at which we should pour the liquid into the container is

$$k\pi rl = \frac{256 + 375\pi}{250\pi\sqrt{281}} \cdot \pi \cdot \frac{25}{8} \cdot \frac{5\sqrt{281}}{8} = \frac{256 + 375\pi}{128} \approx 11.204\text{ cm}^3/\text{min}$$