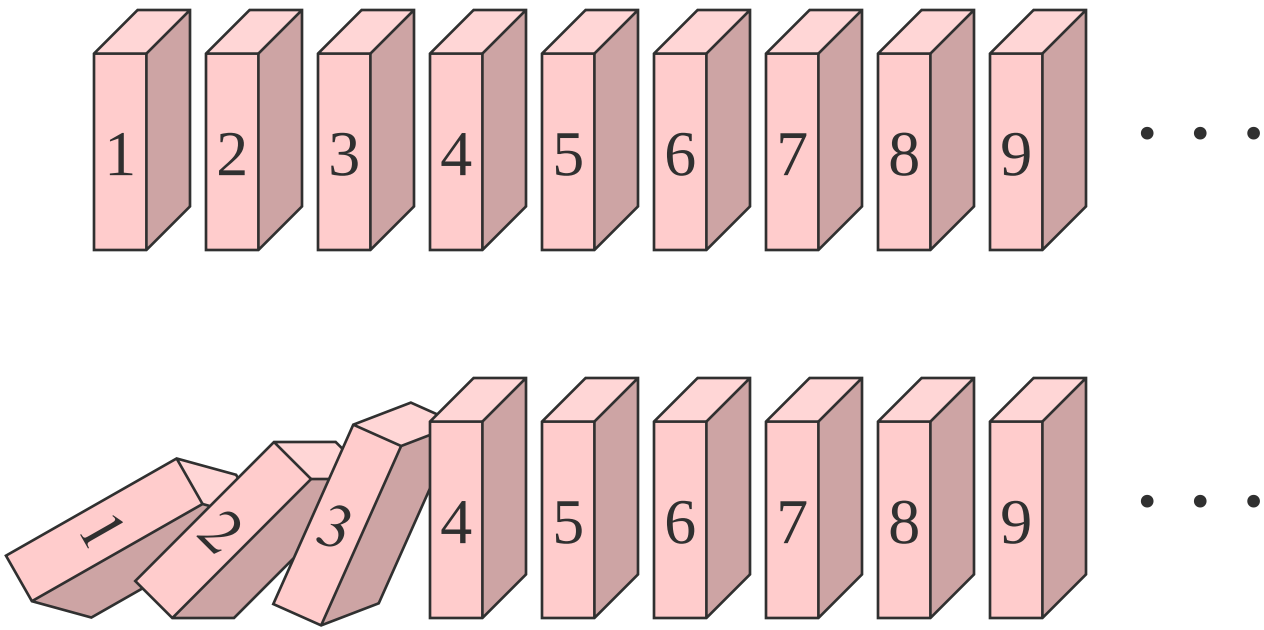
Mathematical Induction

- Some theorems state that $P(n)$ is true for some value n, say $n = 1$.
- Other (more general!) theorems state that $P(n)$ is true for ***all*** positive integers n.
- This requires a proof of infinite cases!

Dominos Example

- Consider the problem of showing if an infinite collection of dominos will all fall if the first one is pushed.



Dominos Example

- *Let P*(1) be the proposition that the *first domino* will fall.
- Let P(n) be the proposition that the nth domino will fall.
- Let each domino be ***exactly*** the same distance apart.
- How do we know if they will ALL fall?

Inductive Proof

- We want to prove we can knock all the domino's down:
  1. (*Basis Step*): I can push over the first one.
  2. (*The Inductive Step*): For all the domino's, if domino n falls, then domino $n + 1$ will also fall
- Therefore, I can knock down ALL the domino's. (by knocking over the first!)

Dominos Example

- If we can show:
  - That we can knock the first one over
  - That the distance between any two adjacent dominos is the right distance so that if one falls, so will the next.
- Then we KNOW that if we *push over the first one*, they *all must fall*!

Mathematical Induction Example

- Let $P(n)$ be the proposition:

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$$

- Prove the theorem that:

$$\forall n: P(n).$$

Inductive Proof Example

- Since there are infinite positive integers, we <u>cannot</u> simply substitute each possible value of n and check.
- However suppose that:

  1. (basis) we can show $P(1)$ is true, and
  2. (induction) if $P(k)$ holds for some positive integer n, then it also holds for $P(k + 1)$.

- Then $P(n)$ must hold for ***all*** $n \geq 1$!

Mathematical Induction

- $P(1)$ is called the ***basis***
- P(k) is called the ***inductive hypothesis***.
- If we can show then $P(1)$ is true,

  and

  if we can show $P(k)$ being true implies the *next case* $P(k+1)$ is also true,

  then $\forall i: P(i)$.

Inductive Proof Example

1. Basis: $P(1)$

Prove $P(1)$: Let $n = 1$. Then

$$\sum_{i=1}^{1} i = \frac{1(1+1)}{2}$$

or 1 = 1, which is true!

Inductive Proof Example

2. Induction: $P(k) \rightarrow P(k + 1)$.

Assume $P(k)$ is true for some value k (***Inductive Hypothesis***):

$$\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$$

Prove that $P(k + 1)$ will also be true:

$$\sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}$$

Inductive Proof Example

Prove that $P(k) \rightarrow P(k+1)$

$P(k+1)$:
$$\sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}$$

$$1 + 2 + \ldots + k + (k+1) = \frac{(k+1)(k+2)}{2}$$

Rewrite

$$\sum_{i=1}^{k} i + (k+1) = \frac{(k+1)(k+2)}{2}$$

Rewrite

Inductive Proof Example

$$\sum_{i=1}^{k} i + (k+1) = \frac{(k+1)(k+2)}{2}$$

$P(k+1)$
(want to prove)

$$\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$$

$P(k)$
(Assumed true)

$$\frac{k(k+1)}{2} + (k+1) = \frac{(k+1)(k+2)}{2}$$

Substitute using $P(k)$ (Inductive Hypothesis)

Inductive Proof Example

$$\frac{k(k+1)}{2}+(k+1)=\frac{(k+1)(k+2)}{2}$$

Showing this true will prove $P(k+1)$

$$\frac{k(k+1)}{2}+(k+1)=\frac{k(k+1)}{2}+\frac{2(k+1)}{2}$$

$$=\frac{k(k+1)+2(k+1)}{2}$$

$$=\frac{(k+2)(k+1)}{2}$$

$$=\frac{(k+1)(k+2)}{2}$$

Same!

Inductive Proof Example

- Since the proposition $P(1)$ is true, and since the proposition $P(k + 1)$ is always true if true if $P(k)$ is true, the proposition $P(n)$ is true ***for all n ≥ 1***.

Mathematical Induction

- *Mathematical Induction* allows us to prove all cases simply by proving only two subcases:
  1. Basis Step: The proposition is true for $P(1)$.
  2. The Inductive Step: If the proposition $P(k)$ is true, then the proposition $P(k+1)$ is true, e.g.

  $P(k) \rightarrow P(k+1)$.

The inductive hypothesis $P(k)$ is used to show that $P(k+1)$ is true

Mathematical Induction

- Formally:
- $[P(1) \wedge \forall i{:}(P(i) \rightarrow P(i+1))] \rightarrow \forall i{:}\ P(i)$.

- Note we ARE NOT showing $P(n)$ is true, rather that IF $P(n)$ is true, then $P(n+1)$ is too.

Inductive Proof Example

- Suppose we wish to prove

$$n < 2^n$$

for all positive integers.

Inductive Proof Example

- Proof that $n < 2^n$ for all $n \geq 1$

1. ***Basis***: Let $n = 1$.

   Then $1 < 2^1$ or $1 < 2$. True!

Inductive Proof Example

Proof that $n < 2^n$ for all $n \geq 1$

2. ***Induction:***

Inductive Hypothesis:

Assume $k < 2^k$ for some positive integer $k \geq 1$

Prove:

that $k + 1 < 2^{k+1}$ is true

Inductive Proof Example

- Prove $k + 1 < 2^k + 1$ using $k < 2^k$

Assume $k < 2^k$

$k + 1 < 2^k + 1$ (*add 1 to each side*)

$2^k + 1 \leq 2^k + 2^k$

$2^k + 2^k = 2^{k+1}$

Thus $k + 1 < 2^{k+1}$

Examples

1. Prove: $1^2 + 2^2 + 3^2 + \ldots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$

2. Prove: $\displaystyle\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

3. Prove: $1 + 2 + 2^2 + 2^3 + \ldots 2^n = 2^{n+1} - 1$

Examples

4. Use the fact that $|x+y| \le |x| + |y|$ for all real numbers to prove that:

$$|x_1 + x_2 + \ldots + x_n| \le |x_1| + |x_2| + \ldots + |x_n|$$

For all real numbers x^1, x^2, ..., x^n

Recursive Definitions

- A ***recursive definition*** consists of two parts.

  1) ***A basis clause***. This tells us that certain elements belong to the set in question.

  2) ***An inductive clause***. This tells us how to use elements that are in the set to get to other elements that are in the set.

Recursive Definitions

- Example: Define a function f: I $\rightarrow$ I as follows:
  1. (Basis) $f(0) = 2$
  2. (Induction) $f(n+1) = 3f(n) - 3$
- Use this definition:
  - $f(0) = 2$
  - $f(1) = 3f(0)-3 = 3(2) - 3 = 6 - 3 = 3$
  - $f(2) = 3f(1)-3 = 3(3) - 3 = 9 - 3 = 6$
  - $f(3) = 3f(2)-3 = 3(6) - 3 = 18 - 3 = 15$

Fibonacci numbers

- Define a function f: I $\rightarrow$ I as follows:

1. (Basis) $f(0) = 0, \quad f(1) = 1$
2. (Induction) $f(n) = f(n\text{-}1) + f(n\text{-}2)$

Recursively defined sets

- Consider a definition of the set S

1. (Basis) $3 \in S$
2. (Induction) if $x \in S$ and $y \in S$ then $x + y \in S$

- Consider a definition of the set S

1. (Basis) $2 \in S$ and $5 \in S$
2. (Induction) if $x \in S$ and $y \in S$ then $x \bullet y \in S$

Set of well-formed expressions

- Let V be the set of real numbers and valid variable names.

  1) $V \subset S$

  2) If $x, y \in S$ then

     (i) $(-x) \in S$

     (ii) $(x + y) \in S$

     (iii) $(x - y) \in S$

     (iv) $(x / y) \in S$

     (v) $(x * y) \in S$

Recursive Definitions

- The basis gives the basic building blocks
- The inductive clause tells how the pieces can be assembled.

Recursive Definitions

- *Example:*

1. Basis: $f(0) = 3$
2. Induction: $f(n+1) = 2\,f(n) + 3$

- Find: $f(0)$, $f(1)$, $f(2)$, $f(3)$, and $f(4)$

Recursive Definitions

- ***Factorials*** - $f(n) = n!$
- Basis: $f(0) = 1$
- Induction: $f(n+1) = (n + 1)\, f(n)$

- Find: $f(0)$, $f(1)$, $f(2)$, $f(3)$, and $f(4)$

Recursive Definitions

- ***Exponents*** - define a^n where a is a nonzero real and n is a non-negative integer.
- Basis: $a^0 = 1$
- Induction: $a^{n+1} = a^n \cdot a$

Recursive Definitions

- ***Sequences*** **-** Give the recursive definitions of the sequence $\{a_n\}$, where

$$n = 1, 2, 3, 4, \ldots$$

a. $a_n = 6n$

b. $a_n = 2n + 1$

c. $a_n = 10^n$

d. $a_n = 5$