# Propositional Logic

Dr. Son P. Nguyen

UEL
VNU-HCMC

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# Definition: Declarative Sentence

## Definition

A declarative sentence is a sentence that declares a fact.

Examples of declarative sentences:

- Nadal will win the Aussie open 2015.
- $2 + 2 = 4$

Examples of sentences that are not declarative sentences:

- What is your name ?
- Take the cookies.

# Definition: Proposition

**Definition**

A proposition is a declarative sentence that is either true or false, but not both.

Examples of declarative sentences that are **not** propositions:

- $x + 2 = 5$

Examples of declarative sentences that are propositions:

- Sydney is the capital of Australia
- Germany won World Cup 2014.

# Notation

We use letters to denote propositional variables (or statement variables), that is, variables that represent propositions, just as letters are used to denote numerical variables. The conventional letters used for propositional variables are $p, q, r, s, \dots$ The area of logic that deals with propositions is called the propositional calculus or propositional logic.

**Definition**

The **truth value** of a proposition is true (denoted T) if it is a true proposition; the truth value of a proposition is false (denoted F) otherwise.

# Definition: Compound Proposition

Many mathematical statements are constructed by combining one or more propositions. New propositions, called compound propositions, are formed from existing propositions using logical operators.

# Definition: Negation

**Definition**

Let $p$ be a proposition. The compound proposition

"it is not the case that p"

is an other proposition, called the **negation** of $p$, and denoted $\neg p$. The truth value of the negation of $p$ is the opposite of the truth value of $p$. The proposition $\neg p$ is read "not p".

# Truth Table of the Negation

A **truth table** presents the relations between the truth value of many propositions involved in a compound proposition. This table has a row for each possible truth value of the propositions.
Truth table for the negation $\neg p$ of the proposition $p$:

| $p$ | $\neg p$ |
|---|---|
| T | F |
| F | T |

# Definition: Conjunction

**Definition**

Let $p$ and $q$ be propositions. The compound proposition

$$\text{“}p \text{ and } q\text{”}$$

denoted $p \wedge q$, is true when both $p$ and $q$ are true and false otherwise. This compound proposition $p \wedge q$ is called the **conjunction** of $p$ and $q$.

Truth table for the conjunction $p \wedge q$ of the propositions $p$ and $q$:

| $p$ | $q$ | $p \wedge q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | F |

# Definition: Disjunction

## Definition

Let $p$ and $q$ be propositions. The compound proposition

"$p$ or $q$"

denoted $p \vee q$, is false when both $p$ and $q$ are false and true otherwise. This compound proposition $p \vee q$ is called the **disjunction** of $p$ and $q$.

Truth table for the disjunction $p \vee q$ of the propositions $p$ and $q$:

| $p$ | $q$ | $p \wedge q$ |
|---|---|---|
| T | T | T |
| T | F | T |
| F | T | T |
| F | F | F |

# Definition: Exclusive Disjunction

**Definition**

Let $p$ and $q$ be propositions. The compound proposition

$$\text{“}p \text{ exclusive or } q\text{”}$$

denoted $p \oplus q$, is true when exactly one of $p$ and $q$ is true and false otherwise. This compound proposition $p \oplus q$ is called the **exclusive disjunction** of $p$ and $q$.

Truth table for the disjunction $p \oplus q$ of the propositions $p$ and $q$:

| $p$ | $q$ | $p \oplus q$ |
|---|---|---|
| T | T | F |
| T | F | T |
| F | T | T |
| F | F | F |

# Definition: Exclusive Disjunction

**Definition**

Let $p$ and $q$ be propositions. The compound proposition

"if $p$, then $q$"

denoted $p \to q$, is false when $p$ is true and $q$ is false, and is true otherwise. This compound proposition $p \to q$ is called the **implication** (or **conditional statement**) of $p$ and $q$.
In this implication, $p$ is called the hypothesis (or antecedent or premise) and $q$ is called the conclusion (or consequence).

# Truth Table of the Implication

Truth table for the implication $p \to q$ of the propositions $p$ and $q$:

| $p$ | $q$ | $p \to q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | T |
| F | F | T |

Remarks:

- The implication $p \to q$ is false only when $p$ is true and $q$ is false.
- The implication $p \to q$ is true when $p$ is false whatever the truth value of $q$.

# The implication

Variety of terminology is used to express the implication $p \to q$.

- "if p, then q";
- "p implies q";
- "q if p";
- "p only if q";
- "q when p";
- "p is sufficient for q";
- "a sufficient condition for q is p";
- "q follows from p";
- "q whenever p".

# The Implication

In natural language, there is a relationship between the hypothesis and the conclusion of an implication. In mathematical reasoning, we consider conditional statements of a more general sort that we use in English. The implication

"If today is Friday, then $2 + 3 = 6$"

is true every day except Friday, even though $2 + 3 = 6$ is false. The mathematical concept of a conditional statement is independent of a cause-and-effect relationship between hypothesis and conclusion. We only parallel English usage to make it easy to use and remember.

# Definitions: Converse, Contrapositive and Inverse

We can form some new conditional statements starting with the implication $p \to q$. There are three related implications that occur so often that they have special names.

- The converse of $p \to q$ is the proposition $q \to p$.
- The inverse of $p \to q$ is the proposition $\neg p \to \neg q$.
- The contrapositive of $p \to q$ is the proposition $\neg q \to \neg p$.

**Remember the contrapositive**. The contrapositive $\neg q \to \neg p$ of the implication $p \to q$ always has the same truth value as $p \to q$.

# Definition: Biconditional Statement

**Definition**

Let p and q be propositions. The compound proposition

"p if and only if q",

denoted $p \leftrightarrow q$, is true when p and q have the same truth value, and is false otherwise. This compound proposition $p \leftrightarrow q$ is called the **biconditional statement** (or the **bi-implication**) of p and q.

Note that the biconditional statement $p \leftrightarrow q$ is true when both implications $p \to q$ and $q \to p$ are true and is false otherwise. There are some other common ways to express $p \leftrightarrow q$:

- "p is necessary and sufficient for q";
- "if p then q, and conversely";
- "p iff q".

# Truth Table of the Bi-Implication

Truth table for the bi-implication $p \leftrightarrow q$ of the propositions p and q:

| $p$ | $q$ | $p \leftrightarrow q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | T |