Elementary counting

Dr. Son P. Nguyen

UEL
VNU-HCMC

October 31, 2015

The following algorithm sorts a sequence of numbers into a nondecreasing sequence

```
(1)  for i = 1 to n − 1
(2)       for j = i + 1 to n
(3)            if (A[i] > A[j])
(4)                 exchange A[i] and A[j]
```

How many times is the comparison $A[i] > A[j]$ made in Line 3 ?

Sets

Concepts

- A set is a collection of objects.
- Two sets are disjoint if the share no common member.
- A family of sets $A_1, \ldots, A_n$ is called *mutually disjoint* if any two of them are disjoint.

Sum principle:

The size of a union of a family of mutually disjoint finite sets is the sum of the sizes of the sets.

Sets

Concepts

- A set is a collection of objects.
- Two sets are disjoint if the share no common member.
- A family of sets $A_1, \ldots, A_n$ is called *mutually disjoint* if any two of them are disjoint.

Sum principle:

The size of a union of a family of mutually disjoint finite sets is the sum of the sizes of the sets.

Example

The following algorithm multiplies 2 matrices

```
(1) for i = 1 to r
(2)         for j = 1 to m
(3)                 S = 0
(4)                 for k = 1 to n
(5)                         S = S + A[i, k] * B[k, j]
(6)                 C[i, j] = S
```

How many multiplications (expressed in terms of r, m, and n) does this code carry out in line 5 ?

Product principle

The size of a union of m disjoint sets, each of size n, is mn.

More examples

Example

Consider the following longer piece of pseudocode that sorts a list of numbers and then counts big gaps in the list. (For this exercise, a big gap is a place where a number in the list is more than twice the previous number.)

```
 (1)  for i = 1 to n − 1
 (2)       minval = A[i]
 (3)       minindex = i
 (4)       for j = i to n
 (5)            if (A[j] < minval)
 (6)                 minval = A[j]
 (7)                 minindex = j
 (8)       exchange A[i] and A[minindex]
 (9)  bigjump = 0
(10)  for i = 2 to n
(11)       if (A[i] > 2 * A[i − 1])
(12)            bigjump = bigjump + 1
```

How many comparisons does the pseudocode make in Lines 5 and 11?

In class exercises

Do 11, 13, 15 on page 9, 10

Using the sum and the product principle

Example 1.2-1

A password for a certain computer system is supposed to be between four and eight characters long and composed of lowercase and/or uppercase letters. How many passwords are possible? What counting principles did you use? Estimate the percentage of the possible passwords that have exactly four letters.

Product principle

Principle 1.4 **(Product Principle, Version 2)**

If a set S of lists of length m has the properties that

1. there are i_1 different first elements of lists in S, and
2. for each $j > 1$ and each choice of the first $j - 1$ elements of a list in S, there are i_j choices of elements in position j of those lists,

then there are $i_1 i_2 \cdots i_m = \prod_{k=1}^{m} i_k$ lists in S.

Recall: The notion of a function or a mapping

Exercise 1.2-2 Write down all the functions from the two-element set $\{1, 2\}$ to the two-element set $\{a, b\}$.

Exercise 1.2-3 How many functions are there from a two-element set to a three-element set?

Exercise 1.2-4 How many functions are there from a three-element set to a two-element set?

Bijection principle

Example 1.2-7

The following loop is part of a program to determine the number of triangles formed by n points in the plane.

```
(1)  trianglecount = 0
(2)  for i = 1 to n
(3)       for j = i+1 to n
(4)            for k = j+1 to n
(5)                 if points i, j, and k are not collinear
(6)                      trianglecount = trianglecount + 1
```

Among all iterations of line 5 of the pseudocode, what is the total number of times this line checks three points to see if they are collinear?

Bijection principle

Two sets have the same size if and only if there is a one-to-one function from one set onto the other.

Theorem

Theorem 1.1 The number of k-element permutations of an n-element set is

$$A_n^k = n^{\underline{k}} = \frac{n!}{(n-k)!}$$

Theorem

Theorem 1.2 For integers n and k with $0 \leq k \leq n$, the number of k-element subsets of an n-element set is

$$C_n^k = \frac{n^{\underline{k}}}{k!} = \frac{n!}{(n-k)!\,k!}$$

Permutations and combinations

Theorem

Theorem 1.1 The number of k-element permutations of an n-element set is

$$A_n^k = n^{\underline{k}} = \frac{n!}{(n-k)!}$$

Theorem

Theorem 1.2 For integers n and k with $0 \leq k \leq n$, the number of k-element subsets of an n-element set is

$$C_n^k = \frac{n^{\underline{k}}}{k!} = \frac{n!}{(n-k)!\,k!}$$

Exercises

Do Ex 9,11,13,15 on page 20

Binomial coefficients

- Understand the idea behind the formula

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

- The binomial theorem

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$$

- The key point is to look at the formula in view of counting

Examples

- Suppose we have k labels of one kind and $n - k$ labels of another. In how many different ways can we apply these labels to n objects ?
- Show that if we have k_1 labels of one kind, k_2 labels of a second kind, and $k_3 = n - k_1 - k_2$ labels of a third kind, then there are $n!/(k_1!k_2!k_3!)$ ways to apply these labels to n objects.

Examples

- Suppose we have k labels of one kind and $n-k$ labels of another. In how many different ways can we apply these labels to n objects ?
- Show that if we have k_1 labels of one kind, k_2 labels of a second kind, and $k_3 = n - k_1 - k_2$ labels of a third kind, then there are $n!/(k_1!k_2!k_3!)$ ways to apply these labels to n objects.
- What is the coefficient of $x^{k_1}y^{k_2}z^{k_3}$ in $(x+y+z)^n$?

Examples

- Suppose we have k labels of one kind and $n-k$ labels of another. In how many different ways can we apply these labels to n objects ?
- Show that if we have k_1 labels of one kind, k_2 labels of a second kind, and $k_3 = n - k_1 - k_2$ labels of a third kind, then there are $n!/(k_1!k_2!k_3!)$ ways to apply these labels to n objects.
- What is the coefficient of $x^{k_1}y^{k_2}z^{k_3}$ in $(x+y+z)^n$?

Exercises

Do Ex 6,9,12 on page 31

Example

How many ways can we arrange 3 people around a circle ? 5 people ? n people ?

overcounting principle

1. Use a *wrong* procedure to count, which yields a number that's too high.
2. Then figure out how wrong our answer was, and reduce it to the correct number.

Using equivalence relations in counting (cont.)

Example

1. You have two colors blue and green. How many ways are there to paint the corners of a *scalene* triangle ?
2. How many ways are there to paint the corners of a *equilateral* triangle ?

Example

You have two colors, red and blue. How many ways are there to color the corners of a square ?

Remark

You can ask similar questions for more complex *regular polygons*. But, the answers will not be easy to see. You need a more advanced tool, namely *group actions*

Examples

Example

1. How many “words” can be form from the letters of “Happy” ?
2. How many “words” can be form from the letters of “Mississippi” ?
3. How many “words” can be form from the letters of “alabama” ?

Bookcase arrangement

We have k books to arrange on the n shelves of a bookcase. The order in which the books appear on a shelf matters, and each shelf can hold all the books. We will assume that as the books are placed on the shelves, they are pushed as far to the left as they will go. Thus, all that matters is the order in which the books appear. When book i is placed on a shelf, it can go between two books already there or to the left or right of all the books on that shelf. In how many ways can we place k distinct books on n shelves ?

Hint: Place one book at a time to the shelves

Multiset

Hamburgers

There are 25 hamburgers to be distributed among 10 people. We allow some people to have no burger. How many different ways ?

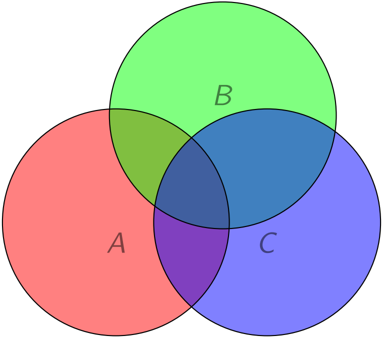
Remark

- This is an example of *multisets* when we label the burgers with the names of the corresponding people receiving them.
- As an equation

$$x_1 + \cdots + x_{10} = 25 \qquad x_i \geq 0, \quad x_i : \text{ integer}$$

Example

Suppose there are 50 beads in a drawer: 25 are glass, 30 are red, 20 are spherical, 18 are red glass, 12 are glass spheres, 15 are red spheres, and 8 are red glass spheres. How many beads are neither red, nor glass, nor spheres ?



Exercise 1

A noted vexillologist tells you that 30 of the 50 U.S. state flags have blue as a background color, twelve have stripes, 26 exhibit a plant or animal, nine have both blue in the background and stripes, 23 have both blue in the background and feature a plant or animal, and three have both stripes and a plant or animal. One of the flags in this last category (California) does not have any blue in the background. How many state flags have no blue in the background, no stripes, and no plant or animal featured ?

Exercise 2

Suppose 50 socks lie in a drawer. Each one is either white or black, ankle-high or knee-high, and either has a hole or doesn't. 22 socks are white, four of these have a hole, and one of these four is knee-high. Ten white socks are knee-high, ten black socks are knee-high, and five knee-high socks have a hole. Exactly three ankle-high socks have a hole. Determine the number of black, ankle-high socks with no holes.

Exercise 3

The buffet line at a local steakhouse has 35 dishes. Sixteen dishes contain meat, fourteen dishes are fried, and of the dishes with meat, eight contain vegetables and seven are fried. Of the fried dishes, five contain a vegetable. Just two dishes are fried and contain both meat and a vegetable, and ten dishes (principally in the dessert section) contain neither meat nor a vegetable and are not fried. Determine how many dishes contain vegetables.

'

Exercise 4

A sneaky registrar reports the following information about a group of 400 students. There are 180 taking a math class, 200 taking an English class, 160 taking a biology class, and 250 in a foreign language class. 80 are enrolled in both math and English, 90 in math and biology, 120 in math and a foreign language, 70 in English and biology, 140 in English and a foreign language, and 60 in biology and a foreign language. Also, there are 25 in math, English, and a foreign language, 30 in math, English, and biology, 40 in math, biology, and a foreign language, and fifteen in English, biology, and a foreign language. Finally, the sum of the number of students with a course in all four subjects, plus the number of students with a course in none of the four subjects, is 100. Determine the number of students that are enrolled in all four subjects simultaneously:math, biology, English, and a foreign language.