Equivalence Relations

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Definition: Equivalence Relations

Definition

A relation on a set A is called an **equivalence relation** if it is reflexive, symmetric and transitive.

Definition

Two elements a and b that are related by an equivalence relation are called **equivalent**. The notation $a \sim b$ is often used to denote that a and b are equivalent elements with respect to a particular equivalence relation.

Review: Modular Arithmetic

Let a an integer and m a positive integer with $m > 1$. The notation $a \bmod m$ is the remainder when a is divided by m.

In other words $a \bmod m$ is the integer r such that $a = qm + r$ where $0 \leq r < m$.

Definition

If a and b are integers and m is a positive integer, then a **is congruent to** b **modulo** m if m divides $a - b$. Notation: $a \equiv b \pmod{m}$.

Theorem

Let a and b be integers, and let m be a positive integer. Then $a \equiv b \pmod{m}$ if and only if a and b have the same remainder when divided by m, i.e., $a \bmod m = b \bmod m$.

Example of an Equivalence Relation

Let m be a positive integer with $m > 1$. The relation

$$R = \{(a, b) \mid a \equiv b \pmod{m}\}$$

is an equivalence relation on the set of integers.

Definition: Equivalence Classes

Definition

Let R be an equivalence relation on a set A. The set of all elements that are related to an elements a of A is called the **equivalence class** of a. The equivalence class of a with respect to R is denoted by $[a]_R$. When only one relation is under consideration, we can delete the subscript R and write $[a]$ for this equivalence class.

In other words, if R is an equivalence relation on a set A, the equivalence class of the element a is

$$[a]_R = \{s \in A \,|\, (a, s) \in R\}.$$

If $b \in [a]_R$, then b is called a **representative** of this equivalence class.

Equivalence Classes

Theorem

Let R be an equivalence relation on a set A. These statements for elements a and b of A are equivalent:

i) $a\,R\,b$,

ii) $[a] = [b]$,

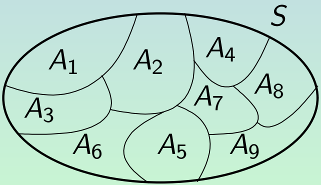
iii) $[a] \cap [b] \neq \emptyset$.

Definition

A **partition** of a set S is a collection of disjoint non empty subsets of S that have S as their union.

In other words, the collection of subsets A_i, $i \in I$ (where I is an index set) forms a partition of S if and only if

- $A_i \neq \emptyset \quad \forall i \in I$,
- $A_i \cap A_j = \emptyset$, for $i \neq j$,
- $\bigcup_{i \in I} A_i = S$.



Partitions and Equivalence Relations

Theorem

Let R be an equivalence relation on a set S. Then the equivalence classes of R form a partition of S.

Conversely, given a partition $\{A_i \mid i \in I\}$ of the set S, there is an equivalence relation R that has the sets A_i, $i \in I$, as its equivalence classes.