Reconstruction of a term in the right-hand side of parabolic equations

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Abstract

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1 Introduction

Consider a physical domain $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}^+$ be bounded with the boundary Γ and donate the cylinder $Q = \Omega \times (0, T]$ and lateral surface area $S = \Gamma \times (0, T]$ where $T > 0$.
Consider the heat equation

$$\frac{\partial u}{\partial t} - \sum_{i,j=1}^{d} \frac{\partial}{\partial x_j}\left(a_{ji}(x,t)\frac{\partial u}{\partial x_i}\right) = F(x,t), \quad (x,t) \in Q, \tag{1.1}$$

with the initial and Dirichlet conditions, respectively

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{1.2}$$

$$u(x,t) = 0, \quad (x,t) \in S, \tag{1.3}$$

where

$$a_{ij} \in L^\infty(Q),\, a_{ij} = a_{ji},\ \forall i,j \in \{1,2,...,d\},$$

$$\lambda_1 \|\xi\|^2 \le \sum_{i,j=1}^{d} a_{ij}\xi_i\xi_j \le \lambda_2 \|\xi\|^2,\ \forall \xi \in \mathbb{R}^d,$$

$$u_0 \in H_0^1(\Omega),\ F \in L^2(0,T;\, H^{-1}(\Omega)),$$

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with λ_1 v λ_2 are positive constants.

The direct problem is to determine u when all data $a_{ji}, i, j = \overline{1,d}$, u_0, φ and F in eqs. (1.1) to (1.3) are given. On the other hand, the inverse problem (IP) is to identify a missed data such as the right hand side F when some additional observations on the solution u are available.

We consider the right hand side of the equation (1.1) following the form $F(x,t) = f(.)q(x,t) + g(x,t)$, where $q(x,t)$, $g(x,t)$ are given and $f(.)$ can be either $f(x,t)$, $f(x)$ or $f(t)$. We have different inverse problems depending on either the form of F or the observation on the solution u:

- IP1: Find $f(x,t)$ if $u(x,t)$ is given on Q [12, 15].
- IP2: Find $f(x)$ if $u(x,T)$ is given on Ω [1, 2, 17, 19].
- IP3: Find $f(t)$ if $\int_\Omega w(x)u(x,t)dx$ and $w(x) > 0, \forall x \in \Omega$ are given [11, 13, 5]. This observation called *integral observation*. Furthermore, an observation derives from integral observation called *point observation* if $w(x)$ is a dirac delta function $\delta(x - x_0)$, so that $\int_\Omega \delta(x - x_0)u(x,t)dx = u(x_0,t), x_0$ is a point in Ω [18, 3, 4]. Beside that, find $f(x,t)$ or $f(x)$ if some integral or point observations are available [6].

Donate w is the value of the observation that given and $\ell u(f)$ is the result of the observation based on the solution u we get. In this paper, we only present the case of having many integral observations with N_m is the number of observations, others can be proved similarly. So, to solve this problem, we need to minimize the least square functional [7, 8]

$$J_\gamma(f) = \frac{1}{2}\sum_{k=1}^{N_m} \|\ell_k u(f) - \omega_k\|^2_{L^2(0,T)}.$$

However, this minimization problem is unstable and there might be many minimizers to it. Therefore, we minimize the Tikhonov functional instead

$$J_\gamma(f) = \frac{1}{2}\sum_{k=1}^{N_o} \|\ell_k u(f) - \omega_k\|^2_{L^2(0,T)} + \frac{\gamma}{2}\|f - f^*\|^2_*,$$

with $\gamma > 0$ being a regularization parameter, f^* is an a prior estimation of f and $\|.\|_*$ an appropriate norm.

2 Variational problem

To introduce the concept of weak form, we use the standard Sobolev spaces $H^1(\Omega)$, $H^1_0(\Omega)$, $H^{1,0}(Q)$ and $H^{1,1}(Q)$ [14, 9, 10]. Further, for a Banach space B, we define

$$L^2(0,T;B) = \left\{u : u(t) \in B \text{ a.e } t \in (0,T) \text{ and } \|u\|_{L^2(0,T;\,B)} < \infty\right\},$$

with the norm

$$\|u\|^2_{L^2(0,T;B)} = \int_0^T \|u(t)\|^2_B \, dt.$$

In this paper, we will use an equivalent norm in $L^2\left(0,T;H_0^1(\Omega)\right)$ with the norm

$$\|u\|^2_{L^2(0,T;\,H_0^1(\Omega))} = \int_Q \sum_{i,j=1}^{d} a_{ji}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}dxdt.$$

So, with the duality pairing $\langle .,.\rangle_Q$, the dual norm will be

$$\|u\|_{L^2(0,T;\,H^{-1}(\Omega))} = \sup_{0\neq v\in L^2(0,T;\,H_0^1(\Omega))}\frac{\langle u,v\rangle_Q}{\|v\|_{L^2(0,T;\,H_0^1(\Omega))}}.$$

In the sequel, we shall use the space $W(0,T)$ define as

$$W(0,T) = \left\{u : u \in L^2(0,T;H_0^1(\Omega)), u_t \in L^2\left(0,T;H^{-1}(\Omega)\right)\right\}$$

A week solution in $W(0,T)$ of the problem eqs. (1.1) to (1.3) is a function $u(x,t)\in W(0,T)$ satisfying the identity

$$\int_Q\left[\frac{\partial u}{\partial t}v + \sum_{i,j=1}^{d} a_{ji}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}\right]dxdt = \int_Q Fvdxdt,\ \forall v\in L^2\left(0,T;H^1(\Omega)\right). \tag{2.1}$$

or

$$a(u,v) = \langle F,v\rangle_Q,\ \forall v\in L^2\left(0,T;H_0^1(\Omega)\right).$$

and

$$u(x,0) = u_0,\ x\in\Omega. \tag{2.2}$$

From now on, we donate $X = \{u : u\in W(0,T) : u(x,0) = 0\}$ and $Y = L^2\left(0,T;H_0^1(\Omega)\right)$. Obviously, we have $X\subset Y$. We split $u(x,t) = \overline{u}(x,t)+\overline{u}_0$ for $(x,t)\in Q$ where $\overline{u}_0\in W(0,T)$ is some extension of the given initial datum $u_0\in H_0^1(\Omega)$. Here, we use space-time finite element method [16] and therefore we can prove that there exists a unique solution $\overline{u}\in X$ of the problem eqs. (1.1) to (1.3) that satisfies

$$\|\overline{u}\|_{W(0,T)} \leq c_d\left(\|F\|_{L^2(0,T;H^{-1}(\Omega))} + \|\overline{u}_0\|_{W(0,T)}\right). \tag{2.3}$$

We suppose that F has the form $F(x,t) = f(x,t)q(x,t)+g(x,t)$ with $f\in L^2(Q)$, $q\in L^\infty(Q)$ and $g\in L^2(Q)$. We hope to recover $f(x,t)$ from the observation. Since the solution $u(x,t)$ depends on the function $f(x,t)$, so we denote it by $u(x,t,f)$ or $u(f)$. Identify $f(x,t)$ satisfying

$$\ell_k u(f) = \omega_k,\ \forall k = \overline{1,N_m}.$$

We need to minimize the Tikhonov functional

$$J_\gamma(f) = \frac{1}{2}\sum_{k=1}^{N_m}\|\ell_k u(f)-\omega_k\|^2_{L^2(0,T)} + \frac{\gamma}{2}\|f-f^*\|^2_{L^2(Q)}. \tag{2.4}$$

We will prove that J_γ is Frechet differentiable and drive a formula for its gradient. In doing so, we need the adjoint problem

$$\begin{cases} -\dfrac{\partial p}{\partial t} - \displaystyle\sum_{i,j=1}^{d}\frac{\partial}{\partial x_j}\left(a_{ji}(x,t)\frac{\partial p}{\partial x_i}\right) = \sum_{k=1}^{N_m} w(x)\left(\ell_k u(f)-\omega_k\right), & (x,t)\in Q,\\ u(x,t) = 0, & (x,t)\in S\\ p(x,T) = 0, & x\in\Omega. \end{cases} \tag{2.5}$$

By changing the time direction, meaning $\tilde{p}(x,t) = p(x, T-t)$, we will get a Dirichlet problem for parabolic equations.

Theorem 2.1. *The functional J_γ is Frechet differentiable and its gradient ∇J_γ at f has the form*

$$\nabla J_\gamma(f) = q(x,t)p(x,t) + \gamma\left(f(x,t) - f^*(x,t)\right) \tag{2.6}$$

Proof. By taking a small variation $\delta f \in L^2(Q)$ of f and denoting $\delta u(f) = u(f+\delta f) - u(f)$, we have

$$\begin{aligned}
J_0(f+\delta f) - J_0(f) &= \frac{1}{2}\sum_{k=1}^{N_m} \|\ell_k u(f+\delta f) - \omega_k\|^2_{L^2(0,T)} - \frac{1}{2}\sum_{k=1}^{N_m} \|\ell_k u(f) - \omega_k\|^2_{L^2(0,T)} \\
&= \frac{1}{2}\sum_{k=1}^{N_m} \|\ell_k \delta u(f) + \ell_k u(f) - \omega_k\|^2_{L^2(0,T)} - \frac{1}{2}\sum_{k=1}^{N_m} \|\ell_k u(f) - \omega_k\|^2_{L^2(0,T)} \\
&= \sum_{k=1}^{N_m} \frac{1}{2} \|\ell_k \delta u(f)\|^2_{L^2(0,T)} + \sum_{k=1}^{N_m} \langle \ell_k \delta u(f), \ell_k u(f) - \omega_k \rangle_{L^2(0,T)},
\end{aligned}$$

where $\delta u(f)$ is the solution to this problem

$$\begin{cases}
\dfrac{\partial \delta u}{\partial t} - \displaystyle\sum_{i,j=1}^{d} \frac{\partial}{\partial x_j}\left(a_{ji}(x,t)\frac{\partial \delta u}{\partial x_i}\right) = q(x,t)\delta f, & (x,t) \in Q, \\
\delta u(x,t) = 0, & (x,t) \in S, \\
\delta u(x,0) = 0, & x \in \Omega.
\end{cases} \tag{2.7}$$

Because the priori estimate (2.3) for the direct problem, we have

$$\|\ell_k \delta u(f)\|^2_{L^2(0,T)} = o\left(\|\delta f\|_{L^2(Q)}\right) \text{ when } \|\delta f\|_{L^2(Q)} \to 0. \tag{2.8}$$

What is more, applying the Green formula [...] for (2.5) and (2.7), we get

$$\sum_{k=1}^{N_m} \int_Q \delta u(x,t) w(x) \left(\ell_k u(f) - \omega_k(t)\right) dxdt = \int_Q p(x,t) q(x,t) \delta f(x,t) dxdt \tag{2.9}$$

According to (2.8) and (2.9), we obtain

$$\begin{aligned}
J_0(f+\delta f) - J_0(f) &= \sum_{k=1}^{N_m} \int_Q \delta u(x,t) w(x) \left(\ell_k u(f) - \omega_k(t)\right) ds + o\left(\|\delta f\|_{L^2(Q)}\right) \\
&= \int_Q q(x,t) p(x,t) \delta f(x,t) dxdt + o\left(\|\delta f\|_{L^2(I)}\right) \\
&= \langle qp, \delta f \rangle_{L^2(Q)} + o\left(\|\delta f\|^2_{L(Q)}\right).
\end{aligned}$$

Therefore, we will obtain

$$J_\gamma(f+\delta f) - J_\gamma(f) = \langle qp, \delta f \rangle_{L^2(Q)} + \gamma \langle f - f^*, \delta f \rangle_{L^2(Q)} + o\left(\|\delta f\|^2_{L(Q)}\right).$$

Hence the functional J_γ is Frechet differentiable and its gradient ∇J_γ at f has the form (2.6). The theorem is proved. $\square$

Remark 2.1. *In this theorem, we write the Tikhonov functional for* $F(x,t) = f(x,t)q(x,t) + g(x,t)$. *But when* F *has another form, the penalty term should be modified*

- $F(x,t) = f(x)q(x,t) + g(x,t)$: *the penalty functional is* $\|f - f^*\|_{L^2(\Omega)}$ *and*

$$\nabla J_0(f) = \int_0^T q(x,t)p(x,t)dt.$$

- $F(x,t) = f(t)q(x,t) + g(x,t)$: *the penalty functional is* $\|f - f^*\|_{L^2(0,T)}$ *and*

$$\nabla J_0(f) = \int_\Omega q(x,t)p(x,t)dt.$$

To find f satisfied (2.4), we use the conjugate gradient method (CG). Its iteration follows, we assume that at the kth iteration, we have f^k and then the next iteration will be

$$f^{k+1} = f^k + \alpha_k d^k,$$

with

$$d^k = \begin{cases} -\nabla J_\gamma(f^k), & k = 0, \\ -\nabla J_\gamma(f^k) + \beta_k d^{k-1}, & k > 0, \end{cases}$$

$$\beta_k = \frac{\|\nabla J_\gamma(f^k)\|^2_{L^2(I)}}{\|\nabla J_\gamma(f^{k-1})\|^2_{L^2(I)}},$$

and

$$\alpha_k = \arg\min_{\alpha \ge 0} J_\gamma(f^k + \alpha d^k).$$

To identify α_k, we consider two problems

Problem 2.1. Denote the solution of this problem is $u[f]$

$$\begin{cases} \dfrac{\partial u}{\partial t} - \displaystyle\sum_{i,j=1}^{d} \frac{\partial}{\partial x_j}\left(a_{ji}(x,t)\frac{\partial u}{\partial x_i}\right) = f(x,t)q(x,t), & (x,t) \in Q, \\ u(x,t) = 0, & (x,t) \in S, \\ u(x,0) = 0, & x \in \Omega. \end{cases}$$

Problem 2.2. Denote the solution of this problem is $u(u_0, \varphi)$

$$\begin{cases} \dfrac{\partial u}{\partial t} - \displaystyle\sum_{i,j=1}^{d} \frac{\partial}{\partial x_j}\left(a_{ji}(x,t)\frac{\partial u}{\partial x_i}\right) = g(x,t), & (x,t) \in Q, \\ u(x,t) = 0, & (x,t) \in S, \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases}$$

If we do so, the observation operators have the form $\ell_i u(f) = \ell_i u[f] + \ell_i u(u_0, \varphi) = A_i f + \ell_i u(u_0, \varphi)$, with A_i being bounded linear operators from $L^2(Q)$ to $L^2(0,T)$.

We have

$$\begin{aligned}
J_\gamma(f^k + \alpha d^k) &= \frac{1}{2}\sum_{i=1}^{N_m} \left\|\ell_i u(f^k + \alpha d^k) - \omega_i\right\|^2_{L^2(0,T)} + \frac{\gamma}{2}\left\|f^k + \alpha d^k - f^*\right\|^2_{L^2(Q)} \\
&= \frac{1}{2}\sum_{i=1}^{N_m} \left\|\alpha A_i d^k + A_i f^k + \ell_i u(u_0,\varphi) - \omega_i\right\|^2_{L^2(0,T)} + \frac{\gamma}{2}\left\|f^k + \alpha d^k - f^*\right\|^2_{L^2(Q)} \\
&= \frac{1}{2}\sum_{i=1}^{N_m} \left\|\alpha A_i d^k + \ell_i u(f^k) - \omega_i\right\|^2_{L^2(0,T)} + \frac{\gamma}{2}\left\|f^k + \alpha d^k - f^*\right\|^2_{L^2(Q)}.
\end{aligned}$$

Differentiating $J_\gamma(f^k + \alpha d^k)$ with respect to α, we get

$$\begin{aligned}
\frac{\partial J_\gamma(f^k + \alpha d^k)}{\partial \alpha} &= \alpha \sum_{i=1}^{N_m} \left\|A_i d^k\right\|^2_{L^2(0,T)} + \sum_{i=1}^{N_m} \left\langle A_i d^k, \ell_i u(f^k) - \omega_i\right\rangle_{L^2(0,T)} \\
&\quad + \gamma\alpha \left\|d^k\right\|^2_{L^2(Q)} + \gamma\left\langle d^k, f^k - f^*\right\rangle_{L^2(Q)}.
\end{aligned}$$

Putting $\dfrac{\partial J_\gamma(f^k + \alpha d^k)}{\partial \alpha} = 0$, we obtain

$$\begin{aligned}
\alpha_k &= -\frac{\sum_{i=1}^{N_m} \left\langle A_i d^k, \ell_i u(f^k) - \omega_i\right\rangle_{L^2(0,T)} + \gamma\left\langle d^k, f^k - f^*\right\rangle_{L^2(Q)}}{\sum_{i=1}^{N_m} \left\|A_i d^k\right\|^2_{L^2(0,T)} + \gamma\left\|d^k\right\|^2_{L^2(Q)}} \\
&= -\frac{\sum_{i=1}^{N_m} \left\langle d^k, A_i^*\left(\ell_i u(f^k) - \omega_i\right)\right\rangle_{L^2(Q)} + \gamma\left\langle d^k, f^k - f^*\right\rangle_{L^2(Q)}}{\sum_{i=1}^{N_m} \left\|A_i d^k\right\|^2_{L^2(0,T)} + \gamma\left\|d^k\right\|^2_{L^2(Q)}} \\
&= -\frac{\sum_{i=1}^{N_m} \left\langle d^k, A_i^*\left(\ell_i u(f^k) - \omega_i\right) + \gamma(f^k - f^*)\right\rangle_{L^2(Q)}}{\sum_{i=1}^{N_m} \left\|A_i d^k\right\|^2_{L^2(0,T)} + \gamma\left\|d^k\right\|^2_{L^2(Q)}} \\
&= -\frac{\left\langle d^k, \nabla J_\gamma(f^k)\right\rangle_{L^2(Q)}}{\sum_{i=1}^{N_m} \left\|A_i d^k\right\|^2_{L^2(0,T)} + \gamma\left\|d^k\right\|^2_{L^2(Q)}}.
\end{aligned}$$

Because of $d^k = r^k + \beta_k d^{k-1}$, $r^k = -\nabla J_\gamma(f^k)$ and $\left\langle r^k, d^{k-1}\right\rangle_{L^2(I)} = 0$, we get

$$\alpha_k = \frac{\left\|r^k\right\|^2_{L^2(Q)}}{\sum_{i=1}^{N_m} \left\|A_i d^k\right\|^2_{L^2(0,T)} + \gamma\left\|d^k\right\|^2_{L^2(Q)}}.$$

CG algorithm

1. Set $k = 0$, initiate f^0.

2. For $k = 0, 1, 2, ...$ Calculate

$$r^k = -\nabla J_\gamma(f^k).$$

   Update

$$d^k = \begin{cases} r^k, & k = 0, \\ r^k + \beta_k d^{k-1}, & k > 0, \end{cases}$$

$$\beta_k = \frac{\|r^k\|^2_{L^2(Q)}}{\|r^{k-1}\|^2_{L^2(Q)}}.$$

3. Calculate

$$\alpha_k = \frac{\|r^k\|^2_{L^2(Q)}}{\sum_{i=1}^{N_m} \|A_i d^k\|^2_{L^2(0,T)} + \gamma \|d^k\|^2_{L^2(Q)}}.$$

   Update

$$f^{k+1} = f^k + \alpha_k d^k.$$

3 Finite element method

We rewrite the Tikhonov functional

$$\begin{aligned} J_\gamma(f) &= \frac{1}{2}\sum_{i=1}^{N_m} \|\ell_i u[f] + \ell_i u(u_0, \varphi) - \omega_i\|^2_{L^2(0,T)} + \frac{\gamma}{2}\|f - f^*\|^2_{L^2(Q)} \\ &= \frac{1}{2}\sum_{i=1}^{N_m} \|A_i f + \ell_i u(u_0, \varphi) - \omega_i\|^2_{L^2(0,T)} + \frac{\gamma}{2}\|f - f^*\|^2_{L^2(Q)} \\ &= \frac{1}{2}\sum_{i=1}^{N_m} \|A_i f - \hat{\omega}_i\|^2_{L^2(0,T)} + \frac{\gamma}{2}\|f - f^*\|^2_{L^2(Q)}, \end{aligned}$$

with $\hat{\omega}_i = \omega_i - \ell_i u(u_0, \varphi)$.

The solution f^γ of the minimization problem (2.4) is characterized by the first-order optimality condition

$$\nabla J_\gamma(f^\gamma) = \sum_{i=1}^{N_m} A_i^*(A_i f^\gamma - \hat{\omega}_i) + \gamma(f^\gamma - f^*) = 0, \tag{3.1}$$

with $A_i^* : L^2(0,T) \to L^2(Q)$ is the adjoint operator of A_i defined by $\sum_{i=1}^{N_m} A_i^*(\ell_i u(f) - \omega_i) = p$ where p is the solution of the adjoint problem (2.5).

We will approximate (3.1) by space-time finite element method. In fact, we will approximate A_k and A_k^* as follows.

3.1 Finite element approximate of A_k, A_k^*

We suppose that finite spaces $W_h \subset W(0,T)$, $X_h \subset X$ and $Y_h \subset Y$, we assume that $X_h \subset Y_h$. The Galerkin-Petrov discretization of the variational problem (2.1) is to find $\overline{u}_h \in X_h$ such that

$$a(\overline{u}_h, v_h) = \langle F, v_h \rangle_Q - a(\overline{u}_0, v_h), \forall v_h \in Y_h. \tag{3.2}$$

For the space-time domain $Q = \Omega \times I \subset \mathbb{R}^{d+1}$, we consider a sequence of admissible decompositions Q_h into shape regular simplicity finite element q_l

$$Q_h = \cup_{l=1}^N \bar{q}_l.$$

Denote $\{(x_k, t_k)\}_{k=1}^M$ is a set of nodes $(x_k, t_k) \in \mathbb{R}^{d+1}$. We introduce a reference element $q \in \mathbb{R}^{d+1}$ which any element q_l can maple to q by using

$$\begin{pmatrix} x \\ t \end{pmatrix} = \begin{pmatrix} x_k \\ t_k \end{pmatrix} + J_l \begin{pmatrix} \xi \\ \tau \end{pmatrix}, \begin{pmatrix} \xi \\ \tau \end{pmatrix} \in q.$$

with Δ_l is the volume of q_l

$$\Delta_l = \int_{q_l} dxdt = \det J_l \int_q d\xi d\tau = |q| \det J_l,$$

and the local mesh width

$$h_l = \Delta_l^{\frac{1}{d+1}}, \ h := \max_{l=1,\dots,N} h_l.$$

Note that

$$|q| = \begin{cases} \dfrac{1}{2}, & d = 1, \\ \dfrac{1}{6}, & d = 2. \end{cases}$$

The discrete variational problem (3.2) admits a unique solution $\overline{u}_h \in X_h$. Let $u_h = \overline{u}_h + \overline{u}_{0,h} \in W_h$. Hence, the discrete version of the optimal control problem (2.4) will be

$$J_{\gamma,h}(f) = \frac{1}{2} \sum_{i=1}^{N_m} \|A_{i,h} f - \hat{\omega}_{i,h}\|^2_{L^2(0,T)} + \frac{\gamma}{2} \|f - f^*\|^2_{L^2(Q)} \to \min.$$

Let f_h^γ be the solution of this problem is characterized by the variational equation

$$\nabla J_{\gamma,h}(f_h^\gamma) = \sum_{i=1}^{N_m} A_{i,h}^*(A_{i,h} f^\gamma - \hat{\omega}_{i,h}) + \gamma(f_h^\gamma - f^*) = 0, \tag{3.3}$$

where $A_{i,h}^*$ is the adjoint operator of $A_{i,h}$. But it is hardly to find $A_{i,h}^*$ from $A_{i,h}$ in practice. So we define a proximate $\hat{A}_{i,h}^*$ of A_i^* instead. In deed, we have $\sum_{i=1}^{N_m} \hat{A}_{i,h}^* \phi_i = p_h$, where $\phi_i = \ell_i u(f) - \omega_i$ and p_h is the approximate solution of adjoint problem (2.5). Therefore, the equation above will be

$$\nabla J_{\gamma,h}(f_h^\gamma) \simeq \nabla J_{\gamma,h}(\hat{f}_h^\gamma) = \sum_{i=1}^{N_m} \hat{A}_{i,h}^*(A_{i,h} \hat{f}^\gamma - \hat{\omega}_{i,h}) + \gamma(\hat{f}_h^\gamma - f^*) = 0, \tag{3.4}$$

Moreover, the observation will have noise in practice, so instead of ω, we only get ω^δ satisfying

$$\left\|\omega - \omega^\delta\right\|_{L^2(S_1)} \le \delta.$$

Therefore, instead of getting $\hat{f}_h^\gamma$ that satisfies the equation (3.5), we will get $\hat{f}_h^{\gamma,\delta}$ satisfying

$$\nabla J_{\gamma,h}\left(\hat{f}_h^{\gamma,\delta}\right) = \sum_{i=1}^{N_m} \hat{A}_{i,h}^*(A_{i,h}\hat{f}_h^{\gamma,\delta} - \hat{\omega}_{i,h}^\delta) + \gamma(\hat{f}_h^{\gamma,\delta} - f^*) = 0, \tag{3.5}$$

with $\hat{\omega}_{i,h}^\delta = \omega^\delta - \ell_i u_h(u_0,\varphi)$.

3.2 Convergence results

Theorem 3.1. *Let $u(x,t)$ be the solution of variational problem* (2.1) - (2.2) *and $\overline{u}_h(x,t)$ be the solution for* (3.2) *and $u_h(x,t) = \overline{u}_h(x,t) + \overline{u}_{0,h}(x,t)$. Then there holds the error estimate*

$$\|u - u_h\|_{L^2(0,T;\, H_0^1(\Omega))} \le ch\,|\overline{u}|_{H^2(\Omega)}\,. \tag{3.6}$$

and

$$\|u - u_h\|_{L^2(Q)} \le ch^2\,|u|_{H^2(\Omega)}\,. \tag{3.7}$$

What is more,

$$\left\|\sum_{i=1}^{N_m}\left(A_i^* - \hat{A}_{i,h}^*\right)\phi_i\right\|_{L^2(Q)}^2 = \int_Q (p - p_h)^2 dxdt = \|p - p_h\|_{L^2(Q)}^2$$

$$\Rightarrow \left\|\sum_{i=1}^{N_m}\left(A_i^* - A_{i,h}^*\right)\phi_i\right\|_{L^2(Q)} \le ch^2. \tag{3.8}$$

Let $u_h[f]$ v $u_h(u_0,\varphi)$ are the approximate solutions of **Problems** 2.1 and **Problems** 2.2 by using space-time finite element method. We define A_h of A is $A_h f = \ell u_h[f]$ and $\hat{\omega}_h = \omega - \ell u_h(u_0,\varphi)$. We have

$$\left\|\sum_{i=1}^{N_m}\left(A_i - A_{i,h}\right)f\right\|_{L^2(0,T)}^2 = \sum_{i=1}^{N_m}\|\ell_i u[f] - \ell_i u_h[f]\|_{L^2(0,T)}^2 \le \sum_{i=1}^{N_m}\|w_i\|_{L^2(\Omega)}^2\,\|u[f] - u_h[f]\|_{L^2(Q)}^2$$

$$\Rightarrow \left\|\sum_{i=1}^{N_m}\left(A_i - A_{i,h}\right)f\right\|_{L^2(0,T)} \le ch^2 \tag{3.9}$$

and

$$\left\|\sum_{i=1}^{N_m}\left(\hat{\omega}_i - \hat{\omega}_{i,h}\right)\right\|_{L^2(0,T)}^2 = \sum_{i=1}^{N_m}\|\ell_i u(u_0,\varphi) - \ell_i u_h(u_0,\varphi)\|_{L^2(0,T)}^2 \le \sum_{i=1}^{N_m}\|w_i\|_{L^2(\Omega)}^2\,\|u(u_0,\varphi) - u_h(u_0,\varphi)\|_{L^2(Q)}^2$$

$$\Rightarrow \left\|\sum_{i=1}^{N_m}\left(\hat{\omega}_i - \hat{\omega}_{i,h}\right)\right\|_{L^2(0,T)} \le ch^2 \tag{3.10}$$

Theorem 3.2. *Let f^γ and $\hat{f}_h^\gamma$ are the solution of variational problems* (3.1) *and* (3.4)*, respectively. Then there hold a error estimate*

$$\left\| f^\gamma - \hat{f}_h^\gamma \right\|_{L^2(Q)} \leq ch^2. \tag{3.11}$$

Proof. From equations (3.1) and (3.4), we will have

$$\begin{aligned}
\gamma\left(f^\gamma - \hat{f}_h^\gamma\right) &= \sum_{i=1}^{N_m} \hat{A}_{i,h}^* \left(A_{i,h}\hat{f}_h^\gamma - \hat{\omega}_{i,h}\right) - \sum_{i=1}^{N_m} A_i^* \left(A_i f^\gamma - \hat{\omega}_i\right) \\
&= \sum_{i=1}^{N_m} \left(\hat{A}_{i,h}^* - A_i^*\right)\left(A_{i,h}\hat{f}_h^\gamma - \hat{\omega}_{i,h}\right) + \sum_{i=1}^{N_m} A_i^* A_{i,h}\left(\hat{f}_h^\gamma - f^\gamma\right) \\
&\quad + \sum_{i=1}^{N_m} A_i^*\left(A_{i,h} - A_i\right) f^\gamma + \sum_{i=1}^{N_m} A_i^*\left(\hat{\omega}_i - \hat{\omega}_{i,h}\right)
\end{aligned}$$

According to (3.8), (3.9) and (3.10), we have

$$\begin{aligned}
&\left\| \sum_{i=1}^{N_m} \left(\hat{A}_{i,h}^* - A_i^*\right)\left(A_{i,h}\hat{f}_h^\gamma - \hat{\omega}_{i,h}\right) \right\|_{L^2(0,T)} \leq ch^2, \\
&\left\| \sum_{i=1}^{N_m} A_i^*\left(A_{i,h} - A_i\right) f^\gamma \right\|_{L^2(0,T)} \leq ch^2, \\
&\left\| \sum_{i=1}^{N_m} A_i^*\left(\hat{\omega}_i - \hat{\omega}_{i,h}\right) \right\|_{L^2(I)} \leq ch^2.
\end{aligned}$$

We take apart this

$$\sum_{i=1}^{N_m} A_i^* A_{i,h}\left(\hat{f}_h^\gamma - f^\gamma\right) = \sum_{i=1}^{N_m} A_i^*\left(A_{i,h} - A_i\right)\left(\hat{f}_h^\gamma - f^\gamma\right) + \sum_{i=1}^{N_m} A_i^* A_i\left(\hat{f}_h^\gamma - f^\gamma\right).$$

Moreover, we have

$$\begin{aligned}
&\left\langle \sum_{i=1}^{N_m} A_i^*\left(A_{i,h} - A_i\right)\left(\hat{f}_h^\gamma - f^\gamma\right), f^\gamma - \hat{f}_h^\gamma \right\rangle_{L^2(0,T)} \leq ch^2 \left\| f^\gamma - \hat{f}_h^\gamma \right\|_{L^2(Q)}^2, \\
&\left\langle \sum_{i=1}^{N_m} A_i^* A_i\left(\hat{f}_h^\gamma - f^\gamma\right), f^\gamma - \hat{f}_h^\gamma \right\rangle_{L^2(I)} = -\sum_{i=1}^{N_m} \left\| A_i\left(f^\gamma - \hat{f}_h^\gamma\right) \right\|_{L^2(0,T)}^2 < 0.
\end{aligned}$$

The theorem is proved. □

Remark 3.1. *Let f^γ and $\hat{f}_h^\gamma$ are the solution of variational problems* (3.1) *and* (3.5)*, respectively. Then there hold a error estimate*

$$\left\| f^\gamma - \hat{f}_h^{\gamma,\delta} \right\|_{L^2(Q)} \leq c(h^2 + \delta). \tag{3.12}$$

4 Numerical results

In all examples in this section, we choose the domain $\Omega = (0,1) \times (0,1)$, $T = 1$ and $a_{ij}(x,t) = \delta_{ij}$. For the temperature we take the exact solution be given by

$$u(x,t) = e^t (x_1 - x_1^2) \sin(\pi x_2).$$

We would like to reconstruct function f with several forms of F following

- Example 1: $F(x,t) = f(x,t)q(x,t) + g(x,t)$ for IP1 (Example 1.1) with $f(x,t) = \sin(\pi x_1)(x_2 - x_2^2)(t^2+1)$,
- Example 2: $F(x,t) = f(x)q(x,t) + g(x,t)$ for IP2 (Example 2.1) and IP3 (Example 2.2) with $f(x) = \sin(\pi x_1)(x_2 - x_2^2)$,
- Example 3: $F(x,t) = f(t)q(x,t) + g(x,t)$ for IP3 with following functions
  1. $f(t) = \begin{cases} 2t, & t \in [0, 0.5], \\ 2(1-t), & t \in [0.5, 1], \end{cases}$ for Example 3.1
  2. $f(t) = \begin{cases} 1, & t \in [0.25, 0.75], \\ 0, & t \notin [0.25, 0.75], \end{cases}$ for Example 3.2

We use a uniform decomposition of the domain Q into $65^3 = 274,625$ nodes and $6 \times 64^3 = 1,572,864$ finite elements. We take $q(x,t) = x_1x_2 + t + 1$, initial guess $f^* = 0, \gamma = 10^{-5}$ and level noise $\delta = 1\%$.

Example 1.1

We reconstruct $f(x,t)$ with observation in the whole domain.

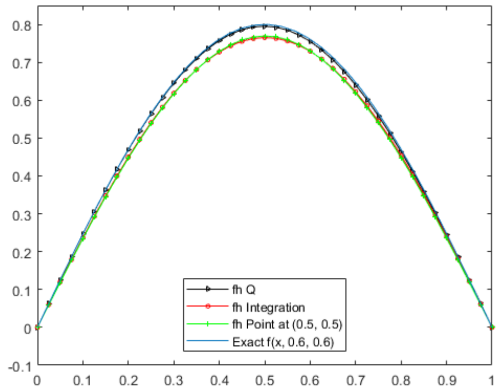

Figure 1: The exact $f(x_p,t)$, $x_p = (0.5, 0.5)$ and the numerical solution of Example 1.1.

Example 2.1

We reconstruct $f(x)$ with observation is the final overdetermination.

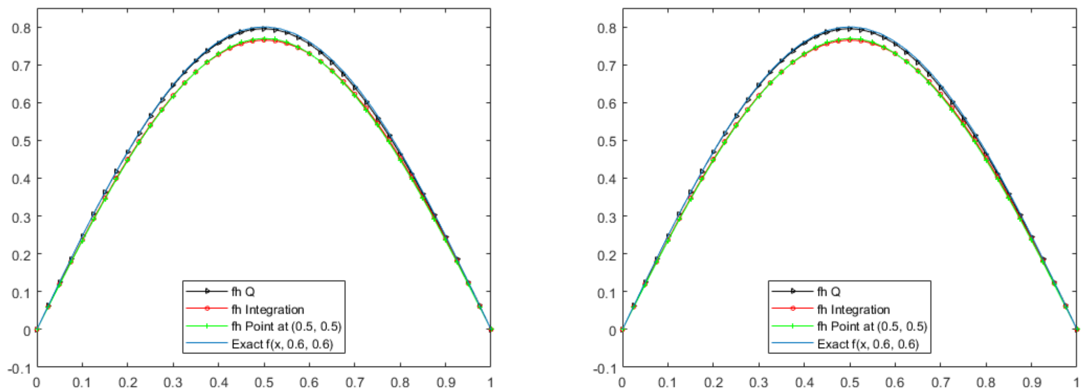

Figure 2: The exacts $f(x_1, 0.5)$, $f(0.5, x_2)$ and the numerical solutions of Example 2.1.

Example 3.1 and 3.2

We reconstruct $f(t)$ with an integral observation $w(x) = x_1^2 + x_2^2 + 1$ or a point observation $x_0 = (0.48, 0.48)$.

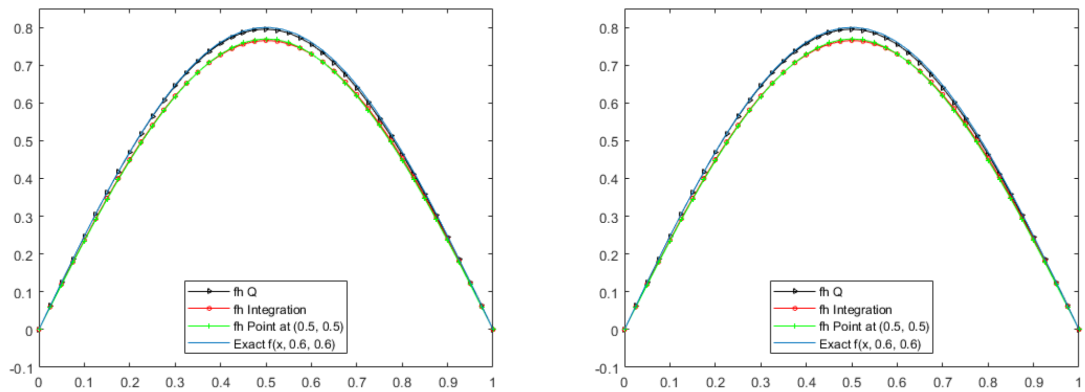

Figure 3: The exact and numerical solution of Example 3.1: integral observation (left) and point observation (right).

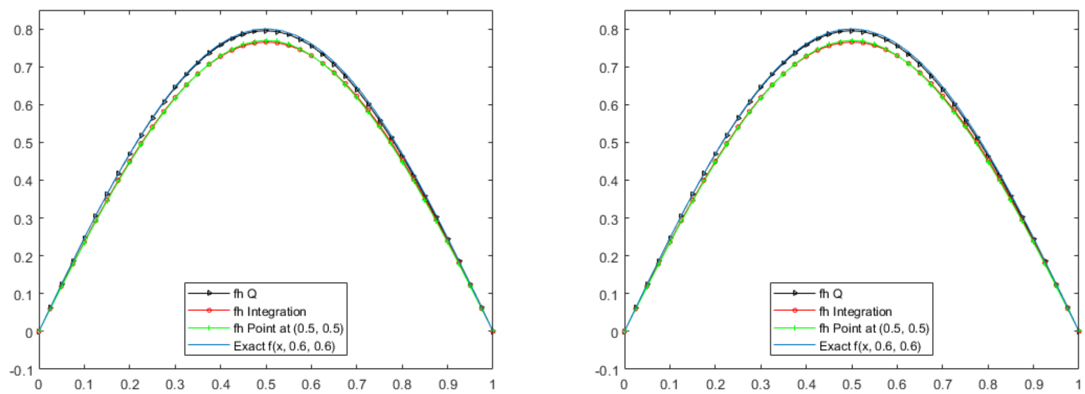

Figure 4: The exact and numerical solution of Example 3.2: integral observation (left) and point observation (right).

Example 2.2

We reconstruct $f(x)$ with 9 points described as follows

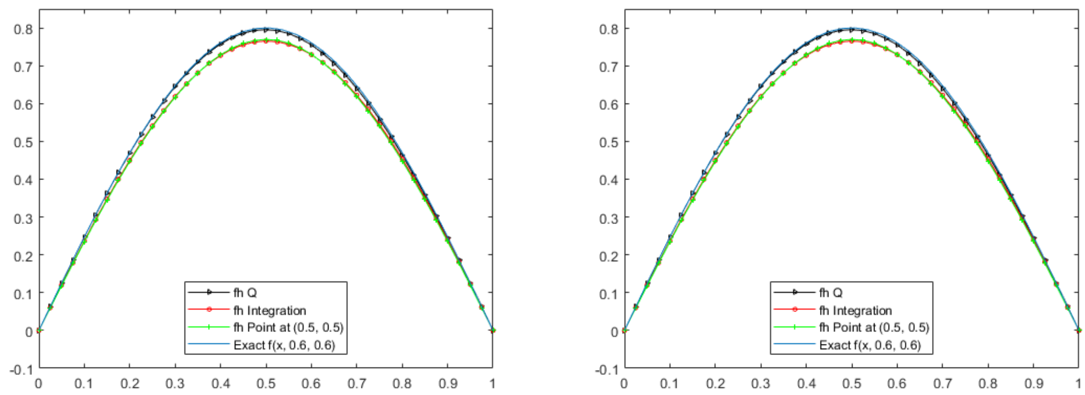

Figure 5: Observation points (left) and the exact $f(x_p, t)$, $x_p = (0.5, 0.5)$ and numerical solution of Example 2.2 (right).

5 Conclusion

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