1 Introduction

Consider a physical domain $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}^+$ be bounded with the boundary Γ and donate $Q := \Omega \times (0, T)$ and $S := \Gamma \times (0, T)$ with $T > 0$ given.

Consider the heat equation

$$\frac{\partial u}{\partial t} - \sum_{i,j=1}^{d} \frac{\partial}{\partial x_j}\left(a_{ji}(x,t)\frac{\partial u}{\partial x_i}\right) = F(x,t), \quad (x,t) \in Q, \tag{1.1}$$

with the initial and Dirichlet conditions, respectively

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{1.2}$$

$$u(x,t) = \varphi(x,t), \quad (x,t) \in S \tag{1.3}$$

where

$$a_{ij} \in L^\infty(Q),\ a_{ij} = a_{ji},\ \forall i,j \in \{1,2,...,d\},$$
$$\lambda_1 \left\|\xi\right\|^2 \le \sum_{i,j=1}^{d} a_{ij}\xi_i\xi_j \le \lambda_2 \left\|\xi\right\|^2,\ \forall \xi \in \mathbb{R}^d,$$
$$\varphi \in L^2(S),\ u_0 \in L^2(\Omega),\ F \in L^2(0, T; H^{-1}(\Omega)),$$

with λ_1 và λ_2 are positive constants.

The direct problem is to determine u when all data $a_{ji}, i,j = \overline{1,d}$, u_0, φ and F in eqs. (1.1) to (1.3) are given. On the other hand, the inverse problem (IP) is to identify a missed data such as the right hand side F when some additional observation on the solution u are available.

Consider the right hand side of equation (1.1) following the form $F(x,t) = f(.)q(x,t) + g(x,t)$, with q, g are given and $f(.)$ is either $f(x,t)$, $f(x)$ or $f(t)$. Denote N_o is the number of measurements and $\ell_k u = \omega_k, k = \overline{1, N_0}$ is the value of kth observation on the solution u we get. We have different inverse problems depending on either the form of F or the observation on the solution u:

- IP1: Find $f(.)$ if u is given on Q. So that $N_o = 1$ and $\ell_k u(x,t) = u(x,t) = \omega_k(x,t),\ (x,t) \in Q$.
- IP2: Find $f(.)$ if $\ell_k u = \displaystyle\int_\Omega w_k(x)u(x,t)dx = \omega_k(t)$ and $w_k(x) > 0, \forall x \in \Omega$ are given. This observation called *integral observation*. Furthermore, an observation derives from integral observation called *point observation* if $w_k(x)$ is a dirac delta function $\delta_k(x - x_k)$, so that $\ell_k(t) = \displaystyle\int_\Omega \delta_k(x - x_k)u(x,t)dx = u(x_k,t) = \omega_k(t)$.

To solve this problem, we need to minimize the least square functional

$$J_\gamma(f) = \frac{1}{2}\sum_{k=1}^{N_o} \left\|\ell_k u(f) - \omega_k\right\|^2_{L^2(*)}.$$

However, this minimization problem is unstable and there might be many minimizers to it. Therefore, we minimize the Tikhonov functional instead

$$J_\gamma(f) = \frac{1}{2}\sum_{k=1}^{N_o} \left\|\ell_k u(f) - \omega_k\right\|^2_{L^2(*)} + \frac{\gamma}{2}\left\|f - f^*\right\|^2_{L^2(**)},$$

with $\gamma > 0$ being a regularization parameter, f^* is an a prior estimation of f and $\|.\|_{L^2(*)}$ and $\|.\|_{L^2(**)}$ respectively depends on $w_k(*)$ and $f(**)$ appropriately.

2 Variational problem

To introduce the concept of weak form, we use the standard Sobolev spaces $H^1(\Omega)$, $H_0^1(\Omega)$, $H^{1,0}(Q)$ and $H^{1,1}(Q)$. Further, for a Banach space B, we define

$$L^2(0,T;B) = \left\{ u : u(t) \in B \text{ a.e } t \in (0,T) \text{ and } \|u\|_{L^2(0,T;B)} < \infty \right\},$$

with the norm

$$\|u\|_{L^2(0,T;B)} = \int_0^T \|u(t)\|_B^2 \, dt.$$

In the sequel, we shall use the space $W(0,T)$ define as

$$W(0,T) = \left\{ u : u \in L^2(0,T;H^1(\Omega)), u_t \in L^2\left(0,T;\left(H^1(\Omega)\right)'\right) \right\}$$

Suppose that $F \in L^2(Q)$, a week solution in $W(0,T)$ of the problem eqs. (1.1) to (1.3) is a function $u(x,t) \in W(0,T)$ satisfying the identity

$$\int_Q \left[\frac{\partial u}{\partial t} v + \sum_{i,j=1}^{d} a_{ji} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} \right] dxdt = \int_Q F v dxdt, \ \forall v \in L^2\left(0,T;H^1(\Omega)\right). \tag{2.1}$$

and

$$u(x,0) = u_0, \ x \in \Omega. \tag{2.2}$$

$$\|u\|_{L^2(0,T;B)} \le c_d \left(\|F\|_{L^2(Q)} + \|u_0\|_{L^2(\Omega)} + \|\varphi\|_{L^2(S)} \right) \tag{2.3}$$

We suppose that F has the form $F(x,t) = f(x,t)q(x,t) + g(x,t)$ with $f \in L^2(Q)$, $q \in L^\infty(Q)$ and $g \in L^2(Q)$. We hope to recover $f(x,t)$ from the observation. Since the solution $u(x,t)$ depends on the function $f(x,t)$, so we denote it by $u(x,t,f)$ or $u(f)$. Identify $f(x,t)$ satisfying

$$\ell_k u(f) = \omega_k, \ \forall k = \overline{1, N_o}$$

where $\ell_k u(f)$ is the observation on the solution depending on f. From now on, we suppose to solve IP1 problem, so that we need to minimize the Tikhonov functional

$$J_\gamma(f) = \frac{1}{2} \sum_{k=1}^{N_o} \|\ell_k u(f) - \omega_k\|_{L^2(Q)}^2 + \frac{\gamma}{2} \|f - f^*\|_{L^2(Q)}^2. \tag{2.4}$$

We will prove that J_γ is Frechet differentiable and drive a formula for its gradient. In doing so, we need the adjoint problem

$$\begin{cases} -\dfrac{\partial p}{\partial t} - \displaystyle\sum_{i,j=1}^{d} \frac{\partial}{\partial x_j} \left(a_{ji}(x,t) \frac{\partial p}{\partial x_i} \right) = \sum_{k=1}^{N_o} \ell_k u(f) - \omega_k, & (x,t) \in Q, \\ u(x,t) = 0, & (x,t) \in S \\ p(x,T) = 0, & x \in \Omega. \end{cases} \tag{2.5}$$

By changing the time direction, meaning $\overline{p}(x,t) = p(x, T-t)$, we will get a Dirichlet problem for parabolic equations.

Theorem 2.1. *The functional J_γ is Frechet differentiable and its gradient ∇J_γ at f has the form*

$$\nabla J_\gamma(f) = q(x,t)p(x,t) + \gamma\left(f(x,t) - f^*(x,t)\right) \tag{2.6}$$

Proof. By taking a small variation $\delta f \in L^2(Q)$ of f and denoting $\delta u(f) = u(f+\delta f) - u(f)$, we have

$$\begin{aligned} J_0(f+\delta f) - J_0(f) &= \frac{1}{2}\left\|\ell u(f+\delta f) - \omega\right\|^2_{L^2(Q)} - \frac{1}{2}\left\|\ell u(f) - \omega\right\|^2_{L^2(Q)} \\ &= \frac{1}{2}\left\|\ell \delta u(f) + \ell u(f) - \omega\right\|^2_{L^2(Q)} - \frac{1}{2}\left\|\ell u(f) - \omega\right\|^2_{L^2(Q)} \\ &= \frac{1}{2}\left\|\ell \delta u(f)\right\|^2_{L^2(Q)} + \langle \ell\delta u(f), \ell u(f) - \omega\rangle_{L^2(Q)} \end{aligned}$$

where $\delta u(f)$ is the solution to the problem

$$\begin{cases} \dfrac{\partial \delta u}{\partial t} - \displaystyle\sum_{i,j=1}^{d} \frac{\partial}{\partial x_j}\left(a_{ji}(x,t)\frac{\partial \delta u}{\partial x_i}\right) = q(x,t)\delta f, & (x,t) \in Q, \\ \delta u(x,t) = 0, & (x,t) \in S, \\ \delta u(x,0) = 0, & x \in \Omega. \end{cases} \tag{2.7}$$

Because the priori estimate (2.3) for the direct problem, we have

$$\left\|\ell \delta u(f)\right\|^2_{L^2(Q)} = o\left(\|\delta f\|_{L^2(Q)}\right) \text{ when } \|\delta f\|_{L^2(Q)} \to 0.$$

What is more, applying the Green formula for (2.5) and (2.7), we get

$$\int_Q \delta u\left(\ell u(f) - \omega\right) dxdt = \int_Q p(x,t)q(x,t)\delta f(x,t) dxdt$$

Therefore,

$$\begin{aligned} J_0(f+\delta f) - J_0(f) &= \int_Q \delta u\left(\ell u - \omega\right) ds + o\left(\|\delta f\|_{L^2(Q)}\right) \\ &= \int_Q qp\delta f dxdt + o\left(\|\delta f\|_{L^2(I)}\right) \\ &= \langle qp, \delta f\rangle_{L^2(Q)} + o\left(\|\delta f\|^2_{L(Q)}\right). \end{aligned}$$

It is easy to get this

$$J_\gamma(f+\delta f) - J_\gamma(f) = \langle qp, \delta f\rangle_{L^2(Q)} + \gamma\langle f - f^*, \delta f\rangle_{L^2(Q)} + o\left(\|\delta f\|^2_{L(Q)}\right).$$

Hence the functional J_γ is Frechet differentiable and its gradient ∇J_γ at f has the form (2.5). The theorem is proved. □

Remark 2.1. *In this theorem, we write the Tikhonov functional for $F(x,t) = f(x,t)q(x,t) + g(x,t)$. But when F has another structure, the penalty term should be modified*

- $F(x,t) = f(x)q(x,t) + g(x,t)$: *the penalty functional is* $\|f - f^*\|_{L^2(\Omega)}$ *and*

$$\nabla J_0(f) = \int_0^T q(x,t)p(x,t)dt.$$

- $F(x,t) = f(t)q(x,t) + g(x,t)$: *the penalty functional is* $\|f - f^*\|_{L^2(0,T)}$ *and*

$$\nabla J_0(f) = \int_\Omega q(x,t)p(x,t)dt.$$

To find f satisfied (2.4), we use the conjugate gradient method (CG). It proceeds as follows: assume that at the kth iteration, we have f^k. Then the next iteration is

$$f^{k+1} = f^k + \alpha_k d^k,$$

with

$$d^k = \begin{cases} -\nabla J_\gamma(f^k), & k = 0, \\ -\nabla J_\gamma(f^k) + \beta_k d^{k-1}, & k > 0, \end{cases}$$

$$\beta_k = \frac{\left\|\nabla J_\gamma(f^k)\right\|^2_{L^2(I)}}{\left\|\nabla J_\gamma(f^{k-1})\right\|^2_{L^2(I)}},$$

and

$$\alpha_k = \arg\min_{\alpha \geq 0} J_\gamma(f^k + \alpha d^k).$$

To identify α_k, we consider two problems

Problem 2.1. Denote the solution of this problem is $u[f]$

$$\begin{cases} \dfrac{\partial u}{\partial t} - \displaystyle\sum_{i,j=1}^d \frac{\partial}{\partial x_j}\left(a_{ji}(x,t)\frac{\partial u}{\partial x_i}\right) = f(t)h(x,t), & (x,t) \in Q, \\ \dfrac{\partial u}{\partial \mathcal{N}}(x,t) = 0, & (x,t) \in S, \\ u(x,0) = 0, & x \in \Omega. \end{cases}$$

Problem 2.2. Denote the solution of this problem is $u(u_0, \varphi)$

$$\begin{cases} \dfrac{\partial u}{\partial t} - \displaystyle\sum_{i,j=1}^d \frac{\partial}{\partial x_j}\left(a_{ji}(x,t)\frac{\partial u}{\partial x_i}\right) = g(x,t), & (x,t) \in Q, \\ \dfrac{\partial u}{\partial \mathcal{N}}(x,t) = \varphi(x,t), & (x,t) \in S, \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases}$$

If so, the observation operators have the form $\ell u(f) = \ell u[f] + \ell u(u_0,\varphi) = Af + \ell u(u_0,\varphi)$, with A being bounded linear operators from $L^2(Q)$ to $L^2(Q)$.

We have

$$\begin{aligned}J_\gamma(f^k+\alpha d^k) &= \frac{1}{2}\left\|\ell u(f^k+\alpha d^k)-\omega\right\|^2_{L^2(S_1)}+\frac{\gamma}{2}\left\|f^k+\alpha d^k-f^*\right\|^2_{L^2(I)}\\ &= \frac{1}{2}\left\|\alpha A d^k + Af^k+\ell u(u_0,\varphi)-\omega\right\|^2_{L^2(S_1)}+\frac{\gamma}{2}\left\|f^k+\alpha d^k-f^*\right\|^2_{L^2(I)}\\ &= \frac{1}{2}\left\|\alpha A d^k+\ell u(f^k)-\omega\right\|^2_{L^2(S_1)}+\frac{\gamma}{2}\left\|f^k+\alpha d^k-f^*\right\|^2_{L^2(I)}.\end{aligned}$$

Differentiating $J_\gamma(f^k+\alpha d^k)$ with respect to α, we get

$$\begin{aligned}\frac{\partial J_\gamma(f^k+\alpha d^k)}{\partial\alpha} &= \alpha\left\|Ad^k\right\|^2_{L^2(S_1)}+\left\langle Ad^k,\ell u(f^k)-\omega\right\rangle_{L^2(S_1)}\\ &\quad+\gamma\alpha\left\|d^k\right\|^2_{L^2(I)}+\gamma\left\langle d^k,f^k-f^*\right\rangle_{L^2(I)}.\end{aligned}$$

Putting $\dfrac{\partial J_\gamma(f^k+\alpha d^k)}{\partial\alpha}=0$, we obtain

$$\begin{aligned}\alpha_k &= -\frac{\left\langle Ad^k,\ell u(f^k)-\omega\right\rangle_{L^2(S_1)}+\gamma\left\langle d^k,f^k-f^*\right\rangle_{L^2(I)}}{\|Ad^k\|^2_{L^2(S_1)}+\gamma\|d^k\|^2_{L^2(I)}}\\ &= -\frac{\left\langle d^k,A^*\left(\ell u(f^k)-\omega\right)\right\rangle_{L^2(I)}+\gamma\left\langle d^k,f^k-f^*\right\rangle_{L^2(I)}}{\|Ad^k\|^2_{L^2(S_1)}+\gamma\|d^k\|^2_{L^2(I)}}\\ &= -\frac{\left\langle d^k,A^*\left(\ell u(f^k)-\omega\right)+\gamma(f^k-f^*)\right\rangle_{L^2(I)}}{\|Ad^k\|^2_{L^2(S_1)}+\gamma\|d^k\|^2_{L^2(I)}}\\ &= -\frac{\left\langle d^k,\nabla J_\gamma(f^k)\right\rangle_{L^2(I)}}{\|Ad^k\|^2_{L^2(S_1)}+\gamma\|d^k\|^2_{L^2(I)}}.\end{aligned}$$

Because of $d^k=r^k+\beta_k d^{k-1}$, $r^k=-\nabla J_\gamma(f^k)$ and $\left\langle r^k,d^{k-1}\right\rangle_{L^2(I)}=0$, we get

$$\alpha_k=\frac{\left\|r^k\right\|^2_{L^2(I)}}{\|Ad^k\|^2_{L^2(S_1)}+\gamma\|d^k\|^2_{L^2(I)}}.$$

CG algorithm

1. Set $k=0$, initiate f^0.

2. For $k=0,1,2,\ldots.$ Calculate

$$r^k=-\nabla J_\gamma(f^k).$$

Update

$$d^k=\begin{cases} r^k, & k=0,\\ r^k+\beta_k d^{k-1}, & k>0,\end{cases}$$

$$\beta_k=\frac{\left\|r^k\right\|^2_{L^2(Q)}}{\|r^{k-1}\|^2_{L^2(Q)}}.$$

3. Calculate

$$\alpha_k = \frac{\left\|r^k\right\|^2_{L^2(Q)}}{\|Ad^k\|^2_{L^2(S_1)} + \gamma \left\|d^k\right\|^2_{L^2(Q)}}.$$

Update

$$f^{k+1} = f^k + \alpha_k d^k.$$

3 Finite element method

We rewrite the Tikhonov functional

$$\begin{aligned}
J_\gamma(f) &= \frac{1}{2}\left\|\ell u[f] + \ell u(u_0,\varphi) - \omega\right\|^2_{L^2(S_1)} + \frac{\gamma}{2}\left\|f - f^*\right\|^2_{L^2(I)} \\
&= \frac{1}{2}\left\|Af + \ell u(u_0,\varphi) - \omega\right\|^2_{L^2(S_1)} + \frac{\gamma}{2}\left\|f - f^*\right\|^2_{L^2(I)} \\
&= \frac{1}{2}\left\|Af - \hat{\omega}\right\|^2_{L^2(S_1)} + \frac{\gamma}{2}\left\|f - f^*\right\|^2_{L^2(I)},
\end{aligned}$$

with $\hat{\omega} = \omega - \ell u(u_0, \varphi)$.

The solution f^γ of the minimization problem (2.4) is characterized by the first-order optimality condition

$$\nabla J_\gamma(f^\gamma) = A^*(Af^\gamma - \hat{\omega}) + \gamma(f^\gamma - f^*) = 0, \tag{3.1}$$

with $A^* : L^2(Q) \to L^2(Q)$ is the adjoint operator of A defined by $A^*\phi = p$ where $\phi = \ell u(f) - \omega$ and p is the solution of the adjoint problem (2.5).

3.1 Finite element approximate of A, A^*

We will approximate (3.1) by using the space-time finite element method with finite spaces $X_h \subset Y_h$. For the space-time domain $Q = \Omega \times I \subset \mathbb{R}^{d+1}$, we consider a sequence of admissible decompositions Q_h into shape regular simplicity finite element q_l

$$Q_h = \cup_{l=1}^N \bar{q}_l.$$

Denote $\{(x_k, t_k)\}_{k=1}^M$ is a set of nodes $(x_k, t_k) \in \mathbb{R}^{d+1}$. We introduce a reference element $q \in \mathbb{R}^{d+1}$ which any element q_l can maple to q by using

$$\begin{pmatrix} x \\ t \end{pmatrix} = \begin{pmatrix} x_k \\ t_k \end{pmatrix} + J_l \begin{pmatrix} \xi \\ \tau \end{pmatrix}, \ \begin{pmatrix} \xi \\ \tau \end{pmatrix} \in q.$$

with Δ_l is the volume of q_l

$$\Delta_l = \int_{q_l} dxdt = \det J_l \int_q d\xi d\tau = |q| \det J_l,$$

and the local mesh width

$$h_l = \Delta_l^{\frac{1}{d+1}}, \ h := \max_{l=1,\dots,N} h_l.$$

Note that

$$|q| = \begin{cases} \dfrac{1}{2}, & d = 1, \\ \dfrac{1}{6}, & d = 2. \end{cases}$$

$$\int_Q \left[\frac{\partial u_h}{t} v_h + \sum_{i,j=1}^{d} a_{ji} \frac{\partial u_h}{\partial x_i} \frac{\partial v_h}{\partial x_j} \right] dxdt = \int_Q F v_h dxdt + \int_S \varphi v_h dsdt, \forall v_h \in Y_h. \tag{3.2}$$

and

$$u_h(x, 0) = u_0, x \in \Omega. \tag{3.3}$$

The discrete variational problem (3.2) admits a unique solution $u_h \in X_h$. Hence, the discrete version of the optimal control problem will be

$$J_{\gamma,h}(f) = \frac{1}{2} \|A_h f - \hat{\omega}_h\|^2_{L^2(S_1)} + \frac{\gamma}{2} \|f - f^*\|^2_{L^2(I)}.$$

Let f_h^γ be the solution of this problem is characterized by the variational equation

$$\nabla J_{\gamma,h}(f_h^\gamma) = A_h^*(A_h f^\gamma - \hat{\omega}_h) + \gamma(f_h^\gamma - f^*) = 0, \tag{3.4}$$

where A_h^* is the adjoint oeprator of A_h. But it is hardly to find A_h^* from A_h in practice. So we define a proximate $\hat{A}_h^*$ of A^* instead. In deed, we have $\hat{A}_h^* \phi = p_h$, with p_h is the approximate solution of adjoint problem (2.5). Therefore, the equation above will be

$$\nabla J_{\gamma,h}(f_h^\gamma) \simeq \nabla J_{\gamma,h}(\hat{f}_h^\gamma) = \hat{A}_h^*(A_h \hat{f}^\gamma - \hat{\omega}_h) + \gamma(\hat{f}_h^\gamma - f^*) = 0, \tag{3.5}$$

Moreover, the observation will have noise in pratice, so instead of $\omega(x, t)$, we only get $\omega^\delta(x, t)$ satisfy

$$\left\|\omega - \omega^\delta\right\|_{L^2(S_1)} \leq \delta.$$

Therefore, instead of getting $\hat{f}_h^\gamma$ satisfies the equation (3.5), we will get $\hat{f}_h^{\gamma,\delta}$ satisfies

$$\nabla J_{\gamma,h} \left(\hat{f}_h^{\gamma,\delta} \right) = \hat{A}_h^*(A_h \hat{f}_h^{\gamma,\delta} - \hat{\omega}_h^\delta) + \gamma(\hat{f}_h^{\gamma,\delta} - f^*) = 0, \tag{3.6}$$

with $\hat{\omega}_h^\delta = \omega^\delta - \ell u_h(u_0, \varphi)$.

3.2 Convergence results

Theorem 3.1. *Let $u(x,t)$ be the solution of variational problem* (2.1) - (2.2) *and $u_h(x,t)$ be the solution for* (3.2) - (3.3) *Then there holds the error estimate*

$$\|u - u_h\|_{L^2(Q)} \leq \|u - u_h\|_{L^2(I;\, H^1(\Omega))} \leq c_2 h_{xt} |u|_{H^2(\Omega)}. \tag{3.7}$$

What is more,

$$\left\| \left(A^* - \hat{A}_h^* \right) \phi \right\|_{L^2(Q)}^2 = \int_Q (p - p_h)^2 dxdt = \|p - p_h\|_{L^2(Q)}^2$$

$$\Rightarrow \|(A^* - A_h^*) \, q\|_{L^2(I)} \le c_5 h. \tag{3.8}$$

Let $u_h[f]$ và $u_h(u_0, \varphi)$ are the approximate solutions of **Problems** 2.1 and **Problems** 2.2 by using space-time finite element method. We define A_h of A is $A_h f = \ell u_h[f]$ and $\hat{\omega}_h = \omega - \ell u_h(u_0, \varphi)$. We have

$$\|(A - A_h) \, f\|_{L^2(Q)} = \|\ell u[f] - \ell u_h[f]\|_{L^2(\Omega)} \le \|u[f] - u_h[f]\|_{L^2(Q)} \le c_3 h, \tag{3.9}$$

and

$$\|\hat{\omega} - \hat{\omega}_h\|_{L^2(Q)} = \|u(u_0, \varphi) - u_h(u_0, \varphi)\|_{L^2(Q)} \le c_4 h. \tag{3.10}$$

Theorem 3.2. *Let f^γ and $\hat{f}_h^\gamma$ are the solution of variational problems* (3.1) *and* (3.5)*, respectively. Then there hold a error estimate*

$$\left\| f^\gamma - \hat{f}_h^\gamma \right\|_{L^2(Q)} \le c_6 h. \tag{3.11}$$

Proof. From equations (3.1) and (3.5), we will have

$$\begin{aligned}
\gamma \left(f^\gamma - \hat{f}_h^\gamma \right) &= \hat{A}_h^* \left(A_h \hat{f}_h^\gamma - \hat{\omega}_h \right) - A^* \left(A f^\gamma - \hat{\omega} \right) \\
&= \left(\hat{A}_h^* - A^* \right) \left(A_h \hat{f}_h^\gamma - \hat{\omega}_h \right) + A^* A_h \left(\hat{f}_h^\gamma - f^\gamma \right) \\
&\quad + A^* \left(A_h - A \right) f^\gamma + A^* \left(\hat{\omega} - \hat{\omega}_h \right)
\end{aligned}$$

According to (3.8), (3.9) and (3.10), we have

$$\begin{aligned}
&\left\| \left(\hat{A}_h^* - A^* \right) \left(A_h \hat{f}_h^\gamma - \hat{\omega}_h \right) \right\|_{L^2(I)} \le c_7 h_{xt}, \\
&\|A^* \left(A_h - A \right) f^\gamma\|_{L^2(I)} \le c_8 h_{xt}, \\
&\|A^* \left(\hat{\omega} - \hat{\omega}_h \right)\|_{L^2(I)} \le c_9 h_{xt}.
\end{aligned}$$

We take apart this

$$A^* A_h \left(\hat{f}_h^\gamma - f^\gamma \right) = A^* \left(A_h - A \right) \left(\hat{f}_h^\gamma - f^\gamma \right) + A^* A \left(\hat{f}_h^\gamma - f^\gamma \right).$$

Moreover, we have

$$\begin{aligned}
&\left\langle A^* \left(A_h - A \right) \left(\hat{f}_h^\gamma - f^\gamma \right), f^\gamma - \hat{f}_h^\gamma \right\rangle_{L^2(I)} \le c_{10} h_{xt}^2 \left\| f^\gamma - \hat{f}_h^\gamma \right\|_{L^2(I)}^2, \\
&\left\langle A^* A \left(\hat{f}_h^\gamma - f^\gamma \right), f^\gamma - \hat{f}_h^\gamma \right\rangle_{L^2(I)} = - \left\| A \left(f^\gamma - \hat{f}_h^\gamma \right) \right\|_{L^(S_1)}^2 < 0.
\end{aligned}$$

The theorem is proved. □

Remark 3.1. *Let f^γ and $\hat{f}_h^\gamma$ are the solution of variational problems* (3.1) *and* (3.5)*, respectively. Then there hold a error estimate*

$$\left\| f^\gamma - \hat{f}_h^{\gamma,\delta} \right\|_{L^2(I)} \le c_{11} (h + \delta). \tag{3.12}$$

4 Numerical results

Let $\Omega = [0,1]^2$, $T = 1$ and $a(x,t) = 1 + x^2 + y^2 + t^2$ and we want to re-simulate the process of conduction heat transfer and identifying the heat source

$$u(x,t) = \mathrm{e}^t \sin(\pi x)\sin(\pi y).$$

The form of the heat source

$$F(x,t) = f(x,t)h(x,t) + g(x,t),$$

where

$$h(x,t) = 2 + x^2 + y^2 + t^2.$$

Set $f^* = 0$, $\gamma = 10^{-6}$.
$f(.)$:

$$f(x,t) = \phi(t)\sin(\pi x)\sin(\pi y)$$

where

- Example 1: $\phi(t) = \sin(\pi t)$
- Example 2: $\phi(t) = \begin{cases} 2t, & t \in [0, 0.5], \\ 2(1-t), & t \in [0.5, 1], \end{cases}$
- Example 3: $\phi(t) = \begin{cases} 1, & t \in [0.25, 0.75], \\ 0, & t \notin [0.25, 0.75], \end{cases}$

5 Conclusion

References