Numerical simulation of heat transfer problem by Freefem++ software

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Abstract

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1 Introduction

Let $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}^+$ be a bounded domain with a boundary Γ and endow the cylinder $Q = \Omega \times (0, T]$ and lateral surface area $S = \Gamma \times (0, T]$ where $T > 0$. Consider the heat equation:

$$\frac{\partial u(x,t)}{\partial t} + \mathcal{L}u(x,t) = F(x,t), \quad (x,t) \in Q, \tag{1.1}$$

with the Dirichlet boundary and initial conditions, respectively

$$u(x,t) = u_D(x,t), \quad (x,t) \in S, \tag{1.2}$$
$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{1.3}$$

where

$$\mathcal{L}u = -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{ji}\frac{\partial u}{\partial x_j}\right) + a_0 u,$$
$$a_{ji} \in L^\infty(Q),\ a_{ij} = a_{ji},\ \forall i,j \in \{1,2,...,d\},$$
$$\lambda_1 \|\xi\|^2 \le \sum_{i,j=1}^{d} a_{ij}\xi_i\xi_j \le \lambda_2 \|\xi\|^2,\ \forall \xi \in \mathbb{R}^d,$$
$$a_0 \in L^\infty(Q),\ 0 \le a_0(x,t) \le \mu_1,\ (x,t) \in Q,$$
$$u_0 \in L^2(\Omega),\ u_D \in L^2(S),$$

with λ_1 and λ_2 are positive constants and $\mu_1 \ge 0$.
The problem is that to determine u when all data a_{ji}, a_0, u_0, u_D and F in (1.1) - (1.2) - (1.3) are given called ***direct problem***.

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2 Numerical method

2.1 Variational problem

Find $u(.,t) \in H^1(\Omega)$ such that

$$\left\langle \frac{\partial u}{\partial t}, v \right\rangle + a\,(u,v) = \langle F, v\rangle\,,\ \forall v \in H^1(\Omega), \tag{2.1}$$

$$u(x,0) = u_0(x),\ x \in \Omega, \tag{2.2}$$

where

$$a\,(u,v) = \int_\Omega \left[\sum_{i,j=1}^{d} a_{ji}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} + a_0 uv\right] dx,$$

$$\langle \varphi, v\rangle = \int_\Omega \varphi v dx.$$

2.2 Finite element method

Now we present a fully discrete finite element approximation for the variational problem (2.1) by the Crank-Nicolson method as follows:
For *spatial approximation*, let $\mathcal{T}_h$ be a triangulation of Ω and define a piecewise linear finite element space $V_h \subset H^1(\Omega)$ by

$$V_h = \left\{v_h : v_h \in C(\overline{\Omega}), v_h|_K \in P_1(K), \forall K \in \mathcal{T}_h\right\},$$

where $P_1(K)$ is a continuous piecewise linear polynomial on the element K.
For *temporal discretization*, discrete $[0,T]$ uniformly into M steps, $t_n = n\Delta t$, $n = 0, 1, \dots, M$ with the time step size $\Delta t = T/M$. We define a function $\varphi(x,t)$ and

$\varphi(x, t_n) = \varphi^n(x)$.
Find $u_h^n \in V_h$ for $n = 1, 2, \ldots, M$ such that

$$\langle d_t u_h^n, v_h \rangle + a\left(\theta u_h^n + (1-\theta)u_h^{n-1}, v_h\right) \\ = \left\langle \theta F^n + (1-\theta)F^{n-1}, v_h \right\rangle, \ \forall v_h \in V_h, \quad (2.3)$$

and the initial condition

$$u_h^0 = u_0, \quad (2.4)$$

where $d_t u_h^n = \dfrac{u_h^n - u_h^{n-1}}{\Delta t}$, $n = 1, 2, ..., M$.

The discrete variational problem (2.3) admits a unique solution $u_h^n \in V_h$, see (**??**). Let $u_h(x,t)$ be the linear interpolation of u_h^n with respect to t. For $x \in \Omega$, $t \in [t_{n-1}, t_n]$, we have

$$u_h(x,t) = \frac{t - t_{n-1}}{\Delta t} u_h^{n-1} + \frac{t_n - t}{\Delta t} u_h^n.$$

Theorem 2.1. *Let $u(x,t)$ be the solution of variational problem (2.1) - (2.2) and $u_h^n \in V_h$ for $n = 1, 2, \ldots, M$ be the solution for (2.3). Then there holds the error estimate, see* (**??**)

$$\|u_h - u\|_{L^2(Q)} = O\left(h^2 + \Delta t^2\right), \quad (2.5)$$

where h is the mesh size.

3 Tests and discussion

3.1 Error evaluation with exact solution

We study a numerical experiment with the exact solution of heat equation and evaluate the error convergence. Consider a square $[0,1] \times [0,1]$. Find $u(x,y,t)$ satisfy

$$\frac{\partial u}{\partial t} - \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = (1+2t)\sin(\pi x)\sin(\pi y) \quad (3.1)$$

with the initial and boundary conditions

$$u(x,y,0) = 0 \quad \text{and} \quad u|_\Gamma = 0.$$

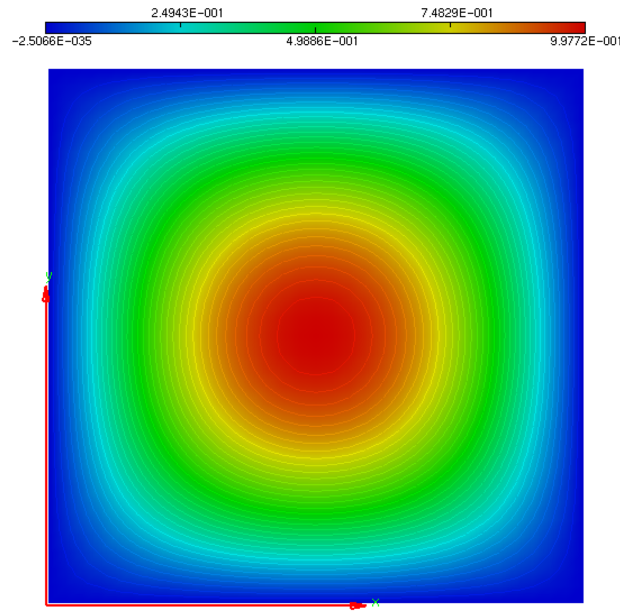
The exact solution is

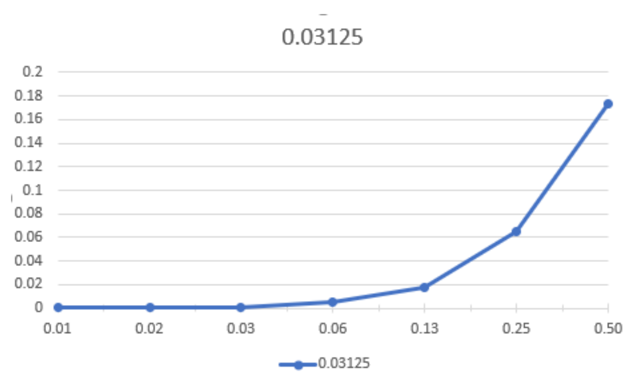
$$u = \sin(\pi x)\sin(\pi y)t,$$

Different cases of mesh size and time step length were studied to show the dependent of error on the mesh smoothness. We also use two different schemes

- Backward Euler scheme
- Crank-Nicolson scheme

The approximate solution at final time is illustrated below.





3.2 A problem of thermal engineering

In the aspect of thermal engineering, the simulations of heat transfer attend in several applications. In [...] the author mentioned a realistic heat problem which can be solve by simulating the heat transfer process. Assuming there are two types of material with different thermal conductivity and price. We would like to build a thermal resistance wall from composite plate of the two materials that provide optimal thermal resistance properties and satisfies economical conditions. Let V is the total volume of the plate, V_e and V_c are the volume of expensive and cheap material respectively, then the ratio

$$\mu = \frac{V_e}{V} = \frac{V_e}{V_c + V_e}$$

is fixed. Consider a rectangular room Ω_r has the size $L_x \times L_y$. At the center of the left wall located a radiator that keep the local temperature at T_r, denote as Γ_r, otherwise as Γ_α. The thermal flux through the walls is expected to be zero, equivalent to the Neumann boundary condition

$$\frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \Gamma_\alpha.$$

The right corner of the room is where we place our composite thermal resistance plate. Let l_x is the width of the plate. The right wall is consider the heat source and gain the Dirichlet boundary condition

$$u = u_{ext} \quad \text{on} \quad \Gamma_{ext}.$$

Finally, let κ_a, κ_c and κ_e is correspondingly the thermal conductivity coefficient of air, our cheap and expensive material. Based on our goal of minimizing the temperature inside the room, the cost function can be formulated as follow

$$J = \frac{1}{|\Omega_a|} \int_0^T \int_{\Omega_a} u dx$$

This problem belong to the set of shape optimization problems or more generally is an inverse heat problem. These kinds of problems take part in large amount of engineering applications. An approach is to solve as much acceptable input cases as possible then determine which one is the optimal solution. This way requires numerous computations.

3.3 Numerical experiment of inverse heat problem

FreeFem++ provide an efficient tool to handle inverse problems for partial differential equations by using the C++ optimization module IPOPT. To be more accurate, the module support multiple languages in solving optimization problems. The application of IPOPT for solving inverse heat problem is shown below.
To use the IPOPT module we have to add command load "ipopt"
We need to find the expression of objective function J and its gradient ∇J respective with the input control function f. The optimal f is achieved via IPOPT loop using a conjugate gradient method solver such as Newton-Raphson method.
J and gradJ
This is an illustrating of the expected control f and its approximation using IPOPT.

4 Conclusion

References