Numerical simulation of heat transfer problem by Freefem++ software

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Abstract

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1 Introduction

Let $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}^+$ be a bounded domain with a boundary Γ and endow the cylinder $Q = \Omega \times (0, T]$ and lateral surface area $S = \Gamma \times (0, T]$ where $T > 0$.

Consider the heat equation:

$$\frac{\partial u(x,t)}{\partial t} + \mathcal{L}u(x,t) = F(x,t), \quad (x,t) \in Q, \tag{1.1}$$

with the Dirichlet boundary and initial conditions, respectively

$$u(x,t) = u_D(x,t), \quad (x,t) \in S, \tag{1.2}$$
$$u(x,0) = u_0(x), \qquad x \in \Omega, \tag{1.3}$$

where

$$\begin{aligned}
&\mathcal{L}u = -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{ji}\frac{\partial u}{\partial x_j}\right) + a_0 u,\\
&a_{ji} \in L^\infty(Q),\ a_{ij} = a_{ji},\ \forall i,j \in \{1,2,...,d\},\\
&\lambda_1 \|\xi\|^2 \le \sum_{i,j=1}^{d} a_{ij}\xi_i\xi_j \le \lambda_2 \|\xi\|^2,\ \forall \xi \in \mathbb{R}^d,\\
&a_0 \in L^\infty(Q),\ 0 \le a_0(x,t) \le \mu_1,\ (x,t) \in Q,\\
&u_0 \in L^2(\Omega),\ u_D \in L^2(S),
\end{aligned}$$

with λ_1 and λ_2 are positive constants and $\mu_1 \ge 0$.
The problem is that to determine u when all data a_{ji}, a_0, u_0, u_D and F in (1.1) - (1.2) - (1.3) are given called ***direct problem***.

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2 Numerical method

2.1 Variational problem

Multiplying (1.1) by an efficient smooth test function v, integrating over Ω and then applying Green's formula, see [], leads to the problem: Find $u(.,t) \in H^1(\Omega)$ such that

$$\left\langle \frac{\partial u}{\partial t}, v \right\rangle + a(u,v) = \langle F, v \rangle,\ \forall v \in H^1(\Omega), \tag{2.1}$$

$$u(x,0) = u_0(x),\ x \in \Omega, \tag{2.2}$$

where

$$a(u,v) = \int_\Omega \left[\sum_{i,j=1}^{d} a_{ji}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} + a_0 uv\right] dx,$$

$$\langle F, v \rangle = \int_\Omega F v dx.$$

2.2 Finite element method

Now we present a fully discrete finite element approximation for the variational problem (2.1) as follows:

- For *spatial approximation*, let $\mathcal{T}_h$ be a triangulation of Ω and define a piecewise linear finite element space $V_h \subset H^1(\Omega)$ by

  $$V_h = \left\{v_h : v_h \in C(\overline{\Omega}), v_h|_K \in P_1(K), \forall K \in \mathcal{T}_h\right\},$$

  where $P_1(K)$ is a continuous piecewise linear polynomial on the element K.

- For *temporal discretization*, discrete $[0,T]$ uniformly into M steps, $t_n = n\Delta t$, $n = 0, 1, \ldots, M$ with the time step size $\Delta t = T/M$. We define a function $u(x, t_n) = u^n(x)$.

Find $u_h^n \in V_h$ for $n = 1, 2, \ldots, M$ with $\theta \in [0,1]$ such that

$$\begin{aligned}\langle d_t u_h^n, v_h\rangle + a\left(\theta u_h^n + (1-\theta)u_h^{n-1}, v_h\right) \\ = \left\langle \theta F^n + (1-\theta)F^{n-1}, v_h\right\rangle,\ \forall v_h \in V_h,\end{aligned} \tag{2.3}$$

and the initial condition

$$u_h^0 = u_0, \tag{2.4}$$

where $d_t u_h^n = \dfrac{u_h^n - u_h^{n-1}}{\Delta t}$, $n = 1, 2, ..., M$.

The discrete variational problem (2.3) -(2.4) admits a unique solution $u_h^n \in V_h$, see []. Let $u_h(x,t)$ be the linear interpolation of u_h^n with respect to t. Therefore, for $x \in \Omega$, $t \in [t_{n-1}, t_n]$, we have

$$u_h(x,t) = \frac{t - t_{n-1}}{\Delta t} u_h^{n-1} + \frac{t_n - t}{\Delta t} u_h^n.$$

Theorem 2.1. *Let $u(x,t)$ be the solution of variational problem (2.1) - (2.2) and $u_h^n \in V_h$ for $n = 1, 2, \ldots, M$ be the solution for (2.3) - (2.4). Then there holds the error estimate, see []*

$$\|u_h - u\|_{L^2(Q)} = \begin{cases} O\left(h^2 + \Delta t\right), & \theta = \{0, 1\}, \\ O\left(h^2 + \Delta t^2\right), & \theta = \{0.5\}, \end{cases} \tag{2.5}$$

where h is the mesh size.

2.3 Types of heat source

- Point source The type of heat source resembles a tiny object produce thermal conduction inside the domain. The point source locates at a fixed coordinate (x_0, y_0) and thermal conduction capacity $h(t)$. These component form the right hand side of heat equation

  $$F(x,y,t) = \delta_{x_0 y_0}(x,y) \times h(t)$$

  Where $\delta_{x_0 y_0}$ is the dirac delta function use for locating the heat point source. For instance, the dirac delta function can be like:

  $$\delta_{x_0 y_0} = \frac{n}{\cosh(n(x - x_0))^2} \times \frac{n}{\cosh(n(y - y_0))^2}$$

  “

- Wall source The fixed temperature walls is the most basic way to resemble a heat source. This type of wall is respective with the Diriclet boundary condition. In freefem++ language
  +on(BoundaryLabel,u=g);
  The other type of wall source provide the heat flux through the wall, respective with Neumann boundary condition
  +...

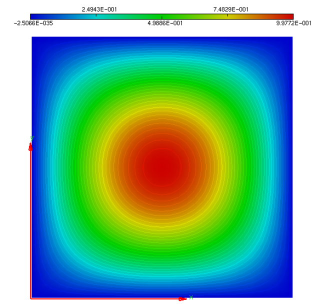


Figure 1: Illustration of chosen dirac delta function.

3 Tests and discussion

3.1 Error evaluation with exact solution

We study a numerical experiment with the exact solution of heat equation and evaluate the error convergence. Consider a square $[0,1] \times [0,1]$. Find $u(x,y,t)$ satisfy

$$\frac{\partial u}{\partial t} - \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = (1+2t)\sin(\pi x)\sin(\pi y) \tag{3.1}$$

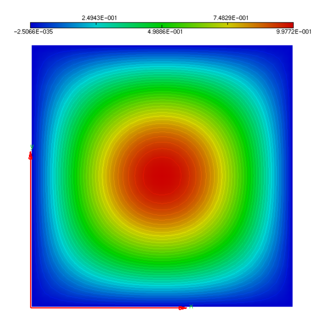
with the initial and boundary conditions

$$u(x,y,0) = 0 \quad \text{and} \quad u|_\Gamma = 0.$$

The exact solution is

$$u = \sin(\pi x)\sin(\pi y)t,$$

The approximate solution at final time is illustrated below. Different cases of mesh size and time step

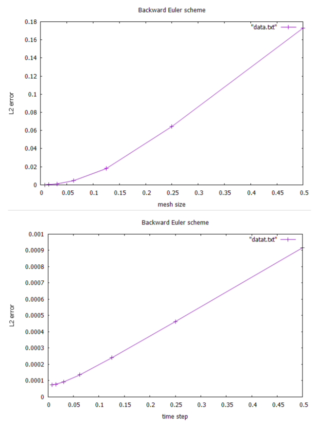


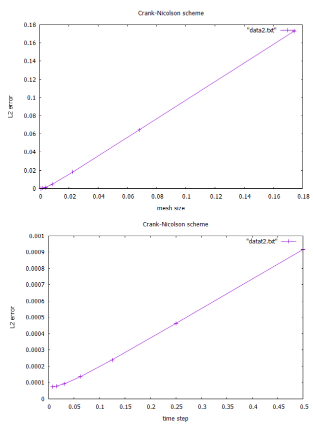
length were studied to show the dependent of error on the mesh smoothness. We also use two different schemes

- Backward Euler scheme
- Crank-Nicolson scheme

3.2 A problem of thermal engineering

We apply the numerical simulations of heat transfer into designing heat sink. Assuming a hot CPU inside





a rectangular room fill with air. Let $u = u_{hot}$ inside CPU region and $u = u_{air}$ on air region, respectively Ω_c and Ω_a, on the initial time. Our goal is to design a heat sink stick on the CPU to lower its temperature. The heat sink region, let say Ω_s, has thermal conductivity coefficient κ_s. Similarly let κ_a and κ_c are respectively the thermal conductivity coefficients inside air and CPU region. Technically, κ_a is small compare to κ_c and κ_s due to nature conduction of air. Furthermore, to provide cooling ability, $\kappa_s > \kappa_c$. Our heat transfer simulations of different heat sink shapes are shown below. The visualization using medit software.

It is obvious that the fin shape heat sink provide

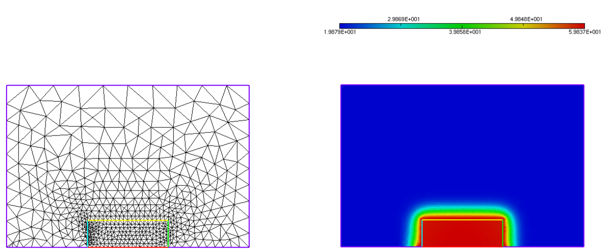


Figure 2: Thermal conduction with no heat sink. $u_{min} = 58.4$, $u_{max} = 59.8$.

better thermal radiator ability compare to a normal rectangular shape, plus requires less material.

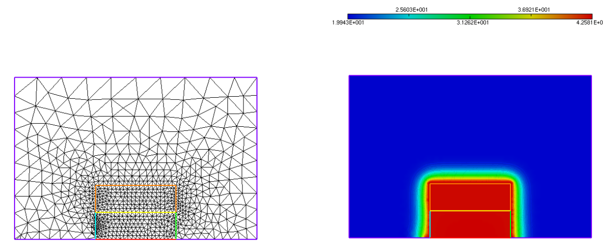


Figure 3: Thermal conduction with rectangular shape heat sink. $u_{min} = 42.3$, $u_{max} = 42.6$.

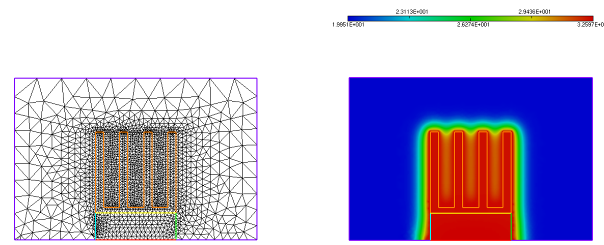


Figure 4: Thermal conduction with fin shape heat sink. $u_{min} = 32.4$, $u_{max} = 32.6$.

3.3 Numerical experiment of optimal control problem

3.3.1 Optimal control problem

In engineering, sometimes we want to know how much heat source provided to receive heat $u(x,t)$ in a physical domain Ω in a time period $[0,T]$ equals or approximates with $\hat{u}(x,t)$, $(x,t) \in Q$. We suppose that heat source following the form $F(x,t) = f(x,t) + q(x,t)$. This leads to optimize the functional

$$J(q) = \frac{1}{2} \|u - \hat{u}\|^2_{L^2(Q)} + \frac{\gamma}{2} \|q\|^2_{L^2(Q)}, \tag{3.2}$$

where q being the control variable and $\gamma > 0$ being a regularization parameter.

To solve this problem, we use FreeFem++ software which provides an efficient tool called IPOPT. It is designed to perform optimal control problems, for more details see at []. To use this optimizer, we need to include the *ff-Ipopt* dynamic library. The parameters including the objective function $J(f)$ and its gradient $\nabla J(f)$ following, see []

$$\nabla J(q) = z(x,t) + \gamma q(x,t), \tag{3.3}$$

where $z(x,t)$ is the solution of the adjoint problem

$$\begin{cases} -\dfrac{\partial z(x,t)}{\partial t} + \mathcal{L}z(x,t) = u - \hat{u}, & (x,t) \in Q, \\ z(x,t) = 0, & (x,t) \in S \\ z(x,T) = 0, & x \in \Omega. \end{cases} \tag{3.4}$$

3.3.2 Numerical results

In this section, we experiment an example as in []. We use finite element method with $\theta = 0.5$ known for Crank-Nicolson method to solve the direct problem (1.1) and adjoint problem (3.4). Consider $\Omega = (0,1)^2$ and $T = 0.1$ and homogeneous Dirichlet boundary condition. The right hand side f, the desired state $\hat{u}$ and the initial condition u_0 such that

$$\begin{aligned} f(x,t) &= -\pi^4 w_b(x,T), \\ \hat{u}(x,t) &= \frac{b^2-5}{2+b}\pi^2 w_b(x,t) + 2\pi^2 w_b(x,T), \\ u_0(x) &= \frac{-1}{2+b}\pi^2 w_b(x,0), \end{aligned}$$

where $w_b(x,t) = e^{b\pi^2 t}\sin(\pi x_1)\sin(\pi x_2),\ b \in \mathbb{R}$.

We chose the regularization parameter $\gamma = \pi^{-4}$ and the optimal solution triple $(\bar{q}, \bar{u}, \bar{z})$ of the optimal control problem (3.2) is given by

$$\begin{aligned} \bar{q}(x,t) &= -\pi^4 \left[w_b(x,t) - w_b(x,T) \right], \\ \bar{u}(x,t) &= \frac{-1}{2+b}\pi^2 w_b(x,t), \\ \bar{z}(x,t) &= w_b(x,t) - w_b(x,T). \end{aligned}$$

First, we consider the behavior of the error for a sequence of discretization with decreasing size of the time steps and a fixed spatial triangulation with $N = 1089$ nodes. Second, we examine the behavior of the error under refinement of the spatial triangulation for $M = 1024$ time steps. We choose the free parameter b to be $-\sqrt{5}$.

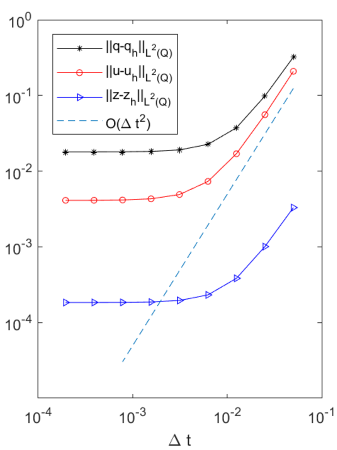


Figure 5: Refinement of the time steps for $N = 1089$ spatial nodes

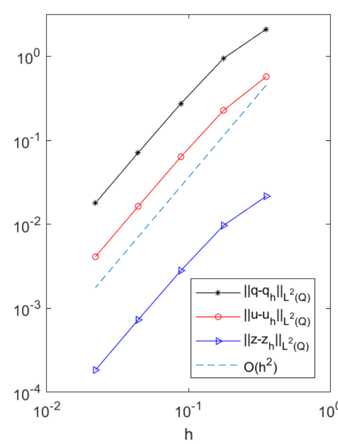


Figure 6: Refinement of the spatial triangulation for $M = 1024$ time steps

4 Conclusion

References