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November 9, 2018

Abstract

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Keywords: Inverse source problems, least squares method, Tikhonov regularization, space-time finite element method, conjugate gradient method.

1 Introduction and Problem setting

There is a lot of physical phenomena in nature such as [...] . Especially, heat transfer is, in mathematics, described by parabolic equation with its right hand side is the source heat. To be more detailed, let $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}^+$ be a bounded domain with a boundary Γ and endow the cylinder $Q = \Omega \times (0, T]$ and lateral surface area $S = \Gamma \times (0, T]$ where $T > 0$.

Consider the heat equation:

$$\frac{\partial u(x,t)}{\partial t} - \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{ji}(x,t)\frac{\partial u(x,t)}{\partial x_j}\right) + a_0(x,t)u(x,t) = F(x,t), \quad (x,t) \in Q, \tag{1.1}$$

with the initial and Dirichlet conditions, respectively

$$u(x,0) = u_0(x), \qquad x \in \Omega, \tag{1.2}$$

$$u(x,t) = u_D(x,t), \quad (x,t) \in S, \tag{1.3}$$

where

$$\begin{aligned}
&a_{ji} \in L^\infty(Q),\, a_{ij} = a_{ji},\ \forall i,j \in \{1,2,...,d\},\\
&\lambda_1 \|\xi\|^2 \le \sum_{i,j=1}^{d} a_{ij}\xi_i\xi_j \le \lambda_2 \|\xi\|^2,\ \forall \xi \in \mathbb{R}^d,\\
&a_0 \in L^\infty(Q),\ 0 \le a_0(x,t) \le \mu_1,\ (x,t) \in Q\\
&u_0 \in L^2(\Omega),\ u_D \in L^2(S),
\end{aligned}$$

with λ_1 and λ_2 are positive constants and $\mu_1 \ge 0$.

The direct problem is to determine u when all data $a_{ji}, i,j = \overline{1,d}$, a_0, u_0, u_D and F in eqs. (1.1)

to (1.3) are given. On the other hand, the inverse problem is to identify a missed data such as the right hand side F when some additional observations on the solution u are available.
We consider the right hand side of the equation (1.1) following the form $F(x,t) = f(x,t)q(x,t)+g(x,t)$, where $q(x,t)$, $g(x,t)$ are given. Find $f(x,t)$ if $\omega(x,t) = u(x,t)$ is given on Q. So, to solve this problem, we need to minimize the least square functional [7, 8]

$$J_\gamma(f) = \frac{1}{2}\left\|u(f) - \omega\right\|^2_{L^2(Q)}.$$

However, this minimization problem is unstable and there might be many minimizers to it. Therefore, we minimize the Tikhonov functional instead

$$J_\gamma(f) = \frac{1}{2}\left\|u(f) - \omega\right\|^2_{L^2(Q)} + \frac{\gamma}{2}\left\|f - f^*\right\|^2_{L^2(Q)},$$

with $\gamma > 0$ being a regularization parameter, f^* is an a prior estimation of f.

2 Variational problem

To introduce the concept of weak form, we use the standard Sobolev spaces $H^1(\Omega)$, $H^1_0(\Omega)$, $H^{1,0}(Q)$ and $H^{1,1}(Q)$ [14, 9, 10]. Further, for a Banach space B, we define

$$L^2(0,T;B) = \left\{u : u(t) \in B \text{ a.e } t \in (0,T) \text{ and } \|u\|_{L^2(0,T;\,B)} < \infty\right\},$$

with the norm

$$\|u\|^2_{L^2(0,T;B)} = \int_0^T \|u(t)\|^2_B \, dt.$$

In the sequel, we shall use the space $W(0,T)$ define as

$$W(0,T) = \left\{u : u \in L^2(0,T;H^1_0(\Omega)), u_t \in L^2\left(0,T;H^{-1}(\Omega)\right)\right\}$$

The solution of eqs. (1.1) to (1.3) is understood in the weak sense as follows: Suppose that $F \in L^2(Q)$ and a week solution in $W(0,T)$ of the problem eqs. (1.1) to (1.3) is a function $u(x,t) \in W(0,T)$ satisfying the identity

$$\int_Q \left[\frac{\partial u}{\partial t} v + \sum_{i,j=1}^{d} a_{ji}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} + a_0 uv\right] dx = \int_Q F v dx, \ \forall v \in L^2\left(0,T;H^1(\Omega)\right). \tag{2.1}$$

and

$$u(x,0) = u_0, \ x \in \Omega. \tag{2.2}$$

Following [...], we can prove that there exists a unique solution $u \in W(0,T)$ of the problem eqs. (1.1) to (1.3). Furthermore, there is a positive c_d independent of a_{ij}, a_0, u_0, u_D and F that satisfies

$$\|u\|_{W(0,T)} \le c_d\left(\|F\|_{L^2(\Omega)} + \|u_0\|_{L^2(\Omega)} + \|u_D\|_{L^2(S)}\right). \tag{2.3}$$

We have the form $F(x,t) = f(x,t)q(x,t) + g(x,t)$ with $f \in L^2(Q)$, $q \in L^\infty(Q)$ and $g \in L^2(Q)$ and hope to recover $f(x,t)$ from the observation $\omega(x,t)$. Since the solution $u(x,t)$ depends on the function $f(x,t)$, so we denote it by $u(x,t,f)$ or $u(f)$. Identify $f(x,t)$ satisfying

$$u(f) = \omega(x,t).$$

We need to minimize the Tikhonov functional

$$J_\gamma(f) = \frac{1}{2}\|u(f) - \omega\|^2_{L^2(Q)} + \frac{\gamma}{2}\|f - f^*\|^2_{L^2(Q)}. \tag{2.4}$$

We will prove that J_γ is Frechet differentiable and drive a formula for its gradient. In doing so, we need the adjoint problem

$$\begin{cases} -\dfrac{\partial p}{\partial t} - \displaystyle\sum_{i,j=1}^{d} \frac{\partial}{\partial x_j}\left(a_{ji}(x,t)\frac{\partial p}{\partial x_i}\right) + a_0 p = u(f) - \omega, & (x,t) \in Q, \\ u(x,t) = 0, & (x,t) \in S \\ p(x,T) = 0, & x \in \Omega. \end{cases} \tag{2.5}$$

By changing the time direction, meaning $\tilde{p}(x,t) = p(x,T-t)$, we will get a Dirichlet problem for parabolic equations.

Theorem 2.1. *The functional J_γ is Frechet differentiable and its gradient ∇J_γ at f has the form*

$$\nabla J_\gamma(f) = q(x,t)p(x,t) + \gamma\left(f(x,t) - f^*(x,t)\right) \tag{2.6}$$

Proof. By taking a small variation $\delta f \in L^2(Q)$ of f and denoting $\delta u(f) = u(f+\delta f) - u(f)$, we have

$$\begin{aligned} J_0(f+\delta f) - J_0(f) &= \frac{1}{2}\|u(f+\delta f) - \omega\|^2_{L^2(Q)} - \frac{1}{2}\|u(f) - \omega\|^2_{L^2(Q)} \\ &= \frac{1}{2}\|\delta u(f) + u(f) - \omega\|^2_{L^2(Q)} - \frac{1}{2}\|u(f) - \omega\|^2_{L^2(Q)} \\ &= \frac{1}{2}\|\delta u(f)\|^2_{L^2(Q)} + \langle \delta u(f), u(f) - \omega\rangle_{L^2(Q)}, \end{aligned}$$

where $\delta u(f)$ is the solution to this problem

$$\begin{cases} \dfrac{\partial \delta u}{\partial t} - \displaystyle\sum_{i,j=1}^{d} \frac{\partial}{\partial x_j}\left(a_{ji}(x,t)\frac{\partial \delta u}{\partial x_i}\right) + a_0 \delta u = q(x,t)\delta f, & (x,t) \in Q, \\ \delta u(x,t) = 0, & (x,t) \in S, \\ \delta u(x,0) = 0, & x \in \Omega. \end{cases} \tag{2.7}$$

Because the priori estimate (2.3) for the direct problem, we have

$$\|\delta u(f)\|^2_{L^2(Q)} = o\left(\|\delta f\|_{L^2(Q)}\right) \text{ when } \|\delta f\|_{L^2(Q)} \to 0. \tag{2.8}$$

What is more, applying the Green formula for (2.5) and (2.7), we get

$$\int_Q \delta u(x,t)\left(u(f) - \omega(t)\right) dxdt = \int_Q p(x,t)q(x,t)\delta f(x,t) dxdt. \tag{2.9}$$

According to (2.8) and (2.9), we obtain

$$\begin{aligned}
J_0(f+\delta f)-J_0(f) &= \int_Q \delta u(x,t)\left(u(f)-\omega(t)\right)dxdt + o\left(\|\delta f\|_{L^2(Q)}\right)\\
&= \int_Q q(x,t)p(x,t)\delta f(x,t)dxdt + o\left(\|\delta f\|_{L^2(Q)}\right)\\
&= \langle qp,\delta f\rangle_{L^2(Q)} + o\left(\|\delta f\|^2_{L(Q)}\right).
\end{aligned}$$

Therefore, we will obtain

$$J_\gamma(f+\delta f)-J_\gamma(f) = \langle qp,\delta f\rangle_{L^2(Q)} + \gamma\langle f-f^*,\delta f\rangle_{L^2(Q)} + o\left(\|\delta f\|^2_{L(Q)}\right).$$

Hence the functional J_γ is Frechet differentiable and its gradient ∇J_γ at f has the form (2.6). The theorem is proved. □

3 Conjugate gradient method

To find f satisfied (2.4), we use the conjugate gradient method (CG). Its iteration follows, we assume that at the kth iteration, we have f^k and then the next iteration will be

$$f^{k+1} = f^k + \alpha_k d^k,$$

with

$$d^k = \begin{cases} -\nabla J_\gamma(f^k), & k=0,\\ -\nabla J_\gamma(f^k)+\beta_k d^{k-1}, & k>0,\end{cases}$$

$$\beta_k = \frac{\left\|\nabla J_\gamma(f^k)\right\|^2_{L^2(Q)}}{\|\nabla J_\gamma(f^{k-1})\|^2_{L^2(Q)}},$$

and

$$\alpha_k = \arg\min_{\alpha\geq 0} J_\gamma(f^k+\alpha d^k).$$

To identify α_k, we consider two problems

Problem 3.1. Denote the solution of this problem is $u[f]$

$$\begin{cases} \dfrac{\partial u}{\partial t} - \displaystyle\sum_{i,j=1}^{d}\frac{\partial}{\partial x_j}\left(a_{ji}(x,t)\frac{\partial u}{\partial x_i}\right) + a_0(x,t)u = f(x,t)q(x,t), & (x,t)\in Q,\\ u(x,t)=0, & (x,t)\in S,\\ u(x,0)=0, & x\in\Omega.\end{cases}$$

Problem 3.2. Denote the solution of this problem is $u(u_0,u_D)$

$$\begin{cases} \dfrac{\partial u}{\partial t} - \displaystyle\sum_{i,j=1}^{d}\frac{\partial}{\partial x_j}\left(a_{ji}(x,t)\frac{\partial u}{\partial x_i}\right) + a_0(x,t)u = g(x,t), & (x,t)\in Q,\\ u(x,t)=u_D(x,t), & (x,t)\in S,\\ u(x,0)=u_0(x), & x\in\Omega.\end{cases}$$

If we do so, the observation operators have the form $u(f) = u[f] + u(u_0, u_D) = Af + u(u_0, u_D)$, with A being bounded linear operator from $L^2(Q)$ to $L^2(Q)$.

We have

$$\begin{aligned} J_\gamma(f^k + \alpha d^k) &= \frac{1}{2}\left\|u(f^k + \alpha d^k) - \omega\right\|^2_{L^2(Q)} + \frac{\gamma}{2}\left\|f^k + \alpha d^k - f^*\right\|^2_{L^2(Q)} \\ &= \frac{1}{2}\left\|\alpha A d^k + A f^k + u(u_0, u_D) - \omega\right\|^2_{L^2(Q)} + \frac{\gamma}{2}\left\|f^k + \alpha d^k - f^*\right\|^2_{L^2(Q)} \\ &= \frac{1}{2}\left\|\alpha A d^k + u(f^k) - \omega\right\|^2_{L^2(Q)} + \frac{\gamma}{2}\left\|f^k + \alpha d^k - f^*\right\|^2_{L^2(Q)}. \end{aligned}$$

Differentiating $J_\gamma(f^k + \alpha d^k)$ with respect to α, we get

$$\begin{aligned} \frac{\partial J_\gamma(f^k + \alpha d^k)}{\partial \alpha} &= \alpha\left\|Ad^k\right\|^2_{L^2(Q)} + \left\langle Ad^k, u(f^k) - \omega\right\rangle_{L^2(Q)} \\ &\quad + \gamma\alpha\left\|d^k\right\|^2_{L^2(Q)} + \gamma\left\langle d^k, f^k - f^*\right\rangle_{L^2(Q)}. \end{aligned}$$

Putting $\dfrac{\partial J_\gamma(f^k + \alpha d^k)}{\partial \alpha} = 0$, we obtain

$$\begin{aligned} \alpha_k &= -\frac{\left\langle Ad^k, u(f^k) - \omega\right\rangle_{L^2(Q)} + \gamma\left\langle d^k, f^k - f^*\right\rangle_{L^2(Q)}}{\left\|Ad^k\right\|^2_{L^2(Q)} + \gamma\left\|d^k\right\|^2_{L^2(Q)}} \\ &= -\frac{\left\langle d^k, A^*\left(u(f^k) - \omega\right)\right\rangle_{L^2(Q)} + \gamma\left\langle d^k, f^k - f^*\right\rangle_{L^2(Q)}}{\left\|Ad^k\right\|^2_{L^2(Q)} + \gamma\left\|d^k\right\|^2_{L^2(Q)}} \\ &= -\frac{\left\langle d^k, A^*\left(u(f^k) - \omega\right) + \gamma(f^k - f^*)\right\rangle_{L^2(Q)}}{\left\|Ad^k\right\|^2_{L^2(Q)} + \gamma\left\|d^k\right\|^2_{L^2(Q)}} \\ &= -\frac{\left\langle d^k, \nabla J_\gamma(f^k)\right\rangle_{L^2(Q)}}{\left\|Ad^k\right\|^2_{L^2(Q)} + \gamma\left\|d^k\right\|^2_{L^2(Q)}}. \end{aligned}$$

Because of $d^k = r^k + \beta_k d^{k-1}$, $r^k = -\nabla J_\gamma(f^k)$ and $\left\langle r^k, d^{k-1}\right\rangle_{L^2(Q)} = 0$, we get

$$\alpha_k = \frac{\left\|r^k\right\|^2_{L^2(Q)}}{\left\|Ad^k\right\|^2_{L^2(Q)} + \gamma\left\|d^k\right\|^2_{L^2(Q)}}.$$

Thus, the CG algorithm is set up by following loop:

CG algorithm

1. Set $k = 0$, initiate f^0.

2. For $k = 0, 1, 2, ...$ Calculate

$$r^k = -\nabla J_\gamma(f^k).$$

   Update

$$d^k = \begin{cases} r^k, & k = 0, \\ r^k + \beta_k d^{k-1}, & k > 0, \end{cases}$$

$$\beta_k = \frac{\left\|r^k\right\|^2_{L^2(Q)}}{\|r^{k-1}\|^2_{L^2(Q)}}.$$

3. Calculate

$$\alpha_k = \frac{\left\|r^k\right\|^2_{L^2(Q)}}{\left\|Ad^k\right\|^2_{L^2(Q)} + \gamma \left\|d^k\right\|^2_{L^2(Q)}}.$$

Update

$$f^{k+1} = f^k + \alpha_k d^k.$$

4 Finite element discretization

We rewrite the Tikhonov functional

$$\begin{aligned} J_\gamma(f) &= \frac{1}{2} \left\|u[f] + u(u_0, u_D) - \omega\right\|^2_{L^2(Q)} + \frac{\gamma}{2} \left\|f - f^*\right\|^2_{L^2(Q)} \\ &= \frac{1}{2} \left\|Af + u(u_0, u_D) - \omega\right\|^2_{L^2(Q)} + \frac{\gamma}{2} \left\|f - f^*\right\|^2_{L^2(Q)} \\ &= \frac{1}{2} \left\|Af - \hat{\omega}\right\|^2_{L^2(Q)} + \frac{\gamma}{2} \left\|f - f^*\right\|^2_{L^2(Q)}, \end{aligned}$$

with $\hat{\omega} = \omega - u(u_0, u_D)$.

The solution f^γ of the minimization problem (2.4) is characterized by the first-order optimality condition

$$\nabla J_\gamma(f^\gamma) = A^*(Af^\gamma - \hat{\omega}) + \gamma(f^\gamma - f^*) = 0, \tag{4.1}$$

with $A^* : L^2(Q) \to L^2(Q)$ is the adjoint operator of A defined by $A^*\left(u(f) - \omega\right) = p$ where p is the solution of the adjoint problem (2.5).

We will approximate (4.1) by finite element method. In fact, we will approximate A and A^* as follows.

4.1 Approximation of A, A^*

We present a fully discrete finite element approximation for the variational problem above. We discrete Ω into finite elements with mesh T_h and define the piecewise linear finite element space $V_h \subset H^1(\Omega)$ by

$$V_h = \left\{v_h : v_h \in C(\overline{\Omega}), v_h|_K \in P_1(K), \forall K \in T_k\right\},$$

where $P_1(K)$ is the space of linear polynomials on the element K. For fully discretization, we introduce a uniform partition of the integral $[0, T]$:

$$0 = t_0 < t_1 < \cdots < T_M = T, \text{ where } t_n = n\Delta t, n = 0, 1, , ..., M \text{ with the time step size } \Delta t = T/M.$$

Let

$$a^n(w,v) = \int_\Omega \sum_{i,j=1}^{d} a_{ji}^n(x) \frac{\partial w}{\partial x_i}\frac{\partial v}{\partial x_j}dx + \int_\Omega a_0^n(x) wv dx,$$

for $w, v \in H^1(\Omega)$ and a function $\varphi(x,t)$, define $\varphi^n(x) = \varphi(x,t_n)$. Then $a^n(.,.) : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ is the bounded bilinear form and $H^1(\Omega)$-elliptic,

$$a^n(v,v) \geq c_a \|v\|^2_{H^1(\Omega)}, \ \forall v \in H^1(\Omega). \tag{4.2}$$

We now define the fully discrete finite element approximation for the variational problem (2.1) by the Crank-Nicolson method as follows: Find $u_h^n \in V_h$ for $n = 1, 2, ..., M$ such that

$$\langle d_t u_h^n, v\rangle_{L^2(\Omega)} + a^n(u_h^n, v) = \langle F^n, v\rangle_{L^2(\Omega)}. \tag{4.3}$$

and

$$\langle u_h^0, v\rangle_{L^2(\Omega)} = \langle u^0, v\rangle_{L^2(\Omega)} \tag{4.4}$$

where $d_t u_h^n = \dfrac{u_h^n - u_h^{n-1}}{\Delta t}$, $n = 1, 2, ..., M$.
The discrete variational problem (4.3) admits a unique solution $u_h^n \in V_h$. Let $u_h(x,t)$ be the linear interpolation of u_h^n with respect to t. Hence, the discrete version of the optimal control problem (2.4) will be

$$J_{\gamma,h}(f) = \frac{1}{2}\|A_h f - \hat{\omega}_h\|^2_{L^2(Q)} + \frac{\gamma}{2}\|f - f^*\|^2_{L^2(Q)} \to \min.$$

Let f_h^γ be the solution of this problem is characterized by the variational equation

$$\nabla J_{\gamma,h}(f_h^\gamma) = A_h^*(A_h f^\gamma - \hat{\omega}_h) + \gamma(f_h^\gamma - f^*) = 0, \tag{4.5}$$

where A_h^* is the adjoint operator of A_h. But it is hardly to find A_h^* from A_h in practice, so we define a proximate $\hat{A}_h^*$ of A^* instead. Therefore, we suppose that $\hat{A}_h^*\phi = p_h$, where $\phi = u(f) - \omega$ and p_h is the approximate solution of adjoint problem (2.5). Therefore, the equation above will be

$$\nabla J_{\gamma,h}(f_h^\gamma) \simeq \nabla J_{\gamma,h}(\hat{f}_h^\gamma) = \hat{A}_h^*(A_h \hat{f}^\gamma - \hat{\omega}_h) + \gamma(\hat{f}_h^\gamma - f^*) = 0, \tag{4.6}$$

Moreover, the observation will have noise in practice, so instead of ω, we suppose that only get ω^δ satisfying

$$\left\|\omega - \omega^\delta\right\|_{L^2(Q)} \leq \delta.$$

Therefore, instead of getting $\hat{f}_h^\gamma$ that satisfies the equation (4.7), we will get $\hat{f}_h^{\gamma,\delta}$ satisfying

$$\nabla J_{\gamma,h}\left(\hat{f}_h^{\gamma,\delta}\right) = \hat{A}_h^*(A_h \hat{f}_h^{\gamma,\delta} - \hat{\omega}_h^\delta) + \gamma(\hat{f}_h^{\gamma,\delta} - f^*) = 0, \tag{4.7}$$

with $\hat{\omega}_h^\delta = \omega^\delta - u_h(u_0, u_D)$.

4.2 Convergence results

Theorem 4.1. *Let $u(x,t)$ be the solution of variational problem* (2.1) - (2.2) *and $u_h^n \in V_h$ for $n = 1, 2, ..., M$ be the solution for* (4.3). *Then there holds the error estimate*

$$\|u_h - u\|_{L^2(Q)} = O\left(h_x^2 + \Delta t^2\right). \tag{4.8}$$

Proof. For $\phi \in H^1(\Omega)$, we define the elliptic projection $R_h : H^1(\Omega) \to V_h$ as the unique solution of the variational problem

$$a(R_h\phi, v_h) = a(\phi, v_h), \ \forall v_h \in V_h.$$

There holds the error estimate, see[...],

$$\|R_h v - v\|_{L^2(\Omega)} + h_x \|R_h v - v\|_{H^1(\Omega)} \le ch^s \|v\|_{H^s(\Omega)}, \ s = \{1, 2\}. \tag{4.9}$$

The idea here is to split the error form into two terms following

$$u_h - u = (u_h - R_h u) + (R_h u - u) = \varphi - \rho.$$

In general, we consider that $t = t_n$, therefore we have $\varphi^n = u_h^n - R_h u^n$ and

$$\begin{aligned}
\langle d_t\varphi^n, v\rangle + \frac{1}{2}\alpha(\varphi^n + \varphi^{n-1}, v) &= \langle d_t u_h^n, v\rangle + \frac{1}{2}\alpha(u_h^n + u_h^{n-1}, v) \\
&\quad - \langle R_h d_t u^n, v\rangle - \frac{1}{2}\alpha(R_h u^n + R_h u^{n-1}, v) \\
&= \frac{1}{2}\left\langle f^n + f^{n-1}, v\right\rangle - \langle R_h d_t u^n, v\rangle - \frac{1}{2}\alpha(u^n + u^{n-1}, v) \\
&= \left\langle \frac{\frac{\partial u^n}{\partial t} + \frac{\partial u^{n-1}}{\partial t}}{2} - R_h d_t u^n, v \right\rangle \\
&= \left\langle \frac{\frac{\partial u^n}{\partial t} + \frac{\partial u^{n-1}}{\partial t}}{2} - d_t u^n, v \right\rangle + \langle d_t u^n - R_h d_t u^n, v\rangle \\
&= \left\langle \frac{\frac{\partial u^n}{\partial t} + \frac{\partial u^{n-1}}{\partial t}}{2} - d_t u^n, v \right\rangle + \langle d_t \rho^n, v\rangle.
\end{aligned}$$

Suppose that $v = \varphi^n + \varphi^{n-1} \in V_N$, we obtain

$$\begin{aligned}
&\left\langle d_t\varphi^n, \varphi^n + \varphi^{n-1}\right\rangle + \frac{1}{2}\alpha(\varphi^n + \varphi^{n-1}, \varphi^n + \varphi^{n-1}) \\
&\quad = \left\langle d_t\rho^n, \varphi^n + \varphi^{n-1}\right\rangle + \left\langle \frac{\frac{\partial u^n}{\partial t} + \frac{\partial u^{n-1}}{\partial t}}{2} - d_t u^n, \varphi^n + \varphi^{n-1} \right\rangle
\end{aligned} \tag{4.10}$$

In the right hand side of (4.10), we analyze two terms, the first is

$$\begin{aligned}
&\chi_{1,n} = d_t\rho^n = d_t u^n - R_h d_t u^n \\
&\Rightarrow \|\chi_{1,n}\|_{L^2(\Omega)} = \|d_t u^n - R_h d_t u^n\|_{L^2(\Omega)} \le ch_x^2 \|d_t u^n\|_{H^2(\Omega)} = \frac{ch_x^2}{\Delta t}\left\|\int_{t_{n-1}}^{t_n} u_t ds\right\|_{H^2(\Omega)} \\
&\qquad \le \frac{ch_x^2}{\Delta t}\sqrt{\int_{t_{n-1}}^{t_n} 1^2 dt}\sqrt{\int_{t_{n-1}}^{t_n} \|u_t\|_{H^2(\Omega)}^2 ds} = \frac{ch_x^2}{\sqrt{\Delta t}}\|u_t\|_{L^2(t_{n-1}, t_n;\, H^2(\Omega))}.
\end{aligned}$$

And the second is

$$\chi_{2,n} = \frac{\frac{\partial u^n}{\partial t} + \frac{\partial u^{n-1}}{\partial t}}{2} - d_t u^n = \frac{\frac{\partial u^n}{\partial t} + \frac{\partial u^{n-1}}{\partial t}}{2} - \frac{u^n - u^{n-1}}{\Delta t}$$
$$= \frac{\frac{\partial u}{\partial t}(t_n) + \frac{\partial u}{\partial t}(t_n - \Delta t)}{2} - \frac{u(t_n) - u(t_n - \Delta t)}{\Delta t}.$$

We have Taylor expansion of a function

$$f(x+h) = f(x) + \sum_{n=1}^{N} \frac{f^{(n)}(x)}{n!} h^n + \int_x^{x+h} \frac{f^{(N+1)}(s)}{N!}(x+h-s)^N ds.$$

Due to Taylor expansion, we have

$$\frac{\partial u}{\partial t}(.,t_n - \Delta t) = \frac{\partial u^{n-1}}{\partial t} = \frac{\partial u^n}{\partial t} - \Delta t u_{tt}^n + \int_{t_n}^{t_n - \Delta t} u_{ttt}(.,s)(t_n - \Delta t - s) ds,$$
$$u(.,t_n - \Delta t) = u^{n-1} = u^n - \Delta t u_t^n + \frac{\Delta t^2}{2} u_{tt}^n + \int_{t_n}^{t_n - \Delta t} \frac{u_{ttt}(.,s)}{2}(t_k - \Delta t - s)^2 ds.$$

$$\Rightarrow \chi_{2,n} = \frac{1}{2}\left(2\frac{\partial u^n}{\partial t} - \Delta t u_{tt}^n + \int_{t_n}^{t_n - \Delta t} u_{ttt}(.,s)(t_k - \Delta t - s) ds\right)$$
$$- \frac{1}{\Delta t}\left(\Delta t u_t^n - \frac{\Delta t^2}{2} u_{tt}^n - \int_{t_n}^{t_n - \Delta t} \frac{u_{ttt}(.,s)}{2}(t_n - \Delta t - s)^2 ds\right)$$
$$= \frac{1}{2}\int_{t_{n-1}}^{t_n} (s - t_{n-1}) u_{ttt}(.,s) ds - \frac{1}{2\Delta t}\int_{t_{n-1}}^{t_n} (s - t_{n-1})^2 u_{ttt}(.,s) ds$$
$$= \frac{1}{2}\int_{t_{n-1}}^{t_n} \left(s - t_{n-1} - \frac{(s - t_{n-1})^2}{\Delta t}\right) u_{ttt}(.,s) ds.$$

$$\Rightarrow \|\chi_{2,n}\|_{L^2(\Omega)} = \frac{1}{2}\left\|\int_{t_{n-1}}^{t_n} \left(s - t_{n-1} - \frac{(s - t_{n-1})^2}{\Delta t}\right) u_{ttt}(.,s) ds\right\|_{L^2(\Omega)}$$
$$\leq \frac{1}{2}\sqrt{\int_{t_{n-1}}^{t_n} \left(s - t_{n-1} - \frac{(s - t_{n-1})^2}{\Delta t}\right)^2 ds}\ \sqrt{\int_{t_{n-1}}^{t_n} \|u_{ttt}\|_{L^2(\Omega)}^2 ds}$$
$$= \frac{1}{2}\sqrt{\frac{\Delta t^3}{30}}\ \|u_{ttt}\|_{L^2(t_{n-1},t_n;\,\Omega)}.$$

Hence, we obtain

$$\left\langle \chi_{1,n}, \varphi^n + \varphi^{n-1} \right\rangle + \left\langle \chi_{2,n}, \varphi^n + \varphi^{n-1} \right\rangle$$
$$\leq \left[\frac{ch_x^2}{\sqrt{\Delta t}} \|u_t\|_{L^2(t_{n-1},t_n;\,H^2(\Omega))} + \sqrt{\frac{\Delta t^3}{30}}\ \|u_{ttt}\|_{L^2(t_{n-1},t_n;\,\Omega)}\right] \left\|\varphi^n + \varphi^{n-1}\right\|_{L^2(\Omega)}. \quad (4.11)$$

On the other hand, due to (4.2), we estimate the left hand side of (4.10) such that

$$\left\langle d_t\varphi^n, \varphi^n + \varphi^{n-1} \right\rangle + \frac{1}{2}\alpha(\varphi^n + \varphi^{n-1}, \varphi^n + \varphi^{n-1})$$
$$\geq \frac{1}{\Delta t}\left[\|\varphi^n\|_{L^2(\Omega)}^2 - \left\|\varphi^{n-1}\right\|_{L^2(\Omega)}^2\right] + c_a \left\|\varphi^n + \varphi^{n-1}\right\|_{L^2(\Omega)}^2. \quad (4.12)$$

According to (4.10) - (4.11) - (4.12) for $n = 1, 2, \ldots, M$

$$\begin{aligned}
&\left\|\varphi^M\right\|^2_{L^2(\Omega)} - \left\|\varphi^0\right\|^2_{L^2(\Omega)} + 2\Delta t \sum_{n=1}^{M} c_a \left\|\varphi^n + \varphi^{n-1}\right\|^2_{L^2(\Omega)} \\
&\qquad \leq \sum_{n=1}^{M} \left[c h_x^2 \left\|u_t\right\|_{L^2(t_{n-1},t_n;\, H^2(\Omega))} + \frac{\Delta t^2}{2\sqrt{30}} \left\|u_{ttt}\right\|_{L^2(t_{n-1},t_n;\, \Omega)} \right] \sqrt{\Delta t} \left\|\varphi^n + \varphi^{n-1}\right\|_{L^2(\Omega)} \\
&\qquad \leq C\delta h_x^4 \sum_{n=1}^{M} \left\|u_t\right\|^2_{L^2(t_{n-1},t_n;\, H^2(\Omega))} + \delta \Delta t \sum_{n=1}^{M} \left\|\varphi^n + \varphi^{n-1}\right\|^2_{L^2(\Omega)} \\
&\qquad + C\delta \Delta t^4 \sum_{n=1}^{M} \left\|u_{ttt}\right\|^2_{L^2(t_{n-1},t_n;\, \Omega)} + \delta \Delta t \sum_{n=1}^{M} \left\|\varphi^n + \varphi^{n-1}\right\|^2_{L^2(\Omega)}.
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \sum_{n=1}^{M} \Delta t \left\|\varphi^n + \varphi^{n-1}\right\|^2 &\leq C h_x^4 \left\|u_t\right\|^2_{L^2(0,T;\, H^2(\Omega))} + C\Delta t^4 \left\|u_{ttt}\right\|^2_{L^2(0,T;\, \Omega)} + \left\|\varphi^0\right\|^2 \\
&\leq C h_x^4 \left\|u_t\right\|^2_{L^2(0,T;\, H^2(\Omega))} + C\Delta t^4 \left\|u_{ttt}\right\|^2_{L^2(0,T;\, \Omega)} + C h_x^4 \left\|u^0\right\|^2_{H^2(\Omega)}.
\end{aligned}$$

For $x \in \Omega$, $t \in [t_{n-1}, t_n]$, we have

$$\begin{aligned}
u_h(x,t) &= \frac{t - t_{n-1}}{\Delta t} u_h^{n-1} + \frac{t_n - t}{\Delta t} u_h^n, \\
R_h u(x,t) &= \frac{t - t_{n-1}}{\Delta t} R_h u^{n-1} + \frac{t_n - t}{\Delta t} R_h u^n.
\end{aligned}$$

$$\begin{aligned}
&\Rightarrow \int_{t_{n-1}}^{t_n} \left\|u_h - R_h u\right\|^2_{L^2(\Omega)} dt \\
&\qquad = \int_{t_{n-1}}^{t_n} \left\| \frac{t - t_{n-1}}{\Delta t} \left(u_h^{n-1} - R u^{n-1}\right) + \frac{t_n - t}{\Delta t} \left(u_h^n - R u^n\right) \right\|^2_{L^2(\Omega)} dt \\
&\qquad = \int_{t_{n-1}}^{t_n} \left\| \frac{t - t_{n-1}}{\Delta t} \varphi^{n-1} + \frac{t_n - t}{\Delta t} \varphi^n \right\|^2_{L^2(\Omega)} dt, \quad \varphi^n = u_h^n - R_h u^n \\
&\qquad = \int_{t_{n-1}}^{t_n} \left\| \frac{1}{2} \left(\varphi^n + \varphi^{n-1}\right) + \frac{t_n + t_{n-1} - 2t}{2\Delta t} \left(\varphi^n - \varphi^{n-1}\right) \right\|^2_{L^2(\Omega)} dt \\
&\qquad \leq \frac{1}{2} \int_{t_{n-1}}^{t_n} \left\|\varphi^n + \varphi^{n-1}\right\|^2_{L^2(\Omega)} dt + \int_{t_{n-1}}^{t_n} \left\| \frac{t_n + t_{n-1} - 2t}{2\Delta t} \left(\varphi^n - \varphi^{n-1}\right) \right\|^2_{L^2(\Omega)} dt \\
&\qquad \leq \frac{1}{2} \Delta t \left\|\varphi^n + \varphi^{n-1}\right\|^2_{L^2(\Omega)} + \sqrt{\int_{t_{n-1}}^{t_n} \left(\frac{t_n + t_{n-1} - 2t}{2} \right)^2 dt} \left\|d_t u_h^n - R_h d_t u_h^n\right\|^2 \\
&\qquad \leq \frac{1}{2} \Delta t \left\|\varphi^n + \varphi^{n-1}\right\|^2_{L^2(\Omega)} + C h_x^4 \sqrt{\Delta t} \left\|u_{h,t}\right\|^2_{L^2(t_{n-1},t_n;\, H^2(\Omega))}.
\end{aligned}$$

Hence,

$$\int_0^T \left\|u_h - R_h u\right\|^2_{L^2(\Omega)} dt \leq \frac{1}{2} \sum_{n=1}^{M} \Delta t \left\|\varphi^n + \varphi^{n-1}\right\|^2 + C h_x^4 \sqrt{\Delta t} \left\|u_{h,t}\right\|^2_{L^2(0,T;\, H^2(\Omega))} = O(h_x^4 + \Delta t^4).$$

$$\Rightarrow \left\|u_h - R_h u\right\|_{L^2(Q)} = O(h_x^2 + \Delta t^2).$$

Due to (4.9), we obtain

$$\left\|R_h u - u\right\|_{L^2(Q)} = O(h_x^2 + \Delta t^2).$$

Therefore, the theory is proved. $\square$

What is more,

$$\left\| \left(A^* - \hat{A}_h^* \right) \phi \right\|_{L^2(Q)}^2 = \int_Q (p - p_h)^2 dx = \|p - p_h\|_{L^2(Q)}^2$$

$$\Rightarrow \|(A^* - A_h^*)\,\phi\|_{L^2(Q)} \le c(h_x^2 + \Delta t^2). \tag{4.13}$$

Let $u_h[f]$ and $u_h(u_0, u_D)$ are the approximate solutions of **Problems** 3.1 and **Problems** 3.2 by using finite element method. We define A_h of A is $A_h f = \ell u_h[f]$ and $\hat{\omega}_h = \omega - \ell u_h(u_0, u_D)$. We have

$$\|(A - A_h)\,f\|_{L^2(Q)}^2 = \|u[f] - u_h[f]\|_{L^2(Q)}^2$$

$$\Rightarrow \|(A - A_h)\,f\|_{L^2(Q)} \le c(h_x^2 + \Delta t^2) \tag{4.14}$$

and

$$\|(\hat{\omega} - \hat{\omega}_h)\|_{L^2(Q)}^2 = \|u(u_0, u_D) - u_h(u_0, u_D)\|_{L^2(Q)}^2$$

$$\Rightarrow \|(\hat{\omega} - \hat{\omega}_h)\|_{L^2(Q)} \le c(h_x^2 + \Delta t^2) \tag{4.15}$$

Theorem 4.2. *Let f^γ and $\hat{f}_h^\gamma$ are the solution of variational problems* (4.1) *and* (4.6), *respectively. Then there hold a error estimate*

$$\left\| f^\gamma - \hat{f}_h^\gamma \right\|_{L^2(Q)} \le c(h_x^2 + \Delta t^2). \tag{4.16}$$

Proof. From equations (4.1) and (4.6), we will have

$$\begin{aligned}
\gamma \left(f^\gamma - \hat{f}_h^\gamma \right) &= \hat{A}_{i,h}^* \left(A_h \hat{f}_h^\gamma - \hat{\omega}_h \right) - A_i^* \left(A f^\gamma - \hat{\omega} \right) \\
&= \left(\hat{A}_{i,h}^* - A_i^* \right) \left(A_h \hat{f}_h^\gamma - \hat{\omega}_h \right) + A_i^* A_h \left(\hat{f}_h^\gamma - f^\gamma \right) \\
&\quad + A^* \left(A_h - A \right) f^\gamma + A_i^* \left(\hat{\omega} - \hat{\omega}_h \right)
\end{aligned}$$

According to (4.13), (4.14) and (4.15), we have

$$\begin{aligned}
&\left\| \left(\hat{A}_{i,h}^* - A_i^* \right) \left(A_h \hat{f}_h^\gamma - \hat{\omega}_h \right) \right\|_{L^2(Q)} \le c(h_x^2 + \Delta t^2), \\
&\|A_i^* \left(A_h - A \right) f^\gamma\|_{L^2(Q)} \le c(h_x^2 + \Delta t^2), \\
&\|A_i^* \left(\hat{\omega} - \hat{\omega}_h \right)\|_{L^2(Q)} \le c(h_x^2 + \Delta t^2).
\end{aligned}$$

We take apart this

$$A_i^* A_h \left(\hat{f}_h^\gamma - f^\gamma \right) = A_i^* \left(A_h - A \right) \left(\hat{f}_h^\gamma - f^\gamma \right) + A_i^* A \left(\hat{f}_h^\gamma - f^\gamma \right).$$

Moreover, we have

$$\begin{aligned}
&\left\langle A_i^* \left(A_h - A \right) \left(\hat{f}_h^\gamma - f^\gamma \right), f^\gamma - \hat{f}_h^\gamma \right\rangle_{L^2(Q)} \le ch^2 \left\| f^\gamma - \hat{f}_h^\gamma \right\|_{L^2(Q)}^2, \\
&\left\langle A_i^* A \left(\hat{f}_h^\gamma - f^\gamma \right), f^\gamma - \hat{f}_h^\gamma \right\rangle_{L^2(Q)} = - \left\| A \left(f^\gamma - \hat{f}_h^\gamma \right) \right\|_{L^2(Q)}^2 < 0.
\end{aligned}$$

The theorem is proved. $\square$

Remark 4.1. *Let f^γ and $\hat{f}_h^\gamma$ are the solution of variational problems* (4.1) *and* (4.7)*, respectively. Then there hold a error estimate*

$$\left\| f^\gamma - \hat{f}_h^{\gamma,\delta} \right\|_{L^2(Q)} \leq c(h_x^2 + \Delta t^2 + \delta). \tag{4.17}$$

5 Numerical results

6 Conclusion

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