Numerical simulation of heat transfer problem by Freefem++ software

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Abstract

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1. Introduction

Let $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}^+$ be a bounded domain with a boundary Γ and endow the cylinder $Q = \Omega \times (0,\, T]$ and lateral surface area $S = \Gamma \times (0,\, T]$ where $T > 0$.
Consider the heat equation:

$$\frac{\partial u(x,t)}{\partial t} + \mathcal{L}u(x,t) = F(x,t), \quad (x,t) \in Q, \tag{1.1}$$

with the Dirichlet boundary and initial conditions, respectively

$$u(x,t) = u_D(x,t), \quad (x,t) \in S, \tag{1.2}$$
$$u(x,0) = u_0(x), \qquad x \in \Omega, \tag{1.3}$$

where

$$\mathcal{L}u(x,t) = -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{ji}(x,t)\frac{\partial u(x,t)}{\partial x_j}\right) + a_0(x,t)u(x,t)$$

$$a_{ji} \in L^\infty(Q),\ a_{ij} = a_{ji},\ \forall i,j \in \{1,2,...,d\},$$

$$\lambda_1 \|\xi\|^2 \leqslant \sum_{i,j=1}^{d} a_{ij}\xi_i\xi_j \leqslant \lambda_2 \|\xi\|^2,\ \forall \xi \in \mathbb{R}^d,$$

$$a_0 \in L^\infty(Q),\ 0 \leqslant a_0(x,t) \leqslant \mu_1,\ (x,t) \in Q$$

$$u_0 \in L^2(\Omega),\ u_D \in L^2(S),$$

with λ_1 and λ_2 are positive constants and $\mu_1 \geqslant 0$.
The problem is that to determine u when all data a_{ji}, a_0, u_0, u_D and F in (1.1)-(1.2)-(1.3) are given called ***direct problem***. But in practice, we miss one of the data above such as the right hand side F of (1.1) known for heat source. The problem identifying F when some additional observations on the solution u available called ***inverse problem***. We suppose that the heat source following the form $F(x,t) = f(x,t)q(x,t) + g(x,t)$, where $q(x,t)$, $g(x,t)$ are given. Find $f(x,t)$ if $\omega(x,t) = u(x,t)$ is given on Q.
Suppose that $F, f, g \in L^2(Q)$ and $q \in L^\infty(Q)$ and hope to recover $f(x,t)$ from the observation $\omega(x,t)$. Since the solution $u(x,t)$ depends on the function $f(x,t)$, so we denote it by $u(x,t,f)$ or $u(f)$. Identify $f(x,t)$ satisfying

$$u(f) = \omega(x,t).$$

We need to minimize the least square functional []

$$J_0(f) = \frac{1}{2}\|u(f) - \omega\|^2_{L^2(Q)}.$$

However, this minimization problem is unstable and there might be many minimizers to it. Therefore, we minimize the Tikhonov functional instead

$$J_\gamma(f) = \frac{1}{2}\|u(f) - \omega\|^2_{L^2(Q)} + \frac{\gamma}{2}\|f - f^*\|^2_{L^2(Q)}, \tag{1.4}$$

where $\gamma > 0$ being a regularization parameter, f^* is an a prior estimation of f.
Additionally, we use the space $W(0,T)$ define as

$$W(0,T) = \left\{u : u \in L^2(0,T;H^1(\Omega)), u_t \in L^2\left(0,T;H^{-1}(\Omega)\right)\right\}.$$

A weak solution in $W(0,T)$ of the equations (1.1)-(1.2) is a function $u(x,t) \in W(0,T)$ satisfying the identity

$$\int_Q \left[\frac{\partial u}{\partial t}v + \sum_{i,j=1}^{d} a_{ji}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} + a_0 uv\right] dxdt = \int_Q Fv dxdt, \tag{1.5}$$

and

$$u(x,0) = u_0,\ x \in \Omega, \tag{1.6}$$

where $v \in L^2\left(0,T;H^1(\Omega)\right)$. Following [...], we can prove that there exists a unique solution $u \in W(0,T)$ of the problem eqs. (1.1) to (1.3).

2. Direct problem

We discrete Ω into finite elements with mesh $\mathcal{T}_h$ and define the piecewise linear finite element space $V_h \subset H^1(\Omega)$ by

$$V_h = \left\{v_h : v_h \in C(\overline{\Omega}), v_h|_K \in P_1(K), \forall K \in \mathcal{T}_k\right\},$$

where $P_1(K)$ is the space of linear polynomials on the element K. For fully discretization, we introduce a uniform partition of the integral $[0,T]$:

$$0 = t_0 < t_1 < \cdots < t_M = T,$$

where $t_n = n\Delta t$, $n = 0, 1, \dots, M$ with the time step size $\Delta t = T/M$.
Let

$$a^n(w,v) = \int_\Omega \sum_{i,j=1}^{d} a_{ji}^n(x) \frac{\partial w}{\partial x_i}\frac{\partial v}{\partial x_j} dx + \int_\Omega a_0^n(x) wv dx,$$

for $w, v \in H^1(\Omega)$. Define $f^n(x) = f(x, t_n)$ and $a^n(.,.) : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ is the bounded bilinear form and $H^1(\Omega)$-elliptic,

$$a^n(v,v) \geqslant c_a \|v\|^2_{H^1(\Omega)}, \ \forall v \in H^1(\Omega). \tag{2.1}$$

We now present the fully discrete finite element approximation for the variational problem (1.5) by the Crank-Nicolson method as follows: Find $u_h^n \in V_h$ for $n = 1, 2, \dots, M$ such that

$$\langle d_t u_h^n, v\rangle + a^n\left(\frac{u_h^n + u_h^{n-1}}{2}, v\right) = \left\langle \frac{F^n + F^{n-1}}{2}, v\right\rangle, \tag{2.2}$$

and

$$\langle u_h^0, v\rangle_{L^2(\Omega)} = \langle u^0, v\rangle_{L^2(\Omega)} \tag{2.3}$$

where $\langle .,.\rangle = \langle .,.\rangle_{L^2(\Omega)}$ and $d_t u_h^n = \dfrac{u_h^n - u_h^{n-1}}{\Delta t}$, $n = 1, 2, ..., M$.
The discrete variational problem (2.2) admits a unique solution $u_h^n \in V_h$. Let $u_h(x,t)$ be the linear interpolation of u_h^n with respect to t.

Theorem 2.1. *Let $u(x,t)$ be the solution of variational problem (1.5) - (1.6) and $u_h^n \in V_h$ for $n = 1, 2, \dots, M$ be the solution for (2.2). Then there holds the error estimate*

$$\|u_h - u\|_{L^2(Q)} = O\left(h^2 + \Delta t^2\right), \tag{2.4}$$

where h is the mesh size.

Proof. For $\phi \in H^1(\Omega)$, we define the elliptic projection $R_h : H^1(\Omega) \to V_h$ as the unique solution of the variational problem

$$a(R_h\phi, v_h) = a(\phi, v_h), \ \forall v_h \in V_h.$$

Denote $\|.\|_{H^s(\Omega)} = \|.\|_s$ and $\|.\|_{L^2(t_{n-1},t_n; H^s(\Omega))} = \|.\|_{n,s}$.
There holds the error estimate, see[...] with $s = \{1, 2\}$,

$$\|R_h v - v\|_0 + h\|R_h v - v\|_1 \leqslant ch^s \|v\|_s. \tag{2.5}$$

The idea here is split the error form into two terms following

$$u_h - u = \underbrace{(u_h - R_h u)}_{\varphi} - \underbrace{(u - R_h u)}_{\rho}.$$

In general, we consider $t = t_n$, therefore we have

$$\begin{aligned}
&\langle d_t\varphi^n, v\rangle + \frac{1}{2} a(\varphi^n + \varphi^{n-1}, v) = \\
&= \langle d_t u_h^n, v\rangle + \frac{1}{2} a(u_h^n + u_h^{n-1}, v) \\
&\qquad - \frac{1}{2} a(R_h u^n + R_h u^{n-1}, v) - \langle R_h d_t u^n, v\rangle \\
&= \frac{1}{2}\langle F^n + F^{n-1}, v\rangle \\
&\qquad - \frac{1}{2} a(u^n + u^{n-1}, v) - \langle R_h d_t u^n, v\rangle \\
&= \frac{1}{2}\left\langle \frac{\partial u^n}{\partial t} + \frac{\partial u^{n-1}}{\partial t}, v\right\rangle - \langle R_h d_t u^n, v\rangle \\
&= \frac{1}{2}\left\langle \frac{\partial u^n}{\partial t} + \frac{\partial u^{n-1}}{\partial t} - 2d_t u^n, v\right\rangle + \langle d_t\rho^n, v\rangle.
\end{aligned}$$

Suppose that $v = \varphi^n + \varphi^{n-1} \in V_N$, we obtain

$$\begin{aligned}
&\left\langle d_t\varphi^n, \varphi^n + \varphi^{n-1}\right\rangle + \frac{1}{2} a(\varphi^n + \varphi^{n-1}, \varphi^n + \varphi^{n-1}) \\
&\quad = \left\langle d_t\rho^n, \varphi^n + \varphi^{n-1}\right\rangle \\
&\quad + \frac{1}{2}\left\langle \frac{\partial u^n}{\partial t} + \frac{\partial u^{n-1}}{\partial t} - 2d_t u^n, \varphi^n + \varphi^{n-1}\right\rangle
\end{aligned} \tag{2.6}$$

In the right hand side of (2.6), we analyze two terms, the first is

$$\chi_{1,n} = d_t\rho^n = d_t u^n - R_h d_t u^n$$

$$\begin{aligned}
\Rightarrow \|\chi_{1,n}\|_0 &= \|d_t u^n - R_h d_t u^n\|_0 \\
&\leqslant ch^2 \|d_t u^n\|_2 = \frac{ch^2}{\Delta t}\left\| \int_{t_{n-1}}^{t_n} u_t ds\right\|_2 \\
&\leqslant \frac{ch^2}{\Delta t}\sqrt{\int_{t_{n-1}}^{t_n} 1^2 dt}\sqrt{\int_{t_{n-1}}^{t_n} \|u_t\|_2^2 ds} \\
&= \frac{ch^2}{\sqrt{\Delta t}}\|u_t\|_{n,2}.
\end{aligned}$$

And the second is

$$\begin{aligned}
\chi_{2,n} &= \frac{\partial u^n}{\partial t} + \frac{\partial u^{n-1}}{\partial t} - 2d_t u^n \\
&= u_t(t_n) + u_t(t_{n-1}) - \frac{2}{\Delta t}\left[u(t_n) - u(t_{n-1})\right].
\end{aligned}$$

We have Taylor expansion for a function $f(y)$, $y = x + \delta x$

$$f(y) = f(x) + \sum_{n=1}^{N} \frac{f^{(n)}(x)}{n!}(\delta x)^n + \int_x^y \frac{f^{(N+1)}(s)}{N!}(y-s)^N ds.$$

Due to Taylor expansion, we have

$$u_t(t_{n-1}) = u_t^n - \Delta t u_{tt}^n + \int_{t_n}^{t_{n-1}} u_{ttt}(s)(t_{n-1} - s) ds,$$

$$u(t_{n-1}) = u^n - \Delta t u_t^n + \frac{\Delta t^2}{2} u_{tt}^n + \int_{t_n}^{t_{n-1}} \frac{u_{ttt}(s)}{2}(t_{n-1} - s)^2 ds.$$

$$\begin{aligned}
\Rightarrow \chi_{2,n} &= 2u_t^n - \Delta t u_{tt}^n + \int_{t_n}^{t_{n-1}} u_{ttt}(s)(t_{n-1} - s) ds \\
&\quad - \left(2u_t^n - \Delta t u_{tt}^n - \int_{t_n}^{t_{n-1}} \frac{u_{ttt}(s)}{\Delta t}(t_{n-1} - s)^2 ds\right) \\
&= \int_{t_{n-1}}^{t_n}\left(s - t_{n-1} - \frac{(s - t_{n-1})^2}{\Delta t}\right) u_{ttt}(s) ds.
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \|\chi_{2,n}\|_0 &= \left\|\int_{t_{n-1}}^{t_n}\left(s - t_{n-1} - \frac{(s - t_{n-1})^2}{\Delta t}\right) u_{ttt}(.,s) ds\right\|_0 \\
&\leqslant \sqrt{\int_{t_{n-1}}^{t_n}\left(s - t_{n-1} - \frac{(s - t_{n-1})^2}{\Delta t}\right)^2 ds}\sqrt{\int_{t_{n-1}}^{t_n}\|u_{ttt}\|_0^2 ds} \\
&= \sqrt{\frac{\Delta t^3}{30}}\|u_{ttt}\|_{n,0}.
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
&\left\langle \chi_{1,n}, \varphi^n + \varphi^{n-1}\right\rangle + \frac{1}{2}\left\langle \chi_{2,n}, \varphi^n + \varphi^{n-1}\right\rangle \\
&\quad \leqslant \left[\frac{ch^2}{\sqrt{\Delta t}}\|u_t\|_{n,2} + \frac{1}{2}\sqrt{\frac{\Delta t^3}{30}}\|u_{ttt}\|_{n,0}\right]\left\|\varphi^n + \varphi^{n-1}\right\|_0.
\end{aligned} \tag{2.7}$$

On the other hand, due to (2.1), we estimate the left hand side of (2.6) such that

$$\langle d_t\varphi^n, \varphi^n+\varphi^{n-1}\rangle + \frac{1}{2}a(\varphi^n+\varphi^{n-1}, \varphi^n+\varphi^{n-1}) \geqslant \frac{1}{\Delta t}\left[\|\varphi^n\|_0^2 - \left\|\varphi^{n-1}\right\|_0^2\right] + c_a\left\|\varphi^n+\varphi^{n-1}\right\|_0^2. \quad (2.8)$$

According to (2.6) - (2.7) - (2.8) for $n = 1, 2, \ldots, M$, we have

$$\begin{aligned}
&\left\|\varphi^M\right\|_0^2 - \left\|\varphi^0\right\|_0^2 + 2\Delta t\sum_{n=1}^{M} c_a\left\|\varphi^n+\varphi^{n-1}\right\|_0^2 \\
&\leqslant \sum_{n=1}^{M}\left[ch^2\|u_t\|_{n,2} + \frac{\Delta t^2}{2\sqrt{30}}\|u_{ttt}\|_{n,0}\right]\sqrt{\Delta t}\left\|\varphi^n+\varphi^{n-1}\right\|_0 \\
&\leqslant c\epsilon h^4\sum_{n=1}^{M}\|u_t\|_{n,2}^2 + \epsilon\Delta t\sum_{n=1}^{M}\left\|\varphi^n+\varphi^{n-1}\right\|_0^2 \\
&+ \frac{\epsilon\Delta t^4}{120}\sum_{n=1}^{M}\|u_{ttt}\|_{n,0}^2 + \epsilon\Delta t\sum_{n=1}^{M}\left\|\varphi^n+\varphi^{n-1}\right\|_0^2.
\end{aligned}$$

$$\begin{aligned}
&\Rightarrow \sum_{n=1}^{M}\Delta t\left\|\varphi^n+\varphi^{n-1}\right\|_0^2 \\
&\leqslant Ch^4\|u_t\|_{L^2(0,T;\,H^2(\Omega))}^2 + C\Delta t^4\|u_{ttt}\|_{L^2(0,T;\,\Omega)}^2 + \left\|\varphi^0\right\|_0^2 \\
&\leqslant Ch^4\|u_t\|_{L^2(0,T;\,H^2(\Omega))}^2 + C\Delta t^4\|u_{ttt}\|_{L^2(0,T;\,\Omega)}^2 + Ch^4\left\|u^0\right\|_2^2.
\end{aligned}$$

For $x\in\Omega,\ t\in[t_{n-1}, t_n]$, we have

$$\begin{aligned}
u_h(x,t) &= \frac{t-t_{n-1}}{\Delta t}u_h^{n-1} + \frac{t_n-t}{\Delta t}u_h^n, \\
R_hu(x,t) &= \frac{t-t_{n-1}}{\Delta t}R_hu^{n-1} + \frac{t_n-t}{\Delta t}R_hu^n.
\end{aligned}$$

$$\begin{aligned}
&\Rightarrow \int_{t_{n-1}}^{t_n}\|u_h-R_hu\|_0^2\,dt \\
&= \int_{t_{n-1}}^{t_n}\left\|\frac{t-t_{n-1}}{\Delta t}\varphi^{n-1} + \frac{t_n-t}{\Delta t}\varphi^n\right\|_0^2 dt \\
&= \frac{1}{4}\int_{t_{n-1}}^{t_n}\left\|\varphi^n+\varphi^{n-1} + \frac{t_n+t_{n-1}-2t}{\Delta t}\left(\varphi^n-\varphi^{n-1}\right)\right\|_0^2 dt \\
&\leqslant \frac{1}{4}\int_{t_{n-1}}^{t_n}\left\|\varphi^n+\varphi^{n-1}\right\|_0^2 + \left\|\frac{t_n+t_{n-1}-2t}{\Delta t}\left(\varphi^n-\varphi^{n-1}\right)\right\|_0^2 dt \\
&\leqslant \frac{\Delta t}{4}\left\|\varphi^n+\varphi^{n-1}\right\|_0^2 \\
&\quad + \frac{1}{4}\int_{t_{n-1}}^{t_n}(t_n+t_{n-1}-2t)^2\,dt\left\|d_tu_h^n - R_hd_tu_h^n\right\|_0^2 \\
&\leqslant \frac{\Delta t}{4}\left\|\varphi^n+\varphi^{n-1}\right\|_0^2 + Ch^4\Delta t\|u_{h,t}\|_{n,2}^2.
\end{aligned}$$

Hence,

$$\begin{aligned}
\int_0^T\|u_h-R_hu\|_0^2\,dt &\leqslant \frac{1}{4}\sum_{n=1}^{M}\Delta t\left\|\varphi^n+\varphi^{n-1}\right\|_0^2 \\
&\quad + Ch^4\Delta t\|u_{h,t}\|_{L^2(0,T;\,H^2(\Omega))}^2 \\
&= O(h^4+\Delta t^4).
\end{aligned}$$

$$\Rightarrow \|u_h-R_hu\|_{L^2(Q)} = O(h^2+\Delta t^2).$$

Due to (2.5), we obtain

$$\|R_hu-u\|_{L^2(Q)} = O(h^2+\Delta t^2).$$

Therefore, the theory is proved. □

3. Inverse problem

3.1. conjugate gradient method

We will prove that J_γ is Frechet differentiable and drive a formula for its gradient.

Theorem 3.1. *The functional J_γ is Frechet differentiable and its gradient ∇J_γ at f has the form*

$$\nabla J_\gamma(f) = q(x,t)p(x,t) + \gamma\left(f(x,t) - f^*(x,t)\right), \quad (3.1)$$

where p(x, t) is the solution of the adjoint problem

$$\begin{cases}
-\dfrac{\partial p(x,t)}{\partial t} + \mathcal{L}p(x,t) = u(f) - \omega, & (x,t)\in Q, \\
u(x,t) = 0, & (x,t)\in S \\
p(x,T) = 0, & x\in\Omega.
\end{cases} \quad (3.2)$$

To find f satisfied (1.4), we use the conjugate gradient method (CG). Its iteration follows, we assume that at the kth iteration, we have f^k and then the next iteration will be

$$f^{k+1} = f^k + \alpha_kd^k,$$

with

$$d^k = \begin{cases} -\nabla J_\gamma(f^k), & k=0, \\ -\nabla J_\gamma(f^k) + \beta_kd^{k-1}, & k>0, \end{cases}$$

$$\beta_k = \frac{\left\|\nabla J_\gamma(f^k)\right\|_{L^2(Q)}^2}{\left\|\nabla J_\gamma(f^{k-1})\right\|_{L^2(Q)}^2},$$

and

$$\alpha_k = \arg\min_{\alpha\geqslant 0} J_\gamma(f^k+\alpha d^k).$$

To identify α_k, we consider two problems

Problem 3.1. Denote the solution of this problem is $u[f]$

$$\begin{cases}
\dfrac{\partial u(x,t)}{\partial t} + \mathcal{L}u(x,t) = f(x,t)q(x,t), & (x,t)\in Q, \\
u(x,t) = 0, & (x,t)\in S, \\
u(x,0) = 0, & x\in\Omega.
\end{cases}$$

Problem 3.2. Denote the solution of this problem is $u(u_0, u_D)$

$$\begin{cases}
\dfrac{\partial u(x,t)}{\partial t} + \mathcal{L}u(x,t) = g(x,t), & (x,t)\in Q, \\
u(x,t) = u_D(x,t), & (x,t)\in S, \\
u(x,0) = u_0(x), & x\in\Omega.
\end{cases}$$

If do so, the observation operators have the form $u(f) = u[f] + u(u_0,u_D) = Af + u(u_0,u_D)$, with A being bounded linear operator from $L^2(Q)$ to $L^2(Q)$. We have

$$\begin{aligned}
J_\gamma(f^k+\alpha d^k) &= \frac{1}{2}\left\|u(f^k+\alpha d^k) - \omega\right\|_{L^2(Q)}^2 + \frac{\gamma}{2}\left\|f^k+\alpha d^k - f^*\right\|_{L^2(Q)}^2 \\
&= \frac{1}{2}\left\|\alpha Ad^k + Af^k + u(u_0,u_D) - \omega\right\|_{L^2(Q)}^2 + \frac{\gamma}{2}\left\|f^k+\alpha d^k - f^*\right\|_{L^2(Q)}^2 \\
&= \frac{1}{2}\left\|\alpha Ad^k + u(f^k) - \omega\right\|_{L^2(Q)}^2 + \frac{\gamma}{2}\left\|f^k+\alpha d^k - f^*\right\|_{L^2(Q)}^2.
\end{aligned}$$

Differentiating $J_\gamma(f^k + \alpha d^k)$ with respect to α, we get

$$\frac{\partial J_\gamma(f^k+\alpha d^k)}{\partial \alpha} = \alpha \left\| Ad^k \right\|^2_{L^2(Q)} + \left\langle Ad^k, u(f^k) - \omega \right\rangle_{L^2(Q)} + \gamma\alpha \left\| d^k \right\|^2_{L^2(Q)} + \gamma \left\langle d^k, f^k - f^* \right\rangle_{L^2(Q)}.$$

Putting $\frac{\partial J_\gamma(f^k+\alpha d^k)}{\partial \alpha} = 0$, we obtain

$$\begin{aligned}\alpha_k &= -\frac{\left\langle Ad^k, u(f^k) - \omega \right\rangle_{L^2(Q)} + \gamma \left\langle d^k, f^k - f^* \right\rangle_{L^2(Q)}}{\left\| Ad^k \right\|^2_{L^2(Q)} + \gamma \left\| d^k \right\|^2_{L^2(Q)}} \\ &= -\frac{\left\langle d^k, A^* \left(u(f^k) - \omega \right) \right\rangle_{L^2(Q)} + \gamma \left\langle d^k, f^k - f^* \right\rangle_{L^2(Q)}}{\left\| Ad^k \right\|^2_{L^2(Q)} + \gamma \left\| d^k \right\|^2_{L^2(Q)}} \\ &= -\frac{\left\langle d^k, A^* \left(u(f^k) - \omega \right) + \gamma (f^k - f^*) \right\rangle_{L^2(Q)}}{\left\| Ad^k \right\|^2_{L^2(Q)} + \gamma \left\| d^k \right\|^2_{L^2(Q)}} \\ &= -\frac{\langle d^k, \nabla J_\gamma(f^k) \rangle_{L^2(Q)}}{\left\| Ad^k \right\|^2_{L^2(Q)} + \gamma \left\| d^k \right\|^2_{L^2(Q)}}.\end{aligned}$$

Because of $d^k = r^k + \beta_k d^{k-1}$, $r^k = -\nabla J_\gamma(f^k)$ and $\left\langle r^k, d^{k-1} \right\rangle_{L^2(Q)} = 0$, we get

$$\alpha_k = \frac{\|r^k\|^2_{L^2(Q)}}{\left\| Ad^k \right\|^2_{L^2(Q)} + \gamma \left\| d^k \right\|^2_{L^2(Q)}}.$$

Thus, the CG algorithm is set up by following loop:

CG algorithm

1. Set $k = 0$, initiate f^0.
2. For $k = 0, 1, 2, ...$ Calculate
$$r^k = -\nabla J_\gamma(f^k).$$
Update
$$d^k = \begin{cases} r^k, & k = 0, \\ r^k + \beta_k d^{k-1}, & k > 0, \end{cases}$$
$$\beta_k = \frac{\|r^k\|^2_{L^2(Q)}}{\|r^{k-1}\|^2_{L^2(Q)}}.$$
3. Calculate
$$\alpha_k = \frac{\|r^k\|^2_{L^2(Q)}}{\left\| Ad^k \right\|^2_{L^2(Q)} + \gamma \left\| d^k \right\|^2_{L^2(Q)}}.$$
Update
$$f^{k+1} = f^k + \alpha_k d^k.$$

3.2. Finite element discretization

We rewrite the Tikhonov functional

$$\begin{aligned}J_\gamma(f) &= \frac{1}{2} \|u[f] + u(u_0, u_D) - \omega\|^2_{L^2(Q)} + \frac{\gamma}{2} \|f - f^*\|^2_{L^2(Q)} \\ &= \frac{1}{2} \|Af - \hat{\omega}\|^2_{L^2(Q)} + \frac{\gamma}{2} \|f - f^*\|^2_{L^2(Q)},\end{aligned}$$

with $\hat{\omega} = \omega - u(u_0, u_D)$.

The solution f^γ of the minimization problem (2.4) is characterized by the first-order optimality condition

$$\nabla J_\gamma(f^\gamma) = A^*(Af^\gamma - \hat{\omega}) + \gamma(f^\gamma - f^*) = 0, \tag{3.3}$$

with $A^* : L^2(Q) \to L^2(Q)$ is the adjoint operator of A defined by $A^*(u(f) - \omega) = p$ where p is the solution of the adjoint problem (3.2).

We will approximate (3.3) by finite element method. In fact, we will approximate A and A^* as follows.

3.3. Approximation of A, A^*

We use the fully discrete finite element approximation for the variational problem (**??**) by the Crank-Nicolson method. Hence, the discrete version of the optimal control problem (2.4) will be

$$J_{\gamma,h}(f) = \frac{1}{2} \|A_h f - \hat{\omega}_h\|^2_{L^2(Q)} + \frac{\gamma}{2} \|f - f^*\|^2_{L^2(Q)} \to \min.$$

Let f^γ_h be the solution of this problem is characterized by the variational equation

$$\nabla J_{\gamma,h}(f^\gamma_h) = A^*_h(A_h f^\gamma - \hat{\omega}_h) + \gamma(f^\gamma_h - f^*) = 0, \tag{3.4}$$

where A^*_h is the adjoint operator of A_h. But it is hardly to find A^*_h from A_h in practice, so we define a proximate $\hat{A}^*_h$ of A^* instead. Therefore, we suppose that $\hat{A}^*_h \phi = p_h$, where $\phi = u(f) - \omega$ and p_h is the approximate solution of adjoint problem (3.2). Therefore, the equation above will be

$$\nabla J_{\gamma,h}(f^\gamma_h) \simeq \nabla J_{\gamma,h}(\hat{f}^\gamma_h) = \hat{A}^*_h(A_h \hat{f}^\gamma - \hat{\omega}_h) + \gamma(\hat{f}^\gamma_h - f^*) = 0, \tag{3.5}$$

Moreover, the observation will have noise in practice, so instead of ω, we suppose that only get ω^δ satisfying

$$\left\| \omega - \omega^\delta \right\|_{L^2(Q)} \leqslant \delta.$$

Therefore, instead of getting $\hat{f}^\gamma_h$ that satisfies the equation (3.6), we will get $\hat{f}^{\gamma,\delta}_h$ satisfying

$$\nabla J_{\gamma,h}\left(\hat{f}^{\gamma,\delta}_h\right) = \hat{A}^*_h(A_h \hat{f}^{\gamma,\delta}_h - \hat{\omega}^\delta_h) + \gamma(\hat{f}^{\gamma,\delta}_h - f^*) = 0, \tag{3.6}$$

with $\hat{\omega}^\delta_h = \omega^\delta - u_h(u_0, u_D)$.

Theorem 3.2. *Let f^γ and $\hat{f}^\gamma_h$ are the solution of variational problems (3.3) and (3.5), respectively. Then there hold a error estimate*

$$\left\| f^\gamma - \hat{f}^\gamma_h \right\|_{L^2(Q)} \leqslant c(h^2 + \Delta t^2). \tag{3.7}$$

Remark 3.1. *Let f^γ and $\hat{f}^\gamma_h$ are the solution of variational problems (3.3) and (3.6), respectively. Then there hold a error estimate*

$$\left\| f^\gamma - \hat{f}^{\gamma,\delta}_h \right\|_{L^2(Q)} \leqslant c(h^2 + \Delta t^2 + \delta). \tag{3.8}$$

4. Tests and discussion

4.1. Exact solution

4.2. A problem of thermal engineering

4.3. Numerical experiment of inverse problem

5. Conclusion

References