Numerical simulation of heat transfer problem by Freefem++ software

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Abstract

Heat transfer via conduction in solid objects is an important part in thermal engineering and solid mechanics. Its mathematical expression, the parabolic equation, also received large amount of attention by mathematician whose numerous researches have been studied to solve the equation numerically. **For physicist interest**, software such as Ansys, Solidwork provides sufficient simulation features, but also lack of mathematical clarity for mathematicians. For such purpose, we use Freefem++ software in study of heat transfer which produces both mathematic structures and powerful simulation qualification.

Keywords:

1 Introduction

The parabolic equation arises in the modelings of a number of phenomena such as heat transfer and financial process. In which, the solving of boundary value problem act as major role.

In the aspect of thermal engineering, parabolic partial differential equation or heat equation is used to describe time dependent heat transfer process via conduction in solid objects. Its solution is the thermal distribution or variation in temperature inside the being observed domain. Wide range in such field of engineering require the simulation of heat transfer and numerically solving of heat equation.

Freefem++ provides efficient tools in solving and modeling PDE systems, where parabolic equation is one of the most basics. Although it has build in visualize tool, extend module medit provides powerful visualization features. Our work focus on the application of Freefem++ in various problems of the heat equation.

First we approach the common boundary value problem. As it is the most basic problem of parabolic equation, numerous researches have been done. Although Freefem++ does not have build-in-solver for such case, time discrete schemes could be applied. The detail method and Freefem++ implementation are discussed.

Various numerical experiments have been done to show Freefem++ ability in finite element analysis. We studied the error convergence with exact solution in a specific case. Also, a shape optimization problem arises in thermal engineering was briefly mentioned. Lastly, an optimal control problem of parabolic equation was performed using module IPOPT.

2 Setting of problem

Let $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}^+$ be a bounded domain with a boundary Γ and denote the cylinder $Q = \Omega \times (0, T]$ and the lateral surface area $S = \Gamma \times [0, T]$ where $T > 0$.

Consider the heat equation:

$$\frac{\partial u(x,t)}{\partial t} + \mathcal{L}u(x,t) = F(x,t), \quad (x,t) \in Q, \tag{2.1}$$

with the Dirichlet boundary and initial conditions, respectively

$$u(x,t) = 0, \quad (x,t) \in S, \tag{2.2}$$

$$u(x,0) = u_0(x), \qquad x \in \Omega, \tag{2.3}$$

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where

$$\mathcal{L}u = -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{ji}\frac{\partial u}{\partial x_j}\right) + a_0 u,$$
$$a_{ji} \in L^\infty(Q),\, a_{ij} = a_{ji},\, \forall i,j \in \{1,2,...,d\},$$
$$\lambda_1 \|\xi\|^2 \leq \sum_{i,j=1}^{d} a_{ij}\xi_i\xi_j \leq \lambda_2 \|\xi\|^2,\, \forall \xi \in \mathbb{R}^d,$$
$$a_0 \in L^\infty(Q),\ 0 \leq a_0(x,t) \leq \mu_1,\ (x,t) \in Q,$$
$$u_0 \in L^2(\Omega),$$

with λ_1 and λ_2 are positive constants and $\mu_1 \geq 0$.

The problem is that to determine $u(x,t)$ when all data $a_{ji}(x,t)$, $a_0(x,t)$, $u_0(x)$ and $F(x,t)$ in equations (2.1) - (2.2) - (2.3) are given.

3 Numerical method

3.1 Variational problem

Multiplying (2.1) by an efficient smooth test function v, integrating over Ω and then applying Green's formula, see [9], leads to the problem: Find $u(.,t) \in H_0^1(\Omega)$ such that

$$\left\langle \frac{\partial u}{\partial t}, v\right\rangle + a(u,v) = \langle F, v\rangle,\ \forall v \in H_0^1(\Omega), \tag{3.1}$$

$$u(x,0) = u_0(x),\ x \in \Omega, \tag{3.2}$$

where

$$a(u,v) = \int_\Omega \left[\sum_{i,j=1}^{d} a_{ji}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} + a_0 uv\right] dx,$$

$$\langle F, v\rangle = \int_\Omega F v dx.$$

Following [6, 5, 2, 7], we can prove that there exists a unique solution of the problem (3.1) - (3.2). We approximate this solution by finite element method as follows.

3.2 Finite element method

Now we present a fully discrete finite element approximation for the variational problem (3.1) as follows:

- For *spatial approximation*, let $\mathcal{T}_h$ be a triangulation of Ω and define a piecewise linear finite element space $V_h \subset H_0^1(\Omega)$ by

  $$V_h = \left\{v_h : v_h \in C(\overline{\Omega}), v_h|_K \in P_1(K), \forall K \in \mathcal{T}_h\right\},$$

  where $P_1(K)$ is a continuous piecewise linear polynomial on the element K.
- For *temporal discretization*, discrete $[0,T]$ uniformly into M steps, where $t_n = n\Delta t$, $n = 0,1,\dots,M$ with the time step size $\Delta t = T/M$. Denote a function $u(x,t_n) = u^n(x)$.

Therefore, the problem is to find $u_h^n \in V_h$ for $n = 1,2,\dots,M$ with $\theta \in [0,1]$ such that

$$\begin{aligned}\langle d_t u_h^n, v_h\rangle &+ a\left(\theta u_h^n + (1-\theta)u_h^{n-1}, v_h\right)\\ &= \left\langle \theta F^n + (1-\theta)F^{n-1}, v_h\right\rangle,\ \forall v_h \in V_h,\end{aligned} \tag{3.3}$$

and the initial condition

$$u_h^0 = u_0, \tag{3.4}$$

where $d_t u_h^n = \dfrac{u_h^n - u_h^{n-1}}{\Delta t}$, $n = 1,2,...,M$.

We have different method depending on θ such as backward Euler ($\theta = 1$) and Crank-Nicolson ($\theta = 0.5$). The discrete variational problem (3.3) -(3.4) admits a unique solution $u_h^n \in V_h$. Let $u_h(x,t)$ be the linear interpolation of u_h^n with respect to t. Therefore, for $x \in \Omega$, $t \in [t_{n-1}, t_n]$, we have

$$u_h(x,t) = \frac{t - t_{n-1}}{\Delta t}u_h^{n-1} + \frac{t_n - t}{\Delta t}u_h^n.$$

Theorem 3.1. *Let $u(x,t)$ be the solution of variational problem (3.1) - (3.2) and $u_h^n \in V_h$ for $n = 1,2,\dots,M$ be the solution for (3.3) - (3.4). Then there holds the error estimate, see [6]*

$$\|u_h - u\|_{L^2(Q)} = \begin{cases} O\left(h^2 + \Delta t\right), & \theta = \{1\}, \\ O\left(h^2 + \Delta t^2\right), & \theta = \{0.5\}, \end{cases} \tag{3.5}$$

where h is the mesh size.

3.3 Types of heat source

- Point source
  The type of heat source resembles a tiny object produce thermal conduction inside the domain. The point source locates at a fixed coordinate (x_0, y_0) and thermal conduction capacity $h(t)$. These component form the right hand side of heat equation

  $$F(x,y,t) = \delta_{x_0y_0}(x,y) \times h(t)$$

  Where $\delta_{x_0y_0}$ is the dirac delta function use for locating the heat point source. For instance, the dirac delta function can be like:

  $$\delta_{x_0y_0} = \frac{n}{\cosh(n(x-x_0))^2} \times \frac{n}{\cosh(n(y-y_0))^2}$$

- Wall source: Assuming the heat source attaches to a part of domain wall. Thermal conduction provide by that heat source can be express by

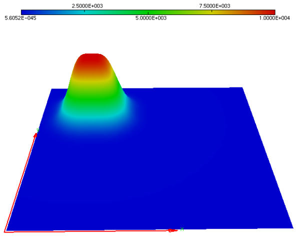

Figure 1: Illustration of chosen dirac delta function.

various way.
The fixed temperature walls is the most basic way to resemble a heat source. This type of wall is respective with the Diriclet boundary condition.
The other type of wall source provide the heat flux through the wall, respective with Neumann boundary condition.

4 Tests and discussion

4.1 Error evaluation with exact solution

We study a numerical experiment with the exact solution of heat equation and evaluate the error convergence. Consider a square $[0,1]\times[0,1]$. Find $u(x,y,t)$ satisfied

$$\frac{\partial u}{\partial t}-\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=(1+2\pi^2)\sin(\pi x)\sin(\pi y)\exp(t), \tag{4.1}$$

with the initial and boundary conditions

$$u(x,y,0)=\sin(\pi x)\sin(\pi y)\exp(t) \quad \text{and} \quad u|_\Gamma = 0.$$

The exact solution is

$$u=\sin(\pi x)\sin(\pi y)\exp(t),$$

Different cases of mesh size and time step length were studied to show the dependent of error on the mesh smoothness.

4.2 A problem of thermal engineering

We apply the numerical simulations of heat transfer into designing heat sink. Assuming a hot CPU inside a rectangular room fill with air. Let $u=u_{hot}$ inside CPU region and $u=u_{air}$ on air region, respectively Ω_c and Ω_a, on the initial time. Our goal is to design a heat sink stick on the CPU to lower its temperature. The heat sink region, denoted by Ω_s, has thermal conductivity coefficient κ_s. Similarly, let κ_a and κ_c be respectively the thermal conductivity coefficients inside air and CPU region. Technically, κ_a is small compare

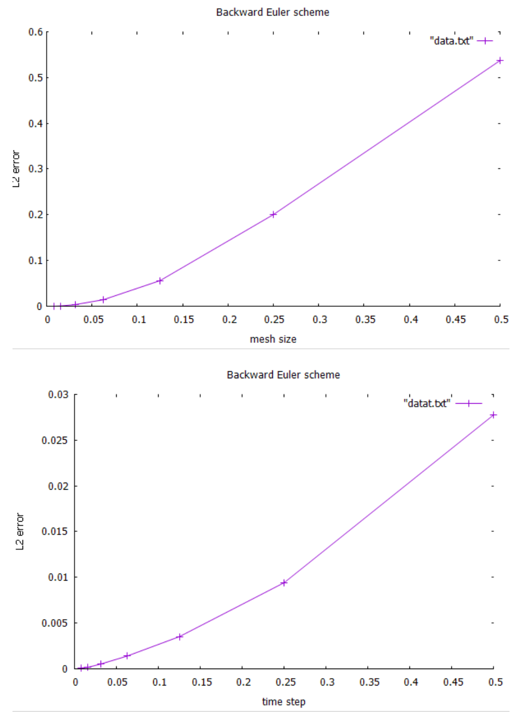

Figure 2: L^2 error convergence of Backward Euler scheme dependency on mesh size and time step.

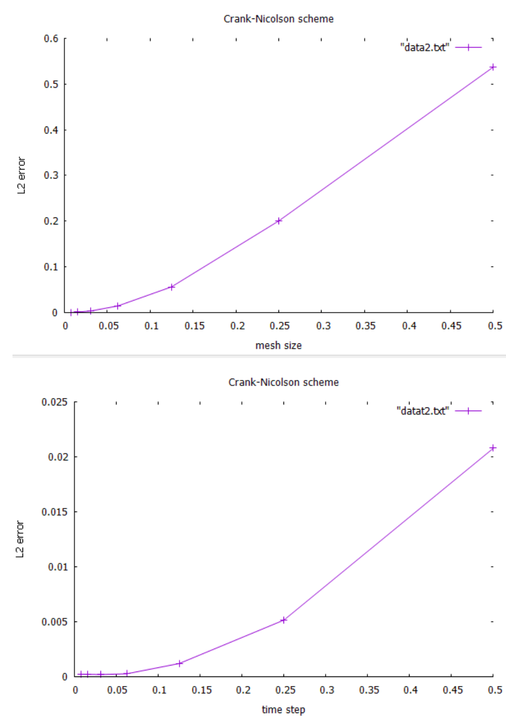

Figure 3: L^2 error convergence of Crank-Nicolson scheme dependency on mesh size and time step.

to κ_c and κ_s due to nature conduction of air. Furthermore, to provide cooling ability, $\kappa_s > \kappa_c$. The visualization using medit software.

Set $T = 1s$, $\kappa_a = 0.01$, $\kappa_c = 1$, $\kappa_s = 100$. The tem-

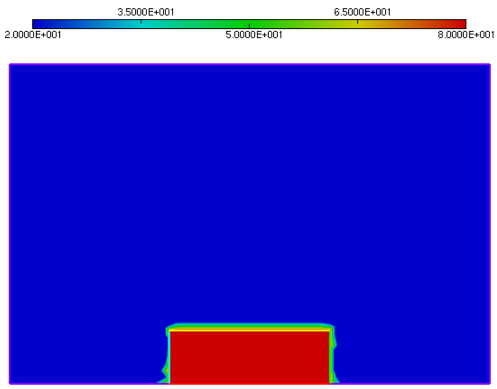

Figure 4: Thermal distribution at initial state. $u|_{\Omega_c} = 80$, $u|_{\Omega_a \cup \Omega_s} = 20$.

perature distribution at final time T of different heat sink shapes are shown below.

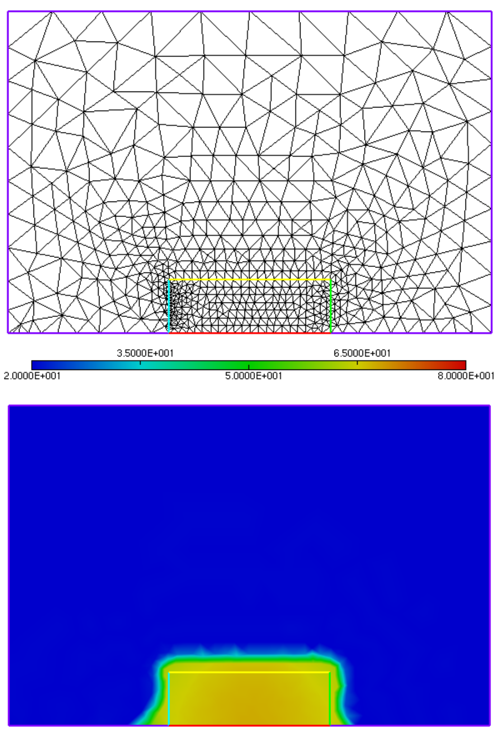

Figure 5: Thermal conduction with no heat sink. $u_{min} = 63.1$, $u_{max} = 67.3$.

4.3 Numerical experiment of optimal control problem

In engineering, sometimes we want to know how much heat source provided to receive heat $u(x,t)$ in a physical domain Ω in a time period $[0,T]$ equals or

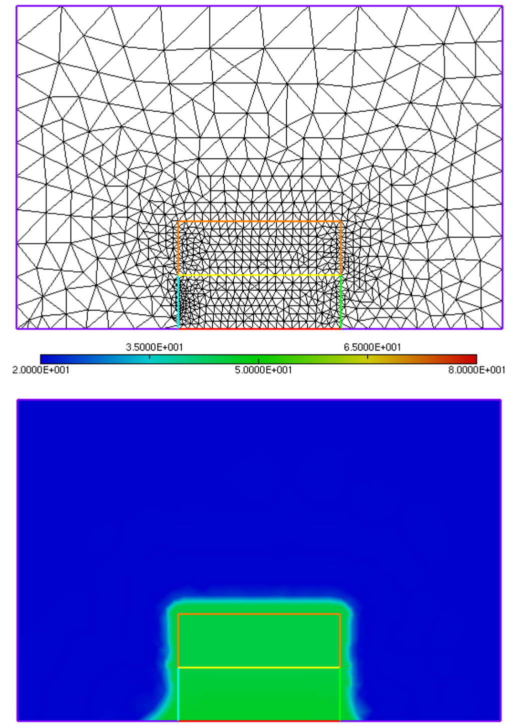

Figure 6: Thermal conduction with rectangular shape heat sink. $u_{min} = 44.8$, $u_{max} = 46.9$.

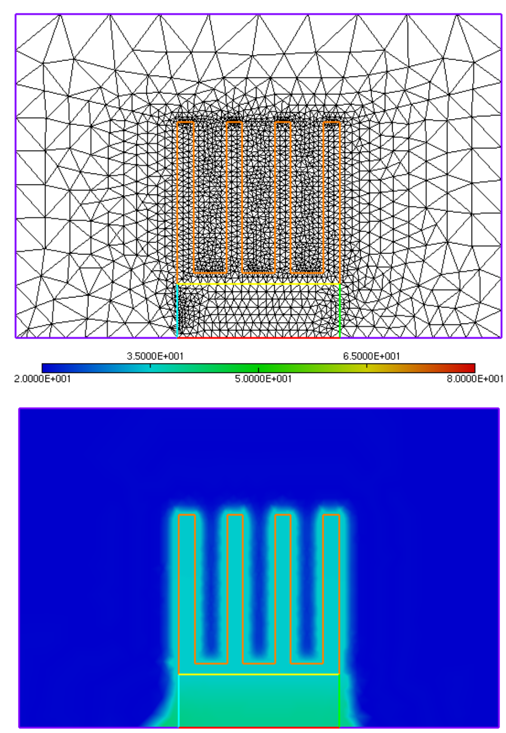

Figure 7: Thermal conduction with fin shape heat sink. $u_{min} = 34.9$, $u_{max} = 40.1$.

approximates with $\hat{u}(x,t)$, $(x,t) \in Q$. We suppose that heat source has the form $F(x,t) = f(x,t) + q(x,t)$. This leads to optimize the functional, see [3, 4],

$$J(q) = \frac{1}{2}\left\|u - \hat{u}\right\|^2_{L^2(Q)} + \frac{\gamma}{2}\left\|q\right\|^2_{L^2(Q)}, \tag{4.2}$$

where q being the control variable and $\gamma > 0$ being a regularization parameter.

To solve this problem, we use FreeFem++ software which provides an efficient tool called IPOPT. It is designed to perform optimal control problems, for more details see at []. To use this optimizer, we need to include the *ff-Ipopt* dynamic library. The parameters including the objective function $J(f)$ and its gradient $\nabla J(f)$ following

$$\nabla J(q) = z(x,t) + \gamma q(x,t), \tag{4.3}$$

where $z(x,t)$ is the solution of the adjoint problem

$$\begin{cases} -\dfrac{\partial z(x,t)}{\partial t} + \mathcal{L}z(x,t) = u - \hat{u}, & (x,t) \in Q, \\ z(x,t) = 0, & (x,t) \in S \\ z(x,T) = 0, & x \in \Omega. \end{cases} \tag{4.4}$$

The gradient $\nabla J(f)$ and adjoint problem will be derived the same as [1]. Now we will experiment the example as in [8]. For the error's sake, we will use Crank-Nicolson method ($\theta = 0.5$) to solve the direct problem (2.1) and adjoint problem (4.4). Consider $\Omega = (0,1)^2$ and $T = 0.1$ and homogeneous Dirichlet boundary condition. The right hand side f, the desired state $\hat{u}$ and the initial condition u_0 such that

$$\begin{aligned} f(x,t) &= -\pi^4 w_b(x,T), \\ \hat{u}(x,t) &= \frac{b^2-5}{2+b}\pi^2 w_b(x,t) + 2\pi^2 w_b(x,T), \\ u_0(x) &= \frac{-1}{2+b}\pi^2 w_b(x,0), \end{aligned}$$

where $w_b(x,t) = e^{b\pi^2 t}\sin(\pi x_1)\sin(\pi x_2),\ b \in \mathbb{R}$.

We chose the regularization parameter $\gamma = \pi^{-4}$ and the optimal solution triple $(\bar{q}, \bar{u}, \bar{z})$ of the optimal control problem (4.2) is given by

$$\begin{aligned} \bar{q}(x,t) &= -\pi^4\left[w_b(x,t) - w_b(x,T)\right], \\ \bar{u}(x,t) &= \frac{-1}{2+b}\pi^2 w_b(x,t), \\ \bar{z}(x,t) &= w_b(x,t) - w_b(x,T). \end{aligned}$$

First, we consider the behavior of the error for a sequence of discretization with decreasing size of the time steps and a fixed spatial triangulation with $N = 1089$ nodes. Second, we examine the behavior of the error under refinement of the spatial triangulation for $M = 1024$ time steps. We choose the free parameter b to be $-\sqrt{5}$.

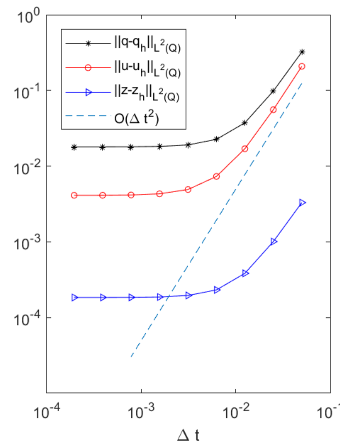

Figure 8: Refinement of the time steps for $N = 1089$ spatial nodes

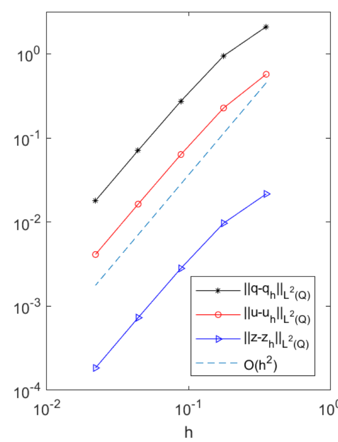

Figure 9: Refinement of the spatial triangulation for $M = 1024$ time steps

5 Conclusion

References

[1] D.N.Hao, P.X.Thanh, D.Lesnic, and M.Ivanchov. Determination of a source in the heat equation from integral observations. *Journal of Computational and Applied Mathematics*, 264:8298, 2014.

[2] Lawrence C. Evans. *Partial differential equations*. American Mathematical Society, 1998.

[3] D.N. Hao. A noncharacteristic cauchy problem for linear parabolic equations ii: A variational method. *Numer. Funct. Anal. Optim.*, 13:541564, 1992.

[4] D.N. Hao. A noncharacteristic cauchy problem for linear parabolic equations iii: A variational method and its approximation schemes. *Numer. Funct. Anal. Optim.*, 13:565583, 1992.

[5] Wloka J. *Partial Differential Equations*. Cambridge University Press, New York, 1987.

[6] Claes Johnson. *Numerical Solution of Partial Differential Equations by the Finite Element Method*. Cambridge University Press, New York, 1987.

[7] Stieg Larsson and Vidar Thomee. *Partial Differential Equations with Numerical Methods*. Springer Science & Business Media, 2003.

[8] Dominik Meidner and Boris Vexler. A priori error estimates for space-time finite element discretization of parabolic optimal control problems part i: Problems without control constraints. *Society for Industrial and Applied Mathematics*, 47(3:11501177, 2008.

[9] Fredi Troltzsch. *Optimal Control of Partial Differential Equations: Theory, Methods, and Applications*. American Mathematical Soc, 2005.