A Multifeatured Objective Function for Highly Generalized Computer Adaptive Item Selection

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Abstract

Computer adaptive testing (CAT) is an increasingly common mode of test administration offering improved test security, better measurement precision, and the potential for shorter testing experiences. This paper presents a new CAT algorithm based on a multifeatured objective function that is highly generalizable to multiple types of testing conditions to support principled assessment design. The behavior of the algorithm and the types of tests it can build is a function of user-defined constraints, thus freeing software developers from needing to alter code to achieve those variants. The paper comprehensively defines the algorithm, its components, and provides a highly detailed computational implementation. The paper consolidates all information needed to build and scale a CAT such that software developers and psychometricians can use this document as a self-contained resource to build and deploy an operational CAT platform.

Keywords: computer adaptive testing; computational psychometrics; item response theory

Introduction

The main purpose of computer adaptive testing (CAT) is to construct a unique test form for an examinee by adapting test items around their unique level of ability on the construct being measured

(Reckase, Ju, & Kim, 2019). In theory, the form that yields the maximum precision for the examinee is one where the test information function (TIF) is maximized at the examinee's true level of ability (Babcock & Weiss, 2009) and is best in the sense that it ensures a psychometric alignment between examinee ability and statistical characteristics of the item (Chang, 2015). Because true ability varies by individual, this implies a unique test form for each examinee where test items are carefully curated by a selection algorithm for administration. When item selection is tailored to examinees, an adaptive test can provide more information regarding examinee ability with fewer items relative to a fixed form test (van der Linden, 2005b; Yao, 2019) and by providing a unique form, test security is improved as examinees in similarly proctored settings will have forms with different test items when item exposure is well-controlled (Yasuda, Hull, & Mae, 2022).

A challenge in adaptive testing is that the true ability is latent and the best indicator of the true ability, the observed ability, changes as examinees provide responses to new items as they progress through a test. In general, few items provide very little information regarding the true ability and more items provide more information (van der Linden & Pashley, 2010). As a consequence, an adaptive test can iteratively improve its choice of items with a selection algorithm as more information is provided about the examinee in order to form the best possible test form for the individual examinee.

Implicit in the description above is the idea that the term "best" is well-understood for item selection (Kingsbury & Zara, 1989) and that some transparent machinery is available to choose those items from a test bank. In fact, "best" can be used differently depending on the test purpose. Therefore, it's key to begin with an operational definition of "best" and then create a mechanism that chooses the best items to uniquely construct a test form for an examinee.

All adaptive algorithms share the common feature that they adapt items to the ability of the examinee (van der Linden, 2005a). In this way, they minimize the individual error variance, thus providing greater precision at the examinee and aggregate levels. However, adaptive algorithms can vary in how they define best in the sense of choosing the items for administration (Ho & Dodd, 2012). For instance, if there is no test blueprint, then "best" might be considered exclusively on the psychometric properties of the items vis-à-vis the individual tester. If there is a content blueprint with requirements to choose items from different dimensions of that blueprint, then perhaps "best" can be defined to include content characteristics of the items. If both factors are important (i.e., psychometric and content balance), then the best item is one where the algorithm simultaneously considers both the measurement properties of the items and the non-statistical content characteristics of those items to identify which items are best on both dimensions at each step.

Perhaps the idea of "best" shouldn't be strictly defined by the capabilities of the algorithm. Test designers should follow theories of principled assessment design (Mislevy & Riconscente, 2006) and the algorithm itself should be able to administer any type of test form developers can imagine, a concept that has been formerly well-articulated (Stocking & Swanson, 1993). In this case, "best" is situationally defined and the algorithm is nothing more than a means to an end and is free to select items from any construct-relevant criteria the test developers wish to impose. This generalized flexibility is one means by which the validity claims regarding test scores arising from CATs can be enhanced (Luecht, De Champlain, & Nungester, 1997; Wise, 2023).

There are many examples of operational CATs used across the education, selection, licensure, and certification fields. The Armed Services Vocational Aptitude Battery (ASVAB) is one large-scale example with military selection applications (DMDC, 2008), the Smarter Balanced Consortium deploys millions of adaptive tests across states in the U.S. in educational testing applications

(Cohen & Albright, 2014), and the National Council Licensure Examination (NCLEX) uses adaptive tests for licensure and certification for medical professionals (Seo, 2017). Some jurisdictions in the U.S. have even codified CATs as a required mode of test administration for educational testing (Florida, 2023; Virginia, 2021).

The vast literature on adaptive testing commonly addresses a particular component within the CAT algorithm, such as item exposure (Huebner, Wang, Quinlan, & Seubert, 2015; Leung, Chang, & Hau, 2002; Sympson & Hetter, 1985), ability estimation (Bock & Mislevy, 1982; T. A. Schmitt & Walker, 2010), information functions (Chang & Ying, 1996), or optimization approaches (Choi, Moellering, Li, & van der Linden, 2016). Less common in the literature are thorough, end-to-end, descriptions of the inner workings of a computer adaptive engine providing enough detail such that an independent team can identify all necessary psychometric and computational components in a complete, self-contained location to build a CAT platform for operational use.

Most often, descriptions of CAT algorithms provide a procedural schema describing steps for the algorithm but lack a highly detailed computational algorithm and decision details needed to deploy operational infrastructure (Wainer, Dorans, Flaugher, Green, & Mislevy, 2000). For example, some describe adaptive technologies and their heuristics (Straetmans & Eggen, 1998) but do not provide thorough algorithmic descriptions. Others provide computational details but limit focus to a particular computational component, such as exposure control or optimal methods for computing psychometric information (Ma et al., 2023). One example provides seemingly complete details for an item selection algorithm using Monte Carlo methods (Belov, Armstrong, & Weissman, 2008), though details on its scalability and computational complexity are not evaluated. Hence, we cannot know the degree to which this is scalable for large-scale design, but Monte Carlo methods are in general very computationally time consuming.

Purpose and Organization

This paper intends to serve as an end-to-end resource bringing transparency to a new multifeatured item selection algorithm. The ideas proposed in this paper combine psychometric and test design concepts that simultaneously fit within a highly generalizable item adaptive objective function. Our work parameterizes this idea as optimizing a linear function consisting of user-defined constraints as function inputs, thus allowing the item selection algorithm to generally select items based on criteria entirely determined by test developers. In other words, the goal is to construct a generalized item selection algorithm that behaves very differently based on the testing objectives. Some examples of those objectives could be: 1) adapt only on ability to minimize individual error, 2) simultaneously adapt on ability and provide content or feature balancing to match constraints defined in a test blueprint, 3) only control for feature constraints in which case the test form is linear-on-the-fly (LOFT), 4) limit the algorithm to administer items from a subset of the bank without bifurcating the bank such as when item enemies might exist, 5) administer a collection of items nested within any common grouping structure, and 6) flexibly administer operational and experimental field test items spiraled together.

More generally, our motivation is to build and demonstrate a generalizable item selection algorithm that can broadly support an array of custom test formats, designs, and developer constraints simply by changing inputs passed to the algorithm. For instance, different content dimension targets, cognitive complexity demands, item enemy targets, types of items administered, psycho-

metric measurement targets, among others can all be managed within our proposed framework without having to alter software code to achieve those variants. We describe how other typical CAT approaches are special cases of our proposed approach.

Our specific aim is to achieve four objectives in this manuscript. First, we propose and fully describe a generalizable CAT algorithm. Second, we include complete end-to-end requirements for deploying a CAT of this form for operational testing. The intent is that all information needed for complete construction of a CAT engine is self-contained within this manuscript so a software development and psychometric team could collaborate to build and deploy an operational instance of a CAT by following the guidance contained here. Third, we describe a scalable computational approach for implementation of the ideas such that the algorithm can be deployed in large settings with minimal latency at the test taker level. Fourth, we demonstrate the behavior of the algorithm with some test trial cases to document its behavior in simulated settings.

The Generalized Item Selection Algorithm

Let the general item selection function for item $j = \{1, \ldots, J\}$ for examinee $i = \{1, \ldots, P\}$ at step $t = \{1, \ldots, T\}$ needed for test feature $g = \{1, \ldots, G\}$ be denoted as

$$\lambda_{jit} = \pi_j I_j^{'}(\widehat{\theta}_{it}) + \sum_{g=1}^{G} \upsilon_g w_{jg} m_{igt} + \sum_{g=1}^{G} h_g w_{jg} \alpha_{igt}, \tag{1}$$

where the function is a weighted linear composite of three additive components on the right hand side including: 1) the rescaled item information function, 2) the value of the content balancing constraint parameter needed to fulfill other test developer features, and 3) a stochastic content parameter to randomly boost and potentially vary the administrative sequence of test items by dimension. Table 1 provides an overview and brief description of the notation used in Equation (1). However, each term in this objective function is fully described in complete detail in the remaining sections.

In this context the term λ_{jit} is a scalar-valued statistic describing the potential value of an item to fulfill the constraints necessary to meet test form criteria for a given examinee at each iteration of the test process. The maximum value of this objective function, $\max \lambda_{jit}$, is the item with the most potential at step t to fulfill the objectives of the test selection algorithm.

Throughout this manuscript we differentiate between two terms, *algorithmic* details and *decision* details. The term "algorithmic details" is used to mean the specific components in Equation (1) and "decision details" are how those ideas are implemented. For example, the item information function is an algorithmic component of the objective function. How item information is computed is a decision detail.

Computational Implementation

The scalar-based expression in Equation (1) is useful for general consideration of the idea, but not a useful computational representation. We can rewrite the objective function as

$$\boldsymbol{\lambda}_{it} = f(\boldsymbol{\pi}, \boldsymbol{I}'(\boldsymbol{\theta}_{it}), \boldsymbol{W}, \boldsymbol{m}) = \boldsymbol{\pi} \odot \boldsymbol{I}'(\boldsymbol{\theta}_{it}) + \boldsymbol{W}\boldsymbol{m}, \tag{2}$$

Parameter	Description
π_j	User-defined importance weight associated with information for the jth item.
$I_j^{'}(\widehat{\theta}_{it})$	Item information function for item j evaluated at the ability estimate for examinee i at step t rescaled to be on same unit space as m_{igt}.
υ_g	User-defined importance feature weight associated with dimension g of test blueprint.
w_{jg}	Indicator value linking item j to feature g of a test blueprint.
m_{igt}	Value of content feature for examinee i at step t for feature g.
h_g	Binary indictor to toggle content sequence parameter on ($h_g = 1$) or off ($h_g = 0$).
α_{igt}	Random draw from a uniform distribution to balance content administration.
$\widehat{\theta}_{it}$	Estimated ability for examinee i at step t.

Table 1: Notation for CAT Algorithm in Equation (1)

where the space and dimensions are $\boldsymbol{\pi} = \{\pi_1, \ldots, \pi_J\} \in \mathbb{R}^J_{[0,\infty]}$, $\boldsymbol{I'}(\boldsymbol{\theta}_{it}) = \{I_1^{'}(\widehat{\theta}_{it}), \ldots, I_J^{'}(\widehat{\theta}_{it})\} \in \mathbb{R}^J_{[0,1]}$, $\boldsymbol{W} \in \mathbb{R}^{J\times G}_{[0,1]}$ is an indicator matrix where the rows are the test items in the bank and the columns are values linking the jth item with the gth item feature, h_g is a binary indicator, and $\boldsymbol{m} = \{(\upsilon_1 m_{i1t} + h_1\alpha_{1it}), \ldots, (\upsilon_G m_{iGt} + h_G\alpha_{Git})\} \in \mathbb{R}^G_+$. Here the notation $\mathbb{R}_{[a,b]}$ is used to mean a real on the interval $[a, b]$ including endpoints, $\boldsymbol{\pi} \odot \boldsymbol{I'}(\boldsymbol{\theta}_{it})$ is the Hadamard product, and $\boldsymbol{Wm}$ is the typical matrix product. Others have proposed managing content constraints alongside psychometric information within the objective function (Mao & Xin, 2013; Stocking & Swanson, 1993) and this algorithm aligns with that notion in spirit, but differs substantially in computational implementation.

Note that $\boldsymbol{W}$ has dimensions $J \times G$, meaning a test item can be linked to many different features. In doing so, it is important to normalize the rows in the matrix $\boldsymbol{W}$ to account for the number of features an item is associated with. Without the normalization, items linked to many features would be more attractive to the item selection algorithm given that the resulting value for λ_{jit} is the row sum over all columns in this matrix. Consequently, the rows of the matrix $\boldsymbol{W}$, denoted as $\boldsymbol{w}_j = \{w_{j1}, w_{j2}, \ldots, w_{jG}\}$, are normalized such that $\sum_{g=1}^{G} w_{jg} = 1$. Note that the elements of $\boldsymbol{W}$, before normalization, assume values of

$$w_{jg} = \begin{cases} 1 \text{ if item } j \text{ is associated with feature } g, \\ 0 \text{ otherwise.} \end{cases}$$

Equation (2) could treat $\pi_j = \pi \; \forall \; j$ as a constant information weight applied in the same way to all items. However, its treatment as a vector offers broader application where different weights for information could be applied such that they vary over some types of items. This could be useful in situations where the psychometric properties of items could be exploited as weights. For example, it could be useful to make use of the item-specific variances, σ_j^2, and use their reciprocals as weights such as $\boldsymbol{\pi} = \{\frac{1}{\sqrt{\sigma_1^2}}, \frac{1}{\sqrt{\sigma_2^2}}, \ldots, \frac{1}{\sqrt{\sigma_J^2}}\}$ mimicking weighting strategies used in survey design for complex samples (Bell et al., 2012; Wolter, 1985). Though interesting, some caution

on this strategy is needed as this would clearly cause for the algorithm to prefer items with smaller standard errors and lead to some potential item overexposure. A second potential use of the vector $\boldsymbol{\pi}$ could be to set $\pi_j = 1$ for operational items and then set $\pi_j = 0$ for experimental field test items. In this way, the algorithm will randomly spiral in field test items alongside operational items. A brief example of this is later described.

Equation (2) is simply a better computational expression of Equation (1) and in the same way yields a vector of values $\boldsymbol{\lambda}_{it} = \{\lambda_{1it}, \lambda_{2it}, \ldots, \lambda_{Jit}\}$. These item-specific values are the values used for item selection with the maximum value describing the best item needed at each step for item selection. A very simple didactic example for computing Equation (2) is provided in Appendix B along with R code.

Optimization Approach

The goal of the objective function is to yield a value denoting which item is best for test administration. Our objective function and constraints could be formulated as a mixed integer linear programming (MILP) problem as commonly used in the automated test assembly literature (van der Linden, 2005c). This has been demonstrated for the shadow test method (Choi et al., 2016) and the weighted deviation method (Stocking & Swanson, 1993), both using MILP for optimization. However, MILP is computationally expensive, challenging to implement in applied settings, and is not guaranteed to find an optimal solution. Additionally, the mathematical complexity for linear programming almost always requires an external commercial or open-source optimizer instead of optimization code written directly in closed form or code written by the development team building the CAT engine (Koch, Berthold, Pedersen, & Vanaret, 2022).

Here we propose optimization in a different way using a sort-and-search algorithm and later show how to reduce the search space to achieve a more scalable option. First, the input values in $f(\boldsymbol{\pi}, \boldsymbol{I}'(\boldsymbol{\theta}_{it}), \boldsymbol{W}, \boldsymbol{m})$ are treated as known, fixed quantities for computing the elements of $\boldsymbol{\lambda}_{it}$. We do not search over the parameter space of $f(\boldsymbol{\pi}, \boldsymbol{I}'(\boldsymbol{\theta}_{it}), \boldsymbol{W}, \boldsymbol{m})$ and find which of their values cause for $\boldsymbol{\lambda}_{it}$ to reach a maximum. Instead, the input values for the known elements are used to directly compute Equation (2) and the aim is to find the maximum value in the set $\boldsymbol{\lambda}_{it}$, or notationally $\max f(\boldsymbol{\pi}, \boldsymbol{I}'(\boldsymbol{\theta}_{it}), \boldsymbol{W}, \boldsymbol{m})$.

When $\boldsymbol{\lambda}_{it}$ is sorted in descending order, the first item in the vector is best for selection in the sense that it contributes best to the objectives defined by the test developer (e.g., psychometric information and content feature constraints). It is generally the case that sorting algorithms have computational complexity $\mathcal{O}(n \log n)$, where n is the number of elements in the vector $\boldsymbol{\lambda}_{it}$. We later show how the vector size can be reduced to minimize the search space for computing the values in $\boldsymbol{\lambda}_{it}$ and then also for sorting this vector.

Tuning Parameters

What makes the objective function configurable is the set of tuning parameters that allow for the different components of the function to be emphasized differently depending on the needs of the testing application. The values for π_j and υ_g are scalar-valued importance weights defined as $\mathbb{R}_{0 \leq \omega \leq \infty} = \{\omega \in \mathbb{R} | 0 \leq \omega \leq \infty\}$ which is used to mean any importance weights in the set $\omega \in \{\pi_j, \upsilon_g\}$ can be any positive real number, including 0. Note that importance weights of

$\pi_j = 0 \; \forall \; j$ cause for the algorithm to be either driven entirely by content constraints (linear on the fly) and when $\upsilon_g = 0 \; \forall \; g$ the algorithm is purely adaptive on ability and all content constraints are ignored. The term h_g is a binary indicator allowing for the random sequence component to be toggled as needed. The most primitive form of an adaptive algorithm is a special case of Equation (1) when $\{\upsilon_g, h_g\} = 0 \; \forall \; g$ and $\pi_j = 1 \; \forall \; j$.

Note that these weights are relative to each other. If $\upsilon_g < \pi_j$, for any particular j and g then the algorithm places greater emphasis on adapting on the measurement properties than it does on the content balancing features for that particular dimension. If $\pi_j < \upsilon_g$, then the algorithm would place more emphasis on the content features in g of the test than it would on adapting towards the individual's ability. Last, if $\pi_j = \upsilon_g \; \forall \; j$ and g, then equal emphasis is placed on both. Finally, note that the feature weights are subscripted, indicating they are configurable by content feature giving the algorithm greater preference in some feature areas and less in others.

These weights are real numbers to reflect gradations of their importance and can be configured in multiple ways to account for custom test design. For example, $\pi_j = 1$ and $\upsilon_g = 1$ might be considered default values or values of $\pi_j = 2$ and $\upsilon_g = .5$ might be used to weight information with greater importance than blueprint features.

Methods for Adapting on Examinee Ability

The first term in Equation (2) is the item information function used to adapt on examinee ability. This is typically done when items are drawn from a calibrated item pool and the item information function is evaluated at the provisional ability estimate and is then used to determine the next best candidate item (Chen, Ankenmann, & Chang, 2000; Han, 2009). Achieving this can be accomplished using multiple approaches as noted below.

One of the greater challenges in CAT is that the provisional ability estimate early in the test is a poor approximation of the true latent trait given that the estimate is based on a small subset of test items (Chen et al., 2000). That is, for initial item selection we effectively have no information regarding the examinee's true ability. However, the asymptotic properties of the estimator, $\widehat{\theta}$, imply that as more items are administered at each additional step t, the observed estimate approaches its true value, θ, and is asymptotically consistent, $\lim_{t \to \infty} \Pr\left(|\widehat{\theta} - \theta|\right) > e = 0$. For this reason, evaluating the item information function at the point estimate $\widehat{\theta}$ early in the test may prove less than ideal because that point estimate is swamped with measurement error (Chang, 2015).

However, as more items are administered, perhaps the point estimate can be plugged in to compute item information. This suggests a possible switching strategy could be employed to account for the uncertainty in the ability estimate early in the test but then switch to a method that directly uses the point estimate later in the test. Below we describe three approaches for computing item information and then describe how switching between them could be an advantage to balance computational overhead with statistical precision.

Maximum Fisher Information

The item information function for a consistent estimator is generally defined from item response theory (IRT) for binary items as

$$I_j(\widehat{\theta}_{it}) = D^2 a_j^2 \left[\frac{\left(\Pr_j \left(\widehat{\theta}_{it} \right) - c_j \right)^2}{\left(1 - c_j \right)^2} \right] \left[\frac{1 - \Pr_j \left(\widehat{\theta}_{it} \right)}{\Pr_j \left(\widehat{\theta}_{it} \right)} \right], \tag{3}$$

where $\Pr_j(\widehat{\theta}_i)$ is the probability of response given the triplet of $\{a_j, b_j, c_j\}$ denoting the IRT parameters of item slope, item location, and item guessing, respectively (Lord, 1980). The Maximum Fisher Information (MFI) computes item information from the provisional point estimate obtained at each step of the algorithm and items can be sorted according to the MFI choosing the item providing the most (or largest) MFI given the provisional ability estimate (Chang & Ying, 1996).

Integrating over the Information Function

The MFI assumes the provisional point estimate is a good measure of examinee ability, which is most likely untrue in early stages of the test. Rather than using the point estimate, a second option is to account for the uncertainty in the point estimate and calculate information averaged over the function as

$$I_j(\widetilde{\theta}_{it}) = \int_{\widehat{\theta}_{it} - \delta_t}^{\widehat{\theta}_{it} + \delta_t} D^2 a_j^2 \left[\frac{\left(\Pr_j \left(\theta_{it} \right) - c_j \right)^2}{\left(1 - c_j \right)^2} \right] \left[\frac{1 - \Pr_j \left(\theta_{it} \right)}{\Pr_j \left(\theta_{it} \right)} \right] d\theta_{it}. \tag{4}$$

The limits for integration can be formed using confidence intervals from the condititional standard errors, $\sigma(\widehat{\theta}_{it})$, as $\delta_t = \Phi^{-1}(p)\sigma(\widehat{\theta}_{it})$ where $\Phi^{-1}(\cdot)$ is the inverse of the normal cumulative distribution function and p is the nominal coverage probability. Here we can form lower and upper bounds for integration using the conditional standard error of measurement obtained at each step (Veerkamp & Berger, 1994).

Evaluation of the integral may be computationally expensive. This can be mitigated in two ways. First, we can use this only for the first t steps of the test where t is completely configurable and then the algorithm switches to using the MFI following the first t steps. Second, we can approximate the integral using Gauss-Legendre quadrature with a small number of quadrature points (See Appendix A).

Kullback-Leibler Information

A third approach is via the Kullback-Leibler information method (Cover & Thomas, 1991), who informally describe this as a distance metric between two distributions. It has been found to be potentially useful in adaptive testing (Belov & Armstrong, 2011; Chen et al., 2000) and for binary items takes the form

$$KL_j \left(\widehat{\theta}_{it} \right) = \int_{\widehat{\theta}_{it} - \delta_t}^{\widehat{\theta}_{it} + \delta_t} \sum_{l=0}^{1} \log \left[\frac{\Pr \left(x_j = l | \theta \right)}{\Pr \left(x_j = l \middle| \widehat{\theta}_{it} \right)} \right] \Pr \left(x_j = l | \theta \right) d\theta, \tag{5}$$

where x_j is the response to item j and θ is true ability (Chang & Ying, 1996). These authors choose $\delta_t = \frac{p}{\sqrt{t}}$ where p is chosen to reflect a nominal coverage probability. This is an impressive implementation relying on the well-documented asymptotic properties of the estimator where the

integration limits becomes smaller with increasing t. Our implementation is similar, but instead of using $\frac{p}{\sqrt{t}}$, we draw the limits of integration using δ_t in the same way described for Equation (4). We again approximate this integral using Gauss-Legendre quadrature as shown in Appendix A.

Information Switching Strategy

The KL procedure and the method for averaging over the information function have greater computational expense than directly computing the MFI as they both involve an integral. However, they make better use of the uncertainty in the estimate than the MFI. Balancing the two ideas suggests a trade-off approach where one could rely on the asymptotic properties of the estimator and begin early item selection with either the more expensive KL or averaging functions for some initial item selection and then later in the test rely more on the observed estimate for computing information.

This idea is originally credited to Chang and Ying (1996). However, our understanding of their approach is that they compute KL over all items, but with increasing values of t, the limits of integration around the observed estimate becomes smaller, thus treating the observed score as a better estimate at each step. This remains computationally expensive as each step still involves evaluating the integral.

An efficient computational alternative, but theoretically consistent, approach would be to switch to the MFI from either the KL or the average information function after a fixed number of items is administered given the asymptotic expectations for $\widehat{\theta}$. For example, we might initially compute item information using KL for t steps and then switch to the MFI or, rather than depending on fixed t, switch to the MFI after reaching a desired level of precision.

Rescaling Item Information and a Note on Notation

In the preceding section the notation $I_j(\widehat{\theta}_{it})$, $I_j(\widetilde{\theta}_{it})$, and $KL_j\left(\widehat{\theta}_{it}\right)$ is used to describe different ways to compute information for item j. This is useful to denote the differences in the approach. Going forward, we now generically use $I_j(\widehat{\theta}_{it})$ to describe a method for computing information and describe in narrative which approach is used for computing information simply for the convenience of notation.

Because this is a multiple featured objective function, the item information is on a different scale than the content constraint features presented below. When CAT algorithms use only $I_j(\widehat{\theta}_{it})$ for choosing items, no rescaling is needed. However, Equation (1) contains multiple components with scale metrics that differ from each other. As such, we need to rescale the item information (and the content indices) to be on the same unit scale so that one does not have greater influence than the other simply as a function of scale differences. The min-max feature scaling method is used to rescale the information function to be used in the objective function is

$$I_j^{'}(\widehat{\theta}_{it}) = \frac{I_j(\widehat{\theta}_{it}) - I(\widehat{\theta}_{it,\min})}{I(\widehat{\theta}_{it,\max}) - I(\widehat{\theta}_{it,\min})} \tag{6}$$

where $I(\widehat{\theta}_{it,\max})$ and $I(\widehat{\theta}_{it,\min})$ denote the maximum and minimum item information functions of all items evaluated for examinee i at step t, respectively. The rescaling in Equation (6) is also used for the content feature so both item and content information are rescaled to be on the unit space.

Content Balancing

The second and third components of the algorithm are related to the content targets to be administered if a test blueprint exists. This component is often critical to support claims of construct validity from an adaptive test. While the term "content balance" is used here, it is perhaps more suitable to refer to them as "feature constraints." These components provide the framework by which expert test developers can appropriately define the construct to be measured and then implementation via the following priority index ensures the test form balances both the psychometric and construct-relevant components simultaneously.

The first content component described here is an index that determines which aspects of the test blueprint should be prioritized at each step while the second component is intended to introduce some randomization such that items have a "boost" to ensure the forms reach their blueprint minimum or maximum expectations.

Content Index

The content balancing *priority index* parameter can be generally defined as in Equation (7), which contributes to the objective function at step t. The term priority index is used to mean that the resulting value contributes to the item selection function adding some priority to improve the chance that item j associated with feature g will be chosen at step t.

$$m_{igt} = \begin{cases} 1 + \frac{N_l - n_{igt}}{N_l} & \text{if } n_{igt} < N_l \\ \frac{N_u - n_{igt}}{N_u - N_l} & \text{if } N_l \leq n_{igt} < N_u \\ -p & \text{otherwise.} \end{cases} \tag{7}$$

The values N_l and N_u represent blueprint minimum and maximum values for a particular area of the blueprint. Hence, they are considered boundaries for a particular test constraint and are used to mean the range of items on the test form necessary to fulfill that constraint, and the value n_{igt} is the number of items with feature g that have been administered to examinee i up to step t.

The value of m_{igt} initiates at 2 $\forall$ g when $n_{igt} = 0$ and the value approaches unity as $n_{igt} \to N_l$. Once a test feature is satisfied by meeting its minimum requirements (i.e., when $n_{igt} = N_l$), the function of m_{igt} resets to 1 and then the value approaches 0 as $n_{igt} \to N_u$. The parameter includes a penalty once the algorithm achieves the maximum, thus lowering the algorithm's preference to use items with this attribute in future selections. This penalty value p might assume a default set to $p = -1$ or perhaps any other negative number. In doing so, the item is not eliminated from potential use, but it adds a negative constant to m_{igt}, which will lower its position in the sorted vector $\boldsymbol{\lambda}_{it}$. Hence, when $N_u = n_{igt}$, the chance that such an item would be chosen is minimized and the algorithm will prevent test blueprint requirements from being exceeded.

The way in which the priority index operates is such that items below their test requirement minimum always have a larger contribution to the objective function than items between their minimum and maximum. That is, items are most attractive to the algorithm when they are below their minimum blueprint requirement and decrease in attractiveness as they approach their maximum.

The function m_{igt} resembles the constraint CAT (CCAT) of Kingsbury and Zara (1989). However, the normalization process described for the rows in the matrix $\boldsymbol{W}$ walks this approach closer

in principle to other ways in which content balancing might occur with items containing multiple content features that are not mutually exclusive (Segall & Davey, 1995; Stocking & Swanson, 1993). In this way, if each item in the test bank is a member of one and only one content feature, then $w_{jg} = 1$ and the CCAT is a special case of the approach we implement. If an item belongs to multiple content features, then $0 < w_{jg} < 1$ is a normalized value to ensure items that disproportionality share many content features relative to others would not be systematically preferred by the algorithm if these normalized weights were not used.

The allowance for a minimum and maximum creates a blueprint boundary to allow for some variability in the nature of the test form at the subtest level, although an overall test length feature is implemented. For instance, we could enforce a total test length of N but have variability at the subset level. In this way, examinees' i and i' would both have tests of length N. But, they would have different numbers of items at the subtest level making up that total test length. If every test is expected to be exactly the same in terms of length at the subtest level, then $N_l = N_u$ would create tests with different items, but with the same number of items at each lower level.

The priority index, m_{igt}, is also rescaled at each step as in Equation (6) so the values are on the same scale as the item information function. Without this rescaling, the value of the priority index decreases with increasing number of items administered and the information function would then have a more dominant role in the objective function. Note that if $\sum_{g=1}^{G} \upsilon_g = 0$, (i.e., all blueprint weights are 0) then feature rescaling is not performed on the content index. In this case, the denominator in the rescaling would be 0 and min/max feature scaling would be undefined.

Content Sequence Balance Parameter

The term m_{igt} in the objective function could cause for the algorithm to be somewhat deterministic and more systematically prefer items with larger values of m_{igt} during item administration with certain types of blueprints. To potentially mitigate this effect, the term α_{igt} is introduced as a stochastic parameter to randomly vary the preference of the algorithm for different areas of the content as the test progresses. This parameter adds a small stochastic component to the function at each iteration of the test which has the effect of randomly varying the chances for an item with feature g of being selected at step t.

Some approaches, such as the modified multinomial model (Leung, Chang, & Hau, 2003), have been proposed to deal with the effects of predictable sequencing. Our approach is a probabilistic stochastic component added to the objective function for each item implemented as

$$\alpha_{igt} = \begin{cases} u_{ig} \text{ if } u_{ig} < m_{igt} \\ 0 \text{ otherwise,} \end{cases} \tag{8}$$

where $u_{ig} \sim U(0, 2)$. Note that we draw from $U(0, 2)$ to be commensurate with the values produced by the content balance index, m_{igt}. The term h_g is a binary indicator allowing this parameter to be toggled on or off as desired. Note, that as m_{igt} tends to 0, the probability that u_g adds to this component also decreases given that $\Pr(u_{ig} < m_{igt}) = \frac{m_{igt}}{2}$ from properties of the uniform cumulative distribution function. The effect on the item selection algorithm then will generally be to add a random boost to the value of the objective function for items in content domains with larger values of m_{igt}.

A second way to consider implementing the stochastic component would be to randomly add a

term to any content area that has not yet reached its minimum value. This would be implemented as

$$\alpha_{igt} = \begin{cases} u_{ig} \text{ if } m_{igt} > 1 \\ 0 \text{ otherwise.} \end{cases} \tag{9}$$

The implementation in Equation (9) randomly boosts the algorithm's preference for items that are below the content blueprint minimum whereas the implementation in Equation (8) will tend to randomly boost the algorithm's preference for items in a dimension that is yet to reach its maximum.

Exposure Control

Item exposure control is perhaps one of the most widely studied topics in CAT. Here, we limit consideration to only two methods for implementation in our work, but other methods could easily be incorporated as decision variables for building a CAT engine. The first approach requires no prior knowledge about the items or the population of examinees and is simple to implement. The second is an IRT-based approximation to prevent items with large a-parameters from being selected too often. We acknowledge that the Sympson-Hetter method (Sympson & Hetter, 1985) is an important exposure control method in the CAT literature. Its implementation requires some work to establish an exposure control parameter whereas the randomesque requires no prior knowledge or exposure limits on items. For this reason, we focus on the two methods below, but the Sympson-Hetter method could very easily fit into our algorithmic framework.

Randomesque Approach

The randomesque approach is implemented by computing the value of λ_{jit} for all items in the bank at the current, provisional, value of $\widehat{\theta}$ which forms a vector, sorted in descending order, $\boldsymbol{\lambda}_{it} = \{\lambda_{1it},\ \lambda_{2it,} \ldots, \lambda_{Nit}\}'$ where N is the number of (remaining) items in the bank. In this way, the first item in the vector provides the most information and the final item in the vector provides the least. Implementation begins by setting a configurable parameter $w = \{1, 2, 3, \ldots, N\}$ and randomly choose one item from $\{\lambda_{1it},\ \lambda_{2it,} \ldots, \lambda_{wit}\}$. If $w = 1$, the algorithm always chooses the top, best item, and if $w = N$ the algorithm is effectively taking a random draw from all available items in the bank. Pragmatically, setting $w = 5$ (or approximately so) ensures some balance between randomization and selection of the best next item (Georgiadou, Triantafillou, & Economides, 2007; Kingsbury & Zara, 1989). This randomization also adds a security feature for examinees in similar proctored settings. This approach mitigates the potential for individuals in similar settings to engage in answer copying since items are partially random even for those with similar ability estimates.

The Impact of Item Slopes

All things being equal, items with large a-parameters provide the most psychometric information conditional on θ and algorithms using IRT-based information functions are likely to overexpose those items (Hau & Chang, 2001). Some approaches consider stratification methods to form bins

such that items with smaller slopes are administered earlier in the test and items with larger slopes would be administered later (Huebner, Wang, Daly, & Pinkelman, 2018). We propose a simpler method to minimize exposure risk of items with large a-parameters by provisionally setting $a_j = 1 \; \forall \; j$ as a pseudo-value only when computing the item information for purposes of item selection. That is, we use this approximation in item selection, but use the estimated (actual) a-parameter when generating ability estimates, both provisional and final. This approximation on the a-parameter is one additional exposure control parameter that mitigates overexposure of highly informative items in certain populations with similar abilities.

Provisional Scoring Options

A CAT progressively updates the examinee ability estimate and its precision after a response is submitted to each new item (Wang & Vispoel, 1998). With IRT, this is generally based on concepts of maximum likelihood estimation (MLE) or Bayesian methods (Bock & Mislevy, 1982; Spray & Reckase, 1996), with the latter being preferred. The challenge with MLE scoring in CAT is that ability estimates cannot be provided for response strings with perfect answers or completely wrong answers as the likelihood function has no maximum. Some tricks can be implemented for MLE methods to adjust score strings or to supplement the response with a fake item response, sometimes referred to as "fences" (Han, 2016). Instead, Bayesian methods do not suffer from this same problem and can be used to generate ability estimates in a broader case of scenarios.

The choice between MLE and Bayesian approaches, however, is not purely the computational advantage, but also based on the theoretical properties of the estimators themselves commonly using the bias/variance trade-off as consideration. MLE methods tend to provide less bias (and in some cases no bias) than Bayesian approaches, but at the expense of larger overall mean squared error (MSE). Bayesian methods are biased but are also known to have smaller overall MSE. However, estimators with smaller MSE will, on average, be closer to the true parameter than an unbiased estimator with larger MSE (Greene, 2000).

Termination Criteria

Termination of the algorithm may occur under multiple conditions (Babcock & Weiss, 2009), two of which are considered here. The first option is to stop when $\sigma(\widehat{\theta}_{it}) < \varepsilon$ where ε is a user-defined tolerance for stopping and $\sigma(\widehat{\theta}_{it})$ denotes a conditional standard error of measurement. In this way, the algorithm may terminate without meeting test developer constraints. A second option is to stop when the test reaches a fixed length. In this way, all test developer constraints have been satisfied. It is also possible to combine them both such that the algorithm repeatedly draws items from content dimensions until the precision target and blueprint targets have both been reached.

Item Selection Process

With all individual components of the algorithm now described, we can provide a high-level schema for item selection. Initial item selection begins by drawing any item from the bank with

equal probability. After this, the value of the objective function is iteratively reevaluated and updates the values of $I_j^{'}(\widehat{\theta}_{it})$, m_{igt}, and a_{gt} and then the value of λ_{jit} is computed at step t for all remaining items in the bank. Item selection then proceeds according to the procedural schema in Algorithm (1).

Algorithm 1 Procedural Schema for CAT Algorithm

Input: Choose initial item j from bank and administer to examinee i at step $t = 0$.

1: Form provisional ability estimate after examinee responds, $\widehat{\theta}_{it}$.
2: Update m_{igt} based on item administered at step t and then update $t = t + 1$.
3: Recompute $I_j^{'}(\widehat{\theta}_{it})$ given new estimate, $\widehat{\theta}_{it}$.
4: Compute $\boldsymbol{\lambda}_{it}$ using Equation (2) for remaining items in the bank excluding previously administered items.
5: Sort vector $\boldsymbol{\lambda}_{it}$ in descending order.
6: Apply exposure control procedure and select one item.
7: If termination criteria are satisfied, stop. Else, return to (1).
8: After test termination, generate final $\widehat{\theta}_{it}$ for reporting.

Runtime Considerations for Scaling the Algorithm

Choi et al (2016) explore an improved computation of the shadow test and make the claim that cloud-based infrastructures for computing are "scalable without limit." This is true in the sense that cloud-based structures provide virtually limitless access to fast computers. This is untrue in the sense that is comes with increasing monetary cost and when building operational environments for industry, that monetary cost indeed has limits. Consequently, it is important to find the best algorithm for computing and only then pass that into larger cloud-based systems.

It has been shown that mixed integer linear programs have time complexity of $n^n 2^{\mathcal{O}(n)}$ (Dadush, Eisenbrand, & Rothvoss, 2023) but sorting algorithms as used here have complexity $\mathcal{O}(n \log(n))$, letting n in this case represent the collection of inputs used for the model with the latter having faster time complexity given that is has a smaller constant multiplier of n instead of n^n. Because the computational complexity of this algorithm is proportional to the number of elements in n, building a scalable variant of this algorithm is then based on finding ways to reduce the search space (Doran, 2023).

Equation (1) simultaneously computes the information function and the content priority index at each step over all items in the bank for each examinee to provide the overall value of λ_{jit}. It is already an efficient implementation given that we treat the inputs as fixed values and the values for λ_{jit} are directly realized. However, computing information for all J items in Equation (1) has some computational cost and the sorting algorithm itself has cost. Both of those issues can be addressed to reduce the search space and improve computational speed.

We begin with the notion that realizing computational speed is not a function of bigger computing environments, per se. The first objective is to analyze the complexity of the algorithm and then make use of distributed computing concepts or cloud-based structures for faster computing. Hence, we could make use of two common ideas. One could be to precompute values of information for all items at fixed increments of θ and use a precompute, store, and recall approach to

create a lookup table. This could be useful, but given that $\theta \in \mathbb{R}$, these values would never be exact for the examinee. Another option is to use parallel processing and distribute the computing of information for J items over multiple cores. This has significant potential, but the split, apply, combine problem brings with it some overhead of reassembling components and this should be evaluated (Wickham, 2011).

Because the algorithm's runtime is proportional to its inputs—specifically computing over the number of items in the test bank, reducing the number of computations over the space of J reduces overhead substantially. Here we describe an option to reduce the number of inputs for computational scaling.

A Feasible Computational Approximation with Reduced Inputs

Our proposed computational option is to compute λ_{jit} in smaller components instead of simultaneously over the entire search space. Those steps could be outlined as two steps. First compute the value of the function based initially on the content features over all items in the bank

$$\lambda^1_{jit} = \sum_{g=1}^{G} v_g w_{jg} m_{igt} + \sum_{g=1}^{G} h_g w_{jg} \alpha_{igt}.$$

Following this step, sort the items with the configurable parameter $s = \{1, 2, \ldots, S\}$ as $\lambda^1_{jit} = \{\lambda^1_{1it}, \lambda^1_{2it}, \ldots, \lambda^1_{sit}\}$ and then compute the information value only for the items in this small subset $\lambda^2_{jit} = \pi_j I'_j(\widehat{\theta}_{it}) + \lambda^1_{jit} \; \forall \; j \in \{1, \ldots, s\}$.

Update the sorted values based on λ^2_{jit} and then implement the randomesque approach to choose from the smaller set in the new space. This approximation significantly reduces the computational overhead relative to the more costly component (i.e., the information function) by reducing the space over which it is computed. Specifically, we now compute item information only for S items at each step rather than over all J items and when $S \ll J$ the cost savings is substantial. Additionally, the sort algorithm searches over a smaller space. Consequently, the idea realizes two computational cost savings. However, this approach implicitly places greater emphasis on the content features of the test than the measurement properties. There could be items with larger values for information outside the set of λ^2_{jit} that would be ignored with smaller values for s.

If the content features are not used by the algorithm for choosing items, an approximation to reduce the search space only based on the item measurement properties could be to initially compute the item information for all items in the bank. Then, after t steps, use an approximate b-matching procedure (Han, 2018) to compute the item information function only for the subset of items in some range of the provisional ability estimate. The b-matching process is equivalent to using the item information with the Rasch model, but with more general IRT models this would only be an approximation to reduce the search space.

To illustrate, consider that an initial ability estimate is obtained after administering t items, $\widehat{\theta}_{it}$. Now, because items and examinees are on the same scale, locate items in the bank where $b_j \in \{\widehat{\theta}_{itl}, \ldots, \widehat{\theta}_{itu}\}$ where the terms l and u are used to mean some lower and upper boundaries around the provisional ability. Then, compute information only for items within this space for next item selection. This technique also implicitly assumes a common a-parameter and would also help with exposure control.

Demonstrating the Model Under Variable Conditions

Some behaviors of the algorithm using a simulated item bank and blueprint are provided in this section. Because the algorithm is designed to be highly generalizable to various testing conditions, we can only provide a few reasonable permutations to explore. For this reason, we have also provided a complete testing environment for this algorithm in a web-based application for others to explore. All simulations and ideas expressed in this manuscript can be independently verified using this tool with the same item bank and blueprints. Additionally, users can test their own unique item banks, blueprint, and various constraints to further explore the ideas discussed. This brings transparency to this algorithm and its behavior for others to explore beyond the descriptions provided in this manuscript. The web-based application is available at `https://catsim.shinyapps.io/humrro_cat_engine/`.

Simulated Data

The following simulations are based on an item bank of 1000 items from a 2-parameter logistic model items with a-parameters generated from $a_j \sim U[.5, 1.5]$ and $b_j \sim U[-4, 4]$. Items are equally assigned to one of four content features and a blueprint (BP) with minimum (N_l) and maximum (N_u) values as in Equation (7) is created as provided in Table (2).

Dimension	Minimum	Maximum
Dimension 1	5	15
Dimension 2	5	20
Dimension 3	5	10
Dimension 4	5	15

Table 2: Blueprint for Simulation

The simulation generates 100 replicates at each true value of θ_q between $q = \{-3, -2.5, \ldots, 2.5, 3\}$ in increments of .5. Hence, the total number of replicates is 1,300. Using this uniform distribution permits for us to capture and report the root mean squared error and the bias at each θ_q. The observed response to each item is generated by first computing the true probability of response

$$\Pr(\theta_q) = \frac{1}{1 + \exp[a_j(\theta_q - b_j)]},$$

and then computing an observed binary response, x_{ijt}, for examinee i to item j at step t

$$x_{ijt} = \begin{cases} 1 \text{ if } \Pr(\theta_q) < u_{ijt} \\ 0 \text{ otherwise,} \end{cases}$$

where $u_{ijt} \sim U[0, 1]$. The fixed length simulations using the content blueprint are run with a test of length 50 drawing from the various blueprint dimensions. The statement "100% blueprint match" is used to mean every one of the simulated individual tests has 50 total items and has item counts within the minimum and maximum boundaries for each dimension of the blueprint. Any deviation from this expectation is considered to not meet blueprint. In all simulations, we compute the EAP as the provisional estimate and the final score is the MLE.

We do not report runtimes in our work for good reason. First, these times are local to the machine used and computational time depends on factors unique to that machine. Hence, reporting such results could be misleading, but almost certainly could not be reproducible. Second, we describe the time complexity of the various algorithms used and that metric is invariant to the machine and is the accepted manner by which algorithm time is evaluated.

Exploring Constraints on the Tuning Parameters

We begin with a few logic tests to evaluate whether the algorithm as proposed returns results that are consistent with the theory of adaptive testing in general and whether the tuning parameters can be modified to yield the results expected from the algorithm's description. Specifically, we explore four conditions and capture a few measures of adaptivity. The first measure will be the root mean squared error (RMSE) at each true value of θ_q. The second will be a ratio of test information functions (TIFs) compared to a theoretical maximum (Wyse & McBride, 2021). The theoretical maximum is determined by computing item information at each true value θ_q and taking their sum over the 50 most informative items at each θ_q. This is only a "theoretical" maximum in the sense of what this specific simulated bank could support.

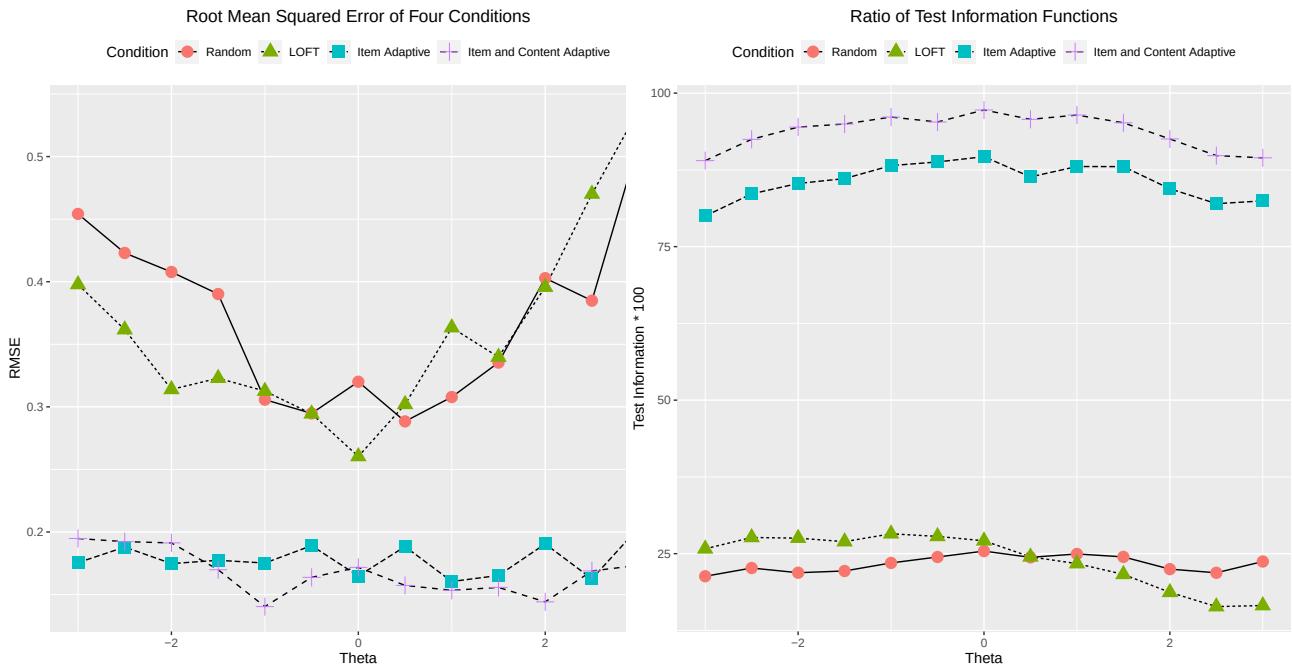
The first condition will be a test that chooses items from the bank entirely randomly. This serves as a general baseline condition that has no adaptiveness to either the examinee or the test blueprint. Hence, under this condition we would expect large RMSE relative to adaptive conditions and we expect match to blueprint at rates only equal to random chance. The second condition is a linear on the fly (LOFT) variant that adheres to a content blueprint. In this condition we would expect the algorithm to choose items based on their content features and return a blueprint match rate of 100%. Because there is no adapting to examinee ability, LOFT chooses items randomly within each content domain feature as a stratified random sampling approach, and so we expect the RMSE to be comparable in magnitude to the random condition. The third condition is purely item adaptive and ignores content constraints. In this condition we expect match to blueprint rates equal to what would occur under random chance and we anticipate very small RMSE relative to non-adaptive conditions as items are now selected based on examinee ability. Our fourth condition is both item adaptive and content adaptive. Here we expect 100% match to blueprint and, like the item adaptive condition, we expect RMSEs that are comparable in size to the item adaptive condition.

Table (3) provides a description of the constraints imposed on the algorithm tuning parameters to yield each result. Additionally, the table shows the observed blueprint match rate from each of the simulated conditions.

Tuning Parameters	Condition	Observed BP Match Rate
$\pi = 0, \upsilon_g = 0 \; \forall \; g, h_g = 0$	Random Condition (Condition 1)	0%
$\pi = 0, \upsilon_g = 1 \; \forall \; g, h_g = 1$	Linear on the Fly (LOFT) (Condition 2)	100%
$\pi = 1, \upsilon_g = 0 \; \forall \; g, h_g = 0$	Item Adaptive (Condition 3)	100%
$\pi = 1, \upsilon_g = 1 \; \forall \; g, h_g = 1$	Item and Content Adaptive (Condition 4)	100%

Table 3: Tuning Parameter Configurations for Simulation

Figure (1a) shows the RMSE for each tested condition at the increments of θ_q used in the



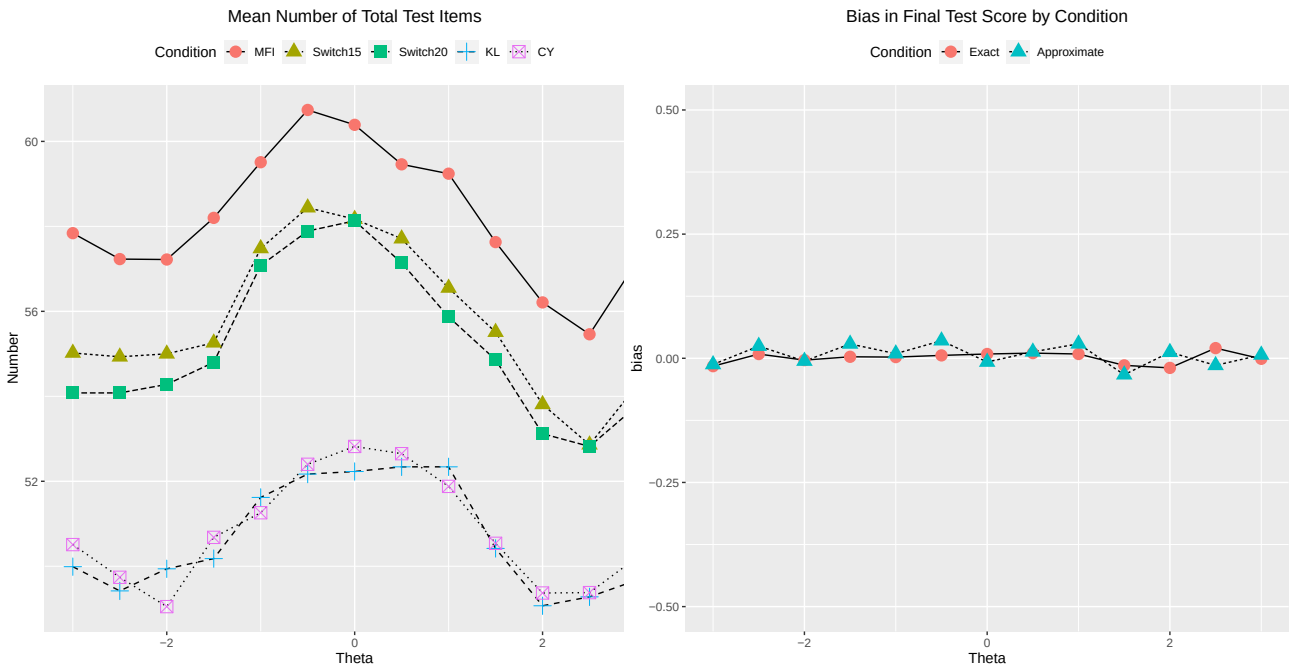
(a) *RMSE of Four Simulated Conditions.* (b) *Ratio of Four Simulated TIFs.*

Figure 1: Two Measures of Adaptivity

simulation and Figure (1b) shows the ratio of the TIFs. Results in both figures match *a priori* expectations for the RMSE and TIF ratios such that RMSEs for item adaptive conditions are smaller than non-adaptive and the TIF ratios approach 100% for the adaptive conditions. Both suggest that the item selection algorithm described in Equation (1) achieves its two primary objectives. First, it is appropriately adapting on examinee ability in Conditions (3) and (4) as observed in the smaller RMSE and larger TIF ratios. Second, small changes to the tuning parameters achieve very different testing conditions and the match to blueprint rates indicate the content priority index achieves its objectives to adhere to a content blueprint.

The Potential of Information Switching

A second question explored is whether the notion of switching information after t steps offers benefits. The idea here is that ability estimates early in the item selection process have large amounts of uncertainty. Hence, using the MFI may be less than ideal as it treats the point estimate as a well-known quantity. The average information function and KL procedure account for some of the uncertainty in the estimate. In this portion of the simulation, we implement a stopping criterion using the conditional standard errors of measurement, $\sigma(\widehat{\theta})$, such that each test terminates once $\sigma(\widehat{\theta}_{it}) < .2$. Here five conditions are explored to assess this question. Condition 1 (MFI) uses the MFI for computing item information for all items administered. Condition 2 (Switch15) uses the KL procedure for the first 15 items and then switches to the MFI. Condition 3 (Switch20) uses the KL procedure for the first 20 items and then switches to the MFI. Condition 4 (CY) computes the KL information demonstrated by Chang and Ying (1996) where we use $d = .95$ as the coverage probability. Last, Condition 5 (KL) uses the KL procedure for all items and always uses a one



(a) *Mean Total Number of Test Items by Condition.* (b) *Observed Bias by Condition.*

Figure 2: Impact of Switching and Pseudo Item Slope Impact on Bias

standard error boundary around the provisional ability estimate. This concept has been explored to assess recovery on the estimated latent trait (Chang & Ying, 1996), but here we explore its impact on the test length.

Both Conditions 1 and 5 intend to provide upper and lower bound limits in terms of what we might expect in terms of total test length, given limits of this specific item bank. If Conditions 2 or 3 offer any benefit, we expect to observe that the test would reach its termination criteria with a fewer number of test items relative to Condition 1 and approach the test length observed in Condition 5. Figure (2a) shows the mean total number of test items for each of the conditions over values of θ_q. Here we observe that over the entire range of ability, Condition 2 reaches the termination criteria with fewer test items relative to Condition 1, typically between 2 to 3 fewer items, effectively reducing test length by about 4% relative to Condition 1. Additionally, we observe Condition 3 even further reduces test length beyond Condition 2 by about one more test item. It is useful to note that the CY condition seems on par with the KL procedure even though it tends to rely on the observed score more as the test increases in length.

These simulations suggest the idea of switching how information is computed helps to reduce test length by capturing the uncertainty in the ability estimate early in the test. We almost certainly do not want to compute the KL for all items as this is computationally expensive relative to the MFI, although doing so seems to provide the shortest test length. Hence, by choosing a value of t where switching occurs, we approach the lower bound result of Condition 5 while reducing computational overhead as we switch to the MFI. This is a promising result for test developers wishing to even further reduce test length when termination depends on the standard errors of measurement.

Exposure Control with Approximate Item Slopes

One of the decision variables described is a proposed constraint on the a-parameters to use an approximation in lieu of the actual a-parameter. The idea is that using a_j will cause for the algorithm to prefer certain items and lead to overexposure. Instead, we propose to set $a_j = 1$ for purposes of choosing items, but to use a_j when performing the actual scoring, both provisional and final scores.

This section explores the overexposure by running the simulation with two conditions. In Condition 1, the actual a-parameter is used and we measure the proportion of the bank that is used across all replicates in the simulation. In Condition 2, the approximate a-parameter is used for item selection and the proportion of the bank is again measured. Additionally, the final test score is captured as the maximum likelihood estimate under both conditions and the bias between both conditions is considered.

Figure (2b) illustrates the bias at all true values of θ_q between both conditions. This figure shows that both conditions have no real differences in bias indicating the approximate a-parameter does not tend to compromise the final test score in a negative way. Additionally, and importantly, the approximate a-parameter tends to use a larger proportion of the item bank. In Condition 1, the simulation shows the exact a-parameter results in usage of 83% of the bank and the approximate a-parameter results in usage of 96.4% of the bank.

Item Enemy Administration

Item enemies are a set of items that cannot be administered to an examinee on the same test form. Constructing fixed forms to avoid item enemies is straightforward. It may be somewhat more challenging in adaptive scenarios. Here we demonstrate that the proposed algorithm can handle item enemy detection as a feature constraint without needing to created a bifurcated item bank. While we demonstrate this using item enemies, really the intent is to show that the feature constraints can be any arbitrary grouping structure defined by test developers. The potential here is virtually unlimited.

The example is constructed by forming a cluster of ten items that are considered to be enemies with each other. Practically speaking, this means if one of these ten items is administered on a test form, then none of the remaining nine can appear on the same test form because they are enemies. We do this by forming a binary indicator in the item metadata associated with each item in the bank denoting a 1 if the items are in the enemy cluster group and 0 otherwise. Second, we modify the test blueprint to have a new feature constraint as shown in Table (4).

These new minimums and maximums establish a feature constraint that are simply passed to the selection algorithm. The min/max of 0 to 1 for Group 1 instructs the algorithm to choose no more than 1 item from within this cluster, but it can optionally choose none. In other words, if the algorithm chooses an item from within Group 1, it cannot administer any more items from within this same cluster, thus avoiding multiple items from within this group from being administered on the same test form. If it chooses 0 items from within this cluster, then all 50 items would come from Group 2, which allows items from within that group to be administered on the same test form.

We again run the simulation with 1,300 total replicates using this new feature constraint. The simulation results in a 100% match to blueprint over all replicates and also no test form contains more than one item from Group 1. Figure (3) shows the proportion of simulated forms that contain

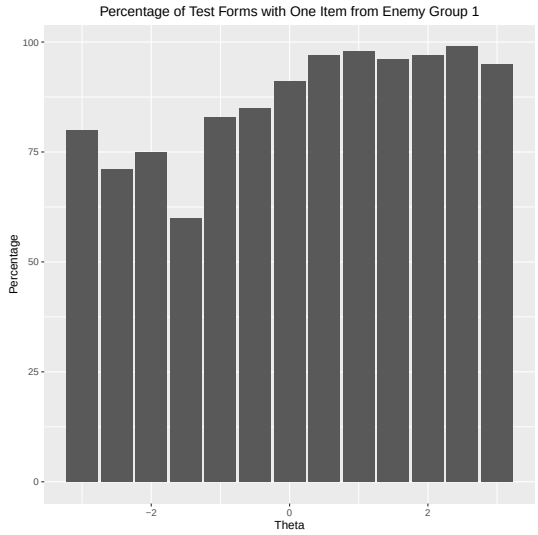


Figure 3: *Percentage of Test Forms with One Enemy Item*

Dimension	Minimum	Maximum
Dimension 1	5	15
Dimension 2	5	20
Dimension 3	5	10
Dimension 4	5	15
Group 1	0	1
Group 2	49	50

Table 4: Blueprint for Item Enemy Simulation

one item from the enemy group, which ranges from 60% to 99%. If the minimum of the blueprint were set to 1 for Group 1, then the algorithm would be forced to always choose at least one item from this group. Setting it to 0 offers the flexibility only to choose an item from this group if they are best according to other constraint criteria.

A Note on Items Nested in Common Groups

The modified blueprint used for the enemy example provides greater insight into how the algorithm can behave in complex ways. The term "Enemy Group" is used only as a descriptor for the example. However, Group 1 and Group 2 can in principle be any conceivable grouping of items desired. These items could be administered in a way that resembles what is demonstrated for the enemy group example with adjustable min/max values. The Group 1 min/max in the enemy example is set to 0 and 1, respectively, to ensure enemies were not administered together. However, had we set the maximum to some number larger than 1, it would cause for multiple items from within this group to be administered together. Hence, by extension, this concept applies to the administration of field test items. These items would form their own grouping in the item metadata, the min/max values would be configured, and then set $\pi_j = 0$ for any item in this group since they have no IRT statistics and information cannot be computed.

Discussion

This manuscript provides thorough details of a generalizable item adaptive algorithm that can be used to fit an array of custom test designs. The results of the simulations indicate its behavior is consistent with the stated expectations demonstrating it can simultaneously adapt on ability and feature constraints of various forms. For example, the feature index can prioritize items from any grouping structure such as content domains, item types, field test items, item enemies, cognitive

demands, and so on. The primary benefit of this idea is that the algorithm can be used as a means to an end allowing test developers to formally define the construct and the algorithm simply implements their design. This differs from other CAT implementations where test banks might be designed to exclude item enemies, or a separate field test algorithm might be used to seed experimental items, or where content balancing is achieved by administering adaptive tests separately by content dimension. It is only possible to test a few ways in which the algorithm can operate and so the manuscript is complimented with a web-based application that implements this algorithm so that interested users can test and explore the potential it has for operational or research-based scenarios.

Second, we offer a thorough and complete treatment of CAT so that an expert psychometric and software development team can collaborate to build an instance of an operational CAT using the information contained entirely within this manuscript. Specifically, Equation (2) could be implemented directly or the alternative variant that reduces the search space can be used. The goal is to specifically propose a computational framework that can be deployed for operational testing such that item refresh latency rates on the screen are not bogged down by complex computational overhead. Not only does latency disrupt an examinee's experience, but the vendor providing the test platform may also be subject to financial damages if they occur, what are commonly termed "liquidated damages".

While we purport that the algorithm is highly configurable and useful for many different testing situations, we note that these claims are based on our experiences and the typical uses cited in the literature. The world of item types and optimization approaches with machine learning technologies is rapidly changing and our claims are based on our understanding of item types commonly used as of this writing. Additionally, the ideas explored here are based on unidimensional IRT models. While potentially a limit, multidimensional IRT to date has not become common in large scale assessment and unidimensional IRT models continue to be the most common in real world applications.

Going forward, a few ideas can be further explored. One idea to consider is whether optimization can be performed with complexity better than $\mathcal{O}(n \log n)$. This is not a current bottleneck, but systems deploying millions of tests could realize faster item refresh speeds if this could be improved. Of course, one might consider throwing these algorithms into larger computational environments such as big cloud-based structures as these can be used to rapidly compute things. This is true, but comes with added real monetary cost and so we recommend that the first goal be to find the more optimal algorithm and only then pass the algorithm into a bigger computing environment (Doran, 2023). A second idea to explore is if other ways to compute item information could be used in the information switching idea to even further reduce test length. The idea we demonstrated has promise, and with concerns over testing time being generally too long, this could help reduce that concern. In particular, the newer notion of through year designs in education could benefit from shorter, more precise tests, to provide diagnostic information at the examinee level. In fact, the State of Virginia in the U.S. passed legislation in 2021 requiring through year designs with a CAT (Virginia, 2021).

Perhaps a lofty third idea could be to explore if CAT algorithms can be combined with automated item generation (AIG), item forecasting, and personalized learning platforms to be a complete environment for tailored assessment and instruction. Imagine an examinee sitting with a device that automatically generates a test item with forecasted item parameters, administers the item, collects and scores the examinee response, uses AIG to develop a new item based on the

examinee's response to the prior item, and iterates this process until completion of the test. Then, once the test is complete, offers up tailored, personalized learning content based on the way in which the examinee performed on the test. Future integrations of large language models within psychometric platforms has tremendous promise.

Appendix A: Some Computational Statistics

We can approximate the integrals expressed in Equation (4) and Equation (5) using Gauss-Legendre quadrature as follows. Let θ_q be the node at quadrature point q and w_q is its corresponding weight.

$$I_j(\widetilde{\theta}_{it}) \simeq \left(\frac{u-l}{2}\right)\sum_{q=1}^{Q} D^2 a_j^2 \left[\frac{\left(\Pr_j(\theta_q)-c_j\right)^2}{\left(1-c_j\right)^2}\right]\left[\frac{1-\Pr_j(\theta_q)}{\Pr_j(\theta_q)}\right] w_q$$

$$KL_j\left(\widehat{\theta}_{it}\right) \simeq \left(\frac{u-l}{2}\right)\sum_{q=1}^{Q}\left(\sum_{l=0}^{1}\log\left[\frac{\Pr\left(x_j=l|\theta_q\right)}{\Pr\left(x_j=l\middle|\widehat{\theta}_{it}\right)}\right]\Pr\left(x_j=l|\theta_q\right)w_q\right)$$

For both integrals, the limits of integration fall outside the $[-1,1]$ space, so the change of variable is applied as $\theta_q = .5(u-l)(f_q+1)+l$ where l and u are the lower and upper bounds for integration, respectively, and f_q is the original node at point q.

The simulations capture the bias and mean squared error as $bias(\widehat{\theta}) = N^{-1}\sum_{k=1}^{K}\left(\widehat{\theta}_{k,q}-\theta_q\right)$ and $mse(\widehat{\theta}) = N^{-1}\sum_{k=1}^{K}\left(\widehat{\theta}_{k,q}-\theta_q\right)^2$ where $\widehat{\theta}_{k,q}$ is the estimated value at replicate k.

Appendix B: Simplified Computational Example

The R code in this appendix provides a simple, minimal example of Equation (2) for item selection. Here we compute only the item information and content priority index and assume $h_g = 0$ just for this simple example. For this illustration, assume there is a blueprint with two features (Dimension 1 and Dimension 2), that the examinee is at step $t = 11$ and that six items have been administered in Dimension 1 and four items have been administered in Dimension 2 and the current provisional ability estimate is $\widehat{\theta}_{it} = 0$.

```
### Item Information Function as MFI
itemInfo <- function(theta, a, b, c, D = 1.7){
  p <- c + (1 - c) / (1 + exp(-D * a * (theta - b)))
  q <- 1 - p
  result <- (D^2*a^2*(q/p * ((p - c)/(1-c))^2))
  result
}

### Generate five item parameters for 2PL
aParm <- runif(5, .5, 1.5); bParm <- runif(5, -4, 4); cParm <- 0

### Compute item information for theta = 0
info <- itemInfo(0, aParm, bParm, cParm)

### Assume minimum = 7, maximum = 10 for Dimension 1 of blueprint
N1.u <- 10; N1.l <- 7
```

```
### Assume minimum = 1, maximum = 5 for Dimension 2 of blueprint
N2.u <- 5; N2.l <- 1

### Assume 6 items administered so far in Dimension 1
### Compute content priority index for Dimension 1
m1 <- (N1.u - 6)/(N1.u - N1.l)

### Assume 4 items administered so far in Dimension 2
### Compute content priority index for Dimension 2
m2 <- (N2.u - 4)/(N2.u - N2.l)
mVector <- c(m1,m2)

### Do min/max rescaling
theInfos <- (info - min(info))/(max(info) - min(info))
mVector <- (mVector - min(mVector))/(max(mVector) - min(mVector))

### Assume items 1, 3, 5 are linked to Dimension 1 and items 2 and 4
    are linked to Dimension 2
### Create W matrix as in Equation (2)
W <- matrix(c(1,0,1,0,1,0,1,0,1,0),5,2)

### Use inputs to compute lambda as shown in Equation (2)
### Assume tuning weights for information and content are all 1
info.weight = 1; bp.weights <- c(1,1); lambdas <- as.vector(info.
   weight * theInfos + W %*% (mVector * bp.weights))

### Sort the vector. First item is "best" for selection.
sort(lambdas, decreasing = TRUE)
```

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