1. $\sum_{i=1}^{n}(y_i-\bar{y})^2=\sum_{i=1}^{n}(y_i-\hat{y}_i+\hat{y}_i-\bar{y})^2$

$=\sum_{i=1}^{n}(y_i-\hat{y}_i)^2+\sum_{i=1}^{n}(\hat{y}_i-\bar{y})^2+2\sum_{i=1}^{n}(y_i-\hat{y}_i)(\hat{y}_i-\bar{y})$

$\because \hat{y}_i=\hat{w}x_i+\hat{b}=\frac{S_{xy}}{S_{xx}}x_i+\bar{y}-\frac{S_{xy}}{S_{xx}}\bar{x}$.（其中，$S_{xx}=\sum_{i=1}^{n}(x_i-\bar{x})^2$，$S_{xy}=\sum_{i=1}^{n}(x_i-\bar{x})(y_i-\bar{y})$）

$\therefore \sum_{i=1}^{n}(y_i-\hat{y}_i)(\hat{y}-\bar{y})$

$=\sum_{i=1}^{n}(y_i-\frac{S_{xy}}{S_{xx}}x_i-\bar{y}+\frac{S_{xy}}{S_{xx}}\bar{x})(x_i-\bar{x})\frac{S_{xy}}{S_{xx}}$.

$=\sum_{i=1}^{n}[(y_i-\bar{y})(x_i-\bar{x})-(x_i-\bar{x})^2\cdot\frac{S_{xy}}{S_{xx}}]\cdot\frac{S_{xy}}{S_{xx}}$.

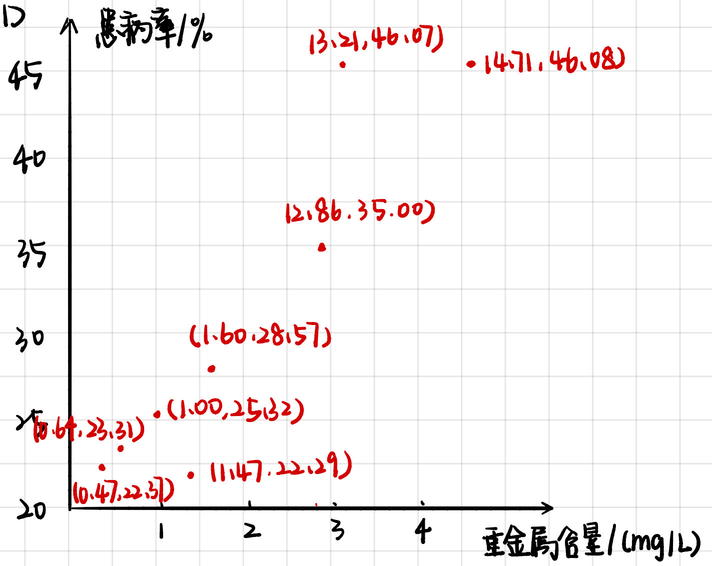
$=[\sum_{i=1}^{n}(y_i-\bar{y})(x_i-\bar{x})-\sum_{i=1}^{n}(x_i-\bar{x})^2\cdot\frac{S_{xy}}{S_{xx}}]\frac{S_{xy}}{S_{xx}}$

$=[S_{xy}-\frac{S_{xy}}{S_{xx}}\cdot S_{xx}]\cdot\frac{S_{xy}}{S_{xx}}$

$=0$

故 $\sum_{i=1}^{n}(y_i-\bar{y})^2=\sum_{i=1}^{n}(y_i-\hat{y}_i)^2+\sum_{i=1}^{n}(\hat{y}_i-\bar{y})^2$

2.(1)



(2) $\bar{x}=\frac{1}{8}\sum_{i=1}^{8}x_i=1.9950$

$\bar{y}=\frac{1}{8}\sum_{i=1}^{8}y_i=31.1263$

$\overline{xy}=\frac{1}{8}\sum_{i=1}^{8}x_iy_i=74.2815$

$\overline{x^2}=\frac{1}{8}\sum_{i=1}^{8}x_i^2=5.8774$

$\hat{w}=\frac{\overline{xy}-\bar{x}\cdot\bar{y}}{\overline{x^2}-(\bar{x})^2}=6.4218$

$\hat{b}=\bar{y}-\hat{w}\bar{x}=18.3147$

故 $\hat{y}=\hat{w}x+\hat{b}=6.4218x+18.3147$

$r^2=\frac{\sum_{i=1}^{8}(\hat{y}_i-\bar{y})^2}{\sum_{i=1}^{8}(y_i-\bar{y})^2}=0.8718$.

3、

(1) 不适用.

因为最小二乘法中对距离的定义是 y 和 $\hat{y}$ 的竖直距离，因此前提是观测值 x 准确.

(2) $l_i \triangleq (x_i, y_i)^T$ 到 $(\vec{x_0}, \vec{n})$ 的距离

$$= |(\vec{x_i}-\vec{x_0})^T\vec{n}|, \quad i=1..n$$

$$\text{则} L = \sum_{i=1}^{n} l_i^2 = \sum_{i=1}^{n}[(\vec{x_i}-\vec{x_0})^T\vec{n}]^2$$

(3) $$\frac{\partial L}{\partial \vec{x_0}} = 2\sum_{i=1}^{n}[\vec{n}^T(\vec{x_i}-\vec{x_0})](-\vec{n})$$

$$= 2\vec{n}^T(\sum_{i=1}^{n}\vec{x_i}-n\vec{x_0})(-\vec{n}) \triangleq 0.$$

$$\Rightarrow \vec{x_0} = \frac{1}{n}\vec{x_i}$$

$$L = \sum_{i=1}^{n}[\vec{n}^T(\vec{x_i}-\vec{x_0})(\vec{x_i}-\vec{x_0})^T\vec{n}]$$

$$= \vec{n}^T\sum_{i=1}^{n}(\vec{x_i}-\vec{x_0})(\vec{x_i}-\vec{x_0})^T\vec{n}$$

$$\triangleq \vec{n}^T A\vec{n}$$

由 $A^T=A$ 知 A为实对称矩阵.

故存在正交阵Q，使得 $(Q^TQ=QQ^T=I)$

$$Q^TAQ = \begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix} \triangleq \Lambda$$

则 $A = Q\Lambda Q^T$

则 $L = \vec{n}^TQ\Lambda Q^T\vec{n}$

记 $\vec{n}^TQ=[n_1\ n_2]$ 则 $L = [n_1\ n_2]\begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}\begin{bmatrix}n_1\\ n_2\end{bmatrix}$

$$= \lambda_1 n_1^2 + \lambda_2 n_2^2$$

又 $\because n_1^2+n_2^2 = \vec{n}^TQ\cdot Q^T\vec{n} = \vec{n}^T\cdot\vec{n} = ||\vec{n}||^2 = 1$

$$\therefore L = \lambda_1 n_1^2 + \lambda_2(1-n_1^2)$$

$$= (\lambda_1-\lambda_2)n_1^2 + \lambda_2.$$

① $\lambda_1 > \lambda_2$.

则 $L_{min} = \lambda_2$. $n_1=0, n_2=\pm 1$

② $\lambda_1 \le \lambda_2$

则 $L_{min} = \lambda_1$, $n_1=\pm 1, n_2=0$.

其中，由 $Q^T\vec{n} = \begin{bmatrix}n_1\\ n_2\end{bmatrix}$ 可得 $\vec{n} = Q\begin{bmatrix}n_1\\ n_2\end{bmatrix}$