1. $I=\int_{1}^{1.6}\frac{x}{x^2-4}dx \qquad f(x)=\frac{x}{x^2-4}$

$$\begin{tikzcd}
T_0^{(0)} \arrow[dr] & & & \\
T_0^{(1)} \arrow[r] & T_1^{(0)} \arrow[ddr] & & \\
T_0^{(2)} \arrow[dr] & & & \\
T_0^{(3)} \arrow[r] \arrow[dr] & T_1^{(1)} \arrow[r] \arrow[dr] & T_2^{(0)} \arrow[dr] & \\
T_0^{(4)} \arrow[r] & T_1^{(2)} \arrow[r] & T_2^{(1)} \arrow[r] & T_3^{(0)}
\end{tikzcd}$$

根据 $T_m^{(k)}=\frac{4^m}{4^m-1}T_{m-1}^{(k+1)}-\frac{1}{4^m-1}T_{m-1}^{(k)}$

$$T_3^{(0)}=\frac{4^3}{4^3-1}T_2^{(1)}-\frac{1}{4^3-1}T_2^{(0)}$$
$$=\frac{64}{63}T_2^{(1)}-\frac{1}{63}T_2^{(0)}$$

从 $T_0^{(0)}$ 开始外推：

$$T_0^{(0)}=\frac{1}{2}(f(1.6)+f(1))(1.6-1)=-0.433333$$
$$T_0^{(1)}=\left(\frac{1}{4}f(1)+\frac{1}{2}f\left(\frac{1+1.6}{2}\right)+\frac{1}{4}f(1.6)\right)(1.6-1)=-0.385498$$
$$T_0^{(2)}=\left(\frac{1}{8}f(1)+\cdots+\frac{1}{8}f(1.6)\right)(1.6-1)=-0.371799$$
$$T_0^{(3)}=\left(\frac{1}{16}\cdots\right)(1.6-1)=-0.368202$$
$$T_0^{(4)}=\left(\frac{1}{32}\cdots\right)(1.6-1)=-0.367290$$

根据递推公式得 $T_3^{(0)}=-0.366986$

2、 $I=\int_0^1 \frac{2}{\sqrt[3]{x}}dx = \int_0^1 2x^{-\frac{1}{3}}dx = 3x^{\frac{2}{3}}\Big|_0^1 = 3$

n=3时，使用复化辛普生公式：

$$I=\sum_{k=0}^{2}\int_{x_k}^{x_{k+1}} f(x)dx = \frac{\frac{b-a}{3}}{6}\Big[f(a)+2\sum_{k=1}^{2}f(x_k)+4\sum_{k=0}^{2}f(x_k+\frac{h}{2})+f(b)\Big]$$

$$h=\frac{b-a}{3}=\frac{1}{3}$$

其中 $a=0$. $b=1$ $x_1=a+h=\frac{1}{3}$ $x_2=\frac{2}{3}$ $x_0=0$ $x_3=1$

可看出 x_0处为奇点，但已知：

$$\int_0^1 \frac{\cos 2x}{\sqrt[3]{x}}dx < \int_0^1 \frac{2}{\sqrt[3]{x}}dx$$

令 $\frac{\cos 2x}{\sqrt[3]{x}} = \frac{1}{\sqrt[3]{x}} - \frac{1-\cos 2x}{\sqrt[3]{x}}$

说明奇点不会造成影响，~~令 $f(x_0)=f(x_1)$~~

算得：$\lim\limits_{x\to 0}\left(\frac{1-\cos 2x}{\sqrt[3]{x}}\right)\approx 0$ 即可求奇点

$$\Rightarrow \int_0^1 f(x)dx = \int_0^1 \frac{1}{\sqrt[3]{x}}dx - \int_0^1 \frac{1-\cos 2x}{\sqrt[3]{x}}dx$$

$$= 1.5-\frac{1}{18}\Big[f(0)+2\big(f(\tfrac{1}{3})+f(\tfrac{2}{3})\big)+4\big(f(\tfrac{1}{6})+f(\tfrac{1}{2})+f(\tfrac{5}{6})\big)+f(1)\Big]$$

$$=0.880063$$

3、① 使用高斯积分：

3阶精度：变换自变量区间：

$$x=\frac{t+1}{2} \quad t\in[-1,1]$$

$$\int_0^1 f(x)dx = \int_{-1}^1 f\left(\frac{t+1}{2}\right)d\frac{t+1}{2} = \int_{-1}^1 \frac{1}{2}f\left(\frac{t+1}{2}\right)dt$$

$$=\frac{1}{2}\int_{-1}^1 F(t)dt$$

$t_0=-\frac{1}{\sqrt{3}}$ $t_1=\frac{1}{\sqrt{3}}$

$$=\frac{1}{2}\left(A_0F(t_0)+A_1F(t_1)\right)$$

$A_0=\int_{-1}^1 \frac{t-t_1}{t_0-t_1}dt=1$ $A_1=1$

$\therefore \int_0^1 f(x)dx = \frac{1}{2}F(t_0) + \frac{1}{2}F(t_1)$

$= \frac{1}{2}f(x_0) + \frac{1}{2}f(x_1)$

$x_0 = \frac{t_0+1}{2} = \frac{1-\frac{1}{\sqrt{3}}}{2}$

$x_1 = \frac{1+\frac{1}{\sqrt{3}}}{2}$

$a = \frac{1}{2}$

② ~~使用高斯积分：~~ ~~对 $f(x)$ 进行泰勒展开：~~ ~~在x=0处~~

$\cancel{f(x) = f(0) + f'(x)(x-0) + \frac{f''(x)}{2}(x-0)^2 + O(x^3)}$

$\cancel{= f(0) + f'(x)x + \frac{f''(x)}{2}(x-0)^2 + O(x^3)}$

$\therefore \cancel{\int_0^h f(x)dx = \int_0^h (f(0) + f'(x)x + O(x^2))dx}$

$\cancel{=}$

根据代数精度定义：$\int_0^h x\,dx = \frac{h^2}{2} \approx \frac{h^2}{2} + 0$

$\int_0^h x^2dx = \frac{h^3}{3} \approx \frac{h^3}{2} - 2ah^3 \Rightarrow a = \frac{1}{12}$ 时精度最高为3

$\int_0^h x^3 dx = \frac{h^4}{4} \approx \frac{h^4}{2} + ah^2(-3h^2)$

$\cancel{\varphi = \frac{\pi}{2}(t+1)}$

~~4.~~ $\cancel{\int_0^{\pi} \varphi^2 \sin\varphi\, d\varphi = \int_{-1}^{1} \frac{\pi^2}{4}(t+1)^2 \sin\frac{\pi}{2}(t+1)\frac{\pi}{2}dt}$

$\cancel{= \int_{-1}^{1} \frac{\pi^3}{8}(t+1)^2 \sin\frac{\pi}{2}(t+1)dt}$

$\cancel{= \left(\int_{-1}^{-\frac{1}{2}} \frac{\pi^3}{8}(t+1)^2 \sin(t+1)\frac{\pi}{2}dt\right.}$

$\cancel{\left. + \int_{-\frac{1}{2}}^{0} + \int_0^{\frac{1}{2}} + \int_{\frac{1}{2}}^{1}\right)}$

4. $\int_0^{\pi} \varphi^2 \sin\varphi \, d\varphi = \left(\int_0^{\frac{\pi}{4}} + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} + \int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} + \int_{\frac{3\pi}{4}}^{\pi}\right) \varphi^2 \sin\varphi \, d\varphi$

$$= \int_{-1}^{1} \left(\frac{\pi}{8} + \frac{\pi}{8}t\right)^2 \sin\left(\frac{\pi}{8} + \frac{\pi}{8}t\right) \frac{\pi}{8} dt$$

$$+ \int_{-1}^{1} \left(\frac{3}{8}\pi + \frac{\pi}{8}t\right)^2 \sin\left(\frac{3}{8}\pi + \frac{\pi}{8}t\right)^2 \frac{\pi}{8} dt$$

$$+ \int_{-1}^{1} \left(\frac{5}{8}\pi + \frac{\pi}{8}t\right)^2 \sin\left(\frac{5}{8}t + \frac{\pi}{8}t\right) \frac{\pi}{8} dt$$

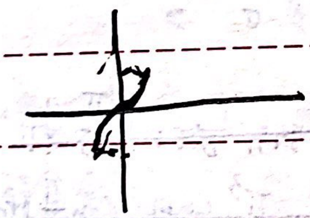
$$+ \int_{-1}^{1} \left(\frac{7}{8}\pi + \frac{\pi}{8}t\right)^2 \sin\left(\frac{7}{8}\pi + \frac{\pi}{8}t\right) \frac{\pi}{8} dt$$

$$= f_0\left(-\frac{1}{\sqrt{3}}\right) + f_0\left(\frac{1}{\sqrt{3}}\right) + f_1\left(-\frac{1}{\sqrt{3}}\right) + f_1\left(\frac{1}{\sqrt{3}}\right) + f_2\left(-\frac{1}{\sqrt{3}}\right) + f_2\left(\frac{1}{\sqrt{3}}\right) + f_3\left(-\frac{1}{\sqrt{3}}\right) + f_3\left(\frac{1}{\sqrt{3}}\right)$$

$$= 0.00178364 + 0.0873043 + 0.288861 + 0.763098 + 1.166644 + 1.532697 + 1.450698 + 0.578750$$

$$= 5.869836$$

5.



~~$y = \tan x$ 令 $x = t$~~

~~$y = \tan t$~~

~~$t \in [-\frac{\pi}{4}, \frac{\pi}{4}]$~~

$$L = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sqrt{1 + ((\tan x)')^2} \, dx = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sqrt{1 + \frac{1}{(\cos x)^4}} \, dx$$

使用复化梯形公式时误差为

$$R = -\frac{\pi}{24} h^2 f''(\eta)$$

又 $f''(x) = \left(\sqrt{1 + \frac{1}{(\cos x)^4}}\right)'' =$

$$2\sin x^2 (\cos x^4 + 1)^{-\frac{3}{2}} + (\cos x^4 + 1)^{-\frac{1}{2}} (6\cos x^{-4} \sin x^2 + 2\cos x^{-2})$$

$$\therefore R = -\frac{\pi}{24}h^2 \left\{ 2\sin\eta^2 (\cos\eta^4+1)^{-\frac{3}{2}} + (\cos\eta^4+1)^{-\frac{1}{2}} (6\cos\eta^{-4}\sin\eta^2 + 2\cos\eta^{-2}) \right.$$

其中 $\eta \in (-\frac{\pi}{4}, \frac{\pi}{4})$