埃尔米特插值唯一性证明

- 证明其唯一性。反证法：

假设$H_{2n+1}(x_i)=y_i, H'_{2n+1}(x_i)=y_i'$.假设还存在一个2n+1次多项式$G_{2n+1}(x)$满足$G_{2n+1}(x_i)=y_i, G'_{2n+1}(x_i)=y_i'$。

则$H(x_i)-G(x_i)=0,\forall i=0,\cdots n$。所以，由罗尔定理，存在$\xi_i\in(x_{i-1},x_i)$，满足$H'(\xi_i)-G'(\xi_i)=0$。所以$H'-G'$共有n个这样的零点$\xi_1\in(x_0,x_1),\cdots,\xi_n\in(x_{n-1},x_n)$。

同时，$H'(x_i)=G'(x_i)=y_i',\forall i=0,\cdots n$。所以$x_0,\cdots,x_n$也是$H'(x)-G'(x)$的零点。所以，$H'(x)-G'(x)$共有2n+1个零点。

$H'(x)-G'(x)$是2n次多项式，有2n+1个零点$\Leftrightarrow H'(x)-G'(x)\equiv 0, \Leftrightarrow H-G=const$。又因$H(x_i)-G(x_i)=0, \therefore H\equiv G$

参考牛顿法设计埃尔米特插值

- 节点逐个增加：$(x_0,\dots,x_{n-1})\to(x_0,\dots,x_n)$

$$H_{2n-1}(x)\to H_{2n+1}(x)=H_{2n-1}(x)+g(x)$$

g(x)为2n+1阶多项式，且在$x_0,\dots,x_{n-1}$点，

$$H_{2n+1}(x_i)=H_{2n-1}(x_i)\to g(x_i)=0$$

$$H'_{2n+1}(x_i)=H'_{2n-1}(x_i)\to g'(x_i)=0$$

所以，$x_0,\dots,x_{n-1}$是g(x)的二阶零点。

$$g(x)=(ax+b)\prod_{i=0}^{n-1}(x-x_i)^2=(Ax+B)l_n^2(x)$$

求(a,b)/(A,B)满足

$$H_{2n+1}(x_n)=H_{2n-1}(x_n)+g(x_n)=f(x_n)$$

$$H'_{2n+1}(x_n)=H'_{2n-1}(x_n)+g'(x_n)=f'(x_n)$$

- 小m法求解三次样条插值

假定$S_j'(x_j)=m_j$，$S_j'(x_{j+1})=m_{j+1}$，j=0,…,n-1。在每个子区间上做三次Hermite插值，有

$$S_j(x)=y_j\alpha_j(x)+y_{j+1}\alpha_{j+1}(x)+m_j\beta_j(x)+m_{j+1}\beta_{j+1}(x), x\in[x_j,x_{j+1}]$$

$$S_j''(x)=\frac{6x-2x_j-4x_{j+1}}{h_j^2}m_j+\frac{6x-4x_j-2x_{j+1}}{h_j^2}m_{j+1}+\frac{6(x_j+x_{j+1}-2x)}{h_j^3}(y_{j+1}-y_j)$$

$$S_j''(x_j+0)=-\frac{4}{h_j}m_j-\frac{2}{h_j}m_{j+1}+\frac{6}{h_j^2}(y_{j+1}-y_j)$$

$$S_{j-1}''(x_j-0)=\frac{2}{h_{j-1}}m_{j-1}+\frac{4}{h_{j-1}}m_j-\frac{6}{h_{j-1}^2}(y_j-y_{j-1})$$

$$S_j''(x_j+0)=S_{j-1}''(x_j-0)\Leftrightarrow$$

$$\lambda_j m_{j-1}+2m_j+\mu_j m_{j+1}=d_j, j=1,\cdots,n-1$$

$$\lambda_j=h_j(h_{j-1}+j_j)^{-1}, \mu_j=1-\lambda_j,$$

$$d_j=2(\lambda_j f[x_{j-1},x_j]+\mu_j f[x_j,x_{j+1}]).$$

加两个边界条件，n+1个未知量{mj}，n+1个方程