

What is Arrow's Impossibility Theorem?

Jesse He

OSU What Is?

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Definitions

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A *total preorder* or *preference order* \leq on a set X is a binary relation such that for all $x, y, z \in X$,

1. If $x \leq y$ and $y \leq z$ then $x \leq z$
2. $x \leq y$ or $y \leq x$ (and hence $x \leq x$)

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Then denote by $\mathcal{L}(A)$ the set of preference orders on A . Each voter $v \in V$ has an associated preference ordering \leq_v , and a *preference profile* (or *situation*) $P = (\leq_1, \dots, \leq_N) \in \mathcal{L}(A)^V$ is an N -tuple of preference orders.

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A *social welfare function* is a function $F : \mathcal{L}(A)^V \rightarrow \mathcal{L}(A)$. We denote the resulting ordering by $\leq_{F(P)}$, or just \leq .

More definitions

Let $a_i, a_j \in A$.

1. Unanimity (Pareto efficiency)

If $a_i <_v a_j$ (that is, $a_i \leq_v a_j$ and $a_j \not\leq_v a_i$) for each $v \in V$, then $a_i \leq_F a_j$.

2. Independence of irrelevant alternatives (IIA)

If $P = (\leq_v), P' = (\leq'_v) \in \mathcal{L}(A)^V$ such that for all $v \in V$, $a_i \leq_v a_j$ iff $a_i \leq'_v a_j$, then $a_i \leq_{F(P)} a_j$ iff $a_i \leq_{F(P')} a_j$.

3. Non-dictatorship

There is no $v \in V$ such that for all $P \in \mathcal{L}(A)^V$, $a_i <_v a_j$ implies $a_i <_{F(P)} a_j$.

The statement

Theorem (Arrow's Impossibility Theorem)

There is no social welfare function that satisfies all three conditions.

Proof

Suppose F satisfies unanimity and IIA.

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By unanimity, we begin with $a_i > a_j$ and end with $a_i < a_j$. Denote the first voter whose swap changes the social preference ordering by v_{ij} , the (i, j) -th *pivotal voter*. By IIA, this definition does not depend on P .

Proof

Let $P' = (\leq'_v) \in \mathcal{L}(A)^V$ such that $a_i <'_v a_k <'_v a_j$ for $v = 1, 2, \dots, v_{ij} - 1$ and $a_k <'_v a_j <'_v a_i$ for $v = v_{ij}, v_{ij} + 1, \dots, N$.

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Then $a_k <' a_j <' a_i$ by the definition of v_{ij} and by unanimity.

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Let $P'' = (\leq''_v) \in \mathcal{L}(A)^V$ such that $a_i <'' a_j$ and $a_i <'' a_k$ for $v = 1, \dots, v_{ij} - 1$, $a_k <_{v_{ij}} a_i <_{v_{ij}} a_j$, and $a_j <'' a_i$ and $a_k <'' a_i$ for $v = v_{ij} + 1, \dots, N$.

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Let $P' = (\leq'_v) \in \mathcal{L}(A)^V$ such that $a_i <'_v a_k <'_v a_j$ for $v = 1, 2, \dots, v_{ij} - 1$ and $a_k <'_v a_j <'_v a_i$ for $v = v_{ij}, v_{ij} + 1, \dots, N$. Then $a_k <' a_j <' a_i$ by the definition of v_{ij} and by unanimity.

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Then by IIA we see that $a_k <_{v_{ij}} a_j$ implies $a_k < a_j$ for all $i \neq j \neq k$.

Proof

	1	...	$v_{ij} - 1$	v_{ij}	$v_{ij} + 1$...	N
P'	a_j	...	a_j	a_i	a_i	...	a_i
	a_k	...	a_k	a_j	a_j	...	a_j
	a_i	...	a_i	a_k	a_k	...	a_k
P''	\square	...	\square	a_j	a_i	...	a_i
	$a_j \square$...	$a_j \square$	a_i	\square	...	\square
	\square	...	\square	a_k	$a_j \square$...	$a_j \square$
	a_i	...	a_i		\square	...	\square

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P'	a_j	...	a_j	a_i	a_i	...	a_i
	a_k	...	a_k	a_j	a_j	...	a_j
	a_i	...	a_i	a_k	a_k	...	a_k
P''	\square	...	\square	a_j	a_i	...	a_i
	$a_j \square$...	$a_j \square$	a_i	\square	...	\square
	\square	...	\square	a_k	$a_j \square$...	$a_j \square$
	a_i	...	a_i		\square	...	\square

Thus $a_k < a_j$ does not change as long as $a_k <_{v_{ij}} a_j$, so $v_{ij} \leq v_{jk}$.
 Further, $v_{kj} \leq v_{ij}$.

Proof

	1	...	$v_{ij} - 1$	v_{ij}	$v_{ij} + 1$...	N
P'	a_j	...	a_j	a_i	a_i	...	a_i
	a_k	...	a_k	a_j	a_j	...	a_j
	a_i	...	a_i	a_k	a_k	...	a_k
P''	\square	...	\square	a_j	a_i	...	a_i
	$a_j \square$...	$a_j \square$	a_i	\square	...	\square
	\square	...	\square	a_k	$a_j \square$...	$a_j \square$
	a_i	...	a_i		\square	...	\square

Thus $a_k < a_j$ does not change as long as $a_k <_{v_{ij}} a_j$, so $v_{ij} \leq v_{jk}$. Further, $v_{kj} \leq v_{ij}$. But $j \neq k$ are arbitrary, so $v_{kj} \leq v_{ij} \leq v_{jk}$ and $v_{jk} \leq v_{kj}$, so $v_{kj} = v_{ij} = v_{jk}$.

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	a_k	...	a_k	a_j	a_j	...	a_j
	a_i	...	a_i	a_k	a_k	...	a_k
P''	\square	...	\square	a_j	a_i	...	a_i
	$a_j \square$...	$a_j \square$	a_i	\square	...	\square
	\square	...	\square	a_k	$a_j \square$...	$a_j \square$
	a_i	...	a_i		\square	...	\square

Thus $a_k < a_j$ does not change as long as $a_k <_{v_{ij}} a_j$, so $v_{ij} \leq v_{jk}$. Further, $v_{kj} \leq v_{ij}$. But $j \neq k$ are arbitrary, so $v_{kj} \leq v_{ij} \leq v_{jk}$ and $v_{jk} \leq v_{kj}$, so $v_{kj} = v_{ij} = v_{jk}$.

Hence v_{ij} is a dictator.



Winning sets

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Consider a situation where $a > b$; by IIA, this depends only on the set $U \subseteq V$ of voters who prefer a to b . Call U a “winning set” for a over b if $a > b$ iff $a >_u b$ for each $u \in U$.

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We want to be able to find a suitable collection $\mathcal{U} \subseteq \mathcal{P}(V)$ such that if $\{v : a <_v b\} \in \mathcal{U}$, then $a < b$. That is, we want \mathcal{U} to be a collection of winning sets.

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Additional votes for an ordering should only help that ordering; if $U \in \mathcal{U}$ and $U \subseteq W \subseteq V$, then $W \in \mathcal{U}$, so in particular $V \in \mathcal{U}$.

Winning sets

Let $U = \{u : a <_u b\}$ and $W = \{w : b <_w c\}$, and suppose $U, W \in \mathcal{U}$. Then $a < b$ and $b < c$, hence $a < b < c$.

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It may be that $U \cap W = \{v : a <_v c\}$; as when

$$\{v : c <_v a <_v b\} = U \setminus W,$$

$$\{v : b <_v c <_v a\} = W \setminus U,$$

$$\{v : c <_v b <_v a\} = (U \cup W)^c.$$

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Thus it must be in fact that $U \cap W \in \mathcal{U}$.

Winning sets

So we have a collection $\mathcal{U} \subseteq \mathcal{P}(V)$ such that

1. $\emptyset \notin \mathcal{U}$.
2. If $U \in \mathcal{U}$ and $U \subseteq W \subseteq V$, then $W \in \mathcal{U}$.
3. If $U, W \in \mathcal{U}$, then $U \cap W \in \mathcal{U}$.
4. For each $U \subseteq V$, either $U \in \mathcal{U}$ or $U^c \in \mathcal{U}$.

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The collection \mathcal{U} is an *ultrafilter* on V !

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Our collection \mathcal{U} of winning sets is a *principal ultrafilter*.

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There are no nonprincipal ultrafilters on finite sets.

We could in fact create a nondictatorial Pareto IIA social welfare function by constructing a *nonprincipal ultrafilter*, provided we have infinite voters and the Axiom of Choice.

Ultraproducts

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Given an index set I , a family of “structures” (in our case, an ordered set) $(M_i)_{i \in I}$, and an ultrafilter \mathcal{U} on I , we define the equivalence relation $a \sim b$ if $\{i \in I : a_i = b_i\} \in \mathcal{U}$. Then the *ultraproduct* is the quotient structure $\prod_{i \in I} M_i / \sim$, sometimes written $\prod_{i \in I} M_i / \mathcal{U}$.

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Let σ be a signature, \mathcal{U} an ultrafilter on a set I , and let $(M_i)_{i \in I}$ be a family of σ -structures. Let $M = \prod_{i \in I} M_i / \mathcal{U}$. Then for each $a^1, \dots, a^n \in \prod_{i \in I} M_i$, where $a^k = (a_i^k)_{i \in I}$, and every σ -formula ϕ ,

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$$M \models \phi[[a^1], \dots, [a^n]] \iff \{i \in I : M_i \models \phi[a_i^1, \dots, a_i^n]\} \in \mathcal{U}.$$

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$$M \models \phi[[a^1], \dots, [a^n]] \iff \{i \in I : M_i \models \phi[a_i^1, \dots, a_i^n]\} \in \mathcal{U}.$$

Proof.

Bonus Exercise. □

A Big Black Box

Theorem (Uniqueness of winning ultrafilters)

1. *For each Pareto IIA social welfare function F there is a unique ultrafilter \mathcal{U}_F such that for each $U \in \mathcal{U}_F$, each $a, b \in A$, and any preference profile P , if $\{u : a \leq_u b\} \in \mathcal{U}_F$ then $a \leq_{F(P)} b$.*
2. *For each ultrafilter \mathcal{U}_F on V there exists such an F .*

Some more black boxes

Fact (Fact 1)

For any atomless measure space (X, \mathcal{M}, μ) , for all $\varepsilon > 0$, there is a finite partition $X = \bigsqcup_{i=1}^r C_i$, such that each $C_i \in \mathcal{M}$ and $\mu(C_i) < \varepsilon$.

Fact (Fact 2)

Let \mathcal{U} be an ultrafilter on a set X . For any finite partition $X = \bigsqcup_{i=1}^r C_i$, exactly one $C_i \in \mathcal{U}$.

Arbitrarily small coalitions

In the case where V is infinite, consider an atomless measure space $(V, \mathcal{V}, \lambda)$.

Theorem

For any $\varepsilon > 0$ there exists $C \in \mathcal{V}$, $C \neq \emptyset$, $\lambda(C) < \varepsilon$ such that for any $a, b \in A$ and any situation P , if $\{v : a \leq_c b\} \subseteq C$ then $a \leq_{F(P)} b$.

Proof.

Apply the first fact to $(V, \mathcal{V}, \lambda)$. Then we have a partition $X = \bigsqcup_{i=1}^r C_i$ such that each $C_i \in \mathcal{M}$ and $\mu(C_i) < \varepsilon$. Take the ultrafilter \mathcal{U}_F guaranteed by the uniqueness of winning ultrafilters and apply the second fact. □

Stone-Čech electorate

Consider V as a discrete topological space and suppose the set A of alternatives is finite. Then $\mathcal{L}(A)$ is finite, so it is also a discrete topological space.

We have the following commutative diagram:

$$\begin{array}{ccc}
 V & \xrightarrow{i_V} & \beta V \\
 & \searrow P & \downarrow \beta P \\
 & & \mathcal{L}(A)
 \end{array}$$

This is the universal property for βV , the Stone-Čech compactification of V .

Note that V is dense in βV .

One more black box

Fact

Let \mathcal{U} be an ultrafilter on a topological space X . Then \mathcal{U} has a (unique) limit in βX .

A hidden dictator

Theorem

There exists a point $\hat{v} \in \beta V$ such that for all $a, b \in A$ and $P \in \mathcal{L}(A)^V$, if $a \leq_{\hat{v}} b$ then $a \leq_{F(P)} b$.

Proof.

We know that there is a unique winning ultrafilter \mathcal{U}_F . By the previous fact, \mathcal{U}_F has a unique limit $\hat{v} \in \beta V$. Then $\beta P(\hat{v}) = \mathcal{U}_F \lim_v P(v)$. Since $\mathcal{L}(A)$ is discrete, there is some $U \in \mathcal{U}_F$ with $P(U) = \beta P(\hat{v})$. By the properties of \mathcal{U}_F , U is a winning set, and hence \hat{v} dictates F . □

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