The History of Curvature

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OSU Reading Classics

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(John Marshall White III's Birthday Observed)

Lines Don't Curve; Circles Do

Aristotle (384 - 322 BCE) made the astute observation that lines do not curve, but circles curve uniformly at each point. Thus, he classified plane loci into three categories: linear, circular, and mixed [1].

Apollonius of Perga (c. 240 BCE - c. 190 BCE) made (implicitly) the first calculation of curvature when considering the problem of drawing normal segments to conic sections, recognizing that such a normal segment was unique.

Nicole Oresme

Nicole Oresme (c. 1320 - 1382) was a French philosopher in the Late Middle Ages whose *Tractatus de Configurationibus Qualitatum et Motuum* (Treatise on the Configurations of Qualities and Motions) contained the first characterization of *Curvitas*, noting that the curvature of a circle is inversely proportional to its radius [1].

Attempting to generalize to other curves, Oresme stated that if two curves are tangent to a line at the same point from the same side, then the "smaller" of the two should have more curvature than the larger.

Oresme's *Tractatus* also contained a number of other important discoveries, such as an early use of rectangular coordinates, a computation of geometric series, and is credited with the first proof of the divergence of the harmonic series [2].

Kepler's Circle of Curvature

Johannes Kepler (1571 - 1630) was the first to suggest approximating a curve by its *circle of curvature*, the "closest" circle to the curve at a point [1].

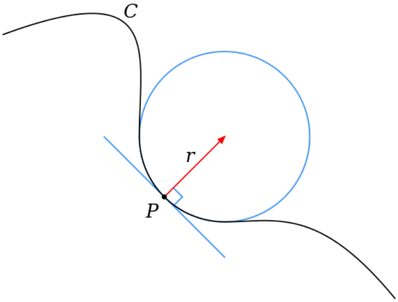


Figure: An osculating circle [3].

Huygens' Construction

Christiaan Huygens (1629 - 1695) later gave a general way to produce the circle of curvature in 1673 in *Horologium Oscillatorium: Sive de Motu Pendulorum ad Horologia Aptato Demonstrationes Geometricae* (The Pendulum Clock: or Geometrical Demonstrations Concerning the Motion of Pendula as Applied to Clocks) using involutes and evolutes.

Begin with a curve (the *evolute*) whose tangent lines are all on one side, and to this attach a flexible string which is pulled taught and then unwound, tracing the *involute*.

Assuming the string is always tangent to the evolute, Huygens shows that it will always be normal to the involute, and that the involute and evolute are unique.

Huygens' Construction (cont.)

The radius of curvature of the involute, then, is the distance up the string from the involute to the evolute. (Apollonius was unknowingly calculating this in the special case of conics.)

Notice that this requires the *evolute* of the curve to be given in advance.

Huygens calculated the radii of curvature for some special cases (notably the cycloid) and attempts to provide a general method, but was not aware of Newton's and Leibniz's calculus and so failed to give a general solution [1].

Newton's Method of Fluxions

Isaac Newton (1642 - 1727[1]) used calculus to describe the center and radius of curvature for a curve at a point in Problem V of his posthumously published work *The Method of Fluxions and Infinite Series: with its Application to the Geometry of Curve-Lines* [4].

Like Huygens and Kepler, Newton reduces the problem to finding the radius of the osculating circle.

[1] Newton died in 1726 according to the Julian calendar, but in 1727 according the the Gregorian calendar; since England had not yet adopted the Gregorian calendar, Newton's death was recorded as being in 1726 but is usually adjusted to 1727 despite his birth usually being listed exclusively by the Julian date.

Newton's Method of Fluxions (cont.)

If a Circle touches any Curve on its concave ſide, in any given Point, and if it be of ſuch magnitude, that no other tangent Circle can be interſcribed in the Angles of Contact near that Point, that Circle will be of the ſame Curvature as the Curve is of, in that Point of Contact. [...]

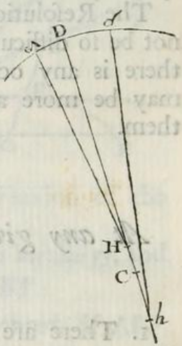
Therefore the Center of Curvature to any Point of a Curve, is the Center of a Circle equally curved. And thus the Radius, or Semidiameter of Curvature is part of the Perpendicular to the Curve, which is terminated at that Center. [...]

And the Proportion of Curvature at different Points will be known from the Proportion of Curvature of aequi-curve Circles, or from the reciprocal Proportion of the Radii of Curvature.

Newton's Method of Fluxions (cont.)

Newton then describes a general method for producing the center of convergence [4]:

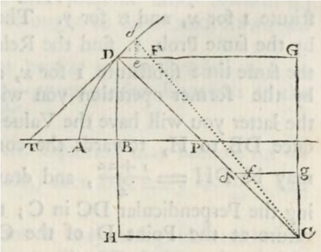
7. Imagine therefore that at three Points of the Curve δ, D, and *d*, Perpendiculars are drawn, of which thoſe that are at D and δ meet in H, and thoſe that are at D and *d* meet in *h*: And the Point D being in the middle, if there is a greater Curvity at the part Dδ than at D*d*, then DH will be leſs than *dh*. But by how much the Perpendiculars δH and *dh* are nearer the intermediate Perpendicular, ſo much the leſs will the diſtance be of the Points H and *h*: And at laſt when the Perpendiculars meet, thoſe Points will coincide. Let them coincide in the Point C, then will C be the Center of Curvature, at the Point D of the Curve, on which the Perpendiculars ſtand; which is manifeſt of itſelf.



Newton's Method of Fluxions (cont.)

Using his construction, Newton goes on to show that putting $x = AB$ with $\dot{x} = 1$ and $y = BD$ with $z = \dot{y}$ in the following figure [4] the radius of convergence is given by

$$DC = \frac{(1+z^2)^{3/2}}{\dot{z}}$$



In modern notation, the radius of curvature of a plane curve $y = y(x)$ is

$$R = \left| \frac{(1 + y'(x)^2)^{3/2}}{y''(x)} \right|.$$

Newton's Contemporaries

Gottfried Wilhelm Leibniz (1646 - 1716) is credited with the term "osculating circle" (or *Circulus Osculans*) in his 1686 paper *Meditation Nova de Natura Anguli Contactus et Osculi* (New Meditation on the Nature of Angles of Contact and Osculation) [1, 5].

Jacob Bernoulli (1655 - 1705) also worked on the curvature of general curves, particularly in his general construction of the evolute of a curve $c(t)$, which is given by

$$E(t) = c(t) + R(t)n(t)$$

where $R(t)$ is the radius of curvature at $c(t)$ and $n(t)$ is the unit normal vector towards the center of curvature.

Johann Bernoulli (1667 - 1748) also studied curvature, including the radius of curvature of a curve in polar coordinates.

Parameterization by Arc Length

Recall that the length of a smooth curve $\gamma : [a, b] \to \mathbb{R}^n$ is given by

$$L(\gamma) = \int_a^b \|\dot{\gamma}(t)\| \, dt.$$

Then the *arc length* of γ is the function

$$s(t) = \int_a^t \|\dot{\gamma}(t)\| \, dt.$$

This is a smooth bijection from $[a, b]$ to $[0, L(\gamma)]$. Writing $t(s)$ for the inverse we have the *arc length parameterization* of γ by

$$\gamma(s) = \gamma(t(s)).$$

Such a curve has the useful property that $\|\dot{\gamma}(s)\| \equiv 1$.

Euler and the Second Derivative

Leonhard Euler (1707 - 1783) proved the following in 1736 [5]:

Theorem

Let $\gamma(s)$ be parameterized by arc length. Then the (unsigned) curvature κ is given by the magnitude of the acceleration:

$$\kappa(\gamma; s) = \|\ddot{\gamma}(s)\| .$$

Surfaces

Thus far, we have seen mostly the theory of curves in the plane, which can be generalized to curves in Euclidean 3-space.

However, the geometry of *surfaces* emerges around Euler's 1760 paper *Recherches sur la courbure des surfaces* (Research on the curvature of surfaces) [5, 6], and will eventually lead to a result of Gauss which would inspire his student Riemann to found the modern theory which bears his name.

Surfaces in $\mathbb{R}^3$

Definition

A *regular surface* S is a subset $S \subset \mathbb{R}^3$ such that at each point $p \in S$ there is an open neighborhood $p \in U \subset S$ with $V \subset \mathbb{R}^2$ and a homeomorphism $f : V \to U$ such that

- $f(u, v)$, as a function $f : V \to \mathbb{R}^3$ is smooth
- for each $x \in V$, $\partial_u f$ and $\partial_v f$ are linearly independent in $\mathbb{R}^3$.

This definition using *local charts* is equivalent to the definition using *Monge patches* (as locally the graph of a smooth function $h : V \to \mathbb{R}$) and the definition using implicit functions; this follows from the *(generalized) implicit function theorem*.

We denote the *tangent plane* to S at p by $T_pS := \operatorname{span}\{\partial_u f, \partial_v f\}$.

Curvature(s) of a Surface

Let $S \subset \mathbb{R}^3$ be a regular surface and consider *normal planes* P_X to S (which contain the normal vector n and a vector X tangent to S at P) which each specify a curve on S.

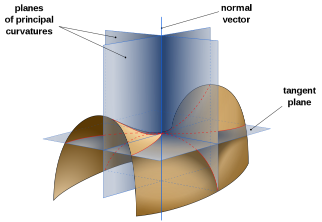


Figure: Planes of principal curvature at a saddle point of a surface.

Then the *normal curvature* κ_X of S with respect to P_X is the curvature of the unit speed curve whose image is $S \cap P_X$.

Euler and the Principal Curvature(s)

Theorem (Euler, 1760)

Suppose S is a regular surface with $p \in S$ such that κ_X is not constant.
Then there are two unit tangent vectors $X_1, X_2 \in T_pS$ with $k_1 = \kappa_{X_1} = \max_{X \in T_pS} \kappa_X$ and $k_2 = \kappa_{X_2} = \min_{X \in T_pS} \kappa_X$ (the principal curvatures of S at p) and that $X_1 \perp X_2$. Further, if $X \in T_pS$ is a unit vector at angle θ with X_1, then

$$\kappa_X = k_1 \cos^2 \theta + k_2 \sin^2 \theta.$$

Gaussian Curvature and the *Theorema Egregium*

Definition

The *Gaussian curvature* K at p is the product $K = k_1 k_2$ of the principal curvatures.

Carl Friedrich Gauss (1777 - 1855) proved that "If a curved surface is developed upon any other surface whatever, the measure of curvature in each point remains unchanged."

In modern language,

Theorem (*Theorema Egregium*; Gauss, 1827)

The Gaussian curvature K of a regular surface S is invariant under local isometry.

Riemann's *Habilitationsvortag*

Gauss's *Theorema Egregium* suggests that curvature can be studied as an *intrinsic* property of a surface rather than the *extrinsic* way in which it is embedded in Euclidean space.

In addition, the search for a rigorous model of non-Euclidean geometry led to the discovery that the hyperbolic plane cannot be properly embedded as a surface in $\mathbb{R}^3$.

Bernhard Riemann (1826 - 1866), in his 1854 *Habilitationsvortag* "*Ueber die Hypothesen, welche der Geometrie zu Grunde liegen*" (On the Hypotheses which Lie at the Bases of Geometry) laid the groundwork for what would become modern differential geometry [5, 7].

Riemann's Notion of an *n*-ply Extended Magnitude

Riemann describes (continuous) "*Mannigfaltigkeit* (manifoldness)" as arising from an "*n*-ply extended magnitude," which we would today recognize as a description of $\mathbb{R}^n$.

He then discusses solution sets to certain polynomial equations as having manifoldness.

Riemann then goes on to describe "line-elements," and calls a manifoldness in variables x_i "flat" if the line-elements s are given by

$$s = \sqrt{\sum_i x_i^2}$$

and describes curvature again as a "deviation from flatness."

Connecting to Poincaré

Henri Poincaré (1854 - 1912) is also credited with an early notion of a *smooth manifold* (variété différentielle) in *Analysis Situs* (1895).

Poincaré describes two manifolds defined by graphs of functions $\theta(y)$ and $\theta'(y')$ and requires that on the intersection the coordiantes y depend continuously differentiably on y' and vice versa.

Hermann Weyl (1885 - 1955) gave an intrisic definition for differentiable manifold in a lecture course on Riemann surfaces in 1911-1912, and Hassler Whitney's (1907 - 1989) eponymous embedding theorem shows the equivalence of these definitions.

Smooth Manifolds

Definition

A *(topological) n-manifold M* is a topological space which is locally homeomorphic to $\mathbb{R}^n$. That is, for each $p \in M$ there is a neighborhood $U \ni p$ which is homeomorphic to an open subset of $\mathbb{R}^n$. We may also require that M be Hausdorff and second countable.

Definition

A *smooth structure* (or *atlas*) on M is a collection of *charts* $(U_\alpha, \varphi_\alpha)$ such that $\varphi_\alpha : U_\alpha \to \mathbb{R}^n$ is a homeomorphism onto its image and the *transition functions* $\varphi_\alpha \circ \varphi_\beta^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ are smooth. A topological manifold M endowed with a smooth structure is said to be a *smooth* (or *differentiable*) *manifold*.

Riemannian Manifolds

Definition

To each point $p \in M$ the *tangent space at* p, denoted T_pM, is the vector space of "directional derivatives" at p. The collection of all such tangent spaces, appropriately topologized according to the smooth structure of M, is the *tangent bundle* TM.

Definition

A *Riemannian metric* g on a smooth manifold M is a "smoothly varying" inner product $g_p : T_pM \times T_pM \to \mathbb{R}$. We say a smooth manifold M endowed with such a metric is a *Riemannian manifold*.

Vector Fields and the Lie Bracket

Definition

A *(tangent) vector field* X on M is a smoothly varying assignment to each $p \in M$ a vector $X_p \in T_pM$. We denote by $\Gamma(TM)$ the set of all such vector fields on M.

Definition

The *Lie bracket* $[X, Y]$ of two vector fields $X, Y \in \Gamma(TM)$ is the vector field given by

$$[X, Y]f = X(Yf) - Y(Xf)$$

where we write Xf to mean "the directional derivative of f in the direction X" at each point $p \in M$.

The Levi-Civita Connection

Definition

A *connection* on TM is a bilinear map $\nabla : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$ written $(X, Y) \mapsto \nabla_X Y$ such that for any smooth function $f : M \to \mathbb{R}$,

1. $\nabla_{fX} Y = f \nabla_X Y$
2. $\nabla_X fY = (\partial_X f) Y + f \nabla_X Y$.

Theorem

Every Riemannian manifold (M, g) has a unique connection ∇ (the Levi-Civita connection) which is

1. **Metric compatible:** $X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$
2. **Torsion free:** $\nabla_X Y - \nabla_Y X = [X, Y]$.

The Curvature Tensor

Definition

The *curvature tensor* of a Riemannian manifold is the map $R : \Gamma(TM) \times \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$ defined by

$$R(X, Y)Z \coloneqq \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

Sectional Curvature

Definition

Given two linearly independent vectors $u, v \in T_pM$, the *sectional curvature* is given by

$$K(u,v) := \frac{\langle R_p(u,v)v, u\rangle}{\langle u,u\rangle \langle v,v\rangle - \langle u,v\rangle^2}.$$

It can be shown that the Riemannian curvature tensor can be recovered from the sectional curvature.

Notice that in the case $\dim M = 2$, the (unique) sectional curvature of M is the Gaussian curvature.

Isotropic and Constant Sectional Curvature

Definition

If the sectional curvature K of M is at each point independent of the choice of u, v then M is said to be *isotropic*. If in addition the sectional curvature is equal at each point of M we say M has *constant sectional curvature*.

Lemma ((Friedrich) Schur)

An isotropic connected manifold of dimension > 2 has constant sectional curvature.

Of particular interest in Riemannian geometry are the so-called *model spaces* $\mathbb{R}^n$, $\mathbb{S}^n$, and $\mathbb{H}^n$, which have constant sectional curvature 0, 1, and -1, respectively.

The Model Spaces $\mathbb{R}^n$, $\mathbb{S}^n$, and $\mathbb{H}^n$

Consider standard Euclidean space with the usual inner product $(\mathbb{R}^n, g^{\mathrm{can}})$.

Now, in $(\mathbb{R}^{n+1}, g^{\mathrm{can}})$ denote by $\mathbb{S}^n$ the usual unit sphere

$$\mathbb{S}^n := \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}.$$

Consider now $\mathbb{R}^{n,1}$, which is $\mathbb{R}^n$ with the *Lorentzian* form

$$g^{\mathrm{Lor}}(x, y) = \left(\sum_{i=1}^{n} x_i y_i\right) - x_{n+1} y_{n+1}$$

(the *pseudo-Riemannian* manifold $(\mathbb{R}^{n,1}, g^{\mathrm{Lor}})$ is sometimes called *Minkowski space*) and denote by $\mathbb{H}^n$ the hyperbolic space

$$\mathbb{H}^n := \{x \in \mathbb{R}^{n+1} \mid \|x\|^{\mathrm{Lor}} = 1\}.$$

The Model Spaces $\mathbb{R}^n$, $\mathbb{S}^n$, and $\mathbb{H}^n$

Scaling the unit sphere $\mathbb{S}^n$ and hyperbolic space $\mathbb{H}^n$, we can actually obtain a space of constant sectional curvature k for any $k \in \mathbb{R}$; for $k > 0$, the sphere $\mathbb{S}^n(k^{-1/2})$ of radius $k^{-1/2}$ has constant curvature k and hyperbolic space $\mathbb{H}^n(k^{-1/2})$ has constant sectional curvature $-k$.

We denote such manifolds by M_k^n.

Notice that M_k^n has diameter $D_k = 2k^{-1/2}$ if $k > 0$ and ∞ otherwise.

Theorems on the Model Spaces

Theorem

Any complete, connected, simply connected manifold M of constant sectional curvature k is isometric to the model space M_k^n.

Theorem (Killing (1891)-Hopf (1926))

Any complete, connected n-manifold M with constant sectional curvature k (a so-called space form) has universal cover M_k^n. That is, M is a quotient of M_k^n by a free and properly discontinuous group action.

On Geodesic Metric Spaces

One interesting feature of spaces of constant curvature k is the behavior of (geodesic) triangles, and we can use this to introduce a notion of "curvature" in a more general setting.

Definition

Let (X, d) be a metric space. A *geodesic segment* is an arc-length parametrized continuous curve $\gamma : [a, b] \to X$ such that

$$d(x, y) = L(\gamma) := \sup \left\{ \sum_{i=1}^{r} d(\gamma(t_{i-1}), \gamma(t_i)) \mid t_i \text{ partitions } [a, b] \right\}.$$

where $x = \gamma(a)$ and $y = \gamma(b)$.
A metric space (X, d) is said to be *geodesic* if any $x, y \in X$ can be connected by a geodesic segment.

Triangles and the CAT(k) Inequality

Suppose $\Delta \subset X$ is a geodesic triangle with vertices p, q, r and consider a *comparison triangle* $\Delta' \subset M_k^2$ with vertices p', q', r' such that $d(p,q) = d(p',q')$, $d(q,r) = d(q',r')$, and $d(r,p) = d(r',p')$.

Notice that each point $x \in \Delta$ has a corresponding point $x' \in \Delta'$.

Δ is said to satisfy the CAT(k) *inequality* if for $x, y \in X$, $d_X(x,y) \leq d_{M_k^2}(x',y')$.

Definition

A geodesic metric space (X,d) is said to be a CAT(k) *space*[2] if every geodesic triangle $\Delta \subset X$ with perimeter $\leq 2D_k$ satisfies the CAT(k) inequality.

[2] CAT(k) was coined by Mikhail Gromov in 1987 as an acronym for Élie **C**artan, Aleksandr Danilovich **A**leksandrov and Victor Andreevich **T**oponogov. Aleksandrov originally called such spaces $\mathfrak{R}_k$ *domains*.

CAT(k) Spaces and Curvature “Bounded Above”

Definition

We say that a (not necessarily geodesic) metric space (X, d) has *curvature* $\leq k$ if every point in X has a geodesically convex CAT(k) neighborhood.

An important case is the case $k = 0$, which was studied by Jacques Hadamard (1865 - 1963). Complete CAT(0) spaces are sometimes called *Hadamard spaces*.

References

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