People

Marie Jean Antoine Nicolas de Caritat, Marquis de Condorcet (1743-1794)

SE France

- French mathematician, philosopher
- Inspector general of Paris mint (1774)
- "Essai sur le calcul intégral" (1785)
- Worked with Euler
- "Essai sur l'application de l'analyse à la probabilité des décisions rendues à la pluralité des voix" (1785)

Kenneth Arrow (1921-2017)

- "A Difficulty in the Concept of Social Welfare" (1950)
  - ↳ Arrow's Impossibility Theorem

Terminology (Part 1)

Let A be a set of outcomes, with a set of full linear orderings (preference orderings) $L(A)$, and let V be a set of N voters.

Def: A preference ordering $\geq_i \in L(A)$, (prefers or is indifferent) $>_i$ (strictly prefers), $=_i$ (indifferent) is associated with voter V_i.

Def: A preference profile is an N-tuple of preference orderings $(\geq_1, \dots, \geq_N) \in L(A)^N$

Def: A (strict) social welfare function $F: L(A)^N \to L(A)$ is a deterministic function defined for each $(\geq_1, \dots, \geq_N) \in L(A)^N$ which aggregates voter preferences into exactly one preference order on A.

(This is sometimes referred to as a constitution)

Remark: It may be convenient to consider a society S consisting of a set of social outcomes A, a set of N voters V, and their respective preference profile $P = (\geq_1, \dots, \geq_N) \in L(A)^N$.

E.g. Take A to be the set of U.S. presidential candidates and V to be the American electorate with a particular preference profile.

We live in a society.

Voting system criteria

Majority Criterion: If a is the maximum of A w.r.t a majority of $\geq_i \in P$ then a is the maximum of A w.r.t. $F(\geq_1, \dots, \geq_N)$

Condorcet Criterion: If $X \geq_i X' \; \forall X' \in A$, for a majority of $\geq_i$ then a is the maximum of A w.r.t $F(\geq_1, \dots, \geq_N)$. We call a the Condorcet winner.

(That is, a wins if a wins every pairwise election under the majority criterion).

Remark: Condorcet $\Rightarrow$ Majority

Non-Dictatorship: There is no $V_i \in V$ s.t. $\geq_i = F(P) \; \forall P \in L(A)^N$.

Independence of Irrelevant Alternatives (IIA): Suppose $a, b, c \in A$, and $\geq$, $\geq'$ arise from $P = (\geq_1, \dots, \geq_N)$, $P' = (\geq_1', \dots, \geq_N')$ resp. Then if $\forall i \; \forall a, b$ we have $a \geq_i b$ iff $a \geq_i' b$, then $\geq = \geq'$.

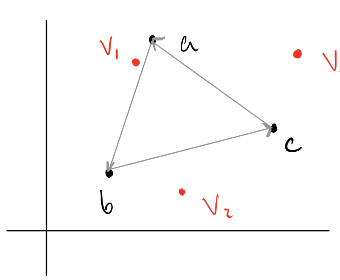
Pareto Efficiency / Unanimity: If $a \geq_i b$ for each $\geq_i \in P$ then $a \geq b$ w.r.t $F(\geq_1, \dots, \geq_N)$.

Condorcet Paradox: There exists a society with no Condorcet winner.

Proof: Suppose $A = \{a, b, c\}$ and V has 3 voters

V_1	$a \geq_1 b \geq_1 c$	a wins pairwise v.s. b but loses to c.
V_2	$b \geq_2 c \geq_2 a$	b wins pairwise v.s. c but loses to a.
V_3	$c \geq_3 a \geq_3 b$	c wins pairwise v.s. a but loses to b.

Remark: We can represent our society as a plot in $\mathbb{R}^n$



Arrow's Theorem

Let A be a set of at least 3 alternatives, V a set of N voters with preference profile P.

The following conditions are incompatible:

(i) Non-Dictatorship

(ii) Independence of Irrelevant Alternatives

(iii) Pareto Efficiency / Unanimity

Lemma: Choose $b \in A$ and suppose $\forall \geq_i$ we have either $b = \min(A)$ or $b = \max(A)$ w.r.t. $\geq_i$. Then either $b = \min(A)$ or $b = \max(A)$ w.r.t. $\geq$.

Proof: Suppose we have $a \geq b \geq c$. Now for each $\geq_i$, produce $\geq_i'$ s.t. $c >_i' a$. By IIA, we still have $a \geq' b$ and $b \geq' c$. But then we have $a \geq c > a$, a contradiction. $\square$

Proof of Theorem: (Geneakoplos, 2004)

Suppose that $\forall i$, $b = \min(A)$ w.r.t. $\geq_i$. Clearly b is the minimum w.r.t. $\geq$. Now, starting from $i=1$, move b so $b = \max(A)$ w.r.t. $\geq_i$ until the placement of b changes (by unanimity this happens before N). By the lemma, b moves to the top of $\geq$. Call the pivotal voter V_n and denote the profile before V_n moves b by P_1 and the profile immediately after by P_2.

Now choose $a \in A$ from a pair $\{a, c\}$. We WTS V_n is a partial dictator for a over c. Have V_n move a above b in $\geq_n$ and let every other $\geq_i$ arrange a, c arbitrarily, leaving b in its extreme position. Call this P_3. By IIA, we must have $a \geq b \geq c$. Then By IIA, $\geq$ agrees with $\geq_n$ on $\{a, c\}$, so V_n is a partial dictator for a over c.

Now consider alternative c, and place c at the bottom of each $\geq_i$. By a similar argument above, there is a pivotal voter V_n' for each pair $\{\alpha, \beta\} \not\ni c$. But $\{a, b\}$ is such a pair, and we know V_n is a partial dictator for $\{a, b\}$, so $V_n = V_n'$. This holds for each $\{\alpha, \beta\}$ pair, so V_n must be a dictator. $\square$

Notes: The above result holds for a ranked system, i.e. $F: L(A)^N \to L(A)$.

One can consider a cardinal system where each voter has a preference function $f_i : A \to S \subseteq \mathbb{R}$ and a social preference function which averages over f_i.

Arrow's Theorem applies to ranked voting but Gibbard's Theorem (1973) extends the result to cardinal systems.

Arrow's Theorem can be "fixed" by relaxing criteria

- Two alternatives (simple majority)
- Infinite voters (construct F using ultrafilters)
- Relaxing IIA (susceptible to strategic manipulation)
- Relaxing Transitivity (a veto exists)
- Relaxing Pareto (either a dictator or an inverse dictator)

References:

[1] Arrow, Kenneth J. "A Difficulty in the Concept of Social Welfare" Journal of Political Economy 58, (1950)

[2] Black, Duncan, "On the Rationale of Group Decision-making", Journal of Political Economy 56, (1948)

[3] Geanakoplos, John. "Three Brief Proofs of Arrow's Impossibility Theorem," Economic Theory 26 (2004)

[4] Wikipedia. "Condorcet Criterion", "Condorcet Paradox", "Arrow's Impossibility Theorem"