A History of Computational Linear Algebra

The Theory of Tables of Numbers through Time

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OSU Reading Classics

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Sheaves of wheat

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Systems of simultaneous linear equations are known to have been computed as far back as ancient Mesopotamia, but we will begin with the first matrices. Consider the following problem:

Problem

"The yield of 3 sheaves of superior grain, 2 sheaves of medium grain, and 1 sheaf of inferior grain is 39 pieces of bread.
The yield of 2 sheaves of superior grain, 3 sheaves of medium grain, and 1 sheaf of inferior grain is 34 pieces of bread.
The yield of 1 sheaf of superior grain, 2 sheaves of medium grain, and 3 sheaves of inferior grain is 26 pieces of bread.
What is the yield of superior, inferior, and medium grains?"

Sheaves of wheat

With x representing superior grain, y representing medium grain, and z representing inferior grain, we have a system of linear equations:

$$\begin{aligned} x + 2y + \ \ z &= 39 \\ 2x + 3y + \ \ z &= 34 \\ 1x + 2y + 3z &= 26 \end{aligned}$$

which we can solve with Gaussian elimination.

However, the Chinese mathematicians who wrote this problem did not have the benefit of Gauss's work, since this was published in the *Nine Chapters on the Art of Calculation* about 600 years before Gauss's birth.[6]

Sheaves of wheat

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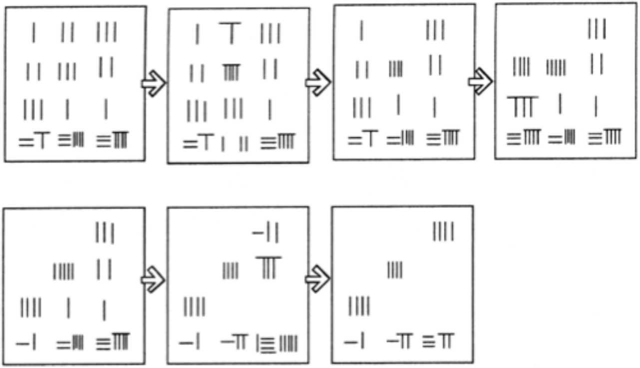

Adapted from Horng Wann-Sheng by Randy K. Schwartz [5]

Sheaves of wheat

In more familiar notation,

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 39 \\ 2 & 3 & 1 & 34 \\ 1 & 2 & 3 & 26 \end{array}\right] \to \left[\begin{array}{ccc|c} 3 & 2 & 1 & 39 \\ 0 & 5 & 1 & 26 \\ 1 & 2 & 3 & 26 \end{array}\right] \to \left[\begin{array}{ccc|c} 3 & 2 & 1 & 39 \\ 0 & 5 & 1 & 26 \\ 0 & 4 & 8 & 39 \end{array}\right] \to \left[\begin{array}{ccc|c} 3 & 2 & 1 & 39 \\ 0 & 5 & 1 & 26 \\ 0 & 0 & 36 & 99 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 3 & 2 & 1 & 39 \\ 0 & 5 & 1 & 26 \\ 0 & 0 & 4 & 11 \end{array}\right] \to \left[\begin{array}{ccc|c} 12 & 8 & 0 & 145 \\ 0 & 4 & 0 & 17 \\ 0 & 0 & 4 & 11 \end{array}\right] \to \left[\begin{array}{ccc|c} 4 & 0 & 0 & 37 \\ 0 & 4 & 0 & 17 \\ 0 & 0 & 4 & 11 \end{array}\right]$$

so $x = \frac{37}{4}$, $y = \frac{17}{4}$, $z = \frac{11}{4}$.

Gaussian elimination, 600 years before Gauss

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The method appeared later in Newton's notes in 1670, and methods of elimination and matrix reduction were developed in Europe throughout the 18th and 19th centuries including Gauss's contribution in 1811[2][1].

Determinate and indeterminate systems

Definition

A system of linear equations is said to be underspecified if there are fewer equations than unknowns.
If there are more equations than unknowns, the system is said to be overspecified.

Definition

A system of equations is determinate if there is a unique solution.
If there is more than one solution, it is indeterminate.

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Under what conditions is a linear system of equations determinate?

Discovery

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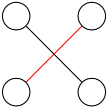
Seki illustrates the 2×2 case with the following picture:



where the red line corresponds to a positive ("constructive") product and the black to a negative ("destructive") product.

Laplace exapansion, half a century before Laplace

Seki describes the method for calculating determinants up to the 5×5 case in his *Fukudai* in 1683, and by 1710 Seki and his contemporaries had described a full, correct description of the Laplace expansion.

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In Europe, Gabriel Cramer used the determinant in 1750 to describe closed-form solutions to systems of linear equations in terms of their coefficients (what we now call Cramer's rule), and other European mathematicians would continue to study determinants throughout the 18th and 19th centuries.

Singular and nonsingular matrices

Cramer noted that system whose determinant was zero did not have a unique solution, and moreover used his closed-form solutions to characterize when a system had zero, one, or infinitely many solutions.

Euler noticed in the same year that a system of n equations in n unknowns need not have a unique solution:

$$3x - 2y = 5$$
$$6x - 4y = 10$$

which led to the idea of "inclusive dependence" (what we now refer to as *linear dependence*)[2].

Algebra with tables of numbers

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Given a matrix A and a scalar λ, the scalar product is $\lambda A = (\lambda a_{ij})$.

Algebra with tables of numbers

The $n \times n$ identity matrix is the matrix

$$I = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

with the property that $AI = IA = A$ for any matrix A (where I is understood to be of the proper size).

The inverse A^{-1} of an $n \times n$ matrix A is the matrix such that $AA^{-1} = A^{-1} = A$.

Abstract lists of numbers

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For completeness's sake, I will include some basic definitions and results from linear algebra.

Vectors

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Definition

A vector space over a field $\mathbb{F}$ is a set V with operations addition $+ : V \times V \longrightarrow V$ and scalar multiplication $\bullet : \mathbb{F} \times V \longrightarrow V$ such that

- For all $u, v, w \in V$, $u + (v + w) = (u + v) + w$
- For all $u, v \in V$, $u + v = v + u$
- There exists $0_V \in V$ such that for all $v \in V$, $v + 0_V = v$
- For all $v \in V$, there exists $-v \in V$ so $v + (-v) = 0_V$
- For all $\lambda, \mu \in \mathbb{F}$, $v \in V$, $\lambda(\mu v) = (\lambda\mu)v$
- For all $v \in V$, $1_{\mathbb{F}} v = v$
- For all $\lambda \in \mathbb{F}$, $u, v \in V$, $\lambda(u + v) = \lambda u + \lambda v$
- For all $\lambda, \mu \in \mathbb{F}$, $v \in V$, $(\lambda + \mu)v = \lambda v + \mu v$

Vectors

Definition

A set $U \subseteq V$ is a linear subspace of V if U is a vector space under the operations on V.

Proposition

$U \subseteq V$ is a linear subspace of V if and only if U is closed under vector addition and scalar multiplication.

Linear combinations

Definition

A linear combination of a set $S \subseteq V$ is a finite sum

$$\sum_{v \in S} \alpha_v v$$

where $\alpha_v \in \mathbb{F}$ and only finitely many are nonzero.
The set of all linear combinations of S is the span of S.

Linear dependence and independence

Definition

A set $S \subseteq V$ is linearly dependent if there exist $(\alpha_v)_{v \in V}$, finitely many nonzero, not all zero, such that

$$0 = \sum_{v \in S} \alpha_v v$$

If S is not linearly dependent we say it is linearly independent.

Basis and dimension

Definition

A basis for a vector space V is a maximal linearly independent set $S \subseteq V$.
Equivalently, S is linearly independent and $\operatorname{span}(S) = V$.

Definition

The dimension of V is the cardinality of a basis for V.

Tables of numbers acting on lists of numbers

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Definition

A function $T : V \longrightarrow W$ between vector spaces is linear if for all $u, v \in V$, $\lambda, \mu \in \mathbb{F}$,

$$T(\lambda v + \mu u) = \lambda T(v) + \mu T(u).$$

Frequently we write Tv to mean $T(v)$.

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$$T(\lambda v + \mu u) = \lambda T(v) + \mu T(u).$$

Frequently we write Tv to mean $T(v)$. A linear map can be encoded as a matrix by choosing a basis for V and then writing the images of each basis vector in terms of the basis.

Kernels and images

Definition

Let U, V be vector spaces and let $T : V \longrightarrow W$ be a linear map. Then

$$\ker(T) = \{v \in V \mid Tv = 0_W\}, \ \operatorname{img}(T) = \{Tv | v \in V\}.$$

Proposition

$\ker(M)$ and $\operatorname{img}(M)$ are linear subspaces.

The characteristic polynomial

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Cayley gave a proof of the Cayley-Hamilton theorem in 1858 (Frobenius gave a full proof in 1878):

Theorem (Cayley-Hamilton, 1858)

Any square matrix M satisfies its characteristic polynomial.

Eigenvalues, eigenvectors, and eigenspaces

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Definition

Let $T : V \longrightarrow V$ be a linear map. We say $\lambda \in \mathbb{F}$ is an eigenvalue of T if there is some $v \in V$, $v \neq 0_V$ such that

$$Tv = \lambda v.$$

The vector v is said to be an eigenvector of T associated with λ, and the set of all such vectors is called the eigenspace associated with λ.

Cauchy coined the term “characteristic root” for the idea of eigenvalues.

Eigenvalues and the characteristic polynomial

Proposition

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Proposition

$$\det(M) = \prod_{\lambda \in \mathrm{Spec}(M)} \lambda$$

Eigenvalue algorithms

There are a number of algorithms to find the spectrum of a matrix:

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Some algorithms are iterative, producing a sequence of values that converges to an eigenvalue, like Jacobi's algorithm, power iteration, or the QR algorithm.

Others involve direct calculation, but there are no general (non-naive) algorithms to directly calculate the spectrum of an arbitrary matrix.

Power Iteration

One of the earliest computational algorithms for identifying eigenvalues is power iteration (sometimes von Mises algorithm).

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Given a matrix $M : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, generate a random vector $b_0 = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ and repeatedly apply M, then renormalize:

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Then if M has a largest eigenvalue λ with (unit) eigenvector v and $\langle b_0, v \rangle > 0$, the sequence (b_k) will converge to v, and the sequence of Rayleigh quotients

$$\mu_k = \frac{\langle b_k, Mb_k \rangle}{\|b_k\|} \xrightarrow[k \to \infty]{} \lambda.$$

Generalized eigenspaces

Definition

Let V be a vector space, let $T : V \longrightarrow V$ be a linear map, and let λ be an eigenvalue of T. The generalized eigenspace associated with λ is the subspace
$E_\lambda = \{v \in V \mid (T - \lambda I)^n v = 0_V \text{ for some } n \in \mathbb{N}\}$.

Theorem (Jordan Decomposition, 1870)

If V is finite-dimensional and $T : V \longrightarrow V$ has all its eigenvalues in $\mathbb{F}$, (i.e. its characteristic polynomial splits), then we can decompose V into its generalized eigenspaces

$$V = \bigoplus_{\lambda \in \mathrm{Spec}(T)} E_\lambda$$

Jordan canonical form

The Jordan canonical form expresses the spectrum and generalized eigenspaces of a transformation:

$$\mathrm{JCF}(T) = \begin{pmatrix} J_0 & 0 & \dots & 0 \\ 0 & J_1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & J_n \end{pmatrix}$$

where $\mathrm{Spec}(T) = \{\lambda_0, \dots, \lambda_n\}$ and each Jordan block J_i is of the form

$$J_i = \begin{pmatrix} \lambda_i & 1 & 0 & \dots & 0 \\ 0 & \lambda_i & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_i \end{pmatrix}$$

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Gather all the Jordan bases for each E_λ into a Jordan basis for T.

The $J_0 \ker(S)$ trick

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To verify that the result of such a computation is correct, we can test each Jordan block, beginning with J_0:

Let $S = T - \lambda_0 I$. Then $E_{\lambda_0} = \ker(S^{n-1})$, so we apply the restriction of $T_{E_{\lambda_0}} = J_0$ to the kernel of S to verify that

$$J_0 \ker(S) = (J_0)^2 \ker(S^2) = \cdots = (J_0)^n \ker(S^n) = \{0_V\}$$

and then we can repeat this "$J_0 \ker(S)$ trick" for each Jordan block.

- Intern et al.
- Christine Andrews-Larson. "Roots of Linear Algebra: An Historical Exploration of Linear Systems". In: *PRIMUS* (Apr. 2015).
- Israel Kleiner. *A History of Abstract Algebra*. Birkhäuser Boston, 2007.
- Yoshio Mikami. "On the Japanese Theory of Determinants". In: *Isis* Vol. 2.1 (1914).
- Randy K. Schwartz. "A Classic from China: The Nine Chapters - Matrices". In: *Convergence* (Dec. 2018).
- B. L. van der Waerden. *Geometry and Algebra in Ancient Civilizations*. New York, New York: Springer-Verlag, 1983.