What is Arrow's Impossibility Theorem?

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OSU What Is?

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Definitions

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Each voter $v \in V$ has an associated preference ordering $\leq_v$, and a *preference profile* (or *situation*) $P = (\leq_1, \ldots, \leq_N) \in \mathcal{L}(A)^V$ is an N-tuple of preference orders.

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A *social welfare function* is a function $F : \mathcal{L}(A)^V \to \mathcal{L}(A)$. We denote the resulting ordering by $\leq_{F(P)}$, or just $\leq$.

More definitions

Let $a_i, a_j \in A$.

1. Unanimity (Pareto efficiency)

   If $a_i <_v a_j$ (that is, $a_i \leq_v a_j$ and $a_j \not\leq_v a_i$) for each $v \in V$, then $a_i \leq_F a_j$.

2. Independence of irrelevant alternatives (IIA)

   If $P = (\leq_v), P' = (\leq'_v) \in \mathcal{L}(A)^V$ such that for all $v \in V$, $a_i \leq_v a_j$ iff $a_i \leq'_v a_j$, then $a_i \leq_{F(P)} a_j$ iff $a_i \leq_{F(P')} a_j$.

3. Non-dictatorship

   There is no $v \in V$ such that for all $P \in \mathcal{L}(A)^V$, $a_i <_v a_j$ implies $a_i <_{F(P)} a_j$.

The statement

Theorem (Arrow's Impossibility Theorem)

There is no social welfare function that satisfies all three conditions.

Proof

Suppose F satisfies unanimity and IIA.

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By unanimity, we begin with $a_i > a_j$ and end with $a_i < a_j$. Denote the first voter whose swap changes the social preference ordering by v_{ij}, the (i, j)-th *pivotal voter*. By IIA, this definition does not depend on P.

Proof

Let $P' = (\leq'_v) \in \mathcal{L}(A)^V$ such that $a_i <'_v a_k <'_v a_j$ for $v = 1, 2, \ldots, v_{ij} - 1$ and $a_k <'_v a_j <'_v a_i$ for $v = v_{ij}, v_{ij} + 1, \ldots, N$.

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Let $P'' = (\leq''_v) \in \mathcal{L}(A)^V$ such that $a_i <'' a_j$ and $a_i <'' a_k$ for $v = 1, \ldots, v_{ij} - 1$, $a_k <_{v_{ij}} a_i <_{v_{ij}} a_j$, and $a_j <'' a_i$ and $a_k <'' a_i$ for $v = v_{ij} + 1, \ldots, N$.

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Let $P' = (\leq'_v) \in \mathcal{L}(A)^V$ such that $a_i <'_v a_k <'_v a_j$ for $v = 1, 2, \ldots, v_{ij} - 1$ and $a_k <'_v a_j <'_v a_i$ for $v = v_{ij}, v_{ij} + 1, \ldots, N$. Then $a_k <' a_j <' a_i$ by the definition of v_{ij} and by unanimity.

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Then by IIA we see that $a_k <_{v_{ij}} a_j$ implies $a_k < a_j$ for all $i \neq j \neq k$.

Proof

	1	…	$v_{ij}-1$	v_{ij}	$v_{ij}+1$	…	N
P'	a_j	…	a_j	a_i	a_i	…	a_i
	a_k	…	a_k	a_j	a_j	…	a_j
	a_i	…	a_i	a_k	a_k	…	a_k
P''	$\square$	…	$\square$	a_j	a_i	…	a_i
	$a_j\square$	…	$a_j\square$	a_i	$\square$	…	$\square$
	$\square$	…	$\square$	a_k	$a_j\square$	…	$a_j\square$
	a_i	…	a_i		$\square$	…	$\square$

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P'	a_j	…	a_j	a_i	a_i	…	a_i
	a_k	…	a_k	a_j	a_j	…	a_j
	a_i	…	a_i	a_k	a_k	…	a_k
P''	$\square$	…	$\square$	a_j	a_i	…	a_i
	$a_j\square$	…	$a_j\square$	a_i	$\square$	…	$\square$
	$\square$	…	$\square$	a_k	$a_j\square$	…	$a_j\square$
	a_i	…	a_i		$\square$	…	$\square$

Thus $a_k < a_j$ does not change as long as $a_k <_{v_{ij}} a_j$, so $v_{ij} \leq v_{jk}$.
Further, $v_{kj} \leq v_{ij}$.

Proof

	1	$\dots$	$v_{ij}-1$	v_{ij}	$v_{ij}+1$	$\dots$	N
	a_j	$\dots$	a_j	a_i	a_i	$\dots$	a_i
P'	a_k	$\dots$	a_k	a_j	a_j	$\dots$	a_j
	a_i	$\dots$	a_i	a_k	a_k	$\dots$	a_k
	$\square$	$\dots$	$\square$	a_j	a_i	$\dots$	a_i
	$a_j\square$	$\dots$	$a_j\square$	a_i	$\square$	$\dots$	$\square$
P''	$\square$	$\dots$	$\square$	a_k	$a_j\square$	$\dots$	$a_j\square$
	a_i	$\dots$	a_i		$\square$	$\dots$	$\square$

Thus $a_k < a_j$ does not change as long as $a_k <_{v_{ij}} a_j$, so $v_{ij} \leq v_{jk}$. Further, $v_{kj} \leq v_{ij}$. But $j \neq k$ are arbitrary, so $v_{kj} \leq v_{ij} \leq v_{jk}$ and $v_{jk} \leq v_{kj}$, so $v_{kj} = v_{ij} = v_{jk}$.

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	a_j	$\dots$	a_j	a_i	a_i	$\dots$	a_i
P'	a_k	$\dots$	a_k	a_j	a_j	$\dots$	a_j
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	$\square$	$\dots$	$\square$	a_j	a_i	$\dots$	a_i
	$a_j\square$	$\dots$	$a_j\square$	a_i	$\square$	$\dots$	$\square$
P''	$\square$	$\dots$	$\square$	a_k	$a_j\square$	$\dots$	$a_j\square$
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Thus $a_k < a_j$ does not change as long as $a_k <_{v_{ij}} a_j$, so $v_{ij} \leq v_{jk}$. Further, $v_{kj} \leq v_{ij}$. But $j \neq k$ are arbitrary, so $v_{kj} \leq v_{ij} \leq v_{jk}$ and $v_{jk} \leq v_{kj}$, so $v_{kj} = v_{ij} = v_{jk}$.

Hence v_{ij} is a dictator. $\square$

Winning sets

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Consider a situation where $a > b$; by IIA, this depends only on the set $U \subseteq V$ of voters who prefer a to b. Call U a "winning set" for a over b if $a > b$ iff $a >_u b$ for each $u \in U$.

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We want to be able to find a suitable collection $\mathcal{U} \subseteq \mathcal{P}(V)$ such that if $\{v : a <_v b\} \in \mathcal{U}$, then $a < b$. That is, we want $\mathcal{U}$ to be a collection of winning sets.

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Further, voting is decisive: if $U \in \mathcal{U}$, then $U^c \notin \mathcal{U}$, and if $U \notin \mathcal{U}$, then $U^c \in \mathcal{U}$.

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Additional votes for an ordering should only help that ordering; if $U \in \mathcal{U}$ and $U \subseteq W \subseteq V$, then $W \in \mathcal{U}$, so in particular $V \in \mathcal{U}$.

Winning sets

Let $U = \{u : a <_u b\}$ and $W = \{w : b <_w c\}$, and suppose $U, W \in \mathcal{U}$. Then $a < b$ and $b < c$, hence $a < b < c$.

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Then $U \cap W \subseteq \{v : a <_v c\}$, and $\{v : a <_v c\} \in \mathcal{U}$.

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It may be that $U \cap W = \{v : a <_v c\}$; as when

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\begin{aligned}
\{v : c <_v a <_v b\} &= U \setminus W, \\
\{v : b <_v c <_v a\} &= W \setminus U, \\
\{v : c <_v b <_v a\} &= (U \cup V)^c.
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Thus it must be in fact that $U \cap W \in \mathcal{U}$.

Winning sets

So we have a collection $\mathcal{U} \subseteq \mathcal{P}(V)$ such that

1. $\varnothing \notin \mathcal{U}$.
2. If $U \in \mathcal{U}$ and $U \subseteq W \subseteq V$, then $W \in \mathcal{U}$.
3. If $U, W \in \mathcal{U}$, then $U \cap W \in \mathcal{U}$.
4. For each $U \subseteq V$, either $U \in \mathcal{U}$ or $U^c \in \mathcal{U}$.

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4. For each $U \subseteq V$, either $U \in \mathcal{U}$ or $U^c \in \mathcal{U}$.

The collection $\mathcal{U}$ is an *ultrafilter* on V!

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Our collection $\mathcal{U}$ of winning sets is a *principal ultrafilter*.

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Theorem (Arrow's Impossibility Theorem (Version 2))

There are no nonprincipal ultrafilters on finite sets.

We could in fact create a nondictatorial Pareto IIA social welfare function by constructing a *nonprincipal ultrafilter*, provided we have infinite voters and the Axiom of Choice.

Ultraproducts

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Given an index set I, a family of "structures" (in our case, an ordered set) $(M_i)_{i\in I}$, and an ultrafilter $\mathcal{U}$ on I, we define the equivalence relation $a \sim b$ if $\{i \in I : a_i = b_i\} \in \mathcal{U}$. Then the *ultraproduct* is the quotient structure $\prod_{i\in I} M_i/\sim$, sometimes written $\prod_{i\in I} M_i/\mathcal{U}$.

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Let σ be a signature, $\mathcal{U}$ an ultrafilter on a set I, and let $(M_i)_{i\in I}$ be a family of σ-structures. Let $M = \prod_{i\in I} M_i/\mathcal{U}$. Then for each $a^1, \ldots, a^n \in \prod_{i\in I} M_i$, where $a^k = (a_i^k)_{i\in I}$, and every σ-formula ϕ,

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$$M \models \phi[[a^1], \ldots, [a^n]] \iff \{i \in I : M_i \models \phi[a_i^1, \ldots, a_i^n]\} \in \mathcal{U}.$$

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Bonus Exercise. □

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1. *For each Pareto IIA social welfare function F there is a unique ultrafilter $\mathcal{U}_F$ such that for each $U \in \mathcal{U}_F$, each $a, b \in A$, and any preference profile P, if $\{u : a \leq_u b\} \in \mathcal{U}_F$ then $a \leq_{F(P)} b$.*

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2. *For each ultrafilter $\mathcal{U}_F$ on V there exists such an F.*

Some more black boxes

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Fact (Fact 1)

For any atomless measure space $(X, \mathcal{M}, \mu)$*, for all* $\varepsilon > 0$*, there is a finite partition* $X = \bigsqcup_{i=1}^{r} C_i$*, such that each* $C_i \in \mathcal{M}$ *and* $\mu(C_i) < \varepsilon$*.*

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Fact (Fact 1)

For any atomless measure space $(X, \mathcal{M}, \mu)$, *for all* $\varepsilon > 0$, *there is a finite partition* $X = \bigsqcup_{i=1}^{r} C_i$, *such that each* $C_i \in \mathcal{M}$ *and* $\mu(C_i) < \varepsilon$.

Fact (Fact 2)

Let $\mathcal{U}$ *be an ultrafilter on a set* X. *For any finite partition* $X = \bigsqcup_{i=1}^{r} C_i$, *exactly one* $C_i \in \mathcal{U}$.

Arbitrarily small coalitions

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For any $\varepsilon > 0$ there exists $C \in \mathcal{V}$, $C \neq \varnothing$, $\lambda(C) < \varepsilon$ such that for any $a, b \in A$ and any situation P, if $\{v : a \leq_c b\} \subseteq C$ then $a \leq_{F(P)} b$.

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For any $\varepsilon > 0$ there exists $C \in \mathcal{V}$, $C \neq \varnothing$, $\lambda(C) < \varepsilon$ such that for any $a, b \in A$ and any situation P, if $\{v : a \leq_c b\} \subseteq C$ then $a \leq_{F(P)} b$.

Proof.

Apply the first fact to $(V, \mathcal{V}, \lambda)$. Then we have a partition $X = \bigsqcup_{i=1}^{r} C_i$ such that each $C_i \in \mathcal{M}$ and $\mu(C_i) < \varepsilon$. Take the ultrafilter $\mathcal{U}_F$ guaranteed by the uniqueness of winning ultrafilters and apply the second fact. □

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We have the following commutative diagram:

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Note that V is dense in βV.

One more black box

Fact

Let $\mathcal{U}$ be an ultrafilter on a topological space X. Then $\mathcal{U}$ has a (unique) limit in βX.

A hidden dictator

Theorem

There exists a point $\hat{v} \in \beta V$ *such that for all* $a, b \in A$ *and* $P \in \mathcal{L}(A)^V$, *if* $a \leq_{\hat{v}} b$ *then* $a \leq_{F(P)} b$.

Proof.

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We know that there is a unique winning ultrafilter $\mathcal{U}_F$.

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We know that there is a unique winning ultrafilter $\mathcal{U}_F$.
By the previous fact, $\mathcal{U}_F$ has a unique limit $\hat{v} \in \beta V$.

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We know that there is a unique winning ultrafilter $\mathcal{U}_F$.
By the previous fact, $\mathcal{U}_F$ has a unique limit $\hat{v} \in \beta V$.
Then $\beta P(\hat{v}) = \mathcal{U}_{\mathcal{F}} \lim_v P(v)$. Since $\mathcal{L}(A)$ is discrete, there is some $U \in \mathcal{U}_F$ with $P(U) = \beta P(\hat{v})$.

A hidden dictator

Theorem

There exists a point $\hat{v} \in \beta V$ such that for all $a, b \in A$ and $P \in \mathcal{L}(A)^V$, if $a \leq_{\hat{v}} b$ then $a \leq_{F(P)} b$.

Proof.

We know that there is a unique winning ultrafilter $\mathcal{U}_F$.
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By the properties of $\mathcal{U}_F$, U is a winning set, and hence $\hat{v}$ dictates F. $\square$

References

- Kenneth J. Arrow. "A Difficulty in the Concept of Social Welfare". In: *Journal of Political Economy* 58.4 (1950).
- Alan P Kirman and Dieter Sondermann. "Arrow's theorem, many agents, and invisible dictators". In: *Journal of Economic Theory* 5.2 (1972).
- David Pierce. *Ultraproducts*. 2012.
- Ning Neil Yu. "A one-shot proof of Arrow's impossibility theorem". In: *Economic Theory* 50.2 (2012).
- Intern et al.