The History of Curvature

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OSU Reading Classics

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Lines Don't Curve; Circles Do

Aristotle (384 - 322 BCE) made the astute observation that lines do not curve, but circles curve uniformly at each point. Thus, he classified plane loci into three categories: linear, circular, and mixed. [1]

Apollonius of Perga (c. 240 BCE - c. 190 BCE) made (implicitly) the first calculation of curvature when considering the problem of drawing normal segments to conic sections, recognizing that such a normal segment was unique.

Nicole Oresme

Nicole Oresme (c. 1320 - 1382) was a French philosopher in the Late Middle Ages whose *Tractatus de Configurationibus Qualitatum et Motuum* (Treatise on the Configurations of Qualities and Motions) contained the first characterization of *Curvitas*, noting that the curvature of a circle is inversely proportional to its radius. [1]

Attempting to generalize to other curves, Oresme stated that if two curves are tangent to a line at the same point from the same side, then the "smaller" of the two should have more curvature than the larger.

Oresme's *Tractatus* also contained a number of other important discoveries, such as an early use of rectangular coordinates, a computation of geometric series, and is credited with the first proof of the divergence of the harmonic series. [2]

Kepler's Circle of Curvature

Johannes Kepler (1571 - 1630) was the first to suggest approximating a curve by its *circle of curvature*, the "closest" circle to the curve at a point. [1]

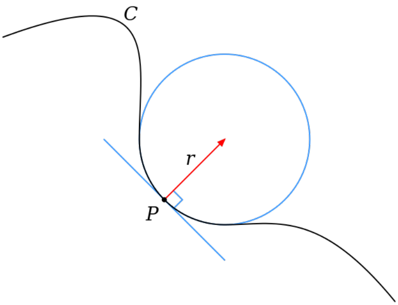


Figure: An osculating circle [3]

Huygens' Construction

Christiaan Huygens (1629 - 1695) later gave a general way to produce the circle of curvature in 1673 in *Horologium Oscillatorium: Sive de Motu Pendulorum ad Horologia Aptato Demonstrationes Geometricae* (The Pendulum Clock: or Geometrical Demonstrations Concerning the Motion of Pendula as Applied to Clocks) using involutes and evolutes.

Begin with a curve (the *evolute*) whose tangent lines are all on one side, and to this attach a flexible string which is pulled taught and then unwound, tracing the *involute*.

Assuming the string is always tangent to the evolute, Huygens shows that it will always be normal to the involute, and that the involute and evolute are unique.

Huygens' Construction (cont.)

The radius of curvature of the involute, then, is the distance up the string from the involute to the evolute. (Apollonius was unknowingly calculating this in the special case of conics.)

Notice that this requires the *evolute* of the curve to be given in advance.

Huygens calculated the radii of curvature for some special cases (notably the cycloid) and attempts to provide a general method, but was not aware of Newton's and Leibniz's calculus and so failed to give a general solution. [1]

Newton's Method of Fluxions

Isaac Newton (1642 - 1727[1]) used calculus to describe the center and radius of curvature for a curve at a point in Problem V of his posthumously published work *The Method of Fluxions and Infinite Series: with its Application to the Geometry of Curve-Lines*. [4]

Like Huygens and Kepler, Newton reduces the problem to finding the radius of the osculating circle.

[1] Newton died in 1726 according to the Julian calendar, but in 1727 according the the Gregorian calendar; since England had not yet adopted the Gregorian calendar, Newton's death was recorded as being in 1726 but is usually adjusted to 1727 despite his birth usually being listed exclusively by the Julian date.

Newton's Method of Fluxions (cont.)

If a Circle touches any Curve on its concave ſide, in any given Point, and if it be of ſuch magnitude, that no other tangent Circle can be interſcribed in the Angles of Contact near that Point, that Circle will be of the ſame Curvature as the Curve is of, in that Point of Contact. [...]

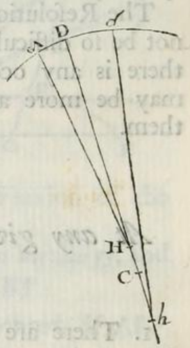
Therefore the Center of Curvature to any Point of a Curve, is the Center of a Circle equally curved. And thus the Radius, or Semidiameter of Curvature is part of the Perpendicular to the Curve, which is terminated at that Center. [...]

And the Proportion of Curvature at different Points will be known from the Proportion of Curvature of aequi-curve Circles, or from the reciprocal Proportion of the Radii of Curvature.

Newton's Method of Fluxions (cont.)

Newton then describes a general method for producing the center of convergence: [4]

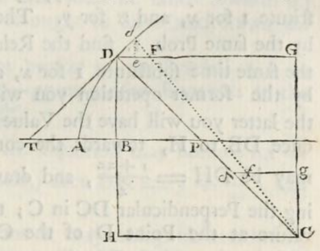
7. Imagine therefore that at three Points of the Curve δ, D, and *d*, Perpendiculars are drawn, of which thoſe that are at D and δ meet in H, and thoſe that are at D and *d* meet in *h*: And the Point D being in the middle, if there is a greater Curvity at the part Dδ than at D*d*, then DH will be leſs than *dh*. But by how much the Perpendiculars δH and *dh* are nearer the intermediate Perpendicular, ſo much the leſs will the diſtance be of the Points H and *h*: And at laſt when the Perpendiculars meet, thoſe Points will coincide. Let them coincide in the Point C, then will C be the Center of Curvature, at the Point D of the Curve, on which the Perpendiculars ſtand; which is manifeſt of itſelf.



Newton's Method of Fluxions (cont.)

Using his construction, Newton goes on to show that putting $x = AB$ with $\dot{x} = 1$ and $y = BD$ with $z = \dot{y}$ in the following figure [4] the radius of convergence is given by

$$DC = \frac{(1 + z^2)^{3/2}}{\dot{z}}$$



In modern notation, the radius of curvature of a plane curve $y = y(x)$ is

$$R = \left| \frac{(1 + y'(x)^2)^{3/2}}{y''(x)} \right| .$$

Newton's Contemporaries

Gottfried Wilhelm Leibniz (1646 - 1716) is credited with the term "osculating circle" (or *Circulus Osculans*) in his 1686 paper *Meditation Nova de Natura Anguli Contactus et Osculi* (New Meditation on the Nature of Angles of Contact and Osculation). [1][5]

Jacob Bernoulli (1655 - 1705) also worked on the curvature of general curves, particularly in his general construction of evolutes from a curve: the evolute of a curve $c(t)$ is given by

$$E(t) = c(t) + R(t)n(t)$$

where $R(t)$ is the radius of curvature at $c(t)$ and $n(t)$ is the unit normal vector towards the center of curvature.

Johann Bernoulli (1667 - 1748) also studied curvature, including the radius of curvature of a curve in polar coordinates.

Parameterization by Arc Length

Recall that the length of a smooth curve $\gamma : [a, b] \to \mathbb{R}^n$ is given by

$$L(\gamma) = \int_a^b \|\dot{\gamma}(t)\| \, dt.$$

Then the *arc length* of γ is the function

$$s(t) = \int_a^t \|\dot{\gamma}(t)\| \, dt.$$

This is a smooth bijection from $[a, b]$ to $[0, L(\gamma)]$. Writing $t(s)$ for the inverse we have the *arc length parameterization* of γ by

$$\gamma(s) = \gamma(t(s)).$$

Such a curve has the useful property that $\|\dot{\gamma}(s)\| \equiv 1$.

Euler and the Second Derivative

Leonhard Euler (1707 - 1783) proved the following in 1736: [5]

Theorem

Let $\gamma(s)$ be parameterized by arc length. Then the (unsigned) curvature κ is given by the magnitude of the acceleration:

$$\kappa(\gamma; s) = \|\ddot{\gamma}(s)\|.$$

Frenet and Serret

Jean Frédéric Frenet (1816 - 1900) and Joseph Alfred Serret (1819 - 1885) derived the following formulas in 1847 and 1851:

Theorem

Let $\gamma : [a, b] \to \mathbb{R}^3$ be parameterized by arc length, $T = \dot{\gamma}$, $N = \frac{dT}{ds} / \left\| \frac{dT}{ds} \right\|$, and $B = T \times N$; denote the curvature by κ and torsion by τ. Then

$$\frac{dT}{ds} = \kappa N, \ \frac{dN}{ds} = -\kappa T + \tau B, \ \frac{dB}{ds} = -\tau N.$$

These are the *Frenet-Serret formulas*, and together T, N, B, κ, τ (the *Frenet-Serret apparatus*) specify γ up to rigid motions of $\mathbb{R}^3$.

Smooth Manifolds

Definition

A *(topological) n-manifold* M is a topological space which is locally homeomorphic to $\mathbb{R}^n$. That is, for each $p \in M$ there is a neighborhood $U \ni p$ which is homeomorphic to an open subset of $\mathbb{R}^n$. We may also require that M be Hausdorff and second countable.

Definition

A *smooth structure* (or *atlas*) on M is a collection of *charts* $(U_\alpha, \varphi_\alpha)$ such that $\varphi_\alpha : U_\alpha \to \mathbb{R}^n$ is a homeomorphism onto its image and the *transition functions* $\varphi_\alpha \circ \varphi_\beta^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ are smooth. A topological manifold M endowed with a smooth structure is said to be a *smooth* (or *differentiable*) *manifold*.

Riemannian Manifolds

Definition

To each point $p \in M$ the *tangent space at p*, denoted T_pM, is the vector space of "directional derivatives" at p. The collection of all such tangent spaces, appropriately topologized according to the smooth structure of M, is the *tangent bundle TM*.

Definition

A *Riemannian metric g* on a smooth manifold M is a "smoothly varying" inner product $g_p : T_pM \times T_pM \to \mathbb{R}$. We say a smooth manifold M endowed with such a metric is a *Riemannian manifold*.

Vector Fields and the Lie Bracket

Definition

A *(tangent) vector field* X on M is a smoothly varying assignment to each $p \in M$ a vector $X_p \in T_pM$. We denote by $\Gamma(TM)$ the set of all such vector fields on M.

Definition

The *Lie bracket* $[X, Y]$ of two vector fields $X, Y \in \Gamma(TM)$ is the vector field given by

$$[X, Y]f = X(Yf) - Y(Xf)$$

where we write Xf to mean "the directional derivative of f in the direction X" at each point $p \in M$.

The Levi-Cività Connection

Definition

A *connection* on TM is a bilinear map $\nabla : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$ written $(X, Y) \mapsto \nabla_X Y$ such that for any smooth function $f : M \to \mathbb{R}$,

1. $\nabla_{fX} Y = f \nabla_X Y$
2. $\nabla_X fY = (\partial_X f) Y + f \nabla_X Y$.

Theorem

Every Riemannian manifold (M, g) has a unique connection ∇ (the Levi-Cività connection) which is

1. **Metric compatible:** $X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$
2. **Torsion free:** $\nabla_X Y - \nabla_Y X = [X, Y]$.

The Curvature Tensor

Definition

The *curvature tensor* of a Riemannian manifold is the map $R : \Gamma(TM) \times \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$ defined by

$$R(X, Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

An Aside on Holonomy

The curvature tensor measures the *holonomy* of a connection, or the failure of the *parallel transport* around a loop to preserve the direction of a tangent vector, by parallel translating a vector around an infinitesimal parallelogram given by X and Y.

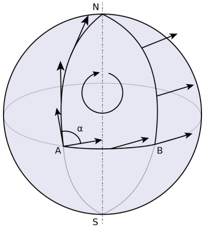


Figure: Holonomy around a geodesic triangle on the sphere.[2]

[2]Created by Fred the Oyster on Wikipedia Commons, used and shared under (CC BY-SA 4.0).

An Aside on Holonomy (cont.)

The holonomy $\mathrm{Hol}_p(M)$ at a point p forms a *Lie subgroup* of $O(T_pM)$; the *restricted holonomy* $\mathrm{Hol}_p^0(M)$ is the subgroup of holonomy considering only contractible loops and is related to curvature by the following theorem:

Theorem (Ambrose-Singer (1953))

The Lie algebra $\mathfrak{hol}_p^0(M) = T_{\mathrm{id}} \mathrm{Hol}_p^0(M)$ *is the Lie subalgebra of* $\mathrm{End}(T_pM)$ *generated by* $R(v,w)$ *for* $v, w \in T_pM$*.*

Holonomy also gives rise to the remarkable De Rham Decomposition theorem (1952), which states roughly that holonomy can be used to locally decompose a simply connected manifold as the product of “nice” submanifolds, and further that this decomposition can be made global if M is complete.

Ricci and Scalar Curvatures

Definition

The *Ricci tensor* of a Riemannian manifold is the map $\mathrm{Ric} : \Gamma(TM) \times \Gamma(TM) \to C^\infty(M)$ defined by

$$\mathrm{Ric}(Y, Z) := \mathrm{tr}(X \mapsto R(X, Y)Z).$$

Definition

The *scalar curvature* of a Riemannian manifold is the trace of the Ricci tensor with respect to the metric:

$$\mathrm{Scal} := \mathrm{tr}_g \mathrm{Ric}\,.$$

Sectional Curvature

Definition

Given two linearly independent vectors $u, v \in T_pM$, the *sectional curvature* is given by

$$K(u, v) := \frac{\langle R_p(u, v)v, u \rangle}{\langle u, u \rangle \langle v, v \rangle - \langle u, v \rangle^2}.$$

It can be shown that the Riemannian curvature tensor can be recovered from the sectional curvature.

Notice that in the case $\dim M = 2$, the (unique) sectional curvature of M is the Gaussian curvature.

Isotropic and Constant Sectional Curvature

Definition

If the sectional curvature K of M is at each point independent of the choice of u, v then M is said to be *isotropic*. If in addition the sectional curvature is equal at each point of M we say M has *constant sectional curvature*.

Lemma ((Friedrich) Schur)

An isotropic connected manifold of dimension > 2 has constant sectional curvature.

Of particular interest in Riemannian geometry are the so-called *model spaces* $\mathbb{R}^n$, $\mathbb{S}^n$, and $\mathbb{H}^n$, which have constant sectional curvature 0, 1, and -1, respectively.

The Model Spaces $\mathbb{R}^n$, $\mathbb{S}^n$, and $\mathbb{H}^n$

Consider standard Euclidean space with the usual inner product $(\mathbb{R}^n, g^{\text{can}})$.

Now, in $(\mathbb{R}^{n+1}, g^{\text{can}})$ denote by $\mathbb{S}^n$ the usual unit sphere

$$\mathbb{S}^n := \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}.$$

Consider now $\mathbb{R}^{n,1}$, which is $\mathbb{R}^n$ with the *Lorentzian* form

$$g^{\text{Lor}}(x, y) = \left(\sum_{i=1}^{n} x_i y_i\right) - x_{n+1} y_{n+1}$$

and denote by $\mathbb{H}^n$ the hyperbolic space

$$\mathbb{H}^n := \{x \in \mathbb{R}^{n+1} \mid \|x\|^{\text{Lor}} = 1\}.$$

The Model Spaces $\mathbb{R}^n$, $\mathbb{S}^n$, and $\mathbb{H}^n$

Scaling the unit sphere $\mathbb{S}^n$ and hyperbolic space $\mathbb{H}^n$, we can actually obtain a space of constant sectional curvature κ for any $\kappa \in \mathbb{R}$; for $\kappa > 0$, the sphere $\mathbb{S}^n(\kappa^{-1/2})$ of radius $\kappa^{-1/2}$ has constant curvature κ and hyperbolic space $\mathbb{H}^n(\kappa^{-1/2})$ has constant sectional curvature $-\kappa$.

We denote such manifolds by M^n_κ.

Theorems on the Model Spaces

Theorem

Any complete, connected, simply connected manifold M of constant sectional curvature κ is isometric to the model space M_κ^n.

Theorem (Killing-Hopf)

Any complete, connected n-manifold M with constant sectional curvature κ (a so-called space form) has universal cover M_κ^n. That is, M is a quotient of M_κ^n by a free and properly discontinuous group action.

References

[1] J. L. Coolidge. "The Unsatisfactory Story of Curvature". In: *The American Mathematical Monthly* 59.6 (1952), pp. 375–379. ISSN: 00029890, 19300972. URL: http://www.jstor.org/stable/2306807.

[2] Stefan Kirschner. "Nicole Oresme". In: *The Stanford Encyclopedia of Philosophy*. Ed. by Edward N. Zalta. Fall 2021. Metaphysics Research Lab, Stanford University, 2021.

[3] Intern et al.

[4] Isaac Newton and John Colson. *The Method of Fluxions and Infinite Series; With Its Application to the Geometry of Curve-Lines*. London : Printed by Henry Woodfall; and sold by John Nourse, 1736.

[5] John McCleary. *Geometry from a Differentiable Viewpoint*. Cambridge University Press, 2012.