1 Simplicial Complexes.

1.1 Definitions and Preliminaries.

Definition 1.1.1. Let $V = \{v_0, \ldots, v_n\}$ be a finite set. An *abstract simplicial complex* K on V is a collection of nonempty finite subsets of V such that

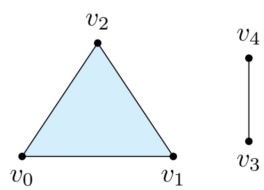
(a) for each $v_i \in V$, $\{v_i\} \in K$.

(b) for any $\sigma \in K$, if $\varnothing \neq \tau \subseteq \sigma$ then $\tau \in K$.

The set $V(K) = V$ is said to be the *vertex* set of K.

Remark. Abstract simplicial complexes are combinatorial representations of *geometric simplicial complexes*. In particular, any geometric simplicial complex can be expressed as an abstract simplicial complex, and every abstract simplicial complex K has a *geometric realization* $|K|$:[1] choose an affinely independent embedding $f : V(K) \longrightarrow \mathbf{R}^n$, and identify each simplex $\sigma \in K$ with the geometric simplex spanned by $f(\sigma)$. For example,

$$K = \{\{v_0\}, \{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}, \{v_0, v_1\}, \{v_0, v_2\}, \{v_1, v_2\}, \{v_3, v_4\}, \{v_0, v_1, v_2\}\}$$



Definition 1.1.2. Let K and L be simplicial complexes. A function $f : V(K) \longrightarrow V(L)$ is a *simplicial map* if for any simplex $\sigma = \{v_0, \ldots, v_n\} \in K$, the images $\{f(v_0), \ldots, f(v_n)\}$ is a simplex in L. That is, $f(\sigma) \in L$. A bijective simplicial map is a *simplicial isomorphism*, and we say K and L are *isomorphic*, $K \cong L$, if there is a simplicial isomorphism between them.

Remark. The collection of simplicial complexes together with simplicial maps forms the category **SCpx** of simplicial complexes. Note that even though we are only considering finite simplicial complexes, the collection of simplicial complexes is a proper class.

Definition 1.1.3. A set σ with $|\sigma| = n+1$ is said to be an *n-simplex*. The *dimension* $\dim(K)$ of a simplicial complex K is the maximum of the dimension of its simplices. The *n-skeleton* $K^{(n)}$ of K is the set of all n-simplices in K (in particular $V(K) = K^{(0)}$), and the *c-vector* of K is the tuple $c_K \coloneqq (|K^{(0)}|, \ldots, |K^{(\dim(K))}|)$. A simplicial complex $L \subseteq K$ is said to be a *subcomplex* of K, and for a simplex $\sigma \in K$ we denote by $\bar{\sigma}$ the *subcomplex generated by* σ (note that $\bar{\sigma} = \mathcal{P}(\sigma)$). For two simplices $\sigma, \tau \in K$ with $\tau \subseteq \sigma$, we say that τ is a *face* of σ and σ is a *coface* of τ.

Definition 1.1.4. Let K be a simplicial complex. Then $\sigma^{(n)}$ denotes an n-simplex in K, and write $\tau < \sigma^{(n)}$ to mean $\tau \subsetneq \sigma^{(n)}$; $\dim(\sigma) - \dim(\tau)$ is the *codimension of* τ *with respect to* σ. The *boundary* of a simplex σ is $\partial_K(\sigma) \coloneqq \partial(\sigma) = \{\tau \in K^{(\dim(\sigma)-1)} : \tau < \sigma\}$. A maximal simplex in K, that is, a simplex σ which is not properly contained in any other simplex in K, is said to be a *facet* of K.

Proposition 1.1.5. Let K be a simplicial complex and let σ be an n-simplex in K. Then $|\partial(\sigma)| = \dim(\sigma)+1$.

Proof. Recall that $|\sigma| = n+1$. Each face $\tau \in \partial(\sigma)$ is precisely a subset of σ with $|\tau| = |\sigma| - 1 = n$, of which there are $\binom{n+1}{n} = n+1 = \dim(\sigma)+1$. ☐

Definition 1.1.6. Let V be a finite set and $\mathcal{H} \subseteq \mathcal{P}(V) \setminus \{\varnothing\}$. The *simplicial complex generated by* $\mathcal{H}$, denoted by $\langle \mathcal{H} \rangle$, is the smallest simplicial complex containing $\mathcal{H}$.

Proposition 1.1.7. If $\mathcal{H}$ is a simplicial complex then $\langle \mathcal{H} \rangle = \mathcal{H}$.

[1] Wikipedia. *Abstract simplicial complex — Wikipedia, The Free Encyclopedia.* 2021. URL: https://en.wikipedia.org/wiki/Abstract_simplicial_complex.

Proof. By definition, $\mathcal{H} \subseteq \langle \mathcal{H} \rangle$. Since $\mathcal{H}$ is a simplicial complex containing $\mathcal{H}$, $\langle \mathcal{H} \rangle \subseteq \mathcal{H}$. Thus $\langle \mathcal{H} \rangle = \mathcal{H}$. □

Proposition 1.1.8. If K is a simplicial complex and $\sigma \in K$, then $\bar{\sigma} = \langle \{\sigma\} \rangle$.

Proof. By definition, $\bar{\sigma} \subseteq \langle \{\sigma\} \rangle$, since $\langle \{\sigma\} \rangle$ is a simplicial complex which contains σ. Similarly, $\bar{\sigma}$ is a simplicial complex containing σ, and hence $\langle \{\sigma\} \rangle \subseteq \bar{\sigma}$. Thus $\bar{\sigma} = \langle \{\sigma\} \rangle$. □

Some Examples.

Let $V_n = \{v_0, \ldots, v_n\}$ throughout.

Example 1.1.9. We will (through some abuse of notation) denote by 0 the *trivial simplicial complex* $\langle \{V_0\} \rangle$. We may further abuse notation by identifying the simplicial complex 0 with its point 0.

Example 1.1.10. The simplicial complex $\Delta^n := \mathcal{P}(V_n) \setminus \{\varnothing\}$ is the (combinatorial) *n-simplex*.

The simplicial complex $\Delta S^n := \Delta^{n+1} \setminus V_{n+1}$ is the *simplicial n-sphere*. Its geometric realization is homeomorphic to the usual n-sphere S^n; it may be convenient to identify ΔS^n with S^n

Example 1.1.11. Any simplicial complex K of dimension 1 forms a *graph* $(V(K), K^{(1)})$.

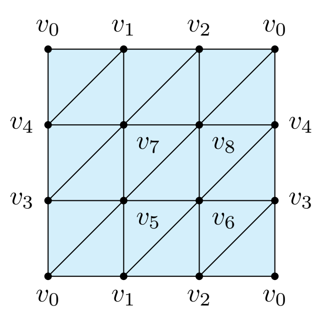
Definition 1.1.12. Given a simplicial complex K, let $v \notin V(K)$ and define the *cone over* K by

$$CK := K \cup \{v\} \cup \{\sigma \cup \{v\} : \sigma \in K\}.$$

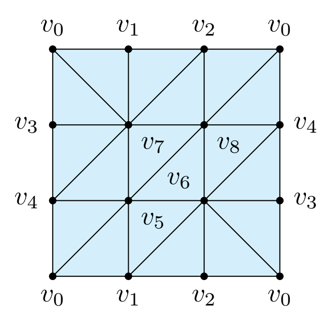
Proposition 1.1.13. The cone over a simplicial complex K is a simplicial complex.

Proof. Let $\sigma \in CK$. We wish to show that any subset $\tau \subseteq \sigma$ is an element of CK. If $v \notin \sigma$, then $\sigma \in K$ and so $\tau \in K \subseteq CK$. If $v \in \sigma$, then let $\sigma' = \sigma \setminus \{v\} \in K$. If $v \notin \tau$, then $\tau \subseteq \sigma'$ and hence $\tau \in CK$. If $v \notin \tau$, then $\tau \setminus \{v\} \subseteq \sigma' \in K$, so $\tau \setminus \{v\} \in K$ and hence $\tau = (\tau \setminus \{v\}) \cup \{v\} \in CK$. □

Example 1.1.14. As with the n-sphere, common topological spaces can be *triangulated* and represented as simplicial complexes:



The torus T^2, $c_{T^2} = (9, 27, 18)$



The Klein bottle K, $c_K = (9, 27, 18)$

Definition 1.1.15. Let K be a simplicial complex and let $n = \dim(K)$, and put $c_i = |K^{(i)}|$. The *Euler characteristic* $\chi(K)$ of K is given by

$$\chi(K) = \sum_{i=0}^{n} (-1)^i c_i(K).$$

1.2 Simple Homotopy.

Definition 1.2.1. Let K be a simplicial complex and suppose we have $\sigma^{(d-1)} < \tau^{(d)} \in K$ such that τ is the only coface of σ. Then the simplicial complex $K \setminus \{\sigma, \tau\}$ is an *elementary collapse* of K, denoted by $K \searrow K \setminus \{\sigma, \tau\}$. Now suppose $\sigma^{(d-1)} < \tau^{(d)} \notin K$, where every other face of τ is in K. Then $K \nearrow K \cup \{\sigma, \tau\}$ in an *elementary expansion* of K. The pair $\{\sigma, \tau\}$ is a *free pair* of simplices. Two simplicial complexes K and L are said to be *simple homotopy equivalent* or of the same *simple homotopy type*, denoted K Kern.05pt$\nearrow\!\searrow$ L if there is a series of elementary collapses and expansions from K to L. If K Kern.05pt$\nearrow\!\searrow$ 0, K is said to have the *simple homotopy type of a point*.

Remark. This is a refinement of the typical notion of homotopy equivalence. [2]

Proposition 1.2.2. Let $\{\sigma, \tau\}$ be a free pair of a simplicial complex K. Then $K \setminus \{\sigma, \tau\}$ is a simplicial complex.

Proof. Since τ is the only coface of σ, it must be that τ is a facet of K (for otherwise σ would have another coface), so we may remove τ; we may also remove σ because τ is the only coface of σ, so σ is a facet of $K \setminus \{\tau\}$, hence $K \setminus \{\sigma, \tau\}$ is a simplicial complex. □

Proposition 1.2.3. Simple homotopy equivalence of simplicial complexes is an equivalence relation.

Proof. Let K be a simplicial complex. Of course K is simple homotopy equivalent to itself, since doing no collapses or expansions preserves K. Now let L be a simplicial complex and K Kern.05pt$\nearrow\!\searrow$ L. Then each elementary collapse can be undone by an elementary expansion and vice versa, so L Kern.05pt$\nearrow\!\searrow$ K. Finally, let H be a simplicial complex and suppose K Kern.05pt$\nearrow\!\searrow$ L and L Kern.05pt$\nearrow\!\searrow$ H. Then there is a series of elementary collapses from K to L, and from L to H, so simply concatenating these yields a series of elementary collapses and expansions from K to H. Hence K Kern.05pt$\nearrow\!\searrow$ H and so simple homotopy equivalence is an equivalence relation. □

Definition 1.2.4. A property $\alpha(K)$ of a simplicial complex K is a *simple homotopy invariant* if for any L Kern.05pt$\nearrow\!\searrow$ K, we have $\alpha(L) = \alpha(K)$.

Proposition 1.2.5. Th Euler characteristic $\chi(K)$ is a simple homotopy invariant.

Proof. Let $\{\sigma^{(d-1)}, \tau^{(d)}\}$ be a free pair in K and let $K' = K \setminus \{\sigma, \tau\}$. Then if $c_K = (c_0, \ldots, c_n)$ then the elementary collapse $K \searrow K'$ has $c_{\{}K'\} = (c_0, \ldots, c_{d-1} - 1, c_d - 1, \ldots, c_n)$ so $\chi(K') = \chi(K) + (-1)^{d-1} + (-1)^d = \chi(K) + 1 - 1 = \chi(K)$. Similarly, if K' is the result of an elementary expansion then $\chi(K') = \chi(K)$. By induction, $\chi(K)$ is invariant under any finite number of elementary collapses and expansions, so χ is a simple homotopy invariant. □

Definition 1.2.6. A simplicial complex K is *collapsible* if there is a sequence of elementary collapses

$$K = K_0 \searrow \cdots \searrow K_n = 0.$$

Definition 1.2.7. Let K and L be (disjoint) simplicial complexes. Their *join* $K * L$ is the simplicial complex

$$K * L \coloneqq K \sqcup \mathcal{L} \sqcup \{\sigma \sqcup \tau : \sigma \in K, \tau \in L\}.$$

Remark. The cone over K is a special case of a join: $CK \cong K * 0$.

Definition 1.2.8. Let $v, w \notin K$, $v \neq w$. The *suspension* of a simplicial complex K is defined by $\Sigma K \coloneqq K * \{v, w\}$.

Proposition 1.2.9. Let σ be a facet of a simplicial complex K. Let $CK = K * \{v\}$. Then $CK \setminus \{\sigma, \sigma \cup \{v\}\} = C(K \setminus \{\sigma\})$.

[2] More details (and generality) can be found in Marshall M. Cohen's *A Course in Simple-Homotopy Theory.*

Proof.

$$\begin{aligned} CK \setminus \{\sigma, \sigma \cup \{v\}\} &= (K * \{v\}) \setminus \{\sigma, \sigma \cup \{v\}\} \\ &= (K \cup \{v\} \cup \{\tau \cup \{v\} : \tau \in K\}) \setminus \{\sigma, \sigma \cup \{v\}\} \\ &= (K \setminus \{\sigma\}) \cup \{v\} \cup \{\tau \cup \{v\} : \tau \in (K \setminus \{\sigma\})\} \\ &= (K \setminus \{\sigma\}) * \{v\} \\ &= C(K \setminus \{\sigma\}) \end{aligned}$$

□

Proposition 1.2.10. The cone over any simplicial complex is collapsible.

Proof. By induction on $n = |K|$. Of course for $n = 0$, $CK = C0$ is collapsible. Now suppose that for some $n \geq 1$, any cone over n simplices is collapsible. Then let K be a simplicial complex with $n + 1$ simplices, and consider $CK = K * \{v\}$. Notice that if σ is a facet of K then $\{\sigma, \sigma \cup \{v\}\}$ is a free pair in CK, so $CK \searrow CK \setminus \{\sigma, \sigma \cup \{v\}\}$. But $CK \setminus \{\sigma, \sigma \cup \{v\}\} = C(K \setminus \{\sigma\})$ is a cone over n simplices and hence is contractible by the inductive hypothesis. Thus $CK \searrow C(K \setminus \{\sigma\}) \searrow 0$ so by induction every cone is collapsible. □

Proposition 1.2.11. If $K \searrow H$ and $H \searrow L$, then K Kern.05pt$\nearrow\searrow$ H Kern.05pt$\nearrow\searrow$ L. □