# Tuning, Temperament, Timbre, and Twos

## Why Rectifying Resonant Ratios Requires Roots

Jesse He

OSU Reading Classics

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# Pythagoras, Strings, and the Square Root of Two

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This became the basis for the octave, which is the basis for most musical scales across different cultures.

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This legend is demonstrably false.

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Because of their obsession with integer ratios, the Pythagoreans were allegedly rather displeased to find, for example, that $\sqrt{2}$ is not rational.

# Ptolemy's Intense Diatonic Scale

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Claudius Ptolemy (c.100-c.170 CE), in *Harmonics*, described the intense (or syntonic) diatonic scale, which became the basis for the modern Western major scale.

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$$\sum_{n=1}^{\infty} \frac{1}{n}$$

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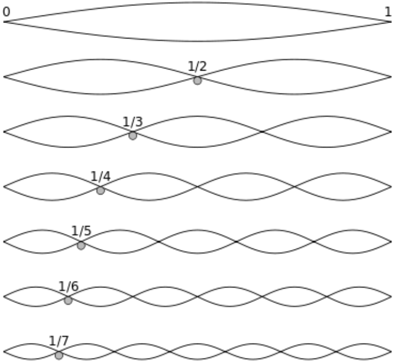
Because the frequency is inversely proportional to period, the period of each harmonic is some fraction of $f$.



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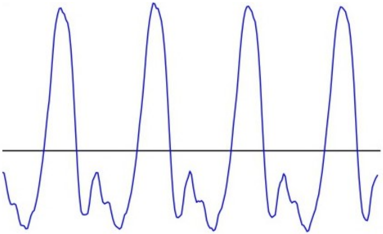
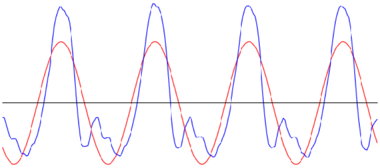
We can, however, use the machinery of Fourier analysis to decompose a sound into its component wavelengths!

# Timbre and Fourier Analysis

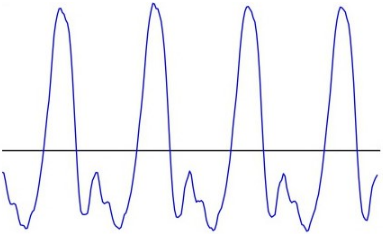
$$y \approx 2.189 \sin(x)$$

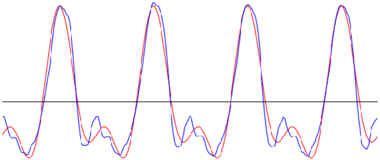




# Timbre and Fourier Analysis

$$y \approx 2.189\sin(x) + 1.256\sin(2x)$$





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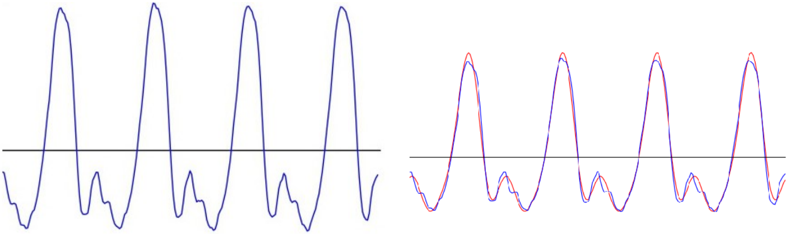
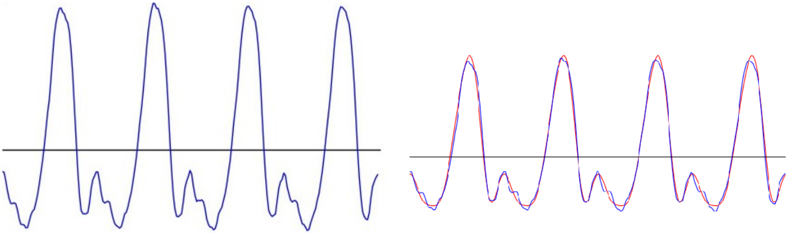
$$y \approx 2.189\sin(x) + 1.256\sin(2x) + 0.459\sin(3x)$$



Oh? You're approaching me?

# Timbre and Fourier Analysis

$$y \approx 2.189\sin(x) + 1.256\sin(2x) + 0.459\sin(3x) + 0.182\sin(4x)$$



# Classical Just Intonation

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By defining each note by an integer ratio of a fixed frequency, we can guarantee the consonance that Pythagoras and Ptolemy observed.

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Another way to define the scale is to define notes by powers of one or two specific ratios.

This gives rise to *syntonic temperament*. If we base our scale around powers of 2 and $\frac{3}{2}$, we get Pythagorean tuning (also known as 3-limit just intonation).

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As an example, we will work out the interval between F$\sharp$ and C$\sharp$ in Pythagorean tuning, which we expect to be $\frac{3}{2}$:

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We will do our calculations with respect to the note C, whose ratio to itself is of course 1. Since Pythagorean tuning is based on $\frac{3}{2}$ (in music, a perfect fifth), we will use the circle of fifths:

C-G-D-A-E-B-F♯-C♯-G♯-D♯-A♯-F-C

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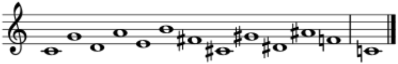

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Since F♯ is 6 fifths above C, its ratio to C is $\left(\frac{3}{2}\right)^6$. To normalize the octave, we divide by 2 three times to get $\left(\frac{3}{2}\right)^6 \left(\frac{1}{2}\right)^3$

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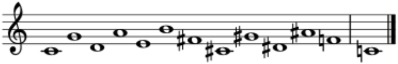

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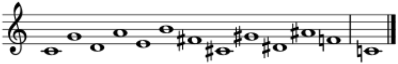

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Well, we can also move backwards: C♯ is 5 fifths *below* C, so after normalizing octaves, we get that it should be $\left(\frac{2}{3}\right)^5 2^4$

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So we end up with two possible values for C$\sharp$: $\left(\frac{3}{2}\right)^7 \left(\frac{1}{2}\right)^3$ or $\left(\frac{2}{3}\right)^5 2^4$. (The difference between these is known as the Pythagorean comma).

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But then the ratio between F$\sharp$ and C$\sharp$ is

$$\frac{\left(\frac{3}{2}\right)^6\left(\frac{1}{2}\right)^3}{\left(\frac{2}{3}\right)^5 2^4} = \frac{2^9 3^{-5}}{3^6 2^{-9}} = \frac{2^{18}}{3^{11}} = \frac{262144}{177141} \approx 1.47981 \neq \frac{3}{2}$$

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This is not a simple integer ratio.

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So even though the ratios look simpler, inside this scale there are two wolves!

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The interval between C and F$\sharp$ is particularly dissonant, earning it the historical nickname “Diabolus in Musica” (the Devil in Music).

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This is known as a *comma pump*.

# Benedetti's Puzzle

| | | | | | |
|---|---|---|---|---|---|
| Octave | 2 | G | A | - | G |
| Fifth | $\frac{3}{2}$ | D | - | E | - |
| Major Third | $\frac{5}{4}$ | G | - | C | - |

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"We are compell'd to use an occult Temperament, and to sing these imperfect Intervals, from doing which less offence arises."

To paraphrase, "I, Gianbattista Benedetti, have a dream…"

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The idea was to distort a few intervals in order to spread any discrepancy across several intervals.

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Zhu developed a twelve-tone equal temperament based on the twelfth root of 2 in 1584:

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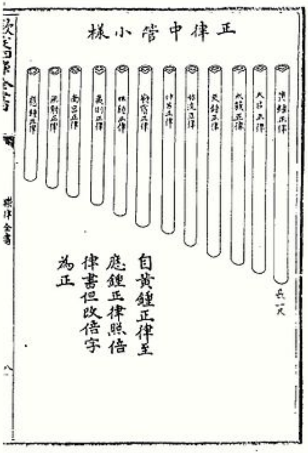


“I have founded a new system. I establish one foot as the number from which the others are to be extracted, and using proportions I extract them. Altogether one has to find the exact figures for the pitch-pipers in twelve operations.”

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Using the twelth root of two intuitively makes sense: we want to divide the octave (whose frequency is twice the fundamental's) into twelve equal parts, so $\sqrt[12]{2}$ should be the basis of our division.

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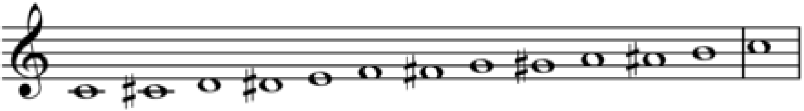

| C | C♯ | D | D♯ | E | F | F♯ | G | G♯ | A | A♯ | B | C |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $1$ | $2^{\frac{1}{12}}$ | $2^{\frac{1}{6}}$ | $2^{\frac{1}{4}}$ | $2^{\frac{1}{3}}$ | $2^{\frac{5}{12}}$ | $2^{\frac{1}{2}}$ | $2^{\frac{7}{12}}$ | $2^{\frac{2}{3}}$ | $2^{\frac{3}{4}}$ | $2^{\frac{5}{6}}$ | $2^{\frac{11}{12}}$ | $2$ |

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