MATH547: Stochastic Processes

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January 7, 2025

Abstract

1 Introduction

Definition 1 (Probability space).

1.1 Independence

Probability theory is about finding some sort of independence.

Definition 2 (Independence).
Independence of Events
Events $A_1, A_2, A_3, \dots, A_n$ are independent if and only if:
For any collection $I \subseteq \{1, 2, \dots, n\}$

$$P\left(\bigcap_{j \in I} A_j\right) = \prod_{j \in I} P(A_j).$$

Independence of Random Variables
For random variables $X_1, X_2, \dots X_n$, taking values in state spaces $S_1, S_2, \dots S_n$, these are independent if and only if:
for any $E_1 \subseteq S_1, E_2 \subset S_2, \dots E_n \subset S_n$ the events $A_j = \{X_j \in E_j\}$ for $1 \leq j \leq n$ are independent.

1.2 Important theorem

Theorem 1 (Law of large number (weak)). *If $\{X_n\}_{n=0}^{\infty}$ are real-valued, independent, identically distributed random variables and $\mathbb{E}(X_i) = \mu < \infty$, then*

$$\frac{1}{n}\sum_{j=1}^{n} X_j \to_{n\to\infty}^{\mathbb{P}} = \mathbb{E}(Xi)$$

Definition 3 (in-probability concergence).

Theorem 2 (Law of large number (strong)). *If $\{X_n\}_{n=0}^{\infty}$ are real-valued, independent, identically distributed random variables and $\mathbb{E}(X_i) = \mu < \infty$, then*

$$\frac{1}{n}\sum_{j=1}^{n} X_j \to_{n\to\infty}^{a.s.} = \mathbb{E}(Xi)$$

Large number theorem tells us value will converge to the expected value. Then central limit theorem tells us how close it will be.

Theorem 3 (Central limit theorem). *If $\{X_n\}_{n=0}^{\infty}$ are iid. random variables, with $\mathbb{E}(X_i) = \mu < \infty$ and $Var(X_i) = \sigma^2 < \infty$, then $\forall t \in \mathbb{R}$,*

$$\lim_{n\to\infty} P\left(\frac{\sum_{j=1}^{n} X_j - n\mu}{\sigma\sqrt{n}} \le t\right) = \Phi(t)$$

where $\Phi(t)$ is the standard normal distribution function.

2 Stochastic Processes

A stochastic process is a family of random variables indexed by natural numbers (time).

Definition 4 (Stochastic process). *A stochastic process is a collection of random variables $\{X_t\}_{t\in\mathbb{N}_0}$ where t represents time.*

Basic examples:
1. iid. random variables (e.g. iid. dice rolls)
2. Random walk

Definition 5 (Countable state space markov chain). *A stochastic process $\{X_j\}_{j\in\mathbb{N}_0}$ satisfies the Markov property.*

References