MATH247: Honours Applied Linear Algebra

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Abstract

1 *Preliminaries*

1.1 *Fields*

Definition 1 (Field). *A (nonempty) set with two (inner) operations, addition and multiplication:*

$$\cdot : K \times K \mapsto K, \cdot(x, y) = x \cdot y$$
$$+ : K \times K \mapsto K, +(x, y) = x + y$$

is called a field if the following axioms hold for all $x, y, z \in K$:

$(F1)$ $x + (y + z) = (x + y) + z$, $x(yz) = (xy)z$, *(Associativity)*
$(F2)$ $x + y = y + x$, $xy = yx$, *(Commutativity)*
$(F3)$ $x + (y + z) = (x + y) + z$, $x(yz) = (xy)z$, *(Distributivity)*
$(F4)$ $\exists o \in K$ *such that* $x + o = x, \exists e \in K$ *such that* $x \cdot e = x$, *(Neutral elements)*
$(F5a)$ $\exists a \in K$ *such that* $x + a = o$, *(Additive inverses)*
$(F5b)$ $\exists b \in K$ *such that* $x \cdot b = e$, *(Multiplicative inverses)*

Subfields Write $(K, +, \cdot)$ to point out notation for a field.

Example 1 ($\mathbb{F}_2$ - *finite field*)

Let $K = \{0, 1\}$ be a set with $1 \neq 0$, and define the operations $+, \cdot$ as follows:

$$0 + x = x + 0, \quad 0 + 1 = 1, \quad 1 + 0 = 1, \quad 1 + 1 = 0$$
$$0 \cdot 0 = 0, \quad 0 \cdot 1 = 0, \quad 1 \cdot 0 = 0, \quad 1 \cdot 1 = 1$$

Theorem 1. *Let $(K, +, \cdot)$ be a field. Then for all x, y, z:*

(a) $x + y = x + z \Rightarrow y = z$ $(Cancellation)$
(b) $xy = xz \Rightarrow y = z, \forall x \in K \setminus \{0\}$
(c) $x \cdot o = o$
(d) $x \cdot y = o \Rightarrow x = o \vee y = o$ *(Free of zero divisors)*

Proof. (a) Let a be the odd

1.2 The field of complex number

Complex numbers $\mathbb{C}$ is born out of necessity to solve equations like

$$x^2 + t = 0.$$

Definition 2 (Complex number). *We set*

$$\mathbb{C} = \{a + bi | a, b \in \mathbb{R}\}$$

where "i" is the imaginary unit

$$i^2 = -1 \quad (\star).$$

Using ordinary unit for addition and multiplication in $\mathbb{R}$ *and* $(\star)$ *,* $(\mathbb{C}, +, \cdot)$ *becomes a field containing* $\mathbb{R}$ *as a subfield.*

Given $z = a + bi \in \mathbb{C}$ we define:

$$\begin{aligned} &= a - bi \quad \text{conjugate} \\ R(z) &= a \\ Jm() &= b \end{aligned}$$

Theorem 2. *Let* $u, v \in \mathbb{C}$

References