

Condensed mathematics I

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1 Condensed Sets

Definition 1.1. Let \mathcal{C} be a category. A Grothendieck topology on \mathcal{C} consists of: for each object X in \mathcal{C} , there is a collection $\text{Cov}(X)$ of sets $\{X_i \rightarrow X\}_{i \in I}$, satisfying the following three axioms:

- (i) If $V \rightarrow X$ is an isomorphism, then $\{V \rightarrow X\} \in \text{Cov}(X)$.
- (ii) If $\{X_i \rightarrow X\}_{i \in I} \in \text{Cov}(X)$ and $Y \rightarrow X$ is any arrow in \mathcal{C} , then the fiber products $X_i \times_X Y$ exist and $\{X_i \times_X Y \rightarrow Y\}_{i \in I} \in \text{Cov}(Y)$.
- (iii) If $\{X_i \rightarrow X\}_{i \in I} \in \text{Cov}(X)$ and for each $i \in I$, $\{V_{ij} \rightarrow X_i\}_{j \in I_i} \in \text{Cov}(X_i)$, then $\{V_{ij} \rightarrow X\}_{i \in I, j \in I_i} \in \text{Cov}(X)$.

We call elements of $\text{Cov}(X)$ coverings.

Definition 1.2. A site is a category \mathcal{C} together with a Grothendieck topology.

Example 1.3. Let $\mathcal{C} = \text{ProFin}$, the category of all profinite sets. For $\{X_i \rightarrow Y\}_{i \in I}$ be a covering, we mean I is a finite index and $\coprod_{i \in I} X_i \rightarrow Y$ is a surjection. We also call maps $\{X_i \rightarrow Y\}_{i \in I}$ finite jointly surjective families of maps.

Now, for the category ProFin together its coverings, we call it the proétale site of a point and denote it by $*_{\text{proét}}$.

Definition 1.4. (i) For any site \mathcal{C} , we call a functor

$$\mathcal{F} : \mathcal{C}^{\text{op}} \rightarrow \text{Set}$$

a presheaf of sets.

- (ii) For a presheaf of sets $\mathcal{F} : \mathcal{C}^{\text{op}} \rightarrow \text{Set}$, if for any $X \in \mathcal{C}$ and any covering $\{X_i \rightarrow X\}_{i \in I} \in \text{Cov}(X)$, we have

$$\mathcal{F}(X) \xrightarrow{\sim} \text{Eq}\left(\prod_{i \in I} \mathcal{F}(X_i) \rightrightarrows \prod_{i, j \in I} \mathcal{F}(X_i \times_X X_j)\right).$$

Then we call \mathcal{F} a sheaf of sets.

Definition 1.5. A condensed set T is a sheaf of sets on $*_{\text{proét}}$, i.e. a functor $T : *_{\text{proét}}^{\text{op}} \rightarrow \text{Set}$ satisfying the sheaf condition.

Remark 1.6. (i) Concretely, a condensed set T is a functor $T : \text{ProFin}^{\text{op}} \rightarrow \text{Set}$, satisfying $T(\emptyset) = *$ and

- For any profinite sets S_1, S_2 , the natural map

$$T(S_1 \sqcup S_2) \longrightarrow T(S_1) \times T(S_2)$$

is a bijection.

- For any surjection $S' \twoheadrightarrow S$ of profinite sets with fiber product $S' \times_S S'$ and two projections $p_1, p_2 : S' \times_S S' \rightarrow S'$, the map

$$T(S) \xrightarrow{\sim} \{x \in T(S') \mid p_1^*(x) = p_2^*(x) \in T(S' \times_S S')\}$$

is a bijection. In other words, T maps the pullback diagram

$$\begin{array}{ccc} S' \times_S S' & \xrightarrow{p_2} & S' \\ p_1 \downarrow & & \downarrow \\ S' & \longrightarrow & S \end{array}$$

to a pullback diagram

$$\begin{array}{ccc} T(S' \times_S S') & \xleftarrow{p_2^*} & T(S') \\ p_1^* \uparrow & & \uparrow \\ T(S') & \xleftarrow{\quad} & T(S) \end{array}$$

- (ii) The category ProFin of all profinite sets is a large category.

Definition 1.7. κ is an uncountable strong limit cardinal if κ is uncountable and for any $\lambda < \kappa$, we have $2^\lambda < \kappa$.

Example 1.8. For any limit cardinal λ , i.e. if $\kappa < \lambda$, then $\kappa + 1 < \lambda$. We define

$$\sqsubset_0 = \aleph_0, \dots, \sqsubset_{\alpha+1} = 2^{\sqsubset_\alpha},$$

and let

$$\sqsubset_\lambda = \bigcup_{\alpha < \lambda} \sqsubset_\alpha,$$

then we can show that \sqsubset_λ is an uncountable strong limit cardinal.

Notation. We let $\kappa\text{-ProFin}$ denote the category of all κ -small profinite sets, i.e. profinite sets whose cardinal less equal than κ . Let $\text{Cond}_\kappa(\text{Set}) = \text{Sh}(\kappa\text{-ProFin}, \text{Set})$.

Remark 1.9. If $\kappa' > \kappa$ are two uncountable strong limit cardinals, and denote the inclusion by $i : \kappa\text{-ProFin} \hookrightarrow \kappa'\text{-ProFin}$, then we have a forgetful functor

$$\text{Cond}_{\kappa'}(\text{Set}) \longrightarrow \text{Cond}_\kappa(\text{Set}); \quad T \mapsto T \circ i.$$

This forgetful functor admits a left adjoint $F : \text{Cond}_\kappa(\text{Set}) \longrightarrow \text{Cond}_{\kappa'}(\text{Set})$. F is fully faithful and F commutes with all colimits and all finite limits.

We define

$$\text{Cond}(\text{Set}) = \bigcup_{\kappa} \text{Cond}_\kappa(\text{Set}) = \varinjlim_{\kappa} \text{Cond}_\kappa(\text{Set}).$$

Example 1.10. Let Top denote the category of all topological spaces. For each $T \in \text{Top}$, we can define $\underline{T} \in \text{Cond}(\text{Set})$ as follows:

$$\underline{T} : \text{ProFin}^{\text{op}} \longrightarrow \text{Set}; S \mapsto \underline{T}(S) = \text{Cont}(S, T) = \{\text{continuous maps from } S \text{ to } T\}.$$

We need to check that \underline{T} is a condensed set:

- (i) $\underline{T}(S_1 \sqcup S_2) = \text{Cont}(S_1 \sqcup S_2, T) = \text{Cont}(S_1, T) \times \text{Cont}(S_2, T) = \underline{T}(S_1) \times \underline{T}(S_2)$.
- (ii) For any surjection $g : S' \twoheadrightarrow S$, let $p_1, p_2 : S' \times_S S' \rightarrow S'$ be the two projections. We need to show the following map is a bijection:

$$\text{Cont}(S, T) \xrightarrow{\sim} \{h : S' \rightarrow T \mid hp_1 = hp_2 : S' \times_S S' \rightarrow T\}; f \mapsto f \circ g.$$

Because g is surjective, it is easy to show this map is an injection.

Now, for any $h : S' \rightarrow T$ with $hp_1 = hp_2$, from the universal property of pushout (in our situation, the pullback square is also a pushout), we can find a unique f , s.t. the diagram commutes.

$$\begin{array}{ccc} S' \times_S S' & \xrightarrow{p_2} & S' \\ p_1 \downarrow & & \downarrow g \\ S' & \xrightarrow{g} & S \\ & \searrow h & \downarrow h \\ & & T \end{array}$$

(Note: In the original image, there is a dashed arrow from S to T labeled f , and a solid arrow from S' to T labeled h . The diagram is a pushout square with an additional arrow from S to T .)

Definition 1.11. Let $X \in \text{Top}$. The following are equivalent definition:

- (i) $X \in \text{Top}$ is compactly generated;
- (ii) If for any compact Hausdorff space S with a map $S \rightarrow X$, if the composition $S \rightarrow X \rightarrow Y$ is continuous, then $X \rightarrow Y$ is continuous;
- (iii) $A \subset X$ is closed if and only if for any compact space K with a map $f : K \rightarrow X$, $f^{-1}(A) \subset K$ is closed.

Remark 1.12. (i) If a topological space X is compact Hausdorff, then X is compactly generated.

- (ii) Let \mathbf{CGTop} denote the category of all compactly generated spaces and let \mathbf{CHaus} denote the category of all compact Hausdorff spaces.

Definition 1.13. For a category \mathcal{C} , $P \in \mathcal{C}$ is a projective object if for any epimorphism $Y \twoheadrightarrow X$ and a morphism $P \rightarrow X$, there is a lift

$$\begin{array}{ccc} & P & \\ \swarrow \exists & \downarrow & \\ Y & \twoheadrightarrow & X \end{array}$$

Definition 1.14. In the category \mathbf{CHaus} , we call its projective objects as extremally disconnected Hausdorff spaces.

- Remark 1.15.** (i) Equivalently a compact Hausdorff space S is extremally disconnected if any surjection $S' \twoheadrightarrow S$ from a compact Hausdorff space splits.
- (ii) Extremally disconnected Hausdorff spaces are profinite sets, i.e. $\mathbf{ExDisc} \subset \mathbf{ProFin}$. Here, \mathbf{ExDisc} denote the category of all extremally disconnected Hausdorff spaces.

Remark 1.16. We have two adjunctions.

- (i) $\mathbf{Top} \begin{smallmatrix} \xrightarrow{\beta} \\ \xleftarrow{i} \end{smallmatrix} \mathbf{CHaus}$, i.e. $\beta \dashv i$.

Where

$$i : \mathbf{CHaus} \rightarrow \mathbf{Top}; X \mapsto X$$

and

$$\beta : \mathbf{Top} \rightarrow \mathbf{CHaus}$$

is the Stone-Cech compactification of topological spaces.

For any $X \in \mathbf{Top}$, we define $\beta X \in \mathbf{CHaus}$ as follows:

for any $Y \in \mathbf{CHaus}$ with a map $f : X \rightarrow Y$, there exists a unique map $\beta X \rightarrow Y$ so that the diagram commutes.

$$\begin{array}{ccc} X & \xrightarrow{i_X} & \beta X \\ & \searrow f & \swarrow \exists! \\ & Y & \end{array}$$

In fact, we can use the ultrafilter to construct βX concretely. And by this construction, we can show that

$$|\beta X| \leq 2^{2^{|X|}}.$$

- (ii) $\mathbf{CGTop} \begin{smallmatrix} \xrightarrow{i} \\ \xleftarrow{c} \end{smallmatrix} \mathbf{Top}$, i.e. $i \dashv c$.

Where

$$i : \mathbf{CGTop} \rightarrow \mathbf{Top}; X \mapsto X$$

and

$$c : \text{Top} \rightarrow \text{CGTop}; X \mapsto X^{\text{cg}}.$$

We define X^{cg} as follows:

- As a set, $X^{\text{cg}} = X$.
- The topology of X^{cg} is given by the quotient topology of

$$\coprod_{S \rightarrow X} \coprod_{S \in \text{CHaus}} S \longrightarrow X.$$

- Proposition 1.17.** (i) The functor $\text{Top} \rightarrow \text{Cond}_\kappa(\text{Set}); T \mapsto \underline{T}$ is a faithful functor.
- (ii) When the above functor restricted to the full subcategory $\kappa\text{-CGTop}$ of all κ -compactly generated spaces, functor $\kappa\text{-CGTop} \rightarrow \text{Cond}_\kappa(\text{Set}); T \mapsto \underline{T}$ is a fully faithful functor.
- (iii) The functor $\text{Top} \rightarrow \text{Cond}_\kappa(\text{Set}); T \mapsto \underline{T}$ admits a left adjoint $\text{Cond}_\kappa(\text{Set}) \rightarrow \text{Top}; \underline{T} \mapsto T(*)_{\text{top}}$. Here, $T(*)_{\text{top}}$ means the underlying set $T(*)$ equipped with the quotient topology of $\sqcup_{S \rightarrow T} S \rightarrow T(*)$, where the disjoint union runs over all κ -small profinite sets S with a map to T , i.e. an element of $T(S)$. Moreover, we have $\underline{T} \cong T^{\kappa\text{-cg}}$.

2 Condensed abelian groups

Definition 2.1. A condensed abelian group T is a sheaf of abelian groups on $*_{\text{proét}}$, i.e. a functor $T : *_{\text{proét}}^{\text{op}} \rightarrow \text{Ab}$ satisfying the sheaf condition. And we denote the category of all condensed abelian groups by $\text{Cond}(\text{Ab})$.

Definition 2.2 (Grothendieck's axioms). Let \mathcal{C} be an abelian category.

(AB3) All colimits exist.

(AB3*) All limits exist.

(AB4) Arbitrary direct sums are exact.

(AB4*) Arbitrary products are exact.

(AB5) Filtered colimits are exact.

(AB6) For any index set J and filtered categories I_j , $j \in J$, with functors $I_j \rightarrow \text{Cond}(\text{Ab})$; $i \mapsto M_i$, the natural map

$$\varinjlim_{(i_j \in I_j)_j, j \in J} \prod M_{i_j} \longrightarrow \prod_{j \in J} \varinjlim_{i_j \in I_j} M_{i_j}$$

is an isomorphism.

Definition 2.3. Let \mathcal{C} be an abelian category. $M \in \mathcal{C}$ is compact if $\text{Hom}(M, -)$ commutes with filtered colimits, i.e. $\text{Hom}(M, \varinjlim_i N_i) \cong \varinjlim_i \text{Hom}(M, N_i)$.

Theorem 2.4. (i) $\text{Cond}(\text{Ab})$ is an abelian category which satisfies Grothendieck's axioms (AB3), (AB4), (AB5), (AB6), (AB3*) and (AB4*).

(ii) $\text{Cond}(\text{Ab})$ is generated by compact projective objects.

Corollary 2.5. There is an adjunction:

$$\text{Cond}_\kappa(\text{Set}) \rightleftarrows \text{Cond}_\kappa(\text{Ab}) .$$

Where $\text{Cond}_\kappa(\text{Ab}) \rightarrow \text{Cond}_\kappa(\text{Set})$ is the forgetful functor and

$$\text{Cond}_\kappa(\text{Set}) \rightarrow \text{Cond}_\kappa(\text{Ab}); T \mapsto \mathbb{Z}[T].$$

Here, $\mathbb{Z}[T] := (S \mapsto \mathbb{Z}[T(S)])^{\text{sh}}$.

Remark 2.6. (i) For $S \in \text{ExDisc}$ and $M \in \text{Cond}(\text{Ab})$, we have

$$\text{Hom}_{\text{Cond}(\text{Ab})}(\mathbb{Z}[S], M) \cong \text{Hom}_{\text{Cond}(\text{Set})}(\underline{S}, M) \cong M(S).$$

Proof: We define the map:

$$\mu : \text{Hom}_{\text{Cond}(\text{Set})}(\underline{S}, M) \longrightarrow M(S); \alpha \mapsto \alpha(S)(1_S),$$

and the map

$$\lambda : M(S) \longrightarrow \text{Hom}_{\text{Cond}(\text{Set})}(\underline{S}, M); x \mapsto \lambda(x),$$

where for $\lambda(x) : \underline{S} \longrightarrow M$,

$$\lambda(x)(T) : \text{Cont}(T, S) \longrightarrow M(T); f \mapsto M(f)(x).$$

One can check that μ and λ are inverse to each other, hence

$$\text{Hom}_{\text{Cond}(\text{Set})}(\underline{S}, M) \cong M(S).$$

□

(ii) For any $S \in \text{ExDisc}$, $\mathbb{Z}[S] \in \text{Cond}(\text{Ab})$ is a compact and projective object.

Proof:

Compactness.

$$\text{Hom}(\mathbb{Z}[S], \varinjlim M_i) = (\varinjlim M_i)(S) = \varinjlim M_i(S) = \varinjlim \text{Hom}(\mathbb{Z}[S], M_i).$$

Projectiveness. For any exact sequence $M' \rightarrow M \rightarrow M''$ in $\text{Cond}(\text{Ab})$, the sequence

$$M'(S) \rightarrow M(S) \rightarrow M''(S)$$

is exact, i.e.

$$\text{Hom}(\mathbb{Z}[S], M') \rightarrow \text{Hom}(\mathbb{Z}[S], M) \rightarrow \text{Hom}(\mathbb{Z}[S], M'')$$

is exact, so $\mathbb{Z}[S]$ is projective.

□

(iii) $\text{Cond}(\text{Ab})$ has enough projectives.

Proposition 2.7. We have two equivalences.

$$(i) \text{ Shv}(\kappa\text{-CHaus}) \xrightarrow{\sim} \text{Shv}(\kappa\text{-ProFin}); T \mapsto T|_{\kappa\text{-ProFin}}.$$

$$(ii) \text{ Shv}(\kappa\text{-ProFin}) \xrightarrow{\sim} \text{Shv}(\kappa\text{-ExDisc}); T \mapsto T|_{\kappa\text{-ExDisc}}.$$

Remark 2.8. In order a presheaf of sets T to be a sheaf of sets, by definition, we need to check the sheaf condition in ProFin . Now, from the equivalence $\text{Shv}(\kappa\text{-ProFin}) \xrightarrow{\sim} \text{Shv}(\kappa\text{-ExDisc})$,

we only need to check the sheaf condition in ExDisc. In this case, the condition(ii) is automatic: T maps the pullback diagram

$$\begin{array}{ccc} S' \times_S S' & \xrightarrow{p_2} & S' \\ p_1 \downarrow & & \downarrow \\ S' & \longrightarrow & S \end{array}$$

to a pullback diagram

$$\begin{array}{ccc} T(S' \times_S S') & \xleftarrow{p_2^*} & T(S') \\ p_1^* \uparrow & & \uparrow \\ T(S') & \xleftarrow{\quad} & T(S) \end{array}$$

This is because any cover of extremally disconnected sets splits. Specifically, the diagram

$$\begin{array}{ccc} S' \times_S S' & \longrightarrow & S' \\ \downarrow & & \downarrow \\ S' & \xrightarrow{g} & S \\ & \xleftarrow{f} & \end{array}$$

can implies the following diagram:

$$\begin{array}{ccccc} T(S' \times_S S') & \xleftarrow{\quad} & T(S') & & \\ \uparrow & & \uparrow & \nearrow & \\ T(S') & \xleftarrow{T(g)} & T(S) & \xleftarrow{\exists} & X \\ & \xleftarrow{T(f)} & & \nwarrow & \\ & & & & \end{array}$$

which means it is a pullback diagram.

Property. There are some properties of the category $\text{Cond}(\text{Ab})$ of condensed abelian groups.

- (i) $\text{Cond}(\text{Ab})$ has a symmetric monoidal tensor products $- \otimes -$, where for $M, N \in \text{Cond}(\text{Ab})$, $M \otimes N = (S \mapsto M(S) \otimes N(S))^{\text{sh}}$.
- (ii) Functor $\text{Cond}(\text{Set}) \rightarrow \text{Cond}(\text{Ab}); T \mapsto \mathbb{Z}[T]$ is symmetric monoidal with respect to the product and the tensor product, i.e. $\mathbb{Z}[T_1 \times T_2] = \mathbb{Z}[T_1] \otimes \mathbb{Z}[T_2]$.

Proof:

- (iii) For $T \in \text{Cond}(\text{Set})$, $\mathbb{Z}[T] \in \text{Cond}(\text{Ab})$ is flat.

Proof: We need to show $- \otimes \mathbb{Z}[T] : \text{Cond}(\text{Ab}) \rightarrow \text{Cond}(\text{Ab})$ is an exact functor.

Take an exact sequence in $\text{Cond}(\text{Ab})$:

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0.$$

For any $S \in \text{ExDisc}$, we have an exact sequence:

$$0 \longrightarrow X(S) \longrightarrow Y(S) \longrightarrow Z(S) \longrightarrow 0.$$

Tensoring with the free abelian group $\mathbb{Z}[T(S)]$, we get an exact sequence:

$$0 \longrightarrow X(S) \otimes \mathbb{Z}[T(S)] \longrightarrow Y(S) \otimes \mathbb{Z}[T(S)] \longrightarrow Z(S) \otimes \mathbb{Z}[T(S)] \longrightarrow 0,$$

i.e.

$$0 \longrightarrow (X \otimes \mathbb{Z}[T])(S) \longrightarrow (Y \otimes \mathbb{Z}[T])(S) \longrightarrow (Z \otimes \mathbb{Z}[T])(S) \longrightarrow 0.$$

Hence the sequence

$$0 \longrightarrow X \otimes \mathbb{Z}[T] \longrightarrow Y \otimes \mathbb{Z}[T] \longrightarrow Z \otimes \mathbb{Z}[T] \longrightarrow 0.$$

exact and $\mathbb{Z}[T]$ is flat. □

- (iv) Given any $M, N \in \text{Cond}(\text{Ab})$, we can give the group of homomorphisms $\text{Hom}(M, N)$ the structure of condensed abelian groups via the following definition, for any $S \in \text{ExDisc}$,

$$\underline{\text{Hom}}(M, N)(S) := \text{Hom}(\mathbb{Z}[S] \otimes M, N).$$

So we define an internal Hom-functor object.

- (v) There is an adjunction. For $P, M, N \in \text{Cond}(\text{Ab})$, we have an isomorphism of abelian groups:

$$\text{Hom}(P, \underline{\text{Hom}}(M, N)) \cong \text{Hom}(P \otimes M, N).$$

Proof: First, if $P = \mathbb{Z}[S]$ for some $S \in \text{ExDisc}$, then

$$\text{Hom}(P, \underline{\text{Hom}}(M, N)) = \text{Hom}(\mathbb{Z}[S], \underline{\text{Hom}}(M, N)) = \underline{\text{Hom}}(M, N)(S) = \text{Hom}(\mathbb{Z}[S] \otimes M, N).$$

Now, for general $P \in \text{Cond}(\text{Ab})$, we can write $P = \varinjlim \mathbb{Z}[S_i]$, so

$$\begin{aligned}
\text{Hom}(P, \underline{\text{Hom}}(M, N)) &= \text{Hom}(\varinjlim \mathbb{Z}[S_i], \underline{\text{Hom}}(M, N)) \\
&= \varprojlim \text{Hom}(\mathbb{Z}[S_i], \underline{\text{Hom}}(M, N)) \\
&= \varprojlim \text{Hom}(\mathbb{Z}[S_i] \otimes M, N) \\
&= \text{Hom}(\varinjlim \mathbb{Z}[S_i] \otimes M, N) \\
&= \text{Hom}(P \otimes M, N).
\end{aligned}$$

□

- (vi) As $\text{Cond}(\text{Ab})$ has enough projectives, one can form the derived category $D(\text{Cond}(\text{Ab}))$. If $P \in \text{Cond}(\text{Ab})$ is compact and projective, then $P[0] \in D(\text{Cond}(\text{Ab}))$ is a compact object of the derived category, i.e. $\text{Hom}(P, -)$ commutes with arbitrary direct sums. In particular, $D(\text{Cond}(\text{Ab}))$ is compactly generated.

- (vii) Similarly, in the derived category $D(\text{Cond}(\text{Ab}))$, we have the adjunction:

$$\text{Hom}(P, R\underline{\text{Hom}}(M, N)) \cong \text{Hom}(P \otimes^L M, N).$$

- (viii) Let $\mathcal{D}(\text{Cond}(\text{Ab}))$ denote the derived ∞ -category of $\text{Cond}(\text{Ab})$ and $\mathcal{D}(\text{Ab})$ denote the derived ∞ -category of Ab , then there is an equivalence

$$\mathcal{D}(\text{Cond}(\text{Ab})) \cong \text{Cond}(\mathcal{D}(\text{Ab})).$$

3 $D(R)$

Definition 3.1. An ∞ -category is a simplicial set \mathcal{C} which satisfies the following extension condition:

$$\begin{array}{ccc} \Lambda_i^n & \longrightarrow & \mathcal{C} \\ \downarrow & \nearrow \exists f & \\ \Delta^n & & \end{array} \quad 0 < \forall i < n.$$

Definition 3.2. Let \mathcal{C} be an ∞ -category. A zero object of \mathcal{C} is an object which is both initial and final. We say that \mathcal{C} is pointed if \mathcal{C} contains a zero object.

Definition 3.3. Let \mathcal{C} be a pointed ∞ -category. A triangle in \mathcal{C} is a diagram $\Delta^1 \times \Delta^1 \rightarrow \mathcal{C}$ depicted as

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow g \\ 0 & \longrightarrow & Z \end{array}$$

where 0 is a zero object in \mathcal{C} .

We say a triangle in \mathcal{C} is a fiber sequence if it is a pullback and say a triangle in \mathcal{C} is a cofiber sequence if it is a pushout.

We generally indicate a triangle by specifying only the pair of maps $X \xrightarrow{f} Y \xrightarrow{g} Z$.

Remark 3.4. Let \mathcal{C} be a pointed ∞ -category. A triangle in \mathcal{C} consists of the following data:

- (i) A pair of morphisms $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ in \mathcal{C} .
- (ii) A 2-simplex in \mathcal{C} corresponding to a diagram

$$\begin{array}{ccc} & Y & \\ f \nearrow & & \searrow g \\ X & \xrightarrow{h} & Z \end{array}$$

in \mathcal{C} , which identifies h with the composition $g \circ f$.

- (iii) A 2-simplex

$$\begin{array}{ccc} & 0 & \\ \nearrow & & \searrow \\ X & \xrightarrow{h} & Z \end{array}$$

in \mathcal{C} , which we view as anullhomotopy of h .

Definition 3.5. Let \mathcal{C} be a pointed ∞ -category containing a morphism $g : X \rightarrow Y$.

A fiber of g is a fiber sequence

$$\begin{array}{ccc} W & \longrightarrow & X \\ \downarrow & & \downarrow g \\ 0 & \longrightarrow & Y \end{array}$$

and we denote $W = \text{fib}(g)$.

Dually, a cofiber of g is a cofiber sequence

$$\begin{array}{ccc} X & \xrightarrow{g} & Y \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & Z \end{array}$$

and we denote $Z = \text{cofib}(g)$.

Definition 3.6. An ∞ -category \mathcal{C} is stable if it satisfies the following conditions:

- (i) There exists a zero object $0 \in \mathcal{C}$.
- (ii) Every morphism in \mathcal{C} admits a fiber and a cofiber.
- (iii) A triangle in \mathcal{C} is a fiber sequence if and only if it is a cofiber sequence.

Remark 3.7. (i) For a stable ∞ -category \mathcal{C} , we define the suspension functor $\Sigma : \mathcal{C} \rightarrow \mathcal{C}$ and the loop functor $\Omega : \mathcal{C} \rightarrow \mathcal{C}$ as follows:

$$\Sigma(X) := \text{cofib}(X \rightarrow 0)$$

and

$$\Omega(X) := \text{fib}(0 \rightarrow X).$$

- (ii) For a stable ∞ -category \mathcal{C} , there is a homotopy equivalence:

$$\text{Map}_{\mathcal{C}}(\Sigma X, Y) \xrightarrow{\sim} \text{Map}_{\mathcal{C}}(X, \Omega Y).$$

Besides, the unit $X \rightarrow \Omega \Sigma(X)$ and $\Sigma \Omega(Y) \rightarrow Y$ counit are isomorphic.

Definition 3.8. Let R be a commutative ring, the ∞ -category $D(R)$ is a stable ∞ -category with all colimits, generated (as a cocomplete stable ∞ -category) by a distinguished compact object 1 , satisfying

$$\pi_0 \text{Map}(1, 1) = R^{\text{op}}, \quad \pi_0 \text{Map}(\Sigma^d 1, 1) = 0, \quad \forall d \neq 0.$$

For $X, Y \in D(R)$, we define

$$[X, Y] := \pi_0 \text{Map}(X, Y)$$

and

$$[X, Y]_d := [\Sigma^d X, Y] = [X, \Omega^d Y].$$

Remark 3.9. (i) From the definition of $D(R)$, we have

$$[1, 1] = \pi_0 \text{Map}(1, 1) = R^{\text{op}}, \quad [1, 1]_d = \pi_0 \text{Map}(\Sigma^d 1, 1) = 0, \quad \forall d \neq 0.$$

(ii) If $d \geq 0$, we have

$$[X, Y]_d = \pi_d \text{Map}(X, Y).$$

(iii) Claim: In $D(R)$, for any integer d , we have $[X, Y]_d \in \text{Ab}$.

Proof: First, if $d \geq 2$, $[X, Y]_d = \pi_d \text{Map}(X, Y) \in \text{Ab}$. For any $d \in \mathbb{Z}$,

$$[X, Y]_d = [\Sigma^d X, Y] = [\Sigma^{d-2} X, Y]_2 \in \text{Ab}.$$

(iv) For a stable ∞ -category, a fiber sequence $X \rightarrow Y \rightarrow Z$ is at the same time a cofiber sequence, and vice versa. Hence, we will call it a fiber-cofiber sequence.

(v) For a fiber-cofiber sequence $X \rightarrow Y \rightarrow Z$ in $D(R)$, we can induce a new fiber-cofiber sequence $Y \rightarrow Z \rightarrow \Sigma X$.

(vi) Given a fiber-cofiber sequence $X \rightarrow Y \rightarrow Z$ and any $A \in D(R)$, we can induce two long exact sequences:

$$\cdots \longrightarrow [A, X]_d \longrightarrow [A, Y]_d \longrightarrow [A, Z]_d \longrightarrow [A, X]_{d-1} \longrightarrow \cdots$$

$$\cdots \longrightarrow [X, A]_{d+1} \longrightarrow [Z, A]_d \longrightarrow [Y, A]_d \longrightarrow [X, A]_d \longrightarrow \cdots$$

(vii) Assume

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

is a pushout-pullback square in $D(R)$, then we can produce a triangle $A \rightarrow B \oplus C \rightarrow D$.

With this, we can induce a long exact sequence.

Definition 3.10. For any integer d , we define a functor $H_d : D(R) \rightarrow \text{Mod}_R$; $X \mapsto [1, X]_d$.

Remark 3.11. (i) We already know $[1, X]_d \in \text{Ab}$. And we need to show $[1, X]_d$ is an R -module.

In fact, we have

$$\text{Map}(\Sigma^d 1, \Sigma^d 1) \times \text{Map}(\Sigma^d 1, X) \rightarrow \text{Map}(\Sigma^d 1, X),$$

applying the functor π_0 , we get:

$$\pi_0 \text{Map}(\Sigma^d 1, \Sigma^d 1) \times \pi_0 \text{Map}(\Sigma^d 1, X) = \pi_0(\text{Map}(\Sigma^d 1, \Sigma^d 1) \times \text{Map}(\Sigma^d 1, X)) \rightarrow \pi_0(\text{Map}(\Sigma^d 1, X)),$$

i.e.

$$R^{\text{op}} \times [1, X]_d \rightarrow [1, X]_d,$$

which implies $[1, X]_d \in \text{Mod}_R$.

(ii) $H_d : D(R) \rightarrow \text{Mod}_R$; $X \mapsto [1, X]_d$ is a representable functor and $\Sigma^d 1$ represents H_d .

Lemma 3.12. (i) $H_d(\prod_i X_i) = \prod_i H_d(X_i)$.

(ii) $H_d(\oplus_i X_i) = \oplus_i H_d(X_i)$.

(iii) $H_d(\varinjlim X_i) = \varinjlim H_d(X_i)$.

(iv) For a sequence of maps $\cdots \rightarrow X_n \rightarrow X_{n-1} \rightarrow \cdots$, we have a Milnor sequence:

$$0 \longrightarrow \varprojlim^1 H_{d+1}(X_n) \longrightarrow H_d(\varprojlim X_n) \longrightarrow \varprojlim H_d(X_n) \longrightarrow 0.$$

Proposition 3.13. $f : X \rightarrow Y$ in $D(R)$ is an isomorphism if and only if $H_d(f) : H_d(X) \xrightarrow{\sim} H_d(Y)$, for $\forall d \in \mathbb{Z}$.

Proof. Take $Z = \text{cofib}(X \xrightarrow{f} Y)$, then $X \rightarrow Y \rightarrow Z$ is a fiber-cofiber sequence, and we can induce a long exact sequence

$$\cdots \rightarrow H_d(X) \rightarrow H_d(Y) \rightarrow H_d(Z) \rightarrow H_{d-1}(X) \rightarrow \cdots.$$

It suffices to show: if $Z \in D(R)$ with $H_d(Z) = 0$, $\forall d \in \mathbb{Z}$, then $Z = 0$.

Consider the full subcategory of $D(R)$:

$$\mathcal{C} = \{A \in D(R) \mid [\Sigma^d A, Z] = 0, \forall d \in \mathbb{Z}\}.$$

Observe that:

- $1 \in \mathcal{C}$.
- \mathcal{C} is stable under colimits. This is because

$$[\Sigma^d \text{colim } A_i, Z] = [\text{colim } \Sigma^d A_i, Z] = \lim [\Sigma^d A_i, Z] = 0.$$

- \mathcal{C} is stable under cofibers.

By definition of $D(R)$, we know $D(R)$ is generated as a cocomplete stable ∞ -category by 1. Hence, $D(R) = \mathcal{C}$. Then by Yoneda's lemma, $Z = 0$. \square

Proposition 3.14. Let $X \in D(R)$, then there exists $Y \in D(R)$ with a map $f : Y \rightarrow X$, s.t.

(i) $H_d(Y) = 0, \forall d < 0$.

(ii) $H_d(f) : H_d(Y) \xrightarrow{\sim} H_d(X)$ are isomorphisms, $\forall d \geq 0$.

Proof. We first prove: there exists a sequence of maps $Y_0 \rightarrow Y_1 \rightarrow Y_2 \rightarrow \dots$ in $D(R)_{/X}$, s.t. for any $n \geq 0$, $H_d(Y_n) = 0, d < 0$ and $H_d(Y_n \rightarrow X)$ are isomorphisms if $0 \leq d < n$ and is a surjection if $d = n$.

We prove this by induction.

First, $n = 0$. Let $Y_0 = \bigoplus_I 1$ for $I = \text{cartinal of } H_0(X)$, then the map $Y_0 \rightarrow X$ can induce a surjection $H_0(Y_0) = R^{\oplus I} \rightarrow H_0(X)$ and for $d < 0$, $H_d(Y_0) = 0$.

Now we assume that there exists a sequence

$$Y_0 \rightarrow Y_1 \rightarrow Y_2 \rightarrow \dots \rightarrow Y_{n-1}$$

in $D(R)_{/X}$ satisfying the assumption.

Let $F = \text{fib}(Y_{n-1} \rightarrow X)$, then $F \rightarrow Y_{n-1} \rightarrow X$ is a fiber-cofiber sequence. We can find an index I , s.t. $\Sigma^{n-1} \oplus_I 1 \rightarrow F$ can induce a surjection $H_{n-1}(\Sigma^{n-1} \oplus_I 1) \rightarrow H_{n-1}(F)$. Then let $Y_n = \text{cofib}(\Sigma^{n-1} \oplus_I 1 \rightarrow F \rightarrow Y_{n-1})$, hence $\Sigma^{n-1} \oplus_I 1 \rightarrow Y_{n-1} \rightarrow Y_n$ is also a fiber-cofiber sequence. Now, we check it satisfies the requirements.

(a) $d < 0$. The fiber-cofiber sequence $\Sigma^{n-1} \oplus_I 1 \rightarrow Y_{n-1} \rightarrow Y_n$ can induce a long exact sequence:

$$\dots \rightarrow H_{-1}(\Sigma^{n-1} \oplus_I 1) \rightarrow H_{-1}(Y_{n-1}) \rightarrow H_{-1}(Y_n) \rightarrow H_{-2}(\Sigma^{n-1} \oplus_I 1) \rightarrow H_{-2}(Y_{n-1}) \rightarrow \dots$$

Since for $k < 0$, $H_k(Y_{n-1}) = 0$, we know $H_d(Y_n) = H_{d-1}(\Sigma^{n-1} \oplus_I 1) = 0 (d < 0)$.

(b) First, there exists a map $Y_n \rightarrow X$, this is because

$$\begin{array}{ccccc} \Sigma^{n-1} \oplus_I 1 & \longrightarrow & Y_{n-1} & \longrightarrow & Y_n \\ \downarrow & & \parallel & & \downarrow \exists \\ F & \longrightarrow & Y_{n-1} & \longrightarrow & X \end{array}$$

and $H_d(\Sigma^{n-1} \oplus_I 1) = 0, \forall d \neq n-1$, then by

$$\dots \rightarrow H_{n-2}(\Sigma^{n-1} \oplus_I 1) \rightarrow H_{n-2}(Y_{n-1}) \rightarrow H_{n-2}(Y_n) \rightarrow H_{n-3}(\Sigma^{n-1} \oplus_I 1) \rightarrow \dots,$$

it implies that $0 \leq \forall d \leq n-2$, $H_d(Y_n) \cong H_d(Y_{n-1}) \cong H_d(X)$.

We have the following diagram:

$$\begin{array}{ccccccccccc}
H_n(Y_{n-1}) & \twoheadrightarrow & H_n(Y_n) & \twoheadrightarrow & H_{n-1}(\Sigma^{n-1} \oplus_I 1) & \twoheadrightarrow & H_{n-1}(Y_{n-1}) & \twoheadrightarrow & H_{n-1}(Y_n) & \twoheadrightarrow & H_{n-2}(\Sigma^{n-1} \oplus_I 1) & \twoheadrightarrow & H_{n-2}(Y_{n-1}) \\
\parallel & & \downarrow & & \downarrow & & \parallel & & \downarrow \sim & & \downarrow \sim & & \parallel \\
H_n(Y_{n-1}) & \twoheadrightarrow & H_n(X) & \longrightarrow & H_{n-1}(F) & \longrightarrow & H_{n-1}(Y_{n-1}) & \twoheadrightarrow & H_{n-1}(X) & \longrightarrow & H_{n-2}(F) & \longrightarrow & H_{n-2}(Y_{n-1})
\end{array}$$

By five's lemma, we can show $H_{n-1}(Y_n) \xrightarrow{\sim} H_{n-1}(X)$ and $H_n(Y_n) \twoheadrightarrow H_n(X)$.

Now, for $Y_0 \rightarrow Y_1 \rightarrow Y_2 \rightarrow \cdots$, we take $Y = \varinjlim Y_n$, and hence we can get a map $Y \rightarrow X$.

By $H_d(\varinjlim Y_n) = \varinjlim H_d(Y_n)$, for $d < 0$, $H_d(Y) = 0$, and for $d \geq 0$,

$$H_d(Y) = \varinjlim (H_d(Y_0) \rightarrow H_d(Y_1) \cdots \rightarrow H_d(Y_{d+1}) \rightarrow H_d(Y_{d+2}) \rightarrow \cdots) = H_d(X).$$

□

Proposition 3.15. For $X \in D(R)$, the following are equivalent:

- (i) $H_d(X) = 0$, $\forall d < 0$.
- (ii) X is generated by 1 under colimits.
- (iii) There exists a sequence of maps $X_0 \rightarrow X_1 \rightarrow X_2 \rightarrow \cdots$ with $X = \varinjlim X_i$, where for each i , the cofiber $\text{cofib}(X_{i-1} \rightarrow X_i)$ is of the form $\Sigma^i \oplus_I 1$.

Proof. (i) \implies (iii). By previous proposition, for $X \in D(R)$, there exists a map $f : Y \rightarrow X$ with $H_d(Y) = 0$ for $d < 0$ and $H_d(f)$ are isomorphisms for $d \geq 0$. Then, for $d < 0$,

$$H_d(X) = H_d(Y) = 0.$$

Hence, for any $d \in \mathbb{Z}$, $H_d(f)$ are isomorphisms. Thus, $f : Y \xrightarrow{\sim} X$.

From the construction of Y , we know there is a sequence of maps $X_0 \rightarrow X_1 \rightarrow X_2 \rightarrow \cdots$ with $\varinjlim X_i = Y \cong X$.

By the fiber-cofiber sequence $\Sigma^{n-1} \oplus_I 1 \rightarrow X_{n-1} \rightarrow X_n$, we can get a new fiber-cofiber sequence $X_{n-1} \rightarrow X_n \rightarrow \Sigma^n \oplus_I 1$, i.e. $\text{cofib}(X_{n-1} \rightarrow X_n) = \Sigma^n \oplus_I 1$.

(iii) \implies (ii).

We have $X_{i-1} \rightarrow X_i \rightarrow \Sigma^i \oplus_I 1$, which gives a new fiber-cofiber sequence: $\Sigma^{i-1} \oplus_I 1 \rightarrow X_{i-1} \rightarrow X_i$. Then $X_i = \text{cofib}(\Sigma^{i-1} \oplus_I 1 \rightarrow X_{i-1}) = \text{colim}(0 \leftarrow \Sigma^{i-1} \oplus_I 1 \rightarrow X_{i-1})$.

Now, $X_1 = \text{cofib}(\oplus_I 1 \rightarrow X_0) = \text{cofib}(\oplus_I 1 \rightarrow \oplus_J 1) = \text{colim}(0 \leftarrow \oplus_I 1 \rightarrow \oplus_J 1)$. Hence, each X_i is generated by 1 under colimits. Finally, $X = \text{colim } X_i$ is also generated by 1 under colimits.

(ii) \implies (i).

Arbitrary colimits can be written in terms of pushouts and filtered colimits. And H_d commutes with filtered colimits. So it suffices to show that for A, B, C with $H_d(A) = H_d(B) = H_d(C) = 0$, $\forall d < 0$, then for the pushout $D = \text{colim}(C \leftarrow A \rightarrow B)$, $H_d(D) = 0$, $\forall d < 0$.

This is because we can get a null-composite sequence $A \rightarrow B \oplus C \rightarrow D$, and induce a long exact sequence

$$\cdots \rightarrow H_d(A) \rightarrow H_d(B) \oplus H_d(C) \rightarrow H_d(D) \rightarrow \cdots$$

which implies $H_d(D) = 0$, $\forall d < 0$. □

Definition 3.16. (i) $D(R)_{\geq 0} := \{X \in D(R) \mid H_d(X) = 0, \forall d < 0\}$.

(ii) $D(R)_{< 0} := \{X \in D(R) \mid H_d(X) = 0, \forall d \geq 0\}$.

(iii) $\tau_{\geq 0} : D(R) \rightarrow D(R)_{\geq 0}; X \mapsto \tau_{\geq 0}(X) := Y$, which is constructed in Proposition 3.14.

Now, given any map $Z \rightarrow X$ in $D(R)$, we can get a commutative diagram:

$$\begin{array}{ccc} \tau_{\geq 0}(Z) & \xrightarrow{\exists!} & \tau_{\geq 0}(X) \\ \downarrow & & \downarrow \\ Z & \longrightarrow & X \end{array}$$

Hence, $\tau_{\geq 0} : D(R) \rightarrow D(R)_{\geq 0}$ is a functor.

Proposition 3.17. $D(R)_{\geq 0} \xrightleftharpoons[\tau_{\geq 0}]{i} D(R) \xrightleftharpoons[i]{\tau_{< 0}} D(R)_{< 0}$, i.e. $i \dashv \tau_{\geq 0}$ and $\tau_{< 0} \dashv i$.

Corollary 3.18. For $X \in D(R)$, we have

$$X \cong \varprojlim \tau_{\leq n}(X) \quad \text{and} \quad X \cong \varinjlim \tau_{\geq -n}(X).$$

Proof. We have a Milnor sequence:

$$0 \longrightarrow \varprojlim^1 H_{d+1}(\tau_{\leq n} X) \longrightarrow H_d(\varprojlim \tau_{\leq n} X) \longrightarrow \varprojlim H_d(\tau_{\leq n} X) \longrightarrow 0.$$

For $n \gg 0$, we have $H_d(\tau_{\leq n} X) = H_d(X)$, hence $\varprojlim H_d(\tau_{\leq n} X) = H_d(X)$.

And $\{H_{d+1}(\tau_{\leq n} X)\}_{n \in \mathbb{Z}}$ satisfies the Mittag-Leffler condition, hence $\varprojlim^1 H_{d+1}(\tau_{\leq n} X) = 0$.

Therefore, from the above short exact sequence, we have

$$H_d(\varprojlim \tau_{\leq n} X) \cong H_d(X), \quad \forall d \in \mathbb{Z},$$

which implies $X \cong \varprojlim \tau_{\leq n} X$.

For another isomorphism, from

$$H_d(\varinjlim \tau_{\geq -n} X) = \varprojlim H_d(\tau_{\geq -n} X) = H_d(X), \quad \forall d \in \mathbb{Z},$$

one can show $X \cong \varinjlim \tau_{\geq -n}(X)$. □

Definition 3.19. For any map $f : X \rightarrow Y$ in $D(R)$, we define its kernel to be

$$\ker(f) := \tau_{\geq 0} \text{fib}(X \rightarrow Y)$$

and its cokernel to be

$$\text{coker}(f) := \tau_{\leq 0} \text{cofib}(X \rightarrow Y).$$

Proposition 3.20. Let $D(R)_0 = \{X \in D(R) \mid H_d(X) = 0, \forall d \neq 0\}$.

- (i) There is an isomorphism $H_0 : D(R)_0 \xrightarrow{\sim} \text{Mod}_R$.
- (ii) Any object in $D(R)_0$ can be written as of the form $\text{coker}(\oplus_I 1 \rightarrow \oplus_J 1)$.
- (iii) $H_0 : D(R)_0 \rightarrow \text{Mod}_R$ is an exact functor.
- (iv) $H_0 : D(R)_0 \rightarrow \text{Mod}_R$ commutes with direct sums.

Proof. (ii) For $X \in D(R)_0$, there exists $f : Y \rightarrow X$ with $H_d(Y) = 0, \forall d < 0$ and $H_d(f)$ are isomorphisms, $\forall d \geq 0$.

By the construction of Y_1 , $Y_1 = \text{cofib}(\oplus_I 1 \rightarrow \oplus_J 1)$.

On the other hand, $X \cong \tau_{\leq 0} Y_1$. Hence,

$$X \cong \tau_{\leq 0} \text{cofib}(\oplus_I 1 \rightarrow \oplus_J 1) = \text{coker}(\oplus_I 1 \rightarrow \oplus_J 1).$$

- (iii) In order to show that H_0 preserves exact sequences, it suffices to show H_0 preserves kernels and cokernels.

For any map $f : X \rightarrow Y$ in $D(R)_0$, applying functor $\tau_{\geq 0}$ to sequence $\text{fib}(f) \rightarrow X \rightarrow Y$, we get a fiber-cofiber sequence

$$\ker(f) = \tau_{\geq 0} \text{fib}(f) \rightarrow X \rightarrow Y.$$

And it induces a long exact sequence

$$0 = H_1(Y) \rightarrow H_0(\ker(f)) \rightarrow H_0(X) \rightarrow H_0(Y) \rightarrow \cdots$$

Hence, $H_0(\ker(f)) = \ker(H_0(X) \rightarrow H_0(Y))$.

Dually, we can prove $H_0(\text{coker}(f)) = \text{coker}(H_0(X) \rightarrow H_0(Y))$.

□

Remark 3.21. $1 \in D(R)_0$ is compact and projective.

Proof: Compactness is the definition.

For the projectiveness, we need to show that any epimorphism $X \twoheadrightarrow 1$ splits.

Let $F = \text{fib}(X \rightarrow 1)$. Consider $F \rightarrow X \rightarrow 1$. Then $H_{-1}(F) = 0$.

By $[M, N]_d = \text{Ext}_R^{-d}(M, N)$, we get $\text{Ext}_R^{-1}(1, F) = [1, F]_{-1} = H_{-1}(F) = 0$. Hence $X \twoheadrightarrow 1$ splits. □

Definition 3.22. (i) A filtered object of $D(R)$ is an object in $\text{Fun}(\mathbb{Z}_{\leq}, D(R))$, i.e.

$$\cdots \longrightarrow F(n-1) \rightarrow F(n) \rightarrow F(n+1) \rightarrow \cdots$$

(ii) A filtered object F is convergent if $\varprojlim F(n) = 0$.

(iii) $F(\infty) := \varinjlim F(n)$. Call it the underlying object of F .

(iv) The n -th associated graded $\text{gr}_n(F) := \text{cofib}(F(n-1) \rightarrow F(n)) \triangleq F(n)/F(n-1)$.

Now, giving a convergent filtered object $F : \mathbb{Z}_{\leq} \rightarrow D(R)$, s.t. $\text{gr}_n(F) \in D(R)_n$, $\forall n$, we can define an R -module M_n :

$$H_n : D(R)_n \rightarrow \text{Mod}_R; \text{gr}_n(F) \mapsto H_n(\text{gr}_n(F)) \triangleq M_n.$$

From the sequence

$$F(n-1)/F(n-2) \longrightarrow F(n)/F(n-2) \longrightarrow F(n)/F(n-1) \longrightarrow \Sigma(F(n-1)/F(n-2)),$$

we get a map $d : H_n(\text{gr}_n(F)) \longrightarrow H_n(\Sigma \text{gr}_{n-1}(F))$, i.e. $d : M_n \longrightarrow M_{n-1}$.

One can check $d^2 = 0$. Hence, given a convergent filtered object F , s.t. $\text{gr}_n(F) \in D(R)_n$, we define a chain complex of R -modules M_* .

We denote $\text{Fun}(\mathbb{Z}_{\leq}, D(R))_{\text{cx}} = \{F \in \text{Fun}(\mathbb{Z}_{\leq}, D(R)) \mid F \text{ convergent}\}$.

Proposition 3.23. (i) $\text{Fun}(\mathbb{Z}_{\leq}, D(R))_{\text{cx}} \xrightarrow{\sim} \text{Ch}_R$; $F \mapsto M_*$.

(ii) $H_n(F(\infty)) = H_n(M_*)$, $\forall n$.

4 $D(\mathbb{Z})$

Definition 4.1. Let $X \in \text{Top}$. A sieve on X is a set \mathfrak{U} of open subsets of X , s.t. if $V \in \mathfrak{U}$ and $V' \subset V$, then $V' \in \mathfrak{U}$. If $U = \cup_{V \in \mathfrak{U}} V$, we say that the sieve \mathfrak{U} covers U .

Definition 4.2. (i) Let $X \in \text{Top}$. Let $\mathcal{F} \in \text{PSh}(X, D(\mathbb{Z}))$ be a presheaf with values in $D(\mathbb{Z})$, i.e. $\mathcal{F} \in \text{Fun}(\text{Op}(X)^{\text{op}}, D(\mathbb{Z}))$. We say \mathcal{F} is a sheaf if for all sieves \mathfrak{U} on X covering $U \in \text{Op}(X)$, we have

$$\mathcal{F}(U) \xrightarrow{\sim} \varprojlim_{V \in \mathfrak{U}^{\text{op}}} \mathcal{F}(V).$$

(ii) For $U \in \text{Op}(X)$, one define $h_U \in \text{PSh}(X, D(\mathbb{Z}))$ via

$$h_U(V) = \begin{cases} * & V \subset U \\ \emptyset & \text{otherwise} \end{cases}$$

(iii) For a sieve \mathfrak{U} , one define $h_{\mathfrak{U}} \in \text{PSh}(X, D(\mathbb{Z}))$ via

$$h_{\mathfrak{U}}(V) = \begin{cases} * & V \in \mathfrak{U} \\ \emptyset & V \notin \mathfrak{U} \end{cases}$$

Proposition 4.3. Let $\mathcal{F} \in \text{PSh}(X, D(\mathbb{Z}))$, then \mathcal{F} is a sheaf if and only if it satisfies:

(i) $\mathcal{F}(\emptyset) = *$.

(ii) For any open subsets $V, V' \in \text{Op}(X)$,

$$\mathcal{F}(V \cup V') \xrightarrow{\sim} \mathcal{F}(V) \times_{\mathcal{F}(V \cap V')} \mathcal{F}(V').$$

(iii) For any sieve \mathfrak{U} , $\mathcal{F}(\varinjlim_{V \in \mathfrak{U}} V) \xrightarrow{\sim} \varprojlim_{V \in \mathfrak{U}^{\text{op}}} \mathcal{F}(V)$.

Remark 4.4.

$$\mathbb{Z}[h_U](V) := \mathbb{Z}[h_U(V)] = \begin{cases} \mathbb{Z} & V \subseteq U \\ 0 & V \not\subseteq U \end{cases}$$

$$\mathbb{Z}[h_{\mathfrak{U}}](V) := \mathbb{Z}[h_{\mathfrak{U}}(V)] = \begin{cases} \mathbb{Z} & V \in \mathfrak{U} \\ 0 & V \notin \mathfrak{U} \end{cases}$$

$$\text{Map}(\mathbb{Z}[h_U], \mathcal{F}) = \text{Map}(\mathbb{Z}, \mathcal{F}(U)).$$

$$\begin{aligned}
\text{Map}(\mathbb{Z}[h_{\mathfrak{U}}], \mathcal{F}) &= \varprojlim_{V \in \mathfrak{U}^{\text{op}}} \text{Map}(\mathbb{Z}[h_V], \mathcal{F}) \\
&= \varprojlim_{V \in \mathfrak{U}^{\text{op}}} \text{Map}(\mathbb{Z}, \mathcal{F}(V)) \\
&= \text{Map}(\mathbb{Z}, \varprojlim_{V \in \mathfrak{U}^{\text{op}}} \mathcal{F}(V)) \\
&= \text{Map}(\mathbb{Z}, \mathcal{F}(\varinjlim_{V \in \mathfrak{U}} V)).
\end{aligned}$$

Proposition 4.5. $\text{PSh}(X, D(\mathbb{Z})) \xrightleftharpoons[i]{\text{sh}} \text{Sh}(X, D(\mathbb{Z}))$; $\mathcal{F} \mapsto \mathcal{F}^{\text{sh}}$. Moreover, $\mathcal{F}^{\text{sh}} = 0$ iff \mathcal{F} lies in the stable co-complete subcategory generated by $\text{cofib}(\mathbb{Z}[h_{\mathfrak{U}}] \rightarrow \mathbb{Z}[h_U])$ for all sieves \mathfrak{U} covering U .

Definition 4.6. For $\mathcal{F} \in \text{PSh}(X, D(\mathbb{Z}))$, define $H_n(\mathcal{F}) \in \text{PSh}(X, \text{Ab})$ by $H_n(\mathcal{F})(U) = H_n(\mathcal{F}(U))$.

With this presheaf $H_n(\mathcal{F}) \in \text{PSh}(X, \text{Ab})$, one can sheafify it to get a sheaf $H_n(\mathcal{F})^{\text{sh}} \in \text{Sh}(X, \text{Ab})$.

Proposition 4.7. Let $\mathcal{F} \in \text{PSh}(X, D(\mathbb{Z}))$.

- (i) If $\mathcal{F}^{\text{sh}} = 0$, then $H_n(\mathcal{F})^{\text{sh}} = 0$, $\forall n \in \mathbb{Z}$.
- (ii) If \mathcal{F} is bounded above and $H_n(\mathcal{F})^{\text{sh}} = 0$, $\forall n \in \mathbb{Z}$, then $\mathcal{F}^{\text{sh}} = 0$.

Corollary 4.8. Let $\mathcal{F} \rightarrow \mathcal{G}$ be a map in $\text{PSh}(X, D(\mathbb{Z}))$.

- (i) If $\mathcal{F}^{\text{sh}} \xrightarrow{\sim} \mathcal{G}^{\text{sh}}$, then $H_n(\mathcal{F})^{\text{sh}} \xrightarrow{\sim} H_n(\mathcal{G})^{\text{sh}}$, $\forall n \in \mathbb{Z}$.
- (ii) If \mathcal{F} and \mathcal{G} are bounded above, and $H_n(\mathcal{F})^{\text{sh}} \xrightarrow{\sim} H_n(\mathcal{G})^{\text{sh}}$, $\forall n \in \mathbb{Z}$, then $\mathcal{F}^{\text{sh}} \xrightarrow{\sim} \mathcal{G}^{\text{sh}}$.

Corollary 4.9. Let $\mathcal{F} \rightarrow \mathcal{G}$ be a map in $\text{PSh}(X, D(\mathbb{Z}))$ and \mathcal{F}, \mathcal{G} are bounded above, then

$$\mathcal{F}^{\text{sh}} \xrightarrow{\sim} \mathcal{G} \iff \begin{cases} \mathcal{G} \text{ is a sheaf.} \\ H_n(\mathcal{F})^{\text{sh}} \xrightarrow{\sim} H_n(\mathcal{G})^{\text{sh}}, \forall n \in \mathbb{Z}. \end{cases}$$

Definition 4.10.

Proposition 4.11.

5 The t-structure on valued sheaves

Definition 5.1. A t-structure on a stable ∞ -category \mathcal{C} is a pair $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ of full sub- ∞ -categories of \mathcal{C} that are stable under equivalences and satisfy:

- (T1) The suspension functor Σ and the loop functor Ω restrict to $\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0}$ resp. are fully faithful functors $\Sigma : \mathcal{C}_{\geq 0} \rightarrow \mathcal{C}_{\geq 0}$ and $\Omega : \mathcal{C}_{\leq 0} \rightarrow \mathcal{C}_{\leq 0}$.
- (T2) If $X \in \mathcal{C}_{\geq 0}$ and $Y \in \mathcal{C}_{\leq 0}$, then $\text{Map}(X, \Omega Y) \simeq *$.
- (T3) For every $X \in \mathcal{C}$, there exists a fiber sequence

$$X' \longrightarrow X \longrightarrow X''$$

with $X' \in \mathcal{C}_{\geq 0}$ and $X'' \in \mathcal{C}_{\leq -1} := \Omega \mathcal{C}_{\leq 0}$.

We call $\mathcal{C}_{\geq 0}$ and $\mathcal{C}_{\leq 0}$ the connective and coconnective parts of the t-structure.

Given $n \in \mathbb{Z}$, we define $\mathcal{C}_{\geq n} := \Sigma^n \mathcal{C}_{\geq 0} \subset \mathcal{C}$ and $\mathcal{C}_{\leq n} := \Sigma^n \mathcal{C}_{\leq 0} \subset \mathcal{C}$, where for $n < 0$, we have $\Sigma^n = \Omega^{-n}$.

The inclusions $i : \mathcal{C}_{\geq m} \rightarrow \mathcal{C}$ and $s : \mathcal{C}_{\leq n} \rightarrow \mathcal{C}$ admit adjoint functors

$$\mathcal{C}_{\geq m} \begin{matrix} \xrightarrow{i} \\ \xleftarrow{r} \end{matrix} \mathcal{C} \begin{matrix} \xrightarrow{p} \\ \xleftarrow{s} \end{matrix} \mathcal{C}_{\leq n}.$$

In particular, the full sub- ∞ -category $\mathcal{C}_{\geq m} \subset \mathcal{C}$ is closed under colimits, and the full sub- ∞ -category $\mathcal{C}_{\leq n} \subset \mathcal{C}$ is closed under limits. From the adjoint pairs, we can form their counit and unit, and we get

$$\tau_{\geq 0} X = (i \circ r)(X) \xrightarrow{\epsilon} X \xrightarrow{\eta} \tau_{\leq -1} X = (s \circ p)(X).$$

The composition of the two maps is a point in the anima $\text{Map}(\tau_{\geq 0} X, \tau_{\leq -1} X) \simeq *$. So the composite map automatically admits a null-homotopy, which is unique, up to contractible ambiguity. We have the following commutative diagram:

$$\begin{array}{ccc} \mathcal{C}_{\leq m} \cap \mathcal{C}_{\geq n} & \begin{matrix} \xrightarrow{i} \\ \xleftarrow{r} \end{matrix} & \mathcal{C}_{\leq m} \\ \begin{matrix} \uparrow p \\ \downarrow s \end{matrix} & & \begin{matrix} \uparrow p \\ \downarrow s \end{matrix} \\ \mathcal{C}_{\geq n} & \begin{matrix} \xrightarrow{i} \\ \xleftarrow{r} \end{matrix} & \mathcal{C} \end{array}$$

The canonical map

$$p \circ r \xrightarrow{\eta \circ p \circ r} r \circ i \circ p \circ r \simeq r \circ p \circ i \circ r \xrightarrow{r \circ p \circ \epsilon} r \circ p$$

is an equivalence.

We say the full sub- ∞ -category

$$\mathcal{C}^\heartsuit := \mathcal{C}_{\geq 0} \cap \mathcal{C}_{\leq 0} \subset \mathcal{C}$$

is the heart of the t-structure. For the functor

$$\pi_0 := \tau_{\geq 0} \circ \tau_{\leq 0} \simeq \tau_{\leq 0} \circ \tau_{\geq 0} : \mathcal{C} \rightarrow \mathcal{C}^\heartsuit,$$

we call it the zeroth homotopy functor. The functor π_0 is additive, but is NOT exact. Instead, for all $n \in \mathbb{Z}$, we define

$$\pi_d : \mathcal{C} \longrightarrow \mathcal{C}^\heartsuit$$

to be $\pi_d = \pi_0 \circ \Omega^d$, and call it the d th homotopy functor. Now, a fiber sequence

$$Z \xrightarrow{g} Y \xrightarrow{f} X$$

in \mathcal{C} gives rise to a long exact sequence

$$\cdots \longrightarrow \pi_{d+1}(X) \longrightarrow \pi_d(Z) \longrightarrow \pi_d(Y) \longrightarrow \pi_d(X) \longrightarrow \cdots$$

in the heart \mathcal{C}^\heartsuit .

If $f : Y \rightarrow X$ is an equivalence, then $f : \pi_d(Y) \rightarrow \pi_d(X)$ is an isomorphism for all $d \in \mathbb{Z}$, but the opposite is generally not the case.

Now, for the stable ∞ -category $D(\mathbb{Z})$, we defined homology functors $H_d : D(\mathbb{Z}) \rightarrow \text{Mod}_{\mathbb{Z}}$ for all $d \in \mathbb{Z}$ by

$$H_d(X) \simeq \pi_0 \text{Map}(\Sigma^d 1, X) \simeq \pi_0 \text{Map}(1, \Omega^d X).$$

$D(\mathbb{Z})$ admits a t-structure $(D(\mathbb{Z})_{\geq 0}, D(\mathbb{Z})_{\leq 0})$, where the connective part $D(\mathbb{Z})_{\geq 0}$ is spanned by those X for which $H_d(X) \simeq 0$, for $d < 0$, and the coconnective part $D(\mathbb{Z})_{\leq 0}$ is spanned by those X for which $H_d(X) \simeq 0$, for $d > 0$. The zeroth homology functor

$$H_0 : D(\mathbb{Z})^\heartsuit \longrightarrow \text{Mod}_{\mathbb{Z}}$$

is an equivalence of (abelian) categories. We have $H_d \simeq H_0 \circ \pi_d$, so the functors H_d and π_d encode the same information.

Proposition 5.2. Let $X \in \text{Top}$, and let \mathcal{C} be a stable ∞ -category. A t-structure on \mathcal{C}

induces a t-structure on the stable ∞ -category $\mathcal{P}(X, \mathcal{C})$ of \mathcal{C} -valued presheaves on X , where the coconnective part $\mathcal{P}(X, \mathcal{C})_{\leq 0} \simeq \mathcal{P}(X, \mathcal{C}_{\leq 0})$, and where the connective part $\mathcal{P}(X, \mathcal{C})_{\geq 0}$ is spanned by those \mathcal{F} such that

$$\mathrm{Map}(\mathcal{F}, \Omega \mathcal{G}) \simeq *$$

for all $\mathcal{G} \in \mathcal{P}(X, \mathcal{C}_{\leq 0})$.

A functor $f : \mathcal{D} \rightarrow \mathcal{C}$ between stable ∞ -categories is exact iff it is left exact iff it is right exact.

An exact functor $f : \mathcal{D} \rightarrow \mathcal{C}$ between stable ∞ -categories with t-structures is left t-exact if $f(\mathcal{D}_{\leq 0}) \subset \mathcal{D}_{\leq 0}$, and it is right t-exact if $f(\mathcal{D}_{\geq 0}) \subset \mathcal{D}_{\geq 0}$. It is t-exact if it is both left t-exact and right t-exact. If $f : \mathcal{D} \rightarrow \mathcal{C}$ admits right adjoint functor $g : \mathcal{C} \rightarrow \mathcal{D}$, then f is right t-exact iff g is left t-exact.

Theorem 5.3. Let $X \in \mathrm{Top}$ and \mathcal{C} a presentable stable ∞ -category.

- (1) The sheafification functor $\mathrm{ass}_X : \mathcal{P}(X, \mathcal{C}) \rightarrow \mathrm{Sh}(X, \mathcal{C})$ is t-exact, and the inclusion functor $\iota_X : \mathrm{Sh}(X, \mathcal{C}) \rightarrow \mathcal{P}(X, \mathcal{C})$ is left t-exact.
- (2) The composite functor

$$\mathrm{Sh}(X, \mathcal{C}^\heartsuit) \xrightarrow{\iota_X^\heartsuit} \mathcal{P}(X, \mathcal{C}^\heartsuit) \simeq \mathcal{P}(X, \mathcal{C})^\heartsuit \xrightarrow{\mathrm{ass}_X} \mathrm{Sh}(X, \mathcal{C})^\heartsuit$$

is an equivalence of categories.

Write π_0^p and π_0^s for the homotopy functors associated with the t-structure on presheaves and sheaves. Since ass_X is both exact and t-exact, we obtain a commutative square

$$\begin{array}{ccc} \mathcal{P}(X, \mathcal{C}) & \xrightarrow{\pi_0^p} & \mathcal{P}(X, \mathcal{C})^\heartsuit \\ \downarrow \mathrm{ass}_X & & \downarrow \mathrm{ass}_X \\ \mathrm{Sh}(X, \mathcal{C}) & \xrightarrow{\pi_0^s} & \mathrm{Sh}(X, \mathcal{C})^\heartsuit \end{array}$$

6 Sheaf

Lemma 6.1. If \mathcal{A} is bounded above, i.e. $\exists d \in \mathbb{Z}$, s.t. $H_n(\mathcal{A}) = 0$, for all $n > d$, then \mathcal{A}^{sh} is also bounded above.

Question. For finite sets X, X' with $X' \rightarrow X$ surjective and split, then

$$0 \rightarrow \mathbb{Z}[X] \rightarrow \mathbb{Z}[X'] \rightarrow \mathbb{Z}[X' \times_X X'] \rightarrow \mathbb{Z}[X' \times_X X' \times_X X'] \rightarrow \cdots$$

is exact.

Lemma 6.2. Arbitrary limits and filtered colimits preserves $D(\mathbb{Z})_{\leq d}$.

Proof. First we show $D(\mathbb{Z})_{\leq d}$ is closed under filtered colimits. Assume $X_i \in D(\mathbb{Z})_{\leq d}, i \in I$, then

$$H_n(\varinjlim X_i) = \varinjlim H_n(X_i) = 0, \text{ for any } n > d.$$

Hence $\varinjlim X_i \in D(\mathbb{Z})_{\leq d}$. Then we show $D(\mathbb{Z})_{\leq d}$ is closed under arbitrary limits. Assume $X_i \in D(\mathbb{Z})_{\leq d}, n > d$, then

$$\begin{aligned} H_n(\lim X_i) &= [\Sigma^n 1, \lim X_i] \\ &= \pi_0 \text{Map}(\Sigma^n 1, \lim X_i) \\ &= \pi_0 \lim \text{Map}(\Sigma^n 1, X_i) \\ &= \pi_0 \lim * \\ &= \pi_0 * \\ &= 0. \end{aligned}$$

Hence, $\lim X_i \in D(\mathbb{Z})_{\leq d}$. □

Problem. What is the relation between $\pi_n(\lim X_i)$ and $\lim \pi_n(X_i)$. Similarly, the relation between $\pi_n(\text{colim } X_i)$ and $\text{colim } \pi_n(X_i)$.

Definition 6.3. We define the singular homology functor to be the composite of

$$\text{Top} \rightarrow \text{Cond}(\text{Set}) \hookrightarrow \text{Cond}(\text{An}) \rightarrow \text{An},$$

and denote it by $h : \text{Top} \rightarrow \text{An}$, where $\text{Top} \rightarrow \text{Cond}(\text{Set}), X \mapsto \underline{X}$; $\text{Cond}(\text{An}) \rightarrow \text{An}$ is the left adjoint of $\text{An} \hookrightarrow \text{Cond}(\text{An})$.

Definition 6.4. For the forgetful functor $D(\mathbb{Z})_{\geq 0} \simeq \text{Ani}(\text{Ab}) \rightarrow \text{Ani}(\text{Set}) \simeq \text{An}$, it has a left

adjoint, and we denote it by

$$\mathbb{Z}[-] : \text{Ani}(\text{Set}) \rightarrow \text{Ani}(\text{Ab}); S \mapsto \mathbb{Z}[S].$$

Definition 6.5. For $X \in \text{Top}$, we define its singular homology object to be

$$\mathbb{Z}[h(X)] \in \text{Ani}(\text{Ab}) \simeq D(\mathbb{Z})_{\geq 0} \subset D(\mathbb{Z}).$$

Lemma 6.6. Assume $\mathcal{A} \in \text{Sh}(X, D(\mathbb{Z}))$, $H_n(\mathcal{A}) = 0, \forall n > d$, and $H_d(\mathcal{A}) \neq 0$, then $H_d(\mathcal{A})$ is a sheaf.

Proof. For $H_d(\mathcal{A}) \in \text{PSh}(X, \text{Ab})$, we need to check $H_d(\mathcal{A}) \in \text{Sh}(X, \text{Ab})$.

By denition, $H_d(\mathcal{A})(U) = H_d(\mathcal{A}(U)) = H_d(\varprojlim \mathcal{A}(V))$. By the Milnor's sequence, we have

$$0 \longrightarrow \varprojlim^1 H_{d+1}(\mathcal{A}(V)) \longrightarrow H_d(\varprojlim \mathcal{A}(V)) \longrightarrow \varprojlim H_d(\mathcal{A}(V)) \longrightarrow 0.$$

Because $H_{d+1}(\mathcal{A}) = 0$, so the left term of this short exact sequence is 0, hence

$$H_d(\mathcal{A})(U) = H_d(\varprojlim \mathcal{A}(V)) = \varprojlim H_d(\mathcal{A}(V)) = \varprojlim H_d(\mathcal{A})(V).$$

Hence, $H_d(\mathcal{A}) \in \text{Sh}(X, \text{Ab})$. □

Proposition 6.7. Let $\mathcal{C}_0 \subset \mathcal{C}$ be a full subcategory, then the following full subcategories of \mathcal{C} agree:

- the full subcategory generated under (small) colimits by \mathcal{C}_0 ;
- the full subcategory generated under filtered colimits and finite colimits by \mathcal{C}_0 ;
- the full subcategory generated under sifted colimits and finite products by \mathcal{C}_0 .

7 Animation

Theorem 7.1 (Yoneda). Let \mathcal{C} be an ∞ -category, the functor

$$\mathcal{C} \hookrightarrow \text{Fun}(\mathcal{C}^{\text{op}}, \text{An}); X \mapsto (Y \mapsto \text{Hom}_{\mathcal{C}}(Y, X))$$

is fully faithful.

Remark 7.2. For S to be an anima, we mean S is an ∞ -category; while S to be a Kan complex, we mean S is a 1-category.

Let \mathcal{C} be a category which admits all small colimits.

Recall an object $X \in \mathcal{C}$ is compact (also called finitely presented) if $\text{Hom}(X, -)$ commutes with filtered colimits.

An object $X \in \mathcal{C}$ is projective if $\text{Hom}(X, -)$ commutes with reflexive coequalizers (coequalizers of parallel arrows $Y \rightrightarrows Z$ with a simultaneous section $Z \rightarrow Y$ of both maps).

Taken together, an object $X \in \mathcal{C}$ is compact projective if $\text{Hom}(X, -)$ commutes with filtered colimits and reflexive coequalizers, equivalently, $\text{Hom}(X, -)$ commutes with 1-sifted colimits.

Let $\mathcal{C}^{\text{cp}} \subset \mathcal{C}$ be the full subcategory of compact projective objects. There is a fully faithful embedding $\text{sInd}(\mathcal{C}^{\text{cp}}) \rightarrow \mathcal{C}$.

If \mathcal{C} is generated under small colimits by \mathcal{C}^{cp} , then the functor is an equivalence:

$$\text{sInd}(\mathcal{C}^{\text{cp}}) \cong \mathcal{C}.$$

If \mathcal{C}^{cp} is small, then

$$\text{sInd}(\mathcal{C}^{\text{cp}}) \subset \text{Fun}((\mathcal{C}^{\text{cp}})^{\text{op}}, \text{Set})$$

is exactly the full subcategory of functors that take finite coproducts in \mathcal{C}^{cp} to products in Set .

Example 7.3. (i) If $\mathcal{C} = \text{Set}$, then $\mathcal{C}^{\text{cp}} = \text{FinSet}$, which generates \mathcal{C} under small colimits.

(ii) If $\mathcal{C} = \text{Ab}$, then $\mathcal{C}^{\text{cp}} = \text{FinFreeAb}$, which generates \mathcal{C} under small colimits.

(iii) If $\mathcal{C} = \text{Ring}$, then $\mathcal{C}^{\text{cp}} = \{\text{retracts of } \mathbb{Z}[X_1, \dots, X_n]\}$, which generates \mathcal{C} under small colimits.

(iv) If $\mathcal{C} = \text{Cond}(\text{Set})$, then $\mathcal{C}^{\text{cp}} = \text{ExDisc}$, which generates \mathcal{C} under small colimits.

(v) If $\mathcal{C} = \text{Cond}(\text{Ab})$, then $\mathcal{C}^{\text{cp}} = \{\text{direct summands of } \mathbb{Z}[S] \mid S \in \text{ExDisc}\}$, which generates \mathcal{C} under small colimits.

(vi) $\mathcal{C} = \text{Cond}(\text{Ring})$, then $\mathcal{C}^{\text{cp}} = \{\text{retracts of } \mathbb{Z}[\mathbb{N}[S]] \mid S \in \text{ExDisc}\}$, which generates \mathcal{C} under small colimits.

Definition 7.4. Let \mathcal{C} be a category that admits all small colimits and \mathcal{C} is generated under small colimits by \mathcal{C}^{cp} . The animation of \mathcal{C} is the ∞ -category $\text{Ani}(\mathcal{C})$ freely generated under sifted colimits by \mathcal{C}^{cp} .

Example 7.5. If $\mathcal{C} = \text{Set}$, then $\text{Ani}(\mathcal{C}) = \text{Ani}(\text{Set}) \triangleq \text{Ani}$ is the ∞ -category of animated sets, or anima in a short.

Any anima has a set of connected components, giving a functor $\pi_0 : \text{Ani} \rightarrow \text{Set}$, which has a fully faithful right adjoint $\text{Set} \hookrightarrow \text{Ani}$.

Given an anima A with a point $a \in A$ (meaning a map $a : * \rightarrow A$), one can define groups $\pi_i(A, a)$, for $i \geq 1$ and for $i \geq 2$, $\pi_i(A, a) \in \text{Ab}$.

An anima A is i -truncated if $\pi_j(A, a) = 0$, $\forall a \in A$ and $\forall j > i$. Then A is 0-truncated if and only if it is in the essential image of $\text{Set} \hookrightarrow \text{Ani}$.

The inclusion of i -truncated anima into all anima has a left adjoint $\tau_{\leq i}$. For all anima A , the natural map

$$A \xrightarrow{\sim} \lim \tau_{\leq i} A$$

is an equivalence.

Picking any $a \in A$ and $i \geq 1$, the fiber of $\tau_{\leq i} A \rightarrow \tau_{\leq i-1} A$ over the image of a is an Eilenberg-MacLane anima $K(\pi_i(A, a), i)$. Here, an Eilenberg-MacLane anima $K(\pi, i)$ with $i \geq 1$ and π a group that is abelian if $i > 0$, is a pointed connected anima with $\pi_j = 0$ for $j \neq i$ and $\pi_i = \pi$. In fact, the ∞ -category of pointed connected anima (A, a) with $\pi_j(A, a) = 0$ for $j \neq i$ is equivalent to Grp when $i = 1$, and to Ab when $i \geq 2$.

Remark 7.6. There are several ways to describe $\text{Ani}(\mathcal{C})$.

- (i) $\text{Ani}(\mathcal{C})$ is the full sub- ∞ -category of objects in $\text{Fun}((\mathcal{C}^{\text{cp}})^{\text{op}}, \text{Ani})$ taking finite disjoint unions to finite products.
- (ii) $\text{Ani}(\mathcal{C})$ is the ∞ -category obtained from $\text{Simp}(\mathcal{C})$ by inverting weak equivalences.

Definition 7.7. Let \mathcal{C} be an ∞ -category that admits all small colimits. For any uncountable strong limit cardinal κ , the ∞ -category $\text{Cond}_{\kappa}(\mathcal{C})$ of κ -condensed objects of \mathcal{C} is the category of contravariant functors from $\kappa\text{-ExDisc}$ to \mathcal{C} that take finite coproducts to finite products. And we define

$$\text{Cond}(\mathcal{C}) := \bigcup_{\kappa} \text{Cond}_{\kappa}(\mathcal{C}).$$

Proposition 7.8. Let \mathcal{C} be a category that is generated under small colimits by \mathcal{C}^{cp} . Then $\text{Cond}(\mathcal{C})$ is still generated under small colimits by its compact projective objects, and there

is a natural equivalence of ∞ -categories

$$\mathrm{Cond}(\mathrm{Ani}(\mathcal{C})) \cong \mathrm{Ani}(\mathrm{Cond}(\mathcal{C})).$$

Definition 7.9. Let \mathcal{C} be some site.

(i) A presheaf of anima is a functor

$$\mathcal{F} : \mathbf{N}(\mathcal{C}^{\mathrm{op}}) \longrightarrow \mathrm{Ani}.$$

(ii) A sheaf of anima is a presheaf of anima \mathcal{F} , s.t. for all coverings $\{f_i : X_i \rightarrow X\}_{i \in I}$, one has

$$\mathcal{F}(X) \xrightarrow{\sim} \lim(\prod_i \mathcal{F}(X_i) \rightrightarrows \prod_{i,j} \mathcal{F}(X_i \times_X X_j) \rightrightarrows \cdots).$$

(iii) A hypercomplete sheaf of anima is a sheaf of anima \mathcal{F} , s.t. for all hypercovers $X_\bullet \rightarrow X$, the map

$$\mathcal{F}(X) \xrightarrow{\sim} \lim \mathcal{F}(X_\bullet) = \lim (\mathcal{F}(X_0) \rightrightarrows \mathcal{F}(X_1) \rightrightarrows \cdots)$$

is an equivalence.

Definition 7.10. The ∞ -category of condensed anima is given by

- The ∞ -category of hypercomplete sheaves of anima on \mathbf{CHaus} .
- The ∞ -category of hypercomplete sheaves of anima on \mathbf{ProFin} .
- The ∞ -category of hypercomplete sheaves of anima on \mathbf{ExDisc} , i.e. of functors

$$\mathbf{ExDisc}^{\mathrm{op}} \longrightarrow \mathrm{Ani}$$

taking finite disjoint unions to finite products.

$$\begin{array}{ccc} \mathbf{CW} & \subset & \mathrm{Cond}(\mathbf{Set}) \\ \cap & & \cap \\ \mathbf{Ani} & \subset & \mathrm{Cond}(\mathbf{Ani}) \end{array}$$

Definition 7.11. $X \in \mathrm{Cond}(\mathbf{Ani})$ is

- discrete, if X in the essential image of Ani .
- static, if X in the essential image of $\text{Cond}(\text{Set})$.

8 Condensed Cohomology

Definition 8.1. Let $X \in \text{Cond}$, $M \in \text{Cond}(\text{Ab})$, we define the global section of M on X to be

$$\Gamma_{\text{cond}}(X, M) := \text{Hom}_{\text{Cond}}(X, M) = \text{Hom}_{\text{Cond}(\text{Ab})}(\mathbb{Z}[X], M) \in \text{Ab},$$

and we define the condensed cohomology to be

$$R\Gamma_{\text{cond}}(X, M) := R\text{Hom}_{\text{Cond}(\text{Ab})}(\mathbb{Z}[X], M),$$

i.e.

$$H_{\text{cond}}^i(X, M) := \text{Ext}_{\text{Cond}(\text{Ab})}^i(\mathbb{Z}[X], M).$$

Lemma 8.2. For $X \in \text{ExDisc}$, the functor $\Gamma_{\text{cond}}(X, -) : \text{Cond}(\text{Ab}) \rightarrow \text{Ab}$ is exact, hence, for any $M \in \text{CondAb}$, $H_{\text{cond}}^i(X, M) = 0, \forall i \geq 1$.

Proof. We have $\Gamma_{\text{cond}}(X, -) = \text{Hom}_{\text{Cond}(\text{Ab})}(\mathbb{Z}[X], -)$, and for $X \in \text{ExDisc}$, $\mathbb{Z}[X]$ is projective, hence $\Gamma_{\text{cond}}(X, -) : \text{Cond}(\text{Ab}) \rightarrow \text{Ab}$ is exact. \square

Question. How to compute $H_{\text{cond}}^i(X, M)$?

From the definition, we need to find a projective resolution of $\mathbb{Z}[X]$.

For $X \in \text{CHaus}$, we pick a hypercover $X_{\bullet} \rightarrow X$, where each $X_i \in \text{ExDisc}$, for this hypercover, applying $\mathbb{Z}[-]$, then we get a projective resolution of $\mathbb{Z}[X]$:

$$\cdots \longrightarrow \mathbb{Z}[X_2] \longrightarrow \mathbb{Z}[X_1] \longrightarrow \mathbb{Z}[X_0] \longrightarrow \mathbb{Z}[X] \longrightarrow 0.$$

By definition, we have

$$\begin{aligned} H_{\text{cond}}^i(X, M) &= \text{Ext}_{\text{Cond}(\text{Ab})}^i(\mathbb{Z}[X], M) \\ &= H^i(0 \rightarrow \text{Hom}_{\text{Cond}(\text{Ab})}(\mathbb{Z}[X_0], M) \rightarrow \text{Hom}_{\text{Cond}(\text{Ab})}(\mathbb{Z}[X_1], M) \rightarrow \cdots) \\ &= H^i(0 \rightarrow \Gamma_{\text{cond}}(X_0, M) \rightarrow \Gamma_{\text{cond}}(X_1, M) \rightarrow \Gamma_{\text{cond}}(X_2, M) \rightarrow \cdots). \end{aligned}$$

Theorem 8.3 (Dyckhoff, 1976). For any $X \in \text{CHaus}$, there are natural isomorphisms:

$$H_{\text{cond}}^i(X, \mathbb{Z}) \cong H_{\text{sh}}^i(X, \mathbb{Z}), \quad \forall i \geq 0.$$

Proof. 1) Assume $X \in \text{Fin}$, then

$$H_{\text{cond}}^i(X, \mathbb{Z}) = \begin{cases} \Gamma_{\text{cond}}(X, \mathbb{Z}) = C(X, \mathbb{Z}) & i = 0 \\ 0 & i > 0 \end{cases}$$

This comes from Lemma 8.2. On the other hand,

$$H_{\text{sh}}^i(X, \mathbb{Z}) = \check{H}^i(X, \mathbb{Z}) = \begin{cases} C(X, \mathbb{Z}) & i = 0 \\ 0 & i > 0 \end{cases}$$

This comes from by computing Čech cohomology. For a finite set X , take the cover $\mathcal{U} = \{x \rightarrow X\}_{x \in X}$, then $\mathcal{C}^0(\mathcal{U}, \mathbb{Z}) = \mathcal{C}^1(\mathcal{U}, \mathbb{Z}) = \cdots = \mathbb{Z}^X$, and because \mathcal{U} is a refinement of any cover, we have

$$\check{H}^i(X, \mathbb{Z}) = \check{H}^i(\mathcal{U}, \mathbb{Z}) = \begin{cases} \mathbb{Z}^X = C(X, \mathbb{Z}) & i = 0 \\ 0 & i > 0 \end{cases}$$

Therefore, for a finite set X , $H_{\text{cond}}^i(X, \mathbb{Z}) \cong H_{\text{sh}}^i(X, \mathbb{Z})$, $\forall i \geq 0$.

2) $X \in \text{ProFin}$, hence we can write $X = \varprojlim_j X^j$, $X^j \in \text{Fin}$.

$$H_{\text{sh}}^i(X, \mathbb{Z}) = \check{H}^i(X, \mathbb{Z}) = \varprojlim_j \check{H}^i(X^j, \mathbb{Z}) = \begin{cases} \varprojlim_j C(X^j, \mathbb{Z}) = C(X, \mathbb{Z}) & i = 0 \\ 0 & i > 0 \end{cases}$$

On the other hand, We compute $H_{\text{cond}}^i(X, \mathbb{Z})$, $i \geq 0$.

For $X \in \text{ProFin}$, pick a hypercover $X_{\bullet} \rightarrow X$ with each $X_i \in \text{ExDisc}$, and for each X^j , pick a finite hypercover $X_{\bullet}^j \rightarrow X^j$, s.t. $\varprojlim_j X_n^j = X_n$. Since X^j is finite, we have

$$H_{\text{cond}}^i(X^j, \mathbb{Z}) = \begin{cases} \Gamma(X^j, \mathbb{Z}) & i = 0 \\ 0 & i > 0 \end{cases}$$

And we know

$$H_{\text{cond}}^i(X^j, \mathbb{Z}) = H^i(0 \longrightarrow \Gamma(X_0^j, \mathbb{Z}) \longrightarrow \Gamma(X_1^j, \mathbb{Z}) \longrightarrow \Gamma(X_2^j, \mathbb{Z}) \rightarrow \cdots),$$

hence we have an exact sequence:

$$0 \longrightarrow \Gamma(X^j, \mathbb{Z}) \longrightarrow \Gamma(X_0^j, \mathbb{Z}) \longrightarrow \Gamma(X_1^j, \mathbb{Z}) \longrightarrow \Gamma(X_2^j, \mathbb{Z}) \rightarrow \cdots$$

Applying the exact functor \varinjlim_j to this exact sequence, we get an exact sequence:

$$0 \longrightarrow \varinjlim_j \Gamma(X^j, \mathbb{Z}) \longrightarrow \varinjlim_j \Gamma(X_0^j, \mathbb{Z}) \longrightarrow \varinjlim_j \Gamma(X_1^j, \mathbb{Z}) \longrightarrow \varinjlim_j \Gamma(X_2^j, \mathbb{Z}) \longrightarrow \cdots,$$

i.e.

$$0 \longrightarrow \Gamma(X, \mathbb{Z}) \longrightarrow \Gamma(X_0, \mathbb{Z}) \longrightarrow \Gamma(X_1, \mathbb{Z}) \longrightarrow \Gamma(X_2, \mathbb{Z}) \longrightarrow \cdots.$$

Hence,

$$H_{\text{cond}}^i(X, \mathbb{Z}) = \begin{cases} \Gamma(X, \mathbb{Z}) & i = 0 \\ 0 & i > 0 \end{cases}$$

3) $X \in \text{CHaus}$.

Consider a morphism of topoi $(\alpha^{-1}, \alpha_*) : \text{Sh}(\text{CHaus}/X) \rightarrow \text{Sh}(X)$. For $\mathcal{F} \in \text{Sh}(\text{CHaus}/X)$, $\alpha_* \mathcal{F}$ is given by

$$U \mapsto \varprojlim_{V \subset U, V \text{ is closed in } X} \mathcal{F}(V \hookrightarrow S).$$

We have the following diagram:

$$\begin{array}{ccc} \text{Sh}(\text{CHaus}/X) & \xrightarrow{\alpha_*} & \text{Sh}(X) \\ & \searrow \Gamma_{\text{cond}}(X, -) & \swarrow \Gamma_{\text{sh}}(X, -) \\ & \text{Set} & \end{array}$$

This is because $\forall Y \in \text{Sh}(\text{CHaus}/X)$,

$$\begin{aligned} \Gamma_{\text{sh}}(X, \alpha_* Y) &= \alpha_* Y(X) = \varprojlim_{V \subset X, V \text{ is closed in } X} Y(V) \\ &= \varprojlim_V \text{Hom}_{\text{cond}}(V, Y) = \text{Hom}_{\text{cond}}(\varinjlim_V V, Y) \\ &= \text{Hom}_{\text{cond}}(X, Y) = \Gamma_{\text{cond}}(X, Y). \end{aligned}$$

And this diagram can induce a diagram:

$$\begin{array}{ccc} D(\text{Ab}(\text{CHaus}/X)) & \xrightarrow{R\alpha_*} & D(\text{Ab}(X)) \\ & \searrow R\Gamma_{\text{cond}}(X, -) & \swarrow R\Gamma_{\text{sh}}(X, -) \\ & D(\text{Ab}) & \end{array}$$

Claim: $R\alpha_* \mathbb{Z} \cong \mathbb{Z}$ in $D(\text{Ab}(X))$.

With this claim, we can show

$$\begin{aligned}
H_{\text{cond}}^i(X, \mathbb{Z}) &= H^i(R\Gamma_{\text{cond}}(X, \mathbb{Z})) \\
&= H^i(R\Gamma_{\text{sh}}(X, -) \circ R\alpha_*\mathbb{Z}) \\
&= H^i(R\Gamma_{\text{sh}}(X, \mathbb{Z})) \\
&= H_{\text{sh}}^i(X, \mathbb{Z}).
\end{aligned}$$

Hence, it suffices to show this claim. We have a map $\mathbb{Z} \rightarrow R\alpha_*\mathbb{Z}$ in $D(\text{Ab}(X))$. In order to show this is an isomorphism, it suffices to check on each stacks.

Fix $s \in S$,

$$\begin{aligned}
(R\alpha_*\mathbb{Z})_s &= \varinjlim_{s \in U \text{ open}} R\Gamma(U, R\alpha_*\mathbb{Z}) \\
&= \varinjlim_{s \in U \text{ open}} R\Gamma_{\text{cond}}(U, \mathbb{Z}) \\
&= \varinjlim_{s \in V \text{ closed}} R\Gamma_{\text{cond}}(V, \mathbb{Z}).
\end{aligned}$$

Pick a hypercover $S_\bullet \rightarrow S$ with $S_i \in \text{ExDisc}$. Then for each closed V , $(S_n \times_X V)_{n \geq 0} \rightarrow V$ is a hypercover. Hence,

$$R\Gamma_{\text{cond}}(V, \mathbb{Z}) \cong (0 \longrightarrow \Gamma(S_0 \times_X V, \mathbb{Z}) \longrightarrow \Gamma(S_1 \times_X V, \mathbb{Z}) \longrightarrow \cdots).$$

Thus, we have

$$\begin{aligned}
(R\alpha_*\mathbb{Z})_s &= \varinjlim_{s \in V \text{ closed}} R\Gamma_{\text{cond}}(V, \mathbb{Z}) \\
&\cong \varinjlim_{s \in V \text{ closed}} (0 \longrightarrow \Gamma(S_0 \times_X V, \mathbb{Z}) \longrightarrow \Gamma(S_1 \times_X V, \mathbb{Z}) \longrightarrow \cdots) \\
&\cong (0 \longrightarrow \varinjlim_{s \in V \text{ closed}} \Gamma(S_0 \times_X V, \mathbb{Z}) \longrightarrow \varinjlim_{s \in V \text{ closed}} \Gamma(S_1 \times_X V, \mathbb{Z}) \longrightarrow \cdots) \\
&\cong (0 \longrightarrow \Gamma(S_0 \times_X \{s\}, \mathbb{Z}) \longrightarrow \Gamma(S_1 \times_X \{s\}, \mathbb{Z}) \longrightarrow \cdots) \\
&\cong R\Gamma_{\text{cond}}(\{s\}, \mathbb{Z}) \\
&\cong \mathbb{Z},
\end{aligned}$$

which finishes our proof. □

Example 8.4. Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, for $\mathbb{T}^I \in \text{CHaus}$, we have $H^n(\mathbb{T}^I, \mathbb{Z}) = \wedge^n(\mathbb{Z}^{\oplus I})$.

Proof. First, we have

$$H^n(\mathbb{T}, \mathbb{Z}) = \begin{cases} \mathbb{Z} & n = 0, 1 \\ 0 & \text{else} \end{cases}$$

i.e. $H^*(\mathbb{T}, \mathbb{Z}) = \wedge(\mathbb{Z})$.

Claim: $H^*(\mathbb{T}^n, \mathbb{Z}) = \wedge(\mathbb{Z}^{\oplus n})$.

We can prove it by induction on n . $n = 1$ is proved above.

By Kunneth theorem, we can show that for $H^*(X, \mathbb{Z})$ finitely generated free in each degree, we have $H^*(X \times Y, \mathbb{Z}) \cong H^*(X, \mathbb{Z}) \otimes H^*(Y, \mathbb{Z})$. Hence, we have

$$\begin{aligned} H^*(\mathbb{T}^n, \mathbb{Z}) &= H^*(\mathbb{T}^{n-1}, \mathbb{Z}) \otimes H^*(\mathbb{T}, \mathbb{Z}) \\ &= \wedge(\mathbb{Z}^{\oplus(n-1)}) \otimes \wedge(\mathbb{Z}) \\ &= \wedge(\mathbb{Z}^{\oplus n}). \end{aligned}$$

In order to prove the general case, there is a fact that for $S \in \text{CHaus}$, $S = \varprojlim_j S_j$, then

$$H^n(S, \mathbb{Z}) = \varinjlim_j H^n(S_j, \mathbb{Z}).$$

Hence,

$$\begin{aligned} H^n(\mathbb{T}^I, \mathbb{Z}) &= H^n(\varprojlim_{J \subset I \text{ finite}} \mathbb{T}^J, \mathbb{Z}) \\ &= \varinjlim_{J \subset I \text{ finite}} H^n(\mathbb{T}^J, \mathbb{Z}) \\ &= \varinjlim_{J \subset I \text{ finite}} \wedge^n(\mathbb{Z}^{\oplus J}) \\ &= \wedge^n(\mathbb{Z}^{\oplus I}). \end{aligned}$$

□

9 Locally compact abelian groups

Notation. Let TopAb be the category of all Hausdorff topological abelian groups and LCAb be the category of all locally compact abelian groups.

Proposition 9.1. Let $A, B \in \text{TopAb}$ and assume that $A \in \text{CGTop}$. Then there is a natural isomorphism of condensed abelian groups

$$\underline{\text{Hom}}(\underline{A}, \underline{B}) \cong \underline{\text{Hom}}(\underline{A}, \underline{B}).$$

Theorem 9.2 (Eilenberg-MacLane, Breen, Deligne resolution). For any abelian group A , there is a functorial resolution

$$\cdots \longrightarrow \bigoplus_{j=1}^{n_i} \mathbb{Z}[A^{r_{i,j}}] \longrightarrow \cdots \longrightarrow \mathbb{Z}[A^3] \oplus \mathbb{Z}[A^2] \longrightarrow \mathbb{Z}[A^2] \longrightarrow \mathbb{Z}[A] \longrightarrow A \rightarrow 0.$$

Remark 9.3. Such functorial ensures that it works for abelian group objects in any topos.

Lemma 9.4. Let $A^{\bullet,\bullet}$ be a double complex and $A^\bullet = \text{Tot}(A^{\bullet,\bullet})$ be its total complex, then there is a spectral sequence

$$E_1^{p,q} = H^q(A^{\bullet,p}) \implies H^{p+q}(A^\bullet).$$

Lemma 9.5. For a complex of abelian groups $M^\bullet \in D(\mathbb{Z})$, let

$$0 \longrightarrow M^\bullet \longrightarrow A^{\bullet,1} \longrightarrow A^{\bullet,2} \longrightarrow A^{\bullet,3} \longrightarrow \cdots$$

be an exact sequence in $D(\mathbb{Z})$, then for the double complex $A^{\bullet,\bullet}$, there is a quasi-isomorphism

$$M^\bullet \xrightarrow{\sim} \text{Tot}(A^{\bullet,\bullet}).$$

Corollary 9.6. For any condensed abelian groups A, M and an extremally disconnected space S , there is a spectral sequence

$$E_1^{p,q} = \prod_{j=1}^{n_p} H^q(A^{r_{p,j}} \times S, M) \implies \underline{\text{Ext}}^{p+q}(A, M)(S),$$

that is functorial in A, M and S .

Proof. For $A \in \text{Cond}(\text{Ab})$, consider its EMBD resolution

$$\cdots \longrightarrow \bigoplus_{j=1}^{n_i} \mathbb{Z}[A^{r_{i,j}}] \longrightarrow \cdots \longrightarrow \mathbb{Z}[A^3] \oplus \mathbb{Z}[A^2] \longrightarrow \mathbb{Z}[A^2] \longrightarrow \mathbb{Z}[A] \longrightarrow A \rightarrow 0,$$

then apply $- \otimes \mathbb{Z}[S]$, which is an exact functor since $\mathbb{Z}[S]$ is flat, we get the resolution of $A \otimes \mathbb{Z}[S]$

$$\cdots \longrightarrow \bigoplus_{j=1}^{n_i} \mathbb{Z}[A^{r_{i,j}} \times S] \cdots \longrightarrow \mathbb{Z}[A^3 \times S] \oplus \mathbb{Z}[A^2 \times S] \longrightarrow \mathbb{Z}[A^2 \times S] \longrightarrow \mathbb{Z}[A \times S] \longrightarrow A \otimes \mathbb{Z}[S] \rightarrow 0,$$

then apply $R\text{Hom}(-, M)$, we get

$$0 \longrightarrow R\text{Hom}(A \otimes \mathbb{Z}[S], M) \longrightarrow R\text{Hom}(\mathbb{Z}[A \times S], M) \longrightarrow R\text{Hom}(\mathbb{Z}[A^2 \times S], M) \longrightarrow \cdots,$$

i.e.

$$0 \longrightarrow R\text{Hom}(A, M)(S) \longrightarrow R\Gamma(A \times S, M) \longrightarrow R\Gamma(A^2 \times S, M) \longrightarrow \cdots,$$

which is an exact sequence in $D(\mathbb{Z})$. By lemma 9.4 and lemma 9.5, we have

$$E_1^{p,q} = H^q\left(\bigoplus_{j=1}^{n_p} R\Gamma(A^{r_{p,j}} \times S, M)\right) \implies H^{p+q}(\text{Tot}(\bigoplus_{j=1}^{n_\bullet} R\Gamma(A^{r_{\bullet,j}} \times S, M)))$$

and

$$R\text{Hom}(A, M)(S) \simeq \text{Tot}(\bigoplus_{j=1}^{n_\bullet} R\Gamma(A^{r_{\bullet,j}} \times S, M)),$$

hence

$$E_1^{p,q} = \prod_{j=1}^{n_p} H^q(A^{r_{p,j}} \times S, M) \implies \text{Ext}^{p+q}(A, M)(S).$$

□

Lemma 9.7. In the category of abelian groups, if the following diagram is exact for each arrow

$$\begin{array}{ccccccc} 0 & \longrightarrow & M^\bullet & \longrightarrow & A^{\bullet,1} & \longrightarrow & A^{\bullet,2} \longrightarrow \cdots, \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & N^\bullet & \longrightarrow & B^{\bullet,1} & \longrightarrow & B^{\bullet,2} \longrightarrow \cdots \end{array}$$

and if for any $j \geq 1$, we have $A^{\bullet,j} \cong B^{\bullet,j}$, then $\text{Tot}(A^{\bullet,\bullet}) \cong \text{Tot}(B^{\bullet,\bullet})$. Furthermore, by $M^\bullet \cong \text{Tot}(A^{\bullet,\bullet})$ and $N^\bullet \cong \text{Tot}(B^{\bullet,\bullet})$, we can get $M^\bullet \cong N^\bullet$.

Theorem 9.8. Assume I is any set, denote the compact condensed abelian group $\prod_I \mathbb{T}$ by \mathbb{T}^I .

(i) For any discrete abelian group M , we have

$$R\text{Hom}(\mathbb{T}^I, M) = M^{\oplus I}[-1],$$

where $M^{\oplus I}[-1] \rightarrow R\mathbf{Hom}(\mathbb{T}^I, M)$ is induced by

$$M[-1] = R\mathbf{Hom}(\mathbb{Z}[1], M) \longrightarrow R\mathbf{Hom}(\mathbb{T}, M) \xrightarrow{p_i^*} R\mathbf{Hom}(\mathbb{T}^I, M),$$

where $p_i : \mathbb{T}^I \longrightarrow \mathbb{T}$ is the projection to the i -th factor, $i \in I$.

$$(ii) \ R\mathbf{Hom}(\mathbb{T}^I, \mathbb{R}) = 0.$$

Proof.

(i) We first prove the case I is a one element set, i.e.

$$R\mathbf{Hom}(\mathbb{T}, M) = M[-1].$$

From the exact sequence $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{R} \rightarrow \mathbb{T} \rightarrow 0$, we have $\mathbb{R} \rightarrow \mathbb{T} \rightarrow \mathbb{Z}[1]$, hence

$$M[-1] = R\mathbf{Hom}(\mathbb{Z}[1], M) \longrightarrow R\mathbf{Hom}(\mathbb{T}, M) \longrightarrow R\mathbf{Hom}(\mathbb{R}, M).$$

In order to show $R\mathbf{Hom}(\mathbb{T}, M) = M[-1]$, it suffices to show $R\mathbf{Hom}(\mathbb{R}, M) = 0$.

Claim: $R\mathbf{Hom}(\mathbb{R}, M) = 0$.

For 0 and \mathbb{R} , we take its EMBD resolution:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \bigoplus_{j=1}^{n_i} \mathbb{Z}[\mathbb{R}^{r_{i,j}}] & \longrightarrow & \cdots & \longrightarrow & \mathbb{Z}[\mathbb{R}] \longrightarrow \mathbb{R} \longrightarrow 0 \\ & & \downarrow & & & & \downarrow \\ \cdots & \longrightarrow & \bigoplus_{j=1}^{n_i} \mathbb{Z}[0^{r_{i,j}}] & \longrightarrow & \cdots & \longrightarrow & \mathbb{Z}[0] \longrightarrow 0 \longrightarrow 0, \end{array}$$

apply $R\mathbf{Hom}(-, M)(S)$, we get

$$\begin{array}{ccccccc} 0 & \longrightarrow & R\mathbf{Hom}(0, M)(S) & \longrightarrow & R\mathbf{Hom}(\mathbb{Z}[0], M)(S) & \longrightarrow & \cdots \longrightarrow R\mathbf{Hom}(\bigoplus_{j=1}^{n_i} \mathbb{Z}[0^{r_{i,j}}], M)(S) \cdots \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & R\mathbf{Hom}(\mathbb{R}, M)(S) & \longrightarrow & R\mathbf{Hom}(\mathbb{Z}[\mathbb{R}], M)(S) & \longrightarrow & \cdots \longrightarrow R\mathbf{Hom}(\bigoplus_{j=1}^{n_i} \mathbb{Z}[\mathbb{R}^{r_{i,j}}], M)(S) \cdots, \end{array}$$

i.e.

$$\begin{array}{ccccccc} 0 & \longrightarrow & 0 & \longrightarrow & R\Gamma(S, M) & \longrightarrow & \cdots \longrightarrow \bigoplus_{j=1}^{n_i} R\Gamma(S, M) \cdots \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & R\mathbf{Hom}(\mathbb{R}, M)(S) & \longrightarrow & R\Gamma(\mathbb{R} \times S, M) & \longrightarrow & \cdots \longrightarrow \bigoplus_{j=1}^{n_i} R\Gamma(\mathbb{R}^{r_{i,j}} \times S, M) \cdots, \end{array}$$

Then by lemma 9.7, in order to show $R\mathbf{H}\mathbf{om}(\mathbb{R}, M) = 0$, it suffices to show

$$R\Gamma(S, M) = R\Gamma(S \times \mathbb{R}^r, M).$$

We know $S \times \mathbb{R}^r = \varinjlim S \times [-N, N]^r$, then

$$\begin{aligned} R\Gamma(S \times \mathbb{R}^r, M) &= R\Gamma(\varinjlim S \times [-N, N]^r, M) \\ &= \varprojlim R\Gamma(S \times [-N, N]^r, M) \\ &= \varprojlim R\Gamma(S, M) \\ &= R\Gamma(S, M). \end{aligned}$$

Here, $\varprojlim R\Gamma(S \times [-N, N]^r, M) = \varprojlim R\Gamma(S, M)$ comes from the fact that for constant sheaf, its sheaf cohomology is homotopy-invariant.

Secondly, assume I is a finite set, then

$$R\mathbf{H}\mathbf{om}(\mathbb{T}^I, M) = R\mathbf{H}\mathbf{om}(\mathbb{T}^{\oplus I}, M) = \prod_I R\mathbf{H}\mathbf{om}(\mathbb{T}, M) = \prod_I M[-1] = M^{\oplus I}[-1].$$

Finally, assume I is any set. Then we can write \mathbb{T}^I as

$$\mathbb{T}^I = \varprojlim_{J \subset I, J \text{ finite}} \mathbb{T}^J.$$

For any finite set J , we have

$$\begin{array}{ccccccc} 0 \longrightarrow & R\mathbf{H}\mathbf{om}(\mathbb{T}^J, M)(S) & \longrightarrow & R\Gamma(\mathbb{T}^J \times S, M) & \longrightarrow \cdots \longrightarrow & \bigoplus_{j=1}^{n_i} R\Gamma((\mathbb{T}^J)^{r_{i,j}} \times S, M) & \cdots \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \longrightarrow & R\mathbf{H}\mathbf{om}(\mathbb{T}^I, M)(S) & \longrightarrow & R\Gamma(\mathbb{T}^I \times S, M) & \longrightarrow \cdots \longrightarrow & \bigoplus_{j=1}^{n_i} R\Gamma((\mathbb{T}^I)^{r_{i,j}} \times S, M) & \cdots, \end{array}$$

apply the exact functor $\varinjlim_{J \subset I}$ to the first arrow, we get

$$\begin{array}{ccccccc} 0 \longrightarrow & \varinjlim_{J \subset I} R\mathbf{H}\mathbf{om}(\mathbb{T}^J, M)(S) & \longrightarrow & \varinjlim_{J \subset I} R\Gamma(\mathbb{T}^J \times S, M) & \longrightarrow \cdots \longrightarrow & \bigoplus_{j=1}^{n_i} \varinjlim_{J \subset I} R\Gamma((\mathbb{T}^J)^{r_{i,j}} \times S, M) & \cdots \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \longrightarrow & R\mathbf{H}\mathbf{om}(\mathbb{T}^I, M)(S) & \longrightarrow & R\Gamma(\mathbb{T}^I \times S, M) & \longrightarrow \cdots \longrightarrow & \bigoplus_{j=1}^{n_i} R\Gamma((\mathbb{T}^I)^{r_{i,j}} \times S, M) & \cdots, \end{array}$$

In order to show

$$\varinjlim_{J \subset I} R\mathbf{H}\mathbf{om}(\mathbb{T}^J, M)(S) \cong R\mathbf{H}\mathbf{om}(\mathbb{T}^I, M)(S),$$

it suffices to show

$$\varinjlim_{J \subset I} R\Gamma((\mathbb{T}^J)^{r_{i,j}} \times S, M) \cong R\Gamma((\mathbb{T}^I)^{r_{i,j}} \times S, M).$$

This is true, because $\varinjlim_{J \subset I} (\mathbb{T}^J)^{r_{i,j}} \times S \cong (\mathbb{T}^I)^{r_{i,j}} \times S$. Therefore,

$$\begin{aligned} R\mathbf{Hom}(\mathbb{T}^I, M) &\cong \varinjlim_{J \subset I} R\mathbf{Hom}(\mathbb{T}^J, M) \\ &\cong \varinjlim_{J \subset I} M^{\oplus J}[-1] \\ &\cong M^{\oplus I}[-1]. \end{aligned}$$

□

Corollary 9.9. $R\mathbf{Hom}(\mathbb{R}, \mathbb{R}) \cong \mathbb{R}$.

Proof. From the exact sequence $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{R} \rightarrow \mathbb{T} \rightarrow 0$, we have

$$R\mathbf{Hom}(\mathbb{T}, \mathbb{R}) \rightarrow R\mathbf{Hom}(\mathbb{R}, \mathbb{R}) \rightarrow R\mathbf{Hom}(\mathbb{Z}, \mathbb{R}).$$

By Theorem 9.8, we know $R\mathbf{Hom}(\mathbb{T}, \mathbb{R}) = 0$, hence $R\mathbf{Hom}(\mathbb{R}, \mathbb{R}) \cong R\mathbf{Hom}(\mathbb{Z}, \mathbb{R}) \cong \mathbb{R}$. □

Corollary 9.10. For any locally compact abelian groups A and B , $R\mathbf{Hom}(A, B)$ is centered at 0 and 1, i.e. $\underline{\mathrm{Ext}}^i(A, B) = 0$, $\forall i \geq 2$.

Proof. By the structure theorem of locally compact abelian groups, it suffices to prove for A and B being compact groups and discrete groups.

(i) A is a discrete group.

Claim: There is an exact sequence: $0 \rightarrow \oplus_I \mathbb{Z} \rightarrow \oplus_J \mathbb{Z} \rightarrow A \rightarrow 0$.

This is because we can construct a surjective homomorphism $\oplus_A \mathbb{Z} \rightarrow A$, and take its kernel, and we know the submodule of a free \mathbb{Z} -module is free, hence $\ker(\oplus_A \mathbb{Z} \rightarrow A) = \oplus_I \mathbb{Z}$, for some I . Thereby, $0 \rightarrow \oplus_I \mathbb{Z} \rightarrow \oplus_A \mathbb{Z} \rightarrow A \rightarrow 0$ is exact.

By the short exact sequence $0 \rightarrow \oplus_I \mathbb{Z} \rightarrow \oplus_J \mathbb{Z} \rightarrow A \rightarrow 0$, we can get a long exact sequence:

$$\begin{aligned} 0 &\longrightarrow \mathbf{Hom}(A, B) \longrightarrow \mathbf{Hom}(\oplus_J \mathbb{Z}, B) \longrightarrow \mathbf{Hom}(\oplus_I \mathbb{Z}, B) \\ &\longrightarrow \underline{\mathrm{Ext}}^1(A, B) \longrightarrow \underline{\mathrm{Ext}}^1(\oplus_J \mathbb{Z}, B) \longrightarrow \underline{\mathrm{Ext}}^1(\oplus_I \mathbb{Z}, B) \\ &\longrightarrow \underline{\mathrm{Ext}}^2(A, B) \longrightarrow \cdots \end{aligned}$$

Because $\oplus_I \mathbb{Z} \in \mathrm{Cond}(\mathrm{Ab})$ is projective, we have $\underline{\mathrm{Ext}}^i(\oplus_I \mathbb{Z}, B) = 0$, $\forall i \geq 1$. Hence $\underline{\mathrm{Ext}}^i(A, B) = 0$, $\forall i \geq 2$.

(ii) A is a compact group.

By Pontrgagin duality, there is a short exact sequence

$$0 \rightarrow A \rightarrow \mathbb{T}^I \rightarrow \mathbb{T}^J \rightarrow 0,$$

and it can induce a long exact sequence

$$\begin{aligned} 0 &\longrightarrow \underline{\mathrm{Hom}}(\mathbb{T}^J, B) \longrightarrow \underline{\mathrm{Hom}}(\mathbb{T}^I, B) \longrightarrow \underline{\mathrm{Hom}}(A, B) \\ &\longrightarrow \underline{\mathrm{Ext}}^1(\mathbb{T}^J, B) \longrightarrow \underline{\mathrm{Ext}}^1(\mathbb{T}^I, B) \longrightarrow \underline{\mathrm{Ext}}^1(A, B) \\ &\longrightarrow \underline{\mathrm{Ext}}^2(\mathbb{T}^J, B) \longrightarrow \underline{\mathrm{Ext}}^2(\mathbb{T}^I, B) \longrightarrow \underline{\mathrm{Ext}}^2(A, B) \\ &\longrightarrow \dots \end{aligned}$$

In order to show $\underline{\mathrm{Ext}}^i(A, B) = 0$, $\forall i \geq 2$, it suffices to show

$$\underline{\mathrm{Ext}}^i(\mathbb{T}^I, B) = 0, \forall i \geq 2, \forall I.$$

(a) B is a discrete group.

In this case, we have $R\underline{\mathrm{Hom}}(\mathbb{T}^I, B) = B^{\oplus I}[-1]$, which is centered at 1, hence $\underline{\mathrm{Ext}}^i(\mathbb{T}^I, B) = 0$, $\forall i \geq 2$, $\forall I$.

(b) B is a compact group.

In this case, we have a short exact sequence $0 \rightarrow B \rightarrow \mathbb{T}^{I'} \rightarrow \mathbb{T}^{J'} \rightarrow 0$, and it induces a long exact sequence:

$$\begin{aligned} 0 &\longrightarrow \underline{\mathrm{Hom}}(\mathbb{T}^I, B) \longrightarrow \underline{\mathrm{Hom}}(\mathbb{T}^I, \mathbb{T}^{I'}) \longrightarrow \underline{\mathrm{Hom}}(\mathbb{T}^I, \mathbb{T}^{J'}) \\ &\longrightarrow \underline{\mathrm{Ext}}^1(\mathbb{T}^I, B) \longrightarrow \underline{\mathrm{Ext}}^1(\mathbb{T}^I, \mathbb{T}^{I'}) \longrightarrow \underline{\mathrm{Ext}}^1(\mathbb{T}^I, \mathbb{T}^{J'}) \\ &\longrightarrow \underline{\mathrm{Ext}}^2(\mathbb{T}^I, B) \longrightarrow \dots \end{aligned}$$

Now, we compute $\underline{\mathrm{Ext}}^i(\mathbb{T}^I, \mathbb{T})$. For the short exact sequence $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{R} \rightarrow \mathbb{T} \rightarrow 0$, we have a long exact sequence:

$$\begin{aligned} 0 &\longrightarrow \underline{\mathrm{Hom}}(\mathbb{T}^I, \mathbb{Z}) \longrightarrow \underline{\mathrm{Hom}}(\mathbb{T}^I, \mathbb{R}) \longrightarrow \underline{\mathrm{Hom}}(\mathbb{T}^I, \mathbb{T}) \\ &\longrightarrow \underline{\mathrm{Ext}}^1(\mathbb{T}^I, \mathbb{Z}) \longrightarrow \underline{\mathrm{Ext}}^1(\mathbb{T}^I, \mathbb{R}) \longrightarrow \underline{\mathrm{Ext}}^1(\mathbb{T}^I, \mathbb{T}) \\ &\longrightarrow \underline{\mathrm{Ext}}^2(\mathbb{T}^I, \mathbb{Z}) \longrightarrow \dots \end{aligned}$$

Since $R\underline{\mathrm{Hom}}(\mathbb{T}^I, \mathbb{R}) = 0$ and $R\underline{\mathrm{Hom}}(\mathbb{T}^I, \mathbb{Z}) = \mathbb{Z}^{\oplus I}[-1]$, we have $\underline{\mathrm{Ext}}^i(\mathbb{T}^I, \mathbb{T}) = 0$, $\forall i \geq 1$, hence $\underline{\mathrm{Ext}}^i(\mathbb{T}^I, \mathbb{T}^J) = 0$, $\forall i \geq 1$, $\forall J$. Thus $\underline{\mathrm{Ext}}^i(\mathbb{T}^I, B) = 0$, $\forall i \geq 2$.

□

10 Solid Abelian Groups

Definition 10.1. For $S \in \text{ProFin}$, write $S = \varprojlim S_i$, where $S_i \in \text{Fin}$, we define the solid free abelian group

$$\mathbb{Z}[S]^{\blacksquare} := \varprojlim \mathbb{Z}[S_i].$$

We call $\mathbb{Z}[S]^{\blacksquare}$ the solidification of $\mathbb{Z}[S]$.

Remark 10.2.

$$\mathbb{Z}[S]^{\blacksquare} = \varprojlim \mathbb{Z}[S_i] = \varprojlim \underline{\text{Hom}}(C(S_i, \mathbb{Z}), \mathbb{Z}) = \underline{\text{Hom}}(\varprojlim C(S_i, \mathbb{Z}), \mathbb{Z}) = \underline{\text{Hom}}(C(S, \mathbb{Z}), \mathbb{Z}).$$

Proposition 10.3. For $S \in \text{ProFin}$, there exists some set I , s.t. $C(S, \mathbb{Z}) \cong \mathbb{Z}^{\oplus I}$, i.e. $C(S, \mathbb{Z})$ is a free abelian group.

Remark 10.4. (i) From the above proposition, we have

$$\mathbb{Z}[S]^{\blacksquare} = \underline{\text{Hom}}(C(S, \mathbb{Z}), \mathbb{Z}) = \underline{\text{Hom}}(\mathbb{Z}^{\oplus I}, \mathbb{Z}) = \mathbb{Z}^I.$$

(ii)

Definition 10.5. A condensed abelian group $X \in \text{Cond}(\text{Ab})$ is solid, if for any $S \in \text{ProFin}$, one has

$$\text{Hom}(\mathbb{Z}[S], X) \cong \text{Hom}(\mathbb{Z}[S]^{\blacksquare}, X).$$

A complex of condensed abelian groups $C \in D(\text{Cond}(\text{Ab}))$ is solid, if for any $S \in \text{ProFin}$, one has

$$R\text{Hom}(\mathbb{Z}[S], C) \cong R\text{Hom}(\mathbb{Z}[S]^{\blacksquare}, C).$$

Now, we need to check $\mathbb{Z}[S]^{\blacksquare}$ is indeed a solid condensed abelian group.

Proposition 10.6. For $S, T \in \text{ProFin}$, we have

$$R\text{Hom}(\mathbb{Z}[S], \mathbb{Z}[T]^{\blacksquare}) \cong R\text{Hom}(\mathbb{Z}[S]^{\blacksquare}, \mathbb{Z}[T]^{\blacksquare}).$$

Proof. Assume $\mathbb{Z}[S]^{\blacksquare} = \mathbb{Z}^I$ and $\mathbb{Z}[T]^{\blacksquare} = \mathbb{Z}^J$ for some sets I and J . Since the functors $R\text{Hom}(\mathbb{Z}[S], -)$ and $R\text{Hom}(\mathbb{Z}[S]^{\blacksquare}, -)$ commute with products, it suffices to show

$$R\text{Hom}(\mathbb{Z}[S], \mathbb{Z}) \cong R\text{Hom}(\mathbb{Z}[S]^{\blacksquare}, \mathbb{Z})$$

The left hand side is $R\text{Hom}(\mathbb{Z}[S], \mathbb{Z}) \cong R\Gamma(S, \mathbb{Z}) = C(S, \mathbb{Z}) = \mathbb{Z}^{\oplus I}$.

Now, consider the short exact sequence $0 \rightarrow \mathbb{R}^I \rightarrow \mathbb{Z}^I \rightarrow \mathbb{T}^I \rightarrow 0$. From theorem 9.8, We know

$$R\text{Hom}(\mathbb{T}^I, \mathbb{Z}) = \mathbb{Z}^{\oplus I}[-1].$$

And by the adjoint relation, we have

$$R\mathrm{Hom}(\mathbb{R}^I, \mathbb{Z}) \cong R\mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^I, R\mathrm{Hom}(\mathbb{R}, \mathbb{Z})) = 0.$$

Hence, $R\mathrm{Hom}(\mathbb{Z}[S]^{\blacksquare}, \mathbb{Z}) \cong R\mathrm{Hom}(\mathbb{Z}^I, \mathbb{Z}) \cong \mathbb{Z}^{\oplus I}$. And this finishes our proof. \square

Lemma 10.7. Let \mathcal{A} be a cocomplete abelian category, and $\mathcal{A}_0 \subseteq \mathcal{A}$ be the full subcategory of compact projective generators. Assume $F : \mathcal{A}_0 \rightarrow \mathcal{A}$ is an additive functor with a natural transformation $\mathrm{id}_{\mathcal{A}_0} \Rightarrow F$, satisfying the following property:

For any $X \in \mathcal{A}_0$, any $Y, Z \in \mathcal{A}$ which can be written as direct sums of objects in the image of F , i.e. $Y = \bigoplus_{i \in I} F(X_i)$ and $Z = \bigoplus_{j \in J} F(X_j)$, and for any map $f : Y \rightarrow Z$ with kernel $K \in \mathcal{A}$, the map

$$R\mathrm{Hom}(F(X), K) \rightarrow R\mathrm{Hom}(X, K)$$

is an isomorphism.

Let

$$\mathcal{A}_F = \{Y \in \mathcal{A} \mid \mathrm{Hom}(F(X), Y) \cong \mathrm{Hom}(X, Y), \forall X \in \mathcal{A}_0\} \subseteq \mathcal{A}$$

and

$$D_F(\mathcal{A}) = \{C \in D(\mathcal{A}) \mid R\mathrm{Hom}(F(X), C) \cong R\mathrm{Hom}(X, C), \forall X \in \mathcal{A}_0\} \subseteq D(\mathcal{A})$$

Then:

- (i) - $\mathcal{A}_F \subseteq \mathcal{A}$ is an abelian subcategory stable under limits, colimits and extensions.
- The objects $F(X), X \in \mathcal{A}_0$ are compact projective generators.
- The inclusion $\mathcal{A}_F \hookrightarrow \mathcal{A}$ admits a left adjoint $L : \mathcal{A} \rightarrow \mathcal{A}_F$, which is the unique colimit-preserving extension of $F : \mathcal{A}_0 \rightarrow \mathcal{A}_F$.
- (ii) - The functor $D(\mathcal{A}_F) \rightarrow D(\mathcal{A})$ is fully faithful and $D(\mathcal{A}_F) \cong D_F(\mathcal{A})$.
- $C \in D(\mathcal{A})$ lies in $D_F(\mathcal{A})$ iff $H^i(C) \in \mathcal{A}_F$.
- The above functor F has a left derived functor, which is the left adjoint of $D_F(\mathcal{A}) \hookrightarrow D(\mathcal{A})$.

Lemma 10.8. We take the above lemma's notation.

- (i) For any C with the form $\bigoplus_{i \in I} F(X_i), X_i \in \mathcal{A}_0$, one has

$$R\mathrm{Hom}(F(X), C) \cong R\mathrm{Hom}(X, C), \forall X \in \mathcal{A}_0.$$

(ii) For any C with the form $\ker(\bigoplus_{i \in I} F(X_i) \rightarrow \bigoplus_{j \in J} F(Y_j))$, $X_i, Y_j \in \mathcal{A}_0$, one has

$$R\mathrm{Hom}(F(X), C) \cong R\mathrm{Hom}(X, C), \quad \forall X \in \mathcal{A}_0.$$

(iii) For any C with the form $\mathrm{coker}(\bigoplus_{i \in I} F(X_i) \rightarrow \bigoplus_{j \in J} F(Y_j))$, $X_i, Y_j \in \mathcal{A}_0$, one has

$$R\mathrm{Hom}(F(X), C) \cong R\mathrm{Hom}(X, C), \quad \forall X \in \mathcal{A}_0.$$

(iv) For any right bounded complex C with each term C_i having the form $\bigoplus_{j \in I_i} F(X_{i_j})$, one has

$$R\mathrm{Hom}(F(X), C) \cong R\mathrm{Hom}(X, C), \quad \forall X \in \mathcal{A}_0.$$

Then (iv) \implies (iii) \iff (ii) \implies (i).

Proof. (ii) \implies (i). Just take $J = \emptyset$, which is exactly (i).

(ii) \iff (iii). For any $f : Y \rightarrow Z$, with $Y = \bigoplus_{i \in I} F(X_i)$ and $Z = \bigoplus_{j \in J} F(Y_j)$, applying functors $R\mathrm{Hom}(X, -)$ and $R\mathrm{Hom}(F(X), -)$ to the exact sequence:

$$0 \rightarrow \ker(f) \rightarrow Y \rightarrow Z \rightarrow \mathrm{coker}(f) \rightarrow 0,$$

one get

$$\begin{array}{ccccccc} R\mathrm{Hom}(F(X), \ker(f)) & \longrightarrow & R\mathrm{Hom}(F(X), Y) & \longrightarrow & R\mathrm{Hom}(F(X), Z) & \longrightarrow & R\mathrm{Hom}(F(X), \mathrm{coker}(f)) \\ \downarrow & & \downarrow \cong & & \downarrow \cong & & \downarrow \\ R\mathrm{Hom}(X, \ker(f)) & \longrightarrow & R\mathrm{Hom}(X, Y) & \longrightarrow & R\mathrm{Hom}(X, Z) & \longrightarrow & R\mathrm{Hom}(X, \mathrm{coker}(f)) \end{array}$$

By five lemma, we can show

$$R\mathrm{Hom}(F(X), \ker(f)) \cong R\mathrm{Hom}(X, \ker(f))$$

\iff

$$R\mathrm{Hom}(F(X), \mathrm{coker}(f)) \cong R\mathrm{Hom}(X, \mathrm{coker}(f)).$$

Hence, (ii) \iff (iii).

(iv) \implies (ii). For any $f : Y \rightarrow Z$, with $Y = \bigoplus_{i \in I} F(X_i)$ and $Z = \bigoplus_{j \in J} F(Y_j)$. Denote $K = \ker(f)$. Take the resolution of K :

$$\cdots \rightarrow B_1 \rightarrow B_0 \rightarrow K \rightarrow 0,$$

where each $B_i \in \mathcal{A}_0$. Now, take $C = [0 \rightarrow Y \rightarrow Z \rightarrow 0]$, by assumption, we have

$$R\mathrm{Hom}(F(B_\bullet), C) \cong R\mathrm{Hom}(B_\bullet, C).$$

Hence,

$$\begin{array}{ccc} B_\bullet & \longrightarrow & F(B_\bullet) \\ \downarrow & \nearrow \exists! & \\ K & \hookrightarrow & \end{array}$$

That is, $K \cong B_\bullet$ is the retract of $F(B_\bullet)$. Thus,

$$R\mathrm{Hom}(X, K) \cong R\mathrm{Hom}(X, F(B_\bullet)) \cong R\mathrm{Hom}(F(X), F(B_\bullet)) \cong R\mathrm{Hom}(F(X), K).$$

□

Theorem 10.9. (i) - The category $\mathrm{Solid} \subset \mathrm{Cond}(\mathrm{Ab})$ of solid abelian groups is an abelian subcategory stable under limits, colimits and extensions.

(ii)

Definition 10.10. (i) For $M, N \in \mathrm{Solid}$, define $M \otimes^\blacksquare N := (M \otimes N)^\blacksquare$.

(ii) For $C, D \in D(\mathrm{Solid})$, define $C \otimes^{L\blacksquare} D := (C \otimes^L D)^{L\blacksquare}$.

Theorem 10.11. (i) The solidification functor $\mathrm{Cond}(\mathrm{Ab}) \rightarrow \mathrm{Solid}$; $M \mapsto M^\blacksquare$ is symmetric monoidal, i.e.

$$(M \otimes N)^\blacksquare \cong M^\blacksquare \otimes^\blacksquare N^\blacksquare.$$

(ii) The solidification functor $D(\mathrm{Cond}(\mathrm{Ab})) \rightarrow D(\mathrm{Solid})$; $C \mapsto C^{L\blacksquare}$ is symmetric monoidal, i.e.

$$(C \otimes^L D)^{L\blacksquare} \cong C^{L\blacksquare} \otimes^{L\blacksquare} D^{L\blacksquare}.$$

(iii) $\otimes^{L\blacksquare}$ is the left derived functor of \otimes^\blacksquare .

Proof. (i) By definition, we need to show:

$$(M \otimes N)^\blacksquare \xrightarrow{\sim} (M^\blacksquare \otimes N^\blacksquare)^\blacksquare.$$

This can be written as the composition:

$$(M \otimes N)^\blacksquare \longrightarrow (M^\blacksquare \otimes N)^\blacksquare \longrightarrow (M^\blacksquare \otimes N^\blacksquare)^\blacksquare.$$

Hence, it is enough to prove

$$(M \otimes N)^{\blacksquare} \xrightarrow{\sim} (M^{\blacksquare} \otimes N)^{\blacksquare}.$$

(With this isomorphism, we can also show that the second map is an isomorphism). Since the tensor functor and the solidification functor commute with colimits, then we can assume $M = \mathbb{Z}[S]$ and $N = \mathbb{Z}[T]$.

It reduces to show:

$$\mathbb{Z}[S \times T]^{\blacksquare} \xrightarrow{\sim} (\mathbb{Z}[S]^{\blacksquare} \otimes \mathbb{Z}[T])^{\blacksquare}.$$

Equivalently, for any $A \in \text{Solid}$,

$$\underline{\text{Hom}}((\mathbb{Z}[S]^{\blacksquare} \otimes \mathbb{Z}[T])^{\blacksquare}, A) \cong \underline{\text{Hom}}(\mathbb{Z}[S \times T]^{\blacksquare}, A).$$

Since A is solid, we have:

$$\underline{\text{Hom}}((\mathbb{Z}[S]^{\blacksquare} \otimes \mathbb{Z}[T])^{\blacksquare}, A) \cong \underline{\text{Hom}}(\mathbb{Z}[S]^{\blacksquare} \otimes \mathbb{Z}[T], A)$$

and

$$\underline{\text{Hom}}(\mathbb{Z}[S \times T]^{\blacksquare}, A) \cong \underline{\text{Hom}}(\mathbb{Z}[S \times T], A).$$

By computation:

$$\begin{aligned} \underline{\text{Hom}}(\mathbb{Z}[S]^{\blacksquare} \otimes \mathbb{Z}[T], A) &\cong \underline{\text{Hom}}(\mathbb{Z}[S]^{\blacksquare}, \underline{\text{Hom}}(\mathbb{Z}[T], A)) \\ &\cong \underline{\text{Hom}}(\mathbb{Z}[S], \underline{\text{Hom}}(\mathbb{Z}[T], A)) \\ &\cong \underline{\text{Hom}}(\mathbb{Z}[S] \otimes \mathbb{Z}[T], A) \\ &\cong \underline{\text{Hom}}(\mathbb{Z}[S \times T], A). \end{aligned}$$

Thus, $\underline{\text{Hom}}((\mathbb{Z}[S]^{\blacksquare} \otimes \mathbb{Z}[T])^{\blacksquare}, A) \cong \underline{\text{Hom}}(\mathbb{Z}[S \times T]^{\blacksquare}, A)$.

(ii) Similar to the proof of (i).

(iii)

Remark 10.12. In Solid , \otimes^{\blacksquare} is the left adjoint of $\underline{\text{Hom}}$:

$$\text{Hom}(M \otimes^{\blacksquare} N, P) \cong \text{Hom}((M \otimes N)^{\blacksquare}, P) \cong \text{Hom}(M \otimes N, P) \cong \text{Hom}(M, \underline{\text{Hom}}(N, P)).$$

Proposition 10.13. (i) If $X \in \text{CHaus}$, then $\mathbb{Z}[X]^{L\blacksquare} = R\underline{\text{Hom}}(R\Gamma(X, \mathbb{Z}), \mathbb{Z})$.

In particular, if $X \in \text{ProFin} \subseteq \text{CHaus}$, then $\mathbb{Z}[X]^{L\blacksquare} = \mathbb{Z}[X]^{\blacksquare}$.

(ii) If X is a CW space, then $\mathbb{Z}[X]^{L\blacksquare} = C_{\bullet}(X)$.

This shows that the derived solidification of a condensed abelian group can sit in all nonnegative homological degrees.

Proposition 10.14. (i) $\mathbb{R}^{L\blacksquare} = 0$.

$$(ii) \mathbb{Z}^I \otimes^{L\blacksquare} \mathbb{Z}^J = \mathbb{Z}^{I \times J}.$$

$$(iii) \mathbb{Z}_p \otimes^{L\blacksquare} \mathbb{Z}_p = \mathbb{Z}_p.$$

$$(iv) \mathbb{Z}_p \otimes^{L\blacksquare} \mathbb{Z}_l = \mathbb{Z}_p. (p \neq l)$$

Proof. (i) By Yoneda's lemma, it suffices to show: for any $C \in D(\text{Solid})$, one has

$$R\text{Hom}(\mathbb{R}^{L\blacksquare}, C) = R\text{Hom}(\mathbb{R}, C) = 0.$$

Since $C = \varprojlim C_{\geq n}$, and $R\text{Hom}(\mathbb{R}, -)$ commutes with limits, it reduces to the case C is a right bounded complex. And for a right bounded complex C , one has $C = \varprojlim C_{\leq n}$, it reduces to the case C is a bounded complex.

Hence it suffices to show: for any $X \in \text{Solid}$, one has $R\text{Hom}(\mathbb{R}, X) = 0$.

We know for any object $X \in \text{Solid}$, we can write X as the colimit of objects of the form $\bigoplus_{j \in J} \mathbb{Z}^{I_j}$. And we know taking all colimits is equivalent to taking all cokernels and all filtered colimits.

Since \mathbb{R} is pseudo-coherent, we get

$$R\text{Hom}(\mathbb{R}, \varinjlim \bigoplus_{i \in J_j} \mathbb{Z}^{I_{i,j}}) = \varinjlim R\text{Hom}(\mathbb{R}, \bigoplus_{i \in J_j} \mathbb{Z}^{I_{i,j}}) = \varinjlim \bigoplus_{i \in J_j} R\text{Hom}(\mathbb{R}, \mathbb{Z}^{I_{i,j}}) = 0.$$

Let $f : X \rightarrow Y$, $X = \bigoplus_{i \in I} \mathbb{Z}^{I_i}$ and $Y = \bigoplus_{j \in J} \mathbb{Z}^{I_j}$, then from $R\text{Hom}(\mathbb{R}, X) = 0$ and $R\text{Hom}(\mathbb{R}, Y) = 0$, we know $R\text{Hom}(\mathbb{R}, \text{coker}(f)) = 0$.

Thus, we finish our proof.

(ii) Assume $\mathbb{Z}^I = \mathbb{Z}[S]^{\blacksquare} = \underline{\text{Hom}}(C(S, \mathbb{Z}), \mathbb{Z})$, $\mathbb{Z}^J = \mathbb{Z}[T]^{\blacksquare} = \underline{\text{Hom}}(C(T, \mathbb{Z}), \mathbb{Z})$, for some $S, T \in \text{ProFin}$. Then

$$\begin{aligned} \mathbb{Z}[S \times T]^{\blacksquare} &= \underline{\text{Hom}}(C(S \times T, \mathbb{Z}), \mathbb{Z}) \\ &= \underline{\text{Hom}}(C(S, \mathbb{Z}) \otimes C(T, \mathbb{Z}), \mathbb{Z}) \\ &= \underline{\text{Hom}}(C(S, \mathbb{Z}), \underline{\text{Hom}}(C(T, \mathbb{Z}), \mathbb{Z})) \\ &= \underline{\text{Hom}}(C(S, \mathbb{Z}), \mathbb{Z}^J) \\ &= \underline{\text{Hom}}(C(S, \mathbb{Z}), \mathbb{Z})^J \\ &= \mathbb{Z}^{I \times J}. \end{aligned}$$

Thus, we have

$$\mathbb{Z}^I \otimes^{L\blacksquare} \mathbb{Z}^J = \mathbb{Z}[S]^{\blacksquare} \otimes^{L\blacksquare} \mathbb{Z}[T]^{\blacksquare} = (\mathbb{Z}[S] \otimes^L \mathbb{Z}[T])^{L\blacksquare} = \mathbb{Z}[S \times T]^{L\blacksquare} = \mathbb{Z}^{I \times J}.$$

(iii)

(iv)

□