Condensed mathematics I

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1 Condensed Sets

Definition 1.1. Let $\mathcal{C}$ be a category. A Grothendieck topology on $\mathcal{C}$ consists of: for each object X in $\mathcal{C}$, there is a collection $\mathrm{Cov}(X)$ of sets $\{X_i \to X\}_{i\in I}$, satisfying the following three axioms:

(i) If $V \to X$ is an isomorphism, then $\{V \to X\} \in \mathrm{Cov}(X)$.

(ii) If $\{X_i \to X\}_{i\in I} \in \mathrm{Cov}(X)$ and $Y \to X$ is any arrow in $\mathcal{C}$, then the fiber products $X_i \times_X Y$ exist and $\{X_i \times_X Y \to Y\}_{i\in I} \in \mathrm{Cov}(Y)$.

(iii) If $\{X_i \to X\}_{i\in I} \in \mathrm{Cov}(X)$ and for each $i \in I$, $\{V_{ij} \to X_i\}_{j\in I_i} \in \mathrm{Cov}(X_i)$, then $\{V_{ij} \to X\}_{i\in I, j\in I_i} \to \mathrm{Cov}(X)$.

We call elements of $\mathrm{Cov}(X)$ coverings.

Definition 1.2. A site is a category $\mathcal{C}$ together with a Grothendieck topology.

Example 1.3. Let $\mathcal{C} = \mathrm{ProFin}$, the category of all profinite sets. For $\{X_i \to Y\}_{i\in I}$ be a covering, we mean I is a finite index and $\coprod_{i\in I} X_i \to Y$ is a surjection. We also call maps $\{X_i \to Y\}_{i\in I}$ finite jointly surjective families of maps.
Now, for the category ProFin together its coverings, we call it the proétale site of a point and denote it by $*_{\text{proét}}$.

Definition 1.4. (i) For any site $\mathcal{C}$, we call a functor

$$\mathcal{F} : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$$

a presheaf of sets.

(ii) For a presheaf of sets $\mathcal{F} : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$, if for any $X \in \mathcal{C}$ and any covering $\{X_i \to X\}_{i\in I} \in \mathrm{Cov}(X)$, we have

$$\mathcal{F}(X) \xrightarrow{\sim} \mathrm{Eq}\Big(\prod_{i\in I} \mathcal{F}(X_i) \rightrightarrows \prod_{i,j\in I} \mathcal{F}(X_i \times_X X_j)\Big).$$

Then we call $\mathcal{F}$ a sheaf of sets.

Definition 1.5. A condensed set T is a sheaf of sets on $*_{\text{proét}}$, i.e. a functor $T : *_{\text{proét}}^{\mathrm{op}} \to \mathrm{Set}$ satisfying the sheaf condition.

Remark 1.6. (i) Concretely, a condensed set T is a functor $T : \mathrm{ProFin}^{\mathrm{op}} \to \mathrm{Set}$, satisfying $T(\emptyset) = *$ and

- For any profinite sets S_1, S_2, the natural map

$$T(S_1 \sqcup S_2) \longrightarrow T(S_1) \times T(S_2)$$

is a bijection.

- For any surjection $S' \twoheadrightarrow S$ of profinite sets with fiber product $S' \times_S S'$ and two projections $p_1, p_2 : S' \times_S S' \to S'$, the map

$$T(S) \xrightarrow{\sim} \{x \in T(S') |\ p_1^*(x) = p_2^*(x) \in T(S' \times_S S')\}$$

is a bijection. In other words, T maps the pullback diagram

$$\begin{tikzcd} S' \times_S S' \arrow[r, "p_2"] \arrow[d, "p_1"'] & S' \arrow[d] \\ S' \arrow[r] & S \end{tikzcd}$$

to a pullback diagram

$$\begin{tikzcd} T(S' \times_S S') & T(S') \arrow[l, "p_2^*"'] \\ T(S') \arrow[u, "p_1^*"] & T(S) \arrow[l] \arrow[u] \end{tikzcd}$$

(ii) The category ProFin of all profinite sets is a large category.

Definition 1.7. κ is an uncountable strong limit cardinal if κ is uncountable and for any $\lambda < \kappa$, we have $2^\lambda < \kappa$.

Example 1.8. For any limit cardinal λ, i.e. if $\kappa < \lambda$, then $\kappa + 1 < \lambda$. We define

$$\beth_0 = \aleph_0, \cdots, \beth_{\alpha+1} = 2^{\beth_\alpha},$$

and let

$$\beth_\lambda = \bigcup_{\alpha < \lambda} \beth_\alpha,$$

then we can show that $\beth_\lambda$ is an uncountable strong limit cartinal.

Notation. We let κ-ProFin denote the category of all κ-small profinite sets, i.e. profinite sets whose cardinal less equal than κ. Let $\mathrm{Cond}_\kappa(\mathrm{Set}) = \mathrm{Sh}(\kappa\text{-ProFin}, \mathrm{Set})$.

Remark 1.9. If $\kappa' > \kappa$ are two uncountable strong limit cardinals, and denote the inclusion by $i : \kappa\text{-ProFin} \hookrightarrow \kappa'\text{-ProFin}$, then we have a forgetful functor

$$\mathrm{Cond}_{\kappa'}(\mathrm{Set}) \longrightarrow \mathrm{Cond}_\kappa(\mathrm{Set});\ T \mapsto T \circ i.$$

This forgetful functor admits a left adjoint $F : \mathrm{Cond}_\kappa(\mathrm{Set}) \longrightarrow \mathrm{Cond}_{\kappa'}(\mathrm{Set})$. F is fully faithful and F commutes with all colimits and all finite limits.
We define

$$\mathrm{Cond}(\mathrm{Set}) = \bigcup_\kappa \mathrm{Cond}_\kappa(\mathrm{Set}) = \varinjlim_\kappa \mathrm{Cond}_\kappa(\mathrm{Set}).$$

Example 1.10. Let Top denote the category of all topological spaces. For each $T \in \mathrm{Top}$, we can define $\underline{T} \in \mathrm{Cond}(\mathrm{Set})$ as follows:

$$\underline{T} : \mathrm{ProFin}^{\mathrm{op}} \longrightarrow \mathrm{Set};\ S \mapsto \underline{T}(S) = \mathrm{Cont}(S, T) = \{\text{continuous maps from } S \text{ to } T\}.$$

We need to check that $\underline{T}$ is a condensed set:

(i) $\underline{T}(S_1 \sqcup S_2) = \mathrm{Cont}(S_1 \sqcup S_2, T) = \mathrm{Cont}(S_1, T) \times \mathrm{Cont}(S_2, T) = \underline{T}(S_1) \times \underline{T}(S_2)$.

(ii) For any surjection $g : S' \twoheadrightarrow S$, let $p_1, p_2 : S' \times_S S' \to S'$ be the two projections. We need to show the following map is a bijection:

$$\mathrm{Cont}(S, T) \xrightarrow{\sim} \{h : S' \to T \mid hp_1 = hp_2 : S' \times_S S' \to T\};\ f \mapsto f \circ g.$$

Because g is surjective, it is easy to show this map is an injection.
Now, for any $h : S' \to T$ with $hp_1 = hp_2$, from the universal property of pushout(in our situation, the pullback square is also a pushout), we can find a unique f, s.t. the diagram commutes.

$$\begin{tikzcd}
S' \times_S S' \arrow[r, "p_2"] \arrow[d, "p_1"'] & S' \arrow[d, two heads, "g"'] \arrow[ddr, "h"] & \\
S' \arrow[r, two heads, "g"] \arrow[drr, "h"'] & S \arrow[dr, dashed, "\exists!\, f"] & \\
& & T
\end{tikzcd}$$

Definition 1.11. Let $X \in \mathrm{Top}$. The following are equivalent definition:

(i) $X \in \mathrm{Top}$ is compactly generated;

(ii) If for any compact Hausdorff space S with a map $S \to X$, if the composition $S \to X \to Y$ is continuous, then $X \to Y$ is continuous;

(iii) $A \subset X$ is closed if and only if for any compact space K with a map $f : K \to X$, $f^{-1}(A) \subset K$ is closed.

Remark 1.12. (i) If a topological space X is compact Hausdorff, then X is compactly generated.

(ii) Let CGTop denote the category of all compactly generated spaces and let CHaus denote the category of all compact Hausdorff spaces.

Definition 1.13. For a category $\mathcal{C}$, $P \in \mathcal{C}$ is a projective object if for any epimorphism $Y \twoheadrightarrow X$ and a morphism $P \to X$, there is a lift

$$\begin{tikzcd} & P \arrow[dl, dashed, "\exists"'] \arrow[d] \\ Y \arrow[r, two heads] & X \end{tikzcd}$$

Definition 1.14. In the category CHaus, we call its projective objects as extremally disconnected Hausdorff spaces.

Remark 1.15. (i) Equivalently a compact Hausdorff space S is extremally disconnected if any surjection $S' \twoheadrightarrow S$ from a compact Hausdorff space splits.

(ii) Extremally disconnected Hausdorff spaces are profinite sets, i.e. ExDisc $\subset$ ProFin. Here, ExDisc denote the category of all extremally disconnected Hausdorff spaces.

Remark 1.16. We have two adjunctions.

(i) $\mathrm{Top} \underset{i}{\overset{\beta}{\rightleftarrows}} \mathrm{CHaus}$, i.e. $\beta \dashv i$.
Where

$$i : \mathrm{CHaus} \to \mathrm{Top};\ X \mapsto X$$

and

$$\beta : \mathrm{Top} \to \mathrm{CHaus}$$

is the Stone-Cech compactification of topological spaces.
For any $X \in$ Top, we define $\beta X \in$ CHaus as follows:
for any $Y \in$ CHaus with a map $f : X \to Y$, there exists a unique map $\beta X \to Y$ so that the diagram commutes.

$$\begin{tikzcd} X \arrow[rr, "i_X"] \arrow[dr, "f"'] & & \beta X \arrow[dl, dashed, "\exists!"] \\ & Y & \end{tikzcd}$$

In fact, we can use the ultrafilter to construct βX concretely. And by this construction, we can show that

$$|\beta X| \leq 2^{2^{|X|}}.$$

(ii) $\mathrm{CGTop} \underset{c}{\overset{i}{\rightleftarrows}} \mathrm{Top}$, i.e. $i \dashv c$.
Where

$$i : \mathrm{CGTop} \to \mathrm{Top};\ X \mapsto X$$

and

$$c : \mathrm{Top} \to \mathrm{CGTop};\ X \mapsto X^{\mathrm{cg}}.$$

We define X^{cg} as follows:

- As a set, $X^{\mathrm{cg}} = X$.
- The topology of X^{cg} is given by the quotient topology of

$$\coprod_{S \to X \ S \in \mathrm{CHaus}} S \longrightarrow X.$$

Proposition 1.17. (i) The functor $\mathrm{Top} \to \mathrm{Cond}_\kappa(\mathrm{Set});\ T \mapsto \underline{T}$ is a faithful functor.

(ii) When the above functor restricted to the full subcategory κ-CGTop of all κ-compactly generated spaces, functor $\kappa\text{-CGTop} \to \mathrm{Cond}_\kappa(\mathrm{Set});\ T \mapsto \underline{T}$ is a fully faithful functor.

(iii) The functor $\mathrm{Top} \to \mathrm{Cond}_\kappa(\mathrm{Set});\ T \mapsto \underline{T}$ admits a left adjoint $\mathrm{Cond}_\kappa(\mathrm{Set}) \to \mathrm{Top};\ T \mapsto T(*)_{\mathrm{top}}$. Here, $T(*)_{\mathrm{top}}$ means the underlying set $T(*)$ equipped with the quotient topology of $\sqcup_{S \to T} S \to T(*)$, where the disjoint union runs over all κ-small profinite sets S with a map to T, i.e. an element of $T(S)$. Moreover, we have $\underline{T}(*)_{\mathrm{top}} \cong T^{\kappa\text{-cg}}$.

2 Condensed abelian groups

Definition 2.1. A condensed abelian group T is a sheaf of abelian groups on $*_{\text{proét}}$, i.e. a functor $T : *_{\text{proét}}^{\text{op}} \to \text{Ab}$ satisfying the sheaf condition. And we denote the category of all condensed abelian groups by Cond(Ab).

Definition 2.2 (Grothendieck's axioms)**.** Let $\mathcal{C}$ be an abelian category.

(AB3) All colimits exist.

(AB3*) All limits exist.

(AB4) Arbitrary direct sums are exact.

(AB4*) Arbitrary products are exact.

(AB5) Filtered colimits are exact.

(AB6) For any index set J and filtered categories I_j, $j \in J$, with functors $I_j \to \text{Cond(Ab)}$; $i \mapsto M_i$, the natural map
$$\varinjlim_{(i_j \in I_j)_j} \prod_{j \in J} M_{i_j} \longrightarrow \prod_{j \in J} \varinjlim_{i_j \in I_j} M_{i_j}$$
is an isomorphism.

Definition 2.3. Let $\mathcal{C}$ be an abelian category. $M \in \mathcal{C}$ is compact if $\text{Hom}(M, -)$ commutes with filtered colimits, i.e. $\text{Hom}(M, \varinjlim_i N_i) \cong \varinjlim_i \text{Hom}(M, N_i)$.

Theorem 2.4. (i) Cond(Ab) is an abelian category which satisfies Grothendieck's axioms (AB3), (AB4), (AB5), (AB6), (AB3*) and (AB4*).

(ii) Cond(Ab) is generated by compact projective objects.

Corollary 2.5. There is an adjunction:
$$\text{Cond}_\kappa(\text{Set}) \rightleftarrows \text{Cond}_\kappa(\text{Ab}) .$$
Where $\text{Cond}_\kappa(\text{Ab}) \longrightarrow \text{Cond}_\kappa(\text{Set})$ is the forgetful functor and
$$\text{Cond}_\kappa(\text{Set}) \longrightarrow \text{Cond}_\kappa(\text{Ab});\ T \mapsto \mathbb{Z}[T].$$
Here, $\mathbb{Z}[T] := (S \mapsto \mathbb{Z}[T(S)])^{\text{sh}}$.

Remark 2.6. (i) For $S \in \text{ExDisc}$ and $M \in \text{Cond(Ab)}$, we have
$$\text{Hom}_{\text{Cond(Ab)}}(\mathbb{Z}[S], M) \cong \text{Hom}_{\text{Cond(Set)}}(\underline{S}, M) \cong M(S).$$

Proof: We define the map:

$$\mu : \mathrm{Hom}_{\mathrm{Cond(Set)}}(\underline{S}, M) \longrightarrow M(S);\ \alpha \mapsto \alpha(S)(1_S),$$

and the map

$$\lambda : M(S) \longrightarrow \mathrm{Hom}_{\mathrm{Cond(Set)}}(\underline{S}, M);\ x \mapsto \lambda(x),$$

where for $\lambda(x) : \underline{S} \longrightarrow M$,

$$\lambda(x)(T) : \mathrm{Cont}(T, S) \longrightarrow M(T);\ f \mapsto M(f)(x).$$

One can check that μ and λ are inverse to each other, hence

$$\mathrm{Hom}_{\mathrm{Cond(Set)}}(\underline{S}, M) \cong M(S).$$

$\square$

(ii) For any $S \in \mathrm{ExDisc}$, $\mathbb{Z}[S] \in \mathrm{Cond(Ab)}$ is a compact and projective object.

Proof:

Compactness.

$$\mathrm{Hom}(\mathbb{Z}[S], \varinjlim M_i) = (\varinjlim M_i)(S) = \varinjlim M_i(S) = \varinjlim \mathrm{Hom}(\mathbb{Z}[S], M_i).$$

Projectiveness. For any exact sequence $M' \to M \to M''$ in Cond(Ab), the sequence

$$M'(S) \to M(S) \to M''(S)$$

is exact, i.e.

$$\mathrm{Hom}(\mathbb{Z}[S], M') \to \mathrm{Hom}(\mathbb{Z}[S], M) \to \mathrm{Hom}(\mathbb{Z}[S], M'')$$

is exact, so $\mathbb{Z}[S]$ is projective. $\square$

(iii) Cond(Ab) has enough projectives.

Proposition 2.7. We have two equivalences.

(i) $\mathrm{Shv}(\kappa\text{-CHaus}) \xrightarrow{\sim} \mathrm{Shv}(\kappa\text{-ProFin});\ T \mapsto T|_{\kappa\text{-ProFin}}$.

(ii) $\mathrm{Shv}(\kappa\text{-ProFin}) \xrightarrow{\sim} \mathrm{Shv}(\kappa\text{-ExDisc});\ T \mapsto T|_{\kappa\text{-ExDisc}}$.

Remark 2.8. In order a presheaf of sets T to be a sheaf of sets, by definition, we need to check the sheaf condition in ProFin. Now, from the equivalence $\mathrm{Shv}(\kappa\text{-ProFin}) \xrightarrow{\sim} \mathrm{Shv}(\kappa\text{-ExDisc})$, we only need to check the sheaf condition in ExDisc. In this case, the condition(ii) is automatic: T maps the pullback diagram

$$\begin{tikzcd} S' \times_S S' \arrow[r, "p_2"] \arrow[d, "p_1"'] & S' \arrow[d] \\ S' \arrow[r] & S \end{tikzcd}$$

to a pullback diagram

$$\begin{tikzcd} T(S' \times_S S') & T(S') \arrow[l, "p_2^*"'] \\ T(S') \arrow[u, "p_1^*"] & T(S) \arrow[l] \arrow[u] \end{tikzcd}$$

This is because any cover of extremally disconnected sets splits. Specifically, the diagram

$$\begin{tikzcd} S' \times_S S' \arrow[r] \arrow[d] & S' \arrow[d] \\ S' \arrow[r, "g"] & S \arrow[l, bend left, "f"] \end{tikzcd}$$

can implies the following diagram:

$$\begin{tikzcd} T(S' \times_S S') & T(S') \arrow[l] & \\ T(S') \arrow[u] \arrow[r, bend right, "T(f)"'] & T(S) \arrow[l, "T(g)"'] \arrow[u] & \\ & & X \arrow[ul, dotted, "\exists"'] \arrow[ull, bend left] \arrow[uul, bend right] \end{tikzcd}$$

which means it is a pullback diagram.

Property. There are some properties of the category Cond(Ab) of condensed abelian groups.

(i) Cond(Ab) has a symmetric monoidal tensor products $-\otimes-$, where for $M, N \in \text{Cond(Ab)}$, $M \otimes N = (S \mapsto M(S) \otimes N(S))^{\text{sh}}$.

(ii) Functor $\text{Cond(Set)} \to \text{Cond(Ab)}$; $T \mapsto \mathbb{Z}[T]$ is symmetric monoidal with respect to the product and the tensor product, i.e. $\mathbb{Z}[T_1 \times T_2] = \mathbb{Z}[T_1] \otimes \mathbb{Z}[T_2]$.
Proof:

(iii) For $T \in \text{Cond(Set)}$, $\mathbb{Z}[T] \in \text{Cond(Ab)}$ is flat.
Proof: We need to show $- \otimes \mathbb{Z}[T] : \text{Cond(Ab)} \to \text{Cond(Ab)}$ is an exact functor.

Take an exact sequence in Cond(Ab):

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0.$$

For any $S \in \mathrm{ExDisc}$, we have an exact sequence:

$$0 \longrightarrow X(S) \longrightarrow Y(S) \longrightarrow Z(S) \longrightarrow 0.$$

Tensoring with the free abelian group $\mathbb{Z}[T(S)]$, we get an exact sequence:

$$0 \longrightarrow X(S) \otimes \mathbb{Z}[T(S)] \longrightarrow Y(S) \otimes \mathbb{Z}[T(S)] \longrightarrow Z(S) \otimes \mathbb{Z}[T(S)] \longrightarrow 0,$$

i.e.

$$0 \longrightarrow (X \otimes \mathbb{Z}[T])(S) \longrightarrow (Y \otimes \mathbb{Z}[T])(S) \longrightarrow (Z \otimes \mathbb{Z}[T])(S) \longrightarrow 0.$$

Hence the sequence

$$0 \longrightarrow X \otimes \mathbb{Z}[T] \longrightarrow Y \otimes \mathbb{Z}[T] \longrightarrow Z \otimes \mathbb{Z}[T] \longrightarrow 0.$$

exact and $\mathbb{Z}[T]$ is flat. $\square$

(iv) Given any $M, N \in \mathrm{Cond(Ab)}$, we can give the group of homomorphisms $\mathrm{Hom}(M, N)$ the structure of condensed abelian groups via the following definition, for any $S \in \mathrm{ExDisc}$,

$$\underline{\mathrm{Hom}}(M, N)(S) := \mathrm{Hom}(\mathbb{Z}[S] \otimes M, N).$$

So we define an internal Hom-functor object.

(v) There is an adjunction. For $P, M, N \in \mathrm{Cond(Ab)}$, we have an isomorphism of abelian groups:

$$\mathrm{Hom}(P, \underline{\mathrm{Hom}}(M, N)) \cong \mathrm{Hom}(P \otimes M, N).$$

Proof: First, if $P = \mathbb{Z}[S]$ for some $S \in \mathrm{ExDisc}$, then

$$\mathrm{Hom}(P, \underline{\mathrm{Hom}}(M, N)) = \mathrm{Hom}(\mathbb{Z}[S], \underline{\mathrm{Hom}}(M, N)) = \underline{\mathrm{Hom}}(M, N)(S) = \mathrm{Hom}(\mathbb{Z}[S] \otimes M, N).$$

Now, for general $P \in \mathrm{Cond}(\mathrm{Ab})$, we can write $P = \varinjlim \mathbb{Z}[S_i]$, so

$$\begin{aligned}
\mathrm{Hom}(P, \underline{\mathrm{Hom}}(M,N)) &= \mathrm{Hom}(\varinjlim \mathbb{Z}[S_i], \underline{\mathrm{Hom}}(M,N)) \\
&= \varprojlim \mathrm{Hom}(\mathbb{Z}[S_i], \underline{\mathrm{Hom}}(M,N)) \\
&= \varprojlim \mathrm{Hom}(\mathbb{Z}[S_i] \otimes M, N) \\
&= \mathrm{Hom}(\varinjlim \mathbb{Z}[S_i] \otimes M, N) \\
&= \mathrm{Hom}(P \otimes M, N).
\end{aligned}$$

□

(vi) As Cond(Ab) has enough projectives, one can form the derived category $D(\mathrm{Cond}(\mathrm{Ab}))$. If $P \in \mathrm{Cond}(\mathrm{Ab})$ is compact and projective, then $P[0] \in D(\mathrm{Cond}(\mathrm{Ab}))$ is a compact object of the dereived category, i.e. $\mathrm{Hom}(P, -)$ commutes with arbitrary direct sums. In particular, $D(\mathrm{Cond}(\mathrm{Ab}))$ is compactly generated.

(vii) Similarly, in the derived category $D(\mathrm{Cond}(\mathrm{Ab}))$, we have the adjunction:

$$\mathrm{Hom}(P, R\underline{\mathrm{Hom}}(M,N)) \cong \mathrm{Hom}(P \otimes^L M, N).$$

(viii) Let $\mathcal{D}(\mathrm{Cond}(\mathrm{Ab}))$ denote the derived ∞-category of Cond(Ab) and $\mathcal{D}(\mathrm{Ab})$ denote the derived ∞-category of Ab, then there is an equivalence

$$\mathcal{D}(\mathrm{Cond}(\mathrm{Ab})) \cong \mathrm{Cond}(\mathcal{D}(\mathrm{Ab})).$$

3 $D(R)$

Definition 3.1. An ∞-category is a simplicial set $\mathcal{C}$ which satisfies the following extension condition:

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, hook] & \mathcal{C} \\ \Delta^n \arrow[ur, dotted, "\exists f"'] & \end{tikzcd} \qquad 0 < \forall i < n.$$

Definition 3.2. Let $\mathcal{C}$ be an ∞-category. A zero object of $\mathcal{C}$ is an object which is both initial and final. We say that $\mathcal{C}$ is pointed if $\mathcal{C}$ contains a zero object.

Definition 3.3. Let $\mathcal{C}$ be a pointed ∞-category. A triangle in $\mathcal{C}$ is a diagram $\Delta^1 \times \Delta^1 \to \mathcal{C}$ depicted as

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d, "g"] \\ 0 \arrow[r] & Z \end{tikzcd}$$

where 0 is a zero object in $\mathcal{C}$.

We say a triangle in $\mathcal{C}$ is a fiber sequence if it is a pullback and say a triangle in $\mathcal{C}$ is a cofiber sequence if it is a pushout.

We generally indicate a triangle by specifying only the pair of maps $X \xrightarrow{f} Y \xrightarrow{g} Z$.

Remark 3.4. Let $\mathcal{C}$ be a pointed ∞-category. A triangle in $\mathcal{C}$ consists of the following data:

(i) A pair of morphisms $f : X \to Y$ and $g : Y \to Z$ in $\mathcal{C}$.

(ii) A 2-simplex in $\mathcal{C}$ corresponding to a diagram

$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, "h"] & & Z \end{tikzcd}$$

in $\mathcal{C}$, which identifies h with the composition $g \circ f$.

(iii) A 2-simplex

$$\begin{tikzcd} & 0 \arrow[dr] & \\ X \arrow[ur] \arrow[rr, "h"] & & Z \end{tikzcd}$$

in $\mathcal{C}$, which we view as anullhomotopy of h.

Definition 3.5. Let $\mathcal{C}$ be a pointed ∞-category containing a morphism $g : X \longrightarrow Y$.

A fiber of g is a fiber sequence

$$\begin{tikzcd} W \arrow[r] \arrow[d] & X \arrow[d, "g"] \\ 0 \arrow[r] & Y \end{tikzcd}$$

and we denote $W = \mathrm{fib}(g)$.

Dually, a cofiber of g is a cofiber sequence

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d] & Y \arrow[d] \\ 0 \arrow[r] & Z \end{tikzcd}$$

and we denote $Z = \mathrm{cofib}(g)$.

Definition 3.6. An ∞-category $\mathcal{C}$ is stable if it satisfies the following conditions:

(i) There exists a zero object $0 \in \mathcal{C}$.

(ii) Every morphism in $\mathcal{C}$ admits a fiber and a cofiber.

(iii) A triangle in $\mathcal{C}$ is a fiber sequence if and only if it is a cofiber sequence.

Remark 3.7. (i) For a stable ∞-category $\mathcal{C}$, we define the suspension functor $\Sigma : \mathcal{C} \longrightarrow \mathcal{C}$ and the loop functor $\Omega : \mathcal{C} \longrightarrow \mathcal{C}$ as follows:

$$\Sigma(X) := \mathrm{cofib}(X \longrightarrow 0)$$

and

$$\Omega(X) := \mathrm{fib}(0 \longrightarrow X).$$

(ii) For a stable ∞-category $\mathcal{C}$, there is a homotopy equivalence:

$$\mathrm{Map}_{\mathcal{C}}(\Sigma X, Y) \xrightarrow{\sim} \mathrm{Map}_{\mathcal{C}}(X, \Omega Y).$$

Besides, the unit $X \longrightarrow \Omega\Sigma(X)$and $\Sigma\Omega(Y) \longrightarrow Y$counit are isomorphic.

Definition 3.8. Let R be a commutative ring, the ∞-category $D(R)$ is a stable ∞-category with all colimits, generated (as a cocomplete stable ∞-category) by a distinguished compact object 1, satisfying

$$\pi_0\mathrm{Map}(1,1) = R^{\mathrm{op}}, \qquad \pi_0\mathrm{Map}(\Sigma^d 1, 1) = 0, \ \forall d \neq 0.$$

For $X, Y \in D(R)$, we define

$$[X, Y] := \pi_0\mathrm{Map}(X, Y)$$

and

$$[X, Y]_d := [\Sigma^d X, Y] = [X, \Omega^d Y].$$

Remark 3.9. (i) From the definition of $D(R)$, we have

$$[1, 1] = \pi_0 \mathrm{Map}(1, 1) = R^{\mathrm{op}}, \qquad [1, 1]_d = \pi_0 \mathrm{Map}(\Sigma^d 1, 1) = 0, \ \forall d \neq 0.$$

(ii) If $d \geq 0$, we have

$$[X, Y]_d = \pi_d \mathrm{Map}(X, Y).$$

(iii) Claim: In $D(R)$, for any integer d, we have $[X, Y]_d \in \mathrm{Ab}$.
Proof: First, if $d \geq 2$, $[X, Y]_d = \pi_d \mathrm{Map}(X, Y) \in \mathrm{Ab}$. For any $d \in \mathbb{Z}$,

$$[X, Y]_d = [\Sigma^d X, Y] = [\Sigma^{d-2} X, Y]_2 \in \mathrm{Ab}.$$

(iv) For a stable ∞-category, a fiber sequence $X \to Y \to Z$ is at the same time a cofiber sequence, and vice versa. Hence, we will call it a fiber-cofiber sequence.

(v) For a fiber-cofiber sequence $X \to Y \to Z$ in $D(R)$, we can induce a new fiber-cofiber sequence $Y \to Z \to \Sigma X$.

(vi) Given a fiber-cofiber sequence $X \to Y \to Z$ and any $A \in D(R)$, we can induce two long exact sequences:

$$\cdots \longrightarrow [A, X]_d \longrightarrow [A, Y]_d \longrightarrow [A, Z]_d \longrightarrow [A, X]_{d-1} \longrightarrow \cdots$$

$$\cdots \longrightarrow [X, A]_{d+1} \longrightarrow [Z, A]_d \longrightarrow [Y, A]_d \longrightarrow [X, A]_d \longrightarrow \cdots.$$

(vii) Assume

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

is a pushout-pullback square in $D(R)$, then we can produce a triangle $A \to B \oplus C \to D$. With this, we can induce a long exact sequence.

Definition 3.10. For any integer d, we define a functor $H_d : D(R) \to \mathrm{Mod}_R; \ X \mapsto [1, X]_d$.

Remark 3.11. (i) We already know $[1, X]_d \in \mathrm{Ab}$. And we need to show $[1, X]_d$ is an R-module.
In fact, we have

$$\mathrm{Map}(\Sigma^d 1, \Sigma^d 1) \times \mathrm{Map}(\Sigma^d 1, X) \to \mathrm{Map}(\Sigma^d 1, X),$$

applying the functor π_0, we get:

$$\pi_0\mathrm{Map}(\Sigma^d 1, \Sigma^d 1)\times\pi_0\mathrm{Map}(\Sigma^d 1, X) = \pi_0(\mathrm{Map}(\Sigma^d 1, \Sigma^d 1)\times\mathrm{Map}(\Sigma^d 1, X)) \to \pi_0(\mathrm{Map}(\Sigma^d 1, X)),$$

i.e.

$$R^{\mathrm{op}} \times [1, X]_d \to [1, X]_d,$$

which implies $[1, X]_d \in \mathrm{Mod}_R$.

(ii) $H_d : D(R) \to \mathrm{Mod}_R;\ X \mapsto [1, X]_d$ is a representable functor and $\Sigma^d 1$ represents H_d.

Lemma 3.12. (i) $H_d(\prod_i X_i) = \prod_i H_d(X_i)$.

(ii) $H_d(\oplus_i X_i) = \oplus_i H_d(X_i)$.

(iii) $H_d(\varinjlim X_i) = \varinjlim H_d(X_i)$.

(iv) For a sequence of maps $\cdots \to X_n \to X_{n-1} \to \cdots$, we have a Milnor sequence:

$$0 \longrightarrow \varprojlim{}^1 H_{d+1}(X_n) \longrightarrow H_d(\varprojlim X_n) \longrightarrow \varprojlim H_d(X_n) \longrightarrow 0.$$

Proposition 3.13. $f : X \to Y$ in $D(R)$ is an isomorphism if and only if $H_d(f) : H_d(X) \xrightarrow{\sim} H_d(Y)$, for $\forall d \in \mathbb{Z}$.

Proof. Take $Z = \mathrm{cofib}(X \xrightarrow{f} Y)$, then $X \to Y \to Z$ is a fiber-cofiber sequence, and we can induce a long exact sequence

$$\cdots \to H_d(X) \to H_d(Y) \to H_d(Z) \to H_{d-1}(X) \to \cdots.$$

It suffices to show: if $Z \in D(R)$ with $H_d(Z) = 0,\ \forall d \in \mathbb{Z}$, then $Z = 0$.
Consider the full subcategory of $D(R)$:

$$\mathcal{C} = \{A \in D(R) |\ [\Sigma^d A, Z] = 0,\ \forall d \in \mathbb{Z}\}.$$

Observe that:

- $1 \in \mathcal{C}$.
- $\mathcal{C}$ is stable under colimits. This is because

$$[\Sigma^d \mathrm{colim}\ A_i, Z] = [\mathrm{colim}\ \Sigma^d A_i, Z] = \lim\ [\Sigma^d A_i, Z] = 0.$$

- $\mathcal{C}$ is stable under cofibers.

By definition of $D(R)$, we know $D(R)$ is generated as a cocomplete stable ∞-category by 1. Hence, $D(R) = \mathcal{C}$. Then by Yoneda's lemma, $Z = 0$. $\square$

Proposition 3.14. Let $X \in D(R)$, then there exists $Y \in D(R)$ with a map $f : Y \to X$, s.t.

(i) $H_d(Y) = 0,\ \forall d < 0$.

(ii) $H_d(f) : H_d(Y) \xrightarrow{\sim} H_d(X)$ are isomorphisms, $\forall d \geq 0$.

Proof. We first prove: there exists a sequence of maps $Y_0 \to Y_1 \to Y_2 \to \cdots$ in $D(R)_{/X}$, s.t. for any $n \geq 0$, $H_d(Y_n) = 0,\ d < 0$ and $H_d(Y_n \to X)$ are isomorphisms if $0 \leq d < n$ and is a surjection if $d = n$.
We prove this by induction.
First, $n = 0$. Let $Y_0 = \oplus_I 1$ for $I =$ cartinal of $H_0(X)$, then the map $Y_0 \to X$ can induce a surjection $H_0(Y_0) = R^{\oplus I} \twoheadrightarrow H_0(X)$ and for $d < 0$, $H_d(Y_0) = 0$.
Now we assume that there exists a sequence

$$Y_0 \to Y_1 \to Y_2 \to \cdots \to Y_{n-1}$$

in $D(R)_{/X}$ satisfying the assumption.
Let $F = \mathrm{fib}(Y_{n-1} \to X)$, then $F \to Y_{n-1} \to X$ is a fiber-cofiber sequence. We can find an index I, s.t. $\Sigma^{n-1} \oplus_I 1 \to F$ can induce a surjection $H_{n-1}(\Sigma^{n-1} \oplus_I 1) \twoheadrightarrow H_{n-1}(F)$. Then let $Y_n = \mathrm{cofib}(\Sigma^{n-1} \oplus_I 1 \to F \to Y_{n-1})$, hence $\Sigma^{n-1} \oplus_I 1 \to Y_{n-1} \to Y_n$ is also a fiber-cofiber sequence. Now, we check it satisfies the requirements.

(a) $d < 0$. The fiber-cofiber sequence $\Sigma^{n-1} \oplus_I 1 \to Y_{n-1} \to Y_n$ can induce a long exact sequence:

$$\cdots \to H_{-1}(\Sigma^{n-1} \oplus_I 1) \to H_{-1}(Y_{n-1}) \to H_{-1}(Y_n) \to H_{-2}(\Sigma^{n-1} \oplus_I 1) \to H_{-2}(Y_{n-1}) \to \cdots .$$

Since for $k < 0$, $H_k(Y_{n-1}) = 0$, we know $H_d(Y_n) = H_{d-1}(\Sigma^{n-1} \oplus_I 1) = 0 (d < 0)$.

(b) First, there exists a map $Y_n \to X$, this is because

$$\begin{tikzcd}
\Sigma^{n-1} \oplus_I 1 \arrow[r] \arrow[d] & Y_{n-1} \arrow[r] \arrow[d, equal] & Y_n \arrow[d, dashed, "\exists"] \\
F \arrow[r] & Y_{n-1} \arrow[r] & X
\end{tikzcd}$$

and $H_d(\Sigma^{n-1} \oplus_I 1) = 0$, $\forall d \neq n-1$, then by

$$\cdots \to H_{n-2}(\Sigma^{n-1} \oplus_I 1) \to H_{n-2}(Y_{n-1}) \to H_{n-2}(Y_n) \to H_{n-3}(\Sigma^{n-1} \oplus_I 1) \to \cdots ,$$

it implies that $0 \leq \forall d \leq n-2,\ H_d(Y_n) \cong H_d(Y_{n-1}) \cong H_d(X)$.
We have the following diagram:

$$\begin{tikzcd}
H_n(Y_{n-1}) \arrow[r] \arrow[d, equal] & H_n(Y_n) \arrow[r] \arrow[d, two heads, dotted] & H_{n-1}(\Sigma^{n-1} \oplus_I 1) \arrow[r] \arrow[d, two heads] & H_{n-1}(Y_{n-1}) \arrow[r] \arrow[d, equal] & H_{n-1}(Y_n) \arrow[r] \arrow[d, dotted, "\sim"] & H_{n-2}(\Sigma^{n-1} \oplus_I 1) \arrow[r] \arrow[d, "\sim"] & H_{n-2}(Y_{n-1}) \arrow[d, equal] \\
H_n(Y_{n-1}) \arrow[r] & H_n(X) \arrow[r] & H_{n-1}(F) \arrow[r] & H_{n-1}(Y_{n-1}) \arrow[r] & H_{n-1}(X) \arrow[r] & H_{n-2}(F) \arrow[r] & H_{n-2}(Y_{n-1})
\end{tikzcd}$$

By five's lemma, we can show $H_{n-1}(Y_n) \xrightarrow{\sim} H_{n-1}(X)$ and $H_n(Y_n) \twoheadrightarrow H_n(X)$.
Now, for $Y_0 \to Y_1 \to Y_2 \to \cdots$, we take $Y = \varinjlim Y_n$, and hence we can get a map $Y \to X$.
By $H_d(\varinjlim Y_n) = \varinjlim H_d(Y_n)$, for $d < 0$, $H_d(Y) = 0$, and for $d \geq 0$,

$$H_d(Y) = \varinjlim \left(H_d(Y_0) \to H_d(Y_1) \cdots \to H_d(Y_{d+1}) \to H_d(Y_{d+2}) \to \cdots\right) = H_d(X).$$

$\square$

Proposition 3.15. For $X \in D(R)$, the following are equivalent:

(i) $H_d(X) = 0$, $\forall d < 0$.

(ii) X is generated by 1 under colimits.

(iii) There exists a sequence of maps $X_0 \to X_1 \to X_2 \to \cdots$ with $X = \varinjlim X_i$, where for each i, the cofiber $\operatorname{cofib}(X_{i-1} \to X_i)$ is of the form $\Sigma^i \oplus_I 1$.

Proof. (i) $\Longrightarrow$ (iii). By previous proposition, for $X \in D(R)$, there exists a map $f : Y \to X$ with $H_d(Y) = 0$ for $d < 0$ and $H_d(f)$ are isomorphisms for $d \geq 0$. Then, for $d < 0$,

$$H_d(X) = H_d(Y) = 0.$$

Hence, for any $d \in \mathbb{Z}$, $H_d(f)$ are isomorphisms. Thus, $f : Y \xrightarrow{\sim} X$.
From the construction of Y, we know there is a sequence of maps $X_0 \to X_1 \to X_2 \to \cdots$ with $\varinjlim X_i = Y \cong X$.
By the fiber-cofiber sequence $\Sigma^{n-1} \oplus_I 1 \to X_{n-1} \to X_n$, we can get a new fiber-cofiber sequence $X_{n-1} \to X_n \to \Sigma^n \oplus_I 1$, i.e. $\operatorname{cofib}(X_{n-1} \to X_n) = \Sigma^n \oplus_I 1$.
(iii) $\Longrightarrow$ (ii).
We have $X_{i-1} \to X_i \to \Sigma^i \oplus_I 1$, which gives a new fiber-cofiber sequence: $\Sigma^{i-1} \oplus_I 1 \to X_{i-1} \to X_i$. Then $X_i = \operatorname{cofib}(\Sigma^{i-1} \oplus_I 1 \to X_{i-1}) = \operatorname{colim}(0 \leftarrow \Sigma^{i-1} \oplus_I 1 \to X_{i-1})$.
Now, $X_1 = \operatorname{cofib}(\oplus_I 1 \to X_0) = \operatorname{cofib}(\oplus_I 1 \to \oplus_J 1) = \operatorname{colim}(0 \leftarrow \oplus_I 1 \to \oplus_J 1)$. Hence, each X_i is generated by 1 under colimits. Finally, $X = \operatorname{colim} X_i$ is also generated by 1 under colimits.
(ii) $\Longrightarrow$ (i).

Arbitrary colimits can be written in terms of pushouts and filtered colimits. And H_d commutes with filtered colimits. So it suffices to show that for A, B, C with $H_d(A) = H_d(B) = H_d(C) = 0,\ \forall d < 0$, then for the pushout $D = \mathrm{colim}(C \leftarrow A \rightarrow B)$, $H_d(D) = 0,\ \forall d < 0$.
This is because we can get a null-composite sequence $A \to B \oplus C \to D$, and induce a long exact sequence

$$\cdots \to H_d(A) \to H_d(B) \oplus H_d(C) \to H_d(D) \to \cdots$$

which implies $H_d(D) = 0,\ \forall d < 0$. □

Definition 3.16. (i) $D(R)_{\geq 0} := \{X \in D(R) \mid H_d(X) = 0,\ \forall d < 0\}$.

(ii) $D(R)_{<0} := \{X \in D(R) \mid H_d(X) = 0,\ \forall d \geq 0\}$.

(iii) $\tau_{\geq 0} : D(R) \to D(R)_{\geq 0}; X \mapsto \tau_{\geq 0}(X) := Y$, which is constructed in Proposition 3.14.

Now, given any map $Z \to X$ in $D(R)$, we can get a commutative diagram:

$$\begin{tikzcd} \tau_{\geq 0}(Z) \arrow[r, dashed, "\exists!"] \arrow[d] & \tau_{\geq 0}(X) \arrow[d] \\ Z \arrow[r] & X \end{tikzcd}$$

Hence, $\tau_{\geq 0} : D(R) \to D(R)_{\geq 0}$ is a functor.

Proposition 3.17. $D(R)_{\geq 0} \underset{\tau_{\geq 0}}{\overset{i}{\rightleftarrows}} D(R) \underset{i}{\overset{\tau_{<0}}{\rightleftarrows}} D(R)_{<0}$, i.e. $i \dashv \tau_{\geq 0}$ and $\tau_{<0} \dashv i$.

Corollary 3.18. For $X \in D(R)$, we have

$$X \cong \varprojlim \tau_{\leq n}(X) \quad \text{and} \quad X \cong \varinjlim \tau_{\geq -n}(X).$$

Proof. We have a Milnor sequence:

$$0 \longrightarrow {\varprojlim}^1 H_{d+1}(\tau_{\leq n} X) \longrightarrow H_d(\varprojlim \tau_{\leq n} X) \longrightarrow \varprojlim H_d(\tau_{\leq n} X) \longrightarrow 0.$$

For $n \gg 0$, we have $H_d(\tau_{\leq n} X) = H_d(X)$, hence $\varprojlim H_d(\tau_{\leq n} X) = H_d(X)$.
And $\{H_{d+1}(\tau_{\leq n} X)\}_{n \in \mathbb{Z}}$ satisfies the Mittag-Leffler condition, hence ${\varprojlim}^1 H_{d+1}(\tau_{\leq n} X) = 0$. Therefore, from the above short exact sequence, we have

$$H_d(\varprojlim \tau_{\leq n} X) \cong H_d(X),\ \forall d \in \mathbb{Z},$$

which impies $X \cong \varprojlim \tau_{\leq n} X$.

For another isomorphsim, from

$$H_d(\varinjlim \tau_{\geq -n} X) = \varprojlim H_d(\tau_{\geq -n} X) = H_d(X), \ \forall d \in \mathbb{Z},$$

one can show $X \cong \varinjlim \tau_{\geq -n}(X)$. $\square$

Definition 3.19. For any map $f : X \to Y$ in $D(R)$, we define its kernel to be

$$\ker(f) := \tau_{\geq 0}\mathrm{fib}(X \to Y)$$

and its cokernel to be

$$\mathrm{coker}(f) := \tau_{\leq 0}\mathrm{cofib}(X \to Y).$$

Proposition 3.20. Let $D(R)_0 = \{X \in D(R) \mid H_d(X), \ \forall d \neq 0\}$.

(i) There is an isomorphism $H_0 : D(R)_0 \xrightarrow{\sim} \mathrm{Mod}_R$.

(ii) Any object in $D(R)_0$ can be written as of the form $\mathrm{coker}(\oplus_I 1 \to \oplus_J 1)$.

(iii) $H_0 : D(R)_0 \longrightarrow \mathrm{Mod}_R$ is an exact functor.

(iv) $H_0 : D(R)_0 \longrightarrow \mathrm{Mod}_R$ commutes with direct sums.

Proof. (ii) For $X \in D(R)_0$, there exists $f : Y \to X$ with $H_d(Y) = 0, \ \forall d < 0$ and $H_d(f)$ are isomorphisms, $\forall d \geq 0$.
By the construction of Y_1, $Y_1 = \mathrm{cofib}(\oplus_I 1 \to \oplus_J 1)$.
On the other hand, $X \cong \tau_{\leq 0} Y_1$. Hence,

$$X \cong \tau_{\leq 0}\mathrm{cofib}(\oplus_I 1 \to \oplus_J 1) = \mathrm{coker}(\oplus_I 1 \to \oplus_J 1).$$

(iii) In order to show that H_0 preserves exact sequences, it suffices to show H_0 preserves kernels and cokernels.
For any map $f : X \to Y$ in $D(R)_0$, applying functor $\tau_{\geq 0}$ to sequence $\mathrm{fib}(f) \to X \to Y$, we get a fiber-cofiber sequence

$$\ker(f) = \tau_{\geq 0}\mathrm{fib}(f) \to X \to Y.$$

And it induces a long exact sequence

$$0 = H_1(Y) \to H_0(\ker(f)) \to H_0(X) \to H_0(Y) \to \cdots .$$

Hence, $H_0(\ker(f)) = \ker(H_0(X) \to H_0(Y))$.
Dually ,we can prove $H_0(\mathrm{coker}(f)) = \mathrm{coker}(H_0(X) \to H_0(Y))$.

$\square$

Remark 3.21. $1 \in D(R)_0$ is compact and projective.

Proof: Compactness is the definition.

For the projectiveness, we need to show that any epimorphism $X \twoheadrightarrow 1$ splits.

Let $F = \mathrm{fib}(X \to 1)$. Consider $F \to X \to 1$. Then $H_{-1}(F) = 0$.

By $[M, N]_d = \mathrm{Ext}_R^{-d}(M, N)$, we get $\mathrm{Ext}_R^{-1}(1, F) = [1, F]_{-1} = H_{-1}(F) = 0$. Hence $X \twoheadrightarrow 1$ splits. $\square$

Definition 3.22. (i) A filtered object of $D(R)$ is an object in $\mathrm{Fun}(\mathbb{Z}_{\leq}, D(R))$, i.e.

$$\cdots \longrightarrow F(n-1) \to F(n) \to F(n+1) \to \cdots .$$

(ii) A filtered object F is convergent if $\varprojlim F(n) = 0$.

(iii) $F(\infty) := \varinjlim F(n)$. Call it the underlying object of F.

(iv) The n-th associated graded $\mathrm{gr}_n(F) := \mathrm{cofib}(F(n-1) \to F(n)) \triangleq F(n)/F(n-1)$.

Now, giving a convergent filtered object $F : \mathbb{Z}_{\leq} \to D(R)$, s.t. $\mathrm{gr}_n(F) \in D(R)_n,\ \forall n$, we can define an R-module M_n:

$$H_n : D(R)_n \to \mathrm{Mod}_R; \mathrm{gr}_n(F) \mapsto H_n(\mathrm{gr}_n(F)) \triangleq M_n.$$

From the sequence

$$F(n-1)/F(n-2) \longrightarrow F(n)/F(n-2) \longrightarrow F(n)/F(n-1) \longrightarrow \Sigma(F(n-1)/F(n-2)),$$

we get a map $d : H_n(\mathrm{gr}_n(F)) \longrightarrow H_n(\Sigma \mathrm{gr}_{n-1}(F))$, i.e. $d : M_n \longrightarrow M_{n-1}$.

One can check $d^2 = 0$. Hence, given a convergent filtered object F, s.t. $\mathrm{gr}_n(F) \in D(R)_n$, we define a chain complex of R-modules M_*.

We denote $\mathrm{Fun}(\mathbb{Z}_{\leq}, D(R))_{\mathrm{cx}} = \{F \in \mathrm{Fun}(\mathbb{Z}_{\leq}, D(R)) \mid F \text{ convergent}\}$.

Proposition 3.23. (i) $\mathrm{Fun}(\mathbb{Z}_{\leq}, D(R))_{\mathrm{cx}} \xrightarrow{\sim} \mathrm{Ch}_R;\ F \mapsto M_*$.

(ii) $H_n(F(\infty)) = H_n(M_*),\ \forall n$.

4 $D(\mathbb{Z})$

Definition 4.1. Let $X \in \text{Top}$. A sieve on X is a set $\mathfrak{U}$ of open subsets of X, s.t. if $V \in \mathfrak{U}$ and $V' \subset V$, then $V' \in \mathfrak{U}$. If $U = \cup_{V \in \mathfrak{U}} V$, we say that the sieve $\mathfrak{U}$ covers U.

Definition 4.2. (i) Let $X \in \text{Top}$. Let $\mathcal{F} \in \text{PSh}(X, D(\mathbb{Z})$ be a presheaf with values in $D(\mathbb{Z})$, i.e. $\mathcal{F} \in \text{Fun}(\text{Op}(X)^{\text{op}}, D(\mathbb{Z}))$. We say $\mathcal{F}$ is a sheaf if for all sieves $\mathfrak{U}$ on X covering $U \in \text{Op}(X)$, we have

$$\mathcal{F}(U) \xrightarrow{\sim} \varprojlim_{V \in \mathfrak{U}^{\text{op}}} \mathcal{F}(V).$$

(ii) For $U \in \text{Op}(X)$, one define $h_U \in \text{PSh}(X, D(\mathbb{Z}))$ via

$$h_U(V) = \begin{cases} * & V \subset U \\ \emptyset & \text{otherwise} \end{cases}$$

(iii) For a sieve $\mathfrak{U}$, one define $h_{\mathfrak{U}} \in \text{PSh}(X, D(\mathbb{Z}))$ via

$$h_{\mathfrak{U}}(V) = \begin{cases} * & V \in \mathfrak{U} \\ \emptyset & V \notin \mathfrak{U} \end{cases}$$

Proposition 4.3. Let $\mathcal{F} \in \text{PSh}(X, D(\mathbb{Z}))$, then $\mathcal{F}$ is a sheaf if and only if it satisfies:

(i) $\mathcal{F}(\emptyset) = *$.

(ii) For any open subsets $V, V' \in \text{Op}(X)$,

$$\mathcal{F}(V \cup V') \xrightarrow{\sim} \mathcal{F}(V) \times_{\mathcal{F}(V \cap V')} \mathcal{F}(V').$$

(iii) For any sieve $\mathfrak{U}$, $\mathcal{F}(\varinjlim_{V \in \mathfrak{U}} V) \xrightarrow{\sim} \varprojlim_{V \in \mathfrak{U}^{\text{op}}} \mathcal{F}(V)$.

Remark 4.4.

$$\mathbb{Z}[h_U](V) := \mathbb{Z}[h_U(V)] = \begin{cases} \mathbb{Z} & V \subseteq U \\ 0 & V \nsubseteq U \end{cases}$$

$$\mathbb{Z}[h_{\mathfrak{U}}](V) := \mathbb{Z}[h_{\mathfrak{U}}(V)] = \begin{cases} \mathbb{Z} & V \in \mathfrak{U} \\ 0 & V \notin \mathfrak{U} \end{cases}$$

$$\text{Map}(\mathbb{Z}[h_U], \mathcal{F}) = \text{Map}(\mathbb{Z}, \mathcal{F}(U)).$$

$$
\begin{aligned}
\operatorname{Map}(\mathbb{Z}[h_{\mathfrak{U}}], \mathcal{F}) &= \varprojlim_{V \in \mathfrak{U}^{\mathrm{op}}} \operatorname{Map}(\mathbb{Z}[h_V], \mathcal{F}) \\
&= \varprojlim_{V \in \mathfrak{U}^{\mathrm{op}}} \operatorname{Map}(\mathbb{Z}, \mathcal{F}(V)) \\
&= \operatorname{Map}(\mathbb{Z}, \varprojlim_{V \in \mathfrak{U}^{\mathrm{op}}} \mathcal{F}(V)) \\
&= \operatorname{Map}(\mathbb{Z}, \mathcal{F}(\varinjlim_{V \in \mathfrak{U}} V)).
\end{aligned}
$$

Proposition 4.5. $\mathrm{PSh}(X, D(\mathbb{Z})) \underset{i}{\overset{\mathrm{sh}}{\rightleftarrows}} \mathrm{Sh}(X, D(\mathbb{Z}))$; $\mathcal{F} \mapsto \mathcal{F}^{\mathrm{sh}}$. Moreover, $\mathcal{F}^{\mathrm{sh}} = 0$ iff $\mathcal{F}$ lies in the stable co-complete subcategory generated by $\mathrm{cofib}(\mathbb{Z}[h_{\mathfrak{U}}] \to \mathbb{Z}[h_U])$ for all sieves $\mathfrak{U}$ covering U.

Definition 4.6. For $\mathcal{F} \in \mathrm{PSh}(X, D(\mathbb{Z}))$, define $H_n(\mathcal{F}) \in \mathrm{PSh}(X, \mathrm{Ab})$ by $H_n(\mathcal{F})(U) = H_n(\mathcal{F}(U))$.
With this presheaf $H_n(\mathcal{F}) \in \mathrm{PSh}(X, \mathrm{Ab})$, one can sheafify it to get a sheaf $H_n(\mathcal{F})^{\mathrm{sh}} \in \mathrm{Sh}(X, \mathrm{Ab})$.

Proposition 4.7. Let $\mathcal{F} \in \mathrm{PSh}(X, D(\mathbb{Z}))$.

(i) If $\mathcal{F}^{\mathrm{sh}} = 0$, then $H_n(\mathcal{F})^{\mathrm{sh}} = 0, \ \forall n \in \mathbb{Z}$.

(ii) If $\mathcal{F}$ is bounded above and $H_n(\mathcal{F})^{\mathrm{sh}} = 0, \ \forall n \in \mathbb{Z}$, then $\mathcal{F}^{\mathrm{sh}} = 0$.

Corollary 4.8. Let $\mathcal{F} \to \mathcal{G}$ be a map in $\mathrm{PSh}(X, D(\mathbb{Z}))$.

(i) If $\mathcal{F}^{\mathrm{sh}} \xrightarrow{\sim} \mathcal{G}^{\mathrm{sh}}$, then $H_n(\mathcal{F})^{\mathrm{sh}} \xrightarrow{\sim} H_n(\mathcal{G})^{\mathrm{sh}}, \ \forall n \in \mathbb{Z}$.

(ii) If $\mathcal{F}$ and $\mathcal{G}$ are bounded above, and $H_n(\mathcal{F})^{\mathrm{sh}} \xrightarrow{\sim} H_n(\mathcal{G})^{\mathrm{sh}}, \ \forall n \in \mathbb{Z}$, then $\mathcal{F}^{\mathrm{sh}} \xrightarrow{\sim} \mathcal{G}^{\mathrm{sh}}$.

Corollary 4.9. Let $\mathcal{F} \to \mathcal{G}$ be a map in $\mathrm{PSh}(X, D(\mathbb{Z}))$ and $\mathcal{F}, \mathcal{G}$ are bounded above, then

$$
\mathcal{F}^{\mathrm{sh}} \xrightarrow{\sim} \mathcal{G} \quad \iff \quad \begin{cases} \mathcal{G} \text{ is a sheaf.} \\ H_n(\mathcal{F})^{\mathrm{sh}} \xrightarrow{\sim} H_n(\mathcal{G})^{\mathrm{sh}}, \ \forall n \in \mathbb{Z}. \end{cases}
$$

Definition 4.10.

Proposition 4.11.

5 The t-structure on valued sheaves

Definition 5.1. A t-structure on a stable ∞-category $\mathcal{C}$ is a pair $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ of full sub-∞-categories of $\mathcal{C}$ that are stable under equivalences and satisfy:

(T1) The suspension functor Σ and the loop functor Ω restrict to $\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0}$ resp. are fully faithful functors $\Sigma : \mathcal{C}_{\geq 0} \to \mathcal{C}_{\geq 0}$ and $\Omega : \mathcal{C}_{\leq 0} \to \mathcal{C}_{\leq 0}$.

(T2) If $X \in \mathcal{C}_{\geq 0}$ and $Y \in \mathcal{C}_{\leq 0}$, then $\mathrm{Map}(X, \Omega Y) \simeq *$.

(T3) For every $X \in \mathcal{C}$, there exists a fiber sequence

$$X' \longrightarrow X \longrightarrow X''$$

with $X' \in \mathcal{C}_{\geq 0}$ and $X'' \in \mathcal{C}_{\leq -1} := \Omega \mathcal{C}_{\leq 0}$.

We call $\mathcal{C}_{\geq 0}$ and $\mathcal{C}_{\leq 0}$ the connective and coconnective parts of the t-structure.

Given $n \in \mathbb{Z}$, we define $\mathcal{C}_{\geq n} := \Sigma^n \mathcal{C}_{\geq 0} \subset \mathcal{C}$ and $\mathcal{C}_{\leq n} := \Sigma^n \mathcal{C}_{\leq 0} \subset \mathcal{C}$, where for $n < 0$, we have $\Sigma^n = \Omega^{-n}$.

The inclusions $i : \mathcal{C}_{\geq m} \to \mathcal{C}$ and $s : \mathcal{C}_{\leq n} \to \mathcal{C}$ admit adjoint functors

$$\begin{tikzcd} \mathcal{C}_{\geq m} \arrow[r, shift left, "i"] & \mathcal{C} \arrow[l, shift left, "r"] \arrow[r, shift left, "p"] & \mathcal{C}_{\leq n} \arrow[l, shift left, "s"] \end{tikzcd}.$$

In particular, the full sub-∞-category $\mathcal{C}_{\geq m} \subset \mathcal{C}$ is closed under colimits, and the full sub-∞-category $\mathcal{C}_{\leq n} \subset \mathcal{C}$ is closed under limits. From the adjoint pairs, we can form their counit and unit, and we get

$$\tau_{\geq 0} X = (i \circ r)(X) \xrightarrow{\epsilon} X \xrightarrow{\eta} \tau_{\leq -1} X = (s \circ p)(X).$$

The composition of the two maps is a point in the anima $\mathrm{Map}(\tau_{\geq 0} X, \tau_{\leq -1} X) \simeq *$. So the composite map automatically admits a null-homotopy, which is unique, up to contractible ambiguity. We have the following commutative diagram:

$$\begin{tikzcd} \mathcal{C}_{\leq m} \cap \mathcal{C}_{\geq n} \arrow[r, shift left, "i"] \arrow[d, shift left, "s"] & \mathcal{C}_{\leq m} \arrow[l, shift left, "r"] \arrow[d, shift left, "s"] \\ \mathcal{C}_{\geq n} \arrow[u, shift left, "p"] \arrow[r, shift left, "i"] & \mathcal{C} \arrow[l, shift left, "r"] \arrow[u, shift left, "p"] \end{tikzcd}$$

The canonical map

$$p \circ r \xrightarrow{\eta \circ p \circ r} r \circ i \circ p \circ r \simeq r \circ p \circ i \circ r \xrightarrow{r \circ p \circ \epsilon} r \circ p$$

is an equivalence.

We say the full sub-∞-category

$$\mathcal{C}^{\heartsuit} := \mathcal{C}_{\geq 0} \cap \mathcal{C}_{\leq 0} \subset \mathcal{C}$$

is the heart of the t-structure. For the functor

$$\pi_0 := \tau_{\geq 0} \circ \tau_{\leq 0} \simeq \tau_{\leq 0} \circ \tau_{\geq 0} : \mathcal{C} \to \mathcal{C}^{\heartsuit},$$

we call it the zeroth homotopy functor. The functor π_0 is additive, but is NOT exact. Instead, for all $n \in \mathbb{Z}$, we define

$$\pi_d : \mathcal{C} \longrightarrow \mathcal{C}^{\heartsuit}$$

to be $\pi_d = \pi_0 \circ \Omega^d$, and call it the dth homotopy functor. Now, a fiber sequence

$$Z \xrightarrow{g} Y \xrightarrow{f} X$$

in $\mathcal{C}$ gives rise to a long exact sequence

$$\cdots \longrightarrow \pi_{d+1}(X) \longrightarrow \pi_d(Z) \longrightarrow \pi_d(Y) \longrightarrow \pi_d(X) \longrightarrow \cdots$$

in the heart $\mathcal{C}^{\heartsuit}$.

If $f : Y \to X$ is an equivalence, then $f : \pi_d(Y) \to \pi_d(X)$ is an isomorphism for all $d \in \mathbb{Z}$, but the opposite is generally not the case.

Now, for the stable ∞-category $D(\mathbb{Z})$, we defined homology functors $H_d : D(\mathbb{Z}) \to \mathrm{Mod}_{\mathbb{Z}}$ for all $d \in \mathbb{Z}$ by

$$H_d(X) \simeq \pi_0 \mathrm{Map}(\Sigma^d 1, X) \simeq \pi_0 \mathrm{Map}(1, \Omega^d X).$$

$D(\mathbb{Z})$ admits a t-structure $(D(\mathbb{Z})_{\geq 0}, D(\mathbb{Z})_{\leq 0})$, where the connective part $D(\mathbb{Z})_{\geq 0}$ is spanned by those X for which $H_d(X) \simeq 0$, for $d < 0$, and the coconnective part $D(\mathbb{Z})_{\leq 0}$ is spanned by those X for which $H_d(X) \simeq 0$, for $d > 0$. The zeroth homology functor

$$H_0 : D(\mathbb{Z})^{\heartsuit} \longrightarrow \mathrm{Mod}_{\mathbb{Z}}$$

is an equivalence of (abelian) categories. We have $H_d \simeq H_0 \circ \pi_d$, so the functors H_d and π_d encode the same information.

Proposition 5.2. Let $X \in \mathrm{Top}$, and let $\mathcal{C}$ be a stable ∞-category. A t-structure on $\mathcal{C}$

induces a t-structure on the stable ∞-category $\mathcal{P}(X,\mathcal{C})$ of $\mathcal{C}$-valued presheaves on X, where the coconnective part $\mathcal{P}(X,\mathcal{C})_{\leq 0} \simeq \mathcal{P}(X,\mathcal{C}_{\leq 0})$, and where the connective part $\mathcal{P}(X,\mathcal{C})_{\geq 0}$ is spanned by those $\mathcal{F}$ such that

$$\mathrm{Map}(\mathcal{F},\Omega\mathcal{G}) \simeq *$$

for all $\mathcal{G} \in \mathcal{P}(X,\mathcal{C}_{\leq 0})$.

A functor $f : \mathcal{D} \to \mathcal{C}$ between stable ∞-categories is exact iff it is left exact iff it is right exact.

An exact funcor $f : \mathcal{D} \to \mathcal{C}$ between stable ∞-categories with t-structures is left t-exact if $f(\mathcal{D}_{\leq 0}) \subset \mathcal{D}_{\leq 0}$, and it is right t-exact if $f(\mathcal{D}_{\geq 0}) \subset \mathcal{D}_{\geq 0}$. It is t-exact if it is both left t-exact and right t-exact. If $f : \mathcal{D} \to \mathcal{C}$ admits right adjoint functor $g : \mathcal{C} \to \mathcal{D}$, then f is right t-exact iff g is left t-exact.

Theorem 5.3. Let $X \in \mathrm{Top}$ and $\mathcal{C}$ a presentable stable ∞-category.

(1) The sheafification functor $\mathrm{ass}_X : \mathcal{P}(X,\mathcal{C}) \to \mathrm{Sh}(X,\mathcal{C})$ is t-exact, and the inclusion functor $\iota_X : \mathrm{Sh}(X,\mathcal{C}) \to \mathcal{P}(X,\mathcal{C})$ is left t-exact.

(2) The composite functor

$$\mathrm{Sh}(X,\mathcal{C}^{\heartsuit}) \xrightarrow{\iota_X^{\heartsuit}} \mathcal{P}(X,\mathcal{C}^{\heartsuit}) \simeq \mathcal{P}(X,\mathcal{C})^{\heartsuit} \xrightarrow{\mathrm{ass}_X} \mathrm{Sh}(X,\mathcal{C})^{\heartsuit}$$

is an equivalence of categories.

Write π_0^p and π_0^s for the homotopy functors associated with the t-structure on presheaves and sheaves. Since ass_X is both exact and t-exact, we obtain a commutative square

$$\begin{tikzcd}
\mathcal{P}(X,\mathcal{C}) \arrow[r, "\pi_0^p"] \arrow[d, "\mathrm{ass}_X"] & \mathcal{P}(X,\mathcal{C})^{\heartsuit} \arrow[d, "\mathrm{ass}_X"] \\
\mathrm{Sh}(X,\mathcal{C}) \arrow[r, "\pi_0^s"] & \mathrm{Sh}(X,\mathcal{C})^{\heartsuit}
\end{tikzcd}$$

6 Sheaf

Lemma 6.1. If $\mathcal{A}$ is bounded above, i.e. $\exists d \in \mathbb{Z}$, s.t. $H_n(\mathcal{A}) = 0$,for all $n > d$, then $\mathcal{A}^{\text{sh}}$ is also bounded above.

Question. For finite sets X, X' with $X' \to X$ surjective and split, then

$$0 \to \mathbb{Z}[X] \to \mathbb{Z}[X'] \to \mathbb{Z}[X' \times_X X'] \to \mathbb{Z}[X' \times_X X' \times_X X'] \to \cdots$$

is exact.

Lemma 6.2. Arbitrary limits and filtered colimits preserves $D(\mathbb{Z})_{\leq d}$.

Proof. First we show $D(\mathbb{Z})_{\leq d}$ is closed under filtered colimits. Assume $X_i \in D(\mathbb{Z})_{\leq d}, i \in I$,then

$$H_n(\varinjlim X_i) = \varinjlim H_n(X_i) = 0, \text{ for any } n > d.$$

Hence $\varinjlim X_i \in D(\mathbb{Z})_{\leq d}$. Then we show $D(\mathbb{Z})_{\leq d}$ is closed under arbitrary limits. Assume $X_i \in D(\mathbb{Z})_{\leq d}, n > d$, then

$$\begin{aligned} H_n(\lim X_i) &= [\Sigma^n 1, \lim X_i] \\ &= \pi_0 \mathrm{Map}(\Sigma^n 1, \lim X_i) \\ &= \pi_0 \lim \mathrm{Map}(\Sigma^n 1, X_i) \\ &= \pi_0 \lim * \\ &= \pi_0 * \\ &= 0. \end{aligned}$$

Hence, $\lim X_i \in D(\mathbb{Z})_{\leq d}$. $\square$

Problem. What is the relation between $\pi_n(\lim X_i)$ and $\lim \pi_n(X_i)$. Similarly, the relation between $\pi_n(\mathrm{colim} X_i)$ and $\mathrm{colim} \pi_n(X_i)$.

Definition 6.3. We define the singular homology functor to be the composite of

$$\mathrm{Top} \to \mathrm{Cond(Set)} \hookrightarrow \mathrm{Cond(An)} \to \mathrm{An},$$

and denote it by $h : \mathrm{Top} \to \mathrm{An}$, where $\mathrm{Top} \to \mathrm{Cond(Set)}, X \mapsto \underline{X}; \mathrm{Cond(An)} \to \mathrm{An}$ is the left adjoint of $\mathrm{An} \hookrightarrow \mathrm{Cond(An)}$.

Definition 6.4. For the forgetful functor $D(\mathbb{Z})_{\geq 0} \simeq \mathrm{Ani(Ab)} \to \mathrm{Ani(Set)} \simeq \mathrm{An}$, it has a left adjoint, and we denote it by

$$\mathbb{Z}[-] : \mathrm{Ani}(\mathrm{Set}) \to \mathrm{Ani}(\mathrm{Ab});\ S \mapsto \mathbb{Z}[S].$$

Definition 6.5. For $X \in \mathrm{Top}$, we define its singular homology object to be

$$\mathbb{Z}[h(X)] \in \mathrm{Ani}(\mathrm{Ab}) \simeq D(\mathbb{Z})_{\geq 0} \subset D(\mathbb{Z}).$$

Lemma 6.6. Assume $\mathcal{A} \in \mathrm{Sh}(X, D(\mathbb{Z}))$, $H_n(\mathcal{A}) = 0, \forall n > d$, and $H_d(\mathcal{A}) \neq 0$, then $H_d(\mathcal{A})$ is a sheaf.

Proof. For $H_d(\mathcal{A}) \in \mathrm{PSh}(X, \mathrm{Ab})$, we need to check $H_d(\mathcal{A}) \in \mathrm{Sh}(X, \mathrm{Ab})$.

By denition, $H_d(\mathcal{A})(U) = H_d(\mathcal{A}(U)) = H_d(\varprojlim \mathcal{A}(V))$. By the Milnor's sequence, we have

$$0 \longrightarrow \varprojlim{}^1 H_{d+1}(\mathcal{A}(V)) \longrightarrow H_d(\varprojlim \mathcal{A}(V)) \longrightarrow \varprojlim H_d(\mathcal{A}(V)) \longrightarrow 0.$$

Because $H_{d+1}(\mathcal{A}) = 0$, so the left term of this short exact sequence is 0, hence

$$H_d(\mathcal{A})(U) = H_d(\varprojlim \mathcal{A}(V)) = \varprojlim H_d(\mathcal{A}(V)) = \varprojlim H_d(\mathcal{A})(V).$$

Hence, $H_d(\mathcal{A}) \in \mathrm{Sh}(X, \mathrm{Ab})$. $\square$

Proposition 6.7. Let $\mathcal{C}_0 \subset \mathcal{C}$ be a full subcategory, then the following full subcategories of $\mathcal{C}$ agree:

- the full subcategory generated under (small) colimits by $\mathcal{C}_0$;
- the full subcategory generated under filtered colimits and finite colimits by $\mathcal{C}_0$;
- the full subcategory generated under sifted colimits and finite producnts by $\mathcal{C}_0$.

7 Animation

Theorem 7.1 (Yoneda)**.** Let $\mathcal{C}$ be an ∞-category, the functor

$$\mathcal{C} \hookrightarrow \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{An}); \; X \mapsto (Y \mapsto \mathrm{Hom}_{\mathcal{C}}(Y, X))$$

is fully faithful.

Remark 7.2. For S to be an anima, we mean S is an ∞-category; while S to be a Kan complex, we mean S is a 1-category.

Let $\mathcal{C}$ be a category which admits all small colimits.
Recall an object $X \in \mathcal{C}$ is compact (also called finitely presented) if $\mathrm{Hom}(X, -)$ commutes with filtered colimits.
An object $X \in \mathcal{C}$ is projective if $\mathrm{Hom}(X, -)$ commutes with reflexive coequalizers (coequalizers of parallel arrows $Y \rightrightarrows Z$ with a simultaneous section $Z \to Y$ of both maps).
Taken together, an object $X \in \mathcal{C}$ is compact projective if $\mathrm{Hom}(X, -)$ commutes with filtered colimits and reflexive coequalizers, equivalently, $\mathrm{Hom}(X, -)$ commutes with 1-sifted colimits.

Let $\mathcal{C}^{\mathrm{cp}} \subset \mathcal{C}$ be the full subcategory of compact projective objects. There is a fully faithful embedding $\mathrm{sInd}(\mathcal{C}^{\mathrm{cp}}) \longrightarrow \mathcal{C}$.
If $\mathcal{C}$ is generated under small colimits by $\mathcal{C}^{\mathrm{cp}}$, then the functor is an equivalence:

$$\mathrm{sInd}(\mathcal{C}^{\mathrm{cp}}) \cong \mathcal{C}.$$

If $\mathcal{C}^{\mathrm{cp}}$ is small, then

$$\mathrm{sInd}(\mathcal{C}^{\mathrm{cp}}) \subset \mathrm{Fun}((\mathcal{C}^{\mathrm{cp}})^{\mathrm{op}}, \mathrm{Set})$$

is exactly the full subcategory of functors that take finite coproducts in $\mathcal{C}^{\mathrm{cp}}$ to products in Set.

Example 7.3. (i) If $\mathcal{C} = \mathrm{Set}$, then $\mathcal{C}^{\mathrm{cp}} = \mathrm{FinSet}$, which generates $\mathcal{C}$ under small colimits.

(ii) If $\mathcal{C} = \mathrm{Ab}$, then $\mathcal{C}^{\mathrm{cp}} = \mathrm{FinFreeAb}$, which generates $\mathcal{C}$ under small colimits.

(iii) If $\mathcal{C} = \mathrm{Ring}$, then $\mathcal{C}^{\mathrm{cp}}$=$\{$retracts of $\mathbb{Z}[X_1, \cdots, X_n]\}$, which generates $\mathcal{C}$ under small colimits.

(iv) If $\mathcal{C} = \mathrm{Cond(Set)}$, then $\mathcal{C}^{\mathrm{cp}} = \mathrm{ExDisc}$, which generates $\mathcal{C}$ under small colimits.

(v) If $\mathcal{C} = \mathrm{Cond(Ab)}$, then $\mathcal{C}^{\mathrm{cp}} = \{$direct summands of $\mathbb{Z}[S] \mid S \in \mathrm{ExDisc}\}$, which generates $\mathcal{C}$ under small colimits.

(vi) $\mathcal{C} = \mathrm{Cond(Ring)}$, then $\mathcal{C}^{\mathrm{cp}} = \{\text{retracts of } \mathbb{Z}[\mathbb{N}[S]] \mid S \in \mathrm{ExDisc}\}$, which generates $\mathcal{C}$ under small colimits.

Definition 7.4. Let $\mathcal{C}$ be a category that admits all small colimits and $\mathcal{C}$ is generated under small colimits by $\mathcal{C}^{\mathrm{cp}}$. The animation of $\mathcal{C}$ is the ∞-category $\mathrm{Ani}(\mathcal{C})$ freely generated under sifted colimits by $\mathcal{C}^{\mathrm{cp}}$.

Example 7.5. If $\mathcal{C} = \mathrm{Set}$, then $\mathrm{Ani}(\mathcal{C}) = \mathrm{Ani(Set)} \triangleq \mathrm{Ani}$ is the ∞-category of animated sets, or anima in a short.
Any anima has a set of connected components, giving a functor $\pi_0 : \mathrm{Ani} \to \mathrm{Set}$, which has a fully faithful right adjoint $\mathrm{Set} \hookrightarrow \mathrm{Ani}$.
Given an anima A with a point $a \in A$ (meaning a map $a : * \to A$), one can define groups $\pi_i(A, a)$, for $i \geq 1$ and for $i \geq 2$, $\pi_i(A, a) \in \mathrm{Ab}$.
An anima A is i-truncated if $\pi_j(A, a) = 0$, $\forall a \in A$ and $\forall j > i$. Then A is 0-truncated if and only if it is in the essential image of $\mathrm{Set} \hookrightarrow \mathrm{Ani}$.
The inclusion of i-truncated anima into all anima has a left adjoint $\tau_{\leq i}$. For all anima A, the natural map

$$A \xrightarrow{\sim} \lim \tau_{\leq i} A$$

is an equivalence.
Picking any $a \in A$ and $i \geq 1$, the fiber of $\tau_{\leq i} A \to \tau_{\leq i-1} A$ over the image of a is an Eilenberg-Maclane anima $K(\pi_i(A, a), i)$. Here, an Eilenberg-Maclane anima $K(\pi, i)$ with $i \geq 1$ and π a group that is abelian if $i > 0$, is a pointed connected anima with $\pi_j = 0$ for $j \neq i$ and $\pi_i = \pi$. In fact, the ∞-category of pointed connected anima (A, a) with $\pi_j(A, a) = 0$ for $j \neq i$ is equivalent to Grp when $i = 1$, and to Ab when $i \geq 2$.

Remark 7.6. There are several ways to describe $\mathrm{Ani}(\mathcal{C})$.

(i) $\mathrm{Ani}(\mathcal{C})$ is the full sub-∞-category of objects in $\mathrm{Fun}((\mathcal{C}^{\mathrm{cp}})^{\mathrm{op}}, \mathrm{Ani})$ taking finite disjoint unions to finite products.

(ii) $\mathrm{Ani}(\mathcal{C})$ is the ∞-category obtained from $\mathrm{Simp}(\mathcal{C})$ by inverting weak equivalences.

Definition 7.7. Let $\mathcal{C}$ be an ∞-category that admits all small colimits. For any uncountable strong limit cardinal κ, the ∞-category $\mathrm{Cond}_\kappa(\mathcal{C})$ of κ-condensed objects of $\mathcal{C}$ is the category of contravariant functors from κ-ExDisc to $\mathcal{C}$ that take finite coproducts to finite products.
And we define

$$\mathrm{Cond}(\mathcal{C}) := \bigcup_{\kappa} \mathrm{Cond}_\kappa(\mathcal{C}).$$

Proposition 7.8. Let $\mathcal{C}$ be a category that is generated under small colimits by $\mathcal{C}^{\mathrm{cp}}$. Then $\mathrm{Cond}(\mathcal{C})$ is still generated under small colimits by its compact projective objects, and there is a natural equivalence of ∞-categories

$$\mathrm{Cond}(\mathrm{Ani}(\mathcal{C})) \cong \mathrm{Ani}(\mathrm{Cond}(\mathcal{C})).$$

Definition 7.9. Let $\mathcal{C}$ be some site.

(i) A presheaf of anima is a functor

$$\mathcal{F} : \mathrm{N}(\mathcal{C}^{\mathrm{op}}) \longrightarrow \mathrm{Ani}.$$

(ii) A sheaf of anima is a presheaf of anima $\mathcal{F}$, s.t. for all coverings $\{f_i : X_i \to X\}_{i \in I}$, one has

$$\mathcal{F}(X) \xrightarrow{\sim} \lim\big(\textstyle\prod_i \mathcal{F}(X_i) \rightrightarrows \prod_{i,j} \mathcal{F}(X_i \times_X X_j) \substack{\rightarrow\\ \rightarrow\\ \rightarrow} \cdots \big).$$

(iii) A hypercomplete sheaf of anima is a sheaf of anima $\mathcal{F}$, s.t. for all hypercovers $X_\bullet \to X$, the map

$$\mathcal{F}(X) \xrightarrow{\sim} \lim \mathcal{F}(X_\bullet) = \lim \big(\mathcal{F}(X_0) \rightrightarrows \mathcal{F}(X_1) \substack{\rightarrow\\ \rightarrow\\ \rightarrow} \cdots \big)$$

is an equivalence.

Definition 7.10. The ∞-category of condensed anima is given by

- The ∞-category of hypercomplete sheaves of anima on CHaus.
- The ∞-category of hypercomplete sheaves of anima on ProFin.
- The ∞-category of hypercomplete sheaves of anima on ExDisc, i.e. of functors

$$\mathrm{ExDisc}^{\mathrm{op}} \longrightarrow \mathrm{Ani}$$

taking finite disjoint unions to finite products.

$$\begin{array}{ccc} \mathrm{CW} & \subset & \mathrm{Cond(Set)} \\ \cap & & \cap \\ \mathrm{Ani} & \subset & \mathrm{Cond(Ani)} \end{array}$$

Definition 7.11. $X \in \mathrm{Cond(Ani)}$ is

- discrete, if X in the essential image of Ani.

- static, if X in the essential image of Cond(Set).

8 Condensed Cohomology

Definition 8.1. Let $X \in \text{Cond}, M \in \text{Cond(Ab)}$, we define the global section of M on X to be

$$\Gamma_{\text{cond}}(X, M) := \text{Hom}_{\text{Cond}}(X, M) = \text{Hom}_{\text{Cond(Ab)}}(\mathbb{Z}\left[X\right], M) \in \text{Ab},$$

and we define the condensed cohomology to be

$$R\Gamma_{\text{cond}}(X, M) := R\text{Hom}_{\text{Cond(Ab)}}(\mathbb{Z}\left[X\right], M),$$

i.e.

$$H^i_{\text{cond}}(X, M) := \text{Ext}^i_{\text{Cond(Ab)}}((\mathbb{Z}\left[X\right], M).$$

Lemma 8.2. For $X \in \text{ExDisc}$, the functor $\Gamma_{\text{cond}}(X, -) : \text{Cond(Ab)} \to \text{Ab}$ is exact, hence, for any $M \in \text{CondAb}, H^i_{\text{cond}}(X, M) = 0, \forall i \geq 1$.

Proof. We have $\Gamma_{\text{cond}}(X, -) = \text{Hom}_{\text{Cond(Ab)}}(\mathbb{Z}\left[X\right], -)$, and for $X \in \text{ExDisc}$, $\mathbb{Z}\left[X\right]$ is projective, hence $\Gamma_{\text{cond}}(X, -) : \text{Cond(Ab)} \to \text{Ab}$ is exact. □

Question. How to compute $H^i_{\text{cond}}(X, M)$?

From the definition, we need to find a projective resolution of $\mathbb{Z}\left[X\right]$.

For $X \in \text{CHaus}$, we pick a hypercover $X_\bullet \to X$, where each $X_i \in \text{ExDisc}$, for this hypercover, applying $\mathbb{Z}\left[-\right]$, then we get a projective resolution of $\mathbb{Z}\left[X\right]$:

$$\cdots \longrightarrow \mathbb{Z}\left[X_2\right] \longrightarrow \mathbb{Z}\left[X_1\right] \longrightarrow \mathbb{Z}\left[X_0\right] \longrightarrow \mathbb{Z}\left[X\right] \longrightarrow 0.$$

By definition, we have

$$\begin{aligned} H^i_{\text{cond}}(X, M) &= \text{Ext}^i_{\text{Cond(Ab)}}(\mathbb{Z}\left[X\right], M) \\ &= H^i(0 \to \text{Hom}_{\text{Cond(Ab)}}(\mathbb{Z}\left[X_0\right], M) \to \text{Hom}_{\text{Cond(Ab)}}(\mathbb{Z}\left[X_1\right], M) \to \cdots) \\ &= H^i(0 \to \Gamma_{\text{cond}}(X_0, M) \to \Gamma_{\text{cond}}(X_1, M) \to \Gamma_{\text{cond}}(X_2, M) \to \cdots). \end{aligned}$$

Theorem 8.3 (Dyckhoff,1976)**.** For any $X \in \text{CHaus}$, there are natural isomorphisms:

$$H^i_{\text{cond}}(X, \mathbb{Z}) \cong H^i_{\text{sh}}(X, \mathbb{Z}), \ \forall i \geq 0.$$

Proof. 1) Assume $X \in \text{Fin}$, then

$$H^i_{\text{cond}}(X, \mathbb{Z}) = \begin{cases} \Gamma_{\text{cond}}(X, \mathbb{Z}) = C(X, \mathbb{Z}) & i = 0 \\ 0 & i > 0 \end{cases}$$

This comes from Lemma 8.2. On the other hand,

$$H^i_{\text{sh}}(X, \mathbb{Z}) = \check{H}^i(X, \mathbb{Z}) = \begin{cases} C(X, \mathbb{Z}) & i = 0 \\ 0 & i > 0 \end{cases}$$

This comes from by computing Cech cohomology. For a finite set X, take the cover $\mathcal{U} = \{x \to X\}_{x \in X}$, then $\mathcal{C}^0(\mathcal{U}, \mathbb{Z}) = \mathcal{C}^1(\mathcal{U}, \mathbb{Z}) = \cdots = \mathbb{Z}^X$, and because $\mathcal{U}$ is a refinement of any cover, we have

$$\check{H}^i(X, \mathbb{Z}) = \check{H}^i(\mathcal{U}, \mathbb{Z}) = \begin{cases} \mathbb{Z}^X = C(X, \mathbb{Z}) & i = 0 \\ 0 & i > 0 \end{cases}$$

Therefore, for a finite set X, $H^i_{\text{cond}}(X, \mathbb{Z}) \cong H^i_{\text{sh}}(X, \mathbb{Z}),\ \forall i \geq 0$.

2) $X \in \text{ProFin}$, hence we can write $X = \varprojlim_j X^j,\ X^j \in \text{Fin}$.

$$H^i_{\text{sh}}(X, \mathbb{Z}) = \check{H}(X, \mathbb{Z}) = \varinjlim_j \check{H}(X_j, \mathbb{Z}) = \begin{cases} \varinjlim_j C(X_j, \mathbb{Z}) = C(X, \mathbb{Z}) & i = 0 \\ 0 & i > 0 \end{cases}$$

On the other hand, We compute $H^i_{\text{cond}}(X, \mathbb{Z}),\ i \geq 0$.
For $X \in \text{ProFin}$, pick a hypercover $X_\bullet \to X$ with each $X_i \in \text{ExDisc}$, and for each X^j, pick a finite hypercover $X^j_\bullet \to X^j$, s.t. $\varprojlim_j X^j_n = X_n$. Since X^j is finite, we have

$$H^i_{\text{cond}}(X^j, \mathbb{Z}) = \begin{cases} \Gamma(X^j, \mathbb{Z}) & i = 0 \\ 0 & i > 0 \end{cases}$$

And we know

$$H^i_{\text{cond}}(X^j, \mathbb{Z}) = H^i(0 \longrightarrow \Gamma(X^j_0, \mathbb{Z}) \longrightarrow \Gamma(X^j_1, \mathbb{Z}) \longrightarrow \Gamma(X^j_2, \mathbb{Z}) \to \cdots),$$

hence we have an exact sequence:

$$0 \longrightarrow \Gamma(X^j, \mathbb{Z}) \longrightarrow \Gamma(X^j_0, \mathbb{Z}) \longrightarrow \Gamma(X^j_1, \mathbb{Z}) \longrightarrow \Gamma(X^j_2, \mathbb{Z}) \to \cdots.$$

Applying the exact functor $\varinjlim_j$ to this exact sequence, we get an exact sequence:

$$0 \longrightarrow \varinjlim_j \Gamma(X^j, \mathbb{Z}) \longrightarrow \varinjlim_j \Gamma(X_0^j, \mathbb{Z}) \longrightarrow \varinjlim_j \Gamma(X_1^j, \mathbb{Z}) \longrightarrow \varinjlim_j \Gamma(X_2^j, \mathbb{Z}) \to \cdots,$$

i.e.

$$0 \longrightarrow \Gamma(X, \mathbb{Z}) \longrightarrow \Gamma(X_0, \mathbb{Z}) \longrightarrow \Gamma(X_1, \mathbb{Z}) \longrightarrow \Gamma(X_2, \mathbb{Z}) \to \cdots.$$

Hence,

$$H^i_{\mathrm{cond}}(X, \mathbb{Z}) = \begin{cases} \Gamma(X, \mathbb{Z}) & i = 0 \\ 0 & i > 0 \end{cases}$$

3) $X \in \mathrm{CHaus}$.

Consider a morphism of topoi $(\alpha^{-1}, \alpha_*) : \mathrm{Sh}(\mathrm{CHaus}/X) \to \mathrm{Sh}(X)$. For $\mathcal{F} \in \mathrm{Sh}(\mathrm{CHaus}/X)$, $\alpha_*\mathcal{F}$ is given by

$$U \mapsto \varprojlim_{V \subset U,\ V \text{ is closed in } X} \mathcal{F}(V \hookrightarrow S).$$

We have the following diagram:

$$\begin{tikzcd} \mathrm{Sh}(\mathrm{CHaus}/X) \arrow[rr, "\alpha_*"] \arrow[dr, "\Gamma_{\mathrm{cond}}(X,-)"'] & & \mathrm{Sh}(X) \arrow[dl, "\Gamma_{\mathrm{sh}}(X,-)"] \\ & \mathrm{Set} & \end{tikzcd}$$

This is because $\forall\ Y \in \mathrm{Sh}(\mathrm{CHaus}/X)$,

$$\begin{aligned} \Gamma_{\mathrm{sh}}(X, \alpha_* Y) = \alpha_* Y(X) &= \varprojlim_{V \subset U,\ V \text{ is closed in } X} Y(V) \\ &= \varprojlim_V \mathrm{Hom}_{\mathrm{cond}}(V, Y) = \mathrm{Hom}_{\mathrm{cond}}(\varinjlim_V V, Y) \\ &= \mathrm{Hom}_{\mathrm{cond}}(X, Y) = \Gamma_{\mathrm{cond}}(X, Y). \end{aligned}$$

And this diagram can induce a diagram:

$$\begin{tikzcd} D(\mathrm{Ab}(\mathrm{CHaus}/X)) \arrow[rr, "R\alpha_*"] \arrow[dr, "R\Gamma_{\mathrm{cond}}(X,-)"'] & & D(\mathrm{Ab}(X)) \arrow[dl, "R\Gamma_{\mathrm{sh}}(X,-)"] \\ & D(\mathrm{Ab}) & \end{tikzcd}$$

Claim: $R\alpha_*\mathbb{Z} \cong \mathbb{Z}$ in $D(\mathrm{Ab}(X))$.

With this claim, we can show

$$\begin{aligned}
H^i_{\text{cond}}(X,\mathbb{Z}) &= H^i(R\Gamma_{\text{cond}}(X,\mathbb{Z})) \\
&= H^i(R\Gamma_{\text{sh}}(X,-)\circ R\alpha_*\mathbb{Z}) \\
&= H^i(R\Gamma_{\text{sh}}(X,\mathbb{Z})) \\
&= H^i_{\text{sh}}(X,\mathbb{Z}).
\end{aligned}$$

Hence, it suffices to show this claim. We have a map $\mathbb{Z}\to R\alpha_*\mathbb{Z}$ in $D(\text{Ab}(X))$. In order to show this is an isomorphism, it suffices to check on each stacks.
Fix $s\in S$,

$$\begin{aligned}
(R\alpha_*\mathbb{Z})_s &= \varinjlim_{s\in U \text{ open}} R\Gamma(U,R\alpha_*\mathbb{Z}) \\
&= \varinjlim_{s\in U \text{ open}} R\Gamma_{\text{cond}}(U,\mathbb{Z}) \\
&= \varinjlim_{s\in V \text{ closed}} R\Gamma_{\text{cond}}(V,\mathbb{Z}).
\end{aligned}$$

Pick a hypercover $S_\bullet\to S$ with $S_i\in\text{ExDisc}$. Then for each closed V, $(S_n\times_X V)_{n\geq 0}\to V$ is a hypercover. Hence,

$$R\Gamma_{\text{cond}}(V,\mathbb{Z})\cong(0\longrightarrow\Gamma(S_0\times_X V,\mathbb{Z})\longrightarrow\Gamma(S_1\times_X V,\mathbb{Z})\longrightarrow\cdots).$$

Thus, we have

$$\begin{aligned}
(R\alpha_*\mathbb{Z})_s &= \varinjlim_{s\in V \text{ closed}} R\Gamma_{\text{cond}}(V,\mathbb{Z}) \\
&\cong \varinjlim_{s\in V \text{ closed}} (0\longrightarrow\Gamma(S_0\times_X V,\mathbb{Z})\longrightarrow\Gamma(S_1\times_X V,\mathbb{Z})\longrightarrow\cdots) \\
&\cong (0\longrightarrow \varinjlim_{s\in V \text{ closed}}\Gamma(S_0\times_X V,\mathbb{Z})\longrightarrow \varinjlim_{s\in V \text{ closed}}\Gamma(S_1\times_X V,\mathbb{Z})\longrightarrow\cdots) \\
&\cong (0\longrightarrow\Gamma(S_0\times_X \{s\},\mathbb{Z})\longrightarrow\Gamma(S_1\times_X \{s\},\mathbb{Z})\longrightarrow\cdots) \\
&\cong R\Gamma_{\text{cond}}(\{s\},\mathbb{Z}) \\
&\cong \mathbb{Z},
\end{aligned}$$

which finishes our proof. $\square$

Example 8.4. Let $\mathbb{T}=\mathbb{R}/\mathbb{Z}$, for $\mathbb{T}^I\in\text{CHaus}$, we have $H^n(\mathbb{T}^I,\mathbb{Z})=\wedge^n(\mathbb{Z}^{\oplus I})$.

Proof. First, we have

$$H^n(\mathbb{T}, \mathbb{Z}) = \begin{cases} \mathbb{Z} & n = 0, 1 \\ 0 & \text{else} \end{cases}$$

i.e. $H^*(\mathbb{T}, \mathbb{Z}) = \wedge(\mathbb{Z})$.

Claim: $H^*(\mathbb{T}^n, \mathbb{Z}) = \wedge(\mathbb{Z}^{\oplus n})$.

We can prove it by induction on n. $n = 1$ is proved above.

By Kunneth theorem, we can show that for $H^*(X, \mathbb{Z})$ finitely generated free in each degree, we have $H^*(X \times Y, \mathbb{Z}) \cong H^*(X, \mathbb{Z}) \otimes H^*(Y, \mathbb{Z})$. Hence, we have

$$\begin{aligned} H^*(\mathbb{T}^n, \mathbb{Z}) &= H^*(\mathbb{T}^{n-1}, \mathbb{Z}) \otimes H^*(\mathbb{T}, \mathbb{Z}) \\ &= \wedge(\mathbb{Z}^{\oplus(n-1)}) \otimes \wedge(\mathbb{Z}) \\ &= \wedge(\mathbb{Z}^{\oplus n}). \end{aligned}$$

In order to prove the general case, there is a fact that for $S \in \text{CHaus}$, $S = \varprojlim_j S_j$, then $H^n(S, \mathbb{Z}) = \varinjlim_j H^n(S_j, \mathbb{Z})$.

Hence,

$$\begin{aligned} H^n(\mathbb{T}^I, \mathbb{Z}) &= H^n(\varprojlim_{J \subset I \text{ finite}} \mathbb{T}^J, \mathbb{Z}) \\ &= \varinjlim_{J \subset I \text{ finite}} H^n(\mathbb{T}^J, \mathbb{Z}) \\ &= \varinjlim_{J \subset I \text{ finite}} \wedge^n (\mathbb{Z}^{\oplus J}) \\ &= \wedge^n(\mathbb{Z}^{\oplus I}). \end{aligned}$$

□

9 Locally compact abelian groups

Notation. Let TopAb be the category of all Hausdorff topological abelian groups and LCAb be the category of all locally compact abelian groups.

Proposition 9.1. Let $A, B \in$ TopAb and assume that $A \in$ CGTop. Then there is a natural isomorphism of condensed abelian groups

$$\underline{\mathrm{Hom}}(\underline{A}, \underline{B}) \cong \underline{\mathrm{Hom}(A, B)}.$$

Theorem 9.2 (Eilenberg-Maclane,Breen,Deligne resolution)**.** For any abelian group A, there is a functorial resolution

$$\cdots \longrightarrow \bigoplus_{j=1}^{n_i} \mathbb{Z}[A^{r_{i,j}}] \longrightarrow \cdots \longrightarrow \mathbb{Z}[A^3] \oplus \mathbb{Z}[A^2] \longrightarrow \mathbb{Z}[A^2] \longrightarrow \mathbb{Z}[A] \longrightarrow A \to 0.$$

Remark 9.3. Such functorial ensures that it works for abelian group objects in any topos.

Lemma 9.4. Let $A^{\bullet,\bullet}$ be a double complex and $A^{\bullet} = \mathrm{Tot}(A^{\bullet,\bullet})$ be its total complex, then there is a spectral sequence

$$E_1^{p,q} = H^q(A^{\bullet,p}) \Longrightarrow H^{p+q}(A^{\bullet}).$$

Lemma 9.5. For a comlpex of abelian groups $M^{\bullet} \in D(\mathbb{Z})$, let

$$0 \longrightarrow M^{\bullet} \longrightarrow A^{\bullet,1} \longrightarrow A^{\bullet,2} \longrightarrow A^{\bullet,3} \longrightarrow \cdots$$

be an exact sequence in $D(\mathbb{Z})$, then for the double complex $A^{\bullet,\bullet}$, there is a quasi-isomorphism

$$M^{\bullet} \xrightarrow{\sim} \mathrm{Tot}(A^{\bullet,\bullet}).$$

Corollary 9.6. For any condensed abelian groups A, M and an extremally disconnected space S, there is a spectral sequence

$$E_1^{p,q} = \prod_{j=1}^{n_p} H^q(A^{r_{p,j}} \times S, M) \Longrightarrow \underline{\mathrm{Ext}}^{p+q}(A, M)(S),$$

that is functorial in A, M and S.

Proof. For $A \in \mathrm{Cond(Ab)}$, consider its EMBD resolution

$$\cdots \longrightarrow \bigoplus_{j=1}^{n_i} \mathbb{Z}[A^{r_{i,j}}] \longrightarrow \cdots \longrightarrow \mathbb{Z}[A^3] \oplus \mathbb{Z}[A^2] \longrightarrow \mathbb{Z}[A^2] \longrightarrow \mathbb{Z}[A] \longrightarrow A \to 0,$$

then apply $- \otimes \mathbb{Z}[S]$, which is an exact functor since $\mathbb{Z}[S]$ is flat, we get the resolution of $A \otimes \mathbb{Z}[S]$

$$\cdots \longrightarrow \bigoplus_{j=1}^{n_i} \mathbb{Z}[A^{r_{i,j}} \times S] \cdots \longrightarrow \mathbb{Z}[A^3 \times S] \oplus \mathbb{Z}[A^2 \times S] \longrightarrow \mathbb{Z}[A^2 \times S] \longrightarrow \mathbb{Z}[A \times S] \longrightarrow A \otimes \mathbb{Z}[S] \to 0,$$

then apply $R\mathrm{Hom}(-, M)$, we get

$$0 \longrightarrow R\mathrm{Hom}(A \otimes \mathbb{Z}[S], M) \longrightarrow R\mathrm{Hom}(\mathbb{Z}[A \times S], M) \longrightarrow R\mathrm{Hom}(\mathbb{Z}[A^2 \times S], M) \longrightarrow \cdots,$$

i.e.

$$0 \longrightarrow R\underline{\mathrm{Hom}}(A, M)(S) \longrightarrow R\Gamma(A \times S, M) \longrightarrow R\Gamma(A^2 \times S, M) \longrightarrow \cdots,$$

which is an exact sequence in $D(\mathbb{Z})$. By lemma 9.4 and lemma 9.5, we have

$$E_1^{p,q} = H^q(\bigoplus_{j=1}^{n_p} R\Gamma(A^{r_{p,j}} \times S, M)) \Longrightarrow H^{p+q}(\mathrm{Tot}(\bigoplus_{j=1}^{n_\bullet} R\Gamma(A^{r_{\bullet,j}} \times S, M)))$$

and

$$R\underline{\mathrm{Hom}}(A, M)(S) \simeq \mathrm{Tot}(\bigoplus_{j=1}^{n_\bullet} R\Gamma(A^{r_{\bullet,j}} \times S, M)),$$

hence

$$E_1^{p,q} = \prod_{j=1}^{n_p} H^q(A^{r_{p,j}} \times S, M) \Longrightarrow \underline{\mathrm{Ext}}^{p+q}(A, M)(S).$$

□

Lemma 9.7. In the category of abelian groups, if the following diagram is exact for each arrow

$$\begin{tikzcd}
0 \arrow[r] & M^\bullet \arrow[r] \arrow[d] & A^{\bullet,1} \arrow[r] \arrow[d] & A^{\bullet,2} \arrow[r] \arrow[d] & \cdots, \\
0 \arrow[r] & N^\bullet \arrow[r] & B^{\bullet,1} \arrow[r] & B^{\bullet,2} \arrow[r] & \cdots
\end{tikzcd}$$

and if for any $j \geq 1$, we have $A^{\bullet,j} \cong B^{\bullet,j}$, then $\mathrm{Tot}(A^{\bullet,\bullet}) \cong \mathrm{Tot}(B^{\bullet,\bullet})$. Furthermore, by $M^\bullet \cong \mathrm{Tot}(A^{\bullet,\bullet})$ and $N^\bullet \cong \mathrm{Tot}(B^{\bullet,\bullet})$, we can get $M^\bullet \cong N^\bullet$.

Theorem 9.8. Assume I is any set, denote the compact condensed abelian group $\prod_I \mathbb{T}$ by $\mathbb{T}^I$.

(i) For any discrete abelian group M, we have

$$R\underline{\mathrm{Hom}}(\mathbb{T}^I, M) = M^{\oplus I}[-1],$$

where $M^{\oplus I}[-1] \to R\underline{\mathrm{Hom}}(\mathbb{T}^I, M)$ is induced by

$$M[-1] = R\underline{\mathrm{Hom}}(\mathbb{Z}[1], M) \longrightarrow R\underline{\mathrm{Hom}}(\mathbb{T}, M) \xrightarrow{p_i^*} R\underline{\mathrm{Hom}}(\mathbb{T}^I, M),$$

where $p_i : \mathbb{T}^I \longrightarrow \mathbb{T}$ is the projection to the i-th factor, $i \in I$.

(ii) $R\underline{\mathrm{Hom}}(\mathbb{T}^I, \mathbb{R}) = 0$.

Proof.

(i) We first prove the case I is a one element set, i.e.

$$R\underline{\mathrm{Hom}}(\mathbb{T}, M) = M[-1].$$

From the exact sequence $0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{T} \to 0$, we have $\mathbb{R} \to \mathbb{T} \to \mathbb{Z}[1]$, hence

$$M[-1] = R\underline{\mathrm{Hom}}(\mathbb{Z}[1], M) \longrightarrow R\underline{\mathrm{Hom}}(\mathbb{T}, M) \longrightarrow R\underline{\mathrm{Hom}}(\mathbb{R}, M).$$

In order to show $R\underline{\mathrm{Hom}}(\mathbb{T}, M) = M[-1]$, it suffices to show $R\underline{\mathrm{Hom}}(\mathbb{R}, M) = 0$.

Claim: $R\underline{\mathrm{Hom}}(\mathbb{R}, M) = 0$.

For 0 and $\mathbb{R}$, we take its EMBD resolution:

$$\begin{tikzcd}
\cdots \arrow[r] & \bigoplus_{j=1}^{n_i} \mathbb{Z}[\mathbb{R}^{r_{i,j}}] \arrow[r] \arrow[d] & \cdots \arrow[r] & \mathbb{Z}[\mathbb{R}] \arrow[r] \arrow[d] & \mathbb{R} \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[r] & \bigoplus_{j=1}^{n_i} \mathbb{Z}[0^{r_{i,j}}] \arrow[r] & \cdots \arrow[r] & \mathbb{Z}[0] \arrow[r] & 0 \arrow[r] & 0,
\end{tikzcd}$$

apply $R\underline{\mathrm{Hom}}(-, M)(S)$, we get

$$\begin{tikzcd}
0 \arrow[r] & R\underline{\mathrm{Hom}}(0, M)(S) \arrow[r] \arrow[d] & R\underline{\mathrm{Hom}}(\mathbb{Z}[0], M)(S) \arrow[r] \arrow[d] & \cdots \arrow[r] & R\underline{\mathrm{Hom}}(\bigoplus_{j=1}^{n_i} \mathbb{Z}[0^{r_{i,j}}], M)(S) \cdots \arrow[d] \\
0 \arrow[r] & R\underline{\mathrm{Hom}}(\mathbb{R}, M)(S) \arrow[r] & R\underline{\mathrm{Hom}}(\mathbb{Z}[\mathbb{R}], M)(S) \arrow[r] & \cdots \arrow[r] & R\underline{\mathrm{Hom}}(\bigoplus_{j=1}^{n_i} \mathbb{Z}[\mathbb{R}^{r_{i,j}}], M)(S) \cdots ,
\end{tikzcd}$$

i.e.

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d] & R\Gamma(S, M) \arrow[r] \arrow[d] & \cdots \arrow[r] & \bigoplus_{j=1}^{n_i} R\Gamma(S, M) \cdots \arrow[d] \\
0 \arrow[r] & R\underline{\mathrm{Hom}}(\mathbb{R}, M)(S) \arrow[r] & R\Gamma(\mathbb{R} \times S, M) \arrow[r] & \cdots \arrow[r] & \bigoplus_{j=1}^{n_i} R\Gamma(\mathbb{R}^{r_{i,j}} \times S, M) \cdots ,
\end{tikzcd}$$

Then by lemma 9.7, in order to show $R\underline{\mathrm{Hom}}(\mathbb{R}, M) = 0$, it suffices to show

$$R\Gamma(S, M) = R\Gamma(S \times \mathbb{R}^r, M).$$

We know $S \times \mathbb{R}^r = \varinjlim S \times [-N, N]^r$, then

$$\begin{aligned} R\Gamma(S \times \mathbb{R}^r, M) &= R\Gamma(\varinjlim S \times [-N, N]^r, M) \\ &= \varprojlim R\Gamma(S \times [-N, N]^r, M) \\ &= \varprojlim R\Gamma(S, M) \\ &= R\Gamma(S, M). \end{aligned}$$

Here, $\varprojlim R\Gamma(S \times [-N, N]^r, M) = \varprojlim R\Gamma(S, M)$ comes from the fact that for constant sheaf, its sheaf cohomology is homotopy-invariant.
Secondly, assume I is a finite set, then

$$R\underline{\mathrm{Hom}}(\mathbb{T}^I, M) = R\underline{\mathrm{Hom}}(\mathbb{T}^{\oplus I}, M) = \prod_I R\underline{\mathrm{Hom}}(\mathbb{T}, M) = \prod_I M[-1] = M^{\oplus I}[-1].$$

Finally, assume I is any set. Then we can write $\mathbb{T}^I$ as

$$\mathbb{T}^I = \varprojlim_{J \subset I, J \text{ finite}} \mathbb{T}^J.$$

For any finite set J, we have

$$\begin{tikzcd} 0 \arrow[r] & R\underline{\mathrm{Hom}}(\mathbb{T}^J, M)(S) \arrow[r] \arrow[d] & R\Gamma(\mathbb{T}^J \times S, M) \arrow[r] \arrow[d] & \cdots \arrow[r] & \bigoplus_{j=1}^{n_i} R\Gamma((\mathbb{T}^J)^{r_{i,j}} \times S, M) \cdots \arrow[d] \\ 0 \arrow[r] & R\underline{\mathrm{Hom}}(\mathbb{T}^I, M)(S) \arrow[r] & R\Gamma(\mathbb{T}^I \times S, M) \arrow[r] & \cdots \arrow[r] & \bigoplus_{j=1}^{n_i} R\Gamma((\mathbb{T}^I)^{r_{i,j}} \times S, M) \cdots , \end{tikzcd}$$

apply the exact functor $\varinjlim_{J \subset I}$ to the first arrow, we get

$$\begin{tikzcd} 0 \arrow[r] & \varinjlim_{J \subset I} R\underline{\mathrm{Hom}}(\mathbb{T}^J, M)(S) \arrow[r] \arrow[d] & \varinjlim_{J \subset I} R\Gamma(\mathbb{T}^J \times S, M) \arrow[r] \arrow[d] & \cdots \arrow[r] & \bigoplus_{j=1}^{n_i} \varinjlim_{J \subset I} R\Gamma((\mathbb{T}^J)^{r_{i,j}} \times S, M) \cdots \arrow[d] \\ 0 \arrow[r] & R\underline{\mathrm{Hom}}(\mathbb{T}^I, M)(S) \arrow[r] & R\Gamma(\mathbb{T}^I \times S, M) \arrow[r] & \cdots \arrow[r] & \bigoplus_{j=1}^{n_i} R\Gamma((\mathbb{T}^I)^{r_{i,j}} \times S, M) \cdots , \end{tikzcd}$$

In order to show

$$\varinjlim_{J \subset I} R\underline{\mathrm{Hom}}(\mathbb{T}^J, M)(S) \cong R\underline{\mathrm{Hom}}(\mathbb{T}^I, M)(S),$$

it suffices to show

$$\varinjlim_{J\subset I} R\Gamma((\mathbb{T}^J)^{r_{i,j}} \times S, M) \cong R\Gamma((\mathbb{T}^I)^{r_{i,j}} \times S, M).$$

This is true, because $\varprojlim_{J\subset I} (\mathbb{T}^J)^{r_{i,j}} \times S \cong (\mathbb{T}^I)^{r_{i,j}} \times S$. Therefore,

$$\begin{aligned} R\underline{\mathrm{Hom}}(\mathbb{T}^I, M) &\cong \varinjlim_{J\subset I} R\underline{\mathrm{Hom}}(\mathbb{T}^J, M) \\ &\cong \varinjlim_{J\subset I} M^{\oplus J}[-1] \\ &\cong M^{\oplus I}[-1]. \end{aligned}$$

□

Corollary 9.9. $R\underline{\mathrm{Hom}}(\mathbb{R}, \mathbb{R}) \cong \mathbb{R}$.

Proof. From the exact sequence $0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{T} \to 0$, we have

$$R\underline{\mathrm{Hom}}(\mathbb{T}, \mathbb{R}) \to R\underline{\mathrm{Hom}}(\mathbb{R}, \mathbb{R}) \to R\underline{\mathrm{Hom}}(\mathbb{Z}, \mathbb{R}).$$

By Theorem 9.8, we know $R\underline{\mathrm{Hom}}(\mathbb{T}, \mathbb{R}) = 0$, hence $R\underline{\mathrm{Hom}}(\mathbb{R}, \mathbb{R}) \cong R\underline{\mathrm{Hom}}(\mathbb{Z}, \mathbb{R}) \cong \mathbb{R}$. □

Corollary 9.10. For any locally compact abelian groups A and B, $R\underline{\mathrm{Hom}}(A, B)$ is centered at 0 and 1, i.e. $\underline{\mathrm{Ext}}^i(A, B) = 0,\ \forall i \geq 2$.

Proof. By the structure theorem of locally compact abelian groups, it suffices to prove for A and B being compact groups and discrete groups.

(i) A is a discrete group.
Claim: There is an exact sequence: $0 \to \oplus_I \mathbb{Z} \to \oplus_J \mathbb{Z} \to A \to 0$.
This is because we can construct a surjective homomorphism $\oplus_A \mathbb{Z} \to A$, and take its kernel, and we know the submodule of a free $\mathbb{Z}$-module is free, hence $\ker(\oplus_A \mathbb{Z} \to A) = \oplus_I \mathbb{Z}$, for some I. Thereby, $0 \to \oplus_I \mathbb{Z} \to \oplus_A \mathbb{Z} \to A \to 0$ is exact.
By the short exact sequence $0 \to \oplus_I \mathbb{Z} \to \oplus_J \mathbb{Z} \to A \to 0$, we can get a long exact sequence:

$$\begin{aligned} 0 &\longrightarrow \underline{\mathrm{Hom}}(A, B) \longrightarrow \underline{\mathrm{Hom}}(\oplus_J \mathbb{Z}, B) \longrightarrow \underline{\mathrm{Hom}}(\oplus_I \mathbb{Z}, B) \\ &\longrightarrow \underline{\mathrm{Ext}}^1(A, B) \longrightarrow \underline{\mathrm{Ext}}^1(\oplus_J \mathbb{Z}, B) \longrightarrow \underline{\mathrm{Ext}}^1(\oplus_I \mathbb{Z}, B) \\ &\longrightarrow \underline{\mathrm{Ext}}^2(A, B) \longrightarrow \cdots . \end{aligned}$$

Because $\oplus_I \mathbb{Z} \in \mathrm{Cond}(\mathrm{Ab})$ is projective, we have $\underline{\mathrm{Ext}}^i(\oplus_I \mathbb{Z}, B) = 0,\ \forall i \geq 1$. Hence $\underline{\mathrm{Ext}}^i(A, B) = 0,\ \forall i \geq 2$.

(ii) A is a compact group.

By Pontrgagin duality, there is a short exact sequence

$$0 \to A \to \mathbb{T}^I \to \mathbb{T}^J \to 0,$$

and it can induce a long exact sequence

$$\begin{aligned} 0 &\longrightarrow \underline{\mathrm{Hom}}(\mathbb{T}^J, B) \longrightarrow \underline{\mathrm{Hom}}(\mathbb{T}^I, B) \longrightarrow \underline{\mathrm{Hom}}(A, B) \\ &\longrightarrow \underline{\mathrm{Ext}}^1(\mathbb{T}^J, B) \longrightarrow \underline{\mathrm{Ext}}^1(\mathbb{T}^I, B) \longrightarrow \underline{\mathrm{Ext}}^1(A, B) \\ &\longrightarrow \underline{\mathrm{Ext}}^2(\mathbb{T}^J, B) \longrightarrow \underline{\mathrm{Ext}}^2(\mathbb{T}^I, B) \longrightarrow \underline{\mathrm{Ext}}^2(A, B) \\ &\longrightarrow \cdots \end{aligned}$$

In order to show $\underline{\mathrm{Ext}}^i(A, B) = 0,\ \forall i \geq 2$, it suffices to show

$$\underline{\mathrm{Ext}}^i(\mathbb{T}^I, B) = 0,\ \forall i \geq 2,\ \forall I.$$

(a) B is a discrete group.

In this case, we have $R\underline{\mathrm{Hom}}(\mathbb{T}^I, B) = B^{\oplus I}[-1]$, which is centered at 1, hence $\underline{\mathrm{Ext}}^i(\mathbb{T}^I, B) = 0,\ \forall i \geq 2,\ \forall I$.

(b) B is a compact group.

In this case, we have a short exact sequence $0 \to B \to \mathbb{T}^{I'} \to \mathbb{T}^{J'} \to 0$, and it induces a long exact sequence:

$$\begin{aligned} 0 &\longrightarrow \underline{\mathrm{Hom}}(\mathbb{T}^I, B) \longrightarrow \underline{\mathrm{Hom}}(\mathbb{T}^I, \mathbb{T}^{I'}) \longrightarrow \underline{\mathrm{Hom}}(\mathbb{T}^I, \mathbb{T}^{J'}) \\ &\longrightarrow \underline{\mathrm{Ext}}^1(\mathbb{T}^I, B) \longrightarrow \underline{\mathrm{Ext}}^1(\mathbb{T}^I, \mathbb{T}^{I'}) \longrightarrow \underline{\mathrm{Ext}}^1(\mathbb{T}^I, \mathbb{T}^{J'}) \\ &\longrightarrow \underline{\mathrm{Ext}}^2(\mathbb{T}^I, B) \longrightarrow \cdots . \end{aligned}$$

Now, we compute $\underline{\mathrm{Ext}}^i(\mathbb{T}^I, \mathbb{T})$. For the short exact sequence $0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{T} \to 0$, we have a long exact sequence:

$$\begin{aligned} 0 &\longrightarrow \underline{\mathrm{Hom}}(\mathbb{T}^I, \mathbb{Z}) \longrightarrow \underline{\mathrm{Hom}}(\mathbb{T}^I, \mathbb{R}) \longrightarrow \underline{\mathrm{Hom}}(\mathbb{T}^I, \mathbb{T}) \\ &\longrightarrow \underline{\mathrm{Ext}}^1(\mathbb{T}^I, \mathbb{Z}) \longrightarrow \underline{\mathrm{Ext}}^1(\mathbb{T}^I, \mathbb{R}) \longrightarrow \underline{\mathrm{Ext}}^1(\mathbb{T}^I, \mathbb{T}) \\ &\longrightarrow \underline{\mathrm{Ext}}^2(\mathbb{T}^I, \mathbb{Z}) \longrightarrow \cdots . \end{aligned}$$

Since $R\underline{\mathrm{Hom}}(\mathbb{T}^I, \mathbb{R}) = 0$ and $R\underline{\mathrm{Hom}}(\mathbb{T}^I, \mathbb{Z}) = \mathbb{Z}^{\oplus I}[-1]$, we have $\underline{\mathrm{Ext}}^i(\mathbb{T}^I, \mathbb{T}) = 0,\ \forall i \geq 1$, hence $\underline{\mathrm{Ext}}^i(\mathbb{T}^I, \mathbb{T}^J) = 0,\ \forall i \geq 1,\ \forall J$. Thus $\underline{\mathrm{Ext}}^i(\mathbb{T}^I, B) = 0,\ \forall i \geq 2$.

□

10 Solid Abelian Groups

Definition 10.1. For $S \in \text{ProFin}$, write $S = \varprojlim S_i$, where $S_i \in \text{Fin}$, we define the solid free abelian group

$$\mathbb{Z}[S]^{\blacksquare} := \varprojlim \mathbb{Z}[S_i].$$

We call $\mathbb{Z}[S]^{\blacksquare}$ the solidification of $\mathbb{Z}[S]$.

Remark 10.2.
$\mathbb{Z}[S]^{\blacksquare} = \varprojlim \mathbb{Z}[S_i] = \varprojlim \underline{\text{Hom}}(C(S_i, \mathbb{Z}), \mathbb{Z}) = \underline{\text{Hom}}(\varinjlim C(S_i, \mathbb{Z}), \mathbb{Z}) = \underline{\text{Hom}}(C(S, \mathbb{Z}), \mathbb{Z})$.

Proposition 10.3. For $S \in \text{ProFin}$, there exists some set I, s.t. $C(S, \mathbb{Z}) \cong \mathbb{Z}^{\oplus I}$, i.e. $C(S, \mathbb{Z})$ is a free abelian group.

Remark 10.4. (i) From the above proposition, we have

$$\mathbb{Z}[S]^{\blacksquare} = \underline{\text{Hom}}(C(S, \mathbb{Z}), \mathbb{Z}) = \underline{\text{Hom}}(\mathbb{Z}^{\oplus I}, \mathbb{Z}) = \mathbb{Z}^{I}.$$

(ii)

Definition 10.5. A condensed abelian group $X \in \text{Cond(Ab)}$ is solid, if for any $S \in \text{ProFin}$, one has

$$\text{Hom}(\mathbb{Z}[S], X) \cong \text{Hom}(\mathbb{Z}[S]^{\blacksquare}, X).$$

A complex of condensed abelian groups $C \in D(\text{Cond(Ab)})$ is solid, if for any $S \in \text{ProFin}$, one has

$$R\text{Hom}(\mathbb{Z}[S], C) \cong R\text{Hom}(\mathbb{Z}[S]^{\blacksquare}, C).$$

Now, we need to check $\mathbb{Z}[S]^{\blacksquare}$ is indeed a solid condensed abelian group.

Proposition 10.6. For $S, T \in \text{ProFin}$, we have

$$R\text{Hom}(\mathbb{Z}[S], \mathbb{Z}[T]^{\blacksquare}) \cong R\text{Hom}(\mathbb{Z}[S]^{\blacksquare}, \mathbb{Z}[T]^{\blacksquare}).$$

Proof. Assume $\mathbb{Z}[S]^{\blacksquare} = \mathbb{Z}^I$ and $\mathbb{Z}[T]^{\blacksquare} = \mathbb{Z}^J$ for some sets I and J. Since the functors $R\text{Hom}(\mathbb{Z}[S], -)$ and $R\text{Hom}(\mathbb{Z}[S]^{\blacksquare}, -)$ commute with products, it suffices to show

$$R\text{Hom}(\mathbb{Z}[S], \mathbb{Z}) \cong R\text{Hom}(\mathbb{Z}[S]^{\blacksquare}, \mathbb{Z})$$

The left hand side is $R\text{Hom}(\mathbb{Z}[S], \mathbb{Z}) \cong R\Gamma(S, \mathbb{Z}) = C(S, \mathbb{Z}) = \mathbb{Z}^{\oplus I}$.
Now, consider the short exact sequence $0 \to \mathbb{R}^I \to \mathbb{Z}^I \to \mathbb{T}^I \to 0$. From theorem 9.8, We know

$$R\text{Hom}(\mathbb{T}^I, \mathbb{Z}) = \mathbb{Z}^{\oplus I}[-1].$$

And by the adjoint relation, we have

$$R\mathrm{Hom}(\mathbb{R}^I, \mathbb{Z}) \cong R\mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^I, R\underline{\mathrm{Hom}}(\mathbb{R}, \mathbb{Z})) = 0.$$

Hence, $R\mathrm{Hom}(\mathbb{Z}[S]^{\blacksquare}, \mathbb{Z}) \cong R\mathrm{Hom}(\mathbb{Z}^I, \mathbb{Z}) \cong \mathbb{Z}^{\oplus I}$. And this finishes our proof. □

Lemma 10.7. Let $\mathcal{A}$ be a cocomplete abelian category, and $\mathcal{A}_0 \subseteq \mathcal{A}$ be the full subcategory of compact projective generators. Assume $F : \mathcal{A}_0 \to \mathcal{A}$ is an additive functor with a natural transformation $\mathrm{id}_{\mathcal{A}_0} \implies F$, satisfying the following property:

For any $X \in \mathcal{A}_0$, any $Y, Z \in \mathcal{A}$ which can be written as direct sums of objects in the image of F, i.e. $Y = \bigoplus_{i\in I} F(X_i)$ and $Z = \bigoplus_{j\in J} F(X_j)$, and for any map $f : Y \to Z$ with kernel $K \in \mathcal{A}$, the map

$$R\mathrm{Hom}(F(X), K) \to R\mathrm{Hom}(X, K)$$

is an isomorphism.

Let

$$\mathcal{A}_F = \{Y \in \mathcal{A} \mid \mathrm{Hom}(F(X), Y) \cong \mathrm{Hom}(X, Y), \forall X \in \mathcal{A}_0\} \subseteq \mathcal{A}$$

and

$$D_F(\mathcal{A}) = \{C \in D(\mathcal{A}) \mid R\mathrm{Hom}(F(X), C) \cong R\mathrm{Hom}(X, C), \forall X \in \mathcal{A}_0\} \subseteq D(\mathcal{A})$$

Then:

(i)
- $\mathcal{A}_F \subseteq \mathcal{A}$ is an abelian subcategory stable under limits, colimits and extensions.
- The objects $F(X), X \in \mathcal{A}_0$ are compact projective generators.
- The inclusion $\mathcal{A}_F \hookrightarrow \mathcal{A}$ admits a left adjoint $L : \mathcal{A} \to \mathcal{A}_F$, which is the unique colimit-preserving extension of $F : \mathcal{A}_0 \to \mathcal{A}_F$.

(ii)
- The functor $D(\mathcal{A}_F) \to D(\mathcal{A})$ is fully faithful and $D(\mathcal{A}_F) \cong D_F(\mathcal{A})$.
- $C \in D(\mathcal{A})$ lies in $D_F(\mathcal{A})$ iff $H^i(C) \in \mathcal{A}_F$.
- The above functor F has a left derived functor, which is the left adjoint of $D_F(\mathcal{A}) \hookrightarrow D(\mathcal{A})$.

Lemma 10.8. We take the above lemma's notation.

(i) For any C with the form $\bigoplus_{i\in I} F(X_i), X_i \in \mathcal{A}_0$, one has

$$R\mathrm{Hom}(F(X), C) \cong R\mathrm{Hom}(X, C),\ \forall X \in \mathcal{A}_0.$$

(ii) For any C with the form $\ker(\bigoplus_{i\in I} F(X_i) \to \bigoplus_{j\in J} F(Y_j))$, $X_i, Y_j \in \mathcal{A}_0$, one has

$$R\mathrm{Hom}(F(X), C) \cong R\mathrm{Hom}(X, C), \ \forall X \in \mathcal{A}_0.$$

(iii) For any C with the form $\mathrm{coker}(\bigoplus_{i\in I} F(X_i) \to \bigoplus_{j\in J} F(Y_j))$, $X_i, Y_j \in \mathcal{A}_0$, one has

$$R\mathrm{Hom}(F(X), C) \cong R\mathrm{Hom}(X, C), \ \forall X \in \mathcal{A}_0.$$

(iv) For any right bounded complex C with each term C_i having the form $\bigoplus_{j\in I_i} F(X_{i_j})$, one has

$$R\mathrm{Hom}(F(X), C) \cong R\mathrm{Hom}(X, C), \ \forall X \in \mathcal{A}_0.$$

Then (iv) $\implies$ (iii) $\iff$ (ii) $\implies$ (i).

Proof. (ii) $\implies$ (i). Just take $J = \emptyset$, which is exactly (i).
(ii) $\iff$ (iii). For any $f : Y \to Z$, with $Y = \bigoplus_{i\in I} F(X_i)$ and $Z = \bigoplus_{j\in J} F(Y_j)$, applying functors $R\mathrm{Hom}(X, -)$ and $R\mathrm{Hom}(F(X), -)$ to the exact sequence:

$$0 \to \ker(f) \to Y \to Z \to \mathrm{coker}(f) \to 0,$$

one get

$$\begin{tikzcd} R\mathrm{Hom}(F(X), \ker(f)) \arrow[r] \arrow[d] & R\mathrm{Hom}(F(X), Y) \arrow[r] \arrow[d, "\cong"] & R\mathrm{Hom}(F(X), Z) \arrow[r] \arrow[d, "\cong"] & R\mathrm{Hom}(F(X), \mathrm{coker}(f)) \arrow[d] \\ R\mathrm{Hom}(X, \ker(f)) \arrow[r] & R\mathrm{Hom}(X, Y) \arrow[r] & R\mathrm{Hom}(X, Z) \arrow[r] & R\mathrm{Hom}(X, \mathrm{coker}(f)) \end{tikzcd}$$

By five lemma, we can show

$$R\mathrm{Hom}(F(X), \ker(f)) \cong R\mathrm{Hom}(X, \ker(f))$$

$\iff$

$$R\mathrm{Hom}(F(X), \mathrm{coker}(f)) \cong R\mathrm{Hom}(X, \mathrm{coker}(f)).$$

Hence, (ii) $\iff$ (iii).
(iv) $\implies$ (ii). For any $f : Y \to Z$, with $Y = \bigoplus_{i\in I} F(X_i)$ and $Z = \bigoplus_{j\in J} F(Y_j)$. Denote $K = \ker(f)$. Take the resolution of K:

$$\cdots \to B_1 \to B_0 \to K \to 0,$$

where each $B_i \in \mathcal{A}_0$. Now, take $C = [0 \to Y \to Z \to 0]$, by assumption, we have

$$R\mathrm{Hom}(F(B_\bullet), C) \cong R\mathrm{Hom}(B_\bullet, C).$$

Hence,

$$\begin{tikzcd} B_\bullet \arrow[r] \arrow[d] & F(B_\bullet) \arrow[dl, dotted, "\exists!"] \\ K & \end{tikzcd}$$

That is, $K \cong B_\bullet$ is the retract of $F(B_\bullet)$. Thus,

$$R\mathrm{Hom}(X, K) \cong R\mathrm{Hom}(X, F(B_\bullet)) \cong R\mathrm{Hom}(F(X), F(B_\bullet)) \cong R\mathrm{Hom}(F(X), K).$$

$\square$

Theorem 10.9. (i) - The category Solid $\subset$ Cond(Ab) of solid abelian groups is an abelian subcategory stable under limits, colimits and extensions.

(ii)

Definition 10.10. (i) For $M, N \in$ Solid, define $M \otimes^{\blacksquare} N := (M \otimes N)^{\blacksquare}$.

(ii) For $C, D \in D(\text{Solid})$, define $C \otimes^{L\blacksquare} D := (C \otimes^L D)^{L\blacksquare}$.

Theorem 10.11. (i) The solidification functor Cond(Ab) $\to$ Solid; $M \mapsto M^{\blacksquare}$ is symmetric monoidal, i.e.

$$(M \otimes N)^{\blacksquare} \cong M^{\blacksquare} \otimes^{\blacksquare} N^{\blacksquare}.$$

(ii) The solidification functor $D(\text{Cond(Ab)}) \to D(\text{Solid})$; $C \mapsto C^{L\blacksquare}$ is symmetric monoidal, i.e.

$$(C \otimes^L D)^{L\blacksquare} \cong C^{L\blacksquare} \otimes^{L\blacksquare} D^{L\blacksquare}.$$

(iii) $\otimes^{L\blacksquare}$ is the left derived functor of $\otimes^{\blacksquare}$.

Proof. (i) By definition, we need to show:

$$(M \otimes N)^{\blacksquare} \xrightarrow{\sim} (M^{\blacksquare} \otimes N^{\blacksquare})^{\blacksquare}.$$

This can be written as the composition:

$$(M \otimes N)^{\blacksquare} \longrightarrow (M^{\blacksquare} \otimes N)^{\blacksquare} \longrightarrow (M^{\blacksquare} \otimes N^{\blacksquare})^{\blacksquare}.$$

Hence, it is enough to prove

$$(M \otimes N)^{\blacksquare} \xrightarrow{\sim} (M^{\blacksquare} \otimes N)^{\blacksquare}.$$

(With this isomorphism, we can also show that the second map is an isomorphism). Since the tensor functor and the solidification functor commute with colimits, then we can assume $M = \mathbb{Z}[S]$ and $N = \mathbb{Z}[T]$.
It reduces to show:

$$\mathbb{Z}[S \times T]^{\blacksquare} \xrightarrow{\sim} (\mathbb{Z}[S]^{\blacksquare} \otimes \mathbb{Z}[T])^{\blacksquare}.$$

Equivalently, for any $A \in \text{Solid}$,

$$\underline{\text{Hom}}((\mathbb{Z}[S]^{\blacksquare} \otimes \mathbb{Z}[T])^{\blacksquare}, A) \cong \underline{\text{Hom}}(\mathbb{Z}[S \times T]^{\blacksquare}, A).$$

Since A is solid, we have:

$$\underline{\text{Hom}}((\mathbb{Z}[S]^{\blacksquare} \otimes \mathbb{Z}[T])^{\blacksquare}, A) \cong \underline{\text{Hom}}(\mathbb{Z}[S]^{\blacksquare} \otimes \mathbb{Z}[T], A)$$

and

$$\underline{\text{Hom}}(\mathbb{Z}[S \times T]^{\blacksquare}, A) \cong \underline{\text{Hom}}(\mathbb{Z}[S \times T], A).$$

By computation:

$$\begin{aligned}
\underline{\text{Hom}}(\mathbb{Z}[S]^{\blacksquare} \otimes \mathbb{Z}[T], A) &\cong \underline{\text{Hom}}(\mathbb{Z}[S]^{\blacksquare}, \underline{\text{Hom}}(\mathbb{Z}[T], A)) \\
&\cong \underline{\text{Hom}}(\mathbb{Z}[S], \underline{\text{Hom}}(\mathbb{Z}[T], A)) \\
&\cong \underline{\text{Hom}}(\mathbb{Z}[S] \otimes \mathbb{Z}[T], A) \\
&\cong \underline{\text{Hom}}(\mathbb{Z}[S \times T], A).
\end{aligned}$$

Thus, $\underline{\text{Hom}}((\mathbb{Z}[S]^{\blacksquare} \otimes \mathbb{Z}[T])^{\blacksquare}, A) \cong \underline{\text{Hom}}(\mathbb{Z}[S \times T]^{\blacksquare}, A)$.

(ii) Similar to the proof of (i).

(iii)

Remark 10.12. In Solid, $\otimes^{\blacksquare}$ is the left adjoint of $\underline{\text{Hom}}$:

$$\text{Hom}(M \otimes^{\blacksquare} N, P) \cong \text{Hom}((M \otimes N)^{\blacksquare}, P) \cong \text{Hom}(M \otimes N, P) \cong \text{Hom}(M, \underline{\text{Hom}}(N, P)).$$

Proposition 10.13. (i) If $X \in \text{CHaus}$, then $\mathbb{Z}[X]^{L\blacksquare} = R\underline{\text{Hom}}(R\Gamma(X, \mathbb{Z}), \mathbb{Z})$.
In particular, if $X \in \text{ProFin} \subseteq \text{CHaus}$, then $\mathbb{Z}[X]^{L\blacksquare} = \mathbb{Z}[X]^{\blacksquare}$.

(ii) If X is a CW space, then $\mathbb{Z}[X]^{L\blacksquare} = C_{\bullet}(X)$.

This shows that the derived solidification of a condensed abelian group can sit in all nonnegative homological degrees.

Proposition 10.14. (i) $\mathbb{R}^{L\blacksquare} = 0$.

(ii) $\mathbb{Z}^I \otimes^{L\blacksquare} \mathbb{Z}^J = \mathbb{Z}^{I\times J}$.

(iii) $\mathbb{Z}_p \otimes^{L\blacksquare} \mathbb{Z}_p = \mathbb{Z}_p$.

(iv) $\mathbb{Z}_p \otimes^{L\blacksquare} \mathbb{Z}_l = \mathbb{Z}_p$. $(p \neq l)$

Proof. (i) By Yoneda's lemma, it suffices to show: for any $C \in D(\text{Solid})$, one has

$$R\text{Hom}(\mathbb{R}^{L\blacksquare}, C) = R\text{Hom}(\mathbb{R}, C) = 0.$$

Since $C = \varprojlim C_{\geq n}$, and $R\text{Hom}(\mathbb{R}, -)$ commutes with limits, it reduces to the case C is a right bounded complex. And for a right bounded complex C, one has $C = \varprojlim C_{\leq n}$, it reduces to the case C is a bounded complex.
Hence it suffices to show: for any $X \in \text{Solid}$, one has $R\text{Hom}(\mathbb{R}, X) = 0$.
We know for any object $X \in \text{Solid}$, we can write X as the colimit of objects of the form $\bigoplus_{j\in J} \mathbb{Z}^{I_j}$. And we know taking all colimits is equivalent to taking all cokernels and all filtered colimits.
Since $\mathbb{R}$ is pseudo-coherent, we get

$$R\text{Hom}(\mathbb{R}, \varinjlim \bigoplus_{i\in J_j} \mathbb{Z}^{I_{i,j}}) = \varinjlim R\text{Hom}(\mathbb{R}, \bigoplus_{i\in J_j} \mathbb{Z}^{I_{i,j}}) = \varinjlim \bigoplus_{i\in J_j} R\text{Hom}(\mathbb{R}, \mathbb{Z}^{I_{i,j}}) = 0.$$

Let $f : X \to Y$, $X = \bigoplus_{i\in I} \mathbb{Z}^{I_i}$ and $Y = \bigoplus_{j\in J} \mathbb{Z}^{I_j}$, then from $R\text{Hom}(\mathbb{R}, X) = 0$ and $R\text{Hom}(\mathbb{R}, Y) = 0$, we know $R\text{Hom}(\mathbb{R}, \text{coker}(f)) = 0$.
Thus, we finish our proof.

(ii) Assume $\mathbb{Z}^I = \mathbb{Z}[S]^{\blacksquare} = \underline{\text{Hom}}(C(S,\mathbb{Z}),\mathbb{Z})$, $\mathbb{Z}^J = \mathbb{Z}[T]^{\blacksquare} = \underline{\text{Hom}}(C(T,\mathbb{Z}),\mathbb{Z})$, for some S, $T \in \text{ProFin}$. Then

$$\begin{aligned}
\mathbb{Z}[S\times T]^{\blacksquare} &= \underline{\text{Hom}}(C(S\times T,\mathbb{Z}),\mathbb{Z}) \\
&= \underline{\text{Hom}}(C(S,\mathbb{Z}) \otimes C(T,\mathbb{Z}),\mathbb{Z}) \\
&= \underline{\text{Hom}}(C(S,\mathbb{Z}), \underline{\text{Hom}}(C(T,\mathbb{Z}),\mathbb{Z})) \\
&= \underline{\text{Hom}}(C(S,\mathbb{Z}), \mathbb{Z}^J) \\
&= \underline{\text{Hom}}(C(S,\mathbb{Z}), \mathbb{Z})^J \\
&= \mathbb{Z}^{I\times J}.
\end{aligned}$$

Thus, we have

$$\mathbb{Z}^I \otimes^{L\blacksquare} \mathbb{Z}^J = \mathbb{Z}[S]^{\blacksquare} \otimes^{L\blacksquare} \mathbb{Z}[T]^{\blacksquare} = (\mathbb{Z}[S] \otimes^{L} \mathbb{Z}[T])^{L\blacksquare} = \mathbb{Z}[S \times T]^{L\blacksquare} = \mathbb{Z}^{I \times J}.$$

(iii)

(iv)

$\square$