An and Sp

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1 An and Sp

1.1 An and An_*

Definition 1.1. Denote the category of all Kan complexes by Kan, and define an ∞-category An $= N(\mathrm{Kan})$, we call An the ∞-category of anima, and its objects are called anima.
Similarly, denote the category of all pointed Kan complexes by Kan_*, and define an ∞-category $\mathrm{An}_* = N(\mathrm{Kan}_*)$, we call An_* the ∞-category of pointed anima, and its objects are called pointed anima.

Definition 1.2. For $X \in \mathrm{An}_*$, define $\Sigma X := \mathrm{cofib}(X \to 0)$ and $\Omega X := \mathrm{fib}(0 \to X)$, i.e. we have a pushout square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0' \arrow[r] & \Sigma X \end{tikzcd}$$

and a pullback square

$$\begin{tikzcd} \Omega X \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0' \arrow[r] & X \end{tikzcd}$$

We call $\Sigma : \mathrm{An}_* \to \mathrm{An}_*$ the suspension functor and $\Omega : \mathrm{An}_* \to \mathrm{An}_*$ the loop space functor.

Proposition 1.3. For $X, Y \in \mathrm{An}_*$, we have a natural equivalence:

$$\mathrm{Map}_{\mathrm{An}_*}(\Sigma X, Y) \simeq \mathrm{Map}_{\mathrm{An}_*}(X, \Omega Y).$$

Proof.

$$\begin{aligned} \mathrm{Map}_{\mathrm{An}_*}(\Sigma X, Y) &\simeq \mathrm{Map}_{\mathrm{Fun}(\Lambda^2_0, \mathrm{An}_*)}(0 \leftarrow X \to 0, Y \leftarrow Y \to Y) \\ &\simeq \mathrm{Map}_{\mathrm{An}_*}(0, Y) \times_{\mathrm{Map}_{\mathrm{An}_*}(X,Y)} \mathrm{Map}_{\mathrm{An}_*}(X, Y) \times_{\mathrm{Map}_{\mathrm{An}_*}(X,Y)} \mathrm{Map}_{\mathrm{An}_*}(0, Y) \\ &\simeq \mathrm{Map}_{\mathrm{Fun}(\Lambda^2_2, \mathrm{An}_*)}(X \to X \leftarrow X, 0 \to Y \leftarrow 0) \\ &\simeq \mathrm{Map}_{\mathrm{An}_*}(X, \Omega Y). \end{aligned}$$

$\square$

Remark 1.4. An_* is generated under colimits by $\{S^n : n \geq 0\}$.

Proposition 1.5. The ∞-category An admits all small limits and colimits.

Proof. Ref Theorem[4.3.3.7].

Proposition 1.6. (1) The forgetful functor $\mathrm{An}_* \to \mathrm{An}$ commutes with filtered colimits.

(2) In the ∞-category An, filtered colimits commute with limits.

(3) $\pi_0 : \mathrm{An} \to \mathrm{Set}$ commutes with all colimits, and therefore

$$\pi_n : \mathrm{An}_* \xrightarrow{\Omega^n} \mathrm{An}_* \xrightarrow{\mathrm{fgt}} \mathrm{An} \xrightarrow{\pi_0} \mathrm{Set}$$

commutes with filtered colimits, for any $n \geq 1$.

1.2 Sp

Definition 1.7. A spectrum is $E = \{E_n, \delta_n : E_n \xrightarrow{\sim} \Omega E_{n+1}\}_{n\in\mathbb{Z}}$, where $E_n \in \mathrm{An}_*$, for all $n \in \mathbb{Z}$. We denote the ∞-category of all spectra by Sp.

We have a pair of adjoint functors $(\Sigma^\infty, \Omega^\infty) : \mathrm{An}_* \to \mathrm{Sp}$, here we mean $\Sigma^\infty \dashv \Omega^\infty$. We can define them as follows:

$$\Omega^\infty : \mathrm{Sp} \to \mathrm{An}_*;\ \{E_n, \delta_n : E_n \xrightarrow{\sim} \Omega E_{n+1}\}_{n\in\mathbb{Z}} \mapsto E_0,$$

and

$$\Sigma^\infty : \mathrm{An}_* \to \mathrm{Sp};\ X \mapsto \Sigma^\infty X = \{Q\Sigma^n X, Q\Sigma^n X \xrightarrow{\sim} \Omega Q \Sigma^{n+1} X\}_{n\in\mathbb{Z}}.$$

Here, $Q : \mathrm{An}_* \to \mathrm{An}_*$ is defined by

$$QX := \mathrm{colim}(X \to \Omega\Sigma X \to \Omega^2\Sigma^2 X \to \cdots).$$

Remark 1.8. We have:

$$\begin{aligned}\Omega Q \Sigma X &\simeq \Omega\,\mathrm{colim}(\Sigma X \to \Omega\Sigma\Sigma X \to \Omega^2\Sigma^2\Sigma X \to \cdots)\\ &\simeq \mathrm{colim}(\Omega\Sigma X \to \Omega^2\Sigma^2 X \to \cdots)\\ &\simeq QX.\end{aligned}$$

Example 1.9. (1) For $A \in \mathrm{Ab}$, we define $HA \in \mathrm{Sp}$ to be

$$\{K(A,n), \delta : K(A,n) \xrightarrow{\sim} K(A, n+1)\}_{n\geq 0}.$$

In fact, $H : \mathrm{Ab} \to \mathrm{Sp}$ is a fully faithful functor. And we have $\Omega^\infty HA = A$.

(ii) The sphere spectrum is defined to be $\mathbb{S} := \Sigma^\infty S^0$, where $S^0 \in \mathrm{An}_*$.

Definition 1.10. The suspension functor Σ and loop space functor Ω on Sp are defined as follows:

$$\Sigma : \mathrm{Sp} \to \mathrm{Sp};\ E = \{E_n\}_{n\in\mathbb{Z}} \mapsto \Sigma E = \{E_{n+1}\}_{n\in\mathbb{Z}},$$

and

$$\Omega : \mathrm{Sp} \to \mathrm{Sp};\ E = \{E_n\}_{n\in\mathbb{Z}} \mapsto \Omega E = \{E_{n-1}\}_{n\in\mathbb{Z}}.$$

Hence from the definition we know that Σ and Ω are inverse to each other, so we can denote by $\Omega = \Sigma^{-1}$ and $\Sigma = \Omega^{-1}$.

Definition 1.11. For $E = \{E_n, \delta_n : E_n \xrightarrow{\sim} \Omega E_{n+1}\}_{n\in\mathbb{Z}} \in \mathrm{Sp}$ and $F = \{F_n, \delta_n : F_n \xrightarrow{\sim} \Omega F_{n+1}\}_{n\in\mathbb{Z}} \in \mathrm{Sp}$, define

$$\begin{aligned}\mathrm{Map}_{\mathrm{Sp}}(E, F) := &\lim_n (\cdots \leftarrow \mathrm{Map}_{\mathrm{An}_*}(E_0, F_0) \leftarrow \mathrm{Map}_{\mathrm{An}_*}(E_1, F_1) \leftarrow \cdots) \\ = &\lim_n \mathrm{Map}_{\mathrm{An}_*}(E_n, F_n).\end{aligned}$$

We next show that Sp has all limits and all colimits.

Proposition 1.12. The ∞-category Sp admits all limits and filtered colimits. More concretely,

(1) Assume $E : I \to \mathrm{Sp}$ is a limit daigram, then $\lim_I E$ exists, and is given by

$$\{\lim_I E(i)_n, \delta_n : \lim_I E(i)_n \xrightarrow{\sim} \Omega \lim_I E(i)_{n+1}\}.$$

(2) Assume $E : I \to \mathrm{Sp}$ is a filtered colimit daigram, then $\operatorname{colim}_I E$ exists, and is given by

$$\{\operatorname{colim}_I E(i)_n, \delta_n : \operatorname{colim}_I E(i)_n \xrightarrow{\sim} \Omega \operatorname{colim}_I E(i)_{n+1}\}.$$

Proof. We have

$$\begin{aligned}\mathrm{Map}_{\mathrm{Sp}}(F, \{\lim_I E(i)_n\}) &\simeq \lim_n \mathrm{Map}_{\mathrm{An}_*}(F_n, \lim_I E(i)_n) \\ &\simeq \lim_n \lim_I \mathrm{Map}_{\mathrm{An}_*}(F_n, E(i)_n) \\ &\simeq \lim_I \mathrm{Map}_{\mathrm{Sp}}(F, E(i)),\end{aligned}$$

which means $\{\lim_I E(i)_n\}$ is the limit of $E(i)$.
For the filtered colimit case, the proof is similar. $\square$

Since the functor $\Omega : \mathrm{An}_* \to \mathrm{An}_*$ commutes with all limits and filtered colimits, we can define the limits and filtered colimits in Sp pointwise. However, since $\Omega : \mathrm{An}_* \to \mathrm{An}_*$ is not commute with all colimits, we cannot define the colimits in Sp pointwise.

In order to show that Sp has all colimits, we introduce the concept of prespectrum.

Definition 1.13. (1) A prespectrum is $E = \{E_n, \delta_n : \Sigma E_n \to E_{n+1}\}_{n\in\mathbb{Z}}$, where $E_n \in \mathrm{An}_*$, for all $n \in \mathbb{Z}$. We denote the ∞-category of all prespectra by PSp.

(2) For a prespectrum $\{E_n, \delta_n : \Sigma E_n \to E_{n+1}\}_{n\in\mathbb{Z}}$, we define its associated spectrum to be

$$\mathrm{colim}(\Sigma^\infty E_0 \to \Omega\Sigma^\infty E_1 \to \Omega^2\Sigma^\infty E_2 \to \cdots) = \mathrm{colim}(\Omega^n\Sigma^\infty E_n).$$

Example 1.14. For $X \in \mathrm{An}_*$, we have a prespectrum $\{\Sigma^n X, \Sigma\Sigma^n X \to \Sigma^{n+1}X\}$. Its associated spectrum is $\Sigma^\infty X$, since

$$\mathrm{colim}(\Omega^n\Sigma^\infty\Sigma^n X) = \mathrm{colim}(\Omega^n\Sigma^n\Sigma^\infty X) = \mathrm{colim}(\Sigma^\infty X) = \Sigma^\infty X.$$

Lemma 1.15. For $E \in \mathrm{Sp}$, we have $E \simeq \mathrm{colim}(\Omega^n\Sigma^\infty E_n)$.

Proof. For any $F \in \mathrm{Sp}$,

$$\begin{aligned}
\mathrm{Map}_{\mathrm{Sp}}(\mathrm{colim}\,\Omega^n\Sigma^\infty E_n, F) &\simeq \lim \mathrm{Map}_{\mathrm{Sp}}(\Omega^n\Sigma^\infty E_n, F)\\
&\simeq \lim \mathrm{Map}_{\mathrm{An}_*}(E_n, \Omega^\infty\Sigma^n F)\\
&\simeq \lim \mathrm{Map}_{\mathrm{An}_*}(E_n, F_n)\\
&\simeq \mathrm{Map}_{\mathrm{Sp}}(E, F),
\end{aligned}$$

then by Yoneda, we have $E \simeq \mathrm{colim}(\Omega^n\Sigma^\infty E_n)$. □

Proposition 1.16. Sp has all colimits.

Proof. Assume $E : I \to \mathrm{Sp}$ is a colimit diagram, then we form a prespectrum $\{\mathrm{colim}_I\, E(i)_n, \delta_n : \Sigma\mathrm{colim}_I\, E(i)_n \to \mathrm{colim}_I\, E(i)_{n+1}\}$. We claim that its associated spectrum is the colimit of the diagram $E : I \to \mathrm{Sp}$:

$$\begin{aligned}
\mathrm{Map}_{\mathrm{Sp}}(\mathrm{colim}_n\,\Omega^n\Sigma^\infty\mathrm{colim}_I\, E(i)_n, F) &\simeq \lim_n \mathrm{Map}_{\mathrm{Sp}}(\Sigma^\infty\mathrm{colim}_I\, E(i)_n, \Sigma^n F)\\
&\simeq \lim_n \mathrm{Map}_{\mathrm{An}_*}(\mathrm{colim}_I\, E(i)_n, \Omega^\infty\Sigma^n F)\\
&\simeq \lim_n \lim_I \mathrm{Map}_{\mathrm{An}_*}(E(i)_n, F_n)\\
&\simeq \lim_I \lim_n \mathrm{Map}_{\mathrm{An}_*}(E(i)_n, F_n)\\
&\simeq \lim_I \mathrm{Map}_{\mathrm{Sp}}(E(i), F).
\end{aligned}$$

Hence $\underset{n}{\mathrm{colim}}\, \Omega^n \Sigma^\infty \underset{I}{\mathrm{colim}}\, E(i)_n \in \mathrm{Sp}$ is the colimit of the diagram $E : I \to \mathrm{Sp}$. Therefore, Sp has all colimits. $\square$

Proposition 1.17. For $X \in \mathrm{An}_*$, and $E \in \mathrm{Sp}$, we have a natural equivalence:

$$\mathrm{Map}_{\mathrm{Sp}}(\Sigma^\infty X, E) \simeq \mathrm{Map}_{\mathrm{An}_*}(X, \Omega^\infty E).$$

Proof.

$$\begin{aligned}
\mathrm{Map}_{\mathrm{Sp}}(\Sigma^\infty X, E) &\simeq \lim_n \mathrm{Map}_{\mathrm{An}_*}((\Sigma^\infty X)_n, E_n) \\
&\simeq \lim_n \mathrm{Map}_{\mathrm{An}_*}(Q\Sigma^n X, E_n) \\
&\simeq \lim_n \mathrm{Map}_{\mathrm{An}_*}(\underset{m}{\mathrm{colim}}\, \Omega^m \Sigma^{n+m} X, E_n) \\
&\simeq \lim_n \lim_m \mathrm{Map}_{\mathrm{An}_*}(\Omega^m \Sigma^{n+m} X, E_n) \\
&\simeq \mathrm{Map}_{\mathrm{An}_*}(X, E_0) \\
&\simeq \mathrm{Map}_{\mathrm{An}_*}(X, \Omega^\infty E).
\end{aligned}$$

$\square$

Example 1.18. Assume $K \in \mathrm{An}_*$ is finite, then we have

$$\begin{aligned}
\mathrm{Map}_{\mathrm{Sp}}(\Sigma^\infty K, \Sigma^\infty X) &\simeq \mathrm{Map}_{\mathrm{An}_*}(K, \Omega^\infty \Sigma^\infty X) \\
&\simeq \mathrm{Map}_{\mathrm{An}_*}(K, \mathrm{colim}\, \Omega^k \Sigma^k X) \\
&\simeq \mathrm{colim}\, \mathrm{Map}_{\mathrm{An}_*}(K, \Omega^k \Sigma^k X) \\
&\simeq \mathrm{colim}\, \mathrm{Map}_{\mathrm{An}_*}(\Sigma^k K, \Sigma^k X).
\end{aligned}$$

Proposition 1.19 (Stability)**.** In the ∞-category Sp, a square

$$\begin{tikzcd}
X_0 \arrow[r] \arrow[d] & X_1 \arrow[d] \\
X_2 \arrow[r] & X_{12}
\end{tikzcd}$$

is a pullback iff it is a pushout.

Definition 1.20. (i) For $E, F \in \mathrm{Sp}$, $[E, F] := \pi_0 \mathrm{Map}_{\mathrm{Sp}}(E, F)$.

(ii) For $E \in \mathrm{Sp}$, define $\pi_n E := [\Sigma^n \mathbb{S}, E] = \pi_n \mathrm{Map}_{\mathrm{Sp}}(\mathbb{S}, E)$.

(iii) $E \in \mathrm{Sp}$ is a connective spectrum, if $\pi_n E \simeq 0$, $\forall n < 0$. We denote the ∞-category of connective spectra by $\mathrm{Sp}_{\geq 0}$.

For any $E \in \mathrm{Sp}$, we have

$$\mathrm{Map}_{\mathrm{Sp}}(\mathbb{S}, E) \simeq \mathrm{Map}_{\mathrm{An}_*}(S^0, \Omega^\infty E) \simeq \Omega^\infty E.$$

So $\pi_n E \simeq \pi_n \mathrm{Map}_{\mathrm{Sp}}(\mathbb{S}, E) \simeq \pi_n \Omega^\infty E$.

Example 1.21. For $X \in \mathrm{An}_*$, $\Sigma^\infty X \in \mathrm{Sp}$, then

$$\pi_n(\Sigma^\infty X) = \begin{cases} 0 & n < 0 \\ \pi_n^s(X) & n \geq 0 \end{cases}$$

Proof. For $n \geq 0$, we have:

$$\begin{aligned} \pi_n(\Sigma^\infty X) &= \pi_n(\Omega^\infty \Sigma^\infty X) = \pi_n(\operatorname{colim}_k \Omega^k \Sigma^k X) \\ &= \operatorname{colim}_k \pi_n(\Omega^k \Sigma^k X) = \operatorname{colim}_k \pi_{n+k}(\Sigma^k X) = \pi_n^s(X). \end{aligned}$$

Definition 1.22. (i) For $E, F \in \mathrm{Sp}$, define $\mathrm{map}(E, F) \in \mathrm{Sp}$ as follows:

$$\mathrm{map}(E, F) := \{\mathrm{Map}_{\mathrm{Sp}}(E, \Sigma^n F), \delta_n : \mathrm{Map}_{\mathrm{Sp}}(E, \Sigma^n F) \xrightarrow{\sim} \Omega \mathrm{Map}_{\mathrm{Sp}}(E, \Sigma^{n+1} F)\}_{n \in \mathbb{Z}}.$$

(ii) For $E, F \in \mathrm{Sp}$, define $E \otimes F \in \mathrm{Sp}$ as follows:

$$E \otimes F := \operatorname{colim}_{n,m} \Omega^{n+m} \Sigma^\infty (E_n \wedge F_m).$$

Remark 1.23. By definition, we know that for $E, F \in \mathrm{Sp}$, $E \otimes F \simeq F \otimes E$.

Proposition 1.24. For $E, F, K \in \mathrm{Sp}$, there is a natural equivalence:

$$\mathrm{Map}_{\mathrm{Sp}}(E \otimes F, K) \simeq \mathrm{Map}_{\mathrm{Sp}}(E, \mathrm{map}(F, K)).$$

Proof.

$$\begin{aligned}
\mathrm{Map}_{\mathrm{Sp}}(E \otimes F, K) &\simeq \mathrm{Map}_{\mathrm{Sp}}(\operatorname*{colim}_{n,m} \Omega^{n+m}\Sigma^{\infty}(E_n \wedge F_m), K) \\
&\simeq \lim_{n,m} \mathrm{Map}_{\mathrm{Sp}}(\Omega^{n+m}\Sigma^{\infty}(E_n \wedge F_m), K) \\
&\simeq \lim_{n,m} \mathrm{Map}_{\mathrm{Sp}}(\Sigma^{\infty}(E_n \wedge F_m), \Sigma^{n+m}K) \\
&\simeq \lim_{n,m} \mathrm{Map}_{\mathrm{An}_*}(E_n \wedge F_m, \Omega^{\infty}\Sigma^{n+m}K) \\
&\simeq \lim_{n,m} \mathrm{Map}_{\mathrm{An}_*}(E_n, \mathrm{Map}_{\mathrm{An}_*}(F_m, \Omega^{\infty}\Sigma^{n+m}K)) \\
&\simeq \lim_{n,m} \mathrm{Map}_{\mathrm{An}_*}(E_n, \mathrm{Map}_{\mathrm{An}_*}(F_m, (\Sigma^{n}K)_m)) \\
&\simeq \lim_{n} \mathrm{Map}_{\mathrm{An}_*}(E_n, \lim_{m} \mathrm{Map}_{\mathrm{An}_*}(F_m, (\Sigma^{n}K)_m)) \\
&\simeq \lim_{n} \mathrm{Map}_{\mathrm{An}_*}(E_n, \mathrm{Map}_{\mathrm{Sp}}(F, \Sigma^{n}K)) \\
&\simeq \lim_{n} \mathrm{Map}_{\mathrm{An}_*}(E_n, \mathrm{map}(F, K)_n) \\
&\simeq \mathrm{Map}_{\mathrm{Sp}}(E, \mathrm{map}(F, K)).
\end{aligned}$$

$\square$

Proposition 1.25. For $X, Y \in \mathrm{An}_*$,

$$\Sigma^{\infty}X \otimes \Sigma^{\infty}Y \simeq \Sigma^{\infty}(X \wedge Y)$$

.

Proof. For any $E \in \mathrm{Sp}$,

$$\begin{aligned}
\mathrm{Map}_{\mathrm{Sp}}(\Sigma^{\infty}X \otimes \Sigma^{\infty}Y, E) &\simeq \mathrm{Map}_{\mathrm{Sp}}(\Sigma^{\infty}X, \mathrm{map}(\Sigma^{\infty}Y, E)) \\
&\simeq \mathrm{Map}_{\mathrm{An}_*}(X, \Omega^{\infty}\mathrm{map}(\Sigma^{\infty}Y, E)) \\
&\simeq \mathrm{Map}_{\mathrm{An}_*}(X, \mathrm{Map}_{\mathrm{Sp}}(\Sigma^{\infty}Y, E)) \\
&\simeq \mathrm{Map}_{\mathrm{An}_*}(X, \mathrm{Map}_{\mathrm{An}_*}(Y, \Omega^{\infty}E)) \\
&\simeq \mathrm{Map}_{\mathrm{An}_*}(X \wedge Y, \Omega^{\infty}E) \\
&\simeq \mathrm{Map}_{\mathrm{An}_*}(\Sigma^{\infty}(X \wedge Y), E),
\end{aligned}$$

then by Yoneda's lemma, we have: $\Sigma^{\infty}X \otimes \Sigma^{\infty}Y \simeq \Sigma^{\infty}(X \wedge Y)$. $\square$