Condensed mathematics

何力

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1 Condensed Sets

Definition 1.1. Let $\mathcal{C}$ be a category. A Grothendieck topology on $\mathcal{C}$ consists of: for each object X in $\mathcal{C}$, there is a collection $\text{Cov}(X)$ of sets $\{X_i \to X\}_{i\in I}$, satisfying the following three axioms:

(i) If $V \to X$ is an isomorphism, then $\{V \to X\} \in \text{Cov}(X)$.

(ii) If $\{X_i \to X\}_{i\in I} \in \text{Cov}(X)$ and $Y \to X$ is any arrow in $\mathcal{C}$, then the fiber products $X_i \times_X Y$ exist and $\{X_i \times_X Y \to Y\}_{i\in I} \in \text{Cov}(Y)$.

(iii) If $\{X_i \to X\}_{i\in I} \in \text{Cov}(X)$ and for each $i \in I$, $\{V_{ij} \to X_i\}_{j\in I_i} \in \text{Cov}(X_i)$, then $\{V_{ij} \to X\}_{i\in I, j\in I_i} \to \text{Cov}(X)$.

We call elements of $\text{Cov}(X)$ coverings.

Definition 1.2. A site is a category $\mathcal{C}$ together with a Grothendieck topology.

Example 1.3. Let $\mathcal{C} = \text{ProFin}$, the category of all profinite sets. For $\{X_i \to Y\}_{i\in I}$ be a covering, we mean I is a finite index and $\coprod_{i\in I} X_i \to Y$ is a surjection. We also call maps $\{X_i \to Y\}_{i\in I}$ finite jointly surjective families of maps.
Now, for the category ProFin together its coverings, we call it the proétale site of a point and denote it by $*_{\text{proét}}$.

Definition 1.4. (i) For any site $\mathcal{C}$, we call a functor

$$\mathcal{F} : \mathcal{C}^{\text{op}} \to \text{Set}$$

a presheaf of sets.

(ii) For a presheaf of sets $\mathcal{F} : \mathcal{C}^{\text{op}} \to \text{Set}$, if for any $X \in \mathcal{C}$ and any covering $\{X_i \to X\}_{i\in I} \in \text{Cov}(X)$, we have

$$\mathcal{F}(X) \xrightarrow{\sim} \text{Eq}(\prod_{i\in I} \mathcal{F}(X_i) \rightrightarrows \prod_{i,j\in I} \mathcal{F}(X_i \times_X X_j)).$$

Then we call $\mathcal{F}$ a sheaf of sets.

Definition 1.5. A condensed set T is a sheaf of sets on $*_{\text{proét}}$, i.e. a functor $T : *_{\text{proét}}^{\text{op}} \to \text{Set}$ satisfying the sheaf condition.

Remark 1.6. (i) Concretely, a condensed set T is a functor $T : \mathrm{ProFin}^{\mathrm{op}} \to \mathrm{Set}$, satisfying $T(\emptyset) = *$ and

- For any profinite sets S_1, S_2, the natural map

$$T(S_1 \sqcup S_2) \longrightarrow T(S_1) \times T(S_2)$$

is a bijection.

- For any surjection $S' \twoheadrightarrow S$ of profinite sets with fiber product $S' \times_S S'$ and two projections $p_1, p_2 : S' \times_S S' \to S'$, the map

$$T(S) \xrightarrow{\sim} \{x \in T(S') |\, p_1^*(x) = p_2^*(x) \in T(S' \times_S S')\}$$

is a bijection. In other words, T maps the pullback diagram

$$\begin{tikzcd} S' \times_S S' \arrow[r, "p_2"] \arrow[d, "p_1"'] & S' \arrow[d] \\ S' \arrow[r] & S \end{tikzcd}$$

to a pullback diagram

$$\begin{tikzcd} T(S' \times_S S') & T(S') \arrow[l, "p_2^*"'] \\ T(S') \arrow[u, "p_1^*"] & T(S) \arrow[l] \arrow[u] \end{tikzcd}$$

(ii) The category ProFin of all profinite sets is a large category.

Definition 1.7. κ is an uncountable strong limit cardinal if κ is uncountable and for any $\lambda < \kappa$, we have $2^\lambda < \kappa$.

Example 1.8. For any limit cardinal λ, i.e. if $\kappa < \lambda$, then $\kappa + 1 < \lambda$. We define

$$\sqsupset_0 = \aleph_0, \cdots, \sqsubset_{\alpha+1} = 2^{\sqsubset_\alpha},$$

and let

$$\sqsubset_\lambda = \bigcup_{\alpha < \lambda} \sqsubset_\alpha,$$

then we can show that $\sqsubset_\lambda$ is an uncountable strong limit cartinal.

Notation. We let κ-ProFin denote the category of all κ-small profinite sets, i.e. profinite sets whose cardinal less equal than κ. Let $\text{Cond}_\kappa(\text{Set}) = \text{Sh}(\kappa\text{-ProFin}, \text{Set})$.

Remark 1.9. If $\kappa' > \kappa$ are two uncountable strong limit cardinals, and denote the inclusion by $i : \kappa\text{-ProFin} \hookrightarrow \kappa'\text{-ProFin}$, then we have a forgetful functor

$$\text{Cond}_{\kappa'}(\text{Set}) \longrightarrow \text{Cond}_\kappa(\text{Set});\ T \mapsto T \circ i.$$

This forgetful functor admits a left adjoint $F : \text{Cond}_\kappa(\text{Set}) \longrightarrow \text{Cond}_{\kappa'}(\text{Set})$. F is fully faithful and F commutes with all colimits and all finite limits.
We define

$$\text{Cond}(\text{Set}) = \bigcup_\kappa \text{Cond}_\kappa(\text{Set}) = \varinjlim_\kappa \text{Cond}_\kappa(\text{Set}).$$

Example 1.10. Let Top denote the category of all topological spaces. For each $T \in \text{Top}$, we can define $\underline{T} \in \text{Cond}(\text{Set})$ as follows:

$$\underline{T} : \text{ProFin}^{\text{op}} \longrightarrow \text{Set};\ S \mapsto \underline{T}(S) = \text{Cont}(S, T) = \{\text{continuous maps from } S \text{ to } T\}.$$

We need to check that $\underline{T}$ is a condensed set:

(i) $\underline{T}(S_1 \sqcup S_2) = \text{Cont}(S_1 \sqcup S_2, T) = \text{Cont}(S_1, T) \times \text{Cont}(S_2, T) = \underline{T}(S_1) \times \underline{T}(S_2)$.

(ii) For any surjection $g : S' \twoheadrightarrow S$, let $p_1, p_2 : S' \times_S S' \to S'$ be the two projections. We need to show the following map is a bijection:

$$\text{Cont}(S, T) \xrightarrow{\sim} \{h : S' \to T \mid hp_1 = hp_2 : S' \times_S S' \to T\};\ f \mapsto f \circ g.$$

Because g is surjective, it is easy to show this map is an injection.
Now, for any $h : S' \to T$ with $hp_1 = hp_2$, from the universal property of pushout(in our situation, the pullback square is also a pushout), we can find a unique f, s.t. the diagram commutes.

$$\begin{tikzcd}
S' \times_S S' \arrow[r, "p_2"] \arrow[d, "p_1"'] & S' \arrow[d, two heads, "g"'] \arrow[ddr, "h"] & \\
S' \arrow[r, two heads, "g"] \arrow[drr, "h"'] & S \arrow[dr, dashed, "\exists! f"] & \\
& & T
\end{tikzcd}$$

Definition 1.11. Let $X \in$ Top. The following are equivalent definition:

(i) $X \in$ Top is compactly generated;

(ii) If for any compact Hausdorff space S with a map $S \to X$, if the composition $S \to X \to Y$ is continuous, then $X \to Y$ is continuous;

(iii) $A \subset X$ is closed if and only if for any compact space K with a map $f : K \to X$, $f^{-1}(A) \subset K$ is closed.

Remark 1.12. (i) If a topological space X is compact Hausdorff, then X is compactly generated.

(ii) Let CGTop denote the category of all compactly generated spaces and let CHaus denote the category of all compact Hausdorff spaces.

Definition 1.13. For a category $\mathcal{C}$, $P \in \mathcal{C}$ is a projective object if for any epimorphism $Y \twoheadrightarrow X$ and a morphism $P \to X$, there is a lift

$$\begin{tikzcd} & P \arrow[d] \arrow[dl, dashed, "\exists"'] \\ Y \arrow[r, two heads] & X \end{tikzcd}$$

Definition 1.14. In the category CHaus, we call its projective objects as extremally disconnected Hausdorff spaces.

Remark 1.15. (i) Equivalently a compact Hausdorff space S is extremally disconnected if any surjection $S' \twoheadrightarrow S$ from a compact Hausdorff space splits.

(ii) Extremally disconnected Hausdorff spaces are profinite sets, i.e. ExDisc $\subset$ ProFin. Here, ExDisc denote the category of all extremally disconnected Hausdorff spaces.

Remark 1.16. We have two adjunctions.

(i) $\text{Top} \underset{i}{\overset{\beta}{\rightleftarrows}} \text{CHaus}$, i.e. $\beta \dashv i$.
Where

$$i : \text{CHaus} \to \text{Top};\ X \mapsto X$$

and

$$\beta : \text{Top} \to \text{CHaus}$$

is the Stone-Cech compactification of topological spaces.

For any $X \in$ Top, we define $\beta X \in$ CHaus as follows:

for any $Y \in$ CHaus with a map $f : X \to Y$, there exists a unique map $\beta X \to Y$ so that the diagram commutes.

$$\begin{tikzcd} X \arrow[rr, "i_X"] \arrow[dr, "f"'] & & \beta X \arrow[dl, dashed, "\exists!"] \\ & Y & \end{tikzcd}$$

In fact, we can use the ultrafilter to construct βX concretely. And by this construction, we can show that

$$|\beta X| \leq 2^{2^{|X|}}.$$

(ii) CGTop $\underset{c}{\overset{i}{\rightleftarrows}}$ Top , i.e. $i \dashv c$.

Where

$$i : \text{CGTop} \to \text{Top};\ X \mapsto X$$

and

$$c : \text{Top} \to \text{CGTop};\ X \mapsto X^{\text{cg}}.$$

We define X^{cg} as follows:

- As a set, $X^{\text{cg}} = X$.
- The topology of X^{cg} is given by the quotient topology of

$$\coprod_{S \to X\ S \in \text{CHaus}} S \longrightarrow X.$$

Proposition 1.17. (i) The functor Top $\to \text{Cond}_\kappa(\text{Set});\ T \mapsto \underline{T}$ is a faithful functor.

(ii) When the above functor restricted to the full subcategory κ-CGTop of all κ-compactly generated spaces, functor κ-CGTop $\to \text{Cond}_\kappa(\text{Set});\ T \mapsto \underline{T}$ is a fully faithful functor.

(iii) The functor Top $\to \text{Cond}_\kappa(\text{Set});\ T \mapsto \underline{T}$ admits a left adjoint $\text{Cond}_\kappa(\text{Set}) \to$ Top; $T \mapsto T(*)_{\text{top}}$. Here, $T(*)_{\text{top}}$ means the underlying set $T(*)$ equipped with the quotient topology of $\sqcup_{S \to T} S \to T(*)$, where the disjoint union runs over all

κ-small profinite sets S with a map to T, i.e. an element of $T(S)$. Moreover, we have $\underline{T}(*)_{\text{top}} \cong T^{\kappa\text{-cg}}$.

2 Condensed abelian groups

Definition 2.1. A condensed abelian group T is a sheaf of abelian groups on $*_{\text{proét}}$, i.e. a functor $T : *_{\text{proét}}^{\text{op}} \to \text{Ab}$ satisfying the sheaf condition. And we denote the category of all condensed abelian groups by Cond(Ab).

Definition 2.2 (Grothendieck's axioms)**.** Let $\mathcal{C}$ be an abelian category.

(AB3) All colimits exist.

(AB3*) All limits exist.

(AB4) Arbitrary direct sums are exact.

(AB4*) Arbitrary products are exact.

(AB5) Filtered colimits are exact.

(AB6) For any index set J and filtered categories $I_j,\ j \in J$, with functors $I_j \to \text{Cond(Ab)};\ i \mapsto M_i$, the natural map

$$\varinjlim_{(i_j \in I_j)_j} \prod_{j \in J} M_{i_j} \longrightarrow \prod_{j \in J} \varinjlim_{i_j \in I_j} M_{i_j}$$

is an isomorphism.

Definition 2.3. Let $\mathcal{C}$ be an abelian category. $M \in \mathcal{C}$ is compact if $\text{Hom}(M, -)$ commutes with filtered colimits, i.e. $\text{Hom}(M, \varinjlim_i N_i) \cong \varinjlim_i \text{Hom}(M, N_i)$.

Theorem 2.4. (i) Cond(Ab) is an abelian category which satisfies Grothendieck's axioms (AB3), (AB4), (AB5), (AB6), (AB3*) and (AB4*).

(ii) Cond(Ab) is generated by compact projective objects.

Corollary 2.5. There is an adjunction:

$$\text{Cond}_\kappa(\text{Set}) \rightleftarrows \text{Cond}_\kappa(\text{Ab})\ .$$

Where $\text{Cond}_\kappa(\text{Ab}) \longrightarrow \text{Cond}_\kappa(\text{Set})$ is the forgetful functor and

$$\text{Cond}_\kappa(\text{Set}) \longrightarrow \text{Cond}_\kappa(\text{Ab});\ T \mapsto \mathbb{Z}[T].$$

Here, $\mathbb{Z}[T] := (S \mapsto \mathbb{Z}[T(S)])^{\text{sh}}$.

Remark 2.6. (i) For $S \in$ ExDisc and $M \in$ Cond(Ab), we have

$$\mathrm{Hom}_{\mathrm{Cond(Ab)}}(\mathbb{Z}[S], M) \cong \mathrm{Hom}_{\mathrm{Cond(Set)}}(\underline{S}, M) \cong M(S).$$

Proof: We define the map:

$$\mu : \mathrm{Hom}_{\mathrm{Cond(Set)}}(\underline{S}, M) \longrightarrow M(S);\ \alpha \mapsto \alpha(S)(1_S),$$

and the map

$$\lambda : M(S) \longrightarrow \mathrm{Hom}_{\mathrm{Cond(Set)}}(\underline{S}, M);\ x \mapsto \lambda(x),$$

where for $\lambda(x) : \underline{S} \longrightarrow M$,

$$\lambda(x)(T) : \mathrm{Cont}(T, S) \longrightarrow M(T);\ f \mapsto M(f)(x).$$

One can check that μ and λ are inverse to each other, hence

$$\mathrm{Hom}_{\mathrm{Cond(Set)}}(\underline{S}, M) \cong M(S).$$

□

(ii) For any $S \in$ ExDisc, $\mathbb{Z}[S] \in$ Cond(Ab) is a compact and projective object.

Proof:

Compactness.

$$\mathrm{Hom}(\mathbb{Z}[S], \varinjlim M_i) = (\varinjlim M_i)(S) = \varinjlim M_i(S) = \varinjlim \mathrm{Hom}(\mathbb{Z}[S], M_i).$$

Projectiveness. For any exact sequence $M' \to M \to M''$ in Cond(Ab), the sequence

$$M'(S) \to M(S) \to M''(S)$$

is exact, i.e.

$$\mathrm{Hom}(\mathbb{Z}[S], M') \to \mathrm{Hom}(\mathbb{Z}[S], M) \to \mathrm{Hom}(\mathbb{Z}[S], M'')$$

is exact, so $\mathbb{Z}[S]$ is projective. □

(iii) Cond$($Ab$)$ has enough projectives.

Proposition 2.7. We have two equivalences.

(i) Shv(κ-CHaus) $\xrightarrow{\sim}$ Shv(κ-ProFin); $T \mapsto T|_{\kappa\text{-ProFin}}$.

(ii) Shv(κ-ProFin) $\xrightarrow{\sim}$ Shv(κ-ExDisc); $T \mapsto T|_{\kappa\text{-ExDisc}}$.

Remark 2.8. In order a presheaf of sets T to be a sheaf of sets, by definition, we need to check the sheaf condition in ProFin. Now, from the equivalence Shv(κ-ProFin) $\xrightarrow{\sim}$ Shv(κ-ExDisc), we only need to check the sheaf condition in ExDisc. In this case, the condition(ii) is automatic: T maps the pullback diagram

$$\begin{tikzcd} S' \times_S S' \arrow[r, "p_2"] \arrow[d, "p_1"'] & S' \arrow[d] \\ S' \arrow[r] & S \end{tikzcd}$$

to a pullback diagram

$$\begin{tikzcd} T(S' \times_S S') & T(S') \arrow[l, "p_2^*"'] \\ T(S') \arrow[u, "p_1^*"] & T(S) \arrow[l] \arrow[u] \end{tikzcd}$$

This is because any cover of extremally disconnected sets splits. Specifically, the diagram

$$\begin{tikzcd} S' \times_S S' \arrow[r] \arrow[d] & S' \arrow[d] \\ S' \arrow[r, "g"] & S \arrow[l, bend left, "f"] \end{tikzcd}$$

can implies the following diagram:

$$\begin{tikzcd} T(S' \times_S S') & T(S') \arrow[l] & \\ T(S') \arrow[u] \arrow[r, bend right, "T(f)"'] & T(S) \arrow[l, "T(g)"'] \arrow[u] & \\ & & X \arrow[ull, bend left] \arrow[ul, dotted, "\exists"'] \arrow[uul, bend right] \end{tikzcd}$$

which means it is a pullback diagram.

Property. There are some properties of the category Cond(Ab) of condensed abelian groups.

(i) Cond(Ab) has a symmetric monoidal tensor products $- \otimes -$, where for $M, N \in$ Cond(Ab), $M \otimes N = (S \mapsto M(S) \otimes N(S))^{\mathrm{sh}}$.

(ii) Functor Cond(Set) $\to$ Cond(Ab); $T \mapsto \mathbb{Z}[T]$ is symmetric monoidal with respect to the product and the tensor product, i.e. $\mathbb{Z}[T_1 \times T_2] = \mathbb{Z}[T_1] \otimes \mathbb{Z}[T_2]$.
Proof:

(iii) For $T \in$ Cond(Set), $\mathbb{Z}[T] \in$ Cond(Ab) is flat.
Proof: We need to show $- \otimes \mathbb{Z}[T]$: Cond(Ab) $\to$ Cond(Ab) is an exact functor. Take an exact sequence in Cond(Ab):

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0.$$

For any $S \in$ ExDisc, we have an exact sequence:

$$0 \longrightarrow X(S) \longrightarrow Y(S) \longrightarrow Z(S) \longrightarrow 0.$$

Tensoring with the free abelian group $\mathbb{Z}[T(S)]$, we get an exact sequence:

$$0 \longrightarrow X(S) \otimes \mathbb{Z}[T(S)] \longrightarrow Y(S) \otimes \mathbb{Z}[T(S)] \longrightarrow Z(S) \otimes \mathbb{Z}[T(S)] \longrightarrow 0,$$

i.e.

$$0 \longrightarrow (X \otimes \mathbb{Z}[T])(S) \longrightarrow (Y \otimes \mathbb{Z}[T])(S) \longrightarrow (Z \otimes \mathbb{Z}[T])(S) \longrightarrow 0.$$

Hence the sequence

$$0 \longrightarrow X \otimes \mathbb{Z}[T] \longrightarrow Y \otimes \mathbb{Z}[T] \longrightarrow Z \otimes \mathbb{Z}[T] \longrightarrow 0.$$

exact and $\mathbb{Z}[T]$ is flat. $\square$

(iv) Given any $M, N \in$ Cond(Ab), we can give the group of homomorphisms Hom(M, N) the structure of condensed abelian groups via the following definition, for any $S \in$ ExDisc,

$$\underline{\mathrm{Hom}}(M, N)(S) := \mathrm{Hom}(\mathbb{Z}[S] \otimes M, N).$$

So we define an internal Hom-functor object.

(v) There is an adjunction. For $P, M, N \in$ Cond(Ab), we have an isomorphism of abelian groups:

$$\mathrm{Hom}(P, \underline{\mathrm{Hom}}(M, N)) \cong \mathrm{Hom}(P \otimes M, N).$$

Proof: First, if $P = \mathbb{Z}[S]$ for some $S \in$ ExDisc, then

$$\mathrm{Hom}(P, \underline{\mathrm{Hom}}(M, N)) = \mathrm{Hom}(\mathbb{Z}[S], \underline{\mathrm{Hom}}(M, N)) = \underline{\mathrm{Hom}}(M, N)(S) = \mathrm{Hom}(\mathbb{Z}[S] \otimes M, N).$$

Now, for general $P \in$ Cond(Ab), we can write $P = \varinjlim \mathbb{Z}[S_i]$, so

$$\begin{aligned} \mathrm{Hom}(P, \underline{\mathrm{Hom}}(M, N)) &= \mathrm{Hom}(\varinjlim \mathbb{Z}[S_i], \underline{\mathrm{Hom}}(M, N)) \\ &= \varprojlim \mathrm{Hom}(\mathbb{Z}[S_i], \underline{\mathrm{Hom}}(M, N)) \\ &= \varprojlim \mathrm{Hom}(\mathbb{Z}[S_i] \otimes M, N) \\ &= \mathrm{Hom}(\varinjlim \mathbb{Z}[S_i] \otimes M, N) \\ &= \mathrm{Hom}(P \otimes M, N). \end{aligned}$$

$\square$

(vi) As Cond(Ab) has enough projectives, one can form the derived category $D(\text{Cond(Ab)})$. If $P \in$ Cond(Ab) is compact and projective, then $P[0] \in D(\text{Cond(Ab)})$ is a compact object of the dereived category, i.e. $\mathrm{Hom}(P, -)$ commutes with arbitrary direct sums. In particular, $D(\text{Cond(Ab)})$ is compactly generated.

(vii) Similarly, in the derived category $D(\text{Cond(Ab)})$, we have the adjunction:

$$\mathrm{Hom}(P, R\underline{\mathrm{Hom}}(M, N)) \cong \mathrm{Hom}(P \otimes^L M, N).$$

(viii) Let $\mathcal{D}(\text{Cond(Ab)})$ denote the derived ∞-category of Cond(Ab) and $\mathcal{D}(\text{Ab})$ denote the derived ∞-category of Ab, then there is an equivalence

$$\mathcal{D}(\text{Cond(Ab)}) \cong \text{Cond}(\mathcal{D}(\text{Ab})).$$

3 $D(R)$

Definition 3.1. An ∞-category is a simplicial set $\mathcal{C}$ which satisfies the following extension condition:

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, hook] & \mathcal{C} \\ \Delta^n \arrow[ur, dotted, "\exists f"'] & \end{tikzcd} \qquad 0 < \forall i < n.$$

Definition 3.2. Let $\mathcal{C}$ be an ∞-category. A zero object of $\mathcal{C}$ is an object which is both initial and final. We say that $\mathcal{C}$ is pointed if $\mathcal{C}$ contains a zero object.

Definition 3.3. Let $\mathcal{C}$ be a pointed ∞-category. A triangle in $\mathcal{C}$ is a diagram $\Delta^1 \times \Delta^1 \to \mathcal{C}$ depicted as

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d, "g"] \\ 0 \arrow[r] & Z \end{tikzcd}$$

where 0 is a zero object in $\mathcal{C}$.

We say a triangle in $\mathcal{C}$ is a fiber sequence if it is a pullback and say a triangle in $\mathcal{C}$ is a cofiber sequence if it is a pushout.

We generally indicate a triangle by specifying only the pair of maps $X \xrightarrow{f} Y \xrightarrow{g} Z$.

Remark 3.4. Let $\mathcal{C}$ be a pointed ∞-category. A triangle in $\mathcal{C}$ consists of the following data:

(i) A pair of morphisms $f : X \to Y$ and $g : Y \to Z$ in $\mathcal{C}$.

(ii) A 2-simplex in $\mathcal{C}$ corresponding to a diagram

$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, "h"] & & Z \end{tikzcd}$$

in $\mathcal{C}$, which identifies h with the composition $g \circ f$.

(iii) A 2-simplex

$$\begin{tikzcd} & 0 \arrow[dr] & \\ X \arrow[ur] \arrow[rr, "h"] & & Z \end{tikzcd}$$

in $\mathcal{C}$, which we view as anullhomotopy of h.

Definition 3.5. Let $\mathcal{C}$ be a pointed ∞-category containing a morphism $g : X \longrightarrow Y$. A fiber of g is a fiber sequence

$$\begin{tikzcd} W \arrow[r] \arrow[d] & X \arrow[d, "g"] \\ 0 \arrow[r] & Y \end{tikzcd}$$

and we denote $W = \text{fib}(g)$.

Dually, a cofiber of g is a cofiber sequence

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d] & Y \arrow[d] \\ 0 \arrow[r] & Z \end{tikzcd}$$

and we denote $Z = \text{cofib}(g)$.

Definition 3.6. An ∞-category $\mathcal{C}$ is stable if it satisfies the following conditions:

(i) There exists a zero object $0 \in \mathcal{C}$.

(ii) Every morphism in $\mathcal{C}$ admits a fiber and a cofiber.

(iii) A triangle in $\mathcal{C}$ is a fiber sequence if and only if it is a cofiber sequence.

Remark 3.7. (i) For a stable ∞-category $\mathcal{C}$, we define the suspension functor $\Sigma : \mathcal{C} \longrightarrow \mathcal{C}$ and the loop functor $\Omega : \mathcal{C} \longrightarrow \mathcal{C}$ as follows:

$$\Sigma(X) := \text{cofib}(X \longrightarrow 0)$$

and

$$\Omega(X) := \text{fib}(0 \longrightarrow X).$$

(ii) For a stable ∞-category $\mathcal{C}$, there is a homotopy equivalence:

$$\text{Map}_{\mathcal{C}}(\Sigma X, Y) \xrightarrow{\sim} \text{Map}_{\mathcal{C}}(X, \Omega Y).$$

Besides, the unit $X \longrightarrow \Omega\Sigma(X)$and $\Sigma\Omega(Y) \longrightarrow Y$counit are isomorphic.

Definition 3.8. Let R be a commutative ring, the ∞-category $D(R)$ is a stable ∞-category with all colimits, generated (as a cocomplete stable ∞-category) by a distinguished compact object 1, satisfying

$$\pi_0\mathrm{Map}(1,1) = R^{\mathrm{op}}, \qquad \pi_0\mathrm{Map}(\Sigma^d 1, 1) = 0,\ \forall d \neq 0.$$

For $X, Y \in D(R)$, we define

$$[X,Y] := \pi_0\mathrm{Map}(X,Y)$$

and

$$[X,Y]_d := [\Sigma^d X, Y] = [X, \Omega^d Y].$$

Remark 3.9. (i) From the definition of $D(R)$, we have

$$[1,1] = \pi_0\mathrm{Map}(1,1) = R^{\mathrm{op}}, \qquad [1,1]_d = \pi_0\mathrm{Map}(\Sigma^d 1, 1) = 0,\ \forall d \neq 0.$$

(ii) If $d \geq 0$, we have

$$[X,Y]_d = \pi_d\mathrm{Map}(X,Y).$$

(iii) Claim: In $D(R)$, for any integer d, we have $[X,Y]_d \in \mathrm{Ab}$.
Proof: First, if $d \geq 2$, $[X,Y]_d = \pi_d\mathrm{Map}(X,Y) \in \mathrm{Ab}$. For any $d \in \mathbb{Z}$,

$$[X,Y]_d = [\Sigma^d X, Y] = [\Sigma^{d-2} X, Y]_2 \in \mathrm{Ab}.$$

(iv) For a stable ∞-category, a fiber sequence $X \to Y \to Z$ is at the same time a cofiber sequence, and vice versa. Hence, we will call it a fiber-cofiber sequence.

(v) For a fiber-cofiber sequence $X \to Y \to Z$ in $D(R)$, we can induce a new fiber-cofiber sequence $Y \to Z \to \Sigma X$.

(vi) Given a fiber-cofiber sequence $X \to Y \to Z$ and any $A \in D(R)$, we can induce two long exact sequences:

$$\cdots \longrightarrow [A,X]_d \longrightarrow [A,Y]_d \longrightarrow [A,Z]_d \longrightarrow [A,X]_{d-1} \longrightarrow \cdots$$

$$\cdots \longrightarrow [X,A]_{d+1} \longrightarrow [Z,A]_d \longrightarrow [Y,A]_d \longrightarrow [X,A]_d \longrightarrow \cdots .$$

(vii) Assume

$$\begin{array}{ccc} A & \longrightarrow & B \\ \big\downarrow & & \big\downarrow \\ C & \longrightarrow & D \end{array}$$

is a pushout-pullback square in $D(R)$, then we can produce a triangle $A \to B \oplus C \to D$. With this, we can induce a long exact sequence.

Definition 3.10. For any integer d, we define a functor $H_d : D(R) \to \mathrm{Mod}_R;\ X \mapsto [1, X]_d$.

Remark 3.11. (i) We already know $[1, X]_d \in \mathrm{Ab}$. And we need to show $[1, X]_d$ is an R-module.

In fact, we have

$$\mathrm{Map}(\Sigma^d 1, \Sigma^d 1) \times \mathrm{Map}(\Sigma^d 1, X) \to \mathrm{Map}(\Sigma^d 1, X),$$

applying the functor π_0, we get:

$$\pi_0 \mathrm{Map}(\Sigma^d 1, \Sigma^d 1) \times \pi_0 \mathrm{Map}(\Sigma^d 1, X) = \pi_0(\mathrm{Map}(\Sigma^d 1, \Sigma^d 1) \times \mathrm{Map}(\Sigma^d 1, X)) \to \pi_0(\mathrm{Map}(\Sigma^d 1, X)),$$

i.e.

$$R^{\mathrm{op}} \times [1, X]_d \to [1, X]_d,$$

which implies $[1, X]_d \in \mathrm{Mod}_R$.

(ii) $H_d : D(R) \to \mathrm{Mod}_R;\ X \mapsto [1, X]_d$ is a representable functor and $\Sigma^d 1$ represents H_d.

Lemma 3.12. (i) $H_d(\prod_i X_i) = \prod_i H_d(X_i)$.

(ii) $H_d(\oplus_i X_i) = \oplus_i H_d(X_i)$.

(iii) $H_d(\varinjlim X_i) = \varinjlim H_d(X_i)$.

(iv) For a sequence of maps $\cdots \to X_n \to X_{n-1} \to \cdots$, we have a Milnor sequence:

$$0 \longrightarrow \varprojlim{}^1 H_{d+1}(X_n) \longrightarrow H_d(\varprojlim X_n) \longrightarrow \varprojlim H_d(X_n) \longrightarrow 0.$$

Proposition 3.13. $f : X \to Y$ in $D(R)$ is an isomorphism if and only if $H_d(f) : H_d(X) \xrightarrow{\sim} H_d(Y)$, for $\forall d \in \mathbb{Z}$.

Proof. Take $Z = \mathrm{cofib}(X \xrightarrow{f} Y)$, then $X \to Y \to Z$ is a fiber-cofiber sequence, and we can induce a long exact sequence

$$\cdots \to H_d(X) \to H_d(Y) \to H_d(Z) \to H_{d-1}(X) \to \cdots .$$

It suffices to show: if $Z \in D(R)$ with $H_d(Z) = 0,\ \forall d \in \mathbb{Z}$, then $Z = 0$.
Consider the full subcategory of $D(R)$:

$$\mathcal{C} = \{A \in D(R) |\ [\Sigma^d A, Z] = 0,\ \forall d \in \mathbb{Z}\}.$$

Observe that:

- $1 \in \mathcal{C}$.
- $\mathcal{C}$ is stable under colimits. This is because

$$[\Sigma^d \text{colim } A_i, Z] = [\text{colim } \Sigma^d A_i, Z] = \lim\, [\Sigma^d A_i, Z] = 0.$$

- $\mathcal{C}$ is stable under cofibers.

By definition of $D(R)$, we know $D(R)$ is generated as a cocomplete stable ∞-category by 1. Hence, $D(R) = \mathcal{C}$. Then by Yoneda's lemma, $Z = 0$. $\square$

Proposition 3.14. Let $X \in D(R)$, then there exists $Y \in D(R)$ with a map $f : Y \to X$, s.t.

(i) $H_d(Y) = 0,\ \forall d < 0$.

(ii) $H_d(f) : H_d(Y) \xrightarrow{\sim} H_d(X)$ are isomorphisms, $\forall d \geq 0$.

Proof. We first prove: there exists a sequence of maps $Y_0 \to Y_1 \to Y_2 \to \cdots$ in $D(R)_{/X}$, s.t. for any $n \geq 0$, $H_d(Y_n) = 0,\ d < 0$ and $H_d(Y_n \to X)$ are isomorphisms if $0 \leq d < n$ and is a surjection if $d = n$.
We prove this by induction.
First, $n = 0$. Let $Y_0 = \oplus_I 1$ for $I =$ cartinal of $H_0(X)$, then the map $Y_0 \to X$ can induce a surjection $H_0(Y_0) = R^{\oplus I} \twoheadrightarrow H_0(X)$ and for $d < 0$, $H_d(Y_0) = 0$.
Now we assume that there exists a sequence

$$Y_0 \to Y_1 \to Y_2 \to \cdots \to Y_{n-1}$$

in $D(R)_{/X}$ satisfying the assumption.

Let $F = \text{fib}(Y_{n-1} \to X)$, then $F \to Y_{n-1} \to X$ is a fiber-cofiber sequence. We can find an index I, s.t. $\Sigma^{n-1} \oplus_I 1 \to F$ can induce a surjection $H_{n-1}(\Sigma^{n-1} \oplus_I 1) \twoheadrightarrow H_{n-1}(F)$. Then let $Y_n = \text{cofib}(\Sigma^{n-1} \oplus_I 1 \to F \to Y_{n-1})$, hence $\Sigma^{n-1} \oplus_I 1 \to Y_{n-1} \to Y_n$ is also a fiber-cofiber sequence. Now, we check it satisfies the requirements.

(a) $d < 0$. The fiber-cofiber sequence $\Sigma^{n-1} \oplus_I 1 \to Y_{n-1} \to Y_n$ can induce a long exact sequence:

$$\cdots \to H_{-1}(\Sigma^{n-1} \oplus_I 1) \to H_{-1}(Y_{n-1}) \to H_{-1}(Y_n) \to H_{-2}(\Sigma^{n-1} \oplus_I 1) \to H_{-2}(Y_{n-1}) \to \cdots .$$

Since for $k < 0$, $H_k(Y_{n-1}) = 0$, we know $H_d(Y_n) = H_{d-1}(\Sigma^{n-1} \oplus_I 1) = 0 (d < 0)$.

(b) First, there exists a map $Y_n \to X$, this is because

$$\begin{tikzcd}
\Sigma^{n-1} \oplus_I 1 \arrow[r] \arrow[d] & Y_{n-1} \arrow[r] \arrow[d, equal] & Y_n \arrow[d, dotted, "\exists"] \\
F \arrow[r] & Y_{n-1} \arrow[r] & X
\end{tikzcd}$$

and $H_d(\Sigma^{n-1} \oplus_I 1) = 0, \forall d \neq n-1$, then by

$$\cdots \to H_{n-2}(\Sigma^{n-1} \oplus_I 1) \to H_{n-2}(Y_{n-1}) \to H_{n-2}(Y_n) \to H_{n-3}(\Sigma^{n-1} \oplus_I 1) \to \cdots ,$$

it implies that $0 \leq \forall d \leq n-2,\ H_d(Y_n) \cong H_d(Y_{n-1}) \cong H_d(X)$.

We have the following diagram:

$$\begin{tikzcd}
H_n(Y_{n-1}) \arrow[r] \arrow[d, equal] & H_n(Y_n) \arrow[r] \arrow[d, dotted, two heads] & H_{n-1}(\Sigma^{n-1} \oplus_I 1) \arrow[r] \arrow[d, two heads] & H_{n-1}(Y_{n-1}) \arrow[r] \arrow[d, equal] & H_{n-1}(Y_n) \arrow[r] \arrow[d, dotted, "\sim"] & H_{n-2}(\Sigma^{n-1} \oplus_I 1) \arrow[r] \arrow[d, "\sim"] & H_{n-2}(Y_{n-1}) \arrow[d, equal] \\
H_n(Y_{n-1}) \arrow[r] & H_n(X) \arrow[r] & H_{n-1}(F) \arrow[r] & H_{n-1}(Y_{n-1}) \arrow[r] & H_{n-1}(X) \arrow[r] & H_{n-2}(F) \arrow[r] & H_{n-2}(Y_{n-1})
\end{tikzcd}$$

By five's lemma, we can show $H_{n-1}(Y_n) \xrightarrow{\sim} H_{n-1}(X)$ and $H_n(Y_n) \twoheadrightarrow H_n(X)$.

Now, for $Y_0 \to Y_1 \to Y_2 \to \cdots$, we take $Y = \varinjlim Y_n$, and hence we can get a map $Y \to X$.

By $H_d(\varinjlim Y_n) = \varinjlim H_d(Y_n)$, for $d < 0$, $H_d(Y) = 0$, and for $d \geq 0$,

$$H_d(Y) = \varinjlim \left(H_d(Y_0) \to H_d(Y_1) \cdots \to H_d(Y_{d+1}) \to H_d(Y_{d+2}) \to \cdots\right) = H_d(X).$$

$\square$

Proposition 3.15. For $X \in D(R)$, the following are equivalent:

(i) $H_d(X) = 0, \forall d < 0$.

(ii) X is generated by 1 under colimits.

(iii) There exists a sequence of maps $X_0 \to X_1 \to X_2 \to \cdots$ with $X = \varinjlim X_i$, where for each i, the cofiber $\mathrm{cofib}(X_{i-1} \to X_i)$ is of the form $\Sigma^i \oplus_I 1$.

Proof. (i) $\implies$ (iii). By previous proposition, for $X \in D(R)$, there exists a map $f : Y \to X$ with $H_d(Y) = 0$ for $d < 0$ and $H_d(f)$ are isomorphisms for $d \geq 0$. Then, for $d < 0$,

$$H_d(X) = H_d(Y) = 0.$$

Hence, for any $d \in \mathbb{Z}$, $H_d(f)$ are isomorphisms. Thus, $f : Y \xrightarrow{\sim} X$.
From the construction of Y, we know there is a sequence of maps $X_0 \to X_1 \to X_2 \to \cdots$ with $\varinjlim X_i = Y \cong X$.
By the fiber-cofiber sequence $\Sigma^{n-1} \oplus_I 1 \to X_{n-1} \to X_n$, we can get a new fiber-cofiber sequence $X_{n-1} \to X_n \to \Sigma^n \oplus_I 1$, i.e. $\mathrm{cofib}(X_{n-1} \to X_n) = \Sigma^n \oplus_I 1$.
(iii) $\implies$ (ii).
We have $X_{i-1} \to X_i \to \Sigma^i \oplus_I 1$, which gives a new fiber-cofiber sequence: $\Sigma^{i-1} \oplus_I 1 \to X_{i-1} \to X_i$. Then $X_i = \mathrm{cofib}(\Sigma^{i-1} \oplus_I 1 \to X_{i-1}) = \mathrm{colim}(0 \leftarrow \Sigma^{i-1} \oplus_I 1 \to X_{i-1})$.
Now, $X_1 = \mathrm{cofib}(\oplus_I 1 \to X_0) = \mathrm{cofib}(\oplus_I 1 \to \oplus_J 1) = \mathrm{colim}(0 \leftarrow \oplus_I 1 \to \oplus_J 1)$. Hence, each X_i is generated by 1 under colimits. Finally, $X = \mathrm{colim}\, X_i$ is also generated by 1 under colimits.
(ii) $\implies$ (i).
Arbitrary colimits can be written in terms of pushouts and filtered colimits. And H_d commutes with filtered colimits. So it suffices to show that for A, B, C with $H_d(A) = H_d(B) = H_d(C) = 0$, $\forall d < 0$, then for the pushout $D = \mathrm{colim}(C \leftarrow A \to B)$, $H_d(D) = 0$, $\forall d < 0$.
This is because we can get a null-composite sequence $A \to B \oplus C \to D$, and induce a long exact sequence

$$\cdots \to H_d(A) \to H_d(B) \oplus H_d(C) \to H_d(D) \to \cdots$$

which implies $H_d(D) = 0, \ \forall d < 0$. $\square$

Definition 3.16. (i) $D(R)_{\geq 0} := \{X \in D(R) \mid H_d(X) = 0, \ \forall d < 0\}$.

(ii) $D(R)_{<0} := \{X \in D(R) \mid H_d(X) = 0, \ \forall d \geq 0\}$.

(iii) $\tau_{\geq 0} : D(R) \to D(R)_{\geq 0}; X \mapsto \tau_{\geq 0}(X) := Y$, which is constructed in Proposition 3.14.

Now, given any map $Z \to X$ in $D(R)$, we can get a commutative diagram:

$$\begin{tikzcd} \tau_{\geq 0}(Z) \arrow[r, dashed, "\exists!"] \arrow[d] & \tau_{\geq 0}(X) \arrow[d] \\ Z \arrow[r] & X \end{tikzcd}$$

Hence, $\tau_{\geq 0} : D(R) \to D(R)_{\geq 0}$ is a functor.

Proposition 3.17. $D(R)_{\geq 0} \underset{\tau_{\geq 0}}{\overset{i}{\rightleftarrows}} D(R) \underset{i}{\overset{\tau_{<0}}{\rightleftarrows}} D(R)_{<0}$, i.e. $i \dashv \tau_{\geq 0}$ and $\tau_{<0} \dashv i$.

Corollary 3.18. For $X \in D(R)$, we have

$$X \cong \varprojlim \tau_{\leq n}(X) \quad \text{and} \quad X \cong \varinjlim \tau_{\geq -n}(X).$$

Proof. We have a Milnor sequence:

$$0 \longrightarrow \varprojlim{}^1 H_{d+1}(\tau_{\leq n} X) \longrightarrow H_d(\varprojlim \tau_{\leq n} X) \longrightarrow \varprojlim H_d(\tau_{\leq n} X) \longrightarrow 0.$$

For $n \gg 0$, we have $H_d(\tau_{\leq n} X) = H_d(X)$, hence $\varprojlim H_d(\tau_{\leq n} X) = H_d(X)$.
And $\{H_{d+1}(\tau_{\leq n} X)\}_{n \in \mathbb{Z}}$ satisfies the Mittag-Leffler condition, hence $\varprojlim{}^1 H_{d+1}(\tau_{\leq n} X) = 0$. Therefore, from the above short exact sequence, we have

$$H_d(\varprojlim \tau_{\leq n} X) \cong H_d(X), \ \forall d \in \mathbb{Z},$$

which impies $X \cong \varprojlim \tau_{\leq n} X$.
For another isomorphsim, from

$$H_d(\varinjlim \tau_{\geq -n} X) = \varinjlim H_d(\tau_{\geq -n} X) = H_d(X), \ \forall d \in \mathbb{Z},$$

one can show $X \cong \varinjlim \tau_{\geq -n}(X)$. $\square$

Definition 3.19. For any map $f : X \to Y$ in $D(R)$, we define its kernel to be

$$\ker(f) := \tau_{\geq 0}\mathrm{fib}(X \to Y)$$

and its cokernel to be

$$\mathrm{coker}(f) := \tau_{\leq 0}\mathrm{cofib}(X \to Y).$$

Proposition 3.20. Let $D(R)_0 = \{X \in D(R) \mid H_d(X),\ \forall d \neq 0\}$.

(i) There is an isomorphism $H_0 : D(R)_0 \xrightarrow{\sim} \mathrm{Mod}_R$.

(ii) Any object in $D(R)_0$ can be written as of the form $\mathrm{coker}(\oplus_I 1 \to \oplus_J 1)$.

(iii) $H_0 : D(R)_0 \longrightarrow \mathrm{Mod}_R$ is an exact functor.

(iv) $H_0 : D(R)_0 \longrightarrow \mathrm{Mod}_R$ commutes with direct sums.

Proof. (ii) For $X \in D(R)_0$, there exists $f : Y \to X$ with $H_d(Y) = 0,\ \forall d < 0$ and $H_d(f)$ are isomorphisms, $\forall d \geq 0$.
By the construction of Y_1, $Y_1 = \mathrm{cofib}(\oplus_I 1 \to \oplus_J 1)$.
On the other hand, $X \cong \tau_{\leq 0} Y_1$. Hence,

$$X \cong \tau_{\leq 0}\mathrm{cofib}(\oplus_I 1 \to \oplus_J 1) = \mathrm{coker}(\oplus_I 1 \to \oplus_J 1).$$

(iii) In order to show that H_0 preserves exact sequences, it suffices to show H_0 preserves kernels and cokernels.
For any map $f : X \to Y$ in $D(R)_0$, applying functor $\tau_{\geq 0}$ to sequence $\mathrm{fib}(f) \to X \to Y$, we get a fiber-cofiber sequence

$$\ker(f) = \tau_{\geq 0}\mathrm{fib}(f) \to X \to Y.$$

And it induces a long exact sequence

$$0 = H_1(Y) \to H_0(\ker(f)) \to H_0(X) \to H_0(Y) \to \cdots.$$

Hence, $H_0(\ker(f)) = \ker(H_0(X) \to H_0(Y))$.
Dually ,we can prove $H_0(\mathrm{coker}(f)) = \mathrm{coker}(H_0(X) \to H_0(Y))$.

$\square$

Remark 3.21. $1 \in D(R)_0$ is compact and projective.
Proof: Compactness is the definition.
For the projectiveness, we need to show that any epimorphism $X \twoheadrightarrow 1$ splits.
Let $F = \mathrm{fib}(X \to 1)$. Consider $F \to X \to 1$. Then $H_{-1}(F) = 0$.
By $[M, N]_d = \mathrm{Ext}_R^{-d}(M, N)$, we get $\mathrm{Ext}_R^{-1}(1, F) = [1, F]_{-1} = H_{-1}(F) = 0$. Hence $X \twoheadrightarrow 1$ splits. $\square$

Definition 3.22. (i) A filtered object of $D(R)$ is an object in $\mathrm{Fun}(\mathbb{Z}_{\leq}, D(R))$, i.e.

$$\cdots \longrightarrow F(n-1) \to F(n) \to F(n+1) \to \cdots .$$

(ii) A filtered object F is convergent if $\varprojlim F(n) = 0$.

(iii) $F(\infty) := \varinjlim F(n)$. Call it the underlying object of F.

(iv) The n-th associated graded $\mathrm{gr}_n(F) := \mathrm{cofib}(F(n-1) \to F(n)) \triangleq F(n)/F(n-1)$.

Now, giving a convergent filtered object $F : \mathbb{Z}_{\leq} \to D(R)$, s.t. $\mathrm{gr}_n(F) \in D(R)_n,\ \forall n$, we can define an R-module M_n:

$$H_n : D(R)_n \to \mathrm{Mod}_R; \mathrm{gr}_n(F) \mapsto H_n(\mathrm{gr}_n(F)) \triangleq M_n.$$

From the sequence

$$F(n-1)/F(n-2) \longrightarrow F(n)/F(n-2) \longrightarrow F(n)/F(n-1) \longrightarrow \Sigma(F(n-1)/F(n-2)),$$

we get a map $d : H_n(\mathrm{gr}_n(F)) \longrightarrow H_n(\Sigma\mathrm{gr}_{n-1}(F))$, i.e. $d : M_n \longrightarrow M_{n-1}$.
One can check $d^2 = 0$. Hence, given a convergent filtered object F, s.t. $\mathrm{gr}_n(F) \in D(R)_n$, we define a chain complex of R-modules M_*.
We denote $\mathrm{Fun}(\mathbb{Z}_{\leq}, D(R))_{\mathrm{cx}} = \{F \in \mathrm{Fun}(\mathbb{Z}_{\leq}, D(R)) \mid F \text{ convergent}\}$.

Proposition 3.23. (i) $\mathrm{Fun}(\mathbb{Z}_{\leq}, D(R))_{\mathrm{cx}} \xrightarrow{\sim} \mathrm{Ch}_R;\ F \mapsto M_*$.

(ii) $H_n(F(\infty)) = H_n(M_*)$, $\forall n$.

4 $D(\mathbb{Z})$

Definition 4.1. Let $X \in$ Top. A sieve on X is a set $\mathfrak{U}$ of open subsets of X, s.t. if $V \in \mathfrak{U}$ and $V' \subset V$, then $V' \in \mathfrak{U}$. If $U = \cup_{V \in \mathfrak{U}} V$, we say that the sieve $\mathfrak{U}$ covers U.

Definition 4.2. (i) Let $X \in$ Top. Let $\mathcal{F} \in$ PSh$(X, D(\mathbb{Z})$ be a presheaf with values in $D(\mathbb{Z})$, i.e. $\mathcal{F} \in$ Fun(Op$(X)^{\text{op}}, D(\mathbb{Z})$). We say $\mathcal{F}$ is a sheaf if for all sieves $\mathfrak{U}$ on X covering $U \in$ Op(X), we have

$$\mathcal{F}(U) \xrightarrow{\sim} \varprojlim_{V \in \mathfrak{U}^{\text{op}}} \mathcal{F}(V).$$

(ii) For $U \in$ Op(X), one define $h_U \in$ PSh$(X, D(\mathbb{Z}))$ via

$$h_U(V) = \begin{cases} * & V \subset U \\ \emptyset & \text{otherwise} \end{cases}$$

(iii) For a sieve $\mathfrak{U}$, one define $h_{\mathfrak{U}} \in$ PSh$(X, D(\mathbb{Z}))$ via

$$h_{\mathfrak{U}}(V) = \begin{cases} * & V \in \mathfrak{U} \\ \emptyset & V \notin \mathfrak{U} \end{cases}$$

Proposition 4.3. Let $\mathcal{F} \in$ PSh$(X, D(\mathbb{Z}))$, then $\mathcal{F}$ is a sheaf if and only if it satisfies:

(i) $\mathcal{F}(\emptyset) = *$.

(ii) For any open subsets $V, V' \in$ Op(X),

$$\mathcal{F}(V \cup V') \xrightarrow{\sim} \mathcal{F}(V) \times_{\mathcal{F}(V \cap V')} \mathcal{F}(V').$$

(iii) For any sieve $\mathfrak{U}$, $\mathcal{F}(\varinjlim_{V \in \mathfrak{U}} V) \xrightarrow{\sim} \varprojlim_{V \in \mathfrak{U}^{\text{op}}} \mathcal{F}(V)$.

Remark 4.4.

$$\mathbb{Z}[h_U](V) := \mathbb{Z}[h_U(V)] = \begin{cases} \mathbb{Z} & V \subseteq U \\ 0 & V \nsubseteq U \end{cases}$$

$$\mathbb{Z}[h_{\mathfrak{U}}](V) := \mathbb{Z}[h_{\mathfrak{U}}(V)] = \begin{cases} \mathbb{Z} & V \in \mathfrak{U} \\ 0 & V \notin \mathfrak{U} \end{cases}$$

$$\text{Map}(\mathbb{Z}[h_U], \mathcal{F}) = \text{Map}(\mathbb{Z}, \mathcal{F}(U)).$$

$$\begin{aligned} \text{Map}(\mathbb{Z}[h_{\mathfrak{U}}], \mathcal{F}) &= \varprojlim_{V \in \mathfrak{U}^{\text{op}}} \text{Map}(\mathbb{Z}[h_V], \mathcal{F}) \\ &= \varprojlim_{V \in \mathfrak{U}^{\text{op}}} \text{Map}(\mathbb{Z}, \mathcal{F}(V)) \\ &= \text{Map}(\mathbb{Z}, \varprojlim_{V \in \mathfrak{U}^{\text{op}}} \mathcal{F}(V)) \\ &= \text{Map}(\mathbb{Z}, \mathcal{F}(\varinjlim_{V \in \mathfrak{U}} V)). \end{aligned}$$

Proposition 4.5. $\text{PSh}(X, D(\mathbb{Z})) \underset{i}{\overset{\text{sh}}{\rightleftarrows}} \text{Sh}(X, D(\mathbb{Z}))$; $\mathcal{F} \mapsto \mathcal{F}^{\text{sh}}$. Moreover, $\mathcal{F}^{\text{sh}} = 0$ iff $\mathcal{F}$ lies in the stable co-complete subcategory generated by cofib$(\mathbb{Z}[h_{\mathfrak{U}}] \to \mathbb{Z}[h_U])$ for all sieves $\mathfrak{U}$ covering U.

Definition 4.6. For $\mathcal{F} \in \text{PSh}(X, D(\mathbb{Z}))$, define $H_n(\mathcal{F}) \in \text{PSh}(X, \text{Ab})$ by $H_n(\mathcal{F})(U) = H_n(\mathcal{F}(U))$.

With this presheaf $H_n(\mathcal{F}) \in \text{PSh}(X, \text{Ab})$, one can sheafify it to get a sheaf $H_n(\mathcal{F})^{\text{sh}} \in \text{Sh}(X, \text{Ab})$.

Proposition 4.7. Let $\mathcal{F} \in \text{PSh}(X, D(\mathbb{Z}))$.

(i) If $\mathcal{F}^{\text{sh}} = 0$, then $H_n(\mathcal{F})^{\text{sh}} = 0,\ \forall n \in \mathbb{Z}$.

(ii) If $\mathcal{F}$ is bounded above and $H_n(\mathcal{F})^{\text{sh}} = 0,\ \forall n \in \mathbb{Z}$, then $\mathcal{F}^{\text{sh}} = 0$.

Corollary 4.8. Let $\mathcal{F} \to \mathcal{G}$ be a map in $\text{PSh}(X, D(\mathbb{Z}))$.

(i) If $\mathcal{F}^{\text{sh}} \xrightarrow{\sim} \mathcal{G}^{\text{sh}}$, then $H_n(\mathcal{F})^{\text{sh}} \xrightarrow{\sim} H_n(\mathcal{G})^{\text{sh}},\ \forall n \in \mathbb{Z}$.

(ii) If $\mathcal{F}$ and $\mathcal{G}$ are bounded above, and $H_n(\mathcal{F})^{\text{sh}} \xrightarrow{\sim} H_n(\mathcal{G})^{\text{sh}},\ \forall n \in \mathbb{Z}$, then $\mathcal{F}^{\text{sh}} \xrightarrow{\sim} \mathcal{G}^{\text{sh}}$.

Corollary 4.9. Let $\mathcal{F} \to \mathcal{G}$ be a map in $\text{PSh}(X, D(\mathbb{Z}))$ and $\mathcal{F}, \mathcal{G}$ are bounded above, then

$$\mathcal{F}^{\text{sh}} \xrightarrow{\sim} \mathcal{G} \quad \iff \quad \begin{cases} \mathcal{G} \text{ is a sheaf.} \\ H_n(\mathcal{F})^{\text{sh}} \xrightarrow{\sim} H_n(\mathcal{G})^{\text{sh}},\ \forall n \in \mathbb{Z}. \end{cases}$$

Definition 4.10.

Proposition 4.11.

5 The t-structure on valued sheaves

Definition 5.1. A t-structure on a stable ∞-category $\mathcal{C}$ is a pair $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ of full sub-∞-categories of $\mathcal{C}$ that are stable under equivalences and satisfy:

(T1) The suspension functor Σ and the loop functor Ω restrict to $\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0}$ resp. are fully faithful functors $\Sigma : \mathcal{C}_{\geq 0} \to \mathcal{C}_{\geq 0}$ and $\Omega : \mathcal{C}_{\leq 0} \to \mathcal{C}_{\leq 0}$.

(T2) If $X \in \mathcal{C}_{\geq 0}$ and $Y \in \mathcal{C}_{\leq 0}$, then $\mathrm{Map}(X, \Omega Y) \simeq *$.

(T3) For every $X \in \mathcal{C}$, there exists a fiber sequence

$$X' \longrightarrow X \longrightarrow X''$$

with $X' \in \mathcal{C}_{\geq 0}$ and $X'' \in \mathcal{C}_{\leq -1} := \Omega \mathcal{C}_{\leq 0}$.

We call $\mathcal{C}_{\geq 0}$ and $\mathcal{C}_{\leq 0}$ the connective and coconnective parts of the t-structure.

Given $n \in \mathbb{Z}$, we define $\mathcal{C}_{\geq n} := \Sigma^n \mathcal{C}_{\geq 0} \subset \mathcal{C}$ and $\mathcal{C}_{\leq n} := \Sigma^n \mathcal{C}_{\leq 0} \subset \mathcal{C}$, where for $n < 0$, we have $\Sigma^n = \Omega^{-n}$.

The inclusions $i : \mathcal{C}_{\geq m} \to \mathcal{C}$ and $s : \mathcal{C}_{\leq n} \to \mathcal{C}$ admit adjoint functors

$$\begin{tikzcd} \mathcal{C}_{\geq m} \arrow[r, shift left, "i"] & \mathcal{C} \arrow[l, shift left, "r"] \arrow[r, shift left, "p"] & \mathcal{C}_{\leq n} \arrow[l, shift left, "s"] \end{tikzcd} .$$

In particular, the full sub-∞-category $\mathcal{C}_{\geq m} \subset \mathcal{C}$ is closed under colimits, and the full sub-∞-category $\mathcal{C}_{\leq n} \subset \mathcal{C}$ is closed under limits. From the adjoint pairs, we can form their counit and unit, and we get

$$\tau_{\geq 0} X = (i \circ r)(X) \xrightarrow{\epsilon} X \xrightarrow{\eta} \tau_{\leq -1} X = (s \circ p)(X).$$

The composition of the two maps is a point in the anima $\mathrm{Map}(\tau_{\geq 0} X, \tau_{\leq -1} X) \simeq *$. So the composite map automatically admits a null-homotopy, which is unique, up to contractible ambiguity. We have the following commutative diagram:

$$\begin{tikzcd} \mathcal{C}_{\leq m} \cap \mathcal{C}_{\geq n} \arrow[r, shift left, "i"] \arrow[d, shift left, "s"] & \mathcal{C}_{\leq m} \arrow[l, shift left, "r"] \arrow[d, shift left, "s"] \\ \mathcal{C}_{\geq n} \arrow[u, shift left, "p"] \arrow[r, shift left, "i"] & \mathcal{C} \arrow[l, shift left, "r"] \arrow[u, shift left, "p"] \end{tikzcd}$$

The canonical map

$$p \circ r \xrightarrow{\eta \circ p \circ r} r \circ i \circ p \circ r \simeq r \circ p \circ i \circ r \xrightarrow{r \circ p \circ \epsilon} r \circ p$$

is an equivalence.

We say the full sub-∞-category

$$\mathcal{C}^{\heartsuit} := \mathcal{C}_{\geq 0} \cap \mathcal{C}_{\leq 0} \subset \mathcal{C}$$

is the heart of the t-structure. For the functor

$$\pi_0 := \tau_{\geq 0} \circ \tau_{\leq 0} \simeq \tau_{\leq 0} \circ \tau_{\geq 0} : \mathcal{C} \to \mathcal{C}^{\heartsuit},$$

we call it the zeroth homotopy functor. The functor π_0 is additive, but is NOT exact. Instead, for all $n \in \mathbb{Z}$, we define

$$\pi_d : \mathcal{C} \longrightarrow \mathcal{C}^{\heartsuit}$$

to be $\pi_d = \pi_0 \circ \Omega^d$, and call it the dth homotopy functor. Now, a fiber sequence

$$Z \xrightarrow{g} Y \xrightarrow{f} X$$

in $\mathcal{C}$ gives rise to a long exact sequence

$$\cdots \longrightarrow \pi_{d+1}(X) \longrightarrow \pi_d(Z) \longrightarrow \pi_d(Y) \longrightarrow \pi_d(X) \longrightarrow \cdots$$

in the heart $\mathcal{C}^{\heartsuit}$.

If $f : Y \to X$ is an equivalence, then $f : \pi_d(Y) \to \pi_d(X)$ is an isomorphism for all $d \in \mathbb{Z}$, but the opposite is generally not the case.

Now, for the stable ∞-category $D(\mathbb{Z})$, we defined homology functors $H_d : D(\mathbb{Z}) \to \mathrm{Mod}_{\mathbb{Z}}$ for all $d \in \mathbb{Z}$ by

$$H_d(X) \simeq \pi_0 \mathrm{Map}(\Sigma^d 1, X) \simeq \pi_0 \mathrm{Map}(1, \Omega^d X).$$

$D(\mathbb{Z})$ admits a t-structure $(D(\mathbb{Z})_{\geq 0}, D(\mathbb{Z})_{\leq 0})$, where the connective part $D(\mathbb{Z})_{\geq 0}$ is spanned by those X for which $H_d(X) \simeq 0$, for $d < 0$, and the coconnective part $D(\mathbb{Z})_{\leq 0}$ is spanned by those X for which $H_d(X) \simeq 0$, for $d > 0$. The zeroth homology functor

$$H_0 : D(\mathbb{Z})^{\heartsuit} \longrightarrow \mathrm{Mod}_{\mathbb{Z}}$$

is an equivalence of (abelian) categories. We have $H_d \simeq H_0 \circ \pi_d$, so the functors H_d and π_d encode the same information.

Proposition 5.2. Let $X \in \mathrm{Top}$, and let $\mathcal{C}$ be a stable ∞-category. A t-structure on $\mathcal{C}$ induces a t-structure on the stable ∞-category $\mathcal{P}(X, \mathcal{C})$ of $\mathcal{C}$-valued presheaves on X, where the coconnective part $\mathcal{P}(X, \mathcal{C})_{\leq 0} \simeq \mathcal{P}(X, \mathcal{C}_{\leq 0})$, and where the connective part $\mathcal{P}(X, \mathcal{C})_{\geq 0}$ is spanned by those $\mathcal{F}$ such that

$$\mathrm{Map}(\mathcal{F}, \Omega\mathcal{G}) \simeq *$$

for all $\mathcal{G} \in \mathcal{P}(X, \mathcal{C}_{\leq 0})$.

A functor $f : \mathcal{D} \to \mathcal{C}$ between stable ∞-categories is exact iff it is left exact iff it is right exact.

An exact funcor $f : \mathcal{D} \to \mathcal{C}$ between stable ∞-categories with t-structures is left t-exact if $f(\mathcal{D}_{\leq 0}) \subset \mathcal{D}_{\leq 0}$, and it is right t-exact if $f(\mathcal{D}_{\geq 0}) \subset \mathcal{D}_{\geq 0}$. It is t-exact if it is both left t-exact and right t-exact. If $f : \mathcal{D} \to \mathcal{C}$ admits right adjoint functor $g : \mathcal{C} \to \mathcal{D}$, then f is right t-exact iff g is left t-exact.

Theorem 5.3. Let $X \in \mathrm{Top}$ and $\mathcal{C}$ a presentable stable ∞-category.

(1) The sheafification functor $\mathrm{ass}_X : \mathcal{P}(X, \mathcal{C}) \to \mathrm{Sh}(X, \mathcal{C})$ is t-exact, and the inclusion functor $\iota_X : \mathrm{Sh}(X, \mathcal{C}) \to \mathcal{P}(X, \mathcal{C})$ is left t-exact.

(2) The composite functor

$$\mathrm{Sh}(X, \mathcal{C}^{\heartsuit}) \xrightarrow{\iota_X^{\heartsuit}} \mathcal{P}(X, \mathcal{C}^{\heartsuit}) \simeq \mathcal{P}(X, \mathcal{C})^{\heartsuit} \xrightarrow{\mathrm{ass}_X} \mathrm{Sh}(X, \mathcal{C})^{\heartsuit}$$

is an equivalence of categories.

Write π_0^p and π_0^s for the homotopy functors associated with the t-structure on presheaves and sheaves. Since ass_X is both exact and t-exact, we obtain a commutative square

$$\begin{tikzcd}
\mathcal{P}(X,\mathcal{C}) \arrow[r, "\pi_0^p"] \arrow[d, "\mathrm{ass}_X"] & \mathcal{P}(X,\mathcal{C})^{\heartsuit} \arrow[d, "\mathrm{ass}_X"] \\
\mathrm{Sh}(X,\mathcal{C}) \arrow[r, "\pi_0^s"] & \mathrm{Sh}(X,\mathcal{C})^{\heartsuit}
\end{tikzcd}$$

6 Sheaf

Lemma 6.1. If $\mathcal{A}$ is bounded above, i.e. $\exists d \in \mathbb{Z}$, s.t. $H_n(\mathcal{A}) = 0$,for all $n > d$, then $\mathcal{A}^{\text{sh}}$ is also bounded above.

Question. For finite sets X, X' with $X' \to X$ surjective and split, then

$$0 \to \mathbb{Z}[X] \to \mathbb{Z}[X'] \to \mathbb{Z}[X' \times_X X'] \to \mathbb{Z}[X' \times_X X' \times_X X'] \to \cdots$$

is exact.

Lemma 6.2. Arbitrary limits and filtered colimits preserves $D(\mathbb{Z})_{\leq d}$.

Proof. First we show $D(\mathbb{Z})_{\leq d}$ is closed under filtered colimits. Assume $X_i \in D(\mathbb{Z})_{\leq d}, i \in I$,then

$$H_n(\varinjlim X_i) = \varinjlim H_n(X_i) = 0, \text{ for any } n > d.$$

Hence $\varinjlim X_i \in D(\mathbb{Z})_{\leq d}$. Then we show $D(\mathbb{Z})_{\leq d}$ is closed under arbitrary limits. Assume $X_i \in D(\mathbb{Z})_{\leq d}, n > d$, then

$$\begin{aligned} H_n(\lim X_i) &= [\Sigma^n 1, \lim X_i] \\ &= \pi_0 \text{Map}(\Sigma^n 1, \lim X_i) \\ &= \pi_0 \lim \text{Map}(\Sigma^n 1, X_i) \\ &= \pi_0 \lim * \\ &= \pi_0 * \\ &= 0. \end{aligned}$$

Hence, $\lim X_i \in D(\mathbb{Z})_{\leq d}$. $\square$

Problem. What is the relation between $\pi_n(\lim X_i)$ and $\lim \pi_n(X_i)$. Similarly, the relation between $\pi_n(\text{colim} X_i)$ and $\text{colim} \pi_n(X_i)$.

Definition 6.3. We define the singular homology functor to be the composite of

$$\text{Top} \to \text{Cond(Set)} \hookrightarrow \text{Cond(An)} \to \text{An},$$

and denote it by h : Top $\to$ An, where Top $\to$ Cond(Set), $X \mapsto \underline{X}$; Cond(An) $\to$ An is the left adjoint of An $\hookrightarrow$ Cond(An).

Definition 6.4. For the forgetful functor $D(\mathbb{Z})_{\geq 0} \simeq$ Ani(Ab) $\to$ Ani(Set) $\simeq$ An, it has a left adjoint, and we denote it by

$$\mathbb{Z}[-] : \text{Ani(Set)} \to \text{Ani(Ab)};\ S \mapsto \mathbb{Z}[S].$$

Definition 6.5. For $X \in$ Top, we define its singular homology object to be

$$\mathbb{Z}[h(X)] \in \text{Ani(Ab)} \simeq D(\mathbb{Z})_{\geq 0} \subset D(\mathbb{Z}).$$

Lemma 6.6. Assume $\mathcal{A} \in \mathrm{Sh}(X, D(\mathbb{Z}))$, $H_n(\mathcal{A}) = 0, \forall n > d$, and $H_d(\mathcal{A}) \neq 0$, then $H_d(\mathcal{A})$ is a sheaf.

Proof. For $H_d(\mathcal{A}) \in \mathrm{PSh}(X, \mathrm{Ab})$, we need to check $H_d(\mathcal{A}) \in \mathrm{Sh}(X, \mathrm{Ab})$.

By denition, $H_d(\mathcal{A})(U) = H_d(\mathcal{A}(U)) = H_d(\varprojlim \mathcal{A}(V))$. By the Milnor's sequence, we have

$$0 \longrightarrow \varprojlim{}^1 H_{d+1}(\mathcal{A}(V)) \longrightarrow H_d(\varprojlim \mathcal{A}(V)) \longrightarrow \varprojlim H_d(\mathcal{A}(V)) \longrightarrow 0.$$

Because $H_{d+1}(\mathcal{A}) = 0$, so the left term of this short exact sequence is 0, hence

$$H_d(\mathcal{A})(U) = H_d(\varprojlim \mathcal{A}(V)) = \varprojlim H_d(\mathcal{A}(V)) = \varprojlim H_d(\mathcal{A})(V).$$

Hence, $H_d(\mathcal{A}) \in \mathrm{Sh}(X, \mathrm{Ab})$. $\square$

Proposition 6.7. Let $\mathcal{C}_0 \subset \mathcal{C}$ be a full subcategory, then the following full subcategories of $\mathcal{C}$ agree:

- the full subcategory generated under (small) colimits by $\mathcal{C}_0$;
- the full subcategory generated under filtered colimits and finite colimits by $\mathcal{C}_0$;
- the full subcategory generated under sifted colimits and finite produncts by $\mathcal{C}_0$.

7 Animation

Theorem 7.1 (Yoneda)**.** Let $\mathcal{C}$ be an ∞-category, the functor

$$\mathcal{C} \hookrightarrow \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{An});\ X \mapsto (Y \mapsto \mathrm{Hom}_{\mathcal{C}}(Y, X))$$

is fully faithful.

Remark 7.2. For S to be an anima, we mean S is an ∞-category; while S to be a Kan complex, we mean S is a 1-category.

Let $\mathcal{C}$ be a category which admits all small colimits.
Recall an object $X \in \mathcal{C}$ is compact (also called finitely presented) if $\mathrm{Hom}(X, -)$ commutes with filtered colimits.
An object $X \in \mathcal{C}$ is projective if $\mathrm{Hom}(X, -)$ commutes with reflexive coequalizers (coequalizers of parallel arrows $Y \rightrightarrows Z$ with a simultaneous section $Z \to Y$ of both maps).
Taken together, an object $X \in \mathcal{C}$ is compact projective if $\mathrm{Hom}(X, -)$ commutes with filtered colimits and reflexive coequalizers, equivalently, $\mathrm{Hom}(X, -)$ commutes with 1-sifted colimits.

Let $\mathcal{C}^{\mathrm{cp}} \subset \mathcal{C}$ be the full subcategory of compact projective objects. There is a fully faithful embedding $\mathrm{sInd}(\mathcal{C}^{\mathrm{cp}}) \longrightarrow \mathcal{C}$.
If $\mathcal{C}$ is generated under small colimits by $\mathcal{C}^{\mathrm{cp}}$, then the functor is an equivalence:

$$\mathrm{sInd}(\mathcal{C}^{\mathrm{cp}}) \cong \mathcal{C}.$$

If $\mathcal{C}^{\mathrm{cp}}$ is small, then

$$\mathrm{sInd}(\mathcal{C}^{\mathrm{cp}}) \subset \mathrm{Fun}((\mathcal{C}^{\mathrm{cp}})^{\mathrm{op}}, \mathrm{Set})$$

is exactly the full subcategory of functors that take finite coproducts in $\mathcal{C}^{\mathrm{cp}}$ to products in Set.

Example 7.3.

(i) If $\mathcal{C} = \mathrm{Set}$, then $\mathcal{C}^{\mathrm{cp}} = \mathrm{FinSet}$, which generates $\mathcal{C}$ under small colimits.

(ii) If $\mathcal{C} = \mathrm{Ab}$, then $\mathcal{C}^{\mathrm{cp}} = \mathrm{FinFreeAb}$, which generates $\mathcal{C}$ under small colimits.

(iii) If $\mathcal{C} = \text{Ring}$, then $\mathcal{C}^{\text{cp}}$={retracts of $\mathbb{Z}[X_1, \cdots, X_n]$}, which generates $\mathcal{C}$ under small colimits.

(iv) If $\mathcal{C} = \text{Cond(Set)}$, then $\mathcal{C}^{\text{cp}} = \text{ExDisc}$, which generates $\mathcal{C}$ under small colimits.

(v) If $\mathcal{C} = \text{Cond(Ab)}$, then $\mathcal{C}^{\text{cp}} = \{\text{direct summands of } \mathbb{Z}[S] \mid S \in \text{ExDisc}\}$, which generates $\mathcal{C}$ under small colimits.

(vi) $\mathcal{C} = \text{Cond(Ring)}$, then $\mathcal{C}^{\text{cp}} = \{\text{retracts of } \mathbb{Z}[\mathbb{N}[S]] \mid S \in \text{ExDisc}\}$, which generates $\mathcal{C}$ under small colimits.

Definition 7.4. Let $\mathcal{C}$ be a category that admits all small colimits and $\mathcal{C}$ is generated under small colimits by $\mathcal{C}^{\text{cp}}$. The animation of $\mathcal{C}$ is the ∞-category $\text{Ani}(\mathcal{C})$ freely generated under sifted colimits by $\mathcal{C}^{\text{cp}}$.

Example 7.5. If $\mathcal{C} = \text{Set}$, then $\text{Ani}(\mathcal{C}) = \text{Ani(Set)} \triangleq \text{Ani}$ is the ∞-category of animated sets, or anima in a short.
Any anima has a set of connected components, giving a functor $\pi_0 : \text{Ani} \to \text{Set}$, which has a fully faithful right adjoint $\text{Set} \hookrightarrow \text{Ani}$.
Given an anima A with a point $a \in A$ (meaning a map $a : * \to A$), one can define groups $\pi_i(A, a)$, for $i \geq 1$ and for $i \geq 2$, $\pi_i(A, a) \in \text{Ab}$.
An anima A is i-truncated if $\pi_j(A, a) = 0,\ \forall a \in A$ and $\forall j > i$. Then A is 0-truncated if and only if it is in the essential image of $\text{Set} \hookrightarrow \text{Ani}$.
The inclusion of i-truncated anima into all anima has a left adjoint $\tau_{\leq i}$. For all anima A, the natural map

$$A \xrightarrow{\sim} \lim \tau_{\leq i} A$$

is an equivalence.
Picking any $a \in A$ and $i \geq 1$, the fiber of $\tau_{\leq i} A \to \tau_{\leq i-1} A$ over the image of a is an Eilenberg-Maclane anima $K(\pi_i(A, a), i)$. Here, an Eilenberg-Maclane anima $K(\pi, i)$ with $i \geq 1$ and π a group that is abelian if $i > 0$, is a pointed connected anima with $\pi_j = 0$ for $j \neq i$ and $\pi_i = \pi$.
In fact, the ∞-category of pointed connected anima (A, a) with $\pi_j(A, a) = 0$ for $j \neq i$ is equivalent to Grp when $i = 1$, and to Ab when $i \geq 2$.

Remark 7.6. There are several ways to describe $\text{Ani}(\mathcal{C})$.

(i) $\mathrm{Ani}(\mathcal{C})$ is the full sub-∞-category of objects in $\mathrm{Fun}((\mathcal{C}^{\mathrm{cp}})^{\mathrm{op}}, \mathrm{Ani})$ taking finite disjoint unions to finite products.

(ii) $\mathrm{Ani}(\mathcal{C})$ is the ∞-category obtained from $\mathrm{Simp}(\mathcal{C})$ by inverting weak equivalences.

Definition 7.7. Let $\mathcal{C}$ be an ∞-category that admits all small colimits. For any uncountable strong limit cardinal κ, the ∞-category $\mathrm{Cond}_\kappa(\mathcal{C})$ of κ-condensed objects of $\mathcal{C}$ is the category of contravariant functors from κ-ExDisc to $\mathcal{C}$ that take finite coproducts to finite products.

And we define

$$\mathrm{Cond}(\mathcal{C}) := \bigcup_\kappa \mathrm{Cond}_\kappa(\mathcal{C}).$$

Proposition 7.8. Let $\mathcal{C}$ be a category that is generated under small colimits by $\mathcal{C}^{\mathrm{cp}}$. Then $\mathrm{Cond}(\mathcal{C})$ is still generated under small colimits by its compact projective objects, and there is a natural equivalence of ∞-categories

$$\mathrm{Cond}(\mathrm{Ani}(\mathcal{C})) \cong \mathrm{Ani}(\mathrm{Cond}(\mathcal{C})).$$

Definition 7.9. Let $\mathcal{C}$ be some site.

(i) A presheaf of anima is a functor

$$\mathcal{F} : \mathrm{N}(\mathcal{C}^{\mathrm{op}}) \longrightarrow \mathrm{Ani}.$$

(ii) A sheaf of anima is a presheaf of anima $\mathcal{F}$, s.t. for all coverings $\{f_i : X_i \to X\}_{i \in I}$, one has

$$\mathcal{F}(X) \xrightarrow{\sim} \lim\Big(\textstyle\prod_i \mathcal{F}(X_i) \rightrightarrows \prod_{i,j} \mathcal{F}(X_i \times_X X_j) \substack{\rightarrow\\[-1em] \rightarrow \\[-1em] \rightarrow} \cdots \Big).$$

(iii) A hypercomplete sheaf of anima is a sheaf of anima $\mathcal{F}$, s.t. for all hypercovers $X_\bullet \to X$, the map

$$\mathcal{F}(X) \xrightarrow{\sim} \lim \mathcal{F}(X_\bullet) = \lim \Big(\mathcal{F}(X_0) \rightrightarrows \mathcal{F}(X_1) \substack{\rightarrow\\[-1em] \rightarrow \\[-1em] \rightarrow} \cdots \Big)$$

is an equivalence.

Definition 7.10. The ∞-category of condensed anima is given by

- The ∞-category of hypercomplete sheaves of anima on CHaus.
- The ∞-category of hypercomplete sheaves of anima on ProFin.
- The ∞-category of hypercomplete sheaves of anima on ExDisc, i.e. of functors

$$\mathrm{ExDisc}^{\mathrm{op}} \longrightarrow \mathrm{Ani}$$

  taking finite disjoint unions to finite products.

$$\begin{array}{ccc} \mathrm{CW} & \subset & \mathrm{Cond(Set)} \\ \cap & & \cap \\ \mathrm{Ani} & \subset & \mathrm{Cond(Ani)} \end{array}$$

Definition 7.11. $X \in \mathrm{Cond(Ani)}$ is

- discrete, if X in the essential image of Ani.
- static, if X in the essential image of Cond(Set).

8 Condensed Cohomology

Definition 8.1. Let $X \in$ Cond, $M \in$ Cond(Ab), we define the global section of M on X to be

$$\Gamma_{\text{cond}}(X, M) := \text{Hom}_{\text{Cond}}(X, M) = \text{Hom}_{\text{Cond(Ab)}}(\mathbb{Z}[X], M) \in \text{Ab},$$

and we define the condensed cohomology to be

$$R\Gamma_{\text{cond}}(X, M) := R\text{Hom}_{\text{Cond(Ab)}}(\mathbb{Z}[X], M),$$

i.e.

$$H^i_{\text{cond}}(X, M) := \text{Ext}^i_{\text{Cond(Ab)}}((\mathbb{Z}[X], M).$$

Lemma 8.2. For $X \in$ ExDisc, the functor $\Gamma_{\text{cond}}(X, -)$: Cond(Ab) $\to$ Ab is exact, hence, for any $M \in$ CondAb, $H^i_{\text{cond}}(X, M) = 0, \forall i \geq 1$.

Proof. We have $\Gamma_{\text{cond}}(X, -) = \text{Hom}_{\text{Cond(Ab)}}(\mathbb{Z}[X], -)$, and for $X \in$ ExDisc, $\mathbb{Z}[X]$ is projective, hence $\Gamma_{\text{cond}}(X, -)$: Cond(Ab) $\to$ Ab is exact. $\square$

Question. How to compute $H^i_{\text{cond}}(X, M)$?

From the definition, we need to find a projective resolution of $\mathbb{Z}[X]$.

For $X \in$ CHaus, we pick a hypercover $X_\bullet \to X$, where each $X_i \in$ ExDisc, for this hypercover, applying $\mathbb{Z}[-]$, then we get a projective resolution of $\mathbb{Z}[X]$:

$$\cdots \longrightarrow \mathbb{Z}[X_2] \longrightarrow \mathbb{Z}[X_1] \longrightarrow \mathbb{Z}[X_0] \longrightarrow \mathbb{Z}[X] \longrightarrow 0.$$

By definition, we have

$$\begin{aligned}
H^i_{\text{cond}}(X, M) &= \text{Ext}^i_{\text{Cond(Ab)}}(\mathbb{Z}[X], M) \\
&= H^i(0 \to \text{Hom}_{\text{Cond(Ab)}}(\mathbb{Z}[X_0], M) \to \text{Hom}_{\text{Cond(Ab)}}(\mathbb{Z}[X_1], M) \to \cdots) \\
&= H^i(0 \to \Gamma_{\text{cond}}(X_0, M) \to \Gamma_{\text{cond}}(X_1, M) \to \Gamma_{\text{cond}}(X_2, M) \to \cdots).
\end{aligned}$$

Theorem 8.3 (Dyckhoff,1976)**.** For any $X \in$ CHaus, there are natural isomorphisms:

$$H^i_{\text{cond}}(X, \mathbb{Z}) \cong H^i_{\text{sh}}(X, \mathbb{Z}), \ \forall i \geq 0.$$

Proof. 1) Assume $X \in$ Fin, then

$$H^i_{\text{cond}}(X, \mathbb{Z}) = \begin{cases} \Gamma_{\text{cond}}(X, \mathbb{Z}) = C(X, \mathbb{Z}) & i = 0 \\ 0 & i > 0 \end{cases}$$

This comes from Lemma 8.2. On the other hand,

$$H^i_{\text{sh}}(X, \mathbb{Z}) = \check{H}^i(X, \mathbb{Z}) = \begin{cases} C(X, \mathbb{Z}) & i = 0 \\ 0 & i > 0 \end{cases}$$

This comes from by computing Cech cohomology. For a finite set X, take the cover $\mathcal{U} = \{x \to X\}_{x \in X}$, then $\mathcal{C}^0(\mathcal{U}, \mathbb{Z}) = \mathcal{C}^1(\mathcal{U}, \mathbb{Z}) = \cdots = \mathbb{Z}^X$, and because $\mathcal{U}$ is a refinement of any cover, we have

$$\check{H}^i(X, \mathbb{Z}) = \check{H}^i(\mathcal{U}, \mathbb{Z}) = \begin{cases} \mathbb{Z}^X = C(X, \mathbb{Z}) & i = 0 \\ 0 & i > 0 \end{cases}$$

Therefore, for a finite set X, $H^i_{\text{cond}}(X, \mathbb{Z}) \cong H^i_{\text{sh}}(X, \mathbb{Z}),\ \forall i \geq 0$.

2) $X \in$ ProFin, hence we can write $X = \varprojlim_j X^j,\ X^j \in$ Fin.

$$H^i_{\text{sh}}(X, \mathbb{Z}) = \check{H}(X, \mathbb{Z}) = \varinjlim_j \check{H}(X_j, \mathbb{Z}) = \begin{cases} \varinjlim_j C(X_j, \mathbb{Z}) = C(X, \mathbb{Z}) & i = 0 \\ 0 & i > 0 \end{cases}$$

On the other hand, We compute $H^i_{\text{cond}}(X, \mathbb{Z}),\ i \geq 0$.

For $X \in$ ProFin, pick a hypercover $X_\bullet \to X$ with each $X_i \in$ ExDisc, and for each X^j, pick a finite hypercover $X^j_\bullet \to X^j$, s.t. $\varprojlim_j X^j_n = X_n$. Since X^j is finite, we have

$$H^i_{\text{cond}}(X^j, \mathbb{Z}) = \begin{cases} \Gamma(X^j, \mathbb{Z}) & i = 0 \\ 0 & i > 0 \end{cases}$$

And we know

$$H^i_{\text{cond}}(X^j, \mathbb{Z}) = H^i(0 \longrightarrow \Gamma(X^j_0, \mathbb{Z}) \longrightarrow \Gamma(X^j_1, \mathbb{Z}) \longrightarrow \Gamma(X^j_2, \mathbb{Z}) \to \cdots),$$

hence we have an exact sequence:

$$0 \longrightarrow \Gamma(X^j, \mathbb{Z}) \longrightarrow \Gamma(X^j_0, \mathbb{Z}) \longrightarrow \Gamma(X^j_1, \mathbb{Z}) \longrightarrow \Gamma(X^j_2, \mathbb{Z}) \to \cdots.$$

Applying the exact functor $\varinjlim_j$ to this exact sequence, we get an exact sequence:

$$0 \longrightarrow \varinjlim_j \Gamma(X^j, \mathbb{Z}) \longrightarrow \varinjlim_j \Gamma(X_0^j, \mathbb{Z}) \longrightarrow \varinjlim_j \Gamma(X_1^j, \mathbb{Z}) \longrightarrow \varinjlim_j \Gamma(X_2^j, \mathbb{Z}) \to \cdots,$$

i.e.

$$0 \longrightarrow \Gamma(X, \mathbb{Z}) \longrightarrow \Gamma(X_0, \mathbb{Z}) \longrightarrow \Gamma(X_1, \mathbb{Z}) \longrightarrow \Gamma(X_2, \mathbb{Z}) \to \cdots.$$

Hence,

$$H^i_{\text{cond}}(X, \mathbb{Z}) = \begin{cases} \Gamma(X, \mathbb{Z}) & i = 0 \\ 0 & i > 0 \end{cases}$$

3) $X \in$ CHaus.

Consider a morphism of topoi (α^{-1}, α_*) : Sh(CHaus$/X$) $\to$ Sh(X). For $\mathcal{F} \in$ Sh(CHaus$/X$), $\alpha_*\mathcal{F}$ is given by

$$U \mapsto \varprojlim_{V \subset U,\ V \text{ is closed in } X} \mathcal{F}(V \hookrightarrow S).$$

We have the following diagram:

$$\begin{tikzcd} \mathrm{Sh}(\mathrm{CHaus}/X) \arrow[rr, "\alpha_*"] \arrow[dr, "\Gamma_{\text{cond}}(X,-)"'] & & \mathrm{Sh}(X) \arrow[dl, "\Gamma_{\text{sh}}(X,-)"] \\ & \mathrm{Set} & \end{tikzcd}$$

This is because $\forall\, Y \in$ Sh(CHaus$/X$),

$$\begin{aligned} \Gamma_{\text{sh}}(X, \alpha_* Y) = \alpha_* Y(X) &= \varprojlim_{V \subset U,\ V \text{ is closed in } X} Y(V) \\ &= \varprojlim_V \mathrm{Hom}_{\text{cond}}(V, Y) = \mathrm{Hom}_{\text{cond}}(\varinjlim_V V, Y) \\ &= \mathrm{Hom}_{\text{cond}}(X, Y) = \Gamma_{\text{cond}}(X, Y). \end{aligned}$$

And this diagram can induce a diagram:

$$\begin{tikzcd} D(\mathrm{Ab}(\mathrm{CHaus}/X)) \arrow[rr, "R\alpha_*"] \arrow[dr, "R\Gamma_{\text{cond}}(X,-)"'] & & D(\mathrm{Ab}(X)) \arrow[dl, "R\Gamma_{\text{sh}}(X,-)"] \\ & D(\mathrm{Ab}) & \end{tikzcd}$$

Claim: $R\alpha_*\mathbb{Z} \cong \mathbb{Z}$ in $D(\mathrm{Ab}(X))$.
With this claim, we can show

$$\begin{aligned} H^i_{\text{cond}}(X,\mathbb{Z}) &= H^i(R\Gamma_{\text{cond}}(X,\mathbb{Z})) \\ &= H^i(R\Gamma_{\text{sh}}(X,-)\circ R\alpha_*\mathbb{Z}) \\ &= H^i(R\Gamma_{\text{sh}}(X,\mathbb{Z})) \\ &= H^i_{\text{sh}}(X,\mathbb{Z}). \end{aligned}$$

Hence, it suffices to show this claim. We have a map $\mathbb{Z} \to R\alpha_*\mathbb{Z}$ in $D(\mathrm{Ab}(X))$. In order to show this is an isomorphism, it suffices to check on each stacks.
Fix $s \in S$,

$$\begin{aligned} (R\alpha_*\mathbb{Z})_s &= \varinjlim_{s\in U \text{ open}} R\Gamma(U, R\alpha_*\mathbb{Z}) \\ &= \varinjlim_{s\in U \text{ open}} R\Gamma_{\text{cond}}(U,\mathbb{Z}) \\ &= \varinjlim_{s\in V \text{ closed}} R\Gamma_{\text{cond}}(V,\mathbb{Z}). \end{aligned}$$

Pick a hypercover $S_\bullet \to S$ with $S_i \in \mathrm{ExDisc}$. Then for each closed V, $(S_n \times_X V)_{n\geq 0} \to V$ is a hypercover. Hence,

$$R\Gamma_{\text{cond}}(V,\mathbb{Z}) \cong (0 \longrightarrow \Gamma(S_0\times_X V,\mathbb{Z}) \longrightarrow \Gamma(S_1 \times_X V,\mathbb{Z}) \longrightarrow \cdots).$$

Thus, we have

$$\begin{aligned} (R\alpha_*\mathbb{Z})_s &= \varinjlim_{s\in V \text{ closed}} R\Gamma_{\text{cond}}(V,\mathbb{Z}) \\ &\cong \varinjlim_{s\in V \text{ closed}} (0 \longrightarrow \Gamma(S_0\times_X V,\mathbb{Z}) \longrightarrow \Gamma(S_1 \times_X V,\mathbb{Z}) \longrightarrow \cdots) \\ &\cong (0 \longrightarrow \varinjlim_{s\in V \text{ closed}} \Gamma(S_0\times_X V,\mathbb{Z}) \longrightarrow \varinjlim_{s\in V \text{ closed}} \Gamma(S_1 \times_X V,\mathbb{Z}) \longrightarrow \cdots) \\ &\cong (0 \longrightarrow \Gamma(S_0\times_X \{s\},\mathbb{Z}) \longrightarrow \Gamma(S_1 \times_X \{s\},\mathbb{Z}) \longrightarrow \cdots) \\ &\cong R\Gamma_{\text{cond}}(\{s\},\mathbb{Z}) \\ &\cong \mathbb{Z}, \end{aligned}$$

which finishes our proof. $\square$

Example 8.4. Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, for $\mathbb{T}^I \in$ CHaus, we have $H^n(\mathbb{T}^I, \mathbb{Z}) = \wedge^n(\mathbb{Z}^{\oplus I})$.

Proof. First, we have

$$H^n(\mathbb{T}, \mathbb{Z}) = \begin{cases} \mathbb{Z} & n = 0, 1 \\ 0 & \text{else} \end{cases}$$

i.e. $H^*(\mathbb{T}, \mathbb{Z}) = \wedge(\mathbb{Z})$.
Claim: $H^*(\mathbb{T}^n, \mathbb{Z}) = \wedge(\mathbb{Z}^{\oplus n})$.
We can prove it by induction on n. $n = 1$ is proved above.
By Kunneth theorem, we can show that for $H^*(X, \mathbb{Z})$ finitely generated free in each degree, we have $H^*(X \times Y, \mathbb{Z}) \cong H^*(X, \mathbb{Z}) \otimes H^*(Y, \mathbb{Z})$. Hence, we have

$$\begin{aligned} H^*(\mathbb{T}^n, \mathbb{Z}) &= H^*(\mathbb{T}^{n-1}, \mathbb{Z}) \otimes H^*(\mathbb{T}, \mathbb{Z}) \\ &= \wedge(\mathbb{Z}^{\oplus(n-1)}) \otimes \wedge(\mathbb{Z}) \\ &= \wedge(\mathbb{Z}^{\oplus n}). \end{aligned}$$

In order to prove the general case, there is a fact that for $S \in$ CHaus, $S = \varprojlim_j S_j$, then $H^n(S, \mathbb{Z}) = \varinjlim_j H^n(S_j, \mathbb{Z})$.
Hence,

$$\begin{aligned} H^n(\mathbb{T}^I, \mathbb{Z}) &= H^n(\varprojlim_{J \subset I \text{ finite}} \mathbb{T}^J, \mathbb{Z}) \\ &= \varinjlim_{J \subset I \text{ finite}} H^n(\mathbb{T}^J, \mathbb{Z}) \\ &= \varinjlim_{J \subset I \text{ finite}} \wedge^n(\mathbb{Z}^{\oplus J}) \\ &= \wedge^n(\mathbb{Z}^{\oplus I}). \end{aligned}$$

$\square$

9 Locally compact abelian groups

Notation. Let TopAb be the category of all Hausdorff topological abelian groups and LCAb be the category of all locally compact abelian groups.

Proposition 9.1. Let $A, B \in$ TopAb and assume that $A \in$ CGTop. Then there is a natural isomorphism of condensed abelian groups

$$\underline{\mathrm{Hom}}(\underline{A}, \underline{B}) \cong \underline{\mathrm{Hom}(A, B)}.$$

Theorem 9.2 (Eilenberg-Maclane,Breen,Deligne resolution)**.** For any abelian group A, there is a functorial resolution

$$\cdots \longrightarrow \bigoplus_{j=1}^{n_i} \mathbb{Z}[A^{r_{i,j}}] \longrightarrow \cdots \longrightarrow \mathbb{Z}[A^3] \oplus \mathbb{Z}[A^2] \longrightarrow \mathbb{Z}[A^2] \longrightarrow \mathbb{Z}[A] \longrightarrow A \to 0.$$

Remark 9.3. Such functorial ensures that it works for abelian group objects in any topos.

Lemma 9.4. Let $A^{\bullet,\bullet}$ be a double complex and $A^\bullet = \mathrm{Tot}(A^{\bullet,\bullet})$ be its total complex, then there is a spectral sequence

$$E_1^{p,q} = H^q(A^{\bullet,p}) \Longrightarrow H^{p+q}(A^\bullet).$$

Lemma 9.5. For a comlpex of abelian groups $M^\bullet \in D(\mathbb{Z})$, let

$$0 \longrightarrow M^\bullet \longrightarrow A^{\bullet,1} \longrightarrow A^{\bullet,2} \longrightarrow A^{\bullet,3} \longrightarrow \cdots$$

be an exact sequence in $D(\mathbb{Z})$, then for the double complex $A^{\bullet,\bullet}$, there is a quasi-isomorphism

$$M^\bullet \xrightarrow{\sim} \mathrm{Tot}(A^{\bullet,\bullet}).$$

Corollary 9.6. For any condensed abelian groups A, M and an extremally disconnected space S, there is a spectral sequence

$$E_1^{p,q} = \prod_{j=1}^{n_p} H^q(A^{r_{p,j}} \times S, M) \Longrightarrow \underline{\mathrm{Ext}}^{p+q}(A, M)(S),$$

that is functorial in A, M and S.

Proof. For $A \in$ Cond(Ab), consider its EMBD resolution

$$\cdots \longrightarrow \bigoplus_{j=1}^{n_i} \mathbb{Z}[A^{r_{i,j}}] \longrightarrow \cdots \longrightarrow \mathbb{Z}[A^3] \oplus \mathbb{Z}[A^2] \longrightarrow \mathbb{Z}[A^2] \longrightarrow \mathbb{Z}[A] \longrightarrow A \to 0,$$

then apply $- \otimes \mathbb{Z}[S]$, which is an exact functor since $\mathbb{Z}[S]$ is flat, we get the resolution of $A \otimes \mathbb{Z}[S]$

$$\cdots \longrightarrow \bigoplus_{j=1}^{n_i} \mathbb{Z}[A^{r_{i,j}} \times S] \cdots \longrightarrow \mathbb{Z}[A^3 \times S] \oplus \mathbb{Z}[A^2 \times S] \longrightarrow \mathbb{Z}[A^2 \times S] \longrightarrow \mathbb{Z}[A \times S] \longrightarrow A \otimes \mathbb{Z}[S] \to 0,$$

then apply $R\mathrm{Hom}(-, M)$, we get

$$0 \longrightarrow R\mathrm{Hom}(A \otimes \mathbb{Z}[S], M) \longrightarrow R\mathrm{Hom}(\mathbb{Z}[A \times S], M) \longrightarrow R\mathrm{Hom}(\mathbb{Z}[A^2 \times S], M) \longrightarrow \cdots,$$

i.e.

$$0 \longrightarrow R\underline{\mathrm{Hom}}(A, M)(S) \longrightarrow R\Gamma(A \times S, M) \longrightarrow R\Gamma(A^2 \times S, M) \longrightarrow \cdots,$$

which is an exact sequence in $D(\mathbb{Z})$. By lemma 9.4 and lemma 9.5, we have

$$E_1^{p,q} = H^q(\bigoplus_{j=1}^{n_p} R\Gamma(A^{r_{p,j}} \times S, M)) \Longrightarrow H^{p+q}(\mathrm{Tot}(\bigoplus_{j=1}^{n_\bullet} R\Gamma(A^{r_{\bullet,j}} \times S, M)))$$

and

$$R\underline{\mathrm{Hom}}(A, M)(S) \simeq \mathrm{Tot}(\bigoplus_{j=1}^{n_\bullet} R\Gamma(A^{r_{\bullet,j}} \times S, M)),$$

hence

$$E_1^{p,q} = \prod_{j=1}^{n_p} H^q(A^{r_{p,j}} \times S, M) \Longrightarrow \underline{\mathrm{Ext}}^{p+q}(A, M)(S).$$

$\square$

Lemma 9.7. In the category of abelian groups, if the following diagram is exact for each arrow

$$\begin{tikzcd} 0 \arrow[r] & M^\bullet \arrow[r] \arrow[d] & A^{\bullet,1} \arrow[r] \arrow[d] & A^{\bullet,2} \arrow[r] \arrow[d] & \cdots, \\ 0 \arrow[r] & N^\bullet \arrow[r] & B^{\bullet,1} \arrow[r] & B^{\bullet,2} \arrow[r] & \cdots \end{tikzcd}$$

and if for any $j \geq 1$, we have $A^{\bullet,j} \cong B^{\bullet,j}$, then $\mathrm{Tot}(A^{\bullet,\bullet}) \cong \mathrm{Tot}(B^{\bullet,\bullet})$. Furthermore, by

$M^\bullet \cong \mathrm{Tot}(A^{\bullet,\bullet})$ and $N^\bullet \cong \mathrm{Tot}(B^{\bullet,\bullet})$, we can get $M^\bullet \cong N^\bullet$.

Theorem 9.8. Assume I is any set, denote the compact condensed abelian group $\prod_I \mathbb{T}$ by $\mathbb{T}^I$.

(i) For any discrete abelian group M, we have

$$R\underline{\mathrm{Hom}}(\mathbb{T}^I, M) = M^{\oplus I}[-1],$$

where $M^{\oplus I}[-1] \to R\underline{\mathrm{Hom}}(\mathbb{T}^I, M)$ is induced by

$$M[-1] = R\underline{\mathrm{Hom}}(\mathbb{Z}[1], M) \longrightarrow R\underline{\mathrm{Hom}}(\mathbb{T}, M) \xrightarrow{p_i^*} R\underline{\mathrm{Hom}}(\mathbb{T}^I, M),$$

where $p_i : \mathbb{T}^I \longrightarrow \mathbb{T}$ is the projection to the i-th factor, $i \in I$.

(ii) $R\underline{\mathrm{Hom}}(\mathbb{T}^I, \mathbb{R}) = 0$.

Proof.

(i) We first prove the case I is a one element set, i.e.

$$R\underline{\mathrm{Hom}}(\mathbb{T}, M) = M[-1].$$

From the exact sequence $0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{T} \to 0$, we have $\mathbb{R} \to \mathbb{T} \to \mathbb{Z}[1]$, hence

$$M[-1] = R\underline{\mathrm{Hom}}(\mathbb{Z}[1], M) \longrightarrow R\underline{\mathrm{Hom}}(\mathbb{T}, M) \longrightarrow R\underline{\mathrm{Hom}}(\mathbb{R}, M).$$

In order to show $R\underline{\mathrm{Hom}}(\mathbb{T}, M) = M[-1]$, it suffices to show $R\underline{\mathrm{Hom}}(\mathbb{R}, M) = 0$.

Claim: $R\underline{\mathrm{Hom}}(\mathbb{R}, M) = 0$.

For 0 and $\mathbb{R}$, we take its EMBD resolution:

$$\begin{tikzcd}
\cdots \arrow[r] & \bigoplus_{j=1}^{n_i} \mathbb{Z}[\mathbb{R}^{r_{i,j}}] \arrow[r] \arrow[d] & \cdots \arrow[r] & \mathbb{Z}[\mathbb{R}] \arrow[r] \arrow[d] & \mathbb{R} \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[r] & \bigoplus_{j=1}^{n_i} \mathbb{Z}[0^{r_{i,j}}] \arrow[r] & \cdots \arrow[r] & \mathbb{Z}[0] \arrow[r] & 0 \arrow[r] & 0,
\end{tikzcd}$$

apply $R\underline{\mathrm{Hom}}(-, M)(S)$, we get

$$\begin{tikzcd}
0 \arrow[r] & R\underline{\mathrm{Hom}}(0, M)(S) \arrow[r] \arrow[d] & R\underline{\mathrm{Hom}}(\mathbb{Z}[0], M)(S) \arrow[r] \arrow[d] & \cdots \arrow[r] & R\underline{\mathrm{Hom}}(\bigoplus_{j=1}^{n_i} \mathbb{Z}[0^{r_{i,j}}], M)(S) \cdots \arrow[d] \\
0 \arrow[r] & R\underline{\mathrm{Hom}}(\mathbb{R}, M)(S) \arrow[r] & R\underline{\mathrm{Hom}}(\mathbb{Z}[\mathbb{R}], M)(S) \arrow[r] & \cdots \arrow[r] & R\underline{\mathrm{Hom}}(\bigoplus_{j=1}^{n_i} \mathbb{Z}[\mathbb{R}^{r_{i,j}}], M)(S) \cdots ,
\end{tikzcd}$$

i.e.

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d] & R\Gamma(S, M) \arrow[r] \arrow[d] & \cdots \arrow[r] & \bigoplus_{j=1}^{n_i} R\Gamma(S, M) \cdots \arrow[d] \\
0 \arrow[r] & R\underline{\mathrm{Hom}}(\mathbb{R}, M)(S) \arrow[r] & R\Gamma(\mathbb{R} \times S, M) \arrow[r] & \cdots \arrow[r] & \bigoplus_{j=1}^{n_i} R\Gamma(\mathbb{R}^{r_{i,j}} \times S, M) \cdots ,
\end{tikzcd}$$

Then by lemma 9.7, in order to show $R\underline{\mathrm{Hom}}(\mathbb{R}, M) = 0$, it suffices to show

$$R\Gamma(S, M) = R\Gamma(S \times \mathbb{R}^r, M).$$

We know $S \times \mathbb{R}^r = \varinjlim S \times [-N, N]^r$, then

$$\begin{aligned}
R\Gamma(S \times \mathbb{R}^r, M) &= R\Gamma(\varinjlim S \times [-N, N]^r, M) \\
&= \varprojlim R\Gamma(S \times [-N, N]^r, M) \\
&= \varprojlim R\Gamma(S, M) \\
&= R\Gamma(S, M).
\end{aligned}$$

Here, $\varprojlim R\Gamma(S \times [-N, N]^r, M) = \varprojlim R\Gamma(S, M)$ comes from the fact that for constant sheaf, its sheaf cohomology is homotopy-invariant.

Secondly, assume I is a finite set, then

$$R\underline{\mathrm{Hom}}(\mathbb{T}^I, M) = R\underline{\mathrm{Hom}}(\mathbb{T}^{\oplus I}, M) = \prod_I R\underline{\mathrm{Hom}}(\mathbb{T}, M) = \prod_I M[-1] = M^{\oplus I}[-1].$$

Finally, assume I is any set. Then we can write $\mathbb{T}^I$ as

$$\mathbb{T}^I = \varprojlim_{J \subset I, J \text{ finite}} \mathbb{T}^J.$$

For any finite set J, we have

$$\begin{tikzcd}
0 \arrow[r] & R\underline{\mathrm{Hom}}(\mathbb{T}^J, M)(S) \arrow[r] \arrow[d] & R\Gamma(\mathbb{T}^J \times S, M) \arrow[r] \arrow[d] & \cdots \arrow[r] & \bigoplus_{j=1}^{n_i} R\Gamma((\mathbb{T}^J)^{r_{i,j}} \times S, M) \cdots \arrow[d] \\
0 \arrow[r] & R\underline{\mathrm{Hom}}(\mathbb{T}^I, M)(S) \arrow[r] & R\Gamma(\mathbb{T}^I \times S, M) \arrow[r] & \cdots \arrow[r] & \bigoplus_{j=1}^{n_i} R\Gamma((\mathbb{T}^I)^{r_{i,j}} \times S, M) \cdots ,
\end{tikzcd}$$

apply the exact functor $\varinjlim_{J\subset I}$ to the first arrow, we get

$$\begin{tikzcd}
0 \arrow[r] & \varinjlim_{J\subset I} R\underline{\mathrm{Hom}}(\mathbb{T}^J, M)(S) \arrow[r] \arrow[d] & \varinjlim_{J\subset I} R\Gamma(\mathbb{T}^J \times S, M) \arrow[r] \arrow[d] & \cdots \arrow[r] & \bigoplus_{j=1}^{n_i} \varinjlim_{J\subset I} R\Gamma((\mathbb{T}^J)^{r_{i,j}} \times S, M) \cdots \arrow[d] \\
0 \arrow[r] & R\underline{\mathrm{Hom}}(\mathbb{T}^I, M)(S) \arrow[r] & R\Gamma(\mathbb{T}^I \times S, M) \arrow[r] & \cdots \arrow[r] & \bigoplus_{j=1}^{n_i} R\Gamma((\mathbb{T}^I)^{r_{i,j}} \times S, M) \cdots ,
\end{tikzcd}$$

In order to show

$$\varinjlim_{J\subset I} R\underline{\mathrm{Hom}}(\mathbb{T}^J, M)(S) \cong R\underline{\mathrm{Hom}}(\mathbb{T}^I, M)(S),$$

it suffices to show

$$\varinjlim_{J\subset I} R\Gamma((\mathbb{T}^J)^{r_{i,j}} \times S, M) \cong R\Gamma((\mathbb{T}^I)^{r_{i,j}} \times S, M).$$

This is true, because $\varprojlim_{J\subset I} (\mathbb{T}^J)^{r_{i,j}} \times S \cong (\mathbb{T}^I)^{r_{i,j}} \times S$. Therefore,

$$\begin{aligned}
R\underline{\mathrm{Hom}}(\mathbb{T}^I, M) &\cong \varinjlim_{J\subset I} R\underline{\mathrm{Hom}}(\mathbb{T}^J, M) \\
&\cong \varinjlim_{J\subset I} M^{\oplus J}[-1] \\
&\cong M^{\oplus I}[-1].
\end{aligned}$$

$\square$

Corollary 9.9. $R\underline{\mathrm{Hom}}(\mathbb{R}, \mathbb{R}) \cong \mathbb{R}$.

Proof. From the exact sequence $0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{T} \to 0$, we have

$$R\underline{\mathrm{Hom}}(\mathbb{T}, \mathbb{R}) \to R\underline{\mathrm{Hom}}(\mathbb{R}, \mathbb{R}) \to R\underline{\mathrm{Hom}}(\mathbb{Z}, \mathbb{R}).$$

By Theorem 9.8, we know $R\underline{\mathrm{Hom}}(\mathbb{T}, \mathbb{R}) = 0$, hence $R\underline{\mathrm{Hom}}(\mathbb{R}, \mathbb{R}) \cong R\underline{\mathrm{Hom}}(\mathbb{Z}, \mathbb{R}) \cong$

$\mathbb{R}$. $\square$

Corollary 9.10. For any locally compact abelian groups A and B, $R\underline{\text{Hom}}(A, B)$ is centered at 0 and 1, i.e. $\underline{\text{Ext}}^i(A, B) = 0, \ \forall i \geq 2$.

Proof. By the structure theorem of locally compact abelian groups, it suffices to prove for A and B being compact groups and discrete groups.

(i) A is a discrete group.
Claim: There is an exact sequence: $0 \to \oplus_I \mathbb{Z} \to \oplus_J \mathbb{Z} \to A \to 0$.
This is because we can construct a surjective homomorphism $\oplus_A \mathbb{Z} \to A$, and take its kernel, and we know the submodule of a free $\mathbb{Z}$-module is free, hence ker$(\oplus_A \mathbb{Z} \to A) = \oplus_I \mathbb{Z}$, for some I. Thereby, $0 \to \oplus_I \mathbb{Z} \to \oplus_A \mathbb{Z} \to A \to 0$ is exact.
By the short exact sequence $0 \to \oplus_I \mathbb{Z} \to \oplus_J \mathbb{Z} \to A \to 0$, we can get a long exact sequence:

$$\begin{aligned} 0 &\longrightarrow \underline{\text{Hom}}(A, B) \longrightarrow \underline{\text{Hom}}(\oplus_J \mathbb{Z}, B) \longrightarrow \underline{\text{Hom}}(\oplus_I \mathbb{Z}, B) \\ &\longrightarrow \underline{\text{Ext}}^1(A, B) \longrightarrow \underline{\text{Ext}}^1(\oplus_J \mathbb{Z}, B) \longrightarrow \underline{\text{Ext}}^1(\oplus_I \mathbb{Z}, B) \\ &\longrightarrow \underline{\text{Ext}}^2(A, B) \longrightarrow \cdots . \end{aligned}$$

Because $\oplus_I \mathbb{Z} \in$ Cond(Ab) is projective, we have $\underline{\text{Ext}}^i(\oplus_I \mathbb{Z}, B) = 0, \ \forall i \geq 1$. Hence $\underline{\text{Ext}}^i(A, B) = 0, \ \forall i \geq 2$.

(ii) A is a compact group.
By Pontrgagin duality, there is a short exact sequence

$$0 \to A \to \mathbb{T}^I \to \mathbb{T}^J \to 0,$$

and it can induce a long exact sequence

$$\begin{aligned} 0 &\longrightarrow \underline{\text{Hom}}(\mathbb{T}^J, B) \longrightarrow \underline{\text{Hom}}(\mathbb{T}^I, B) \longrightarrow \underline{\text{Hom}}(A, B) \\ &\longrightarrow \underline{\text{Ext}}^1(\mathbb{T}^J, B) \longrightarrow \underline{\text{Ext}}^1(\mathbb{T}^I, B) \longrightarrow \underline{\text{Ext}}^1(A, B) \\ &\longrightarrow \underline{\text{Ext}}^2(\mathbb{T}^J, B) \longrightarrow \underline{\text{Ext}}^2(\mathbb{T}^I, B) \longrightarrow \underline{\text{Ext}}^2(A, B) \\ &\longrightarrow \cdots \end{aligned}$$

In order to show $\underline{\mathrm{Ext}}^i(A, B) = 0,\ \forall i \geq 2$, it suffices to show

$$\underline{\mathrm{Ext}}^i(\mathbb{T}^I, B) = 0,\ \forall i \geq 2,\ \forall I.$$

(a) B is a discrete group.

In this case, we have $R\underline{\mathrm{Hom}}(\mathbb{T}^I, B) = B^{\oplus I}[-1]$, which is centered at 1, hence $\underline{\mathrm{Ext}}^i(\mathbb{T}^I, B) = 0,\ \forall i \geq 2,\ \forall I$.

(b) B is a compact group.

In this case, we have a short exact sequence $0 \to B \to \mathbb{T}^{I'} \to \mathbb{T}^{J'} \to 0$, and it induces a long exact sequence:

$$\begin{aligned} 0 &\longrightarrow \underline{\mathrm{Hom}}(\mathbb{T}^I, B) \longrightarrow \underline{\mathrm{Hom}}(\mathbb{T}^I, \mathbb{T}^{I'}) \longrightarrow \underline{\mathrm{Hom}}(\mathbb{T}^I, \mathbb{T}^{J'}) \\ &\longrightarrow \underline{\mathrm{Ext}}^1(\mathbb{T}^I, B) \longrightarrow \underline{\mathrm{Ext}}^1(\mathbb{T}^I, \mathbb{T}^{I'}) \longrightarrow \underline{\mathrm{Ext}}^1(\mathbb{T}^I, \mathbb{T}^{J'}) \\ &\longrightarrow \underline{\mathrm{Ext}}^2(\mathbb{T}^I, B) \longrightarrow \cdots . \end{aligned}$$

Now, we compute $\underline{\mathrm{Ext}}^i(\mathbb{T}^I, \mathbb{T})$. For the short exact sequence $0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{T} \to 0$, we have a long exact sequence:

$$\begin{aligned} 0 &\longrightarrow \underline{\mathrm{Hom}}(\mathbb{T}^I, \mathbb{Z}) \longrightarrow \underline{\mathrm{Hom}}(\mathbb{T}^I, \mathbb{R}) \longrightarrow \underline{\mathrm{Hom}}(\mathbb{T}^I, \mathbb{T}) \\ &\longrightarrow \underline{\mathrm{Ext}}^1(\mathbb{T}^I, \mathbb{Z}) \longrightarrow \underline{\mathrm{Ext}}^1(\mathbb{T}^I, \mathbb{R}) \longrightarrow \underline{\mathrm{Ext}}^1(\mathbb{T}^I, \mathbb{T}) \\ &\longrightarrow \underline{\mathrm{Ext}}^2(\mathbb{T}^I, \mathbb{Z}) \longrightarrow \cdots . \end{aligned}$$

Since $R\underline{\mathrm{Hom}}(\mathbb{T}^I, \mathbb{R}) = 0$ and $R\underline{\mathrm{Hom}}(\mathbb{T}^I, \mathbb{Z}) = \mathbb{Z}^{\oplus I}[-1]$, we have $\underline{\mathrm{Ext}}^i(\mathbb{T}^I, \mathbb{T}) = 0,\ \forall i \geq 1$, hence $\underline{\mathrm{Ext}}^i(\mathbb{T}^I, \mathbb{T}^J) = 0,\ \forall i \geq 1,\ \forall J$. Thus $\underline{\mathrm{Ext}}^i(\mathbb{T}^I, B) = 0,\ \forall i \geq 2$.

$\square$

Appendix: Resolutions

Definition 9.11. Let $\mathcal{A}$ be a Grothendieck abelian category and $X \in D(\mathcal{A})$ is a complex.

(i) Let $n \in \mathbb{Z}_{\geq 0}$. X is n-pseudocoherent if

  (a) X is bounded above, i.e. for $i \gg 0$, $H^i(X) = 0$.

  (b) For $i = 0, 1, \cdots, n-1$, $\mathrm{Ext}^i(X, -) : \mathcal{A} \to \mathrm{Ab}$ commutes with filtered colimits.

  (c) $\mathrm{Ext}^n(X, -) : \mathcal{A} \to \mathrm{Ab}$ commutes with filtered union.

(ii) X is pseudocoherent if

  (a) X is bounded above, i.e. for $i \gg 0$, $H^i(X) = 0$.

  (b) For any $i \geq 0$, $\mathrm{Ext}^i(X, -) : \mathcal{A} \to \mathrm{Ab}$ commutes with filtered colimits.

  X is pseudocoherent iff $\forall n$, X is n-pseudocoherent.

(iii) Let $n \in \mathbb{Z}_{\geq 0}$ and $\mathcal{A}_0$ is a family of compact projective objects of $\mathcal{A}$. X is $n, \mathcal{A}_0$-pseudocoherent if:
there exists a bounded cochain complex $P^\bullet$, and a map $\varphi : P^\bullet \to X$, s.t. each P^i is the finite direct sum of elements of $\mathcal{A}_0$, and if $i > -n$, $H^i(\varphi)$ is an isomorphism; if $i = -n$, $H^i(\varphi)$ is a surjection.

(iv) Let $\mathcal{A}_0$ be a family of compact projective objects of $\mathcal{A}$. X is $\mathcal{A}_0$-pseudocoherent if:
there exists a bounded cochain complex $P^\bullet$, and a map $\varphi : P^\bullet \to X$, which is a quasi-isomorphism, s.t. each P^i is the finite direct sum of elements of $\mathcal{A}_0$.
X is $\mathcal{A}_0$-pseudocoherent iff $\forall n$, X is $n, \mathcal{A}_0$-pseudocoherent.

Definition 9.12. Let $\mathcal{A}$ be a Grothendieck abelian category and $A \in \mathcal{A}$. A is finitely generated if A is 0-pseudocoherent.

Lemma 9.13. Let $\mathcal{A}$ be a Grothendieck abelian category and $A \in \mathcal{A}$. Consider:

(i) A is finitely generated.

(ii) A is the quotient of a compact object.

(iii) If there is a surjection $\bigoplus_{i \in I} B_i \twoheadrightarrow A$, then there exists a finite subset $J \subseteq I$, s.t. $\bigoplus_{i \in J} B_i \twoheadrightarrow A$.

(iv) Let B be the filtered colimit of B_i, then $\varinjlim \operatorname{Hom}(A, B_i) \hookrightarrow \operatorname{Hom}(A, B)$.

Then (i) $\iff$ (ii) $\iff$ (iii) $\implies$ (iv).

Proof. (ii) $\implies$ (i). Assume $C \twoheadrightarrow A$ and C is compact. Take any filtered union $B = \bigcup B_i$. Since $B_i \hookrightarrow B$, then $\operatorname{Hom}(A, B_i) \hookrightarrow \operatorname{Hom}(A, B)$, thus $\varinjlim \operatorname{Hom}(A, B_i) \hookrightarrow \varinjlim \operatorname{Hom}(A, B)$.

In order to show this is a surjection, we take any map $A \to B$, since C is compact, $C \to A \to B$ factors through some B_i, hence $A \to B$ factors through B_i, which implies

$$\varinjlim \operatorname{Hom}(A, B_i) \twoheadrightarrow \varinjlim \operatorname{Hom}(A, B).$$

Then $\varinjlim \operatorname{Hom}(A, B_i) \cong \varinjlim \operatorname{Hom}(A, B)$, i.e. A is finitely generated.

(i) $\implies$ (iii). We write

$$A = \operatorname{Im}(\bigoplus_{i \in I} B_i \to A) = \bigcup_{J \subset I, J \text{ finite}} \operatorname{Im}(\bigoplus_{i \in J} B_i \to A),$$

then

$$\operatorname{Hom}(A, A) = \varinjlim \operatorname{Hom}(A, \operatorname{Im}(\bigoplus_{i \in J} B_i \to A)),$$

hence id_A factors through $\operatorname{Im}(\bigoplus_{i \in J} B_i \to A)$ for some J, thus $\bigoplus_{i \in J} B_i \twoheadrightarrow A$.

(iii) $\implies$ (ii). We can write A as the quotient of the direct sum of a family of compact objects, i.e $\bigoplus_{i \in I} B_i \twoheadrightarrow A$, B_i are compact. By (iii), $\exists$ finite subset $J \subset I$, s.t. $\bigoplus_{i \in J} B_i \twoheadrightarrow A$. And the finite direct sum of compact objects is still compact, hence A is the quotient of a compact object.

(ii) $\implies$ (iv). Assume $C \twoheadrightarrow A$ with C compact and $B = \varinjlim B_i$. Then we have the following diagram

$$\begin{tikzcd}
\operatorname{Hom}(A, B) \arrow[r, hook] \arrow[d] & \operatorname{Hom}(C, B) \arrow[d, equal] \\
\varinjlim \operatorname{Hom}(A, B_i) \arrow[r, hook] & \operatorname{Hom}(C, B)
\end{tikzcd}$$

which gives $\operatorname{Hom}(A, B) \hookrightarrow \varinjlim \operatorname{Hom}(A, B_i)$. $\square$

10 Solid Abelian Groups

Definition 10.1. For $S \in$ ProFin, write $S = \varprojlim S_i$, where $S_i \in$ Fin, we define the solid free abelian group

$$\mathbb{Z}[S]^{\blacksquare} := \varprojlim \mathbb{Z}[S_i].$$

We call $\mathbb{Z}[S]^{\blacksquare}$ the solidification of $\mathbb{Z}[S]$.

Remark 10.2.
$\mathbb{Z}[S]^{\blacksquare} = \varprojlim \mathbb{Z}[S_i] = \varprojlim \underline{\mathrm{Hom}}(C(S_i, \mathbb{Z}), \mathbb{Z}) = \underline{\mathrm{Hom}}(\varinjlim C(S_i, \mathbb{Z}), \mathbb{Z}) = \underline{\mathrm{Hom}}(C(S, \mathbb{Z}), \mathbb{Z})$.

Proposition 10.3. (i) For $S \in$ ProFin, there exists some set I, s.t. $C(S, \mathbb{Z}) \cong \mathbb{Z}^{\oplus I}$, i.e. $C(S, \mathbb{Z})$ is a free abelian group.

(ii) We have

$$\mathbb{Z}[S]^{\blacksquare} = \underline{\mathrm{Hom}}(C(S, \mathbb{Z}), \mathbb{Z}) = \underline{\mathrm{Hom}}(\mathbb{Z}^{\oplus I}, \mathbb{Z}) = \mathbb{Z}^{I}.$$

Definition 10.4. A condensed abelian group $X \in$ Cond(Ab) is solid, if for any $S \in$ ProFin, one has

$$\mathrm{Hom}(\mathbb{Z}[S], X) \cong \mathrm{Hom}(\mathbb{Z}[S]^{\blacksquare}, X).$$

A complex of condensed abelian groups $C \in D(\text{Cond(Ab)})$ is solid, if for any $S \in$ ProFin, one has

$$R\mathrm{Hom}(\mathbb{Z}[S], C) \cong R\mathrm{Hom}(\mathbb{Z}[S]^{\blacksquare}, C).$$

Now, we need to check $\mathbb{Z}[S]^{\blacksquare}$ is indeed a solid condensed abelian group.

Proposition 10.5. For $S, T \in$ ProFin, we have

$$R\mathrm{Hom}(\mathbb{Z}[S], \mathbb{Z}[T]^{\blacksquare}) \cong R\mathrm{Hom}(\mathbb{Z}[S]^{\blacksquare}, \mathbb{Z}[T]^{\blacksquare}).$$

Proof. Assume $\mathbb{Z}[S]^{\blacksquare} = \mathbb{Z}^{I}$ and $\mathbb{Z}[T]^{\blacksquare} = \mathbb{Z}^{J}$ for some sets I and J. Since the functors $R\mathrm{Hom}(\mathbb{Z}[S], -)$ and $R\mathrm{Hom}(\mathbb{Z}[S]^{\blacksquare}, -)$ commute with products, it suffices to show

$$R\mathrm{Hom}(\mathbb{Z}[S], \mathbb{Z}) \cong R\mathrm{Hom}(\mathbb{Z}[S]^{\blacksquare}, \mathbb{Z})$$

The left hand side is $R\mathrm{Hom}(\mathbb{Z}[S], \mathbb{Z}) \cong R\Gamma(S, \mathbb{Z}) = C(S, \mathbb{Z}) = \mathbb{Z}^{\oplus I}$.
Now, consider the short exact sequence $0 \to \mathbb{R}^{I} \to \mathbb{Z}^{I} \to \mathbb{T}^{I} \to 0$. From theorem 9.8,

We know

$$R\mathrm{Hom}(\mathbb{T}^I, \mathbb{Z}) = \mathbb{Z}^{\oplus I}[-1].$$

And by the adjoint relation, we have

$$R\mathrm{Hom}(\mathbb{R}^I, \mathbb{Z}) \cong R\mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^I, R\underline{\mathrm{Hom}}(\mathbb{R}, \mathbb{Z})) = 0.$$

Hence, $R\mathrm{Hom}(\mathbb{Z}[S]^{\blacksquare}, \mathbb{Z}) \cong R\mathrm{Hom}(\mathbb{Z}^I, \mathbb{Z}) \cong \mathbb{Z}^{\oplus I}$. And this finishes our proof. $\square$

Lemma 10.6. Let $\mathcal{A}$ be a cocomplete abelian category, and $\mathcal{A}_0 \subseteq \mathcal{A}$ be the full subcategory of compact projective generators. Assume $F : \mathcal{A}_0 \to \mathcal{A}$ is an additive functor with a natural transformation $\mathrm{id}_{\mathcal{A}_0} \implies F$, satisfying the following property:

For any $X \in \mathcal{A}_0$, any $Y, Z \in \mathcal{A}$ which can be written as direct sums of objects in the image of F, i.e. $Y = \bigoplus_{i\in I} F(X_i)$ and $Z = \bigoplus_{j\in J} F(X_j)$, and for any map $f : Y \to Z$ with kernel $K \in \mathcal{A}$, the map

$$R\mathrm{Hom}(F(X), K) \to R\mathrm{Hom}(X, K)$$

is an isomorphism.

Let

$$\mathcal{A}_F = \{Y \in \mathcal{A} \mid \mathrm{Hom}(F(X), Y) \cong \mathrm{Hom}(X, Y), \forall X \in \mathcal{A}_0\} \subseteq \mathcal{A}$$

and

$$D_F(\mathcal{A}) = \{C \in D(\mathcal{A}) \mid R\mathrm{Hom}(F(X), C) \cong R\mathrm{Hom}(X, C), \forall X \in \mathcal{A}_0\} \subseteq D(\mathcal{A})$$

Then:

(i)
- $\mathcal{A}_F \subseteq \mathcal{A}$ is an abelian subcategory stable under limits, colimits and extensions.
- The objects $F(X), X \in \mathcal{A}_0$ are compact projective generators.
- The inclusion $\mathcal{A}_F \hookrightarrow \mathcal{A}$ admits a left adjoint $L : \mathcal{A} \to \mathcal{A}_F$, which is the unique colimit-preserving extension of $F : \mathcal{A}_0 \to \mathcal{A}_F$.

(ii) - The functor $D(\mathcal{A}_F) \to D(\mathcal{A})$ is fully faithful and $D(\mathcal{A}_F) \cong D_F(\mathcal{A})$.

- $C \in D(\mathcal{A})$ lies in $D_F(\mathcal{A})$ iff $H^i(C) \in \mathcal{A}_F$.

- The above functor F has a left derived functor, which is the left adjoint of $D_F(\mathcal{A}) \hookrightarrow D(\mathcal{A})$.

Proof. (i) $\mathcal{A}_F$ is stable under limits:

$\mathrm{Hom}(FX, \lim Y_i) \cong \lim \mathrm{Hom}(FX, Y_i) \cong \lim \mathrm{Hom}(X, Y_i) \cong \mathrm{Hom}(X, \lim Y_i)$.

$\mathcal{A}_F$ is stable under colimits:

It suffices to show $\mathcal{A}_F$ is stable under cokernels and direct sums.

For any map $f : Y \to Z$ in $\mathcal{A}_F$. We can find a surjection $\bigoplus_{i\in I} P_i \twoheadrightarrow Z$, which factors through $\bigoplus_{i\in I} F(P_i)$, hence $\bigoplus_{i\in I} F(P_i) \twoheadrightarrow Z$. Assume the pullback diagram:

$$\begin{tikzcd}
X \arrow[r, "g"] \arrow[d] & \bigoplus_{i\in I} F(P_i) \arrow[r] \arrow[d] & \mathrm{coker}(g) \\
Y \arrow[r, "f"] & Z \arrow[r] & \mathrm{coker}(f)
\end{tikzcd}$$

By the pullback we know $X \twoheadrightarrow Y$. With this, one can show $\mathrm{coker}(g) = \mathrm{coker}(f)$. Hence, we may replace Z by $\bigoplus_{i\in I} F(P_i)$. With the same reason, one can also replace Y by the object of the form $\bigoplus_{j\in J} F(Q_j)$. Therefore, we assume $f : \bigoplus_{j\in J} F(Q_j) \to \bigoplus_{i\in I} F(P_i)$. We already know $\ker(f) \in \mathcal{A}_F$. From the following lemma 10.7, $\mathrm{coker}(f) \in \mathcal{A}_F$. Thus, $\mathcal{A}_F$ is stable under cokernels.

Moreover, the objects of $\mathcal{A}_F$ are precisely the cokernels of maps $f : Y \to Z$ between objects $Y, Z \in \mathcal{A}$ that are direct sums of objects in the image of F.

Hence by

$$\bigoplus_{i\in I} \mathrm{coker}(Y_i \to Z_i) \cong \mathrm{coker}(\bigoplus_{i\in I} Y_i \to \bigoplus_{i\in I} Z_i),$$

we know $\mathcal{A}_F$ is stable under direct sums.

Thus, $\mathcal{A}_F$ is stable under colimits.

Now assume $0 \to X \to Y \to Z \to 0$ is exact with $X, Z \in \mathcal{A}_F$. Then X, Z can be written as $\mathrm{coker}(A_1 \to A_2)$ and $\mathrm{coker}(B_1 \to B_2)$, where A_1, A_2, B_1, B_2 are the direct sums of objects in the image of F. We form the diagram

$$\begin{tikzcd}
0 \arrow[r] & A_1 \arrow[r] \arrow[d] & A_1 \oplus B_1 \arrow[r] \arrow[d] & B_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_2 \arrow[r] \arrow[d, two heads] & A_2 \oplus B_2 \arrow[r] \arrow[d, two heads] & B_2 \arrow[r] \arrow[d, two heads] & 0 \\
0 \arrow[r] & \mathrm{coker}(A_1 \to A_2) \arrow[r] & Y \arrow[r] & \mathrm{coker}(B_1 \to B_2) \arrow[r] & 0
\end{tikzcd}$$

Then the extension $Y = \mathrm{coker}(A_1 \oplus B_1 \to A_2 \oplus B_2) \in \mathcal{A}_F$.

For each $Y_i \in \mathcal{A}_F$, $X \in \mathcal{A}_0$, from the isomorphisms,

$$\mathrm{Hom}(FX, \mathrm{colim}\, Y_i) \cong \mathrm{Hom}(X, \mathrm{colim}\, Y_i) \cong \mathrm{colim}\, \mathrm{Hom}(X, Y_i) \cong \mathrm{colim}\, \mathrm{Hom}(FX, Y_i),$$

we know FX is compact projective.

We define a functor $L : \mathcal{A} \to \mathcal{A}_F$. For any $X \in \mathcal{A}$, $X = \mathrm{colim}\ X_i$, $X_i \in \mathcal{A}_0$, hence define $L(X) := \mathrm{colim}\ F(X_i)$. By the isomorphisms

$$\mathrm{Hom}(LX, Y) \cong \mathrm{Hom}(\mathrm{colim}\ F(X_i), Y) \cong \lim \mathrm{Hom}(F(X_i), Y) \cong \lim \mathrm{Hom}(X_i, Y) \cong \mathrm{Hom}(X, Y),$$

we know L is the left adjoint of the inclusion $\mathcal{A}_F \hookrightarrow \mathcal{A}$. Besides, by the construction, we know L agrees with F on $\mathcal{A}_0$ and is the unique colimit-preserving extension of F.

In fact, one can show for any $X \in \mathcal{A}$, take a resolution $B \to A \to X \to 0$, where $A, B \in \mathcal{A}_0$, then $L(X) \cong \mathrm{coker}(FB \to FA)$.

(ii) The functor $D(\mathcal{A}_F) \to D(\mathcal{A})$ is fully faithful:
It suffices to show for any $X \in \mathcal{A}_0$ and any $C \in D(\mathcal{A}_F)$, there is an isomorphism:

$$R\mathrm{Hom}_{D(\mathcal{A}_F)}(FX, C) \to R\mathrm{Hom}_{D(\mathcal{A})}(FX, C) \cong R\mathrm{Hom}_{D(\mathcal{A})}(X, C).$$

Since $R\mathrm{Hom}_{D(\mathcal{A}_F)}(FX, -)$ and $R\mathrm{Hom}_{D(\mathcal{A})}(X, -)$ commute with limits, we may assume C is bounded, and hence assume C is concentrated at degree 0, i.e. $C =$

$Y[0]$, where $Y \in \mathcal{A}_F$, then it suffices to show:

$$R\mathrm{Hom}_{D(\mathcal{A}_F)}(FX, Y) \cong R\mathrm{Hom}_{D(\mathcal{A})}(X, Y).$$

By taking the cohomology, it reduces to show for any $i \geq 0$,

$$\mathrm{Ext}^i_{\mathcal{A}_F}(FX, Y) \cong \mathrm{Ext}^i_{\mathcal{A}}(X, Y).$$

Since $FX \in \mathcal{A}_F$ is projective and $X \in \mathcal{A}$ is projective, then for $i > 0$, $\mathrm{Ext}^i_{\mathcal{A}_F}(FX, Y) \cong \mathrm{Ext}^i_{\mathcal{A}}(X, Y) \cong 0$. For $i = 0$, because $Y \in \mathcal{A}_F$, we have $\mathrm{Hom}_{\mathcal{A}_F}(FX, Y) \cong \mathrm{Hom}_{\mathcal{A}}(X, Y)$.

Denote

$$D'_F(\mathcal{A}) := \{C \in D(\mathcal{A}) \mid H^i(C) \in \mathcal{A}_F, \ \forall i\}.$$

Claim: $D(\mathcal{A}_F) \subset D'_F(\mathcal{A}) = D_F(\mathcal{A})$.
First, it is obvious that $D(\mathcal{A}_F) \subset D'_F(\mathcal{A})$.
Then we prove $D'_F(\mathcal{A}) \subset D_F(\mathcal{A})$.
If $C \in D'_F(\mathcal{A})$ is bounded, then we can reduce to the case $C = Y[0]$, where $Y \in \mathcal{A}_F$. From the isomorphism $R\mathrm{Hom}(FX, Y) \cong R\mathrm{Hom}(X, Y)$, we know $C = Y[0] \in D_F(\mathcal{A})$.
If $C \in D'_F(\mathcal{A})$ is right bounded, then $C = \varprojlim \tau_{\leq n} C$, where $\tau_{\leq n} C$ are bounded. Hence we have

$$\begin{aligned} R\mathrm{Hom}(FX, C) &\cong R\mathrm{Hom}(FX, \varprojlim \tau_{\leq n} C) \cong \varprojlim R\mathrm{Hom}(FX, \tau_{\leq n} C) \\ &\cong \varprojlim R\mathrm{Hom}(X, \tau_{\leq n} C) \cong R\mathrm{Hom}(X, C), \end{aligned}$$

which means $C \in D_F(\mathcal{A})$.
In general, consider the truncation $C_{\geq n} = [\cdots \to C_{n+1} \to C_n \to 0 \to \cdots]$ which is right bounded. Then $C = \varinjlim C_{\geq n}$. Since FX, X are compact, we have

$$R\mathrm{Hom}(FX, C) \cong \varinjlim R\mathrm{Hom}(FX, C_{\geq n}) \cong \varinjlim R\mathrm{Hom}(X, C_{\geq n}) \cong R\mathrm{Hom}(X, C),$$

which means $C \in D_F(\mathcal{A})$.
On the other hand, $D_F(\mathcal{A})$ is generated by $F(X)[0]$ for $X \in \mathcal{A}_0$ and $F(X)[0] \in$

$D'_F(\mathcal{A})$, then $D'_F(\mathcal{A}) = D_F(\mathcal{A})$.

Finally, we show $D(\mathcal{A}_F) \cong D_F(\mathcal{A})$.

The functor $D(\mathcal{A}_F) \to D(\mathcal{A})$ factors over $D(\mathcal{A}_F) \to D'_F(\mathcal{A}) = D_F(\mathcal{A})$ and induces an equivalence on hearts. As it is fully faithful and commutes with all products and direct sums, then it is an equivalence. The inclusion $D_F(\mathcal{A}) \subset D(\mathcal{A})$ admits a left adjoint, which necessarily commutes with direct sums and by definition, takes $X \in \mathcal{A}_0$ to $F(X) \in \mathcal{A}_0$, so is the left derived functor of $L : \mathcal{A} \to \mathcal{A}_F$. $\square$

Lemma 10.7. We take the above lemma's notation.

(i) For any C with the form $\bigoplus_{i\in I} F(X_i), X_i \in \mathcal{A}_0$, one has

$$R\mathrm{Hom}(F(X), C) \cong R\mathrm{Hom}(X, C),\ \forall X \in \mathcal{A}_0.$$

(ii) For any C with the form $\ker(\bigoplus_{i\in I} F(X_i) \to \bigoplus_{j\in J} F(Y_j))$, $X_i, Y_j \in \mathcal{A}_0$, one has

$$R\mathrm{Hom}(F(X), C) \cong R\mathrm{Hom}(X, C),\ \forall X \in \mathcal{A}_0.$$

(iii) For any C with the form $\mathrm{coker}(\bigoplus_{i\in I} F(X_i) \to \bigoplus_{j\in J} F(Y_j))$, $X_i, Y_j \in \mathcal{A}_0$, one has

$$R\mathrm{Hom}(F(X), C) \cong R\mathrm{Hom}(X, C),\ \forall X \in \mathcal{A}_0.$$

(iv) For any right bounded complex C with each term C_i having the form $\bigoplus_{j\in I_i} F(X_{i_j})$, one has

$$R\mathrm{Hom}(F(X), C) \cong R\mathrm{Hom}(X, C),\ \forall X \in \mathcal{A}_0.$$

Then (iv) $\Longrightarrow$ (iii) $\Longleftrightarrow$ (ii) $\Longrightarrow$ (i).

Proof. (ii) $\Longrightarrow$ (i). Just take $J = \emptyset$, which is exactly (i).
(ii) $\Longleftrightarrow$ (iii). For any $f : Y \to Z$, with $Y = \bigoplus_{i\in I} F(X_i)$ and $Z = \bigoplus_{j\in J} F(Y_j)$, applying functors $R\mathrm{Hom}(X, -)$ and $R\mathrm{Hom}(F(X), -)$ to the exact sequence:

$$0 \to \ker(f) \to Y \to Z \to \mathrm{coker}(f) \to 0,$$

one get

$$\begin{tikzcd}
R\mathrm{Hom}(F(X), \ker(f)) \arrow[r] \arrow[d] & R\mathrm{Hom}(F(X), Y) \arrow[r] \arrow[d, "\cong"] & R\mathrm{Hom}(F(X), Z) \arrow[r] \arrow[d, "\cong"] & R\mathrm{Hom}(F(X), \mathrm{coker}(f)) \arrow[d] \\
R\mathrm{Hom}(X, \ker(f)) \arrow[r] & R\mathrm{Hom}(X, Y) \arrow[r] & R\mathrm{Hom}(X, Z) \arrow[r] & R\mathrm{Hom}(X, \mathrm{coker}(f))
\end{tikzcd}$$

By five lemma, we can show

$$R\mathrm{Hom}(F(X), \ker(f)) \cong R\mathrm{Hom}(X, \ker(f))$$

$\Longleftrightarrow$

$$R\mathrm{Hom}(F(X), \mathrm{coker}(f)) \cong R\mathrm{Hom}(X, \mathrm{coker}(f)).$$

Hence, (ii) $\iff$ (iii).

(iv) $\implies$ (ii). For any $f : Y \to Z$, with $Y = \bigoplus_{i\in I} F(X_i)$ and $Z = \bigoplus_{j\in J} F(Y_j)$. Denote $K = \ker(f)$. Take the resolution of K:

$$\cdots \to B_1 \to B_0 \to K \to 0,$$

where each $B_i \in \mathcal{A}_0$. Now, take $C = [0 \to Y \to Z \to 0]$, by assumption, we have

$$R\mathrm{Hom}(F(B_\bullet), C) \cong R\mathrm{Hom}(B_\bullet, C).$$

Hence,

$$\begin{tikzcd} B_\bullet \arrow[r] \arrow[d] & F(B_\bullet) \arrow[dl, dotted, "\exists!"] \\ K & \end{tikzcd}$$

That is, $K \cong B_\bullet$ is the retract of $F(B_\bullet)$. Thus,

$$R\mathrm{Hom}(X, K) \cong R\mathrm{Hom}(X, F(B_\bullet)) \cong R\mathrm{Hom}(F(X), F(B_\bullet)) \cong R\mathrm{Hom}(F(X), K).$$

$\square$

Theorem 10.8. (i) - The category Solid $\subset$ Cond(Ab) of solid abelian groups is an abelian subcategory stable under limits, colimits and extensions.

- Solid$^{\text{cp}} = \{\mathbb{Z}^I \mid I \text{ is any set}\}$.
- The inclusion Solid $\subset$ Cond(Ab) admits a left adjoint

$$\text{Cond(Ab)} \to \text{Solid};\ M \mapsto M^{\blacksquare},$$

which is the unique colimit-preserving extension of $\mathbb{Z}[S] \to \mathbb{Z}[S]^{\blacksquare}$.

(ii) - The functor $D(\text{Solid}) \to D(\text{Cond(Ab)})$ is fully faithful and its essential image are precisely the solid objects of $D(\text{Cond(Ab)})$.

- An object $C \in D(\text{Cond(Ab)})$ is solid iff all $H^i(C) \in$ Solid.
- The inclusion functor $D(\text{Solid}) \to D(\text{Cond(Ab)})$ admits a left adjoint

$$D(\text{Cond(Ab)}) \to D(\text{Solid});\ C \mapsto C^{L\blacksquare},$$

which is the left derived functor of Cond(Ab) $\to$ Solid; $M \mapsto M^{\blacksquare}$.

Proposition 10.9. For an extremally disconnected space $S \in$ ExDisc and a chain complex

$$C : \cdots \to C_1 \to C_0 \to 0,$$

where each $C_i = \bigoplus_{j \in I_i} \mathbb{Z}^{I_{i,j}}$, we have

$$R\underline{\mathrm{Hom}}(\mathbb{Z}[S]^{\blacksquare}, C) \cong R\underline{\mathrm{Hom}}(\mathbb{Z}[S], C).$$

Proof. Case 1. C is concentrated in degree 0, i.e. $C = \bigoplus_{j \in J} \mathbb{Z}^{I_j}$. Since $\mathbb{Z}[S]$ is compact projective, we have

$$\begin{aligned} R\underline{\mathrm{Hom}}(\mathbb{Z}[S], C) &= R\underline{\mathrm{Hom}}(\mathbb{Z}[S], \bigoplus_{j \in J} \mathbb{Z}^{I_j}) \\ &= \bigoplus_{j \in J} R\underline{\mathrm{Hom}}(\mathbb{Z}[S], \mathbb{Z}^{I_j}) \\ &= \bigoplus_{j \in J} R\underline{\mathrm{Hom}}(\mathbb{Z}[S], \mathbb{Z})^{I_j} \\ &= \bigoplus_{j \in J} C(S, \mathbb{Z})^{I_j} \\ &= \bigoplus_{j \in J} (\mathbb{Z}^{\oplus K})^{I_j}. \end{aligned}$$

It suffices to show:

$$R\underline{\mathrm{Hom}}(\mathbb{Z}[S]^{\blacksquare}, C) \cong \bigoplus_{j \in J} (\mathbb{Z}^{\oplus K})^{I_j}.$$

We know $\mathbb{Z}[S]^{\blacksquare} = R\underline{\mathrm{Hom}}(C(S, \mathbb{Z}), \mathbb{Z}) = \mathbb{Z}^K$. Then it suffices to show:

$$R\underline{\mathrm{Hom}}(\mathbb{Z}^K, \bigoplus_{j \in J} \mathbb{Z}^{I_j}) \cong \bigoplus_{j \in J} (\mathbb{Z}^{\oplus K})^{I_j}.$$

Consider the short exact sequence: $0 \to \mathbb{Z}^K \to \mathbb{R}^K \to \mathbb{T}^K \to 0$. Since $\mathbb{Z}$ and $\mathbb{T}$ are pseudocoherent, $\mathbb{R}$ is pseudocoherent. Then

$$R\underline{\mathrm{Hom}}(\mathbb{R}, \bigoplus_{j \in J} \mathbb{Z}^{I_j}) \cong \bigoplus_{j \in J} R\underline{\mathrm{Hom}}(\mathbb{R}, \mathbb{Z}^{I_j}) \cong \bigoplus_{j \in J} R\underline{\mathrm{Hom}}(\mathbb{R}, \mathbb{Z})^{I_j} \cong 0.$$

From this, we can get:

$$R\underline{\mathrm{Hom}}(\mathbb{R}^K, \bigoplus_{j\in J} \mathbb{Z}^{I_j}) \cong R\underline{\mathrm{Hom}}_{\mathbb{R}}(\mathbb{R}^K, R\underline{\mathrm{Hom}}(\mathbb{R}, \bigoplus_{j\in J} \mathbb{Z}^{I_j})) \cong 0.$$

And because $\mathbb{T}^K$ is pseudocoherent, we have:

$$R\underline{\mathrm{Hom}}(\mathbb{T}^K, \bigoplus_{j\in J} \mathbb{Z}^{I_j}) \cong \bigoplus_{j\in J} R\underline{\mathrm{Hom}}(\mathbb{T}^K, \mathbb{Z})^{I_j} \cong \bigoplus_{j\in J} (\mathbb{Z}^{\oplus K})^{I_j}[-1].$$

Therefore, we have:

$$R\underline{\mathrm{Hom}}(\mathbb{Z}^K, \bigoplus_{j\in J} \mathbb{Z}^{I_j}) \cong \bigoplus_{j\in J} (\mathbb{Z}^{\oplus K})^{I_j}.$$

Case 2. C is bounded. It is obvious from case 1.

Case 3. For the general complex $C : \cdots \to C_1 \to C_0 \to 0$. Consider the short exact sequence:

$$0 \to C_{\leq n} \to C \to C_{>n} \to 0.$$

It suffices to show: $R\underline{\mathrm{Hom}}(\mathbb{Z}[S]^{\blacksquare}, C_{>n})$ and $R\underline{\mathrm{Hom}}(\mathbb{Z}[S], C_{>n})$ are concentrated at degree $\geq n$. This is because for any n, the cofiber of

$$R\underline{\mathrm{Hom}}(\mathbb{Z}[S]^{\blacksquare}, C) \to R\underline{\mathrm{Hom}}(\mathbb{Z}[S], C).$$

is concentrated at $\geq n$, hence the cofiber is 0.

For $R\underline{\mathrm{Hom}}(\mathbb{Z}[S], C_{>n})$, since $\mathbb{Z}[S]$ projective, $R\underline{\mathrm{Hom}}(\mathbb{Z}[S], C_{>n})$ is concentrated at $\geq n$. Hence, we need to prove $R\underline{\mathrm{Hom}}(\mathbb{Z}[S]^{\blacksquare}, C_{>n}) = R\underline{\mathrm{Hom}}(\mathbb{Z}^K, C_{>n})$ is concentrated at $\geq n$, which is equivalent to prove $R\underline{\mathrm{Hom}}(\mathbb{Z}^K, C)$ is concentrated at ≥ -1.

Claim 1: For any K, $R\underline{\mathrm{Hom}}(\mathbb{T}^K, C)$ is concentrated at ≥ -2.

Claim 2: For any K, $R\underline{\mathrm{Hom}}(\mathbb{R}^K, C) = 0$.

This is because we have $R\underline{\mathrm{Hom}}(\mathbb{R}^K, C) = \mathrm{colim}\ R\underline{\mathrm{Hom}}(\mathbb{R}^K, C_{\leq n})$, and

$$R\underline{\mathrm{Hom}}(\mathbb{R}^K, C_{\leq n}) = R\underline{\mathrm{Hom}}_{\mathbb{R}}(\mathbb{R}^K, R\mathrm{Hom}_{\mathbb{R}}(\mathbb{R}, C_{\leq n})) = 0.$$

Now, from these two claims, $R\underline{\mathrm{Hom}}(\mathbb{R}^K, C) = 0$ and $R\underline{\mathrm{Hom}}(\mathbb{T}^K, C)$ is concentrated at degree ≥ -2, we get $R\underline{\mathrm{Hom}}(\mathbb{Z}^K, C) = R\underline{\mathrm{Hom}}(\mathbb{T}^K, C)[1]$ is concentrated at degree ≥ -1.

Hence, it suffices to prove Claim 1.

We denote $C_{\mathbb{R},i} = \bigoplus_{J_i} \mathbb{R}^{I_{i,j}}$, $C_{\mathbb{T},i} = \bigoplus_{J_i} \mathbb{T}^{I_{i,j}}$, and form complexes

$$C_{\mathbb{R}} : \cdots \to C_{\mathbb{R},i} \to \cdots \to C_{\mathbb{R},1} \to C_{\mathbb{R},0} \to 0$$

and

$$C_{\mathbb{T}} : \cdots \to C_{\mathbb{T},i} \to \cdots \to C_{\mathbb{T},1} \to C_{\mathbb{T},0} \to 0.$$

There is an exact sequence $0 \to C \to C_{\mathbb{R}} \to C_{\mathbb{T}} \to 0$.

Therefore, we can reduce to prove the following claim.

Claim: $R\underline{\mathrm{Hom}}(\mathbb{T}^K, C_{\mathbb{R}})$ and $R\underline{\mathrm{Hom}}(\mathbb{T}^K, C_{\mathbb{T}})$ are concentrated at ≥ -1.

We know $R\underline{\mathrm{Hom}}(\mathbb{T}^K, C_{\mathbb{R}}) = R\lim R\underline{\mathrm{Hom}}(\mathbb{T}^K, \tau_{<n}C_{\mathbb{R}})$, where $\tau_{<n}C_{\mathbb{R}}$ is

$$0 \to \ker(C_{\mathbb{R},n} \to C_{\mathbb{R},n-1}) \to C_{\mathbb{R},n} \to \cdots \to C_{\mathbb{R},0} \to 0,$$

and $R\underline{\mathrm{Hom}}(\mathbb{T}^K, C_{\mathbb{T}}) = R\lim R\underline{\mathrm{Hom}}(\mathbb{T}^K, \tau_{<n}C_{\mathbb{T}})$, where $\tau_{<n}C_{\mathbb{T}}$ is

$$0 \to \ker(C_{\mathbb{T},n} \to C_{\mathbb{T},n-1}) \to C_{\mathbb{T},n} \to \cdots \to C_{\mathbb{T},0} \to 0.$$

Let $M_{\mathbb{R}} = \ker(\bigoplus_{j\in J} \mathbb{R}^{I_j} \to \bigoplus_{j\in J'} \mathbb{R}^{I'_j})$ and $M_{\mathbb{T}} = \ker(\bigoplus_{j\in J} \mathbb{T}^{I_j} \to \bigoplus_{j\in J'} \mathbb{T}^{I'_j})$. Because $C_{\mathbb{R},i} = \ker(C_{\mathbb{R},i} \to 0)$ and $C_{\mathbb{T},i} = \ker(C_{\mathbb{T},i} \to 0)$ also have the form of $M_{\mathbb{R}}$ and $M_{\mathbb{T}}$, it suffices to show $R\underline{\mathrm{Hom}}(\mathbb{T}^K, M_{\mathbb{R}})$ and $R\underline{\mathrm{Hom}}(\mathbb{T}^K, M_{\mathbb{T}})$ are concentrated at degree ≥ -1.

Since $\mathbb{T}^K$ is pseudocoherent, $R\underline{\mathrm{Hom}}(\mathbb{T}^K, -)$ commutes with filtered colimits, hence we can assume J is finite. Then assume

$$M_{\mathbb{R}} = \ker(\mathbb{R}^I \to \bigoplus_{j\in J'} \mathbb{R}^{I'_j}),\ M_{\mathbb{T}} = \ker(\mathbb{T}^I \to \bigoplus_{j\in J'} \mathbb{T}^{I'_j}).$$

Besides, we can also assume J' is finite. Hence let

$$M_{\mathbb{R}} = \ker(\mathbb{R}^I \to \mathbb{R}^{I'}),\ M_{\mathbb{T}} = \ker(\mathbb{T}^I \to \mathbb{T}^{I'}).$$

Now, as a topological group, $M_{\mathbb{T}} = \ker(\mathbb{T}^I \to \mathbb{T}^{I'})$ is compact, and $\mathbb{T}^K$ is compact, by Corollary 9.10, the cohomology of $R\underline{\mathrm{Hom}}(\mathbb{T}^K, M_{\mathbb{T}})$ is concentrated at 0 and 1, hence its homology is concentrated at ≥ -1.

Claim: $M_{\mathbb{R}}$ is a direct summand of $\mathbb{R}^I$.

From this claim, by $R\underline{\mathrm{Hom}}(\mathbb{T}^K, \mathbb{R}^I) = R\underline{\mathrm{Hom}}(\mathbb{T}^K, \mathbb{R})^I = 0$, we have $R\underline{\mathrm{Hom}}(\mathbb{T}^K, M_{\mathbb{R}}) = 0$.

Then, it suffices to prove the above claim. This is because, for $\mathbb{R}$-linear map $\mathbb{R}^{\oplus I'} \to \mathbb{R}^{\oplus I}$ is the composition of a split surjection and a split injection. Then by taking the duality $\underline{\mathrm{Hom}}(-, \mathbb{R})$, the dual map $\mathbb{R}^I \to \mathbb{R}^{I'}$ is split. $\square$

Definition 10.10. (i) For $M, N \in \text{Solid}$, define $M \otimes^{\blacksquare} N := (M \otimes N)^{\blacksquare}$.

(ii) For $C, D \in D(\text{Solid})$, define $C \otimes^{L\blacksquare} D := (C \otimes^{L} D)^{L\blacksquare}$.

Theorem 10.11. (i) The solidification functor Cond(Ab) $\to$ Solid; $M \mapsto M^{\blacksquare}$ is symmetric monoidal, i.e.

$$(M \otimes N)^{\blacksquare} \cong M^{\blacksquare} \otimes^{\blacksquare} N^{\blacksquare}.$$

(ii) The solidification functor $D(\text{Cond(Ab)}) \to D(\text{Solid})$; $C \mapsto C^{L\blacksquare}$ is symmetric monoidal, i.e.

$$(C \otimes^{L} D)^{L\blacksquare} \cong C^{L\blacksquare} \otimes^{L\blacksquare} D^{L\blacksquare}.$$

(iii) $\otimes^{L\blacksquare}$ is the left derived functor of $\otimes^{\blacksquare}$.

Proof. (i) By definition, we need to show:

$$(M \otimes N)^{\blacksquare} \xrightarrow{\sim} (M^{\blacksquare} \otimes N^{\blacksquare})^{\blacksquare}.$$

This can be written as the composition:

$$(M \otimes N)^{\blacksquare} \longrightarrow (M^{\blacksquare} \otimes N)^{\blacksquare} \longrightarrow (M^{\blacksquare} \otimes N^{\blacksquare})^{\blacksquare}.$$

Hence, it is enough to prove

$$(M \otimes N)^{\blacksquare} \xrightarrow{\sim} (M^{\blacksquare} \otimes N)^{\blacksquare}.$$

(With this isomorphism, we can also show that the second map is an isomorphism). Since the tensor functor and the solidification functor commute with colimits, then we can assume $M = \mathbb{Z}[S]$ and $N = \mathbb{Z}[T]$.
It reduces to show:

$$\mathbb{Z}[S \times T]^{\blacksquare} \xrightarrow{\sim} (\mathbb{Z}[S]^{\blacksquare} \otimes \mathbb{Z}[T])^{\blacksquare}.$$

Equivalently, for any $A \in \text{Solid}$,

$$\underline{\text{Hom}}((\mathbb{Z}[S]^{\blacksquare} \otimes \mathbb{Z}[T])^{\blacksquare}, A) \cong \underline{\text{Hom}}(\mathbb{Z}[S \times T]^{\blacksquare}, A).$$

Since A is solid, we have:

$$\underline{\mathrm{Hom}}((\mathbb{Z}[S]^{\blacksquare} \otimes \mathbb{Z}[T])^{\blacksquare}, A) \cong \underline{\mathrm{Hom}}(\mathbb{Z}[S]^{\blacksquare} \otimes \mathbb{Z}[T], A)$$

and

$$\underline{\mathrm{Hom}}(\mathbb{Z}[S \times T]^{\blacksquare}, A) \cong \underline{\mathrm{Hom}}(\mathbb{Z}[S \times T], A).$$

By computation:

$$\begin{aligned}
\underline{\mathrm{Hom}}(\mathbb{Z}[S]^{\blacksquare} \otimes \mathbb{Z}[T], A) &\cong \underline{\mathrm{Hom}}(\mathbb{Z}[S]^{\blacksquare}, \underline{\mathrm{Hom}}(\mathbb{Z}[T], A)) \\
&\cong \underline{\mathrm{Hom}}(\mathbb{Z}[S], \underline{\mathrm{Hom}}(\mathbb{Z}[T], A)) \\
&\cong \underline{\mathrm{Hom}}(\mathbb{Z}[S] \otimes \mathbb{Z}[T], A) \\
&\cong \underline{\mathrm{Hom}}(\mathbb{Z}[S \times T], A).
\end{aligned}$$

Thus, $\underline{\mathrm{Hom}}((\mathbb{Z}[S]^{\blacksquare} \otimes \mathbb{Z}[T])^{\blacksquare}, A) \cong \underline{\mathrm{Hom}}(\mathbb{Z}[S \times T]^{\blacksquare}, A)$.

(ii) Similar to the proof of (i).

(iii)

Remark 10.12. In Solid, $\otimes^{\blacksquare}$ is the left adjoint of $\underline{\mathrm{Hom}}$:

$$\mathrm{Hom}(M \otimes^{\blacksquare} N, P) \cong \mathrm{Hom}((M \otimes N)^{\blacksquare}, P) \cong \mathrm{Hom}(M \otimes N, P) \cong \mathrm{Hom}(M, \underline{\mathrm{Hom}}(N, P)).$$

Proposition 10.13. (i) If $X \in$ CHaus, then $\mathbb{Z}[X]^{L\blacksquare} = R\underline{\mathrm{Hom}}(R\Gamma(X, \mathbb{Z}), \mathbb{Z})$.
In particular, if $X \in$ ProFin $\subseteq$ CHaus, then $\mathbb{Z}[X]^{L\blacksquare} = \mathbb{Z}[X]^{\blacksquare}$.

(ii) If X is a CW space, then $\mathbb{Z}[X]^{L\blacksquare} = C_\bullet(X)$.
This shows that the derived solidification of a condensed abelian group can sit in all nonnegative homological degrees.

Proposition 10.14. (i) $\mathbb{R}^{L\blacksquare} = 0$.

(ii) $\mathbb{Z}^I \otimes^{L\blacksquare} \mathbb{Z}^J = \mathbb{Z}^{I \times J}$.

(iii) $\mathbb{Z}_p \otimes^{L\blacksquare} \mathbb{Z}_p = \mathbb{Z}_p$.

(iv) $\mathbb{Z}_p \otimes^{L\blacksquare} \mathbb{Z}_\ell = 0$. $(p \neq \ell)$

Proof. (i) By Yoneda's lemma, it suffices to show: for any $C \in D(\text{Solid})$, one has

$$R\text{Hom}(\mathbb{R}^{L\blacksquare}, C) = R\text{Hom}(\mathbb{R}, C) = 0.$$

Since $C = \varprojlim C_{\geq n}$, and $R\text{Hom}(\mathbb{R}, -)$ commutes with limits, it reduces to the case C is a right bounded complex. And for a right bounded complex C, one has $C = \varprojlim C_{\leq n}$, it reduces to the case C is a bounded complex.

Hence it suffices to show: for any $X \in \text{Solid}$, one has $R\text{Hom}(\mathbb{R}, X) = 0$.

We know for any object $X \in \text{Solid}$, we can write X as the colimit of objects of the form $\bigoplus_{j \in J} \mathbb{Z}^{I_j}$. And we know taking all colimits is equivalent to taking all cokernels and all filtered colimits.

Since $\mathbb{R}$ is pseudo-coherent, we get

$$R\text{Hom}(\mathbb{R}, \varinjlim \bigoplus_{i \in J_j} \mathbb{Z}^{I_{i,j}}) = \varinjlim R\text{Hom}(\mathbb{R}, \bigoplus_{i \in J_j} \mathbb{Z}^{I_{i,j}}) = \varinjlim \bigoplus_{i \in J_j} R\text{Hom}(\mathbb{R}, \mathbb{Z}^{I_{i,j}}) = 0.$$

Let $f : X \to Y$, $X = \bigoplus_{i \in I} \mathbb{Z}^{I_i}$ and $Y = \bigoplus_{j \in J} \mathbb{Z}^{I_j}$, then from $R\text{Hom}(\mathbb{R}, X) = 0$ and $R\text{Hom}(\mathbb{R}, Y) = 0$, we know $R\text{Hom}(\mathbb{R}, \text{coker}(f)) = 0$.

Thus, we finish our proof.

(ii) Assume $\mathbb{Z}^I = \mathbb{Z}[S]^\blacksquare = \underline{\text{Hom}}(C(S, \mathbb{Z}), \mathbb{Z})$, $\mathbb{Z}^J = \mathbb{Z}[T]^\blacksquare = \underline{\text{Hom}}(C(T, \mathbb{Z}), \mathbb{Z})$, for some S, $T \in \text{ProFin}$. Then

$$\begin{aligned}
\mathbb{Z}[S \times T]^\blacksquare &= \underline{\text{Hom}}(C(S \times T, \mathbb{Z}), \mathbb{Z}) \\
&= \underline{\text{Hom}}(C(S, \mathbb{Z}) \otimes C(T, \mathbb{Z}), \mathbb{Z}) \\
&= \underline{\text{Hom}}(C(S, \mathbb{Z}), \underline{\text{Hom}}(C(T, \mathbb{Z}), \mathbb{Z})) \\
&= \underline{\text{Hom}}(C(S, \mathbb{Z}), \mathbb{Z}^J) \\
&= \underline{\text{Hom}}(C(S, \mathbb{Z}), \mathbb{Z})^J \\
&= \mathbb{Z}^{I \times J}.
\end{aligned}$$

Thus, we have

$$\mathbb{Z}^I \otimes^{L\blacksquare} \mathbb{Z}^J = \mathbb{Z}[S]^\blacksquare \otimes^{L\blacksquare} \mathbb{Z}[T]^\blacksquare = (\mathbb{Z}[S] \otimes^L \mathbb{Z}[T])^{L\blacksquare} = \mathbb{Z}[S \times T]^{L\blacksquare} = \mathbb{Z}^{I \times J}.$$

(iii) We write $\mathbb{Z}_p = \mathbb{Z}[[x]]/(x-p)$, then

$$\begin{aligned}\mathbb{Z}_p \otimes^{L\blacksquare} \mathbb{Z}_p &= \mathbb{Z}[[x]]/(x-p) \otimes^{L\blacksquare} \mathbb{Z}[[y]]/(y-p) \\ &= \mathbb{Z}[[x,y]]/(x-p, y-p) \\ &= \mathbb{Z}[[x,y]]/(x-p, x-y) \\ &= \mathbb{Z}_p.\end{aligned}$$

(iv) We write $\mathbb{Z}_p = \mathbb{Z}[[x]]/(x-p)$ and $\mathbb{Z}_\ell = \mathbb{Z}[[y]]/(y-\ell)$, then

$$\begin{aligned}\mathbb{Z}_p \otimes^{L\blacksquare} \mathbb{Z}_\ell &= \mathbb{Z}[[x]]/(x-p) \otimes^{L\blacksquare} \mathbb{Z}[[y]]/(y-\ell) \\ &= \mathbb{Z}[[x,y]]/(x-p, y-\ell).\end{aligned}$$

Since p and ℓ are coprime, pick $a, b \in \mathbb{Z}$, s.t. $ap + b\ell = 1$, then $a(p-x) + b(\ell - y) = 1 - ax - by$ is invertible in $\mathbb{Z}[[x,y]]$. Hence $(x-p, y-\ell) = \mathbb{Z}[[x,y]]$, i.e. $\mathbb{Z}_p \otimes^{L\blacksquare} \mathbb{Z}_\ell = 0$.

$\square$

11 Analytic rings

Definition 11.1. A pre-analytic ring $\mathcal{A}$ is a triple $(\underline{\mathcal{A}}, \mathcal{A}[-], \alpha)$, where:

- $\underline{\mathcal{A}}$ is a condensed ring, which is called the underlying condensed ring of the pre-analytic ring $\mathcal{A}$.
- The functor $\mathcal{A}[-] : \text{ExDisc} \to \text{Mod}(\underline{\mathcal{A}});\ S \mapsto \mathcal{A}[S]$ preserves finite colimits, where $\text{Mod}(\underline{\mathcal{A}})$ is the category of $\underline{\mathcal{A}}$-modules in Cond(Ab).
- $\alpha : \underline{\mathcal{A}}[-] \to \mathcal{A}[-]$ is a natural transformation.

$\underline{\mathcal{A}}[S]$ is called the free $\underline{\mathcal{A}}$-module on S, and $\mathcal{A}[S]$ is called the free $\mathcal{A}$-module on S.

Definition 11.2. A map of pre-analytic rings $\mathcal{A} \to \mathcal{B}$ is s.t.

- $\underline{\mathcal{A}} \to \underline{\mathcal{B}}$ is a map of condensed rings.
- For any $S \in \text{ExDisc}$, $\mathcal{A}[S] \to \mathcal{B}[S]$ is an $\underline{\mathcal{A}}$-linear map and is natural in S and commutes with the map from S.

Example 11.3. (i) For any condensed ring R, $R = (R, R[-], \text{id})$ is a pre-analytic ring.

(ii) The pre-analytic ring $\mathbb{Z}_\blacksquare$.
Take the underlying condensed ring to be $\underline{\mathbb{Z}_\blacksquare} = \mathbb{Z}$, and for any $S \in \text{ExDisc}$, take $\mathbb{Z}_\blacksquare[S] := \mathbb{Z}[S]^\blacksquare = \varprojlim \mathbb{Z}[S_i]$, where $S = \varprojlim S_i$, each S_i is finite.

(iii) The pre-analytic ring $\mathbb{Z}_{p,\blacksquare}$.
Take the underlying condensed ring to be $\underline{\mathbb{Z}_{p,\blacksquare}} = \mathbb{Z}_p$, and for any $S \in \text{ExDisc}$, take $\mathbb{Z}_{p,\blacksquare}[S] := \varprojlim \mathbb{Z}_p[S_i]$, where $S = \varprojlim S_i$, each S_i is finite.

(iv) For any finitely generated $\mathbb{Z}$-algebra R, we can define a pre-analytic ring $R_\blacksquare$.
The underlying condensed ring is $\underline{R_\blacksquare} = R$ and $R_\blacksquare[S] := \varprojlim R[S_i]$, where $S = \varprojlim S_i$, each S_i is finite.

(v) Let R, A be finitely generated $\mathbb{Z}$-algebra, $R \to A$ is a map. We define a pre-analytic ring $(A, R)_\blacksquare$.
$\underline{(A, R)_\blacksquare} := A$ and $(A, R)_\blacksquare[S] := R_\blacksquare[S] \otimes_R A$.
In particular, for the pre-analytic ring $(A, \mathbb{Z})_\blacksquare$, $\underline{(A, \mathbb{Z})_\blacksquare} = A$ and $(A, \mathbb{Z})_\blacksquare[S] = \mathbb{Z}_\blacksquare[S] \otimes_{\mathbb{Z}} A$.

(vi) The pre-analytic ring $(\mathbb{Q}_p, \mathbb{Z}_p)_{\blacksquare}$.

$\underline{(\mathbb{Q}_p, \mathbb{Z}_p)_{\blacksquare}} := \mathbb{Q}_p$ and $(\mathbb{Q}_p, \mathbb{Z}_p)_{\blacksquare}[S] := \mathbb{Z}_{p,\blacksquare}[S] \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

(vii) For $0 < p \leq 1$, we define the pre-analytic ring $\mathbb{R}_{\ell^p}$.

$\underline{\mathbb{R}_{\ell^p}} := \mathbb{R}$ and $\mathbb{R}_{\ell^p}[S] := \bigcup_{r>0} \varprojlim \mathbb{R}[S_i]_{\ell^p \leq r}$, where $S = \varprojlim S_i$, each S_i is finite.

Here, $\mathbb{R}[S_i]_{\ell^p \leq r} := \{\sum r_j x_j \in \mathbb{R}[S_i] \mid r_j \in \mathbb{R}, x_j \in S_i, \sum |r_j|^p \leq r\}$.

(vii) For $0 < p \leq 1$, we define the pre-analytic ring $\mathbb{R}_{\ell^{<p}}$.

$\underline{\mathbb{R}_{\ell^{<p}}} := \mathbb{R}$ and $\mathbb{R}_{\ell^{<p}}[S] := \varinjlim_{q<p} \mathbb{R}_{\ell^q}[S]$.

Definition 11.4. An analytic ring is a pre-analytic ring $\mathcal{A}$, s.t. for any complex

$$C : \cdots \to C_i \to \cdots \to C_1 \to C_0 \to 0$$

in $D(\mathrm{Mod}(\underline{\mathcal{A}}))$, each $C_i = \bigoplus_{j \in I_i} \mathcal{A}[T_{i,j}]$, $T_{i,j} \in \mathrm{ExDisc}$, the map

$$R\underline{\mathrm{Hom}}_{\underline{\mathcal{A}}}(\mathcal{A}[S], C) \xrightarrow{\sim} R\underline{\mathrm{Hom}}_{\underline{\mathcal{A}}}(\underline{\mathcal{A}}[S], C)$$

is an isomorphism.

12 Notes on 12.24

Definition 12.1. For a pair (A, R), we mean $R \to A$, where A, R are $\mathbb{Z}$-algebra of finite type. For any map $f : (A, R) \to (B, S)$, consider the decomposition

$$\begin{tikzcd}
(A,R) \arrow[r, "\overline{f}"] \arrow[dr, "f"'] & (B,R) \arrow[d, "j"] \\
& (B,S)
\end{tikzcd}$$

We define $f_! := \overline{f}_* j_! : D((B,S)_\blacksquare) \to D((A,R)_\blacksquare)$. Since $\overline{f}_*$ and $j_!$ commute with all colimits, then $f_!$ commutes with all colimits, hence $f_!$ admits a right adjoint $f^! : D((A,R)_\blacksquare) \to D((B,S)_\blacksquare)$.

Proposition 12.2. For $f : (A, R) \to (B, S)$ and $g : (B, S) \to (C, T)$, we have $(gf)_! = f_! g_!$, hence by adjunction, we have $(gf)^! = g^! f^!$.

Proof. We have the following diagram

$$\begin{tikzcd}
(A,R) \arrow[r, "\overline{f}"] \arrow[dr, "f"'] & (B,R) \arrow[r, "\overline{g}'"] \arrow[d, "j"] & (C,R) \arrow[d, "j'"] \\
& (B,S) \arrow[r, "\overline{g}"] \arrow[dr, "g"'] & (C,S) \arrow[d, "k"] \\
& & (C,T)
\end{tikzcd}$$

We know $f_! = \overline{f}_* j_!$, $g_! = \overline{g}_* k_!$ and $(gf)_! = (\overline{g}'\overline{f})_*(kj')_! = \overline{f}_*\overline{g}'_* j'_! k_!$, in order to show $(gf)_! = f_! g_!$, it suffices to show: $j_! \overline{g}_* = \overline{g}'_* j'_!$. This is because, for any $M \in D((B,S)_\blacksquare)$,

$$\begin{aligned}
\overline{g}'_* j'_! M &= \overline{g}'_*(M \otimes^L_{(S,R)_\blacksquare} j'_! S) \\
&= \overline{g}_* M \otimes^L_{(S,R)_\blacksquare} j_! S \\
&= j_! \overline{g}_* M.
\end{aligned}$$

□

Theorem 12.3. Assume $f : R \to A$ is a map of $\mathbb{Z}$-algebra of finite types. View it as $f : (R, R) \to (A, A)$, then we get $f_! = \overline{f}_* j_! : D(A_\blacksquare) \to D(R_\blacksquare)$, and its right adjoint $f^! : D(R_\blacksquare) \to D(A_\blacksquare)$.

(1) (i) $f_!$ preserves pseudo-coherent objects.

(ii) $f^!$ preserves discrete objects.

(iii) [Projection formula] $f_!(f^*M \otimes^L_{A_\blacksquare} N) \cong M \otimes^L_{R_\blacksquare} f_!N$.

(2) Assume $f : R \to A$ is of finite Tor dimension, then

(i) $f_!$ preserves compact objects.

(ii) $f^!$ preserves all colimits.

(iii) $f^!$ preserves perfect complexes.

(iv) For any $M \in D(R_\blacksquare)$, $f^!M = f^*M \otimes^L_{A_\blacksquare} f^!R$, i.e. $f^!$ is the twist of f^* by $f^!R$.

Proof. (1).(i).and (2).(i).
Since the pseudo-coherent objects of $D(A_\blacksquare)$ are right bounded complexes which each term has the form A^I, and the compact objects are the direct summands of bounded pseudo-coherent objects, in order to show $f_!$ preserves pseudo-coherent objects, it suffices to show $f_!A^I$ is pseudo-coherent, and if f is of finite Tor dimension, then $f_!A^I$ is compact. As before, we can reduce the case $f : R \to R[X] = A$ and the case $f : R \twoheadrightarrow A$.
Case 1. $f : R \to R[X] = A$. Then

$$\begin{aligned} f_!A^I &= \overline{f}_*j_!A^I \\ &= \overline{f}_*j_!j^*(R^I \otimes_R A) \\ &= R^I \otimes_R A \otimes^L_{(A,R)_\blacksquare} (A_\infty/A)[-1] \\ &= R^I \otimes^L_{R_\blacksquare} (A_\infty/A)[-1] \\ &= R^{I\times\omega}[-1] \end{aligned}$$

is a compact object in $D(R_\blacksquare)$.
Case 2. $f : R \twoheadrightarrow A$. Then $f_! = f_*$, and $f_!A^I = f_*A^I = A^I$ viewed as an R-module. Since R is Noetherian ring, we can take the resolution of A by finite free R-modules $C_\bullet \to A$, hence $A^I \cong C_\bullet^I \in D(R_\bullet)$ is pseudo-coherent.
If for general $f : R \to A$, f is of finite Tor dimension, then when we reduce to the case $f : R \twoheadrightarrow A$, it is also of finite Tor dimension. In this case, we can that $C_\bullet \to A$ so that $C_\bullet$ is a bounded complex, then $f_!A^I \in D(R_\blacksquare)$ is compact.
(1).(ii). From the adjunction we know $f^!M = R\underline{\mathrm{Hom}}_R(f_!A, M)$. If $M]inD(R_\blacksquare)$ is

discrete, since $f_!A$ is pseudo-coherent and $R\underline{\mathrm{Hom}}_R(A^I, M) = M^{\oplus I}$, then $f^!M \in D(A_\blacksquare)$ discrete.

(1).(iii). $\forall P \in D(R_\blacksquare)$,

$$\begin{aligned} R\underline{\mathrm{Hom}}_R(f_!(f^*M \otimes^L_{A_\blacksquare} N), P) &= R\underline{\mathrm{Hom}}_A(f^*M \otimes^L_{A_\blacksquare} N, f^!P) \\ &= R\underline{\mathrm{Hom}}_A(f^*M, R\underline{\mathrm{Hom}}_A(N, f^!P)) \\ &= R\underline{\mathrm{Hom}}_R(M, f_*R\underline{\mathrm{Hom}}_A(N, f^!P)) \\ &= R\underline{\mathrm{Hom}}_R(M, R\underline{\mathrm{Hom}}_R(f_!N, P)) \\ &= R\underline{\mathrm{Hom}}_R(M \otimes^L_{R_\blacksquare} f_!N, P), \end{aligned}$$

then by Yoneda's lemma, we have $f_!(f^*M \otimes^L_{A_\blacksquare} N) = M \otimes^L_{R_\blacksquare} f_!N$.

(2).(ii). Since $f_!$ preserves compact objects, and $D(A_\blacksquare)$ is compactly generated, then $f^!$ commutes with filtered colimits.

And since $f^!$ is a right adjoint, it preserves finite limits. We know $D(R_\blacksquare)$ and $D(A_\blacksquare)$ are stable ∞-category, a functor preserves finite limits if and only if it preserves finite colimits. Hence $f^!$ preserves finite colimits.

Thus, $f^!$ preserves all colimits.

(2).(iii). It suffices to show: $f^!R$ is a perfect A-complex.

Case 1. $f : R \to R[X] = A$. Then $f^!R = R\underline{\mathrm{Hom}}_R(f_!A, R) = R\underline{\mathrm{Hom}}_R((A_\infty/A)[-1], R) = A[1]$ is a perfect A-complex.

Case 2. $f : R \twoheadrightarrow A$. Since f is of finite Tor dimension, then $f_!A = f_*A \in D(R)$ is perfect. Hence $f^!R = R\underline{\mathrm{Hom}}_R(f_!A, R) \in D(R)$ is perfect. We know that a complex is perfect if and only if it is pseudo-coherent and of finite Tor dimension, therefore, $f^!R \in D(A)$ is perfect.

(2).(iv). First, we construct such a map. From the counit

$$f_!f^!R \to R,$$

we can get

$$M = R\underline{\mathrm{Hom}}_R(R, M) \to R\underline{\mathrm{Hom}}_R(f_!f^!R, M) = f_*R\underline{\mathrm{Hom}}_A(f^!R, f^!M),$$

then

$$f^*M \to R\underline{\mathrm{Hom}}_A(f^!R, f^!M),$$

and finally we have:

$$f^*M \otimes^L_{A_\blacksquare} f^!R \to f^!M.$$

Since both sides commutes with all colimits, and $D(R_\blacksquare)$ is generated under colimits by R^I, hence it suffices to show:

$$f^*R^I \otimes^L_{A_\blacksquare} f^!R = f^!R^I.$$

This is because

$$\begin{aligned} f^*R^I \otimes^L_{A_\blacksquare} f^!R &= A^I \otimes^L_{A_\blacksquare} f^!R \\ &= (f^!R)^I \\ &= f^!R^I. \end{aligned}$$

□

Proposition 12.4. $f : R \to A$ is a ring mao which is the base change of a finitely generated map of finite Tor dimension between Noetherian rings, consider the flat ring map $g : R \to S$, and

$$\begin{tikzcd} R \arrow[r, "f"] \arrow[d, "g"'] & A \arrow[d, "g'"] \\ S \arrow[r, "f'"] & B \end{tikzcd}$$

where $B = A \otimes_R S$. Then

$$g'^*f^! \cong f'^!g^* : D(R_\blacksquare) \to D(B_\blacksquare).$$

Proof. Similar as before, we can reduce to the case $A = R[T]$ and the case $f : R \twoheadrightarrow A$. We have, for any $M \in D(R_\blacksquare)$,

$$g'^*f^!M = g'^*(f^*M \otimes^L_{A_\blacksquare} f^!R) = g'^*f^*M \otimes^L_{B_\blacksquare} g'^*f^!R$$

and

$$f'^!g^*M = f'^*g^*M \otimes^L_{B_\blacksquare} f'^!S = g'^*f^*M \otimes^L_{B_\blacksquare} f'^!g^*R.$$

Hence, it suffices to show: $g'^*f^!R \cong f'^!g^*R$.

Case 1. $f : R \to A = R[T]$. Then $f^!R = A[1]$ and $f'^!S = B[1]$, hence

$$g'^*f^!R = g'^*A[1] = A[1] \otimes^L_{A_\blacksquare} B_\blacksquare = B[1] = f'^!g^*R.$$

Case 2. $f : R \twoheadrightarrow A$, then $f_! = f_*$ and $f^!R = R\underline{\mathrm{Hom}}_R(A, R)$. Similarly, $f'^!g^*R = f'^!S = R\underline{\mathrm{Hom}}_S(B, S)$. Hence, it suffices to show

$$g'^*f^!R = g'^*R\underline{\mathrm{Hom}}_R(A, R) = R\underline{\mathrm{Hom}}_R(A, R) \otimes^L_{A_\blacksquare} B_\blacksquare = R\underline{\mathrm{Hom}}_S(B, S).$$

This is true since g is flat. □

Proposition 12.5. Assume $f : R \to A$ is a regular closed immersion of pure codimension c and $I = \mathrm{Ker}(f)$, then

$$f^!R = \mathrm{det}_A(I/I^2)^*[-c].$$

Proof. We know $A = R/(f_1, \cdots, f_c)$, where $f_1, \cdots, f_c$ is a regular sequence. By Koszul complex:

$$0 \to \bigwedge^c R^c \to \bigwedge^{c-1} R^c \to \cdots \to \bigwedge^1 R^c \to R \to A = R/(f_1, \cdots, f_c) \to 0,$$

we can get

$$f^!R = R\underline{\mathrm{Hom}}_R(A, R) = A[-c].$$

We also have $\mathrm{det}_A(I/I^2) \cong A$, then $f^!R = \mathrm{det}_A(I/I^2)^*[-c]$, which is independent of the choice of $f_1, \cdots, f_c$. □

Proposition 12.6. Assume $f : R \to A$ is smooth of relative dimension d, then

$$f^!R = \mathrm{det}_A(\Omega^1_{A/R})[d].$$

Proof. We have already known $f^!R$ is a line bundle concentrated in degree d. Let $g : A \otimes_R A \to A; a \otimes b \mapsto ab$, corresponding the diagonal $\Delta_f : \mathrm{Spec}\ A \hookrightarrow \mathrm{Spec}\ A \times_{\mathrm{Spec}\ R}$

Spec A. Then g is a regular closed immersion of dimension d. Consider

$$\begin{tikzcd}
R \arrow[r, "f"] \arrow[d, "f"'] & A \arrow[d, "p_2"'] \arrow[ddr, equal] & \\
A \arrow[r, "p_1"] \arrow[drr, equal] & A \otimes_R A \arrow[dr, "g"] & \\
& & A
\end{tikzcd}$$

We have:

$$\begin{aligned}
f^! R &= g^! p_1^! f^! R \\
&= g^! (p_1^* f^! R \otimes^L_{(A \otimes_R A)_\blacksquare} p_1^! A) \\
&= g^! (p_1^* f^! R \otimes^L_{(A \otimes_R A)_\blacksquare} p_2^* f^! R) \\
&= g^* (p_1^* f^! R \otimes^L_{(A \otimes_R A)_\blacksquare} p_2^* f^! R) \otimes^L_{A_\blacksquare} g^! (A \otimes_R A) \\
&= g^* p_1^* f^! R \otimes^L_{A_\blacksquare} g^* p_2^* f^! R \otimes^L_{A_\blacksquare} g^! (A \otimes_R A) \\
&= f^! R \otimes^L_{A_\blacksquare} f^! R \otimes^L_{A_\blacksquare} g^! (A \otimes_R A).
\end{aligned}$$

Since $f^! R \in D(A_\blacksquare)$ is invertible, then

$$f^! R = (g^! (A \otimes_R A))^*.$$

Since g is a regular closed immersion of codimension d, then $g^!(A \otimes_R A) = \det_A(I/I^2)^*[-d]$, where $I = \mathrm{Ker}(A \otimes_R A \to A)$. Thus,

$$f^! R = (g^! (A \otimes_R A))^* = \det_A(I/I^2)[d] = \det_A(\Omega^1_{A/R})[d].$$

$\square$

13 Hypercompletion and Hypercovering

13.1 Grothendieck topologies

Proposition 13.1. Let $L : \mathcal{X} \to \mathcal{Y}$ be a localization of ∞-categories. Suppose that $\mathcal{X}$ admits finite limits. TFAE.

(1) The functor L is left exact.

(2) For every pullback diagram

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r] & Y \end{tikzcd}$$

in $\mathcal{X}$ s.t. Lf is an equivalence in $\mathcal{Y}$, Lf' is also an equivalence in $\mathcal{Y}$.

Proof. (1) $\implies$ (2).
Since L left exact, i.e. L preserves finite limits, L sends a pullback diagram to a pullback diagram, and the pullback of an equivalence is an equivalence, we can get (2).
(2) $\implies$ (1).
Let $S = \{\text{morphism } f \text{ in } \mathcal{X} \mid Lf \text{ is an equivalence}\}$. We may assume $\mathcal{Y} = S^{-1}\mathcal{X}$.
Since the final object $1_{\mathcal{X}} \in \mathcal{X}$ is S-local, we have $L(1_{\mathcal{X}}) \simeq 1_{\mathcal{Y}}$.
Then it suffices to show L commutes with pullbacks. Consider

$$\begin{tikzcd} X \times_Y Z \arrow[r] \arrow[d] & Z \arrow[d] \\ X \arrow[r] & Y \end{tikzcd} \qquad \begin{tikzcd} L(X \times_Y Z) \arrow[r] \arrow[d] & LZ \arrow[d] \\ LX \arrow[r] & LY \end{tikzcd} .$$

Since LX, LY, LZ are S-local, we get the pullback $LX \times_{LY} LZ$ is S-local. We claim that the map $f : X \times_Y Z \to LX \times_{LY} LZ$ is in S, then from this claim and $LX \times_{LY} LZ$ is S-local, we get

$$L(f) : L(X \times_Y Z) \xrightarrow{\sim} L(LX \times_{LY} LZ) \simeq LX \times_{LY} LZ.$$

Write f as

$$X \times_Y Z \xrightarrow{f'} X \times_{LY} Z \to LX \times_{LY} Z \to LX \times_{LY} LZ.$$

Since $X \times_{LY} Z \to LX \times_{LY} Z$ is the pullback of $X \to LX$ and $LX \times_{LY} Z \to LX \times_{LY} LZ$

is the pullback of $Z \to LZ$, and we know that $X \to LX, Z \to LZ$ are in S, then $X \times_{LY} Z \to LX \times_{LY} Z, LX \times_{LY} Z \to LX \times_{LY} LZ$ are in S, thus it suffices to show $f' : X \times_Y Z \to X \times_{LY} Z$ is in S.

We know that

$$\begin{tikzcd} X \times_Y Z \arrow[r] \arrow[d, "f'"] & Y \times_Y Y = Y \arrow[d, "f''"] \\ X \times_{LY} Z \arrow[r] & Y \times_{LY} Y \end{tikzcd}$$

is a pullback, in order to show $f' \in S$, it suffices to show $f'' \in S$. Here $f'' : Y \to Y \times_{LY} Y$ is a diagonal. For the following diagram:

$$\begin{tikzcd} Y \arrow[dr] \arrow[drr, equal] \arrow[ddr, equal] & & \\ & Y \times_{LY} Y \arrow[r] \arrow[d] & Y \arrow[d] \\ & Y \arrow[r] & LY \end{tikzcd}$$

Since $Y \to LY \in S$, we get $Y \times_{LY} Y \to Y \in S$, and since $Y \xrightarrow{''} Y \times_{LY} Y \to Y$ is the $1_Y \in S$, by two out of three, we know that $f'' : Y \to Y \times_{LY} Y \in S$. $\square$

Definition 13.2. Let $\mathcal{X}$ be a presentable ∞-category and let $\overline{S}$ be a strongly saturated class of morphisms of $\mathcal{X}$. We call $\overline{S}$ is topological, if it satisfies:

(1) There exists $S \subset \overline{S}$ consisting of monomorphism s.t. S generates $\overline{S}$ as a strongly saturated class of morphisms.

(2) $\overline{S}$ is stable under pullback.

We call a localization $L : \mathcal{X} \to \mathcal{Y}$ is a topological localization if $\overline{S} = \{f : X \to Y \text{ in } \mathcal{X} \mid Lf \text{ is an equivalence}\}$ is topological.

Theorem 13.3 (HTT 6.2.1.7)**.** Let $\mathcal{X}$ be a presentable ∞-category. Then every topological localization $L : \mathcal{X} \to \mathcal{Y}$ is an accessible left exact localization.

Definition 13.4. Let $\mathcal{C}$ be an ∞-category.

- A sieve on $\mathcal{C}$ is a full subcategory $\mathcal{C}^{(0)} \subset \mathcal{C}$ having the property that if $f : C \to D$ is a morphism in $\mathcal{C}$ and $D \in \mathcal{C}^{(0)}$, then $C \in \mathcal{C}^{(0)}$.

- If $C \in \mathcal{C}$, then a sieve on C is a sieve on the ∞-category $\mathcal{C}_{/C}$.

Definition 13.5 (Grothendieck topology)**.** A Grothendieck topology on an ∞-category $\mathcal{C}$ consists of a specification, for each $C \in \mathcal{C}$, of a collection of sieves on C, which we refer to as covering sieves. Covering sieves are required to satisfies:

(1) If $C \in \mathcal{C}$, then the sieve $\mathcal{C}_{/C} \subset \mathcal{C}_{/C}$ is a covering sieve.

(2) If $f : C \to D$ is a morphism in $\mathcal{C}$, and $\mathcal{C}^{(0)}_{/D} \subset \mathcal{C}_{/D}$ is a covering sieve on D, then $f^*\mathcal{C}^{(0)}_{/D} := \mathcal{C}_{/C} \times_{\mathcal{C}_{/D}} \mathcal{C}^{(0)}_{/D} \subset \mathcal{C}_{/C}$ is a covering sieve on C.

(3) Let $C \in \mathcal{C}$, $\mathcal{C}^{(0)}_{/C}$ is a covering sieve on C, and $\mathcal{C}^{(1)}_{/C}$ an arbitrary sieve on C. Suppose that for each $f : D \to C \in \mathcal{C}^{(0)}_{/C}$, the pullback $f^*\mathcal{C}^{(1)}_{/C}$ is a covering sieve on D, then $\mathcal{C}^{(1)}_{/C}$ is a covering sieve on C.

For each $U \in \mathcal{P}(\mathcal{C})$, let $\mathcal{C}^{(0)}(U) \subset \mathcal{C}$ be the full subcategory spanned by $\{C \in \mathcal{C} \mid U(C) \neq \emptyset\}$. Then $\mathcal{C}^{(0)}(U)$ is a sieve on $\mathcal{C}$.
This is because for any $f : C \to D$ in $\mathcal{C}$, with $D \in \mathcal{C}^{(0)}(U)$, we have $\emptyset \neq U(D) \to U(C)$. Hence $U(C) \neq \emptyset$, so $C \in \mathcal{C}^{(0)}(U)$.
In fact, we have the following proposition.

Proposition 13.6. For every small ∞-category $\mathcal{C}$, we have a bijection:

$$\begin{aligned}\{\text{equivalence classes of } (-1)\text{-truncated objects of } \mathcal{P}(\mathcal{C})\} &\xrightarrow{\sim} \{\text{all sieves on } \mathcal{C}\}; \\ U &\mapsto \quad \mathcal{C}^{(0)}(U).\end{aligned}$$

We can also state the relative version of the above construction.
Let $C \in \mathcal{C}$ be an object and $i : U \to j(C)$ be a monomorphism in $\mathcal{P}(\mathcal{C})$. Let $\mathcal{C}_{/C}(U)$ denote the full subcategory of $\mathcal{C}$ spanned by those objects $f : D \to C$ of $\mathcal{C}_{/C}$ s.t. there exists a commutative triangle

$$\begin{tikzcd} j(D) \arrow[rr, "j(f)"] \arrow[dr] & & j(C) \\ & U \arrow[ur, "i"] & \end{tikzcd}$$

Then, we have:

- $\mathcal{C}_{/C}(U)$ is a sieve on C.
- if $i : U \to j(C)$ and $i' : U' \to j(C)$ are equivalent subobjects of $j(C)$, then $\mathcal{C}_{/C}(U) = \mathcal{C}_{/C}(U')$.

Proposition 13.7 (HTT 6.2.2.5)**.** Let $\mathcal{C}$ be a small ∞-category and $C \in \mathcal{C}$. Let $j : \mathcal{C} \to \mathcal{P}(\mathcal{C})$ be the Yoneda embedding. Then we have a bijection

$$\begin{aligned} \mathrm{Sub}(j(C)) &\xrightarrow{\sim} \{\text{all sieves on } \mathcal{C}\}; \\ (i : U \to j(C)) &\mapsto \quad \mathcal{C}_{/C}(U). \end{aligned}$$

Definition 13.8. Let $\mathcal{C}$ be a (small) ∞-category with a Grothendieck topology. Let S be the collection of all monomorphisms $U \to j(C)$ which correspond to covering sieves $\mathcal{C}^{(0)}_{/C} \subset \mathcal{C}_{/C}$.
An object $\mathcal{F} \in \mathcal{P}(\mathcal{C})$ is a sheaf if $\mathcal{F}$ is S-local.
Denote $\mathrm{Shv}(\mathcal{C}) := S^{-1}\mathcal{P}(\mathcal{C})$, i.e. $\mathrm{Shv}(\mathcal{C})$ its the full subcategory of $\mathcal{P}(\mathcal{C})$ spanned by S-local objects.

Proposition 13.9 (HTT 6.2.2.7)**.** Let $\mathcal{C}$ be a (small) ∞-category with a Grothendieck topology. Then $\mathrm{Shv}(\mathcal{C})$ is a topological localization of $\mathcal{P}(\mathcal{C})$. In particular, $\mathrm{Shv}(\mathcal{C})$ is an ∞-topos.

Proposition 13.10 (HTT 6.2.2.9)**.** Let $\mathcal{C}$ be a small ∞-cateogry. There is a bijection correspondence between Grothendieck topologies on $\mathcal{C}$ and topological localization of $\mathcal{P}(\mathcal{C})$.

In order to the later use, here we record a few propositions about effective epimorphisms.

Proposition 13.11 (HTT 6.2.3.7)**.** Let $\mathcal{X}$ be an ∞-topos.

- Equivalence $X \to Y$ is an effective epimorphism.
- If $F : \mathcal{X} \to \mathcal{Y}$ is a left exact functor, and $f : U \to X$ is an effective epimorphism, then $F(f)$ is an effective epimorphism.

Proposition 13.12 (HTT 6.2.3.12)**.** Let $\mathcal{X}$ be an ∞-topos, assume a diagram

$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, "h"] & & Z \end{tikzcd}$$

(1) If f, g are effective epimorphisms, then h is an effective epimorphism.

(2) If h is an effective epimorphism, then g is an effective epimorphism.

Proposition 13.13 (HTT 6.5.1.18)**.** Let $f : X \to Y$ be a morphism in an ∞-topos $\mathcal{X}$, $\delta : X \to X \times_Y X$ the associated diagonal morphism, $n \geq 0$. The following are equivalent:

(1) $f : X \to Y$ is n-connective.

(2) $\delta : X \to X \times_Y X$ is $(n-1)$-connective and $f : X \to Y$ is an effective epimorphism.

13.2 hypercompletion

Definition 13.14. Let $\mathcal{X}$ be an ∞-topos, and let S be the class of ∞-connective morphisms.

- $X \in \mathcal{X}$ is hypercomplete if X is local with respect to S.
- $\mathcal{X}^\wedge := S^{-1}\mathcal{X} \subset \mathcal{X}$.

There are some properties about $\mathcal{X}^\wedge$, we record in the following proposition.

Proposition 13.15. Let $\mathcal{X}$ be an ∞-topos.

(1) $\mathcal{X}^\wedge \subset \mathcal{X}$ is an accessible left exact localization.

(2) $\mathcal{X}^\wedge$ is an ∞-topos.

(3) $\mathcal{X}$ is hypercomplete if every ∞-connective morphism in $\mathcal{X}$ is an equivalence.

(4) Let $n < \infty$, then $\tau_{\leq n}\mathcal{X} \subset \mathcal{X}^\wedge$.

(5) Let $L : \mathcal{X} \to \mathcal{X}^\wedge$ be the left adjoint to the inclusion $\mathcal{X}^\wedge \subset \mathcal{X}$, and let $X \in \mathcal{X}$ be s.t. $LX \in \mathcal{X}^\wedge$ is an ∞-connective object. Then $LX \in \mathcal{X}^\wedge$ is a final object.

(6) $\mathcal{X}^\wedge$ is hypercomplete.

Proof. (1) From HTT 6.5.2.8, we know that S is strongly saturated and of small generation. From HTT 5.4.4.15, we deduce that $\mathcal{X}^\wedge \subset \mathcal{X}$ is an accessible localization. From HTT 6.5.1.16, we know S is stable under pullbacks, then from Proposition 13.1, we know $\mathcal{X}^\wedge \subset \mathcal{X}$ is a left exact localization.

(2) By the definition of ∞-topos, we know there is an accessible left exact localization functor $\mathcal{P}(\mathcal{C}) \to \mathcal{X}$. Since $L : \mathcal{X} \to \mathcal{X}^\wedge$ is also an accessible left exact localization functor, we conclude that $\mathcal{P}(\mathcal{C}) \to \mathcal{X} \to \mathcal{X}^\wedge$ is an accessible left exact localization functor, hence $\mathcal{X}^\wedge$ is an ∞-topos.

(4) From HTT 6.5.1.14, we know that an n-truncated object of $\mathcal{X}$ is local with respect to every n-connective morphism of $\mathcal{X}$, in particular, every ∞-connective morphism of $\mathcal{X}$, therefore $\tau_{\leq n}\mathcal{X} \subset \mathcal{X}^\wedge$.

(5) For each $n < \infty$, we have

$$1_{\mathcal{X}} \simeq \tau_{\leq n}^{\mathcal{X}^\wedge} LX \simeq L\tau_{\leq n}^{\mathcal{X}} X \simeq \tau_{\leq n}^{\mathcal{X}} X,$$

where the first one is because $LX \in \mathcal{X}^\wedge$ is an ∞-connective object, the second one comes from HTT 5.5.6.28 and the third one is because $\tau_{\leq n}^{\mathcal{X}} X \in \mathcal{X}^\wedge$ and the localization functor is idempotent.

Therefore $X \in \mathcal{X}$ is an ∞-connective object, then we have an ∞-connective $X \to 1_{\mathcal{X}}$, which implies $LX \to L1_{\mathcal{X}}$ is an equivalence, hence $LX \in \mathcal{X}^\wedge$ is a final object.

(6) Let $f : X \to Y$ be an ∞-connective morphism in $\mathcal{X}^\wedge$, then $f : X \to Y \in (\mathcal{X}^\wedge)_{/Y} \simeq (\mathcal{X}_{/Y})^\wedge$ is an ∞-connective object, hence $Lf \simeq f$ is a final object of $(\mathcal{X}_{/Y})^\wedge$, so $f : X \to Y$ is an equivalence.

Therefore, $\mathcal{X}^\wedge$ is hypercomplete.

□

Proposition 13.16. Let $\mathcal{X}, \mathcal{Y}$ be ∞-topoi, and let $f^* : \mathcal{X} \to \mathcal{Y}$ be a left exact colimit-preserving functor. The following are equivalent:

(1) For every monomorphism u in $\mathcal{X}$, if f^*u in $\mathcal{Y}$ is an equivalence, then u is an equivalence in $\mathcal{X}$.

(2) For every morphism u in $\mathcal{X}$, if f^*u in $\mathcal{Y}$ is an equivalence, then u is ∞-connective in $\mathcal{X}$.

Proof. (2) $\implies$ (1).
If u is a monomorphism and f^*u is an equivalence, then u is ∞-connective. In particular, u is an effective epimorphism, and since u is a monomorphism, we get u is an equivalence.
(1) $\implies$ (2).
Let u be any morphism on $\mathcal{X}$, s.t. f^*u is an equivalence. We prove by induction on n that u is n-connective.
$[n = 0]$. Choose a factorization

$$\begin{tikzcd} & Y \arrow[dr, "u''"] & \\ X \arrow[ur, "u'"] \arrow[rr, "u"] & & Z \end{tikzcd}$$

where u' is an effective epimorphism and u'' is a monomorphism. Since f^*u is an equivalence, in particular f^*u is an effective epimorphism, then f^*u'' is an effective epimorphism.

Since f^* is left exact, u'' is a monomorphism implies that f^*u'' is a monomorphism, hence f^*u'' is an equivalenece. Applying (1), we conclude that u'' is an equivalence. In particular, u'' is an effective epimorphism, then $u = u'' \circ u'$ is an effective epimorphism, i.e. 0-connective.

$[n > 0]$. By Proposition 13.13, it suffices to show that the diagonal $\delta : X \to X \times_Z X$ is $(n-1)$-connective. By the inductive hypothesis, it suffices to show $f^*(\delta)$ is an equivalence in $\mathcal{Y}$.

Since f^* is left exact, then $f^*(\delta) = (f^*X \to f^*(X \times_Z X) = f^*X \times_{f^*Z} f^*X)$. This is the diagonal map associated to $f^*(u) : f^*X \to f^*Z$, which is an equivalence, therefore the diagonal $f^*(\delta)$ is an equivalence. $\square$

Definition 13.17. Let $\mathcal{X}$ be an ∞-topos and let $\mathcal{Y} \subset \mathcal{X}$ be an accessible left exact localization of $\mathcal{X}$. We call $\mathcal{Y}$ is a cotopological localization of $\mathcal{X}$ it the left adjoint $L : \mathcal{X} \to \mathcal{Y}$ to $\mathcal{Y} \subset \mathcal{X}$ satisfies the equivalent conditions of Proposition 13.16.

Remark 13.18. Let $f^* : \mathcal{X} \to \mathcal{Y}$ be the left adjoint of a geometric morphism between ∞-topoi and f^* is left exact, and suppose f^* satisfies the quivalent conditions of Proposition 13.16. Let $u : X \to Z$ be a morphism in $\mathcal{X}$ and choose a factorization

$$\begin{tikzcd} & Y \arrow[dr, "u''"] & \\ X \arrow[ur, "u'"] \arrow[rr, "u"] & & Z \end{tikzcd}$$

where u' is an effective epimorphism and u'' is a monomorphism. Then

- u'' is an equivalence iff $f^*(u'')$ is an equivalence.

  Since f^* is left exact, u'' is a monomorphism implies that $f^*(u'')$ is a monomorphism, and also from Proposition 13.11, u'' is an effective epimorphism implies that $f^*(u'')$ is an effective epimorphism. Therefore, u'' is an equivalence implies that $f^*(u'')$ is an equivalence.

  Conversely, this is just the equivalent conditions of Proposition 13.16. $\square$

- u is an effective epimorphism iff $f^*(u)$ is an effective epimorphism.

Proof. From Proposition 13.11, we know: since f^* is left exact, u is an effective epimorphism $\implies f^*(u)$ is an effective epimorphism.
Conversely, if $f^*(u)$ is an effective epimorphism, then $f^*(u'')$ is an effective epimorphism. Since u'' is a monomorphism, we get $f^*(u'')$ is a monomorphism, therefore $f^*(u'')$ is an equivalence, then we claim that u'' is an equivalence. Therefore $u = u'' \circ u'$ is an effective epimorphism. $\square$

The hypercompletion $\mathcal{X}^\wedge$ of an ∞-topos $\mathcal{X}$ can be characterized as the maximal cotopological localization of $\mathcal{X}$. That is, the cotopological localization which is obtained by inverting as many morphisms as possible.
The next result says that, every localization can be obtained by combining topological localization and cotopological localization.

Proposition 13.19. Let $\mathcal{X}$ be an ∞-topos and let $\mathcal{X}'' \subset \mathcal{X}$ be an accessible left exact localization of $\mathcal{X}$. Then there exists a topological localization $\mathcal{X}' \subset \mathcal{X}$ s.t. $\mathcal{X}'' \subset \mathcal{X}'$ is a cotopological localization of $\mathcal{X}'$.

Proof. Let $L : \mathcal{X} \to \mathcal{X}''$ be the left adjoint of $\mathcal{X}'' \subset \mathcal{X}$, let

$$S = \{\text{monomorphisms } u \text{ in } \mathcal{X} \mid Lu \text{ is an equivalence}\},$$

and let $\mathcal{X}' := S^{-1}\mathcal{X}$. Since L is left exact, and the pullback of equivalence is an equivalence, we know S is stable under pullbacks and therefore determines a topological localization of $\mathcal{X}$.
We have $\mathcal{X}'' \subset \mathcal{X}' = S^{-1}\mathcal{X}$, i.e. for any $X \in \mathcal{X}''$, X is S-local. This is because for any monomorphism $u : Y \to Z$ in S, we have

$$\mathrm{Map}(Z, X) \simeq \mathrm{Map}(LZ, X) \xrightarrow{\simeq} \mathrm{Map}(LY, X) \simeq \mathrm{Map}(Y, X),$$

the left and right equivalences are because L is a left adjoint, and the middle one is because $Lu : LY \to LZ$ is an equivalence.
We need to show $\mathcal{X}'' \subset \mathcal{X}' = S^{-1}\mathcal{X}$ is a cotopological localization.
First, the restriction $L|\mathcal{X}'$ exhibits $\mathcal{X}''$ as an accessible left exact localization of $\mathcal{X}'$.
Second, assume a monomorphism $u : X \to Z$ in $\mathcal{X}'$ with Lu is an equivalence. Then u is a monomorphism in $\mathcal{X}$, so $u \in S$. Then for any $Z \in \mathcal{X}' = S^{-1}\mathcal{X}$, we have

$\mathrm{Map}(Y, Z) \xrightarrow{\sim} \mathrm{Map}(X, Z)$. Then by Yoneda we know that $u : X \to Y$ is an equivalence.

Hence, $\mathcal{X}'' \subset \mathcal{X}'$ is a cotopological localization of $\mathcal{X}'$. $\square$

13.3 hypercoverings

Definition 13.20. Let $\Delta^{\leq n}$ be the full subcategory of Δ spanned by the set of objects $\{[0],[1],\cdots,[n]\}$. And let $j:\Delta^{\leq n}\hookrightarrow\Delta$ be the inclusion functor. For any presentable ∞-category $\mathcal{X}$, we can induce a functor

$$j^*:\mathcal{X}_\Delta=\mathrm{Fun}(\Delta^{\mathrm{op}},\mathcal{X})\to\mathrm{Fun}((\Delta^{\leq n})^{\mathrm{op}},\mathcal{X}).$$

The functor j^* has a left adjoint $j_!$ and a right adjoint j_*. Therefore, for any $X\in\mathcal{X}_\Delta$, we have a triangle

$$j_!j^*X\to X\to j_*j^*X.$$

We define the n-skeleton functor $\mathrm{sk}_n:\mathcal{X}_\Delta\to\mathcal{X}_\Delta$ to be $\mathrm{sk}_nX:=j_!j^*X$ and the n-coskeleton functor $\mathrm{cosk}_n:\mathcal{X}_\Delta\to\mathcal{X}_\Delta$ to be $\mathrm{cosk}_nX:=j_*j^*X$.

Definition 13.21. Let $\mathcal{X}$ be an ∞-topos.

- A simplicial object $U_\bullet\in\mathcal{X}_\Delta$ is a hypercovering, if for any $n\geq 0$. the unit map

$$U_n\to(\mathrm{cosk}_{n-1}U_\bullet)_n$$

  is an effective epimorphism.

- $U_\bullet$ is an effective hypercovering of $\mathcal{X}$ if $|U_\bullet|$ is a final object of $\mathcal{X}$.

Lemma 13.22. Let $\mathcal{X}$ be an ∞-topos, and let $U_\bullet$ be a simplicial object in $\mathcal{X}$. Let $L:\mathcal{X}\to\mathcal{X}^\wedge$ be a left adjoint to the inclusion. The following are equivalent:

(1) The simplicial object $U_\bullet$ is a hypercovering of $\mathcal{X}$.

(2) The simplicial object $L\circ U_\bullet$ is a hypercovering of $\mathcal{X}^\wedge$.

Proof. Since L is left exact, we claim that $L\circ\mathrm{cosk}_nU_\bullet=\mathrm{cosk}_n(L\circ U_\bullet)$.

Now, from Remark 13.18, we have

$$
\begin{aligned}
& U_\bullet \in \mathcal{X}_\Delta \text{ is a hypercovering,} \\
\iff & \forall n \geq 0, U_n \to (\mathrm{cosk}_{n-1} U_\bullet)_n \text{ is an effective epimorphism,} \\
\iff & \forall n \geq 0, (L \circ U_\bullet)_n = LU_n \to L(\mathrm{cosk}_{n-1} U_\bullet)_n = (L \circ \mathrm{cosk}_{n-1} U_\bullet)_n = (\mathrm{cosk}_{n-1}(L \circ U_\bullet))_n \\
& \text{is an effective epimorphism,} \\
\iff & L \circ U_\bullet \in (\mathcal{X}^\wedge)_\Delta \text{ is a hypercovering.}
\end{aligned}
$$

□

Lemma 13.23. Let $\mathcal{X}$ be an ∞-topos and let U be an ∞-connective object of $\mathcal{X}$. Let $U_\bullet$ be the constant simplicial object with value U. Then $U_\bullet$ is a hypercovering of $\mathcal{X}$.

Proof. From Lemma 13.22, in order to show the constant simplicial object $U_\bullet$ of $\mathcal{X}$ is a hypercovering, it suffices to show the constant simplicial object $L \circ U_\bullet$ of $\mathcal{X}^\wedge$ is a hypercovering.
Hence we can reduce to the case where $\mathcal{X}$ is hypercomplete. Then $U \simeq 1_{\mathcal{X}}$. So $U_\bullet$ is equivalent to the constant functor with value $1_{\mathcal{X}}$, hence $U_\bullet$ is a final object of $\mathcal{X}_\Delta$.
For each $n \geq 0$, the coskeleton functor $\mathrm{cosk}_{n-1} = j_* j^*$ preserves limits, so $\mathrm{cosk}_{n-1} U_\bullet$ is also a final object of $\mathcal{X}_\Delta$. Therefore, the unit map $U_\bullet \to \mathrm{cosk}_{n-1} U_\bullet$ is an equivalence, which implies that for each $n \geq 0$, $U_n \to (\mathrm{cosk}_{n-1} U_\bullet)_n$ is an effective epimorphism, hence $U_\bullet$ is a hypercovering of $\mathcal{X}$. □

Lemma 13.24 (HTT 6.5.3.11)**.** Let $\mathcal{X}$ be an ∞-topos and let $U_\bullet$ be a hypercovering of $\mathcal{X}$. Then the colimit $|U_\bullet|$ is ∞-connective, i.e. the geometric realization of a hypercovering is ∞-connective.

Theorem 13.25. Let $\mathcal{X}$ be an ∞-topos. The following are equivalent:

(1) For every $X \in \mathcal{X}$, every hypercovering $U_\bullet$ of $\mathcal{X}_{/X}$ is effective.

(2) The ∞-topos $\mathcal{X}$ is hypercomplete.

Proof. (2) $\implies$ (1).
Assume $\mathcal{X}$ is hypercomplete. Let $X \in \mathcal{X}$ and $U_\bullet$ is a hypercovering of $\mathcal{X}_{/X}$. The from Lemma 13.24, we know that $|U_\bullet|$ is an ∞-connective object of $\mathcal{X}_{/X}$. Since $\mathcal{X}$ is hypercomplete, the ∞-connective morphism $|U_\bullet| \to X$ is an equivalence, hence $|U_\bullet|$ is a

final object of $\mathcal{X}_{/X}$, so that the hypercovering $U_\bullet$ is effective.

(1) $\implies$ (2).

Suppose (1) is satisfied. Let $f : U \to X$ be an ∞-connective morphism in $\mathcal{X}$, and let $f_\bullet$ be the constant simplicial object of $\mathcal{X}_{/X}$ with value f. Then by Lemma 13.23, we know that $f_\bullet$ is a hypercovering of $\mathcal{X}_{/X}$. From the condition of (1), the hypercovering $f_\bullet$ is effective, hence $|f_\bullet| \simeq f$ is a final object of $\mathcal{X}_{/X}$, i.e. $f : U \to X$ is an equivalence. $\square$

Corollary 13.26 (HTT 6.5.3.13)**.** Let $\mathcal{X}$ be an ∞-topos. For each $X \in \mathcal{X}$ and each hypercovering $U_\bullet$ of $\mathcal{X}_{/X}$, let $|U_\bullet|$ be the geometric realization of $U_\bullet$. Let S donote the collection of all such morphisms $|U_\bullet|$. Then $\mathcal{X}^\wedge = S^{-1}\mathcal{X}$, i.e. $X \in \mathcal{X}$ is hypercomplete iff X is S-local.

14 CondAn

Proposition 14.1. Cond(Sp(An)) $\simeq$ Sp(Cond(An)) $\simeq$ Sp(An(Cond)).

Definition 14.2. Let $\mathcal{C}$ be any ∞-category, define $\mathrm{Stab}(\mathcal{C}) := \mathrm{Sp}(\mathcal{C}_*)$.

Proposition 14.3.

- $\mathrm{Stab}(\mathrm{An}) = \mathrm{Sp}$.
- $\mathrm{Stab}(\mathrm{CondAn}) = \mathrm{CondSp}$.

Definition 14.4. Let $\mathcal{C}$ be a presentable ∞-category.

- We define the condensed object of $\mathcal{C}$ to be the hypercomplete sheaf on CHaus with values in $\mathcal{C}$.
- $\mathrm{Cond}(\mathcal{C}) := \mathrm{Shv}(\mathrm{CHaus}, \mathcal{C})^{\wedge}$.
- If $\mathcal{C} = \mathrm{An}$, $\mathrm{CondAn} := \mathrm{Shv}(\mathrm{CHaus}, \mathrm{An})^{\wedge}$.
  If $\mathcal{C} = \mathrm{Sp}$, $\mathrm{CondSp} := \mathrm{Shv}(\mathrm{CHaus}, \mathrm{Sp})^{\wedge}$.

Definition 14.5. Let $\mathcal{V}$ be a Bénabou cosmos. Let $\mathcal{C}$ be a $\mathcal{V}$-enriched category.

(1) A Bénabou cosmos is a complete, cocomplete, closed symmetric monoidal category.

(2) A cotensorring of $\mathcal{C}$ over $\mathcal{V}$ is

- A functor
$$[-,-] : \mathcal{V}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}.$$
- For each $v \in \mathcal{V}$, $c_1, c_2 \in \mathcal{C}$, a natural isomorphism of the form
$$\mathcal{V}(v, \mathcal{C}(c_1, c_2)) \simeq \mathcal{C}(c_1, [v, c_2]).$$

(3) A tensoring of $\mathcal{C}$ over $\mathcal{V}$ is

- A functor
$$- \otimes - : \mathcal{V} \times \mathcal{C} \to \mathcal{C}.$$
- For each $v \in \mathcal{V}$, $c_1, c_2 \in \mathcal{C}$, a natural isomorphism of the form
$$\mathcal{C}(v \otimes c_1, c_2) \simeq \mathcal{V}(v, \mathcal{C}(c_1, c_2)).$$

Proposition 14.6. If $\mathcal{C}$ is $\mathcal{V}$-tensored and cotensored, then for any fixed $v \in \mathcal{V}$, we have a pair of adjoint functors:

$$v \otimes - : \mathcal{C} \longrightarrow \mathcal{C} : [v, -].$$

Proof. This is because

$$\mathcal{C}(v \otimes c_1, c_2) \simeq \mathcal{V}(v, \mathcal{C}(c_1, c_2)) \simeq \mathcal{C}(c_1, [v, c_2]).$$

□

Example 14.7. The closed symmetric monoidal category $\mathcal{V}$ is tensored and cotensored over itself, with tensoring being its tensor product and cotensoring being its internal hom.

Definition 14.8. Let $\mathcal{C}$ be a site.

- A presheaf of anima is a functor $\mathcal{F}\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{An}$.
- A sheaf of anima is a presheaf of anima $\mathcal{F}$, s.t. for all covers $\{f_i\colon X_i \to X\}_{i \in I}$,

$$\mathcal{F}(X) \xrightarrow{\sim} \lim(\prod_i \mathcal{F}(X_i) \rightrightarrows \prod_{i,j} \mathcal{F}(X_i \times_X X_j) \prod_{i,j,k} \mathcal{F}(X_i \times_X X_j \times_X X_k) \cdots)$$

- A hypercomplete sheaf of anima is a sheaf of anima $\mathcal{F}$, s.t. for all hypercovers $X_\bullet \to X$, the following map is an equivalence,

$$\mathcal{F}(X) \xrightarrow{\sim} \lim_{\Delta} \mathcal{F}(X_\bullet) \simeq \lim(\mathcal{F}(X_0) \rightrightarrows \mathcal{F}(X_1)\mathcal{F}(X_2) \cdots).$$

Definition 14.9. The ∞-category of condensed anima is given by one of the following ∞-categories:

- the ∞-category of hypercomplete sheaves of anima on CHaus;
- the ∞-category of hypercomplete sheaves of anima on ProFin;
- the ∞-category of sheaves of anima on ExDisc.

We denote the ∞-category of condensed anima by $\mathrm{CondAn} \simeq \mathrm{Shv}(\mathrm{CHaus}, \mathrm{An})^\wedge \simeq \mathrm{Shv}(\mathrm{ProFin}, \mathrm{An})^\wedge \simeq \mathrm{Shv}(\mathrm{ExDisc}, \mathrm{An})$.

More generally, we can define the condensed objects of a presentable ∞-category.

Definition 14.10. For a presentable ∞-category $\mathcal{C}$, the condensed objects of $\mathcal{C}$ is

- a hypercomplete sheaf on CHaus with values in $\mathcal{C}$;
- a hypercomplete sheaf on ProFin with values in $\mathcal{C}$;
- a complete sheaf on ExDisc with values in $\mathcal{C}$.

We denote the ∞-category of condensed objects in $\mathcal{C}$ by $\mathrm{Cond}(\mathcal{C}) \simeq \mathrm{Shv}(\mathrm{CHaus}, \mathcal{C})^\wedge \simeq \mathrm{Shv}(\mathrm{ProFin}, \mathcal{C})^\wedge \simeq \mathrm{Shv}(\mathrm{ExDisc}, \mathcal{C})$.

Proposition 14.11. Let $\mathcal{C}$ be a presentable ∞-category.

(1) $\mathcal{C}$ is naturally An-enriched, tensored and cotensored.

(2) If $\mathcal{C}$ is stable, then $\mathcal{C}$ is naturally Sp-enriched, tensored and cotensored.

Proof. (1) Ref HTT 4.4.4.9.

Proposition 14.12. Let $\mathcal{C}$ be a presentable ∞-category.

(1) $\mathrm{Cond}(\mathcal{C})$ is naturally CondAn-enriched, tensored and cotensored.

(2) If $\mathcal{C}$ is stable, then $\mathrm{Cond}(\mathcal{C})$ is naturally CondSp-enriched, tensored and cotensored.

Definition 14.13. Let $\mathcal{C}$ be a cocomplete ∞-category, and let $\mathcal{C}^{\mathrm{cp}}$ be all compact projective objects of $\mathcal{C}$, and assume $\mathcal{C}^{\mathrm{cp}}$ is small, we define an animated object of $\mathcal{C}$ is a contravairant functor from $\mathcal{C}^{\mathrm{cp}}$ to An, taking finite disjoint unions to finite products, and the ∞-category of all animated objects of $\mathcal{C}$ is denoted by $\mathrm{An}(\mathcal{C})$.

Example 14.14.

- $\mathcal{C} = \mathrm{Set}$, then $\mathcal{C}^{\mathrm{cp}} = \mathrm{FinSet}$, and $\mathrm{An}(\mathrm{Set}) = \mathrm{An}$.
- $\mathcal{C} = \mathrm{Cond}$, then $\mathcal{C}^{\mathrm{cp}} = \mathrm{ExDisc}$, and $\mathrm{An}(\mathrm{Cond}) = \mathrm{CondAn}$.
- $\mathcal{C} = \mathrm{Ab}$, then $\mathcal{C}^{\mathrm{cp}} = \{\text{finite free abelian groups}\}$, and $\mathrm{An}(\mathrm{Ab}) = \mathcal{D}_{\geq 0}(\mathbb{Z})$.
- If $\mathcal{C}$ is an abelian category, generated by $\mathcal{C}^{\mathrm{cp}}$ under colimits, then $\mathrm{An}(\mathcal{C}) = \mathcal{D}_{\geq 0}(\mathcal{C})$.

Proposition 14.15. Assume $\mathcal{C}$ is a category generated by $\mathcal{C}^{\mathrm{cp}}$ under colimits, then

(1) $\mathrm{Cond}(\mathcal{C})$ is generated by $\mathrm{Cond}(\mathcal{C})^{\mathrm{cp}}$ under colimits.

(2) $\mathrm{An}(\mathrm{Cond}(\mathcal{C})) \simeq \mathrm{Cond}(\mathrm{An}(\mathcal{C}))$. In particular, $\mathrm{An}(\mathrm{Cond}) \simeq \mathrm{Cond}(\mathrm{An})$.

We have a pair of adjoint functors

$$(-)^{\mathrm{disc}} \colon \mathrm{An} \rightleftarrows \mathrm{CondAn} \colon \Gamma.$$

They are given as follows:

$$\Gamma \colon \mathrm{CondAn} \longrightarrow \mathrm{An}; X \mapsto X(*),$$

and

$$(-)^{\mathrm{disc}} \colon \mathrm{An} \longrightarrow \mathrm{CondAn}; A \mapsto A^{\mathrm{disc}},$$

where $A^{\mathrm{disc}} : \mathrm{ExDisc}^{\mathrm{op}} \to \mathrm{An}; X = \varprojlim X_i \mapsto \varinjlim \mathrm{Map}_{\mathrm{An}}(X_i, A)$, here each X_i is a finite set.

One the other hand, we have a pair of adjoint functors between CondAn and Cond:

$$\pi_0 \colon \mathrm{CondAn} \rightleftarrows \mathrm{Cond}.$$

In particular, the inculsion functors $\mathrm{Cond} \longrightarrow \mathrm{CondAn}$ and $\mathrm{An} \longrightarrow \mathrm{CondAn}$ are fully faithful, hence we have the following definitions.

Definition 14.16. The object in $\mathrm{An} \subset \mathrm{CondAn}$ is called a discrete object, the object in $\mathrm{Cond} \subset \mathrm{CondAn}$ is called a static object.

Proposition 14.17. $\mathrm{An} \cap \mathrm{Cond} = \mathrm{Set}$.

Proof. It is clear that $\mathrm{Set} \subset \mathrm{An} \cap \mathrm{Cond}$.

Now for $X \in \mathrm{CondAn}$, X is discrete means that $X \simeq X(*)$ and X is static means that $X \simeq \pi_0 X$, so we have

$$X \simeq \pi_0 X \simeq \pi_0 X(*),$$

so X is a set. $\square$

Proposition 14.18. $X \in$ CondAn is discrete if and only if for any $S \in$ ExDisc, $s \in S$,

$$\operatorname*{colim}_{s \in V \text{cpt}} X(V) \simeq X(\{s\}).$$

Proof. 待证

Definition 14.19.
- For $X \in$ CW, we can view it as a topological space, and hence view it as a condensed set, and hence view it as a condensed anima.
- For $X \in$ CW, its homotopy type also can be viewed as an anima, hence a condensed anima, denoted by $\mathrm{h}(X)$.

The following proposition gives the relation between X and $\mathrm{h}(X)$.

Proposition 14.20. For a CW complex X, we have an initial map to anima $X \longrightarrow \mathrm{h}(X)$, i.e. for any map $X \longrightarrow Y$ with Y being an anima, there exists a unique map from $\mathrm{h}(X) \longrightarrow Y$, s.t.

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & Y . \\ & \mathrm{h}(X) \arrow[ur, dotted, "\exists!"'] & \end{tikzcd}$$

Proof. Since X is the filtered colimits of its finite subcomplexes, hence X is the colimit of some closed cells D^n.
In order to show $X \longrightarrow \mathrm{h}(X)$ is an initial map to anima, it suffices to show

$$\mathrm{Hom}_{\mathrm{An}}(\mathrm{h}(X), Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{CondAn}}(X, Y)$$

is an equivalence, for any discrete condensed anima Y.
Since the functors $\mathrm{h}(-)$ and $\mathrm{Hom}(-, Y)$ commute with all colimits, it suffices to show

$$Y \simeq \mathrm{Hom}_{\mathrm{An}}(*, Y) \simeq \mathrm{Hom}_{\mathrm{An}}(\mathrm{h}(D^n), Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{CondAn}}(D^n, Y).$$

Y as an anima, we can write it as $Y \simeq \varprojlim \tau_{\leq d} Y$. Since $\mathrm{Hom}(D^n, -)$ commutes with limits, we reduce Y to the case Y is d-truncated. From the fiber sequence

$$(\pi_d Y)[d] \longrightarrow \tau_{\leq d} Y \longrightarrow \tau_{\leq d-1} Y,$$

and $\mathrm{Hom}(D^n, -)$ preserves fiber sequence, by induction, it suffices to show, for Y, whose homotopy group concentrated at d, we have $\mathrm{Hom}_{\mathrm{CondAn}}(D^n, Y) \simeq Y$.

Theorem 14.21. We have pairs of adjoint functors:

$$(\Sigma^\infty, \Omega^\infty) : \mathrm{CondAn}_* \rightleftarrows \mathrm{CondSp};$$
$$(\Sigma^\infty_+, \Omega^\infty) : \mathrm{CondAn} \rightleftarrows \mathrm{CondSp}.$$

Usually, we donote $\Sigma^\infty_+ = \mathbb{S}[-] : \mathrm{CondAn} \longrightarrow \mathrm{CondSp}$.

Example 14.22. For $X, Y \in \mathrm{CondAn}$, we have:

- $\mathbb{S}[X \sqcup Y] = \mathbb{S}[X] \oplus \mathbb{S}[Y]$.
- $\mathbb{S}[X \times Y] = \mathbb{S}[X] \otimes \mathbb{S}[Y]$.

15 SolidSp

Lemma 15.1. (1) In Sp, we have $\mathbb{S}^I \otimes_{\mathbb{S}} \mathbb{Z} \simeq \mathbb{Z}^I$, for any set I.

(2) In CondSp, we have $\mathbb{S}^I \otimes_{\mathbb{S}} \mathbb{Z} \simeq \mathbb{Z}^I$, for any set I.

Proof. (1). Note that for any $M \in$ Sp, we have a map $\mathbb{S}^I \otimes_{\mathbb{S}} M \to M^I$.
If $M = \mathbb{S}$, then this map is an equivalence.
If M is a finite limit of $\mathbb{S}$, since the functor $\mathbb{S}^I \otimes_{\mathbb{S}} -$ commutes with colimits, and in the stable ∞-cateogory Sp, finite products are the same as finite coproducts and pullbacks are tha same as pushouts, hence the functor $\mathbb{S}^I \otimes_{\mathbb{S}} -$ commutes with finite limits, so we have $\mathbb{S}^I \otimes_{\mathbb{S}} M \simeq M^I$.
For $\mathbb{Z} \in$ Sp, in every dimension, $\mathbb{Z}$ has only finitely many cells, so for any n, there exists M_n, which is a finite limit of $\mathbb{S}$, with a map $M_n \to \mathbb{Z}$, s.t. the cofiber $Q_n = \mathrm{cofib}(M_n \to \mathbb{Z})$ is $(n+1)$-connective.
For M_n, we have $\mathbb{S}^I \otimes_{\mathbb{S}} M_n \simeq M_n^I$. Now, we have

$$\begin{tikzcd}
\mathbb{S}^I \otimes_{\mathbb{S}} M_n \arrow[r] \arrow[d] & M_n^I \arrow[r] \arrow[d] & 0 \arrow[d] \\
\mathbb{S}^I \otimes_{\mathbb{S}} \mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Z}^I \arrow[r] \arrow[d] & \mathrm{cofib}(\mathbb{S}^I \otimes_{\mathbb{S}} \mathbb{Z} \to \mathbb{Z}^I) \arrow[d] \\
\mathbb{S}^I \otimes_{\mathbb{S}} Q_n \arrow[r] & Q_n^I \arrow[r] & \mathrm{cofib}(\mathbb{S}^I \otimes_{\mathbb{S}} Q_n \to Q_n^I)
\end{tikzcd}$$

so

$$\mathrm{cofib}(\mathbb{S}^I \otimes_{\mathbb{S}} \mathbb{Z} \to \mathbb{Z}^I) \simeq \mathrm{cofib}(\mathbb{S}^I \otimes_{\mathbb{S}} Q_n \to Q_n^I).$$

We know Q_n^I is $(n+1)$-connective and $\mathbb{S}^I \otimes_{\mathbb{S}} Q_n$ is also $(n+1)$-connective, hence $\mathrm{cofib}(\mathbb{S}^I \otimes_{\mathbb{S}} Q_n \to Q_n^I)$ is $(n+1)$-connective, which means $\mathrm{cofib}(\mathbb{S}^I \otimes_{\mathbb{S}} \mathbb{Z} \to \mathbb{Z}^I)$ is $(n+1)$-connective, for any n. Since Sp is left complete, we conclude that $\mathbb{S}^I \otimes_{\mathbb{S}} \mathbb{Z} \simeq \mathbb{Z}^I$.
(2). Just check it pointwise. □

Definition 15.2. (1) For $S \in$ ProFin, write $S = \lim S_i, S_i \in$ Fin, define the solid free spectrum

$$\mathbb{S}[S]^{\blacksquare} := \lim \mathbb{S}[S_i].$$

(2) For $X \in$ CondAn, $Y \in$ CondSp, we define $\Gamma(X, Y) := \mathrm{map}(\mathbb{S}[X], Y)$.

Proposition 15.3. For $S \in$ ProFin, $S = \lim S_i$, $S_i \in$ Fin, there exists some set I, s.t.

(1) $\varinjlim \Gamma(S_i, \mathbb{S}) \simeq \mathbb{S}^{\oplus I}$.

(2) $\mathbb{S}[S]^{\blacksquare} \simeq \mathbb{S}^I$.

(3) $\Gamma(S, \mathbb{S}) \simeq \mathbb{S}^{\oplus I}$.

Proof. (1) We have a map $\mathbb{S}^{\oplus I} \longrightarrow \varinjlim \Gamma(S_i, \mathbb{S})$. This is a map between connective spectra. We have

$$
\begin{aligned}
\mathrm{cofib}(\mathbb{S}^{\oplus I} \longrightarrow \varinjlim \Gamma(S_i, \mathbb{S})) \otimes \mathbb{Z} &\simeq \mathrm{cofib}(\mathbb{S}^{\oplus I} \otimes \mathbb{Z} \longrightarrow \varinjlim \Gamma(S_i, \mathbb{S}) \otimes \mathbb{Z}) \\
&\simeq \mathrm{cofib}(\mathbb{Z}^{\oplus I} \longrightarrow \varinjlim \Gamma(S_i, \mathbb{Z})) \\
&\simeq \mathrm{cofib}(\mathbb{Z}^{\oplus I} \longrightarrow \Gamma(S, \mathbb{Z})) \\
&\simeq 0.
\end{aligned}
$$

Then $\mathrm{cofib}(\mathbb{S}^{\oplus I} \longrightarrow \varinjlim \Gamma(S_i, \mathbb{S})) \simeq 0$. Hence, $\varinjlim \Gamma(S_i, \mathbb{S}) \simeq \mathbb{S}^{\oplus I}$.

(2)

$$
\begin{aligned}
\mathbb{S}[S]^{\blacksquare} &\simeq \lim \mathbb{S}[S_i] \\
&\simeq \lim \mathrm{map}(\Gamma(S_i, \mathbb{S}), \mathbb{S}) \\
&\simeq \mathrm{map}(\mathrm{colim}\, \Gamma(S_i, \mathbb{S}), \mathbb{S}) \\
&\simeq \mathrm{map}(\mathbb{S}^{\oplus I}, \mathbb{S}) \\
&\simeq \mathbb{S}^I.
\end{aligned}
$$

(3) For any $M \in$ CondSp, we have a map $M^{\oplus I} \longrightarrow \Gamma(S, M)$.
If $M \simeq \mathbb{Z}$, then $M^{\oplus I} \simeq \Gamma(S, M)$.
For any $n > 0$, $\tau_{\leq n}\mathbb{S}$ has finite homotopy groups in deg $\leq n$. Hence $\tau_{\leq n}\mathbb{S}$ can be generated under finite colimits by $\mathbb{Z}$. Hence

$$
\begin{aligned}
\Gamma(S, \mathbb{S}) &\simeq \Gamma(S, \lim \tau_{\leq n}\mathbb{S}) \simeq \lim \Gamma(S, \tau_{\leq n}\mathbb{S}) \\
&\simeq \lim\, (\tau_{\leq n}\mathbb{S})^{\oplus I} \simeq \lim \tau_{\leq n}\mathbb{S}^{\oplus I} \simeq \mathbb{S}^{\oplus I}.
\end{aligned}
$$

□

Definition 15.4. The ∞-category of solid spectra is defined to be

$$\mathrm{SolidSp} := \{X \in \mathrm{CondSp} \mid \mathrm{map}(\mathbb{S}[S], X) \simeq \mathrm{map}(\mathbb{S}[S]^{\blacksquare}, X), \forall S \in \mathrm{ProFin}\},$$

the full subcategory of CondSp.

Theorem 15.5. (1) For any $S \in$ ProFin, $\mathbb{S}[S]^{\blacksquare} \in$ SolidSp.

(2) For any $X \in$ CondSp,

$$X \in \mathrm{SolidSp} \iff \pi_n X \in \mathrm{Solid}, \forall n \in \mathbb{Z}.$$

(3) SolidSp $\subset$ CondSp is closed under limits, colimits and extensions.

(4) SolidSp $\subset$ CondSp has a left adjoint $(-)^{\blacksquare} : \mathrm{CondSp} \longrightarrow \mathrm{SolidSp}$.

(5) SolidSp naturally inherits a t-structure from CondSp and the heart of SolidSp is equivalent to Solid.

Proof. (1). We need to prove:

$$\mathrm{map}(\mathbb{S}[S], \mathbb{S}[T]^{\blacksquare}) \simeq \mathrm{map}(\mathbb{S}[S]^{\blacksquare}, \mathbb{S}[T]^{\blacksquare}).$$

Since $\mathbb{S}[T]^{\blacksquare} \simeq \mathbb{S}^J$ for some set J, it suffices to show:

$$\mathrm{map}(\mathbb{S}[S], \mathbb{S}) \simeq \mathrm{map}(\mathbb{S}[S]^{\blacksquare}, \mathbb{S}).$$

Firstly, $\mathrm{map}(\mathbb{S}[S], \mathbb{S}) \simeq \Gamma(S, \mathbb{S}) \simeq \mathbb{S}^{\oplus I}$, for some set I. Hence, it suffices to show:

$$\mathrm{map}(\mathbb{S}[S]^{\blacksquare}, \mathbb{S}) \simeq \mathrm{map}(\mathbb{S}^I, \mathbb{S}) \simeq \mathbb{S}^{\oplus I}.$$

For any $M \in$ CondSp, consider the map $M^{\oplus I} \longrightarrow \mathrm{map}(\mathbb{S}^I, M)$.
If $M \simeq \mathbb{Z}$, then

$$\mathrm{map}(\mathbb{S}^I, \mathbb{Z}) \simeq \mathrm{Hom}_{\mathbb{Z}}(\mathbb{S}^I \otimes \mathbb{Z}, \mathbb{Z}) \simeq \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^I, \mathbb{Z}) \simeq \mathbb{Z}^{\oplus I}.$$

If $M \simeq \tau_{\leq n}\mathbb{S}$, which is a finite colimit of $\mathbb{Z}$, then $\mathrm{map}(\mathbb{S}^I, M) \simeq M^{\oplus I}$. Therefore, we have

$$\begin{aligned}\operatorname{map}(\mathbb{S}^I, \mathbb{S}) &\simeq \operatorname{map}(\mathbb{S}^I, \lim \tau_{\leq n}\mathbb{S}) \simeq \lim \operatorname{map}(\mathbb{S}^I, \tau_{\leq n}\mathbb{S}) \\ &\simeq \lim\, (\tau_{\leq n}\mathbb{S})^{\oplus I} \simeq \lim \tau_{\leq n}\mathbb{S}^{\oplus I} \simeq \mathbb{S}^{\oplus I}.\end{aligned}$$

Now, we define

$$\mathrm{SolidSp}' := \{X \in \mathrm{CondSp} \mid \pi_n X \in \mathrm{Solid}, \forall n\},$$

We want to show $\mathrm{SolidSp}' = \mathrm{SolidSp}$.

(i) $\mathrm{SolidSp}' \subset \mathrm{SolidSp}$.

Proof. For $X \in \mathrm{SolidSp}'$, we need to prove

$$\operatorname{map}(\mathbb{S}[S], X) \simeq \operatorname{map}(\mathbb{S}[S]^{\blacksquare}, X), \forall S \in \mathrm{ProFin}.$$

Since $X \simeq \lim \tau_{\leq n}X$, we may assume X is bounded above.
If $X \in \mathrm{CondSp}_{<-n}$, then $\operatorname{map}(\mathbb{S}[S], X), \operatorname{map}(\mathbb{S}[S]^{\blacksquare}, X) \in \mathrm{CondSp}_{<-n}$, so we can check the equivalence on each homotopy group. So we can assume X is bounded.
Now we have $X \simeq \bigoplus_{\alpha\in A} \pi_\alpha X[\alpha]$, where A is finite. So we may assume X is solid, then

$$\operatorname{map}(\mathbb{S}[S], X) \simeq \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}[S], X) \simeq \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}[S]^{\blacksquare}, X) \simeq \operatorname{map}(\mathbb{S}[S]^{\blacksquare}, X).$$

□

(ii) $\mathrm{SolidSp}'$ is closed under limits, colimits and extensions.

Proof. Since π_n commutes with limits and filtered colimits, then $\mathrm{SolidSp}'$ is closed under limits and filtered colimits.
On the other hand, in the stable ∞-category CondSp, finite limits are the same as finite colimits, $\mathrm{SolidSp}'$ is closed under finite limits implies that $\mathrm{SolidSp}'$ is closed under finite colimits. Therefore, $\mathrm{SolidSp}'$ is closed under all colimits.
Assume $X \longrightarrow Y \longrightarrow Z$ is an exact triangle with $X, Z \in \mathrm{SolidSp}'$, then we have a long exact sequence

$$\cdots \longrightarrow \pi_{i+1}Z \xrightarrow{h_{i+1}} \pi_i X \xrightarrow{f_i} \pi_i Y \xrightarrow{g_i} \pi_i Z \xrightarrow{h_i} \pi_{i-1}X \longrightarrow \cdots,$$

which induces an exact sequence

$$0 \longrightarrow \operatorname{coker}(h_{i+1}) \longrightarrow \pi_i Y \longrightarrow \ker(h_i) \longrightarrow 0.$$

Now, $X, Z \in \mathrm{SolidSp}' \implies \forall i, \pi_i X, \pi_i Z \in \mathrm{Solid}$. Since Solid is closed under kernels and cokernels, we have $\operatorname{coker}(h_{i+1}), \ker(h_i) \in \mathrm{Solid}$, and since Solid is closed under extensions, we have $\pi_i Y \in \mathrm{Solid}$, hence $Y \in \mathrm{SolidSp}'$. □

(iii) $\mathrm{SolidSp} \subset \mathrm{SolidSp}'$.

Proof. For $X \in \mathrm{CondSp}$, we have

$$X \simeq \operatorname*{colim}_{c \in \mathcal{C}} \mathbb{S}[S_c][-n_c], \quad S_c \in \mathrm{ProFin},$$

where $\mathcal{C}$ is an index category. Denote $X^{\blacksquare} \simeq \operatorname*{colim}_{c \in \mathcal{C}} \mathbb{S}[S_c]^{\blacksquare}[-n_c]$.
Since $\mathbb{S}[S_c]^{\blacksquare} \simeq \mathbb{S}^I \in \mathrm{SolidSp}'$ and $\mathrm{SolidSp}'$ is closed under colimits, we have

$$X^{\blacksquare} \simeq \operatorname*{colim}_{c \in \mathcal{C}} \mathbb{S}[S_c]^{\blacksquare}[-n_c] \in \mathrm{SolidSp}' \subset \mathrm{SolidSp}.$$

If $X \in \mathrm{SolidSp}$, for any $Y \in \mathrm{SolidSp}$, we have

$$\begin{aligned} \mathrm{map}(X^{\blacksquare}, Y) &\simeq \mathrm{map}(\operatorname*{colim}_{c \in \mathcal{C}} \mathbb{S}[S_c]^{\blacksquare}[-n_c], Y) \\ &\simeq \lim_{c \in \mathcal{C}} \mathrm{map}(\mathbb{S}[S_c]^{\blacksquare}, Y)[n_c] \\ &\simeq \lim_{c \in \mathcal{C}} \mathrm{map}(\mathbb{S}[S_c], Y)[n_c] \\ &\simeq \mathrm{map}(\operatorname*{colim}_{c \in \mathcal{C}} \mathbb{S}[S_c][-n_c], Y) \\ &\simeq \mathrm{map}(X, Y). \end{aligned}$$

Hence $X \simeq X^{\blacksquare} \in \mathrm{SolidSp}'$. Therefore, $\mathrm{SolidSp} \subset \mathrm{SolidSp}'$. □

Proposition 15.6. For a CW complex X, we have

$$\mathbb{S}[X]^{\blacksquare} \simeq \mathbb{S}[\mathrm{h}(X)],$$

which is a discrete condensed spectrum.

Proof. We have a map $X \longrightarrow \mathrm{h}(X)$, which induces $\mathbb{S}[X] \longrightarrow \mathbb{S}[\mathrm{h}(X)]$. Since $\mathbb{S}[\mathrm{h}(X)] \in \mathrm{CondSp}$ is discrete, hence solid, thus it induces a map $\mathbb{S}[X]^{\blacksquare} \longrightarrow \mathbb{S}[\mathrm{h}(X)]$.

Claim: $\mathrm{cofib}(\mathbb{S}[X]^{\blacksquare} \to \mathbb{S}[\mathrm{h}(X)]) \otimes \mathbb{Z} \simeq 0$.

With this claim, we can say that $\mathbb{S}[X]^{\blacksquare} \simeq \mathbb{S}[\mathrm{h}(X)]$.

- $\mathbb{S}[X]^{\blacksquare} \otimes \mathbb{Z} \simeq \mathbb{Z}[X]^{\blacksquare}$.

  This is because we can write $\mathbb{S}[X]^{\blacksquare} \simeq \operatorname*{colim}_{c \in \mathcal{C}} \mathbb{S}[S_c]^{\blacksquare}[-n_c]$, $S_c \in \mathrm{ProFin}$, then

$$\mathbb{S}[X]^{\blacksquare} \otimes \mathbb{Z} \simeq \operatorname*{colim}_{c \in \mathcal{C}} (\mathbb{S}[S_c]^{\blacksquare} \otimes \mathbb{Z})[-n_c] \simeq \operatorname*{colim}_{c \in \mathcal{C}} \mathbb{Z}[S_c]^{\blacksquare}[-n_c] \simeq \mathbb{Z}[X]^{\blacksquare}.$$

- $\mathbb{S}[\mathrm{h}(X)] \otimes \mathbb{Z} \simeq \mathbb{Z}[\mathrm{h}(X)]$.

- $\mathbb{Z}[X]^{\blacksquare} \simeq \mathbb{Z}[\mathrm{h}(X)]$.

  This is a result from [Sch19] Example 6.5.

□

Definition 15.7. Let $X \in \mathrm{CondSp}$.

- For any integer n, we define $X/n := \mathrm{cofib}(X \xrightarrow{n} X)$.

- The p-completion of X is defined to be $X_p := \varprojlim X/p^n$.

In particular, we have $\mathbb{S}/n = \mathrm{cofib}(\mathbb{S} \xrightarrow{n} \mathbb{S})$ and $\mathbb{S}_p = \varprojlim \mathbb{S}/p^n$.

Proposition 15.8. (1) $\mathbb{S}_p \otimes^{\blacksquare} \mathbb{S}_p \simeq \mathbb{S}_p$.

(1) $\mathbb{S}_p \otimes^{\blacksquare} \mathbb{S}_\ell \simeq 0 (p \neq \ell)$.

Proof. Recall, we have

$$\mathbb{Z}_p \otimes^{\blacksquare} \mathbb{Z}_p \simeq \mathbb{Z}_p, \quad \mathbb{Z}_p \otimes^{\blacksquare} \mathbb{Z}_\ell \simeq 0 (p \neq \ell).$$

Claim: $\mathbb{S}_p \otimes \mathbb{Z} \simeq \mathbb{Z}_p$.

The proof for this claim is similar to the proof for $\mathbb{S}^I \otimes \mathbb{Z} \simeq \mathbb{S}^I$.

With this claim, we have

$$(\mathbb{S}_p \otimes^{\blacksquare} \mathbb{S}_p) \otimes \mathbb{Z} \simeq (\mathbb{S}_p \otimes \mathbb{Z}) \otimes^{\blacksquare} (\mathbb{S}_p \otimes \mathbb{Z}) \simeq \mathbb{Z}_p \otimes^{\blacksquare} \mathbb{Z}_p \simeq \mathbb{Z}_p \simeq \mathbb{S}_p \otimes \mathbb{Z},$$

$$(\mathbb{S}_p \otimes^{\blacksquare} \mathbb{S}_\ell) \otimes \mathbb{Z} \simeq (\mathbb{S}_p \otimes \mathbb{Z}) \otimes^{\blacksquare} (\mathbb{S}_\ell \otimes \mathbb{Z}) \simeq \mathbb{Z}_p \otimes^{\blacksquare} \mathbb{Z}_\ell \simeq 0,$$

hence

$$\mathbb{S}_p \otimes^{\blacksquare} \mathbb{S}_p \simeq \mathbb{S}_p, \quad \mathbb{S}_p \otimes^{\blacksquare} \mathbb{S}_\ell \simeq 0.$$

□

Open Problem 1. For $X \in \mathrm{CHaus}$, we have

$$\mathbb{S}[X]^{\blacksquare} \simeq \mathrm{map}(\Gamma(X, \mathbb{S}), \mathbb{S}).$$

Question. In the ∞-category Sp, we have

$$\mathrm{map}(\lim_{\Delta} X_\bullet, \mathbb{S}) \simeq \operatorname*{colim}_{\Delta} \mathrm{map}(X_\bullet, \mathbb{S}).$$