

Condensed mathematics

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1 Condensed Sets

Definition 1.1. Let \mathcal{C} be a category. A Grothendieck topology on \mathcal{C} consists of: for each object X in \mathcal{C} , there is a collection $\text{Cov}(X)$ of sets $\{X_i \rightarrow X\}_{i \in I}$, satisfying the following three axioms:

- (i) If $V \rightarrow X$ is an isomorphism, then $\{V \rightarrow X\} \in \text{Cov}(X)$.
- (ii) If $\{X_i \rightarrow X\}_{i \in I} \in \text{Cov}(X)$ and $Y \rightarrow X$ is any arrow in \mathcal{C} , then the fiber products $X_i \times_X Y$ exist and $\{X_i \times_X Y \rightarrow Y\}_{i \in I} \in \text{Cov}(Y)$.
- (iii) If $\{X_i \rightarrow X\}_{i \in I} \in \text{Cov}(X)$ and for each $i \in I$, $\{V_{ij} \rightarrow X_i\}_{j \in I_i} \in \text{Cov}(X_i)$, then $\{V_{ij} \rightarrow X\}_{i \in I, j \in I_i} \in \text{Cov}(X)$.

We call elements of $\text{Cov}(X)$ coverings.

Definition 1.2. A site is a category \mathcal{C} together with a Grothendieck topology.

Example 1.3. Let $\mathcal{C} = \text{ProFin}$, the category of all profinite sets. For $\{X_i \rightarrow Y\}_{i \in I}$ be a covering, we mean I is a finite index and $\coprod_{i \in I} X_i \rightarrow Y$ is a surjection. We also call maps $\{X_i \rightarrow Y\}_{i \in I}$ finite jointly surjective families of maps.

Now, for the category ProFin together its coverings, we call it the proétale site of a point and denote it by $*_{\text{proét}}$.

Definition 1.4. (i) For any site \mathcal{C} , we call a functor

$$\mathcal{F} : \mathcal{C}^{\text{op}} \rightarrow \text{Set}$$

a presheaf of sets.

- (ii) For a presheaf of sets $\mathcal{F} : \mathcal{C}^{\text{op}} \rightarrow \text{Set}$, if for any $X \in \mathcal{C}$ and any covering $\{X_i \rightarrow X\}_{i \in I} \in \text{Cov}(X)$, we have

$$\mathcal{F}(X) \xrightarrow{\sim} \text{Eq}\left(\prod_{i \in I} \mathcal{F}(X_i) \rightrightarrows \prod_{i, j \in I} \mathcal{F}(X_i \times_X X_j)\right).$$

Then we call \mathcal{F} a sheaf of sets.

Definition 1.5. A condensed set T is a sheaf of sets on $*_{\text{proét}}$, i.e. a functor $T : *_{\text{proét}}^{\text{op}} \rightarrow \text{Set}$ satisfying the sheaf condition.

Remark 1.6. (i) Concretely, a condensed set T is a functor $T : \text{ProFin}^{\text{op}} \rightarrow \text{Set}$, satisfying $T(\emptyset) = *$ and

- For any profinite sets S_1, S_2 , the natural map

$$T(S_1 \sqcup S_2) \longrightarrow T(S_1) \times T(S_2)$$

is a bijection.

- For any surjection $S' \twoheadrightarrow S$ of profinite sets with fiber product $S' \times_S S'$ and two projections $p_1, p_2 : S' \times_S S' \rightarrow S'$, the map

$$T(S) \xrightarrow{\sim} \{x \in T(S') \mid p_1^*(x) = p_2^*(x) \in T(S' \times_S S')\}$$

is a bijection. In other words, T maps the pullback diagram

$$\begin{array}{ccc} S' \times_S S' & \xrightarrow{p_2} & S' \\ p_1 \downarrow & & \downarrow \\ S' & \longrightarrow & S \end{array}$$

to a pullback diagram

$$\begin{array}{ccc} T(S' \times_S S') & \xleftarrow{p_2^*} & T(S') \\ p_1^* \uparrow & & \uparrow \\ T(S') & \xleftarrow{\quad} & T(S) \end{array}$$

- (ii) The category ProFin of all profinite sets is a large category.

Definition 1.7. κ is an uncountable strong limit cardinal if κ is uncountable and for any $\lambda < \kappa$, we have $2^\lambda < \kappa$.

Example 1.8. For any limit cardinal λ , i.e. if $\kappa < \lambda$, then $\kappa + 1 < \lambda$. We define

$$\sqsubset_0 = \aleph_0, \dots, \sqsubset_{\alpha+1} = 2^{\sqsubset_\alpha},$$

and let

$$\sqsubset_\lambda = \bigcup_{\alpha < \lambda} \sqsubset_\alpha,$$

then we can show that \sqsubset_λ is an uncountable strong limit cardinal.

Notation. We let κ -ProFin denote the category of all κ -small profinite sets, i.e. profinite sets whose cardinal less equal than κ . Let $\text{Cond}_\kappa(\text{Set}) = \text{Sh}(\kappa\text{-ProFin}, \text{Set})$.

Remark 1.9. If $\kappa' > \kappa$ are two uncountable strong limit cardinals, and denote the inclusion by $i : \kappa\text{-ProFin} \hookrightarrow \kappa'\text{-ProFin}$, then we have a forgetful functor

$$\text{Cond}_{\kappa'}(\text{Set}) \longrightarrow \text{Cond}_\kappa(\text{Set}); T \mapsto T \circ i.$$

This forgetful functor admits a left adjoint $F : \text{Cond}_\kappa(\text{Set}) \longrightarrow \text{Cond}_{\kappa'}(\text{Set})$. F is fully faithful and F commutes with all colimits and all finite limits.

We define

$$\text{Cond}(\text{Set}) = \bigcup_{\kappa} \text{Cond}_\kappa(\text{Set}) = \varinjlim_{\kappa} \text{Cond}_\kappa(\text{Set}).$$

Example 1.10. Let Top denote the category of all topological spaces. For each $T \in \text{Top}$, we can define $\underline{T} \in \text{Cond}(\text{Set})$ as follows:

$$\underline{T} : \text{ProFin}^{\text{op}} \longrightarrow \text{Set}; S \mapsto \underline{T}(S) = \text{Cont}(S, T) = \{\text{continuous maps from } S \text{ to } T\}.$$

We need to check that \underline{T} is a condensed set:

- (i) $\underline{T}(S_1 \sqcup S_2) = \text{Cont}(S_1 \sqcup S_2, T) = \text{Cont}(S_1, T) \times \text{Cont}(S_2, T) = \underline{T}(S_1) \times \underline{T}(S_2)$.
- (ii) For any surjection $g : S' \twoheadrightarrow S$, let $p_1, p_2 : S' \times_S S' \rightarrow S'$ be the two projections. We need to show the following map is a bijection:

$$\text{Cont}(S, T) \xrightarrow{\sim} \{h : S' \rightarrow T \mid hp_1 = hp_2 : S' \times_S S' \rightarrow T\}; f \mapsto f \circ g.$$

Because g is surjective, it is easy to show this map is an injection.

Now, for any $h : S' \rightarrow T$ with $hp_1 = hp_2$, from the universal property of pushout(in our situation, the pullback square is also a pushout), we can find a unique f , s.t. the diagram commutes.

$$\begin{array}{ccc} S' \times_S S' & \xrightarrow{p_2} & S' \\ p_1 \downarrow & & \downarrow g \\ S' & \xrightarrow{g} & S \\ & \searrow h & \downarrow h \\ & & T \end{array}$$

(Note: In the original image, there is a dashed arrow from S to T labeled f , and a solid arrow from S' to T labeled h . The diagram shows the commutativity of the pushout square.)

Definition 1.11. Let $X \in \text{Top}$. The following are equivalent definition:

- (i) $X \in \text{Top}$ is compactly generated;
- (ii) If for any compact Hausdorff space S with a map $S \rightarrow X$, if the composition $S \rightarrow X \rightarrow Y$ is continuous, then $X \rightarrow Y$ is continuous;
- (iii) $A \subset X$ is closed if and only if for any compact space K with a map $f : K \rightarrow X$, $f^{-1}(A) \subset K$ is closed.

Remark 1.12. (i) If a topological space X is compact Hausdorff, then X is compactly generated.

- (ii) Let CGTop denote the category of all compactly generated spaces and let CHaus denote the category of all compact Hausdorff spaces.

Definition 1.13. For a category \mathcal{C} , $P \in \mathcal{C}$ is a projective object if for any epimorphism $Y \twoheadrightarrow X$ and a morphism $P \rightarrow X$, there is a lift

$$\begin{array}{ccc} & P & \\ \swarrow \exists & \downarrow & \\ Y & \twoheadrightarrow & X \end{array}$$

Definition 1.14. In the category CHaus , we call its projective objects as extremally disconnected Hausdorff spaces.

Remark 1.15. (i) Equivalently a compact Hausdorff space S is extremally disconnected if any surjection $S' \twoheadrightarrow S$ from a compact Hausdorff space splits.

- (ii) Extremally disconnected Hausdorff spaces are profinite sets, i.e. $\text{ExDisc} \subset \text{ProFin}$. Here, ExDisc denote the category of all extremally disconnected Hausdorff spaces.

Remark 1.16. We have two adjunctions.

$$(i) \quad \text{Top} \xrightleftharpoons[i]{\beta} \text{CHaus}, \text{ i.e. } \beta \dashv i.$$

Where

$$i : \text{CHaus} \rightarrow \text{Top}; X \mapsto X$$

and

$$\beta : \text{Top} \rightarrow \text{CHaus}$$

is the Stone-Cech compactification of topological spaces.

For any $X \in \mathbf{Top}$, we define $\beta X \in \mathbf{CHaus}$ as follows:

for any $Y \in \mathbf{CHaus}$ with a map $f : X \rightarrow Y$, there exists a unique map $\beta X \rightarrow Y$ so that the diagram commutes.

$$\begin{array}{ccc} X & \xrightarrow{i_X} & \beta X \\ & \searrow f & \swarrow \exists! \\ & Y & \end{array}$$

In fact, we can use the ultrafilter to construct βX concretely. And by this construction, we can show that

$$|\beta X| \leq 2^{2^{|X|}}.$$

(ii) $\mathbf{CGTop} \xrightleftharpoons[c]{i} \mathbf{Top}$, i.e. $i \dashv c$.

Where

$$i : \mathbf{CGTop} \rightarrow \mathbf{Top}; X \mapsto X$$

and

$$c : \mathbf{Top} \rightarrow \mathbf{CGTop}; X \mapsto X^{\text{cg}}.$$

We define X^{cg} as follows:

- As a set, $X^{\text{cg}} = X$.
- The topology of X^{cg} is given by the quotient topology of

$$\coprod_{S \rightarrow X} S \longrightarrow X, \quad S \in \mathbf{CHaus}$$

Proposition 1.17. (i) The functor $\mathbf{Top} \rightarrow \mathbf{Cond}_\kappa(\mathbf{Set}); T \mapsto \underline{T}$ is a faithful functor.

(ii) When the above functor restricted to the full subcategory $\kappa\text{-CGTop}$ of all κ -compactly generated spaces, functor $\kappa\text{-CGTop} \rightarrow \mathbf{Cond}_\kappa(\mathbf{Set}); T \mapsto \underline{T}$ is a fully faithful functor.

(iii) The functor $\mathbf{Top} \rightarrow \mathbf{Cond}_\kappa(\mathbf{Set}); T \mapsto \underline{T}$ admits a left adjoint $\mathbf{Cond}_\kappa(\mathbf{Set}) \rightarrow \mathbf{Top}; T \mapsto T(*)_{\text{top}}$. Here, $T(*)_{\text{top}}$ means the underlying set $T(*)$ equipped with the quotient topology of $\sqcup_{S \rightarrow T} S \rightarrow T(*)$, where the disjoint union runs over all

κ -small profinite sets S with a map to T , i.e. an element of $T(S)$. Moreover, we have $\underline{T}(*)_\text{top} \cong T^{\kappa\text{-cg}}$.

2 Condensed abelian groups

Definition 2.1. A condensed abelian group T is a sheaf of abelian groups on $*_{\text{proét}}$, i.e. a functor $T : *_{\text{proét}}^{\text{op}} \rightarrow \text{Ab}$ satisfying the sheaf condition. And we denote the category of all condensed abelian groups by $\text{Cond}(\text{Ab})$.

Definition 2.2 (Grothendieck's axioms). Let \mathcal{C} be an abelian category.

(AB3) All colimits exist.

(AB3*) All limits exist.

(AB4) Arbitrary direct sums are exact.

(AB4*) Arbitrary products are exact.

(AB5) Filtered colimits are exact.

(AB6) For any index set J and filtered categories I_j , $j \in J$, with functors $I_j \rightarrow \text{Cond}(\text{Ab})$; $i \mapsto M_i$, the natural map

$$\varinjlim_{(i_j \in I_j)_j} \prod_{j \in J} M_{i_j} \longrightarrow \prod_{j \in J} \varinjlim_{i_j \in I_j} M_{i_j}$$

is an isomorphism.

Definition 2.3. Let \mathcal{C} be an abelian category. $M \in \mathcal{C}$ is compact if $\text{Hom}(M, -)$ commutes with filtered colimits, i.e. $\text{Hom}(M, \varinjlim_i N_i) \cong \varinjlim_i \text{Hom}(M, N_i)$.

Theorem 2.4. (i) $\text{Cond}(\text{Ab})$ is an abelian category which satisfies Grothendieck's axioms (AB3), (AB4), (AB5), (AB6), (AB3*) and (AB4*).

(ii) $\text{Cond}(\text{Ab})$ is generated by compact projective objects.

Corollary 2.5. There is an adjunction:

$$\text{Cond}_\kappa(\text{Set}) \rightleftarrows \text{Cond}_\kappa(\text{Ab}) .$$

Where $\text{Cond}_\kappa(\text{Ab}) \rightarrow \text{Cond}_\kappa(\text{Set})$ is the forgetful functor and

$$\text{Cond}_\kappa(\text{Set}) \rightarrow \text{Cond}_\kappa(\text{Ab}); T \mapsto \mathbb{Z}[T].$$

Here, $\mathbb{Z}[T] := (S \mapsto \mathbb{Z}[T(S)])^{\text{sh}}$.

Remark 2.6. (i) For $S \in \text{ExDisc}$ and $M \in \text{Cond}(\text{Ab})$, we have

$$\text{Hom}_{\text{Cond}(\text{Ab})}(\mathbb{Z}[S], M) \cong \text{Hom}_{\text{Cond}(\text{Set})}(\underline{S}, M) \cong M(S).$$

Proof: We define the map:

$$\mu : \text{Hom}_{\text{Cond}(\text{Set})}(\underline{S}, M) \longrightarrow M(S); \alpha \mapsto \alpha(S)(1_S),$$

and the map

$$\lambda : M(S) \longrightarrow \text{Hom}_{\text{Cond}(\text{Set})}(\underline{S}, M); x \mapsto \lambda(x),$$

where for $\lambda(x) : \underline{S} \longrightarrow M$,

$$\lambda(x)(T) : \text{Cont}(T, S) \longrightarrow M(T); f \mapsto M(f)(x).$$

One can check that μ and λ are inverse to each other, hence

$$\text{Hom}_{\text{Cond}(\text{Set})}(\underline{S}, M) \cong M(S).$$

□

(ii) For any $S \in \text{ExDisc}$, $\mathbb{Z}[S] \in \text{Cond}(\text{Ab})$ is a compact and projective object.

Proof:

Compactness.

$$\text{Hom}(\mathbb{Z}[S], \varinjlim M_i) = (\varinjlim M_i)(S) = \varinjlim M_i(S) = \varinjlim \text{Hom}(\mathbb{Z}[S], M_i).$$

Projectiveness. For any exact sequence $M' \rightarrow M \rightarrow M''$ in $\text{Cond}(\text{Ab})$, the sequence

$$M'(S) \rightarrow M(S) \rightarrow M''(S)$$

is exact, i.e.

$$\text{Hom}(\mathbb{Z}[S], M') \rightarrow \text{Hom}(\mathbb{Z}[S], M) \rightarrow \text{Hom}(\mathbb{Z}[S], M'')$$

is exact, so $\mathbb{Z}[S]$ is projective.

□

(iii) $\text{Cond}(\text{Ab})$ has enough projectives.

Proposition 2.7. We have two equivalences.

$$(i) \operatorname{Shv}(\kappa\text{-CHaus}) \xrightarrow{\sim} \operatorname{Shv}(\kappa\text{-ProFin}); T \mapsto T|_{\kappa\text{-ProFin}}.$$

$$(ii) \operatorname{Shv}(\kappa\text{-ProFin}) \xrightarrow{\sim} \operatorname{Shv}(\kappa\text{-ExDisc}); T \mapsto T|_{\kappa\text{-ExDisc}}.$$

Remark 2.8. In order a presheaf of sets T to be a sheaf of sets, by definition, we need to check the sheaf condition in ProFin . Now, from the equivalence $\operatorname{Shv}(\kappa\text{-ProFin}) \xrightarrow{\sim} \operatorname{Shv}(\kappa\text{-ExDisc})$, we only need to check the sheaf condition in ExDisc . In this case, the condition(ii) is automatic: T maps the pullback diagram

$$\begin{array}{ccc} S' \times_S S' & \xrightarrow{p_2} & S' \\ p_1 \downarrow & & \downarrow \\ S' & \longrightarrow & S \end{array}$$

to a pullback diagram

$$\begin{array}{ccc} T(S' \times_S S') & \xleftarrow{p_2^*} & T(S') \\ p_1^* \uparrow & & \uparrow \\ T(S') & \xleftarrow{\quad} & T(S) \end{array}$$

This is because any cover of extremally disconnected sets splits. Specifically, the diagram

$$\begin{array}{ccc} S' \times_S S' & \longrightarrow & S' \\ \downarrow & & \downarrow \\ S' & \xrightarrow{g} & S \\ & \xleftarrow{f} & \end{array}$$

can implies the following diagram:

$$\begin{array}{ccccc} T(S' \times_S S') & \xleftarrow{\quad} & T(S') & & \\ \uparrow & & \uparrow & \nearrow & \\ T(S') & \xleftarrow{T(g)} & T(S) & \xleftarrow{\exists} & X \\ & \xleftarrow{T(f)} & & \nwarrow & \end{array}$$

which means it is a pullback diagram.

Property. There are some properties of the category $\operatorname{Cond}(\operatorname{Ab})$ of condensed abelian groups.

(i) $\text{Cond}(\text{Ab})$ has a symmetric monoidal tensor products $- \otimes -$, where for $M, N \in \text{Cond}(\text{Ab})$, $M \otimes N = (S \mapsto M(S) \otimes N(S))^{\text{sh}}$.

(ii) Functor $\text{Cond}(\text{Set}) \rightarrow \text{Cond}(\text{Ab})$; $T \mapsto \mathbb{Z}[T]$ is symmetric monoidal with respect to the product and the tensor product, i.e. $\mathbb{Z}[T_1 \times T_2] = \mathbb{Z}[T_1] \otimes \mathbb{Z}[T_2]$.

Proof:

(iii) For $T \in \text{Cond}(\text{Set})$, $\mathbb{Z}[T] \in \text{Cond}(\text{Ab})$ is flat.

Proof: We need to show $- \otimes \mathbb{Z}[T] : \text{Cond}(\text{Ab}) \rightarrow \text{Cond}(\text{Ab})$ is an exact functor.

Take an exact sequence in $\text{Cond}(\text{Ab})$:

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0.$$

For any $S \in \text{ExDisc}$, we have an exact sequence:

$$0 \longrightarrow X(S) \longrightarrow Y(S) \longrightarrow Z(S) \longrightarrow 0.$$

Tensoring with the free abelian group $\mathbb{Z}[T(S)]$, we get an exact sequence:

$$0 \longrightarrow X(S) \otimes \mathbb{Z}[T(S)] \longrightarrow Y(S) \otimes \mathbb{Z}[T(S)] \longrightarrow Z(S) \otimes \mathbb{Z}[T(S)] \longrightarrow 0,$$

i.e.

$$0 \longrightarrow (X \otimes \mathbb{Z}[T])(S) \longrightarrow (Y \otimes \mathbb{Z}[T])(S) \longrightarrow (Z \otimes \mathbb{Z}[T])(S) \longrightarrow 0.$$

Hence the sequence

$$0 \longrightarrow X \otimes \mathbb{Z}[T] \longrightarrow Y \otimes \mathbb{Z}[T] \longrightarrow Z \otimes \mathbb{Z}[T] \longrightarrow 0.$$

exact and $\mathbb{Z}[T]$ is flat. □

(iv) Given any $M, N \in \text{Cond}(\text{Ab})$, we can give the group of homomorphisms $\text{Hom}(M, N)$ the structure of condensed abelian groups via the following definition, for any $S \in \text{ExDisc}$,

$$\underline{\text{Hom}}(M, N)(S) := \text{Hom}(\mathbb{Z}[S] \otimes M, N).$$

So we define an internal Hom-functor object.

- (v) There is an adjunction. For $P, M, N \in \text{Cond}(\text{Ab})$, we have an isomorphism of abelian groups:

$$\text{Hom}(P, \underline{\text{Hom}}(M, N)) \cong \text{Hom}(P \otimes M, N).$$

Proof: First, if $P = \mathbb{Z}[S]$ for some $S \in \text{ExDisc}$, then

$$\text{Hom}(P, \underline{\text{Hom}}(M, N)) = \text{Hom}(\mathbb{Z}[S], \underline{\text{Hom}}(M, N)) = \underline{\text{Hom}}(M, N)(S) = \text{Hom}(\mathbb{Z}[S] \otimes M, N).$$

Now, for general $P \in \text{Cond}(\text{Ab})$, we can write $P = \varinjlim \mathbb{Z}[S_i]$, so

$$\begin{aligned} \text{Hom}(P, \underline{\text{Hom}}(M, N)) &= \text{Hom}(\varinjlim \mathbb{Z}[S_i], \underline{\text{Hom}}(M, N)) \\ &= \varprojlim \text{Hom}(\mathbb{Z}[S_i], \underline{\text{Hom}}(M, N)) \\ &= \varprojlim \text{Hom}(\mathbb{Z}[S_i] \otimes M, N) \\ &= \text{Hom}(\varinjlim \mathbb{Z}[S_i] \otimes M, N) \\ &= \text{Hom}(P \otimes M, N). \end{aligned}$$

□

- (vi) As $\text{Cond}(\text{Ab})$ has enough projectives, one can form the derived category $D(\text{Cond}(\text{Ab}))$.

If $P \in \text{Cond}(\text{Ab})$ is compact and projective, then $P[0] \in D(\text{Cond}(\text{Ab}))$ is a compact object of the derived category, i.e. $\text{Hom}(P, -)$ commutes with arbitrary direct sums. In particular, $D(\text{Cond}(\text{Ab}))$ is compactly generated.

- (vii) Similarly, in the derived category $D(\text{Cond}(\text{Ab}))$, we have the adjunction:

$$\text{Hom}(P, R\underline{\text{Hom}}(M, N)) \cong \text{Hom}(P \otimes^L M, N).$$

- (viii) Let $\mathcal{D}(\text{Cond}(\text{Ab}))$ denote the derived ∞ -category of $\text{Cond}(\text{Ab})$ and $\mathcal{D}(\text{Ab})$ denote the derived ∞ -category of Ab , then there is an equivalence

$$\mathcal{D}(\text{Cond}(\text{Ab})) \cong \text{Cond}(\mathcal{D}(\text{Ab})).$$

3 $D(R)$

Definition 3.1. An ∞ -category is a simplicial set \mathcal{C} which satisfies the following extension condition:

$$\begin{array}{ccc} \Lambda_i^n & \longrightarrow & \mathcal{C} \\ \downarrow & \nearrow \exists f & \\ \Delta^n & & \end{array} \quad 0 < \forall i < n.$$

Definition 3.2. Let \mathcal{C} be an ∞ -category. A zero object of \mathcal{C} is an object which is both initial and final. We say that \mathcal{C} is pointed if \mathcal{C} contains a zero object.

Definition 3.3. Let \mathcal{C} be a pointed ∞ -category. A triangle in \mathcal{C} is a diagram $\Delta^1 \times \Delta^1 \rightarrow \mathcal{C}$ depicted as

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow g \\ 0 & \longrightarrow & Z \end{array}$$

where 0 is a zero object in \mathcal{C} .

We say a triangle in \mathcal{C} is a fiber sequence if it is a pullback and say a triangle in \mathcal{C} is a cofiber sequence if it is a pushout.

We generally indicate a triangle by specifying only the pair of maps $X \xrightarrow{f} Y \xrightarrow{g} Z$.

Remark 3.4. Let \mathcal{C} be a pointed ∞ -category. A triangle in \mathcal{C} consists of the following data:

- (i) A pair of morphisms $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ in \mathcal{C} .
- (ii) A 2-simplex in \mathcal{C} corresponding to a diagram

$$\begin{array}{ccc} & Y & \\ f \nearrow & & \searrow g \\ X & \xrightarrow{h} & Z \end{array}$$

in \mathcal{C} , which identifies h with the composition $g \circ f$.

- (iii) A 2-simplex

$$\begin{array}{ccc} & 0 & \\ \nearrow & & \searrow \\ X & \xrightarrow{h} & Z \end{array}$$

in \mathcal{C} , which we view as anullhomotopy of h .

Definition 3.5. Let \mathcal{C} be a pointed ∞ -category containing a morphism $g : X \longrightarrow Y$.

A fiber of g is a fiber sequence

$$\begin{array}{ccc} W & \longrightarrow & X \\ \downarrow & & \downarrow g \\ 0 & \longrightarrow & Y \end{array}$$

and we denote $W = \text{fib}(g)$.

Dually, a cofiber of g is a cofiber sequence

$$\begin{array}{ccc} X & \xrightarrow{g} & Y \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & Z \end{array}$$

and we denote $Z = \text{cofib}(g)$.

Definition 3.6. An ∞ -category \mathcal{C} is stable if it satisfies the following conditions:

- (i) There exists a zero object $0 \in \mathcal{C}$.
- (ii) Every morphism in \mathcal{C} admits a fiber and a cofiber.
- (iii) A triangle in \mathcal{C} is a fiber sequence if and only if it is a cofiber sequence.

Remark 3.7. (i) For a stable ∞ -category \mathcal{C} , we define the suspension functor $\Sigma : \mathcal{C} \longrightarrow \mathcal{C}$ and the loop functor $\Omega : \mathcal{C} \longrightarrow \mathcal{C}$ as follows:

$$\Sigma(X) := \text{cofib}(X \longrightarrow 0)$$

and

$$\Omega(X) := \text{fib}(0 \longrightarrow X).$$

- (ii) For a stable ∞ -category \mathcal{C} , there is a homotopy equivalence:

$$\text{Map}_{\mathcal{C}}(\Sigma X, Y) \xrightarrow{\sim} \text{Map}_{\mathcal{C}}(X, \Omega Y).$$

Besides, the unit $X \longrightarrow \Omega \Sigma(X)$ and $\Sigma \Omega(Y) \longrightarrow Y$ counit are isomorphic.

Definition 3.8. Let R be a commutative ring, the ∞ -category $D(R)$ is a stable ∞ -category with all colimits, generated (as a cocomplete stable ∞ -category) by a distin-

guished compact object 1, satisfying

$$\pi_0 \mathbf{Map}(1, 1) = R^{\text{op}}, \quad \pi_0 \mathbf{Map}(\Sigma^d 1, 1) = 0, \quad \forall d \neq 0.$$

For $X, Y \in D(R)$, we define

$$[X, Y] := \pi_0 \mathbf{Map}(X, Y)$$

and

$$[X, Y]_d := [\Sigma^d X, Y] = [X, \Omega^d Y].$$

Remark 3.9. (i) From the definition of $D(R)$, we have

$$[1, 1] = \pi_0 \mathbf{Map}(1, 1) = R^{\text{op}}, \quad [1, 1]_d = \pi_0 \mathbf{Map}(\Sigma^d 1, 1) = 0, \quad \forall d \neq 0.$$

(ii) If $d \geq 0$, we have

$$[X, Y]_d = \pi_d \mathbf{Map}(X, Y).$$

(iii) Claim: In $D(R)$, for any integer d , we have $[X, Y]_d \in \mathbf{Ab}$.

Proof: First, if $d \geq 2$, $[X, Y]_d = \pi_d \mathbf{Map}(X, Y) \in \mathbf{Ab}$. For any $d \in \mathbb{Z}$,

$$[X, Y]_d = [\Sigma^d X, Y] = [\Sigma^{d-2} X, Y]_2 \in \mathbf{Ab}.$$

(iv) For a stable ∞ -category, a fiber sequence $X \rightarrow Y \rightarrow Z$ is at the same time a cofiber sequence, and vice versa. Hence, we will call it a fiber-cofiber sequence.

(v) For a fiber-cofiber sequence $X \rightarrow Y \rightarrow Z$ in $D(R)$, we can induce a new fiber-cofiber sequence $Y \rightarrow Z \rightarrow \Sigma X$.

(vi) Given a fiber-cofiber sequence $X \rightarrow Y \rightarrow Z$ and any $A \in D(R)$, we can induce two long exact sequences:

$$\cdots \longrightarrow [A, X]_d \longrightarrow [A, Y]_d \longrightarrow [A, Z]_d \longrightarrow [A, X]_{d-1} \longrightarrow \cdots$$

$$\cdots \longrightarrow [X, A]_{d+1} \longrightarrow [Z, A]_d \longrightarrow [Y, A]_d \longrightarrow [X, A]_d \longrightarrow \cdots$$

(vii) Assume

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

is a pushout-pullback square in $D(R)$, then we can produce a triangle $A \rightarrow B \oplus C \rightarrow D$. With this, we can induce a long exact sequence.

Definition 3.10. For any integer d , we define a functor $H_d : D(R) \rightarrow \mathbf{Mod}_R$; $X \mapsto [1, X]_d$.

Remark 3.11. (i) We already know $[1, X]_d \in \mathbf{Ab}$. And we need to show $[1, X]_d$ is an R -module.

In fact, we have

$$\mathbf{Map}(\Sigma^d 1, \Sigma^d 1) \times \mathbf{Map}(\Sigma^d 1, X) \rightarrow \mathbf{Map}(\Sigma^d 1, X),$$

applying the functor π_0 , we get:

$$\pi_0 \mathbf{Map}(\Sigma^d 1, \Sigma^d 1) \times \pi_0 \mathbf{Map}(\Sigma^d 1, X) = \pi_0(\mathbf{Map}(\Sigma^d 1, \Sigma^d 1) \times \mathbf{Map}(\Sigma^d 1, X)) \rightarrow \pi_0(\mathbf{Map}(\Sigma^d 1, X)),$$

i.e.

$$R^{\text{op}} \times [1, X]_d \rightarrow [1, X]_d,$$

which implies $[1, X]_d \in \mathbf{Mod}_R$.

(ii) $H_d : D(R) \rightarrow \mathbf{Mod}_R$; $X \mapsto [1, X]_d$ is a representable functor and $\Sigma^d 1$ represents H_d .

Lemma 3.12. (i) $H_d(\prod_i X_i) = \prod_i H_d(X_i)$.

(ii) $H_d(\oplus_i X_i) = \oplus_i H_d(X_i)$.

(iii) $H_d(\varinjlim X_i) = \varinjlim H_d(X_i)$.

(iv) For a sequence of maps $\cdots \rightarrow X_n \rightarrow X_{n-1} \rightarrow \cdots$, we have a Milnor sequence:

$$0 \longrightarrow \varprojlim^1 H_{d+1}(X_n) \longrightarrow H_d(\varprojlim X_n) \longrightarrow \varprojlim H_d(X_n) \longrightarrow 0.$$

Proposition 3.13. $f : X \rightarrow Y$ in $D(R)$ is an isomorphism if and only if $H_d(f) : H_d(X) \xrightarrow{\sim} H_d(Y)$, for $\forall d \in \mathbb{Z}$.

Proof. Take $Z = \text{cofib}(X \xrightarrow{f} Y)$, then $X \rightarrow Y \rightarrow Z$ is a fiber-cofiber sequence, and we can induce a long exact sequence

$$\cdots \rightarrow H_d(X) \rightarrow H_d(Y) \rightarrow H_d(Z) \rightarrow H_{d-1}(X) \rightarrow \cdots .$$

It suffices to show: if $Z \in D(R)$ with $H_d(Z) = 0, \forall d \in \mathbb{Z}$, then $Z = 0$.

Consider the full subcategory of $D(R)$:

$$\mathcal{C} = \{A \in D(R) \mid [\Sigma^d A, Z] = 0, \forall d \in \mathbb{Z}\}.$$

Observe that:

- $1 \in \mathcal{C}$.
- \mathcal{C} is stable under colimits. This is because

$$[\Sigma^d \text{colim } A_i, Z] = [\text{colim } \Sigma^d A_i, Z] = \lim [\Sigma^d A_i, Z] = 0.$$

- \mathcal{C} is stable under cofibers.

By definition of $D(R)$, we know $D(R)$ is generated as a cocomplete stable ∞ -category by 1. Hence, $D(R) = \mathcal{C}$. Then by Yoneda's lemma, $Z = 0$. \square

Proposition 3.14. Let $X \in D(R)$, then there exists $Y \in D(R)$ with a map $f : Y \rightarrow X$, s.t.

- (i) $H_d(Y) = 0, \forall d < 0$.
- (ii) $H_d(f) : H_d(Y) \xrightarrow{\sim} H_d(X)$ are isomorphisms, $\forall d \geq 0$.

Proof. We first prove: there exists a sequence of maps $Y_0 \rightarrow Y_1 \rightarrow Y_2 \rightarrow \cdots$ in $D(R)_{/X}$, s.t. for any $n \geq 0$, $H_d(Y_n) = 0, d < 0$ and $H_d(Y_n \rightarrow X)$ are isomorphisms if $0 \leq d < n$ and is a surjection if $d = n$.

We prove this by induction.

First, $n = 0$. Let $Y_0 = \bigoplus_I 1$ for $I = \text{cartinal of } H_0(X)$, then the map $Y_0 \rightarrow X$ can induce a surjection $H_0(Y_0) = R^{\oplus I} \twoheadrightarrow H_0(X)$ and for $d < 0$, $H_d(Y_0) = 0$.

Now we assume that there exists a sequence

$$Y_0 \rightarrow Y_1 \rightarrow Y_2 \rightarrow \cdots \rightarrow Y_{n-1}$$

in $D(R)_X$ satisfying the assumption.

Let $F = \text{fib}(Y_{n-1} \rightarrow X)$, then $F \rightarrow Y_{n-1} \rightarrow X$ is a fiber-cofiber sequence. We can find an index I , s.t. $\Sigma^{n-1} \oplus_I 1 \rightarrow F$ can induce a surjection $H_{n-1}(\Sigma^{n-1} \oplus_I 1) \twoheadrightarrow H_{n-1}(F)$. Then let $Y_n = \text{cofib}(\Sigma^{n-1} \oplus_I 1 \rightarrow F \rightarrow Y_{n-1})$, hence $\Sigma^{n-1} \oplus_I 1 \rightarrow Y_{n-1} \rightarrow Y_n$ is also a fiber-cofiber sequence. Now, we check it satisfies the requirements.

(a) $d < 0$. The fiber-cofiber sequence $\Sigma^{n-1} \oplus_I 1 \rightarrow Y_{n-1} \rightarrow Y_n$ can induce a long exact sequence:

$$\cdots \rightarrow H_{-1}(\Sigma^{n-1} \oplus_I 1) \rightarrow H_{-1}(Y_{n-1}) \rightarrow H_{-1}(Y_n) \rightarrow H_{-2}(\Sigma^{n-1} \oplus_I 1) \rightarrow H_{-2}(Y_{n-1}) \rightarrow \cdots$$

Since for $k < 0$, $H_k(Y_{n-1}) = 0$, we know $H_d(Y_n) = H_{d-1}(\Sigma^{n-1} \oplus_I 1) = 0 (d < 0)$.

(b) First, there exists a map $Y_n \rightarrow X$, this is because

$$\begin{array}{ccccc} \Sigma^{n-1} \oplus_I 1 & \longrightarrow & Y_{n-1} & \longrightarrow & Y_n \\ \downarrow & & \parallel & & \downarrow \exists \\ F & \longrightarrow & Y_{n-1} & \longrightarrow & X \end{array}$$

and $H_d(\Sigma^{n-1} \oplus_I 1) = 0, \forall d \neq n-1$, then by

$$\cdots \rightarrow H_{n-2}(\Sigma^{n-1} \oplus_I 1) \rightarrow H_{n-2}(Y_{n-1}) \rightarrow H_{n-2}(Y_n) \rightarrow H_{n-3}(\Sigma^{n-1} \oplus_I 1) \rightarrow \cdots,$$

it implies that $0 \leq \forall d \leq n-2, H_d(Y_n) \cong H_d(Y_{n-1}) \cong H_d(X)$.

We have the following diagram:

$$\begin{array}{ccccccccccc} H_n(Y_{n-1}) & \twoheadrightarrow & H_n(Y_n) & \twoheadrightarrow & H_{n-1}(\Sigma^{n-1} \oplus_I 1) & \twoheadrightarrow & H_{n-1}(Y_{n-1}) & \twoheadrightarrow & H_{n-1}(Y_n) & \twoheadrightarrow & H_{n-2}(\Sigma^{n-1} \oplus_I 1) & \twoheadrightarrow & H_{n-2}(Y_{n-1}) \\ \parallel & & \downarrow \exists & & \downarrow & & \parallel & & \downarrow \sim & & \downarrow \sim & & \parallel \\ H_n(Y_{n-1}) & \twoheadrightarrow & H_n(X) & \longrightarrow & H_{n-1}(F) & \longrightarrow & H_{n-1}(Y_{n-1}) & \twoheadrightarrow & H_{n-1}(X) & \longrightarrow & H_{n-2}(F) & \longrightarrow & H_{n-2}(Y_{n-1}) \end{array}$$

By five's lemma, we can show $H_{n-1}(Y_n) \xrightarrow{\sim} H_{n-1}(X)$ and $H_n(Y_n) \twoheadrightarrow H_n(X)$.

Now, for $Y_0 \rightarrow Y_1 \rightarrow Y_2 \rightarrow \cdots$, we take $Y = \varinjlim Y_n$, and hence we can get a map $Y \rightarrow X$.

By $H_d(\varinjlim Y_n) = \varinjlim H_d(Y_n)$, for $d < 0$, $H_d(Y) = 0$, and for $d \geq 0$,

$$H_d(Y) = \varinjlim (H_d(Y_0) \rightarrow H_d(Y_1) \rightarrow \cdots \rightarrow H_d(Y_{d+1}) \rightarrow H_d(Y_{d+2}) \rightarrow \cdots) = H_d(X).$$

□

Proposition 3.15. For $X \in D(R)$, the following are equivalent:

- (i) $H_d(X) = 0, \forall d < 0$.
- (ii) X is generated by 1 under colimits.
- (iii) There exists a sequence of maps $X_0 \rightarrow X_1 \rightarrow X_2 \rightarrow \cdots$ with $X = \varinjlim X_i$, where for each i , the cofiber $\text{cofib}(X_{i-1} \rightarrow X_i)$ is of the form $\Sigma^i \oplus_I 1$.

Proof. (i) \implies (iii). By previous proposition, for $X \in D(R)$, there exists a map $f : Y \rightarrow X$ with $H_d(Y) = 0$ for $d < 0$ and $H_d(f)$ are isomorphisms for $d \geq 0$. Then, for $d < 0$,

$$H_d(X) = H_d(Y) = 0.$$

Hence, for any $d \in \mathbb{Z}$, $H_d(f)$ are isomorphisms. Thus, $f : Y \xrightarrow{\sim} X$.

From the construction of Y , we know there is a sequence of maps $X_0 \rightarrow X_1 \rightarrow X_2 \rightarrow \cdots$ with $\varinjlim X_i = Y \cong X$.

By the fiber-cofiber sequence $\Sigma^{n-1} \oplus_I 1 \rightarrow X_{n-1} \rightarrow X_n$, we can get a new fiber-cofiber sequence $X_{n-1} \rightarrow X_n \rightarrow \Sigma^n \oplus_I 1$, i.e. $\text{cofib}(X_{n-1} \rightarrow X_n) = \Sigma^n \oplus_I 1$.

(iii) \implies (ii).

We have $X_{i-1} \rightarrow X_i \rightarrow \Sigma^i \oplus_I 1$, which gives a new fiber-cofiber sequence: $\Sigma^{i-1} \oplus_I 1 \rightarrow X_{i-1} \rightarrow X_i$. Then $X_i = \text{cofib}(\Sigma^{i-1} \oplus_I 1 \rightarrow X_{i-1}) = \text{colim}(0 \leftarrow \Sigma^{i-1} \oplus_I 1 \rightarrow X_{i-1})$.

Now, $X_1 = \text{cofib}(\oplus_I 1 \rightarrow X_0) = \text{cofib}(\oplus_I 1 \rightarrow \oplus_J 1) = \text{colim}(0 \leftarrow \oplus_I 1 \rightarrow \oplus_J 1)$.

Hence, each X_i is generated by 1 under colimits. Finally, $X = \text{colim } X_i$ is also generated by 1 under colimits.

(ii) \implies (i).

Arbitrary colimits can be written in terms of pushouts and filtered colimits. And H_d commutes with filtered colimits. So it suffices to show that for A, B, C with $H_d(A) = H_d(B) = H_d(C) = 0, \forall d < 0$, then for the pushout $D = \text{colim}(C \leftarrow A \rightarrow B)$, $H_d(D) = 0, \forall d < 0$.

This is because we can get a null-composite sequence $A \rightarrow B \oplus C \rightarrow D$, and induce a long exact sequence

$$\cdots \rightarrow H_d(A) \rightarrow H_d(B) \oplus H_d(C) \rightarrow H_d(D) \rightarrow \cdots$$

which implies $H_d(D) = 0, \forall d < 0$. □

- Definition 3.16.** (i) $D(R)_{\geq 0} := \{X \in D(R) \mid H_d(X) = 0, \forall d < 0\}$.
(ii) $D(R)_{< 0} := \{X \in D(R) \mid H_d(X) = 0, \forall d \geq 0\}$.
(iii) $\tau_{\geq 0} : D(R) \rightarrow D(R)_{\geq 0}; X \mapsto \tau_{\geq 0}(X) := Y$, which is constructed in Proposition 3.14.

Now, given any map $Z \rightarrow X$ in $D(R)$, we can get a commutative diagram:

$$\begin{array}{ccc} \tau_{\geq 0}(Z) & \xrightarrow{\exists!} & \tau_{\geq 0}(X) \\ \downarrow & & \downarrow \\ Z & \longrightarrow & X \end{array}$$

Hence, $\tau_{\geq 0} : D(R) \rightarrow D(R)_{\geq 0}$ is a functor.

Proposition 3.17. $D(R)_{\geq 0} \xrightleftharpoons[\tau_{\geq 0}]{i} D(R) \xrightleftharpoons[i]{\tau_{< 0}} D(R)_{< 0}$, i.e. $i \dashv \tau_{\geq 0}$ and $\tau_{< 0} \dashv i$.

Corollary 3.18. For $X \in D(R)$, we have

$$X \cong \varprojlim \tau_{\leq n}(X) \quad \text{and} \quad X \cong \varinjlim \tau_{\geq -n}(X).$$

Proof. We have a Milnor sequence:

$$0 \longrightarrow \varprojlim^1 H_{d+1}(\tau_{\leq n} X) \longrightarrow H_d(\varprojlim \tau_{\leq n} X) \longrightarrow \varprojlim H_d(\tau_{\leq n} X) \longrightarrow 0.$$

For $n \gg 0$, we have $H_d(\tau_{\leq n} X) = H_d(X)$, hence $\varprojlim H_d(\tau_{\leq n} X) = H_d(X)$.

And $\{H_{d+1}(\tau_{\leq n} X)\}_{n \in \mathbb{Z}}$ satisfies the Mittag-Leffler condition, hence $\varprojlim^1 H_{d+1}(\tau_{\leq n} X) = 0$. Therefore, from the above short exact sequence, we have

$$H_d(\varprojlim \tau_{\leq n} X) \cong H_d(X), \forall d \in \mathbb{Z},$$

which implies $X \cong \varprojlim \tau_{\leq n} X$.

For another isomorphism, from

$$H_d(\varinjlim \tau_{\geq -n} X) = \varinjlim H_d(\tau_{\geq -n} X) = H_d(X), \forall d \in \mathbb{Z},$$

one can show $X \cong \varinjlim \tau_{\geq -n}(X)$. □

Definition 3.19. For any map $f : X \rightarrow Y$ in $D(R)$, we define its kernel to be

$$\ker(f) := \tau_{\geq 0} \text{fib}(X \rightarrow Y)$$

and its cokernel to be

$$\text{coker}(f) := \tau_{\leq 0} \text{cofib}(X \rightarrow Y).$$

Proposition 3.20. Let $D(R)_0 = \{X \in D(R) \mid H_d(X) = 0, \forall d \neq 0\}$.

- (i) There is an isomorphism $H_0 : D(R)_0 \xrightarrow{\sim} \text{Mod}_R$.
- (ii) Any object in $D(R)_0$ can be written as of the form $\text{coker}(\oplus_I 1 \rightarrow \oplus_J 1)$.
- (iii) $H_0 : D(R)_0 \rightarrow \text{Mod}_R$ is an exact functor.
- (iv) $H_0 : D(R)_0 \rightarrow \text{Mod}_R$ commutes with direct sums.

Proof. (ii) For $X \in D(R)_0$, there exists $f : Y \rightarrow X$ with $H_d(Y) = 0, \forall d < 0$ and $H_d(f)$ are isomorphisms, $\forall d \geq 0$.

By the construction of Y_1 , $Y_1 = \text{cofib}(\oplus_I 1 \rightarrow \oplus_J 1)$.

On the other hand, $X \cong \tau_{\leq 0} Y_1$. Hence,

$$X \cong \tau_{\leq 0} \text{cofib}(\oplus_I 1 \rightarrow \oplus_J 1) = \text{coker}(\oplus_I 1 \rightarrow \oplus_J 1).$$

- (iii) In order to show that H_0 preserves exact sequences, it suffices to show H_0 preserves kernels and cokernels.

For any map $f : X \rightarrow Y$ in $D(R)_0$, applying functor $\tau_{\geq 0}$ to sequence $\text{fib}(f) \rightarrow X \rightarrow Y$, we get a fiber-cofiber sequence

$$\ker(f) = \tau_{\geq 0} \text{fib}(f) \rightarrow X \rightarrow Y.$$

And it induces a long exact sequence

$$0 = H_1(Y) \rightarrow H_0(\ker(f)) \rightarrow H_0(X) \rightarrow H_0(Y) \rightarrow \cdots.$$

Hence, $H_0(\ker(f)) = \ker(H_0(X) \rightarrow H_0(Y))$.

Dually, we can prove $H_0(\text{coker}(f)) = \text{coker}(H_0(X) \rightarrow H_0(Y))$.

□

Remark 3.21. $1 \in D(R)_0$ is compact and projective.

Proof: Compactness is the definition.

For the projectiveness, we need to show that any epimorphism $X \twoheadrightarrow 1$ splits.

Let $F = \text{fib}(X \rightarrow 1)$. Consider $F \rightarrow X \rightarrow 1$. Then $H_{-1}(F) = 0$.

By $[M, N]_d = \text{Ext}_R^{-d}(M, N)$, we get $\text{Ext}_R^{-1}(1, F) = [1, F]_{-1} = H_{-1}(F) = 0$. Hence $X \twoheadrightarrow 1$ splits. □

Definition 3.22. (i) A filtered object of $D(R)$ is an object in $\text{Fun}(\mathbb{Z}_{\leq}, D(R))$, i.e.

$$\cdots \longrightarrow F(n-1) \rightarrow F(n) \rightarrow F(n+1) \rightarrow \cdots$$

(ii) A filtered object F is convergent if $\varprojlim F(n) = 0$.

(iii) $F(\infty) := \varinjlim F(n)$. Call it the underlying object of F .

(iv) The n -th associated graded $\text{gr}_n(F) := \text{cofib}(F(n-1) \rightarrow F(n)) \triangleq F(n)/F(n-1)$.

Now, giving a convergent filtered object $F : \mathbb{Z}_{\leq} \rightarrow D(R)$, s.t. $\text{gr}_n(F) \in D(R)_n$, $\forall n$, we can define an R -module M_n :

$$H_n : D(R)_n \rightarrow \text{Mod}_R; \text{gr}_n(F) \mapsto H_n(\text{gr}_n(F)) \triangleq M_n.$$

From the sequence

$$F(n-1)/F(n-2) \longrightarrow F(n)/F(n-2) \longrightarrow F(n)/F(n-1) \longrightarrow \Sigma(F(n-1)/F(n-2)),$$

we get a map $d : H_n(\text{gr}_n(F)) \longrightarrow H_n(\Sigma \text{gr}_{n-1}(F))$, i.e. $d : M_n \longrightarrow M_{n-1}$.

One can check $d^2 = 0$. Hence, given a convergent filtered object F , s.t. $\text{gr}_n(F) \in D(R)_n$, we define a chain complex of R -modules M_* .

We denote $\text{Fun}(\mathbb{Z}_{\leq}, D(R))_{\text{cx}} = \{F \in \text{Fun}(\mathbb{Z}_{\leq}, D(R)) \mid F \text{ convergent}\}$.

Proposition 3.23. (i) $\text{Fun}(\mathbb{Z}_{\leq}, D(R))_{\text{cx}} \xrightarrow{\sim} \text{Ch}_R; F \mapsto M_*$.

$$(ii) \ H_n(F(\infty)) = H_n(M_*), \ \forall n.$$

4 $D(\mathbb{Z})$

Definition 4.1. Let $X \in \mathbf{Top}$. A sieve on X is a set \mathfrak{U} of open subsets of X , s.t. if $V \in \mathfrak{U}$ and $V' \subset V$, then $V' \in \mathfrak{U}$. If $U = \bigcup_{V \in \mathfrak{U}} V$, we say that the sieve \mathfrak{U} covers U .

Definition 4.2. (i) Let $X \in \mathbf{Top}$. Let $\mathcal{F} \in \mathbf{PSh}(X, D(\mathbb{Z}))$ be a presheaf with values in $D(\mathbb{Z})$, i.e. $\mathcal{F} \in \mathbf{Fun}(\mathbf{Op}(X)^{\text{op}}, D(\mathbb{Z}))$. We say \mathcal{F} is a sheaf if for all sieves \mathfrak{U} on X covering $U \in \mathbf{Op}(X)$, we have

$$\mathcal{F}(U) \xrightarrow{\sim} \varprojlim_{V \in \mathfrak{U}^{\text{op}}} \mathcal{F}(V).$$

(ii) For $U \in \mathbf{Op}(X)$, one define $h_U \in \mathbf{PSh}(X, D(\mathbb{Z}))$ via

$$h_U(V) = \begin{cases} * & V \subset U \\ \emptyset & \text{otherwise} \end{cases}$$

(iii) For a sieve \mathfrak{U} , one define $h_{\mathfrak{U}} \in \mathbf{PSh}(X, D(\mathbb{Z}))$ via

$$h_{\mathfrak{U}}(V) = \begin{cases} * & V \in \mathfrak{U} \\ \emptyset & V \notin \mathfrak{U} \end{cases}$$

Proposition 4.3. Let $\mathcal{F} \in \mathbf{PSh}(X, D(\mathbb{Z}))$, then \mathcal{F} is a sheaf if and only if it satisfies:

(i) $\mathcal{F}(\emptyset) = *$.

(ii) For any open subsets $V, V' \in \mathbf{Op}(X)$,

$$\mathcal{F}(V \cup V') \xrightarrow{\sim} \mathcal{F}(V) \times_{\mathcal{F}(V \cap V')} \mathcal{F}(V').$$

(iii) For any sieve \mathfrak{U} , $\mathcal{F}(\varinjlim_{V \in \mathfrak{U}} V) \xrightarrow{\sim} \varprojlim_{V \in \mathfrak{U}^{\text{op}}} \mathcal{F}(V)$.

Remark 4.4.

$$\mathbb{Z}[h_U](V) := \mathbb{Z}[h_U(V)] = \begin{cases} \mathbb{Z} & V \subseteq U \\ 0 & V \not\subseteq U \end{cases}$$

$$\mathbb{Z}[h_{\mathfrak{U}}](V) := \mathbb{Z}[h_{\mathfrak{U}}(V)] = \begin{cases} \mathbb{Z} & V \in \mathfrak{U} \\ 0 & V \notin \mathfrak{U} \end{cases}$$

$$\mathrm{Map}(\mathbb{Z}[h_U], \mathcal{F}) = \mathrm{Map}(\mathbb{Z}, \mathcal{F}(U)).$$

$$\begin{aligned} \mathrm{Map}(\mathbb{Z}[h_{\mathfrak{U}}], \mathcal{F}) &= \varprojlim_{V \in \mathfrak{U}^{\mathrm{op}}} \mathrm{Map}(\mathbb{Z}[h_V], \mathcal{F}) \\ &= \varprojlim_{V \in \mathfrak{U}^{\mathrm{op}}} \mathrm{Map}(\mathbb{Z}, \mathcal{F}(V)) \\ &= \mathrm{Map}(\mathbb{Z}, \varprojlim_{V \in \mathfrak{U}^{\mathrm{op}}} \mathcal{F}(V)) \\ &= \mathrm{Map}(\mathbb{Z}, \mathcal{F}(\varinjlim_{V \in \mathfrak{U}} V)). \end{aligned}$$

Proposition 4.5. $\mathrm{PSh}(X, D(\mathbb{Z})) \xrightleftharpoons[i]{\mathrm{sh}} \mathrm{Sh}(X, D(\mathbb{Z}))$; $\mathcal{F} \mapsto \mathcal{F}^{\mathrm{sh}}$. Moreover, $\mathcal{F}^{\mathrm{sh}} = 0$ iff \mathcal{F} lies in the stable co-complete subcategory generated by $\mathrm{cofib}(\mathbb{Z}[h_{\mathfrak{U}}] \rightarrow \mathbb{Z}[h_U])$ for all sieves \mathfrak{U} covering U .

Definition 4.6. For $\mathcal{F} \in \mathrm{PSh}(X, D(\mathbb{Z}))$, define $H_n(\mathcal{F}) \in \mathrm{PSh}(X, \mathrm{Ab})$ by $H_n(\mathcal{F})(U) = H_n(\mathcal{F}(U))$.

With this presheaf $H_n(\mathcal{F}) \in \mathrm{PSh}(X, \mathrm{Ab})$, one can sheafify it to get a sheaf $H_n(\mathcal{F})^{\mathrm{sh}} \in \mathrm{Sh}(X, \mathrm{Ab})$.

Proposition 4.7. Let $\mathcal{F} \in \mathrm{PSh}(X, D(\mathbb{Z}))$.

- (i) If $\mathcal{F}^{\mathrm{sh}} = 0$, then $H_n(\mathcal{F})^{\mathrm{sh}} = 0$, $\forall n \in \mathbb{Z}$.
- (ii) If \mathcal{F} is bounded above and $H_n(\mathcal{F})^{\mathrm{sh}} = 0$, $\forall n \in \mathbb{Z}$, then $\mathcal{F}^{\mathrm{sh}} = 0$.

Corollary 4.8. Let $\mathcal{F} \rightarrow \mathcal{G}$ be a map in $\mathrm{PSh}(X, D(\mathbb{Z}))$.

- (i) If $\mathcal{F}^{\mathrm{sh}} \xrightarrow{\sim} \mathcal{G}^{\mathrm{sh}}$, then $H_n(\mathcal{F})^{\mathrm{sh}} \xrightarrow{\sim} H_n(\mathcal{G})^{\mathrm{sh}}$, $\forall n \in \mathbb{Z}$.
- (ii) If \mathcal{F} and \mathcal{G} are bounded above, and $H_n(\mathcal{F})^{\mathrm{sh}} \xrightarrow{\sim} H_n(\mathcal{G})^{\mathrm{sh}}$, $\forall n \in \mathbb{Z}$, then $\mathcal{F}^{\mathrm{sh}} \xrightarrow{\sim} \mathcal{G}^{\mathrm{sh}}$.

Corollary 4.9. Let $\mathcal{F} \rightarrow \mathcal{G}$ be a map in $\mathrm{PSh}(X, D(\mathbb{Z}))$ and \mathcal{F}, \mathcal{G} are bounded above, then

$$\mathcal{F}^{\mathrm{sh}} \xrightarrow{\sim} \mathcal{G} \iff \begin{cases} \mathcal{G} \text{ is a sheaf.} \\ H_n(\mathcal{F})^{\mathrm{sh}} \xrightarrow{\sim} H_n(\mathcal{G})^{\mathrm{sh}}, \forall n \in \mathbb{Z}. \end{cases}$$

Definition 4.10.

Proposition 4.11.

5 The t-structure on valued sheaves

Definition 5.1. A t-structure on a stable ∞ -category \mathcal{C} is a pair $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ of full sub- ∞ -categories of \mathcal{C} that are stable under equivalences and satisfy:

(T1) The suspension functor Σ and the loop functor Ω restrict to $\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0}$ resp. are fully faithful functors $\Sigma : \mathcal{C}_{\geq 0} \rightarrow \mathcal{C}_{\geq 0}$ and $\Omega : \mathcal{C}_{\leq 0} \rightarrow \mathcal{C}_{\leq 0}$.

(T2) If $X \in \mathcal{C}_{\geq 0}$ and $Y \in \mathcal{C}_{\leq 0}$, then $\text{Map}(X, \Omega Y) \simeq *$.

(T3) For every $X \in \mathcal{C}$, there exists a fiber sequence

$$X' \longrightarrow X \longrightarrow X''$$

with $X' \in \mathcal{C}_{\geq 0}$ and $X'' \in \mathcal{C}_{\leq -1} := \Omega \mathcal{C}_{\leq 0}$.

We call $\mathcal{C}_{\geq 0}$ and $\mathcal{C}_{\leq 0}$ the connective and coconnective parts of the t-structure.

Given $n \in \mathbb{Z}$, we define $\mathcal{C}_{\geq n} := \Sigma^n \mathcal{C}_{\geq 0} \subset \mathcal{C}$ and $\mathcal{C}_{\leq n} := \Sigma^n \mathcal{C}_{\leq 0} \subset \mathcal{C}$, where for $n < 0$, we have $\Sigma^n = \Omega^{-n}$.

The inclusions $i : \mathcal{C}_{\geq m} \rightarrow \mathcal{C}$ and $s : \mathcal{C}_{\leq n} \rightarrow \mathcal{C}$ admit adjoint functors

$$\mathcal{C}_{\geq m} \begin{matrix} \xrightarrow{i} \\ \xleftarrow{r} \end{matrix} \mathcal{C} \begin{matrix} \xrightarrow{p} \\ \xleftarrow{s} \end{matrix} \mathcal{C}_{\leq n}.$$

In particular, the full sub- ∞ -category $\mathcal{C}_{\geq m} \subset \mathcal{C}$ is closed under colimits, and the full sub- ∞ -category $\mathcal{C}_{\leq n} \subset \mathcal{C}$ is closed under limits. From the adjoint pairs, we can form their counit and unit, and we get

$$\tau_{\geq 0} X = (i \circ r)(X) \xrightarrow{\epsilon} X \xrightarrow{\eta} \tau_{\leq -1} X = (s \circ p)(X).$$

The composition of the two maps is a point in the anima $\text{Map}(\tau_{\geq 0} X, \tau_{\leq -1} X) \simeq *$. So the composite map automatically admits a null-homotopy, which is unique, up to contractible ambiguity. We have the following commutative diagram:

$$\begin{array}{ccc} \mathcal{C}_{\leq m} \cap \mathcal{C}_{\geq n} & \begin{matrix} \xrightarrow{i} \\ \xleftarrow{r} \end{matrix} & \mathcal{C}_{\leq m} \\ p \uparrow \downarrow s & & p \uparrow \downarrow s \\ \mathcal{C}_{\geq n} & \begin{matrix} \xrightarrow{i} \\ \xleftarrow{r} \end{matrix} & \mathcal{C} \end{array}$$

The canonical map

$$p \circ r \xrightarrow{\eta \circ p \circ r} r \circ i \circ p \circ r \simeq r \circ p \circ i \circ r \xrightarrow{r \circ p \circ \epsilon} r \circ p$$

is an equivalence.

We say the full sub- ∞ -category

$$\mathcal{C}^\heartsuit := \mathcal{C}_{\geq 0} \cap \mathcal{C}_{\leq 0} \subset \mathcal{C}$$

is the heart of the t-structure. For the functor

$$\pi_0 := \tau_{\geq 0} \circ \tau_{\leq 0} \simeq \tau_{\leq 0} \circ \tau_{\geq 0} : \mathcal{C} \rightarrow \mathcal{C}^\heartsuit,$$

we call it the zeroth homotopy functor. The functor π_0 is additive, but is NOT exact. Instead, for all $n \in \mathbb{Z}$, we define

$$\pi_d : \mathcal{C} \longrightarrow \mathcal{C}^\heartsuit$$

to be $\pi_d = \pi_0 \circ \Omega^d$, and call it the d th homotopy functor. Now, a fiber sequence

$$Z \xrightarrow{g} Y \xrightarrow{f} X$$

in \mathcal{C} gives rise to a long exact sequence

$$\cdots \longrightarrow \pi_{d+1}(X) \longrightarrow \pi_d(Z) \longrightarrow \pi_d(Y) \longrightarrow \pi_d(X) \longrightarrow \cdots$$

in the heart \mathcal{C}^\heartsuit .

If $f : Y \rightarrow X$ is an equivalence, then $f : \pi_d(Y) \rightarrow \pi_d(X)$ is an isomorphism for all $d \in \mathbb{Z}$, but the opposite is generally not the case.

Now, for the stable ∞ -category $D(\mathbb{Z})$, we defined homology functors $H_d : D(\mathbb{Z}) \rightarrow \text{Mod}_{\mathbb{Z}}$ for all $d \in \mathbb{Z}$ by

$$H_d(X) \simeq \pi_0 \mathbf{Map}(\Sigma^d 1, X) \simeq \pi_0 \mathbf{Map}(1, \Omega^d X).$$

$D(\mathbb{Z})$ admits a t-structure $(D(\mathbb{Z})_{\geq 0}, D(\mathbb{Z})_{\leq 0})$, where the connective part $D(\mathbb{Z})_{\geq 0}$ is spanned by those X for which $H_d(X) \simeq 0$, for $d < 0$, and the coconnective part $D(\mathbb{Z})_{\leq 0}$ is spanned by those X for which $H_d(X) \simeq 0$, for $d > 0$. The zeroth homology functor

$$H_0 : D(\mathbb{Z})^\heartsuit \longrightarrow \mathbf{Mod}_{\mathbb{Z}}$$

is an equivalence of (abelian) categories. We have $H_d \simeq H_0 \circ \pi_d$, so the functors H_d and π_d encode the same information.

Proposition 5.2. Let $X \in \mathbf{Top}$, and let \mathcal{C} be a stable ∞ -category. A t-structure on \mathcal{C} induces a t-structure on the stable ∞ -category $\mathcal{P}(X, \mathcal{C})$ of \mathcal{C} -valued presheaves on X , where the coconnective part $\mathcal{P}(X, \mathcal{C})_{\leq 0} \simeq \mathcal{P}(X, \mathcal{C}_{\leq 0})$, and where the connective part $\mathcal{P}(X, \mathcal{C})_{\geq 0}$ is spanned by those \mathcal{F} such that

$$\mathrm{Map}(\mathcal{F}, \Omega \mathcal{G}) \simeq *$$

for all $\mathcal{G} \in \mathcal{P}(X, \mathcal{C}_{\leq 0})$.

A functor $f : \mathcal{D} \rightarrow \mathcal{C}$ between stable ∞ -categories is exact iff it is left exact iff it is right exact.

An exact functor $f : \mathcal{D} \rightarrow \mathcal{C}$ between stable ∞ -categories with t-structures is left t-exact if $f(\mathcal{D}_{\leq 0}) \subset \mathcal{D}_{\leq 0}$, and it is right t-exact if $f(\mathcal{D}_{\geq 0}) \subset \mathcal{D}_{\geq 0}$. It is t-exact if it is both left t-exact and right t-exact. If $f : \mathcal{D} \rightarrow \mathcal{C}$ admits right adjoint functor $g : \mathcal{C} \rightarrow \mathcal{D}$, then f is right t-exact iff g is left t-exact.

Theorem 5.3. Let $X \in \mathbf{Top}$ and \mathcal{C} a presentable stable ∞ -category.

- (1) The sheafification functor $\mathrm{ass}_X : \mathcal{P}(X, \mathcal{C}) \rightarrow \mathrm{Sh}(X, \mathcal{C})$ is t-exact, and the inclusion functor $\iota_X : \mathrm{Sh}(X, \mathcal{C}) \rightarrow \mathcal{P}(X, \mathcal{C})$ is left t-exact.
- (2) The composite functor

$$\mathrm{Sh}(X, \mathcal{C}^\heartsuit) \xrightarrow{\iota_X^\heartsuit} \mathcal{P}(X, \mathcal{C}^\heartsuit) \simeq \mathcal{P}(X, \mathcal{C})^\heartsuit \xrightarrow{\mathrm{ass}_X} \mathrm{Sh}(X, \mathcal{C})^\heartsuit$$

is an equivalence of categories.

Write π_0^p and π_0^s for the homotopy functors associated with the t-structure on presheaves and sheaves. Since ass_X is both exact and t-exact, we obtain a commutative square

$$\begin{array}{ccc} \mathcal{P}(X, \mathcal{C}) & \xrightarrow{\pi_0^p} & \mathcal{P}(X, \mathcal{C})^\heartsuit \\ \downarrow \text{ass}_X & & \downarrow \text{ass}_X \\ \text{Sh}(X, \mathcal{C}) & \xrightarrow{\pi_0^s} & \text{Sh}(X, \mathcal{C})^\heartsuit \end{array}$$

6 Sheaf

Lemma 6.1. If \mathcal{A} is bounded above, i.e. $\exists d \in \mathbb{Z}$, s.t. $H_n(\mathcal{A}) = 0$, for all $n > d$, then \mathcal{A}^{sh} is also bounded above.

Question. For finite sets X, X' with $X' \rightarrow X$ surjective and split, then

$$0 \rightarrow \mathbb{Z}[X] \rightarrow \mathbb{Z}[X'] \rightarrow \mathbb{Z}[X' \times_X X'] \rightarrow \mathbb{Z}[X' \times_X X' \times_X X'] \rightarrow \dots$$

is exact.

Lemma 6.2. Arbitrary limits and filtered colimits preserves $D(\mathbb{Z})_{\leq d}$.

Proof. First we show $D(\mathbb{Z})_{\leq d}$ is closed under filtered colimits. Assume $X_i \in D(\mathbb{Z})_{\leq d}$, $i \in I$, then

$$H_n(\varinjlim X_i) = \varinjlim H_n(X_i) = 0, \text{ for any } n > d.$$

Hence $\varinjlim X_i \in D(\mathbb{Z})_{\leq d}$. Then we show $D(\mathbb{Z})_{\leq d}$ is closed under arbitrary limits. Assume $X_i \in D(\mathbb{Z})_{\leq d}$, $n > d$, then

$$\begin{aligned} H_n(\lim X_i) &= [\Sigma^n 1, \lim X_i] \\ &= \pi_0 \mathbf{Map}(\Sigma^n 1, \lim X_i) \\ &= \pi_0 \lim \mathbf{Map}(\Sigma^n 1, X_i) \\ &= \pi_0 \lim * \\ &= \pi_0 * \\ &= 0. \end{aligned}$$

Hence, $\lim X_i \in D(\mathbb{Z})_{\leq d}$. □

Problem. What is the relation between $\pi_n(\lim X_i)$ and $\lim \pi_n(X_i)$. Similarly, the relation between $\pi_n(\text{colim } X_i)$ and $\text{colim } \pi_n(X_i)$.

Definition 6.3. We define the singular homology functor to be the composite of

$$\text{Top} \rightarrow \text{Cond}(\text{Set}) \hookrightarrow \text{Cond}(\text{An}) \rightarrow \text{An},$$

and denote it by $h : \text{Top} \rightarrow \text{An}$, where $\text{Top} \rightarrow \text{Cond}(\text{Set})$, $X \mapsto \underline{X}$; $\text{Cond}(\text{An}) \rightarrow \text{An}$ is the left adjoint of $\text{An} \hookrightarrow \text{Cond}(\text{An})$.

Definition 6.4. For the forgetful functor $D(\mathbb{Z})_{\geq 0} \simeq \text{Ani}(\text{Ab}) \rightarrow \text{Ani}(\text{Set}) \simeq \text{An}$, it has a left adjoint, and we denote it by

$$\mathbb{Z}[-] : \text{Ani}(\text{Set}) \rightarrow \text{Ani}(\text{Ab}); S \mapsto \mathbb{Z}[S].$$

Definition 6.5. For $X \in \text{Top}$, we define its singular homology object to be

$$\mathbb{Z}[h(X)] \in \text{Ani}(\text{Ab}) \simeq D(\mathbb{Z})_{\geq 0} \subset D(\mathbb{Z}).$$

Lemma 6.6. Assume $\mathcal{A} \in \text{Sh}(X, D(\mathbb{Z}))$, $H_n(\mathcal{A}) = 0, \forall n > d$, and $H_d(\mathcal{A}) \neq 0$, then $H_d(\mathcal{A})$ is a sheaf.

Proof. For $H_d(\mathcal{A}) \in \text{PSh}(X, \text{Ab})$, we need to check $H_d(\mathcal{A}) \in \text{Sh}(X, \text{Ab})$.

By denition, $H_d(\mathcal{A})(U) = H_d(\mathcal{A}(U)) = H_d(\varprojlim \mathcal{A}(V))$. By the Milnor's sequence, we have

$$0 \longrightarrow \varprojlim^1 H_{d+1}(\mathcal{A}(V)) \longrightarrow H_d(\varprojlim \mathcal{A}(V)) \longrightarrow \varprojlim H_d(\mathcal{A}(V)) \longrightarrow 0.$$

Because $H_{d+1}(\mathcal{A}) = 0$, so the left term of this short exact sequence is 0, hence

$$H_d(\mathcal{A})(U) = H_d(\varprojlim \mathcal{A}(V)) = \varprojlim H_d(\mathcal{A}(V)) = \varprojlim H_d(\mathcal{A})(V).$$

Hence, $H_d(\mathcal{A}) \in \text{Sh}(X, \text{Ab})$. □

Proposition 6.7. Let $\mathcal{C}_0 \subset \mathcal{C}$ be a full subcategory, then the following full subcategories of \mathcal{C} agree:

- the full subcategory generated under (small) colimits by \mathcal{C}_0 ;
- the full subcategory generated under filtered colimits and finite colimits by \mathcal{C}_0 ;
- the full subcategory generated under sifted colimits and finite products by \mathcal{C}_0 .

7 Animation

Theorem 7.1 (Yoneda). Let \mathcal{C} be an ∞ -category, the functor

$$\mathcal{C} \hookrightarrow \text{Fun}(\mathcal{C}^{\text{op}}, \mathbf{An}); X \mapsto (Y \mapsto \text{Hom}_{\mathcal{C}}(Y, X))$$

is fully faithful.

Remark 7.2. For S to be an anima, we mean S is an ∞ -category; while S to be a Kan complex, we mean S is a 1-category.

Let \mathcal{C} be a category which admits all small colimits.

Recall an object $X \in \mathcal{C}$ is compact (also called finitely presented) if $\text{Hom}(X, -)$ commutes with filtered colimits.

An object $X \in \mathcal{C}$ is projective if $\text{Hom}(X, -)$ commutes with reflexive coequalizers (coequalizers of parallel arrows $Y \rightrightarrows Z$ with a simultaneous section $Z \rightarrow Y$ of both maps). Taken together, an object $X \in \mathcal{C}$ is compact projective if $\text{Hom}(X, -)$ commutes with filtered colimits and reflexive coequalizers, equivalently, $\text{Hom}(X, -)$ commutes with 1-sifted colimits.

Let $\mathcal{C}^{\text{cp}} \subset \mathcal{C}$ be the full subcategory of compact projective objects. There is a fully faithful embedding $\text{sInd}(\mathcal{C}^{\text{cp}}) \longrightarrow \mathcal{C}$.

If \mathcal{C} is generated under small colimits by \mathcal{C}^{cp} , then the functor is an equivalence:

$$\text{sInd}(\mathcal{C}^{\text{cp}}) \cong \mathcal{C}.$$

If \mathcal{C}^{cp} is small, then

$$\text{sInd}(\mathcal{C}^{\text{cp}}) \subset \text{Fun}((\mathcal{C}^{\text{cp}})^{\text{op}}, \mathbf{Set})$$

is exactly the full subcategory of functors that take finite coproducts in \mathcal{C}^{cp} to products in \mathbf{Set} .

Example 7.3. (i) If $\mathcal{C} = \mathbf{Set}$, then $\mathcal{C}^{\text{cp}} = \mathbf{FinSet}$, which generates \mathcal{C} under small colimits.

(ii) If $\mathcal{C} = \mathbf{Ab}$, then $\mathcal{C}^{\text{cp}} = \mathbf{FinFreeAb}$, which generates \mathcal{C} under small colimits.

- (iii) If $\mathcal{C} = \text{Ring}$, then $\mathcal{C}^{\text{cp}} = \{\text{retracts of } \mathbb{Z}[X_1, \dots, X_n]\}$, which generates \mathcal{C} under small colimits.
- (iv) If $\mathcal{C} = \text{Cond}(\text{Set})$, then $\mathcal{C}^{\text{cp}} = \text{ExDisc}$, which generates \mathcal{C} under small colimits.
- (v) If $\mathcal{C} = \text{Cond}(\text{Ab})$, then $\mathcal{C}^{\text{cp}} = \{\text{direct summands of } \mathbb{Z}[S] \mid S \in \text{ExDisc}\}$, which generates \mathcal{C} under small colimits.
- (vi) $\mathcal{C} = \text{Cond}(\text{Ring})$, then $\mathcal{C}^{\text{cp}} = \{\text{retracts of } \mathbb{Z}[\mathbb{N}[S]] \mid S \in \text{ExDisc}\}$, which generates \mathcal{C} under small colimits.

Definition 7.4. Let \mathcal{C} be a category that admits all small colimits and \mathcal{C} is generated under small colimits by \mathcal{C}^{cp} . The animation of \mathcal{C} is the ∞ -category $\text{Ani}(\mathcal{C})$ freely generated under sifted colimits by \mathcal{C}^{cp} .

Example 7.5. If $\mathcal{C} = \text{Set}$, then $\text{Ani}(\mathcal{C}) = \text{Ani}(\text{Set}) \stackrel{\Delta}{=} \text{Ani}$ is the ∞ -category of animated sets, or anima in a short.

Any anima has a set of connected components, giving a functor $\pi_0 : \text{Ani} \rightarrow \text{Set}$, which has a fully faithful right adjoint $\text{Set} \hookrightarrow \text{Ani}$.

Given an anima A with a point $a \in A$ (meaning a map $a : * \rightarrow A$), one can define groups $\pi_i(A, a)$, for $i \geq 1$ and for $i \geq 2$, $\pi_i(A, a) \in \text{Ab}$.

An anima A is i -truncated if $\pi_j(A, a) = 0$, $\forall a \in A$ and $\forall j > i$. Then A is 0-truncated if and only if it is in the essential image of $\text{Set} \hookrightarrow \text{Ani}$.

The inclusion of i -truncated anima into all anima has a left adjoint $\tau_{\leq i}$. For all anima A , the natural map

$$A \xrightarrow{\sim} \lim_{\tau_{\leq i}} A$$

is an equivalence.

Picking any $a \in A$ and $i \geq 1$, the fiber of $\tau_{\leq i} A \rightarrow \tau_{\leq i-1} A$ over the image of a is an Eilenberg-MacLane anima $K(\pi_i(A, a), i)$. Here, an Eilenberg-MacLane anima $K(\pi, i)$ with $i \geq 1$ and π a group that is abelian if $i > 0$, is a pointed connected anima with $\pi_j = 0$ for $j \neq i$ and $\pi_i = \pi$.

In fact, the ∞ -category of pointed connected anima (A, a) with $\pi_j(A, a) = 0$ for $j \neq i$ is equivalent to Grp when $i = 1$, and to Ab when $i \geq 2$.

Remark 7.6. There are several ways to describe $\text{Ani}(\mathcal{C})$.

(i) $\text{Ani}(\mathcal{C})$ is the full sub- ∞ -category of objects in $\text{Fun}((\mathcal{C}^{\text{cp}})^{\text{op}}, \text{Ani})$ taking finite disjoint unions to finite products.

(ii) $\text{Ani}(\mathcal{C})$ is the ∞ -category obtained from $\text{Simp}(\mathcal{C})$ by inverting weak equivalences.

Definition 7.7. Let \mathcal{C} be an ∞ -category that admits all small colimits. For any uncountable strong limit cardinal κ , the ∞ -category $\text{Cond}_\kappa(\mathcal{C})$ of κ -condensed objects of \mathcal{C} is the category of contravariant functors from $\kappa\text{-ExDisc}$ to \mathcal{C} that take finite coproducts to finite products.

And we define

$$\text{Cond}(\mathcal{C}) := \bigcup_{\kappa} \text{Cond}_\kappa(\mathcal{C}).$$

Proposition 7.8. Let \mathcal{C} be a category that is generated under small colimits by \mathcal{C}^{cp} . Then $\text{Cond}(\mathcal{C})$ is still generated under small colimits by its compact projective objects, and there is a natural equivalence of ∞ -categories

$$\text{Cond}(\text{Ani}(\mathcal{C})) \cong \text{Ani}(\text{Cond}(\mathcal{C})).$$

Definition 7.9. Let \mathcal{C} be some site.

(i) A presheaf of anima is a functor

$$\mathcal{F} : \mathbf{N}(\mathcal{C}^{\text{op}}) \longrightarrow \text{Ani}.$$

(ii) A sheaf of anima is a presheaf of anima \mathcal{F} , s.t. for all coverings $\{f_i : X_i \rightarrow X\}_{i \in I}$, one has

$$\mathcal{F}(X) \xrightarrow{\sim} \lim(\prod_i \mathcal{F}(X_i) \rightrightarrows \prod_{i,j} \mathcal{F}(X_i \times_X X_j) \rightrightarrows \cdots).$$

(iii) A hypercomplete sheaf of anima is a sheaf of anima \mathcal{F} , s.t. for all hypercovers $X_\bullet \rightarrow X$, the map

$$\mathcal{F}(X) \xrightarrow{\sim} \lim \mathcal{F}(X_\bullet) = \lim (\mathcal{F}(X_0) \rightrightarrows \mathcal{F}(X_1) \rightrightarrows \cdots)$$

is an equivalence.

Definition 7.10. The ∞ -category of condensed anima is given by

- The ∞ -category of hypercomplete sheaves of anima on \mathbf{CHaus} .
- The ∞ -category of hypercomplete sheaves of anima on \mathbf{ProFin} .
- The ∞ -category of hypercomplete sheaves of anima on \mathbf{ExDisc} , i.e. of functors

$$\mathbf{ExDisc}^{\mathrm{op}} \longrightarrow \mathbf{Ani}$$

taking finite disjoint unions to finite products.

$$\begin{array}{ccc} \mathbf{CW} & \subset & \mathbf{Cond}(\mathbf{Set}) \\ \cap & & \cap \\ \mathbf{Ani} & \subset & \mathbf{Cond}(\mathbf{Ani}) \end{array}$$

Definition 7.11. $X \in \mathbf{Cond}(\mathbf{Ani})$ is

- discrete, if X in the essential image of \mathbf{Ani} .
- static, if X in the essential image of $\mathbf{Cond}(\mathbf{Set})$.

8 Condensed Cohomology

Definition 8.1. Let $X \in \text{Cond}$, $M \in \text{Cond}(\text{Ab})$, we define the global section of M on X to be

$$\Gamma_{\text{cond}}(X, M) := \text{Hom}_{\text{Cond}}(X, M) = \text{Hom}_{\text{Cond}(\text{Ab})}(\mathbb{Z}[X], M) \in \text{Ab},$$

and we define the condensed cohomology to be

$$R\Gamma_{\text{cond}}(X, M) := R\text{Hom}_{\text{Cond}(\text{Ab})}(\mathbb{Z}[X], M),$$

i.e.

$$H_{\text{cond}}^i(X, M) := \text{Ext}_{\text{Cond}(\text{Ab})}^i(\mathbb{Z}[X], M).$$

Lemma 8.2. For $X \in \text{ExDisc}$, the functor $\Gamma_{\text{cond}}(X, -) : \text{Cond}(\text{Ab}) \rightarrow \text{Ab}$ is exact, hence, for any $M \in \text{CondAb}$, $H_{\text{cond}}^i(X, M) = 0, \forall i \geq 1$.

Proof. We have $\Gamma_{\text{cond}}(X, -) = \text{Hom}_{\text{Cond}(\text{Ab})}(\mathbb{Z}[X], -)$, and for $X \in \text{ExDisc}$, $\mathbb{Z}[X]$ is projective, hence $\Gamma_{\text{cond}}(X, -) : \text{Cond}(\text{Ab}) \rightarrow \text{Ab}$ is exact. \square

Question. How to compute $H_{\text{cond}}^i(X, M)$?

From the definition, we need to find a projective resolution of $\mathbb{Z}[X]$.

For $X \in \text{CHaus}$, we pick a hypercover $X_{\bullet} \rightarrow X$, where each $X_i \in \text{ExDisc}$, for this hypercover, applying $\mathbb{Z}[-]$, then we get a projective resolution of $\mathbb{Z}[X]$:

$$\cdots \longrightarrow \mathbb{Z}[X_2] \longrightarrow \mathbb{Z}[X_1] \longrightarrow \mathbb{Z}[X_0] \longrightarrow \mathbb{Z}[X] \longrightarrow 0.$$

By definition, we have

$$\begin{aligned} H_{\text{cond}}^i(X, M) &= \text{Ext}_{\text{Cond}(\text{Ab})}^i(\mathbb{Z}[X], M) \\ &= H^i(0 \rightarrow \text{Hom}_{\text{Cond}(\text{Ab})}(\mathbb{Z}[X_0], M) \rightarrow \text{Hom}_{\text{Cond}(\text{Ab})}(\mathbb{Z}[X_1], M) \rightarrow \cdots) \\ &= H^i(0 \rightarrow \Gamma_{\text{cond}}(X_0, M) \rightarrow \Gamma_{\text{cond}}(X_1, M) \rightarrow \Gamma_{\text{cond}}(X_2, M) \rightarrow \cdots). \end{aligned}$$

Theorem 8.3 (Dyckhoff, 1976). For any $X \in \text{CHaus}$, there are natural isomorphisms:

$$H_{\text{cond}}^i(X, \mathbb{Z}) \cong H_{\text{sh}}^i(X, \mathbb{Z}), \forall i \geq 0.$$

Proof. 1) Assume $X \in \text{Fin}$, then

$$H_{\text{cond}}^i(X, \mathbb{Z}) = \begin{cases} \Gamma_{\text{cond}}(X, \mathbb{Z}) = C(X, \mathbb{Z}) & i = 0 \\ 0 & i > 0 \end{cases}$$

This comes from Lemma 8.2. On the other hand,

$$H_{\text{sh}}^i(X, \mathbb{Z}) = \check{H}^i(X, \mathbb{Z}) = \begin{cases} C(X, \mathbb{Z}) & i = 0 \\ 0 & i > 0 \end{cases}$$

This comes from by computing Čech cohomology. For a finite set X , take the cover $\mathcal{U} = \{x \rightarrow X\}_{x \in X}$, then $\mathcal{C}^0(\mathcal{U}, \mathbb{Z}) = \mathcal{C}^1(\mathcal{U}, \mathbb{Z}) = \cdots = \mathbb{Z}^X$, and because \mathcal{U} is a refinement of any cover, we have

$$\check{H}^i(X, \mathbb{Z}) = \check{H}^i(\mathcal{U}, \mathbb{Z}) = \begin{cases} \mathbb{Z}^X = C(X, \mathbb{Z}) & i = 0 \\ 0 & i > 0 \end{cases}$$

Therefore, for a finite set X , $H_{\text{cond}}^i(X, \mathbb{Z}) \cong H_{\text{sh}}^i(X, \mathbb{Z})$, $\forall i \geq 0$.

2) $X \in \text{ProFin}$, hence we can write $X = \varprojlim_j X^j$, $X^j \in \text{Fin}$.

$$H_{\text{sh}}^i(X, \mathbb{Z}) = \check{H}^i(X, \mathbb{Z}) = \varprojlim_j \check{H}^i(X^j, \mathbb{Z}) = \begin{cases} \varprojlim_j C(X^j, \mathbb{Z}) = C(X, \mathbb{Z}) & i = 0 \\ 0 & i > 0 \end{cases}$$

On the other hand, We compute $H_{\text{cond}}^i(X, \mathbb{Z})$, $i \geq 0$.

For $X \in \text{ProFin}$, pick a hypercover $X_\bullet \rightarrow X$ with each $X_i \in \text{ExDisc}$, and for each X^j , pick a finite hypercover $X_\bullet^j \rightarrow X^j$, s.t. $\varprojlim_j X_n^j = X_n$. Since X^j is finite, we have

$$H_{\text{cond}}^i(X^j, \mathbb{Z}) = \begin{cases} \Gamma(X^j, \mathbb{Z}) & i = 0 \\ 0 & i > 0 \end{cases}$$

And we know

$$H_{\text{cond}}^i(X^j, \mathbb{Z}) = H^i(0 \longrightarrow \Gamma(X_0^j, \mathbb{Z}) \longrightarrow \Gamma(X_1^j, \mathbb{Z}) \longrightarrow \Gamma(X_2^j, \mathbb{Z}) \longrightarrow \cdots),$$

hence we have an exact sequence:

$$0 \longrightarrow \Gamma(X^j, \mathbb{Z}) \longrightarrow \Gamma(X_0^j, \mathbb{Z}) \longrightarrow \Gamma(X_1^j, \mathbb{Z}) \longrightarrow \Gamma(X_2^j, \mathbb{Z}) \longrightarrow \cdots$$

Applying the exact functor \varinjlim_j to this exact sequence, we get an exact sequence:

$$0 \longrightarrow \varinjlim_j \Gamma(X^j, \mathbb{Z}) \longrightarrow \varinjlim_j \Gamma(X_0^j, \mathbb{Z}) \longrightarrow \varinjlim_j \Gamma(X_1^j, \mathbb{Z}) \longrightarrow \varinjlim_j \Gamma(X_2^j, \mathbb{Z}) \longrightarrow \cdots,$$

i.e.

$$0 \longrightarrow \Gamma(X, \mathbb{Z}) \longrightarrow \Gamma(X_0, \mathbb{Z}) \longrightarrow \Gamma(X_1, \mathbb{Z}) \longrightarrow \Gamma(X_2, \mathbb{Z}) \longrightarrow \cdots.$$

Hence,

$$H_{\text{cond}}^i(X, \mathbb{Z}) = \begin{cases} \Gamma(X, \mathbb{Z}) & i = 0 \\ 0 & i > 0 \end{cases}$$

3) $X \in \mathbf{CHaus}$.

Consider a morphism of topoi $(\alpha^{-1}, \alpha_*) : \mathbf{Sh}(\mathbf{CHaus}/X) \rightarrow \mathbf{Sh}(X)$. For $\mathcal{F} \in \mathbf{Sh}(\mathbf{CHaus}/X)$, $\alpha_* \mathcal{F}$ is given by

$$U \mapsto \varprojlim_{V \subset U, V \text{ is closed in } X} \mathcal{F}(V \hookrightarrow S).$$

We have the following diagram:

$$\begin{array}{ccc} \mathbf{Sh}(\mathbf{CHaus}/X) & \xrightarrow{\alpha_*} & \mathbf{Sh}(X) \\ & \searrow \Gamma_{\text{cond}}(X, -) & \swarrow \Gamma_{\text{sh}}(X, -) \\ & \mathbf{Set} & \end{array}$$

This is because $\forall Y \in \mathbf{Sh}(\mathbf{CHaus}/X)$,

$$\begin{aligned} \Gamma_{\text{sh}}(X, \alpha_* Y) &= \alpha_* Y(X) = \varprojlim_{V \subset U, V \text{ is closed in } X} Y(V) \\ &= \varprojlim_V \text{Hom}_{\text{cond}}(V, Y) = \text{Hom}_{\text{cond}}(\varinjlim_V V, Y) \\ &= \text{Hom}_{\text{cond}}(X, Y) = \Gamma_{\text{cond}}(X, Y). \end{aligned}$$

And this diagram can induce a diagram:

$$\begin{array}{ccc} D(\mathbf{Ab}(\mathbf{CHaus}/X)) & \xrightarrow{R\alpha_*} & D(\mathbf{Ab}(X)) \\ & \searrow R\Gamma_{\text{cond}}(X, -) & \swarrow R\Gamma_{\text{sh}}(X, -) \\ & D(\mathbf{Ab}) & \end{array}$$

Claim: $R\alpha_*\mathbb{Z} \cong \mathbb{Z}$ in $D(\text{Ab}(X))$.

With this claim, we can show

$$\begin{aligned}
H_{\text{cond}}^i(X, \mathbb{Z}) &= H^i(R\Gamma_{\text{cond}}(X, \mathbb{Z})) \\
&= H^i(R\Gamma_{\text{sh}}(X, -) \circ R\alpha_*\mathbb{Z}) \\
&= H^i(R\Gamma_{\text{sh}}(X, \mathbb{Z})) \\
&= H_{\text{sh}}^i(X, \mathbb{Z}).
\end{aligned}$$

Hence, it suffices to show this claim. We have a map $\mathbb{Z} \rightarrow R\alpha_*\mathbb{Z}$ in $D(\text{Ab}(X))$. In order to show this is an isomorphism, it suffices to check on each stacks.

Fix $s \in S$,

$$\begin{aligned}
(R\alpha_*\mathbb{Z})_s &= \varinjlim_{s \in U \text{ open}} R\Gamma(U, R\alpha_*\mathbb{Z}) \\
&= \varinjlim_{s \in U \text{ open}} R\Gamma_{\text{cond}}(U, \mathbb{Z}) \\
&= \varinjlim_{s \in V \text{ closed}} R\Gamma_{\text{cond}}(V, \mathbb{Z}).
\end{aligned}$$

Pick a hypercover $S_\bullet \rightarrow S$ with $S_i \in \text{ExDisc}$. Then for each closed V , $(S_n \times_X V)_{n \geq 0} \rightarrow V$ is a hypercover. Hence,

$$R\Gamma_{\text{cond}}(V, \mathbb{Z}) \cong (0 \longrightarrow \Gamma(S_0 \times_X V, \mathbb{Z}) \longrightarrow \Gamma(S_1 \times_X V, \mathbb{Z}) \longrightarrow \cdots).$$

Thus, we have

$$\begin{aligned}
(R\alpha_*\mathbb{Z})_s &= \varinjlim_{s \in V \text{ closed}} R\Gamma_{\text{cond}}(V, \mathbb{Z}) \\
&\cong \varinjlim_{s \in V \text{ closed}} (0 \longrightarrow \Gamma(S_0 \times_X V, \mathbb{Z}) \longrightarrow \Gamma(S_1 \times_X V, \mathbb{Z}) \longrightarrow \cdots) \\
&\cong (0 \longrightarrow \varinjlim_{s \in V \text{ closed}} \Gamma(S_0 \times_X V, \mathbb{Z}) \longrightarrow \varinjlim_{s \in V \text{ closed}} \Gamma(S_1 \times_X V, \mathbb{Z}) \longrightarrow \cdots) \\
&\cong (0 \longrightarrow \Gamma(S_0 \times_X \{s\}, \mathbb{Z}) \longrightarrow \Gamma(S_1 \times_X \{s\}, \mathbb{Z}) \longrightarrow \cdots) \\
&\cong R\Gamma_{\text{cond}}(\{s\}, \mathbb{Z}) \\
&\cong \mathbb{Z},
\end{aligned}$$

which finishes our proof. □

Example 8.4. Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, for $\mathbb{T}^I \in \text{CHaus}$, we have $H^n(\mathbb{T}^I, \mathbb{Z}) = \wedge^n(\mathbb{Z}^{\oplus I})$.

Proof. First, we have

$$H^n(\mathbb{T}, \mathbb{Z}) = \begin{cases} \mathbb{Z} & n = 0, 1 \\ 0 & \text{else} \end{cases}$$

i.e. $H^*(\mathbb{T}, \mathbb{Z}) = \wedge(\mathbb{Z})$.

Claim: $H^*(\mathbb{T}^n, \mathbb{Z}) = \wedge(\mathbb{Z}^{\oplus n})$.

We can prove it by induction on n . $n = 1$ is proved above.

By Kunneth theorem, we can show that for $H^*(X, \mathbb{Z})$ finitely generated free in each degree, we have $H^*(X \times Y, \mathbb{Z}) \cong H^*(X, \mathbb{Z}) \otimes H^*(Y, \mathbb{Z})$. Hence, we have

$$\begin{aligned} H^*(\mathbb{T}^n, \mathbb{Z}) &= H^*(\mathbb{T}^{n-1}, \mathbb{Z}) \otimes H^*(\mathbb{T}, \mathbb{Z}) \\ &= \wedge(\mathbb{Z}^{\oplus(n-1)}) \otimes \wedge(\mathbb{Z}) \\ &= \wedge(\mathbb{Z}^{\oplus n}). \end{aligned}$$

In order to prove the general case, there is a fact that for $S \in \text{CHaus}$, $S = \varprojlim_j S_j$, then

$$H^n(S, \mathbb{Z}) = \varinjlim_j H^n(S_j, \mathbb{Z}).$$

Hence,

$$\begin{aligned} H^n(\mathbb{T}^I, \mathbb{Z}) &= H^n(\varprojlim_{J \subset I \text{ finite}} \mathbb{T}^J, \mathbb{Z}) \\ &= \varinjlim_{J \subset I \text{ finite}} H^n(\mathbb{T}^J, \mathbb{Z}) \\ &= \varinjlim_{J \subset I \text{ finite}} \wedge^n(\mathbb{Z}^{\oplus J}) \\ &= \wedge^n(\mathbb{Z}^{\oplus I}). \end{aligned}$$

□

9 Locally compact abelian groups

Notation. Let TopAb be the category of all Hausdorff topological abelian groups and LCAb be the category of all locally compact abelian groups.

Proposition 9.1. Let $A, B \in \text{TopAb}$ and assume that $A \in \text{CGTop}$. Then there is a natural isomorphism of condensed abelian groups

$$\underline{\text{Hom}}(\underline{A}, \underline{B}) \cong \underline{\text{Hom}}(A, B).$$

Theorem 9.2 (Eilenberg-MacLane, Breen, Deligne resolution). For any abelian group A , there is a functorial resolution

$$\cdots \longrightarrow \bigoplus_{j=1}^{n_i} \mathbb{Z}[A^{r_{i,j}}] \longrightarrow \cdots \longrightarrow \mathbb{Z}[A^3] \oplus \mathbb{Z}[A^2] \longrightarrow \mathbb{Z}[A^2] \longrightarrow \mathbb{Z}[A] \longrightarrow A \rightarrow 0.$$

Remark 9.3. Such functorial ensures that it works for abelian group objects in any topos.

Lemma 9.4. Let $A^{\bullet,\bullet}$ be a double complex and $A^\bullet = \text{Tot}(A^{\bullet,\bullet})$ be its total complex, then there is a spectral sequence

$$E_1^{p,q} = H^q(A^{\bullet,p}) \implies H^{p+q}(A^\bullet).$$

Lemma 9.5. For a complex of abelian groups $M^\bullet \in D(\mathbb{Z})$, let

$$0 \longrightarrow M^\bullet \longrightarrow A^{\bullet,1} \longrightarrow A^{\bullet,2} \longrightarrow A^{\bullet,3} \longrightarrow \cdots$$

be an exact sequence in $D(\mathbb{Z})$, then for the double complex $A^{\bullet,\bullet}$, there is a quasi-isomorphism

$$M^\bullet \xrightarrow{\sim} \text{Tot}(A^{\bullet,\bullet}).$$

Corollary 9.6. For any condensed abelian groups A, M and an extremally disconnected space S , there is a spectral sequence

$$E_1^{p,q} = \prod_{j=1}^{n_p} H^q(A^{r_{p,j}} \times S, M) \implies \underline{\text{Ext}}^{p+q}(A, M)(S),$$

that is functorial in A, M and S .

Proof. For $A \in \text{Cond}(\text{Ab})$, consider its EMBD resolution

$$\cdots \longrightarrow \bigoplus_{j=1}^{n_i} \mathbb{Z}[A^{r_{i,j}}] \longrightarrow \cdots \longrightarrow \mathbb{Z}[A^3] \oplus \mathbb{Z}[A^2] \longrightarrow \mathbb{Z}[A^2] \longrightarrow \mathbb{Z}[A] \longrightarrow A \rightarrow 0,$$

then apply $-\otimes \mathbb{Z}[S]$, which is an exact functor since $\mathbb{Z}[S]$ is flat, we get the resolution of $A \otimes \mathbb{Z}[S]$

$$\cdots \longrightarrow \bigoplus_{j=1}^{n_i} \mathbb{Z}[A^{r_{i,j}} \times S] \cdots \longrightarrow \mathbb{Z}[A^3 \times S] \oplus \mathbb{Z}[A^2 \times S] \longrightarrow \mathbb{Z}[A^2 \times S] \longrightarrow \mathbb{Z}[A \times S] \longrightarrow A \otimes \mathbb{Z}[S] \rightarrow 0,$$

then apply $R\text{Hom}(-, M)$, we get

$$0 \longrightarrow R\text{Hom}(A \otimes \mathbb{Z}[S], M) \longrightarrow R\text{Hom}(\mathbb{Z}[A \times S], M) \longrightarrow R\text{Hom}(\mathbb{Z}[A^2 \times S], M) \longrightarrow \cdots ,$$

i.e.

$$0 \longrightarrow \underline{R\text{Hom}}(A, M)(S) \longrightarrow R\Gamma(A \times S, M) \longrightarrow R\Gamma(A^2 \times S, M) \longrightarrow \cdots ,$$

which is an exact sequence in $D(\mathbb{Z})$. By lemma 9.4 and lemma 9.5, we have

$$E_1^{p,q} = H^q\left(\bigoplus_{j=1}^{n_p} R\Gamma(A^{r_{p,j}} \times S, M)\right) \implies H^{p+q}\left(\text{Tot}\left(\bigoplus_{j=1}^{n_\bullet} R\Gamma(A^{r_{\bullet,j}} \times S, M)\right)\right)$$

and

$$\underline{R\text{Hom}}(A, M)(S) \simeq \text{Tot}\left(\bigoplus_{j=1}^{n_\bullet} R\Gamma(A^{r_{\bullet,j}} \times S, M)\right),$$

hence

$$E_1^{p,q} = \prod_{j=1}^{n_p} H^q(A^{r_{p,j}} \times S, M) \implies \underline{\text{Ext}}^{p+q}(A, M)(S).$$

□

Lemma 9.7. In the category of abelian groups, if the following diagram is exact for each arrow

$$\begin{array}{ccccccc} 0 & \longrightarrow & M^\bullet & \longrightarrow & A^{\bullet,1} & \longrightarrow & A^{\bullet,2} \longrightarrow \cdots, \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & N^\bullet & \longrightarrow & B^{\bullet,1} & \longrightarrow & B^{\bullet,2} \longrightarrow \cdots \end{array}$$

and if for any $j \geq 1$, we have $A^{\bullet,j} \cong B^{\bullet,j}$, then $\text{Tot}(A^{\bullet,\bullet}) \cong \text{Tot}(B^{\bullet,\bullet})$. Furthermore, by

$M^\bullet \cong \text{Tot}(A^{\bullet,\bullet})$ and $N^\bullet \cong \text{Tot}(B^{\bullet,\bullet})$, we can get $M^\bullet \cong N^\bullet$.

Theorem 9.8. Assume I is any set, denote the compact condensed abelian group $\prod_I \mathbb{T}$ by \mathbb{T}^I .

(i) For any discrete abelian group M , we have

$$R\text{Hom}(\mathbb{T}^I, M) = M^{\oplus I}[-1],$$

where $M^{\oplus I}[-1] \rightarrow R\text{Hom}(\mathbb{T}^I, M)$ is induced by

$$M[-1] = R\text{Hom}(\mathbb{Z}[1], M) \longrightarrow R\text{Hom}(\mathbb{T}, M) \xrightarrow{p_i^*} R\text{Hom}(\mathbb{T}^I, M),$$

where $p_i : \mathbb{T}^I \rightarrow \mathbb{T}$ is the projection to the i -th factor, $i \in I$.

(ii) $R\text{Hom}(\mathbb{T}^I, \mathbb{R}) = 0$.

Proof.

(i) We first prove the case I is a one element set, i.e.

$$R\text{Hom}(\mathbb{T}, M) = M[-1].$$

From the exact sequence $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{R} \rightarrow \mathbb{T} \rightarrow 0$, we have $\mathbb{R} \rightarrow \mathbb{T} \rightarrow \mathbb{Z}[1]$, hence

$$M[-1] = R\text{Hom}(\mathbb{Z}[1], M) \longrightarrow R\text{Hom}(\mathbb{T}, M) \longrightarrow R\text{Hom}(\mathbb{R}, M).$$

In order to show $R\text{Hom}(\mathbb{T}, M) = M[-1]$, it suffices to show $R\text{Hom}(\mathbb{R}, M) = 0$.

Claim: $R\text{Hom}(\mathbb{R}, M) = 0$.

For 0 and \mathbb{R} , we take its EMBD resolution:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \bigoplus_{j=1}^{n_i} \mathbb{Z}[\mathbb{R}^{r_{i,j}}] & \longrightarrow & \cdots & \longrightarrow & \mathbb{Z}[\mathbb{R}] \longrightarrow \mathbb{R} \longrightarrow 0 \\ & & \downarrow & & & & \downarrow \\ \cdots & \longrightarrow & \bigoplus_{j=1}^{n_i} \mathbb{Z}[0^{r_{i,j}}] & \longrightarrow & \cdots & \longrightarrow & \mathbb{Z}[0] \longrightarrow 0 \longrightarrow 0, \end{array}$$

apply $R\mathbf{Hom}(-, M)(S)$, we get

$$\begin{array}{ccccccc} 0 & \longrightarrow & R\mathbf{Hom}(0, M)(S) & \longrightarrow & R\mathbf{Hom}(\mathbb{Z}[0], M)(S) & \longrightarrow & \cdots \longrightarrow R\mathbf{Hom}(\bigoplus_{j=1}^{n_i} \mathbb{Z}[0^{r_{i,j}}], M)(S) \cdots \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & R\mathbf{Hom}(\mathbb{R}, M)(S) & \longrightarrow & R\mathbf{Hom}(\mathbb{Z}[\mathbb{R}], M)(S) & \longrightarrow & \cdots \longrightarrow R\mathbf{Hom}(\bigoplus_{j=1}^{n_i} \mathbb{Z}[\mathbb{R}^{r_{i,j}}], M)(S) \cdots, \end{array}$$

i.e.

$$\begin{array}{ccccccc} 0 & \longrightarrow & 0 & \longrightarrow & R\Gamma(S, M) & \longrightarrow & \cdots \longrightarrow \bigoplus_{j=1}^{n_i} R\Gamma(S, M) \cdots \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & R\mathbf{Hom}(\mathbb{R}, M)(S) & \longrightarrow & R\Gamma(\mathbb{R} \times S, M) & \longrightarrow & \cdots \longrightarrow \bigoplus_{j=1}^{n_i} R\Gamma(\mathbb{R}^{r_{i,j}} \times S, M) \cdots, \end{array}$$

Then by lemma 9.7, in order to show $R\mathbf{Hom}(\mathbb{R}, M) = 0$, it suffices to show

$$R\Gamma(S, M) = R\Gamma(S \times \mathbb{R}^r, M).$$

We know $S \times \mathbb{R}^r = \varinjlim S \times [-N, N]^r$, then

$$\begin{aligned} R\Gamma(S \times \mathbb{R}^r, M) &= R\Gamma(\varinjlim S \times [-N, N]^r, M) \\ &= \varprojlim R\Gamma(S \times [-N, N]^r, M) \\ &= \varprojlim R\Gamma(S, M) \\ &= R\Gamma(S, M). \end{aligned}$$

Here, $\varprojlim R\Gamma(S \times [-N, N]^r, M) = \varprojlim R\Gamma(S, M)$ comes from the fact that for constant sheaf, its sheaf cohomology is homotopy-invariant.

Secondly, assume I is a finite set, then

$$R\mathbf{Hom}(\mathbb{T}^I, M) = R\mathbf{Hom}(\mathbb{T}^{\oplus I}, M) = \prod_I R\mathbf{Hom}(\mathbb{T}, M) = \prod_I M[-1] = M^{\oplus I}[-1].$$

Finally, assume I is any set. Then we can write \mathbb{T}^I as

$$\mathbb{T}^I = \varprojlim_{J \subset I, J \text{ finite}} \mathbb{T}^J.$$

For any finite set J , we have

$$\begin{array}{ccccccc}
0 \longrightarrow & R\mathbf{Hom}(\mathbb{T}^J, M)(S) & \longrightarrow & R\Gamma(\mathbb{T}^J \times S, M) & \longrightarrow & \cdots & \longrightarrow \bigoplus_{j=1}^{n_i} R\Gamma((\mathbb{T}^J)^{r_{i,j}} \times S, M) \cdots \\
& \downarrow & & \downarrow & & & \downarrow \\
0 \longrightarrow & R\mathbf{Hom}(\mathbb{T}^I, M)(S) & \longrightarrow & R\Gamma(\mathbb{T}^I \times S, M) & \longrightarrow & \cdots & \longrightarrow \bigoplus_{j=1}^{n_i} R\Gamma((\mathbb{T}^I)^{r_{i,j}} \times S, M) \cdots,
\end{array}$$

apply the exact functor $\varinjlim_{J \subset I}$ to the first arrow, we get

$$\begin{array}{ccccccc}
0 \longrightarrow & \varinjlim_{J \subset I} R\mathbf{Hom}(\mathbb{T}^J, M)(S) & \longrightarrow & \varinjlim_{J \subset I} R\Gamma(\mathbb{T}^J \times S, M) & \longrightarrow & \cdots & \longrightarrow \bigoplus_{j=1}^{n_i} \varinjlim_{J \subset I} R\Gamma((\mathbb{T}^J)^{r_{i,j}} \times S, M) \cdots \\
& \downarrow & & \downarrow & & & \downarrow \\
0 \longrightarrow & R\mathbf{Hom}(\mathbb{T}^I, M)(S) & \longrightarrow & R\Gamma(\mathbb{T}^I \times S, M) & \longrightarrow & \cdots & \longrightarrow \bigoplus_{j=1}^{n_i} R\Gamma((\mathbb{T}^I)^{r_{i,j}} \times S, M) \cdots,
\end{array}$$

In order to show

$$\varinjlim_{J \subset I} R\mathbf{Hom}(\mathbb{T}^J, M)(S) \cong R\mathbf{Hom}(\mathbb{T}^I, M)(S),$$

it suffices to show

$$\varinjlim_{J \subset I} R\Gamma((\mathbb{T}^J)^{r_{i,j}} \times S, M) \cong R\Gamma((\mathbb{T}^I)^{r_{i,j}} \times S, M).$$

This is true, because $\varprojlim_{J \subset I} (\mathbb{T}^J)^{r_{i,j}} \times S \cong (\mathbb{T}^I)^{r_{i,j}} \times S$. Therefore,

$$\begin{aligned}
R\mathbf{Hom}(\mathbb{T}^I, M) &\cong \varinjlim_{J \subset I} R\mathbf{Hom}(\mathbb{T}^J, M) \\
&\cong \varinjlim_{J \subset I} M^{\oplus J}[-1] \\
&\cong M^{\oplus I}[-1].
\end{aligned}$$

□

Corollary 9.9. $R\mathbf{Hom}(\mathbb{R}, \mathbb{R}) \cong \mathbb{R}$.

Proof. From the exact sequence $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{R} \rightarrow \mathbb{T} \rightarrow 0$, we have

$$R\mathbf{Hom}(\mathbb{T}, \mathbb{R}) \rightarrow R\mathbf{Hom}(\mathbb{R}, \mathbb{R}) \rightarrow R\mathbf{Hom}(\mathbb{Z}, \mathbb{R}).$$

By Theorem 9.8, we know $R\mathbf{Hom}(\mathbb{T}, \mathbb{R}) = 0$, hence $R\mathbf{Hom}(\mathbb{R}, \mathbb{R}) \cong R\mathbf{Hom}(\mathbb{Z}, \mathbb{R}) \cong$

\mathbb{R} .

□

Corollary 9.10. For any locally compact abelian groups A and B , $R\text{Hom}(A, B)$ is centered at 0 and 1, i.e. $\underline{\text{Ext}}^i(A, B) = 0$, $\forall i \geq 2$.

Proof. By the structure theorem of locally compact abelian groups, it suffices to prove for A and B being compact groups and discrete groups.

(i) A is a discrete group.

Claim: There is an exact sequence: $0 \rightarrow \oplus_I \mathbb{Z} \rightarrow \oplus_J \mathbb{Z} \rightarrow A \rightarrow 0$.

This is because we can construct a surjective homomorphism $\oplus_A \mathbb{Z} \rightarrow A$, and take its kernel, and we know the submodule of a free \mathbb{Z} -module is free, hence $\ker(\oplus_A \mathbb{Z} \rightarrow A) = \oplus_I \mathbb{Z}$, for some I . Thereby, $0 \rightarrow \oplus_I \mathbb{Z} \rightarrow \oplus_A \mathbb{Z} \rightarrow A \rightarrow 0$ is exact.

By the short exact sequence $0 \rightarrow \oplus_I \mathbb{Z} \rightarrow \oplus_J \mathbb{Z} \rightarrow A \rightarrow 0$, we can get a long exact sequence:

$$\begin{aligned} 0 &\longrightarrow \underline{\text{Hom}}(A, B) \longrightarrow \underline{\text{Hom}}(\oplus_J \mathbb{Z}, B) \longrightarrow \underline{\text{Hom}}(\oplus_I \mathbb{Z}, B) \\ &\longrightarrow \underline{\text{Ext}}^1(A, B) \longrightarrow \underline{\text{Ext}}^1(\oplus_J \mathbb{Z}, B) \longrightarrow \underline{\text{Ext}}^1(\oplus_I \mathbb{Z}, B) \\ &\longrightarrow \underline{\text{Ext}}^2(A, B) \longrightarrow \cdots \end{aligned}$$

Because $\oplus_I \mathbb{Z} \in \text{Cond}(\text{Ab})$ is projective, we have $\underline{\text{Ext}}^i(\oplus_I \mathbb{Z}, B) = 0$, $\forall i \geq 1$. Hence $\underline{\text{Ext}}^i(A, B) = 0$, $\forall i \geq 2$.

(ii) A is a compact group.

By Pontrgagin duality, there is a short exact sequence

$$0 \rightarrow A \rightarrow \mathbb{T}^I \rightarrow \mathbb{T}^J \rightarrow 0,$$

and it can induce a long exact sequence

$$\begin{aligned} 0 &\longrightarrow \underline{\text{Hom}}(\mathbb{T}^J, B) \longrightarrow \underline{\text{Hom}}(\mathbb{T}^I, B) \longrightarrow \underline{\text{Hom}}(A, B) \\ &\longrightarrow \underline{\text{Ext}}^1(\mathbb{T}^J, B) \longrightarrow \underline{\text{Ext}}^1(\mathbb{T}^I, B) \longrightarrow \underline{\text{Ext}}^1(A, B) \\ &\longrightarrow \underline{\text{Ext}}^2(\mathbb{T}^J, B) \longrightarrow \underline{\text{Ext}}^2(\mathbb{T}^I, B) \longrightarrow \underline{\text{Ext}}^2(A, B) \\ &\longrightarrow \cdots \end{aligned}$$

In order to show $\underline{\text{Ext}}^i(A, B) = 0, \forall i \geq 2$, it suffices to show

$$\underline{\text{Ext}}^i(\mathbb{T}^I, B) = 0, \forall i \geq 2, \forall I.$$

(a) B is a discrete group.

In this case, we have $R\underline{\text{Hom}}(\mathbb{T}^I, B) = B^{\oplus I}[-1]$, which is centered at 1, hence $\underline{\text{Ext}}^i(\mathbb{T}^I, B) = 0, \forall i \geq 2, \forall I$.

(b) B is a compact group.

In this case, we have a short exact sequence $0 \rightarrow B \rightarrow \mathbb{T}^{I'} \rightarrow \mathbb{T}^{J'} \rightarrow 0$, and it induces a long exact sequence:

$$\begin{aligned} 0 &\longrightarrow \underline{\text{Hom}}(\mathbb{T}^I, B) \longrightarrow \underline{\text{Hom}}(\mathbb{T}^I, \mathbb{T}^{I'}) \longrightarrow \underline{\text{Hom}}(\mathbb{T}^I, \mathbb{T}^{J'}) \\ &\longrightarrow \underline{\text{Ext}}^1(\mathbb{T}^I, B) \longrightarrow \underline{\text{Ext}}^1(\mathbb{T}^I, \mathbb{T}^{I'}) \longrightarrow \underline{\text{Ext}}^1(\mathbb{T}^I, \mathbb{T}^{J'}) \\ &\longrightarrow \underline{\text{Ext}}^2(\mathbb{T}^I, B) \longrightarrow \cdots \end{aligned}$$

Now, we compute $\underline{\text{Ext}}^i(\mathbb{T}^I, \mathbb{T})$. For the short exact sequence $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{R} \rightarrow \mathbb{T} \rightarrow 0$, we have a long exact sequence:

$$\begin{aligned} 0 &\longrightarrow \underline{\text{Hom}}(\mathbb{T}^I, \mathbb{Z}) \longrightarrow \underline{\text{Hom}}(\mathbb{T}^I, \mathbb{R}) \longrightarrow \underline{\text{Hom}}(\mathbb{T}^I, \mathbb{T}) \\ &\longrightarrow \underline{\text{Ext}}^1(\mathbb{T}^I, \mathbb{Z}) \longrightarrow \underline{\text{Ext}}^1(\mathbb{T}^I, \mathbb{R}) \longrightarrow \underline{\text{Ext}}^1(\mathbb{T}^I, \mathbb{T}) \\ &\longrightarrow \underline{\text{Ext}}^2(\mathbb{T}^I, \mathbb{Z}) \longrightarrow \cdots \end{aligned}$$

Since $R\underline{\text{Hom}}(\mathbb{T}^I, \mathbb{R}) = 0$ and $R\underline{\text{Hom}}(\mathbb{T}^I, \mathbb{Z}) = \mathbb{Z}^{\oplus I}[-1]$, we have $\underline{\text{Ext}}^i(\mathbb{T}^I, \mathbb{T}) = 0, \forall i \geq 1$, hence $\underline{\text{Ext}}^i(\mathbb{T}^I, \mathbb{T}^J) = 0, \forall i \geq 1, \forall J$. Thus $\underline{\text{Ext}}^i(\mathbb{T}^I, B) = 0, \forall i \geq 2$.

□

Appendix: Resolutions

Definition 9.11. Let \mathcal{A} be a Grothendieck abelian category and $X \in D(\mathcal{A})$ is a complex.

(i) Let $n \in \mathbb{Z}_{\geq 0}$. X is n -pseudocoherent if

- (a) X is bounded above, i.e. for $i \gg 0$, $H^i(X) = 0$.
- (b) For $i = 0, 1, \dots, n-1$, $\text{Ext}^i(X, -) : \mathcal{A} \rightarrow \text{Ab}$ commutes with filtered colimits.
- (c) $\text{Ext}^n(X, -) : \mathcal{A} \rightarrow \text{Ab}$ commutes with filtered union.

(ii) X is pseudocoherent if

- (a) X is bounded above, i.e. for $i \gg 0$, $H^i(X) = 0$.
- (b) For any $i \geq 0$, $\text{Ext}^i(X, -) : \mathcal{A} \rightarrow \text{Ab}$ commutes with filtered colimits.

X is pseudocoherent iff $\forall n$, X is n -pseudocoherent.

(iii) Let $n \in \mathbb{Z}_{\geq 0}$ and \mathcal{A}_0 is a family of compact projective objects of \mathcal{A} . X is n, \mathcal{A}_0 -pseudocoherent if:

there exists a bounded cochain complex P^\bullet , and a map $\varphi : P^\bullet \rightarrow X$, s.t. each P^i is the finite direct sum of elements of \mathcal{A}_0 , and if $i > -n$, $H^i(\varphi)$ is an isomorphism; if $i = -n$, $H^i(\varphi)$ is a surjection.

(iv) Let \mathcal{A}_0 be a family of compact projective objects of \mathcal{A} . X is \mathcal{A}_0 -pseudocoherent if: there exists a bounded cochain complex P^\bullet , and a map $\varphi : P^\bullet \rightarrow X$, which is a quasi-isomorphism, s.t. each P^i is the finite direct sum of elements of \mathcal{A}_0 .

X is \mathcal{A}_0 -pseudocoherent iff $\forall n$, X is n, \mathcal{A}_0 -pseudocoherent.

Definition 9.12. Let \mathcal{A} be a Grothendieck abelian category and $A \in \mathcal{A}$. A is finitely generated if A is 0-pseudocoherent.

Lemma 9.13. Let \mathcal{A} be a Grothendieck abelian category and $A \in \mathcal{A}$. Consider:

- (i) A is finitely generated.
- (ii) A is the quotient of a compact object.
- (iii) If there is a surjection $\bigoplus_{i \in I} B_i \twoheadrightarrow A$, then there exists a finite subset $J \subseteq I$, s.t. $\bigoplus_{i \in J} B_i \twoheadrightarrow A$.

(iv) Let B be the filtered colimit of B_i , then $\varinjlim \text{Hom}(A, B_i) \hookrightarrow \text{Hom}(A, B)$.

Then (i) \iff (ii) \iff (iii) \implies (iv).

Proof. (ii) \implies (i). Assume $C \twoheadrightarrow A$ and C is compact. Take any filtered union $B = \bigcup B_i$. Since $B_i \hookrightarrow B$, then $\text{Hom}(A, B_i) \hookrightarrow \text{Hom}(A, B)$, thus $\varinjlim \text{Hom}(A, B_i) \hookrightarrow \varinjlim \text{Hom}(A, B)$.

In order to show this is a surjection, we take any map $A \rightarrow B$, since C is compact, $C \rightarrow A \rightarrow B$ factors through some B_i , hence $A \rightarrow B$ factors through B_i , which implies

$$\varinjlim \text{Hom}(A, B_i) \twoheadrightarrow \varinjlim \text{Hom}(A, B).$$

Then $\varinjlim \text{Hom}(A, B_i) \cong \varinjlim \text{Hom}(A, B)$, i.e. A is finitely generated.

(i) \implies (iii). We write

$$A = \text{Im}\left(\bigoplus_{i \in I} B_i \rightarrow A\right) = \bigcup_{J \subset I, J \text{ finite}} \text{Im}\left(\bigoplus_{i \in J} B_i \rightarrow A\right),$$

then

$$\text{Hom}(A, A) = \varinjlim \text{Hom}(A, \text{Im}(\bigoplus_{i \in J} B_i \rightarrow A)),$$

hence id_A factors through $\text{Im}(\bigoplus_{i \in J} B_i \rightarrow A)$ for some J , thus $\bigoplus_{i \in J} B_i \twoheadrightarrow A$.

(iii) \implies (ii). We can write A as the quotient of the direct sum of a family of compact objects, i.e. $\bigoplus_{i \in I} B_i \twoheadrightarrow A$, B_i are compact. By (iii), \exists finite subset $J \subset I$, s.t. $\bigoplus_{i \in J} B_i \twoheadrightarrow A$. And the finite direct sum of compact objects is still compact, hence A is the quotient of a compact object.

(ii) \implies (iv). Assume $C \twoheadrightarrow A$ with C compact and $B = \varinjlim B_i$. Then we have the following diagram

$$\begin{array}{ccc} \text{Hom}(A, B) & \hookrightarrow & \text{Hom}(C, B) \\ \downarrow & & \parallel \\ \varinjlim \text{Hom}(A, B_i) & \hookrightarrow & \text{Hom}(C, B) \end{array}$$

which gives $\text{Hom}(A, B) \hookrightarrow \varinjlim \text{Hom}(A, B_i)$. □

10 Solid Abelian Groups

Definition 10.1. For $S \in \text{ProFin}$, write $S = \varprojlim S_i$, where $S_i \in \text{Fin}$, we define the solid free abelian group

$$\mathbb{Z}[S]^\blacksquare := \varprojlim \mathbb{Z}[S_i].$$

We call $\mathbb{Z}[S]^\blacksquare$ the solidification of $\mathbb{Z}[S]$.

Remark 10.2.

$$\mathbb{Z}[S]^\blacksquare = \varprojlim \mathbb{Z}[S_i] = \varprojlim \underline{\text{Hom}}(C(S_i, \mathbb{Z}), \mathbb{Z}) = \underline{\text{Hom}}(\varinjlim C(S_i, \mathbb{Z}), \mathbb{Z}) = \underline{\text{Hom}}(C(S, \mathbb{Z}), \mathbb{Z}).$$

Proposition 10.3. (i) For $S \in \text{ProFin}$, there exists some set I , s.t. $C(S, \mathbb{Z}) \cong \mathbb{Z}^{\oplus I}$, i.e. $C(S, \mathbb{Z})$ is a free abelian group.

(ii) We have

$$\mathbb{Z}[S]^\blacksquare = \underline{\text{Hom}}(C(S, \mathbb{Z}), \mathbb{Z}) = \underline{\text{Hom}}(\mathbb{Z}^{\oplus I}, \mathbb{Z}) = \mathbb{Z}^I.$$

Definition 10.4. A condensed abelian group $X \in \text{Cond}(\text{Ab})$ is solid, if for any $S \in \text{ProFin}$, one has

$$\text{Hom}(\mathbb{Z}[S], X) \cong \text{Hom}(\mathbb{Z}[S]^\blacksquare, X).$$

A complex of condensed abelian groups $C \in D(\text{Cond}(\text{Ab}))$ is solid, if for any $S \in \text{ProFin}$, one has

$$R\text{Hom}(\mathbb{Z}[S], C) \cong R\text{Hom}(\mathbb{Z}[S]^\blacksquare, C).$$

Now, we need to check $\mathbb{Z}[S]^\blacksquare$ is indeed a solid condensed abelian group.

Proposition 10.5. For $S, T \in \text{ProFin}$, we have

$$R\text{Hom}(\mathbb{Z}[S], \mathbb{Z}[T]^\blacksquare) \cong R\text{Hom}(\mathbb{Z}[S]^\blacksquare, \mathbb{Z}[T]^\blacksquare).$$

Proof. Assume $\mathbb{Z}[S]^\blacksquare = \mathbb{Z}^I$ and $\mathbb{Z}[T]^\blacksquare = \mathbb{Z}^J$ for some sets I and J . Since the functors $R\text{Hom}(\mathbb{Z}[S], -)$ and $R\text{Hom}(\mathbb{Z}[S]^\blacksquare, -)$ commute with products, it suffices to show

$$R\text{Hom}(\mathbb{Z}[S], \mathbb{Z}) \cong R\text{Hom}(\mathbb{Z}[S]^\blacksquare, \mathbb{Z})$$

The left hand side is $R\text{Hom}(\mathbb{Z}[S], \mathbb{Z}) \cong R\Gamma(S, \mathbb{Z}) = C(S, \mathbb{Z}) = \mathbb{Z}^{\oplus I}$.

Now, consider the short exact sequence $0 \rightarrow \mathbb{R}^I \rightarrow \mathbb{Z}^I \rightarrow \mathbb{T}^I \rightarrow 0$. From theorem 9.8,

We know

$$R\mathrm{Hom}(\mathbb{T}^I, \mathbb{Z}) = \mathbb{Z}^{\oplus I}[-1].$$

And by the adjoint relation, we have

$$R\mathrm{Hom}(\mathbb{R}^I, \mathbb{Z}) \cong R\mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^I, R\mathrm{Hom}(\mathbb{R}, \mathbb{Z})) = 0.$$

Hence, $R\mathrm{Hom}(\mathbb{Z}[S]^{\blacksquare}, \mathbb{Z}) \cong R\mathrm{Hom}(\mathbb{Z}^I, \mathbb{Z}) \cong \mathbb{Z}^{\oplus I}$. And this finishes our proof. \square

Lemma 10.6. Let \mathcal{A} be a cocomplete abelian category, and $\mathcal{A}_0 \subseteq \mathcal{A}$ be the full subcategory of compact projective generators. Assume $F : \mathcal{A}_0 \rightarrow \mathcal{A}$ is an additive functor with a natural transformation $\mathrm{id}_{\mathcal{A}_0} \Rightarrow F$, satisfying the following property:

For any $X \in \mathcal{A}_0$, any $Y, Z \in \mathcal{A}$ which can be written as direct sums of objects in the image of F , i.e. $Y = \bigoplus_{i \in I} F(X_i)$ and $Z = \bigoplus_{j \in J} F(X_j)$, and for any map $f : Y \rightarrow Z$ with kernel $K \in \mathcal{A}$, the map

$$R\mathrm{Hom}(F(X), K) \rightarrow R\mathrm{Hom}(X, K)$$

is an isomorphism.

Let

$$\mathcal{A}_F = \{Y \in \mathcal{A} \mid \mathrm{Hom}(F(X), Y) \cong \mathrm{Hom}(X, Y), \forall X \in \mathcal{A}_0\} \subseteq \mathcal{A}$$

and

$$D_F(\mathcal{A}) = \{C \in D(\mathcal{A}) \mid R\mathrm{Hom}(F(X), C) \cong R\mathrm{Hom}(X, C), \forall X \in \mathcal{A}_0\} \subseteq D(\mathcal{A})$$

Then:

- (i) - $\mathcal{A}_F \subseteq \mathcal{A}$ is an abelian subcategory stable under limits, colimits and extensions.
- The objects $F(X), X \in \mathcal{A}_0$ are compact projective generators.
- The inclusion $\mathcal{A}_F \hookrightarrow \mathcal{A}$ admits a left adjoint $L : \mathcal{A} \rightarrow \mathcal{A}_F$, which is the unique colimit-preserving extension of $F : \mathcal{A}_0 \rightarrow \mathcal{A}_F$.

- (ii) - The functor $D(\mathcal{A}_F) \rightarrow D(\mathcal{A})$ is fully faithful and $D(\mathcal{A}_F) \cong D_F(\mathcal{A})$.
- $C \in D(\mathcal{A})$ lies in $D_F(\mathcal{A})$ iff $H^i(C) \in \mathcal{A}_F$.
- The above functor F has a left derived functor, which is the left adjoint of $D_F(\mathcal{A}) \hookrightarrow D(\mathcal{A})$.

Proof. (i) \mathcal{A}_F is stable under limits:

$$\mathrm{Hom}(FX, \lim Y_i) \cong \lim \mathrm{Hom}(FX, Y_i) \cong \lim \mathrm{Hom}(X, Y_i) \cong \mathrm{Hom}(X, \lim Y_i).$$

\mathcal{A}_F is stable under colimits:

It suffices to show \mathcal{A}_F is stable under cokernels and direct sums.

For any map $f : Y \rightarrow Z$ in \mathcal{A}_F . We can find a surjection $\bigoplus_{i \in I} P_i \twoheadrightarrow Z$, which factors through $\bigoplus_{i \in I} F(P_i)$, hence $\bigoplus_{i \in I} F(P_i) \twoheadrightarrow Z$. Assume the pullback diagram:

$$\begin{array}{ccccc} X & \xrightarrow{g} & \bigoplus_{i \in I} F(P_i) & \longrightarrow & \mathrm{coker}(g) \\ \downarrow & & \downarrow & & \\ Y & \xrightarrow{f} & Z & \longrightarrow & \mathrm{coker}(f) \end{array}$$

By the pullback we know $X \twoheadrightarrow Y$. With this, one can show $\mathrm{coker}(g) = \mathrm{coker}(f)$. Hence, we may replace Z by $\bigoplus_{i \in I} F(P_i)$. With the same reason, one can also replace Y by the object of the form $\bigoplus_{j \in J} F(Q_j)$. Therefore, we assume $f : \bigoplus_{j \in J} F(Q_j) \rightarrow \bigoplus_{i \in I} F(P_i)$. We already know $\ker(f) \in \mathcal{A}_F$. From the following lemma 10.7, $\mathrm{coker}(f) \in \mathcal{A}_F$. Thus, \mathcal{A}_F is stable under cokernels.

Moreover, the objects of \mathcal{A}_F are precisely the cokernels of maps $f : Y \rightarrow Z$ between objects $Y, Z \in \mathcal{A}$ that are direct sums of objects in the image of F .

Hence by

$$\bigoplus_{i \in I} \mathrm{coker}(Y_i \rightarrow Z_i) \cong \mathrm{coker}\left(\bigoplus_{i \in I} Y_i \rightarrow \bigoplus_{i \in I} Z_i\right),$$

we know \mathcal{A}_F is stable under direct sums.

Thus, \mathcal{A}_F is stable under colimits.

Now assume $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$ is exact with $X, Z \in \mathcal{A}_F$. Then X, Z can be written as $\mathrm{coker}(A_1 \rightarrow A_2)$ and $\mathrm{coker}(B_1 \rightarrow B_2)$, where A_1, A_2, B_1, B_2 are the

direct sums of objects in the image of F . We form the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A_1 & \longrightarrow & A_1 \oplus B_1 & \longrightarrow & B_1 & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_2 & \longrightarrow & A_2 \oplus B_2 & \longrightarrow & B_2 & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \text{coker}(A_1 \rightarrow A_2) & \longrightarrow & Y & \longrightarrow & \text{coker}(B_1 \rightarrow B_2) & \longrightarrow & 0
\end{array}$$

Then the extension $Y = \text{coker}(A_1 \oplus B_1 \rightarrow A_2 \oplus B_2) \in \mathcal{A}_F$.

For each $Y_i \in \mathcal{A}_F$, $X \in \mathcal{A}_0$, from the isomorphisms,

$$\text{Hom}(FX, \text{colim } Y_i) \cong \text{Hom}(X, \text{colim } Y_i) \cong \text{colim } \text{Hom}(X, Y_i) \cong \text{colim } \text{Hom}(FX, Y_i),$$

we know FX is compact projective.

We define a functor $L : \mathcal{A} \rightarrow \mathcal{A}_F$. For any $X \in \mathcal{A}$, $X = \text{colim } X_i$, $X_i \in \mathcal{A}_0$, hence define $L(X) := \text{colim } F(X_i)$. By the isomorphisms

$$\text{Hom}(LX, Y) \cong \text{Hom}(\text{colim } F(X_i), Y) \cong \lim \text{Hom}(F(X_i), Y) \cong \lim \text{Hom}(X_i, Y) \cong \text{Hom}(X, Y),$$

we know L is the left adjoint of the inclusion $\mathcal{A}_F \hookrightarrow \mathcal{A}$. Besides, by the construction, we know L agrees with F on \mathcal{A}_0 and is the unique colimit-preserving extension of F .

In fact, one can show for any $X \in \mathcal{A}$, take a resolution $B \rightarrow A \rightarrow X \rightarrow 0$, where $A, B \in \mathcal{A}_0$, then $L(X) \cong \text{coker}(FB \rightarrow FA)$.

(ii) The functor $D(\mathcal{A}_F) \rightarrow D(\mathcal{A})$ is fully faithful:

It suffices to show for any $X \in \mathcal{A}_0$ and any $C \in D(\mathcal{A}_F)$, there is an isomorphism:

$$R\text{Hom}_{D(\mathcal{A}_F)}(FX, C) \rightarrow R\text{Hom}_{D(\mathcal{A})}(FX, C) \cong R\text{Hom}_{D(\mathcal{A})}(X, C).$$

Since $R\text{Hom}_{D(\mathcal{A}_F)}(FX, -)$ and $R\text{Hom}_{D(\mathcal{A})}(X, -)$ commute with limits, we may assume C is bounded, and hence assume C is concentrated at degree 0, i.e. $C =$

$Y[0]$, where $Y \in \mathcal{A}_F$, then it suffices to show:

$$R\mathrm{Hom}_{D(\mathcal{A}_F)}(FX, Y) \cong R\mathrm{Hom}_{D(\mathcal{A})}(X, Y).$$

By taking the cohomology, it reduces to show for any $i \geq 0$,

$$\mathrm{Ext}_{\mathcal{A}_F}^i(FX, Y) \cong \mathrm{Ext}_{\mathcal{A}}^i(X, Y).$$

Since $FX \in \mathcal{A}_F$ is projective and $X \in \mathcal{A}$ is projective, then for $i > 0$, $\mathrm{Ext}_{\mathcal{A}_F}^i(FX, Y) \cong \mathrm{Ext}_{\mathcal{A}}^i(X, Y) \cong 0$. For $i = 0$, because $Y \in \mathcal{A}_F$, we have $\mathrm{Hom}_{\mathcal{A}_F}(FX, Y) \cong \mathrm{Hom}_{\mathcal{A}}(X, Y)$.

Denote

$$D'_F(\mathcal{A}) := \{C \in D(\mathcal{A}) \mid H^i(C) \in \mathcal{A}_F, \forall i\}.$$

Claim: $D(\mathcal{A}_F) \subset D'_F(\mathcal{A}) = D_F(\mathcal{A})$.

First, it is obvious that $D(\mathcal{A}_F) \subset D'_F(\mathcal{A})$.

Then we prove $D'_F(\mathcal{A}) \subset D_F(\mathcal{A})$.

If $C \in D'_F(\mathcal{A})$ is bounded, then we can reduce to the case $C = Y[0]$, where $Y \in \mathcal{A}_F$. From the isomorphism $R\mathrm{Hom}(FX, Y) \cong R\mathrm{Hom}(X, Y)$, we know $C = Y[0] \in D_F(\mathcal{A})$.

If $C \in D'_F(\mathcal{A})$ is right bounded, then $C = \varprojlim \tau_{\leq n} C$, where $\tau_{\leq n} C$ are bounded.

Hence we have

$$\begin{aligned} R\mathrm{Hom}(FX, C) &\cong R\mathrm{Hom}(FX, \varprojlim \tau_{\leq n} C) \cong \varprojlim R\mathrm{Hom}(FX, \tau_{\leq n} C) \\ &\cong \varprojlim R\mathrm{Hom}(X, \tau_{\leq n} C) \cong R\mathrm{Hom}(X, C), \end{aligned}$$

which means $C \in D_F(\mathcal{A})$.

In general, consider the truncation $C_{\geq n} = [\cdots \rightarrow C_{n+1} \rightarrow C_n \rightarrow 0 \rightarrow \cdots]$ which is right bounded. Then $C = \varinjlim C_{\geq n}$. Since FX, X are compact, we have

$$R\mathrm{Hom}(FX, C) \cong \varinjlim R\mathrm{Hom}(FX, C_{\geq n}) \cong \varinjlim R\mathrm{Hom}(X, C_{\geq n}) \cong R\mathrm{Hom}(X, C),$$

which means $C \in D_F(\mathcal{A})$.

On the other hand, $D_F(\mathcal{A})$ is generated by $F(X)[0]$ for $X \in \mathcal{A}_0$ and $F(X)[0] \in$

$D'_F(\mathcal{A})$, then $D'_F(\mathcal{A}) = D_F(\mathcal{A})$.

Finally, we show $D(\mathcal{A}_F) \cong D_F(\mathcal{A})$.

The functor $D(\mathcal{A}_F) \rightarrow D(\mathcal{A})$ factors over $D(\mathcal{A}_F) \rightarrow D'_F(\mathcal{A}) = D_F(\mathcal{A})$ and induces an equivalence on hearts. As it is fully faithful and commutes with all products and direct sums, then it is an equivalence. The inclusion $D_F(\mathcal{A}) \subset D(\mathcal{A})$ admits a left adjoint, which necessarily commutes with direct sums and by definition, takes $X \in \mathcal{A}_0$ to $F(X) \in \mathcal{A}_0$, so is the left derived functor of $L : \mathcal{A} \rightarrow \mathcal{A}_F$. \square

Lemma 10.7. We take the above lemma's notation.

(i) For any C with the form $\bigoplus_{i \in I} F(X_i)$, $X_i \in \mathcal{A}_0$, one has

$$R\mathrm{Hom}(F(X), C) \cong R\mathrm{Hom}(X, C), \quad \forall X \in \mathcal{A}_0.$$

(ii) For any C with the form $\ker(\bigoplus_{i \in I} F(X_i) \rightarrow \bigoplus_{j \in J} F(Y_j))$, $X_i, Y_j \in \mathcal{A}_0$, one has

$$R\mathrm{Hom}(F(X), C) \cong R\mathrm{Hom}(X, C), \quad \forall X \in \mathcal{A}_0.$$

(iii) For any C with the form $\mathrm{coker}(\bigoplus_{i \in I} F(X_i) \rightarrow \bigoplus_{j \in J} F(Y_j))$, $X_i, Y_j \in \mathcal{A}_0$, one has

$$R\mathrm{Hom}(F(X), C) \cong R\mathrm{Hom}(X, C), \quad \forall X \in \mathcal{A}_0.$$

(iv) For any right bounded complex C with each term C_i having the form $\bigoplus_{j \in I_i} F(X_{ij})$, one has

$$R\mathrm{Hom}(F(X), C) \cong R\mathrm{Hom}(X, C), \quad \forall X \in \mathcal{A}_0.$$

Then (iv) \implies (iii) \iff (ii) \implies (i).

Proof. (ii) \implies (i). Just take $J = \emptyset$, which is exactly (i).

(ii) \iff (iii). For any $f : Y \rightarrow Z$, with $Y = \bigoplus_{i \in I} F(X_i)$ and $Z = \bigoplus_{j \in J} F(Y_j)$, applying functors $R\mathrm{Hom}(X, -)$ and $R\mathrm{Hom}(F(X), -)$ to the exact sequence:

$$0 \rightarrow \ker(f) \rightarrow Y \rightarrow Z \rightarrow \mathrm{coker}(f) \rightarrow 0,$$

one get

$$\begin{array}{ccccccc} R\mathrm{Hom}(F(X), \ker(f)) & \longrightarrow & R\mathrm{Hom}(F(X), Y) & \longrightarrow & R\mathrm{Hom}(F(X), Z) & \longrightarrow & R\mathrm{Hom}(F(X), \mathrm{coker}(f)) \\ \downarrow & & \downarrow \cong & & \downarrow \cong & & \downarrow \\ R\mathrm{Hom}(X, \ker(f)) & \longrightarrow & R\mathrm{Hom}(X, Y) & \longrightarrow & R\mathrm{Hom}(X, Z) & \longrightarrow & R\mathrm{Hom}(X, \mathrm{coker}(f)) \end{array}$$

By five lemma, we can show

$$R\mathrm{Hom}(F(X), \ker(f)) \cong R\mathrm{Hom}(X, \ker(f))$$

\iff

$$R\mathrm{Hom}(F(X), \mathrm{coker}(f)) \cong R\mathrm{Hom}(X, \mathrm{coker}(f)).$$

Hence, (ii) \iff (iii).

(iv) \implies (ii). For any $f : Y \rightarrow Z$, with $Y = \bigoplus_{i \in I} F(X_i)$ and $Z = \bigoplus_{j \in J} F(Y_j)$. Denote $K = \ker(f)$. Take the resolution of K :

$$\cdots \rightarrow B_1 \rightarrow B_0 \rightarrow K \rightarrow 0,$$

where each $B_i \in \mathcal{A}_0$. Now, take $C = [0 \rightarrow Y \rightarrow Z \rightarrow 0]$, by assumption, we have

$$R\mathrm{Hom}(F(B_\bullet), C) \cong R\mathrm{Hom}(B_\bullet, C).$$

Hence,

$$\begin{array}{ccc} B_\bullet & \longrightarrow & F(B_\bullet) \\ \downarrow & \nearrow \exists! & \\ K & & \end{array}$$

That is, $K \cong B_\bullet$ is the retract of $F(B_\bullet)$. Thus,

$$R\mathrm{Hom}(X, K) \cong R\mathrm{Hom}(X, F(B_\bullet)) \cong R\mathrm{Hom}(F(X), F(B_\bullet)) \cong R\mathrm{Hom}(F(X), K).$$

□

Theorem 10.8. (i) - The category $\mathrm{Solid} \subset \mathrm{Cond}(\mathrm{Ab})$ of solid abelian groups is an abelian subcategory stable under limits, colimits and extensions.

- $\mathrm{Solid}^{\mathrm{cp}} = \{\mathbb{Z}^I \mid I \text{ is any set}\}.$
- The inclusion $\mathrm{Solid} \subset \mathrm{Cond}(\mathrm{Ab})$ admits a left adjoint

$$\mathrm{Cond}(\mathrm{Ab}) \rightarrow \mathrm{Solid}; M \mapsto M^\blacksquare,$$

which is the unique colimit-preserving extension of $\mathbb{Z}[S] \rightarrow \mathbb{Z}[S]^\blacksquare$.

- (ii) - The functor $D(\mathrm{Solid}) \rightarrow D(\mathrm{Cond}(\mathrm{Ab}))$ is fully faithful and its essential image are precisely the solid objects of $D(\mathrm{Cond}(\mathrm{Ab}))$.
- An object $C \in D(\mathrm{Cond}(\mathrm{Ab}))$ is solid iff all $H^i(C) \in \mathrm{Solid}$.
- The inclusion functor $D(\mathrm{Solid}) \rightarrow D(\mathrm{Cond}(\mathrm{Ab}))$ admits a left adjoint

$$D(\mathrm{Cond}(\mathrm{Ab})) \rightarrow D(\mathrm{Solid}); C \mapsto C^{L\blacksquare},$$

which is the left derived functor of $\text{Cond}(\text{Ab}) \rightarrow \text{Solid}$; $M \mapsto M^\blacksquare$.

Proposition 10.9. For an extremally disconnected space $S \in \text{ExDisc}$ and a chain complex

$$C : \cdots \rightarrow C_1 \rightarrow C_0 \rightarrow 0,$$

where each $C_i = \bigoplus_{j \in I_i} \mathbb{Z}^{I_{i,j}}$, we have

$$R\text{Hom}(\mathbb{Z}[S]^\blacksquare, C) \cong R\text{Hom}(\mathbb{Z}[S], C).$$

Proof. Case 1. C is concentrated in degree 0, i.e. $C = \bigoplus_{j \in J} \mathbb{Z}^{I_j}$. Since $\mathbb{Z}[S]$ is compact projective, we have

$$\begin{aligned} R\text{Hom}(\mathbb{Z}[S], C) &= R\text{Hom}(\mathbb{Z}[S], \bigoplus_{j \in J} \mathbb{Z}^{I_j}) \\ &= \bigoplus_{j \in J} R\text{Hom}(\mathbb{Z}[S], \mathbb{Z}^{I_j}) \\ &= \bigoplus_{j \in J} R\text{Hom}(\mathbb{Z}[S], \mathbb{Z})^{I_j} \\ &= \bigoplus_{j \in J} C(S, \mathbb{Z})^{I_j} \\ &= \bigoplus_{j \in J} (\mathbb{Z}^{\oplus K})^{I_j}. \end{aligned}$$

It suffices to show:

$$R\text{Hom}(\mathbb{Z}[S]^\blacksquare, C) \cong \bigoplus_{j \in J} (\mathbb{Z}^{\oplus K})^{I_j}.$$

We know $\mathbb{Z}[S]^\blacksquare = R\text{Hom}(C(S, \mathbb{Z}), \mathbb{Z}) = \mathbb{Z}^K$. Then it suffices to show:

$$R\text{Hom}(\mathbb{Z}^K, \bigoplus_{j \in J} \mathbb{Z}^{I_j}) \cong \bigoplus_{j \in J} (\mathbb{Z}^{\oplus K})^{I_j}.$$

Consider the short exact sequence: $0 \rightarrow \mathbb{Z}^K \rightarrow \mathbb{R}^K \rightarrow \mathbb{T}^K \rightarrow 0$. Since \mathbb{Z} and \mathbb{T} are pseudocoherent, \mathbb{R} is pseudocoherent. Then

$$R\text{Hom}(\mathbb{R}, \bigoplus_{j \in J} \mathbb{Z}^{I_j}) \cong \bigoplus_{j \in J} R\text{Hom}(\mathbb{R}, \mathbb{Z}^{I_j}) \cong \bigoplus_{j \in J} R\text{Hom}(\mathbb{R}, \mathbb{Z})^{I_j} \cong 0.$$

From this, we can get:

$$R\mathbf{Hom}(\mathbb{R}^K, \bigoplus_{j \in J} \mathbb{Z}^{I_j}) \cong R\mathbf{Hom}_{\mathbb{R}}(\mathbb{R}^K, R\mathbf{Hom}(\mathbb{R}, \bigoplus_{j \in J} \mathbb{Z}^{I_j})) \cong 0.$$

And because \mathbb{T}^K is pseudocoherent, we have:

$$R\mathbf{Hom}(\mathbb{T}^K, \bigoplus_{j \in J} \mathbb{Z}^{I_j}) \cong \bigoplus_{j \in J} R\mathbf{Hom}(\mathbb{T}^K, \mathbb{Z})^{I_j} \cong \bigoplus_{j \in J} (\mathbb{Z}^{\oplus K})^{I_j}[-1].$$

Therefore, we have:

$$R\mathbf{Hom}(\mathbb{Z}^K, \bigoplus_{j \in J} \mathbb{Z}^{I_j}) \cong \bigoplus_{j \in J} (\mathbb{Z}^{\oplus K})^{I_j}.$$

Case 2. C is bounded. It is obvious from case 1.

Case 3. For the general complex $C : \cdots \rightarrow C_1 \rightarrow C_0 \rightarrow 0$. Consider the short exact sequence:

$$0 \rightarrow C_{\leq n} \rightarrow C \rightarrow C_{>n} \rightarrow 0.$$

It suffices to show: $R\mathbf{Hom}(\mathbb{Z}[S]^{\blacksquare}, C_{>n})$ and $R\mathbf{Hom}(\mathbb{Z}[S], C_{>n})$ are concentrated at degree $\geq n$. This is because for any n , the cofiber of

$$R\mathbf{Hom}(\mathbb{Z}[S]^{\blacksquare}, C) \rightarrow R\mathbf{Hom}(\mathbb{Z}[S], C).$$

is concentrated at $\geq n$, hence the cofiber is 0.

For $R\mathbf{Hom}(\mathbb{Z}[S], C_{>n})$, since $\mathbb{Z}[S]$ projective, $R\mathbf{Hom}(\mathbb{Z}[S], C_{>n})$ is concentrated at $\geq n$.

Hence, we need to prove $R\mathbf{Hom}(\mathbb{Z}[S]^{\blacksquare}, C_{>n}) = R\mathbf{Hom}(\mathbb{Z}^K, C_{>n})$ is concentrated at $\geq n$, which is equivalent to prove $R\mathbf{Hom}(\mathbb{Z}^K, C)$ is concentrated at ≥ -1 .

Claim 1: For any K , $R\mathbf{Hom}(\mathbb{T}^K, C)$ is concentrated at ≥ -2 .

Claim 2: For any K , $R\mathbf{Hom}(\mathbb{R}^K, C) = 0$.

This is because we have $R\mathbf{Hom}(\mathbb{R}^K, C) = \text{colim } R\mathbf{Hom}(\mathbb{R}^K, C_{\leq n})$, and

$$R\mathbf{Hom}(\mathbb{R}^K, C_{\leq n}) = R\mathbf{Hom}_{\mathbb{R}}(\mathbb{R}^K, R\mathbf{Hom}_{\mathbb{R}}(\mathbb{R}, C_{\leq n})) = 0.$$

Now, from these two claims, $R\mathbf{Hom}(\mathbb{R}^K, C) = 0$ and $R\mathbf{Hom}(\mathbb{T}^K, C)$ is concentrated at degree ≥ -2 , we get $R\mathbf{Hom}(\mathbb{Z}^K, C) = R\mathbf{Hom}(\mathbb{T}^K, C)[1]$ is concentrated at degree ≥ -1 .

Hence, it suffices to prove Claim 1.

We denote $C_{\mathbb{R},i} = \bigoplus_{J_i} \mathbb{R}^{I_{i,j}}$, $C_{\mathbb{T},i} = \bigoplus_{J_i} \mathbb{T}^{I_{i,j}}$, and form complexes

$$C_{\mathbb{R}} : \cdots \rightarrow C_{\mathbb{R},i} \rightarrow \cdots \rightarrow C_{\mathbb{R},1} \rightarrow C_{\mathbb{R},0} \rightarrow 0$$

and

$$C_{\mathbb{T}} : \cdots \rightarrow C_{\mathbb{T},i} \rightarrow \cdots \rightarrow C_{\mathbb{T},1} \rightarrow C_{\mathbb{T},0} \rightarrow 0.$$

There is an exact sequence $0 \rightarrow C \rightarrow C_{\mathbb{R}} \rightarrow C_{\mathbb{T}} \rightarrow 0$.

Therefore, we can reduce to prove the following claim.

Claim: $R\mathbf{Hom}(\mathbb{T}^K, C_{\mathbb{R}})$ and $R\mathbf{Hom}(\mathbb{T}^K, C_{\mathbb{T}})$ are concentrated at ≥ -1 .

We know $R\mathbf{Hom}(\mathbb{T}^K, C_{\mathbb{R}}) = R\lim R\mathbf{Hom}(\mathbb{T}^K, \tau_{<n} C_{\mathbb{R}})$, where $\tau_{<n} C_{\mathbb{R}}$ is

$$0 \rightarrow \ker(C_{\mathbb{R},n} \rightarrow C_{\mathbb{R},n-1}) \rightarrow C_{\mathbb{R},n} \rightarrow \cdots \rightarrow C_{\mathbb{R},0} \rightarrow 0,$$

and $R\mathbf{Hom}(\mathbb{T}^K, C_{\mathbb{T}}) = R\lim R\mathbf{Hom}(\mathbb{T}^K, \tau_{<n} C_{\mathbb{T}})$, where $\tau_{<n} C_{\mathbb{T}}$ is

$$0 \rightarrow \ker(C_{\mathbb{T},n} \rightarrow C_{\mathbb{T},n-1}) \rightarrow C_{\mathbb{T},n} \rightarrow \cdots \rightarrow C_{\mathbb{T},0} \rightarrow 0.$$

Let $M_{\mathbb{R}} = \ker(\bigoplus_{j \in J} \mathbb{R}^{I_j} \rightarrow \bigoplus_{j \in J'} \mathbb{R}^{I'_j})$ and $M_{\mathbb{T}} = \ker(\bigoplus_{j \in J} \mathbb{T}^{I_j} \rightarrow \bigoplus_{j \in J'} \mathbb{T}^{I'_j})$. Because $C_{\mathbb{R},i} = \ker(C_{\mathbb{R},i} \rightarrow 0)$ and $C_{\mathbb{T},i} = \ker(C_{\mathbb{T},i} \rightarrow 0)$ also have the form of $M_{\mathbb{R}}$ and $M_{\mathbb{T}}$, it suffices to show $R\mathbf{Hom}(\mathbb{T}^K, M_{\mathbb{R}})$ and $R\mathbf{Hom}(\mathbb{T}^K, M_{\mathbb{T}})$ are concentrated at degree ≥ -1 .

Since \mathbb{T}^K is pseudocoherent, $R\mathbf{Hom}(\mathbb{T}^K, -)$ commutes with filtered colimits, hence we can assume J is finite. Then assume

$$M_{\mathbb{R}} = \ker(\mathbb{R}^I \rightarrow \bigoplus_{j \in J'} \mathbb{R}^{I'_j}), \quad M_{\mathbb{T}} = \ker(\mathbb{T}^I \rightarrow \bigoplus_{j \in J'} \mathbb{T}^{I'_j}).$$

Besides, we can also assume J' is finite. Hence let

$$M_{\mathbb{R}} = \ker(\mathbb{R}^I \rightarrow \mathbb{R}^{I'}), \quad M_{\mathbb{T}} = \ker(\mathbb{T}^I \rightarrow \mathbb{T}^{I'}).$$

Now, as a topological group, $M_{\mathbb{T}} = \ker(\mathbb{T}^I \rightarrow \mathbb{T}^{I'})$ is compact, and \mathbb{T}^K is compact, by Corollary 9.10, the cohomology of $R\mathbf{Hom}(\mathbb{T}^K, M_{\mathbb{T}})$ is concentrated at 0 and 1, hence its homology is concentrated at ≥ -1 .

Claim: $M_{\mathbb{R}}$ is a direct summand of \mathbb{R}^I .

From this claim, by $R\mathbf{Hom}(\mathbb{T}^K, \mathbb{R}^I) = R\mathbf{Hom}(\mathbb{T}^K, \mathbb{R})^I = 0$, we have $R\mathbf{Hom}(\mathbb{T}^K, M_{\mathbb{R}}) = 0$.

Then, it suffices to prove the above claim. This is because, for \mathbb{R} -linear map $\mathbb{R}^{\oplus I'} \rightarrow \mathbb{R}^{\oplus I}$ is the composition of a split surjection and a split injection. Then by taking the duality $\mathbf{Hom}(-, \mathbb{R})$, the dual map $\mathbb{R}^I \rightarrow \mathbb{R}^{I'}$ is split. \square

Definition 10.10. (i) For $M, N \in \text{Solid}$, define $M \otimes^{\blacksquare} N := (M \otimes N)^{\blacksquare}$.

(ii) For $C, D \in D(\text{Solid})$, define $C \otimes^{L\blacksquare} D := (C \otimes^L D)^{L\blacksquare}$.

Theorem 10.11. (i) The solidification functor $\text{Cond}(\text{Ab}) \rightarrow \text{Solid}$; $M \mapsto M^{\blacksquare}$ is symmetric monoidal, i.e.

$$(M \otimes N)^{\blacksquare} \cong M^{\blacksquare} \otimes^{\blacksquare} N^{\blacksquare}.$$

(ii) The solidification functor $D(\text{Cond}(\text{Ab})) \rightarrow D(\text{Solid})$; $C \mapsto C^{L\blacksquare}$ is symmetric monoidal, i.e.

$$(C \otimes^L D)^{L\blacksquare} \cong C^{L\blacksquare} \otimes^{L\blacksquare} D^{L\blacksquare}.$$

(iii) $\otimes^{L\blacksquare}$ is the left derived functor of \otimes^{\blacksquare} .

Proof. (i) By definition, we need to show:

$$(M \otimes N)^{\blacksquare} \xrightarrow{\sim} (M^{\blacksquare} \otimes N^{\blacksquare})^{\blacksquare}.$$

This can be written as the composition:

$$(M \otimes N)^{\blacksquare} \longrightarrow (M^{\blacksquare} \otimes N)^{\blacksquare} \longrightarrow (M^{\blacksquare} \otimes N^{\blacksquare})^{\blacksquare}.$$

Hence, it is enough to prove

$$(M \otimes N)^{\blacksquare} \xrightarrow{\sim} (M^{\blacksquare} \otimes N)^{\blacksquare}.$$

(With this isomorphism, we can also show that the second map is an isomorphism).

Since the tensor functor and the solidification functor commute with colimits, then we can assume $M = \mathbb{Z}[S]$ and $N = \mathbb{Z}[T]$.

It reduces to show:

$$\mathbb{Z}[S \times T]^{\blacksquare} \xrightarrow{\sim} (\mathbb{Z}[S]^{\blacksquare} \otimes \mathbb{Z}[T])^{\blacksquare}.$$

Equivalently, for any $A \in \text{Solid}$,

$$\underline{\text{Hom}}((\mathbb{Z}[S]^{\blacksquare} \otimes \mathbb{Z}[T])^{\blacksquare}, A) \cong \underline{\text{Hom}}(\mathbb{Z}[S \times T]^{\blacksquare}, A).$$

Since A is solid, we have:

$$\underline{\mathrm{Hom}}((\mathbb{Z}[S]^\blacksquare \otimes \mathbb{Z}[T])^\blacksquare, A) \cong \underline{\mathrm{Hom}}(\mathbb{Z}[S]^\blacksquare \otimes \mathbb{Z}[T], A)$$

and

$$\underline{\mathrm{Hom}}(\mathbb{Z}[S \times T]^\blacksquare, A) \cong \underline{\mathrm{Hom}}(\mathbb{Z}[S \times T], A).$$

By computation:

$$\begin{aligned} \underline{\mathrm{Hom}}(\mathbb{Z}[S]^\blacksquare \otimes \mathbb{Z}[T], A) &\cong \underline{\mathrm{Hom}}(\mathbb{Z}[S]^\blacksquare, \underline{\mathrm{Hom}}(\mathbb{Z}[T], A)) \\ &\cong \underline{\mathrm{Hom}}(\mathbb{Z}[S], \underline{\mathrm{Hom}}(\mathbb{Z}[T], A)) \\ &\cong \underline{\mathrm{Hom}}(\mathbb{Z}[S] \otimes \mathbb{Z}[T], A) \\ &\cong \underline{\mathrm{Hom}}(\mathbb{Z}[S \times T], A). \end{aligned}$$

Thus, $\underline{\mathrm{Hom}}((\mathbb{Z}[S]^\blacksquare \otimes \mathbb{Z}[T])^\blacksquare, A) \cong \underline{\mathrm{Hom}}(\mathbb{Z}[S \times T]^\blacksquare, A)$.

(ii) Similar to the proof of (i).

(iii)

Remark 10.12. In Solid, \otimes^\blacksquare is the left adjoint of $\underline{\mathrm{Hom}}$:

$$\mathrm{Hom}(M \otimes^\blacksquare N, P) \cong \mathrm{Hom}((M \otimes N)^\blacksquare, P) \cong \mathrm{Hom}(M \otimes N, P) \cong \mathrm{Hom}(M, \underline{\mathrm{Hom}}(N, P)).$$

Proposition 10.13. (i) If $X \in \mathrm{CHaus}$, then $\mathbb{Z}[X]^{L\blacksquare} = R\underline{\mathrm{Hom}}(R\Gamma(X, \mathbb{Z}), \mathbb{Z})$.

In particular, if $X \in \mathrm{ProFin} \subseteq \mathrm{CHaus}$, then $\mathbb{Z}[X]^{L\blacksquare} = \mathbb{Z}[X]^\blacksquare$.

(ii) If X is a CW space, then $\mathbb{Z}[X]^{L\blacksquare} = C_\bullet(X)$.

This shows that the derived solidification of a condensed abelian group can sit in all nonnegative homological degrees.

Proposition 10.14. (i) $\mathbb{R}^{L\blacksquare} = 0$.

$$(ii) \mathbb{Z}^I \otimes^{L\blacksquare} \mathbb{Z}^J = \mathbb{Z}^{I \times J}.$$

$$(iii) \mathbb{Z}_p \otimes^{L\blacksquare} \mathbb{Z}_p = \mathbb{Z}_p.$$

$$(iv) \mathbb{Z}_p \otimes^{L\blacksquare} \mathbb{Z}_\ell = 0. (p \neq \ell)$$

Proof. (i) By Yoneda's lemma, it suffices to show: for any $C \in D(\text{Solid})$, one has

$$R\text{Hom}(\mathbb{R}^{L^\blacksquare}, C) = R\text{Hom}(\mathbb{R}, C) = 0.$$

Since $C = \varprojlim C_{\geq n}$, and $R\text{Hom}(\mathbb{R}, -)$ commutes with limits, it reduces to the case C is a right bounded complex. And for a right bounded complex C , one has $C = \varprojlim C_{\leq n}$, it reduces to the case C is a bounded complex.

Hence it suffices to show: for any $X \in \text{Solid}$, one has $R\text{Hom}(\mathbb{R}, X) = 0$.

We know for any object $X \in \text{Solid}$, we can write X as the colimit of objects of the form $\bigoplus_{j \in J} \mathbb{Z}^{I_j}$. And we know taking all colimits is equivalent to taking all cokernels and all filtered colimits.

Since \mathbb{R} is pseudo-coherent, we get

$$R\text{Hom}(\mathbb{R}, \varinjlim \bigoplus_{i \in J_j} \mathbb{Z}^{I_{i,j}}) = \varinjlim R\text{Hom}(\mathbb{R}, \bigoplus_{i \in J_j} \mathbb{Z}^{I_{i,j}}) = \varinjlim \bigoplus_{i \in J_j} R\text{Hom}(\mathbb{R}, \mathbb{Z}^{I_{i,j}}) = 0.$$

Let $f : X \rightarrow Y$, $X = \bigoplus_{i \in I} \mathbb{Z}^{I_i}$ and $Y = \bigoplus_{j \in J} \mathbb{Z}^{I_j}$, then from $R\text{Hom}(\mathbb{R}, X) = 0$ and $R\text{Hom}(\mathbb{R}, Y) = 0$, we know $R\text{Hom}(\mathbb{R}, \text{coker}(f)) = 0$.

Thus, we finish our proof.

(ii) Assume $\mathbb{Z}^I = \mathbb{Z}[S]^\blacksquare = \underline{\text{Hom}}(C(S, \mathbb{Z}), \mathbb{Z})$, $\mathbb{Z}^J = \mathbb{Z}[T]^\blacksquare = \underline{\text{Hom}}(C(T, \mathbb{Z}), \mathbb{Z})$, for some $S, T \in \text{ProFin}$. Then

$$\begin{aligned} \mathbb{Z}[S \times T]^\blacksquare &= \underline{\text{Hom}}(C(S \times T, \mathbb{Z}), \mathbb{Z}) \\ &= \underline{\text{Hom}}(C(S, \mathbb{Z}) \otimes C(T, \mathbb{Z}), \mathbb{Z}) \\ &= \underline{\text{Hom}}(C(S, \mathbb{Z}), \underline{\text{Hom}}(C(T, \mathbb{Z}), \mathbb{Z})) \\ &= \underline{\text{Hom}}(C(S, \mathbb{Z}), \mathbb{Z}^J) \\ &= \underline{\text{Hom}}(C(S, \mathbb{Z}), \mathbb{Z})^J \\ &= \mathbb{Z}^{I \times J}. \end{aligned}$$

Thus, we have

$$\mathbb{Z}^I \otimes^{L^\blacksquare} \mathbb{Z}^J = \mathbb{Z}[S]^\blacksquare \otimes^{L^\blacksquare} \mathbb{Z}[T]^\blacksquare = (\mathbb{Z}[S] \otimes^L \mathbb{Z}[T])^{L^\blacksquare} = \mathbb{Z}[S \times T]^{L^\blacksquare} = \mathbb{Z}^{I \times J}.$$

(iii) We write $\mathbb{Z}_p = \mathbb{Z}[[x]]/(x - p)$, then

$$\begin{aligned}
\mathbb{Z}_p \otimes^{L\blacksquare} \mathbb{Z}_p &= \mathbb{Z}[[x]]/(x - p) \otimes^{L\blacksquare} \mathbb{Z}[[y]]/(y - p) \\
&= \mathbb{Z}[[x, y]]/(x - p, y - p) \\
&= \mathbb{Z}[[x, y]]/(x - p, x - y) \\
&= \mathbb{Z}_p.
\end{aligned}$$

(iv) We write $\mathbb{Z}_p = \mathbb{Z}[[x]]/(x - p)$ and $\mathbb{Z}_\ell = \mathbb{Z}[[y]]/(y - \ell)$, then

$$\begin{aligned}
\mathbb{Z}_p \otimes^{L\blacksquare} \mathbb{Z}_\ell &= \mathbb{Z}[[x]]/(x - p) \otimes^{L\blacksquare} \mathbb{Z}[[y]]/(y - \ell) \\
&= \mathbb{Z}[[x, y]]/(x - p, y - \ell).
\end{aligned}$$

Since p and ℓ are coprime, pick $a, b \in \mathbb{Z}$, s.t. $ap + b\ell = 1$, then $a(p - x) + b(\ell - y) = 1 - ax - by$ is invertible in $\mathbb{Z}[[x, y]]$. Hence $(x - p, y - \ell) = \mathbb{Z}[[x, y]]$, i.e. $\mathbb{Z}_p \otimes^{L\blacksquare} \mathbb{Z}_\ell = 0$.

□

11 Analytic rings

Definition 11.1. A pre-analytic ring \mathcal{A} is a triple $(\underline{\mathcal{A}}, \mathcal{A}[-], \alpha)$, where:

- $\underline{\mathcal{A}}$ is a condensed ring, which is called the underlying condensed ring of the pre-analytic ring \mathcal{A} .
- The functor $\mathcal{A}[-] : \text{ExDisc} \rightarrow \text{Mod}(\underline{\mathcal{A}}); S \mapsto \mathcal{A}[S]$ preserves finite colimits, where $\text{Mod}(\underline{\mathcal{A}})$ is the category of $\underline{\mathcal{A}}$ -modules in $\text{Cond}(\text{Ab})$.
- $\alpha : \underline{\mathcal{A}}[-] \rightarrow \mathcal{A}[-]$ is a natural transformation.

$\underline{\mathcal{A}}[S]$ is called the free $\underline{\mathcal{A}}$ -module on S , and $\mathcal{A}[S]$ is called the free \mathcal{A} -module on S .

Definition 11.2. A map of pre-analytic rings $\mathcal{A} \rightarrow \mathcal{B}$ is s.t.

- $\underline{\mathcal{A}} \rightarrow \underline{\mathcal{B}}$ is a map of condensed rings.
- For any $S \in \text{ExDisc}$, $\mathcal{A}[S] \rightarrow \mathcal{B}[S]$ is an $\underline{\mathcal{A}}$ -linear map and is natural in S and commutes with the map from S .

Example 11.3. (i) For any condensed ring R , $R = (R, R[-], \text{id})$ is a pre-analytic ring.

(ii) The pre-analytic ring $\mathbb{Z}_{\blacksquare}$.

Take the underlying condensed ring to be $\underline{\mathbb{Z}_{\blacksquare}} = \mathbb{Z}$, and for any $S \in \text{ExDisc}$, take $\mathbb{Z}_{\blacksquare}[S] := \mathbb{Z}[S]^{\blacksquare} = \varprojlim \mathbb{Z}[S_i]$, where $S = \varprojlim S_i$, each S_i is finite.

(iii) The pre-analytic ring $\mathbb{Z}_{p,\blacksquare}$.

Take the underlying condensed ring to be $\underline{\mathbb{Z}_{p,\blacksquare}} = \mathbb{Z}_p$, and for any $S \in \text{ExDisc}$, take $\mathbb{Z}_{p,\blacksquare}[S] := \varprojlim \mathbb{Z}_p[S_i]$, where $S = \varprojlim S_i$, each S_i is finite.

(iv) For any finitely generated \mathbb{Z} -algebra R , we can define a pre-analytic ring R_{\blacksquare} .

The underlying condensed ring is $\underline{R_{\blacksquare}} = R$ and $R_{\blacksquare}[S] := \varprojlim R[S_i]$, where $S = \varprojlim S_i$, each S_i is finite.

(v) Let R, A be finitely generated \mathbb{Z} -algebra, $R \rightarrow A$ is a map. We define a pre-analytic ring $(A, R)_{\blacksquare}$.

$(A, R)_{\blacksquare} := A$ and $(A, R)_{\blacksquare}[S] := R_{\blacksquare}[S] \otimes_R A$.

In particular, for the pre-analytic ring $(A, \mathbb{Z})_{\blacksquare}$, $\underline{(A, \mathbb{Z})_{\blacksquare}} = A$ and $(A, \mathbb{Z})_{\blacksquare}[S] = \mathbb{Z}_{\blacksquare}[S] \otimes_{\mathbb{Z}} A$.

(vi) The pre-analytic ring $(\mathbb{Q}_p, \mathbb{Z}_p)_\blacksquare$.

$$(\mathbb{Q}_p, \mathbb{Z}_p)_\blacksquare := \mathbb{Q}_p \text{ and } (\mathbb{Q}_p, \mathbb{Z}_p)_\blacksquare[S] := \mathbb{Z}_{p,\blacksquare}[S] \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

(vii) For $0 < p \leq 1$, we define the pre-analytic ring \mathbb{R}_{ℓ^p} .

$$\mathbb{R}_{\ell^p} := \mathbb{R} \text{ and } \mathbb{R}_{\ell^p}[S] := \bigcup_{r>0} \varprojlim \mathbb{R}[S_i]_{\ell^p \leq r}, \text{ where } S = \varprojlim S_i, \text{ each } S_i \text{ is finite.}$$

$$\text{Here, } \mathbb{R}[S_i]_{\ell^p \leq r} := \{ \sum r_j x_j \in \mathbb{R}[S_i] \mid r_j \in \mathbb{R}, x_j \in S_i, \sum |r_j|^p \leq r \}.$$

(vii) For $0 < p \leq 1$, we define the pre-analytic ring $\mathbb{R}_{\ell^{<p}}$.

$$\mathbb{R}_{\ell^{<p}} := \mathbb{R} \text{ and } \mathbb{R}_{\ell^{<p}}[S] := \varinjlim_{q < p} \mathbb{R}_{\ell^q}[S].$$

Definition 11.4. An analytic ring is a pre-analytic ring \mathcal{A} , s.t. for any complex

$$C : \cdots \rightarrow C_i \rightarrow \cdots \rightarrow C_1 \rightarrow C_0 \rightarrow 0$$

in $D(\text{Mod}(\underline{\mathcal{A}}))$, each $C_i = \bigoplus_{j \in I_i} \mathcal{A}[T_{i,j}]$, $T_{i,j} \in \text{ExDisc}$, the map

$$R\text{Hom}_{\underline{\mathcal{A}}}(\mathcal{A}[S], C) \xrightarrow{\sim} R\text{Hom}_{\underline{\mathcal{A}}}(\underline{\mathcal{A}}[S], C)$$

is an isomorphism.

12 Notes on 12.24

Definition 12.1. For a pair (A, R) , we mean $R \rightarrow A$, where A, R are \mathbb{Z} -algebra of finite type. For any map $f : (A, R) \rightarrow (B, S)$, consider the decomposition

$$\begin{array}{ccc} (A, R) & \xrightarrow{\bar{f}} & (B, R) \\ & \searrow f & \downarrow j \\ & & (B, S) \end{array}$$

We define $f_! := \bar{f}_* j_! : D((B, S)_{\blacksquare}) \rightarrow D((A, R)_{\blacksquare})$. Since \bar{f}_* and $j_!$ commute with all colimits, then $f_!$ commutes with all colimits, hence $f_!$ admits a right adjoint $f^! : D((A, R)_{\blacksquare}) \rightarrow D((B, S)_{\blacksquare})$.

Proposition 12.2. For $f : (A, R) \rightarrow (B, S)$ and $g : (B, S) \rightarrow (C, T)$, we have $(gf)_! = f_! g_!$, hence by adjunction, we have $(gf)^! = g^! f^!$.

Proof. We have the following diagram

$$\begin{array}{ccccc} (A, R) & \xrightarrow{\bar{f}} & (B, R) & \xrightarrow{\bar{g}'} & (C, R) \\ & \searrow f & \downarrow j & & \downarrow j' \\ & & (B, S) & \xrightarrow{\bar{g}} & (C, S) \\ & & & \searrow g & \downarrow k \\ & & & & (C, T) \end{array}$$

We know $f_! = \bar{f}_* j_!$, $g_! = \bar{g}_* k_!$ and $(gf)_! = (\bar{g}' \bar{f})_*(k j')_! = \bar{f}_* \bar{g}'_* j'_! k_!$, in order to show $(gf)_! = f_! g_!$, it suffices to show: $j_! \bar{g}_* = \bar{g}'_* j'_!$. This is because, for any $M \in D((B, S)_{\blacksquare})$,

$$\begin{aligned} \bar{g}'_* j'_! M &= \bar{g}'_*(M \otimes_{(S, R)_{\blacksquare}}^L j'_! S) \\ &= \bar{g}_* M \otimes_{(S, R)_{\blacksquare}}^L j_! S \\ &= j_! \bar{g}_* M. \end{aligned}$$

□

Theorem 12.3. Assume $f : R \rightarrow A$ is a map of \mathbb{Z} -algebra of finite types. View it as $f : (R, R) \rightarrow (A, A)$, then we get $f_! = \bar{f}_* j_! : D(A_{\blacksquare}) \rightarrow D(R_{\blacksquare})$, and its right adjoint $f^! : D(R_{\blacksquare}) \rightarrow D(A_{\blacksquare})$.

- (1) (i) $f_!$ preserves pseudo-coherent objects.
- (ii) $f^!$ preserves discrete objects.
- (iii) [Projection formula] $f_!(f^*M \otimes_{A_\bullet}^L N) \cong M \otimes_{R_\bullet}^L f_!N$.
- (2) Assume $f : R \rightarrow A$ is of finite Tor dimension, then
 - (i) $f_!$ preserves compact objects.
 - (ii) $f^!$ preserves all colimits.
 - (iii) $f^!$ preserves perfect complexes.
 - (iv) For any $M \in D(R_\bullet)$, $f^!M = f^*M \otimes_{A_\bullet}^L f^!R$, i.e. $f^!$ is the twist of f^* by $f^!R$.

Proof. (1).(i).and (2).(i).

Since the pseudo-coherent objects of $D(A_\bullet)$ are right bounded complexes which each term has the form A^I , and the compact objects are the direct summands of bounded pseudo-coherent objects, in order to show $f_!$ preserves pseudo-coherent objects, it suffices to show $f_!A^I$ is pseudo-coherent, and if f is of finite Tor dimension, then $f_!A^I$ is compact. As before, we can reduce the case $f : R \rightarrow R[X] = A$ and the case $f : R \twoheadrightarrow A$.

Case 1. $f : R \rightarrow R[X] = A$. Then

$$\begin{aligned}
 f_!A^I &= \overline{f}_*j_!A^I \\
 &= \overline{f}_*j_!j^*(R^I \otimes_R A) \\
 &= R^I \otimes_R A \otimes_{(A,R)_\bullet}^L (A_\infty/A)[-1] \\
 &= R^I \otimes_{R_\bullet}^L (A_\infty/A)[-1] \\
 &= R^{I \times \omega}[-1]
 \end{aligned}$$

is a compact object in $D(R_\bullet)$.

Case 2. $f : R \twoheadrightarrow A$. Then $f_! = f_*$, and $f_!A^I = f_*A^I = A^I$ viewed as an R -module. Since R is Noetherian ring, we can take the resolution of A by finite free R -modules $C_\bullet \rightarrow A$, hence $A^I \cong C_\bullet^I \in D(R_\bullet)$ is pseudo-coherent.

If for general $f : R \rightarrow A$, f is of finite Tor dimension, then when we reduce to the case $f : R \twoheadrightarrow A$, it is also of finite Tor dimension. In this case, we can that $C_\bullet \rightarrow A$ so that C_\bullet is a bounded complex, then $f_!A^I \in D(R_\bullet)$ is compact.

(1).(ii). From the adjunction we know $f^!M = R\text{Hom}_R(f_!A, M)$. If $M \in D(R_\bullet)$ is

discrete, since $f_!A$ is pseudo-coherent and $R\mathbf{Hom}_R(A^I, M) = M^{\oplus I}$, then $f^!M \in D(A_\blacksquare)$ discrete.

(1).(iii). $\forall P \in D(R_\blacksquare)$,

$$\begin{aligned} R\mathbf{Hom}_R(f_!(f^*M \otimes_{A_\blacksquare}^L N), P) &= R\mathbf{Hom}_A(f^*M \otimes_{A_\blacksquare}^L N, f^!P) \\ &= R\mathbf{Hom}_A(f^*M, R\mathbf{Hom}_A(N, f^!P)) \\ &= R\mathbf{Hom}_R(M, f_*R\mathbf{Hom}_A(N, f^!P)) \\ &= R\mathbf{Hom}_R(M, R\mathbf{Hom}_R(f_!N, P)) \\ &= R\mathbf{Hom}_R(M \otimes_{R_\blacksquare}^L f_!N, P), \end{aligned}$$

then by Yoneda's lemma, we have $f_!(f^*M \otimes_{A_\blacksquare}^L N) = M \otimes_{R_\blacksquare}^L f_!N$.

(2).(ii). Since $f_!$ preserves compact objects, and $D(A_\blacksquare)$ is compactly generated, then $f^!$ commutes with filtered colimits.

And since $f^!$ is a right adjoint, it preserves finite limits. We know $D(R_\blacksquare)$ and $D(A_\blacksquare)$ are stable ∞ -category, a functor preserves finite limits if and only if it preserves finite colimits. Hence $f^!$ preserves finite colimits.

Thus, $f^!$ preserves all colimits.

(2).(iii). It suffices to show: $f^!R$ is a perfect A -complex.

Case 1. $f : R \rightarrow R[X] = A$. Then $f^!R = R\mathbf{Hom}_R(f_!A, R) = R\mathbf{Hom}_R((A_\infty/A)[-1], R) = A[1]$ is a perfect A -complex.

Case 2. $f : R \twoheadrightarrow A$. Since f is of finite Tor dimension, then $f_!A = f_*A \in D(R)$ is perfect. Hence $f^!R = R\mathbf{Hom}_R(f_!A, R) \in D(R)$ is perfect. We know that a complex is perfect if and only if it is pseudo-coherent and of finite Tor dimension, therefore, $f^!R \in D(A)$ is perfect.

(2).(iv). First, we construct such a map. From the counit

$$f_!f^!R \rightarrow R,$$

we can get

$$M = R\mathbf{Hom}_R(R, M) \rightarrow R\mathbf{Hom}_R(f_!f^!R, M) = f_*R\mathbf{Hom}_A(f^!R, f^!M),$$

then

$$f^*M \rightarrow R\text{Hom}_A(f^!R, f^!M),$$

and finally we have:

$$f^*M \otimes_{A_\bullet}^L f^!R \rightarrow f^!M.$$

Since both sides commutes with all colimits, and $D(R_\bullet)$ is generated under colimits by R^I , hence it suffices to show:

$$f^*R^I \otimes_{A_\bullet}^L f^!R = f^!R^I.$$

This is because

$$\begin{aligned} f^*R^I \otimes_{A_\bullet}^L f^!R &= A^I \otimes_{A_\bullet}^L f^!R \\ &= (f^!R)^I \\ &= f^!R^I. \end{aligned}$$

□

Proposition 12.4. $f : R \rightarrow A$ is a ring map which is the base change of a finitely generated map of finite Tor dimension between Noetherian rings, consider the flat ring map $g : R \rightarrow S$, and

$$\begin{array}{ccc} R & \xrightarrow{f} & A \\ g \downarrow & & \downarrow g' \\ S & \xrightarrow{f'} & B \end{array}$$

where $B = A \otimes_R S$. Then

$$g'^*f^! \cong f'^!g^* : D(R_\bullet) \rightarrow D(B_\bullet).$$

Proof. Similar as before, we can reduce to the case $A = R[T]$ and the case $f : R \twoheadrightarrow A$. We have, for any $M \in D(R_\bullet)$,

$$g'^*f^!M = g'^*(f^*M \otimes_{A_\bullet}^L f^!R) = g'^*f^*M \otimes_{B_\bullet}^L g'^*f^!R$$

and

$$f'^!g^*M = f'^*g^*M \otimes_{B_\bullet}^L f'^!S = g'^*f^*M \otimes_{B_\bullet}^L f'^!g^*R.$$

Hence, it suffices to show: $g'^* f^! R \cong f'^! g^* R$.

Case 1. $f : R \rightarrow A = R[T]$. Then $f^! R = A[1]$ and $f'^! S = B[1]$, hence

$$g'^* f^! R = g'^* A[1] = A[1] \otimes_{A_{\blacksquare}}^L B_{\blacksquare} = B[1] = f'^! g^* R.$$

Case 2. $f : R \twoheadrightarrow A$, then $f_! = f_*$ and $f^! R = R\mathbf{Hom}_R(A, R)$. Similarly, $f'^! g^* R = f'^! S = R\mathbf{Hom}_S(B, S)$. Hence, it suffices to show

$$g'^* f^! R = g'^* R\mathbf{Hom}_R(A, R) = R\mathbf{Hom}_R(A, R) \otimes_{A_{\blacksquare}}^L B_{\blacksquare} = R\mathbf{Hom}_S(B, S).$$

This is true since g is flat. □

Proposition 12.5. Assume $f : R \rightarrow A$ is a regular closed immersion of pure codimension c and $I = \text{Ker}(f)$, then

$$f^! R = \det_A(I/I^2)^*[-c].$$

Proof. We know $A = R/(f_1, \dots, f_c)$, where f_1, \dots, f_c is a regular sequence. By Koszul complex:

$$0 \rightarrow \bigwedge^c R^c \rightarrow \bigwedge^{c-1} R^c \rightarrow \dots \rightarrow \bigwedge^1 R^c \rightarrow R \rightarrow A = R/(f_1, \dots, f_c) \rightarrow 0,$$

we can get

$$f^! R = R\mathbf{Hom}_R(A, R) = A[-c].$$

We also have $\det_A(I/I^2) \cong A$, then $f^! R = \det_A(I/I^2)^*[-c]$, which is independent of the choice of f_1, \dots, f_c . □

Proposition 12.6. Assume $f : R \rightarrow A$ is smooth of relative dimension d , then

$$f^! R = \det_A(\Omega_{A/R}^1)[d].$$

Proof. We have already known $f^! R$ is a line bundle concentrated in degree d . Let $g : A \otimes_R A \rightarrow A; a \otimes b \mapsto ab$, corresponding the diagonal $\Delta_f : \text{Spec } A \hookrightarrow \text{Spec } A \times_{\text{Spec } R} A$

$\text{Spec } A$. Then g is a regular closed immersion of dimension d . Consider

$$\begin{array}{ccc}
 R & \xrightarrow{f} & A \\
 f \downarrow & & \downarrow p_2 \\
 A & \xrightarrow{p_1} & A \otimes_R A \\
 & & \searrow g \\
 & & A
 \end{array}$$

We have:

$$\begin{aligned}
 f^! R &= g^! p_1^! f^! R \\
 &= g^! (p_1^* f^! R \otimes_{(A \otimes_R A)_{\blacksquare}}^L p_1^! A) \\
 &= g^! (p_1^* f^! R \otimes_{(A \otimes_R A)_{\blacksquare}}^L p_2^* f^! R) \\
 &= g^* (p_1^* f^! R \otimes_{(A \otimes_R A)_{\blacksquare}}^L p_2^* f^! R) \otimes_{A_{\blacksquare}}^L g^! (A \otimes_R A) \\
 &= g^* p_1^* f^! R \otimes_{A_{\blacksquare}}^L g^* p_2^* f^! R \otimes_{A_{\blacksquare}}^L g^! (A \otimes_R A) \\
 &= f^! R \otimes_{A_{\blacksquare}}^L f^! R \otimes_{A_{\blacksquare}}^L g^! (A \otimes_R A).
 \end{aligned}$$

Since $f^! R \in D(A_{\blacksquare})$ is invertible, then

$$f^! R = (g^! (A \otimes_R A))^*.$$

Since g is a regular closed immersion of codimension d , then $g^! (A \otimes_R A) = \det_A(I/I^2)^*[-d]$, where $I = \text{Ker}(A \otimes_R A \rightarrow A)$. Thus,

$$f^! R = (g^! (A \otimes_R A))^* = \det_A(I/I^2)[d] = \det_A(\Omega_{A/R}^1)[d].$$

□