T. Y. LAM

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In Memory of Jumei

Preface

"Serre's Conjecture", for the most part of the second half of the 20th century, referred to the famous statement made by J.-P. Serre in 1955, to the effect that one did not know if finitely generated projective modules were free over a polynomial ring $k[x_1, \ldots, x_n]$, where k is a field. This statement was motivated by the fact that the affine scheme defined by $k[x_1, \ldots, x_n]$ is the algebro-geometric analogue of the affine n-space over k. In topology, the n-space is contractible, so there are only trivial bundles over it. *Would the analogue of the latter also hold for the n-space in algebraic geometry?* Since algebraic vector bundles over $\operatorname{Spec} k[x_1, \ldots, x_n]$ correspond to finitely generated projective modules over $k[x_1, \ldots, x_n]$, the question was tantamount to whether such projective modules were free, for any base field k.

It was quite clear that Serre intended his statement as an *open problem* in the sheaf-theoretic framework of algebraic geometry, which was just beginning to emerge in the mid-1950s. Nowhere in his published writings had Serre speculated, one way or another, upon the possible outcome of his problem. However, almost from the start, a surmised positive answer to Serre's problem became known to the world as "Serre's Conjecture". Somewhat later, interest in this "Conjecture" was further heightened by the advent of two new (and closely related) subjects in mathematics: homological algebra, and algebraic K-theory. With cross-fertilizations coming from these new areas, "Serre's Conjecture" entered the 1960s as one of the premier open problems in algebra and affine algebraic geometry.

A brief history of the work done on Serre's Conjecture in the period 1955–1976 is given in the Introductory Survey in pp. 1-7 of the text to follow. In this 20-year period, new techniques and new insights were introduced into the study of projective modules — over general rings as well as polynomial rings. In particular, such work led to the solution of various special cases of Serre's Conjecture. The "Big Time" came in 1976, when Quillen and Suslin, working independently of each other, arrived at the first complete solutions of Serre's Conjecture in full generality; that is, for polynomial rings in any number of variables, over an arbitrary ground field.

Exhilarated by the solution of a conjecture that had fascinated me from my graduate student days, I presented the mathematics surrounding the Quillen-Suslin solutions of Serre's Conjecture in a lecture course in the Winter Quarter of 1977 at the University of California, Berkeley. An expanded version of the notes from that course, with the title "Serre's Conjecture", was later published in Springer's Lecture Notes in Mathematics, Vol. 635, in 1978. These notes (hereafter referred to as "SC") were written in a rather unusual way, in that I tried to accommodate in them two largely different views of the subject, one from the pedagogical angle, and the other from the historical angle. To make the notes pedagogically sound and accessible to novices in the subject, I have included in "SC" all the preparatory material on projective and stably free modules, localization and pasting techniques in basic commutative algebra, as well as some dimension theory and the rudiments of classical algebraic K-theory. On the other hand, to fulfill the historical needs in a full exposition on Serre's Conjecture, I tried to cover in "SC" not only the final solutions to the Conjecture, but also the twenty years of germane mathematical developments that led up to, and culminated in these solutions. This historical view of the subject required a retracing and reassessment of many of the earlier treatments of partial cases of Serre's Conjecture and the special techniques invented to handle them, some of which overlapped with or superseded the others. To weave all of these developments linearly into a text suitable for students and experts alike was certainly a daunting task.

If the writing of "SC" was tough, the rewards for publishing the book were high. The integration of introductory material and historical survey into a single volume turned out to be very well received by the mathematical community. In the last 25 years or so, many students and beginning researchers have used "SC" as the text of choice for learning the basics on Serre's Conjecture and its various solutions. In the meantime, from 1978 on, a majority of the articles in the literature devoted to the continued study of Serre's Conjecture cited "SC" — either for technical information, or as a primary source providing the general framework for the research at hand. To me, this warm reception given to "SC" by the mathematical community has certainly been very gratifying, perhaps even more so than the award of a Steele Prize for Exposition by the American Mathematical Society in 1982.

What I could not have predicted in 1977, however, was the fact that the research on Serre's Conjecture and its generalizations would so actively continue into the 1980s, 1990s, and beyond. Indeed, the level and the intensity of the work on further developments in "SC mathematics" in this period were nothing short of remarkable. I'll say more about this in the introduction to Chapter VIII in this book; in the main, the solution of Serre's Conjecture in 1976 proved to be not just the end of one period of research, but also the beginning of another. The truth of Serre's Conjecture and the various techniques used to solve it lent much momentum and optimism to problems such as Anderson's Conjecture, the Eisenbud-Evans Conjectures, and the Bass-Quillen Conjecture, all of which were "higher forms" of Serre's Conjecture. In addition, the results of Quillen and Suslin directly and indirectly inspired much new work on Serre's Problem in non-noetherian and quantum cases, its K_1-analogue, unimodular rows over general commutative rings, quadratic forms over polynomial rings, and complete intersections, etc. Algorithmic aspects of Serre's Problem and applications of the Quillen-Suslin Theorem to systems theory and signal processing have also brought an influx of new ideas from computational algebraists and electrical engineers.

While many of the papers written on Serre's Problem after 1978 made references to "SC", my Springer Lecture Notes volume had gone out of print since the early 1980s. The fact that over the years "SC" has firmly retained its place in this area of research, along with the urgings of my readers and the encouragement of my publisher, finally gave me the needed impetus to undertake a reissue of "SC".

In planning for this reissue, I have decided early on that my priority was to preserve and to improve the contents of "SC", but not to write a new book on the subject. Thus, in this reissue, all six original chapters of "SC" are kept, but they were substantially expanded to take into account the current state of the material. This resulted in the addition of many new topics, and much more extensive treatments of some of the old ones. For instance, Chapter I now offers a coverage of Kaplansky's K-Hermite rings, as well as an introduction to Whitehead's K_1-group, examples of $\mathrm{E}_n \neq \mathrm{SL}_n$, and Suslin's Normality Theorem. Chapter II is enriched with a detailed discussion on the first syzygy modules of ideals in $k[x, y]$ as an application of Seshadri's Theorem, and Chapter III features four new sections of material on the theory of stably free modules and unimodular rows. The other three chapters of "SC" have also undergone similar improvements and expansions. In addition, this book has a self-contained chapter on the K_1-analogue of Serre's Conjecture, based on the work of Suslin. All of this additional material made for a much broader coverage of Serre's Conjecture. On the other hand, in order to preserve the main strengths of "SC", we have continued to place the same emphasis on pedagogy and history in the current exposition. This is done by the inclusion of many worked examples throughout the text to illustrate the theory, and by a rather extensive updating of the historical notes at the end of the individual chapters.

The many new developments in the continued study of Serre's Conjecture after 1977 presented a dilemma. While parts of these developments that were close enough to one of the chapters of this book are now included, other parts are not. To give a full account for these other parts would, however, require at least several more chapters of exposition, which could easily double this book in size. To avoid this undesirable option, we resorted to a compromise. While unable to present all newer developments with proofs, we offer instead a comprehensive historical survey. This survey, with a long bibliography covering the literature from 1977 to 2005, constitutes Chapter VIII in this book. More about the nature and content of this survey will be said in the introduction to that chapter.

This book has been in preparation for some years, but it is only by a propitious coincidence that it goes to press just in time to celebrate the 50-year anniversary of

Serre's Conjecture — and the 30-year anniversary of its solution by Quillen and Suslin. Following a suggestion of J.-P. Serre, I have renamed this book "Serre's Problem on Projective Modules". The original "SC" title is indeed inappropriate, for a number of reasons. First, as we have observed earlier, Serre only formulated a problem, but never made a conjecture. And, even if there *was* a conjecture to begin with, once it was fully solved, it would have become a "theorem" instead. Although it is still convenient for me to speak of and refer to "Serre's Conjecture" within this text (and I will), these words clearly would no longer constitute a mathematically correct title for this book. Second, while my Springer Lecture Notes were primarily devoted to the solutions of "Serre's Conjecture", the expanded coverage in this book often takes us to problems on projective and stably free modules that go beyond the case of polynomial rings over fields. The choice of the new title of the book reflects this fact. Last but not least, since the late 1980s, the term "Serre's Conjecture" has been quite extensively used by number theorists to refer to a series of important conjectures on modular forms and Galois representations made by Serre in his 1987 Duke Journal paper. Deferring to this more modern use of the term, I am only circumspect in replacing my original "SC" title by something else to avoid an unnecessary confusion. Of course, I was also content to have settled down to a new title of the book that Serre himself had approved of.

This book owes its existence to the mathematical genius of Serre, Quillen, and Suslin: they deserve the highest accolades. That said, it should next be abundantly clear that the body of mathematical knowledge assembled in this monograph is the result of the efforts of many mathematicians in a grand international scale. A quick glance at the bibliographic sections of this book should give an idea of the large number of contributors to this area of mathematics, so I won't attempt to give an incomplete list here. It gives me special pleasure, however, to thank four people who have played special roles in my writing project. To begin with, it was the excellent teaching of Professor Hyman Bass many years ago that had first sparked my interest in Serre's Conjecture. Much of what I knew about SC came from Professor Bass's early writings. Professor Joseph Gubeladze taught me the Swan-Weibel homotopy trick (now posited in an Appendix to Chapter V), and kept me abreast with the current developments in the K-theory of monoid rings. Professor Ravi A. Rao kindly answered countless emails from me (in most cases instantly!) on mathematical topics ranging from Suslin's Problem to Local-Global Principles. His effervescence in discussing the mathematics of unimodular rows over commutative rings has inspired me to an expanded treatment of this topic in Chapter III and Chapter VI. Professor Richard G. Swan was magnanimous in letting me intrude into his retirement years with all kinds of tentative questions and conjectures. His always complete answers to my queries have led to a much improved exposition on stably free modules in Chapter III and symplectic structures in Chapter VII. To all seven mathematicians mentioned above — and the many more too numerous to name, I give my hearty salute.

As always, my family (Chee King, Juwen, Fumei, Juleen, and Tsai Yu) was instrumental in providing the warmth, encouragement and support much needed for this grueling journey into authorland. But this book is special. Holding hands, we dedicate it to the memory of a departed member of our family, Jumei. The writing of

“SC” in 1977 sadly spanned her short life: the real child passed, as if to let the brain child live. On the significant occasion of this reissue of “SC”, may Jumei’s little spirit be with Mom, Dad, and her siblings again and always, until we rejoin her.

T.Y.L.

Berkeley, California
January 1, 2006

Contents

Notes to the Reader

The main text of this book consists of an introductory survey on Serre's Conjecture, followed by eight chapters each containing a number of sections. The chapters are referred to in roman numerals, such as I, II, III, etc. A cross-reference such as IV.3 refers to Section 3 in Chapter IV. Within a given chapter, a reference such as 3.4 refers to the result (lemma, theorem, example, or remark) so labelled in Section 3 of that chapter, while, globally, V.1.6 refers to the result 1.6 in Chapter V. The running heads offer the quickest and most convenient way to tell what chapter and what section a particular page belongs to. This should make it very easy to find any result, such as V.1.6 (Quillen's Patching Theorem!).

There are two main bibliographies for this book. The first one, at the end of Chapter VII, gives the list of 132 papers referred to in the first seven chapters. These are mostly papers published before 1977. Chapter VIII, a survey chapter on the post-1976 developments on Serre's Problem, is followed by its own bibliography, with 606 entries. The papers in these two bibliographies (which are largely non-overlapping) are referred to by the authors' names and the year of the papers' publication, such as [Quillen: 1976], or [Suslin: 1976a, 1976b]. The first bibliography is followed by the Appendix on Complete Intersections, which comes with its own short list of references.

Throughout the text, a good grounding in graduate level algebra is assumed. However, a certain amount of basic commutative algebra is developed (or recalled) in the first chapter, for the reader's convenience. The notations and conventions we use in this book should be fairly standard. Unless otherwise stated, all rings are assumed to have identity, and all modules are assumed unital. Rings are, however, not always assumed to be commutative. For any ring R, $\mathfrak{M}(R)$ shall denote the class of all f.g. (finitely generated) left (sometimes right) R-modules, while $\mathfrak{P}(R)$ shall denote the class of all f.g. projective left (sometimes right) R-modules. These and other basic notations and abbreviations are summarized in the "Partial List of Notations" on the following pages. They will be used pretty consistently (and mostly without further explanation) throughout this book.

Partial List of Notations

$\mathbb{Z}$	the ring of integers
$\mathbb{N}$	the set of natural numbers
$\mathbb{Q}$	the field of rational numbers
$\mathbb{R}$	the field of real numbers
$\mathbb{C}$	the field of complex numbers
$\mathbb{Z}_n$	the ring (or the cyclic group) $\mathbb{Z}/n\,\mathbb{Z}$
$\mathbf{A}_k^n$	affine n-space over k
$\mathbb{P}_k^n$	projective n-space over k
$A \setminus B$	set-theoretic difference
$A \twoheadrightarrow B$	surjective mapping from A onto B
$\dot{k}$, k^*	multiplicative group of the field k
$\mathrm{U}(R)$, R^*	group of units of the ring R
$\mathfrak{M}(R)$	family of finitely generated R-modules
$\mathfrak{P}(R)$	family of finitely generated projective R-modules
$\dim(R)$	(or Krull $\dim(R)$) Krull dimension of the ring R
$\mathrm{Nil}(R)$	the nilradical of the commutative ring R

$\mathrm{rad}(R)$	Jacobson radical of the ring R
$\mathrm{rad}\,(J)$	the radical of an ideal J in a commutative ring
$\mathrm{ht}\,(J)$	the height of the ideal J in a commutative ring
$S^{-1}R$, R_S	localization of R at the multiplicative set S
R_a, $R[a^{-1}]$	localization of R at $\{a^i : i \geqslant 0\}$
$R_{\mathfrak{p}}$	localization of R at the prime ideal $\mathfrak{p}$
$\mathrm{Max}(R)$	maximal ideal spectrum of the commutative ring R
$\mathrm{Spec}(R)$	prime ideal spectrum of the commutative ring R
$D(f)$	set of prime ideals not containing f
$V(J)$	set of prime ideals containing the ideal J
I_n	$n \times n$ identity matrix
e_{ij}	standard matrix units
A^t, A^T	the transpose of a rectangular matrix A
$S_n(\alpha, \beta)$	n th Suslin matrix associated to α and β
$\det(A)$	determinant of the square matrix A
$\mathbb{M}_{r,n}(R)$	space of $r \times n$ matrices over the ring R
$\mathbb{M}_n(R)$	ring of $n \times n$ matrices over the ring R
$\mathrm{GL}_n(R)$	$n \times n$ general linear group over R
$\mathrm{SL}_n(R)$	$n \times n$ special linear group over (commutative) R
$\mathrm{E}_n(R)$	the group of $n \times n$ elementary matrices over R
$\mathrm{GL}_n(R, J)$	$n \times n$ general linear group relative to an ideal $J \subseteq R$
$\mathrm{E}_n(R, J)$	$n \times n$ elementary group relative to an ideal $J \subseteq R$
$\mathrm{GL}(R)$	infinite general linear group over R
$\mathrm{SL}(R)$	infinite special linear group over R
$\mathrm{E}(R)$	infinite elementary group over R
$\mathrm{Um}_n(R)$	the set of length n unimodular rows over R
$f \sim_G g$	conjugacy of $f, g \in \mathrm{Um}_n(R)$ under G-action
$f \sim g$	conjugacy of $f, g \in \mathrm{Um}_n(R)$ under $\mathrm{GL}_n(R)$-action
$\mathrm{Pic}(R)$	Picard group of R
$K_0(R)$	Grothendieck group of f.g. projective R-modules
$K_1(R)$	Whitehead group of R
$SK_1(R)$	special Whitehead group of (commutative) R
$R\,[\mathbf{x}]$	polynomial ring in $\mathbf{x}$ over R $(\mathbf{x} = (x_1, \ldots, x_n))$
$\mathfrak{M}^R\big(R\,[\mathbf{x}]\big)$	modules in $\mathfrak{M}\big(R\,[\mathbf{x}]\big)$ that are extended from R
$\mathfrak{P}^R\big(R[\mathbf{x}]\big)$	modules in $\mathfrak{P}\big(R\,[\mathbf{x}]\big)$ that are extended from R
$R\,[[\mathbf{x}]]$	power series ring in $\mathbf{x}$ over R
$R\,[\mathbf{x}, \mathbf{x}^{-1}]$	Laurent polynomial ring over R
$R\,\langle x\rangle$	localization of $R\,[x]$ at all monic polynomials in x
M^n	$M \oplus \cdots \oplus M$ (n copies)
$M[\mathbf{x}]$	the $R[\mathbf{x}]$-module $R[\mathbf{x}] \otimes_R M$
(M)	isomorphism class of the module M
$[M]$	the element in $K_0(R)$ given by $M \in \mathfrak{P}(R)$
$S^{-1}M$, M_S	localization of the module M at S
M_f	localization of the module M at $\{f^i : i \geqslant 0\}$

$M_{\mathfrak{p}}$	localization of the module M at the prime ideal $\mathfrak{p}$
M^*	the dual $\mathrm{Hom}_R(M,\ R)$ of an R-module M
$M^{\perp}$	orthogonal complement to M
$\mu(M)$	least number of generators for the module M
$\mathrm{pd}(M)$	projective dimension of the module M
rank M	(or rk M) rank of a projective module M
$\mathrm{rank}_{\mathfrak{p}}\, M$	$\mathfrak{p}$-rank of a projective module M ($\mathfrak{p} \in \mathrm{Spec}(R)$)
$\Lambda(M)$	the exterior algebra on the module M
$\Lambda^n(M)$	n th exterior power of the module M
$\mathrm{End}_R(M)$	endomorphism ring of the R-module M
$\mathrm{Hom}_R(M, N)$	group of R-homomorphisms from M to N
$M \otimes_R N$	tensor product of R-modules M and N
$M \perp N$	orthogonal sum of inner product spaces M and N
$P(b_1, \ldots, b_n)$	kernel of the map $R^n \to R$ defined by $(b_1, \ldots, b_n)$

f. g.	finitely generated
f. p.	finitely presentted
p. d.	projective dimension
tr. d.	transcendence degree
LHS	left-hand side
RHS	right-hand side
PIR	(usually commutative) principal ideal ring
PID	(usually commutative) principal ideal domain
UFD	unique factorization domain
IBP	invariant basis property
IPS	inner product space
$\mathrm{Su}(R)_n$	Suslin's Problem for unimodular rows of length $n+1$ over $R[t]$
(E), (E_n)	extension properties defined in (V.1.9)
(BQ_d)	Bass-Quillen Conjecture for regular rings of dimension $\leqslant d$
(BQ'_d)	Bass-Quillen Conjecture for regular local rings of dimension $\leqslant d$
(QQ_d)	Quillen's Question for regular local rings of dimension $\leqslant d+1$
(H)	Hermite Ring Conjecture
(H$'$)	Hermite Ring Conjecture for local base ring

Introduction to Serre's Conjecture: 1955–1976

On p. 243 of his famous article "Faisceaux algébriques cohérents" (FAC, ca. 1955), Serre wrote: "On ignore s'il existe des A-modules projectifs de type fini qui ne soient pas libres" ($A = k[t_1, \ldots, t_n]$, k a field). Shortly thereafter, the freeness of finitely generated projective modules over $k[t_1, \ldots, t_n]$ became known to the mathematical world as "Serre's Conjecture". Serre had in no uncertain terms objected to the fact that what he raised as an open problem was turned into his "conjecture" by world acclaim.(*) However, the fine distinction between "Serre's Problem" and "Serre's Conjecture" may now be safely left to the deliberations of the mathematical historian. Culminating almost twenty years of effort by algebraists, D. Quillen and A. Suslin proved independently, in January of 1976, that finitely generated projective modules over $k[t_1, \ldots, t_n]$ are, indeed, free.

In this book, I try to give a comprehensive account of the mathematics surrounding Serre's Problem — and its many solutions, extensions, and ramifications. Before we plunge into the technical details, let us give first, in this Introduction, a historical overview of the developments on Serre's Problem in the period 1955–1976; that is, from its inception to its first full resolution. One interesting feature about the work in this period is the fact that Serre's Conjecture was solved in one special case after another, by widely different methods and techniques that *did not* seem to generalize readily. Then, after a gestation of 20 years, the Conjecture was solved in full by Quillen and Suslin —- not quite by building on the existing ideas in the special cases, but much rather "by a true flash of genius" that almost seemed to defy all past patterns. This is, however, not a really uncommon occurrence in the annals of great mathematical problems. The fact that Quillen and Suslin happened to have proved Serre's Conjecture in the same month of the same year would seem to suggest simply that, in January of 1976, two long decades after its formulation, the time for solving Serre's Conjecture had finally come.

(*)In a letter to me on April 24, 1978 (right after the appearance of my Springer Lecture Notes), Serre reminisced: "No doubt about that: "Problem" is right. I never wrote anything indicating that I believed or disbelieved in a positive solution. Of course, I could not help that some people called the whole thing a "conjecture", but I objected as often as I could."

To embark on our survey, a good place to start is to ask the obvious question: what was the motivation of Serre's Problem, and why was it raised in FAC, in 1955? In FAC, the modern sheaf-theoretic viewpoint was systematically introduced into algebraic geometry. From this viewpoint, vector bundles over algebraic varieties can be defined as "locally free sheaves"; the trivial bundles simply correspond to the free sheaves. The affine n-space $\mathbf{A}_k^n$ over a field k is the prime ideal spectrum Spec $k[t_1, \ldots, t_n]$. This being an affine scheme on the ring $k[t_1, \ldots, t_n]$, the locally free sheaves are given by the f.g. (finitely generated) projective modules over $k[t_1, \ldots, t_n]$. Thus, the f.g. projective modules over $k[t_1, \ldots, t_n]$ correspond to (algebraic) vector bundles on $\mathbf{A}_k^n$. In geometric language, therefore, Serre's Problem simply asks: *is every vector bundle over* $\mathbf{A}_k^n$ *a trivial bundle*? The "moral impulse" behind this question is that the affine n-space $\mathbf{A}_k^n$ should behave like a "contractible" space in topology, and hence should have only trivial bundles over it.

A different motivation of Serre's Problem comes from Serre's investigation of the complete intersection question for (irreducible) subvarieties V of codimension 2 in $\mathbf{A}_k^n$ [Serre: 1960/61]. This is a question of efficient generation of the height 2 prime ideal $I \subset k[t_1, \ldots, t_n]$ corresponding to V: we would like to know *when* I can be generated by two polynomials. (If so, we say V is a "complete intersection"). Under certain "necessary" assumptions on V, Serre showed that there exists an exact sequence $0 \to A \to P \to I \to 0$, where $A = k[t_1, \ldots, t_n]$, and P is a f.g. projective A-module of rank 2. If we know that P must be *free*, then $P \cong A^2$, and the surjection $P \to I$ above will show that I is generated, as an ideal, by only two polynomials. To take a concrete case, let k be algebraically closed, and let V be a nonsingular irreducible curve in $\mathbf{A}_k^3$, of genus 0 or 1 (i.e. V is either rational or elliptic). If every f.g. projective module of rank 2 over $k[x, y, z]$ is free, then Serre's argument would have implied that the prime ideal defining V can be generated by 2 polynomials; in particular, the given curve V could be "cut out" by two suitable surfaces in 3-space.

Let us now give a historical survey of the development on Serre's Conjecture from 1955 to 1976. This survey, however, is not intended to be complete, and the items below are not exactly arranged in the chronological order. *We shall write, in the following,* $A = k\,[t_1, \ldots, t_n]$ (k *a field*), *and let* P *be a f.g. projective* A*-module.*

(1) If P has rank 1, the freeness of P can be easily deduced from the fact that A is a UFD (a theorem of Gauss).

(2) If $n = 1$, then $A = k\,[t_1]$ is a PID, and, over a PID, it is a standard result that (f.g.) projectives are free. We could have even let k be a division ring here; $k\,[t_1]$ in that case will be a noncommutative PID, and the result on the freeness of projectives over a (noncommutative) PID still applies.

(3) In 1958, Seshadri showed that Serre's Conjecture is true for two variables (i.e. for $A = k\,[t_1, t_2]$). In fact, Seshadri proved that f.g. projectives over $R[t]$ are free if R is any commutative PID.

(4) Sharma, Ojanguren, and Sridharan showed, in 1971, that Serre's Conjecture has a *negative* answer for $n \geqslant 2$ if k was allowed to be a division ring, instead of a field. This shows that f.g. projectives over $R[t]$ need not be free if R is a *noncommutative* PID.

(5) Serre (1957/58) showed that the projective module P must admit a decomposition $P \cong A^r \oplus Q$ where rank $Q \leqslant n$ (and $r \geqslant 0$). This, in essence, reduces Serre's Conjecture (the freeness of P) to the case where rank $P \leqslant n$.

(6) [Bass: 1964] showed that P is indeed free if rank $P > n$. In an earlier work, [Bass: 1963], it was also shown that *non*-finitely generated A-projectives are free.

(7) If P is a *graded* module with respect to the natural graded structure on A, then P is indeed A-free. This result was already contained in the classical book *Homological Algebra* of Cartan and Eilenberg, ca. 1956.

(8) The Theorem of Hilbert-Serre [Serre: 1957/58] states that P must be *stably free*, i.e. $P \oplus A^r \cong A^s$ for suitable integers $r,\ s \geqslant 0$. In view of this, Serre's Problem becomes the following: *does "stably free" imply "free" over* $A = k[t_1, \ldots, t_n]$? In this connection, we introduce the following general definition:

> *A (commutative) ring B is said to be a* Hermite ring *if every f.g. stably free module over B is B-free.*

It is fairly easy to see that B is Hermite iff any row $(b_1, \ldots, b_m) \in B^m$ such that $\sum B \cdot b_i = B$ can be completed to an invertible $m \times m$ matrix over B. In view of this characterization of Hermite rings, (8) above leads to the following completely elementary and purely matrix theoretic formulation of Serre's Problem:

> *If $f_1, \ldots, f_m \in A = k[t_1, \ldots, t_n]$ generate the unit ideal, can we always complete $(f_1, \ldots, f_m)$ to an $m \times m$ matrix over A with determinant in $k \setminus \{0\}$?*

(9) It follows from a theorem in [Bass: 1969] that, to show that a commutative ring B is Hermite, it will suffice to show that all *even rank* stably free B-modules are free. For instance, to answer Serre's Conjecture in the affirmative for $n = 3$, it would be sufficient to show that any *rank* 2 (f.g.) projective module is free over $A = k[t_1, t_2, t_3]$. Using the method of symplectic modules, Bass proved that every f.g. projective A-module P must be at least *self-dual*; that is, $P \cong P^* := \mathrm{Hom}_A(P,\ A)$. See VII.5.

(10) In 1970, Segre claimed that a nonsingular curve in $\mathbf{A}_k^3$ of genus 0 or 1 (k algebraically closed) need not be a complete intersection. This would have implied, in view of Serre's earlier results, that a rank 2 projective over $k[t_1, t_2, t_3]$ need not be free. However, (quoting from [Bass: 1972a]) "Abhyankar has indicated that there are serious deficiencies both in the statements of Segre's results, and in his method of proof. According to Abhyankar's testimony, one should not regard [Segre:

1970] as essentially altering the open status of (Serre's Conjecture for $k[t_1, t_2, t_3]$)."

(11) The confusion caused by Segre's paper was finally resolved in [Murthy-Towber: 1974], in which it was *proved* once and for all that Serre's Conjecture is true for $k[t_1, t_2, t_3]$, with k algebraically closed. The argument of Murthy and Towber made use of "Kleiman's Bertini Theorem", and an explicit construction of three generators for the prime ideals $\mathfrak{p}$ of certain nonsingular curves in $\mathbf{A}_k^3$, due to Abhyankar. The existence of three generators for $\mathfrak{p}$, without assuming k to be algebraically closed, was first proved in [Murthy: 1972]. In retrospect, Murthy's result amounted to a manifestation (in the case of height 2 prime ideals in $k[t_1, t_2, t_3]$) of a conjecture made by Forster in 1964. However, no reference was made to this conjecture in [Murthy: 1972].

(12) M. Roitman proved, in 1974, that if k is algebraically closed, and rank $P = n$ (the number of variables), then P must be free. In the published version [Roitman: 1975], the assumption that k be algebraically closed was relaxed to k being infinite.

(13) Using the powerful method of symplectic modules, A. Suslin and L. N. Vaserstein proved, in 1973/74, that P must be free in the following cases:

(a) if rank $P \geqslant 1 + \frac{n}{2}$;

(b) if $n = 3, 4, 5$; or

(c) if $n = 6$ and k is a finite field.

See [Vaserstein-Suslin: 1974, 1976], or Bass's Bourbaki talk [Bass: 1974]. Concerning these three papers, and especially the result in the case (a) above, the following historical account I gave in another article [(†)] is perhaps still of some interest today:

> "When Suslin and Vaserstein began to make significant progress on Serre's conjecture in the early 70s, the former Soviet Union was still quite isolated mathematically from Europe and from the U.S. In order to make their latest findings known to the West, Suslin and Vaserstein could only communicate them by letter to Bass. I still remember vividly the AMS Annual Meeting in San Francisco in 1974, in which Bass gave an "impromptu" lecture on the most recent Suslin-Vaserstein results on Serre's Conjecture — to a room-full of people eager to find out how close Serre's Conjecture had come to being solved. One of these "From Russia, with Love" results proclaimed the freeness of projective $k[x_1, \ldots, x_n]$-modules "of rank $\geqslant 1 + n/2$". This surely looked wonderful, but a high-school algebra question unwittingly came up: when Suslin wrote "$1 + n/2$" in his letter to Bass, did he mean $1 + \frac{n}{2}$, or could he have meant the more ambitious $(1 + n)/2$? It was anybody's guess .... (As it turned out, Suslin did mean $1 + \frac{n}{2}$, as he, perhaps, should.) Later in June that year, in Paris, Bass was to give a similar lecture on the status of Serre's Conjecture in the "Séminaire Bourbaki". The write-up of this survey lecture subsequently appeared in Bass's article ([Bass: 1974] here), with the

(†) See p. 92 in [Lam: 1999] listed in the references to Chapter VIII.

charming title 'Libération des modules projectifs sur certains anneaux de polynômes'."

(14) Independently of Vaserstein and Suslin, R. Swan proved certain cancellation theorems in 1974 which also led to the case (a) in (13) above if k is infinite, and to the case $n = 4$ if k is infinite of characteristic $\neq 2$. Swan's argument refined that of Roitman, via the use of a Bertini type theorem. An amalgamation of Swan's cancellation methods and Vaserstein-Suslin's symplectic methods is presented in [Swan: 1975].

(15) In January, 1976, D. Quillen and A. Suslin proved independently that Serre's Conjecture is true for *all* n, and for *all* fields k. Their results appeared, respectively, in [Quillen: 1976] and [Suslin: 1976a]. We shall now briefly describe Quillen's solution. For a ring R, let us say that an $R[t_1, \dots, t_n]$-module M is *extended* (*from* R) if there exists an R-module M_0 such that $M \cong R[t_1, \dots, t_n] \otimes_R M_0$. Note that if such an M_0 exists, it must be unique, since

$$\frac{M}{(t_1, \dots, t_n) \cdot M} \cong \frac{R[t_1, \dots, t_n]}{(t_1, \dots, t_n)} \otimes_R M_0 \cong R \otimes_R M_0 \cong M_0.$$

In [Quillen: 1976], the following remarkable local-global criterion for a module M to be extended is obtained:

Quillen's Patching Theorem. *Let R be a commutative ring, and let M be a finitely presented* (*) *$R[t_1, \dots, t_n]$-module. Then M is extended from R iff, for every maximal ideal $\mathfrak{m}$ of R, the localization $M_{\mathfrak{m}}$ is extended from $R_{\mathfrak{m}}$.*

Using this Patching Theorem, together with an earlier result of [Horrocks: 1964], Quillen showed that a f.g. projective module P over $k[t_1, \dots, t_n]$ must be extended from $k[t_1, \dots, t_{n-1}]$. Thus, P is a scalar extension of $P_0 = P/t_nP$, which is (clearly) finitely generated projective over $k[t_1, \dots, t_{n-1}]$. Invoking an inductive hypothesis on n, we may assume that P_0 is *free* over $k[t_1, \dots, t_{n-1}]$, so, upon scalar extension, P is also *free* over $k[t_1, \dots, t_n]$.

The above argument works already if k is a PID, in which case then we get the same conclusion (that P is free over $k[t_1, \dots, t_n]$). If k is, more generally, a Dedekind domain, the above argument yields the conclusion that P is extended from k. This can be pushed one step further, as follows.

Theorem (Quillen-Suslin). *If k is a commutative regular ring of Krull dimension* 2, *then any f.g. projective module P over $k[t_1, \dots, t_n]$ is extended from k.*

Quillen's proof of this consists of a reduction to the case $n = 1$, and then (by the Patching Theorem) to the case where k is local. With these two additional assumptions, P is known to be free by a theorem of Horrocks (1964) and Murthy (1966).

(*) For the definition, see (I.2.9).

The theorem above and the earlier work of Bass prompted the following natural extension of the original Serre Conjecture:

Bass-Quillen Conjecture. *If k is a commutative regular ring of Krull dimension $d < \infty$, then every f.g. projective module P over $k[t_1, \dots, t_n]$ is extended from k.*

Just as above, we can make a reduction to the case where $n = 1$ and k is local. In that special case, we need to show that P is free. From a theorem of Grothendieck, we can show that P is *stably* free. Thus, it seems tempting to raise the following question:

(H) *If R is Hermite, is $R[t]$ necessarily Hermite?*

If the answer to this is 'Yes', then the Bass-Quillen Conjecture would be true! However, (H) seems rather untractable, and there does not seem to exist much evidence for its truth. (This is why we raised (H) in the form of a question, rather than making it a conjecture.) The Bass-Quillen Conjecture, on the other hand, is much more reasonable, since it is a natural extension of Serre's Conjecture, and is known to be true for dimension $d \leqslant 2$ (by the Quillen-Suslin Theorem quoted above). In 1976/77, Lindel-Lütkebohmert and Mohan Kumar have independently proved that the Bass-Quillen Conjecture is true if the regular ring k is a formal power series ring in a finite number of (commuting) variables over a field.

(16) Shortly after Quillen and Suslin gave the first full solutions to Serre's Conjecture in January, 1976, some very ingenious short proofs of the Conjecture became available. Most notable among these are two elementary proofs given, respectively, by Vaserstein and Suslin, using the technique of the completion of unimodular rows. (See Chapter III.) While the discovery of alternative proofs is a rather common occurrence in mathematics after a famous conjecture is solved, the elementary proofs of Vaserstein and Suslin have had two remarkable side effects. First, they made the solution to Serre's Conjecture completely accessible to anyone with a basic knowledge of graduate algebra. Second, they have shown the full potential of the method of unimodular rows, and have thus established the study of unimodular rows as a new interesting theme for research in commutative ring theory.

This concludes our historical survey on the evolution of Serre's Problem from 1955 to 1976 (and only to 1976). What may not have been apparent from the above survey is that attempts at proving Serre's Conjecture in the 1960s have played a major role in stimulating the birth and development of another subject — Algebraic K-Theory. One of the pronounced goals in "classical" algebraic K-theory was, indeed, to provide new techniques and new insights with which to attack (and possibly solve) Serre's Conjecture. It is perhaps a bit of a pity that the final solution of Serre's Conjecture did not seem to have relied on Algebraic K-Theory in a really substantial manner. This, however, should hardly diminish the significant role Serre's Conjecture had played in giving early life to algebraic K-theory, which has since grown into an enormous subject, with relevance reaching far beyond what was envisioned in its original charter.

Some (indeed most) of the main developments sketched in the paragraphs above will be presented in detail in Chapters I-V of this book. This is followed by two more chapters covering some further "analogues" of Serre's Problem, namely, the "K_1–Analogue" (Chapter VI) and the "Quadratic Analogue" (Chapter VII). The work on both of these analogues was also essentially complete by the year 1977, although the "K_1–Analogue" did not appear in time to be taken into account in the original version of this book (which went to press in late 1977). The inclusion of the new Chapter VI now atones for this involuntary omission.

By design, this introductory survey covered the history of Serre's Problem in the 20-year period 1955–1976. In retrospect, however, 1976 was only the landmark year for Serre's Problem, but by no means the time for its grand finale. In fact, the research on Serre's Problem and its many ramifications and generalizations has continued to flourish at a steady pace from 1976 to the present day. A separate survey on the "post Quillen-Suslin" developments on Serre's Problem, from 1977 to the date of writing, is given in Chapter VIII, with a detailed running commentary on the highlights of these developments, followed by an extensive guide to the relevant literature. The discussions in that chapter are, therefore, a natural continuation of the present introductory survey.

The following sage words of J. Ewing et al. in the *Epilogue* of their article on mathematical problems(*) seem to be also a rather fitting description of the transition of Serre's Conjecture from its inception, gestation, to its partial and then full solution, and to the multitudinous developments that were subsequently made possible by its many generalizations.

> *Concepts, examples, methods, and facts continue to be discovered; problems get reformulated, placed in new contexts, better understood, and solved every day. .... Mathematics is alive, and is here to stay.*

Fortuitously, the suppression of the middle (in this case inessential) sentence in this quote has turned it into a little rhyme.

(*) J.H. Ewing, W.H. Gustafson, P.R. Halmos, S.H. Moolgavkar, W.H. Wheeler, and W.P. Ziemer: "American mathematics from 1940 to the day before yesterday", Amer. Math. Monthly **83** (1976), 503–516.

Chapter I

Foundations

§1. Projective Modules

Starting our exposition in the first gear, let us recall the definition of a projective module over a ring R.

Definition. A (left) R-module P is called R-*projective* (or just projective) if $\mathrm{Hom}_R(P, \cdot)$ is an exact functor from (left) R-modules to abelian groups.

Since $\mathrm{Hom}_R(P, \cdot)$ is always left exact, we need only require that the functor $\mathrm{Hom}_R(P, \cdot)$ be right exact in the above definition. In other words, we require that, for any R-epimorphism $g \in \mathrm{Hom}_R(A, B)$ (we write $A \overset{g}{\twoheadrightarrow} B$, following MacLane), and any $f \in \mathrm{Hom}_R(P, B)$, there exists an $f' \in \mathrm{Hom}_R(P, A)$ such that $gf' = f$:

$$\begin{tikzcd} & P \arrow[dl, dashed, "f'"'] \arrow[d, "f"] \\ A \arrow[r, two heads, "g"'] & B \end{tikzcd}$$

For example, *if P is free, then P is projective*. In fact, for P free, it is easy to complete the above diagram by defining f' suitably on a basis of P.

Proposition 1.1. *A direct sum $P = \bigoplus_\alpha P_\alpha$ is projective iff each P_α is projective.*

Proof. We have a natural equivalence of functors

$$\mathrm{Hom}_R\left(\bigoplus_\alpha P_\alpha, \ \cdot \ \right) \cong \prod_\alpha \mathrm{Hom}_R(P_\alpha, \ \cdot \ ).$$

Thus the left-hand-side is exact iff each $\mathrm{Hom}_R(P_\alpha, \cdot)$ is exact. $\square$

Proposition 1.2. *For any R-module P, the following statements are equivalent:*

(1) *P is projective;*
(2) *Any R-epimorphism $A \twoheadrightarrow P$ splits;*
(3) *P is a direct summand of a free R-module.*

Proof. (1) $\Rightarrow$ (2) is clear by applying the definition of projectivity to the diagram

$$\begin{tikzcd} & P \arrow[dl, dashed] \arrow[d, equal] \\ A \arrow[r, two heads] & P \end{tikzcd}$$

(2) $\Rightarrow$ (3). Take a big free module F mapping onto P. By (2) the epimorphism splits, so $F \cong P \oplus Q$ for some Q.

(3) $\Rightarrow$ (1) follows from (1.1.) and the fact that free modules are projective. $\square$

Corollary 1.3. *For any ring R, the following statements are equivalent:*

(1) *All short exact sequences of (left) R-modules split.*
(2) *All (left) R-modules are projective.*

We note, of course, that (1) is one of the many equivalent definitions for R to be an (artinian) semisimple ring.

For the convenience of the reader, we also recall:

Nakayama's Lemma 1.4. *Let $M \in \mathfrak{M}(R)$ (i.e. M is a finitely generated left R-module), and $J = \operatorname{rad} R$ (the Jacobson radical). Then $M = J \cdot M \Rightarrow M = 0$.*

Proof. Let $M = \sum_{i=1}^{r} R \cdot m_i$ with r minimal. Then $M = J \cdot M = \sum_{i=1}^{r} J \cdot m_i$. In particular, we can write $m_1 = j_1 m_1 + \cdots + j_r m_r$ with $j_i \in J$. But then $(1 - j_1)m_1 = j_2 m_2 + \cdots + j_r m_r$. Since $1 - j_1$ is not in any left maximal ideal and hence has a left inverse, we get $m_1 \in \sum_{i=2}^{r} R \cdot m_i$, in contradiction to the choice of r. $\square$

We may record the following alternative (and obviously equivalent) form of (1.4).

Nakayama's Lemma 1.5. *Let $N_0 \subseteq N$ be R-modules with $N/N_0 \in \mathfrak{M}(R)$. Then $N = N_0 + J \cdot N \Rightarrow N = N_0$.*

If we apply (1.4) to modules in $\mathfrak{P}(R)$ (i.e., finitely generated projective left R-modules), we obtain the following remarkable consequence.

Corollary 1.6. *Let "bar" denote reduction modulo $J = \operatorname{rad} R$. Let $Q \in \mathfrak{M}(R)$, $P \in \mathfrak{P}(R)$, and $\gamma \in \operatorname{Hom}_R(Q, P)$. If $\overline{\gamma} : \overline{Q} \to \overline{P}$ is an isomorphism, then γ is an isomorphism. If $Q, P \in \mathfrak{P}(R)$, and $\overline{Q} \cong \overline{P}$ as $\overline{R}$-modules, then $Q \cong P$ as R-modules.*

Proof. By the right-exactness of "bar" ($= R/J \otimes_R \ \cdot$), we have

$$\overline{\operatorname{coker}\gamma} \cong \operatorname{coker}\overline{\gamma} = 0.$$

Since coker $\gamma \in \mathfrak{M}(R)$ (being a quotient of P), Nakayama's Lemma yields coker $\gamma = 0$. Using $P \in \mathfrak{P}(R)$, we see that γ is a *split* surjection. This guarantees that $\overline{\ker\gamma} \cong \ker\overline{\gamma} = 0$. But we also have ker $\gamma \in \mathfrak{M}(R)$ (since it is a *direct* summand of Q), so "Nakayama" again implies ker $\gamma = 0$. Thus, γ is an isomorphism.

For the second conclusion of the corollary, suppose there exists some $\overline{R}$-isomorphism $g : \overline{Q} \to \overline{P}$, where $Q, P \in \mathfrak{P}(R)$. Since Q is R-projective, there exists $\gamma \in \operatorname{Hom}_R(Q, P)$ making the following diagram

$$\begin{tikzcd} Q \arrow[r, two heads] \arrow[d, dashed, "\gamma"'] & \overline{Q} \arrow[d, "g"] \\ P \arrow[r, two heads] & \overline{P} \end{tikzcd}$$

commutative. This means that $\overline{\gamma} = g$, so the paragraph above shows that $\gamma : Q \to P$ is an R-isomorphism. □

Corollary 1.7. *Let* $P \in \mathfrak{P}(R)$*, and* $z_1, \ldots, z_r \in P$*. Then* $\{z_i\}$ *form a free* R*-basis for* P *iff their images* $\{\overline{z}_i\}$ *form a free* $\overline{R}$*-basis for* $\overline{P}$*.*

Proof. For the "if" part, simply apply (1.6) to $\gamma : R^r \to P$ where $\gamma(e_i) = z_i$ for the unit vectors $\{e_i\}$. □

Remark. From the proofs of (1.4)–(1.7), it is clear that these results hold more generally if J is any ideal of R contained in rad R.

We say that a ring R is a *local ring* if $\overline{R} = R/J$ is a division ring for $J = \operatorname{rad} R$, i.e., if R has a unique maximal left (or right) ideal. In this case, *any* $\overline{R}$-module is free (being a vector space over a division ring), so we get:

Corollary 1.8. *Let R be a local ring, and* $J = \operatorname{rad} R$*. Then any* $P \in \mathfrak{P}(R)$ *is free. A set of elements* $z_1, \ldots, z_r \in P$ *form a free* R*-basis for* P *iff their images* $\overline{z}_1, \ldots, \overline{z}_r$ *form a vector space basis for* $\overline{P} = P/J \cdot P$ *over* $\overline{R} = R/J$*.*

Actually, [Kaplansky: 1958] has proved that *any* projective module (without any f.g. assumption) over a local ring is free, but we will not need this more general result in the sequel.

§2. Flat Modules, Faithfully Flat Modules and Finitely Presented Modules

Definition 2.1. *A right R-module N is said to be R-*flat *(or just flat) if $N \otimes_R -$ is an exact functor from left R-modules to abelian groups, i.e., if*

$$M_1 \to M_2 \to M_3 \text{ exact}$$
$$\Longrightarrow N \otimes_R M_1 \to N \otimes_R M_2 \to N \otimes_R M_3 \text{ exact}.$$

*We shall say that N is R-*faithfully flat *if, more strongly,*

$$M_1 \to M_2 \to M_3 \text{ exact}$$
$$\Longleftrightarrow N \otimes_R M_1 \to N \otimes_R M_2 \to N \otimes_R M_3 \text{ exact}.$$

Examples 2.2.

(1) Clearly, a nonzero right free module is always faithfully flat.

(2) A direct sum $\bigoplus_\alpha N_\alpha$ is flat iff each summand N_α is flat.

From (1) and (2), it follows evidently that

(3) Any right projective R-module is R-flat.

(4) (Scalar Extension) If N is R-flat (resp. faithfully flat), and $\phi : R \to R'$ is a ring homomorphism, then $N \otimes_R R'$ is R'-flat (resp. faithfully flat). In fact, let $\mathfrak{E} : M_1 \to M_2 \to M_3$ be any sequence of left R'-modules. We have a natural isomorphism

$$(*) \qquad (N \otimes_R R') \otimes_{R'} M_i \cong N \otimes_R M_i,$$

where M_i on the RHS is viewed as an R-module via the given homomorphism ϕ. Thus, we may identify the sequence $(N \otimes_R R') \otimes_{R'} \mathfrak{E}$ with $N \otimes_R \mathfrak{E}$. From this identification, the desired conclusions follow immediately.

In a similar vein, we can prove the following property:

(5) Let C be a commutative ring, and R, D be two C-algebras. If N is R-flat (resp. faithfully flat), then $D \otimes_C N$ is $D \otimes_C R$-flat (resp. faithfully flat). The proof is analogous to that of (4): if M is any (left) $D \otimes_C R$-module, we observe that there is a natural isomorphism

$$(**) \qquad (D \otimes_C N) \otimes_{D \otimes_C R} M \cong N \otimes_R M,$$

taking $(d \otimes n) \otimes m$ to $n \otimes (d \otimes 1) \cdot m$. Here, M on the RHS is viewed as a left R-module via the homomorphism

$$i : R \to D \otimes_C R, \quad i(r) = 1 \otimes r.$$

The isomorphism $(**)$ enables us essentially to pass from left $D \otimes_C R$-modules back to left R-modules, which easily gives what we want.

Theorem 2.3. *For any right R-module N, the following are equivalent:*

(1) *N is faithfully flat*;
(2) *N is flat, and for any left R-module M, $M \neq 0 \Rightarrow N \otimes_R M \neq 0$*;
(3) *N is flat, and, for any maximal left ideal $\mathfrak{m} \subset R$, $N \cdot \mathfrak{m} \neq N$.*

Proof. (1)$\Rightarrow$(2). Suppose $N \otimes_R M = 0$. Then the exactness of $N \otimes_R 0 \to N \otimes_R M \to N \otimes_R 0$ implies the exactness of $0 \to M \to 0$!

(2)$\Rightarrow$(3). By (2), $R/\mathfrak{m} \neq 0 \Rightarrow N \otimes_R R/\mathfrak{m} \cong N/N \cdot \mathfrak{m} \neq 0$.

(3)$\Rightarrow$(2). To show (2), we may assume that M is cyclic (since N is already flat). Say $M \cong R/I$, where $I \subsetneq R$ is a left ideal. Let $\mathfrak{m}$ be a maximal left ideal containing I. Then $N \otimes_R M \cong N/N \cdot I$ has an epimorphic image $N/N \cdot \mathfrak{m} \neq 0$.

(2)$\Rightarrow$(1). Let $M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3$ be a sequence of left R-modules such that $N \otimes_R M_1 \xrightarrow{f'} N \otimes_R M_2 \xrightarrow{g'} N \otimes_R M_3$ is exact (where $f' = N \otimes f$, $g' = N \otimes g$). Since $N \otimes_R -$ is an exact functor,

$$N \otimes_R \operatorname{im}(gf) \cong \operatorname{im}(g'f') = 0.$$

By (2), this implies that $gf = 0$; that is, $M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3$ is a zero-sequence. Similarly, the exactness of $N \otimes_R -$ guarantees that $N \otimes_R \frac{\ker g}{\operatorname{im} f} \cong \frac{\ker g'}{\operatorname{im} f'} = 0$, and, by (2) again, this implies $\ker g = \operatorname{im} f$. $\square$

Definition. We say that a ring homomorphism $R \xrightarrow{\phi} R'$ is flat (resp. faithfully flat) if R' is R-flat (resp. R-faithfully flat) under the right R-action induced by ϕ.

Note that *if $\phi : R \to R'$ is faithfully flat, ϕ must be a monomorphism.*(*) To see this, we need only show that $R' \otimes_R R \xrightarrow{R' \otimes \phi} R' \otimes_R R'$ is a monomorphism. But $R' \otimes_R R \cong R'$, and $R' \to R' \otimes_R R'$ by $r' \mapsto r' \otimes 1$ has clearly a left inverse (in the category of left R'-modules) given by $r'_1 \otimes r'_2 \to r'_1 r'_2$.

Example 2.4. Let R be a ring, and S be a *central* multiplicative set in R; R_S shall denote the localization of R at S. (Some authors prefer the notation $S^{-1}R$; we shall use this too whenever it proves to be more convenient.) More generally, for any R-module M, we write $M_S \; (\cong R_S \otimes_R M)$ for the localization of M at S. It is a standard fact that $R \to R_S$ *is flat*, i.e., $M \mapsto M_S$ *is an exact functor*. (However, if S contains a non-unit s, then $R \to R_S$ won't be faithfully flat, since $R/R \cdot s \neq 0$ localizes to zero at S.)

Example 2.5. A polynomial extension $R \subset R[t]$ (inclusion homomorphism) is always faithfully flat. This is clear since, for any left R-module M:

$$R[t] \otimes_R M = \left(\bigoplus_i t^i R \right) \otimes_R M = \bigoplus_i \left(t^i \otimes_R M \right).$$

(*) We shall say, in this case, that R' is a *faithfully flat extension* of R.

Example 2.6. If the homomorphisms $R \xrightarrow{\phi} R'$, $R' \xrightarrow{\psi} R''$ are both flat (resp. faithfully flat), then so is $R \xrightarrow{\psi\phi} R''$. For instance, if $S \subseteq R$ is as in (2.4), then $R \to R_S[t_1, \ldots, t_n]$ is flat.

Example 2.7. Let $\mathfrak{A}$ be an ideal in a commutative noetherian ring R, and let $\hat{R}$ be the $\mathfrak{A}$-adic completion of R. Then the natural homomorphism $R \to \hat{R}$ is flat; it is faithfully flat iff $\mathfrak{A} \subseteq \operatorname{rad} R$. It will take us too far afield to give a complete proof of this. Thus, for the details, we refer the reader to [Matsumura: 1970, pp. 170–172].

Proposition 2.8. *Let $R' \supseteq R$ be a faithfully flat extension, and let M be any left R-module. Then $M \in \mathfrak{M}(R)$ iff $R' \otimes_R M \in \mathfrak{M}(R')$.*

Proof. ("If" part): Choose $m_1, \ldots, m_r \in M$ such that $\{1 \otimes m_i\}$ generate $R' \otimes_R M$ as a left R'-module, and let $M_0 = \sum_i R \cdot m_i$. The map $R' \otimes_R M_0 \to R' \otimes_R M$ is onto. Since R' is faithfully flat as a right R-module, $M_0 \to M$ must already be onto. □

As we shall see, an analogous version of this proposition also holds for the class of finitely presented modules, which we now introduce.

Definition 2.9. A left R-module M is said to be *finitely presented* if there exists an exact sequence $R^m \to R^n \to M \to 0$, for suitable natural numbers m, n.

Example. Any $P \in \mathfrak{P}(R)$ is clearly finitely presented.

Example. If R is left noetherian, then any $M \in \mathfrak{M}(R)$ is automatically finitely presented.

Remark 2.10. Let M be finitely presented. If $M' \in \mathfrak{M}(R)$ and $f : M' \to M$ is an R-epimorphism, then $K = \ker f \in \mathfrak{M}(R)$. In fact, take $R^m \to R^n \to M \to 0$ as in (2.9). By the freeness of R^n, we can construct a commutative diagram

$$\begin{tikzcd}
 & R^m \arrow[r] \arrow[d, "g"] & R^n \arrow[r] \arrow[d, "h"] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & K \arrow[r] & M' \arrow[r, "f"] & M \arrow[r] & 0
\end{tikzcd}$$

By an easy diagram chase, we have $\operatorname{coker} g \cong \operatorname{coker} h$. This is f.g. over R (since M' is), so the exact sequence $R^m \xrightarrow{g} K \to \operatorname{coker} g \to 0$ implies that K is also f.g. over R.

Using (2.10), we can now prove the analog of (2.8) for finitely presented modules.

Proposition 2.11. *Let $R' \supseteq R$ be a faithfully flat extension, and let M be any left R-module. Then M is finitely presented over R iff $R' \otimes_R M$ is finitely presented over R'.*

Proof. ("If" part): From (2.8), we know that $M \in \mathfrak{M}(R)$. Thus, there exists an exact sequence $0 \to K \to R^n \to M \to 0$. This tensors up to an exact sequence

$$0 \to R' \otimes_R K \to R'^n \to R' \otimes_R M \to 0.$$

By (2.10), $R' \otimes_R K \in \mathfrak{M}(R')$, so by (2.8) again, we have $K \in \mathfrak{M}(R)$. If we take a suitable R^m that maps onto K, then we get $R^m \to R^n \to M \to 0$. □

One of the nicest features of finitely presented modules is that, often, theorems about such modules can be proved via a reduction to the case of free modules of finite rank. The following general result is a good illustration of the procedures involved.

Proposition 2.12. *Let F_1, F_2 be left exact contravariant functors from $\mathfrak{M}(R)$ to abelian groups, and let $\theta : F_1 \to F_2$ be a natural transformation. If $\theta(R^n) : F_1(R^n) \to F_2(R^n)$ is an isomorphism (resp. monomorphism) for every free module R^n, then $\theta(M) : F_1(M) \to F_2(M)$ is also an isomorphism (resp. monomorphism) for every finitely presented (resp. finitely generated) R-module M.*

Proof. Let M be finitely presented. Applying $F_1 \xrightarrow{\theta} F_2$ to $R^m \to R^n \to M \to 0$, we get a commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & F_1(M) \arrow[r] \arrow[d, "\theta(M)"] & F_1(R^n) \arrow[r] \arrow[d, "\theta(R^n)"] & F_1(R^m) \arrow[d, "\theta(R^m)"] \\
0 \arrow[r] & F_2(M) \arrow[r] & F_2(R^n) \arrow[r] & F_2(R^m)
\end{tikzcd}$$

The theorem now follows from the 4-lemma (i.e., a variant of the 5-lemma). If M is only f.g. instead of finitely presented, we take $R^n \to M \to 0$ and argue just with the square on the left above. □

A typical application of this principle is as follows.

Proposition 2.13. *Let $R \xrightarrow{\phi} R'$ be a flat homomorphism such that R is commutative and $\phi(R)$ lies in the center of R'. Let M be a finitely presented (resp. f.g.) left R-module, and N any left R-module. Then the natural map*

$$\sigma : R' \otimes_R \operatorname{Hom}_R(M, N) \longrightarrow \operatorname{Hom}_{R'}(R' \otimes_R M, R' \otimes_R N)$$

is an isomorphism (resp. monomorphism).

Proof. The "natural" map σ is given by

$$\sigma(r_1' \otimes g)(r_2' \otimes m) = r_1' r_2' \otimes g(m),$$
$$\text{where } r_i' \in R',\ m \in M,\ g \in \operatorname{Hom}_R(M, N).$$

(The commutativity hypotheses on R and $\phi(R)$ guarantee that $\operatorname{Hom}_R(M, N)$ has a well-defined left R-module structure, *and* that the σ above is well-defined.) *Regarding N as fixed*, we set

$$F_1(M) = R' \otimes_R \operatorname{Hom}_R(M, N),$$
$$F_2(M) = \operatorname{Hom}_{R'}(R' \otimes_R M,\ R' \otimes_R N).$$

Clearly, F_1, F_2 are left exact contravariant functors, and the map σ above (for varying M) gives a natural transformation: $F_1 \to F_2$. According to (2.12), it is sufficient to check the conclusion of (2.13) for $M = R^n$, i.e., to check that σ is an isomorphism (resp. monomorphism) for $M = R^n$. But

$$F_1(R^n) = R' \otimes_R \operatorname{Hom}_R(R^n, N) \cong (R' \otimes_R N)^n,$$
$$F_2(R^n) \cong \operatorname{Hom}_{R'}(R'^n,\ R' \otimes_R N) \cong (R' \otimes_R N)^n,$$

so everything is clear. □

We shall now give some applications of (2.13). The following two propositions are part of a general method which has come to be known as "faithfully flat descent."

Proposition 2.14. *Let R' be a faithfully flat extension of a subring R in the center of R'. Let $f : N \to M$ be a homomorphism of left R-modules, with M finitely presented over R. Then $N \xrightarrow{f} M$ is a split surjection of R-modules iff*

$$R' \otimes_R N \xrightarrow{R' \otimes_R f} R' \otimes_R M$$

is a split surjection of R'-modules.

Proof. Since R' is faithfully flat as a right R-module, we know that f is surjective iff $R' \otimes_R f$ is surjective. Moreover, for any surjective f:

$$\begin{aligned}
f \text{ splits} &\Longleftrightarrow \operatorname{Hom}_R(M, N) \to \operatorname{Hom}_R(M, M) \text{ is onto}\\
&\Longleftrightarrow R' \otimes_R \operatorname{Hom}_R(M, N) \to R' \otimes_R \operatorname{Hom}_R(M, M) \text{ is onto}\\
&\overset{(2.13)}{\Longleftrightarrow} \operatorname{Hom}_{R'}(R' \otimes_R M,\ R' \otimes_R N)\\
&\qquad\qquad \to \operatorname{Hom}_{R'}(R' \otimes_R M,\ R' \otimes_R M) \text{ is onto}\\
&\Longleftrightarrow R' \otimes_R f \text{ splits.}
\end{aligned}$$

This proves the last statement in the Proposition. □

Proposition 2.15. *Let R' be a faithfully flat extension of a subring R in the center of R'. Then, for any left R-module M,*

$$M \in \mathfrak{P}(R) \Longleftrightarrow R' \otimes_R M \in \mathfrak{P}(R').$$

Proof. ("If" part) $R' \otimes_R M \in \mathfrak{P}(R')$ implies that $R' \otimes_R M$ is finitely presented, so, by (2.11), the R-module M is also finitely presented. From (2.14), it follows that any R-surjection $N \twoheadrightarrow M$ splits. This implies that $M \in \mathfrak{P}(R)$, by (1.2). □

In the result (2.13), we need the hypothesis that $\phi(R)$ lies in the center of R' to guarantee that σ and its domain $R' \otimes_R \mathrm{Hom}_R(M, N)$ are both well-defined. If, for a *particular* flat homomorphism $\phi : R \to R'$, we can define a suitable substitute for σ and for $R' \otimes_R \mathrm{Hom}_R(M, N)$, then the arguments given in (2.13) can be repeated to give analogous results for the substitute quantities. The following results, (2.13′) and (2.13″), are two important instances of this.

Proposition 2.13′. *For any ring R, let $R[T] = R[t_1, \ldots, t_n]$. Let M be a finitely presented* (*resp. f.g.*) *R-module, and N any R-module. Then the natural map*

$$\sigma : \ \mathrm{Hom}_R(M, N)[T] \to \mathrm{Hom}_{R[T]}(M[T], N[T])$$

is an isomorphism (*resp. monomorphism*).

Here, for any abelian group G, we understand by $G[T]$ the abelian group of all "polynomials" in $t_1, \ldots, t_n$ with coefficients in G. (Of course,

$$M[T] \cong R[T] \otimes_R M, \qquad N[T] \cong R[T] \otimes_R N,$$

so they are $R[T]$-modules.) The "natural" map σ is induced by

$$\sigma\left(t_1^{r_1} \cdots t_n^{r_n} \cdot g\right)\left(t_1^{s_1} \cdots t_n^{s_n} \otimes m\right) = t_1^{r_1+s_1} \cdots t_n^{r_n+s_n} \otimes g(m)$$

for any $g \in \mathrm{Hom}_R(M, N)$ and any $m \in M$. Here, σ is well-defined without any assumptions on R.

Proposition 2.13″. *Let S be a central multiplicative set in a ring R. Then, for any finitely presented (resp. f.g.) R-module M and any R-module N, the natural map*

$$\sigma : \ (\mathrm{Hom}_R(M, N))_S \to \mathrm{Hom}_{R_S}(M_S, N_S)$$

is an isomorphism (*resp. monomorphism*).

Here, $\mathrm{Hom}_R(M, N)$ is no longer an R-module. However, it is a k-module for $k = \mathrm{center}(R)$. Since we assumed that $S \subseteq k$, we can localize $\mathrm{Hom}_R(M, N)$ at S *as a k-module* – this is how we define the domain of σ. The map σ in this case is induced by $\sigma\left(\frac{g}{s}\right)\left(\frac{m}{s'}\right) = \frac{g(m)}{ss'}$, where $s, s' \in S$, $g \in \mathrm{Hom}_R(M, N)$, and $m \in M$. Again, σ is well-defined without any further assumptions.

We record now some useful consequences of (2.13″).

Corollary 2.16. *Keep the hypotheses in* (2.13″) *and assume that both M, N are finitely presented. Suppose $\varphi : M_S \to N_S$ is an R_S-isomorphism. Then there exist $f \in S$ and an R_f-isomorphism $M_f \to N_f$ that localizes to φ.* (*Subscript f denotes localization at the central multiplicative set* $\{f^n : n \geqslant 0\}$.)

Proof. Let $\psi : N_S \to M_S$ be the inverse of φ. By (2.13″), there exists $\Phi : M \to N$ such that $\varphi = \frac{\Phi}{g_1}\left(= \frac{1}{g_1}\Phi_S\right)$ and there exists $\Psi : N \to M$ such that $\psi = \frac{\Psi}{g_2}\left(= \frac{1}{g_2}\Psi_S\right)$. Let $g = g_1 g_2 \in S$, and let

$$\varphi' = \frac{1}{g_1}\Phi \in \mathrm{Hom}_{R_g}(M_g, N_g),$$

$$\psi' = \frac{1}{g_2}\Psi \in \mathrm{Hom}_{R_g}(N_g, M_g).$$

Thus, $\psi'\varphi' \in \mathrm{End}_{R_g}(M_g)$ localizes to the identity in $\mathrm{End}_{R_S}(M_S)$. But

$$\left(\mathrm{End}_{R_g}(M_g)\right)_S \cong \mathrm{End}_{R_S}\left(M_S\right),$$

so $f_1(\psi'\varphi' - 1) = 0$ for some $f_1 \in S$. Therefore, $\psi'\varphi' = \mathrm{Id.}$ on M_{gf_1}. Similarly, we can find $f_2 \in S$ such that $\varphi'\psi' = \mathrm{Id.}$ on N_{gf_2}. Now for $f = gf_1 f_2 \in S$, φ' gives an isomorphism $M_f \to N_f$ that localizes to φ. □

Corollary 2.17. *Let R, S, M be as in* (2.16). *If $M_S \cong R_S^n$, then there exists $f \in S$ such that $M_f \cong R_f^n$.*

Proof. Apply (2.16) to the finitely presented modules M and $N = R^n$. □

§3. Local-Global Methods

In this section, we collect a few standard results which show how we can derive information about modules and module homomorphisms by looking at their localizations. *Throughout this section, we shall take R to be a commutative ring (with identity).* We write Spec R for the set of prime ideals of R, and Max R for the set of maximal ideals of R. (These are called the *prime ideal spectrum* and the *maximal ideal spectrum* of R.) For any $\mathfrak{p} \in \mathrm{Spec}\, R$, and any R-module M, we shall write $M_\mathfrak{p}$ for the localization of M at the multiplicative set $R \setminus \mathfrak{p}$. Note that the localization $R_\mathfrak{p}$ is a commutative local ring with unique maximal ideal $\mathfrak{p}R_\mathfrak{p}$.

Proposition 3.1. *The R-module $\bigoplus_\mathfrak{m} R_\mathfrak{m}$ $(\mathfrak{m} \in \mathrm{Max}\, R)$ is faithfully flat.*

Proof. As we already observed in (2.4), each $R_\mathfrak{m}$ is R-flat, so $\bigoplus_\mathfrak{m} R_\mathfrak{m}$ is also R-flat by (2.2)(2). Since $\mathfrak{m}R_\mathfrak{m} \neq R_\mathfrak{m}$ for each $\mathfrak{m} \in \mathrm{Max}\, R$, it follows from (2.3)(3) that $\bigoplus_\mathfrak{m} R_\mathfrak{m}$ is faithfully flat. (We could equally well have used (2.3)(2) for this. In fact, assume $M_\mathfrak{m} = 0$ for all $\mathfrak{m} \in \mathrm{Max}\, R$. For $m \in M$, $\mathrm{ann}_R(m)$ intersects $R \setminus \mathfrak{m}$ for all $\mathfrak{m} \in \mathrm{Max}\, R$. Thus, $\mathrm{ann}_R(m)$ must be the unit ideal.) □

Unfortunately, $\bigoplus_\mathfrak{m} R_\mathfrak{m}$ is not a ring with unity, and there is no natural map from R to $\bigoplus_\mathfrak{m} R_\mathfrak{m}$, so most of our earlier results on faithfully flat extensions cannot be directly applied here. We might try to replace $\bigoplus_\mathfrak{m} R_\mathfrak{m}$ by $\prod_\mathfrak{m} R_\mathfrak{m}$, which is indeed a ring with unity, with a ring homomorphism $R \to \prod_\mathfrak{m} R_\mathfrak{m}$ given by the "diagonal" map. This, however, doesn't help since direct product *does not* commute with tensor product, and, as a result, $\prod_\mathfrak{m} R_\mathfrak{m}$ need not be R-flat in general.

Nevertheless, we can derive the following consequences of (3.1).

Corollary 3.2. *A sequence of R-modules $M' \xrightarrow{f} M \xrightarrow{g} M''$ is exact iff $M'_{\mathfrak{m}} \xrightarrow{f_{\mathfrak{m}}} M_{\mathfrak{m}} \xrightarrow{g_{\mathfrak{m}}} M''_{\mathfrak{m}}$ is exact for all $\mathfrak{m} \in \operatorname{Max} R$. In particular, $\varphi : N \to M$ is injective (resp. surjective, bijective) iff $\varphi_{\mathfrak{m}} : N_{\mathfrak{m}} \to M_{\mathfrak{m}}$ is injective (resp. surjective, bijective) for all $\mathfrak{m} \in \operatorname{Max} R$.*

Corollary 3.3. *Let $\varphi : N \to M$ be a homomorphism of R-modules, with M finitely presented over R. Then φ is a split surjection iff $\varphi_{\mathfrak{m}} : N_{\mathfrak{m}} \to M_{\mathfrak{m}}$ is a split surjection of $R_{\mathfrak{m}}$-modules for all $\mathfrak{m} \in \operatorname{Max} R$.*

(The proof of this is identical to that of (2.14), with one minor difference: instead of using (2.13), we have to use (2.13$''$).)

Corollary 3.4. *For any finitely presented module P over R, the following statements are equivalent:*

(1) *P is R-projective;*
(2) *$P_{\mathfrak{m}}$ is $R_{\mathfrak{m}}$-projective for all $\mathfrak{m} \in \operatorname{Max} R$;*
(3) *$P_{\mathfrak{p}}$ is $R_{\mathfrak{p}}$-free for all $\mathfrak{p} \in \operatorname{Spec} R$.*

Proof. (1) $\Leftrightarrow$ (2) follows from (3.3), and the fact that P is projective iff any surjection $N \to P$ splits. (3) $\Rightarrow$ (2) is trivial. For (2) $\Rightarrow$ (3), note that if $\mathfrak{p} \subseteq \mathfrak{m} \in \operatorname{Max} R$, then $P_{\mathfrak{p}}$ is obtained by further localizing $P_{\mathfrak{m}}$. Since $P_{\mathfrak{m}}$ is $R_{\mathfrak{m}}$-projective, we conclude that $P_{\mathfrak{p}}$ is $R_{\mathfrak{p}}$-projective, and hence $R_{\mathfrak{p}}$-free by (1.8). □

The difference between the hypothesis of (3.4) and that of (2.15) should not go unnoticed. For the 'if' part of (2.15), we did not need any finiteness assumptions on P, whereas in (3.4), we assumed that P is *finitely presented* over R. *In fact*, (3.4) *is not true in general if P is only assumed to be* f.g. For example, if R is von Neumann regular (that is, $a \in aRa$ for every $a \in R$), then (3.4)(2) is certainly true because all the localizations $R_{\mathfrak{p}}$ are fields (for more details, see the proof of (V.2.14)). But if R is *non-noetherian* and $\mathfrak{A}$ is a non-f.g. ideal, the cyclic module $P = R/\mathfrak{A}$ cannot be R-projective. (If it was, then $0 \to \mathfrak{A} \to R \to P \to 0$ splits, and $\mathfrak{A}$ would have to be principal.)

The implication (1) $\Rightarrow$ (3) in (3.4) leads to the important notion of "rank" for f.g. projective modules:

Definition. If $P \in \mathfrak{P}(R)$, and $\mathfrak{p} \in \operatorname{Spec} R$, we define $\operatorname{rk}_{\mathfrak{p}} P = \operatorname{rank}_{R_{\mathfrak{p}}} P_{\mathfrak{p}}$.

Thus, we get a function $\operatorname{rk} P$: $\operatorname{Spec} R \to \mathbb{Z}$. If we write $\operatorname{rk} P \geqslant r$, we shall mean that $\operatorname{rk}_{\mathfrak{p}} P \geqslant r$ for all $\mathfrak{p} \in \operatorname{Spec} R$. If $\operatorname{rk} P \equiv r$, we shall say that P has (constant) rank r.

On $\operatorname{Spec} R$, we have a natural topology, called the *Zariski topology*, for which the closed sets are given by

$$V(\mathfrak{A}) = \{\, \mathfrak{p} \in \operatorname{Spec} R \mid \mathfrak{p} \supseteq \mathfrak{A} \,\},$$

where $\mathfrak{A}$ is any ideal in R. The complement of such a closed set is

$$\bigcup_{f\in\mathfrak{A}} \{\,\mathfrak{p}\in \operatorname{Spec} R \mid \mathfrak{p}\not\ni f\,\}.$$

Thus, a basis of open sets is given by the sets

$$D(f)=\{\,\mathfrak{p}\in \operatorname{Spec} R \mid \mathfrak{p}\not\ni f\,\}.$$

Note that the Zariski topology on Spec R is always (quasi)-compact. In fact, suppose a family of basic open sets $D(f_\alpha)$ cover Spec R. The family f_α must then generate the unit ideal, and so a finite number of these already generate R. The corresponding $D(f_\alpha)$'s will then cover Spec R.

Proposition 3.5. *If* $P\in\mathfrak{P}(R)$, *then* $\operatorname{rk} P:\operatorname{Spec} R\to\mathbb{Z}$ *is continuous with respect to the Zariski topology on* Spec R, *and the discrete topology on* $\mathbb{Z}$. *In particular,* $\operatorname{rk} P$ *is bounded.*

Proof. Say $\operatorname{rk}_\mathfrak{p} P=n$, i.e., $P_\mathfrak{p}\cong R_\mathfrak{p}^n$. By (2.17), there exists $f\notin\mathfrak{p}$ such that $P_f\cong R_f^n$. Thus $\operatorname{rk} P$ takes constant value n on the neighborhood $D(f)$ of $\mathfrak{p}$. This implies the continuity of $\operatorname{rk} P$, and its boundedness follows from the compactness of the prime spectrum Spec R. □

Remark 3.5'. Actually, the result in (3.5) can be improved into a sharper "iff" statement. Let P be any f.g. R-module (not assumed to be projective) such that $R_\mathfrak{p}$ is $R_\mathfrak{p}$-free for every prime ideal $\mathfrak{p}$ (so that the function $\operatorname{rk} P:\operatorname{Spec} R\to\mathbb{Z}$ is defined). Then $P\in\mathfrak{P}(R)$ *if and only if the function* $\operatorname{rk} P$ *is locally constant* (that is, every point of Spec R has a neighborhood on which $\operatorname{rk} P$ is constant). This statement is a well-known result from Bourbaki's "Commutative Algebra" (Theorem 1 in §5.2 in Chapter II). The "if" part of this result will be used only once in our book (in (IV.5)), so we shall simply refer to Bourbaki's book for this result, instead of using up extra space to prove it here.

Corollary 3.6. *If* R *has no nontrivial idempotents, then every* $P\in\mathfrak{P}(R)$ *has constant rank.*

Proof. The hypothesis on R is tantamount to the fact that Spec R is connected (easy exercise). Thus, any continuous function Spec $R\to\mathbb{Z}$ must be constant. □

This Corollary applies, for instance, to any R that is a (commutative) integral domain. In that case, we can calculate $\operatorname{rk} P$ by using the prime ideal $\mathfrak{p}=(0)$:

$$\operatorname{rk} P=\operatorname{rk}_{(0)} P_{(0)}=\dim_K K\otimes_R P\quad (K=\text{quot. field of } R).$$

This is, of course, the "usual" definition of the rank for finitely generated modules over such integral domains.

We say that a commutative ring R is *semilocal* if R has only a finite number of maximal ideals. The following provides a generalization of (1.8) (in the commutative case).

Theorem 3.7. *Let R be a commutative semilocal ring. If $P \in \mathfrak{P}(R)$ has constant rank n, then $P \cong R^n$.*

Proof. Suppose $\operatorname{Max} R = \{\mathfrak{m}_1, \ldots, \mathfrak{m}_r\}$. For any i $(1 \leqslant i \leqslant r)$, pick $x_{i1}, \ldots, x_{in} \in P$ such that their images in $P_{\mathfrak{m}_i}$ form a free basis for $P_{\mathfrak{m}_i}$. By the Chinese Remainder Theorem, there exist $x_j \in P$ $(1 \leqslant j \leqslant n)$ such that $x_j \equiv x_{ij} \pmod{\mathfrak{m}_i P}$ for all $1 \leqslant i \leqslant r$. Using (1.8.), we see that $x_1, \ldots, x_n$ form a free basis for *each* $P_{\mathfrak{m}_i}$ upon localization. If we define $f : R^n \to P$ by $f(e_j) = x_j$ for all unit vectors e_j, then $f_{\mathfrak{m}_i}$ is an isomorphism for every i. From (3.2), it follows that f itself is an isomorphism, so $P \cong R^n$ as desired. $\square$

For later reference, we shall take this opportunity to make some other useful remarks about semilocal rings.

Proposition 3.8. *A commutative ring R is semilocal iff $R/\operatorname{rad} R$ is artinian.*

Proof. If $R/\operatorname{rad} R$ is artinian, it is a finite direct product of fields. Thus, $\operatorname{Max} R = \operatorname{Max}(R/\operatorname{rad} R)$ is finite. Conversely, if $\operatorname{Max} R = \{\mathfrak{m}_1, \ldots, \mathfrak{m}_r\}$, we have an embedding of R-modules $R/\operatorname{rad} R \to \bigoplus_{i=1}^r R/\mathfrak{m}_i$. Since the RHS has a composition series, the same is true for $R/\operatorname{rad} R$. Thus, $R/\operatorname{rad} R$ satisfies both the ACC and the DCC for its ideals. $\square$

Corollary 3.9. (1) *Any commutative artinian ring is semilocal.*

(2) *If R is a commutative semilocal ring, and A is a commutative R-algebra f.g. as an R-module, then A is also semilocal.*

Proof. (1) follows from the Proposition. For (2), let $J = \operatorname{rad} R$. For any $\mathfrak{m} \in \operatorname{Max} A$, we must have $J \cdot A \subseteq \mathfrak{m}$ (for otherwise $\mathfrak{m} + J \cdot A = A$ implies $\mathfrak{m} = A$ by Nakayama's Lemma). Hence $J \cdot A \subseteq \operatorname{rad} A$. Viewing $A/\operatorname{rad} A$ as a f.g. module over the artinian ring R/J, we see that $A/\operatorname{rad} A$ is artinian. Thus, A is semilocal by (3.8). $\square$

To close this section, we shall record some more results which show how we can "patch together" local information to arrive at global information. These results will play a rather crucial role in some of our later chapters. (Recall that R continues to denote a commutative ring in the current section.)

Proposition 3.10. *Let M be any R-module, and let $f, g \in R$ be two comaximal elements (in the sense that $fR + gR = R$). Then the following is a pullback diagram (so M can be "retrieved" from M_f and M_g):*

$$\begin{tikzcd}
M \arrow[r] \arrow[d] & M_g \arrow[d] \\
M_f \arrow[r] & M_{fg}
\end{tikzcd}$$

Proof. First, $M \to M_f \oplus M_g$ is injective. In fact, if $m \in M$ localizes to 0 in M_f and M_g, then $f^r m = 0 = g^r m$ for some sufficiently large r. But f^r and g^r are still comaximal, so $m = 0 \in M$. For the rest, write

$$S = \{ f^n : n \geqslant 0 \}, \qquad T = \{ g^n : n \geqslant 0 \}.$$

Suppose $\frac{m}{s} \in M_f = M_S$ and $\frac{n}{t} \in M_g = M_T$ localize to the same element in M_{fg}. We may assume that $sn = tm \in M$. (In fact, we have $(s't')(tm - sn) = 0$ for some $s' \in S$, $t' \in T$. Thus, $(tt')(s'm) = (ss')(t'n)$, so we may replace $\frac{m}{s}$ by $\frac{s'm}{s's}$, and $\frac{n}{t}$ by $\frac{t'n}{t't}$.) Write $xs + yt = 1$ $(x, y \in R)$, and set $q = xm + yn \in M$. Then

$$sq = (xs)m + y(sn) = (xs)m + (yt)m = m,$$

and similarly, $tq = n$. Thus, we have $q = \frac{m}{s} \in M_S$, and $q = \frac{n}{t} \in M_T$. □

Corollary 3.11. (Patching Homomorphisms) *Let M, N be R-modules, and let f, g be as above. Suppose we have two homomorphisms*

$$\beta_f : M_f \longrightarrow N_f, \quad \beta_g : M_g \longrightarrow N_g$$

that localize to the same homomorphism $M_{fg} \to N_{fg}$. Then,

(1) *there exists a unique R-homomorphism $\beta : M \to N$ that localizes to β_f and β_g ;*
(2) *furthermore, β is an isomorphism (resp. monomorphism, epimorphism) iff β_f and β_g are both isomorphisms (resp. monomorphisms, epimorphisms).*

Proof. The existence and uniqueness of β amount essentially to the fact that the pullback construction is a functor; see the diagram:

$$\begin{tikzcd}
M \arrow[rr] \arrow[dd] \arrow[dr, dashed, "\beta"] & & M_g \arrow[dr, "\beta_g"] \arrow[dd] & \\
& N \arrow[rr] \arrow[dd] & & N_g \arrow[dd] \\
M_f \arrow[rr] \arrow[dr, "\beta_f"'] & & M_{fg} \arrow[dr] & \\
& N_f \arrow[rr] & & N_{fg}
\end{tikzcd}$$

For (2), the "only if" part is trivial. For the "if" part, the "monomorphism" and the "isomorphism" cases follow from an easy diagram chase. The "epimorphism" case apparently *does not* follow from diagram chase. However, if β_f, β_g are onto, then for any $\mathfrak{p} \in \operatorname{Spec} R = D(f) \cup D(g)$, $\beta_{\mathfrak{p}} : M_{\mathfrak{p}} \to N_{\mathfrak{p}}$ is also onto. Applying (3.2), we do know that β itself must be onto. □

Corollary 3.12. (Patching Modules) *Let f, g be as above. Let Y indexpatching modules be an R_f-module, Z be an R_g-module, such that $Y_g \cong Z_f$ as R_{fg}-modules. Then there exists (up to isomorphism) a unique R-module X such that $X_f \cong Y$ and $X_g \cong Z$. Furthermore,*

(3.13) $Y \in \mathfrak{M}(R_f),\ Z \in \mathfrak{M}(R_g) \Longleftrightarrow X \in \mathfrak{M}(R)$
($\mathfrak{M}(R)$ *means the class of f.g. modules*);

(3.14) *Y and Z finitely presented over R_f and R_g* $\Longleftrightarrow$
X finitely presented over R;

(3.15) $Y \in \mathfrak{P}(R_f),\ Z \in \mathfrak{P}(R_g) \Longleftrightarrow X \in \mathfrak{P}(R)$.

Proof. We identify Y_g with Z_f (using the given isomorphism), and let X be the pullback formed from the following diagram:

$$\begin{tikzcd} X \arrow[r, dashed, "j"] \arrow[d, dashed, "i"'] & Z \arrow[d, "g"] \\ Y \arrow[r] & Y_g \cong Z_f \end{tikzcd}$$

Clearly, X has a natural R-module structure, and, by (3.10), X is our *unique* candidate. Let $i' : X_f \to Y$ be the R_f-homomorphism induced by $i : X \to Y$. *We claim that i' is an isomorphism*:

(1) *i' is injective*. Indeed, if $i'(x/f^n) = 0\ (x \in X)$, then $i(x) = 0$ and so $j(x) \in Z$ localizes to zero in Z_f. This implies that $f^m \cdot j(x) = j(f^m x) = 0$ for some $m \geqslant 0$. Since we have also $i(f^m x) = 0$, the pullback diagram implies that $f^m x = 0$, so $x/f^n = 0 \in X_f$.

(2) *i' is surjective*. Let $y \in Y$, and let $z/f^r \in Z_f$ correspond to $y/1$ under the isomorphism $Y_g \cong Z_f$. Then $f^r y \in Y$ and $z \in Z$ localize to the same element in $Y_g \cong Z_f$, so there exists $x \in X$ with $i(x) = f^r y$ and $j(x) = z$. But then $i'(x/f^r) = y$, as desired.

We have thus proved that $X_f \cong Y$, and, by symmetry, $X_g \cong Z$. For (3.13), (3.14) and (3.15), the implications "$\Leftarrow$" are trivial, so we need only prove "$\Rightarrow$" in each case.

(3) *Assume that $Y \in \mathfrak{M}(R_f)$* and $Z \in \mathfrak{M}(R_g)$. Pick $x_1, \ldots, x_s \in X$ such that $i(x_1), \ldots, i(x_s)$ form R_f-generators for $Y \cong X_f$, and that $j(x_1), \ldots, j(x_s)$ form R_g-generators for $Z \cong X_g$. Let X' be the R-submodule of X generated by $x_1, \ldots, x_s$. Then $X' \subseteq X$ induces isomorphisms $X'_f \to X_f$ and $X'_g \to X_g$, so (3.11) implies that $X = X' \in \mathfrak{M}(R)$.

(4) *Assume that Y, Z are finitely presented, respectively, over R_f* and R_g. By (3.13), there exists a free module $F = R^t\ (t < \infty)$ with an epimorphism $F \twoheadrightarrow X$, say with kernel K. Localizing at f, we have

$$0 \to K_f \to F_f \to X_f\ (\cong Y) \to 0.$$

By (2.10), $K_f \in \mathfrak{M}(R_f)$, and similarly, $K_g \in \mathfrak{M}(R_g)$. By (3.13), we conclude that $K \in \mathfrak{M}(R)$, so X is finitely presented over R.

(5) Assume that $Y \in \mathfrak{P}(R_f)$, $Z \in \mathfrak{P}(R_g)$. To see that X is R-projective, let us show that $\operatorname{Hom}_R(X, A) \xrightarrow{\varepsilon_*} \operatorname{Hom}_R(X, B)$ is *onto* for any R-epimorphism $\varepsilon : A \twoheadrightarrow B$. The domain and range of ε_* are R-modules, so, by (3.11), it is enough to show that $(\varepsilon_*)_f$, $(\varepsilon_*)_g$ are epimorphisms. Localizing at f, we get

$$\big(\operatorname{Hom}_R(X, A)\big)_f \xrightarrow{(\varepsilon_*)_f} \big(\operatorname{Hom}_R(X, B)\big)_f.$$

Since X is finitely presented over R (by (3.15), and by hypotheses), the domain and range of $(\varepsilon_*)_f$ can be identified with $\operatorname{Hom}_{R_f}(X_f, A_f)$ and $\operatorname{Hom}_{R_f}(X_f, B_f)$, by (2.13$''$). Via these identifications, we see that $(\varepsilon_*)_f$ is *onto* since $X_f \cong Y \in \mathfrak{P}(R_f)$. Similarly $(\varepsilon_*)_g$ is onto. $\square$

Remark 3.16. The idea behind (3.10) through (3.15) is that we can construct (and analyze) "global" data on Spec $R = D(f) \cup D(g)$ by "gluing together" local data on $D(f)$ and $D(g)$, as long as these local data are "compatible" on the intersection $D(f) \cap D(g) = D(fg)$. This basic technique is typical of the modern sheaf-theoretic approach to algebraic geometry. Of course, we have to be careful about any global conclusions we want to make from this "gluing" process. For instance, in (3.15), we see that "gluing" f.g. projective modules Y and Z yields a f.g. projective R-module X; but if Y and Z are free, X need not be free. For a trivial example, consider the case where $f,\ g \in R$ are nonzero idempotents, with $f + g = 1$. Here, $X := Rf \in \mathfrak{P}(R)$ localizes to R_f and (0) respectively, which are free; but X itself is not free. This example of X might seem a bit pathological, since the two localizations X_f and X_g are free of *different* ranks. (Here, $D(f) \cap D(g)$ is the empty set.) However, there is no lack of examples of nonfree rank n projective modules X over *commutative domains* R such that X_f and X_g are both free (of rank n), where $f,\ g$ are comaximal in R. For an explicit example of this nature, see (5.13).

Remark 3.17. The results (3.10) to (3.15) could have been presented a bit more generally. Instead of working with the comaximal elements f and g as above, we could have considered two multiplicative sets $S,\ T \subseteq R$ with the property that

(3.18) $\quad s \in S,\ t \in T \Longrightarrow s \textit{ and } t \textit{ are comaximal in } R.$

All the results above remain valid if we replace M_f, M_g by M_S and M_T – and the proofs carry over *verbatim*. Note that, under the hypothesis (3.18), any given $\mathfrak{p} \in \operatorname{Spec} R$ is disjoint either from S or from T, so for any R-module M, $M_{\mathfrak{p}}$ is either a localization of M_S or of M_T.

Greedy-Greedy-Greedy: To what extent do the results above remain valid if R is not assumed commutative, but S and T are assumed to be *central* multiplicative sets satisfying (3.18)?

§4. Stably Free Modules and Hermite Rings

In this section, R will denote an arbitrary ring, unless otherwise specified. For technical reasons, it will be slightly more convenient for us to consider *right* modules instead of left modules in this section. However, module homomorphisms will continue to be written on the left.

Definition 4.1. A (right) R-module P is said to be *stably free of type* m $(0 \leqslant m < \infty)$ if $P \oplus R^m$ is free.(*) A module is said to be *stably free* if it is stably free of type m for some m. (Such a module P is, of course, projective, and if $m = 0$, P is free.)

There is a good reason why we want to restrict the m above to be a *finite* cardinal number. In fact, if we do not impose such a restriction, *any* projective module P would have satisfied the property in (4.1). The proof of this is a famous trick of Eilenberg: let $P \oplus Q = E$ be free, and let $F = E \oplus E \oplus \dots$ (also free). Then,

$$\begin{aligned} P \oplus F &\cong P \oplus E \oplus E \oplus \dots \\ &\cong P \oplus (Q \oplus P) \oplus (Q \oplus P) \oplus \dots \\ &\cong (P \oplus Q) \oplus (P \oplus Q) \oplus \dots \\ &\cong E \oplus E \oplus \dots \\ &\cong F\,! \end{aligned}$$

Another observation we can make about Definition 4.1 is that it is of interest mainly in the case where P is finitely generated. In fact, another beautiful trick, due to Gabel, shows that:

Proposition 4.2. *If P is stably free, but not finitely generated, then P is actually free.*

Proof. Say $P \oplus R^m \cong F = \text{free with basis } e_i,\, i \in I$. Since P is not f.g., I must be an *infinite* set. View P as $\ker(F \overset{f}{\twoheadrightarrow} R^m)$ (for some epimorphism f). For a sufficiently large finite subset $I_0 \subseteq I$, we have $F_0 = \sum_{i \in I_0} e_i \cdot R \xrightarrow{f} R^m$ already onto. Thus $F = P + F_0$. Writing $Q = P \cap F_0$, we have two short exact sequences:

$$\begin{array}{l} 0 \to Q \to P \to P/Q \to 0, \\ 0 \to Q \to F_0 \to R^m \to 0. \end{array}$$

By Noether's Isomorphism Theorem, $P/Q \cong F/F_0 \cong \sum_{i \in I \setminus I_0} e_i \cdot R$. Since $I \setminus I_0$ is infinite, we can write $P/Q \cong R^m \oplus F_1$ for some free module F_1. Now both of the sequences above split, so

$$P \cong Q \oplus (R^m \oplus F_1) \cong F_0 \oplus F_1 = \text{free},$$

as desired. ∎

(*) This terminology is a little bit informal, since the "type" of P is not a uniquely determined integer. (If P has type m, then it also has type $m + k$ for any $k \geqslant 0$.) If we want to associate a uniquely defined nonnegative integer to P, we can define its "minimal type" to be the smallest integer $m \geqslant 0$ such that $P \oplus R^m$ is free.

Because of (4.2), *we shall henceforth restrict our attention to modules* $P \in \mathfrak{M}(R)$. Such a P is stably free of type m iff $P \cong \ker(R^n \overset{f}{\twoheadrightarrow} R^m)$ for a suitable (split) epimorphism f. If M is the $m \times n$ matrix associated with f, then M is right invertible, i.e., there exists an $n \times m$ matrix N such that $MN = I_m$. Conversely, any right invertible $m \times n$ matrix M *defines* a f.g. stably free (right) R-module P of type m, namely,

$$P = \left\{ \alpha = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} : \; M \cdot \alpha = 0 \right\} \quad \text{(the "solution space" of } M).$$

In this way, the study of f.g. stably free right R-modules becomes equivalent to the study of right invertible rectangular matrices over R.

The following furnishes a criterion for the freeness of $P = \ker(R^n \overset{f}{\twoheadrightarrow} R^m)$.

Proposition 4.3. *$P = \ker(R^n \overset{f}{\twoheadrightarrow} R^m)$ is free iff f can be lifted to an isomorphism $\hat{f} : R^n \to R^m \oplus R^r$ (for some r) such that $\pi \hat{f} = f$, where $\pi : R^m \oplus R^r \to R^m$ is the projection onto the first factor.*

$$\begin{tikzcd} & R^m \bigoplus R^r \arrow[d, "\pi"] \\ R^n \arrow[ur, "\hat{f}", "\cong"'] \arrow[r, two heads, "f"'] & R^m \end{tikzcd}$$

Proof. Suppose the isomorphism $\hat{f}$ exists. Then $\ker f \cong \ker \pi = R^r$. Conversely, suppose there exists $g : P \xrightarrow{\cong} R^r$. Write $R^n = Q \oplus P$, so the restriction of f to Q gives an isomorphism $f_0 : Q \to R^m$. Then $f_0 \oplus g : R^n \to R^m \oplus R^r$ clearly gives the desired isomorphism $\hat{f}$. $\square$

Note that in the situation above, $R^n \cong R^m \oplus R^r$ need not imply $n = m + r$, in general. We say that a ring R satisfies the *(right) invariant basis property* (IBP) if, for any integers $s, t \geqslant 0$,

$$R^s \cong R^t \text{ (as right modules)} \implies s = t.$$

For example, a basic theorem in linear algebra says that division rings satisfy this IBP. Now if R, S are two rings for which there exists a ring homomorphism $R \to S$, an obvious tensor product argument shows that, *whenever S satisfies IBP, R* also does. Using the above observations, it is immediate that *local rings satisfy IBP, and that nonzero commutative rings satisfy IBP*. It can also be shown that *nonzero right* *noetherian rings satisfy IBP*; we leave this as an easy exercise.(*) For a ring R of any of the above types, we can infer that $n = m + r$ in the situation of (4.3).

Let us now give the matrix theoretic interpretation of (4.3). Let M denote the $m \times n$ matrix corresponding to f, and let N denote the $(m + r) \times n$ matrix corresponding to $\hat{f}$, if $\hat{f}$ exists. The condition $\pi \hat{f} = f$ says that M *is a submatrix of* N, consisting of its first m rows. The condition that $\hat{f}$ be an isomorphism says that N is a (*not necessarily square*!) invertible matrix; i.e., there exists another matrix N', of size $n \times (m + r)$, such that $NN' = I_{m+r}$, and $N'N = I_n$. Thus, we arrive at the following matrix theoretic version of (4.3).

Proposition 4.3$'$. *For any right invertible $m \times n$ matrix M, the (stably free) solution space of M is free iff M can be completed to an invertible matrix by adding a suitable number of new rows.*

Definition 4.4. We say that $(b_1, \ldots, b_n)$ is a (*right*) *unimodular row* if the row matrix $(b_1, \ldots, b_n)$ is right invertible; i.e. if $\sum_{i=1}^{n} b_i R = R$. The set of such unimodular rows will be denoted (throughout this book) by $\mathrm{Um}_n(R)$.

Corollary 4.5. *For any ring R, the following are equivalent:*

(1) *Any f.g. stably free right R-module is free;*
(1)$'$ *Any f.g. stably free right R-module of type* 1 *is free;*
(2) *Any right unimodular row over R can be completed to an invertible matrix (by adding a suitable number of new rows).*

Definition 4.6. Rings satisfying the above (equivalent) properties (1), (1)$'$, (2) will be called *(right) Hermite rings*.

(We should point out that the term "Hermite ring" (or "H-ring") has been used in different senses by some other authors. For a more detailed discussion on this, see the Appendix to §4, and the *Notes* at the end of this Chapter.)

Examples 4.7.

(0) For any ring R, let $\overline{R} = R/\mathrm{rad}\, R$. If $\overline{R}$ is left or right Hermite, then so is R. (This follows readily from (1.6).)

(1) Any semisimple ring (in the sense of Noether) is left and right Hermite. This is an easy consequence of the Jordan-Hölder Theorem; we leave it as an easy exercise.

(2) If every f.g. projective right R-module is free, R is clearly right Hermite. This applies, for instance, to any local ring (by (1.8)), and to the ring of integers $\mathbb{Z}$ (by the Fundamental Theorem of Abelian Groups).

(3) Any commutative semilocal ring is Hermite, by (3.7). But, just as in (1) and in (2), commutativity is not essential here. The generalization to the noncommutative setting goes as follows. Taking hint from (3.8), we define a (general) ring R to be

(*)For this as well as for other valuable information on IBP, see [Cohn: 1966b].

semilocal if the factor ring $\overline{R} = R/\mathrm{rad}\, R$ is artinian.[†] In this case, $\overline{R}$ is a semisimple ring. From (0) and (1) above, therefore, it follows that any semilocal ring R is (left and right) Hermite. This applies, in particular, to any 1-sided artinian ring.

(4) Any Dedekind domain R is Hermite. In fact, the Steinitz-Chevalley structure theory for f.g. torsionfree modules over R implies that the cancellation law holds in $\mathfrak{P}(R)$. In particular, f.g. stably free modules over R are free, so R is Hermite. For a more general result, see (5.8) below. For more results on completing rectangular matrices into square matrices of prescribed determinants over Dedekind domains (with informative historical remarks), see the paper of Gustafson, Moore and Reiner listed in the references on Chapter VIII.

(5) A commutative ring R is called a *Bézout ring* if every f.g. ideal in R is principal. A theorem of Albrecht (proved, for instance, in (2.29) of the author's book [Lam: 1999]) implies that, over a Bézout *domain* R, any f.g. projective module is free. *In particular, a Bézout domain is Hermite.* Without using Albrecht's theorem, we can also directly verify the Hermite property (4.5)(2) for a Bézout domain R as follows. Given $(b_1, \ldots, b_n) \in \mathrm{Um}_n(R)$ (with say $n \geqslant 3$), write

$$b_1 R + \cdots + b_{n-1} R = dR, \quad \text{with } b_i = dc_i \ (i < n).$$

We may assume $d \neq 0$, in which case $(c_1, \ldots, c_{n-1})$ must be unimodular (since R is a domain). By induction, we may thus assume that there is a matrix $C \in \mathrm{GL}_{n-1}(R)$ with first row $(c_1, \ldots, c_{n-1})$. Since $dR + b_n R = R$, there exists an equation $dx - b_n y = 1$. Then the matrix $B = \begin{pmatrix} d & 0 & b_n \\ 0 & I_{n-2} & 0 \\ y & 0 & x \end{pmatrix}$ has determinant 1. Now $B \begin{pmatrix} C & 0 \\ 0 & 1 \end{pmatrix}$ is invertible, and has first row $(b_1, \ldots, b_n)$.

The proof above is folklore in the subject, and has been rediscovered many times over, as recently as in K. Kedlaya's paper "A p-adic local monodromy theorem", Ann. of Math. **160** (2004), 93–184. This paper, incidentally, offers several new interesting classes of examples of Bézout domains from both p-adic analysis and algebraic geometry.

Example (5) is, in fact, a special case of a result in [Kaplansky: 1949]. According to Kaplansky's result, a Bézout domain is "K-Hermite" (in the sense of (4.23) below), and a K-Hermite ring is always Hermite. This K-Hermite connection will be explained in more detail in the Appendix to this section.

(6) According to [Steger: 1966], a ring R is called "ID" if every idempotent square matrix over R is diagonalizable. In this paper, Steger showed that *any commutative ID ring is Hermite.* This result also implies the one in (5) above, since (it can be shown that) any Bézout domain is ID. We will not prove Steger's results here, since they are not needed for our exposition.

Now let us bring the general linear group $\mathrm{GL}_n(R)$ into play. This group acts by right multiplication on $\mathrm{Um}_n(R)$, the set of right unimodular rows of length n over

[†] We can interpret "artinian" here to mean left artinian or right artinian: these properties are well-known to be equivalent for $\overline{R}$.

R. If two rows $f,\ g \in \mathrm{Um}_n(R)$ are conjugate under this action, we shall write $f \sim g$; this defines an equivalence relation on $\mathrm{Um}_n(R)$. The equivalence classes of $\mathrm{Um}_n(R)$ under $\sim$ are just the orbits of the $\mathrm{GL}_n(R)$-action.

Proposition 4.8. *The orbits of* $\mathrm{Um}_n(R)$ *under the* $\mathrm{GL}_n(R)$*-action are in* 1-1 *correspondence with the isomorphism classes of right* R*-modules* P *for which* $P \oplus R \cong R^n$*. Under this correspondence, the orbit of* $(1, 0, \ldots, 0)$ *corresponds to the isomorphism class of the free module* R^{n-1}.

Proof. To any $(b_1, \ldots, b_n) \in \mathrm{Um}_n(R)$, we can associate $P = P(b_1, \ldots, b_n)$, the "solution space" (i.e., kernel) of the R-homomorphism

$$(b_1, \ldots, b_n):\ R^n \twoheadrightarrow R^1.$$

Such P is a typical module for which $P \oplus R \cong R^n$. Suppose

$$P(b_1, \ldots, b_n) \overset{\beta}{\cong} P(c_1, \ldots, c_n)$$

for another $(c_1, \ldots, c_n) \in \mathrm{Um}_n(R)$. Then we can complete the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & P(b_1,\ldots,b_n) \arrow[r] \arrow[d, "\beta"] & R^n \arrow[r, "{(b_1,\ldots,b_n)}"] \arrow[d, dashed] & R^1 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & P(c_1,\ldots,c_n) \arrow[r] & R^n \arrow[r, "{(c_1,\ldots,c_n)}"] & R^1 \arrow[r] & 0
\end{tikzcd}$$

with a suitable isomorphism $R^n \dashrightarrow R^n$ (note that the rows are *split* exact). If $M \in \mathrm{GL}_n(R)$ denotes the matrix of this isomorphism, we will have

$$(b_1, \ldots, b_n) = (c_1, \ldots, c_n) \cdot M.$$

Conversely, if this equation holds for some $M \in \mathrm{GL}_n(R)$, then the automorphism $R^n \to R^n$ defined by M induces an isomorphism of the two kernels, yielding $P(b_1, \ldots, b_n) \cong P(c_1, \ldots, c_n)$. □

Corollary 4.9. *A row* $(b_1, \ldots, b_n) \in \mathrm{Um}_n(R)$ *is completable to a* square *invertible matrix iff* $P(b_1, \ldots, b_n) \cong R^{n-1}$*, iff* $(b_1, \ldots, b_n) \sim (1, 0, \ldots, 0)$.

Note that, because of the possible failure of IBP, it may happen that $(b_1, \ldots, b_n) \in \mathrm{Um}_n(R)$ can be completed to a *rectangular* invertible matrix (so that $P(b_1, \ldots, b_n)$ is free), but cannot be completed to a *square* invertible matrix. Fortunately, in most applications of the theory of stably free modules, we do not have to deal with such delicate situations. To simplify matters (and to make life easier), *we shall assume, for the rest of this section, that* R *is a commutative ring*. In this case, we have the following very pleasant facts.

Proposition 4.10. (1) *Suppose* $b_1b_1' + \cdots + b_nb_n' = 1$ *in a (commutative) ring* R. *If*

$$P := P(b_1, \ldots, b_n), \quad P' := P(b_1', \ldots, b_n'), \quad \textit{and}$$
$$Q := R^n/(b_1, \ldots, b_n)R, \quad Q' := R^n/(b_1', \ldots, b_n')R,$$

then $P' \cong P^* \cong Q \cong Q'^*$, *where* M^* *denotes the dual of an* R*-module* M *(that is,* $M^* = \mathrm{Hom}_R(P, R)$).

(2) *Suppose* $b_1b_1' + \cdots + b_nb_n' = 1$ *and* $b_1b_1'' + \cdots + b_nb_n'' = 1$ *in* R. *Then*

$$P(b_1', \ldots, b_n') \cong P(b_1'', \ldots, b_n'').$$

Proof. (1) By definition, P is the kernel of the map $\varphi : R^n \to R$ defined by $(b_1, \ldots, b_n)$, and P' is the kernel of the map $\varphi' : R^n \to R$ defined by $(b_1', \ldots, b_n')$. Taking duals on the split exact sequence $0 \to P \to R^n \xrightarrow{\varphi} R \to 0$, we get

$$0 \to R^* \xrightarrow{\varphi^*} (R^n)^* \longrightarrow P^* \to 0.$$

If we use the canonical identifications $R^* = R$ and $(R^n)^* = R^n$, the map φ^* above sends $1 \in R$ to $(b_1, \ldots, b_n) \in R^n$. Since $b_1b_1' + \cdots + b_nb_n' = 1$, this φ^* splits the surjection φ'. Thus,

$$P^* \cong \mathrm{coker}(\varphi^*) \cong \ker(\varphi') \cong P'.$$

Also, $P^* \cong \mathrm{coker}(\varphi^*) = R^n/(b_1, \ldots, b_n)R = Q$. By symmetry, we have $P'^* \cong Q'$, so we are done..

(2) This follows from (1) since both $P(b_1', \ldots, b_n')$ and $P(b_1'', \ldots, b_n'')$ are isomorphic to P^*. $\square$

Remark. (A) In our text, we have viewed $P(b_1, \ldots, b_n)$ as the stably free R-module (of type 1) "associated to" the right unimodular row $(b_1, \ldots, b_n)$. In the commutative case, we could have equally well associated to $(b_1, \ldots, b_n)$ the module $R^n/(b_1, \ldots, b_n)R$, which is also stably free of type 1. The result (4.10)(1) shows that the choice does not matter too much, as these two modules are mutually dual.

(B) In (4.10)(1) above, *if* n *is even, we have in fact* $P' \cong P^* \cong P$; that is, the module P is self-dual. This will be proved later in III.6.7(1). However, if n is odd and $\geqslant 3$, $P' \cong P^*$ need not be isomorphic to P; see III.6.12.

(C) For alternative ways to prove (4.10)(2), see Remark III.6.2(A). Actually, under the hypotheses in (4.10)(2), a stronger conclusion is possible if $n \geqslant 3$: see (III.6.1)(1).

(D) The result (4.10) is by no means special to the type 1 case. We could have stated its higher type analogue by considering an $m \times n$ matrix M, with $MN = I_m$ where N is $n \times m$. We then let $P = \ker(M)$, $P' = \ker(N^t)$, and

$$Q = R^n/(\text{row space of } M), \quad Q' = R^n/(\text{row space of } N^t).$$

The same proof as above gives readily $P' \cong P^* \cong Q \cong Q'^*$. We stated this in (4.10) only in the type 1 case ($m = 1$) since that is the case on which most current work on stably free modules is focused.

The assumption that R is a commutative ring has at least two advantages. First, R satisfies IBP (if $R \neq 0$), so "invertible" matrices automatically mean *square* invertible matrices. Second, we can talk about the *rank* of f.g. projective R-modules. If $P \oplus R^m \cong R^n$, we get $\operatorname{rk} P = n - m$. Using the existence of rank, we may recast some of our earlier results in more convenient forms. For instance, in (4.8), the orbits of $\mathrm{Um}_n(R)$ under the $\mathrm{GL}_n(R)$-action will be in 1-1 correspondence with the isomorphism classes of f.g. stably free R-modules of type 1 and rank $n-1$. In particular, in (4.5), we can repeat the arguments with a suitable rank restriction, to get:

(4.5)$_d$ *For any integer $d \geqslant 0$, the following statements are equivalent* :

(a) *Any f.g. stably free R-module of rank $> d$ is free*;
(b) *Any unimodular row over R of length $\geqslant d+2$ can be completed to a (square) invertible matrix over R*;
(c) *For $n \geqslant d+2$, $\mathrm{GL}_n(R)$ acts transitively on $\mathrm{Um}_n(R)$.*

Refining Definition (4.6), we say that a commutative ring R is *d-Hermite* if it satisfies (a) (or (b), (c)) above. Of course, "0-Hermite" is just "Hermite". Also, since a unimodular row of length 2 is always completable to a matrix of determinant 1, we see that "1-Hermite" is still synonymous with "Hermite."

Remark. In the next chapter, we will show that, *if R is noetherian, and has Krull dimension d, then R is d-Hermite*. This result is due to Bass; a full proof of it will be given in (II.7.3′).

The connection between Hermite rings and Serre's Conjecture is as follows. For $R = k[t_1, \ldots, t_d]$ (k a field), it will be shown that *any $P \in \mathfrak{P}(R)$ is stably free* (see II.5.9). In view of this result, Serre's Problem can be translated into the following question about the polynomial ring $R = k[t_1, \ldots, t_d]$:

> *Is $R = k[t_1, \ldots, t_d]$ a Hermite ring ? Equivalently, can every $(b_1, \ldots, b_n) \in \mathrm{Um}_n(R)$ be completed into an $n \times n$ matrix over R with determinant $\in k \setminus \{0\}$?*

It is well-known that $R = k[t_1, \ldots, t_d]$ is noetherian ("Hilbert Basis Theorem"), and has Krull dimension d (see (II.7)). Thus, by Bass's result quoted in the remark above, we know that R is d-Hermite. To solve Serre's Conjecture in full, we need to improve this statement to: "R is Hermite".

To better prepare ourselves for working with stably free modules over commutative rings, let us prove two basic results, (4.11) and (4.12) below, in the commutative setting. Both of these results are known to be false for noncommutative rings; see the end of this section.

Theorem 4.11. *(R commutative) Suppose $P \oplus R^{n-1} \cong R^n$ (i.e., P is stably free of rank* 1*). Then $P \cong R$.*

Proof. The standard proof of this is based on the use of exterior powers. Let $k \geqslant 2$. For any $\mathfrak{m} \in \operatorname{Max} R$,

$$(\Lambda^k P)_{\mathfrak{m}} \cong \Lambda^k(P_{\mathfrak{m}}) \cong \Lambda^k(R_{\mathfrak{m}}) = 0.$$

Thus, $\Lambda^k P = 0$. We have then

$$\begin{aligned} R &\cong \Lambda^n R^n \cong \Lambda^n(P \oplus R^{n-1}) \cong \bigoplus_{i+j=n} (\Lambda^i P \otimes \Lambda^j R^{n-1}) \\ &\cong \Lambda^1 P \otimes \Lambda^{n-1} R^{n-1} \cong \Lambda^1 P \cong P. \end{aligned}$$

For readers who prefer not to use the exterior powers, we can also offer the following more "interior" proof. Represent P as the solution space of a right invertible $(n-1) \times n$ matrix M. It is sufficient to show that *the maximal minors* $b_1, \ldots, b_n$ of M are unimodular. [For, if $a_1 b_1 + \cdots + a_n b_n = 1$, we can complete M to a matrix of determinant 1 by adding a last row $a_1, \ldots, a_n$, with appropriate signs. By (4.3′), this implies that P is free.] Assume not; then there exists a maximal ideal $\mathfrak{m}$ containing $b_1, \ldots, b_n$. Working over $\overline{R} = R/\mathfrak{m}$, we do have $\overline{P} \cong \overline{R}$, so $\overline{M}$ can be completed to a matrix X in $\mathrm{GL}_n(\overline{R})$. But, expanding $\det X$ along the last row, we get $\det X = \overline{0}$, a contradiction. □

Our next goal is to prove a result of Bass on the existence of unimodular elements in a type 1 stably free module of odd rank. By definition, an element α in a right R-module P is called *unimodular* if αR is free on the singleton basis $\{\alpha\}$ and αR is a direct summand of P. The following two observations are immediate (and are valid over any ring R, with or without commutativity):

(A) $\alpha \in P$ is unimodular iff $f(\alpha) = 1$ for some $f \in \operatorname{Hom}_R(P, R)$. (In particular, an element $\alpha = (a_1, \ldots, a_n)$ in the free right module R^n is unimodular iff α is a *left* unimodular row; that is, iff $\sum Ra_i = R$.)

(B) If $Q = P \oplus X$ for some R-module X and $\alpha \in P$, then α is unimodular in P iff it is unimodular in Q

An easy criterion for the existence of unimodular elements in a stably free module of type 1 is given in (1) below, with a very pleasant application in (2).

Theorem 4.12. *Let R be a commutative ring, and let $P = P(b_1, \ldots, b_n)$, where $(b_1, \ldots, b_n) \in \mathrm{Um}_n(R)$.*

(1) *P has a unimodular element iff $b_1 a_1 + \cdots + b_n a_n = 0$ for some $(a_1, \ldots, a_n) \in \mathrm{Um}_n(R)$.*

(2) *If P has odd rank (i.e. n is even), then P has a unimodular element. (In particular, a type 1 indecomposable stably free module $\ncong R$ must have even rank.)*

(3) *If P has rank 2 (i.e. $n = 3$), then P has a unimodular element iff (b_1, b_2, b_3) is completable.*

Proof. (1) We have $P = \ker(\varphi)$, where $\varphi\colon R^n \to R$ is defined by the row matrix $(b_1, \ldots, b_n)$. Suppose the a_i 's exist. Then the column vector $\alpha := (a_1, \ldots, a_n)^t \in R^n$ lies in P. The assumption that $(a_1, \ldots, a_n) \in \mathrm{Um}_n(R)$ implies that α is unimodular in R^n, so it is also unimodular in P. The converse is proved similarly. (Note that (1) is still formally true in the case where $n = 1$. In this case, $P = 0$, and the two conditions in (1) can hold only when R is the zero ring.)

(2) In order to talk about "rank", we are assuming here that $R \neq 0$. If rank $P = n-1$ is odd, we may write $n = 2m$ for some integer m. Taking

$$(a_1, \ldots, a_n) := (b_2, -b_1, b_4, -b_3, \ldots, b_{2m}, -b_{2m-1}) \in \mathrm{Um}_n(R),$$

we have $\sum b_i a_i = 0$, so the "if" part of (1) applies.

(3) The "if" part is trivial. Conversely, assume P has a unimodular element. Then $P \cong R \oplus P_0$ for some module P_0. Clearly, P_0 is stably free of rank 1, so $P_0 \cong R$ by (4.11). Thus, $P \cong R^2$, and (4.9) implies that (b_1, b_2, b_3) is completable. □

The result (4.12)(2) was first observed by H. Bass, who proved it by using the method of symplectic modules. The simple proof above did not depend on such methods. We note, in passing, that (4.12)(2) is also true for rank P even, provided that $n \geqslant 3$ *and one of the* b_i *'s is a perfect square.* This result, due to Ravi Rao and Raja Sridharan, will be proved later in (III.5.8).

We record the following nice consequence of (4.12)(2).

Corollary 4.13. *A commutative ring* R *is Hermite iff all f.g. stably free* R*-modules of even rank are free. (A similar statement can also be made for* d *-Hermite rings.)*

Proof. ("If" part) Say P is stably free of rank $2m-1$. By hypothesis, $P \oplus R \cong R^{2m}$, so P has type 1. By (4.12)(2), $P \cong R \oplus Q$ for some Q, and, by hypothesis again, $Q \cong R^{2m-2}$. Thus, $P \cong R^{2m-1}$. □

Remark 4.14. Let R be a commutative noetherian ring of finite Krull dimension d. By Bass's result stated in the Remark following $(4.5)_d$, R is d-Hermite. To show that R is Hermite, it would suffice, according to (4.13), to check the freeness of f.g. stably free modules P of rank $2r$ in the finite range $2 \leqslant 2r \leqslant d$. For instance, to show that $R = k[x, y, z]$ is Hermite (k a field), it would suffice to verify that all rank two stably free R-modules are free. This is exactly what was done in [Murthy-Towber: 1974] in the case where k is algebraically closed, which then proved Serre's Conjecture in three variables over such ground fields k.

In contrast to (4.12)(2), if $P \oplus R \cong R^{2m-1}$, P *need not* have a direct summand $\cong R$, or any nontrivial direct summand for that matter. To give an example of this, consider the coordinate ring of the real n-sphere,

$$R_n = \mathbb{R}[t_0, \ldots, t_n]/(t_0^2 + \cdots + t_n^2 - 1).$$

Let $x_0, \ldots, x_n$ be the images of $t_0, \ldots, t_n$ in R_n, and let $\tau^{(n)}$ be the unimodular row $(x_0, \ldots, x_n) \in \mathrm{Um}_{n+1}(R_n)$. The solution space $P^{(n)}$ of $\tau^{(n)}$ defines a f.g. stably free

R_n-module of type 1 and rank n. Note that, since $x_0^2 + \cdots + x_n^2 = 1$, this R_n-module is *self-dual* according to (4.10). It corresponds to the algebraic tangent bundle of the n-sphere S^n, and is free iff the algebraic tangent bundle is trivial. One may ask: *for which values of n will $P^{(n)}$ be R_n-free?*

For $n = 1, 3, 7$, this is indeed the case. In fact, for $n = 1, 3, 7$, it is not difficult to complete $\tau^{(n)}$ to an invertible matrix: just look at

$$\begin{pmatrix} x_0 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ x_1 & -x_0 & x_3 & -x_2 & x_5 & -x_4 & -x_7 & x_6 \\ x_2 & -x_3 & -x_0 & x_1 & x_6 & x_7 & -x_4 & -x_5 \\ x_3 & x_2 & -x_1 & -x_0 & x_7 & -x_6 & x_5 & -x_4 \\ x_4 & -x_5 & -x_6 & -x_7 & -x_0 & x_1 & x_2 & x_3 \\ x_5 & x_4 & -x_7 & x_6 & -x_1 & -x_0 & -x_3 & x_2 \\ x_6 & x_7 & x_4 & -x_5 & -x_2 & x_3 & -x_0 & -x_1 \\ x_7 & -x_6 & x_5 & x_4 & -x_3 & -x_2 & x_1 & -x_0 \end{pmatrix}.$$

The three indicated matrices all have determinant 1 (and are written down by considering the multiplication rule in the complex numbers, the quaternions, and the Cayley numbers, respectively). Thus, we have:

Proposition 4.15. *$P^{(n)}$ is free over R_n for $n = 1, 3, 7$.*

For $n = 2$, however, $P^{(2)}$ is not free over R_2. The best and quickest way to prove this is to use a little bit of topology. Indeed, suppose $\tau^{(2)} = (x_0, x_1, x_2)$ can be completed into $\begin{pmatrix} x_0 & x_1 & x_2 \\ f & g & h \\ f' & g' & h' \end{pmatrix}$, with determinant equal to a unit $e \in R_2$. We think of $f, g, h, e, e^{-1}, \ldots$ as functions on S^2: they are polynomial expressions in the "coordinate functions" x_0, x_1, x_2. Consider the continuous vector field on S^2 given by $v \in S^2 \mapsto (f(v), g(v), h(v)) \in \mathbb{R}^3$. Since

$$e = f' \cdot (x_1 h - x_2 g) - g' \cdot (x_0 h - x_2 f) + h' \cdot (x_0 g - x_1 f)$$

is clearly nowhere zero on S^2, the vector $(f(v), g(v), h(v))$ is nowhere collinear with the vector v. Taking the orthogonal projections onto the tangent planes of S^2, we obtain a continuous vector field of nowhere vanishing tangent vectors on S^2. This is well known to be impossible by elementary topology ("you can't comb all the hairs on a coconut!"). Thus, $P^{(2)}$ cannot be R_2-free.

We claim, further, that $P^{(2)}$ is indecomposable. (This will show that (4.12)(2) does not work for $P \oplus R \cong R^3$.) It is enough to show that R_2 is a UFD. For, if $P^{(2)} = A \oplus B$ (nontrivially), we must have $\operatorname{rk} A = \operatorname{rk} B = 1$. Assuming that R_2 is a UFD, we'll get $A \cong B \cong R_2$ (look ahead at (II.1.3)), and so $P^{(2)} \cong R_2^2$, a contradiction.

Let us now briefly sketch a proof for the fact that R_2 is a UFD.[(*)] Localize R_2 at the prime element $x_2 - 1$, and let $t = 1/(x_2 - 1)$. From $x_2 = (1+t)/t$, a direct calculation shows that $(tx_0)^2 + (tx_1)^2 = -(2t+1)$. Thus, the ring $\mathbb{R}\,[tx_0,\ tx_1]$ contains t. Using this, we check easily that that

$$R_2[t] = \mathbb{R}\,[tx_0,\ tx_1,\ t^{-1}\,].$$

By a transcendence degree consideration, $\mathbb{R}[tx_0, tx_1]$ is isomorphic to a polynomial ring in two variables over $\mathbb{R}$, so $\mathbb{R}\,[tx_0, tx_1, t^{-1}]$ is a UFD. It follows that $R_2[t]$ is a UFD, and a result of Nagata[(†)] implies that R_2 is a UFD. (A similar argument shows that R_n is a UFD for any $n \geqslant 3$.) This completes the proof of:

Proposition 4.16. *$P^{(2)}$ is indecomposable.*

As a matter of fact, *$P^{(n)}$ is indecomposable for all even integers n.* For a simple proof of this using characteristic classes, see e.g. [Swan: 1962, p. 269]. If, however, n is odd and $\geqslant 3$, (4.12)(2) implies that $P^{(n)}$ has a rank 1 free direct summand, and is therefore decomposable.

In topology, a manifold is called *parallelizable* if its tangent bundle is a trivial bundle. A deep result of Bott, Milnor and Kervaire states that S^n is parallelizable iff $n = 1, 3, 7$. It follows immediately that:

Proposition 4.17. *$P^{(n)}$ is not free for $n \neq 1, 3, 7$.*

Nevertheless, it turns out that the "complexifications" of the $P^{(n)}$'s are free *for all n*. We will prove this later in (5.9), after introducing the technique of elementary transformations.

From the above work, we deduce the following interesting example.

Proposition 4.18. *Let B_n denote the localization of the polynomial ring $A_n = \mathbb{R}\,[y_0, \ldots, y_n]$ at the element $y_0^2 + \cdots + y_n^2$. Then for $n \neq 1, 3, 7$, the* (*type* 1) *stably free B_n-module $Q^{(n)}$ defined by the unimodular row $(y_0, \ldots, y_n)$ is not free.*

Proof. The key point here is that we have an $\mathbb{R}$-algebra homomorphism from B_n to R_n given by $y_i \mapsto x_i$. Since $P^{(n)}$ is not free over R_n, it follows that $Q^{(n)}$ is not free over B_n. □

Remark 4.19. By Serre's Conjecture (to be proved later), the polynomial ring $A_n = \mathbb{R}\,[y_0, \ldots, y_n]$ is "projective-free" (f.g. projectives are free), and in particular Hermite. *But B_n possesses neither property.* This shows that the "projective-free"

(*) It is important that we are working over the reals here. If the constants were $\mathbb{C}$ instead of $\mathbb{R}$, then $\mathbb{C}[t_0, t_1, t_2]/\big(t_0^2 + t_1^2 + t_2^2 - 1\big)$ $(\cong \mathbb{C} \otimes_{\mathbb{R}} R_2)$ is obviously *not* a UFD, since $(t_0 + it_1)(t_0 - it_1) = (1+t_2)(1-t_2)$.

(†) Nagata's result states that, if $p \neq 0$ is a prime element in a commutative noetherian domain R, then the localization $R[1/p]$ is a UFD iff R itself is. See [Nagata: 1957].

and Hermite properties for commutative rings are not (in general) preserved by localizations. These properties, incidentally, are also not preserved by the passage to factor rings, since $y_i \mapsto x_i$ defines a natural surjection from A_n to R_n.

What about (4.11) and (4.12)(2) in the case of noncommutative rings? In (II.3), we will show (after [Ojanguren-Sridharan: 1971]) that,

If D is a noncommutative division ring, then $R = D[x, y]$ has a right module P such that $P \oplus R \cong R^2$, but P is non-free.

This shows that *both* (4.11) and (4.12)(2) break down in the noncommutative case!

To conclude this section, we record a freeness criterion for stably free modules from the viewpoint of exterior algebras. This is a fundamental result for the understanding of the structure of stably free modules. However, it will not be needed until we reach Chapter VII. Therefore, we include this result here more or less as a sidelight for this section, with the pointer that it can be read at any time before the reader tackles Chapter VII.

Before coming to the freeness criterion, we first prove the following general fact on the "top exterior power" of a f.g. projective module. (We have already used this fact in the special case of rank 1 projectives in the proof of (4.11).)

Lemma 4.20. *If $P \in \mathfrak{P}(R)$ has constant rank n, then $\Lambda^i(P) = 0$ for all $i > n$, while $\Lambda^n(P)$ is a f.g. projective R-module of rank* 1.

Proof. Of course, $\Lambda^i(P) \in \mathfrak{P}(R)$ for all i. Since Λ^i commutes with localizations, we may localize to assume that R is a local ring, in which case $P \cong R^n$. For the free module R^n, it is well-known that $\Lambda^i(R^n) = 0$ for $i > n$, and that $\Lambda^n(R^n)$ is free of rank 1 (on the basis $e_1 \wedge \cdots \wedge e_n$ where e_i are the unit vectors), so we are done. □

The top exterior power $\Lambda^n(P)$ above is sometimes referred to as the "determinant" of P. In terms of this determinant module, the freeness criterion for a stably free module can be formulated as follows.

Theorem 4.21. *Let P be a f.g. stably free R-module of rank n. Then $\Lambda^n(P) \cong R$, and P is free iff there exist $p_1, \ldots, p_n \in P$ such that $\Lambda^n(P)$ is free on the singleton basis $\{p_1 \wedge \cdots \wedge p_n\}$.*

Proof. Say $P \oplus Q = R^{n+t}$, where $Q \cong R^t$. Then

$$\begin{aligned} R &\cong \Lambda^{n+t}(R^{n+t}) = \Lambda^{n+t}(P \oplus Q) \\ &\cong \bigoplus_{i+j=n+t} \Lambda^i(P) \otimes \Lambda^j(Q) \\ &= \Lambda^n(P) \otimes \Lambda^t(Q) \cong \Lambda^n(P), \end{aligned}$$

as claimed. If P is free on a basis $\{p_1, \ldots, p_n\}$, then, as we have observed in the proof of (4.20), $p_1 \wedge \cdots \wedge p_n$ is a basis for $\Lambda^n(P)$. Conversely, suppose $\Lambda^n(P)$ has a basis $p_1 \wedge \cdots \wedge p_n$ for suitable $p_i \in P$. Fixing a basis $\{q_1, \ldots, q_t\}$ on Q, we see from the isomorphisms displayed above that

$$p_1 \wedge \cdots \wedge p_n \wedge q_1 \wedge \cdots \wedge q_t$$

is a singleton basis on $\Lambda^{n+t}(R^{n+t})$. But the standard basis for $\Lambda^{n+t}(R^{n+t})$ is $e_1 \wedge \cdots \wedge e_{n+t}$ where $\{e_1, \ldots, e_{n+t}\}$ is the unit-vector basis on R^{n+t}. By determinant theory à la Bourbaki, the matrix expressing the p_i, q_i's in terms of $\{e_1, \ldots, e_{n+t}\}$ must then have a determinant in $\mathrm{U}(R)$. This means that $\{p_1, \ldots, p_n, q_1, \ldots, q_t\}$ is itself a basis for R^{n+t}. From this, it follows readily that $\{p_1, \ldots, p_n\}$ is a basis for P. □

Remark 4.22. In the study of the exterior algebra $\Lambda(P)$, elements of the form $p_1 \wedge \cdots \wedge p_k$ in $\Lambda^k(P)$ (where $p_i \in P$) always play an important and very special role. They are called the *decomposable elements* in $\Lambda(P)$.

Appendix to §4

In this Appendix, we offer some clarifications on terminology, and give more examples of Hermite rings (beyond those in (4.7)). We have pointed out, in a paragraph after Def. (4.6), that the term "Hermite ring" has been given other meanings in the literature. Most notably, in his classical paper [Kaplansky: 1949], Kaplansky defined a class of Hermite rings that is not the same as the Hermite rings we defined in (4.6). In this Appendix, we shall review Kaplansky's definition, and compare it with ours. In order to distinguish the two classes of rings under scrutiny, Kaplansky's Hermite rings will be renamed "K-Hermite rings" in the following. We believe this is a reasonable choice of nomenclature, as it honors Kaplansky (as well as Hermite) who first identified this class of rings for study. To avoid unnecessary complications, we will assume that all rings considered in this Appendix are commutative.

Definition 4.23. A commutative ring R is called *K-Hermite* if, for any rectangular matrix $B \in \mathbb{M}_{m,n}(R)$, there exists $Q \in \mathrm{GL}_n(R)$ such that BQ is lower triangular.$^{(*)}$

The following observation of Kaplansky shows that the K-Hermite condition in the above definition can be considerably simplified.

Proposition 4.24. *For any commutative ring R, the following are equivalent:*

(1) *R is K-Hermite.*

(2) *The property in* (4.23) *holds for $m = 1$ and for all n; that is, for any $b_1, \ldots, b_n \in R$, there exists $Q \in \mathrm{GL}_n(R)$ such that*

$$(b_1, \ldots, b_n) \cdot Q = (d, 0, \ldots, 0) \quad \textit{for some } d \in R.$$

$^{(*)}$A rectangular matrix (a_{ij}) is said to be *lower triangular* if $a_{ij} = 0$ for all $i < j$.

(3) *The property in* (4.23) *holds for* $m = 1$ *and* $n = 2$.

Proof. Of course (1) $\Rightarrow$ (3). For (3) $\Rightarrow$ (2), consider any $\beta = (b_1, \ldots, b_n)$ with $n > 2$. Right multiplication by a suitable invertible matrix takes β to $(b_1, \ldots, b_{n-2}, d, 0)$ for some d. Thus, (2) follows by induction on n.

(2) $\Rightarrow$ (1). Consider $B \in \mathbb{M}_{m,n}(R)$. There exists $Q_1 \in \mathrm{GL}_n(R)$ such that BQ_1 has first row (say) $(d, 0, \ldots, 0)$. Let B_1 be the matrix obtained from BQ_1 by deleting the first row and first column. Invoking an inductive hypothesis (on m), we can right multiply B_1 by a matrix in $Q_2 \in \mathrm{GL}_{n-1}(R)$ to bring it to lower triangular form. Now $BQ_1 \operatorname{diag}(1,\ Q_2)$ is also lower triangular. □

The next theorem offers some more characterizations of K-Hermite rings in the language of unimodular rows, due to [Kaplansky: 1949] and [Gillman-Henriksen: 1956].

Theorem 4.25. *For any commutative ring R, the following are equivalent:*

(1) *R is K-Hermite.*

(4) *For any $a, b \in R$, there exists $(r, s) \in \mathrm{Um}_2(R)$ such that $ar + bs = 0$.*

(5) *For any $a, b \in R$, there exist $d \in R$ and $(x, y) \in \mathrm{Um}_2(R)$ such that $(a, b) = d \cdot (x, y)$.* (*In this case, $aR + bR = dR$.*)

(6) *For any $b_1, \ldots, b_n \in R$, there exist $d \in R$ and $(x_1, \ldots, x_n) \in \mathrm{Um}_n(R)$ such that $(b_1, \ldots, b_n) = d \cdot (x_1, \ldots, x_n)$.* (*In this case, $b_1R + \cdots + b_nR = dR$.*)

Proof. (1) $\Leftrightarrow$ (4). First assume (1). Given any $a, b \in R$, $(a, b)\, Q = (*,\ 0)$ for some invertible $Q = \begin{pmatrix} p & r \\ q & s \end{pmatrix}$. Then $rR + sR = R$, and $ar + bs = 0$. The converse is similar, noting that $rR + sR = R$ implies that there exists $Q \in \mathrm{GL}_2(R)$ with second column $(r, s)^t$.

(6) $\Rightarrow$ (5) is clear (by taking $n = 2$).

(5) $\Rightarrow$ (1). Assuming (5), it suffices to check (3) in (4.24). Given $a, b \in R$, write $(a, b) = d \cdot (x, y)$, with, say, $xp + yq = 1$. Then, for $Q = \begin{pmatrix} p & -y \\ q & x \end{pmatrix} \in \mathrm{SL}_2(R)$, we have

$$(a,\ b) \cdot Q = d \cdot (x,\ y)\, Q = d \cdot (1,\ 0) = (d,\ 0).$$

(1) $\Rightarrow$ (6). Assume (1). Given any $b_1, \ldots, b_n \in R$, we have

$$(b_1, \ldots, b_n)\, Q = (d, 0, \ldots, 0) \quad \text{(for some } d \in R \text{ and } Q \in \mathrm{GL}_n(R)).$$

Let $(x_1, \ldots, x_n) \in \mathrm{Um}_n(R)$ be the first row of Q^{-1}. Then

$$(b_1, \ldots, b_n) = (d, 0, \ldots, 0) \cdot Q^{-1} = d \cdot (x_1, \ldots, x_n). \qquad □$$

We shall next show that our Hermite rings defined in (4.6) are more general than Kaplansky's K-Hermite rings. We'll also show that the property (6) above for a K-Hermite ring can be further strengthened.

Theorem 4.26. *Let R be K-Hermite. Then*

(A) *R is both a Hermite ring and a Bézout ring.*

(B) *Whenever $b_1R+\cdots+b_nR=eR$ with $n\geqslant 2$, there exist $(y_1,\ldots,y_n)\in \mathrm{Um}_n(R)$ such that $(b_1,\ldots,b_n)=e\cdot(y_1,\ldots,y_n)$.*

Proof. $(1)\Rightarrow(6)$ in (4.25) shows that R is Bézout. Given any $\beta=(b_1,\ldots,b_n)\in\mathrm{Um}_n(R)$, take $Q\in\mathrm{GL}_n(R)$ such that

$$\beta Q=(d,0,\ldots,0)\quad(\text{for some } d\in R).$$

Then $dR=b_1R+\cdots+b_nR=R$, and so $d\in\mathrm{U}(R)$. After replacing Q by $d^{-1}Q$, we may assume that $d=1$. This shows that R is Hermite, by (4.9). This proves (A).

For (B), we first handle the crucial case $n=2$, following a calculation in [Gillman-Henriksen: 1956]. Using (4.25)(5), we first write $(b_1,b_2)=d\cdot(x_1,x_2)$, where $d\in R$ and $(x_1,x_2)\in\mathrm{Um}_2(R)$. From $e=b_1R+b_2R=dR$, we can write $d=ke$ and $e=\ell d$ (for suitable $k,\ \ell\in R$). Fixing an equation $sx_1+tx_2=1$, let

$$y_1=x_1k+rt\quad\text{and}\quad y_2=x_2k-rs,\ \text{ where } r:=k\ell-1.$$

Since r kills d and e, we have $y_ie=x_ike=x_id=b_i$ (for $i=1,2$). Finally,

$$\begin{aligned}(s\ell-x_2)y_1+(t\ell+x_1)y_2 &= k\ell\,(sx_1+tx_2)-r(sx_1+tx_2)\\ &=(k\ell-r)(sx_1+tx_2)=1\end{aligned}$$

shows that $(y_1,y_2)\in\mathrm{Um}_2(R)$, as desired.

The case of a general $n>2$ in (B) can be done as follows. By (4.25)(6), write

$$(b_1,\ldots,b_{n-1})=d\cdot(x_1,\ldots,x_{n-1}),$$

where $d\in R$ and $(x_1,\ldots,x_{n-1})\in\mathrm{Um}_{n-1}(R)$. Then

$$dR+b_nR=b_1R+\cdots+b_nR=eR.$$

By what we've done above, there exist $(p,q)\in\mathrm{Um}_2(R)$ such that $(d,b_n)=e\cdot(p,q)$. Now

$$(b_1,\ldots,b_{n-1},b_n)=(dx_1,\ldots,dx_{n-1},b_n)=e\cdot(px_1,\ldots,px_{n-1},q),$$

where $px_1R+\cdots+px_{n-1}R+qR=pR+qR=R$, as desired. □

It turns out that, for various classes of commutative rings, "K-Hermite" is actually synonymous with "Bézout". In fact, we have the following nice result of Kaplansky.

Theorem 4.27. *Let R be a commutative ring in which every zero-divisor lies in* $\mathrm{rad}(R)$. *Then R is K-Hermite iff R is Bézout.*

Proof. The "only if" part follows from (4.26)(A). Conversely, assuming R is Bézout, let us check (4.25)(5) for any two elements $a, b \in R$. We may assume $(a, b) \neq (0, 0)$ (for otherwise we can take $d = 0$ and $x = y = 1$). Write $aR + bR = dR$, with, say, $a = dx$, $b = dy$, and $d = ap + bq$. Since $d \neq 0$, $d = d(xp + yq)$ implies (by assumption) that $1 - (xp + yq) \in \mathrm{rad}(R)$. But then

$$xp + yq \in 1 + \mathrm{rad}(R) \subseteq \mathrm{U}(R)$$

yields $xR + yR = R$, as desired. □

Corollary 4.28. *Let R be a commutative domain, or a commutative local ring. Then R is K-Hermite iff R is Bézout.*

Proof. In either case, R clearly satisfies the hypothesis of (4.27). □

Throughout this book, Hermite rings will be taken to be those defined in Def. (4.6), and K-Hermite rings will be those defined in Def. (4.23). From (4.26) and (4.28), we see that *K-Hermite rings are somewhere "between" Bézout domains and Bézout rings.* While K-Hermite rings are always Hermite, the converse is not true in general. For instance, any local ring R is Hermite, but it is K-Hermite (if and) only if it is Bézout. Also, any Dedekind domain R is Hermite (by (4.7)(4)), but R is K-Hermite (if and) only if R is a PID. These (and other) examples show that Kaplansky's K-Hermite rings form a considerably more specialized class than the Hermite rings we defined in (4.6). As far as the completion problem is concerned, Kaplansky showed that K-Hermite rings have, in fact, the following strong Hermite completion property for *not necessarily unimodular* rows.

Theorem 4.29. *Let R be a K-Hermite ring. Then any row $(a_1, \dots, a_n) \in R^n$ can be completed to a square matrix*

$$\begin{pmatrix} a_1, \dots, a_n \\ N \end{pmatrix}$$

whose determinant generates the ideal $a_1R + \cdots + a_nR$. Furthermore, the $(n-1) \times n$ matrix N may be chosen to be itself completable to a matrix in $\mathrm{GL}_n(R)$.

Quite interestingly, Kaplansky did this work exactly 100 years after Hermite did his! However, the completion property in (4.29) does not seem to have too much bearing on the structure of the projective modules over R. Thus, we shall not dwell on (4.29) here, and just refer the reader to [Kaplansky: 1949] for its proof.

It is easy to see that Hermite rings are preserved by direct products, although, as we have seen in (4.19), they are not preserved by the passage to factor rings or by localizations at multiplicative sets. The situation of K-Hermite rings turns out to be much nicer. Indeed, a straightforward check from Def. (4.23) (or from equivalent characterizations of K-Hermite rings) shows the following.

Proposition 4.30. *The class of commutative K-Hermite rings is closed under the formation of direct products, factor rings, and localizations at multiplicative sets.*

While (4.28) shows that K-Hermite domains are synonymous with Bézout domains, there is, in view of (4.30), no lack of examples of K-Hermite rings that are *not* domains. In fact, the following classical result of Kaplansky offers a rather nontrivial class of rings possibly with 0-divisors that are always K-Hermite rings.

Theorem 4.31. *Any principal ideal ring* (PIR) *is K-Hermite (and therefore also Hermite).*

Proof. (Note that a PIR is just a noetherian Bézout ring.) The proof we shall give for this Proposition is a bit round-about, which is why we called the PIR's a *nontrivial* class of K-Hermite ring examples. Unfortunately, I do not know of a more "direct" proof.

By a classical theorem of Krull, any PIR is a finite direct product of PID's and local artinian PIR's. (For a proof of this, see Thm. 14.6 in the book [Brown: 1993].) By (4.28), all factors in this direct product representation are K-Hermite. Therefore, so is R. (For another proof of the theorem, using subdirect product representations instead of direct product representations, see [Sjogren: 2003].) □

Another important class of K-Hermite rings with 0-divisors is given by the von Neumann regular rings. To see this, let us first prove the following lemma on ideals generated by idempotents.

Lemma 4.32. *Let* $d = e + f - ef$ *where* e, f *are idempotents in a commutative ring* R. *Then* $d = d^2$, *and for* $x = 1 - f + ef$, $y = 1 - e + ef$, *we have* $e = dx$, $f = dy$, *and* $xR + yR = R$ *(and therefore* $eR + fR = dR$*).*

Proof. By direct computations, we have $d = d^2$, $ed = e$, and $fd = f$. Thus, $dx = d - f + ef = e$, and similarly, $dy = f$. Finally,

$$ex + (1-e)y = e[1 - f(1-e)] + (1-e)[1 - e + ef] = e + 1 - e = 1$$

shows that $xR + yR = R$. (The fact that $d = d^2$ is not needed below.) □

Recall that a ring R is called *von Neumann regular* if, for every $a \in R$, there exists $r \in R$ such that $a = ara$.

Proposition 4.33. *Let* R *be a commutative von Neumann regular ring. Then, for any* $a \in R$, *there exists* $u \in \mathrm{U}(R)$ *such that* $a = aua$.

Proof. Say $a = ara$, where $r \in R$. Then $e := ar$ is an idempotent with $ea = a$. For $e' = 1 - e$, let $u = er + e'$. Then $(ea + e')u = e + e' = 1$, so $u \in \mathrm{U}(R)$. Since $e'a = e'(ea) = 0$, we have $aua = a(er + e')a = ara = a$, as desired. □

Theorem 4.34. *Let R be a commutative von Neumann regular ring. Then R is K-Hermite (and therefore also Hermite).*

Proof. Let us check the condition (4.25)(5). For $a, b \in R$, use (4.33) to write $a = aua$ and $b = bvb$, where $u, v \in \mathrm{U}(R)$. Then $e = au$ and $f = bv$ are idempotents. Let $d = e + f - ef$, and define $x, y \in R$ as in (4.32). Then $a = eu^{-1} = d(xu^{-1})$ and $b = fv^{-1} = d(yv^{-1})$. Since

$$xu^{-1}R + yv^{-1}R = xR + yR = R,$$

we have verified the condition (4.25)(5). □

To conclude this Appendix, we prove the following strong cancellation theorem over a K-Hermite domain (or equivalently, a Bézout domain). This result may be thought of as a broad generalization of the fact that such a domain is always Hermite.

Theorem 4.35. *Let R be a Bézout domain. Then any free module R^n $(n < \infty)$ is a cancellable object in the category of f.g. R-modules.*

Proof. It suffices to prove this for $n = 1$. Thus, our job is to prove the following. Suppose $M = A \oplus B = A' \oplus C$, where B, C are f.g. R-modules, and $A \cong A' \cong R$; then $B \cong C$. We may assume that there is no inclusion relation between B and C. For, if (say) $B \subseteq C$, then there is a surjection from M/B onto M/C. This must be an isomorphism (since both modules are isomorphic to R, and R is a domain), and hence $B = C$, which is more than we want.

Let $D := B \cap C \subsetneq C$. Projecting M onto A (with kernel B), we get an exact sequence

$$0 \longrightarrow D \longrightarrow C \longrightarrow I \longrightarrow 0,$$

where $I \subseteq A \cong R$. Clearly, I is f.g. and nonzero. Since R is a Bézout domain, we have $I \cong R$. Thus, the above sequence splits, and we have $C \cong D \oplus R$. Similarly, $B \cong D \oplus R$, and hence $C \cong B$. □

Remark 4.36. The above argument, essentially due to [Hsü: 1962], can be used to prove various other cancellation theorems. For instance, if R is a Dedekind domain, we can use a modification of this argument to prove Hsü's result that *any f.g. R-module is cancellable in the category of all (not necessarily f.g.) R-modules.* For an exposition on this proof, see Theorem 5.8 in [Lam: 2004].

§5. Elementary Transformations

Let R be any ring. If $\alpha = (a_1, \dots, a_n)$ and $\beta = (b_1, \dots, b_n)$ are in $\mathrm{Um}_n(R)$ (right unimodular rows), we have agreed to write $\alpha \sim \beta$ to indicate the fact that α and β are conjugate under the right multiplication action of $\mathrm{GL}_n(R)$. More generally, if G is any given subgroup of $\mathrm{GL}_n(R)$, we shall write $\sim_G$ to denote conjugacy of right unimodular rows under the action of G. By a straightforward matrix calculation, we see that, for a right unimodular row $\alpha \in \mathrm{Um}_n(R)$, we have $\alpha \sim_G (1, 0, \dots, 0)$ iff α is completable to a matrix in G. This, in part, generalizes (4.9).

For the purpose of proving theorems about unimodular rows, the most important subgroup of $\mathrm{GL}_n(R)$ to look at is the so-called *group of elementary matrices*. This group, denoted by $\mathrm{E}_n(R)$, is generated by matrices of the form $I + xe_{ij}$, where $x \in R$, $i \neq j$, and e_{ij} are the matrix units. (The matrix $I + xe_{ij}$ is invertible, with inverse $I - xe_{ij}$, $i \neq j$. If R happens to be commutative, $I + xe_{ij}$ belongs even to the special linear group $\mathrm{SL}_n(R)$.) Note that if M is an $n \times n$ matrix, left multiplication of M by $I + xe_{ij}$ corresponds to adding a left x-multiple of the j^{th} row of M to the i^{th} row of M, and right multiplication of M by $I + xe_{ij}$ corresponds to adding a right x-multiple of the i^{th} column of M to the j^{th} column of M. These are, respectively, the *elementary row and column transformations* (or *operations*). It is easy to see that block matrices of the form

$$\begin{pmatrix} I_n & A \\ 0 & I_m \end{pmatrix}, \quad \begin{pmatrix} I_n & 0 \\ B & I_m \end{pmatrix}$$

are both in the group $\mathrm{E}_{n+m}(R)$. Therefore, in transforming a block matrix by elementary row (column) operations (in the above sense), it will be legitimate to apply elementary row (column) block operations to the given matrix. This remark will be used freely without further mention in the following.

We proceed now to prove a few basic facts about elementary transformations, and to provide some examples.

Proposition 5.0. (1) *The group of diagonal matrices in* $\mathrm{GL}_n(R)$ *normalizes the group* $\mathrm{E}_n(R)$.

(2) $\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \in \mathrm{E}_{2n}(R)$. *In particular, we have*

$$\begin{aligned} (\dots, a, \dots, b, \dots) &\sim_{\mathrm{E}_n(R)} (\dots, -b, \dots, a, \dots) \\ &\sim_{\mathrm{E}_n(R)} (\dots, b, \dots, -a, \dots). \end{aligned}$$

Proof. (1) It is enough to show that any $D = \mathrm{diag}(1, \dots, d, \dots, 1)$ ($d =$ unit) normalizes $\mathrm{E}_n(R)$. Suppose d occurs in the k^{th} row. Clearly, for $i \neq j$:

$$D \cdot (I + xe_{ij}) \cdot D^{-1} = \begin{cases} I + xe_{ij} & \text{if } \ i \neq k \neq j, \\ I + dxe_{ij} & \text{if } \ k = i, \\ I + xd^{-1}e_{ij} & \text{if } \ k = j. \end{cases}$$

(2) It suffices to prove $\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \in \mathrm{E}_{2n}(R)$. This follows from the following sequence of block elementary column transformations, where $I = I_n$:

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \longmapsto \begin{pmatrix} I & I \\ -I & 0 \end{pmatrix} \longmapsto \begin{pmatrix} I & 0 \\ -I & I \end{pmatrix} \longmapsto \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}. \qquad \square$$

Whitehead's Lemma 5.1. *For* $A, B \in \mathrm{GL}_n(R)$*, we have*

$$\begin{pmatrix} AB & 0 \\ 0 & I_n \end{pmatrix} \in \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \cdot \mathrm{E}_{2n}(R).$$

Proof. This follows from the (block) column transformations:

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \longmapsto \begin{pmatrix} A & 0 \\ I & B \end{pmatrix} \longmapsto \begin{pmatrix} A & -AB \\ I & 0 \end{pmatrix}$$
$$\longmapsto \begin{pmatrix} AB & A \\ 0 & I \end{pmatrix} \longmapsto \begin{pmatrix} AB & 0 \\ 0 & I \end{pmatrix}.$$

Here, $I = I_n$, and the third transformation was made possible by (5.0)(2). $\square$

Corollary 5.2. (1) *For any* $u_1, \ldots, u_n \in \mathrm{U}(R)$*, we have*

$$\operatorname{diag}(u_1, \ldots, u_n) \in \operatorname{diag}(u_1 \cdots u_n,\ 1, \ldots, 1) \cdot \mathrm{E}_n(R).$$

(2) *If R is a commutative ring, any diagonal matrix in* $\mathrm{SL}_n(R)$ *belongs to* $\mathrm{E}_n(R)$.

Proof. (1) follows from (5.1.) by induction, and (2) follows from (1). $\square$

Proposition 5.3. *If* $(b_1, \ldots, b_n) \in \mathrm{Um}_n(R)$ *contains a right unimodular sub-row of shorter length, then* $(b_1, \ldots, b_n) \sim_{\mathrm{E}_n(R)} (1, 0, \ldots, 0)$.

Proof. Say $(b_{i_1}, \ldots, b_{i_m}) \in \mathrm{Um}_m(R)$, and $i \notin \{i_1, \ldots, i_m\}$. Write $b_{i_1}a_{i_1} + \cdots + b_{i_m}a_{i_m} = 1$. After a sequence of elementary transformations, we may change b_i to

$$b_i - (b_{i_1}a_{i_1} + \cdots + b_{i_m}a_{i_m})(b_i - 1) = 1.$$

Further elementary transformations change the other b_j's to 0. Now finish by (5.0)(2). $\square$

Proposition 5.4. *Let R be either a euclidean domain or a commutative semilocal ring.*

(1) *For $n \geqslant 2$, any* $(a_1, \ldots, a_n) \in \mathrm{Um}_n(R)$ *is* $\sim_{\mathrm{E}_n(R)} (1, 0, \ldots, 0)$.

(2) *Moreover,* $\mathrm{SL}_n(R) = \mathrm{E}_n(R)$.

Proof. (1) *First, assume R is a euclidean domain.* By successive applications of the euclidean algorithm, we may change $(a_1, \ldots, a_n)$ by elementary transformations to $(b_1, \ldots, b_n)$ where some b_i is a unit. We can then finish by (5.3). Now *assume R is a commutative semilocal ring.* Let $\mathfrak{A} = \sum_{i \geqslant 2} R \cdot a_i$, so that $R \cdot a_1 + \mathfrak{A} = R$. As above, it will be enough to show that $a_1 + \mathfrak{A}$ contains a unit. To do this, we may assume that rad $R = 0$. But then R is a finite direct product of fields. Checking in each "coordinate," we may assume that R itself is a field. In this case, the desired conclusion is trivial.

(2) Let $M \in \mathrm{SL}_n(R)$. By (1), we can perform suitable elementary column transformations to bring M to M_1 with first row $(1, 0, \ldots, 0)$. An obvious sequence of row transformations brings M_1 to $M_2 = \begin{pmatrix} 1 & 0 \\ 0 & M' \end{pmatrix}$ where $M' \in \mathrm{SL}_{n-1}(R)$. The proof now proceeds by induction on n. □

Remark 5.5. The above proposition shows, in particular, that euclidean domains and commutative semilocal rings are both Hermite rings. The latter case has already been observed in (4.7)(3), but (5.4) uses a different proof, and yields a stronger conclusion.

One very special feature of $\mathrm{E}_n(R)$ that is often used to great advantage is the fact that *if $R \to S$ is onto, then the induced map* $\mathrm{E}_n(R) \to \mathrm{E}_n(S)$ *is also onto* (although $\mathrm{GL}_n(R) \to \mathrm{GL}_n(S)$ and $\mathrm{SL}_n(R) \to \mathrm{SL}_n(S)$ need not be onto). This observation leads to the following remarkable statement.

Proposition 5.6. *Let R be a commutative ring, with*

$$(a_1, \ldots, a_r, b_1, \ldots, b_s) \in \mathrm{Um}_{r+s}(R), \quad s \geqslant 1.$$

Let $\mathfrak{A} = \sum R \cdot a_i$, and $\overline{R} = R/\mathfrak{A}$. If $(\overline{b}_1, \ldots, \overline{b}_s) \sim_{\mathrm{E}_s(\overline{R})} (\overline{1}, \overline{0}, \ldots, \overline{0})$, then

$$(a_1, \ldots, a_r, b_1, \ldots, b_s) \sim_{\mathrm{E}_{r+s}(R)} (1, 0, \ldots, 0).$$

Proof. Pulling back the elementary transformations which bring $(\overline{b}_1, \ldots, \overline{b}_s)$ to $(\overline{1}, \overline{0}, \ldots, \overline{0})$, we get

$$(a_1, \ldots, a_r, b_1, \ldots, b_s) \sim_{\mathrm{E}_{r+s}(R)} (a_1, \ldots, a_r, b'_1, \ldots, b'_s)$$

where $(b'_1, \ldots, b'_s) \equiv (1, 0, \ldots, 0) \pmod{\mathfrak{A}}$. Performing further elementary transformations, we can bring $(a_1, \ldots, a_r, b'_1, \ldots, b'_s)$ to $(a_1, \ldots, a_r, 1, 0, \ldots, 0)$. Now finish by (5.3). □

Corollary 5.7. *Let R be a commutative ring, and*

$$(a_1, \ldots, a_r, b_1, \ldots, b_s) \in \mathrm{Um}_{r+s}(R), \quad s \geqslant 2.$$

If either (1) $R/\sum R \cdot a_i$ *is a euclidean domain, or* (2) $\sum R \cdot a_i$ *is contained in only finitely many maximal ideals of R, then*

$$(a_1, \ldots, a_r, b_1, \ldots, b_s) \sim_{\mathrm{E}_{r+s}(R)} (1, 0, \ldots, 0).$$

Corollary 5.8. *If R is a noetherian domain of Krull dimension* 1*, then R is Hermite. (For instance, any Dedekind domain is Hermite, as we have pointed out in* (4.7)(4)*.)*

Proof. If $(a_1, \ldots, a_n) \in \mathrm{Um}_n(R)$, $n \geqslant 3$, $a_1 \neq 0$, we can apply (5.7) since a_1 is contained in only finitely many maximal ideals of R (why?). [Actually, a slightly more refined argument shows that (5.8) is true without the assumption that R is a domain: see (II.7.3).] □

Here is another application of (5.6) in the form of an example.

Example 5.9. Let R be a commutative ring with an element i such that $i^2 = -1$. If $x_0^2 + x_1^2 + x_2 y_2 + \cdots + x_n y_n = 1$ in R, then

$$(x_0,\ x_1, \ldots, x_n) \sim_{\mathrm{E}_n(R)} (1, 0, \ldots, 0).$$

Proof. Working modulo $\mathfrak{A} = \sum_{i \geqslant 2} R x_i$, we have

$$(\overline{x}_0,\ \overline{x}_1) \sim_{\mathrm{E}_2(\overline{R})} \left(\overline{x_0 + i x_1},\ \overline{x}_1\right).$$

Since $(\overline{x_0 + i x_1})\,(\overline{x_0 - i x_1}) = \overline{x_0^2 + x_1^2} = \overline{1}$, (5.3) implies that

$$\left(\overline{x_0 + i x_1},\ \overline{x_1}\right) \sim_{\mathrm{E}_2(\overline{R})} (\overline{1},\ \overline{0}).$$

Thus, (5.6) gives the desired result. □

The most concrete case of Example (5.9) is given by the commutative ring $R = \mathbb{C}[x_0, \ldots, x_n]$, with relation $x_0^2 + \cdots + x_n^2 = 1$, i.e., the coordinate ring of the n-sphere over the complexes. The above shows that the solution space of $(x_0, x_1, \ldots, x_n)$ is always free of rank n over R. Geometrically, this says that the *complexification* of the tangent bundle over the real n-sphere is always a trivial bundle.

We take this opportunity to present a few other easy computational examples. None of these examples is really surprising, but it is of interest to see that we can actually carry out some *explicit* matrix completions for unimodular rows.

Example 5.10. This first example is a special case of Prop. 5.3. Let $(a_1, \ldots, a_n) \in \mathrm{Um}_n(R)$, where R is any ring. Then $(0, a_1, \ldots, a_n) \in \mathrm{Um}_{n+1}(R)$ is *always* completable to a matrix in $\mathrm{E}_{n+1}(R)$. To see this, we fix an equation $a_1 b_1 + \cdots + a_n b_n = 1$, and carry out the following sequence of elementary transformations:

$$\begin{aligned}(0, a_1, \ldots, a_n) &\longmapsto (a_1 b_1 + \cdots + a_n b_n, a_1, \ldots, a_n) = (1, a_1, \ldots, a_n)\\ &\longmapsto (1, a_1 - a_1, \ldots, a_n - a_n) = (1, 0, \ldots, 0).\end{aligned}$$

This means that we have multiplied $(0, a_1, \ldots, a_n)$ from the right by the two matrices

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ b_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ b_n & 0 & \cdots & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & -a_1 & \cdots & -a_n \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \quad \text{in } \mathrm{E}_{n+1}(R).$$

Thus, we see that $(0, a_1, \ldots, a_n)$ is completed into the following product matrix

$$\begin{pmatrix} 1 & a_1 & \cdots & a_n \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ -b_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -b_n & 0 & \cdots & 1 \end{pmatrix} = \begin{pmatrix} 0 & a_1 & \cdots & a_n \\ -b_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -b_n & 0 & \cdots & 1 \end{pmatrix}$$

in $\mathrm{E}_{n+1}(R)$, as desired.

Example 5.11. Consider $(a_1, \ldots, a_n) \in R^n$, where $a_1 + \cdots + a_n = 1$. Then $(a_1, \ldots, a_n) \in \mathrm{Um}_n(R)$ is *always* completable to a matrix in $\mathrm{E}_n(R)$, since we can carry out the following elementary transformations:

$$\begin{aligned} (a_1, \ldots, a_n) &\longmapsto (a_1 + \cdots + a_n, a_2, \ldots, a_n) = (1, a_2, \ldots, a_n) \\ &\longmapsto (1, a_2 - a_2, \ldots, a_n - a_n) = (1, 0, \ldots, 0). \end{aligned}$$

Proceeding are in the previous example, we can complete $(a_1, \ldots, a_n)$ into the following product matrix

$$\begin{pmatrix} 1 & a_2 & \cdots & a_n \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ -1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & \cdots & 1 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ -1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & \cdots & 1 \end{pmatrix}$$

in $\mathrm{E}_n(R)$, as desired.

Example 5.12. Let $R = \mathbb{R}[x_0, x_1, \ldots, x_n]$ with the relation $x_0^2 + \cdots + x_n^2 = 1$ be the coordinate ring of the real sphere S^n. We have observed (though not proved) that the unimodular vector $(x_0, x_1, \ldots, x_n) \in \mathrm{Um}_{n+1}(R)$ is not completable (to a matrix in $\mathrm{GL}_{n+1}(R)$) unless $n = 1, 3$, or 7. However, for any n, it is easy to see that $(x_0^2, x_1, \ldots, x_n)$ is completable to a matrix in $\mathrm{E}_{n+1}(R)$. In fact, in view of the sequence of elementary transformations

$$\begin{aligned} (x_0^2, x_1, \ldots, x_n) &\longmapsto (x_0^2 + \cdots + x_n^2,\ x_1, \ldots, x_n) = (1, x_1, \ldots, x_n) \\ &\longmapsto (1,\ x_1 - x_1, \ldots, x_n - x_n) = (1, 0, \ldots, 0), \end{aligned}$$

we arrive quickly at the following completion:

$$\begin{pmatrix} 1 & x_1 & \cdots & x_n \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ -x_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -x_n & 0 & \cdots & 1 \end{pmatrix} = \begin{pmatrix} x_0^2 & x_1 & \cdots & x_n \\ -x_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -x_n & 0 & \cdots & 1 \end{pmatrix} \in \mathrm{E}_{n+1}(R).$$

We invite the reader to check that $(x_0^{r_0}, \dots, x_n^{r_n})$ is likewise completable to a matrix in $\mathrm{E}_{n+1}(R)$ as long as one of the nonnegative integers $r_0, \dots, r_n$ is even. On the other hand, if $n \neq 1$, 3, or 7, and $r_0, \dots, r_n$ are all odd, Murthy and Swan have shown, using topology, that $(x_0^{r_0}, \dots, x_n^{r_n})$ is *not* completable to a matrix in $\mathrm{GL}_{n+1}(R)$; see [Murthy-Swan: 1976].

Example 5.13. Let R be as in the previous example, and consider the localization $A = R\,[s]$, where $s = 1/(x_0 + 1)$. *We claim that* $(x_0, x_1, \dots, x_n) \in \mathrm{Um}_{n+1}(R)$ *is completable to a matrix in* $\mathrm{E}_{n+1}(A)$. Indeed, this can be checked by a sequence of elementary transformations remarkably similar to those in (5.12), but carried over the localized ring A:

$$\begin{aligned}(x_0, x_1, \dots, x_n) &\longmapsto (x_0 + x_1^2 s + \cdots + x_n^2 s,\ x_1, \dots, x_n) = (1, x_1, \dots, x_n)\\ &\longmapsto (1,\ x_1 - x_1, \dots, x_n - x_n) = (1, 0, \dots, 0).\end{aligned}$$

As before, we come up with the following completion over A:

$$\begin{pmatrix} 1 & x_1 & \cdots & x_n \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ -x_1 s & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -x_n s & 0 & \cdots & 1 \end{pmatrix} = \begin{pmatrix} x_0 & x_1 & \cdots & x_n \\ -x_1 s & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -x_n s & 0 & \cdots & 1 \end{pmatrix} \in \mathrm{E}_{n+1}(A).$$

By an obvious change of variables, we see easily that a similar construction also works over the localization $B = R\,[t]$ where $t = 1/(x_0 - 1)$. This shows something interesting, since the two elements $f = x_0 + 1$ and $g = x_0 - 1$, at which we localize respectively, happen to be *comaximal* in R. The sequence $(x_0, x_1, \dots, x_n) \in \mathrm{Um}_{n+1}(R)$ is elementarily completable over *both* of the localizations $A = R_f$ and $B = R_g$, but is not even completable over R, except when $n = 1, 3, 7$. The type 1 stably free R-module P defined by $(x_0, x_1, \dots, x_n)$ localizes to free modules of rank n both at f and at g, but P is not free, except when $n = 1, 3, 7$.

Example 5.14. (Sketch) Let R be any commutative ring with $2 \in \mathrm{U}(R)$, and let $x, y \in R$. Then $\beta = (1 + x,\ 1 - y,\ x(1 + y)) \in \mathrm{Um}_3(R)$, since $x(1 + y) \equiv -2$ modulo $(1 + x)R + (1 - y)R$. This shows that $\beta \sim_{\mathrm{E}_3(R)} (1 + x,\ 1 - y,\ -2)$. Since $2 \in \mathrm{U}(k)$, β is completable to a matrix in $\mathrm{E}_3(R)$, and (in the notation of (4.8)) $P(\beta) \cong R^2$. As in the previous examples, we can complete β to $E = \begin{pmatrix} 1 + x & 1 - y & x(1 + y) \\ 1/2 & 0 & 1 \\ 0 & -1 & x \end{pmatrix} \in \mathrm{E}_3(R)$. (The lone occurrence of a "$1/2$" in E serves as a stark reminder that $2 \in \mathrm{U}(k)$ is needed in this Example!) Since

$$E^{-1} = \begin{pmatrix} 1 & -2x & 1 - y \\ -x/2 & x(1 + x) & (-2 - x + xy)/2 \\ -1/2 & 1 + x & (y - 1)/2 \end{pmatrix},$$

the last two columns of E^{-1} form a free R-basis for $P(\beta)$ (according to the proof of (4.3)).

§6. The Grothendieck Group K_0

We shall give in this section a quick introduction to the theory of the Grothendieck group K_0 for arbitrary rings R. Our coverage will be rather brief since K_0 will not be used too heavily in the sequel. However, we do need the idea of Grothendieck groups in order to discuss an important theorem of Serre in Chapter II, §5. And besides, it will be important to be familiar with the groups K_0R so that in our exposition we will have the proper perspective of algebraic K-theory. These are the main reasons for including here a section on the Grothendieck groups K_0.

For $P \in \mathfrak{P}(R)$, we write (P) for the isomorphism class of P. The *Grothendieck group* K_0R is an additive abelian group generated by the symbols (P) with certain natural relations. To be precise, we let:

$G =$ free abelian group generated by $(P): \ P \in \mathfrak{P}(R)$,

$H =$ subgroup of G generated by $(P \oplus Q) - (P) - (Q): \ P, Q \in \mathfrak{P}(R)$,

$K_0R = G/H$, and

$[P] =$ image of (P) in K_0R.

Thus, we have $[P \oplus Q] = [P] + [Q] \in K_0R$ whenever $P, Q \in \mathfrak{P}(R)$. More generally, if there is an exact sequence

$$(*) \qquad 0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to 0, \quad \text{where } P_i \in \mathfrak{P}(R),$$

then $\sum (-1)^i [P_i] = 0 \in K_0R$. This follows by breaking up $(*)$ into short exact sequences, and using induction on n (noting that all modules involved in this process are f.g. projective).

A general element of K_0R has the form

$$\begin{aligned} z &= [P_1] + \cdots + [P_m] - [Q_1] - \cdots - [Q_n] \\ &= [P] - [Q], \ \text{ where } P = P_1 \oplus \cdots \oplus P_m, \ \ Q = Q_1 \oplus \cdots \oplus Q_n. \end{aligned}$$

If we want, we can choose $Q' \in \mathfrak{P}(R)$ such that $Q \oplus Q' \cong R^t$, and rewrite:

$$z = [P \oplus Q'] - [Q \oplus Q'] = [P_1] - [R^t],$$

where $P_1 = P \oplus Q' \in \mathfrak{P}(R)$.

Note that if $f: R \to S$ is a ring homomorphism, then f induces a well-defined group homomorphism $f_*: K_0R \to K_0S$ such that $f_*[P] = [S \otimes_R P]$, for every $P \in \mathfrak{P}(R)$. From this observation, we check easily that K_0 gives a covariant functor from the category of rings to the category of additive abelian groups.

Proposition 6.1. *For $P, Q \in \mathfrak{P}(R)$, the following are equivalent:*

(1) $[P] = [Q] \in K_0R$;
(2) *There exists $T \in \mathfrak{P}(R)$ such that $P \oplus T \cong Q \oplus T$ (if this is the case, we say that P, Q are stably isomorphic);*
(3) *There exists a natural number t such that $P \oplus R^t \cong Q \oplus R^t$.*

Proof. Obviously, (3) $\Leftrightarrow$ (2) $\Rightarrow$ (1), so we need only assume (1), and prove (2). From (1), we have $(P)-(Q)\in H$ (in the notations above), and so

$$(P)-(Q) = \sum_i \{(P_i \oplus Q_i) - (P_i) - (Q_i)\} \\ - \sum_j \{(P'_j \oplus Q'_j) - (P'_j) - (Q'_j)\},$$

where all modules involved are in $\mathfrak{P}(R)$. By transposition,

$$(P) + \sum_i \{(P_i) + (Q_i)\} + \sum_j (P'_j \oplus Q'_j) \\ = (Q) + \sum_i (P_i \oplus Q_i) + \sum_j \{(P'_j) + (Q'_j)\}.$$

Now, since G is free, $\sum (M_\alpha) = \sum (N_\beta)$ implies $\bigoplus M_\alpha \cong \bigoplus N_\beta$. Therefore, we obtain $P \oplus T \cong Q \oplus T$ where $T = \bigoplus_i \{P_i \oplus Q_i\} \oplus \bigoplus_j \{P'_j \oplus Q'_j\}$. $\square$

Corollary 6.2. *$P \in \mathfrak{P}(R)$ is stably free iff $[P] \in \mathbb{Z}\cdot[R]$. (Thus, $K_0R = \mathbb{Z}\cdot[R]$ iff all f.g. R-projectives are stably free.)*

Proof. Say $P \oplus R^m \cong R^n$. Then clearly $[P]=[R^n]-[R^m]=(n-m)\cdot[R]$. Conversely, assume $[P]=r\cdot[R]$, $r\in\mathbb{Z}$. Pick an integer s such that $r+s\geqslant 0$. We have $[P\oplus R^s]=(r+s)\cdot[R]=[R^{r+s}]\in K_0R$, so by the Proposition, $P\oplus R^s\oplus R^t\cong R^{r+s+t}$ for some t. $\square$

Corollary 6.3. *If $P\in\mathfrak{P}(R)$ admits a finite free resolution, i.e., if there exists*

$$0\to F_n\to F_{n-1}\to\cdots\to F_0\to P\to 0$$

where F_i are free of finite rank, then P is stably free. (If, in addition, R is commutative and $\operatorname{rk} P = 1$*, then $P\cong R$.)*

Proof. $[P]=\sum(-1)^i[F_i]\in\mathbb{Z}\cdot[R]$; now use (6.2) and (4.11). $\square$

Corollary 6.4. *R satisfies IBP (invariant basis property) iff $[R]$ has infinite order in K_0R (i.e., iff $\mathbb{Z}\cdot[R]\cong\mathbb{Z}$).*

Proof. Suppose R has infinite order. Then $R^r\cong R^s$ implies $(r-s)\cdot[R]=0$ in K_0R, so $r=s$. On the other hand, if $m\,[R]=0$ for some $m>0$, then (6.1) yields $R^m\oplus R^t\cong R^t$ for some $t\geqslant 0$, which clearly violates IBP. $\square$

Combining (6.2) and (6.4), we see that *if R satisfies IBP and all f.g. R-projectives are stably free*, then $K_0R\cong\mathbb{Z}$ with generator $[R]$. This holds, for example, for the following classes of rings R:

(1) $R=$ local (see (1.8));
(2) $R=$ commutative semilocal, without nontrivial idempotents (see (3.7));

(3) $R =$ principal ideal domain (PID) (see II.2.4).

If, however, R does not satisfy IBP, K_0R tends to become less predictable. For instance, in this case, K_0R can be the zero group! Fortunately, we do not have to deal with such "bad" rings in our exposition on Serre's Problem.

Remark 6.5. In the case where R is a *commutative* ring, it is easy to see that the tensor product of R-modules induces a multiplicative structure on K_0R that makes K_0R into a commutative ring (with identity $[R]$). Therefore, in this case, we can refer to K_0R as the *Grothendieck ring* of R. One checks readily (for instance, by factoring out a maximal ideal) that any nonzero commutative ring R satisfies IBP, so in this case $[R]$ has infinite additive order in K_0R; or equivalently, the Grothendieck ring K_0R is a ring of characteristic 0.

To close this section, we provide the following result which enables us to conclude "isomorphism" from "stable isomorphism" for projective modules in a special case.

Proposition 6.6. *Let R be a commutative ring, and $P, Q \in \mathfrak{P}(R)$. If* $\operatorname{rk} P = 1$*, then* $[P] = [Q] \in K_0R \Longrightarrow P \cong Q$.

Proof. As we saw in the proof of (4.11), $\Lambda^k P = 0$ for $k \geqslant 2$. Say $P \oplus R^t \cong Q \oplus R^t$ (see (6.1)), which clearly implies that $\operatorname{rk} Q = 1$. Arguing as in (4.11), we have

$$\Lambda^{t+1}\big(P \oplus R^t\big) \cong \bigoplus_{i+j=t+1} \big(\Lambda^i P \oplus \Lambda^j R^t\big) \cong \Lambda^1 P \cong P,$$

and similarly $\Lambda^{t+1}\big(Q \oplus R^t\big) \cong Q$. Thus, $P \cong Q$, as desired. □

In Chapter II, we shall introduce the *Picard group* $\operatorname{Pic}(R)$ of a commutative ring R, which consists of the isomorphism classes of f.g. rank 1 projective R-modules under the tensor product operation. Using the $\operatorname{Pic}(R)$ notation in advance, we can state (6.6) in a somewhat more sleek form, as follows.

Corollary 6.7. *For a commutative ring R, the natural map sending the isomorphism class of P in* $\operatorname{Pic}(R)$ *to $[P] \in K_0R$ defines a group embedding*

$$\varepsilon : \operatorname{Pic}(R) \longrightarrow \mathrm{U}(K_0R),$$

where the RHS *denotes the group of units of the Grothendieck ring K_0R.*

As an aside, we point out that there is also a map $\delta : \operatorname{Pic}(R) \longrightarrow K_0R$, defined by sending the isomorphism class of P in $\operatorname{Pic}(R)$ to $[R] - [P] \in K_0R$. However, while δ is still an embedding (since ε is), it is not in general a homomorphism from $\operatorname{Pic}(R)$ to the Grothendieck group K_0R. We can only conclude that δ is a group homomorphism in some special cases, e.g. when R is a commutative noetherian ring of Krull dimension $\leqslant 1$.

§7. The Whitehead Group K_1

Although we will not need the Whitehead group $K_1(R)$ in our discussions on Serre's Conjecture, it seems appropriate to give a quick exposition on how this group is defined, after our introduction of the Grothendieck group K_0R in §6. The groups K_0R and K_1R (defined for any ring R) constitute the first two groups in algebraic K-theory. Milnor introduced K_2R in 1967, and in a few years Quillen formulated the correct definition of K_nR for all $n \geqslant 0$, thus founding the modern subject of algebraic K-theory.

The definition of the Whitehead group K_1R depends on a certain stabilization process for the classical groups. This is the process whereby we embed the group $\mathrm{GL}_n(R)$ into $\mathrm{GL}_{n+1}(R)$ by identifying a matrix $A \in \mathrm{GL}_n(R)$ with its "suspension" $\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_{n+1}(R)$. Under such an identification, we have clearly $\mathrm{E}_n(R) \subseteq \mathrm{E}_{n+1}(R)$ (and $\mathrm{SL}_n(R) \subseteq \mathrm{SL}_{n+1}(R)$ if R is commutative). With this stabilization process in place, we can thus form the ascending unions

$$\text{(7.1)} \qquad \mathrm{GL}(R) := \bigcup_{n \geqslant 1} \mathrm{GL}_n(R), \quad \text{and} \quad \mathrm{E}(R) := \bigcup_{n \geqslant 1} \mathrm{E}_n(R)$$

(and also $\mathrm{SL}(R)$ if R is commutative). To see the efficacy of these "stable versions" of GL_n and E_n, we begin with the following observation on the commutators of elementary matrices.

Lemma 7.2. *For elements a, b in an arbitrary ring R, we have*

$$[I_n + ae_{ij},\ I_n + be_{jk}] = I_n + abe_{ik} \in \mathrm{E}_n(R)$$

for any three distinct indices i, j, k. (Here, $\{e_{ij}\}$ are the matrix units, and $[\alpha,\ \beta]$ denotes the commutator $\alpha^{-1}\beta^{-1}\alpha\beta$.)

Proof. This follows by a direct matrix computation. □

In particular, if $n \geqslant 3$, we see that any generator $I_n + ae_{ik}$ $(i \neq k)$ for $\mathrm{E}_n(R)$ is a commutator in that group. Therefore, we have:

Corollary 7.3. $\mathrm{E}_n(R) = [\mathrm{E}_n(R), \mathrm{E}_n(R)]$ *for $n \geqslant 3$, and* $\mathrm{E}(R) = [\mathrm{E}(R), \mathrm{E}(R)]$.

Bringing to bear Whitehead's Lemma (5.1), we can now prove the following.

Theorem 7.4. $\mathrm{E}(R) = [\mathrm{GL}(R), \mathrm{GL}(R)]$.

Proof. The inclusion "$\subseteq$" is clear from (7.3). The proof of the reverse inclusion takes full advantage of the stabilization process, as follows. A typical commutator in $\mathrm{GL}(R) = \bigcup_{n \geqslant 1} \mathrm{GL}_n(R)$ can be written as $A^{-1}B^{-1}AB$, where $A, B \in \mathrm{GL}_n(R)$. By Whitehead's Lemma, the following three matrices are in $\mathrm{E}_{2n}(R)$:

$$\begin{pmatrix} A^{-1} & 0 \\ 0 & A \end{pmatrix}, \quad \begin{pmatrix} B^{-1} & 0 \\ 0 & B \end{pmatrix}, \quad \begin{pmatrix} AB & 0 \\ 0 & (AB)^{-1} \end{pmatrix}.$$

Multiplying these together, we see that $\begin{pmatrix} A^{-1}B^{-1}AB & 0 \\ 0 & I_n \end{pmatrix} \in \mathrm{E}_{2n}(R)$, and hence $A^{-1}B^{-1}AB \in \mathrm{E}(R)$. □

The two results above imply that $\mathrm{E}(R)$ is a *normal* subgroup of $\mathrm{GL}(R)$; in fact, it is the commutator subgroup of $\mathrm{GL}(R)$, and is a *perfect group* (i.e. a group that is equal to its own commutator subgroup). This observation leads directly to the definition of the Whitehead group.

Definition 7.5. $K_1R := \mathrm{GL}(R)/\mathrm{E}(R) = \mathrm{GL}(R)^{\mathrm{ab}}$, where for any group G, G^{ab} denotes the commutator quotient group (or "abelianization") of G.

The group K_1R arose from Whitehead's theory of simple homotopy types, where the relevant ring R is the integral group ring of the fundamental group of a topological space. The notation K_1R is due to Bass, who was the first one to envisage that the two groups K_0R and K_1R should constitute the beginning of an "algebraic K-theory" for an arbitrary ring R.

In the case of a commutative ring R, we can define the special Whitehead group

$$SK_1(R) := \mathrm{SL}(R)/\mathrm{E}(R) = \mathrm{SL}(R)^{\mathrm{ab}}. \tag{7.6}$$

If we identify $\mathrm{U}(R)$ with $\mathrm{GL}_1(R) \subseteq \mathrm{GL}(R)$, $\mathrm{GL}(R)$ is a semidirect product of its normal subgroup $\mathrm{SL}(R)$ with the subgroup $\mathrm{U}(R)$. Thus, by quotienting out the normal subgroup $\mathrm{E}(R)$ of $\mathrm{GL}(R)$, we see that

$$K_1(R) \cong SK_1(R) \times \mathrm{U}(R). \tag{7.7}$$

Therefore, assuming that $\mathrm{U}(R)$ is given, the computation of $K_1(R)$ reduces to that of $SK_1(R)$. In particular, if R is such that $\mathrm{SL}_n(R) = \mathrm{E}_n(R)$ for all (sufficiently large) n (e.g. as in (5.4)), then we have $K_1R \cong \mathrm{U}(R)$ where the isomorphism is given by the determinant map. However, as we shall see in the next section, $SK_1(R)$ may not be trivial for a commutative ring R, so K_1R may be larger than $\mathrm{U}(R)$.

The fact that $\mathrm{E}(R) = [\mathrm{GL}(R), \mathrm{GL}(R)]$ was made possible strictly by the stabilization process. For a given n, we cannot expect $\mathrm{E}_n(R)$ to be the commutator subgroup of $\mathrm{GL}_n(R)$. In fact, as we shall see in the next section, $\mathrm{E}_n(R)$ may not even be normal in $\mathrm{GL}_n(R)$. However, we can make the following observation if R is assumed to be commutative. In this case, $\mathrm{GL}_n(R)$ is generated by $\mathrm{SL}_n(R)$ and the (invertible) diagonal matrices. Since the latter matrices normalize $\mathrm{E}_n(R)$ by (5.0)(1), we see that $\mathrm{E}_n(R)$ *is normal in* $\mathrm{GL}_n(R)$ *iff it is normal in* $\mathrm{SL}_n(R)$. Our examples in the next section will show that these conditions may fail, for instance for $n = 2$.

§8. Examples for $\mathrm{SL}_n(R) \neq \mathrm{E}_n(R)$

In (5.4), we have shown that if R is a euclidean domain or a commutative semilocal ring, then $\mathrm{SL}_n(R) = \mathrm{E}_n(R)$ for all $n \geqslant 1$. In this section, we would like to give some examples of (commutative) integral domains R for which $\mathrm{E}_n(R) \subsetneq \mathrm{SL}_n(R)$. As it turns out, some of these examples will have the stronger property that $\mathrm{E}_n(R)$ is not even normal in $\mathrm{SL}_n(R)$ (and $\mathrm{GL}_n(R)$).

One of the earliest examples for $\mathrm{E}_n(R) \subsetneq \mathrm{SL}_n(R)$ appeared in [Cohn: 1966a]. For $R = k[x, y]$ where k is any nonzero commutative ring, Cohn showed that the matrix

$$C = \begin{pmatrix} 1+xy & x^2 \\ -y^2 & 1-xy \end{pmatrix} \in \mathrm{SL}_2(R) \tag{8.1}$$

is not in $\mathrm{E}_2(R)$. We shall give a proof for this fact, following a clever new approach due to Hyungju Park. Note that, after factoring out a maximal ideal of k, we may assume that k is a field.

We work more generally in $R = k[x_1, \ldots, x_n]$, where k is an arbitrary field. Any nonzero polynomial $f \in R$ has the form $f_0 + f_1 + \cdots + f_d$ where f_i is the homogeneous component of (total) degree i, with $f_d \neq 0$. We write $f_d = \mathrm{LF}(f)$ (the "leading form" of f). If $f = 0$, we take, of course, $\mathrm{LF}(f) = 0$. If $A = (a_{ij})$ is a matrix over R, we define $\mathrm{LF}(A)$ to be the matrix $(\mathrm{LF}(a_{ij}))$. With these notations in place, we have the following result from [Park: 1999].

Theorem 8.2. *Suppose* $A = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in \mathrm{E}_2(R) \backslash \mathbb{M}_2(k)$. *Then either one of* $p,\ q,\ r,\ s$ *is zero, or one of the rows of* $\mathrm{LF}(A)$ *is an* R*-multiple of the other row; and similarly for the columns.*

This necessary condition on the matrices in $\mathrm{E}_2(R) \backslash \mathbb{M}_2(k)$ enables us to see quickly that the Cohn matrix C in (8.1) is not in $\mathrm{E}_2(R)$. For, the entries of C are all nonzero, and $\mathrm{LF}(C) = \begin{pmatrix} xy & x^2 \\ -y^2 & -xy \end{pmatrix}$. The two rows here are: $x(y, x)$ and $-y(y, x)$, neither of which is an R-multiple of the other. Thus, (8.2) implies that $C \notin \mathrm{E}_2(R)$!

Proof of (8.2). By symmetry, it suffices to prove the desired conclusion for the rows of $\mathrm{LF}(A)$. Express A in the form $E_1 \cdots E_m$, where each E_i is of the form $\begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 \\ f & 1 \end{pmatrix}$, with $f \in R$. We shall induct on m. If $m = 1$, then A has a zero entry and we are done. For the inductive step, we assume that $m \geqslant 2$ and let

$$A' := E_1 \cdots E_{\mathrm{m}-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{E}_2(R).$$

Case 1. A' *does not have a zero entry.* We first assume $E_m = \begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix}$, in which case

$$A = \begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & af+b \\ c & cf+d \end{pmatrix}. \tag{8.3}$$

If $A' \in \mathbb{M}_2(k)$, then $a, b, c, d \in \mathrm{U}(k)$, and, since $A \notin \mathbb{M}_2(k)$, we must have $f \notin k$, and so

$$\mathrm{LF}(A) = \begin{pmatrix} a & a \cdot \mathrm{LF}(f) \\ c & c \cdot \mathrm{LF}(f) \end{pmatrix}.$$

The first row of this matrix is ac^{-1} times its second row, so we are done. We may thus assume that $A' \notin \mathbb{M}_2(k)$. Then the inductive hypothesis implies, say:

$$(\mathrm{LF}(a),\ \mathrm{LF}(b)) = h \cdot (\mathrm{LF}(c),\ \mathrm{LF}(d)) \tag{8.4}$$

for some $h \in R$. Since $A \notin \mathbb{M}_2(k)$ and $\det(A) = 1$, we see easily that $\det(\mathrm{LF}(A)) = 0$. Therefore, from (8.3) and (8.4), we have

$$\mathrm{LF}(c) \cdot \mathrm{LF}(af+b) = \mathrm{LF}(a) \cdot \mathrm{LF}(cf+d) = h \cdot \mathrm{LF}(c) \cdot \mathrm{LF}(cf+d),$$

which gives $\mathrm{LF}(af+b) = h \cdot \mathrm{LF}(cf+d)$. Using (8.3) and (8.4) again, we deduce that

$$\begin{aligned}(\mathrm{LF}(p),\ \mathrm{LF}(q)) &= (\mathrm{LF}(a),\ \mathrm{LF}(af+b)) \\ &= (h \cdot \mathrm{LF}(c),\ h \cdot \mathrm{LF}(cf+d)) \\ &= h \cdot (\mathrm{LF}(r),\ \mathrm{LF}(s)),\end{aligned}$$

as desired. The case where $E_m = \begin{pmatrix} 1 & 0 \\ f & 1 \end{pmatrix}$ is similar.

Case 2. A' has a zero entry. Let us assume $c = 0$. [The case when another entry of A' is zero can be treated in the same way (as below).] Since $1 = \det(A') = ad$, we have $a, d \in \mathrm{U}(R) = \mathrm{U}(k)$. If $E_m = \begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix}$, then from (8.3), A has a zero $(2, 1)$-entry. If, on the other hand, $E_m = \begin{pmatrix} 1 & 0 \\ f & 1 \end{pmatrix}$, then $A = \begin{pmatrix} a+fb & b \\ fd & d \end{pmatrix}$. We may assume $b \neq 0 \neq f$ (for otherwise A has a zero entry and we are done). In this case, $fb \notin k$ (for otherwise $b, f \in k$ and $A \in \mathbb{M}_2(k)$, contrary to our assumptions). But then

$$\mathrm{LF}(A) = \begin{pmatrix} \mathrm{LF}(f) \cdot \mathrm{LF}(b) & \mathrm{LF}(b) \\ d \cdot \mathrm{LF}(f) & d \end{pmatrix}. \tag{8.5}$$

Since $d \in \mathrm{U}(k)$, we have

$$(\mathrm{LF}(f) \cdot \mathrm{LF}(b),\ \mathrm{LF}(b)) = d^{-1}\mathrm{LF}(b)\,(d \cdot \mathrm{LF}(f),\ d),$$

so the first row of $\mathrm{LF}(A)$ in (8.5) is an R-multiple of its second row, as desired. □

Remark 8.6. For the Cohn matrix C in (8.1), the fact that $C \notin \mathrm{E}_2(R)$ implies that $(1+xy,\ x^2) \in \mathrm{Um}_2(R)$ *cannot* be brought to $(1,\ 0)$ by elementary transformations. For, if $(1+xy,\ x^2)E = (1, 0)$ for some $E \in \mathrm{E}_2(R)$, let

$$(g,\ h) := (-y^2,\ 1-xy) \cdot E.$$

Then $\begin{pmatrix} 1 & 0 \\ g & h \end{pmatrix} = C \cdot E \in \mathrm{SL}_2(R) \Rightarrow h = 1$. But then

$$C = \begin{pmatrix} 1 & 0 \\ g & 1 \end{pmatrix} E^{-1} \in \mathrm{E}_2(R),$$

a contradiction. On the other hand, we shall show later (see (9.10)) that the 3×3 matrix $\begin{pmatrix} C & 0 \\ 0 & 1 \end{pmatrix}$ *does* belong to $\mathrm{E}_3(R)$, so we will have

$$(8.7) \qquad \big(1+xy,\ x^2,\ 0\big) \sim_{\mathrm{E}_3(R)} (1, 0, 0).$$

Remark 8.8. Note that the initial conditions imposed on the matrix A in (8.2) cannot be removed. For, if one of the entries of A is zero, or if all of them are in k, the conclusion on $\mathrm{LF}(A)$ in (8.2) can clearly fail. Park has also pointed out that (8.2) does not have a converse. In other words, if a matrix $A \in \mathrm{SL}_2(R) \setminus \mathbb{M}_2(k)$ has all entries nonzero and $\mathrm{LF}(A)$ has the row and column dependence properties in the conclusion of (8.2), it need not follow that $A \in \mathrm{E}_2(R)$. To see this, we take the Cohn matrix C in (8.1) over $R = k[x, y]$ and modify it into

$$\begin{aligned} A &:= \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} C \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1+xy-y^3 & x+y+x^2+x^2y-xy^2-xy^3 \\ -y^2 & 1-xy-xy^2 \end{pmatrix}. \end{aligned}$$

Then $\mathrm{LF}(A) = \begin{pmatrix} -y^3 & -xy^3 \\ -y^2 & -xy^2 \end{pmatrix}$. Here, the first row is y times the second row, and the second column is x times the first column. And yet, $C \notin \mathrm{E}_2(R) \Rightarrow A \notin \mathrm{E}_2(R)$.

Remark 8.9. For $R = k[x, y]$ as above, we can parlay $\mathrm{E}_2(R) \subsetneq \mathrm{SL}_2(R)$ into the stronger conclusion that $\mathrm{E}_2(R)$ *is not a normal subgroup of* $\mathrm{SL}_2(R)$. In fact, for the Cohn matrix C, a quick calculation yields

$$C \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} C^{-1} =$$

$$\begin{pmatrix} y^2(1+xy)-x^2(1-xy) & x^4+(1+xy)^2 \\ -y^4-(1-xy)^2 & x^2(1-xy)-y^2(1+xy) \end{pmatrix}.$$

If we denote this matrix by $C' \in \mathrm{SL}_2(R)$,

$$\mathrm{LF}(C') = \begin{pmatrix} xy(x^2+y^2) & x^2(x^2+y^2) \\ -y^2(x^2+y^2) & -xy(x^2+y^2) \end{pmatrix}.$$

For $\mathbf{v} = (y,\ x)$, the two rows of $\mathrm{LF}(C')$ are

$$x(x^2+y^2)\,\mathbf{v} \quad \text{and} \quad -\,y(x^2+y^2)\,\mathbf{v},$$

none of which is an R-multiple of the other. By (8.2.), $C' \notin \mathrm{E}_2(R)$. But $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathrm{E}_2(R)$ (by (5.0)(2)), so $C \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} C^{-1} = C' \notin \mathrm{E}_2(R)$ shows that $\mathrm{E}_2(R)$ is not normal in $\mathrm{SL}_2(R)$.

Remark 8.10. It is worth noting that a stronger formulation of Theorem (8.2) is possible. In fact, in [Park: 1999], a "monomial order" is fixed on the set of monomials in $x_1, \ldots, x_n$, with respect to which every polynomial f has a *monomial* leading term, say $\mathrm{LT}(f)$, and a leading term matrix $\mathrm{LT}(A)$ can be associated with any matrix A over $R = k[x_1, \ldots, x_n]$. Park's original theorem was proved in this framework, with the stronger conclusion that in Theorem (8.2), "an R-multiple" is replaced by "a monomial multiple." The same proof given in the text works for this case (and fills a small gap left in the proof of Th. 2.1 in [Park: 1999]$^{(*)}$).

Remark 8.11. In the above example, $R = k[x, y] = k[x][y]$. Replacing $k[x]$ by $\mathbb{Z}$ (for instance), Cohn has also shown that $\mathrm{E}_2(\mathbb{Z}[y]) \neq \mathrm{SL}_2(\mathbb{Z}[y])$, by proving that the matrix

$$\begin{pmatrix} 1+2y & 4 \\ -y^2 & 1-2y \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}[y])$$

does not belong to $\mathrm{E}_2(\mathbb{Z}[y])$.

Next, we shall present a second example of $\mathrm{SL}_n(R) \neq \mathrm{E}_n(R)$ by exploiting some known facts in topology. This example was first pointed out by J. Stallings.

Consider the real coordinate ring of the circle S^1, that is, $R = \mathbb{R}[x, y]$, with the relation $x^2 + y^2 = 1$. Any element of R is a real polynomial function on S^1. We claim that

Proposition 8.12. *The matrix* $\sigma = \begin{pmatrix} x & y \\ -y & x \end{pmatrix} \in \mathrm{SL}_2(R)$ *does not belong to* $\mathrm{E}_2(R)$ (*so in particular* $\mathrm{E}_2(R) \subsetneq \mathrm{SL}_2(R)$).

This would not be easy to prove by pure algebra, but it can be seen by using a little bit of homotopy theory, as follows. We work in $\mathrm{GL}_3(R)$, and, as in §7, view $\mathrm{GL}_2(R)$ as a subgroup of $\mathrm{GL}_3(R)$ by identifying a matrix $A \in \mathrm{GL}_2(R)$ with $\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_3(R)$. In this way, we have $\mathrm{SL}_2(R) \subseteq \mathrm{SL}_3(R)$, and also $\mathrm{E}_2(R) \subseteq \mathrm{E}_3(R)$.

Note that each matrix $\varphi = (\varphi_{ij}) \in \mathrm{SL}_3(R)$ gives rise to a continuous mapping $\varphi_* : S^1 \to \mathrm{SL}_3(\mathbb{R})$ defined by $\varphi_*(t) = (\varphi_{ij}(t))$ (for any $t \in S^1$). This mapping, in turn, determines a homotopy class $[\varphi_*] \in \pi_1(\mathrm{SL}_3(\mathbb{R}))$ (the fundamental group of $\mathrm{SL}_3(\mathbb{R})$). The rule $\varphi \mapsto [\varphi_*]$ clearly gives a group homomorphism

$$\varepsilon : \ \mathrm{SL}_3(R) \to \pi_1(\mathrm{SL}_3(\mathbb{R})). \tag{8.13}$$

Note that the fundamental group on the RHS is *abelian*, since $\mathrm{SL}_3(\mathbb{R})$ is a topological group.

For the matrix $\sigma = \begin{pmatrix} x & y \\ -y & x \end{pmatrix} \in \mathrm{SL}_2(R) \subseteq \mathrm{SL}_3(R)$, *let us prove the stronger conclusion that* $\sigma \notin \mathrm{E}_3(R)$. Indeed, if $\sigma \in \mathrm{E}_3(R)$, then (7.3) yields $\sigma \in [\mathrm{E}_3(R), \mathrm{E}_3(R)]$,

$^{(*)}$This gap necessitates a correction in Park's Theorem 2.1: this result holds only over fields, and not over euclidean domains as stated.

and we would have $1 = \varepsilon(\sigma) = [\sigma_*]$ since $\pi_1(\mathrm{SL}_3(\mathbb{R}))$ is abelian. However, the special orthogonal group $\mathrm{SO}(3)$ is a deformation retract of $\mathrm{SL}_3(\mathbb{R})$. This implies that $\pi_1(\mathrm{SL}_3(\mathbb{R})) \cong \pi_1(\mathrm{SO}(3))$, and the latter is well-known to be $\mathbb{Z}/2\mathbb{Z}$, with the homotopy class $[\sigma_*]$ as its generator.[(†)] This contradiction shows that we must have $\sigma \notin \mathrm{E}_3(R)$.

Remark 8.14. Of course, the argument above shows much more. Along with the stabilized groups $\mathrm{GL}(R)$, $\mathrm{SL}(R)$ and $\mathrm{E}(R)$ introduced in §7, we have also the infinite orthogonal group $\mathrm{SO}(\infty) = \bigcup_{n \geqslant 1} \mathrm{SO}(n)$. Since $\pi_1\big(\mathrm{SO}(\infty)\big)$ is still $\mathbb{Z}/2\mathbb{Z}$, and generated by $[\sigma_*]$, the same argument as above shows that $\sigma \notin \mathrm{E}(R)$. Thus, we have $\mathrm{E}_n(R) \subsetneq \mathrm{SL}_n(R)$ for all $n \geqslant 2$. In fact, the group $SK_1(R)$ (defined to be $\mathrm{SL}(R)/\mathrm{E}(R)$ in §7) turns out to be precisely $\mathbb{Z}/2\mathbb{Z}$, with the image of σ as its generator.

The ring R above is a Dedekind domain, so we see that $\mathrm{SL}_n(R) = \mathrm{E}_n(R)$ *may not hold for Dedekind domains*. One might suspect that this is caused by the presence of a nontrivial ideal class group, so a natural question would be whether $\mathrm{SL}_n(R) = \mathrm{E}_n(R)$ holds for principal ideal domains.

The answer to this question is, however, still negative. In fact, in [Cohn: 1966a], it was shown that, for R the ring of algebraic integers in $\mathbb{Q}\big(\sqrt{-19}\big)$ and $\theta = \big(1+\sqrt{-19}\big)/2 \in R$, the matrix

$$\sigma = \begin{pmatrix} 3-\theta & 2+\theta \\ -3-2\theta & 5-2\theta \end{pmatrix} \tag{8.15}$$

is *not* in $\mathrm{E}_2(R)$, although it is in $\mathrm{SL}_2(R)$, since its determinant is

$$(3-\theta)(5-2\theta) - (2+\theta)(-3-2\theta) = 4\theta^2 - 4\theta + 21 = 1,$$

thanks to the integral equation $\theta^2 - \theta + 5 = 0$ of θ over $\mathbb{Z}$. Now $\mathbb{Q}\big(\sqrt{-19}\big)$ is one of Gauss's nine imaginary quadratic number fields of class number 1, so its ring of integers R is a PID. In view of (5.4)(2), Cohn's result implies, in particular, that R is not a euclidean domain (a fact already observed earlier by Motzkin).

As in (8.9), one can actually show that

(8.16) $\mathrm{E}_2(R)$ *is not normal in* $\mathrm{SL}_2(R)$.

To see this, we first try to "simplify" Cohn's matrix σ. First left multiplying σ by $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \theta & 1 \end{pmatrix}$, we get the matrix $\begin{pmatrix} 1-\theta & 2 \\ 2 & \theta \end{pmatrix}$. Further right multiplying this by $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathrm{E}_2(R)$, we get down to

(†) Of course, we have given out more information than was necessary. All we really need for this proof is the fact that the loop φ_* is not null homotopic.

$$\tau = \begin{pmatrix} 2 & \theta - 1 \\ \theta & -2 \end{pmatrix}.$$

Since the "multipliers" used above are all in $\mathrm{E}_2(R)$, we see that $\tau \in \mathrm{SL}_2(R) \backslash \mathrm{E}_2(R)$, so we may replace Cohn's matrix σ by the simpler matrix[(*)] τ.

Taking hint from (8.9), we zero in on the conjugation

$$\tau \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tau^{-1} = \begin{pmatrix} -2 & -\theta \\ 1-\theta & 2 \end{pmatrix} = -\tau^t \notin \mathrm{E}_2(R).$$

Since $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathrm{E}_2(R)$ by (5.0)(2), this proves (8.16)!

The fact that, for commutative rings R, we have examples of $\mathrm{E}_2(R)$ being not normal in $\mathrm{SL}_2(R)$ is very special for dimension 2. The case $n \geqslant 3$ turns out to be totally different, as we'll see in the next section.

§9. Suslin's Normality Theorem

In 1977, in connection with his work on the special linear group of a polynomial ring, Suslin proved the following very surprising theorem:

> *Over any commutative ring* R*, the group* $\mathrm{E}_n(R)$ *is always normal in* $\mathrm{GL}_n(R)$ *for* $n \geqslant 3$.

In view of the fact that conjugation in $\mathrm{GL}_n(R)$ corresponds to a "change of bases", this theorem has the following wonderful interpretation in terms of the matrix representations of linear endomorphisms of free R-modules. Let $\sigma \in \mathrm{End}_R(F)$, where F is a free R-module of rank $n \geqslant 3$ with two bases $\{e_i\}$ and $\{e_i'\}$. Then the matrix of σ with respect to $\{e_i\}$ is in $\mathrm{E}_n(R)$ if and only if the matrix of σ with respect to $\{e_i'\}$ is in $\mathrm{E}_n(R)$! (By what we have done in the previous section, we know that this does not work in general for $n = 2$.)

In this section, we shall actually prove a stronger, "relative" form of the normality statement above, which we'll call *Suslin's Normality Theorem.* In order to formulate this stronger version of the theorem (following [Suslin: 1977a]), let us first introduce the relative general linear groups and the corresponding relative elementary matrix groups.

Definition 9.1. For any ideal J in a ring R, we define

$$\mathrm{GL}_n(R, J) = \ker\,(\mathrm{GL}_n(R) \to \mathrm{GL}_n(R/J)).$$

Thus, $\mathrm{GL}_n(R, J)$ is a normal subgroup of $\mathrm{GL}_n(R)$, consisting of the invertible $n \times n$ matrices whose diagonal elements are $\equiv 1 \pmod{J}$, and whose off-diagonal elements

[(*)] Of course, it is also possible to apply the methods in [Cohn: 1966a] directly, to show that $\tau \notin \mathrm{E}_2(R)$. Here we started with σ since it was Cohn's original choice of a matrix in $\mathrm{SL}_2(R) \setminus \mathrm{E}_2(R)$.

are in J. The group $\mathrm{E}_n(R, J)$, on the other hand, is defined to be the *normal* subgroup of $\mathrm{E}_n(R)$ generated by the elementary matrices

$$\{I + ae_{ij} : \quad 1 \leqslant i \neq j \leqslant n, \quad a \in J\}; \tag{9.2}$$

that is, $\mathrm{E}_n(R, J)$ is generated by matrices of the form $\gamma(I + ae_{ij})\gamma^{-1}$, where $I + ae_{ij}$ is as in (9.2) and $\gamma \in \mathrm{E}_n(R)$. (Of course, $\mathrm{GL}_n(R, R)$ is just $\mathrm{GL}_n(R)$ and $\mathrm{E}_n(R, R)$ is just $\mathrm{E}_n(R)$, while $\mathrm{GL}_n(R, (0))$ and $\mathrm{E}_n(R, (0))$ are both trivial.)

We have always $\mathrm{E}_n(R, J) \subseteq \mathrm{E}_n(R) \cap \mathrm{GL}_n(R, J)$, though we should not expect an equality here in general. If R happens to be a commutative ring, then

$$\mathrm{E}_n(R, J) \subseteq \mathrm{SL}_n(R, J) := \mathrm{SL}_n(R) \cap \mathrm{GL}_n(R, J). \tag{9.3}$$

For the general discussions in this section, we fix the pair (R, J) as in (9.1), and consider two *rectangular* matrices, $\alpha \in \mathbb{M}_{n,m}(R)$, $\beta \in \mathbb{M}_{m,n}(R)$, where n and m are fixed integers. We begin with the following well known ring-theoretic fact for matrices.

Lemma 9.4. *$I_n + \alpha\beta$ is (left, right) invertible in $\mathbb{M}_n(R)$ iff $I_m + \beta\alpha$ is (left, right) invertible in $\mathbb{M}_m(R)$.*

Proof. By symmetry, it suffices to prove the "only if" part, and we may focus on the left-invertible case. Let $\gamma \in \mathbb{M}_n(R)$ be a left inverse for $I_n + \alpha\beta$. Then

$$\begin{aligned}(I_m - \beta\gamma\alpha)(I_m + \beta\alpha) &= I_m + \beta\alpha - \beta\gamma(\alpha + \alpha\beta\alpha)\\ &= I_m + \beta\alpha - \beta\gamma(I_n + \alpha\beta)\alpha\\ &= I_m + \beta\alpha - \beta\alpha = I_m,\end{aligned}$$

so $I_m + \beta\alpha$ has a left inverse $I_m - \beta\gamma\alpha$. □

Lemma (9.4) will be needed only in the "invertible" case. For later reference, we record the exact relationship obtained above between the inverse γ of $I_n + \alpha\beta$ and the inverse δ of $I_m + \beta\alpha$ (assuming that they exist):

$$\delta = (I_m + \beta\alpha)^{-1} = I_m - \beta\gamma\alpha, \tag{9.5}$$

$$\gamma = (I_n + \alpha\beta)^{-1} = I_n - \alpha\delta\beta. \tag{9.6}$$

For the genesis of these symmetrical formulas, see the discussion on Ex. 1.6 in my book "Exercises in Classical Ring Theory" (2nd ed.), Springer-Verlag, 2003.

With the above background information, we shall now prove

Vaserstein's Lemma 9.7. *For $\beta \in \mathbb{M}_{m,n}(R)$ and $\alpha \in \mathbb{M}_{n,m}(J)$ such that $I_m + \beta\alpha \in \mathrm{GL}_m(R)$, we have*

$$\begin{pmatrix} I_n + \alpha\beta & 0 \\ 0 & (I_m + \beta\alpha)^{-1} \end{pmatrix} \in \mathrm{E}_{n+m}(R, J).$$

Proof. For δ and γ defined in (9.5)–(9.6), note that

$$\gamma \equiv I_n \pmod{J}, \quad \text{and} \quad \delta \equiv I_m \pmod{J}.$$

There will be no danger in suppressing the subscripts for the identity matrices. With this notational simplification, we carry out the following sequence of elementary (block) column transformations:

$$\begin{aligned}\begin{pmatrix} I+\alpha\beta & 0 \\ 0 & \delta \end{pmatrix} &\longmapsto \begin{pmatrix} I+\alpha\beta & -\alpha \\ 0 & \delta \end{pmatrix} \longmapsto \begin{pmatrix} I & -\alpha \\ \delta\beta & \delta \end{pmatrix} \\ &\longmapsto \begin{pmatrix} I & 0 \\ \delta\beta & \delta+\delta\beta\alpha \end{pmatrix} = \begin{pmatrix} I & 0 \\ \delta\beta & I \end{pmatrix} \\ &\longmapsto \begin{pmatrix} I & 0 \\ \delta\beta-\beta & I \end{pmatrix}.\end{aligned}$$

Writing this out in a single matrix equation, we have

$$\text{(9.8)} \quad \begin{pmatrix} I+\alpha\beta & 0 \\ 0 & \delta \end{pmatrix} = \begin{pmatrix} I & 0 \\ (\delta-I)\beta & I \end{pmatrix} \left[\begin{pmatrix} I & 0 \\ \beta & I \end{pmatrix} \begin{pmatrix} I & -\alpha \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ -\beta & I \end{pmatrix} \right] \begin{pmatrix} I & \gamma\alpha \\ 0 & I \end{pmatrix}.$$

Since α, $\gamma\alpha$ and $(\delta-I)\beta$ all have entries in J, each of

$$\begin{pmatrix} I & -\alpha \\ 0 & I \end{pmatrix}, \quad \begin{pmatrix} I & \gamma\alpha \\ 0 & I \end{pmatrix}, \quad \begin{pmatrix} I & 0 \\ (\delta-I)\beta & I \end{pmatrix}$$

is a product of matrices of the form $I+ae_{ij}$ with $a \in J$ (and $i \neq j$). Interpreting the term in brackets on the RHS of (9.8) as a conjugation in $\mathrm{E}_{n+m}(R)$, we conclude that the LHS belongs to $\mathrm{E}_{n+m}(R, J)$. □

Lemma (9.7) comes from [Vaserstein: 1969b]. The proof above is a slight adaptation (to the case of relative groups) of the one given by K. Mukherjea in [Gupta-Murthy: 1980, p. 15]. In the case $J=R$, Mukherjea factorized $\begin{pmatrix} I+\alpha\beta & 0 \\ 0 & \delta \end{pmatrix}$ "symmetrically" into

$$(*) \qquad \begin{pmatrix} I & 0 \\ \delta\beta & I \end{pmatrix} \begin{pmatrix} I & -\alpha \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ -\beta & I \end{pmatrix} \begin{pmatrix} I & \gamma\alpha \\ 0 & I \end{pmatrix},$$

which we have re-expressed in the form (9.8) to push through the argument in the case of a general ideal J. Note that, from

$$\beta(I+\alpha\beta) = (I+\beta\alpha)\beta,$$

we have an identity $\delta\beta = \beta\gamma$. Thus, in $(*)$, the first matrix could have been replaced by $\begin{pmatrix} I & 0 \\ \beta\gamma & I \end{pmatrix}$, so that we would only need the matrix γ (and not δ) in factorizing

$\begin{pmatrix} I+\alpha\beta & 0 \\ 0 & \delta \end{pmatrix}$. In this spirit, the first matrix on the RHS of (9.8) could have been written as $\begin{pmatrix} I & 0 \\ \beta(\gamma - I) & I \end{pmatrix}$.

For the applications to Suslin's Normality Theorem, let us specialize (9.7) to the case where $m = 1$ and $\beta\alpha = 0 \in R$. In this case, α is a column n-vector, β is a row n-vector, and $I_m + \beta\alpha$ is the scalar 1. We restate (9.7) in this special case, with an important refinement (in part (2) below).

Corollary 9.9. *Let* $\beta \in \mathbb{M}_{1,n}(R)$ *and* $\alpha \in \mathbb{M}_{n,1}(J)$ *be such that* $\beta\alpha = 0 \in R$. *Then:*

(1) $I_n + \alpha\beta \in \mathrm{E}_{n+1}(R, J)$ (*under suspension*)*; and*
(2) *If* α *has at least one zero coordinate, then* $I_n + \alpha\beta \in \mathrm{E}_n(R, J)$.

Proof. (2) Assume, for ease of notation, that α has last coordinate zero, say, $\alpha = \begin{pmatrix} \alpha' \\ 0 \end{pmatrix}$, and write $\beta = (\beta', b)$. Then

$$I_n + \alpha\beta = \begin{pmatrix} I_{n-1} + \alpha'\beta' & \alpha' b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} I_{n-1} & \alpha' b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} I_{n-1} + \alpha'\beta' & 0 \\ 0 & 1 \end{pmatrix}.$$

Since $(\beta', b) \cdot \begin{pmatrix} \alpha' \\ 0 \end{pmatrix} = 0$, we have $\beta'\alpha' = 0$, so by part (1) (for $n-1$),

$$I_{n-1} + \alpha'\beta' \in \mathrm{E}_n(R, J) \quad \text{(under suspension)}.$$

On the other hand, $\alpha' \in \mathbb{M}_{n-1,1}(J)$ implies that $\begin{pmatrix} I_{n-1} & \alpha' b \\ 0 & 1 \end{pmatrix} \in \mathrm{E}_n(R, J)$, so we have $I_n + \alpha\beta \in \mathrm{E}_n(R, J)$. □

Example 9.10. As a quick example (in the case $J = R$), take a pair of commuting elements x, y in a ring R, and let $\alpha = (x, -y, 0)^t$ and $\beta = (y, x, 0)$. Since $\beta\alpha = xy - yx = 0$, and α has its third coordinate equal to 0, part (2) of the above corollary implies that the matrix

$$I_3 + \begin{pmatrix} x \\ -y \\ 0 \end{pmatrix} (y, x, 0) = \begin{pmatrix} 1+xy & x^2 & 0 \\ -y^2 & 1-xy & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is in $\mathrm{E}_3(R)$. (We have referred to this fact before in Remark (8.6).) However, if $R = k[x, y]$ where k is any nonzero commutative ring, then by the remark following (8.2),

$$C := \begin{pmatrix} 1+xy & x^2 \\ -y^2 & 1-xy \end{pmatrix} \notin \mathrm{E}_2(R).$$

Thus, we see that it is possible for $\mathrm{E}_2(R) \subsetneqq \mathrm{E}_3(R) \cap \mathrm{GL}_2(R)$. [For another derivation of $C \in \mathrm{E}_3(R)$ via "Mennicke symbols" (and a generalization), see (VI.3.5).]

Returning to the setting of (9.7), we prove now another preparatory lemma in the *commutative* case.

Lemma 9.11. *For any commutative ring R, let α, β be as in* (9.9)*, and let $r \in R$ be any element in the ideal of R generated by the coordinates of β. Then*

(1) *there exists a decomposition $r\alpha = \alpha_1 + \cdots + \alpha_N$ where, for each i, $\alpha_i \in \mathbb{M}_{n,1}(J)$, $\beta\alpha_i = 0$, and α_i has at most two nonzero coordinates.*

(2) *If $n \geqslant 3$, then $I_n + r\alpha\beta \in \mathrm{E}_n(R, J)$.*

Proof. (1) Let $\beta = (b_1, \ldots, b_n)$, and $\alpha = (a_1, \ldots, a_n)^t$ where each $a_i \in J$. By linearity, it suffices for us to handle the case (say) $r = b_n$. Since $0 = \beta\alpha = b_1 a_1 + \cdots + b_n a_n$, we can decompose

$$\begin{aligned} r\alpha &= \begin{pmatrix} b_n a_1 \\ \vdots \\ b_n a_{n-1} \\ -(b_1 a_1 + \cdots + b_{n-1} a_{n-1}) \end{pmatrix} \\ &= a_1 \begin{pmatrix} b_n \\ 0 \\ \vdots \\ -b_1 \end{pmatrix} + a_2 \begin{pmatrix} 0 \\ b_n \\ \vdots \\ -b_2 \end{pmatrix} + \cdots + a_{n-1} \begin{pmatrix} 0 \\ \vdots \\ b_n \\ -b_{n-1} \end{pmatrix}. \end{aligned}$$

Letting α_i be the ith summand in the expression above, we have $\beta\alpha_i = 0$ (by commutativity), and each α_i has at most two nonzero coordinates, both in J, as desired.

(2) Write $r\alpha = \alpha_1 + \cdots + \alpha_N$ as in (1). Then

$$\begin{aligned} I_n + r\alpha\beta &= I_n + (\alpha_1 + \cdots + \alpha_N)\beta \\ &= (I_n + \alpha_1\beta)(I_n + \alpha_2\beta) \cdots (I_n + \alpha_N\beta), \end{aligned}$$

since each $\beta\alpha_i = 0$. As α_i has at most two nonzero coordinates (both belonging to J), and $n \geqslant 3$, α_i must have at least one zero coordinate. Thus, $I_n + \alpha_i\beta \in \mathrm{E}_n(R, J)$ by (9.9)(2), and the equations above give $I_n + r\alpha\beta \in \mathrm{E}_n(R, J)$. $\square$

Let $B = b_1 R + \cdots + b_n R$, and let $P(\beta) \subseteq R^n$ denote (as in §4) the "solution space" of $\beta = (b_1, \ldots, b_n)$.(*) The elements $\gamma_{ij} = b_j e_i - b_i e_j$ $(i < j)$ (where $\{e_i\}$ are the unit vectors) are called the *Koszul elements* in $P(\beta)$. What the above proof showed was that

$$B \cdot \big(P(\beta) \cap J \cdot R^n\big) \subseteq J \cdot P_0(\beta),$$

(*) In §4, we have used this notation only for unimodular vectors β. From now on, we shall use the same notation for any row vector $\beta \in R^n$.

where $P_0(\beta)$ is the submodule of $P(\beta)$ generated by the γ_{ij} 's. Specializing this to the case where β is unimodular, we get the following result (for any commutative ring R).

Corollary 9.12. *Let* $\beta \in \mathrm{Um}_n(R)$. *For any ideal* $J \subseteq R$,

(1) $P(\beta) \cap J \cdot R^n = J \cdot P_0(\beta)$. *In particular,* $P(\beta) = P_0(\beta)$.

(2) *If* $n \geqslant 3$, *then* $I_n + \alpha\beta \in \mathrm{E}_n(R, J)$ *for any column vector* $\alpha \in P(\beta) \cap J \cdot R^n$.

Remark 9.13. Without assuming β to be unimodular, it turns out that there is another case in which $P(\beta) = P_0(\beta)$. This is the case where β is a "regular sequence", which will be treated later in (III.5.13).

Equipped with the crucial Corollary 9.12, we now return to give a proof of the following remarkable result of Suslin.

Suslin's Normality Theorem 9.14. *Let* R *be a commutative ring, and let* $n \geqslant 3$. *Then, for any ideal* $J \subseteq R$, $\mathrm{E}_n(R, J)$ *is a normal subgroup of* $\mathrm{GL}_n(R)$. (*In particular,* $\mathrm{E}_n(R)$ *is a normal subgroup of* $\mathrm{GL}_n(R)$.)

Proof. Since $\mathrm{E}_n(R, J)$ is generated by matrices of the form

$$\tau(I_n + ae_{ij})\tau^{-1} \quad (\tau \in \mathrm{E}_n(R),\ a \in J,\ i \neq j),$$

it suffices to show that, for any $\gamma \in \mathrm{GL}_n(R)$, the conjugate

$$\gamma(I_n + ae_{ij})\gamma^{-1} = I_n + a(\gamma e_{ij}\gamma^{-1}) \quad (a \in J) \tag{9.15}$$

remains in $\mathrm{E}_n(R, J)$ for all $i \neq j$. Let α be the ith column of γ, and β be the jth row of γ^{-1}. Since $i \neq j$, we have $\beta\alpha = 0$, and hence $\beta\,\widetilde{\alpha} = 0$ for $\widetilde{\alpha} := a\,\alpha \in \mathbb{M}_{n,1}(J)$. By easy inspection, $\gamma e_{ij}\gamma^{-1}$ is just $\alpha\beta$, so the matrix in (9.15) is $I_n + \widetilde{\alpha}\,\beta$. Since (obviously) $\beta \in \mathrm{Um}_n(R)$, (9.12)(2) gives

$$\gamma(I + ae_{ij})\gamma^{-1} = I_n + \widetilde{\alpha}\,\beta \in \mathrm{E}_n(R, J),$$

as desired. □

Remark. Note that the *commutativity* of R was used implicitly for both (9.15) and the equation $\beta\,\widetilde{\alpha} = 0$ (as well as in the proof of (9.11) earlier). However, the normality of $\mathrm{E}_n(R)$ in $\mathrm{GL}_n(R)$ (for $n \geqslant 3$) is known to hold more generally, say, for rings R that are module-finite over their centers; see [Tulenbaev: 1979] and [Vaserstein: 1981] in the references on Ch. VIII. For $n \geqslant 3$, A. Bak has also obtained strong results on the nilpotency of the groups $\mathrm{SL}_n(R)/\mathrm{E}_n(R)$ (and the solvability of the groups $\mathrm{GL}_n(R)/\mathrm{E}_n(R)$), for rings R that include commutative noetherian rings of finite Krull dimension; see [Bak: 1991] (and Vaserstein: 1992]) listed in the references on Chapter VIII.

The main applications of (9.14) to the stability theorems in K-theory will be given in III.3 and VI.5. In closing this chapter, we return to Vaserstein's Lemma (9.7) and record a couple more of its consequences. The first one is a refinement of Whitehead's Lemma (5.1) for the relative elementary groups. (Commutativity is not required for these results.)

Proposition 9.16. *Let R be any ring, with an ideal J, and let $\sigma \in \mathrm{GL}_n(R, J)$. Then* $\begin{pmatrix} \sigma & 0 \\ 0 & \sigma^{-1} \end{pmatrix} \in \mathrm{E}_{2n}(R, J)$, *and for any $\tau \in \mathrm{GL}_n(R)$, we have*

$$\begin{pmatrix} \tau\sigma & 0 \\ 0 & I_n \end{pmatrix} \in \begin{pmatrix} \tau & 0 \\ 0 & \sigma \end{pmatrix} \cdot \mathrm{E}_{2n}(R, J). \tag{9.17}$$

Proof. Write $\sigma = I_n + \alpha$ where $\alpha \in \mathbb{M}_n(J)$. Applying (9.7) with $m = n$ and $\beta = I_n$, we have

$$I_n + \alpha\beta = I_n + \beta\alpha = I_n + \alpha = \sigma,$$

and hence $\begin{pmatrix} \sigma & 0 \\ 0 & \sigma^{-1} \end{pmatrix} \in \mathrm{E}_{2n}(R, J)$. From this, we get

$$\begin{pmatrix} \tau & 0 \\ 0 & \sigma \end{pmatrix}^{-1} \begin{pmatrix} \tau\sigma & 0 \\ 0 & I_n \end{pmatrix} = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma^{-1} \end{pmatrix} \in \mathrm{E}_{2n}(R, J),$$

which gives (9.17). □

It is worthwhile to note an explicit equation that expresses $\begin{pmatrix} \sigma & 0 \\ 0 & \sigma^{-1} \end{pmatrix}$ as an element of $\mathrm{E}_{2n}(R, J)$. In fact, by specializing (9.8), we have

(9.18)
$$\begin{pmatrix} \sigma & 0 \\ 0 & \sigma^{-1} \end{pmatrix} = \begin{pmatrix} I & 0 \\ \sigma^{-1} - I & I \end{pmatrix} \left[\begin{pmatrix} I & 0 \\ I & I \end{pmatrix} \begin{pmatrix} I & I - \sigma \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ -I & I \end{pmatrix} \right] \begin{pmatrix} I & I - \sigma^{-1} \\ 0 & I \end{pmatrix}.$$

Of course, such an expression is far from being unique. For instance, the reader can verify quickly that the following expression is also valid:

$$\begin{pmatrix} \sigma & 0 \\ 0 & \sigma^{-1} \end{pmatrix} = \left[\begin{pmatrix} I & I \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ \alpha & I \end{pmatrix} \begin{pmatrix} I & -I \\ 0 & I \end{pmatrix} \right] \begin{pmatrix} I & \sigma^{-1}\alpha \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ -\sigma\alpha & I \end{pmatrix}, \tag{9.19}$$

with $\alpha = \sigma - I_n \in \mathbb{M}_n(J)$.

A nice application of (9.16) to upper triangular matrices is as follows.

Proposition 9.20. *Let R be any ring with an ideal J, and let $\gamma = (a_{ij}) \in \mathbb{M}_n(R)$ be an upper triangular matrix congruent to I_n (mod J). If each a_{ii} is a unit, then*

$$\gamma \in a_{11} \cdots a_{nn} \cdot \mathrm{E}_n(R, J).$$

(*Here, of course,* $\mathrm{U}(R) = \mathrm{GL}_1(R)$ *is viewed as a subgroup of* $\mathrm{GL}_n(R)$ *by the usual suspension process.*)

Proof. Since $a_{ii} \in \mathrm{U}(R)$ and $a_{ij} \in J$ for all $i < j$, we can right multiply γ by suitable matrices $e_{ij}(a)$ with $i < j$ and $a \in J$ to bring γ to the diagonal matrix $\gamma_1 = \mathrm{diag}(a_{11}, \ldots, a_{nn})$. Thus, $\gamma \in \gamma_1 \cdot \mathrm{E}_n(R, J)$. Using now $a_{ii} \in \mathrm{GL}_1(R, J)$, we can apply (9.16) to get

$$\gamma_1 \in a_{11} \cdots a_{nn} \cdot \mathrm{E}_n(R, J),$$

from which we get $\gamma \in a_{11} \cdots a_{nn} \cdot \mathrm{E}_n(R, J)$. □

Note that $a_{ii} \equiv 1 \pmod J$ will imply $a_{ii} \in \mathrm{U}(R)$ if J lies in the Jacobson radical of the ring R. This observation can be used to get a result parallel to (9.20), but applicable to matrices that may not be upper triangular.

Proposition 9.21. *Suppose* $J \subseteq \mathrm{rad}\ R$ (*the Jacobson radical of* R), *and let* $\gamma = (a_{ij}) \in \mathbb{M}_n(R)$ *be congruent to* $I_n \pmod J$. *Then* $\gamma \in (1 + J) \cdot \mathrm{E}_n(R, J)$.

Proof. Using again the properties $a_{ii} \in 1 + J \subseteq \mathrm{U}(R)$ and $a_{ij} \in J$ for all i, j, we can right multiply γ by suitable $e_{ij}(a)$ $(i \neq j,\ a \in J)$ to bring γ to a diagonal matrix $\mathrm{diag}(b_1, \ldots, b_n)$ with $b_i \in 1 + J \subseteq \mathrm{U}(R)$ for all i. (Note that in carrying out the elementary column transformations corresponding to the right multiplications by $e_{ij}(a)$ above, the diagonal entries of γ will change. However, they will always be in $1 + J$, and will thus remain units of R.) Now we can complete the argument as in the proof for (9.20). □

Corollary 9.22. (1) *If* J *is an ideal in a commutative ring* R, *then any upper triangular matrix in* $\mathrm{SL}_n(R, J)$ *is in* $\mathrm{E}_n(R, J)$.

(2) *If* (R, J) *is a* (*possibly noncommutative*) *local ring* (*that is,* J *is the unique maximal left ideal of* R), *then* $\mathrm{GL}_n(R, J) = (1 + J) \cdot \mathrm{E}_n(R, J)$.

(3) *If* (R, J) *is a commutative local ring, then* $\mathrm{SL}_n(R, J) = \mathrm{E}_n(R, J)$.

Proof. (1) follows from (9.20), and (2) follows from (9.21). Finally, (3) follows from (2). □

Some of the above results will be used later in (VI.6). We close by observing that, if we define

$$\mathrm{GL}(R, J) = \bigcup_{n \geqslant 1} \mathrm{GL}_n(R, J), \quad \text{and} \quad \mathrm{E}(R, J) = \bigcup_{n \geqslant 1} \mathrm{E}_n(R, J) \tag{9.23}$$

(for an ideal J in any ring R), then a repetition of our work in §6 (now using (9.16)) shows that

$$\mathrm{E}(R, J) = [\mathrm{E}(R),\ \mathrm{E}(R, J)] = [\mathrm{GL}(R),\ \mathrm{GL}(R, J)]. \tag{9.24}$$

In particular, the factor group

$$K_1(R, J) := \mathrm{GL}(R, J)/\mathrm{E}(R, J) \tag{9.25}$$

makes sense, and is an abelian group. This is the *relative* K_1*-group* in algebraic K-theory. We will, however, not dwell on this matter, as we shall not deal with the group $K_1(R, J)$ in this book.

Notes on Chapter I

The important notion of a projective module has played a vital role in algebra since its introduction (with the advent of the new homological methods) in the 1950s. A different title for this book could have been "Finitely Generated Projective Modules over Polynomial Rings".

Nakayama's Lemma (1.5) is due to Nakayama, Krull and Azumaya. According to [Matsumura: 1970, p. 11], "Priority (of this lemma) is obscure, and although it is usually called the Lemma of Nakayama, late Professor Nakayama did not like the name." Perhaps because of this, Nagata, in his book on Local Rings [Nagata: 1962], used the term "Lemma of Krull-Azumaya".

Flat modules and faithfully flat modules were introduced and studied by Serre in an appendix to his famous GAGA paper Serre: 1955/56]. These two types of modules were extensively treated in Bourbaki's "Algèbre Commutative: Modules Plats, Localization", to which we refer the reader for more complete information. Our treatment of these topics is rather brief, as we focus only on those results that we will have occasion to use later on.

The local-global methods of §3 are standard material in commutative ring theory, but are included here to make the text self-contained. The treatment in this section is considerably simplified as we have already at our disposal the basic technique of faithfully flat modules. The "patching" theorems at the end of §3 are developed *ad hoc*; they would have no doubt looked more 'natural' in the proper setting of sheaves of modules over affine schemes.

The equivalence of the several statements in (4.5) defining a (right) Hermite ring was first observed in [Serre: 1957/58]. Gabel's proof that a nonfinitely generated stably free module must be free actually shows much more; see [Gabel: 1972] and [Lam: 1976]. The term "H-ring" was first introduced in [Lissner: 1965]; the nomenclature is motivated by the classical theorem of Charles Hermite that any row of integers can be completed to an integral matrix whose determinant is their greatest common divisor. I plead guilty to changing "H-ring" to "Hermite ring", and remind the reader once more that the Hermite rings defined in this book are more general than those defined in [Kaplansky: 1949]. The relationship between our Hermite rings and Kaplansky's K-Hermite rings was fully explained in the Appendix to §4. We took advantage of this Appendix to give a quick introduction to (commutative) K-Hermite rings and their principal examples. Actually, Kaplansky's goal in his 1949 paper was to study a class of K-Hermite rings that he called "elementary divisor rings" (or ED rings). These are rings over which every rectangular matrix has a "Smith Normal Form". Commutative von Neumann rings and PIR's are, in fact, examples of ED rings. An old question in this area of study is *whether a Bézout domain is an ED domain*: this question has apparently remained unresolved up to this date.

For other classes of (commutative) rings related to Hermite rings, for instance, outer product rings, Plücker rings, Towber rings, etc., see [Lissner: 1965, 1967], [Lissner-Geramita: 1970], [Towber: 1964, 1968, 1970], [Simis: 1969], as well as as [Kleiner: 1971]. For information on the problem of completing various types of unimodular rows to invertible matrices, see [Lissner: 1967], [Swan-Towber: 1975], [Murthy-Swan: 1976], [Krusemeyer: 1975, 1976], [Suslin: 1977b], [Roitman: 1977a], [Swan: 1978], and (II.6)–(II.7), (III.1)–(III.7), (V.3), (VI.1)–(VI.4), as well as (VIII.5) below. The literature on the *general* behavior (or structure) of stably free modules has remained, however, relatively meager.

In commutative algebra, for any (not necessarily unimodular) $(b_1, \ldots, b_n) \in R^n$, the solution space $P(b_1, \ldots, b_n) \subseteq R^n$ is a familiar object, called the (first) syzygy module of the ideal $J = b_1 R + \cdots + b_n R$. (It is also known as the "module of relations" for the chosen generators $b_1, \ldots, b_n$ of the ideal J.) In §4, we were primarily interested in the unimodular case (where $J = R$), but we have also encountered the non-unimodular case in §9. In either case, the determination of the syzygy module $P(b_1, \ldots, b_n)$ presents an interesting (and challenging) problem in computational commutative algebra. For some concrete computations of syzygy modules, see the last part of §6 in Chapter II.

The duality statement (4.10)(1) seems to be folklore in the subject. As far as I can ascertain, it first appeared explicitly (as Lemma 8.2) in Swan's Queen's Notes (1975). The result (4.11) that, over a commutative ring, a rank 1 stably free module must be free is, again, another piece of the folklore! Bass's Theorem (4.12)(2) that *an odd rank type* 1 *stably free module* P *over a commutative ring must have a unimodular element* first appeared in [Bass: 1969]. In the text, we pointed out that the "odd rank" and the "commutative" assumptions were essential in this statement. The "type 1" assumption on P turned out to be indispensable as well. In fact, without this latter assumption, P can even be indecomposable: see [Swan: 1977]. The *ad hoc* proof in the text for (4.12)(2) is conceptually simpler than Bass's original proof by symplectic methods. However, the latter gives a considerably stronger conclusion:

> *If* $P \oplus L \in \mathfrak{P}(R)$ *admits a symplectic structure* ($\operatorname{rk} L = 1$, R *commutative*), *then* $P \cong L^* \oplus Q$ *for some module* Q.

(Here, L^* denotes the dual of L.) This subsumes (4.12)(2), since, for a given type 1 stably free module P of rank $2m - 1$, we can take $L = R$, for which $P \oplus L \cong R^{2m}$ always admits a (hyperbolic) symplectic structure. For a quick introduction to the theory of symplectic modules, including a proof of the italicized statement above, see VII.5.

Sections 5, 6 and 7 are standard material from algebraic K-theory. For more complete expositions on this subject, see [Bass: 1968], [Swan: 1968], and [Milnor: 1971]. For another quick introduction to K_0 and K_1, see [Lam-Siu: 1975]. Whitehead's Lemma (5.1) is due to J.H.C. Whitehead (in the context of homotopy theory), and made popular in algebra by the work of H. Bass; see, for instance, [Bass: 1964a, 1968]. For some generalizations of Whitehead's Lemma, see [Lam: 1976].

The material in §§7–9 is new to this version of our book. The normality of $\mathrm{E}_n(R)$ in $\mathrm{SL}_n(R)$ for $n \geqslant 3$ over a commutative ring R comes from the important paper [Suslin: 1977a], which followed on the heel of Suslin's proof of Serre's Conjecture. In this paper, using the normality result (9.14) as one of his tools, Suslin proved a very general Stability Theorem for the general linear group over a polynomial ring.

For a precise statement of this theorem, see (III.3.8). In the case of a polynomial ring $A = k[t_1, \dots, t_m]$ over a field k, Suslin's result yields $\mathrm{E}_n(A) = \mathrm{SL}_n(A)$ for any $n \geqslant 3$ (and any $m \geqslant 0$). We shall prove this eventually in (VI.4.5). Note that Suslin's result is the best possible since the existence of the Cohn matrix (8.1) shows that $\mathrm{E}_2(A) \subsetneq \mathrm{SL}_2(A)$ (as long as $m \geqslant 2$).

Chapter II

The "Classical" Results on Serre's Conjecture

§1. The Case of Rank 1 Projectives

It is fairly easy to see that any rank 1 f.g. projective module over $k[t_1, \ldots, t_n]$ (k a field) is free. To give a proof of this, we start with the following characterization of rank 1 projective modules over any (commutative) integral domain.

Lemma 1.1. *Let R be an integral domain, with quotient field K. Let $P \neq 0$ be an R-submodule of K. Then P is projective iff there exists another R-submodule $Q \subseteq K$, such that $P \cdot Q = R$. In this case, P is automatically f.g. over R.*

Proof. "Only if": Choose a suitable free module $F = \bigoplus_{\alpha \in \Lambda} R \cdot e_\alpha$, and homomorphisms $F \underset{g}{\overset{f}{\rightleftarrows}} P$ such that $fg = 1_P$. We have $g(p) = \sum g_\alpha(p) e_\alpha$ for every $p \in P$, where $g_\alpha \in \mathrm{Hom}_R(P, R)$. Each g_α induces $K \otimes_R P \to K \otimes_R R$, or $K \to K$. Thus, g_α is induced by multiplication with some $b_\alpha \in K$, $b_\alpha P \subseteq R$. For each $p \in P$, only a finite number of $g_\alpha(p) = b_\alpha p$ can be nonzero. Since $P \neq 0$, we see that only a finite number of b_α's can be nonzero. After knocking out the indices $\{\alpha : b_\alpha = 0\}$, we may assume that $|\Lambda| < \infty$. Letting $a_\alpha = f(e_\alpha) \in P$, we have $p = fg(p) = p \sum b_\alpha a_\alpha$ for all $p \in P$, so $\sum b_\alpha a_\alpha = 1$. This shows that $P \cdot Q = R$ for $Q = \sum R \cdot b_\alpha$.

"If": Write $\sum b_\alpha a_\alpha = 1$, where $a_\alpha \in P$, $b_\alpha \in Q$, and use a_α, b_α to define the maps f, g above. □

Example 1.2. Let $R = \mathbb{Z}[\sqrt{5}\,]$, and P be the ideal $2R + \big(1 + \sqrt{5}\,\big)R$. A quick calculation shows that there is no R-submodule Q in the quotient field of R such that $P \cdot Q = R$. Thus, P is *not* a projective R-module. On the other hand, if we let $S = \mathbb{Z}\,[(1+\sqrt{5}\,)/2]$ (the full ring of algebraic integers in $\mathbb{Q}\big(\sqrt{5}\,\big)$), then $1 + \sqrt{5} \in 2S$ implies that

$$P \cdot S = 2S + \big(1 + \sqrt{5}\,\big)S = 2S,$$

so $P \cdot S$ becomes free of rank 1 over S.

Theorem 1.3. *Let R be a unique factorization domain* (UFD)*, and $P \in \mathfrak{P}(R)$. Then* $\operatorname{rk} P = 1 \Rightarrow P \cong R$.

Proof. Since P is torsion free, $P \subseteq K \otimes_R P \cong K$. Also, P is f.g., so we may assume that P is an ideal in R. The proof of the Lemma shows that there exist $a_\alpha \in P,\ b_\alpha \in K$, such that $b_\alpha P \subseteq R$ and $\sum b_\alpha a_\alpha = 1$. Write $b_\alpha = \frac{c_\alpha}{d_\alpha}$, where $c_\alpha, d_\alpha \in R$ have no non-unit common factor. Since $\frac{c_\beta}{d_\beta} a_\alpha \in R$, it follows (from unique factorization) that d_β divides a_α for every pair α, β. Let $d = \ell\mathrm{cm}\,\{d_\beta\}$; then $a_\alpha \in R \cdot d \Rightarrow P \subseteq R \cdot d$. On the other hand, clearing the denominators of $1 = \sum \frac{c_\alpha}{d_\alpha} a_\alpha$ by multiplication by d, we see that $d = \sum c_\alpha \frac{d}{d_\alpha} \cdot a_\alpha \in P$. Thus, $P = R \cdot d \cong R$. □

Corollary 1.4. *Let A be a* UFD *(e.g. a* PID*, or a field), $R = A[t_1, \ldots, t_n]$, and $P \in \mathfrak{P}(R)$. Then* $\operatorname{rk} P = 1 \Rightarrow P \cong R$.

Proof. By the theorem of Gauss, R is also a UFD, so we can apply (1.3). □

We note in passing that Theorem 1.3 is also true for any commutative semilocal ring. Indeed, over such a ring R, any f.g. projective module of constant rank is free. This can be easily checked by noting that $R/\operatorname{rad} R$ is a finite direct product of fields, and using (I.1.6). However, (1.3) is not true in general for Hermite rings; for instance, it fails for every Dedekind ring that is not a PID.

§2. The Case of One Variable

If k is a field, the polynomial ring $k[t]$ in one variable over k is a PID. Thus, the truth of Serre's Conjecture in this case follows from the fact that, over a PID, submodules of free modules are always free. This fact will be deduced below from a more general theorem. First we make the following definition:

Definition 2.1. A ring R is said to be left hereditary if every left ideal in R is R-projective. [For example, a PID is (left) hereditary.(*)]

Kaplansky's Theorem 2.2. *If R is a left hereditary ring, then any submodule A of any free module $F = \bigoplus_{\alpha \in \Lambda} R \cdot e_\alpha$ is isomorphic to a direct sum of left R-ideals. In particular, A must be R-projective.*

Note that this leads immediately to the following alternative characterization of left hereditary rings:

Corollary 2.3. *R is left hereditary iff submodules of* (*left*) *R-projectives are projective.*

(*) More generally, any Dedekind domain is hereditary. Conversely, if R is a (commutative) domain and R is hereditary, then R is a Dedekind domain.

Proof of (2.2). Fix a well-ordering $<$ on the indexing set Λ. Let F_α (resp. $\overline{F}_\alpha$) be the submodule of F with basis e_β for all $\beta < \alpha$ (resp. for all $\beta \leqslant \alpha$). Further, let $A_\alpha = A \cap F_\alpha$ and $\overline{A}_\alpha = A \cap \overline{F}_\alpha$. We have

$$\overline{A}_\alpha / A_\alpha \subseteq \overline{F}_\alpha / F_\alpha \cong R \cdot e_\alpha,$$

so, by hypothesis, $\overline{A}_\alpha / A_\alpha$ is R-projective. We may thus write $\overline{A}_\alpha = A_\alpha \oplus I_\alpha$ where I_α is isomorphic to a left R-ideal. We finish by showing that $A = \bigoplus_{\alpha\in\Lambda} I_\alpha$. First we show that the latter sum is direct. Suppose $x_1 + \cdots + x_n = 0$, where $x_i \in I_{\alpha_i} \subseteq \overline{F}_{\alpha_i}$. We may assume the indices are arranged so that $\alpha_1 < \cdots < \alpha_n$. Then $x_1, \ldots, x_{n-1} \in F_{\alpha_n}$. But $F_{\alpha_n} \cap I_{\alpha_n} = 0$, so $x_n = 0$, and by induction all $x_i = 0$. To finish the argument, assume $\sum_\alpha I_\alpha \subsetneq A$. Choose β to be the smallest index for which there exists an element $\overline{a} \in \overline{A}_\beta - \sum_\alpha I_\alpha$. Decompose $\overline{a} = a + x$, where $a \in A_\beta$, $x \in I_\beta$. This a clearly lies in some $\overline{A}_\gamma$ where $\gamma < \beta$. By the minimal choice of β, we have $a \in \sum_\alpha I_\alpha$, whence $\overline{a} = a + x \in \sum_\alpha I_\alpha$, a contradiction. □

The following is another immediate consequence of Kaplansky's Theorem.

Corollary 2.4. *If R is a ring all of whose left ideals are free* (*e.g. a left* PID), *then submodules of free left R-modules are free. In particular, all projective left R-modules are free.*

Serre's original problem was posed for f.g. projectives over $k[t_1, \ldots, t_n]$, where k is a field. In the one variable case, we can now settle Serre's problem even in the case where k is a division ring:

Theorem 2.5. *For any division ring k, any projective (say, left) module over $R = k[t]$ is free.*

Proof. If I is any nonzero left ideal in R, the usual division algorithm argument shows that $I = R \cdot f$, where f is a polynomial in I of the smallest degree. Since R has no zero-divisors, $I \cong R$ as left R-modules. Now apply Corollary 2.4. □

§3. The Case of Noncommutative Base Rings

We have seen in the last section that Serre's Conjecture has an affirmative answer for $R = k[t]$ where k is any division ring. Unfortunately, if we try to generalize this to the case of more than one variable, and keep k to be just a division ring, we begin to run into real obstacles. It turns out that, if k is a division ring *but not a field*, then there exists $P \in \mathfrak{P}(k[t_1, t_2])$ that is not free. By tensoring P up to $R_n = k[t_1, \ldots, t_n]$, we see that R_n admits f.g. non-free projective modules for any $n \geqslant 2$!

The construction of a nonfree projective module over $k[t_1, t_2]$ appeared in [Sharma: 1971] in the case where k is the division ring of the real quaternions, and in [Ojanguren-Sridharan: 1971] in the general case of a noncommutative division ring k. A main consequence of the latter paper is the following.

Theorem 3.1. (Ojanguren-Sridharan) *Let k be a division ring that is not a field, and $R = k[x, y]$. Then there exists a right R-ideal J such that $J \oplus R \cong R^2$ (so J is stably free of type 1, and generated by two elements), but J is not free.*

Before going into the proof of this, we make the following remarks about the result itself.

Remark 3.2. Although this is not our primary goal, the (right) module P above does provide counterexamples to (I.4.11) and (I.4.12)(2) for *noncommutative* rings. (Note that we use right modules again in this section in order to conform with the notations used in (I.4).) Also, for $A = k[x]$, the J in (3.1) provides an example of a f.g. projective right $A[y]$-module that is not extended from A; see (2.5).

Remark 3.3. For certain division rings k, Parimala and Sridharan (1975) have shown that $R = k[x, y]$ may admit *infinitely many* isomorphism classes of rank 1 projective modules. (In this case, the rank of a projective right R-module P is defined to be $\dim_k P/P \cdot (x, y)$.) For instance, k can be taken to be Hamilton’s division ring of quaternions over the reals. We shall return to verify this fact, after learning the technique of Quillen’s Patching Theorem in Chapter V; see V.1.17.

The basic construction idea in [Sharma: 1971] and [Ojanguren-Sridharan: 1971] is the following. Let R be a ring, and let $a, b \in R$ be such that the additive commutator

$$c := ab - ba \in \mathrm{U}(R) \quad \text{(the group of units of } R\text{)}.$$

For any pair of *central* elements $x, y \in R$, we define $\phi : R^2 \to R$ by the rectangular matrix $(x + a, y + b)$; i.e., $\phi(e_1) = x + a$ and $\phi(e_2) = y + b$ for the unit vectors e_1, e_2. The map ϕ is *onto* since

$$\phi \begin{pmatrix} y+b \\ -(x+a) \end{pmatrix} = (x + a, \ y + b) \begin{pmatrix} y+b \\ -(x+a) \end{pmatrix} = ab - ba = c \quad \text{is a unit.}$$

Thus, the solution space $P = P(x + a, y + b) := \ker(\phi)$ is *stably free of type* 1 (since the splitting of ϕ leads to $P \oplus R \cong R^2$). It turns out that, under suitable assumptions on $a, b, x, y \in R$, P is isomorphic to a right ideal J of R, and one can prove that J is *not* free.

To accomplish this, let us assume that $x + a$ and $y + b$ are not 0-divisors in R. Then, the “second coordinate” projection $\pi_2 : R^2 \to R$ maps P isomorphically onto the right ideal

(3.4) $$J := \{\beta \in R : \ (y + b)\beta \in (x + a)R\};$$

and left multiplication by $y + b$ defines an isomorphism from J onto the very nicely expressed right ideal $(x + a)R \cap (y + b)R$.

Since $P \oplus R \cong R^2$, we know J has two generators. It is not hard to find them. Indeed, ϕ is split by the map $\psi : R \to R^2$ defined by

$$\psi(1) = \begin{pmatrix} (y+b)c^{-1} \\ -(x+a)c^{-1} \end{pmatrix}.$$

Thus, P is the image of the projection $\mathrm{Id} - \psi\phi$ on R^2. A direct calculation then shows that J is generated by

(∗)
$\pi_2(\mathrm{Id} - \psi\phi)(e_1) = (x+a)c^{-1}(x+a)$ and $\pi_2(\mathrm{Id} - \psi\phi)(e_2) = 1 + (x+a)c^{-1}(y+b)$.

We are now ready to present the following generalization of Theorem 3.1 due to R. G. Swan. Our exposition here is based on [Swan: 1996].

Theorem 3.5. *Let D be a domain such that $D^2 \cong D^m$ (as right modules) implies $m = 2$, and assume there exist a, $b \in D$ such that $c := ab - ba \in \mathrm{U}(D)$. Let $R = D[x, z_1, \ldots, z_n]$ be a polynomial ring over D. Let $y := f(x, z_1, \ldots, z_n)$ be a polynomial over the center of D such that $f(-a, z_1, \ldots, z_n)$ is a* nonconstant *polynomial (in $D[z_1, \ldots, z_n]$). Then the stably free R-module $P = P(x+a,\ y+b)$ is not free.*

To better appreciate the abstract set-up in (3.5), let us state a special case of it by taking $n = 1$ and $y = f(z_1) = z_1$ (for which all needed hypotheses are trivially satisfied). We get the following:

Corollary 3.6. *Let D be a domain such that $D^2 \cong D^m$ (as right modules) implies $m = 2$, and assume there exist a, $b \in D$ such that $ab - ba \in \mathrm{U}(D)$. Then for $R = D[x, y]$, the stably free R-module $P = P(x+a,\ y+b)$ is not free.*

Note that, if we specialize this to the case where D is a noncommutative division ring k, we get back Theorem 3.1 of Ojanguren and Sridharan — by taking a pair of noncommuting elements $a, b \in k$. (This gives $ab - ba \in k \setminus \{0\} = \mathrm{U}(k)$.) For the specific choices $a = i$, $b = j$ in Hamilton's division ring H of real quaternions, for instance, we get the nonfree stably free module $P(x+i,\ y+j)$ over $H[x, y]$ in [Sharma: 1971].

Coming back to the notations in (3.5), let us now present:

Proof of (3.5). Assume $P \cong R^m$. Reducing this modulo $x, z_1, \ldots, z_n$, we get $D^m \oplus D \cong D^2$, so the hypothesis on D gives $m = 1$. Thus, the right ideal $J = \pi_2(P)$ is *principal*; say, $J = gR$ (for some $g \in R$). By (∗), we have then:

(1) $(x+a)c^{-1}(x+a) = gh$, for some $h \in R$; and

(2) $1 + (x+a)c^{-1}(y+b) = gh'$, for some $h' \in R$.

From (1), we must have $g \in D[x]$, with $\deg g \leqslant 2$. We may assume that g is monic since its leading coefficient divides the unit c^{-1} (and we are free to right-scale g by a unit). We'll see that each of the following three cases leads to a contradiction.

Case 1. $\deg g = 0$. Here, $g = 1$, so (3.4) shows $y + b \in (x+a)R$. Then, modulo $(x+a)R$, we have

$$-b \equiv y = f(x, z_1, \ldots, z_n) \equiv f(-a, z_1, \ldots, z_n).$$

Since neither LHS nor RHS involves x, we must have $-b = f(-a, z_1, \ldots, z_n)$, which is not the case, by assumption.

Case 2. $\deg g = 2$. This forces $h \in \mathrm{U}(D)$, and hence by (1) above:

$$g = (x+a)c^{-1}(x+a)h^{-1} \in (x+a)R.$$

Plugging this into (2), we get $1 \in (x+a)R$, which is impossible.

Case 3. $\deg g = 1$. Here, $g = x + d$ for some $d \in D$. From (3.4), $(y+b)(x+d) \in (x+a)R$. Let $y_0 := f(-a, z_1, \ldots, z_n)$. As in *Case* 1, $y \equiv y_0 \pmod{(x+a)R}$. Therefore, $(y_0+b)(x+d) \in (x+a)R$. But x is central and $ay_0 = y_0 a$, so modulo $(x+a)R$, we have

$$0 \equiv x(y_0+b) + y_0 d + bd \equiv -a(y_0+b) + y_0 d + bd = y_0(d-a) - ab + bd.$$

Since the RHS does not involve x, it must be zero. Now $y_0 \in D[z_1, \ldots, z_n]$ is not a constant, and D is a domain, so (by a degree argument) we must have $d = a$, and $0 = bd - ab = ba - ab = -c$, a contradiction. $\square$

Following Swan, we illustrate the use of (3.5) by a couple of more explicit examples.

Corollary 3.7. *Let D be a noncommutative division ring, and let $R = D[u, v, w]$, with the relation $uv + w^2 = 1$. Then there is a rank* 1 *stably free R-module that is not free.*

Proof. Fix $a, b \in k$ with $ab \neq ba$. One checks easily (after Murthy) that the rule

$$u \mapsto x, \quad v \mapsto -y(2+xy), \quad w \mapsto 1 + xy$$

gives a well-defined ring homomorphism $\sigma : R \to D[x, y]$. By scalar extension along σ, the stably free R-module $P(u+a,\ w+b)$ "becomes" the $D[x, y]$-module $P(x+a,\ 1+xy+b)$. The latter module is not free over $D[x, y]$ (by (3.5) applied with $f(x, y) = 1 + xy$). Therefore, $P(u+a,\ w+b)$ is also not free over $D[u, v, w]$. $\square$

Corollary 3.8. *Let D be as in* (3.7)*, and let $R = D[u, v, w, z_1, \ldots, z_n]$ with the relation $uv + w^2 + q(z_1, \ldots, z_n) = 1$, where q is a polynomial over the center of D with $q(0, \ldots, 0) = 0$. Then there is a rank* 1 *stably free R-module that is not free.*

Proof. Fix a, b as before, and consider $P = P(u+a,\ w+b)$. Here, (3.5) does not apply directly (since R is not a polynomial ring). But if we reduce R modulo $z_1, \ldots, z_n$, it "becomes" the ring in (3.7). Since the reduction of P is not free by (3.7), P is not free either. $\square$

We close this section with a couple of further observations.

Remark 3.9. (1) In [Parimala-Sridharan: 1975], it was pointed out that the domain hypothesis on D in (3.6) is essential for the proof that the module

$$P = P(x+a,\ y+b)$$

constructed there is nonfree. If, instead, we let $D = \mathbb{M}_r(F)$ (matrix algebra over a field F), and construct P as in (3.6) using any pair $a,\ b \in D$ with $ab - ba \in \mathrm{U}(D)$, then P is always *free* of rank 1 over $R = D[x,\ y] \cong \mathbb{M}_r\big(F[x,\ y]\big)$.

(2) Much more information is now available for f.g. projectives over $R = D[t_1, \ldots, t_n]$ for a noncommutative division ring D. For a quick survey on this, see (VIII.8). Most notably, Stafford has proved that, if D has infinite center, then any $P \in \mathfrak{P}(R)$ of rank > 1 is free. Thus, for the module P in (3.6), for instance, we have automatically $P \oplus P \cong R^2$, $P \oplus P \oplus P \cong R^3$, etc., provided that D has an infinite center. In this case, one says that $\mathfrak{P}(R)$ fails to satisfy "separative cancellation", in that we have modules $P,\ Q$ in $\mathfrak{P}(R)$ (here $Q = R$) such that

$$P \oplus P \cong P \oplus Q \cong Q \oplus Q,$$

but $P \ncong Q$.

§4. The Graded Case

It is well known, and not difficult to prove, that if a f.g. projective module P over $k[t_l, \ldots, t_n]$ (k a field) happens to be *graded*, in a suitable sense, then P is in fact free over $k[t_l, \ldots, t_n]$. The present section covers this result. The techniques used are actually valid for graded projective modules over arbitrary graded rings, so we shall take this more general approach.

Definition 4.1. We say that a ring R is *graded* if there is given a direct sum decomposition of R into additive groups: $R = R_0 \oplus R_1 \oplus R_2 \oplus \ldots$, such that $R_n R_m \subseteq R_{n+m}$ for all $m, n \geqslant 0$. (We say that R_i is the *homogeneous component in* R of degree i.)

The quintessential example is $R = R_0[t_1, \ldots, t_d]$, which can be graded by letting R_i be the subgroup of all homogeneous polynomials in $t_1, \ldots, t_d$ of degree i. This grading is uniquely prescribed by the properties that R_0 (the constant polynomials) has degree zero, and that all the variables $\{t_j\}$ have degree one.

Now back to Definition (4.1) above. The degree zero component R_0 is an additive subgroup of R closed under multiplication. In view of the following observation, we know that R_0 is a (unital) *subring* of R:

Lemma 4.2. *The identity* 1 *of* R *belongs to* R_0.

Proof. We may assume $R \neq 0$ for otherwise there is nothing to prove. Write $1 = a_i + a_{i+1} + \dots$, with $a_j \in R_j$, and $a_i \neq 0$. Since $1 = 1^2 = a_i^2 + \dots$, and $a_i^2 \in R_{2i}$, we must have $i = 0$. If $x \in R_j$, $1 \cdot x = x \cdot 1 = x$ clearly implies that $a_0 x = x a_0 = x$, by degree comparison. Thus, a_0 is a multiplicative identity for R, and so we have $a_0 = 1$. □

Let R be as in (4.1). We say that an R-module M is *graded* if there is given a direct sum decomposition of M into additive groups: $M = \bigoplus_{j \in \mathbb{Z}} M_j$, such that $R_i M_j \subseteq M_{i+j}$, for all $i \geqslant 0$ and all $j \in \mathbb{Z}$. Note that in this definition, we do allow M to have "negative degree" components. We say that the graded module M is *bounded from below* if there exists $r \in \mathbb{Z}$ (possibly negative!) such that $M_j = 0$ for all $j < r$. The following proposition says that it is not unreasonable for us to restrict our attention to such modules.

Proposition 4.3. *Any finitely generated graded R-module M is bounded from below.*

Proof. Suppose $m_1, \dots, m_k$ are R-generators for M. Pick r so small that the homogeneous components of $m_1, \dots, m_k$ have degree $\geqslant r$. Then $M = \sum R\, m_i \subseteq \sum_{j \geqslant r} M_j$, as desired. □

For any graded ring $R = R_0 \oplus R_1 \oplus \dots$, it is customary to write $R^+ = R_1 \oplus R_2 \oplus \dots$, which is a 2-sided (graded) ideal in R. For any graded R-module M, we shall write

$$\overline{M} = \frac{M}{R^+ M} \cong \frac{R}{R^+} \otimes_R M,$$

which is a graded module over $R_0 = R_0 \oplus (0) \oplus \cdots$.

Proposition 4.4. *For any graded R-module M that is bounded from below, $\overline{M} = 0$ implies $M = 0$.*

Proof. Say $M = M_r \oplus M_{r+1} \oplus \dots$, then $R^+ M \subseteq M_{r+1} \oplus M_{r+2} \oplus \dots$. If $M = R^+ M$, we must have $M_r = 0$. By induction, it follows that all $M_i = 0$. □

By (4.3), the proposition applies, in particular, to any f.g. graded R-modules. In that context, the proposition may be regarded as a kind of vague analog of Nakayama's Lemma (I.1.4). This analogy can be exploited to yield certain useful results for f.g. graded modules. For instance, we can retrace some of the basic arguments used in the proof of (I.1.6), and obtain the following result in the "graded" setting.

Proposition 4.5. *Let P, Q be f.g. graded R-modules over a graded ring R, with $P \in \mathfrak{P}(R)$. Let $\gamma : Q \to P$ be a degree preserving R-homomorphism (i.e., $\gamma(Q_k) \subseteq P_k$ for every k). Then γ is an isomorphism iff $\overline{\gamma} : \overline{Q} \to \overline{P}$ is an isomorphism.*

Proof. ("If" part.) Let $K = \ker(\gamma)$, and $C = \operatorname{coker}(\gamma)$; these are *graded* R-modules. By the right exactness of "bar," $\overline{C} = 0$. Since C is f.g., (4.3) and (4.4) imply that $C = 0$, i.e., γ is *onto*. But $P \in \mathfrak{P}(R)$, so $0 \to K \to Q \to P \to 0$ splits. We have therefore $0 \to \overline{K} \to \overline{Q} \to \overline{P} \to 0$ exact, and so $\overline{K} = 0$. Now K is f.g. since Q is; we conclude as above that $K = 0$. □

Theorem 4.6. *Let $P \in \mathfrak{P}(R)$ be a graded module over the graded ring $R = R_0 \oplus R_1 \oplus \cdots$. Then P is extended from R_0. More precisely, there is a graded R-module isomorphism $R \otimes_{R_0} \overline{P} \cong P$.*

(As we have already observed, $\overline{P}$ is a graded R_0-module. The tensor product $R \otimes_{R_0} \overline{P}$ on the left is then graded in the usual way: $(R \otimes_{R_0} \overline{P})_k = \sum R_i \otimes_{R_0} \overline{P}_j$ with summation over all i, j such that $i \geqslant 0$, and $i + j = k$.)

Proof. Consider the projection $f : P \to \overline{P}$. Since $P \in \mathfrak{P}(R)$ implies that $\overline{P} \in \mathfrak{P}(R_0)$, f splits as an R_0-epimorphism. Choose $g \in \mathrm{Hom}_{R_0}(\overline{P}, P)$ such that $fg = 1_{\overline{P}}$. The map f is degree preserving; hence, after an obvious modification of g, we may assume that g is also degree preserving. Thus, g induces a degree preserving R-homomorphism $\gamma : Q = R \otimes_{R_0} \overline{P} \to P$. Clearly, $\overline{\gamma} : \overline{Q} \to \overline{P}$ is an isomorphism, so γ is an isomorphism by (4.5). □

Corollary 4.7. *Let $R = R_0[t_1, \ldots, t_n]$ be graded with R_0 having degree* 0 *and each t_i having degree* 1. *Let $P \in \mathfrak{P}(R)$ be a graded R-module. Then P is extended from R_0. If all f.g. projectives over R_0 are free (e.g., R_0 is a field, a* PID, *or a local ring), then P is in fact R-free.*

§5. The Stable Case

This section will be devoted to a detailed proof of the fact that, if $R = k[t_1, \ldots, t_n]$ (k a field), then any $P \in \mathfrak{P}(R)$ is stably free. This provides an affirmative answer to Serre's Conjecture *in the stable sense*. It will be deduced from a more general result of Grothendieck on the K_0 of a polynomial extension of a left regular ring; for this theorem of Grothendieck, see (5.8).

We shall begin this section with an interesting lemma of Swan. To formulate this, let us first recall some of the basic notations introduced earlier. If R is a subring of S, and $M \in \mathfrak{M}(S)$ (i.e., M is a f.g. left S-module), we write $M \in \mathfrak{M}^R(S)$ to indicate the fact that M is extended from R, i.e., that there exists a left R-module M_0 with $S \otimes_R M_0 \cong M$. For the purposes of this section, S will be of the form $R[t]$ for a single variable t. In this case, M_0 will be uniquely determined by M (namely, $M_0 \cong M/tM$), and we have $M \in \mathfrak{P}(R[t])$ iff $M_0 \in \mathfrak{P}(R)$. If this is the case, we shall write $M \in \mathfrak{P}^R(R[t])$.

Swan's Lemma 5.1. *Let R be a left noetherian ring, and let M be an $R[t]$-submodule of some $N \in \mathfrak{M}^R(R[t])$. Then there exists a short exact sequence $0 \to X \xrightarrow{g} Y \xrightarrow{f} M \to 0$, where $X, Y \in \mathfrak{M}^R(R[t])$.*

Proof. Write $N = R[t] \otimes_R N_0$ ($N_0 \in \mathfrak{M}(R)$), and $N_k = \sum_{i=0}^k R \cdot t^i \otimes_R N_0$ (think of elements in N_k as all "polynomials" of degree $\leqslant k$, with "coefficients" from N_0). Let $M_k = M \cap N_k$, $k \geqslant 0$. Since $N_k \in \mathfrak{M}(R)$ and R is left noetherian, we have $M_k \in \mathfrak{M}(R)$ for all k. By the Hilbert Basis Theorem, $R[t]$ is also left noetherian. Thus,

$N \in \mathfrak{M}(R[t])$ implies that $M \in \mathfrak{M}(R[t])$. Now choose an integer n so large that M_{n+1} contains an $R[t]$-generating set for M. Furthermore, choose

$$X = R[t] \otimes_R M_n, \quad \text{and} \quad Y = R[t] \otimes_R M_{n+1}.$$

We can define $f : Y \to M$ by $f(t^i \otimes m) = t^i m$ for every $m \in M_{n+1}$. This is clearly a well-defined $R[t]$-homomorphism. Also, $\mathrm{Im}(f) \supseteq M_{n+1}$, so $\mathrm{Im}(f) = M$ by the choice of n. To construct $g : X \to Y$, we observe that $t N_n \subseteq N_{n+1}$, which implies that $t M_n \subseteq M_{n+1}$. We may thus define g by

$$g\big(t^i \otimes m\big) = t^{i+1} \otimes m - t^i \otimes tm, \quad \text{for all } m \in M_n.$$

It only remains to prove that $0 \to X \xrightarrow{g} Y \xrightarrow{f} M \to 0$ is exact.

(1) For any $m \in M_n$,

$$fg\big(t^i \otimes m\big) = f\big(t^{i+1} \otimes m - t^i \otimes tm\big) = t^{i+1}m - t^{i+1}m = 0,$$

so $fg = 0$.

(2) If $x = t^i \otimes m + t^{i-1} \otimes m' + \dots$, with $m, m', \dots \in M_n$, and $m \neq 0$, then

$$\begin{aligned} g(x) &= \big(t^{i+1} \otimes m - t^i \otimes tm\big) + \big(t^i \otimes m' - t^{i-1} \otimes tm'\big) + \dots \\ &= t^{i+1} \otimes m + t^i \otimes (m' - tm) + \dots \\ &\neq 0, \end{aligned}$$

so g is injective.

(3) Suppose $y = \sum_{i=0}^{r} t^i \otimes m_i \in \ker f$ $(m_i \in M_{n+1})$. We shall show that $y \in \mathrm{Im}(g)$ by induction on r. This is trivial for $r = 0$, so we may assume that $r \geqslant 1$. Write $m_i = \sum_{j=0}^{n+1} t^j \otimes a_{ij}$, where $a_{ij} \in N_0$. Then

$$\begin{aligned} 0 = f(y) &= \textstyle\sum_{i=0}^{r} t^i m_i = \sum_{i=0}^{r} \sum_{j=0}^{n+1} t^{i+j} \otimes a_{ij} \\ &= t^{r+n+1} \otimes a_{r,n+1} + t^{r+n} \otimes (\dots) + \dots . \end{aligned}$$

Thus, $a_{r,n+1} = 0$, i.e., $m_r \in M_n$. Now

$$\begin{aligned} y - g(t^{r-1} \otimes m_r) &= y - t^r \otimes m_r + t^{r-1} \otimes t\, m_r \\ &= \textstyle\sum_{i=0}^{r-1} t^i \otimes m_i' \in \ker(f) \end{aligned}$$

for suitable $m_i' \in M_{n+1}$, so the induction proceeds. □

Definition 5.2. A ring R is said to be *left regular* if (1) R is left noetherian, and (2) every $M \in \mathfrak{M}(R)$ admits a resolution $0 \to P_n \to \cdots \to P_0 \to M \to 0$ where $P_i \in \mathfrak{P}(R)$ (and n depends on M).

Note that, in the presence of (1), the condition (2) is equivalent to saying that: (2′) every $M \in \mathfrak{M}(R)$ has finite (left) projective dimension. However, we prefer to work with (2) rather than (2′), in order to avoid using the theory of projective dimensions of modules, which we have not formally developed in this text.

Examples and Remarks.

(**5.3**) *If R is left noetherian and left hereditary, then R is left regular*. (Indeed, if $M \in \mathfrak{M}(R)$, take a short exact sequence $0 \to M' \to R^n \to M \to 0$ for suitable n. By Kaplansky's Theorem (2.2), and the fact that R is left noetherian, we have $M' \in \mathfrak{P}(R)$, so (5.2)(2) above is checked for a constant value $n = 1$.) In particular, Dedekind domains are always (left) regular.

(**5.4**) *If R is left regular, and S is a central multiplicative set in R, then the localization R_S is also left regular*. (Clearly, R being left noetherian implies the same for R_S. To check (5.2)(2) for R_S, use the fact that any $M \in \mathfrak{M}(R_S)$ "comes from" some $N \in \mathfrak{M}(R)$; we can then simply localize a finite projective resolution for N to obtain one for M.)

(**5.5**) (Stated without proof) *If R is commutative and noetherian, then R is regular iff $R_{\mathfrak{m}}$ is regular for every* $\mathfrak{m} \in \operatorname{Max} R$. The "only if" part is, of course, covered by (5.4), but the "if" part is nontrivial. From the viewpoint of projective dimensions (abbreviated by "pd"), one has the known formula

$$\operatorname{pd}_R M = \sup\{\operatorname{pd}_{R_{\mathfrak{m}}}(M_{\mathfrak{m}}) : \ \mathfrak{m} \in \operatorname{Max} R\}.$$

If all localizations $R_{\mathfrak{m}}$ are regular, one has $\operatorname{pd}_{R_{\mathfrak{m}}} M_{\mathfrak{m}} < \infty$ *for each* $\mathfrak{m}$, but it is not clear that the "sup" over *all* $\mathfrak{m}$ should remain finite. This subtle point calls for a finer analysis that would ultimately invoke the compactness property of the Zariski topology. We do not wish to go into the technical details here since the statement (5.5) will be needed only peripherally in the rest of our exposition.

(**5.6**) (also without proof) For a commutative noetherian local ring $(R, \mathfrak{m})$, regularity in the sense of (5.2) is equivalent to $\dim_{R/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2 = \operatorname{Krull\,dim} R$, which is the "usual" definition of regular local rings in commutative algebra and algebraic geometry.

Swan's Theorem 5.7. *Let R be left regular. Then $R[t]$ is left regular. More precisely, for every $M \in \mathfrak{M}(R[t])$, there exists a resolution*

$$0 \to Z_n \to \cdots \to Z_0 \to M \to 0$$

where $Z_i \in \mathfrak{P}^R(R[t])$ (and n depending on M).

Proof. By the Hilbert Basis Theorem, R being left noetherian implies the same for $R[t]$. Thus, it is sufficient to prove the second conclusion in the theorem. Take a short exact sequence

$$0 \to M' \to R[t]^n \to M \to 0,$$

for a suitable n. It is clearly enough to construct a resolution of the desired type for M'. By Lemma 5.1, there exists a short resolution

$$0 \to X \xrightarrow{g} Y \to M' \to 0,$$

where $X,\ Y \in \mathfrak{M}^R\big(R[t]\big)$. Since R is left regular (and the polynomial extension $R \to R[t]$ is flat), we can find finite resolutions $\mathcal{X} \to X \to 0$, and $\mathcal{Y} \to Y \to 0$, where $\mathcal{X}$ (resp. $\mathcal{Y}$) consists of modules X_i (resp. Y_i) in $\mathfrak{P}^R\big(R[t]\big)$. Using the fact that the X_i 's are $R[t]$-projective, it is easy to lift $g : X \to Y$ to a map of chain complexes $\mathcal{X} \to \mathcal{Y}$, denoted by, say, the same letter g. Our job now is to put together the resolutions $\mathcal{X} \to X \to 0$ and $\mathcal{Y} \to Y \to 0$ via g to obtain a “good” resolution for M'. This is done by a standard device in homological algebra, called the “mapping cone construction.” The details of this are explained by the diagram below:

$$\begin{tikzcd}
(\mathcal{Y}):\ \ldots & Y_3 \arrow[r, "\partial"] \arrow[d] & Y_2 \arrow[r, "\partial"] \arrow[d] & Y_1 \arrow[r, "\partial"] \arrow[d] & Y_0 \arrow[r, "\partial"] \arrow[dd, equal] & Y \arrow[r] & 0 \\
 & Y_3 \arrow[r, "\partial"] & Y_2 \arrow[r, "\partial"] & Y_1 \arrow[dr, "\partial"] & & & \\
(\mathcal{Z}):\ \ldots & \oplus & \oplus & \oplus & Y_0 & & \\
 & X_2 \arrow[r, "-\partial"'] \arrow[uur, "g"] & X_1 \arrow[r, "-\partial"'] \arrow[uur, "g"] & X_0 \arrow[ur, "g"'] & & & \\
(\mathcal{X}):\ \ldots & X_2 \arrow[r, "-\partial"'] \arrow[u] & X_1 \arrow[r, "-\partial"'] \arrow[u] & X_0 \arrow[r, "-\partial"'] \arrow[u] & X \arrow[r] & 0 &
\end{tikzcd}$$

Here, ∂ denotes the boundary maps for $\mathcal{X}$ as well as those for $\mathcal{Y}$. The boundary maps for the middle complex, $\mathcal{Z}$, are defined as indicated. By making the sign change in the boundary maps for the X_i 's, we ensure that $\mathcal{Z}$ is a zero sequence. *We claim now that $\mathcal{Z}$ is exact*. This can be deduced from the long homology sequence, but it is just as easy to make a direct diagram chase for the proof. Say $(y, x) \mapsto 0$ in the complex $\mathcal{Z}$. Then $\partial x = 0 = \partial y + g(x)$. Writing $x = \partial x'$, we have

$$0 = \partial y + g\,\partial x' = \partial(y + g(x')),$$

so $y + g(x') = \partial y'$. But then $(y', -x')$ maps to

$$(\partial y' - g(x'),\ -\partial(-x')) = (y, x),$$

as desired.

Finally, the “homology” of the middle complex $\mathcal{Z}$ at Y_0 is

$$Y_0/(\partial Y_1 + gX_0) \cong Y/gX \cong M',$$

so we have a resolution $\mathcal{Z} \to M' \to 0$. The modules in $\mathcal{Z}$ are $Y_{i+1} \oplus X_i \in \mathfrak{P}^R\big(R[t]\big)$, so this proves (5.7). $\square$

Grothendieck's Theorem 5.8. *Let R be left regular. Then the functorial map*

$$K_0R \to K_0R\,[t_1, \ldots, t_n]$$

is an isomorphism. If every $P \in \mathfrak{P}(R)$ is stably free, then every $Q \in \mathfrak{P}\big(R[t_1, \ldots, t_n]\big)$ is stably free.

Proof. In view of (5.7) (and induction), it is enough to treat the case $K_0R \xrightarrow{i_*} K_0R[t]$. Since $i : R \subseteq R[t]$ splits by a retraction $R[t] \to R$ (sending $t \mapsto 0$ for example), i_* is clearly a (split) injection. Thus, we only need the surjectivity of i_*. If $P \in \mathfrak{P}(R[t])$, we can take (by (5.7)) a resolution:

$$0 \to Z_n \to \cdots \to Z_0 \to P \to 0, \quad Z_i \in \mathfrak{P}^R\big(R[t]\big),$$

and conclude that $[P] = \sum(-1)^i[Z_i] \in \operatorname{im}(i_*)$. This clearly shows that i_* is surjective. Finally, if every $P \in \mathfrak{P}(R)$ is stably free, then $K_0R = \mathbb{Z} \cdot [R]$ by (I.6.2). The surjectivity of i_* implies that $K_0R[t] = \mathbb{Z} \cdot [R[t]]$, so, again by (I.6.2), every $Q \in \mathfrak{P}\big(R[t]\big)$ is stably free. □

Corollary 5.9. *If R is a* PID*, or a left regular local ring, or a commutative regular semilocal ring with no nontrivial idempotents, then every $Q \in \mathfrak{P}\big(R[t_l, \ldots, t_n]\big)$ is stably free.*

Recall the earlier result (I.4.11) that, over any commutative ring, any f.g. stably free module of rank 1 is free. Thus, if R is a commutative ring as in (5.9), any f.g. rank 1 projective over $R[t_1, \ldots, t_n]$ is free. In the special case $R = k$ (a field), this gives an affirmative answer to Serre's Conjecture *in the rank* 1 *case*, independently of Gauss's theorem on the unique factorization of $k[t_1, \ldots, t_n]$.

To close this section, we shall provide an example of a ring (in fact, a commutative local domain) R for which $K_0R \to K_0R[t]$ fails to be an isomorphism. This will serve to demonstrate that some assumptions are necessary on R for the statement in Grothendieck's Theorem (5.8) to be true. The following proposition is due to S. Schanuel; its proof first appeared in [Bass: 1962].

Proposition 5.10. *Let R be an integral domain with quotient field K. Suppose there exists an element $a \in K \backslash R$ such that $a^m \in R$ for all sufficiently large m. Then there exists a rank one module $P \in \mathfrak{P}\big(R[t]\big)$ that is* not *extended from R.*

Assuming this result, it is not difficult to give an example of a local domain R for which $K_0R \to K_0R[t]$ is not onto. For example, take R to be the local ring at the point $(0, 0)$ of the curve $x^3 = y^2$, say over the reals; that is, R is the localization of $\mathbb{R}[t^2, t^3]$ at (t^2, t^3), which has a quotient field $\mathbb{R}\,(t)$. Clearly, $t \notin R$, and for $m \geqslant 2$, $t^m \in R$ (we can write $m = 2j + 3k$ for suitable $j, k \geqslant 0$). By (5.10), there exists a rank one module $P \in \mathfrak{P}\big(R[t]\big)$ that is not extended from R (in particular not free). *We claim that*

$$[P] \notin \operatorname{im}\big(K_0R \xrightarrow{i_*} K_0R[t]\big).$$

In fact, since R is local, $K_0R = \mathbb{Z} \cdot [R]$; hence $\mathrm{im}(i_*) = \mathbb{Z} \cdot [R[t]]$. If $[P] \in \mathrm{im}(i_*)$, P would have to be stably free by (I.6.2), and hence free by (I.4.11), a contradiction.

The above example of the local ring at a cusp shows that *if all f.g. projectives over a ring R are free, the same property may not be possessed by the polynomial ring $R[t]$*. Thus, adding one variable at a time and hoping for the preservation of the "projective ⇒ free" property is definitely "How NOT to prove Serre's Conjecture"!

Proof of (5.10). After replacing a by a suitable power of itself, we may assume that $a^m \in R$ for all $m \geqslant 2$. Let $P = (a^2,\ 1 + at)$, which denotes the $R[t]$-submodule of $K[t]$ generated by a^2 and $1 + at$. Similarly, let $Q = (a^2,\ 1 - at)$. By multiplying out the generators of P and Q, we see that $P \cdot Q \subseteq R[t]$. But $P \cdot Q$ contains a^4 and $1 - a^2t^2$, so it also contains

$$a^4t^4 + (1 + a^2t^2)(1 - a^2t^2) = 1.$$

Thus, $P \cdot Q = R[t]$, so by (1.1), P is $R[t]$-projective (of rank 1). *We claim that P cannot be extended from R.* To prove this, we may assume that R is local. (Simply replace R by a localization $R_{\mathfrak{p}}$ ($\mathfrak{p} \in \mathrm{Spec}\, R$) that avoids a.) Assume $P \in \mathfrak{P}^R(R[t])$; then P must be free of rank 1, so $P = R[t] \cdot b$ for some $b \in P$. Since $a^2 \in P$, we have $b \in K^*$. But $b(1 - at) \in P \cdot Q \subseteq R[t]$ implies $b \in R$. Now

$$1 + at \in P = R[t] \cdot b \subseteq R[t]$$

gives the desired contradiction. $\square$

Remark. The choice of P here is a slight modification of the one used by Schanuel in [Bass: 1962]. This construction can be further generalized. For any $h \in R\,[t]$, we can define the $R[t]$-module

$$P_h := (a^2,\ 1 + ath),$$

and show that $P_h \cdot P_{-h} = R[t]$. This leads to a *family* of rank 1 projectives over $R[t]$ parametrized by $h \in R[t]$. For more details on this construction and its applications, see Section 2 of my book [Lam: 1999].

It turns out that the *converse* of (5.10) is also true! But this fact is not easy to prove, and we will not prove it here. For any commutative ring R, $\mathrm{Pic}(R)$ (the *Picard group* of R) is defined to be the abelian group whose elements are the isomorphism classes $[P]$ of the f.g. projective R-modules of rank 1, with multiplication given by

$$[P] \cdot [Q] = [P \otimes_R Q].$$

(The inverse of a class $[P]$ is given by the class $[P^*]$, where P^* denotes the dual module of P.) Clearly, $R \mapsto \mathrm{Pic}(R)$ gives a covariant functor from the category of commutative rings to the category of abelian groups. In the notation of Picard groups, (5.10) *and its converse* can be stated together as follows.

Theorem 5.11. *For an integral domain R with quotient field K, the following are equivalent*:

(1) *The natural map* $j_* : \mathrm{Pic}(R) \to \mathrm{Pic}\big(R[t_1, \ldots, t_n]\big)$ *is an isomorphism*$^{(*)}$ *for some* $n \geqslant 1$.

(2) j_* *is an isomorphism for all* n.

(3) *For any* $a \in K$, *if* $a^m \in K$ *for all sufficiently large* m, *then* $a \in R$.

(4) *For any* $a \in K$, $a^2, a^3 \in R \Rightarrow a \in R$.

From what is done above, we have already (2) $\Rightarrow$ (1) $\Rightarrow$ (3) $\Leftrightarrow$ (4). To complete the proof of (5.11) would require proving (for instance) (4) $\Rightarrow$ (2).

Recall that an integral domain R is called *normal* if R is integrally closed in its quotient field K. The condition (4) above represents a weak version of normality: in the literature, an integral domain R is said to be *seminormal* if it satisfies (4). For more discussions on this notion and bibliographical information on Theorem 5.11, we refer the reader to VIII.7.

Returning to the functor K_0, let us define a ring R to be K_0*-regular* if the (injective) functorial map

$$i_* : \ K_0 R \to K_0 R\,[t_1, \ldots, t_n]$$

is an isomorphism for all n. In contrast to (5.11), even among integral domains, it still seems unknown exactly what rings are K_0-regular. By Grothendieck's Theorem 5.8, we know that *regular domains* R *are* K_0*-regular.* Note that such domains are normal, since each localization $R_{\mathfrak{p}}$ $(\mathfrak{p} \in \mathrm{Spec}(R))$ is a regular local ring and it is a standard fact that regular local rings are normal. This prompted Murthy to ask *if all noetherian normal domains are* K_0*-regular.* This question was also stated in Chapter II in the earlier version of this book. However, it was answered in the negative by Weibel in 1980. For more details and discussions on this, see VIII.9.

§6. The Case of Two Variables

The affirmative answer to Serre's Conjecture in the 2 variable case (over fields) is due to [Seshadri: 1958]. The result obtained by Seshadri is as follows:

Seshadri's Theorem 6.1. *If* R *is a* PID, *then any* $P \in \mathfrak{P}\big(R[t]\big)$ *is free. In particular, if* k *is a field, any* $P \in \mathfrak{P}\big(k[x, y]\big)$ *is free.*

A few years after Seshadri's paper appeared, [Bass: 1962] and [Serre: 1960/61] extended Seshadri's argument, and obtained the following refinement of (6.1).

$^{(*)}$As in (5.8), j_* here is already a monomorphism, since R is a retract of the polynomial ring $R[t_1, \ldots, t_n]$.

Bass-Serre Theorem 6.2. *If R is a Dedekind domain, then* $\mathfrak{P}(R[t]) = \mathfrak{P}^R(R[t])$ *(i.e. any $P \in \mathfrak{P}(R[t])$ is extended from R).*

Of course, (6.2) implies (6.1). In fact, if $P \in \mathfrak{P}(R[t])$ where R is a PID, we can express P as $R[t] \otimes_R P_0$ for some R-module P_0 (according to (6.2)). But $P_0 \cong P/tP \in \mathfrak{P}(R)$ is R-free since R is a PID, so P is $R[t]$-free.

In the following, *we shall prove* (6.1), *but shall not prove* (6.2). The reason for this is that, in a certain sense, it is “unnecessary” to prove (6.2): if we assume Quillen’s Patching Theorem (V.1.6 below), it can be seen that (6.1) $\Rightarrow$ (6.2). In fact, if R is Dedekind, to check that $P \in \mathfrak{P}(R[t])$ is extended from R, it is sufficient (by Quillen’s result) to check that $P_{\mathfrak{m}}$ is extended from $R_{\mathfrak{m}}$, for every $\mathfrak{m} \in \operatorname{Max} R$. But $R_{\mathfrak{m}}$ is a discrete valuation ring, hence a PID. By (6.1), $P_{\mathfrak{m}}$ is free over $R_{\mathfrak{m}}[t]$; in particular, $P_{\mathfrak{m}}$ is extended from $R_{\mathfrak{m}}$.

The rest of this section will be devoted to the proof of (6.1). Before we go into the details, let us first sketch the main ideas of the proof. For R a PID, let K be the field of fractions of R, i.e., the localization of R at $S = R \setminus \{0\}$. For $P \in \mathfrak{P}(R[t])$, the localization $P_S \in \mathfrak{P}(R_S[t])$ is free, since $R_S[t] = K[t]$ is a PID.

Problem. From the fact that P_S is $R_S[t]$-free, can we (somehow) conclude that P is $R[t]$-free?

In general, of course, a module that is free upon a certain localization may be very far from being free itself. Nevertheless, for R a PID, the answer to the problem above is “YES!” In the following, we shall deduce this from a certain “axiomatic” result: Theorem 6.4. The proof of (6.4) is simply an axiomatization of Seshadri’s original argument. However, there are two advantages to be gained from this axiomatization. First, the proof becomes somewhat easier to follow because of its axiomatic nature. Second, it happens that we will need this axiomatic version again later in another context (namely, in the proof of a theorem of Murthy and Horrocks (IV.6.6)).

Before stating Theorem 6.4, we make the following definition:

Definition 6.3. Let S be a multiplicative set in a commutative ring A. A non-unit $a \in S$ is said to be *S-irreducible* if a cannot be written as $b \cdot c$ where b, c are non-units in S.

Theorem 6.4. *Let A be a commutative noetherian ring, $S \subseteq A$ be a multiplicative set of non zero-divisors, and $n \geqslant 1$. Assume that for every S-irreducible element $b \in S$, $A/(b)$ is a* PID *with the property that* $\mathrm{SL}_n(A/(b)) = \mathrm{E}_n(A/(b))$. *Then, for any $P \in \mathfrak{P}(A)$, $P_S \cong A_S^n \Longrightarrow P \cong A^n$.*

Note that this “axiomatic” result does imply (6.1), in the following way. For R a PID, let $A = R[t]$ and $S = R \setminus \{0\}$. The S-irreducible elements b are simply the (non-zero) prime elements of R. For any such b, $A/(b) = \frac{R}{(b)}[t]$ is a *euclidean domain*, so the hypotheses for (6.4) are satisfied, by (I.5.4).

Proof of (6.4). Since the elements of S are non zero-divisors on P, we can think of P as embedded in P_S. Choose $f_1, \dots, f_n \in P$ that form a free A_S-basis of P_S, and let L be the free A-module $\sum A \cdot f_i$ in P. Since P is f.g. over A, we see that

(6.5) there exists $a \in S$ such that $a \cdot P \subseteq L \subseteq P$.

Among all A-free submodules $L \subseteq P$ for which (6.5) is true, choose one, say L again, that is maximal. (This is possible since P is a *noetherian* module over the noetherian ring A.) This L will be kept fixed in the following. Among all $a \in S$ satisfying (6.5), choose one, say a again, such that (a) is maximal (in the family of all such principal ideals). *We claim that a is a unit in A*. If so, then clearly, $P = L \cong A^n$, as desired.

Assume a is not a unit. By an easy noetherian argument, we can write $a = b \cdot c$ where $b, c \in S$, and b is S-irreducible. Then $(a) \subsetneq (c)$; by the maximal choice of (a), there exists $z \in P$ such that $c \cdot z \notin L$. Since b is not a zero-divisor on P, we have $bc \cdot z \notin b \cdot L$. But

$$bc \cdot z = a \cdot z \in L \cap b \cdot P,$$

so $b \cdot L \subsetneq L \cap b \cdot P$. Let $\overline{A} = A/(b)$, and write "bar" to denote the reduction of modules modulo (b). The inclusion map $g : L \subseteq P$ induces a map $\overline{g} : \overline{L} \to \overline{P}$ with kernel $(L \cap b \cdot P)/b \cdot L \neq 0$. Since $\overline{A}$ is a PID (by hypothesis), im $(\overline{g}) \subseteq \overline{P}$ is $\overline{A}$-free, so $\ker \overline{g}$ is a (free) direct summand of $\overline{L}$. Choose $e_1, \dots, e_n \in L$ so that $\overline{e}_1, \dots, \overline{e}_n$ form a free $\overline{A}$-basis for $\overline{L}$, while $\overline{e}_1, \dots, \overline{e}_m$ $(m \geqslant 1)$ form a free $\overline{A}$-basis for $\ker \overline{g}$. Let $f_1, \dots, f_n$ be a free A-basis for L. There exists an automorphism $\overline{\alpha}$ of $\overline{L}$ such that

$$\overline{\alpha}(\overline{f}_i) = \overline{e}_i, \quad 1 \leqslant i \leqslant n.$$

After scaling the $\overline{e}_i$'s if necessary, we may assume that $\det(\overline{\alpha}) = \overline{1}$. Since $\mathrm{SL}_n(\overline{A}) = \mathrm{E}_n(\overline{A})$ (by hypothesis), there exists $\alpha \in \mathrm{E}_n(A)$ that reduces modulo b to $\overline{\alpha}$. Let $\alpha(f_i) = e_i'$, so $e_1', \dots, e_n'$ form another A-basis of L. We have $\overline{e}_i' = \overline{\alpha}(\overline{f}_i) = \overline{e}_i$ for all i; in particular $\overline{e}_1' \in \ker \overline{g}$. This means that $e_1' = b \cdot e_1''$ for some $e_1'' \in P$. Clearly, $A \cdot e_1'' + \sum_{i \geqslant 2} A \cdot e_i'$ is A-free and, since b is a nonunit, this contains L *properly* – a contradiction to the maximal choice of L. ◻

There is also an alternative proof for Seshadri's Theorem 6.1 that is due to M. Roitman. Some of the ideas in Roitman's proof are close to the ones used above, but the proof is sufficiently different to merit a coverage here.

Roitman's Proof of (6.1). Let R be a PID, with field of fractions K. From (5.9), we know that every $P \in \mathfrak{P}\big(R[t]\big)$ is stably free. Thus, we will be done if we can show that $A = R[t]$ is Hermite.

Let $\alpha = (a_1, \dots, a_n) \in \mathrm{Um}_n(A)$. Since $K[t]$ is Hermite, we can at least complete α to a matrix M in $\mathrm{GL}_n\big(K[t]\big)$. After removing all the denominators from M, we may assume that M has all entries in A, and $\det M = d \in R \setminus \{0\}$. If d is a unit in R, we are done, so we assume d is *not* a unit. Let $b \in R$ be a prime factor of d. Since $\overline{A} = A/b \cong (R/b)[t]$ is a euclidean domain,

$$\overline{\alpha} \sim_{\mathrm{E}_n(\overline{A})} (\overline{1}, \overline{0}, \dots, \overline{0})$$

by (I.5.4). Lifting this to A, we see that there exists $E_1 \in \mathrm{E}_n(A)$ such that

$$\alpha \cdot E_1 \equiv (1, 0, \ldots, 0) \pmod{b}.$$

Let

$$M_1 = M\, E_1 = (a_{ij}), \quad \text{and} \quad N = (a_{ij})_{2 \leqslant i,\, j \leqslant n}\,.$$

Expanding $\det M_1 = d$ along the first row, we see that $\det N \equiv 0 \pmod{b}$. By the theorem of elementary divisors over a PID, we can write

$$\overline{X} \cdot \overline{N} \cdot \overline{Y} = \mathrm{diag}\left(\overline{d}_2, \ldots, \overline{d}_{n-1}, \overline{0}\right), \tag{6.6}$$

where $\overline{X}$, $\overline{Y} \in \mathrm{SL}_{n-1}(\overline{A}) = \mathrm{E}_{n-1}(\overline{A})$ (cf. I.5.4(2)). Lift $\overline{X}$, $\overline{Y}$ to X, $Y \in \mathrm{E}_{n-1}(A)$, and write $E_2 = \begin{pmatrix} 1 & 0 \\ 0 & Y \end{pmatrix}$, $E_3 = \begin{pmatrix} 1 & 0 \\ 0 & X \end{pmatrix}$. Then,

$$M_2 = E_3\, M\, E_1\, E_2 = \begin{pmatrix} a_{11} & (a_{12}, \ldots, a_{1n}) \cdot Y \\ * & XNY \end{pmatrix}, \tag{6.7}$$

where, by (6.6), the last row of XNY consists of multiples of b. Without loss of generality, we may assume that the $(n, 1)$-entry of M_2, say c, is *also* a multiple of b. (This can be accomplished by subtracting c times the first row of M_2 from its last row. This will change c into $c(1 - a_{11}) \in bA$, and it will not change the other features of M_2 (since the row $(a_{12}, \ldots, a_{1n}) \cdot Y$ also consists of multiples of b.) After removing a common factor b from this row, it follows from (6.7) that

$$\beta = (a_{11},\ (a_{12}, \ldots, a_{1n}) \cdot Y)$$

can be completed to a matrix over A of determinant d/b (with one fewer prime factor!). Invoking an inductive hypothesis at this point, we may assume that β can, in fact, be completed to a matrix in $\mathrm{GL}_n(A)$. But

$$\alpha \sim (a_{11},\ a_{12}, \ldots, a_{1n}) \sim (a_{11}, (a_{12}, \ldots, a_{1n})Y) = \beta,$$

so α can also be completed to a matrix in $\mathrm{GL}_n(A)$. □

Note that Roitman's proof above actually gave an explicit algorithmic construction for the completion of unimodular rows over $R[t]$, where R is a PID.

In the next chapter, we shall give a more general proof which shows that, *for R a* PID*, and for m any positive integer, $R[t_1, \ldots, t_m]$ is always Hermite* (see (III.3.1)). This, of course, implies the truth of Serre's Conjecture over PID's, and in particular, over fields.

We should like to remind our readers that the *noncommutative* situation is rather different. For a division ring k that is not a field, we have shown in §3 that there exist nonfree projectives over $k[x, t]$. The ring $R = k[x]$ is a noncommutative PID (in the sense that all 1-sided ideals are principal), though projectives over $R[t]$ need not be free. This shows that the commutativity assumption is essential in Seshadri's Theorem 6.1.

To close this section, we note a remarkable application of Seshadri's theorem (6.1). This is a special result for the kernels of linear functionals on f.g. free modules over $R[t]$, where R is a PID. (In particular, the result is applicable to $k[x, y]$ for any field k.) For the (quickest) proof of this result, we'll need to assume some standard facts from the theory of homological dimensions of modules and rings. The principal fact needed below is that, if R is any PID, $R[t]$ has global dimension $\leqslant 2$. A detailed proof of this fact can be found in (5.36) of [Lam: 1999], so it won't be repeated here.

Theorem 6.8. *Let $A = R\,[t]$, where R is a principal ideal domain. Then the kernel of any $\varphi \in \mathrm{Hom}_A(A^n, A)$ is a free A-module.*

Proof. Let $K = \ker(\varphi)$ and $J = \mathrm{im}(\varphi)$, so we have an exact sequence

$$(*) \qquad 0 \longrightarrow K \longrightarrow A^n \xrightarrow{\varphi} A \longrightarrow A/J \longrightarrow 0.$$

Since A has global dimension $\leqslant 2$, the A-module A/J has projective dimension $\leqslant 2$. From this fact and the long exact sequence $(*)$, it follows that K is a *projective* A-module (see (5.11) in [Lam: 1999]). Thus, Seshadri's theorem (6.1) implies that K is A-free. $\square$

Note that, if the R-homomorphism φ is (6.9) is represented by a row matrix $\beta = (b_1, \ldots, b_n)$, then K is the "solution space" $P(\beta)$ that we have studied in Chapter I. This A-submodule of A^n is often called the (first) *syzygy module* of the ideal $J = b_1 R + \cdots + b_n R \subseteq A$: it provides a (first) measure of the relations that hold among the generators $\{b_i\}$ of J. (The relations among these relations will define the second syzygy module, etc.) The remarkable theorem above says that, in the case $A = R[t]$ where R is a PID, the first syzygy module is always free.

Of course, this is an easy result when R is a field. In this case, $A = R[t]$ is itself a PID, so not only kernels of functionals on A^n are free, but *every* submodule of A^n is free, by (2.4). To illustrate (6.8) in less trivial cases, we offer some examples below where $R = k[x]$ for a field k. In this case, we switch to the more common notation $A = k[x, y]$ (replacing t by y).

Examples 6.9. We start with the simple case where $\varphi : A^2 \to A$ is given by a nonzero row matrix (a, b) over A. In this case, we write $a = ca_0$, $b = cb_0$, where c is a greatest common divisor of a, b. Here, unique factorization shows quickly that $\ker(\varphi)$ is freely generated by $(b_0, -a_0)^T$, so this conclusion would have worked over any unique factorization domain A. For an example in the case $n = 3$ (for $A = k[x, y]$ now), consider the map $\varphi : A^3 \to A$ defined by the row matrix (x^2, xy^3, y^5). Let $(f, g, h)^T \in K = \ker(\varphi)$, so $x^2 f + xy^3 g + y^5 h = 0$. Using unique factorization again, we can write $f = y^3 f_0$ and $h = x h_0$, from which we get $g = -(x f_0 + y^2 h_0)$. The two column vectors

$$u = (y^3,\ -x,\ 0)^T \quad \text{and} \quad v = (0,\ -y^2,\ x)^T$$

in K are obviously independent over A, and we have $(f, g, h)^T = f_0 u + h_0 v$. Thus, $\{u,\ v\}$ *form a free basis for* K. A similar calculation would have worked if we replace $(x^2,\ xy^3,\ y^5)$ by any monomial row matrix of length 3.

Example 6.10. Many more cases of maps $\varphi : A^3 \to A$ can be handled by the technique used in (6.9). For instance, let φ be defined by the row matrix $(1+x,\ 1-y,\ x(1+y))$. If $\mathrm{char}(k) \neq 2$, we have shown in (I.5.14) that this row is *unimodular*, and we have computed there a free basis for $K = \ker(\varphi)$. Now assume $\mathrm{char}(k) = 2$ (and remove all minus signs). In this case,

$$u = (1+y,\ 1+x,\ 0)^T \quad \text{and} \quad v = (0,\ x,\ 1)^T$$

are obviously linearly independent vectors in K. Moreover, for any $(f, g, h)^T \in K$,

$$(1+x)f + (1+y)g + x(1+y)h = 0$$

shows that $f = (1+y) f_0$ for some $f_0 \in A$, and cancellation of $1+y$ gives $g = (1+x) f_0 + xh$. This shows that $(f, g, h)^T = f_0 u + hv$, so $\{u,\ v\}$ form a free basis for K.

Example 6.11. For another example over $A = k[x, y]$, let $\varphi : A^3 \to A$ be defined by the *non-unimodular* row $(x(x-1),\ xy,\ y(y-1))$. Here, by inspection, the two column vectors

$$u = (y,\ 1-x,\ 0)^T, \ \text{ and } \ v = (0,\ 1-y,\ x)^T$$

are linearly independent elements in $K = \ker(\varphi)$. To show that they span K, consider any $(f, g, h)^T \in K$. Proceeding as in (6.9)-(6.10), we see from the equation

$$x(x-1)f + xyg + y(y-1)h = 0, \tag{6.12}$$

that $f = yf_0$ and $h = xh_0$ for some $f_0, h_0 \in A$, whence $g = (1-x) f_0 + (1-y)h_0$. Thus,

$$(f,\ g,\ h)^T = f_0\,(y,\ 1-x,\ 0)^T + h_0\,(0,\ 1-y,\ x)^T = f_0\,u + h_0\,v,$$

showing again that $\{u,\ v\}$ form a free basis for K.

The freeness of the kernel of any linear functional $k[x, y]^n \to k[x, y]$ is a delight in computational commutative algebra, and has led to interesting questions on the explicit computation of the free bases of such kernels. For some information on such computational issues, see the book of Cox, Little, and O’Shea listed in the references section for Chapter VIII. In the next example, we offer a computation of a slightly different flavor, over the polynomial ring on a PID.

Example 6.13. Here, we take the ring $A = \mathbb{Z}[t]$, and let $\varphi : A^3 \to A$ be defined by the row $(2(t+1),\ 4t,\ t^2+3)$. Direct computations lead to the following linearly independent elements

$$u = (2t,\ -(t+1),\ 0)^T \quad \text{and} \quad v = (-(t+3),\ 2,\ 2)^T$$

in $K = \ker(\varphi)$. To see that these span K, consider any $(f, g, h)^T \in K$. The equation

$$2(t+1)f + 4tg + (t^2+3)h = 0 \tag{6.14}$$

implies that $h = 2h_0$ for some $h_0 \in A$. Plugging this into (6.14) and writing t^2+3 as $(t+1)(t+3) - 4t$ give the equation

$$(t+1)\,[f + (t+3)h_0] = 2t\,(-g + 2h_0).$$

Thus, we can write $-g + 2h_0 = (t+1)g_0$ (for some g_0), and get $f + (t+3)h_0 = 2tg_0$, leading to $(f, g, h)^T = g_0 u + h_0 v$. It follows that $\{u,\ v\}$ form a free A-basis for K.

Remark 6.15. Of course, by the truth of Serre's Conjecture, (6.8) remains valid when $R = k[x_1, \ldots, x_r]$ (for a field k) if φ is defined by a *unimodular* row. However, it becomes false without this assumption when $r \geqslant 2$. In this case, $A = k[x_1, \ldots, x_{r+1}]$ with $r+1 \geqslant 3$. Consider the ideal $J = x_1A + x_2A + x_3A$, and let $\varphi : A^3 \to A$ be defined by the row matrix (x_1, x_2, x_3). If $K = \ker(\varphi)$ is free (or just projective), then the sequence

$$0 \longrightarrow K \longrightarrow A^3 \xrightarrow{\varphi} A \longrightarrow A/J \longrightarrow 0 \tag{6.16}$$

shows that A/J has projective dimension $\leqslant 2$. However, it is well known that the A-module A/J has projective dimension 3 (see (5.32) in [Lam: 1999]), a contradiction. This shows that K is not even projective. We'll be able to see later (in III.5.13) that K is generated as an A-module by the three elements

$$(y,\ -x,\ 0)^T,\ (z,\ 0,\ -x)^T,\ \text{and}\ (0,\ z,\ -y)^T$$

in A^3. However, it can be shown that K can't be generated by two elements.

§7. The Case of Big Rank

The main result in this section is that the case of "relatively big" projectives over a polynomial ring $k[t_1, \ldots, t_d]$ (for a field k) can be handled with the help of general methods from commutative algebra. To be more precise, we have the following result from [Bass: 1964a].

Theorem 7.1. *Let $R = k[t_1, \ldots, t_d]$, where k is a field. Then any $P \in \mathfrak{P}(R)$ with* $\operatorname{rk} P > d$ *is free.*

From (5.9), we know already that P must be stably free. Thus, (7.1) is a consequence of the following two results:

Theorem 7.2. *$k[t_1, \ldots, t_d]$ has Krull dimension d.*

Theorem 7.3. *If R is a commutative noetherian ring with Krull dimension $d < \infty$, then R is d-Hermite. (In particular, any commutative noetherian ring of Krull dimension $\leqslant 1$ is Hermite.)*

(Recall that the *Krull dimension* of a commutative ring R is the supremum of integers n for which there exists a prime chain of length n: $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n$ in R.)

To prove the above two results, we need a few standard facts from commutative algebra. For the sake of completeness, their proofs are included below.

Lemma 7.4. *Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_r \in \operatorname{Spec} R$ (R commutative). Suppose $x \in R$ and $\mathfrak{A}$ is an ideal in R. If $x + \mathfrak{A} \subseteq \bigcup_i \mathfrak{p}_i$, then $(x, \mathfrak{A}) \subseteq \mathfrak{p}_i$ for some i.* [*Here, $(x, \mathfrak{A})$ denotes the ideal generated by x and $\mathfrak{A}$.*]

Proof. Choose r minimal for which the lemma is false. We must have $r > 1$, and $\mathfrak{p}_i \not\subseteq \mathfrak{p}_j$ for $i \neq j$. *We claim that* $x \in \bigcap \mathfrak{p}_i$. Indeed, suppose $x \notin \mathfrak{p}_i$ for some i. Then $x + \mathfrak{p}_i\mathfrak{A}$ is disjoint from $\mathfrak{p}_i$, so $x + \mathfrak{p}_i\mathfrak{A} \subseteq \bigcup_{j \neq i} \mathfrak{p}_j$. From the choice of r, we infer that $(x, \mathfrak{p}_i\mathfrak{A}) \subseteq \mathfrak{p}_j$ for some $j \neq i$. Since $\mathfrak{p}_i \not\subseteq \mathfrak{p}_j$, this gives $(x, \mathfrak{A}) \subseteq \mathfrak{p}_j$, a contradiction. Having established the claim, we have then $\mathfrak{A} \subseteq \bigcup_i \mathfrak{p}_i$. Again by the choice of r, we see that, for every i, $\mathfrak{A} \not\subseteq \bigcup_{j \neq i} \mathfrak{p}_j$. Let $y_i \in \mathfrak{A} \setminus \bigcup_{j \neq i} \mathfrak{p}_j$. We must have $y_i \in \mathfrak{p}_i$. If we set $y = \sum y_1 \cdots \hat{y}_i \cdots y_r$, then $y \in \mathfrak{A}$ but y cannot belong to *any* $\mathfrak{p}_i$, a contradiction. $\square$

One small application of (7.4) is that we can use it to give a second proof of a part of (I.5.4):

Corollary 7.5. *If R is a commutative semilocal ring, then any $(a_1, \ldots, a_n) \in \mathrm{Um}_n(R)$ ($n \geqslant 2$) is $\sim_{\mathrm{E}_n(R)} (1, 0, \ldots, 0)$.*

Proof. Let $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$ be the maximal ideals of R. Let $x = a_1$, and $\mathfrak{A} = \sum_{i \geqslant 2} R \cdot a_i$. Since $(x, \mathfrak{A}) = R \not\subseteq \mathfrak{m}_i$ for every i, there exists an element

$$u = a_1 + c_2 a_2 + \cdots + c_n a_n \notin \bigcup_i \mathfrak{m}_i .$$

Such u is a unit of R, so we are done as in the proof of (I.5.4). $\square$

Lemma 7.6. *Let R be a commutative ring. Then*

(1) *any prime ideal in R contains a minimal prime ideal; and*
(2) *if R is noetherian, then the number of minimal prime ideals in R is finite.*

Proof. (1) By Zorn’s lemma, it is enough to show that, if $\{\mathfrak{p}_\alpha\}$ is a family of prime ideals linearly ordered by inclusion, then $I = \bigcap \mathfrak{p}_\alpha$ is prime. Say $a, b \notin I$. Then $a \notin \mathfrak{p}_\alpha$ and $b \notin \mathfrak{p}_\beta$ for some α, β. If, say, $\mathfrak{p}_\alpha \subseteq \mathfrak{p}_\beta$, then clearly $ab \notin \mathfrak{p}_\alpha$ so $ab \notin I$.

(2) Assume R is noetherian. Then *any radical ideal* $\mathfrak{A}$ *in* R $(x^n \in \mathfrak{A} \Rightarrow x \in \mathfrak{A})$ *is a finite intersection of prime ideals*. Indeed, if not, choose a maximal counter-example $\mathfrak{A}$. Clearly $\mathfrak{A}$ is not prime (and $\mathfrak{A} \neq R$), so there exist $b, c \notin \mathfrak{A}$ with $bc \in \mathfrak{A}$. Let

$$\mathfrak{B} = (b, \mathfrak{A}) \supsetneq \mathfrak{A}, \quad \mathfrak{C} = (c, \mathfrak{A}) \supsetneq \mathfrak{A},$$

and let $\mathfrak{B}'$, $\mathfrak{C}'$ be their radicals. We will get a contradiction if we show that $\mathfrak{B}' \cap \mathfrak{C}' = \mathfrak{A}$. The inclusion $\supseteq$ is clear. Conversely, if $x \in \mathfrak{B}' \cup \mathfrak{C}'$, then

$$x^m = by + a_1, \quad x^n = cz + a_2$$

for suitable integers m, n, and $a_i \in \mathfrak{A}$. But then $x^{m+n} \in \mathfrak{A}$ implies that $x \in \mathfrak{A}$, as desired. Now, write Nil R (the nil radical of R) as $\mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_r$. If $\mathfrak{p}$ is a minimal prime of R,

$$\begin{aligned} \mathfrak{p} \supseteq \text{Nil}\, R &\Longrightarrow \mathfrak{p} \supseteq \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_r \supseteq \mathfrak{p}_1 \mathfrak{p}_2 \cdots \mathfrak{p}_r \\ &\Longrightarrow \mathfrak{p} \supseteq \mathfrak{p}_i \text{ (for some } i) \\ &\Longrightarrow \mathfrak{p} = \mathfrak{p}_i . \quad \square \end{aligned}$$

N.B. If we arrange the expression Nil $R = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_r$ so that there are no inclusion relationships among the $\mathfrak{p}_i$'s, then, using (l) above, we see that $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ are *precisely all* the minimal primes of R.

We shall now try to prove (7.3). In fact, we shall restrict ourselves to *elementary transformations* of unimodular rows, and prove the following more precise version of (7.3).

Theorem 7.3′. *Let* R *be a commutative noetherian ring of Krull dimension* $d < \infty$*, and let* $(a_1, \ldots, a_n) \in \mathrm{Um}_n(R)$*. If* $n \geqslant d + 2$*, then*

$$(a_1, \ldots, a_n) \sim_{\mathrm{E}_n(R)} (1, 0, \ldots, 0).$$

Proof. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ be the minimal primes of R. Arguing as in the proof of (7.5), we can find an element

$$(*) \qquad a_1' = a_1 + b_2 a_2 + \cdots + b_n a_n \notin \bigcup_i \mathfrak{p}_i ,$$

so

$$(a_1, \ldots, a_n) \sim_{\mathrm{E}_n(R)} (a_1', a_2, \ldots, a_n).$$

Let $\overline{R} = R/(a_1')$. If $d = 0$, a_1' will be a unit and $n \geqslant 2$, so we are done by (I.5.3). If $d \geqslant 1$, the Krull dimension of $\overline{R}$ is at most $d - 1$ by $(*)$ and (7.6)(1). Invoking an inductive hypothesis at this point, we may assume that

$$(\overline{a}_2, \ldots, \overline{a}_n) \sim_{\mathrm{E}_{n-1}(\overline{R})} (\overline{1}, \overline{0}, \ldots, \overline{0})$$

(where "bar" = modulo (a_1')). Thus, by (I.5.6), $(a_1, \ldots, a_n) \sim_{\mathrm{E}_n(R)} (1, 0, \ldots, 0)$, as desired. $\square$

Note that, since only elementary transformations are used in the conclusion of (7.3′), we can actually get for free the following self-strengthening of this result.

Theorem 7.3″. *For a commutative ring R, let $R' = R/\mathrm{rad}\,(R)$ where $\mathrm{rad}\,(R)$ denotes the Jacobson radical of R, and let $(a_1, \ldots, a_n) \in \mathrm{Um}_n(R)$. If R' is a noetherian ring of Krull dimension $d' < \infty$ and $n \geqslant d' + 2$, then*

$$(a_1, \ldots, a_n) \sim_{\mathrm{E}_n(R)} (1, 0, \ldots, 0).$$

Proof. The point is that $\mathrm{E}_n(R)$ maps *onto* $\mathrm{E}_n(R')$. Thus, applying (7.3′) over R' and lifting the elementary transformations used, we get

$$(a_1, \ldots, a_n) \sim_{\mathrm{E}_n(R)} (1 + r_1, r_2, \ldots, r_n),$$

for suitable elements $r_i \in \mathrm{rad}\,(R)$. Now $1 + r_1$ is a *unit* in R, so by (I.6.3), the right side above can be brought by a sequence of elementary transformations to $(1, 0, \ldots, 0)$. □

The above self-strengthening of (7.3′) is potentially very useful, since R' may be noetherian without R being so, and $\dim\,(R')$ may be considerably smaller than $\dim\,(R)$.

Our next goal will be to prove (7.2). To this end, we shall deal more generally with polynomial rings $R[t]$ over any commutative noetherian ring R. Some of the results established in this generality will also be needed in a later chapter.

Recall that the *height* of a prime $\mathfrak{p} \subset R$ (written ht $\mathfrak{p}$) is the supremum of the integers n for which there exists a prime chain

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n = \mathfrak{p}.$$

Thus, $\mathrm{ht}\ \mathfrak{p} = \text{Krull dim } R_{\mathfrak{p}}$. For R noetherian, ht $\mathfrak{p}$ is always finite, and the Theorem of Krull(*) characterizes it as the smallest integer m such that $\mathfrak{p}$ is a minimal prime over an R-ideal generated by m elements. This fact will be assumed in the proof of the following result.

Proposition 7.7. *Let R be a commutative noetherian ring, $\mathfrak{P}$ a prime ideal in $R[t]$, and $\mathfrak{p} = \mathfrak{P} \cap R$. Then*

$$\mathrm{ht}\,\mathfrak{P} = \begin{cases} \mathrm{ht}\,\mathfrak{p} & \textit{if} \quad \mathfrak{P} = \mathfrak{p}[t], \\ 1 + \mathrm{ht}\,\mathfrak{p} & \textit{if} \quad \mathfrak{P} \supsetneq \mathfrak{p}[t]. \end{cases}$$

Proof. Any prime chain $\mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_r \subsetneq \mathfrak{p}$ extends to

$$\mathfrak{p}_0[t] \subsetneq \cdots \subsetneq \mathfrak{p}_r[t] \subsetneq \mathfrak{p}[t] \subseteq \mathfrak{P},$$

so clearly ht $\mathfrak{P}$ is $\geqslant$ the given values. Now let $m = \mathrm{ht}\,\mathfrak{p}$, and suppose $\mathfrak{p}$ is minimal over $\mathfrak{A} = (a_1, \ldots, a_m)$. Clearly $\mathfrak{p}[t]$ is a minimal prime over $\mathfrak{A}[t]$, so $\mathrm{ht}\,\mathfrak{p}[t] \leqslant m$. Thus

(*) See, e.g. [Nagata: 1962], p. 26.

we have $\operatorname{ht}\mathfrak{P} = \operatorname{ht}\mathfrak{p}$ in the case $\mathfrak{P} = \mathfrak{p}[t]$. Now assume $\mathfrak{P} \supsetneq \mathfrak{p}\,[t]$, say $f \in \mathfrak{P} \setminus \mathfrak{p}[t]$. We will be done if we can show that $\mathfrak{P}$ is a minimal prime over $\mathfrak{A}[t] + f \cdot R[t]$. Let $\mathfrak{P}'$ be a prime between these. Then

$$\mathfrak{A} \subseteq \mathfrak{P}' \cap R \subseteq \mathfrak{P} \cap R = \mathfrak{p},$$

so $\mathfrak{P}' \cap R = \mathfrak{p}$. In particular,

$$\mathfrak{p}[t] \subsetneq \mathfrak{P}' \subseteq \mathfrak{P}.$$

We may factor out $\mathfrak{p}$ to assume that $\mathfrak{p} = 0$. Then R is a domain, and we can localize the prime chain $0 \neq \mathfrak{P}' \subseteq \mathfrak{P}$ with respect to the multiplicative set $S = R \setminus \{0\}$. The localized ring $S^{-1}R[t]$, being a PID, is of Krull dimension 1, so $S^{-1}\mathfrak{P}' = S^{-1}\mathfrak{P}$. But $\mathfrak{P}'$ and $\mathfrak{P}$ are disjoint from S since they both contract to the zero ideal in R. Thus, $\mathfrak{P}' = \mathfrak{P}$, as desired. □

Corollary 7.8. *Let R be a commutative noetherian ring. Then*

$$\operatorname{Krull\,dim}\big(R[t]\big) = 1 + \operatorname{Krull\,dim} R.$$

In particular, Krull dim $k[t_1, \ldots, t_d] = d$ *for any field* k.

Proof. The Proposition gives the inequality $\leqslant$. For the reverse inequality, note that for any prime $\mathfrak{p} \subseteq R$, we have $\mathfrak{p}[t] \subsetneq \mathfrak{p}[t] + t \cdot R[t]$ – and the latter is a prime ideal in $R[t]$. □

As we have already observed in the paragraph preceding (7.7), the proof of Proposition 7.7 (and hence that of 7.8) depends on the Theorem of Krull. If, however, we only want to know Krull dim $k[t_1, \ldots, t_d] = d$ in 7.8 (which is all we need for the purpose of proving (7.1)), it is possible to give a different approach from the viewpoint of the transcendence degree (tr.d.). In fact, for any (commutative) domain A finitely generated over a field k (such A is called an *affine domain*), a well-known result in dimension theory states that

(7.9) $$\operatorname{Krull\,dim} A = \operatorname{tr.d}_k\,(\text{quotient field of } A)$$

(which certainly includes the case Krull dim $k[t_1, \ldots, t_d] = d$). This result can be proved strictly within the framework of the theory of affine domains, independently of Krull's Principal Ideal Theorem for general commutative rings. For such an approach, see, for example, Chapter 7, §7 in [Zariski-Samuel: 1960] (especially the second half of that section, and Theorem 21).

Notes on Chapter II

The results in this chapter are all "classical", in the sense that (with only a couple of exceptions) they all appeared before 1972. The first two sections are, in fact, standard material from ring theory and homological algebra. Strangely enough, the failure of Serre's Conjecture for noncommutative division rings (Theorem 3.1) was not noticed until [Sharma: 1971] and [Ojanguren-Sridharan: 1971]. The results contained in the latter paper turned out to be seminal for the subsequent works of [Parimala-Sridharan: 1975], [Parimala: 1976a, b; 1978] and [Knus-Ojanguren: 1976]. In the earlier version of this book, the checking of the details for the Ojanguren-Sridharan construction was tedious and rather complicated. Here, in §3, we presented a new approach from [Swan: 1996], which is not only much shorter, but also provides a stronger result over a class of noncommutative base domains. Just for the record, we note that the work of Ojanguren and Sridharan was, in part, motivated by cancellation problems for Azumaya algebras.

The graded case of Serre's Conjecture ((4.6) and (4.7)) is contained in [Swan: 1968], in the full generality of graded rings. See also [Bass: 1968]. The basic idea of the stable case of Serre's Conjecture goes back to Hilbert, in the form of Hilbert's Syzygy Theorem. For regular rings R, the isomorphism

$$K_0 R \cong K_0 R[t_1, \dots, t_n]$$

was proved by Grothendieck, who also proved that $K_0 R \cong G_0 R$ for the functor G_0.(*) Grothendieck's Theorem is, in fact, true with $R[t_1, \dots, t_n]$ replaced by any graded regular ring $R \oplus R_1 \oplus R_2 \oplus \dots$; see [Bass: 1968] and [Swan: 1968]. Our proof that R regular $\Longrightarrow R[t]$ regular is based on [Swan: 1975]. The module N over $R[t]$ in (5.1) was assumed to be free in Swan's notes, but it is clear that one need only assume N to be extended. For different proofs of the regularity of $R[t]$, see [Bass: 1968] and [Swan: 1968].

The example (5.10) of a non-extended $R[t]$-module was due to Schanuel, and first appeared in [Bass: 1962]; see also [Simis: 1969]. More information and results on Schanuel's construction can be found in Section 2 of [Lam: 1999]. For related literature on sufficient conditions for projective $R[t]$-modules to be free or extended, see, e.g. [Endô: 1963], [Bass: 1964b], [Murthy: 1965a, b; 1966], [Bass-Murthy: 1967], and [Geramita: 1972]. For an exhaustive survey on questions of $R[t]$-projective modules, K_0-regularity and K_0-stability, see Sections 2, 4, and 5 in [Bass: 1972a]. In particular, the question whether a (noetherian) normal domain need to be K_0-regular was mentioned in this paper. For subsequent developments on this question, and on the important notion of seminormality, see (VIII.7).

The 2-variable case of Serre's Conjecture was affirmed by [Seshadri: 1958], shortly after the formulation of the "Conjecture". The extension of Seshadri's result from PID to Dedekind rings was due independently to [Bass: 1962], and [Serre:

(*) In the literature, Grothendieck's Theorem (5.8) is sometimes attributed to *both* Grothendieck and Serre; see, e.g. [Swan: 1975], p. 6.

1960/61], while the case where the Dedekind ring is the coordinate ring of a nonsingular affine curve over an algebraically closed field was already covered in [Seshadri: 1959]. For a generalization of Bass's work in this period, see [Endô: 1963]. Another derivation of the Bass-Seshadri-Serre result, based on a cancellation theorem of Suslin, is available in [Swan: 1975]. Roitman's proof that $R[t]$ is Hermite for R a PID appeared in [Roitman: 1975], shortly before the full resolution of Serre's Conjecture. For yet another proof of Seshadri's Theorem, given by electrical engineers who applied the theory of polynomial matrices to the study of multi-dimensional systems theory, see the paper [Youla-Gnavi: 1979] listed in the references on Chapter VIII. In the applications of Seshadri's theorem to systems theory, the freeness of the "second syzygies" for f.g. modules over $k[t_1, t_2]$ (for any field k) has the effect of making systems theory *in two dimensions* considerably more manageable and complete.

For R a PID, [Lissner: 1967] showed that $R[t]$ is an "outer product ring". This includes Seshadri's Theorem, since outer product rings are Hermite. (On the other hand, it was shown in [Lissner: 1965] that $k[t_1, \ldots, t_n]$ over a field k is *not* an outer product ring for $n > 2$.) Lissner's result on $R[t]$ was further extended to Dedekind coefficient rings R by [Towber: 1968]; see also [Kleiner: 1971] and [Geramita: 1972].

The result (6.8) on first syzygy modules of ideals over $R[t]$ for a principal ideal domain R is essentially folklore, after the publication of Seshadri's Theorem. With the advent of various software packages for algebraic computations in the last decade, theoretical results such as (6.8) have received renewed interest. Without going into any really algorithmic issues, we thought it appropriate to include a number of concrete ad hoc computations of syzygy modules at the end of §6.

For work on projective modules over various affine subrings of $k[t_1, t_2]$ in the late 1970s, see [Anderson: 1978a, b]. This work led to "Anderson's Conjecture" on subrings of $k[t_1, \ldots, t_n]$ generated by monomials, which turned out to be a harbinger for the extensive later work on projective modules over monoid rings and vector bundles over toric varieties. For more details on this, see VIII.4.

The fact that any $P \in \mathfrak{P}(k[t_1, \ldots, t_d])$ of rank $> d$ contains a free module as direct summand is a special case of a celebrated Splitting Theorem of [Serre: 1957/58]. After the introduction of the stable range theory in [Bass: 1964a], it became well-known that the above P must in fact be free. In the text, we did just enough to show that a d-dimensional noetherian ring must be d-Hermite, but have refrained from a fuller development of the stable range ideas of Bass. For more details on the stable range theory, see [Bass: 1964a, 1968], [Estes-Ohm: 1967], [Vaserstein: 1969, 1971], [Gabel-Geramita: 1974], and [Gabel: 1975], among others.

Chapter III

The Basic Calculus of Unimodular Rows

The first part of this chapter presents two elementary proofs of Serre's Conjecture, due to Suslin and Vaserstein respectively, that were found shortly *after* the first solutions of the Conjecture were given by Quillen and Suslin in 1976. These "elementary" proofs are both formulated in the language of unimodular rows, so their main focus is on *stably free modules* rather than general projective modules. Both proofs are highly ingenious and delightful, and should be of interest to anyone seeking a thorough understanding of Serre's Conjecture and the variety of its many possible solutions. The presentation of these two elementary proofs occupies §1 and §2, which are followed by a section (§3) on the existence of monic polynomials ("Suslin's Monic Polynomial Theorem") in certain polynomial ideals.

Just a short time before the two elementary proofs of Serre's Conjecture were found, Suslin also proved a remarkable theorem on the completability of unimodular rows in a certain "factorial" form. This has come to be known as "Suslin's $n!$ Theorem". We devote §4 to a full coverage of this significant result — and some of its applications to the completion problem of unimodular polynomial vectors. These applications are continued in §5, which consists of some material on the new notion of "sectionable (and presectionable) sequences" in commutative rings. We have included this material since it is expected to be of use in the future investigations on unimodular rows. This presentation is followed by a short section (§6) on a certain graph structure on $\mathrm{Um}_n(R)$, and on duality questions of stably free modules.

Suslin's investigations on the $n!$ theorem also led him to a class of $2^n \times 2^n$ matrices with some very interesting properties. Following Ravi Rao, we call these the *Suslin matrices.* Since this material has never appeared in book form before, we have included a brief introduction to it in §7, to give the reader an idea of the flavor of this part of Suslin's work.

The theme of unimodular rows runs like a red thread through all seven sections of this chapter. Hence no title is more appropriate for this chapter than "The Basic Calculus of Unimodular Rows".

§1. Suslin's Elementary Proof of Serre's Conjecture

In this section, we give an elementary proof for the fact that, for any field k, *the polynomial ring* $k[t_1, \ldots, t_n]$ *is a Hermite ring.* (As we know from the discussions in (I.4), this implies the truth of Serre's Conjecture.) The proof to be given below was contained in a letter from Suslin to Bass in May, 1976. We begin with some preliminary lemmas.

Lemma 1.1. *Let R be a commutative ring, and I be an ideal in $R[t]$ that contains a monic polynomial. Let J be an ideal in R such that $I + J[t] = R[t]$. Then $(R \cap I) + J = R$.*

Proof. Let $S = R[t]/I \supseteq R/R \cap I$, and let $\overline{J}$ be the image of J in $R/R \cap I$. The hypothesis means that $\overline{J} \cdot S = S$. Since S is integral over $R/R \cap I$, the "Going Up" Theorem for integral extensions.(*) implies that $\overline{J} = R/R \cap I$, i.e. $(R \cap I) + J = R$. □

Lemma 1.2. *Let R be a commutative ring, and $f = (f_1, f_2) \in R[t]^2$. Let $c \in R \cap (f_1, f_2)$. Then for any commutative R-algebra A, and $b, b' \in A$,*

$$b \equiv b' \pmod{cA} \Longrightarrow f(b) \sim_{\mathrm{SL}_2} f(b').$$

Note. (1) We have previously used the notation $\sim$ only for unimodular rows, but it is clearly meaningful for arbitrary rows. (2) We write $\sim_{\mathrm{SL}_2}$ instead of $\sim_{\mathrm{SL}_2(A)}$ if no ambiguity about the ring can arise.

Proof of (1.2). Write $c = f_1 g_1 + f_2 g_2$ $(g_1, g_2 \in R[t])$. First, let us assume that c is not a zero-divisor in A, so we can work in the localization $A_c \supseteq A$. The candidate for the required $\mathrm{SL}_2(A)$-matrix is taken to be

$$M = \frac{1}{c}\begin{pmatrix} g_1(b) & -f_2(b) \\ g_2(b) & f_1(b) \end{pmatrix}\begin{pmatrix} f_1(b') & f_2(b') \\ -g_2(b') & g_1(b') \end{pmatrix}. \tag{1.3}$$

To check that this belongs to $\mathrm{SL}_2(A)$, we first observe that det $M = \frac{1}{c^2} \cdot c \cdot c = 1$. Secondly, if we work in A/cA, the product of the two matrices above is $\equiv \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix} \equiv 0$, so M has all entries in A. Finally, we have

$$\begin{aligned} f(b) \cdot M &= \frac{1}{c}(f_1(b), f_2(b))\begin{pmatrix} g_1(b) & -f_2(b) \\ g_2(b) & f_1(b) \end{pmatrix}\begin{pmatrix} f_1(b') & f_2(b') \\ -g_2(b') & g_1(b') \end{pmatrix} \\ &= \frac{1}{c}(c, 0)\begin{pmatrix} f_1(b') & f_2(b') \\ -g_2(b') & g_1(b') \end{pmatrix} \\ &= (f_1(b'), f_2(b')) = f(b'). \end{aligned}$$

(*)See, e.g. [Matsumura: 1970, p. 34].

In the general case where c need not be a non zero-divisor, we can still use the above procedure to get explicit formulas for the entries of M, thus avoiding the use of c in the denominator. In fact, write

$$\begin{aligned} f_i(t+yz) &= f_i(t) + y\phi_i(t, y, z), \\ g_i(t+yz) &= g_i(t) + y\psi_i(t, y, z), \\ b' &= b + cd \quad (d \in A). \end{aligned}$$

We shall try to define $M = (m_{ij})$ over A. The (1, 1) entry in the product of the two matrices in (1.3) is

$$\begin{aligned} &g_1(b)(f_1(b) + c\phi_1(b, c, d)) + f_2(b)(g_2(b) + c\psi_2(b, c, d)) \\ &\quad = c[1 + g_1(b)\phi_1(b, c, d) + f_2(b)\psi_2(b, c, d)]. \end{aligned}$$

Thus, we can *define* the expression in brackets to be our m_{11}. Similarly, we can define m_{12}, m_{21}, m_{22}. We can conclude, on formal grounds, that $\det(m_{ij}) = 1$, and that $f(b) \cdot (m_{ij}) = f(b')$. [Note: c may be a zero-divisor in A, but it is certainly not a zero-divisor as an indeterminate!] □

Note. As a matter of fact, for application to Serre's Conjecture over fields, the case $c =$ non zero-divisor is already sufficient.

Lemma 1.4. *Let R be a commutative ring, and let $f \in R[t]^n$. Then, for any commutative R-algebra A, and any subgroup $G \subseteq \mathrm{GL}_n(A)$,*

$$I = I_{f,A,G} = \{\, c \in R \mid b \equiv b' \pmod{cA} \Longrightarrow f(b) \sim_G f(b') \,\}$$

is always an ideal in R.

Proof. Let $c, c' \in I$, and $r, r' \in R$. To see that $rc + r'c' \in I$, let $b - b' = (rc + r'c')a$, where $b, b', a \in A$. From $b - ra \cdot c = b' + r'a \cdot c'$, we have

$$f(b) \sim_G f(b - ra \cdot c) = f(b' + r'a \cdot c') \sim_G f(b'). \quad \square$$

We shall say that a polynomial $g \in R[t]$ is *unitary* if the leading coefficient of g is a unit in R.

Theorem 1.5. (Suslin) *Let R be a commutative ring, and $f = (f_1, \ldots, f_n)$ $(n \geqslant 2)$ be a unimodular row over $R[t]$, with f_1 unitary. Then, for any commutative R-algebra A, and any $b, b' \in A$, we have $f(b) \sim_G f(b')$ where G is the subgroup of $\mathrm{GL}_n(A)$ generated by $\mathrm{E}_n(A)$ and $\mathrm{SL}_2(A)$.* [*As usual,* $\mathrm{SL}_2(A)$ *is viewed as a subgroup of* $\mathrm{SL}_n(A)$ *via the embedding* $M \mapsto \begin{pmatrix} M & 0 \\ 0 & I_{n-2} \end{pmatrix}$.]

Proof. For f and G as in the statement of the theorem, we want to prove that the ideal $I = I_{f,A,G}$ in (1.4) is the *unit ideal* in R. For any given maximal ideal $\mathfrak{m} \subset R$, it suffices to find an element $c \in I \setminus \mathfrak{m}$. Look at f modulo $(f_1) + \mathfrak{m}[t]$; we get a unimodular row $(\overline{f}_2, \ldots, \overline{f}_n)$ over

$$\frac{R[t]}{(f_1)+\mathfrak{m}[t]} \cong \left(\frac{R}{\mathfrak{m}}[t]\right)\Big/(\overline{f}_1),$$

which is a finite dimensional algebra over the field $R/\mathfrak{m}$, hence a (commutative) semilocal ring (see (I.3.9)). We may assume $n \geqslant 3$ (for otherwise $f(b) \sim_{\mathrm{SL}_2(A)} (1,0) \sim_{\mathrm{SL}_2(A)} f(b'))$. Thus, by (I.5.4), there exists $\overline{M} \in \mathrm{E}_{n-1}\big(R[t]/((f_1)+\mathfrak{m}[t])\big)$ such that

$$(\overline{f}_2, \ldots, \overline{f}_n) \cdot \overline{M} = (\overline{1}, \overline{0}, \ldots, \overline{0}).$$

Lift $\overline{M}$ to $M \in E_{n-1}(R[t])$, and let $(f_2, \ldots, f_n) \cdot M = (g_2, \ldots, g_n) \in R[t]^{n-1}$. Then $g_2 \equiv 1 \bmod \big((f_1)+\mathfrak{m}[t]\big)$, which implies $(f_1, g_2)+\mathfrak{m}[t] = R[t]$. Since f_1 is monic, we can infer from (1.1) that $R \cap (f_1, g_2) + \mathfrak{m} = R$. In particular, there exists an element $c \in R \cap (f_1, g_2)$, $c \notin \mathfrak{m}$. We will be done if we can show that $c \in I = I_{f,A,G}$. To check this, let $b \equiv b' \pmod{cA}$. We have, for $i \geqslant 2$,

$$g_i(b) - g_i(b') \in (b-b') \cdot A \subseteq c \cdot A \subseteq f_1(b)A + g_2(b)A. \tag{1.6}$$

Thus,

$$\begin{aligned}
f(b) &\sim_{\mathrm{E}_n} (f_1(b), g_2(b), g_3(b), \ldots, g_n(b)) \\
&\sim_{\mathrm{E}_n} (f_1(b), g_2(b), g_3(b'), \ldots, g_n(b')) && \text{(by (1.6))} \\
&\sim_{\mathrm{SL}_2} (f_1(b'), g_2(b'), g_3(b'), \ldots, g_n(b')) && \text{(by (1.2))} \\
&\sim_{\mathrm{E}_n} (f_1(b'), f_2(b'), f_3(b'), \ldots, f_n(b')) \\
&= f(b').
\end{aligned}$$

□

The following is a special case of (1.5).

Corollary 1.7. *Let R be a commutative ring, and $f = (f_1, \ldots, f_n)$ $(n \geqslant 2)$ be a unimodular row over $R[t]$, with f_1 unitary. Then $f \sim_G f(0)$ where G is the subgroup of $\mathrm{GL}_n\big(R[t]\big)$ generated by $\mathrm{E}_n\big(R[t]\big)$ and $\mathrm{SL}_2\big(R[t]\big)$.*

Proof. Apply the theorem to $A = R[t]$, and $b = t$, $b' = 0$. □

In fact, the above Corollary is obviously equivalent to (1.5).

Theorem 1.8. *If k is a field, and $A = k[t_1, \ldots, t_d]$, then any $P \in \mathfrak{P}(A)$ is free.*

Proof. By (II.5.9), it suffices to show that A is a Hermite ring. We do this by induction on d. Let $f = (f_1, \ldots, f_n)$ be a unimodular row over A. Say $f_1 \neq 0$. It is a standard fact that

$$\begin{aligned}
&\textit{There exists a change of variables } t_1 \mapsto t_1, \\
&t_i \mapsto t_i + t_1^{r_i} \ (2 \leqslant i \leqslant d), \textit{ such that} \\
&f_1(t_1,\ t_2 + t_1^{r_2}, \ldots) = c \cdot h(t_1, \ldots, t_d), \textit{ where} \\
&c \in k\backslash\{0\}, \textit{ and } h \textit{ is monic as a polynomial in } t_1.
\end{aligned} \tag{1.9}$$

Since the change of variables involved induces a k-algebra automorphism of A, we may as well assume that the original $f_1(t_1, \ldots, t_d)$ is, up to a scalar multiple, monic in $R[t_1]$, where $R = k[t_2, \ldots, t_d]$. The (nonzero) scalar is invertible since k is a field. Thus, by the above corollary, $f \sim_{\mathrm{GL_n}} f(0, t_2, \ldots, t_d)$. By inductive hypothesis, the latter is $\sim_{\mathrm{GL_n}} (1, 0, \ldots, 0)$. □

The observation (1.9) is apparently due to Nagata, in his proof of the Noether Normalization Theorem. For the convenience of the reader, we shall recall the proof here. Let

$$f_1(t_1, \cdots, t_d) = \sum_i a_i t_1^{i_1} \ldots t_d^{i_d}, \text{ so}$$

$$\begin{aligned} f_1\big(t_1, t_2 + t_1^{r_2}, \ldots\big) &= \sum_i a_i t_1^{i_1} \big(t_2 + t_1^{r_2}\big)^{i_2} \cdots \big(t_d + t_1^{r_d}\big)^{i_d} \\ &= \sum_i \big(a_i t_1^{i_1 + r_2 i_2 + \cdots + r_d i_d} + \text{ terms with} \\ &\qquad t_1\text{-deg} < i_1 + r_2 i_2 + \cdots + r_d i_d\big). \end{aligned}$$

We can choose $r_2, \ldots, r_d$ such that the integers $i_1 + r_2 i_2 + \cdots + r_d i_d$ are distinct for all the intervening d-tuples $i = (i_1, \ldots, i_d)$. In fact, if m is an integer greater than max $(i_1, \ldots, i_d)$ for all i, we may choose $r_j = m^{j-1}$, since then the integers $i_1 + r_2 i_2 + \cdots + r_d i_d$ will have different m-adic expansions. Having thus chosen the $r_2, \ldots, r_d$, the monomials $a_i t_1^{i_1 + r_2 i_2 + \cdots + r_d i_d}$ in $f(t_1, t_2 + t_1^{r_2}, \ldots)$ will not cancel out each other, and the one with the highest degree (and with $a_i \neq 0$) will emerge as the leading term in $f(t_1, t_2 + t_1^{r_2}, \ldots)$ as a polynomial in t_1.

Remark 1.10. The proof of (1.8) actually shows that, to take a unimodular row $(f_1, \ldots, f_n)$ over $A = k[t_1, \ldots, t_d]$ to $(1, 0, \ldots, 0)$, we need only use repeatedly matrices from $\mathrm{SL}_2(A)$ and $\mathrm{E}_n(A)$.

§2. Vaserstein's Elementary Proof of Serre's Conjecture

This section is, in essence, independent of the preceding one. We shall give a proof of the following.

Theorem 2.1. *Let R be a commutative ring. Let $f = (f_1, \ldots, f_n) \in \mathrm{Um}_n\big(R[t]\big)$ be such that the leading coefficients of the f_i's generate the unit ideal R. Then $f(t) \sim f(0)$ over $R[t]$.* ($\sim$ always means $\sim_{\mathrm{GL}_n}$.)

This result should be compared with Corollary (1.7). The hypothesis here is weaker, since we only assume that the leading coefficients of the f_i's generate R, whereas, in (1.7), we assumed that one of these leading coefficients is a unit. On the other hand, the conclusion of (1.7) is stronger, because it specified the type of matrices needed to transform $f(t)$ to $f(0)$. However, this finer information is not needed in the proof of Serre's Conjecture in (1.8). Thus, *once we can verify* (2.1) *by another method, we will have obtained a second proof of Serre's Conjecture.*

In contrast to §1, the arguments in this section will be based on local-global techniques.

Lemma 2.2. *Let B be a commutative ring, and S be a multiplicative set in B. Let $\tau(x) \in \mathrm{GL}_n\big(B_S[x]\big)$ be such that $\tau(0) = I_n$. Then there exists a matrix $\hat{\tau}(x) \in \mathrm{GL}_n(B[x])$ such that $\hat{\tau}(x)$ localizes into $\tau(sx)$ (for some $s \in S$), and $\hat{\tau}(0) = I_n$.*

Proof. Say $\tau(x)\mu(x) = I_n$ over $B_S[x]$. We have also $\mu(0) = I_n$, so τ, μ have diagonal entries in $1 + xB_S[x]$, and the other entries in $xB_S[x]$. Since only a finite number of denominators are involved, there exists $s_1 \in S$ such that $\tau(s_1x)$ and $\mu(s_1x)$ are both "defined" over $B[x]$. Let $\tau_1(x)$, $\mu_1(x)$ be matrices over $B[x]$ with $\tau_1(0) = \mu_1(0) = I_n$, localizing into $\tau(s_1x)$ and $\mu(s_1x)$. Then $\beta(x) = \tau_1(x)\mu_1(x)$ localizes into I_n. Since $\beta(0) = I_n$, there exists $s_2 \in S$ such that $\beta(s_2x) = I_n = \tau_1(s_2x)\mu_1(s_2x)$. Now the matrix $\hat{\tau}(x) = \tau_1(s_2x)$ is invertible over $B[x]$, and localizes into $\tau(s_1s_2x)$. Moreover, $\hat{\tau}(0) = \tau_1(0) = I_n$. □

Remark. Analogues of (2.1) and (2.2) (as well as some of the results to follow) also hold when the general linear groups are replaced by the elementary groups. However, the proofs generally require more work, and sometimes the additional hypothesis $n \geqslant 3$ is needed. We will take up these issues later, in VI.1 and VI.2.

Proposition 2.3. *Let R be a commutative ring and S a multiplicative set in R. For $f = (f_1, \ldots, f_n) \in \mathrm{Um}_n\big(R[t]\big)$, the following statements are equivalent:*

(1) *$f(t) \sim f(0)$ over $R_S[t]$;*
(2) *there exists $b \in S$ such that $f(t + bx) \sim f(t)$ over $R[t, x]$.*

Proof. (2) $\Rightarrow$ (1). Go to $R_S[t, x]$, and specialize the given $\sim$ equivalence by $t \mapsto 0$, $x \mapsto b^{-1}t$. We get $f(0 + bb^{-1}t) = f(t) \sim f(0)$ over $R_S[t]$.

(1) $\Rightarrow$ (2). Take $\sigma(t) \in \mathrm{GL}_n\big(R_S[t]\big)$ such that $f(t) \cdot \sigma(t) = f(0)$. Let $\tau(t, x) = \sigma(t + x) \cdot \sigma(t)^{-1} \in \mathrm{GL}_n\big(R_S[t, x]\big)$. Then

$$\begin{aligned} f(t+x) \cdot \tau(t, x) &= f(t+x) \cdot \sigma(t+x)\sigma(t)^{-1} \\ &= f(0) \cdot \sigma(t)^{-1} \\ &= f(t) \quad (\text{in } R_S[t, x]^n). \end{aligned}$$

We now try to "lift" this equation to $R[t, x]^n$. Since $\tau(t, 0) = \sigma(t) \cdot \sigma(t)^{-1} = I_n$, we can apply Lemma 2.2 over $B = R[t]$. By this lemma, we can find $\hat{\tau}(t, x) \in \mathrm{GL}_n\big(R[t, x]\big)$ that localizes to $\tau(t, sx)$ (for some $s \in S$), and $\hat{\tau}(t, 0) = I_n$. Then, in $R[t, x]^n$, we have

$$f(t + sx) \cdot \hat{\tau}(t, x) - f(t) = x \cdot g(t, x)$$

for some row $g(t, x)$ that localizes to 0. Thus, for a suitable $s' \in S$, we get

$$f(t + ss'x) \cdot \hat{\tau}(t, s'x) - f(t) = x \cdot s'g(t, s'x) = 0. \qquad \square$$

Theorem 2.4. *Let R be a commutative ring, and $f = (f_1, \ldots, f_n) \in \mathrm{Um}_n\big(R[t]\big)$. Let*

$$\mathfrak{A} = \{\, a \in R \mid\ f(t) \sim f(0) \text{ over } R_a[t]\,\},$$
$$\mathfrak{B} = \{\, b \in R \mid\ f(t+bx) \sim f(t) \text{ over } R[t,x]\,\}.$$

Then, $\mathfrak{A}$, $\mathfrak{B}$ are ideals in R, with $\mathfrak{A} = \operatorname{rad}\mathfrak{B}$.

Proof. If $b \in \mathfrak{B}$, $c \in R$, substitution of x by cx gives $f(t+bcx) \sim f(t)$, so $bc \in \mathfrak{B}$. Also, if $b, b' \in \mathfrak{B}$, substitution of t by $t + b'x$ gives

$$f(t+(b'+b)x) \sim f(t+b'x) \sim f(t).$$

Thus, $\mathfrak{B}$ is an ideal, and (2.3) implies that $\mathfrak{A} = \operatorname{rad}\mathfrak{B}$. □

We now arrive at the following the following "local-global principle" for the action of the general linear group on unimodular polynomial vectors.

Theorem 2.5. *Let R be a commutative ring, and $f = (f_1, \ldots, f_n) \in \mathrm{Um}_n\big(R[t]\big)$. If $f(t) \sim f(0)$ over $R_{\mathfrak{m}}[t]$ for all maximal ideals $\mathfrak{m} \subset R$, then $f(t) \sim f(0)$ over $R[t]$.*

Proof. Form the ideals, $\mathfrak{A}$, $\mathfrak{B}$ as in (2.4). For any maximal ideal $\mathfrak{m} \subset R$, (2.3) implies that $R \backslash \mathfrak{m}$ contains an element of $\mathfrak{B}$. This shows that $\mathfrak{B} = R$. By (2.4), we have $\mathfrak{A} = R$ too, and hence $f(t) \sim f(0)$ over $R[t]$. □

Having established (2.5), the proof of (2.1) is immediately reduced to the local case. For the local case, a stronger statement is possible, as the following proposition shows.

Proposition 2.6. *Let R be a commutative local ring, and $f = (f_1, \ldots, f_n) \in \mathrm{Um}_n\big(R[t]\big)$, where $n \geqslant 3$, and f_1 is unitary. Then*

$$f(t) \sim_{\mathrm{E}_n(R[t])} f(0) \sim_{\mathrm{E}_n(R)} (1, 0, \ldots, 0).$$

Proof. The proof here uses only results from Chapter I. By (I.5.6), it is sufficient to show that

(2.7) $$(\overline{f}_2, \ldots, \overline{f}_n) \sim_{\mathrm{E}_{n-1}(A)} (\overline{1}, \overline{0}, \ldots, \overline{0}),$$

where $A = R[t]/(f_1)$. Since f_1 is unitary, A is f.g. as an R-module. Thus A is semilocal by (I.3.9), and (2.7) follows from (I.5.4). □

An alternative proof of the above proposition, due to Suslin, will be given in an Appendix to this section. It turns out that, for $n \geqslant 3$ and f_1 unitary, the conclusion $f(t) \sim_{\mathrm{E}_n(R[t])} f(0)$ actually holds over *any* commutative base ring R. The proof of this depends on getting a local-global principle like (2.5) with the group $\mathrm{GL}_n\big(R[t]\big)$ replaced by the smaller group $\mathrm{E}_n\big(R[t]\big)$. This, however, requires considerable work on the group of elementary matrices. We'll return to this theme in VI.2; see VI.2.10.

Let us now make some closing remarks about the results proved in this section vis-à-vis some of our subsequent results to be obtained in Chapters IV–V. First, we take a closer look at the local-global principle (2.5). Recall that a unimodular row $f = (f_1, \dots, f_n) \in \mathrm{Um}_n\big(R[t]\big)$ defines a stably free $R[t]$-module P of type 1 (its solution space). Clearly, $P_0 = P/tP$ is correspondingly the R-stably free module of type 1 defined by $f(0) \in \mathrm{Um}_n(R)$. Moreover,

$$\begin{aligned} P \text{ is extended from } R &\iff P \cong R[t] \otimes_R P_0 \\ &\iff f(t) \sim f(0) \text{ over } R[t] \text{ (by (I.4.8)).} \end{aligned}$$

Using this, we may restate (2.5) as follows:

For any stably free $R[t]$-module P of type 1, P is extended from R iff, for all maximal ideals $\mathfrak{m} \subset R$, $P_{\mathfrak{m}}$ is extended from $R_{\mathfrak{m}}$.

This reformulation shows that (2.5) is a special case of Quillen's Patching Theorem stated in the Introduction section of this book. Quillen's Theorem, however, holds for *all finitely presented $R[t]$-modules P*, whereas (2.5) applies only to stably free $R[t]$-modules of type 1. Nevertheless, the proof of (2.5) is substantially easier than that in the general case (and strong enough to settle Serre's Conjecture over fields). In fact, if we are interested *only* in proving (2.5), parts of the material in (2.3), (2.4) can be skipped, and the proof of (2.5) can be considerably shortened. We have, however, chosen to state (2.3) and (2.4) separately, because they are of interest in their own right.

Finally, a word about the local result (2.6). In the statement of this theorem, we assumed that $f_1 \in R[t]$ is unitary. This implies that f_1 becomes invertible in the ring $R\langle t\rangle$ (= localization of $R[t]$ at monic polynomials) to be introduced in the next chapter. Thus, $f \sim (1, 0, \dots, 0)$ over $R\langle t\rangle$, and (2.6) purports to give $f \sim (1, 0, \dots, 0)$ over $R[t]$ (R local). This is a special case of Horrocks' Theorem in Chapter IV: see (IV.2.1). Again, Horrocks' Theorem is more general – it works for f.g. projective $R[t]$-modules, whereas (2.6) handles only stably free $R[t]$-modules of type 1.

Quillen's proof of Serre's Conjecture is based on his Patching Theorem, and on the theorem of Horrocks. With the results in this section, we can prove Serre's Conjecture using (2.5) and (2.6) instead. The latter proof (which is observed by Vaserstein) is, therefore, close in spirit to Quillen's proof; it may be regarded as a cross-breed of Quillen's methods (in Chapter V) and Suslin's methods in §1.

Some of the material in this section can be extended from the case of unimodular rows to the case of invertible matrices (over a polynomial ring). For more details on this, see §§1–4 of Chapter VI.

Appendix to §2

In this Appendix, we give a proof of (2.6) that is independent of the notion of semilocal rings. This alternative proof is based on an elementary lemma of Suslin on the leading coefficients of polynomials in a polynomial ideal.

Let R be any commutative ring. For any ideal $\mathfrak{A} \subseteq R[t]$ and any integer $m \geqslant 0$, let $\mathfrak{A}_m$ denote the ideal in R consisting of all elements $a \in R$ for which there exists a polynomial $at^m + a_1 t^{m-1} + \cdots \in \mathfrak{A}$.

Suslin's Lemma 2.8. *Let $\mathfrak{A}$ be an ideal in $R[t]$ that contains a unitary polynomial f of degree $m+1$. Then for any $g \in \mathfrak{A}$ with $\deg g \leqslant m$, all coefficients of g lie in $\mathfrak{A}_m$.*

Proof. Suppose that, for *all* $g = b_0 t^m + b_1 t^{m-1} + \cdots + b_m \in \mathfrak{A}$, we have shown that $b_0, \ldots, b_i \in \mathfrak{A}_m$. (We do know this for $i = 0$, by definition). We claim that $b_{i+1} \in \mathfrak{A}_m$ too. To see this, let u be the leading coefficient of f, and form

$$g' = t \cdot g - b_0 u^{-1} \cdot f \overset{\text{def}}{=} b_0' t^m + \cdots + b_i' t^{m-i} + \cdots + b_m' \in \mathfrak{A}.$$

Then $b_i' \in \mathfrak{A}_m$. But $b_i' \equiv b_{i+1} \pmod{b_0}$, so $b_{i+1} \in \mathfrak{A}_m$ also. □

Using Suslin's Lemma, we can re-prove (2.6) easily as follows. Let $(R, \mathfrak{m})$ be a commutative *local* ring, and $f = (f_1, \ldots, f_n) \in \mathrm{Um}_n\big(R[t]\big)$, where $n \geqslant 3$, and some f_i is unitary. Among all f_i's that are unitary, pick one (say f_1) of the smallest degree; we shall induct on $d = \deg f_1$. If $d = 0$, f_1 is a unit in R, so we are done. Assume now $d > 0$. After "dividing" each f_i by f_1, we may assume that $\deg f_i < d$, for all $i \geqslant 2$. Since $f = (f_1, \ldots, f_n)$ remains unimodular mod $\mathfrak{m}$, one of $f_2, \ldots, f_n$ (say f_2) must have a coefficient $c \notin \mathfrak{m}$. By Suslin's Lemma (applied to $\mathfrak{A} = (f_1, f_2)$, $m = d-1$), we can find a polynomial $h_1 f_1 + h_2 f_2$ of degree $d-1$, with leading coefficient c. We have

$$f \sim_{\mathrm{E}_n(R[t])} (f_1, f_2, f_3', \ldots, f_n),$$

where $f_3' = f_3 + h_1 f_1 + h_2 f_2$. We may assume that f_3 is *not* unitary, for otherwise we are done by induction since $\deg f_3 < d$. Now, with f_3 non-unitary, f_3' will have leading coefficient $\in c + \mathfrak{m}$, and hence f_3' is unitary of degree $d-1$. By the inductive hypothesis, we have

$$(f_1, f_2, f_3', \ldots, f_n) \sim_{\mathrm{E}_n(R[t])} (1, 0, \ldots, 0),$$

so the proof is complete. □

§3. Suslin's Monic Polynomial Theorem

In §1, we showed that Suslin's Theorem 1.5 implied that, for any field k, $k[t_1, \ldots, t_m]$ is Hermite, which settles Serre's Conjecture over fields. In this section, we'll show that (1.5) can be used to prove the following stronger result.

Theorem 3.1. *If R is any commutative noetherian ring of Krull dimension $d < \infty$, then, for any m, $R[t_1, \ldots, t_m]$ is d-Hermite.*

For $d = 0$, we get back the field case mentioned above. For $d = 1$, we get the following extension of the theorem of Seshadri.

Corollary. *If R is a* PID*, then any $P \in \mathfrak{P}(R[t_1, \ldots, t_m])$ is free.*

Proof. By (II.5.9), P is stably free. By (3.1), $R[t_1, \ldots, t_m]$ is 1-Hermite, hence Hermite. It follows that P is free. $\square$

The deduction of (3.1) from (1.5) *in the special case $d = 0$* is contained essentially in the observation (1.9) (the Nagata type of change of variables). In order to prove (3.1) for *arbitrary d*, we must somehow try to generalize (1.9). Such a generalization has been obtained by Suslin: see his Monic Polynomial Theorem in (3.3) below. We shall now try to lead up to this result.

Let R be a commutative ring. For any ideal $\mathfrak{A}$ in $R[t]$, let $\ell(\mathfrak{A})$ be the set in R consisting of zero and all the leading coefficients of polynomials in $\mathfrak{A}$. This is clearly an ideal in R (containing $\mathfrak{A} \cap R$). We shall need the following lemma which relates the height of $\ell(\mathfrak{A})$ to the height of $\mathfrak{A}$. (By definition, ht $I = \min \{\text{ht}\, \mathfrak{p}\}$ where $\mathfrak{p}$ ranges over all prime ideals $\supseteq I$. The height of the unit ideal is taken to be ∞.)

Lemma 3.2. *Let R be a commutative noetherian ring, and $\mathfrak{A}$ be an ideal in $R[t]$. Then* $\text{ht}\ \ell(\mathfrak{A}) \geqslant \text{ht}\ \mathfrak{A}$.

Proof. First assume $\mathfrak{A}$ is a prime ideal, $\mathfrak{P}$; let $\mathfrak{p} = \mathfrak{P} \cap R$. If $\mathfrak{P} = \mathfrak{p}[t]$, clearly $\ell(\mathfrak{P}) = \mathfrak{p}$, so $\text{ht}\ \ell(\mathfrak{P}) = \text{ht}\, \mathfrak{p} = \text{ht}\, \mathfrak{p}[t] = \text{ht}\, \mathfrak{P}$ by (II.7.7). If $\mathfrak{P} \supsetneqq \mathfrak{p}[t]$, clearly $\ell(\mathfrak{P}) \supsetneqq \mathfrak{p}$, so, again by (II.7.7), $\text{ht}\, \ell(\mathfrak{P}) > \text{ht}\, \mathfrak{p} = \text{ht}\, \mathfrak{P} - 1$, i.e., $\text{ht}\, \ell(\mathfrak{P}) \geqslant \text{ht}\, \mathfrak{P}$. For the general case, let $\mathfrak{P}_1, \ldots, \mathfrak{P}_r$ be the prime ideals of $R[t]$ that are minimal over $\mathfrak{A}$. (We know that $r < \infty$ by (II.7.6), since $R[t]$ is noetherian by Hilbert's Basis Theorem. Also, we may assume $r \geqslant 1$, for if $\mathfrak{A}$ is the unit ideal, we have $\text{ht}\, \ell(\mathfrak{A}) = \infty = \text{ht}\, \mathfrak{A}$.) From

$$\prod \mathfrak{P}_i \subseteq \bigcap \mathfrak{P}_i = \text{rad}\ \mathfrak{A},$$

we see that $(\prod \mathfrak{P}_i)^N \subseteq \mathfrak{A}$ for some N. Applying "ℓ" to this, and using the fact that $\ell(I) \cdot \ell(J) \subseteq \ell(I \cdot J)$, we get $\prod \ell(\mathfrak{P}_i)^N \subseteq \ell(\mathfrak{A})$. Let $\mathfrak{p}$ be a prime over $\ell(\mathfrak{A})$ with $\text{ht}\, \ell(\mathfrak{A}) = \text{ht}\, \mathfrak{p}$. Then $\ell(\mathfrak{P}_i) \subseteq \mathfrak{p}$ for some i, from which we get

$$\text{ht}\ \mathfrak{A} \leqslant \text{ht}\ \mathfrak{P}_i \leqslant \text{ht}\ \ell(\mathfrak{P}_i) \leqslant \text{ht}\ \mathfrak{p} = \text{ht}\ \ell(\mathfrak{A}). \qquad \square$$

Suslin's Monic Polynomial Theorem 3.3. *Let R be a commutative noetherian ring of Krull dimension $d < \infty$. Let $\mathfrak{A}$ be an ideal in $A = R[t_1, \ldots, t_m]$, with* $\text{ht}\ \mathfrak{A} > d$. *Then there exist new "variables" $s_1, \ldots, s_m \in A$ with $A = R[s_1, \ldots, s_m]$ such that $\mathfrak{A}$ contains a polynomial that is monic as a polynomial in s_1.*

Proof. We induct on m. If $m = 0$, $\text{ht}\, \mathfrak{A} > d$ means $\mathfrak{A} = R$ so the result is trivial. For $m \geqslant 1$, write $B = R[t_1, \ldots, t_{m-1}]$; we view $\mathfrak{A}$ as an ideal in $B[t_m] = A$, and get $\ell(\mathfrak{A}) \subseteq B$. By (3.2), $\text{ht}\ \ell(\mathfrak{A}) \geqslant \text{ht}\ \mathfrak{A} > d$, so, by the inductive hypothesis, we can write $B = R[s_1, \ldots, s_{m-1}]$ such that there exists $g \in \ell(\mathfrak{A})$ that is monic in s_1, say,

$$g = s_1^T + g_{T-1} s_1^{T-1} + \cdots + g_0 \quad (g_i \in R[s_2, \ldots, s_{m-1}]).$$

By definition of $\ell(\mathfrak{A})$, $\mathfrak{A}$ contains a polynomial

$$f = g \cdot t_m^N + b_{N-1} t_m^{N-1} + \cdots + b_0, \quad b_i \in B.$$

Let M be the highest power of s_1 occurring in $b_0, \ldots, b_{N-1}$. Set $t_m = s_m + s_1^K$, so $A = B[t_m] = R[s_1, \ldots, s_m]$. In $b_{N-1} t_m^{N-1} + \cdots + b_0$, s_1 occurs with exponent $\leqslant M + K(N-1)$. On the other hand,

$$g \cdot t_m^N = (s_1^T + g_{T-1} s_1^{T-1} + \cdots + g_0)(s_m + s_1^K)^N$$

is monic of degree $T + KN$ in s_1. For K sufficiently large, we have $T + K \cdot N > M + K(N-1)$, so f is monic as a polynomial in s_1. □

Before we give the application of this theorem, we need another lemma that is based on an earlier argument in (II.7.3′).

Lemma 3.4. *Let R be a commutative noetherian ring, and $(a_1, \ldots, a_n) \in \mathrm{Um}_n(R)$. Then $(a_1, \ldots, a_n) \sim_{\mathrm{E}_n(R)} (a_1', \ldots, a_n')$ where, for any $r \leqslant n$, $\mathrm{ht}\big((a_1', \ldots, a_r')\big) \geqslant r$.*

Proof. Suppose we have already $\mathrm{ht}\big((a_1, \ldots, a_r)\big) \geqslant r$, for some r. Consider $(\overline{a}_{r+1}, \ldots, \overline{a}_n) \in \mathrm{Um}_{n-r}(\overline{R})$, where $\overline{R} = R/(a_1, \ldots, a_r)$. Arguing as in (II.7.3′), we can find an element $a_{r+1}' = a_{r+1} + b_{r+2} a_{r+2} + \cdots + b_n a_n$ such that $\overline{a}_{r+1}'$ avoids all the minimal primes of $\overline{R}$. Then

$$(a_1, \ldots, a_r, a_{r+1}, \ldots) \sim_{\mathrm{E}_n(R)} (a_1, \ldots, a_r, a_{r+1}', \ldots).$$

Let $\mathfrak{p}$ be any prime containing $(a_1, \ldots, a_r, a_{r+1}')$. Since $\overline{\mathfrak{p}}$ cannot be minimal in $\overline{R}$, there exists another prime, $\mathfrak{p}'$, such that $(a_1, \ldots, a_r) \subseteq \mathfrak{p}' \subsetneq \mathfrak{p}$. Thus,

$$\mathrm{ht}\, \mathfrak{p} > \mathrm{ht}\, \mathfrak{p}' \geqslant \mathrm{ht}\, (a_1, \ldots, a_r) \geqslant r,$$

i.e., $\mathrm{ht}\ \mathfrak{p} \geqslant r + 1$. It follows that $\mathrm{ht}\, (a_1, \ldots, a_r, a_{r+1}') \geqslant r + 1$, so the proof proceeds by induction. □

We can now refine our earlier solution of Serre's Conjecture over fields.

Theorem 3.5. *Let R be a commutative noetherian ring, of Krull dimension $d < \infty$, and let $A = R[t_1, \ldots, t_m]$. If $f = (f_1, \ldots, f_n) \in \mathrm{Um}_n(A)$, $n \geqslant d + 2$, then $f \sim_G (1, 0, \ldots, 0)$ where $G \subseteq \mathrm{GL}_n(A)$ is generated by $\mathrm{SL}_2(A)$ and $\mathrm{E}_n(A)$. In particular, A is d-Hermite.*

Proof. We induct on m; the case $m = 0$ is covered by (II.7.3′). To deal with the general case, find $f \sim_{\mathrm{E_n}(A)} (f_1', \ldots, f_n')$ such that $\mathrm{ht}\big((f_1', \ldots, f_r')\big) \geqslant r$ for every $r \leqslant n$. In particular, $\mathfrak{A} = (f_1', \ldots, f_{n-1}')$ has height $\geqslant n - 1 > d$. By Suslin's Monic Polynomial Theorem, there exist new variables $s_1, \ldots, s_m$ such that $A = R[s_1, \ldots, s_m]$, and such that $\mathfrak{A}$ contains $f_1' g_1 + \cdots + f_{n-1}' g_{n-1}$ that is monic as a polynomial in s_1. Using elementary transformations, we may replace f_n' by

$$f_n' + s_1^N (f_1' g_1 + \cdots + f_{n-1}' g_{n-1}),$$

so we may assume that f_n' is already monic as a polynomial in s_1. By (1.7),

$$(f_1', \dots, f_n') \sim_G \big(f_1'(0, s_2, \dots, s_m),\ \dots\ ,\ f_n'(0, s_2, \dots, s_m)\big) \\ \in \mathrm{Um}_n\big(R[s_2, \dots, s_m]\big).$$

By the inductive hypothesis, the latter is $\sim_G (1, 0, \dots, 0)$. □

Combining the above with the earlier normality results (from (I.9)) on the groups $\mathrm{E}_n(A)$ for $n \geqslant 3$, we can get rid of the $\mathrm{SL}_2(A)$ factor in the statement of (3.5), provided that we consider only unimodular rows of length $\geqslant 3$. (This is actually quite an interesting application of the normality results in (I.9)!)

Transitivity Theorem 3.6. *For A as in* (3.5), $\mathrm{E}_n(A)$ *acts transitively on* $\mathrm{Um}_n(A)$ *for all* $n \geqslant \max\{3,\ d+2\}$.

Proof. Since $n \geqslant 3$, $\mathrm{E}_n(A)$ is normal in $\mathrm{GL}_n(A)$. Thus, the group G in (3.5) is now simply $\mathrm{E}_n(A) \cdot \mathrm{SL}_2(A)$. For any $f \in \mathrm{Um}_n(A)$, there exists a matrix $\sigma \in G$ such that $f\sigma = (0, 0, \dots, 1)$. Writing σ in the form $\sigma_0\tau$, where $\sigma_0 \in \mathrm{E}_n(A)$ and $\tau \in \mathrm{SL}_2(A) \subseteq \mathrm{SL}_n(A)$, we have

$$f\sigma_0 = (0, 0, \dots, 1)\tau^{-1} = (0, 0, \dots, 1)$$

since $n \geqslant 3$. This proves the transitivity of $\mathrm{E}_n(A)$ on $\mathrm{Um}_n(A)$. □

We close this section with the following K-theoretic result of Suslin, with the caveat that its proof here is *not* entirely self-contained in the case $d = 0$.

Corollary 3.7. *For A as in* (3.5), *the canonical map* $\mathrm{GL}_{d+1}(A) \to K_1(A)$ *is surjective.*

Proof. We first work with the case $d \geqslant 1$. An element in $K_1(A)$ is represented by a matrix, say, $\alpha \in \mathrm{GL}_n(A)$. If $n \geqslant d+2$, then $n \geqslant 3$ also, so by (3.6), the last row of α can be multiplied by a suitable $\sigma_1 \in \mathrm{E}_n(A)$ to become $(0, 0, \dots, 1)$. Left multiplying $\alpha\sigma_1$ by another suitable matrix $\sigma_2 \in \mathrm{E}_n(A)$, we get $\sigma_2\alpha\sigma_1 = \begin{pmatrix} \alpha' & 0 \\ 0 & 1 \end{pmatrix}$ where $\alpha' \in \mathrm{GL}_{n-1}(A)$. Therefore, the class of α in $K_1(A)$ is represented by the matrix α' of a smaller size, which (by induction) gives what we want.

The case $d = 0$ requires special treatment, since here we would have to show that any element of $K_1(A)$ is represented by a unit of A. This amounts to $SK_1(A) = 0$, which can be deduced by using the standard tools in algebraic K-theory. Since R is noetherian and 0-dimensional, it is a finite direct product of artinian local rings. Therefore, we may assume that R itself is artinian local. Reducing modulo the maximal ideal of R and using the Bass-Heller-Swan Theorem from [Bass: 1968, Ch. 7], one can show without too much difficulty that $SK_1(A) = 0$. □

Since $\varphi_n : \mathrm{GL}_n(A) \to K_1(A)$ is surjective for $n \geqslant d+1$ by (3.7), it is of interest to determine $\ker(\varphi_n)$. We cannot expect this kernel to be $\mathrm{E}_n(A)$ in general. For instance, if R is a field and $m \geqslant 2$, then $d = 0$, but $\mathrm{E}_2(A)$ is not normal in $\mathrm{GL}_2(A)$ by the results in (I.8), so $\mathrm{E}_2(A)$ cannot be the kernel of φ_2. For another example, if R is a Dedekind ring with $\mathrm{E}_2(R)$ not normal in $\mathrm{GL}_2(R)$ (see (I.8.16)), then $d = 1$ and again $\mathrm{E}_2(R)$

cannot be $\ker(\varphi_2)$ for $m = 0$. Excluding the case $n = 2$, however, Suslin has proved the following very general result.

Theorem 3.8. (Suslin's Stability Theorem) *For* A *as in* (3.5)*, the canonical map*

$$\mathrm{GL}_n(A)/\mathrm{E}_n(A) \longrightarrow K_1(A) \tag{3.9}$$

is an isomorphism for all $n \geqslant \max\{3,\ d+2\}$ *(independently of* m*).*

We will not prove this result in its full generality in our exposition. However, we shall prove it later in the special case where the ground ring R is a field. In this case, we'll prove (in VI.4.5) that $\mathrm{SL}_n(A) = \mathrm{E}_n(A)$ for $n \geqslant 3$. From this, it follows that both groups in (3.9) are given by $\mathrm{U}(R)$ for $n \geqslant 3$.

In the case where $m = 0$ (that is, $A = R$), (3.7) goes back to [Bass: 1964a], and (3.8) is due to [Bass: 1968] and [Vaserstein: 1969a, b] for $n \geqslant d+2$. The proof of (3.8) depends ultimately on this special case.

§4. Suslin's n! Theorem

Just prior to giving his first proof of Serre's Conjecture in 1976, Suslin also proved another beautiful theorem on the completion of unimodular rows over arbitrary commutative rings. We shall give an exposition on this result (and its applications) here since it relates well to the material in the previous sections of this chapter, and was discovered also in the same time frame. The statement of the result, from [Suslin: 1977b], is as follows.

Suslin's n! Theorem 4.1. *Let* R *be any commutative ring, and let* $(a_0, a_1, \ldots, a_n) \in \mathrm{Um}_{n+1}(R)$*. Let* $r_0, \ldots, r_n$ *be positive integers such that* $n!$ *divides* $r_0 \cdots r_n$*. Then the unimodular row*

$$\left(a_0^{r_0},\ a_1^{r_1},\ \ldots,\ a_n^{r_n}\right) \tag{4.2}$$

is completable (to a matrix in $\mathrm{GL}_{n+1}(R)$*).*

In the case $n = 2$, it is easy to see that this boils down to statement that (a^2, b, c) is completable for any unimodular row (a, b, c). This was first proved in [Swan-Towber: 1975]. Besides giving a module-theoretic proof of this fact, Swan and Towber also supplied an explicit completion. If $ap + bq + cr = 1$ in R, they gave the completion:

$$\begin{pmatrix} a^2 & b & c \\ b + ar & -r^2 + bpr & -p + qr - bpq \\ c - aq & p + qr + cpr & -q^2 - cpq \end{pmatrix} \in \mathrm{SL}_3(R). \tag{4.3}$$

A simpler (and somewhat more "symmetrical") completion is provided in [Krusemeyer: 1976] and [Suslin: 1977b] via the following determinantal identity[(*)] in $\mathbb{Z}[a, b, c, p, q, r]$ (which can be checked by direct expansion):

$$\text{(4.4)} \qquad \det \begin{pmatrix} a^2 & b & c \\ -b-2ar & r^2 & p-qr \\ -c+2aq & -p-qr & q^2 \end{pmatrix} = (ap+bq+cr)^2.$$

For an explicit derivation of this completion of (a^2, b, c), see the paragraphs following the proof of VII.5.31.

Swan and Towber noted, however, that a unimodular row (a^2, b, c, d) over a commutative ring need not be completable. More generally, they proved the following remarkable fact in [Swan-Towber: 1975, Thm. 3.1]. Let $R = \mathbb{C}[x_0, \ldots, x_{2n+1}]$ with the relation $x_0^2 + \cdots + x_{2n+1}^2 = 1$ be the complex coordinate ring of the sphere S^{2n+1}, and let

$$\text{(4.5)} \qquad z_0 = x_0 + ix_1, \;\; z_1 = x_2 + ix_3, \;\; \ldots \;, \;\; z_n = x_{2n} + ix_{2n+1}$$

in R. Then $z_0\bar{z}_0 + \cdots + z_n\bar{z}_n = 1$ so $(z_0, \ldots, z_n) \in \mathrm{Um}_{n+1}(R)$, *but the unimodular row* $\left(z_0^{r_0}, \ldots, z_n^{r_n}\right)$ *is not completable unless the (positive) exponents* $r_0, \ldots, r_n$ *are such that* $n!$ *divides* $r_0 \cdots r_n$. For instance, for $n = 3$, (z_0^2, z_1, z_2, z_3) is not completable over the complex coordinate ring of S^7. (For a related result, see [Murthy-Swan: 1976, Lemma 9.2].)

The above result of Swan and Towber is shown by using vector bundle theory and the homotopy theory of the spheres, so its proof is beyond the scope of our text. The significance of this result is that it shows the "converse" of (4.1) to be also true; or, in other words, (4.1) is already the best result possible. This is quite interesting since (4.1) was not yet proved (or even conjectured) when Swan and Towber submitted their paper for publication in 1974.

Coming now to (4.1), the first step toward its proof is the following. (For the rest of this section, R denotes a commutative ring.)

Proposition 4.6. *Let* $(a_0, \ldots, a_n) \in \mathrm{Um}_{n+1}(R)$ *be such that* $(\bar{a}_0, \ldots, \bar{a}_{n-1})$ *is completable in* $\bar{R} = R/Ra_n$. *Then*

$$\text{(4.7)} \qquad (a_0, a_1, \ldots, a_{n-1}, a_n^n) \in \mathrm{Um}_{n+1}(R)$$

is completable (*in* R).

Proof. Let $\alpha \in \mathbb{M}_n(R)$ be a matrix with first row $(a_0, \ldots, a_{n-1})$ such that $\bar{\alpha} \in \mathrm{GL}_n(\bar{R})$, and let $\beta \in \mathbb{M}_n(R)$ be such that $\bar{\alpha}\bar{\beta} = I_n$. Then

$$\alpha\beta = I_n + a_n\gamma, \quad \beta\alpha = I_n + a_n\delta$$

[(*)] For a brief indication on the origin of this identity, see the beginning of (VIII.5).

for suitable matrices $\gamma, \delta \in \mathbb{M}_n(R)$. From

$$\begin{pmatrix} \alpha & a_n I_n \\ \delta & \beta \end{pmatrix} \begin{pmatrix} \beta & -a_n I_n \\ -\gamma & \alpha \end{pmatrix} = \begin{pmatrix} I_n & 0 \\ * & I_n \end{pmatrix} \in \mathrm{GL}_{2n}(R),$$

we see that $\begin{pmatrix} \alpha & a_n I_n \\ \delta & \beta \end{pmatrix} \in \mathrm{GL}_{2n}(R)$. Next, we try to "transform" the corner $a_n I_n$ into $\mathrm{diag}(a_n^n, 1, \ldots, 1)$. Let $d = \det(\alpha) \in R$. Since d is a unit in $\bar{R}$, a_n is a unit in R/Rd. Applying Whitehead's Lemma (I.5.1) over the ring R/Rd and using the surjectivity of the natural map $\mathrm{E}_n(R) \to \mathrm{E}_n(R/Rd)$, we find an equation

$$\begin{pmatrix} a_n^n & 0 \\ 0 & I_{n-1} \end{pmatrix} = (a_n I_n)\sigma + d\tau$$

for some $\sigma \in \mathrm{E}_n(R)$ and $\tau \in \mathbb{M}_n(R)$. For $\alpha' = \mathrm{adj}(\alpha)$ (the classical adjoint of α), we have

$$\varphi := \begin{pmatrix} \alpha & a_n I_n \\ \delta & \beta \end{pmatrix} \begin{pmatrix} I_n & \alpha'\tau \\ 0 & \sigma \end{pmatrix} = \begin{pmatrix} \alpha & \lambda \\ \delta & * \end{pmatrix} \in \mathrm{GL}_{2n}(R)$$

where $\lambda = \alpha\alpha'\tau + a_n\sigma = d\tau + a_n\sigma = \begin{pmatrix} a_n^n & 0 \\ 0 & I_{n-1} \end{pmatrix}$. Rewriting φ in the block form $\begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix}$ where α_1 is $n \times (n+1)$ (so that α_1 is α adjoined with the column $(a_n^n, 0, \ldots, 0)^t$), we have $\alpha_2 = \begin{pmatrix} 0 \\ I_{n-1} \end{pmatrix}$. After performing some elementary transformations on the last n rows of φ, we get a new invertible matrix $\varphi' = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3' & 0 \end{pmatrix}$. Since $\alpha_2 = \begin{pmatrix} 0 \\ I_{n-1} \end{pmatrix}$, the submatrix of φ' "complementary" to I_{n-1} (obtained by deleting from φ' the second to the nth rows and the last $n-1$ columns) is an *invertible* $(n+1) \times (n+1)$ matrix with top row $(a_0, \ldots, a_{n-1}, a_n^n)$, as desired. □

By induction on $n \geqslant 1$, we obtain immediately the following crucial special case of Theorem 4.1.

Corollary 4.8. *For any* $(a_0, a_1, \ldots, a_n) \in \mathrm{Um}_{n+1}(R)$, *the "factorial" unimodular row*

$$\left(a_0,\ a_1,\ a_2^2,\ \ldots\ ,\ a_n^n\right)$$

is always completable.

Recall that, for $\rho,\ \rho' \in \mathrm{Um}_{n+1}(R)$, $\rho \sim \rho'$ means that $\rho' = \rho\sigma$ for some $\sigma \in \mathrm{GL}_{n+1}(R)$. It is easy to see that, with (4.8) proven, (4.1) will result from repeated use of the following device of "shifting powers" in a unimodular row.

Proposition 4.9. *Let* $(a_0, \ldots, a_n) \in \mathrm{Um}_{n+1}(R)$. *For any* $r \geqslant 1$ *and* $i \neq j$, *we have*

$$(a_0, \ldots, a_i^r, \ldots, a_n) \sim (a_0, \ldots, a_j^r, \ldots, a_n).$$

Proof. For ease of notation, we give the proof for $i=0$, $j=1$. It is easy to see that

$$f(t) := (a_0^r,\ a_1 + a_0 t,\ a_2,\ \ldots,\ a_n) \in \mathrm{Um}_{n+1}(A)$$

over $A = R[t]$. *We claim that* $f(t) \sim f(0)$ over A. By (2.5), it suffices to check this for a local ring $(R, \mathfrak{m})$. In this case, we may assume $a_0, a_2, \ldots, a_n \in \mathfrak{m}$ (for otherwise the claim is trivial), and so $a_1 \in \mathrm{U}(R)$. These imply easily(*) that $(a_0^r, a_1 + a_0 t) \in \mathrm{Um}_2(A)$, which is completable. Therefore,

$$f(t) \sim (1, 0, \ldots, 0) \sim f(0) \quad (\text{over } A),$$

as claimed. Returning now to the general case, note that $f(t) \sim f(0)$ over A implies that $f(-1) \sim f(0)$ over R; that is,

$$(4.10) \qquad (a_0^r, a_1, a_2, \ldots, a_n) \sim (a_0^r, a_1 - a_0, a_2, \ldots, a_n).$$

Upon changing $a_0 \mapsto a_1$ and $a_1 \mapsto a_1 - a_0$, we get

$$(4.10') \qquad (a_1^r, a_1 - a_0, a_2, \ldots, a_n) \sim (a_1^r, -a_0, a_2, \ldots, a_n).$$

On the other hand, adding $a_1^{r-1} + a_1^{r-2} a_0 + \cdots + a_0^{r-1}$ times $a_1 - a_0$ on the RHS of (4.10) to a_0^r, we get

$$\begin{aligned}(a_0^r, a_1, a_2, \ldots, a_n) &\sim (a_1^r, a_1 - a_0, a_2, \ldots, a_n)\\ &\sim (a_1^r, -a_0, a_2, \ldots, a_n) \quad \text{by (4.10}')\\ &\sim (a_0, a_1^r, a_2, \ldots, a_n) \quad \text{by} \quad (\text{I.5.0})(2),\end{aligned}$$

as desired. □

The work above completes the proof of Suslin's $n!$ Theorem. The first major consequence of this theorem is the following.

Proposition 4.11. *Let* n *be an integer such that* $n! \in \mathrm{U}(R)$, *and let* $\alpha = (a_0, \ldots, a_n) \in \mathrm{Um}_{n+1}(R)$. *If* a_0 *is unipotent (that is, "$1 +$ nilpotent") modulo* $Ra_1 + \cdots + Ra_n$, *then* $\alpha \sim (1, 0, \ldots, 0)$.

Proof. We use the following basic principle in ring theory: *if a ring* S *is such that an integer* $k \in \mathbb{N}$ *is invertible in* S, *then any unipotent element* a_0 *is a* k *th power in* S. This follows since we can use the "binomial expansion"

$$(4.12) \qquad a_0^{1/k} = \big(1 + (a_0 - 1)\big)^{1/k} = \sum \binom{1/k}{i} (a_0 - 1)^i,$$

which is a finite sum in S. (Note that $k \in \mathrm{U}(S)$ implies that all binomial coefficients $\binom{1/k}{i}$ are meaningful in S.(†)) Applying this to $S = R/(Ra_1 + \cdots + Ra_n)$ and $k = n!$,

(*)If a maximal ideal M of $R[t]$ contains a_0^r and $a_1 + a_0 t$, then $a_0 \in M$ and hence $a_1 \in M$, which is impossible.

(†)For *three* different proofs of this fact, see the solution to Ex. 14.9 in the author's exercise book [Lam: 2003].

we see that $\overline{a}_0 = \overline{a}^{\,n!} \in S$ for some $a \in R$. Thus, by (4.1),

$$(a_0, a_1, \ldots, a_n) \sim (a^{n!}, a_1, \ldots, a_n) \sim (1, 0, \ldots, 0). \quad \square$$

To illustrate the utility of this proposition, we shall give a couple of its applications below, following [Suslin: 1977b] and [Swan: 1978]. The first application, dealing with unimodular rows over polynomial rings, works again over any commutative ground ring R with $n! \in \mathrm{U}(R)$. We'll see later that this result is an affirmative special case of a yet unsolved problem of Suslin.

Proposition 4.13. *Let* $f(t) = (f_0(t), \ldots, f_n(t)) \in \mathrm{Um}_{n+1}\big(R[t]\big)$ *be such that* $\deg f_i \leqslant 1$ *for all* i*. If* $n! \in \mathrm{U}(R)$*, then* $f(t)$ *is completable in* $R[t]$ *iff* $f(0)$ *is completable in* R*.*

Proof. We need only prove the "if" part. Say $f_i(t) = a_i + b_i t$ $(a_i, b_i \in R)$ with $f(0) = (a_0, \ldots, a_n)$ completable in R. After transforming f by a matrix from $\mathrm{GL}_{n+1}(R)$, we may assume that $a_0 = 1, a_1 = \cdots = a_n = 0$, so that

$$f = (1 + b_0 t,\ b_1 t, \ldots,\ b_n t) \in \mathrm{Um}_{n+1}\big(R[t]\big).$$

We claim that $b_0^r \in b_1 R + \cdots + b_n R$ *for some* $r \geqslant 1$. For, if otherwise, b_0 would avoid some $\mathfrak{p} \in \operatorname{Spec} R$ containing $b_1, \ldots, b_n$. Letting K be the quotient field of $R/\mathfrak{p}$, we can extend the projection map $R \to R/\mathfrak{p} \subseteq K$ to a ring homomorphism $R[t] \to K$ by sending $t \mapsto -\big(\bar{b}_0\big)^{-1} \in K$. This homomorphism sends all coordinates of f to 0, which is impossible. Having proved our claim, we see in particular that $b_0 t$ is nilpotent modulo $b_1 t R[t] + \cdots + b_n t R[t]$, so we are done by (4.11) (applied to $R[t]$). $\square$

The second application of Proposition 4.11 replaces the polynomial ring $R[t]$ in (4.13) by the Laurent polynomial ring $R[t, t^{-1}]$. Here, we need a *local* assumption on R.

Proposition 4.14. *Let* $f(t) = (f_0(t), \ldots, f_n(t)) \in \mathrm{Um}_{n+1}\big(R[t, t^{-1}]\big)$*, where*

$$f_i(t) = a_i + b_i t \quad (a_i,\ b_i \in R)$$

for all i*. If* R *is local and* $n! \in \mathrm{U}(R)$*, then* $f(t)$ *is completable in* $R[t, t^{-1}]$*.*

Proof. Since R is local, $f(1) \in \mathrm{Um}_{n+1}(R)$ is certainly completable. Thus, after first right multiplying by a suitable matrix in $\mathrm{GL}_{n+1}(R)$, we may assume that $f(1) = (1, 0, \ldots, 0)$; that is,

$$f(t) = \big(1 + b_0(t-1),\ b_1(t-1),\ \ldots,\ b_n(t-1)\big). \tag{4.15}$$

Now, we claim that $[b_0(b_0-1)]^r \in b_1 R + \cdots + b_n R$ *for some* $r \geqslant 1$. For, if otherwise, $b_0(b_0-1)$ would avoid some $\mathfrak{p} \in \operatorname{Spec} R$ containing $b_1, \ldots, b_n$. Letting K be the quotient field of $R/\mathfrak{p}$ as before, we would have $\bar{b}_0 \neq 0, 1$ in K, so we can extend $R \to R/\mathfrak{p} \subseteq K$ to a ring homomorphism $R[t, t^{-1}] \to K$ by sending t to $1 - (\bar{b}_0)^{-1} \neq 0$. This homomorphism sends all coordinates of f to 0, which is impossible. Since R is a local ring, we have one of the following possibilities.

Case 1. $b_0 - 1 \in \mathrm{U}(R)$. Here, $b_0^r \in b_1 R + \cdots + b_n R$. So again $b_0(t-1)$ is nilpotent modulo $\sum_{i=1}^n b_i(t-1)R[t, t^{-1}]$, so we are done by (4.11). (In this case, the completion of $f(t)$ can be carried out already in the subring $R[t-1] = R[t]$.)

Case 2. $b_0 \in \mathrm{U}(R)$. Here, $(b_0 - 1)^r \in b_1 R + \cdots + b_n R$. The isomorphism type of the stably free module defined by (4.15) over $R[t, t^{-1}]$ is certainly unchanged if we scale each of the coordinates of $f(t)$ by the unit t^{-1}, so we can replace $f(t)$ by

$$\begin{aligned} g(t) &= \big(t^{-1} + b_0(1-t^{-1}),\ b_1(1-t^{-1}), \ldots,\ b_n(1-t^{-1})\big) \\ &= \big(1 + (b_0 - 1)(1-t^{-1}),\ b_1(1-t^{-1}),\ \ldots,\ b_n(1-t^{-1})\big), \end{aligned}$$

which enables us to finish as in *Case 1.* (Here, the completion of $g(t)$ can be carried out in the subring $R[1-t^{-1}] = R[t^{-1}]$.) □

Remark 4.16. One might hope that (4.14) would hold for a commutative (but not local) ring R if we add the assumption that $f(1) \in \mathrm{Um}_{n+1}(R)$ is completable over R. However, this is not the case, as can be seen from an example we'll give later in (V.4.16), over the Laurent polynomial ring $R[t, t^{-1}]$ where R is the real coordinate ring of the 2-sphere S^2. (I do not know what happens, however, if we assume that *both* $f(\pm 1) \in \mathrm{Um}_{n+1}(R)$ are completable over R.) As for Proposition 4.13, it begs the question whether the result remains true if the degree hypothesis on the f_i is removed. For R local (which is all that counts), this is "Suslin's Problem", which we will state more formally in (V.3.8). Suslin's Problem is closely related to some problems of a more tentative nature on commutative Hermite rings. All of these issues will be discussed in more detail in the context of the Bass-Quillen Conjecture; see (V.3).

§5. Sectionable Sequences

This section is, in some sense, a continuation of the study of the theme of unimodular elements in stably free modules. The notion of a *sectionable sequence* to be introduced below is based on a careful look at the proof of a certain result of Ravi Rao and Raja Sridharan discovered in the late 1990s. This led to the new material in this section, which is presented here to pave the way to a more general study of sequences over rings that are not necessarily unimodular rows. I want to thank Ravi Rao for informing me of his result, and R. G. Swan for many useful suggestions toward the writing of this section.

To avoid undue difficulties, *we'll continue to assume in this section that all rings under consideration are commutative.* Recall that an element α in an R-module P is called *unimodular* if αR is a direct summand of P that is free on the basis $\{\alpha\}$. In the case where P is a stably free module of type 1 defined by a unimodular sequence, we have a simple criterion I.4.12(1) for the existence of a unimodular element in P. Let us start by first generalizing this situation to an *arbitrary* sequence.

Definition 5.1. A sequence $(b_1, \ldots, b_n)$ in a (commutative) ring R is said to be *sectionable* (resp. *presectionable*) if there exist elements $c_1, \ldots, c_n \in R$ such that $\sum b_i c_i = 0$ and $\sum b_i R = \sum c_i R$ (resp. $\sum b_i R \subseteq \sum c_i R$). By convention, the empty sequence to taken to be (pre)sectionable.

The point behind this definition is that, in case $\beta = (b_1, \ldots, b_n)$ is a *unimodular* sequence in R, then by the result I.4.12(1) referred to above, β is sectionable (resp. presectionable) in the sense of (5.1) iff the stably free module $P(\beta)$ defined by β has a unimodular element. Since a unimodular element in a (f.g.) projective module corresponds to a "section" of the algebraic vector bundle associated to the module, we borrow the term "section" from this sheaf-theoretic context, and use it for arbitrary sequences in rings.

Note that the "(pre)sectionable" concept is primarily of interest for sequences of odd length since we have the following.

Proposition 5.2. *Any sequence $(b_1, \ldots, b_n)$ with n even is sectionable.*

Proof. This is simply based on the idea used in the proof I.4.12(2); namely, if $n = 2m$, we can choose the c_i 's in (5.1) to be $b_2, -b_1, b_4, -b_3, \ldots, b_{2m}, -b_{2m-1}$, in that order. □

There is no lack of examples of sectionable sequences of odd length also. After all, we can resort to taking unimodular sequences $(b_1, \ldots, b_{2m+1})$ that define stably free modules with a nonzero free direct summand. The items (0)-(9) below give many other examples of odd-length sectionable sequences that may not be unimodular. Some of the easier verifications for these examples will be left to the reader.

Examples 5.3. (1) *A singleton sequence $\{b\}$ is (pre)sectionable iff $b^2 = 0$.* The "if" part is trivial. Conversely, if $\{b\}$ is presectionable, take $bR \subseteq cR$ with $bc = 0$. Then $b^2 R \subseteq bcR = 0$ implies $b^2 = 0$, and so $\{b\}$ is also sectionable.

(2) If a, b, c are elements in a ring R such that

$$ab + ubc + vca = 0, \text{ or } a^2 + (u+v)bc = 0, \text{ or } a^2 + ub^2 + vc^2 = 0,$$

where u, v are suitable units in R, then (a, b, c) is sectionable.

(3) If $(b_1, \ldots, b_r)$ and $(b_{r+1}, \ldots, b_n)$ are both (pre)sectionable, then so is the concatenated sequence $(b_1, \ldots, b_r, \ldots, b_n)$.

(4) If $b_1^2 = 0$, then any sequence $(b_1, b_2, \ldots, b_n)$ is sectionable. This follows easily from (1) and (3) above, together with (5.2).

(5) Let $Q \in \mathrm{GL}_n(R)$. Then $(b_1, \ldots, b_n)$ *is (pre)sectionable iff*

$$(b'_1, \ldots, b'_n) := (b_1, \ldots, b_n)\, Q$$

is (pre)sectionable. It suffices to prove the "only if" part. We first work with the presectionable case. Say $\sum b_i R \subseteq \sum c_i R$, with $\sum b_i c_i = 0$. Then, for

$$(c'_1, \ldots, c'_n) := (c_1, \ldots, c_n)\, (Q^{-1})^t\,,$$

we have $\sum b'_i R = \sum b_i R \subseteq \sum c_i R = \sum c'_i R$, and

$$(b'_1, \ldots, b'_n) \cdot (c'_1, \ldots, c'_n)^t = (b_1, \ldots, b_n)\, Q \cdot Q^{-1}\, (c_1, \ldots, c_n)^t = 0,$$

so $(b'_1, \ldots, b'_n)$ is presectionable, as claimed. The sectionable case follows from the same argument, with the inclusion signs above replaced by equality signs.

(6) If $b_1 \in b_2 R + \cdots + b_n R$, then $(b_1, \ldots, b_n)$ is sectionable. (In particular, this conclusion always holds if $(b_2, \ldots, b_n) \in \mathrm{Um}_{n-1}(R)$.) This follows from (4) and (5), since we can perform an elementary transformation on $(b_1, \ldots, b_n)$ to change it into $(0, b_2, \ldots, b_n)$.

(7) Here is a somewhat more sophisticated example. Over the ring $R = \mathbb{Z}[t]$, the sequence $\beta = \big(2t+2,\ 4t,\ t^2+3\big)$, which first appeared in (II.6.13), turns out to be sectionable. The sectionability can be confirmed directly by verifying the equation

$$(*) \qquad (2t+2)R + 4tR + (t^2+3)R = (-t^2-2t-9)R + 4R + (2t+6)R,$$

and checking that the two rows of ideal generators here have dot product zero. We'll leave it to the reader to ponder how an equation such as $(*)$ could have been gotten, with the little hint that this has to do with the computation of the syzygy module $P(\beta)$ presented earlier in (II.6.13).

(8) For any elements $x, y \in R$, the sequence $\beta = (x(x-1),\ xy,\ y(y-1))$ is always sectionable. This follows from the equation

$(**)$

$$x(x-1)R + xyR + y(y-1)R = y(1-y)R + xy(y-1)R + x(x-1-xy)R,$$

where the two rows of ideal generators have dot product zero. This equation is, again, derived by working with the two special vectors $u = (y,\ 1-x,\ 0)^t$ and $v = (0,\ 1-y,\ x)^t$ in $P(\beta)$ (see II.6.11), although we are not claiming that u, v generate the syzygy module $P(\beta)$.

(9) Let R be a (commutative) K-Hermite ring, in the sense of I.4.23. (E.g. R can be a Bézout domain, a PIR, or a von Neumann regular ring: see I.4.28, I.4.30, and I.3.34.) Then, *for $n > 1$, any $\beta = (b_1, \ldots, b_n) \in R^n$ is sectionable.* To see this, recall from (I.4.24)(2) that we can transform β into $(b, 0, \ldots, 0)$ (for some $b \in R$) by the right action of $\mathrm{GL}_n(R)$ on R^n. If $n > 1$, $(b, 0, \ldots, 0)$ is sectionable by (4). Thus, by (5), $\beta = (b_1, \ldots, b_n) \in R^n$ is also sectionable. Another way to see this, independently of (4) and (5) above, is as follows. Apply (I.4.25) to write

$$(b_1, \ldots, b_n) = d \cdot (e_1, \ldots, e_n), \ \text{ where } d \in R, \ \text{ and } (e_1, \ldots, e_n) \in \mathrm{Um}_n(R).$$

This implies $\sum_i b_i R = dR$. By I.4.26, R is also a Hermite ring, so the stably free module defined by $(e_1, \ldots, e_n)$ is free. In particular, it has a unimodular element (since $n > 1$). Thus, there exists $(y_1, \ldots, y_n) \in \mathrm{Um}_n(R)$ such that $\sum e_i\, y_i = 0$. We have then $\sum b_i R = dR = \sum dy_i R$, with

$$\sum b_i dy_i = \sum (de_i)(dy_i) = d^2 \sum e_i\, y_i = 0,$$

so $(b_1, \ldots, b_n)$ is sectionable.

(10) (Specialization) If $(b_1, \ldots, b_n)$ in a ring A is (pre)sectionable, then for any ring homomorphism $f : A \to R$, $(f(b_1), \ldots, f(b_n))$ is (pre)sectionable in R. (Thus, for example, (7) above would have worked if we had replaced $\mathbb{Z}[t]$ by any commutative ring R with a given element $t \in R$.)

The point of introducing sectionable and presectionable sequences is that these new notions provide a larger and more general framework in which to study unimodular rows and unimodular elements in the syzygy modules defined by them. As is often the case, the extra degree of generality gives us a somewhat broader perspective. The next two theorems, (5.4) and (5.6) below, are typical results that demonstrate the utility of these new notions.

Theorem 5.4. *Let $a, b, c \in R$ be such that either $bc \in a^2R$ or $a^2 \in bcR$. Then the sequence (a, b, c) is sectionable. If (a, b, c) happens to be unimodular, then it is completable.*

Proof. In the case where $bc \in a^2R$, write $bc = a^2x$, where $x \in R$. It is easy to check the ideal equation

$$aR + bR + cR = (-b - c - ax)R + (a + c)R + aR.$$

Here, the two rows of ideal generators have dot product

$$a(-b - c - ax) + b(a + c) + ca = -a^2x + bc = 0,$$

so (a, b, c) is sectionable, as claimed. In the case where $a^2 \in bcR$, write $a^2 = bcy$ (with $y \in R$). Here, the ideal equation

$$aR + bR + cR = (-a - b - c)R + (a - c + cy)R + (a + b)R$$

does the corresponding trick, as

$$a(-a - b - c) + b(a - c + yc) + c(a + b) = -a^2 + bcy = 0.$$

Note that the conclusions obtained so far are truly results about sectionable sequences: they made no appeal to any assumptions on unimodularity. Of course, in the case where (a, b, c) happens to be unimodular, once the sectionability of (a, b, c) is proven, its completability follows from I.4.12(3). $\square$

If $(a, b, c) \in \mathrm{Um}_3(R)$, writing $bc = a^2x$ or $a^2 = bcy$ and taking a unimodularity equation $aa' + bb' + cc' = 1$, we can use the steps in the above proof to *explicitly construct* an invertible matrix involving a, b, c, a', b', c' (and x or y) with a first row (a, b, c). We will not do this here, as the end result is not a very pretty sight.

Corollary 5.5. *Any sequence of the form* $(ab,\ a^2r,\ b^2s)$ *or* $(abt,\ a^2,\ b^2)$ *is sectionable. In particular, unimodular rows in these forms are always completable.*

We now come to the result that led us to the introduction of Def. (5.1). This result was first observed in a crucial special case by Ravi Rao and Raja Sridharan, and subsequently slightly extended in the following way by the author.

Theorem 5.6. *Let R be a commutative ring, and let P be the type* 1 *stably free R-module defined by* $(b_1, \dots, b_n) \in \mathrm{Um}_n(R)$. *Let* $r \in [1,\ n]$ *be such that either*

(1) $n - r$ *is odd and* $(b_{r+1}, \dots, b_n)$ *is presectionable over* R*; or*

(2) $n - r$ *is even,* $r > 1$*, and* $(b_1, \dots, b_r)$ *is completable over the factor ring* $R/(b_{r+1}R + \cdots + b_nR)$.

Then P has a unimodular element.

Proof. First assume $n - r$ is odd. In this case, we claim more generally that:

> *As long as* $(b_{r+1}, \dots, b_n)$ *is (pre)sectionable (without any unimodularity assumptions), then so is* $(b_1, \dots, b_r, \dots, b_n)$.

(This is clearly sufficient to prove the theorem in the Case (1).) If n is even, the claim follows from (5.2). If n is odd, then r is even, and so $(b_1, \dots, b_r)$ is sectionable (again by (5.2)). Using (5.3)(3), we see that $(b_1, \dots, b_r, \dots, b_n)$ is (pre)sectionable, as claimed.

Now assume we are in Case (2). In this case, there exists a matrix $M \in \mathbb{M}_r(R)$ with first row $(b_1, \dots, b_r)$ whose determinant reduces to a unit modulo $b_{r+1}R + \cdots + b_nR$. Since $r > 1$, we may assume that $\det(M)$ reduces to 1 modulo $b_{r+1}R + \cdots + b_nR$. Let $(x_1, \dots, x_r)$ be the second row of M, and let $c_1, \dots, c_r$ be the cofactors of the matrix M computed along the elements x_i in the second row. Then we have an equation

$$\text{(A)} \qquad \det(M) = x_1c_1 + \cdots + x_rc_r = 1 + b_{r+1}x_{r+1} + \cdots + b_nx_n$$

for suitable $x_{r+1}, \dots, x_n \in R$. Also, by determinant theory, $b_1c_1 + \cdots + b_rc_r = 0$. Since $n - r$ is *even*, we can augment this equation into

$$\text{(B)} \qquad 0 = b_1c_1 + \cdots + b_rc_r + b_{r+1}(b_{r+2}) + b_{r+2}(-b_{r+1}) + \cdots .$$

But by (A), $(c_1, \dots, c_r, b_{r+2}, -b_{r+1}, \dots) \in \mathrm{Um}_n(R)$. Thus, (B) (together with I.4.12(1)) implies that P has a unimodular element. ∎

Remark 5.7. Case (2) above is definitely *not true* if $r = 1$, as is shown by the following example. Let $n > 1$ be an odd integer, and consider the unimodular sequence

$(x_1, \ldots, x_n)$ over the real coordinate ring of the even sphere S^{n-1}; that is, the ring $R = \mathbb{R}[x_1, \ldots, x_n]$ with the relation $x_1^2 + \cdots + x_n^2 = 1$. The singleton (x_1) is certainly completable over the factor ring $R/(x_2 R + \cdots + x_n R)$. However, we have pointed out before (after the statement of (I.4.16)) that the stably free module defined by $(x_1, \ldots, x_n)$ over R is *indecomposable.* Thus, the conclusion of (5.6)(2) no longer holds. (Of course, the proof of (5.6) in this case made explicit use of $r > 1$.)

Corollary 5.8. *For any $r > 0$, the stably free module defined by any unimodular sequence*

$$\left(a_0^{k_0},\, a_1^{k_1},\, \ldots,\, a_r^{k_r},\, c_1,\, c_2,\, \ldots,\, c_{2s}\right) \quad (k_0,\, \ldots,\, k_r > 0)$$

with $r! \mid k_0 \cdots k_r$ has a unimodular element. For instance, the stably free module defined by any unimodular sequence

$$\left(a_0,\, a_1,\, a_2^2,\, \ldots,\, a_r^r,\, c_1,\, c_2,\, \ldots,\, c_{2s}\right)$$

always has a unimodular element.

Proof. By Suslin's $r!$ theorem, $r! \mid k_0 \cdots k_r$ implies that $\left(a_0^{k_0},\, a_1^{k_1},\, \ldots,\, a_r^{k_r}\right)$ is completable over any commutative ring S with respect to which it is unimodular. Now take

$$S = R/(b_{r+1}R + \cdots + b_n R),$$

and apply (5.6)(2). □

The case $r = 2$ of the last statement in (5.8) was the original observation of Rao and Sridharan. A proof for this case has also appeared in the paper [van der Kallen: 2002] listed in the references on Chapter VIII. Note that the conclusion here is that $P(a_0, a_1, a_2^2, c_1, \ldots, c_{2s})$ has a unimodular element — not that it is free. The freeness will follow if $P(a_0, a_1, a_2, c_1, \ldots, c_{2s})$ itself is assumed to be free: this will be proved later in VII.5.32.

In (5.3), we have given many classes of examples of sectionable sequences of odd length. But surely the reader would expect that, "generically", odd-length sequences should *not* be sectionable. Given what we have already written, the quickest way to see this is to exploit again the example of even spheres in (5.7). Let A be the polynomial ring $\mathbb{R}[x_1, \ldots, x_n]$ where n is odd. By specializing A to the ring R in (5.7) and using (5.3)(10), we see that $(x_1, \ldots, x_n)$ is not sectionable over A. Here, A is a Hermite domain. Thus, we also see that the result (5.3)(9) for K-Hermite rings does not extend to Hermite rings — or even Hermite domains.

As it turned out, the topology used in the work on (5.7) above can be altogether avoided. We shall next sketch some results of a purely algebraic nature that will also imply the non-sectionability of the generic odd-length sequence — and quite a bit more. We start by first recalling the following basic definition from commutative algebra.

Definition 5.9. A finite (ordered) sequence $b_1, \dots, b_n$ in a commutative ring R is called a *regular sequence* (or alternatively, an R*-sequence*) if $\sum b_i R \neq R$ and for each i, b_i is not a 0-divisor in the factor ring $R/(b_1 R + \cdots + b_{i-1} R)$.

The following result, to appear in [Lam-Swan: 2006], gives a quick determination of all *regular* sequences that are sectionable or presectionable, as follows.

Theorem 5.10. *Over a nonzero commutative ring R, a regular sequence $b_1, \dots, b_n$ is presectionable iff it is sectionable, iff n is even.*

This may appear to contradict (5.3)(9) since the latter says that in a K-Hermite ring, *any* finite sequence of length > 1 is sectionable. But of course there is no contradiction, since for a K-Hermite ring, it is easy to see that the only regular sequences are the singletons given by nonunit non 0-divisors, and these are not sectionable by (5.3)(1).

Corollary 5.11. *Let $R = S[x_0, \dots, x_n]$, where n is even and S is a nonzero commutative ring. For any positive integers $k_0, \dots, k_n$, the (odd-length) monomial sequence $\left(x_0^{k_0}, \dots, x_n^{k_n}\right)$ is not presectionable over R.*

Proof. Using $S \neq 0$, we check easily that $x_0^{k_0}, \dots, x_n^{k_n}$ is a regular sequence in the polynomial ring R. Since it has odd length $n+1$, (5.10) implies that this sequence is not presectionable. □

While (5.11) shows that the generic odd-length sequence $(x_0, \dots, x_n)$ is not presectionable, it also shows, for example, that the generic odd-length "factorial row"

$$\left(x_0, x_1, x_2^2, \dots, x_n^n\right)$$

is not presectionable. This contrasts with the fact that any *unimodular* factorial row is (completable and hence) sectionable. What this means is that Suslin's $n!$ Theorem and the Swan-Towber Theorem are special results holding only for unimodular rows, and not for arbitrary sequences in general.

In the following, we shall present some of the basic facts needed for the proof of (5.10). For the full details of this proof, we refer the reader to [Lam-Swan: 2006]. The proposition below will also turn out to be very useful later in Chapter VII. As usual, R denotes a commutative ring.

Proposition 5.12. *Let $M \in \mathbb{M}_n(R)$ be an alternating matrix (that is, M is skew-symmetric with a zero diagonal). If n is odd, then* $\det M = 0$.

Proof. First consider the "generic" $n \times n$ alternating matrix

$$X = \begin{pmatrix} 0 & x_{12} & x_{13} & \cdots \\ -x_{12} & 0 & x_{23} & \cdots \\ -x_{13} & -x_{23} & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

over $\mathbb{Z}[x_{12}, x_{13}, \ldots, x_{n-1,n}]$, where x_{ij} are independent (commuting) indeterminates over $\mathbb{Z}$. If n is odd, taking determinants on the equation $X^t = -X$ yields

$$\det X = (-1)^n \det X = -\det X.$$

Thus, $\det X = 0$. By specialization, the same result follows for our alternating matrix M over R. $\square$

For *any* row $\beta = (b_1, \ldots, b_n)$, recall that $P(\beta)$ denotes the solution space of β; that is, the kernel of the R-homomorphism $R^n \to R$ defined by β. As we have noted in (II.6), $P(\beta)$ is the (first) *syzygy module* of the ideal generated by the b_i's. In the case where β is unimodular, we have obtained a nice description of $P(\beta)$ in terms of the "Koszul elements" in I.9.12(1). In the proposition below, we shall obtain the exact analogue of the last part of I.9.12(1) for *regular* sequences β.

Proposition 5.13. *Let $\beta = (b_1, \ldots, b_n)$ be a regular sequence in R. Then $P(\beta)$ is generated as a (left) R-module by the Koszul elements*

$$\gamma_{ij} := b_j e_i - b_i e_j \in P(\beta) \quad (i < j),$$

where $\{e_i\}$ are the unit vectors in R^n.

Proof. This Proposition amounts essentially to the exactness of the beginning part of the Koszul resolution associated with the ideal B generated by the regular sequence β. With $\varphi : R^n \to B$ defined by $e_i \mapsto b_i$, the Koszul resolution

$$\cdots \longrightarrow \Lambda^2(R^n) \xrightarrow{\psi} R^n \xrightarrow{\varphi} B \longrightarrow 0$$

gives $P(\beta) = \ker(\varphi) = \mathrm{im}(\psi)$. Here, by definition, $\psi(e_i \wedge e_j) = \gamma_{ij}$, so $\mathrm{im}(\psi)$ is precisely the R-submodule of R^n generated by the Koszul elements γ_{ij}.

To make our exposition self-contained, we include here a direct proof of (5.13), using ideas similar to those in the proof of I.9.11(1). Let $\alpha = (a_1, \ldots, a_n) \in P(\beta)$. Then $b_n a_n = -(b_1 a_1 + \cdots + b_{n-1} a_{n-1})$ implies that

$$a_n = x_1 b_1 + \cdots + x_{n-1} b_{n-1}$$

for suitable x_i's. Then $\alpha' := \alpha + \sum_{i<n} x_i \gamma_{in} \in P(\beta)$ has last coordinate zero. Invoking an inductive hypothesis on the vector consisting of the first $n-1$ coordinates of α' (which is orthogonal to the shorter R-sequence $(b_1, \ldots, b_{n-1})$), we can then express α as a linear combination of the γ_{ij}'s. $\square$

Remark 5.14. Note that, without the regular sequence assumption on β, the conclusion in (5.13) may not hold. For instance, let R be the commutative local ring $\mathbb{Q}[x, y]$ with the relations $x^2 = y^2 = xy = 0$. Here, x, y generate the unique maximal ideal $\mathfrak{m}$ of R, but $\beta := (x, y)$ is *not* a regular sequence. Contrary to (5.13), *the Koszul element $\gamma_{12} = (y, -x)$ fails to generate $P(\beta)$*. In fact, $(y, 0) \in P(\beta)$ cannot be of the form $r\,(y,-x)$ for $r\in R$. Otherwise, $rx=0$ implies $r\in\mathfrak{m}$, and $y=ry$ implies $(1-r)y=0$ and hence $y=0$, a contradiction.

While the ring R above has many 0-divisors, it is equally easy to give a ("more legitimate") example where R is a domain. For instance, consider the sequence $\beta=(x^2,\ xy,\ y^2)$ in $R=\mathbb{Q}[x,y]$. Here, the syzygy module $P(\beta)$ contains $(y,-x,0)^T$, which is not an R-linear combination of the γ_{ij}'s since the coordinates of the latter are (zero or) monomials of degree 2.

Assuming the two propositions (5.12) and (5.13), it is not difficult to prove Theorem 5.10. For the details, we refer the reader to [Lam-Swan: 2006]. The following two examples are easy applications of (5.10).

Example 5.15. Let $R=k[x,y]$ where k is a field, and let

$$\beta=\big(1+x,\ 1-y,\ x(1+y)\big)\in R^3.$$

Then β *is sectionable.* [By I.5.14 and (5.3)(5) above, we may replace β by $\beta'=(1+x,\ 1-y,\ -2)$. If $\mathrm{char}(k)=2$, then by (5.3)(4), β' is sectionable. If $\mathrm{char}(k)\neq 2$, then by I.5.14, β' is unimodular and completable, and in particular sectionable.] On the other hand, if we take k to be, say, the ring $\mathbb{Z}$, then β is easily seen to be a regular sequence in R. In this case, (5.10) implies that β is not even presectionable.

Example 5.16. Using (5.10), we can construct some examples of nonsectionable sequences that are neither unimodular nor regular sequences. For instance, let A be the k-algebra generated by x,y,z,x',y',z' with the relation $xx'+yy'+zz'=0$, where k is a field. Then (x,y,z) is neither unimodular nor a regular sequence over A. It is not presectionable since we can specialize A to $R=k[x,y,z]$ by fixing x,y,z and mapping x',y',z' to zero. Since (x,y,z) is a regular sequence over R, it is not presectionable over R by (5.10), and therefore not presectionable over A by (5.3)(10).

We close this section with the following relevant observation.

Remark 5.17. For sequences of length $\geqslant 3$, presectionability is indeed a notion that is *weaker* than sectionability. For instance, let $R=k[x,y]$, where k is any field. Then $(x^3,\ xy^2,\ y^3)$ is presectionable, since

$$x^3R+xy^2R+y^3R\subseteq 0R+yR+(-x)R,$$

with $x^3\cdot 0+xy^2\cdot y+y^3\cdot(-x)=0$. However, it is shown in [Lam-Swan: 2006] that $(x^3,\ xy^2,\ y^3)$ is *not* a sectionable row in R. In this connection, note that, among monomial sequences in $k[x,y]$, this choice of a counterexample is optimal, in the sense that, if we decrease *any* of the exponents in x^3, xy^2 or y^3, the resulting sequence will become sectionable. This can be checked easily by applying (5.3)(6) and (5.4).

§6. Self-Duality of Stably Free Modules

Throughout this section, R denotes a commutative ring. If P is a f.g. projective right R-module, the dual module $P^* = \mathrm{Hom}_R(P, R)$ is a *left* R-module, but it may also be viewed as a right R-module if R is commutative. This makes it possible for us to "compare" the two modules P and P^* in the same module category. For instance, given a f.g. stably free module P, we may ask if P is *self-dual*; that is, if $P \cong P^*$. This would be a weaker question than asking P to be free, since P cannot be free without first being self-dual (upon recalling that $(R^n)^* \cong (R^*)^n \cong R^n$ over a commutative ring R).

Some facts pertinent to the duality question above will be given in this section. We'll be mainly working with stably free modules of "low type", so this section will also serve as a natural place for us to collect various useful facts on such stably free modules.

Recall that, for a subgroup G in $\mathrm{GL}_n(R)$, the notation $\beta \sim_G \gamma$ for unimodular rows $\beta,\ \gamma \in \mathrm{Um}_n(R)$ means that these rows are conjugate under the right G-action on $\mathrm{Um}_n(R)$; that is, $\gamma = \beta \cdot \sigma$ for some $\sigma \in G$. In this section, we'll be especially interested in the case $G = \mathrm{E}_n(R)$ (the group of $n \times n$ elementary matrices), where $n \geqslant 3$. We first record some interesting consequences of (I.9.12) on the action of this group on $\mathrm{Um}_n(R)$ that are due to Suslin, Vaserstein and Roitman.

Proposition 6.1. *Let $G = \mathrm{E}_n(R)$, where R is a commutative ring, and $n \geqslant 3$.*

(1) *Suppose $\beta,\ \gamma_1,\ \gamma_2 \in \mathbb{M}_{1,n}(R)$ are such that $\gamma_1\, \beta^t = \gamma_2\, \beta^t = 1$. Then $\gamma_1 \sim_G \gamma_2$.*

(2) *If $\gamma_1,\ \gamma_2$ are two different rows of a matrix $\sigma \in \mathrm{GL}_n(R)$, then $\gamma_1 \sim_G \gamma_2$.*

(3) *Let $\gamma = (c_1, \ldots, c_n) \in \mathrm{Um}_n(R)$, and $b_1, b_2 \in R$.*

(a) *If $c_1 b_1 + c_2 b_2$ is a unit modulo $c_3 R + \cdots + c_n R$, then*

$$\gamma \sim_G (b_1, b_2, c_3, \ldots, c_n).$$

(b) *If $u \in R$ is a unit modulo $c_3 R + \cdots + c_n R$, then $\gamma \sim_G (u^2 c_1,\ c_2,\ \ldots,\ c_n)$.*

(c) *Let $u \in \mathrm{U}(R)$. Then $u\gamma \sim_G \gamma$ if n is even, and $u\gamma \sim_G (uc_1, c_2, \ldots, c_n)$ if n is odd.*

Proof. (1) Note that the hypotheses imply that

$$\gamma_1,\ \gamma_2,\ \beta \in \mathrm{Um}_n(R), \quad \text{and} \quad (\gamma_2 - \gamma_1)\, \beta^t = 0.$$

Since $n \geqslant 3$, I.9.12(2) (with $J = R$) gives

$$E := I_n + \beta^t\, (\gamma_2 - \gamma_1) \in \mathrm{E}_n(R).$$

We have $\gamma_1 \cdot E = \gamma_1 + (\gamma_1 \beta^t)(\gamma_2 - \gamma_1) = \gamma_1 + (\gamma_2 - \gamma_1) = \gamma_2$, so $\gamma_1 \sim_G \gamma_2$.

(2) We may assume that $\gamma_1,\ \gamma_2$ are the first two rows of σ. Let β be the transpose of the sum of the first two columns of σ^{-1}. Clearly, $\gamma_1\, \beta^t = \gamma_2\, \beta^t = 1$, so (1) gives $\gamma_1 \sim_G \gamma_2$.

(3a) Fix an equation $x(c_1b_1+c_2b_2)+c_3x_3+\cdots+c_nx_n=1$ $(x,\ x_i\in R)$, and let

$$\gamma':=(b_2,\ -b_1,\ c_3,\ldots,\ c_n),\qquad \beta=(x(b_1+c_2),\ x(b_2-c_1),\ x_3,\ldots,\ x_n).$$

Direct computations show that $\gamma\,\beta^t=\gamma'\beta^t=1$. Applying (1) and (I.5.0)(2), we get

$$\gamma\sim_G\gamma'=(b_2,-b_1,c_3,\ldots,c_n)\sim_G(b_1,b_2,c_3,\ldots,c_n).$$

(3b) Fix an equation $c_1x_1+\cdots+c_nx_n=1$. Applying (3a) above twice, we get

$$(\dagger)\qquad \gamma\sim_G(x_1,\ x_2,\ c_3,\ \ldots,\ c_n)\sim_G(uc_1,\ uc_2,\ c_3,\ \ldots,\ c_n).$$

Say $uv\equiv 1\ (\mathrm{mod}\ A)$, where $A=c_3R+\cdots+c_nR$. Over $\overline{R}:=R/A$, Whitehead's Lemma (I.5.1) gives $\mathrm{diag}(u,\ v)\in\mathrm{E}_2(\overline{R})$. Therefore, over $\overline{R}$, right multiplication by $\mathrm{diag}(u,\ v)$ gives

$$(uc_1,\ uc_2)\sim_{\mathrm{E}_2(\overline{R})}(u^2c_1,\ c_2).$$

By the proof of (I.5.6), we can "pull back" this equivalence to get

$$(uc_1,\ uc_2,\ c_3,\ \ldots,\ c_n)\sim_G(u^2c_1,\ c_2,\ c_3,\ \ldots,\ c_n).$$

Combining this with (†) gives what we want.

(3c) The case $n=2m$ follows by applying (†) m times. In the case $n=2m+1$, applying (†) m times gives

$$(c_1,\ c_2,\ \ldots,\ c_n)\sim_G(c_1,\ uc_2,\ uc_3,\ \ldots,\ uc_n).$$

Replacing c_1 by uc_1 gives the desired conclusion. □

Remark 6.2. (A) Note that if we replace the group $G=\mathrm{E}_n(R)$ by the larger group $H=\mathrm{GL}_n(R)$, the results (1) and (2) become much easier. In fact, *two* alternative arguments can be given for (1). The first one is that given in the proof for (I.4.10)(2) (using dual modules). The second is simply to observe that the matrix E used in the proof of (1) above is *unipotent* (that is, identity + nilpotent), so obviously $E\in H$, which gives directly $\gamma_1\sim_H\gamma_2$. Similarly, in the context of (2), the relation $\gamma_1\sim_H\gamma_2$ is trivial, since each γ_i (being completable) is H-conjugate to $(1,0,\ldots,0)$.

(B) We note also that, in (3c) above, if n is *odd,* the unimodular rows $u\gamma\sim_G(uc_1,c_2,\ldots,c_n)$ are not G-equivalent to γ in general (although, of course, they are all equivalent under $\mathrm{GL}_n(R)$). For instance, if $R=\mathbb{R}[x_1,\ldots,x_n]$ with the relation $x_1^2+\cdots+x_n^2=1$ (where n is odd), Vaserstein has pointed out that the unimodular row $\gamma=(x_1,\ldots,x_n)$ is *not* G-equivalent to $-\gamma$.

Applying (6.1) to the stably free modules defined by unimodular sequences, we get the following result, where again $G=\mathrm{E}_n(R)$.

Proposition 6.3. *Let $\beta=(b_1,\ldots,b_n)$ and $\gamma=(c_1,\ldots,c_n)$ be such that $\beta\gamma^t\in\mathrm{U}(R)$, and let P, P' be the stably free modules (of type 1) defined by β, γ respectively.*

(1) *If* n *is even and* $\geqslant 4$*, then* $\beta \sim_G \gamma$*. In particular,* $P \cong P'$*.*

(2) *If* n *is odd and* $\geqslant 3$*, then* $(\beta,\ 0) \sim_{\mathrm{E}_{n+1}(R)} (\gamma,\ 0)$*. In particular,* $P \oplus R \cong P' \oplus R$*.*

Proof. (1) Fixing an equation $x(c_1b_1 + \cdots + c_nb_n) = 1$ where $x \in \mathrm{U}(R)$, we simply apply (6.1)(3a) and change two of the c_i 's at a time until all entries are converted to the b_i 's. For instance, if $n = 4$, then using the above equation twice, we get

$$(c_1, c_2, c_3, c_4) \sim_G (b_1, b_2, c_3, c_4) \sim_G (b_1, b_2, b_3, b_4).$$

After we get $\beta \sim_G \gamma$, (I.4.8) gives $P \cong P'$.

(2) follows by applying (1) to $(\beta,\ 0)$, $(\gamma,\ 0) \in \mathrm{Um}_{n+1}(R)$, which "correspond to" the stably free modules $P \oplus R$ and $P' \oplus R$ respectively. □

The results above strongly suggest that we put a certain graph structure on the set $\mathrm{Um}_n(R)$. Following [Hinson: 1991a] (listed in the references on Chapter VIII), we view the elements of $\mathrm{Um}_n(R)$ as the *vertices* of our graph, and connect two vertices $\beta,\ \gamma$ by an *edge* iff $\beta\gamma^t = 1$; in this case, we write $\beta \leftrightarrow \gamma$. This makes $\mathrm{Um}_n(R)$ into a *graph*, as long as $n \geqslant 2$. Actually, *pseudo-graph* would have been a more proper name for this new structure, since we do have some vertices that are connected to themselves by an edge, namely, those unimodular rows that have "squared length" 1. To simplify the terminology, however, we'll just use here the more traditional term "graph" for this structure.

With this graph structure in place, we can talk about *paths* and *path components* on $\mathrm{Um}_n(R)$. (It is understood that, along a path, two adjacent vertices could be the same.) The following is an interesting result of Hinson relating the $\mathrm{E}_n(R)$-orbits in $\mathrm{Um}_n(R)$ to the path components, in case $n \geqslant 3$.

Theorem 6.4. *Let* $G = \mathrm{E}_n(R)$*, where* $n \geqslant 3$*. Then* $\beta \sim_G \beta'$ *iff* β *is connected to* β' *by a path of even length.*

Proof. The "if" part follows from (6.1)(1). For the "only if" part, it suffices to show that, if β' is obtained from β by a *single* elementary transformation, we can connect one to the other by a path of length 4. Let's say

$$\beta = (b_1, \ldots, b_n), \quad \text{and} \quad \beta' = (b_1 + rb_3,\ b_2, b_3, \ldots, b_n).$$

Fixing an equation $b_1c_1 + \cdots + b_nc_n = 1$, we check directly that

$$\begin{aligned} \beta &\leftrightarrow (c_1,\ \ldots,\ c_n) \\ &\leftrightarrow (b_1 - c_2,\ b_2 + c_1,\ b_3,\ \ldots,\ b_n) \\ &\leftrightarrow (c_1,\ c_2 + rb_3,\ c_3 - r(b_2 + c_1),\ c_4,\ \ldots,\ c_n) \\ &\leftrightarrow (b_1 + rb_3,\ b_2,\ b_3, \ldots,\ b_n). \end{aligned}$$

Note that there is no problem about the intermediate rows belonging to $\mathrm{Um}_n(R)$, since any adjacent pair of rows here have dot product equal to 1. □

For $\beta \in \mathrm{Um}_n(R)$, let us write $[\beta]$ and $\langle\beta\rangle$, respectively, for the $\mathrm{E}_n(R)$-orbit and the path component of β. With these notations, and recalling (6.3)(1), we have:

Theorem 6.5. *Let $n \geqslant 3$. If $\beta,\ \gamma \in \mathrm{Um}_n(R)$ are such that $\beta\gamma^t = 1$, then*

$$\langle\beta\rangle = [\beta] \cup [\gamma].$$

We have $\langle\beta\rangle = [\beta]$ iff there exists $\sigma \in \mathrm{E}_n(R)$ such that $\beta\sigma\beta^t = 1$. This always holds if n is even.

The following is a nice (and very natural) example for $\langle\beta\rangle = [\beta]$, pointed out by Hinson. The remarkable thing about this example is that it works without assuming $n \geqslant 3$.

Proposition 6.6. *Let $n \geqslant 2$. Then $\langle e_1\rangle = [e_1]$, where $e_1 = (1, 0, \dots, 0)$.*

Proof. This follows trivially from (6.5) if $n \geqslant 3$, since we can take $\beta = \gamma = e_1$ in (6.5). This would, however, necessitate a separate argument for $n = 2$. To proceed more uniformly, let us prove the desired result *directly for all* $n \geqslant 2$.

The first step is to prove $\langle e_1\rangle \subseteq [e_1]$. Suppose α can be reached by a path from e_1. If β is the vertex "preceding" α on this path, we may assume that $\beta \in [e_1]$, so there is a matrix $B \in \mathrm{E}_n(R)$ with first row β. Then $\alpha\, B^t = (1, c_2, \dots, c_n)$ for some c_i 's. There certainly exists a matrix $C \in \mathrm{E}_n(R)$ with first row $(1, c_2, \dots, c_n)$. From $\alpha\, B^t = e_1 C$, we have $\alpha = e_1 C(B^t)^{-1} \in [e_1]$, as desired.

It remains to prove that $[e_1] \subseteq \langle e_1\rangle$. For any $M \in \mathrm{GL}_n(R)$, let M_1 denote the first row of M. We claim that:

(†) *For any* $E \in \mathrm{E}_n(R)$, $\langle (EM)_1\rangle = \langle M_1\rangle$.

If this is true, we'll get (upon taking $M = I_n$) $\langle E_1\rangle = \langle e_1\rangle$. Now for $E \in \mathrm{E}_n(R)$, E_1 is a typical element in $[e_1]$, so we are done. To prove the claim (†), it suffices to check the case where $E = I_n + re_{ij}$ ($r \in R$, and $i \neq j$). If $i > 1$, there is nothing to check, so assume $i = 1$. Let $\beta,\ \beta'$ be the first and j th rows of M. *We need to check that β is path-connected to $\beta + r\beta'$.* Let γ be the transpose of the first column of M^{-1}. Then

$$\beta\,\gamma^t = 1, \quad \text{and} \quad (\beta + r\beta')\,\gamma^t = \beta\,\gamma^t = 1.$$

This gives $\beta \leftrightarrow \gamma \leftrightarrow \beta + r\beta'$, which is what we want. □

Remark. Of course, there is nothing special about the choice of e_1 in (6.6). In fact, we have $\langle e_i\rangle = \langle e_1\rangle = [e_1] = [e_i]$ for all i. The easy proof of this is left to the reader.

Much more information about the graph structure on $\mathrm{Um}_n(R)$ (e.g. its diameter, "size" of its path components, and relations to the arithmetic of the ring R) can be found in [Hinson: 1991a]. However, our main goal in this section is to study duality issues of stably free modules, so let us return to this topic now. Bringing (6.3) to bear and recalling the crucial duality fact (I.4.10), we get our first positive result on the self-duality of stably free modules..

Theorem 6.7. *Let P be a f.g. stably free R-module.*

(1) *If P has type 1 and odd rank, then P is self-dual; that is, $P \cong P^*$.*

(2) *If P has type 2 and even rank, then $P \oplus R \cong P^* \oplus R$.*

Proof. (1) In this case, $P \oplus R \cong R^n$ for some *even* integer n. If $n = 2$, (I.4.11) implies that $P \cong R$, which is self-dual. Thus, we may assume that $n \geqslant 4$. Think of P as the kernel of a surjection $R^n \to R$ defined by some $\beta \in \mathrm{Um}_n(R)$, and let γ be such that $\beta\gamma^t = 1$. This γ defines in turn a surjection $R^n \to R$ whose kernel P' is isomorphic to P^* by (I.4.10)(1). But by (6.3)(1), $P \cong P'$, and hence $P \cong P^*$.

(2) In this case, $P_1 := P \oplus R$ is stably free of type 1 and odd rank, so by (1), we have $P_1 \cong P_1^*$. Since $R^* \cong R$, this gives $P \oplus R \cong P^* \oplus R$. □

A second result on self-duality, also in the positive direction, comes from [Bass: 1969].

Theorem 6.8. *Any f.g. stably free module P of rank 2 is self-dual.*

This theorem was not explicitly stated in [Bass: 1969], but it follows readily from the results in that paper. The proof of (6.8) depends on showing that the module P there supports a symplectic structure. Since symplectic structures will only be treated much later (in Chapter VII), we shall postpone this proof to VII.5 (see (VII.5.18)). However, in order to complete our discussions on duality issues in this section, it will be convenient for us to assume the truth of (6.8) here.

The special case of (6.8) for type 1 modules is worth stating separately, since (in view of (I.4.8) and (I.4.10)) it amounts to the following remarkable statement on unimodular rows of length 3.

Corollary 6.9. *If $b_1c_1 + b_2c_2 + b_3c_3 = 1$ over a commutative ring R, then*

$$(b_1, b_2, b_3) \sim_{\mathrm{GL}_3(R)} (c_1, c_2, c_3).$$

Remark. I asked Professor R.G. Swan if an explicit matrix in $\mathrm{GL}_3(R)$ can be found that transforms $\beta = (b_1, b_2, b_3)$ into $\gamma = (c_1, c_2, c_3)$. Professor Swan obliged, pointing out that, for the alternating matrices

$$B = \begin{pmatrix} 0 & -b_3 & b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & b_1 & 0 \end{pmatrix}, \quad \text{and} \quad C = \begin{pmatrix} 0 & c_3 & -c_2 \\ -c_3 & 0 & c_1 \\ c_2 & -c_1 & 0 \end{pmatrix},$$

we have $BC = I_3 - \gamma^t\beta$ and $CB = I_3 - \beta^t\gamma$. Since $\beta B = B\beta^t = 0 = \gamma C = C\gamma^t$, it follows that $B + \gamma^t\gamma \in \mathrm{GL}_3(R)$ with inverse $C + \beta^t\beta$, and

$$\beta\,(B+\gamma^t\gamma)=0+(\beta\,\gamma^t)\,\gamma=\gamma,$$

which gives what we want. On the other hand, Professor Swan has also shown, by a topological example, that there may not exist a matrix $E\in \mathrm{E}_3(R)$ such that $\beta\,E=\gamma$. This contrasts with (6.3)(1), where, for n *even* and $\geqslant 4$, we can transform β to γ by some $E\in \mathrm{E}_n(R)$.

While the matrix construction above was carried out and checked directly without recourse to any symplectic techniques, the ultimate source of the construction is the presence of a symplectic structure on the rank 2 stably free module $P(\gamma)$. This can be seen as follows. The automorphism

$$R\cdot\beta^t\oplus P(\gamma)=R^3\xrightarrow{\ \varphi\ }R^3=R\cdot\gamma^t\oplus P(\beta)$$

defined by the matrix $B+\gamma^t\gamma$ takes β^t to γ^t, and takes $P(\gamma)$ isomorphically onto $P(\beta)$. If we think of the second R^3 above as $(R^3)^*$ (by identifying its standard basis with its dual basis), $P(\beta)$ will then be identified with $P(\gamma)^*$. Thus, we have an induced isomorphism $\varphi: P(\gamma)\to P(\gamma)^*$. For any column vector $\delta\in P(\gamma)$, we have

$$\varphi(\delta)(\delta)=\delta^t\,(B+\gamma^t\gamma)\,\delta=\delta^t B\delta=0,$$

since B is an alternating matrix. Thus, the isomorphism $\varphi: P(\gamma)\to P(\gamma)^*$ corresponds to a symplectic structure on the module $P(\gamma)$ (in the sense of (5.0) in Chapter VII).

Summarizing the results on duality so far, if P is a stably free module of type t and rank r, we can say that P is self-dual at least in the following four cases:

(6.10A) $t=0$. (Here, it follows straight from definition that P is free!)

(6.10B) $r\leqslant 1$. (Here, P is free by (I.4.11).)

(6.10C) $r=2$. (Here, P is self-dual by (6.8).)

(6.10D) r *is odd and* $t=1$. (Here, P is self-dual by (6.7)(1).)

As it turned out, these are *all* the cases in which we can infer the self-duality of P — without imposing further assumptions on the module! This was pointed out by R. G. Swan. In the following, we'll briefly explain how this claim can be verified.

Let $R=\mathbb{C}\,[x_0,\dots,x_{2n+1}]$ with the relation $x_0^2+\cdots+x_{2n+1}^2=1$ be the complex coordinate ring of the sphere S^{2n+1}. We had mentioned this ring earlier in §4 in connection with the converse to Suslin's $n!$ Theorem. As in (4.5), we set

$$z_0=x_0+ix_1,\ \ z_1=x_2+ix_3,\ \ \dots\ ,\ \ z_n=x_{2n}+ix_{2n+1}$$

in R. (These are the "complex coordinate functions" when we view S^{2n+1} as embedded in $\mathbb{R}^{2n+2}=\mathbb{C}^{n+1}$.) Since $z_0\bar{z}_0+\cdots+z_n\bar{z}_n=1$, we have $(z_0,\dots,z_n)\in \mathrm{Um}_{n+1}(R)$. Let

$$(6.11) \qquad P = P(z_0^2,\, z_1, \ldots,\, z_n)$$

be the stably free R-module defined by the unimodular row $(z_0^2,\, z_1, \ldots,\, z_n)$ over R. *We assume in the following that n is even and $\geqslant 4$.*

Proposition 6.12. *In the notations above, the stably free R-module P has type* 1 *and even rank $n \geqslant 4$, and it is not self-dual.*

The proof of this, due to R. G. Swan, uses the homotopy theory of the classifying spaces $\mathrm{BU}(n)$ (namely, the fact that $\pi_{2n+1}(\mathrm{BU}(n+1)) = 0$ and $\pi_{2n+1}(\mathrm{BU}(n)) = \mathbb{Z}/n!\,\mathbb{Z}$). Since we have not developed any homotopy theory in this text, we are not in a position to present Swan's proof here. The fact that even rank stably free modules of type 1 need not be self-dual was also noted independently by M. V. Nori. For a purely algebraic approach to this problem (using Mennicke symbols and Suslin matrices), see the lecture notes [Rao: 2004] listed in the references on Chapter VIII.

A word of explanation is in order on the definition of the module P in (6.11). Putting a square term z_0^2 in the first coordinate on the RHS of (6.11) is not essential. We could have used just z_0 and (6.12) would have worked just as well. (Details of the proof of this can be found in [Swan: 2006] listed in the references on Chapter VIII.) However, the choice of the first coordinate z_0^2 will be needed in a substantial way in the proof of the proposition below.

Proposition 6.13. *For the module P in* (6.11)*, there exists a decomposition $P \cong R \oplus Q$ (for some module Q). The module Q is stably free of type* 2 *and odd rank $n-1 \geqslant 3$, and it is not self-dual.*

Proof. The existence of the decomposition $P \cong R \oplus Q$ follows from (5.8). We have

$$R^2 \oplus Q \cong R \oplus P \cong R^{n+1},$$

so Q is stably free of type 2 and odd rank $n-1 \geqslant 3$. If Q was self-dual, then

$$P^* \cong R^* \oplus Q^* \cong R \oplus Q \cong P,$$

contradicting (6.12). Thus, $Q \not\cong Q^*$. □

By easy inspection, the results (6.12) and (6.13) above covered all the cases beyond (6.10A)–(6.10D). Thus, there are no more cases of self-duality for general stably free modules besides those cited in (6.10A)–(6.10D).

§7. A Touch of Suslin Matrices

In connection with his proof of the $n!$ Theorem, Suslin (1977b) introduced a family of $2^n \times 2^n$-matrices over a commutative ring R, which have come to be known as the *Suslin matrices.* The proof we gave for the $n!$ Theorem in §4 did not make use of Suslin's matrices, so we have not introduced them so far. However, since the appearance of [Suslin: 1977b], Suslin's matrices have found various other applications, and have proved to be a valuable tool in studying the orbit spaces of $\mathrm{Um}_n(R)$ under the actions of both $\mathrm{GL}_n(R)$ and $\mathrm{E}_n(R)$. For these reasons, we'll give in this section a short exposé on the definition and basic properties of Suslin's matrices. Due to space limitations, however, only the beginning part of the theory of Suslin matrices will be presented here; hence the caption of this section.

Throughout this section, R denotes a commutative ring. For any two row vectors $\alpha = (a_0, \ldots, a_n)$ and $\beta = (b_0, \ldots, b_n)$ over R, one constructs a matrix $S_n(\alpha, \beta) \in \mathbb{M}_{2^n}(R)$ inductively (on n) as follows. For $n = 0$, we define $S_0(\alpha, \beta)$ to be the 1×1 matrix (a_0) (independently of b_0). For $n \geqslant 1$, we let $\alpha_1 = (a_1, \ldots, a_n)$, $\beta_1 = (b_1, \ldots, b_n)$, and define

$$S_n(\alpha, \beta) = \begin{pmatrix} a_0 I_{2^{n-1}} & S_{n-1}(\alpha_1, \beta_1) \\ -S_{n-1}(\beta_1, \alpha_1)^t & b_0 I_{2^{n-1}} \end{pmatrix}, \tag{7.1}$$

assuming that the two off-diagonal blocks are already defined. Thus, beginning with $S_0(\alpha, \beta) = (a_0)$, we have

$$S_1(\alpha, \beta) = \begin{pmatrix} a_0 & a_1 \\ -b_1 & b_0 \end{pmatrix}, \quad \text{and} \quad S_2(\alpha, \beta) = \begin{pmatrix} a_0 & 0 & a_1 & a_2 \\ 0 & a_0 & -b_2 & b_1 \\ -b_1 & a_2 & b_0 & 0 \\ -b_2 & -a_1 & 0 & b_0 \end{pmatrix}, \tag{7.2}$$

etc. These Suslin matrices are closely related to the alternating matrices, although we'll make this connection explicit only in the case $n = 2$. In the matrix $S_2(\alpha, \beta)$ above, if we scale the second column by -1 and interchange the first two columns, then scale the fourth row by -1 and interchange the last two rows, we'll get the generic alternating matrix

$$\begin{pmatrix} 0 & a_0 & a_1 & a_2 \\ -a_0 & 0 & -b_2 & b_1 \\ -a_1 & b_2 & 0 & -b_0 \\ -a_2 & -b_1 & b_0 & 0 \end{pmatrix}. \tag{7.2$'$}$$

Clearly, for any n, the entries of $S_n(\alpha, \beta)$ are all of the form $\pm a_i$, $\pm b_i$, or 0. This shows that

$$S_n(\alpha, \beta) = S_n(\alpha, 0) + S_n(0, \beta), \qquad S_n(r\alpha, r\beta) = r S_n(\alpha, \beta) \ \ (r \in R), \tag{7.3}$$

and that $S_n(\alpha, 0)$ (resp. $S_n(0, \beta)$) is R-linear in α (resp. β). From these properties, we quickly deduce the following.

Proposition 7.4. *For a fixed n, all Suslin matrices $S_n(\alpha, \beta)$ form a free R-submodule of $\mathbb{M}_{2^n}(R)$ with basis*

$$\{ S_n(e_i, 0), \ S_n(0, e_i) : \ 0 \leqslant i \leqslant n \},$$

where $\{e_i\}$ *are the unit vectors in* R^{n+1}. *This free* R*-module of matrices, denoted by* $\Sigma_n(R)$*, is called the* Suslin space *in* $\mathbb{M}_{2^n}(R)$. *It is closed under matrix transposition.*

Proof. Only the last statement requires some work, and it will follow as soon as we verify the following transposition formula:

$$(7.5) \qquad S_n(\alpha,\ \beta)^t = S_n(\alpha',\ \beta') \quad \text{where } \alpha' = (a_0,\ -\beta_1),\ \ \beta' = (b_0,\ -\alpha_1).$$

From (7.1) and (7.3), $S_n(\alpha,\ \beta)^t$ is given by

$$\begin{pmatrix} a_0 I_{2^{n-1}} & S_{n-1}(-\beta_1,\ -\alpha_1) \\ S_{n-1}(\alpha_1,\ \beta_1)^t & b_0 I_{2^{n-1}} \end{pmatrix}.$$

By (7.3) again, $S_{n-1}(\alpha_1,\ \beta_1)^t = -\big(S_{n-1}(-\alpha_1,\ -\beta_1)\big)^t$, so the matrix displayed above is just $S_n(\alpha',\ \beta')$ for the α' and β' defined in the Proposition. □

Remark. Of course, $\Sigma_1(R)$ is the whole space $\mathbb{M}_2(R)$. However, for $n > 1$, $\Sigma_n(R)$ is just an R-submodule, but not an R-subalgebra, of $\mathbb{M}_{2^n}(R)$. For instance, in (7.2), the product of two matrices of the form $S_2(\alpha,\ \beta)$ need not have $(1, 2)$-entry equal to zero.

The next result gives the relationship between $S_n(\alpha,\ \beta)$ and $S_n(\beta,\ \alpha)$. Here, the $\alpha_1,\ \beta_1$ notations in (7.1) will remain in force, and $\langle u,\ v\rangle$ denotes the inner product of two row vectors $u,\ v$ of equal length.(*)

Proposition 7.6. $S_n(\alpha,\ \beta) S_n(\beta,\ \alpha)^t = \langle\alpha,\ \beta\rangle\, I_{2^n} = S_n(\beta,\ \alpha)^t S_n(\alpha,\ \beta)$.

Proof. We induct on n (the case $n = 0$ being clear). Assuming the truth of (7.6) for $n-1$, the matrix product on the LHS has the form

$$\begin{pmatrix} (a_0 b_0 + \langle\alpha_1,\ \beta_1\rangle) I_{2^{n-1}} & 0 \\ 0 & (a_0 b_0 + \langle\beta_1,\ \alpha_1\rangle) I_{2^{n-1}} \end{pmatrix} = \langle\alpha,\ \beta\rangle\, I_{2^n},$$

as desired. The second equality in (7.6) can be proved similarly. □

Our next job is to compute the determinant of a Suslin matrix. This is greatly facilitated by the following fact on the determinant of block matrices with a certain commuting property on its blocks.

Lemma 7.7. *Let* $S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, *where* $A, B, C, D \in \mathbb{M}_k(R)$ *are such that* $AB = BA$. *Then* $\det S = \det T$, *where* $T := DA - CB$.

(*)In this section, we prefer to write $\langle u,\ v\rangle$ for the inner product instead of $u\, v^t$, since the notation $\langle u,\ v\rangle$ looks more "symmetrical".

Proof. Let $I = I_k$. Since $BA = AB$, we have $S\begin{pmatrix} I & 0 \\ 0 & A \end{pmatrix} = \begin{pmatrix} A & AB \\ C & DA \end{pmatrix}$. The latter can be brought by (block) elementary column transformations into $\begin{pmatrix} A & 0 \\ C & T \end{pmatrix}$. Thus,

$$(\det S)(\det A) = (\det A)(\det T).$$

If $\det A$ is *not* a 0-divisor in R, we get $\det S = \det T$, as desired. In general, go to the polynomial ring $R[x]$, and consider the block matrix $\begin{pmatrix} A+xI & B \\ C & D \end{pmatrix} \in \mathbb{M}_{2k}\big(R[x]\big)$, where of course $(A+xI)B = B(A+xI)$. Here, $\det(A+xI)$ is a monic polynomial in $R[x]$, so it is not a 0-divisor in $R[x]$. By the case we have settled above,

$$\det \begin{pmatrix} A+xI & B \\ C & D \end{pmatrix} = \det \big(D(A+xI) - CB\big).$$

Now by pure magic, setting $x = 0$ gives the desired result! $\square$

From (7.2), we see easily that

$$\det S_1(\alpha,\ \beta) = a_0b_0 + a_1b_1, \quad \text{and} \quad \det S_2(\alpha,\ \beta) = (a_0b_0 + a_1b_1 + a_2b_2)^2.$$

These formulas generalize nicely to $\det S_n(\alpha,\ \beta)$, although we must ignore the case $n = 0$ (with apologies).

Proposition 7.8. *For* $n \geqslant 1$, *we have the following*:

(1) $\det S_n(\alpha,\ \beta) = \langle \alpha,\ \beta \rangle^{2^{n-1}} \in R$.

(2) $S_n(\alpha,\ \beta) \in \mathrm{GL}_{2^n}(R)$ *iff* $u := \langle \alpha,\ \beta \rangle \in \mathrm{U}(R)$, *in which case*

$$S_n(\alpha,\ \beta)^{-1} = S_n(u^{-1}\beta,\ u^{-1}\alpha)^t \in \Sigma_n(R).$$

(3) *If* $\langle \alpha,\ \beta \rangle = 1$, *then* $S_n(\alpha,\ \beta) \in \mathrm{SL}_{2^n}(R)$, *with*

$$S_n(\alpha,\ \beta)^{-1} = S_n(\beta,\ \alpha)^t \in \Sigma_n(R).$$

Proof. (3) follows from (1) and (2), while (2) follows easily from (1), (7.3), (7.4), and (7.6). To prove (1), we apply the Lemma to (7.1), noting that the upper-left block $a_0 I_{2^{n-1}}$ of $S_n(\alpha,\ \beta)$ in (7.1) commutes with all matrices of the same size. This shows that $\det S_n(\alpha,\ \beta)$ is given by

$$\det \big(a_0b_0 I_{2^{n-1}} + S_{n-1}(\beta_1,\ \alpha_1)^t S_{n-1}(\alpha_1,\ \beta_1)\big) = \det \big((a_0b_0 + \langle \alpha_1,\ \beta_1 \rangle) I_{2^{n-1}}\big),$$

thanks to (7.6) (for $n-1$). The RHS is, of course, just $\langle \alpha,\ \beta \rangle^{2^{n-1}}$. $\square$

Remark 7.9. The first nontrivial case of (7.8)(1) is when $n = 2$. Here, we can replace $S_2(\alpha,\ \beta)$ by the matrix in (7.2)$'$ without changing its determinant. Therefore, (7.8)(1) amounts to the nice classical fact that the generic alternating matrix in (7.2)$'$ has determinant $(a_0b_0 + a_1b_1 + a_2b_2)^2$. This fact will become very handy later in the study of symplectic modules over commutative rings; see VII.5.

Corollary 7.10. *All finite products of Suslin matrices from the set*

$$T = \{\, S_n(\beta,\ \alpha) :\ \langle\beta,\ \alpha\rangle = 1\}$$

form a subgroup of $\mathrm{SL}_{2^n}(R)$. *This group, denoted by* $\mathrm{SUm}_n(R)$, *is called the* special unimodular vector group over R.

Proof. It suffices to show that the set T is closed under inverses. Now the inverse of a typical element $S_n(\beta,\ \alpha)$ from T above has the form $S_n(\alpha,\ \beta)^t$. By (7.8), the latter is given by $S_n(\alpha',\ \beta')$, where (using the notations in (7.5))

$$\langle\alpha',\ \beta'\rangle = \langle(a_0,\ -\beta_1),\ (b_0,\ -\alpha_1)\rangle = \langle\alpha,\ \beta\rangle = 1.$$

This shows that $S_n(\beta,\ \alpha)^{-1} \in T$. □

The fact that $S_n(\alpha,\ \beta)$ belongs to $\mathrm{SL}_{2^n}(R)$ when $\langle\alpha,\ \beta\rangle = 1$ points squarely to the origin of Suslin's matrices. Suslin (1977b) remarked that, in case $\langle\alpha,\ \beta\rangle = 1$, one can perform elementary row and column transformations on the invertible matrix $S_n(\alpha,\ \beta)$ to reduce it (via "de-suspension") to a matrix in $\mathrm{SL}_{n+1}(R)$ with a first row $(a_0, a_1, a_2^2, \ldots, a_n^n)$. This then amounts to a constructive proof (of the essential case (4.8)) of the $n!$ Theorem. We will not present the details of this reduction here, since the ideas involved are rather similar to those used already in our proof of (4.6). A quick look at the special case $n = 2$ will suffice to recall these ideas. In the matrix $S_2(\alpha,\ \beta)$ in (7.2), the upper-left 2×2 block has the form $\mathrm{diag}(a_0,\ a_0)$, where a_0 is invertible modulo the ideal $J = a_1R + a_2R$. Thus, in R/J, we can use elementary transformations to bring $\mathrm{diag}(a_0,\ a_0)$ to $\mathrm{diag}(a_0^2,\ 1)$. With the entry 1 in the (2,2)-position, we can then "get rid of" the entries below the "1" in the second column. Then, deleting the second row and the second column leaves an invertible matrix with a first row $(a_0^2,\ a_1,\ a_2)$. Of course, the reduction steps have to be carried out more carefully, since the first step in the construction was done only modulo the ideal J.(*) But the above sketch gives a good illustration of how the Suslin matrix $S_n(\alpha,\ \beta)$ may be used in a constructive proof of the $n!$ Theorem.

The importance of the Suslin matrices can be seen also from another later work of Suslin. For any field k, let A_n be the commutative k-algebra with $2(n+1)$ generators

$$\{\, x_0,\ \ldots,\ x_n\,;\ y_0,\ \ldots,\ y_n\,\},$$

subject to the relation $\sum_{i=0}^n x_iy_i = 1$. (Thus, $\mathbf{x} = (x_0, \ldots, x_n)$ is a generic unimodular $(n+1)$-vector over A_n, with another generic vector $\mathbf{y} = (y_0, \ldots, y_n)$ "realizing" the unimodularity of $\mathbf{x}$.) The Suslin matrix $S_n(\mathbf{x},\ \mathbf{y}) \in \mathrm{SL}_{2^n}(A_n)$ then defines an element in $\mathrm{SL}(A_n) = \bigcup_i\ \mathrm{SL}_i(A_n)$, and hence also an element in the special K_1-group

(*)A largely similar version of this reduction (in the case $n = 2$) will be given explicitly later, after the proof of VII.5.31, in the context of symplectic structures.

$SK_1(A_n) = \mathrm{SL}(A_n)/\mathrm{E}(A_n)$, denoted by $[S_n(\mathbf{x}, \mathbf{y})]$. Suslin provided the following remarkable computation of this latter group (over any ground field k).

Theorem 7.11. *For $n \geqslant 1$, the group $SK_1(A_n)$ is infinite cyclic, with generator given by the generic Suslin matrix class $[S_n(\mathbf{x}, \mathbf{y})]$. The same holds if the ground field k is replaced by the ring of integers $\mathbb{Z}$.*

A proof of this result can be found in the paper [Suslin: 1982c] (see also [Suslin: 1991] written in a more general setting) listed in the references on Chapter VIII. Using this result, Suslin proved that, over the generic ring A_n and for positive exponents r_i, the unimodular row

$$\left(x_0^{r_0}, \ \ldots, \ x_n^{r_n}\right)$$

is *not* completable (to an invertible matrix) unless $n!$ divides $r_0 \cdots r_n$. This is, of course, the Swan-Towber converse to Suslin's $n!$ theorem that we have mentioned earlier in §4. However, while the Swan-Towber proof of this converse used a significant amount of topology, Suslin's proof using (7.11) was purely algebraic.

Notes on Chapter III

The two elementary proofs of Serre's Conjecture presented in the beginning sections of this chapter were both discovered shortly after the Quillen-Suslin solution in January 1976. Suslin's proof was contained in a letter from him to Bass dated May 2, 1976. I first learned about this proof from a 1976 talk of [L. Roberts: 1976], who learned about this proof from a talk of Murthy.

Our exposition of Vaserstein's elementary proof follows the lecture notes of Ferrand's Bourbaki talk [Ferrand: 1976]. For the original source of this proof, see [Vaserstein: 1976]. In the literature, this proof of Serre's Conjecture has sometimes been fondly referred to as "Vaserstein's 8-line proof" (see, e.g. Math Reviews MR 0472826). We must therefore plead guilty to consuming considerably more than eight lines in our exposition! As was observed in our verbose text, Vaserstein's proof uses a local-global method to reduce the consideration to a local Horrocks-type result (2.6), and is therefore rather close in spirit to Quillen's proof. However, the arguments in Vaserstein's proof are substantially simpler, since one need only deal with type 1 stably free modules (i.e. unimodular rows) in this proof, rather than with general finitely generated projective modules.

Suslin's Monic Polynomial Theorem (3.3) was proved by Suslin several years before the solution of Serre's Conjecture. For coefficient rings of dimension zero, (3.3) boils down essentially to Noether's classical Normalization Theorem, so (3.3) may be viewed as a strong generalization of the latter. Suslin's result has played a crucial role in his work on cancellation theorems over $R[t_1, \ldots, t_n]$, and has led to the affirmation of Serre's Conjecture in some special cases for small values of n, in the period 1973/75. Suslin's Theorem (3.3), as well as other parts of the work of [Vaserstein-Suslin: 1974], was made widely available to the American and European mathematical communities by the Bourbaki talk of [Bass: 1974], and subsequently by the Queen's lecture notes of [Swan: 1975]. See also [Geramita: 1974/76].

The Transitivity Theorem (3.6), its Corollary (3.7), and the spectacular Stability Theorem (3.8) all came from [Suslin: 1977a]. The proof of (3.6) offered here is self-contained, and so is the proof of (3.7) (except when the ground ring has dimension 0). As for the Stability Theorem (3.8), we shall eventually come back to it in the context of the K_1-analogue of Serre's Conjecture. For more details on this, see VI.4.

Suslin's $n!$ Theorem (4.1) is decidedly a highlight in the research work on the completion of unimodular rows, and has important applications to complete intersections; see (VIII.3). Our exposition in §4 follows [Suslin: 1977b] (which is a part of Sulin's doctoral dissertation), and in part also [Gupta-Murthy: 1980] and [Mandal: 1997]. For another proof of the $n!$ Theorem, see [Roitman: 1985, Thm. 4] listed in the references on Chapter VIII. The completion proposition (4.13) on linear polynomial unimodular vectors, due to Suslin and Swan, is a natural application of the $n!$ theorem. From an expository point of view, this result serves advance notice for Suslin's Problem $\mathrm{Su}(R)_n$ to be introduced and discussed later in IV.3.

Given a unimodular row $(a_1, \ldots, a_{2n-1})$, there is also the problem of completing $(0, a_1, \ldots, a_{2n-1})$ to an invertible *alternating* matrix, and this turns out to be related to the problem of completing the unimodular row $\alpha = (a_1^2, a_2, \ldots, a_{2n-1})$. For more information on this (and on unimodular rows in general), see (VII.5.23), (VII.5.31), and (VIII.5). While the unimodular row α above is not completable in general, it is always *sectionable* according to the observation of Rao and Sridharan (the case $r = 2$ in (5.8)). This observation led me to the notion of (possibly non-unimodular) sectionable and presectionable sequences, and to the material in §5. While somewhat experimental in nature, the notion of sectionable and presectionable sequences in rings does seem to have the potential of generating new ideas in working with unimodular rows in general. For instance, the sectionability of the unimodular row α above proves to be a handy tool in the treatment of duality questions on stably free modules, as we saw in the proof of (6.13). The material in §6 on the graph structure on $\mathrm{Um}_n(R)$, taken essentially from Hinson's doctoral thesis at Northwestern University ([Hinson: 1991a], listed in the references on Chapter VIII), has added further color and flavor to the still ongoing research on the theory of unimodular rows. For a more extensive introduction to the theory of sectionable and presectionable sequences in commutative rings, see [Lam-Swan: 2006].

The short exposé on Suslin matrices in §7 is intended to supplement our discussion of Suslin's $n!$ Theorem. The material here is, again, taken largely from [Suslin: 1977b]. A somewhat more detailed coverage on the applications of Suslin matrices can be found in the last part of §5 in Chapter VIII. I take this opportunity to thank Ravi Rao for urging me in no uncertain terms to include at least "A Touch of Suslin Matrices" in our exposition.

Chapter IV

Horrocks' Theorem

§1. Localization at Monic Polynomials

Throughout this section, R shall denote a commutative ring. Starting with the polynomial ring $R[t]$, we shall define a certain very useful ring of quotients of $R[t]$, denoted by $R\langle t\rangle$. The main concern of this chapter will be to study the behavior of f.g. projective $R[t]$-modules upon scalar extension from $R[t]$ to $R\langle t\rangle$.

By definition, *$R\langle t\rangle$ is the localization of $R[t]$ at the multiplicative set S consisting of all the <u>monic</u> polynomials in t*. Note that we get the same ring $R\langle t\rangle$ if we localize at the somewhat larger multiplicative set S' consisting of all *unitary* polynomials (that is, polynomials in t with leading coefficient a unit in R). Since S and S' consist of non zero-divisors of $R[t]$, we may think of $R[t]$ as a subring of $R\langle t\rangle$. In the special case where R is a field k, the localization $k\langle t\rangle$ is, of course, just the quotient field of $k[t]$, i.e., the field of rational functions over k, usually denoted by $k(t)$.

Proposition 1.1. *If R is regular, $R\langle t\rangle$ is also regular.*

Proof. By Swan's Theorem (II.5.7), R regular implies that $R[t]$ is regular. By (II.5.4), we see that the localization $R\langle t\rangle$ of $R[t]$ is also regular. □

Proposition 1.2. *If R is commutative noetherian, then* Krull dim $R\langle t\rangle =$ Krull dim R.

Proof. A prime chain $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n$ in R lifts to a prime chain $\mathfrak{p}_0[t] \subsetneq \mathfrak{p}_1[t] \subsetneq \cdots \subsetneq \mathfrak{p}_n[t]$ in $R[t]$. The prime ideals in the latter chain are all disjoint from the multiplicative set S of monic polynomials, so localization at S yields a prime chain of length n in $R\langle t\rangle$. This shows that Krull dim $R \leqslant$ Krull dim $R\langle t\rangle$. If $d =$ Krull dim R is infinite, we are done, so assume $d < \infty$. By (II.7.8), Krull dim $R[t] = d+1$. To show that Krull dim $R\langle t\rangle = d$ is thus equivalent to showing that any prime ideal $\mathfrak{p} \subset R[t]$ of height $d+1$ localizes to the unit ideal in $R\langle t\rangle$, i.e., that $\mathfrak{p}$ contains a *monic* polynomial. This can be easily deduced from Suslin's Monic Polynomial Theorem

(III.3.3), but we shall give a direct argument here. Let $\mathfrak{p} = \mathfrak{P} \cap R$. By (II.7.7), we have $\mathfrak{p}[t] \subsetneq \mathfrak{P}$, and ht $\mathfrak{p} = d$, so $\mathfrak{p}$ is a *maximal* ideal in R. Let

$$f = a_0 t^n + \cdots + a_n \in \mathfrak{P} \setminus \mathfrak{p}[t];$$

we may choose f such that $a_0 \notin \mathfrak{p}$. Since $\mathfrak{p}$ is maximal, there exists $c = a_0 b - 1 \in \mathfrak{p}$. Then, $\mathfrak{P}$ contains

$$b \cdot f - c \cdot t^n = (a_0 b - c) t^n + \cdots + a_n,$$

which is monic. Thus, $\mathfrak{P}$ localizes to the unit ideal in $R\langle t\rangle$. □

Corollary 1.3. (1) *If R is a regular ring of Krull dimension d, so is $R\langle t\rangle$.*

(2) *If R is a Dedekind domain, so is $R\langle t\rangle$.*

(3) *If R is a* PID*, so is $R\langle t\rangle$.*

Proof. (l) follows from the two propositions. (2) follows from (1) by letting $d = 1$, and taking R to be a domain. For (3), note that a Dedekind domain is a PID iff it is a UFD. Thus, (3) follows from (2) since a localization of a UFD is always a UFD. □

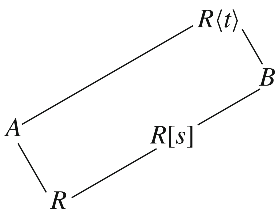
To familiarize ourselves some more with $R\langle t\rangle$, let us also prove the following structure theorem about it.

Proposition 1.4. *Let $A = R[t]$, and B be the subring of $R\langle t\rangle$ consisting of $g(t)/f(t)$ with f monic, and* $\deg g \leqslant \deg f$. *Also, let $s = 1/t \in B$. Then:*

(1) $R\langle t\rangle = A + B = B[\frac{1}{s}]$.

(2) $B = (1 + sR[s])^{-1} R[s]$.

(3) *If R is local, with maximal ideal $\mathfrak{m}$, then $B = R[s]_{(\mathfrak{m},s)}$, and is also local.*



Proof. (1) An arbitrary element of $R\langle t\rangle$ has the form $h(t)/f(t)$ where f is monic. We can "divide" h by f, and write $h = q \cdot f + g$ where $g = 0$ or deg $g <$ deg f. Thus, $h/f = q + g/f \in A + B = R[t] + B$. Since $B \supseteq R$, we conclude that $R\langle t\rangle = B[t] = B[\frac{1}{s}]$.

(2) Of course, $R[s] \subseteq B$. An element in $1 + sR[s]$ has the form

$$\alpha = 1 + b_1 s + \cdots + b_n s^n = \frac{t^n + b_1 t^{n-1} + \cdots + b_n}{t^n},$$

so

$$\alpha^{-1} = \frac{t^n}{t^n + b_1 t^{n-1} + \cdots + b_n} \in B.$$

This shows that $(1 + sR[s])^{-1}R[s] \subseteq B$. Conversely, a typical element of B has the form

$$\beta = \frac{a_0 t^m + \cdots + a_m}{t^n + \cdots + b_{n-1}t + b_n}, \quad \text{where } a_i, b_j \in R,\ m \leqslant n.$$

Multiplying numerator and denominator by s^n, we have

$$\beta = \frac{s^{n-m}(a_0 + \cdots + a_m s^m)}{1 + b_1 s + \cdots + b_n s^n} \in \big(1 + sR[s]\big)^{-1} R[s].$$

(3) Suppose now $(R, \mathfrak{m})$ is a local ring. The multiplicative set $R[s]\backslash(\mathfrak{m}, s)$ consists of $b_0 + b_1 s + \cdots + b_n s^n$ where $b_0 \in R\backslash\mathfrak{m}$ is a unit. Thus, $R[s]_{(\mathfrak{m},s)}$ is the same as the localization $\big(1 + sR[s]\big)^{-1}R[s] = B$. Since $R[s]/(\mathfrak{m}, s) \cong R/\mathfrak{m}$, we see that $(\mathfrak{m},\ s)$ is a maximal ideal, so $R[s]_{(\mathfrak{m},\,s)} = B$ is a local ring. □

We shall now make some remarks about the geometrical interpretation of the ring $R\langle t\rangle$ and the ring B, using the language of the "projective line" $\mathbb{P}^1_R$ over R.

In modern algebraic geometry, the *projective line* $\mathbb{P}^1_R$ over a commutative ring R is obtained by "gluing" two copies of the affine line $\mathbf{A}^1_R = \operatorname{Spec} R[t]$ along a certain open subset. To set this up more formally, form the open sets

$$D(t) = \{\, \mathfrak{p} \subset R[t] : \mathfrak{p} \not\ni t \,\} \subseteq \operatorname{Spec} R[t],$$

$$D(s) = \{\, \mathfrak{p} \subset R[s] : \mathfrak{p} \not\ni s \,\} \subseteq \operatorname{Spec} R[s].$$

Clearly, $D(t)$ is just $\operatorname{Spec} R[t, t^{-1}]$, and $D(s)$ is just $\operatorname{Spec} R[s, s^{-1}]$. We can identify $D(t)$ with $D(s)$ by using $s = 1/t$, and "glue" $\operatorname{Spec} R[t]$ with $\operatorname{Spec} R[s]$ via this identification. The resulting "scheme" is the projective line $\mathbb{P}^1_R$. The following diagrams will help illustrate this gluing process:

$$\begin{tikzcd}
 & & R[s] \arrow[d] \\
 & & R[s]_s \arrow[dd, equal] \\
R[t] \arrow[r] & R[t]_t \arrow[ur, equal] \arrow[dr, equal] & \\
 & & R[t, t^{-1}]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{P}^1_R & & \operatorname{Spec} R[s] \arrow[ll, dashed] \\
 & & D(s) \arrow[u] \arrow[dd, equal] \\
\operatorname{Spec} R[t] \arrow[uu, dashed] & D(t) \arrow[l] \arrow[ur, equal] \arrow[dr, equal] & \\
 & & \operatorname{Spec} R[t, t^{-1}]
\end{tikzcd}$$

We shall just write D for $D(t)$ or $D(s)$ inside the projective line $\mathbb{P}^1_R$. However, to distinguish Spec $R[t]$ and Spec $R[s]$ within $\mathbb{P}^1_R$, we shall write $\mathbf{A}^1_t$ and $\mathbf{A}^1_s$ (suppressing R). Further, we write

$$\mathbf{A}^1_t \backslash D = V(t), \qquad \mathbf{A}^1_s \backslash D = V(s);$$

these are Zariski closed sets in $\mathbf{A}^1_t$ (resp. $\mathbf{A}^1_s$) defined by t (resp. by s).

We can now give nice interpretations of the two rings $R\langle t\rangle$ and B of (1.4) in the above framework.

Proposition 1.5. *For* $B = \big(1 + sR[s]\big)^{-1} R[s] \supseteq R[s]$, Spec $B \subseteq$ Spec $R[s] = \mathbf{A}^1_s$ *is the intersection of all the open sets containing* $V(s)$. *Moreover,*

$$(\text{Spec } B) \cap \mathbf{A}^1_t = \text{Spec } R\,\langle t\rangle\,.$$

Proof. Let $V(\mathfrak{A})$ $\big(\mathfrak{A} \subseteq R[s]\big)$ be a typical closed set disjoint from $V(s)$. Then, $\mathfrak{A}$ and s are comaximal, which means that $\mathfrak{A}$ intersects $1 + sR[s]$. Thus, $\mathfrak{p} \in$ Spec B cannot contain $\mathfrak{A}$. This shows that Spec $B \subseteq \mathbf{A}^1_s \backslash V(\mathfrak{A})$, a typical open set containing $V(s)$. Conversely, if $\mathfrak{p} \notin$ Spec B, $\mathfrak{p}$ must contain some $1 + s \cdot f(s)$. Thus, the open set Spec $R[s] \setminus V(1 + sf(s))$ contains $V(s)$, but misses $\mathfrak{p}$. The above consideration shows that Spec B is the intersection of all the open sets containing $V(s)$. Finally,

$$\begin{aligned}
\mathbf{A}^1_t \cap \text{Spec } B &= D \cap \text{Spec } B \\
&= \{\, \mathfrak{p} \in \text{Spec } B \mid \mathfrak{p} \in D \,\} \\
&= \{\, \mathfrak{p} \in \text{Spec } B \mid \mathfrak{p} \not\ni s \,\} \\
&= \text{Spec } B[s^{-1}] \\
&= \text{Spec } R\,\langle t\rangle \qquad\qquad (\text{by } (1.4)). \qquad \square
\end{aligned}$$

Remark 1.6. Suppose R is local with maximal ideal $\mathfrak{m}$. Then $(\mathfrak{m}, s)$ is clearly the only maximal ideal in Spec $R[s]$ containing s, i.e., $(\mathfrak{m}, s)$ is the unique closed point in $V(s)$. Thus, we speak of $(\mathfrak{m}, s)$ as *the* "closed point at infinity" on $\mathbb{P}^1_R$. The "stalk" of the structural sheaf of $\mathbb{P}^1_R$ at this closed point at infinity is precisely $R[s]_{(\mathfrak{m},s)} = B$, by (1.4)(3)! In the special case where R is a field, B is, of course, just the discrete valuation ring of the $\big(\frac{1}{t}\big)$-adic valuation v_∞ on $R\langle t\rangle$ induced by $v_\infty(g) = -\deg g$ for $g \in R[t]$.

Remark 1.7. Suppose $(R, \mathfrak{m})$ is as in (1.6) above, and let M be the maximal ideal of the local ring B; that is, $M = (\mathfrak{m}, s)B$. *Then, we have* $s \in M \backslash M^2$. For, if $s \in M^2$, then $\alpha s = \sum_i \beta_i \gamma_i$ for some β_i, $\gamma_i \in (\mathfrak{m}, s)$ and $\alpha \in R[s] \setminus (\mathfrak{m}, s)$. Write

$$\alpha = \sum_j a_j s^j, \quad \beta_i = \sum_j b_{ij} s^j, \quad \gamma_i = \sum_j c_{ij} s^j,$$

where a_j, b_{ij}, $c_{ij} \in R$, with b_{i0}, $c_{i0} \in \mathfrak{m}$, and $a_0 \notin \mathfrak{m}$. Comparing the coefficients of s in the equation $\alpha s = \sum_i \beta_i \gamma_i$, we get $a_0 = \sum_i (b_{i1} c_{i0} + b_{i0} c_{i1}) \in \mathfrak{m}$, a contradiction. The observation that $s \notin M^2$ in the local ring (B, M) will be useful later on in (V.3).

§2. Statement of Horrocks' Theorem

The theorem of Horrocks [Horrocks: 1964] is a criterion for a vector bundle over the affine line $\mathbf{A}^1_R$ to be trivial, where R is a commutative local ring. This criterion will be called the "geometric form" of Horrocks' Theorem. Since we have not carefully defined the notion of a vector bundle, our discussion of the geometric form of Horrocks' Theorem will not be entirely self-contained. However, it is possible to give an "algebraic form" of Horrocks' Theorem; the statement and proof of this algebraic form will be completely self-contained. In this section, we shall formulate the two different forms of Horrocks' Theorem, and show, in some detail, that these two forms are equivalent.

Horrocks' Theorem 2.1. (Algebraic Form) *Let R be a commutative local ring, and $P \in \mathfrak{P}\big(R[t]\big)$. If $P\langle t\rangle = R\langle t\rangle \otimes_{R[t]} P$ is $R\langle t\rangle$-free, then P is $R[t]$-free.*

To formulate the geometric form of Horrocks' Theorem, we need the notion of a *vector bundle* – or at least the special case of vector bundles over a projective line, $\mathbb{P}^1_R$. In modern algebraic geometry, a vector bundle is understood to be a *locally free sheaf.* Since $\mathbb{P}^1_R$ is obtained by "gluing" together Spec $R[t]$ and Spec $R[t^{-1}]$ along Spec $R[t, t^{-1}]$, a locally free sheaf on $\mathbb{P}^1_R$ amounts essentially to a locally free sheaf on Spec $R[t]$ and a locally free sheaf on Spec $R[t^{-1}]$ that agree on the "overlap" Spec $R[t, t^{-1}]$. Now, over any affine scheme, Spec A, the locally free sheaves are uniquely determined by their modules of global sections, which are f.g. projective A-modules.(*) *Thus, to specify a vector bundle over $\mathbb{P}^1_R$, we need two projective modules: $P \in \mathfrak{P}(R[t])$, $Q \in \mathfrak{P}(R[t^{-1}])$ together with an isomorphism*

$$R[t, t^{-1}] \otimes_{R[t]} P \cong R[t, t^{-1}] \otimes_{R[t^{-1}]} Q.$$

We can, in fact, use this as a working definition for vector bundles over $\mathbb{P}^1_R$, if we seek to avoid using the undefined terminology of locally free sheaves.

From the above discussion, we see that a vector bundle over $\mathbf{A}^1_R$ given by $P \in \mathfrak{P}\big(R[t]\big)$ extends to a vector bundle on $\mathbb{P}^1_R$ iff there exists $Q \in \mathfrak{P}\big(R[t^{-1}]\big)$ such that

$$R[t, t^{-1}] \otimes_{R[t]} P \cong R[t, t^{-1}] \otimes_{R[t^{-1}]} Q.$$

We can now state:

Horrocks' Theorem 2.2. (Geometric Form) *Let R be a commutative local ring. If a vector bundle B on $\mathbf{A}^1_R$ extends to a vector bundle on $\mathbb{P}^1_R$, then B is a trivial bundle.*

The proof of Horrocks' Theorem, in the form 2.1, will be taken up in the two subsequent sections. For the balance of this section, we shall explain why the two forms of the theorem given above should be viewed as one and the same result. To see this, it suffices to verify the following:

(*)The free sheaves (or trivial bundles) over Spec A correspond to the free A-modules of finite rank.

Proposition 2.3. *For any commutative local ring R, and $P \in \mathfrak{P}(R[t])$, P extends to a vector bundle on $\mathbb{P}^1_R$ iff $P\langle t\rangle = R\langle t\rangle \otimes_{R[t]} P$ is $R\langle t\rangle$-free.*

Proof. For the "only if" part, let us first give a geometric proof. Suppose P extends to a vector bundle, say $\mathfrak{B}$, over $\mathbb{P}^1_R$. Let B be the ring defined in (1.4). Since R is local, B is also local by (1.4)(3), so any f.g. B-projective is free. Thus, the restriction of $\mathfrak{B}$ to Spec B is trivial, and in particular, the restriction of $\mathfrak{B}$ to

$$\text{Spec } B \cap \text{Spec } R[t] = \text{Spec } R\langle t\rangle \qquad \text{(see (1.5))}$$

is trivial. This means that $R\langle t\rangle \otimes_{R[t]} P$ is $R\langle t\rangle$-free. *To proceed more algebraically*, take a suitable $Q \in \mathfrak{P}(R[t^{-1}])$ such that

$$R[t,t^{-1}] \otimes_{R[t]} P \cong R[t,t^{-1}] \otimes_{R[t^{-1}]} Q.$$

Write $s = 1/t$, and consider the following commutative diagram of inclusion maps:

$$\begin{tikzcd}
 & R[s] \arrow[r] \arrow[d] & B \arrow[d] \\
R[t] \arrow[r] & R[t,t^{-1}] \arrow[r] & R\langle t\rangle
\end{tikzcd}$$

We have a sequence of isomorphisms:

$$\begin{aligned}
R\langle t\rangle \otimes_{R[t]} P &\cong R\langle t\rangle \otimes_{R[t,t^{-1}]} (R[t,t^{-1}] \otimes_{R[t]} P) \\
&\cong R\langle t\rangle \otimes_{R[t,t^{-1}]} (R[t,t^{-1}] \otimes_{R[s]} Q) \\
&\cong R\langle t\rangle \otimes_{R[s]} Q \\
&\cong R\langle t\rangle \otimes_{B} (B \otimes_{R[s]} Q).
\end{aligned}$$

As above, $B \otimes_{R[s]} Q$ is B-free, so $R\langle t\rangle \otimes_{R[t]} P$ is $R\langle t\rangle$-free, as desired.

Conversely, assume $R\langle t\rangle \otimes_{R[t]} P$ is $R\langle t\rangle$-free. By (I.2.17), there exists a monic polynomial $f(t) \in R[t]$ such that $P_f = R[t]_f \otimes_{R[t]} P$ is already $R[t]_f$-free. This means that the vector bundle $\widetilde{P}$, defined by P on Spec $R[t]$, has trivial restriction to Spec $R[t]\backslash V(f)$, where $V(f)$ denotes the Zariski closed set in Spec $R[t]$ consisting of primes containing f. We claim that

(2.4) $\qquad$ *$V(f)$ is actually closed in $\mathbb{P}^1_R$.*

If we can show this, then we can "glue" the bundle $\widetilde{P}$ on Spec $R[t]$ with a trivial bundle on the *open* set Spec $R[s]\backslash V(f)$, to obtain a bundle on $\mathbb{P}^1_R$, as desired. To show (2.4), it is enough to show that $V(f) \cap$ Spec $R[s]$ is closed in Spec $R[s]$. This is done essentially by the idea of proof in (1.5). Define $g(s) = t^{-n} f(t) \in R[s]$, where $n = \deg f$. The two elements $f(t)$, $g(s)$ are associates in $R[t, t^{-1}]$, so

$$\begin{aligned} V(f) \cap \operatorname{Spec} R[s] &= V(f) \cap \operatorname{Spec} R[t, t^{-1}] \\ &= V(g) \cap \operatorname{Spec} R[t, t^{-1}]. \end{aligned}$$

Since f is monic in t, we have $g \in 1 + s \cdot R[s]$, so $V(g) \subseteq \operatorname{Spec} R[t, t^{-1}]$. Thus, $V(f) \cap \operatorname{Spec} R[s] = V(g)$ is closed in $\operatorname{Spec} R[s]$, proving the claim (2.4). □

Remark 2.5. The second half of the above proof (the "if" part) works over any commutative ring R (i.e., it does not invoke the hypothesis that R is local). The argument is due to Murthy; see [Bass: 1972a, p. 25], or [Quillen: 1976, Corollary to Th. 2].

The first published proof of Horrocks' Theorem is in [Horrocks: 1964]. The result was stated there in its geometric form (2.2), and proved by a certain filtration argument on vector bundles over $\mathbb{P}^1_R$ (R local). In the special case where R is an (algebraically closed) field, one uses a result of [Grothendieck: 1957], that a vector bundle over $\mathbb{P}^1_R$ must decompose into a direct sum of line bundles. The general case (R local) can then be derived from the field case by reducing the given bundle on $\mathbb{P}^1_R$ modulo the maximal ideal, and then using certain facts from sheaf cohomology theory to pull back the information.

Since we lack preparation on sheaf theory, we shall not go through the geometric proof described above. In 1976, after the solution of Serre's Conjecture, there have been several (successful) attempts at proving Horrocks' Theorem *without* the geometric arguments. These attempts have resulted in a number of purely algebraic proofs of Horrocks' Theorem, in the algebraic form (2.1). Because of the crucial role played by this theorem in the solution of Serre's Conjecture, and also for the sake of completeness, we shall present three of these proofs, which are, respectively, due to R.G. Swan, Paul Roberts, and Nashier-Nichols. These will occupy the next three sections.

§3. Swan's Proof of Horrocks' Theorem

We begin with some preliminary lemmas. The first one is very well-known, but we shall include its proof for the sake of completeness.

Schanuel's Lemma 3.1. *Let R be any ring, and $0 \to K_i \to Q_i \xrightarrow{g_i} A \to 0$ be exact sequences of left R-modules, $i = 1, 2$. If Q_1, Q_2 are R-projective, then $K_1 \oplus Q_2 \cong K_2 \oplus Q_1$.*

Proof. Let X be the pullback of g_1 and g_2, i.e.

$$X = \{\, (x_1, x_2) \in Q_1 \oplus Q_2 \,:\, g_1(x_1) = g_2(x_2)\},$$

and let $p_i : X \to Q_i$ be given by the coordinate projections. By an easy diagram inspection, we see that p_1, p_2 are onto, with $\ker(p_1) \cong K_2$ and $\ker(p_2) \cong K_1$. Since Q_i is projective, p_i must split. Thus, $K_1 \oplus Q_2 \cong X \cong K_2 \oplus Q_1$. □

Next, we shall set up the "Characteristic Sequence" of an R-module endomorphism. Let M be any R-module, and $\alpha \in \mathrm{End}_R M$. We write M_α to denote the $R[t]$-module M with t acting as α, and write $M[t] = R[t] \otimes_R M = \bigoplus_i t^i \otimes M$. There exists a unique $R[t]$-epimorphism $\theta : M[t] \to M_\alpha$ that is the identity on M, namely, $\theta\big(\sum t^i \otimes m_i\big) = \sum \alpha^i m_i$.

Lemma 3.2. (Characteristic Sequence) *Let R be any ring, and let M, α, θ be as above. Then*

$$0 \to M[t] \xrightarrow{t-\alpha} M[t] \xrightarrow{\theta} M_\alpha \to 0 \tag{3.3}$$

is an exact sequence of $R[t]$-modules. (*Here, $t-\alpha$ is an abbreviation for the endomorphism $t \cdot 1_{M[t]} - \alpha[t]$ of $M[t]$.*)

Proof. (1) For $m \in M$,

$$\theta\big((t-\alpha)(1 \otimes m)\big) = \theta(t \otimes m - 1 \otimes \alpha m) = \alpha m - \alpha m = 0,$$

so the sequence is a zero sequence. The injectivity of $t-\alpha$ is proved in the same way as in (II.5.1)(2); namely, if $x = t^i \otimes m + t^{i-1} \otimes m' + \dots$ with $m, m', \dots \in M$, and $m \neq 0$, then

$$(t-\alpha)(x) = t^{i+1} \otimes m + t^i \otimes (m' - \alpha m) + \cdots \neq 0.$$

To see the exactness at the middle of the sequence, consider the surjection from $M[t]/\mathrm{im}(t-\alpha)$ onto M_α. This is clearly a bijection since $m \mapsto 1 \otimes m$ provides an obvious inverse. $\square$

Remark 3.4. In the above, *assume that M is finitely presented as an R-module. Then* (3.3) *splits as a sequence of $R[t]$-modules iff $M = 0$*. In fact, by (I.2.13′),

$$\mathrm{End}_{R[t]}\, M[t] \cong R[t] \otimes_R \mathrm{Hom}_R(M, M) = (\mathrm{End}_R\, M)[t].$$

If $M \neq 0$, then $\mathrm{End}_R\, M \neq 0$, so the "polynomial" $t - \alpha \in (\mathrm{End}_R\, M)[t]$ cannot have a left inverse.

Theorem 3.5. (Towber Presentation) *Let R be any ring, and $P,\ F \in \mathfrak{P}(R[t])$, where $F = F_0[t]$ for some R-module F_0 ($\cong F/tF \in \mathfrak{P}(R)$). Let f be a monic polynomial in the center of $R[t]$, and suppose $P_f \cong F_f$ as $R[t]_f$-modules. Then there exist $M, N \in \mathfrak{P}(R)$, and $u, v \in \mathrm{Hom}_R(M, N)$ such that there are two (split) exact sequences of, respectively, R-modules and $R[t]$-modules:*

$$0 \longrightarrow M \xrightarrow{v} N \longrightarrow F_0 \longrightarrow 0,$$
$$0 \longrightarrow M[t] \xrightarrow{u+vt} N[t] \longrightarrow P \longrightarrow 0,$$

where u and v in the second sequence are abbreviations for $u[t]$ and $v[t]$.

Note. The first sequence induces $0 \to M[t] \xrightarrow{v} N[t] \to F \to 0$, so the two modules P and F have rather "close" presentations by the extended modules $N[t]$ and $M[t]$. *In particular,* $[P] = [F] \in \operatorname{im}\big(K_0R \to K_0R[t]\big)$, *and P and F are stably isomorphic.*

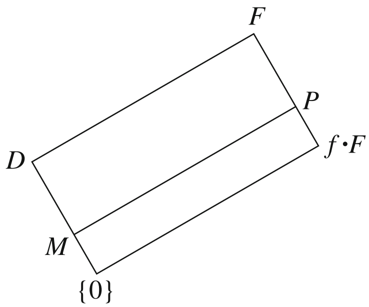
Proof. Since P is finitely presented over $R[t]$, we have by (I.2.13′):

$$\operatorname{Hom}_{R[t]_f}(P_f, F_f) \cong \big(\operatorname{Hom}_{R[t]}(P, F)\big)_f.$$

Thus, we may choose an isomorphism $P_f \cong F_f$ that is the localization of an $R[t]$-homomorphism $\lambda : P \to F$. Since f is not a zero-divisor in $R[t]$, $(\ker \lambda)_f = 0 \Rightarrow \ker \lambda = 0$, so we may think of λ as an inclusion $P \subseteq F$. Also, $\operatorname{coker} \lambda$ is f.g., so

$$(\operatorname{coker} \lambda)_f = 0 \implies f^n \cdot \operatorname{coker} \lambda = 0$$

for some n. After replacing f by f^n, we may assume that $f \cdot F \subseteq P \subseteq F$. Let $d = \deg f$, and $D = F_0 \oplus tF_0 \oplus \cdots \oplus t^{d-1}F_0 \subseteq F$. Then $F = D \oplus f \cdot F$ (direct sum of R-submodules). For $M = D \cap P$, we have:



As R-modules, $M \cong P/f \cdot F$, so the t-action on $P/f \cdot F$ induces an R-endomorphism, say α, on M. By definition, $(t - \alpha)M \subseteq f \cdot F$. Since f is not a zero-divisor, there exists a (unique) R-homomorphism $\beta : M \to F$ such that $(t - \alpha)m = f \cdot \beta(m)$, for all $m \in M$. We have $t \cdot D \subseteq D \oplus f \cdot F_0$, so we must have $\beta(M) \subseteq F_0$. Consider now the following diagram of $R[t]$-modules (without the dotted maps for the moment):

$$\begin{tikzcd}
0 \arrow[r] & M[t] \arrow[rr, "t-\alpha"] \arrow[dd, "\beta"'] \arrow[dr, dotted, "\Phi"] & & M[t] \arrow[r, "\theta"] \arrow[dd, "\sigma"] & M_\alpha \arrow[r] \arrow[dd, "\cong"] & 0 \\
& & N[t] \arrow[ur, dotted] \arrow[dl, dotted] \arrow[dr, dotted, "\Psi"] & & & \\
0 \arrow[r] & F = F_0[t] \arrow[rr, "f\cdot"] & & P \arrow[r] & P/f\cdot F \arrow[r] & 0
\end{tikzcd}$$

Here, the top row is just the Characteristic Sequence (3.3), and σ is the unique $R[t]$-homomorphism such that $\sigma \mid M$ is the inclusion map $M \subseteq P$. By a trivial diagram chase, we see that the left-hand square is a pullback diagram. Let $N = M \oplus F_0$, and let $v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $u = \begin{pmatrix} -\alpha \\ \beta \end{pmatrix}$ in $\mathrm{Hom}_R(M, N)$. We have then

$$0 \longrightarrow M \xrightarrow{v} N \longrightarrow F_0 \longrightarrow 0,$$

and

$$0 \longrightarrow M[t] \xrightarrow{\Phi} N[t] \xrightarrow{\Psi} P \longrightarrow 0,$$

where $\Phi = \begin{pmatrix} t-\alpha \\ \beta \end{pmatrix} = \begin{pmatrix} -\alpha \\ \beta \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} t = u + vt$, and Ψ is the "difference map" $(-\sigma, f\cdot)$. (Note that Ψ is onto since $P = M + f \cdot F$.)

To complete the proof of the Theorem, it remains only to show that $M \cong P/f \cdot F \in \mathfrak{P}(R)$. Since P is f.g. over $R[t]$, $P/f \cdot F$ is f.g. over $R[t]/(f)$. The latter is f.g. over R, so M is f.g. over R. Consider the exact sequence s of R-modules:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P & \longrightarrow & F & \longrightarrow & F/P & \longrightarrow & 0, \\ 0 & \longrightarrow & \frac{P}{f\cdot F} & \longrightarrow & \frac{F}{f\cdot F} & \longrightarrow & F/P & \longrightarrow & 0. \end{array}$$

Here, P, F, and $F/f \cdot F \cong D$ are all R-projective. It follows from Schanuel's Lemma and (I.1.1) that $M \cong P/f \cdot F$ is R-projective. $\square$

Remark. The above proof is close in spirit to the earlier proof of (II.5.1), though it is a bit more sophisticated.

Theorem 3.6. *Let* $(R, \mathfrak{m})$ *be a* (*not necessarily commutative*) *local ring, and* $f(t)$ *be a monic polynomial in the center of* $R[t]$. *If* $P \in \mathfrak{P}(R[t])$ *and* P_f *is* $R[t]_f$*-free, then* P *is* $R[t]$*-free.*

(For R commutative local, this implies the (algebraic form of) Horrocks' Theorem, by (I.2.17).)

Proof. Take a f.g. free R-module F_0 such that $P_f \cong F_f$, where $F = F_0[t]$. By the proof of (3.5), there exist $N = M \oplus F_0 \in \mathfrak{P}(R)$ and a Towber Presentation

$$0 \longrightarrow M \xrightarrow{v} N \longrightarrow F_0 \longrightarrow 0, \tag{3.7}$$

$$0 \longrightarrow M[t] \xrightarrow{\Phi = u + vt} N[t] \longrightarrow P \longrightarrow 0, \tag{3.8}$$

where $v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $u = \begin{pmatrix} -\alpha \\ \beta \end{pmatrix}$ in $\mathrm{Hom}_R(M, M \oplus F_0)$, so $\Phi = \begin{pmatrix} t-\alpha \\ \beta \end{pmatrix}$. If $M = 0$, we will have $P \cong N[t] \cong F_0[t] = F$ as desired, *so assume* $M \neq 0$. By Nakayama's Lemma, $M \neq \mathfrak{m}M$.

We claim that $\overline{\beta} : \overline{M} \to \overline{F}_0$ is not zero, where "bar" denotes going modulo $\mathfrak{m}$. Indeed, suppose $\overline{\beta} = 0$. Reducing (3.8) mod $\mathfrak{m}$, we get $\Phi = \begin{pmatrix} t-\overline{\alpha} \\ 0 \end{pmatrix}$. But (3.8) splits, so $\overline{M}[t] \xrightarrow{t-\overline{\alpha}} \overline{M}[t]$ also splits. Since $\dim_{\overline{R}} \overline{M} < \infty$, this is possible only when $\overline{M} = 0$ (by (3.4)), a contradiction.

Let $x \in M \backslash \mathfrak{m}M$ be such that $\beta(x) = y \in F_0 \backslash \mathfrak{m}F_0$. By (I.1.6), we have decompositions $M = R \cdot x \oplus M'$, $F_0 = R \cdot y \oplus F_0'$ into R-free modules. Choose an automorphism g of M, of the form $\begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix}$ (where $\gamma : M' \to R \cdot x$) such that $\beta g(M') \subseteq F_0'$. (Take γ such that in the following diagram,

$$\begin{tikzcd}
R\cdot x \arrow[r, "1"] & R\cdot x \arrow[r, "\cong"] \arrow[ddr] & R\cdot y \\
\oplus & \oplus & \oplus \\
M' \arrow[r, "1"'] \arrow[uur, "\gamma"] & M' \arrow[r] \arrow[uur] & F_0' \\
\underbrace{\hspace{6em}}_{g} & & \underbrace{\hspace{6em}}_{\beta}
\end{tikzcd}$$

the "rhombus" anticommutes.) After replacing $\Phi = \begin{pmatrix} t-\alpha \\ \beta \end{pmatrix}$ by

$$\hat{\Phi} = (g^{-1} \oplus 1)\, \Phi\, g = \begin{pmatrix} g^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t-\alpha \\ \beta \end{pmatrix} g = \begin{pmatrix} t - g^{-1}\alpha g \\ \beta g \end{pmatrix},$$

we may assume that $\beta(M') \subseteq F_0'$. (Note that clearly $\operatorname{coker} \widehat{\Phi} \cong \operatorname{coker} \Phi \cong P$.)

Now let $\Phi' : M'[t] \to M[t] \oplus F_0'[t]$ be given by $\begin{pmatrix} t-\alpha | M' \\ \beta | M' \end{pmatrix}$. From the diagram

$$\begin{tikzcd}
0 \arrow[r] & M'[t] \arrow[r, "\Phi'"] \arrow[d, phantom, "\cap|"] & M[t] \oplus F_0'[t] \arrow[r] \arrow[d, phantom, "\cap|"] & \operatorname{coker} \Phi' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M[t] \arrow[r, "\Phi"] \arrow[d] & M[t] \oplus F_0[t] \arrow[r] \arrow[d] & P \arrow[r] & 0, \\
& (R\cdot x)[t] \arrow[r, "\cong"] & (R\cdot y)[t] & &
\end{tikzcd}$$

we see easily that $\operatorname{coker} \Phi' \cong P$. If we write

$$N' = M \oplus F_0' = M' \oplus R \cdot x \oplus F_0' \cong M' \oplus F_0,$$

the coordinate maps of $\Phi' : M'[t] \to N[t] \cong M'[t] \oplus F_0[t]$ will be given by $\begin{pmatrix} t-\alpha' \\ \beta' \end{pmatrix}$, where α' is the composition $M' \xrightarrow{\alpha} M \longrightarrow M/R \cdot x \cong M'$, and β' is the sum of

$\beta \mid M'$ and the composition

$$M' \xrightarrow{\alpha} M \longrightarrow M/M' \cong R\cdot x \cong R\cdot y \subseteq F_0.$$

Thus, we obtain new presentations as in (3.7), (3.8), with a "smaller" M. By repeating this process, we can eventually reduce M to zero. $\square$

The possibility of giving a purely algebraic proof of Horrocks' Theorem using Towber's presentation was observed independently by H. Lindel at Münster. Lindel's proof was very close to Swan's, but was couched in matrix terms. For the sake of completeness, we also sketch this proof below.

Keep the notations in the proof of (3.6). After choosing R-bases for M and F_0, we can express $\alpha\colon M\to M$, $\beta\colon M\to F_0$ by matrices (a_{ij}) and (b_{kj}) over R. With respect to the same bases (over $R[t]$), Φ can be expressed as $\phi = \begin{pmatrix} t\cdot I-(a_{ij}) \\ (b_{kj}) \end{pmatrix}$. We will show, by induction on $\mathrm{rk}_R M$ $(=\text{size of } (a_{ij}))$, that *by elementary row and column transformations, we can transform* ϕ *to* $\begin{pmatrix} I \\ 0 \end{pmatrix}$. If so, $P\cong \operatorname{coker}\Phi$ will clearly be isomorphic to $F_0[t]=F$.

The case $M=0$ is trivial, so assume $r=\mathrm{rk}_R\, M>0$. By going modulo $\mathfrak{m}$, we see as before that $(\bar{b}_{kj})\neq 0$. Say b_{11} is a unit in R. Then we can perform the following elementary row transformations on ϕ:

$$\phi \longmapsto \left(\begin{array}{cccc} 1 & A_2 & \dots & A_r \\ 0 & t-a'_{22} & \dots & -a'_{2r} \\ \vdots & \vdots & & \vdots \\ 0 & -a'_{r2} & \dots & t-a'_{rr} \\ \hline b_{11} & b_{12} & \dots & b_{1r} \\ 0 & b'_{22} & \dots & b'_{2r} \\ \vdots & \vdots & & \vdots \\ 0 & b'_{s2} & \dots & b'_{sr} \end{array}\right) \qquad \begin{array}{l}(A_j\in R\oplus R\cdot t,\ \ 2\leqslant j\leqslant r).\\ \quad (a'_{ij}, b'_{ij}\in R)\end{array}$$

$$\longmapsto \left(\begin{array}{cccc} 1 & A_2 & \dots & A_r \\ 0 & t-a'_{22} & \dots & -a'_{2r} \\ \vdots & \vdots & & \vdots \\ 0 & -a'_{r2} & \dots & t-a'_{rr} \\ \hline 0 & B_2 & \dots & B_r \\ 0 & b'_{22} & \dots & b'_{2r} \\ \vdots & \vdots & & \vdots \\ 0 & b'_{s2} & \dots & b'_{sr} \end{array}\right) \qquad (B_j\in R\oplus R\cdot t,\ \ 2\leqslant j\leqslant r).$$

$$\longmapsto \left(\begin{array}{c|ccc} 1 & A_2 & \dots & A_r \\ \cline{2-4} 0 & t-a'_{22} & \dots & -a'_{2r} \\ \vdots & \vdots & & \vdots \\ 0 & -a'_{r2} & \dots & t-a'_{rr} \\ \hline 0 & b_2 & \dots & b_r \\ 0 & b'_{22} & \dots & b'_{2r} \\ \vdots & \vdots & & \vdots \\ 0 & b'_{s2} & \dots & b'_{sr} \end{array}\right) \qquad (b_j \in R,\ 2 \leqslant j \leqslant r).$$

By $r-1$ more elementary *column* transformations, we may bring the A_j's to zero. From there on the induction proceeds. □

§4. Roberts' Proof of Horrocks' Theorem

In this section, we present a completely different proof of Horrocks' Theorem that is due to Paul Roberts (c. 1976). Throughout this section, R shall denote a commutative local ring, with maximal ideal $\mathfrak{m}$, and *we shall write* "bar" *to denote reduction modulo* $\mathfrak{m}$, for R-modules as well as for R-algebras: $\overline{M} = M/\mathfrak{m}\cdot M$. Recall that $\mathfrak{M}(A)$ denotes the category of f.g. (left) A-modules.

To achieve greater generality, we shall present an "axiomatization" of Roberts' ideas. Aside from being more general, this axiomatic rendition of [P. Roberts: 1976] has the further advantage that it shows exactly what are the various assumptions needed for the whole argument to work.

Theorem 4.1. *Let* $(R, \mathfrak{m})$ *be a commutative local ring, and A be an R-algebra (not necessarily commutative). Let S be a multiplicative set of central non zero-divisors in A, and* $n \geqslant 0$ *be a fixed integer. Assume that the following hypotheses hold:*

(1) *For any* $f \in S$, $A/f\cdot A \in \mathfrak{M}(R)$;
(2) $\mathrm{GL}_n(\overline{S^{-1}A}) = \mathrm{im}\,\big(\mathrm{GL}_n(S^{-1}A)\big)\cdot \mathrm{im}\,\big(\mathrm{GL}_n(\overline{A})\big)$;
(3) $S^{-1}A$ *contains an* R*-subalgebra* B *(not necessarily commutative) such that* $S^{-1}A = A + B$ *and* $\mathfrak{m}B \subseteq \mathrm{rad}\, B$.

Let $P \in \mathfrak{M}(A)$ *be such that all elements* $f \in S$ *act on* P *as non zero-divisors. If*

$$\overline{P} \cong \overline{A}^n \quad \text{and} \quad S^{-1}P \cong (S^{-1}A)^n,$$

then $P \cong A^n$.

To motivate this abstract formulation, let us first deduce the algebraic form of Horrocks' Theorem as a special case of this result.

Corollary 4.2. *Let* R *be a commutative local ring, and* $P \in \mathfrak{P}\big(R[t]\big)$. *If* $P\langle t\rangle = R\langle t\rangle \otimes_{R[t]} P$ *is* $R\langle t\rangle$*-free, then* P *is* $R[t]$*-free.*

Proof. We apply (4.1) with $A = R[t]$, and S the set of all monic polynomials. Obviously, these are non zero-divisors, and $S^{-1}A = R\langle t\rangle$ by definition. For hypothesis (1), we note that, for $f \in S$ of degree d, $A/f \cdot A = \sum R \cdot t^i$ over $0 \leqslant i < d$. For hypothesis (2), note that $\overline{S^{-1}A} = \overline{R}\langle t\rangle$. Since this is a field,

$$\mathrm{GL}_n\big(\overline{S^{-1}A}\big) = \mathrm{E}_n\big(\overline{S^{-1}A}\big) \cdot \{\text{nonsingular diagonal matrices}\}.$$

Obviously $\mathrm{E}_n(S^{-1}A) \to \mathrm{E_n}\big(\overline{S^{-1}A}\big)$ is onto, so we can actually infer that $\mathrm{GL}_n(S^{-1}A) \to \mathrm{GL}_n\big(\overline{S^{-1}A}\big)$ is onto if any nonzero element $\alpha \in \overline{R}\langle t\rangle$ can be lifted to a unit in $S^{-1}A = R\langle t\rangle$. But we can write α as $\overline{r} \cdot \overline{g}/\overline{f}$, where $r \in R\backslash\mathfrak{m}$, and f, g are monic. The obvious lift $r \cdot g/f$ is clearly a unit in $R\langle t\rangle$. For hypothesis (3), take $B \subseteq R\langle t\rangle$ to be the ring constructed in (1.4). By (1.4)(3) and (1.4)(1), B is a local ring with maximal ideal $\supseteq \mathfrak{m}B$, and $R\langle t\rangle = A + B$. Thus, all hypotheses in (4.1) are satisfied. Let $P \in \mathfrak{P}\big(R[t]\big)$ with $P\langle t\rangle \cong R\langle t\rangle^n$. We have $\overline{P} \in \mathfrak{P}\big(\overline{R}[t]\big)$ so $\overline{P} \cong \overline{R}[t]^n$ since $\overline{R}[t]$ is a PID. Finally, the projectivity of P implies that all elements $f \in S$ act on P as non zero-divisors. Applying the theorem, we get $P \cong R[t]^n$ $\square$

We shall now give the proof of (4.1). From the hypothesis on P, we can view $P \subseteq S^{-1}P$. Take $x_1, \ldots, x_n \in S^{-1}P$ which form an $S^{-1}A$-basis of $S^{-1}P$, and take $y_1, \ldots, y_n \in P$ such that $\overline{y}_1, \ldots, \overline{y}_n$ form an $\overline{A}$-basis for $\overline{P}$. Let α be the natural map from $\overline{P}$ to $\overline{S^{-1}P}$:

$$\begin{tikzcd}
R \arrow[r] \arrow[d] & B \arrow[d, phantom, "\cap" description] \\
S \subseteq A \arrow[r] \arrow[d] & S^{-1}A \arrow[d] \\
\overline{A} \arrow[r] & \overline{S^{-1}A}
\end{tikzcd}
\qquad
\begin{tikzcd}
y_i \in P \arrow[r, phantom, "\subseteq"] \arrow[d, two heads] & S^{-1}P \ni x_i \arrow[d, two heads] \\
\overline{y}_i \in \overline{P} \arrow[r, "\alpha"] & \overline{S^{-1}P}
\end{tikzcd}$$

We may assume, without loss of generality, that $\alpha(\overline{y}_i) = \overline{x}_i$ *for all* i. In fact, both $\{\overline{x}_i\}$ and $\{\alpha(\overline{y_i})\}$ are $\overline{S^{-1}A}$-bases for $\overline{S^{-1}P}$, so they are related by a matrix in $\mathrm{GL}_n\big(\overline{S^{-1}A}\big)$. By (2), this matrix has the form $\mathrm{im}(\sigma) \cdot \mathrm{im}(\tau)$, where $\sigma \in \mathrm{GL}_n(S^{-1}A)$ and $\tau \in \mathrm{GL}_n\big(\overline{A}\big)$. After changing $\{x_i\}$ by σ, and $\{y_i\}$ by τ^{-1}, we can clearly get what we want.

In $S^{-1}P$, form the A-submodule $P' = \sum A \cdot x_i \cong A^n$. Then P and P' have the same image in $\overline{S^{-1}P}$. Using hypothesis (3), we have

$$S^{-1}P = \sum (A + B) \cdot x_i = P' + Q,$$

where $Q = \sum B \cdot x_i \cong B^n$. Reducing this mod $\mathfrak{m}$, we see that $\overline{S^{-1}P}$ is generated by the images of P and Q.

(4.3) *We claim that, in fact,* $S^{-1}P = P + Q$.

To see this, let C be the R-module $S^{-1}P/(P+Q)$. We already have $\overline{C}=0$, so it suffices (by Nakayama's Lemma) to prove that $C \in \mathfrak{M}(R)$. There is a surjection

$$(P'+P)/P \to (P'+Q)/(P+Q) = C,$$

so it is enough to show that $(P'+P)/P \in \mathfrak{M}(R)$. Take $f \in S$ such that $f \cdot P' \subseteq P$. Then $(P'+P)/P \in \mathfrak{M}(A/f \cdot A)$. By hypothesis (1), we have indeed $(P'+P)/P \in \mathfrak{M}(R)$.

Having established (4.3), we see that $x_i - y_i \in \mathfrak{m} \cdot S^{-1}P = \mathfrak{m}P + \mathfrak{m}Q$, for all i. Thus, after changing x_i mod $\mathfrak{m}Q$ and changing y_i mod $\mathfrak{m}P$, we may assume that $x_i = y_i$ for all i. (In particular, we have now $P' \subseteq P$.) Note that, in view of (I.1.7) *and the hypothesis that* $\mathfrak{m}B \subseteq \operatorname{rad} B$, when we change the x_i 's mod $\mathfrak{m}Q$, the "new" x_i 's will remain a B-basis for Q, and therefore, by scalar extension, remain an $S^{-1}A$-basis for $S^{-1}P$.

By choice of the y_i's, we have $P = \mathfrak{m}P + \sum A \cdot y_i = \mathfrak{m}P + P'$. But by the argument in an earlier paragraph, $P/P' \in \mathfrak{M}(R)$. Thus, Nakayama's Lemma implies that $P = P' \cong A^n$, with a free A-basis $\{x_i\}$. □

Acknowledgement. My thanks are due to Dr. Joseph Gubeladze for suggesting the present form of the hypothesis (2) in the formulation of Theorem 4.1.

§5. Nashier-Nichols' Proof of Horrocks' Theorem

In this section, we present yet another proof of Horrocks' theorem that is due to B. Nashier and W. Nichols, in their joint paper in 1987. (As the reader can see, this is a much later proof, found only some ten years *after* the resolution of Serre's Conjecture.) This proof is rather short, and its overall structure is also relatively easy to explain. One inducts on the rank of the projective $R[t]$-module P in question, using Nakayama's Lemma (and an idea of Roitman) to accomplish the inductive step. To get the induction started, one needs to prove Horrocks' Theorem in the rank 1 case. Here, one uses the basic theory of invertible ideals in $R[t]$, coupled with an ingenious lemma for lifting monic polynomials from $(R/\mathfrak{m})[t]$, where $\mathfrak{m}$ is the unique maximal ideal of the local ring R.

To formulate this proof from [Nashier-Nichols: 1987], let us start with their lemma on lifting monic polynomials. Proceeding more generally, let J be an ideal in a (not necessarily commutative) ring R such that $J \subseteq \operatorname{rad}(R)$, and let $k := R/J$. We shall write "bar" for the projection map from the polynomial ring $R[t]$ to the factor ring $R[t]/J[t]$. Since $R[t]/J[t] \cong k[t]$, it makes sense to talk about "monic polynomials" in $R[t]/J[t]$.

Lemma 5.1. *In the above notations, let I be a left ideal in $R[t]$ containing a monic (polynomial). Then any monic $\gamma \in \overline{I}$ is the image of a suitable monic in I.*

Proof. Say $\gamma = g + J[t]$ $(g \in I)$ is monic of degree r in $k[t]$. Then $m := \deg g \geqslant r$, say $g - at^m + \cdots$. Let us first assume that I contains a monic f of degree $r+1$. If $m = r$, then $a \in 1 + J \subseteq \mathrm{U}(R)$. In this case, $a^{-1}g \in I$ is monic and obviously lifts γ. If $m > r$, then $a \in J$. Here,

$$g' := g - at^{m-r-1}f \in I$$

lifts γ, with deg $g' <$ deg g. Continuing this process, we'll arrive at the desired monic lift for γ.

Next, we show that I *does* contain a monic of degree $r+1$. By assumption, I contains a monic of some degree, and therefore it contains monics of every higher degree. Thus, it suffices to show that, if I contains a monic of degree $n > r+1$, then it contains a monic of degree $n-1$. Since $n - 1 \geqslant r$, $\overline{I}$ contains a monic β of degree $n-1$. By the last paragraph, there is a monic $h \in I$ (necessarily of degree $n-1$) with $h + J[t] = \beta$, so we are done. □

For the balance of this section, we return to the situation of a commutative local ring R with maximal ideal $\mathfrak{m}$. Let k denote the residue field $R/\mathfrak{m}$, and let $A = R[t]$. In the following, "bar" will continue to denote the "mod $\mathfrak{m}$" map, from A to $k[t]$. We'll now prove the following proposition, which will be seen to (essentially) correspond to the rank 1 case of Horrocks' Theorem. (For this proposition, the reader is assumed to be familiar with the basic facts about invertible ideals in a commutative ring.)

Proposition 5.2. *Let I be an invertible ideal of $A = R[t]$. If I contains a monic, then $I = Af$ for some monic $f \in I$. (In particular, $I \cong A$, since f is clearly not a zero-divisor in A.)*

Proof. Of course $I \nsubseteq \mathfrak{m}[t]$, so $\overline{I} \neq 0$. Since $k[t]$ is a PID, $\overline{I}$ is generated by a monic polynomial γ. By (5.1), $\gamma = f + \mathfrak{m}[t]$ for a suitable *monic* $f \in I$. Thus, $I = Af + I \cap \mathfrak{m}[t]$. *We claim that* $I \cap \mathfrak{m}[t] = \mathfrak{m} I$. To see this, let $L = I^{-1}(I \cap \mathfrak{m}[t])$ (where I^{-1} is, of course, formed in the total ring of quotients of A). Then $I\,I^{-1} = A$ yields

$$I\,L = I \cap \mathfrak{m}[t] \subseteq \mathfrak{m}[t].$$

Since I contains a monic, this implies that $L \subseteq \mathfrak{m}[t]$. Therefore, $I \cap \mathfrak{m}[t] \subseteq \mathfrak{m} I$, and so equality holds. Let $M = I/Af$, which is f.g. over A/Af (since I is f.g. over A). The fact that f is monic now implies that M is f.g. over R. Since $\mathfrak{m} M = M$, we have $M = 0$ by Nakayama's Lemma, and hence $I = Af$, as desired. □

Before we come to the proof of Horrocks' Theorem, we'll need just one more result about the polynomial ring $R[t]$ (for $(R, \mathfrak{m})$ local).

Top-Bottom Lemma 5.3. *Let $f = \sum_{i=0}^{n} a_i t^i$, $g = \sum_{j=0}^{m} b_j t^j$ in $A = R[t]$ be such that a_n, $b_0 \in \mathrm{U}(R)$. If either all $a_0, \ldots, a_{n-1} \in \mathfrak{m}$ or all $b_1, \ldots, b_m \in \mathfrak{m}$, then f, g are comaximal in A; that is, $Af + Ag = A$.*

Proof. If otherwise, $f,\ g$ are contained in a maximal ideal $M \subseteq A$. Since $f \in M$ is unitary, $R/R \cap M \subseteq A/M$ is an integral ring extension. But A/M is a field, so by the Cohen-Seidenberg theorem (for instance), $R/R \cap M$ is also a field. This means that $R \cap M = \mathfrak{m}$. We now go into the two cases.

Case 1. All $b_1, \ldots, b_m \in \mathfrak{m}$. Since $g \in M$ and $\mathfrak{m} \subseteq M$, this implies that $b_0 \in M$, which is impossible (since $b_0 \in \mathrm{U}(R)$).

Case 2. All $a_1, \ldots, a_{n-1} \in \mathfrak{m}$. Since $f \in M$ and $\mathfrak{m} \subseteq M$, this implies that $a_n t^n \in M$, and hence $t \in M$ (since $a_n \in \mathrm{U}(R)$). But then $g \in M$ yields the same contradiction $b_0 \in M$ as in *Case 1.* $\square$

Remark. (1) We called (5.3) the "Top-Bottom Lemma" since this result requires a unit assumption on the top coefficient and the bottom coefficient of the two polynomials involved. What we need for the proof of Horrocks' Theorem below is *Case 1* of this Lemma. It turns out that *Case 2* will be needed in the next chapter too; see the proof of V.5.16.

(2) In this Lemma, the *local* assumption (that is, $\mathfrak{m}$ is the *only* maximal ideal in R) is important. For instance, let $R = \mathbb{Z}$ with the maximal ideal $\mathfrak{m} = 2\mathbb{Z}$ (or $\mathfrak{m} = 3\mathbb{Z}$). Then $f = t + 2$ and $g = 3t + 1$ satisfy the hypotheses of (5.3). But $f,\ g$ are *not* comaximal in $R[t]$, since they are contained in the maximal ideal $(5,\ t+2)$.

We are now ready to give our last proof of Horrocks' theorem, which we restate for the convenience of the reader.

Horrocks' Theorem 5.4. *Let* $A = R[t]$ *where* $(R,\ \mathfrak{m})$ *is a commutative local ring. Let* $S \subseteq A$ *be the multiplicative set of monic polynomials, and let* $P \in \mathfrak{P}\big(R[t]\big)$. *If* $P_S \cong A_S^n$, *then* $P \cong A^n$.

Proof. First we note that R (and hence A) has no nontrivial idempotents. Therefore, P is of constant rank n. We shall prove Horrocks' Theorem by induction on n.

To begin the induction, let $n = 1$. Say $\varphi : P_S \to A_S$ is an A_S-isomorphism. Since P is f.g., we may assume (after a slight modification of φ) that $I := \varphi(P) \subseteq A$. Since $I_S = A_S$, I contains an element in S (that is, a monic polynomial). From this, and the fact that $I \cong P$ is projective, we see easily that I is an *invertible* ideal of A. We can thus conclude from (5.2) that $P \cong I \cong A$.

For the inductive step, assume now $n \geqslant 2$. Let $p_1, \ldots, p_n \in P$ be a free A_S-basis for P_S. Since $\overline{A} = k[t]$ is a PID, $\overline{P}$ is $\overline{A}$-free, so by the theorem on elementary divisors over a PID, there exists a free basis $\overline{q}_1, \ldots, \overline{q}_n$ in $\overline{P}$ such $\overline{p}_1 = \alpha\,\overline{q}_2$ for some $\alpha \in k[t]$. Following an idea of Roitman, we set $p := q_1 + t^r p_1 \in P$, where r is to be specified. Then $\overline{p} = \overline{q}_1 + t^r \alpha\,\overline{q}_2$, so $\overline{p},\ \overline{q}_2, \ldots, \overline{q}_n$ form a free $\overline{A}$-basis for $\overline{P}$. Choose $s \in S$ such that $sq_1 = \sum_{i=1}^n h_i p_i$ for some $s_i \in A$. Then

$$sp = (h_1 + st^r)\, p_1 + \sum_{i=2}^{n} h_i p_i. \tag{5.5}$$

Now choose r big enough so that $h_1 + st^r$ is monic. Then, by (5.5), (5.6) $$p,\ p_2,\ \dots,\ p_n \text{ form a free } A_S\text{-basis for } P_S.$$

For the multiplicative set $T := 1 + \mathfrak{m}\,[t] = 1 + \mathfrak{m}\,A$ in A, we claim that

(5.7) $$p,\ q_2,\ \dots,\ q_n \text{ form a free } A_T\text{-basis for } P_T.$$

To see this, first note that $1 + \mathfrak{m}\,A \subseteq \mathrm{U}(A_T)$ implies that $1 + \mathfrak{m}\,A_T \subseteq \mathrm{U}(A_T)$. Therefore, $\mathfrak{m}\,A_T \subseteq \mathrm{rad}(A_T)$. Now

(5.8) $$\frac{P_T}{\mathfrak{m}A_T \cdot P_T} \cong \left(\frac{P}{\mathfrak{m}A \cdot P}\right)_T = \left(\overline{P}\right)_T = \overline{P}.$$

Since $\overline{p},\ \overline{q}_2,\ \dots,\ \overline{q}_n$ form a free basis for $\overline{P}$, (5.7) follows from (I.1.7) (and the ensuing remark).

Now consider the A-module $Q = P/A \cdot p$. From (5.6), $Q_S \cong P_S/A_S \cdot p$ is free of rank $n-1$, and from (5.7), $Q_T \cong P_T/A_T \cdot p$ is also free of rank $n-1$. Since any maximal ideal of A avoids at least one of S and T by the Top-Bottom Lemma (5.3), it follows that Q is locally free of rank $n-1$. Thus, from (I.3.5′), we see that Q is (f.g.) *projective* of rank $n-1$ over R. Since $Q_S \cong A_S^{n-1}$, the inductive hypothesis implies that $Q \cong A^{n-1}$, and therefore $P \cong A \cdot p \oplus Q \cong A^n$, as desired. □

Remark 5.9. In Chapter V, we'll prove a more general "Affine Horrocks' Theorem" (V.2.3), which implies that Horrocks' Theorem (5.4) is actually true for *any* commutative ring R. In view of this, a result such as (5.2) can be seen to be also true for any commutative ring R.

§6. Murthy-Horrocks Theorem

The Murthy-Horrocks Theorem states that, for *any commutative regular local ring* R *of (Krull) dimension* 2, *any* $P \in \mathfrak{P}(R[t])$ *is free*. This important result was first proved in [Horrocks: 1964] with the restriction that R contains a coefficient field,(*) and later proved without this restriction in [Murthy: 1966]. Both of their proofs are based on Horrocks' Theorem in §2, so the result belongs quite naturally to the present chapter.

The presentation in this section follows essentially Murthy's proof, which, besides using Horrocks' Theorem, relies heavily on the consideration of the group of elementary matrices. We shall therefore need some of our earlier results on elementary matrices in (I, §5), as well as some additional facts relating E_n to SL_n.

Lemma 6.1. *Let B be a commutative ring, and let $s \in B$ be a non zero-divisor such that $B/s \cdot B$ is a* PID. *If*

$$\mathrm{SL}_n(B) = \mathrm{E}_n(B), \quad \textit{and} \quad \mathrm{SL}_n(B/s \cdot B) = \mathrm{E}_n(B/s \cdot B)$$

(where n is fixed), then $\mathrm{SL}_n(B_s) = \mathrm{E}_n(B_s)$. *[$B_s$ denotes the localization of B at the multiplicative set $\{s^i : i \geqslant 0\}$.]*

(*) I.e. there exists a subfield of R which maps isomorphically onto the residue class field of R.

Proof. Throughout the proof, we shall view B as a subring of B_s. Let $T \in \mathrm{SL}_n(B_s)$. For m sufficiently large, we have $T_1 = s^m \cdot T \in \mathbb{M}_n(B)$ ($= n \times n$ matrices over B), with $\det T_1 = s^k$ for some $k \geqslant 0$. For such a matrix T_1 over B, we will show, by induction on k, that T_1 can be factored into $E \cdot D$ where $E \in \mathrm{E}_n(B_s)$, and $D \in \mathbb{D}_n(B_s)$ ($= n \times n$ invertible diagonal matrices over B_s). If $k = 0$, then $T_1 \in \mathrm{SL}_n(B) = \mathrm{E}_n(B)$ and we are done. Suppose now $k > 0$. Writing "bar" to denote reduction modulo s, we have $\det \overline{T}_1 = 0$ over the principal ideal domain $\overline{B}$. By the theorem of elementary divisors over $\overline{B}$, there exist $\overline{M}$, $\overline{N} \in \mathrm{SL}_n(\overline{B})$ such that

$$\overline{M}\,\overline{T}_1\,\overline{N} = \mathrm{diag}\left(\bar{b}_1, \ldots, \bar{b}_{n-1}, \bar{0}\right).$$

Since, by hypothesis, $\mathrm{SL}_n(\overline{B}) = \mathrm{E}_n(\overline{B})$, we can find M, $N \in \mathrm{E}_n(B)$ reducing to $\overline{M}$, $\overline{N}$ mod s. Thus, all entries on the n^{th} column of MT_1N are multiples of s, and so

$$T_2 = MT_1N \cdot \mathrm{diag}\left(1, \ldots, 1, s^{-1}\right) \in \mathbb{M}_n(B),$$

with $\det T_2 = s^{k-1}$. By the inductive hypothesis, we can write $T_2 = E' \cdot D'$ where $E' \in \mathrm{E}_n(B_s)$ and $D' \in \mathbb{D}_n(B_s)$, so now

$$T_1 = M^{-1}E'D' \cdot \mathrm{diag}\,(1, \ldots, 1, s) \cdot N^{-1}.$$

But $\mathbb{D}_n(B_s)$ normalizes $\mathrm{E}_n(B_s)$ by (I.5.0). Thus, we can rewrite T_1 as $E \cdot D$, where $E \in \mathrm{E}_n(B_s)$ and $D \in \mathbb{D}_n(B_s)$. This completes the induction argument. Finally, from $T = E \cdot s^{-m}D$, we have $s^{-m}D \in \mathrm{SL}_n(B_s)$. Since

$$\mathbb{D}_n(B_s) \cap \mathrm{SL}_n(B_s) \subseteq \mathrm{E}_n(B_s)$$

by (I.5.2)(2), it follows that $T \in \mathrm{E}_n(B_s)$, as desired. □

The equality $\mathrm{SL}_n = \mathrm{E}_n$ holds for all euclidean domains (by (I.5.4)), but unfortunately not for all PID's, as we have seen in (I.8). Let us say that a PID, R, is *special*, if, for all n, $\mathrm{SL}_n(R) = \mathrm{E}_n(R)$. The following two corollaries furnish some nice examples of these special PID's, and they will play a crucial role in the proof of the Murthy-Horrocks Theorem.

Corollary 6.2. *Let B be a commutative regular local ring of (Krull) dimension* 2, *with maximal ideal generated by π and s* (*see* (II.5.6)). *Then the localization B_s is a special* PID.

Proof. It is well known that (s) is prime. The quotient $B/s \cdot B$ is a commutative noetherian local domain, with a principal maximal ideal (generated by $\pi + s \cdot B$). Thus, $B/s \cdot B$ is a discrete valuation ring. In particular, it is a euclidean domain, and a special PID. Since B is local, $\mathrm{SL}_n(B) = \mathrm{E}_n(B)$ by (I.5.4). Applying (6.1), we obtain $\mathrm{SL}_n(B_s) = \mathrm{E}_n(B_s)$. *It remains to show that B_s is a* PID. To begin with, we know that B_s is a Dedekind ring, since it is a regular domain of dimension 1. To see that B_s is a PID, we can give two different proofs.

First Proof. By the theorem of Auslander and Buchsbaum,(*) the regular local ring B_s is a UFD. Hence its localization B_s is also a UFD. This shows that B_s is a PID.

Since we are dealing only with the 2-dimensional case, we certainly do not need the full force of the deep result of Auslander and Buchsbaum. We shall, therefore, supply the following *ad hoc* proof:

Second Proof. Consider a typical ideal in B_s, say $\mathfrak{A}_s$, where $\mathfrak{A}$ is an ideal in B. Since B is regular and local, there exists a finite resolution

$$0 \to F_m \to F_{m-1} \to \cdots \to F_0 \to \mathfrak{A} \to 0$$

where F_i 's are free R-modules of finite rank. Localizing this at s, we obtain a finite free resolution for $\mathfrak{A}_s$. But $\mathfrak{A}_s$ is B_s-projective (since Dedekind rings are hereditary domains). From (I.6.3), we conclude that $\mathfrak{A}_s$ must be principal. □

Corollary 6.3. *If R is a discrete valuation ring, then $R\langle t\rangle$ is a special* PID.

Proof. Let (π) be the maximal ideal of R. We shall use the subring B in $R\langle t\rangle$ constructed in (1.4). According to (1.4)(3), we have $B = R[s]_{(\pi,s)}$, where $s = 1/t$. Since $R[s]$ is regular of dimension 2, its localization B is exactly as in (6.2). By (6.2), B_s is a special PID. But B_s is just $R\langle t\rangle$ by (1.4)(1). □

Remark. Of course, we also know $R\langle t\rangle$ is a PID from (1.3)(3), so, as far as (6.3) is concerned, the second half of the proof of (6.2) is unnecessary. In the same vein, we may observe that the B in (6.3) *clearly* has the properties invoked for general 2-dimensional regular local rings in (6.2) and its proof. Thus, for the purposes of proving (6.3), we need not invoke any extraneous information at all.

The following two corollaries will not be needed for the proof of the Murthy-Horrocks Theorem. We include them here for the sake of completeness, since they elaborate some of the above results.

Corollary 6.4. *Let S be a multiplicative set in a commutative ring R (not containing* 0). *If R is a special* PID, *so is the localization R_S.*

Proof. Say $T \in \mathrm{SL}_n(R_S)$. There exists $s \in S$ such that $T \in \mathrm{SL}_n(R_s)$. Write $s = up_1^{a_1} \cdots p_k^{a_k}$, where $u \in \mathrm{U}(R)$, and the p_i 's are mutually nonassociate prime elements in R. For $s_0 := p_1 \cdots p_k$, R/s_0R is a direct product of the fields R/p_iR, so we can apply the arguments in the proof of (6.1) to deduce that $\mathrm{SL}_n(R_{s_0}) = \mathrm{E}_n(R_{s_0})$. But clearly, $R_s = R_{s_0}$, so we have

$$T \in \mathrm{SL}_n(R_s) = \mathrm{E}_n(R_s) \subseteq \mathrm{E}_n(R_S),$$

as desired. □

(*) See, e.g., [Zariski-Samuel: 1960], Appendix 7.

Using this, we get an immediate strengthening of (6.2).

Corollary 6.5. *Let B and s be as in* (6.2)*, and let S be any multiplicative set of B containing s. Then B_S is a special* PID.

We come now to the principal result of this section.

Theorem 6.6. (Murthy-Horrocks) *Let R be a commutative regular local ring of* (*Krull*) *dimension* 2. *Then any $Q \in \mathfrak{P}(R[t])$ is free.*

Proof. Let $A = R\langle t\rangle$. By Horrocks' Theorem, it is sufficient to show that $P = A \otimes_{R[t]} Q$ is A-free. Let π, s generate the maximal ideal of R. By (6.3),

$$A/s \cdot A \cong (R/s \cdot R)\langle t\rangle$$

is a special PID. This observation enables us to apply the axiomatic version of Seshadri's Theorem (II.6.4) to $A = R\langle t\rangle$, localized at $S = \{s^i : i \geqslant 0\}$. (Clearly, the only S-irreducible element is s.) According to (II.6.4), to check that P is A-free, it suffices to check that P_S is A_S-free. But P_S is extended from $Q_s \in \mathfrak{P}(R_s[t])$:

$$\begin{array}{ccc} R_s[t] & \longrightarrow & R\langle t\rangle_s = A_s \\ \uparrow & & \uparrow \\ R[t] & \longrightarrow & R\langle t\rangle = A \end{array}$$

so it is enough to show that Q_s is $R_s[t]$-free. From (6.2), we know that R_s is a (special) PID. Thus, Seshadri's Theorem (II.6.1) guarantees that any module in $\mathfrak{P}(R_s[t])$ is free. $\square$

Notes on Chapter IV

Horrocks' Theorem, in its geometric form (2.2), first appeared in [Horrocks: 1964]. The algebraic form (2.1) was described in [Bass: 1972a], which contained Murthy's remark that, for any $P \in \mathfrak{P}(R[t])$, if $P\langle t\rangle$ is $R\langle t\rangle$-free, then P extends to a vector bundle on $\mathbb{P}^1_R$. From this remark, the equivalence of the geometric form and the algebraic form follows.

The Towber Presentation Theorem (3.5) is in [Murthy-Towber: 1974], and [Swan-Towber: 1975]. Our exposition of Swan's algebraic proof of Horrocks' Theorem (based on the Towber Presentation) follows [Ferrand: 1976]. Lindel's proof of Horrocks' Theorem [Lindel: 1977] was actually very close to (but independent of) Swan's. I first learned about Lindel's proof through the notes of [Knus: 1976], which was brought to my attention by A. Rosenberg. Another exposition of Lindel's proof can be found in [Eisenbud: 1977].

Theorem (4.1) is my "axiomatization" of Paul Roberts' proof [P. Roberts: 1976] of Horrocks' Theorem. I first learned about Roberts' proof through the notes of a talk given by [Davis: 1976]. The formulation of an axiomatized form of Roberts' proof in our text turned out to be a fortuitous circumstance, since this axiomatized version has been used and extended later in various ways. For another formulation of Roberts' proof using the viewpoint of idempotent matrices, see [Dicks: 1976].

The last proof of Horrocks' Theorem in §5 was given considerably later in time, in [Nashier-Nichols: 1987]. For another formulation of this proof, see Mandal's Springer Lecture Notes, Vol. 1672, 1997 (although Mandal assumed the base ring to be noetherian in his exposition). The intricate inductive step in this proof, using Nakayama's Lemma, is based partly on the ideas of Roitman and Roy: see [Roitman: 1979], and [Roy: 1980] listed in the bibliographical references for Chapter VIII. At the beginning of the induction, the case of rank 1 projective modules over $R[t]$ was already treated in [Bass-Murthy: 1967], not only for R local but for R commutative in general. On p. 38 of this paper, Bass and Murthy lamented: "we can, unfortunately, do this only for P of rank 1." The case of arbitrary rank (for non-local R), now christened the Affine Horrocks' Theorem, was to wait another ten years for the birth of Quillen's Patching Theorem in 1976! We shall return to complete this part of the story in the next chapter: see (V.2.1) – (V.2.3).

Horrocks' version of the Murthy-Horrocks Theorem 6.6 assumed that the regular local ring of dimension 2 contains a coefficient field [Horrocks: 1964]. The fact that the theorem remains true without this restriction was proved in [Murthy: 1966]; see also Murthy's Tata thesis [Murthy: 1965b]. Murthy's proof utilized at one point a result of Auslander and Goldman, but it is shown in the text that this result can be bypassed.

Chapter V

Quillen's Methods

§1. Quillen's Patching Theorem

This chapter will be devoted mainly to the study of Quillen's proof of Serre's Conjecture and its various applications. The principal tool of Quillen's methods is his celebrated Patching Theorem (1.6), which is a local-global characterization of extended modules over polynomial rings. The main objective in §1 will be to establish this important result.

We shall start with a preliminary theorem about polynomial units. Let E be a (not necessarily commutative) ring, and let $x, y, \ldots$ be independent (commuting) indeterminates over E. We write $E[x, y, \ldots]^*$ to denote the group of invertible elements in $E[x, y, \ldots]$ with constant term 1. If f is a central element in E, we shall write $\big(E[x, y, \ldots]^*\big)_f$ to denote the image of $E[x, y, \ldots]^* \to E_f[x, y, \ldots]^*$.

Theorem 1.1. *Let E be a ring, f a central element in E, and x, y, t be independent indeterminates over E. If $\theta(t) \in E_f[t]^*$, then there exists $k \geqslant 0$ such that*

$$\theta\big((x + f^k y)\, t\big)\, \theta(xt)^{-1} \in \big(E[x, y, t]^*\big)_f.$$

By specializing x and y to central elements in E, we immediately obtain the following:

Corollary 1.2. *Let R be a commutative ring, and E be an R-algebra. Let $f \in R$, and $\theta(t) \in E_f[t]^*$. Then there exists $k \geqslant 0$ such that for any $a, b \in R$ with $a - b \in f^k R$, we have $\theta(at)\, \theta(bt)^{-1} \in \big(E[t]^*\big)_f$.*

Part of the utility of this result is that it enables us to "lift" polynomial units with local coefficients. For example, if we let $b = 0$ and $a = f^k$ in (1.2), we get the conclusion that $\theta(f^k t) \in (E[t]^*)_f$ (for some k).

Corollary 1.3. *Let E be an R-algebra as above, and let f_0, f_1 be two comaximal elements in R* (i.e., $f_0 R + f_1 R = R$). *Then*

$$E_{f_0 f_1}[t]^* = \big(E_{f_1}[t]^*\big)_{f_0} \cdot \big(E_{f_0}[t]^*\big)_{f_1}.$$

Proof. Let $\theta(t) \in E_{f_0 f_1}[t]^*$. We can apply (1.2) to both the localizations $E_{f_1} \to (E_{f_1})_{f_0}$ and $E_{f_0} \to (E_{f_0})_{f_1}$. Pick $k \geqslant 0$ that works for both localizations. For any $b \in R$, write

$$\theta(t) = [\theta(t)\,\theta(bt)^{-1}] \cdot \theta(bt).$$

Since $f_0^k R + f_1^k R = R$, we can pick $b \in f_1^k R$ such that $1 - b \in f_0^k R$. Then

$$\theta(t)\,\theta(bt)^{-1} \in (E_{f_1}[t]^*)_{f_0}, \quad \text{and} \quad \theta(bt)\theta(0)^{-1} \in \big(E_{f_0}[t]^*\big)_{f_1}. \qquad \square$$

We shall now prove (1.1). Parts of the ideas used here will be analogous to those in (III.2.2).

Proof of (1.1). Write $\theta(x+y) - \theta(x) = y \cdot \phi(x, y)$, where $\phi(x, y) \in E_f[x, y]$. For any $r \geqslant 0$, we have

$$\begin{aligned} \theta((x + f^r y)\,t)\,\theta(xt)^{-1} &= 1 + [\theta((x + f^r y)\,t) - \theta(xt)] \cdot \theta(xt)^{-1} \\ &= 1 + f^r yt \cdot \phi(xt, f^r yt) \cdot \theta(xt)^{-1}. \end{aligned} \tag{1.4}$$

Pick r such that $f^r \cdot \phi(x, y)\theta(x)^{-1}$ comes from $E[x, y]$. Thus, there exists $\sigma(x, y, t) \in E[x, y, t]$ such that

$$\theta((x + f^r y)\,t)\,\theta(xt)^{-1} = \sigma(x, y, t)_f. \tag{1.5}$$

Because of (1.4), we may clearly assume that $\sigma(x, y, t) \equiv 1 \pmod{yt}$; but $\sigma(x, y, t)$ need not be invertible in $E[x, y, t]$. We shall now try to adjust the choice of σ. Note that in $E_f[x, y, t]$, the inverse of $\sigma(x, y, t)$ is

$$\theta(xt)\,\theta\big((x + f^r y)\,t\big)^{-1} = \sigma(x + f^r y, -y, t)_f.$$

Thus, if we let

$$\sigma'(x, y, t) = \sigma(x + f^r y,\ -y,\ t) \in E[x, y, t],$$

we will have $\sigma^{-1} = \sigma'$ in $E_f[x, y, t]$. Clearly, $\sigma'(x, y, t) \equiv 1 \pmod{yt}$, so we can write

$$\begin{aligned} \sigma\sigma' &= 1 + yt \cdot \mu_1, \quad \mu_1 \in E[x, y, t], \\ \sigma'\sigma &= 1 + yt \cdot \mu_2, \quad \mu_2 \in E[x, y, t]. \end{aligned}$$

Take a big power, say f^s, such that $f^s \mu_1 = f^s \mu_2 = 0$. It follows that $\sigma(x,\ f^s y,\ t) \in E[x, y, t]^*$ (with inverse $\sigma'(x,\ f^s y,\ t)$). Replacing y by $f^s y$ in (1.5), we get

$$\theta\big((x + f^{r+s} y)\,t\big)\,\theta(xt)^{-1} = \sigma(x, f^s y, t)_f \in \big(E[x, y, t]^*\big)_f. \quad \square$$

Recall that, for any ring A, the notation $M \in \mathfrak{M}^A\big(A[t_1, \ldots, t_n]\big)$ means that the f.g. $A[t_1, \ldots, t_n]$-module M is extended from A, i.e., there exists an A-module N such that $M \cong A[t_1, \ldots, t_n] \otimes_A N$. We have necessarily $N \cong M/(t_1, \ldots, t_n)M \in \mathfrak{M}(A)$. With these notations in place, we are now ready to prove the following general local-global principle of Quillen on extended modules over polynomial rings.

Theorem 1.6. (Quillen's Patching Theorem) *Let R be a commutative ring. Let A be any (not necessarily commutative) R-algebra, and let M be a finitely presented $A[t_1, \dots, t_n]$-module. Then:*

(A_n) $\mathrm{Q}(M) := \{ g \in R : M_g \in \mathfrak{M}^{A_g}\big(A_g[t_1, \dots, t_n]\big)\}$ *is an ideal in R (called the* Quillen ideal *of M).*

(B_n) *If $M_{\mathfrak{m}} \in \mathfrak{M}^{A_{\mathfrak{m}}}\big(A_{\mathfrak{m}}[t_1, \dots, t_n]\big)$ for every maximal ideal $\mathfrak{m} \in \operatorname{Max} R$, then $M \in \mathfrak{M}^A\big(A[t_1, \dots, t_n]\big)$.*

(For applications to Serre's Conjecture, we shall need this theorem only in the case $A = R$. However, the proof for the general case does not require any extra work, so we may as well do the general case here. Of course, the general case has the advantage that it can be applied to a *noncommutative* setting. For such an application, see (1.17) below.)

Proof. Step 1. *We first show that* $(\mathrm{A}_n) \Rightarrow (\mathrm{B}_n)$ (for any n). For this, it suffices to check that, for M as in (B_n), the Quillen ideal $\mathrm{Q}(M)$ formed in (A_n) is the unit ideal, R. Let

$$M' = A[t_1, \dots, t_n] \otimes_A (M/(t_1, \dots, t_n)M),$$

which is a finitely presented $A[t_1, \dots, t_n]$-module extended from A. For any $\mathfrak{m} \in \operatorname{Max} R$, there exists an isomorphism $\phi : M_{\mathfrak{m}} \to M'_{\mathfrak{m}}$. By (I.2.16), ϕ is the localization of an $A_g[t_1, \dots, t_n]$-isomorphism $M_g \to M'_g$, for some $g \in R \setminus \mathfrak{m}$. We have then $g \in \mathrm{Q}(M) \setminus \mathfrak{m}$, and thus $\mathrm{Q}(M) \not\subseteq \mathfrak{m}$. This shows that $\mathrm{Q}(M) = R$, as desired.

Step 2. *If* (A_1) *holds, then* (A_n) *holds for all n*. By induction, we may assume that (A_{n-1}) holds, and hence (B_{n-1}) also holds. Form the set $\mathrm{Q}(M)$ as in (A_n). Clearly, $R \cdot \mathrm{Q}(M) \subseteq \mathrm{Q}(M)$, so it is enough to show that

$$f_0,\ f_1 \in \mathrm{Q}(M) \implies f = f_0 + f_1 \in \mathrm{Q}(M).$$

Let $N = M/t_n M$, which is finitely presented over $A[t_1, \dots, t_{n-1}]$, and

$$L = M/(t_1, \dots, t_n)M,$$

which is finitely presented over A. Applying (A_1) to

$$A[t_1, \dots, t_{n-1}] \longrightarrow A[t_1, \dots, t_{n-1}][t_n],$$

we see that M_f is extended from N_f, which is finitely presented over $A_f[t_1, \dots, t_{n-1}]$. *We claim that* $N_f \in \mathfrak{M}^{A_f}\big(A_f[t_1, \dots, t_{n-1}]\big)$. If so, we will have

$$M_f \in \mathfrak{M}^{A_f}\big(A_f[t_1, \dots, t_n]\big);$$

that is, $f \in \mathrm{Q}(M)$. To show the claim, it suffices to check (thanks to (B_{n-1})) that $(N_f)_{\mathfrak{m}}$ is extended from $(A_f)_{\mathfrak{m}}$, for every $\mathfrak{m} \in \operatorname{Max}(R_f)$. Write $\mathfrak{m} = \mathfrak{p}_f$, where $\mathfrak{p}$ is the contraction of $\mathfrak{m}$ to R. Since $f \notin \mathfrak{p}$, we have $f_i \notin \mathfrak{p}$ for some i, say $i = 0$.

But M_{f_0} is extended from L_{f_0}, so $(N_f)_{\mathfrak{m}} = N_{\mathfrak{p}}$ is extended from $L_{\mathfrak{p}} \in \mathfrak{M}(A_{\mathfrak{p}}) = \mathfrak{M}((A_f)_{\mathfrak{m}})$, as desired.

The above argument can be nicely summarized in the following diagrams of rings and modules:

$$
\begin{array}{ccccc}
 & & A_f[t_1,\dots,t_{n-1}] & \to & A_f[t_1,\dots,t_{n-1}][t_n] \\
 & & \downarrow & & \downarrow \\
A_{\mathfrak{p}} & \to & (A_f)_{\mathfrak{m}}\,[t_1,\dots,t_{n-1}] & \to & (A_f)_{\mathfrak{m}}\,[t_1,\dots,t_{n-1}]\,[t_n] \\
\uparrow & & \uparrow & & \uparrow \\
A_{f_0} & \to & (A_{f_0})\,[t_1,\dots,t_{n-1}] & \to & (A_{f_0})\,[t_1,\dots,t_{n-1}]\,[t_n]
\end{array}
\qquad
\begin{array}{ccccc}
 & & N_f & \mapsto & M_f \\
 & & \downarrow & & \downarrow \\
L & \mapsto & (N_f)_{\mathfrak{m}} & \mapsto & (M_f)_{\mathfrak{m}} \\
\uparrow & & \uparrow & & \uparrow \\
L_{f_0} & \mapsto & N_{f_0} & \mapsto & M_{f_0}
\end{array}
$$

Step 3. We shall now prove (A_1), so assume in the following $n = 1$, and write t for t_1. We must show that

$$f_0,\ f_1 \in \mathrm{Q}(M) \Longrightarrow f = f_0 + f_1 \in \mathrm{Q}(M).$$

After replacing R by R_f, *we may assume that* f_0, f_1 *are comaximal in* R, *and try to show that* $M \cong N[t]$, where $N = M/tM$. Take $A_{f_i}[t]$-isomorphisms $u_i : M_{f_i} \to N_{f_i}[t]$, $i = 0, 1$. After composing this with a suitable automorphism on $N_{f_i}[t]$, we may assume that u_i reduces modulo t to the identify map of N_{f_i}. We have now the top half of the diagram:

$$
\begin{tikzcd}
M_{f_0} \arrow[rr, "\text{loc.}"] \arrow[d, "u_0"'] & & M_{f_0 f_1} \arrow[dl, "(u_0)_{f_1}"'] \arrow[dr, "(u_1)_{f_0}"] & & M_{f_1} \arrow[ll, "\text{loc.}"'] \arrow[d, "u_1"] \\
N_{f_0}[t] \arrow[r, "\text{loc.}"] \arrow[d, "v_0"'] & N_{f_0 f_1}[t] \arrow[rr, dashed, "\theta"] \arrow[dr, "(v_0)_{f_1}"'] & & N_{f_0 f_1}[t] & N_{f_1}[t] \arrow[l, "\text{loc.}"'] \arrow[d, "v_1"] \\
N_{f_0}[t] \arrow[rr, "\text{loc.}"'] & & N_{f_0 f_1}[t] & & N_{f_1}[t] \arrow[ll, "\text{loc.}"] \\
\end{tikzcd}
$$
$$N_{f_0 f_1}[t] \xrightarrow{(v_1)_{f_0}} N_{f_0 f_1}[t]$$

in which there are two isomorphisms from $M_{f_0 f_1}$ to $N_{f_0 f_1}[t]$. If these *happen* to be the same isomorphism, we can conclude, from (I.3.11), that there exists an $A[t]$-isomorphism $M \to N[t]$ (localizing to u_0 and u_1). To complete the proof, the obvious idea is, therefore, to adjust the choices of u_0, u_1, so that $(u_0)_{f_1}$ becomes the same as $(u_1)_{f_0}$. Let $\theta = (u_1)_{f_0} \circ (u_0)_{f_1}^{-1}$. This belongs to $\mathrm{End}_{A_{f_0 f_1}[t]}\big(N_{f_0 f_1}[t]\big)$, which can be identified with $(\mathrm{End}_A N)_{f_0 f_1}[t]$, by (I.2.13′) and (I.2.13″). Write $E = \mathrm{End}_A N$, which is an R-algebra. Since θ reduces to the identity mod t, we have $\theta \in E_{f_0 f_1}[t]^*$. By (1.3), we can write $\theta = (v_1)_{f_0}^{-1} \circ (v_0)_{f_1}$, for suitable

$$v_i \in E_{f_i}[t]^* \subseteq \mathrm{Aut}_{A_{f_i}[t]}\big(N_{f_i[t]}\big).$$

We have now $(v_0u_0)_{f_1} = (v_1u_1)_{f_0}$, so we are done by replacing u_i by v_iu_i for $i = 0, 1$. □

Recall that f.g. projective modules over any ring are always finitely presented. If we specialize Quillen's Patching Theorem to f.g. projective $A[t_1, \dots, t_n]$-modules P, and take $A = R$, we obtain the following criterion for P to be extended (i.e., for $P \in \mathfrak{P}^R(R[t_1, \dots, t_n])$).

Corollary 1.7. *Let R be a commutative ring, and $P \in \mathfrak{P}\big(R[t_1, \dots, t_n]\big)$. Then $P \in \mathfrak{P}^R\big(R[t_1, \dots, t_n]\big)$ iff $P_\mathfrak{m}$ is $R_\mathfrak{m}[t_1, \dots, t_n]$-free for every $\mathfrak{m} \in \operatorname{Max} R$.*

Proof. "If": Free $R_\mathfrak{m}[t_1, \dots, t_n]$-modules are clearly extended from $R_\mathfrak{m}$.
"Only if": $P_\mathfrak{m}$ is extended from $R_\mathfrak{m}$, and it must be extended from

$$P_\mathfrak{m}/(t_1, \dots, t_n)P_\mathfrak{m} \cong (P/(t_1, \dots, t_n)P)_\mathfrak{m}.$$

The latter is $R_\mathfrak{m}$-projective, hence $R_\mathfrak{m}$-free by (I.1.8). Thus, its extension $P_\mathfrak{m}$ must be $R_\mathfrak{m}[t_1, \dots, t_n]$-free. □

It is also possible to give a "stable version" of the above result, as observed in [Swan: 1978]. This stable version gives a certain characterization of the image of Grothendieck groups under polynomial extensions.

Corollary 1.8. *Let R be a commutative ring, and $P \in \mathfrak{P}\big(R[t_1, \dots, t_n]\big)$. Then, $[P] \in \operatorname{im}\big(K_0R \to K_0R[t_1, \dots, t_n]\big)$ iff $P_\mathfrak{m}$ is stably free over $R_\mathfrak{m}[t_1, \dots, t_n]$ for every $\mathfrak{m} \in \operatorname{Max} R$.*

Proof. The "only if" part is clear from the following diagram

$$\begin{tikzcd} K_0R \arrow[r] \arrow[d] & K_0R[t_1, \dots, t_n] \arrow[d] \\ \mathbb{Z} \cdot [R_\mathfrak{m}] = K_0R_\mathfrak{m} \arrow[r] & K_0R_\mathfrak{m}[t_1, \dots, t_n] \end{tikzcd}$$

together with (I.6.2). To prove the converse, we apply Quillen's Patching Theorem together with a "compactness" argument. Let us assume, for every $\mathfrak{m} \in \operatorname{Max} R$, that $P_\mathfrak{m}$ is stably free over $R_\mathfrak{m}[t_1, \dots, t_n]$. Take free modules F, F' of finite rank over $R[t_1, \dots, t_n]$, such that $(P \oplus F')_\mathfrak{m} \cong F_\mathfrak{m}$. (The choices of F, F' depend on $\mathfrak{m}$.) By (I.2.17), we can find $f \in R \setminus \mathfrak{m}$ such that $(P \oplus F')_f \cong F_f$. The set of f's so obtained (for all $\mathfrak{m}$) generates the unit ideal R, so we can find a *finite* subset $\{f_i\}$ such that $\sum R \cdot f_i = R$. Changing notations, we write $(P \oplus F_i')_{f_i} \cong (F_i)_{f_i}$. Let F'' be free over $R[t_1, \dots, t_n]$, of rank = $\max_i$ {rank F_i'}. Then $(P \oplus F'')_{f_i}$ is $R_{f_i}[t_1, \dots, t_n]$-free for all i, and so $(P \oplus F'')_\mathfrak{m}$ is $R_\mathfrak{m}[t_1, \dots, t_n]$-free for all $\mathfrak{m} \in \operatorname{Max} R$. We conclude, as in (1.7), that $P \oplus F''$ is extended from R. Going over to $K_0R[t_1, \dots, t_n]$, it follows that $[P] \in \operatorname{im}\big(K_0R \to K_0R[t_1, \dots, t_n]\big)$, as desired. □

Quillen's Patching Theorem, stated for projective modules in the form (1.7), enables us to go "from local to global" in the study of the extension property. As it turned out, there is a *converse* to this principle (for projective modules) too, so that one can also go "from global to local". Interestingly, the proof of this depends on being able to go "from local to global"! We shall present this converse form of the Patching Theorem below (in Theorem 1.11), following [Roitman: 1979].

To facilitate the formulation to this converse, let us introduce the following convenient terminology, which will also be used freely in the rest of this book.

Definition 1.9. For $n \geqslant 1$, we say that a (not necessarily commutative) ring R has the property (E_n) if every f.g. projective right module over $R[t_1, \dots, t_n]$ is extended from R. If R has the property (E_n) for all n, we'll simply say that it has the property (E).

The letter "E" here comes from the word "extension". Note that, although we have used *right* modules in the definition of the properties (E) and (E_n) above, they are actually left-right symmetric properties. This is clear since we can think of f.g. projective right modules over $R[t_1, \dots, t_n]$ as being "defined" by idempotent matrices over such a polynomial ring. This enables us to express the property (E_n) (and hence also (E)) by a (left-right symmetric) statement on idempotent matrices over $R[t_1, \dots, t_n]$. (This particular way of interpreting the properties (E_n) will turn out to be useful in the proof of (1.11) below.)

Let us also take this opportunity to record some other facts about the properties (E_n). The routine proofs for them are left as exercises.

Remark 1.10A. *If R is a ring over which all f.g. (right) projectives are free, then R has* (E_n) *iff all f.g. (right) projectives over $R[t_1, \dots, t_n]$ are free.*

Remark 1.10B. *For $n \geqslant 1$, if R has the property* (E_{n+1})*, then both R and $R[x]$ have the property* (E_n)*. In particular, if R has the property* (E)*, so does $R[x]$.*

For instance, by (1.10A), Serre's Conjecture amounts to the statement that all fields have the property (E). By (1.7), if a commutative ring R is such that $R_{\mathfrak{m}}$ has (E_n) for all $\mathfrak{m} \in \operatorname{Max} R$, then R itself has (E_n). Roitman's result is the following "converse" of this statement.[†]

Roitman's Theorem 1.11. *Let R be a commutative ring and S be a multiplicative set in R. For any $n \geqslant 1$, if R has* (E_n)*, then so does the localization R_S. In particular, if R the property* (E)*, then so does R_S.*

Proof. Step 1. We'll prove the theorem (for all commutative rings) by induction on n. The beginning step of the induction (the case $n = 1$) will be handled in *Step 2* below.

[†] Since this converse does hold, we can rightfully say that, for commutative rings, (E_n) ($n \geqslant 1$) and (E) are all "local" properties (properties that can be checked locally).

Assuming this case, we carry out the inductive step here. Let $n\geqslant 2$, and assume the theorem is true for $n-1$. To prove the theorem for n, suppose R satisfies (E_n). Clearly, $R[t_1, \dots, t_{n-1}]$ then satisfies (E_1), and so by *Step 2* below,

$$(*)\qquad R[t_1, \dots, t_{n-1}]_S = R_S[t_1, \dots, t_{n-1}] \text{ satisfies } (\mathrm{E}_1).$$

Consider any f.g. projective module over $R_S[t_1, \dots, t_n] = R_S[t_1, \dots, t_{n-1}][t_n]$. By ($*$), P is extended from $R_S[t_1, \dots, t_{n-1}]$. But R satisfies (E_{n-1}) (as we've observed in (1.10B)), so R_S also does (by the inductive hypothesis). Therefore, P is extended from R_S. This checks that R_S satisfies (E_n).

Step 2. Here, we prove that the theorem is true for $n=1$. Assume R satisfies (E_1), and let $P \in \mathfrak{P}\big(R_S[x]\big)$. To check that P is extended from R_S, Quillen's Patching Theorem enables us to replace R_S by $(R_S)_{\mathfrak{m}}$ where $\mathfrak{m} \in \operatorname{Max} R_S$. But $(R_S)_{\mathfrak{m}} = R_{\mathfrak{p}}$ for some $\mathfrak{p} \in \operatorname{Spec} R$. Thus, we may as well assume that $R_S = R_{\mathfrak{p}}$. In this situation, our job is to prove that every f.g. projective module is free over $R_{\mathfrak{p}}[x]$; that is, to prove that, for any k, any idempotent in $e(x) \in \mathbb{M}_k(R_{\mathfrak{p}}[x])$ is conjugate to a standard idempotent $\operatorname{diag}(1, \dots, 1, 0, \dots, 0)$. As $R_{\mathfrak{p}}$ is local, $e(0)$ is conjugate in $\mathbb{M}_k(R_{\mathfrak{p}})$ to a standard idempotent. After a conjugation, we may thus assume that $e(0)$ is a standard idempotent. There exists an element $r \in R \setminus \mathfrak{p}$ such that $e(rx)$ lies in the image of $\mathbb{M}_k(R[x]) \to \mathbb{M}_k(R_{\mathfrak{p}}[x])$. (Note that the constant terms of the entries of $e(x)$ are 0 or 1, both of which lie in R already.) Fix an $e_0(x) \in \mathbb{M}_k(R[x])$ localizing to $e(rx)$ such that $e_0(0)$ is the standard idempotent in question. Then $s(e_0(x)^2 - e_0(x)) = 0$ for some $s \in R \setminus \mathfrak{p}$. As $e_0(0)^2 = e_0(0)$, $e_0(x)^2 - e_0(x)$ has the form $x\varepsilon(x)$ for some matrix $\varepsilon(x)$ over $R[x]$. Now $sx\varepsilon(x) = 0$ implies $s\varepsilon(x) = 0$ (since x is not a 0-divisor in $R[x]$). Thus, $s\varepsilon(sx) = 0$ too, and hence

$$e_0(sx)^2 - e_0(sx) = sx\varepsilon(sx) = 0 \in \mathbb{M}_k(R[x]).$$

But R has the property (E_1), so

$$\sigma(x)^{-1}\, e_0(sx)\, \sigma(x) = e_0(0) \ \text{ for some } \sigma(x) \in \mathrm{GL}_k\big(R[x]\big).$$

In $\mathbb{M}_k\big(R_{\mathfrak{p}}[x]\big)$, this becomes $\sigma(x)^{-1}\, e(rsx)\, \sigma(x) = e(0)$, and therefore

$$\sigma(x/rs)^{-1}\, e(x)\, \sigma(x/rs) = e(0) \in \mathbb{M}_k\big(R_{\mathfrak{p}}[x]\big),$$

which gives what we want. □

The following easy consequence of (1.11) and (1.10B) gives a large supply of rings satisfying the property (E) (for example, by taking A below to be a field).

Corollary 1.12. *If a ring A has the property* (E_{r+n}) (*resp. the property* (E)), *then* $A\,[x_1, \dots, x_r]_S$ *has the property* (E_n) (*resp. the property* (E)) *for any multiplicative set* $S \subseteq A\,[x_1, \dots, x_r]$.

Besides being of intrinsic interest, Roitman's Theorem 1.11 has proved to be important for the later work of Lindel, Vorst, and others. For more details on this, see

§6 and §9 in Chapter VIII. We note, incidentally, that the paper [Roitman: 1979] also contained several useful variants of the definition of the properties (E_n) and (E) (for commutative rings), which we will not repeat here.

Coming back to the fact (mentioned in (1.10B)) that, for any ring R, the condition (E_{n+1}) implies (E_n), one naturally wonders about its converse. The following observation gives various interpretations of a possible converse.

Proposition 1.13. *For a fixed integer* $n \geqslant 1$, *the following four statements are equivalent:*

(1) *Any ring satisfying* (E_n) *also satisfies* (E_{n+1}).

(2) *Any ring satisfying* (E_n) *also satisfies* (E_{n+r}) *for all* $r \geqslant 1$.

(3) *If a ring* R *satisfies* (E_n), *then so does* $R[x]$.

(4) *If a ring* R *satisfies* (E_n), *then so does* $R[x_1, \ldots, x_r]$ *for all* $r \geqslant 1$.

Proof. (1) $\Rightarrow$ (3). Suppose R satisfies (E_n), and consider

$$P \in \mathfrak{P}\big(R[x][t_1, \ldots, t_n]\big) = \mathfrak{P}\big(R[x, t_1, \ldots, t_n]\big).$$

By (1), R satisfies (E_{n+1}), so P is extended from R. But then P is also extended from $R[x]$. This checks that $R[x]$ satisfies (E_n).

(3) $\Rightarrow$ (1). Suppose R satisfies (E_n), and consider

$$P \in \mathfrak{P}\big(R[x, t_1, \ldots, t_n]\big) = \mathfrak{P}\big(R[x][t_1, \ldots, t_n]\big).$$

Since $R[x]$ satisfies (E_n) (by (3)), P is extended from $R[x]$. But R also satisfies (E_1) by (successive applications of) (1.10B), so P is extended from R. This checks that R satisfies (E_{n+1}).

(3) $\Leftrightarrow$ (4). This follows from induction on r.

(1) $\Leftrightarrow$ (2). It suffices to prove the forward implication. Let $\mathcal{E}_n$ be the class of rings satisfying (E_n). Assuming (1) (and using again (1.10B), we have $\mathcal{E}_n = \mathcal{E}_{n+1}$, so by (3) (which is implied by (1)), $\mathcal{E}_{n+1}$ is closed under the formation of polynomial rings in one variable. Invoking (1) $\Leftrightarrow$ (3) for $n+1$ (instead of n), it follows that $\mathcal{E}_{n+1} = \mathcal{E}_{n+2}$. Repeating this argument then shows $\mathcal{E}_n = \mathcal{E}_{n+r}$ for all $r \geqslant 1$. $\square$

In studying the possible truth of the statements in (1.13), the following important special case must be borne in mind.

Remark 1.14. *If a ring* R *has* (E_1), *it need not have* (E_2). For instance, a noncommutative division ring H has (E_1) since $H[x]$ is a right PID (so all f.g. right projectives are free). However, $H[x, y]$ has nonfree projective right ideals by II.3.4, so H does not have (E_2).

The example above means that the four conditions in (1.13) do not hold for $n = 1$. It invites, however, the following tantalizing question:

Question 1.15. *Does there exist an integer* $n \geqslant 2$ *for which the four conditions in* (1.13) *all hold? In other words, does the descending chain*

$$\mathcal{E}_1 \supseteq \mathcal{E}_2 \supseteq \mathcal{E}_3 \supseteq \ldots$$

of families of rings stabilize?

If such an integer $n \geqslant 2$ *does* exist, one would like to determine its smallest possible value. Thus, a "bold form" of (1.15) would be the following:

Question 1.16. *Do the conditions in* (1.13) *all hold for* $n = 2$ *(i.e., is* $\mathcal{E}_2 = \mathcal{E}_3$*)? If we interpret the word "ring" to mean "commutative ring" throughout, do the conditions in* (1.13) *all hold for* $n = 1$ *(i.e., is* $\mathcal{E}_1 = \mathcal{E}_2$*)?*

I do not know of any counterexamples for any of the questions above. Note that, by (1.13), if (1.16) does have affirmative answers, then for general rings, the property (E) would boil down to (E_2), and for commutative rings, (E) would boil down to (E_1). These would be rather remarkable statements (the latter vastly generalizing the Quillen-Suslin Theorem), *if they were true*!

Later in this chapter, we'll return to consider some other problems that are closely related to (the commutative case of) (1.16). For instance, the Quillen Induction Theorem 3.1 below provides a general method for checking the property (E) via the property (E_1) — for certain *families* of commutative rings (instead of for a single commutative ring R).

To further illustrate the use of Quillen's Patching Theorem, let us close this section by giving a quick application of it to the existence of nonfree projectives over the ring $H[x, y]$ mentioned in Remark 1.11 above. To be more specific, let H be Hamilton's division ring of the real quaternions. We shall view the polynomial ring $A := H[x]$ as an algebra over the commutative ring $R = \mathbb{R}[x]$, and apply Quillen's Patching Theorem 1.6 to f.g. projective (say, left) modules over $B := A[y] = H[x, y]$. The notations $R,\ A,\ B$ in this paragraph will be fixed in the following.

Recall from II.3 that, for any pair of noncommuting elements $a, b \in H$, there is a f.g. projective B-module (of rank 1) defined to be the kernel of the surjection $B^2 \to B$ that takes the natural basis of B^2 to $x + a$ and $y + b$. Denoting this projective B-module (as usual) by $P(x + a,\ y + b)$, let us prove the following result from [Parimala-Sridharan: 1975].

Theorem 1.17. *Let* $s,\ t$ *be nonzero real numbers with* $|s| \neq |t|$*. For the standard quaternion generators* $i,\ j \in H$*, one has*

$$P(x + si,\ y + j) \not\cong P(x + ti,\ y + j).$$

In particular, by letting s *range over the positive reals, we get uncountably many mutually nonisomorphic nonfree projectives* $P(x + si,\ y + j)$ *(of rank* 1*) over the polynomial ring* $B = H[x, y]$*.*

Proof. The proof of this in [Parimala-Sridharan: 1975] involves some rather complicated calculations. Here we present a quick proof of (1.17) via Quillen's Patching Theorem, following an argument of Raja Sridharan which appeared on p. 775 of [Swan: 1996].

Assume, for the moment, that $P(x+si,\ y+j)$ and $P(x+ti,\ y+j)$ are both isomorphic to some $P \in \mathfrak{P}(B)$. Note that the elements

$$f := x^2 + s^2, \quad \text{and} \quad g := x^2 + t^2$$

in $R = \mathbb{R}[x]$ are *comaximal*, since $f - g = s^2 - t^2 \neq 0$ is a unit in $\mathbb{R}$. Upon localizing at f, $P \cong P(x+si,\ y+j)$ becomes free (of rank 1) since

$$(x+si)(x-si) = (x-si)(x+si) = x^2 + s^2 = f$$

implies that $x+si$ becomes a unit. In particular, P_f is extended from A_f. By symmetry, P_g is also extended from A_g. Since f, g are comaximal in R, the first part of Quillen's Patching Theorem implies that P itself is extended from A. But $A = H[x]$ is a (noncommutative) principal ideal domain, so any f.g. projective A-module is free. This means that P itself is free (over $B = A[y] = H[x, y]$), which, however, contradicts II.3.6. (Note that $s \neq 0$ implies that si does not commute with j.) $\square$

Remark 1.18. The theorem above left open the case where $|s| = |t| \neq 0$ in $\mathbb{R}$. This case was treated in [Swan: 1996] as well. In fact, in §7 of this paper, Swan showed that, for any nonzero $s \in \mathbb{R}$,

$$P(x+si,\ y+j) \cong P(x-si,\ y+j)$$

as B-modules. Thus, for the purposes of constructing distinct projectives over $B = H[x, y]$, it is sufficient to consider the modules $P(x+si,\ y+j)$ with $s > 0$ in $\mathbb{R}$.

§2. Affine Horrocks' Theorem and Applications

Horrocks' Theorem, as stated in (IV.2.1) and (IV.2.2), is essentially a result about commutative local rings. With Quillen's Patching Theorem in the form (1.7), it is now possible to extend Horrocks' result to the case of *arbitrary* commutative ground rings. From now on, the Horrocks' Theorem given in Chapter IV will be referred to as the *Local Horrocks' Theorem*: in its geometric form, it is a statement about the affine line $\mathbf{A}^1_R$ and the projective line $\mathbb{P}^1_R$ over a "local spectrum" (i.e., over Spec R where R is local). The extension obtained in this section will be a similar statement about $\mathbf{A}^1_R$ and $\mathbb{P}^1_R$ over an arbitrary affine scheme Spec R. Following a suggestion of H. Bass, we shall refer to this extended version as the *Affine Horrocks' Theorem*.

To conform with the earlier developments in (IV.2), let us also state this in geometric terms as well as in algebraic terms.

Affine Horrocks 2.1. (Geometric Form). *Let R be any commutative ring, and let B be a vector bundle on $\mathbf{A}^1_R$ given by $P \in \mathfrak{P}(R[t])$. If B extends to a vector bundle on $\mathbb{P}^1_R$, then B is extended from* Spec R; *that is, $P \in \mathfrak{P}^R(R[t])$.*

Proof. By functoriality, the vector bundle on $\mathbf{A}^1_{R_\mathfrak{m}}$ given by $P_\mathfrak{m}$ clearly extends to a vector bundle on $\mathbb{P}^1_{R_\mathfrak{m}}$, for any $\mathfrak{m} \in \operatorname{Max} R$. By Local Horrocks, $P_\mathfrak{m}$ is $R_\mathfrak{m}[t]$-free, so by (1.7), P is extended from R. □

Recall that, for any commutative ring R, $R\langle t\rangle$ denotes the ring obtained by localizing $R[t]$ at the multiplicative set of all monic polynomials in t.

Affine Horrocks 2.2. (Algebraic Form). *Let R be any commutative ring, and let $P \in \mathfrak{P}(R[t])$. If $P\langle t\rangle := R\langle t\rangle \otimes_{R[t]} P$ is extended from a f.g. projective R-module* (e.g.*, if $P\langle t\rangle$ is $R\langle t\rangle$-free), then $P \in \mathfrak{P}^R(R[t])$.*

Proof. By functoriality, $P_\mathfrak{m}\langle t\rangle$ is extended from some module in $\mathfrak{P}(R_\mathfrak{m})$, so $P_\mathfrak{m}\langle t\rangle$ is $R_\mathfrak{m}\langle t\rangle$-free, for any $\mathfrak{m} \in$ Max R. By Local Horrocks, $P_\mathfrak{m}$ is $R_\mathfrak{m}[t]$-free, so by (1.7), P is extended from R. □

Supplement 2.3. *In* (2.2)*, if $P\langle t\rangle$ is extended from $P_0 \in \mathfrak{P}(R)$, then P is also extended from P_0.* (*In particular, if $P\langle t\rangle$ is $R\langle t\rangle$-free, P must be $R[t]$-free.*(*))

In fact, from (2.2), we know already that P is extended from *some* $Q_0 \in \mathfrak{P}(R)$ (of course, $Q_0 \cong P/tP$). We have

$$P_0\langle t\rangle \ (= R\langle t\rangle \otimes_R P_0) \cong P\langle t\rangle \cong R\langle t\rangle \otimes_{R[t]} (R[t] \otimes_R Q_0) \cong Q_0\langle t\rangle.$$

Thus, it is sufficient to prove the following:

Proposition 2.4. *Let R be any commutative ring, and P_0, $Q_0 \in \mathfrak{P}(R)$. If $P_0\langle t\rangle \cong Q_0\langle t\rangle$ as $R\langle t\rangle$-modules, then $P_0 \cong Q_0$ as R-modules.*

Proof. By (I.2.16), there exists a monic polynomial $f(t)$, say of degree n, such that $P_0[t]_f \cong Q_0[t]_f$. Proceeding as in the proof of (IV.2.4), we set $s = t^{-1}$, and $g(s) = t^{-n} f(t) \in R[s]$. The fact that f is monic in t translates into $g(0) = 1$, so $g(s)$ and s are comaximal in $R[s]$. Consider the modules

$$P_0[s]_g \in \mathfrak{P}\big(R[s]_g\big) \quad \text{and} \quad Q_0[s]_s \in \mathfrak{P}\big(R[s]_s\big).$$

Since f and g are associates in $R[t, t^{-1}]$, we have

(*)This simply means that the Local Horrocks' Theorem (IV.2.1) is true *without* the assumption that R be local!

$$\big(Q_0[s]_s\big)_g = Q_0[t,t^{-1}]_g \cong Q_0[t,t^{-1}]_f = Q_0[t]_{tf},$$
$$\big(P_0[s]_g\big)_s \cong \big(P_0[s]_s\big)_g \cong P_0[t]_{tf},$$

and these are isomorphic as $R[t]_{tf}$-modules:

$$\begin{tikzcd}[column sep=small]
 & R[s]_{sg} = R[t,\, t^{-1}]_f \arrow[dl, dash] \arrow[dr, dash] & \\
R[s]_s = R[t,\, t^{-1}] \arrow[dr, dash] & & R[s]_g \arrow[dl, dash] \\
 & R[s] &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
 & Q_0[s]_{sg} \cong P_0[s]_{gs} \arrow[dl, dash] \arrow[dr, dash] & \\
Q_0[s]_s \arrow[dr, dash, dashed] & & P_0[s]_g \arrow[dl, dash, dashed] \\
 & M &
\end{tikzcd}$$

By (I.3.12), there exists a module $M \in \mathfrak{P}\big(R[s]\big)$ such that $M_g \cong P_0[s]_g$ and $M_s \cong Q_0[s]_s$. The latter implies that $R\langle s\rangle \otimes_{R[s]} M$ is extended from $Q_0 \in \mathfrak{P}(R)$, so by (2.2), M is extended from some $M_0 \in \mathfrak{P}(R)$. Clearly,

$$M_0 \cong \frac{M}{sM} \cong \left(\frac{M}{sM}\right)_g \cong \frac{M_g}{sM_g} \cong \frac{P_0[s]_g}{s \cdot P_0[s]_g} \cong P_0\,;$$

but also

$$M_0 \cong \frac{M}{(s-1)M} \cong \left(\frac{M}{(s-1)M}\right)_s \cong \frac{M_s}{(s-1)M_s} \cong \frac{Q_0[s]_s}{(s-1)\cdot Q_0[s]_s} \cong Q_0,$$

so we get $P_0 \cong Q_0$. □

The proof above was pointed out to me by H. Bass. The same argument can be utilized for something else: it also yields a useful criterion for a module in $\mathfrak{P}\big(R[t,\ t^{-1}]\big)$ to be the extension of some module in $\mathfrak{P}\big(R[s]\big)$. We shall need this criterion later in §4.

Proposition 2.5. *Let R be a commutative ring, and $P \in \mathfrak{P}\big([t,\ t^{-1}]\big)$. Then P is extended from a module in $\mathfrak{P}\big(R[s]\big)$ $(s = t^{-1})$ iff there exists a monic polynomial $f \in R[t]$ such that P_f is extended from a module in $\mathfrak{P}\big(R[s]\big)$.*

Proof. ("If") Say $P_f \cong R[t,\ t^{-1}]_f \otimes_{R[s]} Q$, where $Q \in \mathfrak{P}\big(R[s]\big)$. Using the notations in the proof of (2.4), consider $P \in \mathfrak{P}\big(R[s]_s\big)$ and $Q_g \in \mathfrak{P}\big(R[s]_g\big)$. Since

$$P_g \cong P_f \cong Q_{sg} \cong (Q_g)_s,$$

there exists a module $M \in \mathfrak{P}\big(R[s]\big)$ such that $M_g \cong Q_g$ and $M_s \cong P$ (by (I.3.12)). The latter is precisely what we want. □

(The geometric idea behind this proof is the following. The affine line Spec $R[s]$ has an open cover given by the basic Zariski open sets $D(g) = \{\mathfrak{P} \mid \mathfrak{P} \not\ni g\}$ and $D(s) = \{\mathfrak{P} \mid \mathfrak{P} \not\ni s\}$. The module P defines a bundle over

$$D(s) = \text{Spec } R[s, s^{-1}] = \text{Spec } R[t, t^{-1}],$$

and, by restriction, Q defines a bundle over $D(g)$. The isomorphism $P_f = R[t, t^{-1}]_f \otimes_{R[s]} Q$ says that these two bundles "agree" on

$$D(g) \cap D(s) = \text{Spec } R[t, t^{-1}]_f,$$

so we can "glue" the two bundles together to get a bundle over Spec $R[s]$. This new bundle then extends the one given by P on Spec $R[t, t^{-1}]$.)

There is a nice application of (2.2) which leads to a striking result on unimodular rows over polynomial rings that can be understood by any student familiar with high school algebra (and the definition of a commutative ring). We state it as follows.

Proposition 2.6. *For any commutative ring R, let*

$$u = \big(f_1(t), \ldots, f_n(t)\big) \in \mathrm{Um}_n\big(R[t]\big).$$

If $f_1(t)$ is unitary (i.e. *its leading coefficient is a unit*), *then u is completable* (*to a matrix in* $\mathrm{GL}_n\big(R[t]\big)$).

Proof. Over any commutative ring S, any unimodular row $u \in \mathrm{Um}_n(S)$ defines a stably free S-module of rank $n-1$; namely, $P(u) := \ker(u)$. By I.4.8, this "solution space" is free iff u is completable (over S). Now let $S = R[t]$, and u be as given in the Proposition. After scaling u by a unit of R, we may assume $f_1(t)$ is monic. Since

$$u_{f_1(t)} \in \mathrm{Um}_n\big(R[t]_{f_1(t)}\big)$$

has first entry a unit, $P\big(u_{f_1(t)}\big)$ is free. Thus, $P(u)_{f_1(t)} \cong P\big(u_{f_1(t)}\big)$ is free, and (2.3) implies that $P(u)$ is free, which gives what we want. $\square$

This result could have been deduced from (III.1.7) too. But this would take one or two extra steps, so we derived it more directly from (2.3) instead. Also, in case $n \geqslant 3$ and R is local, (III.2.6) would give a stronger conclusion; namely, u can be completed to a matrix in $\mathrm{E}_n\big(R[t]\big)$. For a single theorem encompassing all of (2.6), (III.1.7), and (III.2.6) in the case $n \geqslant 3$, see VI.2.8.

We now record some K_0-theoretic consequences of (2.2) and (2.4).

Proposition 2.7. *For any commutative ring R, the functorial maps i, j, k and ℓ in the following diagram are all injective:*

$$\begin{tikzcd}
K_0R \arrow[r, "i"] \arrow[dr, "k"'] & K_0R[t] \arrow[r, "j"] \arrow[d, "\ell"] & K_0R[t, t^{-1}] \arrow[dl] \\
& K_0R[t] &
\end{tikzcd}$$

Proof. (1) *i is (split) injective* since R is a (ring-theoretic) retract of $R[t]$ (by $f \longmapsto f(a)$ for any $a \in R$).

(2) *k is injective*. Indeed, suppose $[P_0] - [Q_0] \in \ker k$, where $P_0, Q_0 \in \mathfrak{P}(R)$. We have then $P_0\langle t\rangle \oplus R\langle t\rangle^m \cong Q_0\langle t\rangle \oplus R\langle t\rangle^m$ for a suitable m. By (2.4), we conclude that $P_0 \oplus R^m \cong Q_0 \oplus R^m$, whence $[P_0] = [Q_0] \in K_0R$.

(3) *ℓ is injective*. Indeed, suppose $z = [P_1] - [P_2] \in \ker \ell$, where $P_i \in \mathfrak{P}\big(R[t]\big)$. We may assume, without loss of generality, that P_2 is $R[t]$-free. Moreover, after adding a suitable free module to P_1 and P_2, we may assume that

$$R\langle t\rangle \otimes_{R[t]} P_1 \cong R\langle t\rangle \otimes_{R[t]} P_2,$$

which is $R\langle t\rangle$-free. By (2.2), P_1 is extended from R, so $z = i(z_0)$ for some $z_0 \in K_0R$. But then $k(z_0) = \ell\, i(z_0) = \ell(z) = 0$ implies that $z_0 = 0$ (by (2)); à fortiori, $z = 0$.

(4) *j is injective*. This is clear from (3), since $\ell = g \circ j$ where g is the functorial map $K_0R[t, t^{-1}] \longrightarrow K_0R\langle t\rangle$. □

Remark 2.8. Actually, the injectivity of j was known *before* the Affine Horrocks' Theorem. For a proof of the injectivity of j independently of Affine Horrocks', see [Bass: 1968, p. 669].

To close this section, we shall now give the main application of the Affine Horrocks' Theorem to Serre's Conjecture. It has been known, at least several years *before* the proof of Serre's Conjecture by Quillen and Suslin, that a suitable affine form of Horrocks' Theorem (such as (2.1) or (2.2)) will be sufficient to furnish an affirmative answer to Serre's Conjecture. This observation was made by Murthy, and the details appeared in Bass' survey article "Some Problems in 'Classical' Algebraic K-Theory" [Bass: 1972a, Sec. 4.2]. Since we have established the Affine Horrocks' Theorem, the solution of Serre's Conjecture is now at hand.

Quillen-Suslin Theorem 2.9. *Let R be a field, or more generally, a PID. Then, any $P \in \mathfrak{P}\big(R[t_1, \ldots, t_n]\big)$ is free.*

Proof. Say R is a field. We induct on n, the case $n = 0$ being trivial. For $n > 0$, let $A = R[t_2, \ldots, t_n]$, $t = t_1$, and consider

$$A\,[t] \subseteq R\,\langle t\rangle\,[t_2, \ldots, t_n] \subseteq A\,\langle t\rangle.$$

By the inductive hypothesis, P becomes free upon tensoring up to $R\langle t\rangle\,[t_2,\ldots,t_n]$; in particular, $A\langle t\rangle\otimes_{A[t]}P$ is $A\langle t\rangle$-free. By (2.2), P is extended from $P/t_1P\in\mathfrak{P}(A)$. Again by the inductive hypothesis, P/t_1P is A-free, so P is $A[t]=R[t_1,\ldots,t_n]$-free. If R is a PID (instead of a field), the same inductive proof works, with the extra observation that R is a PID $\Rightarrow$ $R\langle t\rangle$ is a PID (by IV.1.3(3)). $\square$

Remark 2.10. (1) If we use (2.3) instead of (2.2), we would have needed the inductive hypothesis only once, not twice. But the above shows that (2.2) suffices for the proof.

(2) In case R is a PID, the case $n=1$ of (2.9) is Seshadri's Theorem (II.6.1), and the case $n=2$ of (2.9) is a result of Lindel (see [Lindel: 1976]). In the above proof, however, the case $n\leqslant 2$, as well as the case of a general n, follows uniformly from (2.2) and (IV.1.3)(3) without recourse to other results.

Recall that, according to Def. 1.9, a ring R has the property (E_n) if every $P\in\mathfrak{P}\big(R[t_1,\ldots,t_n]\big)$ is extended from R, and R has the property (E) if it has (E_n) for all n. Using these definitions and the Patching Theorem, we can further refine (2.9) as follows.

Theorem 2.11. *Let R be a Dedekind domain. Then* (1) R *has the property* (E), *and* (2) $R[t_1,\ldots,t_n]$ *is a Hermite ring for every* n.

(That R satisfies (E_1) was first proved by Bass and Serre in the early 1960s. We have already covered this case in II.6.2.)

Proof. (1) Let $P\in\mathfrak{P}\big(R[t_1,\ldots,t_n]\big)$, where $n\geqslant 1$. For $\mathfrak{m}\in\operatorname{Max}R$, $R_{\mathfrak{m}}$ is a discrete valuation ring, in particular a PID. By (2.9), $P_{\mathfrak{m}}$ is free over $R_{\mathfrak{m}}[t_1,\ldots,t_n]$, so by (1.7), $P\cong P_0[t_1,\ldots,t_n]$, where $P_0=P/(t_1,\ldots,t_n)P$. This checks that R has the property (E_n) (for every n). In fact, we can describe the projective module P above very explicitly. Since $P_0\in\mathfrak{P}(R)$, we have (by a standard result) $P_0\cong R^r\oplus J$, where J is an ideal in R. Thus, by tensoring up,

$$P\cong R\,[t_1,\ldots,t_n]^r\oplus J\,[t_1,\ldots,t_n].$$

(2) follows from (1) and the fact (I.4.7)(4) that R itself is Hermite. $\square$

In the rest of this section, we'll show that the Patching Theorem also leads to new results on the property (E) in the *non-noetherian* case. First, let us prove a few things about von Neumann regular rings. Recall that a ring R is called *von Neumann regular* if, for every $a\in R$, there exists $x\in R$ such that $a=axa$. (For instance, division rings and Boolean rings are von Neumann regular.) In spite of the use of the word "regular" in this definition, the class of von Neumann regular rings is not to be confused with the class of left regular rings we defined earlier in II.5.2. These two classes have very little in common. In fact, since left regular rings are supposed to be left noetherian, it is easy to see that a ring R is both left regular and von Neumann regular iff R is an (artinian) semisimple ring. In the case where R is commutative, this means that R is just a finite direct product of fields.

Here, we are mainly interested in *commutative* von Neumann regular rings. The following characterization result for such rings is well known; a proof for it is included for the reader's convenience.

Proposition 2.12. *For any commutative ring* R*, the following are equivalent:*

(1) R *is von Neumann regular.*

(2) $R_{\mathfrak{m}}$ *is a field for every* $\mathfrak{m} \in \operatorname{Max} R$.

(3) R *is reduced and has Krull dimension* 0.

Proof. (1) $\Rightarrow$ (2). Consider any $\mathfrak{m} \in \operatorname{Max} R$. For any $a \in \mathfrak{m}$, there exists $x \in R$ with $a(1-ax) = 0$. Since $1 - ax \notin \mathfrak{m}$, $a/1 = 0 \in R_{\mathfrak{m}}$. This means that $\mathfrak{m}R_{\mathfrak{m}} = 0$. Thus, the local ring $R_{\mathfrak{m}}$ is a field.

(2) $\Rightarrow$ (3) is clear, since $\dim R = 0$ and the reducedness of R are properties that can be checked locally (in $R_{\mathfrak{m}}$ for all $\mathfrak{m} \in \operatorname{Max} R$).

(3) $\Rightarrow$ (1). Assume (3). Then, for any $\mathfrak{m} \in \operatorname{Max} R$, the local ring $R_{\mathfrak{m}}$ is also reduced and has dimension 0. Thus, $0 = \operatorname{Nil}(R_{\mathfrak{m}}) = \mathfrak{m}R_{\mathfrak{m}}$, so $R_{\mathfrak{m}}$ is a field. Thus, for any $a \in R$, $aR_{\mathfrak{m}} = a^2R_{\mathfrak{m}}$. This implies that $(aR/a^2R)_{\mathfrak{m}} = 0$ for every $\mathfrak{m} \in \operatorname{Max} R$. It follows that $aR = a^2R$, and so $a = a^2x$ for some $x \in R$ $\square$

We can now prove the following result for 0-dimensional rings, the first part of which is from [Brewer-Costa: 1978] listed in the references on Chapter VIII.

Proposition 2.13. *Let* R *be a commutative ring of Krull dimension* 0*. Then* (1) R *has the property* (E)*, and* (2) $R[t_1, \ldots, t_n]$ *is a Hermite ring for every* n*.*

Proof. (1) Let $A = R[t_1, \ldots, t_n]$, and let $P \in \mathfrak{P}(A)$. To show that P is extended from R, we may assume (thanks to Quillen's Patching Theorem) that R is local, in which case we must show that P is free. Let $k = R/\mathfrak{m}$, where $\mathfrak{m} \in \operatorname{Max} R$. Since $\operatorname{Spec} R = \{\mathfrak{m}\}$, we have $\mathfrak{m} = \operatorname{Nil}(R)$, and so

(2.14) $$\mathfrak{m}\,[t_1, \ldots, t_n] \subseteq \operatorname{Nil}\,(A) \subseteq \operatorname{rad}\,(A).$$

By (2.9), $P/\mathfrak{m}[t_1, \ldots, t_n] \cdot P \in \mathfrak{P}\big(k[t_1, \ldots, t_n]\big)$ is free, so by (I.1.7), P is free, as desired.(*)

For (2), note that, since all prime ideals in R are maximal, we have $\operatorname{rad}(R) = \operatorname{Nil}(R)$. Thus, $\overline{R} = R/\operatorname{rad}(R)$ is reduced and has Krull dimension 0. By (2.12), $\overline{R}$ is von Neumann regular. Therefore, $\overline{R}$ is Hermite, by (I.4.34). It follows from (I.4.7)(0) that R itself is Hermite. Since R has the property (E), this implies that $R[t_1, \ldots, t_n]$ is Hermite for every n. $\square$

(*) We have observed earlier that (I.1.7) applies to ideals contained in the Jacobson radical. (In the present situation, it does turn out that $\operatorname{rad}(A) = \mathfrak{m}[t_1, \ldots, t_n]$. This information is, however, not needed in our proof.)

Corollary 2.15. *If R be a commutative artinian ring, or a commutative von Neumann regular ring, then both conclusions* (1) *and* (2) *in* (2.13) *hold.*

Proof. It suffices to show that $\dim R = 0$. If R is commutative and von Neumann regular, this follows from (2.12). Next, assume R is commutative artinian. If $\dim R > 0$, there would exist prime ideals $\mathfrak{p}_0 \subsetneq \mathfrak{p}$ in R. After replacing R by $R/\mathfrak{p}_0$, we may assume that $\mathfrak{p}_0 = 0$, so that R is an integral domain, and $\mathfrak{p} \neq 0$. But then any $a \in \mathfrak{p} \setminus \{0\}$ creates a strictly decreasing chain $aR \supsetneq a^2R \supsetneq \cdots$, a contradiction. □

In commutative algebra, it is well-known that a (commutative) ring R *is artinian iff* R *is noetherian and* $\dim R = 0$. In the proof of (2.15) above, we used only half of the "only if" part of this statement, for which we have supplied a direct proof.

Remark 2.16. (A) In the case where R is commutative and von Neumann regular, the property (E) for R can also be seen in a slightly different way, as follows. Let $P \in \mathfrak{P}\big(R[t_1, \ldots, t_n]\big)$. By (1.7), it suffices to show that $P_{\mathfrak{m}}$ is free for every $\mathfrak{m} \in \operatorname{Max} R$. This follows from (2.9), since $R_{\mathfrak{m}}$ is a field.

(B) In contrast to (A), (1) in Corollary 2.15 does not hold for *general* (that is, noncommutative) von Neumann regular rings R. For instance, a noncommutative division ring is certainly von Neumann regular, but it does not satisfy (E_n) for any $n \geqslant 2$, according to Remark 1.10B and Remark 1.14.

§3. Quillen Induction, and the Bass-Quillen Conjecture

The first main topic in this section is Quillen Induction, which is a powerful way to test if a given family $\mathfrak{F}$ of commutative rings would have the extension property (E) (defined in (1.9)). This induction principle will be derived here by elaborating and axiomatizing the inductive argument used in the proof of (2.9). Of course, if the open question (1.16) we posed earlier does have an affirmative answer, then all we need to do is to check that each ring in $\mathfrak{F}$ has the property (E_1), which can be checked locally. Given, however, that the answer to (1.16) is unknown for individual rings, the following induction principle is the best general machinery that one can come up with in checking the property (E) via (E_1) — for *families* of commutative rings.

Theorem 3.1. (Quillen Induction) *Let $\mathfrak{F}$ be a class of commutative rings satisfying the properties below:*

(Q_1) : $R \in \mathfrak{F} \implies R\langle t\rangle \in \mathfrak{F}$;
(Q_2) : $R \in \mathfrak{F} \implies R_{\mathfrak{m}} \in \mathfrak{F}$ *for any* $\mathfrak{m} \in \operatorname{Max} R$;
(Q_3) : $R \in \mathfrak{F}$ *and* R *local* $\implies R$ *has the property* (E_1) *(that is, any* $P \in \mathfrak{P}\big(R[t]\big)$ *is free).*

Then, any $R \in \mathfrak{F}$ has the property (E)*; i.e., for all n, any $P \in \mathfrak{P}\big(R[t_1, \ldots, t_n]\big)$ is extended from R.*

Proof. We induct on n. The case $n = 0$ is trivial, and the case $n = 1$ follows from (Q_2), (Q_3), and (1.7). In the following, we shall, therefore, assume $n > 1$. Let $A = R[t_2, \ldots, t_n]$, so $P \in \mathfrak{P}(A[t_1])$. By the inductive hypothesis, it is enough to show that P is extended from A. This will follow from the Affine Horrocks' Theorem (2.2) if we can show that $A\langle t_1\rangle \otimes_{A[t_1]} P$ is extended from a module in $\mathfrak{P}(A)$. [Of course, A need not be in $\mathfrak{F}$, but (2.2) applies to *any* commutative ring.] In view of the inclusions

$$A[t_1] \subseteq R\langle t_1\rangle[t_2, \ldots, t_n] \subseteq A\langle t_1\rangle,$$

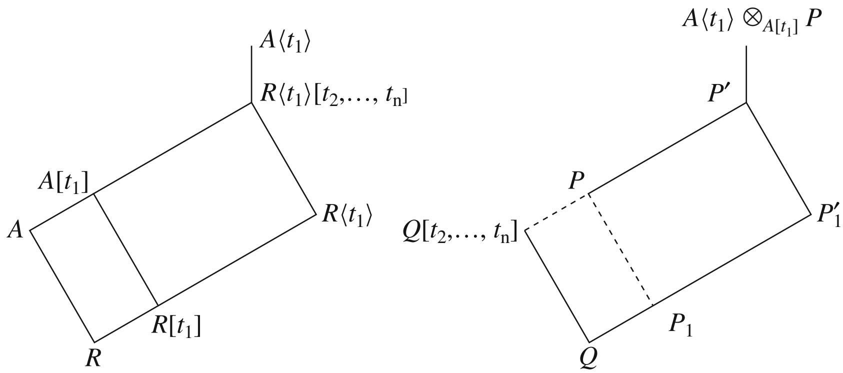
it is sufficient to show that

(3.2) $\quad P' = R\langle t_1\rangle[t_2, \ldots, t_n] \otimes_{A[t_1]} P$ is extended from a module in $\mathfrak{P}(A)$.

Since $R\langle t_1\rangle \in \mathfrak{F}$ (by (Q_1)), the inductive hypothesis implies that P' is the extension of

$$P_1' = P'/(t_2, \ldots, t_n)P' \in \mathfrak{P}\big(R\langle t_1\rangle\big),$$

which is, in turn, the extension of $P_1 = P/(t_2, \ldots, t_n)P \in \mathfrak{P}\big(R[t_1]\big)$:



By the case $n = 1$, P_1 is the extension of

$$Q = P_1/t_1P \cong P/(t_1, \ldots, t_n)P.$$

Thus, P' is the extension of $Q \in \mathfrak{P}(R)$ from R to $R\langle t_1\rangle[t_2, \ldots, t_n]$. In particular, P' is the extension of $Q[t_2, \ldots, t_n] \in \mathfrak{P}(A)$ from A to $R\langle t_1\rangle[t_2, \ldots, t_n]$. This proves (3.2). $\square$

If we apply the above Quillen Induction, respectively, to

$$\mathfrak{F} = \{\text{all fields}\}, \quad \{\text{all PID's}\}, \quad \text{or} \quad \{\text{all Dedekind domains}\},$$

we will get back the results (2.9) and (2.11). [For these inductions, the hypothesis (Q_1) is ensured by (2) and (3) of (IV.1.3).] More ambitiously, we can try to apply Quillen Induction to

$$\mathfrak{F} = \{\text{all commutative regular rings of Krull dimension} \leqslant 2\}.$$

For this class $\mathfrak{F}$, (Q_1) is guaranteed by IV.1.3(1), and (Q_2) by II.5.4. The upshot of the induction process is the following wonderful result, further extending (2.9) and (2.11).

Theorem 3.3. *Any commutative regular ring R of Krull dimension $\leqslant 2$ has the property* (E).

Proof. By the Quillen Induction, it is enough to show that any $P \in \mathfrak{P}\big(R[t]\big)$ is free, where R is any commutative regular local ring of Krull dimension $d \leqslant 2$. If $d \leqslant 1$, R is a discrete valuation ring — in particular, a PID. In this case, we can use Seshadri's Theorem (II.6.1). [Alternatively, we can invoke (2.2) and (IV.1.3(3)).] If $d = 2$, we can use the Murthy-Horrocks Theorem (IV.6.6). $\square$

The above analysis clearly points to the following interesting conjecture, raised by [Bass: 1972a, Sec. 4.1] and by [Quillen: 1976]:

(BQ_d) The Bass-Quillen Conjecture. *Any commutative regular ring R of Krull dimension $\leqslant d$ has the property* (E).

To handle this conjecture, one would naturally hope to proceed as before – this time applying the Quillen Induction to the class of all commutative regular rings of Krull dimension $\leqslant d$. This class satisfies (Q_1) and (Q_2) by the same references quoted earlier. Thus, (BQ_d) can be translated into the following equivalent "local" conjecture:

(BQ'_d) *Any commutative regular local ring R of Krull dimension $\leqslant d$ has the property* (E_1)*; that is, any $P \in \mathfrak{P}\big(R[t]\big)$ is free.*

In connection to this version of the Bass-Quillen Conjecture, we should mention a possible line of attack that was proposed by Quillen in his original (1976) paper solving Serre's Conjecture. Quillen raised the following question:

(QQ_d) *Let (B, M) be a commutative regular local ring of Krull dimension $\leqslant d+1$, and let $s \in B$ be a regular parameter (that is, $s \in M \backslash M^2$). Over the localization B_s, is every f.g. projective module free?*

(Note that the answer to (QQ_1) is "yes" by (IV.6.2).)

The raison d'être for the above question, as cogently pointed out by Quillen, is that *if (QQ_d) is answered affirmatively, then the truth of (BQ'_d) will follow.* Indeed, if $(R, \mathfrak{m})$ is any regular local ring of dimension $e \leqslant d$, then for the local ring $(B, M) \subseteq R\langle t\rangle$ constructed in (IV.1.4(3)) and (IV.1.7), we have $R\langle t\rangle = B_s$, with $s := t^{-1} \in M \backslash M^2$. Since $R[s]$ is regular by (II.5.7), $B = R[s]_{(\mathfrak{m},s)}$ is a regular local ring, of dimension

$$\mathrm{ht}\,(\mathfrak{m}, s) = \mathrm{ht}\,(\mathfrak{m}) + 1 = e + 1 \leqslant d + 1.$$

If the answer to (QQ_d) is "yes", then f.g. projectives are free over $B_s = R\langle t\rangle$. From the Supplement (2.3) to the Affine Horrocks' Theorem, it follows immediately that f.g. projectives are also free over $R[t]$, thus proving (BQ'_d)!

The local Bass-Quillen Conjecture (BQ'_d) seems to be still open (for $d \geqslant 3$) as of this date of writing. However, many special cases of it have been successfully handled, starting with [Lindel-Lütkebohmert: 1977] and [Mohan Kumar: 1979]. In these two papers, (BQ'_d) was shown to be true (for any d) in the case where R is a formal power series ring $k[[x_1, \ldots, x_d]]$ over a field k. (This result will be presented in the last section (§5) of this chapter.) The strongest result to date is that (BQ'_d) *is true* (*for any* d) *if the regular local ring* R *contains a field*; for more information on this, see Remark (5.23)(2) and VIII.6.

In the balance of this section, we shall formulate some more questions, which we'll call "conjectures" for convenience. These are all rather strong statements, at least in the sense that the truth of *any* of them will imply the truth of (BQ_d) (for any d)! There does not seem to exist too much evidence for any of these stronger "conjectures" to be true. However, no counterexamples for them seem to be known either.

We start with the following conjecture (H) and its local version (H′):

(H) *If* R *is a commutative Hermite ring, then* $R[t]$ *is also Hermite.*(*)

(H′) *If* R *is a commutative local ring, then* $R[t]$ *is Hermite.*

There are also the following problems (S_1), (S_2), (S_3) raised in [Swan: 1978]. We'll record them below in the form of "conjectures" as well.

(S_1) *If* R *is a commutative ring, and* $P \in \mathfrak{P}(R[t])$ *is such that* $[P] \in \operatorname{im}(K_0 R \to K_0 R[t])$*, then* P *is extended from* R.

(S_2) *If* R *is a commutative ring, and* P *is a f.g. stably free module over* $R[t]$*, then* P *is extended from* R.

(S_3) *If* R *is a commutative ring, and* $f = (f_1(t), \ldots, f_n(t))$ *is a unimodular row over* $R[t]$ *such that* $f(0) = (1, 0, \ldots, 0)$*, then* f *can be completed to a matrix in* $\mathrm{GL}_n(R[t])$.

We can also define (S'_1), (S'_2), (S'_3) to be the *local* versions of the above, i.e., (S'_i) shall denote the conjecture (S_i) for commutative *local* rings R, $1 \leqslant i \leqslant 3$.

Proposition 3.4. *The conjectures* (H), (H′), (S_1), (S_2), (S_3), (S'_1), (S'_2), *and* (S'_3) *are all equivalent, and the truth of any of these will imply the truth of* (BQ'_d) *(hence that of* (BQ_d) *) for all* d.

Proof. By Grothendieck's Theorem (II.5.8), we have $(\mathrm{S}'_1) \Rightarrow (\mathrm{BQ}'_d)$ for any d. The equivalence of the eight "conjectures" can be seen as follows:

(*)One can also consider the related statement: *"if every* $P \in \mathfrak{P}(R)$ *is free, then every* $Q \in \mathfrak{P}(R[t])$ *is free."* However, this is not true according to the example given after the statement of Proposition II.5.10.

$$
\begin{tikzcd}
(\mathrm{S}_1) \arrow[dd, shift right, "T"'] & (\mathrm{S}_2) \arrow[dd, shift right, "T"'] \arrow[rr, "T"] \arrow[dr, "T"] & & (\mathrm{S}_3) \arrow[dd, "T"'] \\
& & (\mathrm{H}) \arrow[d, "T"] & \\
(\mathrm{S}_1') \arrow[uu, shift right, "Q"'] \arrow[r, Leftrightarrow, "T"'] & (\mathrm{S}_2') \arrow[uu, shift right, "Q"'] \arrow[r, Leftrightarrow, "T"'] & (\mathrm{H}') & (\mathrm{S}_3') \arrow[l]
\end{tikzcd}
$$

Here, T denotes a trivial implication, and Q denotes "by application of the Quillen Patching Theorem". In view of this diagram, we need only explain $(\mathrm{S}_3') \Rightarrow (\mathrm{H}')$. Let R be a commutative local ring, and let $f = (f_1(t), \ldots, f_n(t))$ be a unimodular row over $R[t]$. Let M be an invertible matrix over R such that $f(0) \cdot M = (1, 0, \ldots, 0)$. We have $f(t) \sim g(t) = f(t) \cdot M$ (see the paragraph preceding (I.4.8) for the notation), and, by (S_3'), $g(t) \sim (1, 0, \ldots, 0)$. Thus, $f(t) \sim (1, 0, \ldots, 0)$, as required. $\square$

Of course, we can also get $(\mathrm{H}') \Rightarrow (\mathrm{H})$ directly from Quillen's Patching Theorem, in the form (1.7). In fact, (1.7) has obviously the following implication for Hermite rings.

Remark 3.5. If R is a commutative Hermite ring such that $R_{\mathfrak{m}}[t_1, \ldots, t_n]$ is Hermite for all $\mathfrak{m} \in \operatorname{Max} R$, then $R[t_1, \ldots, t_n]$ is Hermite. (The same statement remains valid if we replace "Hermite" by "d-Hermite".)

In a similar vein, we can prove (3.6) and (3.7) below, which say essentially that the various conjectures made above are true if the rank of P is sufficiently large.

Big Rank Theorem 3.6. *Let R be a commutative noetherian ring of Krull dimension $d < \infty$. Let $P \in \mathfrak{P}\big(R[t_1, \ldots, t_n]\big)$ be such that* $\operatorname{rk} P > d$*, and*

$$[P] \in \operatorname{im}\big(K_0 R \to K_0 R[t_1, \ldots, t_n]\big).$$

Then P is extended from R. (In particular, (S_1) *is true if P has "big rank.")*

Applying this to regular rings and using (II.5.8), we get:

Corollary 3.7. *Let R be a commutative regular ring of Krull dimension $d < \infty$. Then any $P \in \mathfrak{P}\big(R[t_1, \ldots, t_n]\big)$ with* $\operatorname{rk} P > d$ *is extended from R. (*(BQ_d) *is true if P has "big rank.")*

Proof of (3.6). The hypotheses on P are not changed when we localize R to $R_{\mathfrak{m}}$. Thus, by Quillen's Patching Theorem, we may assume that R is local, and hence P is stably free. But, by (III.3.1), $R[t_1, \ldots, t_n]$ is d-Hermite, so P must be free. $\square$

The fact that $R[t_1, \ldots, t_n]$ above is d-Hermite (III.3.1) is a rather deep result. Recall that the proof of this involves, among other things, Suslin's Monic Polynomial Theorem, and the Substitution Principle (III.1.7). Of course, we may now say that the use of (III.1.7) can be superseded by an application of the Affine Horrocks' Theorem (2.2), but the latter is by no means an easier result.

There exists an alternative method for proving (3.6) that depends on Quillen's Patching Theorem and Suslin's Monic Polynomial Theorem, *but not on* (III.1.7) *or the Affine Horrocks' Theorem*. This method was observed by [Roitman: 1977b]; see also [Geramita: 1974/76]). We shall explain Roitman's method below because it provides a somewhat simpler approach to (3.6).

We work in the local setting (thanks to (3.5)). For R a commutative noetherian local ring of Krull dimension d, we want to show that $A = R[t_1, \ldots, t_n]$ is d-Hermite. We shall induct on n; the case $n = 0$ is trivial (I.4.7(2)). Let

$$f = (f_1, \ldots, f_r) \in \mathrm{Um}_r(A) \quad (r \geqslant d + 2),$$

and let P be its solution space. After replacing f by an equivalent unimodular row, and after a suitable change of variables, we may assume, as in the proof of (III.3.5), that f_1 is monic as a polynomial in t_n. To show that P is free, it suffices to show that P is extended from $B = R[t_1, \ldots, t_{n-1}]$. According to Quillen's Patching Theorem, we need only check that $P_{\mathfrak{m}}$ is extended from $B_{\mathfrak{m}}$ for any $\mathfrak{m} \in \mathrm{Max}\, B$. But $P_{\mathfrak{m}}$ is the solution space of $f_{\mathfrak{m}} \in \mathrm{Um}_r\big(B_{\mathfrak{m}}[t_n]\big)$, and $f_1 \in B_{\mathfrak{m}}[t_n]$ is monic in t_n, so $P_{\mathfrak{m}}$ is free by (III.2.6).

This also gives a somewhat simpler approach to (3.7) (in particular to the solution of Serre's Conjecture over fields and PID's), since (3.7) follows from (3.6) (plus Grothendieck's Theorem (II.5.8)).

The two paragraphs above are essentially dispensable since they offered only a minor variation in methods. The more interesting question behind (3.6) remains whether the rank assumption there is really necessary, which takes us back to the various conjectures enunciated in (3.4).

It would, of course, be very nice if these (equivalent) conjectures all hold, but as we have already cautioned, there does not seem to be much evidence for this. There is, however, a somewhat weaker version of these conjectures, originating from [Suslin: 1977b], which is perceived to be much more likely to hold. We close therefore, by quoting the following more cautious version of (H$'$) and (S$'_3$), in the form of a "problem" — after Suslin.

(3.8) Suslin's Problem Su(R)$_n$. *Let R be a commutative local ring. If $n!$ is invertible in R, is every*

$$f(t) = \big(f_0(t), f_1(t), \ldots, f_n(t)\big) \in \mathrm{Um}_{n+1}\big(R[t]\big)$$

completable (*to a matrix in* $\mathrm{GL}_n\big(R[t]\big)$)?

One reason for optimism with $\mathrm{Su}(R)_n$ is that we know it has a "yes" answer in case $\deg f_i(t) \leqslant 1$ for all i (as long as R is a commutative Hermite ring), by (III.4.13). Beyond this, a number of other significant special cases of $\mathrm{Su}(R)_n$ have also been answered affirmatively. For more information on this, see our survey on unimodular vectors in (VIII.5).

Appendix to §3

In this Appendix to the section on the Bass-Quillen Conjecture, we shall give a short exposition on a general functorial principle (for functors from the category of rings to the category of abelian groups) which has come to be known in the literature as the *Swan-Weibel Homotopy Trick*. To motivate this homotopy principle, let us first revisit the basic theme of the Bass-Quillen Conjecture. Recall that a convenient form of this Conjecture is that, *if A is a commutative regular ring of finite Krull dimension, then A has the property* (E_1)*; that is, any $P \in \mathfrak{P}\big(A[t]\big)$ is extended from A.* Let $\mathcal{E}_1$ be the class of commutative rings that have the property (E_1).(*) Of course, any general information we can garner about the family $\mathcal{E}_1$ should be relevant to the full understanding of the Conjectures (BQ_d).

The Swan-Weibel Trick is an intriguing homotopy device that has (among other applications) led to some interesting facts about the class of rings $\mathcal{E}_1$. As far as I can ascertain, this homotopy trick was published for the first time in D.F. Anderson's paper [1978b] on projective modules over monomial subrings of $k[x, y]$. The following is "Lemma 5.7" in that paper.

Swan-Weibel Homotopy Trick 3.9. *Let F be a covariant functor from the category of rings to the category of abelian groups. Let $A = A_0 \oplus A_1 \oplus A_2 \oplus \cdots$ be a positively graded ring* (*for instance, a multivariate polynomial ring over a ground ring A_0, with the total degree grading*)*. If the inclusion map $i : A \to A[t]$ induces an isomorphism $F(i) : F(A) \to F(A[t])$, then the inclusion map $j : A_0 \to A$ also induces an isomorphism $F(j) : F(A_0) \to F(A)$.*

Proof. Let e_0 and e_1 be the ring homomorphisms $A[t] \to A$ obtained by evaluation at 0 and 1 respectively. Note that e_0 and e_1 are ring homomorphisms (since 0 and 1 are *central* elements in A), and each of them provides a ring splitting for i. In a similar vein, the augmentation map $\pi : A \to A_0$ provides a ring splitting for j.

The crux of the proof lies in the use of the *Swan-Weibel map* $h : A \to A[t]$, defined by

$$(3.10) \qquad h(a_0 + a_1 + \cdots + a_n) = a_0 + a_1 t + \cdots + a_n t^n \quad (n \geqslant 0,\ a_i \in A_i).$$

Since t is a central variable, a routine calculation shows that h is a ring homomorphism. (It is actually also a *graded ring homomorphism* if we view $A[t]$ as graded by taking $\deg(A) = 0$ and $\deg(t) = 1$: h just happens to be a natural graded ring embedding $A \to A[t]$ extending the ring embedding $j : A_0 \to A$ in degree 0.) Obviously, we have

$$(3.11) \qquad e_1 h = \mathrm{Id}_A, \quad \text{and} \quad e_0 h = j\pi.$$

By functoriality, the commutative diagram on the left below induces the commutative diagram on the right:

(*)In §1, we have used the same notation for the class of *all rings* satisfying (E_1). In this section, we'll only be working with commutative rings, so we take the liberty of using the same notation $\mathcal{E}_1$ for the commutative family.

$$\begin{array}{ccccc} A & \xrightarrow{h} & A[t] & \xrightarrow{e_1} & A \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle e_0} & & \\ A_0 & \xrightarrow{j} & A & & \end{array} \qquad \begin{array}{ccccc} F(A) & \xrightarrow{F(h)} & F(A[t]) & \xrightarrow{F(e_1)} & F(A) \\ {\scriptstyle F(\pi)}\downarrow & & \downarrow{\scriptstyle F(e_0)} & & \\ F(A_0) & \xrightarrow{F(j)} & F(A). & & \end{array}$$

Here, $F(e_0)$ and $F(e_1)$ are both splittings for $F(i)$, which is (by assumption) an isomorphism. Therefore, $F(e_0)=F(e_1)$, so by (3.11),

$$F(e_0)\circ F(h)=F(e_1)\circ F(h)=\mathrm{Id}_{F(A)}. \tag{3.12}$$

From the right diagram above, it follows that $F(j)$ is *onto*, and therefore, an *isomorphism* (since $F(j)$ is already a split injection — split by $F(\pi)$). □

As is the case with most mathematical tricks, in the proof above not much seemed to be happening, and yet by the end, a great deal has happened: from $F(i)$ being an isomorphism, we have deduced that *all* maps on the right diagram above are isomorphisms! Incidentally, we could also have concluded that $F(h)=F(i)$, since both maps are inverse to the isomorphism $F(e_1)$.

That the result (3.9) is dubbed a "homotopy trick" is perhaps best explained via the case where A is an algebra over the field of real numbers $\mathbb{R}$. In this case, we can extend e_0 and e_1 to a 1-parameter family of evaluation ring homomorphisms

$$e_r: A[t]\to A \quad \text{for} \quad r\in[0,1]\subseteq\mathbb{R}\subseteq A. \tag{3.13}$$

As r varies continuously from 0 to 1, e_rh provides a "homotopy" from the map $e_0h=j\pi: A\to A$ to the identity map $e_1h=\mathrm{Id}_A$. From the fact that $F(i)$ is an isomorphism ("F is homotopy invariant on A"), it is a natural topological conclusion that the "homotopic maps" e_0h and e_1h induce the same homomorphism under F: this is exactly the key step (3.12) that clinched the proof to (3.9). We thank K.Y. Lam for these clarifying remarks on the topological ideas hidden behind the purely algebraic arguments given above.

The universal nature of the homotopy trick (3.9) is, of course, what makes it so striking. Restating it in terms of the functors particularly relevant to us, we have

Corollary 3.14. *Let* $A=A_0\oplus A_1\oplus\cdots$ *be a graded ring for which any of the following* (*natural*) *maps*[(*)]

$$\mathrm{Pic}(A)\to\mathrm{Pic}\big(A[t]\big),\quad K_0(A)\to K_0\big(A[t]\big),\quad K_1(A)\to K_1\big(A[t]\big)$$

is an isomorphism. Then the corresponding map among

$$\mathrm{Pic}(A_0)\to\mathrm{Pic}(A),\quad K_0(A_0)\to K_0(A),\quad K_1(A_0)\to K_1(A)$$

is also an isomorphism.

[(*)] In the case of the functor "Pic", we assume that A is commutative.

These are nice conclusions that can be applied to Pic when A is a seminormal domain (via (II.5.11)), and to K_0 and K_1 when A is a left regular ring (via Grothendieck's Theorem (II.5.8) and a result not proved here, called the Bass-Heller-Swan Theorem). While some of the conclusions in these special cases could have been gotten from other means, they have been obtained here, almost for free, from nothing more than basic functorial principles.

In the case of K_0 and K_1, the conclusions obtained above can also be given suitable "nonstable" versions, in terms of the extendibility of projective modules and the transitivity of GL_n on unimodular rows. The proofs of these nonstable versions can be given along exactly the same lines as in the proof of (3.9).

Now let us come to the property (E_1) and its (nonstable) K_1-analogue.

Proposition 3.15. *Let $A = A_0 \oplus A_1 \oplus \cdots$ be a graded ring.*

(1) *If A satisfies* (E_1) *(i.e. all f.g. projective $A[t]$-modules are extended from A), then all f.g. projective A-modules are extended from A_0.*

(2) *If, for some n, $\mathrm{GL}_n\big(A[t]\big) = \mathrm{GL}_n(A) \cdot \mathrm{E}_n\big(A[t]\big)$, then*

$$\mathrm{GL}_n(A) = \mathrm{GL}_n(A_0) \cdot \mathrm{E}_n(A).$$

Proof. (1) Given any ring homomorphism $f : R \to S$, any left R-module P "extends" to a left S-module $f_*P = S \otimes_R P$, where S is viewed as an (S, R)-bimodule via f. We shall use the same notations as in the proof of (3.9). Given any $P \in \mathfrak{P}(A)$, let $h_*P = Q \in \mathfrak{P}\big(A[t]\big)$. Since Q is extended from A (by assumption), we get as before

$$\begin{aligned} P &\cong (e_1)_*Q \cong (e_0)_*Q \\ &\cong (e_0h)_*P \cong (j\pi)_*P \\ &\cong j_*(\pi_*P). \end{aligned}$$

Thus, P is extended from $\pi_*P \in \mathfrak{P}(A_0)$. (2) is proved similarly. □

The above result was pointed out to me by Joseph Gubeladze, who also made the following remarks. Recall that $\mathcal{E}_1$ denotes the class of commutative rings having the property (E_1). For any commutative ring R and any f.g. commutative torsion-free, cancellative monoid M without nontrivial invertible elements, the monoid ring $R[M]$ admits a positive grading

$$R[M] = R \oplus R_1 \oplus R_2 \oplus \cdots .$$

If R, M are such that $R[M] \in \mathcal{E}_1$, then (3.15) implies that f.g. $R[M]$-projectives are extended from R. Thus, if $\mathcal{M}$ is a class of monoids of the above type such that

$$M \in \mathcal{M} \Longrightarrow M \times \mathbb{N} \in \mathcal{M},$$

then, for a fixed R, *testing if f.g. $R[M]$-projectives are extended from R for all $M \in \mathcal{M}$ is tantamount to testing if $R[M] \in \mathcal{E}_1$ for all $M \in \mathcal{M}$* (noting that $R[M][t] \cong R[M \times \mathbb{N}]$).

The unexpected fact that the above paragraph reveals is that the problem of determining all (commutative) rings in $\mathcal{E}_1$ would include not only a solution to the Bass-Quillen Conjecture, but also a solution to "Anderson's Conjecture" — that f.g. projectives are free over normal monomial subalgebras of $k[t_1, \ldots, t_n]$ (k a field or a PID). Anderson's Conjecture has since been proved by Gubeladze, and there has been very substantial progress on the Bass-Quillen Conjecture (see VIII.4 and VIII.6 respectively), but the general problem of determining the class $\mathcal{E}_1$ has remained largely an unsolved mystery.

§4. Laurent Polynomial Rings

In this section, we shall try to extend the Big Rank Theorem (3.6) to rings of the sort

$$A = R[t_1^{\pm 1}, \ldots, t_m^{\pm 1}, s_1, \ldots, s_n],$$

where t_i, s_j are independent indeterminates, and R is any (commutative) ring. Such a ring A is called a (generalized) *Laurent polynomial ring (or extension) over R*. Note that the subring $R \subseteq A$ is a (ring-theoretic) retract of A; for instance, $t_i \mapsto 1$ and $s_j \mapsto 0$ induces a well defined retraction from A to R. The existence of such a retraction has the following useful consequences:

(4.1) *By functoriality, $K_0 R \to K_0 A$ is always a monomorphism.*
(4.2) *For $P \in \mathfrak{P}(A)$, let*

$$Q = P/(t_1 - 1, \ldots, t_m - 1, s_1, \ldots, s_n) \cdot P \in \mathfrak{P}(R).$$

If rk $P > d$, *then* rk $Q > d$. (In general, if $A \to B$ is any homomorphism of commutative rings, then for any $P \in \mathfrak{P}(A)$, rk $P > d \Rightarrow$ rk $B \otimes_A P > d$.)
(4.3) *Let P, Q be as above. Then $[P] \in \operatorname{im}(K_0 R \to K_0 A)$ iff $[P] = [A \otimes_R Q]$, and $P \cong A \otimes_R M \Rightarrow M \cong Q$.*

We shall also need the following basic fact about the Grothendieck group K_0 of $A = R[t_1^{\pm 1}, \ldots, t_m^{\pm 1}, s_1, \ldots, s_n]$.

Lemma 4.4. *If R is a left regular ring, the map $K_0 R \to K_0 A$ is an isomorphism.*

Proof. In view of (II.5.4), (II.5.7), (II.5.8), it is sufficient to treat the case $m = 1$, $n = 0$, i.e., to treat $K_0 R \to K_0 R[t, t^{-1}]$. This is one-one by (4.1); to see that it is *onto*, we use Grothendieck's Theorem (II.5.8). By this theorem, $K_0 R \to K_0 R[t]$ is an isomorphism. Thus, it suffices to show that $j : K_0 R[t] \to K_0 R[t, t^{-1}]$ is *onto*. Given $P \in \mathfrak{P}\big(R[t, t^{-1}]\big)$, let M be a f.g. $R[t]$-module such that $S^{-1} M \cong P$, where $S = \{t^i : i \geqslant 0\}$. Since $R[t]$ is left regular, there exists a resolution

$$0 \to Q_r \to \cdots \to Q_0 \to M \to 0$$

where $Q_i \in \mathfrak{P}\big(R[t]\big)$. Localizing this at S, we get a resolution

$$(4.5) \qquad 0 \to S^{-1}Q_r \to \cdots \to S^{-1}Q_0 \to S^{-1}M \to 0.$$

Working in $K_0R[t, t^{-1}]$, we obtain from (4.5):

$$[P] = [S^{-1}M] = \sum (-1)^i [S^{-1}Q_i] \in \mathrm{im}\,(j). \qquad \square$$

Remark 4.6. If R is *not* assumed left regular, $j : K_0R[t] \to K_0R[t, t^{-1}]$ may no longer be onto. We recall, however, that this map is always one-one, at least when R is commutative (see (2.7)).

We are now in a position to state the generalization of (3.6) to the case of Laurent polynomial rings.

Big Rank Theorem 4.7. [Swan: 1978] *Let R be a commutative noetherian ring of Krull dimension $d < \infty$, and let*

$$A = R\,[t_1^{\pm 1}, \ldots, t_m^{\pm 1}, s_1, \ldots, s_n].$$

Let $P \in \mathfrak{P}(A)$ be such that $\mathrm{rk}\, P > d$, *and* $[P] \in \mathrm{im}\,(K_0R \to K_0A)$. *Then P is extended from R.*

The proof we shall give for (4.7) is, however, not going to be completely self-contained. We shall need to know that (4.7) has a certain equivalent formulation ((4.7′) below), and it turns out that the proof of this equivalence depends on a result which we have not previously covered. This needed classical result is the following.

Bass' Cancellation Theorem 4.8. *Let R be a commutative noetherian ring of Krull dimension $d < \infty$. Let $Q, Q' \in \mathfrak{P}(R)$ be such that* $\mathrm{rk}\, Q > d$. *If Q, Q' are stably isomorphic* (*in the sense of* (I.6.1))*, then $Q \cong Q'$.*

This is a fairly difficult result (of which, for instance, (II.7.3) is a special case). To avoid taking a detour here, we'll just assume it, referring the reader for its proof to [Bass: 1968, p. 184]. To prove (4.7), we first verify that it is equivalent to the following.

Theorem 4.7′. *Let R, A and P be as in* (4.7)*, and let $P' \in \mathfrak{P}(A)$. If P, P' are stably isomorphic, then $P \cong P'$.*

In fact, assume (4.7) *holds*, and let P, P' be as in (4.7′). By (4.7), $P \cong A \otimes_R Q$, $P' \cong A \otimes_R Q'$ for suitable (uniquely determined) $Q, Q' \in \mathfrak{P}(R)$. Since $K_0R \to K_0A$ is injective, $[P] = [P'] \Rightarrow [Q] = [Q']$. By (4.2), $\mathrm{rk}\, P > d \Rightarrow \mathrm{rk}\, Q > d$. Invoking now (4.8), $[Q] = [Q'] \Rightarrow Q \cong Q' \Rightarrow P \cong P'$.

Conversely, assume that (4.7′) holds. For P as in (4.7), we have $[P] = [A \otimes_R Q]$ for some $Q \in \mathfrak{P}(R)$ (cf. (4.3)). By (4.7′), this implies that $P \cong A \otimes_R Q$, so P is extended from R.

After knowing the equivalence (4.7) $\Leftrightarrow$ (4.7′), we proceed to prove (4.7) by induction on m. The case $m = 0$ is our earlier result (3.6). In the general case, write $t = t_1$, and let

$$C = R\,[t_2^{\pm 1}, \dots, t_m^{\pm 1}, s_1, \dots, s_n], \text{ and}$$
$$B = R\,\langle t\rangle\,[t_2^{\pm 1}, \dots, t_m^{\pm 1}, s_1, \dots, s_n],$$

so $A = C[t, t^{-1}] \subseteq B$:

$$\begin{tikzcd}
 & & & B \\
 & & A \arrow[ur, dash] & \\
 & C[t^{-1}] \arrow[ur, dash] & & \\
C \arrow[ur, dash] & & & R\langle t\rangle \arrow[uuu, dash, "*"'] \\
 & R[t^{-1}] \arrow[uu, dash] \arrow[urr, dash] & & \\
R \arrow[uu, dash, "*"] \arrow[uuur, dash, "*"] \arrow[ur, dash] & & &
\end{tikzcd}$$

The lines marked with the asterisks are Laurent polynomial extensions with $m-1$ t_i-variables, so, as inductive hypothesis, *we may assume that* (4.7) (*and hence* (4.7′)) *holds for these extensions.* (Note that the coefficient ring $R\langle t\rangle$ for B has the same Krull dimension d as R, by (IV.1.2)). For P as in (4.7), we have $[P] = [A \otimes_R Q]$ for $Q \in \mathfrak{P}(R)$ as in (4.3). Going up to B, this gives

$$[B \otimes_A P] = [B \otimes_R Q] \in K_0 B,$$

so $B \otimes_A P \cong B \otimes_R Q$ by (4.7′). (As we have observed earlier in (4.2), $\operatorname{rk} P > d$ implies that $\operatorname{rk} B \otimes_A P > d$.) Since B is the localization of A at monic polynomials $f(t) \in R[t]$, there exists (by (I.2.16)) such an f with $P_f \cong A_f \otimes_R Q$. By (2.5), P is extended from a certain $P' \in \mathfrak{P}\big(C[t^{-1}]\big)$. For this P', we have

$$[P'] - [C[t^{-1}] \otimes_R Q] \in \ker\big(K_0 C[t^{-1}] \xrightarrow{j} K_0 C[t, t^{-1}]\big).$$

Since j is *injective* by (2.7), we get

$$[P'] = [C[t^{-1}] \otimes_R Q] \in \operatorname{im}\big(K_0 R \to K_0 C[t^{-1}]\big).$$

Moreover,

$$\operatorname{rk} P > d \implies \operatorname{rk} Q > d \implies \operatorname{rk} P' = \operatorname{rk} C[t^{-1}] \otimes_R Q > d.$$

By our inductive hypothesis for $R \overset{*}{\subseteq} C[t^{-1}]$, P' is extended from R, so P is also extended from R. This "completes" the proof of (4.7). $\square$

If R is *regular* in (4.7), $K_0 R \to K_0 A$ is an isomorphism by (4.4), so the hypothesis $[P] \in \operatorname{im}(K_0 R \to K_0 A)$ becomes redundant. Thus, we obtain the following analogue of (3.7) for Laurent polynomial rings.

Corollary 4.9. *Let R be a commutative regular ring of Krull dimension $d < \infty$. Then any $P \in \mathfrak{P}(A)$ $(A = R[t_1^{\pm 1}, \ldots, t_m^{\pm 1}, s_1, \ldots, s_n])$ with* $\operatorname{rk} P > d$ *is extended from R.*

In particular, we get:

Corollary 4.10. *If R is a field, then any $P \in \mathfrak{P}(A)$ is free.*

Note that, in proving (4.7), we only need to invoke the Bass Cancellation Theorem for the coefficient rings R and $R\langle t\rangle$ of a *fixed* Krull dimension d. If $R = k$ is a field as in (4.10), the Bass Cancellation Theorem (for R and $R\langle t\rangle$) is, of course, a triviality, so the arguments needed for (4.10) are nevertheless completely self-contained.

It is of interest to note that, if we restrict our attention to the case where $P \in \mathfrak{P}(A)$ has rank 1, then a fairly mild hypothesis on the base ring R (weaker than regularity) will already guarantee that P is extended from R. We quote, without proof, the following result from [Bass-Murthy: 1967].

Bass-Murthy Theorem 4.11. *If R is a noetherian normal domain, and*

$$A = R\,[t_1^{\pm 1}, \ldots, t_m^{\pm 1}, s_1, \ldots, s_n],$$

then any $P \in \mathfrak{P}(A)$ of rank 1 *is extended from (a f.g. rank* 1 *projective over) R.*

This result allows us to improve upon (4.10), as follows.

Corollary 4.12. *If R is a Dedekind domain, then any $P \in \mathfrak{P}(A)$ is extended from R. If R is a PID, then any $P \in \mathfrak{P}(A)$ is free.*

Proof. It is sufficient to treat the case where R is Dedekind. Since A is an integral domain, $r = \operatorname{rk} P$ is a constant. If $r > 1$, we are done by (4.9) (with $d \leqslant 1$). If $r = 1$, we are done by (4.11).

If we are reluctant to assume (4.11), a direct argument is also possible in the case $r = 1$. We simply repeat the proof for (4.7), inducting on the integer m. Here, we use the results (4.4), IV.1.3(2) and I.6.6 (the latter replacing the use of Bass' Cancellation Theorem 4.8). The induction argument on m reduces us to the case $m = 0$, for which we can invoke (2.11). □

To close this section, let us remark that, in the Big Rank Theorem 4.7, the "big rank" hypothesis $\operatorname{rk} P > d$ is indeed essential (in spite of the encouraging result (4.11)). In [Swan: 1978], an example has been constructed in which

(a) *R is a normal noetherian domain, with* $d =$ Krull dim $R = 4$;
(b) $m = 1$, $n = 0$, *so* $A = R[t, t^{-1}]$;
(c) $P \in \mathfrak{P}(A)$, with $\operatorname{rk} P = 2$;
(d) *P is stably free (in particular,* $[P] \in \operatorname{im}(K_0 R \to K_0 A)$);
(e) *$P_{\mathfrak{m}}$ is free over $R_{\mathfrak{m}}[t, t^{-1}]$ for every* $\mathfrak{m} \in \operatorname{Max} R$, *but*
(f) *P is not extended from $R[t]$, or from $R[t^{-1}]$. A fortiori, P is not extended from R.*

Besides showing that (4.7) need not be true without some appropriate rank assumptions, Swan's example also shows, incidentally, the following:

(4.13) *The result* (4.11) *is peculiar for rank* 1 (*by* (a), (c), (f));

(4.14) *Quillen's Patching Theorem* (1.6) *has no analogue for Laurent polynomial rings* (*by* (e), (f)).

Swan's example of R is simply the complex coordinate ring of the 4-sphere S^4:

$$R = \mathbb{C}[x_0, \ldots, x_4] \qquad (x_0^2 + \cdots + x_4^2 = 1).$$

The module $P \in \mathfrak{P}\big(R[t, t^{-1}]\big)$ is taken to be the solution space of the row matrix:

$$\textbf{(4.15)} \qquad \left(1 - (1-t)\frac{1-x_0}{2},\ (1-t)\frac{x_1 + ix_2}{2},\ (1-t)\frac{x_3 + ix_4}{2}\right).$$

This is easily seen to be unimodular over $R[t, t^{-1}]$. (Modulo the $R[t]$-ideal generated by the three entries, we have

$$(1-t)(x_1^2 + x_2^2) \equiv (1-t)(x_3^2 + x_4^2) \equiv 0,$$

so $(1-t)(1-x_0^2) \equiv 0$. But then $1 + x_0 \equiv 0$; hence $1 - (1-t) = t \equiv 0$. In particular, the three entries generate the unit ideal over $R[t, t^{-1}]$.) The properties (a), (b), (c), (d) follow immediately, and (e) can be checked by applying (III.4.14). However, the crucial fact (f) (that P is *not* extended from $R[t]$ or from $R[t^{-1}]$) depends on a rather deep topological argument involving the cohomology of unitary groups and Steenrod operations. (Note that although (4.15) is defined over $R[t]$, it is not a unimodular row over $R[t]$!)

If all we want is an example to show (4.14), an easier construction would have sufficed. Following Bhatwadekar and Rao, we take

$$R = \mathbb{R}[x, y, z] \quad (\text{with } x^2 + y^2 + z^2 = 1)$$

to be the real coordinate ring of the 2-sphere S^2, and $P \in \mathfrak{P}\big(R[t, t^{-1}]\big)$ to be the solution space of the row matrix

$$(4.16) \qquad \big((1+x) + (1-x)t,\ y,\ z\big).$$

A calculation similar to the one in the last paragraph shows that (4.16) is unimodular over $R[t, t^{-1}]$ (though still not over $R[t]$), so $P \in \mathfrak{P}\big(R[t, t^{-1}]\big)$. Applying (III.4.14), we know that (e) holds. However, P is not extended from R. In fact, $P/(t-1)P$ is the stably free R-module defined by $(2, y, z) \in \mathrm{Um}_3(R)$, which is free, but $P/(t+1)P$ is the stably free R-module defined by

$$(2x,\ y,\ z) \sim (x,\ y,\ z) \in \mathrm{Um}_3(R),$$

which is *not* free (from our earlier discussions on the ring R in (I.4)). This is enough to show that $P \notin \mathfrak{P}^R\big(R[t, t^{-1}]\big)$.

The above counterexamples are peculiar to the Laurent polynomial ring $R[t, t^{-1}]$. Recall that, if $A = R[s_1, \ldots, s_n]$ instead, it is still an open question whether rk $P > d$ is essential for the Big Rank Theorem. In fact, that is in part what the conjectures (S_i), (S_i') in §3 were all about. For more information on this, see Chapter VIII, §§5–6.

§5. Power Series Rings

Recall that the Bass-Quillen Conjecture (in the "local" form (BQ'_d) in §3) states that:

If R is a commutative regular local ring, then any $P \in \mathfrak{P}(R[t])$ is free.

The first important result obtained on this Conjecture is that it is true for rings of the form

$$R = k[[X]] = k[[x_1, \ldots, x_d]];$$

that is, formal power series rings in the variables $X = (x_1, \ldots, x_d)$ over a field k. Such a ring R is noetherian(*) of Krull dimension d, with unique maximal ideal generated by $x_1, \ldots, x_d$, so it is a regular local ring by our earlier remark (II.5.6). The present section will be devoted to a proof of the following result, which is due to [Lindel-Lütkebohmert: 1977] and [Mohan Kumar: 1979].

Theorem 5.1. *Let $X = (x_1, \ldots, x_d)$, $T = (t_1, \ldots, t_n)$ and $R = k[[X]]$, where k is a field. Then any $P \in \mathfrak{P}(R[T])$ is free.*

Actually, according to a well-known result of I. S. Cohen,(†) a commutative regular local ring $(R, \mathfrak{m})$ of Krull dimension d is isomorphic to a formal power series ring $k[[X]]$ as in (5.1) iff it enjoys the following two properties:

(A) R is "*equi-characteristic*," i.e., the characteristic of R equals that of its residue class field $R/\mathfrak{m}$;
(B) R *is complete with respect to its* $\mathfrak{m}$-*adic topology*, i.e., the natural map from R to $\varprojlim R/\mathfrak{m}^i$ is an isomorphism.

Assuming Cohen's result, one may thus restate (5.1) by saying that the conclusion there holds for any commutative regular local ring R satisfying the properties (A) and (B).

To prepare ourselves for the proof of (5.1), let us start by recapitulating some of the key properties of power series rings. The most crucial property needed for (5.1) will be the classical Weierstrass Preparation Theorem. We shall now set the stage for this important result.

Definition 5.2. Let ord_{x_d} denote the (rank 1) discrete valuation on $k[[x_d]]$. A power series $F(x_1, \ldots, x_d) \in k[[X]]$ will be said to be *regular in x_d (of degree s)* if $\mathrm{ord}_{x_d} F(0, \ldots, 0, x_d) = s < \infty$. [In other words, F contains a term $a \cdot x_d^s$ $(a \in k\backslash\{0\})$, but no term $b \cdot x_d^i$ with $b \in k\backslash\{0\}$ and $i < s$.]

(*) A proof for R being noetherian is given in (5.10) below.

(†) See [Zariski-Samuel: 1960, p. 307].

Weierstrass Division Theorem 5.3. *Let* $R = k[[X]] = k[[x_1, \ldots, x_d]]$*, and* $R' = k[[x_1, \ldots, x_{d-1}]]$*. Let* $F \in R$ *be regular in* x_d *of degree* s*. Then, for any* $G \in R$*, there exist* $U \in R$ *and* $V \in \sum_{i=0}^{s-1} R' \cdot x_d^i$ *such that* $G = U \cdot F + V$*. Moreover, such* U*,* V *are uniquely determined* (*for given* F *and* G).

(We can easily combine the existence and the uniqueness parts above into one single statement; namely, $R/R \cdot F$ is a free R'-module with basis $\{x_d^i : 0 \leqslant i < s\}$.)

Proof. Induct on d. For $d = 1$, $F = ax_d^s + bx_d^{s+1} + \ldots$ $(a \in k \setminus \{0\})$ is an associate of x_d^s, so $k\,[[x_d]]/(F) = k\,[[x_d]]/(x_d^s)$, which is clearly a k-vector space with basis $\{x_d^i : 0 \leqslant i < s\}$.

By induction, we may assume that the desired result is true for $S = k[[x_2, \ldots, x_d]]$. Let $S' = k\,[[x_2, \ldots, x_{d-1}]]$, and let us write F, G as formal power series in x (not x_d !):

$$F = \sum_{i=0}^{\infty} f_i \cdot x_1^i, \quad G = \sum_{i=0}^{\infty} g_i \cdot x_1^i \quad (f_i,\ g_i \in S).$$

We shall proceed to find $U = \sum_{i=0}^{\infty} u_i \cdot x_1^i$ and $V = \sum_{i=0}^{\infty} v_i \cdot x_1^i$ (with $u_i \in S$ and $v_i \in \sum_{j=0}^{s-1} S' \cdot x_d^j$) such that $G = U \cdot F + V$. The latter amounts to the system of equations:

$$\begin{aligned} g_0 &= u_0 f_0 + v_0 \\ g_1 - u_0 f_1 &= u_1 f_0 + v_1 \\ &\ldots\ldots\ldots \\ g_i - u_0 f_i - \cdots - u_{i-1} f_1 &= u_i f_0 + v_i \\ &\ldots\ldots\ldots \\ &\ldots\ldots\ldots \end{aligned}$$

Since $F(0, \ldots, 0, x_d) = f_0(0, \ldots, 0, x_d)$, f_0 is also regular in x_d of degree s. By the inductive hypothesis, we can solve $u_0 \in S$ and $v_0 \in \sum_{i=0}^{s-1} S' \cdot x_d^i$ uniquely from the first equation. Using this u_0, we can solve $u_1 \in S$ and $v_1 \in \sum_{i=0}^{s-1} S' \cdot x_d^i$ uniquely from the second equation, ... and so on down! This shows that the required U, V exist, and are unique. □

Definition 5.4. A power series $W \in R = k[[X]]$ will be called a *Weierstrass polynomial* (*of degree* s) *in* x_d if it has the form

(5.5) $$W = w_0 + w_1 x_d + \cdots + w_{s-1} x_d^{s-1} + x_d^s,$$

where $w_i \in k[[x_1, \ldots, x_{d-1}]]$ and $w_i(0, \ldots, 0) = 0$ for all i. [More generally, if $(R', \mathfrak{m}')$ is any local ring, we can say that $W \in R'[x]$ is a Weierstrass polynomial of degree s in x (with coefficients in R') if W has the form

$$w_0 + w_1 x + \cdots + w_{s-1} x^{s-1} + x^s$$

where all $w_i \in \mathfrak{m}'$. The situation in (5.5) above is the special case where $R' = k[[x_1, \ldots, x_{d-1}]]$, and $x = x_d$.]

Note that the Weierstrass polynomials W in (5.5) are special instances of power series that are regular in x_d of degree s, since $\mathrm{ord}_{x_d} W(0, \ldots, 0, x_d) = \mathrm{ord}_{x_d} x_d^s = s$. It turns out that, up to units in $k[[X]]$, the Weierstrass polynomials in x_d actually represent *all* the power series regular in x_d. More precisely, we have

Corollary 5.6. (Weierstrass Preparation Theorem) *Suppose $F \in R = k[[X]]$ is regular in x_d of degree s. Then there exists a unique Weierstrass polynomial W in x_d of degree s such that W is an associate of F in R.*

Proof. We apply the Weierstrass Division Theorem to $G = x_d^s$ "divided" by F, say

$$\begin{aligned} x_d^s &= U \cdot F - (w_0 + w_1 x_d + \cdots + w_{s-1} x_d^{s-1}), \\ U &\in R, \qquad w_i \in k[[x_1, \ldots, x_{d-1}]]. \end{aligned} \tag{5.7}$$

Then, $W := w_0 + \cdots + w_{s-1} x_d^{s-1} + x_d^s = U \cdot F$. Here, U must have nonzero constant term (hence a unit in R), otherwise we can't get a term x_d^s from $U \cdot F$. Moreover, from

$$w_0(0) + \cdots + w_{s-1}(0) x_d^{s-1} + x_d^s = U(0, \ldots, 0, x_d)(a x_d^s + \cdots) \quad (a \in k \backslash \{0\}),$$

it is clear that $w_0(0) = \cdots = w_{s-1}(0) = 0$, so W is a Weierstrass polynomial in x_d of degree s. The uniqueness of W follows evidently from the uniqueness part of (5.3), in view of the equation (5.7). □

We shall now explain that, essentially, any given (nonzero) power series in $k[[X]]$ may be construed to be regular in x_d. The precise statement is as follows.

Lemma 5.8. *Given $0 \neq F \in R = k[[X]]$, there always exists a k-algebra automorphism σ of R leaving x_d fixed, such that $\sigma(F)$ is regular in x_d (of some finite degree).*

Proof. Let σ be the k-algebra automorphism of R induced by the change of variables

$$\sigma(x_i) = x_i + x_d^{r_i} \;\; (1 \leqslant i < d) \quad \frown\lll\Cup \quad \sigma(x_d) = x_d,$$

where $\{r_i\}$ are to be determined. (The inverse of σ is induced by $\tau(x_d) = x_d$ and $\tau(x_i) = x_i - x_d^{r_i}$ for $i < d$.) We would like

$$\sigma(F) = F\big(x_1 + x_d^{r_1}, \ldots, x_{d-1} + x_d^{r_{d-1}}, x_d\big)$$

to be regular in x_d, so we want $F\big(x_d^{r_1}, \ldots, x_d^{r_{d-1}}, x_d\big) \neq 0$. Thus, it will be sufficient to prove the following.

Sublemma 5.9. *Let A be any commutative ring. If $0 \neq f(y, z) \in A[[y, z]]$, then there exists $r \geqslant 0$ such that $f(z^r, z) \neq 0$.*

Proof. Order all monomials $y^i z^j$ lexicographically by their exponents (i, j). Let $a \cdot y^{i_0} z^{j_0}$ $(0 \neq a \in A)$ be the term in f that is lexicographically the smallest. We take any $r > j_0$ and it works! In fact, for any *other* nonzero monomial $b \cdot y^i z^j$ in f, the substitution $y \mapsto z^r$ will take it to $b \cdot z^{ri+j}$ with exponent $> ri_0 + j_0$. Thus, $z^{ri_0+j_0}$ is alone of its kind in $f(z^r, z)$, with nonzero coefficient a. □

A useful by-product of the above considerations is:

Corollary 5.10. $R = k[[X]]$ *is noetherian.*

Proof. We induct on the number of variables, d, the case $d = 0$ being trivial. Let $\mathfrak{A}$ be a nonzero ideal in R. To show that $\mathfrak{A}$ is f.g., we may assume, by (5.8), that $\mathfrak{A}$ contains a power series F that is regular in x_d, say of degree s. By (5.3),

$$\mathfrak{A} = R \cdot F + \mathfrak{A} \cap \sum_{i=0}^{s-1} R' \cdot x_d^i \qquad \big(R' = k[[x_1, \ldots, x_{d-1}]]\big).$$

By the inductive hypothesis, R' is noetherian. Hence, $\mathfrak{A} \cap \sum_{i=0}^{s-1} R' \cdot x_d^i$ is a f.g. R'-module, and so $\mathfrak{A}$ is a f.g. R-module.(*) □

Next, we shall establish a technical lemma that will be crucial for the proof of (5.1). The motivation for this abstract lemma will be given in several remarks following its statement.

Descent Lemma 5.11. *Let B be a subring of a ring A, and let W be a central non 0-divisor of A that lies in B, with the property that $B/B \cdot W \to A/A \cdot W$ (induced by the inclusion map) is an isomorphism. Let P be an A-module on which W acts as a non 0-divisor. Let M be a B_W-module such that $P_W \cong A_W \otimes_{B_W} M$. Then there exists a B-module Q such that $P \cong A \otimes_B Q$ and $Q_W \cong M$.*

Remarks 5.12. (1) For instance, if P above is such that P_W is free, then we can conclude that P is extended from some suitable B-module Q. (Moreover, Q may be chosen such that Q_W is free.)

(2) As an important illustration of the lemma, let

$$A = k[[x_1, \ldots, x_d]], \quad B = k[[x_1, \ldots, x_{d-1}]]\,[x_d],$$

and $W \in B$ be a Weierstrass polynomial in x_d. Then (5.3) implies that $A = A \cdot W + B$, and clearly $A \cdot W \cap B = B \cdot W$. Thus, $B/B \cdot W \to A/A \cdot W$ is an isomorphism, as in the hypothesis of (5.11).

(3) Suppose A, B are commutative algebras over a commutative ring k, and that $W \in B \subseteq A$ satisfy the hypotheses of (5.11). Let E be any commutative k-algebra that is flat as a k-module, and let

(*) More generally, it is true that, whenever A is left noetherian, $A[[X]]$ is also left noetherian. This can be shown by an argument similar to that used for the proof of the Hilbert Basis Theorem in the polynomial case. The proof given here for (5.10), albeit less general, is an immediate application of the Weierstrass Preparation Theorem.

$$A' = A \otimes_k E, \quad B' = B \otimes_k E.$$

Then $W \otimes 1 \in B' \to A'$ *also satisfy the hypotheses* of (5.11). In fact, the flatness of E over k implies that $B' \to A'$ is one-one, and that $1 \otimes W$ is a non 0-divisor in A'. The rest is clear from:

$$\begin{array}{ccccc}
B'/B' \cdot (W \otimes 1) = & \dfrac{B \otimes_k E}{B{\cdot}W \otimes_k E} & \cong & \dfrac{B}{B{\cdot}W} & \otimes_k E \\
\Big\downarrow & & & & \Big\downarrow \cong \\
A'/A' \cdot (W \otimes 1) \cong & \dfrac{A \otimes_k E}{A{\cdot}W \otimes_k E} & \cong & \dfrac{A}{A{\cdot}W} & \otimes_k E.
\end{array}$$

For example, let k and $W \in B \subseteq A$ be as in (2). Setting $E = k[T] = k[t_1, \ldots, t_n]$ in the above observation, we see that the Descent Lemma can be applied to

$$W \in B' = k[[x_1, \ldots, x_{d-1}]]\,[x_d, T] \subseteq A' = k[[x_1, \ldots, x_d]]\,[T].$$

This will prove to be important in studying the (projective) modules over $k[[x_1, \ldots, x_d]][T]$.

Proof of (5.11). Since W acts as a non 0-divisor on P, we can think of $P \to P_W$ as an embedding. Let f be the composition

$$P \subseteq P_W \cong A_W \otimes_{B_W} M \cong A \otimes_B M,$$

which is a monomorphism of A-modules. Let N be the A-module defined by the exact sequence

$$(5.13) \qquad 0 \longrightarrow P \xrightarrow{f} A \otimes_B M \xrightarrow{g} N \longrightarrow 0.$$

Localizing this at W, we see that $N_W = 0$. We claim that

$$(5.14) \qquad N \cong A \otimes_B N \text{ as } A\text{-modules.}$$

It suffices to show that $a \otimes_B n = 1 \otimes_B a \cdot n$ for all $a \in A$, $n \in N$, for then $n \mapsto 1 \otimes_B n$ provides the required A-isomorphism in (5.14). Say $W^i \cdot n = 0$. From the hypothesis $A = A \cdot W + B$, we get by induction $A = A \cdot W^i + B$. Thus, we can write $a = a' \cdot W^i + b$ $(a' \in A,\ b \in B)$, whence

$$\begin{aligned} a \otimes_B n = (a' \cdot W^i + b) \otimes_B n &= a' \otimes_B W^i n + 1 \otimes_B bn \\ &= 1 \otimes_B bn = 1 \otimes_B an. \end{aligned}$$

Using (5.14) as an identification, we can rewrite (5.13) as

$$0 \longrightarrow P \xrightarrow{f} A \otimes_B M \xrightarrow{A \otimes_B g_0} A \otimes_B N \longrightarrow 0,$$

where g_0 is the composition $M \to 1 \otimes M \xrightarrow{g} N$. Let Q be the B-module defined by

(5.15) $$0 \longrightarrow Q \longrightarrow M \xrightarrow{g_0} N \longrightarrow 0.$$

Localizing this at W, we see that $Q_W \cong M$, as required. If we can show that

(†) *the sequence* (5.15) *remains exact after tensoring up to* A,

then (5.13) shows that $P \cong A \otimes_B Q$, and we are finished. To show (†), the quickest way is to use a little bit of the theory of Tor, as follows. From (5.15), we get from the Tor-sequence:

$$\mathrm{Tor}_1^B(A, N) \to A \otimes_B Q \to A \otimes_B M \to A \otimes_B N \to 0,$$

so it suffices to show $\mathrm{Tor}_1^B(A, N) = 0$. But from $0 \to B \to A \to A/B \to 0$, we have

$$0 = \mathrm{Tor}_i^B(B, N) \to \mathrm{Tor}_i^B(A, N) \to \mathrm{Tor}_i^B(A/B, N) \qquad (i \geqslant 1),$$

so we need only show that $\mathrm{Tor}_i^B(A/B,\ N) = 0$ for $i \geqslant 1$. Since Tor_i^B commutes with direct limit in the second variable, we may replace N by its f.g. submodules, and hence assume that $W^r \cdot N = 0$ for some r. Then W^r kills $\mathrm{Tor}_i^B(-,\ N) = 0$.(*) But W acts *invertibly* on A/B (from the hypothesis $B/B \cdot W \cong A/A \cdot W$) and hence also on $\mathrm{Tor}_i^B(A/B,\ -)$. It follows that $\mathrm{Tor}_i^B(A/B,\ N) = 0$, as desired. □

We would like to leave to the reader, at this point, the task of finding a proof for (†) *without* using the functor Tor. Such a proof would, typically, involve the use of generators and relations on various modules. It is, by all means, feasible, but it will be considerably longer and more computational. Since the Tor functor is devised exactly to conceptualize and simplify such computations, we have chosen to present the (somewhat more general) Tor proof above.

We are now ready to begin the proof of (5.1). In fact, we shall try to prove a much stronger result, which is essentially due to [Mohan Kumar: 1979]. I am grateful to H. Bass who pointed out to me the following formulation of Mohan Kumar's result. In fact, the exposition of this entire section is very much based on Bass's suggestions.

Theorem 5.16. *Let k be a field, $R = k[[X]] = k[[x_1, \ldots, x_d]]$, and K be the quotient field of R. Let E be any commutative k-algebra. Let $P \in \mathfrak{P}(R \otimes_k E)$, and $P_0 \in \mathfrak{P}(E)$. If $K \otimes_R P \cong K \otimes_k P_0$ as $K \otimes_k E$-modules, then $P \cong R \otimes_k P_0$ as $R \otimes_k E$-modules (and, of course, conversely).*

The formulation of this result is somewhat analogous to that of the Affine Horrocks' Theorem; cf. Supplement (2.3). (In fact, the latter will be used for the proof of (5.16).) The following diagram may be of help in keeping track of the various rings involved:

(*)$\mathrm{Tor}_i^B(-,\ -)$ is a module over the center of B.

$$\begin{tikzcd}[column sep=small]
 & & K \otimes_k E \arrow[d, dash] \\
 & R \otimes_k E \arrow[ur, dash] \arrow[d, dash] & K \\
E \arrow[ur, dash] \arrow[d, dash] & R \arrow[ur, dash] & \\
k \arrow[ur, dash] & &
\end{tikzcd} \tag{5.17}$$

Before we prove the theorem, let us first explore some of its important consequences.

Corollary 5.18. *Keep the notations above. If $K \otimes_R P$ is $K \otimes_k E$-free (resp. stably free), then P is $R \otimes_k E$-free (resp. stably free). If all modules in $\mathfrak{P}(K \otimes_k E)$ are free (resp. stably free), then all modules in $\mathfrak{P}(R \otimes_k E)$ are free (resp. stably free).*

Proof. Simply take P_0 to be a free E-module of suitable rank, and apply the theorem. (In the stably free case, apply the theorem "stably" instead!) □

Corollary 5.19. (= Theorem 5.1) *If k is a field, and $R = k[[X]]$, then any $P \in \mathfrak{P}\big(R[T]\big)$ is free (where $T = (t_1, \ldots, t_n)$).*

Proof. Let E be the polynomial algebra $k[T]$. Then $R \otimes_k E \cong R[T]$ and $K \otimes_k E \cong K[T]$. By the Quillen-Suslin Theorem, all modules in $\mathfrak{P}\big(K[T]\big)$ are free. Now apply (the second part of) the corollary above. □

Remark 5.20. If, instead of the Quillen-Suslin Theorem, we apply (4.10), we will get the analogue of (5.19) for the Laurent polynomial ring $R[t_1, \ldots, t_n; s_1, s_1^{-1}, \ldots, s_m, s_m^{-1}]$.

Returning to (5.16), we'll now give its proof.

Proof of (5.16). The functor $K \otimes_R -$ is just localization of R-modules at the multiplicative set $R \setminus \{0\}$. By (I.2.16), the given isomorphism

$$K \otimes_R P \cong K \otimes_k P_0 \cong K \otimes_R (R \otimes_k P_0)$$

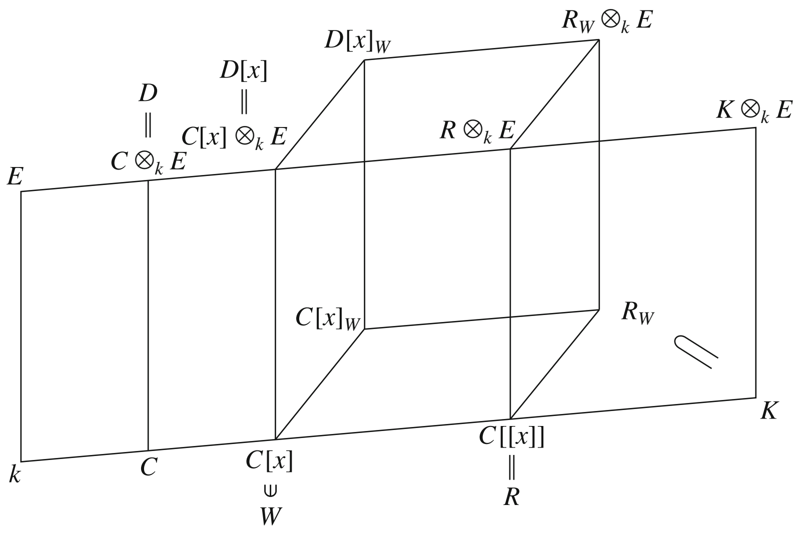
implies that there exists a power series $0 \neq W \in R$ such that

$$P_W \cong (R \otimes_k P_0)_W \cong R_W \otimes_k P_0.$$

By (5.8), and the Weierstrass Preparation Theorem, *we may assume that W is a Weierstrass polynomial in x_d*. To simplify the notations, let us write

$$x = x_d, \quad C = k[[x_1, \ldots, x_{d-1}]], \quad \text{and} \quad R = C[[x]]$$

in the following. Also, let $D = C \otimes_k E$ (a C-algebra) and $P_1 = C \otimes_k P_0 \in \mathfrak{P}(D)$. The earlier diagram (5.17) can thus be enlarged as follows:



From $P_W \cong R_W \otimes_k P_0$, we somehow wish to conclude that $P \cong R \otimes_k P_0$. Recall that the Descent Lemma (5.11) can be applied to the pair of rings $C[[x]] \supseteq C[x] \ni W$ (see (5.12)(2)), so it can also be applied to the pair

$$R \otimes_k E \supseteq C[x] \otimes_k E \ni W \otimes 1,$$

by (5.12)(3). Let $M = P_1[x]_W \in \mathfrak{P}\big(D[x]_W\big)$. Since $P_W \cong R_W \otimes_k P_0$, P_W is the extension of M. By (5.11), there exists a $D[x]$-module Q such that P is the extension of Q, and $Q_W \cong M$. We claim that

(5.21) $\qquad$ *Q is f.g. projective over $D[x]$.*

If we assume this claim, then by the Affine Horrocks' Theorem (in the Supplement form (2.3)), we have

$$Q_W \cong M = P_1[x]_W \Longrightarrow Q \cong P_1[x] \quad (W \in C[x] \text{ is monic!}).$$

Thus, P is the extension of $P_1[x]$, and hence of P_0 all the way from E, which is what we want.

Now let S be the multiplicative set $1 + x\,C[x]$ in $C[x]$. By the Top-Bottom Lemma (IV.5.3), $W \in C[x]$ is comaximal with every polynomial in S. Thus, to prove (5.21), it suffices to check that

$$Q_W \in \mathfrak{P}\big(D[x]_W\big) \quad \text{and} \quad Q_S \in \mathfrak{P}\big(D[x]_S\big),$$

according to (I.3.16). The former is trivial, since (by choice) $Q_W \cong M \in \mathfrak{P}\big(D[x]_W\big)$. For the latter, note that Q_S extends to $P \in \mathfrak{P}\big(R \otimes_k E\big)$. If we can show that

(5.22) $\qquad R \otimes_k E$ *is faithfully flat over* $D[x]_S = C[x]_S \otimes_k E$,

then, by faithfully flat descent (cf. (I.2.15)), we can conclude that $Q_S \in \mathfrak{P}\big(D[x]_S\big)$, as desired.

To wrap up the proof, we must then establish (5.22). For this, we invoke (I.2.2(5)). According to this, (5.22) will follow if we can show that R is faithfully flat over $C[x]_S$. Now,

$$\begin{aligned} R = C[[x]] &= (x)\text{-adic completion of } C[x] \\ &= (x)\text{-adic completion of } C[x]_S \\ &= (x)\text{-adic completion of } C[x]_{(\mathfrak{m},x)}, \end{aligned}$$

where $\mathfrak{m}$ is the maximal ideal of C. Since $(\mathfrak{m}, x)$ is maximal in $C[x]$, the localization $C[x]_{(\mathfrak{m},x)}$ is a *local* ring. By (I.2.7), its (x)-adic completion is faithfully flat over it. This completes the proof of (5.16). $\square$

Remark 5.23. (1) We have seen from the above that the Weierstrass Preparation Theorem provides the key to the proof of Theorem 5.1. Now it is well-known that the Weierstrass Preparation Theorem also holds for $k\{X\} = k\{x_1, \ldots, x_d\}$, the ring of *convergent* power series over an absolute valued field k (see, e.g. [Zariski-Samuel: 1960, pp. 142–145]). Thus, an argument similar to the one given above can be used to prove Theorem 5.1 for projective modules over $k\{X\}[T]$.

(2) As we have observed before, (5.1) settles the Bass-Quillen Conjecture (BQ'_d) for *equi-characteristic complete* regular local rings. The strongest result known to date about (BQ'_d) is that it is true for *all equi-characteristic* regular local rings; that is, all those that contain a field. (In particular, the conjecture (BQ_d) in §3 is true for all regular rings containing a field.) This is proved by a new technique known as the Néron-Popescu desingularization. For more information on this and bibliographical references, see our survey in (VIII.6).

Notes on Chapter V

Quillen's proof and Suslin's proof of Serre's Conjecture were discovered independently, at about the same time, around January, 1976 (to the best of my knowledge). The local-global method exposited in this chapter is that of Quillen. Suslin's proof, on the other hand, does not involve local-global techniques. Using the Towber Presentation as a starting point, Suslin's proof is carried out basically on the global level, and leads directly to the Affine Horrocks' Theorem without recourse to its local version. For details of Suslin's proof, see [Suslin: 1976a] or the exposition in [Ferrand: 1976].

Quillen's theorem on polynomial units ((1.1), (1.2), (1.3)) is an "elementary" result which, according to some people, "should have been in Bourbaki" (but has never appeared before in the literature). The Patching Theorem ((1.6), (1.7)) is an easy and natural consequence of (1.1). It holds even for graded algebras, as was observed by M. P. Murthy. For another proof of the Patching Theorem, see the paper [Roitman: 1979] listed in the references on Chapter VIII. For further generalizations of the Patching Theorem in several different directions, see [Bass-Connell-Wright: 1976, Th. (4.13), (4.14)], and [Raghunathan: 1978].

Quillen's Patching Theorem and Roitman's result (1.11) lent credence to the extension properties (E_n) and (E) in (1.9) as natural and useful *local* properties to study. (Recall that, by "local property", we mean a property that can be checked locally.) A positive answer to Question 1.16 is probably "too good to be true", but I don't know of any counterexamples either. In the commutative case, there is a "stable form" of this question, due to Sharma and Strooker, asking if an isomorphism $K_0R \cong K_0R\,[t]$ would entail an isomorphism $K_0R \cong K_0R\,[t_1, \dots, t_n]$ for every n. For the proper setting in which this question arose, and the relevant bibliographical information, see VIII.9.

The Affine Horrocks' Theorem was dealt with in the rank 1 case as early as in [Bass-Murthy: 1967, §6], [Bass: 1968, XII.10], and was later raised in general as an open question ("Problem X") in [Bass: 1972a]. Bass's paper (p. 24) also contained Murthy's important remark that the Affine Horrocks' Theorem can be used to prove Serre's Conjecture over fields. Given hindsight, it is, of course, clear that the Patching Theorem provides exactly the right tool for deriving the "Affine Horrocks" result. "Affine Horrocks", however, seems to be peculiar to affine schemes. For the failure of its extension to *arbitrary* schemes (or even to punctured local spectra), see [Swan: 1978].

The possibility of using Suslin's Monic Polynomial Theorem together with Quillen's Patching Theorem to prove Serre's Conjecture (or a 'big rank' type theorem) was independently observed by Murthy, [Roitman: 1977b], and [Swan: 1978]. Our exposition of the Big Rank Theorem in the Laurent polynomial case follows closely [Swan: 1978]. Corollary 1.8 is taken from the same reference, where the result is formulated in terms of the functor NK_0 in algebraic K-theory. For an analogue of this result for the functor NK_1, see [Vorst: 1979].

H. Bass has pointed out that the terminology of "Quillen Induction" in our text (§3) is not consistent with its use in [Bass-Connell-Wright: 1976]. Unfortunately, after much effort, I can think of no better name! Thus, I decided to keep the original usage in §3, with the understanding from the reader that "Quillen Induction" is not unique.

The Bass-Quillen Conjecture first appeared as "Problem IX" in [Bass: 1972a, p. 21] and then as a *conjecture* in the preprint version of [Quillen: 1976]. It was retained only as a *question* in the published version of Quillen's paper. The possibility of reducing the case of dimension $\leqslant d$ to the *local* situation is solely due to Quillen. The affirmative answer to (BQ_2) is mentioned in the published versions of both [Quillen: 1976] and [Suslin: 1976a]. Historians on Serre's Problem might note, however, that in the preliminary versions of these papers, (BQ_2) was proved only under certain mild assumptions. Quillen assumed that each localization of the regular ring at a height 2 prime contains a coefficient field, so as to apply Horrocks' theorem on 2-dimensional regular local rings. Suslin, on the other hand, assumed that the regular ring R contains $1/2$ in order to invoke a theorem of [Karoubi: 1973]. In the published versions of their papers, the role of Murthy's Theorem [Murthy: 1966] was pointed out.

Quillen's question (QQ_d) was raised in [Quillen: 1976] as a plausible way to prove the Bass-Quillen Conjecture (BQ'_d). The "conjectures" (S_i) and (S'_i) were also raised only as "questions" in [Swan: 1978], and I plead guilty to adding (H) and (H') to the list. Perhaps I should not have ventured them as conjectures (even for the convenience of exposition), since there does not seem to exist much evidence for their truth. As we have noted in the text, Suslin's Problem $\mathrm{Su}(R)_n$ is a more cautious version of the "conjectures" (H') and (S'_3). It is thought to be more likely to be true, since various special cases of it have been proved in the post-1977 literature. (For more detailed information on this, see VIII.5.)

The power series ring case of the Bass-Quillen Conjecture was answered independently by Lindel-Lütkebohmert in June, 1976, and by Mohan Kumar in December, 1976. Both parties observed that the result also holds for convergent power series rings over absolute valued fields. Our exposition is based on [Lindel-Lütkebohmert: 1977], [Mohan Kumar: 1979], and on suggestions of H. Bass. Of course, these results are now special cases of the general theorem (mentioned in Remark (5.23)(2)) that (BQ'_d) is true in the equi-characteristic case. Nevertheless, the techniques developed for affirming (BQ'_d) for formal power series rings have proved to be useful in various other contexts, so we have chosen to preserve our original exposition in §5 for this version of the book.

The Swan-Weibel Homotopy Trick (3.9) does not seem to be easily available in the expository literature, so we have included it in an Appendix to §3, with a detailed proof accompanied by motivating remarks. Concrete applications such as (3.14) demonstrate the great utility of this general homotopy principle. The result (3.15), proved essentially by the same means, reveals some interesting features of the property (E_1) — and its nonstable K_1-analogue.

In writing up this chapter, I have used, besides the original papers referred to above, notes of lectures given by [Davis: 1976], [Shapiro: 1976], and also the following expository notes: [Eisenbud: 1977], [Dicks: 1976], [Ferrand: 1976], [Geramita: 1974/76], [Knus: 1976], and [Lindel: 1977].

For more information on the work done on Serre's Conjecture in the period 1972–75 (before the Quillen-Suslin solution), especially the important method of symplectic modules, see [Suslin: 1974], [Vaserstein-Suslin: 1974], and the surveys in [Bass: 1974], [Swan: 1975]. For further generalizations of Serre's Problem, see [Raghunathan: 1978], [Raghunathan-Ramanathan: 1984] (principal bundles over affine spaces), [Murthy-Swan: 1976] (vector bundles on affine surfaces), [Bass-Connell-Wright: 1976] (locally polynomial algebras), [Suslin: 1977a] (the K_1-analogue, to be discussed in Chapter VI), and [Parimala: 1976b, 1978], [Knus-Ojanguren: 1977], (the quadratic analogue, to be discussed in Chapter VII). [Bass: 1977]

Chapter VI

K_1–Analogue of Serre's Conjecture

§1. Patching Theorems for GL_n

The goal of this chapter is to report on some of Suslin's results obtained in 1977 on the structure of the general linear groups over polynomial rings with coefficients from a field. These results appeared just as the 1978 edition of "Serre's Conjecture" was going to press, so they could not be included in my exposition then. The issue of this new version of the book finally gave me a chance to atone for this involuntary omission.

Suslin's results may be thought of as the K_1–analogues of the Quillen-Suslin results on f.g. projective modules over polynomial rings $R\,[t_1, \ldots, t_m]$ over a field R. By a general theorem of Bass, Heller and Swan (see, e.g. [Bass: 1968, XII.7]),

$$K_1\big(R\,[t_1, \ldots, t_m]\big) \cong \mathrm{U}\big(R\,[t_1, \ldots, t_m]\big) = \mathrm{U}(R); \tag{1.1}$$

in other words, $\mathrm{SL}\big(R\,[t_1, \ldots, t_m]\big) = \mathrm{E}\big(R\,[t_1, \ldots, t_m]\big)$, where "SL" and "E" denote, respectively, the stablized special linear group and the stabilized group of elementary matrices over the polynomial ring (see I.7). Since the Quillen-Suslin theorem gives the strongest possible *nonstable* version of $K_0\big(R\,[t_1, \ldots, t_m]\big) \cong \mathbb{Z}$, it would be natural to seek also the strongest possible *nonstable* version of (1.1); that is, to determine the smallest integer n_0 such that

$$\mathrm{SL}_n\big(R\,[t_1, \ldots, t_m]\big) = \mathrm{E}_n\big(R\,[t_1, \ldots, t_m]\big) \tag{1.2}$$

holds for all m, all $n \geqslant n_0$, and for all fields R. This, indeed, was Suslin's motivation, and in his spectacular paper [Suslin: 1977a], he showed that $n_0 = 3$. Recall that an example of Cohn discussed earlier in (I.8) showed already that n_0 cannot be 2.

The main focus of the present chapter will be on the proof of (1.2), for all $n \geqslant 3$ and for all fields R. Suslin's results are, however, much more general, and apply to the case where R is any commutative noetherian ring of finite Krull dimension. For a statement of this general stability Theorem, see (III.3.8).

While this chapter is primarily based on Suslin's paper (loc. cit.), we have also added material to it by using two later sources. The first one is the exposition on Suslin's results in [Gupta-Murthy: 1980], from which the material on Mennicke symbols in §3 is taken. The second source is [Rao: 1985], which offers a local-global principle on the action of the elementary group on unimodular polynomial vectors. This is a natural extension of [Suslin: 1977a], so we have included it in §2 below, where we'll also prove Rao's theorem (2.8) on an elementary transformation of "special" unimodular polynomial rows to their top coefficient rows.

In this beginning section, we shall first prove a Patching Theorem for the general linear group of a polynomial ring $R[t]$ (over any commutative ring R), which is directly inspired by Quillen's Patching Theorem for f.g. projective modules over $R[t]$. This patching theorem for GL_n will provide one of the key tools for proving Suslin's Stability Theorem (for fields R). Some of the other tools are the normality results for E_n $(n \geqslant 3)$ over any commutative ring, which were already developed in (I.9), and which, as we have pointed out there, also originated from the same 1977 paper of Suslin. A few of the results in Chapter III will prove to be useful as well. In that chapter, we have already dealt with some of the transitivity and local-global issues concerning unimodular rows over polynomial rings. Here, the main focus will be shifted to similar issues concerning the general linear groups instead.

To state the patching theorem for GL_n, we first recall the definitions of the relative groups $\mathrm{GL}_n(S, J)$ and $\mathrm{E}_n(S, J)$ (introduced in (I.9)). For any ideal J in a ring S, we have

$$\mathrm{GL}_n(S, J) := \ker\big(\mathrm{GL}_n(S) \to \mathrm{GL}_n(S/J)\big), \tag{1.3}$$

and $\mathrm{E}_n(S, J)$ is the normal hull in $\mathrm{E}_n(S)$ of the set of elementary matrices

$$\{I + ae_{ij} : \quad 1 \leqslant i \neq j \leqslant n, \quad a \in J\}. \tag{1.4}$$

We have always $\mathrm{E}_n(S, J) \subseteq \mathrm{GL}_n(S, J) \cap \mathrm{E}_n(S)$, but equality may not hold in general.

To facilitate some of our matrix calculations, it will be useful to have a notation for the matrices in (1.4). Throughout this chapter, let us, therefore, abbreviate the elementary matrix $I + ae_{ij}$ by $e_{ij}(a)$. Whenever this notation is used, it will always be assumed that $i \neq j$.

We begin with the following observation on the relative linear groups.

Lemma 1.5. *Let J be an ideal in a ring S such that a subring R of S maps isomorphically (under projection) onto S/J. Then, for any n, the group*

$$G := \mathrm{GL}_n(S, J) \cap \mathrm{E}_n(S)$$

is generated by matrices of the form $\gamma \cdot e_{ij}(a) \cdot \gamma^{-1}$, where $\gamma \in \mathrm{E}_n(R)$, $a \in J$, and $1 \leqslant i \neq j \leqslant n$. In particular, we have $G = \mathrm{E}_n(S, J)$.

Proof. An element $\sigma \in G$ has the form $\prod_{k=1}^{m} e_{i_k j_k}(s_k)$, where $s_k = r_k + a_k$ for some $r_k \in R$ and $a_k \in J$. Let

$$\gamma_l := \prod_{k=1}^{l} e_{i_k j_k}(r_k) \in \mathrm{E}_n(R) \quad (1 \leqslant l \leqslant m).$$

In view of the isomorphism $R \to S/J$ (given by the projection map), the fact that $\sigma \in \mathrm{GL}_n(S, J)$ amounts to $\gamma_m = I$. Using this, and replacing each $e_{i_k j_k}(s_k)$ by $e_{i_k j_k}(r_k) e_{i_k j_k}(a_k)$, we can rewrite σ as a "telescopic" product

$$\sigma = \prod_{k=1}^{m} [\gamma_k \, e_{i_k j_k}(a_k) \, \gamma_k^{-1}],$$

as desired. □

Remark 1.6. The lemma above will be needed primarily in the case where S is a polynomial ring $R[t]$ and J is the ideal $(t) = tR$. In this case, (1.5) applies, and yields the following useful equations:

$$\begin{aligned} \mathrm{E}_n\big(R[t], (t)\big) &= \mathrm{GL}_n\big(R[t], (t)\big) \cap \mathrm{E}_n\big(R[t]\big) \\ &= \langle\, \gamma \cdot e_{ij}(tf) \cdot \gamma^{-1} \mid \gamma \in \mathrm{E}_n(R),\ f \in R[t]\,\rangle, \end{aligned}$$

where $\langle C \rangle$ *means the group generated by* C.

Our next lemma, somewhat akin to (III.2.2), deals with the relation between $\mathrm{E}_n\big(R[t], (t)\big)$ and $\mathrm{E}_n\big(R_a[t], (t)\big)$, where, as usual, R_a denotes the localization of R at the multiplicative set $\{a^i : i \geqslant 0\}$. In order to work freely with such localizations, we shall now assume, through the rest of this chapter, that R *denotes a commutative ring*.

Lemma 1.7. *Let* $n \geqslant 3$, $a \in R$, *and* $\sigma \in \mathrm{E}_n\big(R_a[t], (t)\big)$. *Then there exists a matrix* $\tau \in \mathrm{E}_n\big(R[t], (t)\big)$ *such that* $\tau_a = \sigma(a^p t)$ *for some integer* p (*where* τ_a *means the image of* τ *under the localization*).

Proof. Thanks to (1.6), it is sufficient to handle the case where

$$\sigma = \gamma e_{ij}(tf) \gamma^{-1} \quad (\gamma \in \mathrm{E_n}(R_a),\ f \in R_a[t]).$$

Let α be the i th column of γ, and β be the j th row of γ^{-1}. Then, as in the proof of (I.9.14),

$$\beta\alpha = 0, \text{ and } \sigma = \gamma e_{ij}(tf)\gamma^{-1} = I + (tf)\,\alpha\beta.$$

Since β is unimodular and $\beta\alpha = 0$, we can use (I.9.11) to decompose α into $\sum_i \alpha_i$ where, for each i, $\beta\alpha_i = 0$ and α_i has at most two nonzero coordinates. Picking a large enough integer s, we can write

$$\beta = \beta'/a^s, \quad \alpha_i = \alpha_i'/a^s, \quad \text{and} \quad f = f'/a^s,$$

where $f' \in R[t]$, β' and α_i' are over R, and for each i, α_i' has at most two nonzero coordinates. Increasing s further if necessary, we may also assume that $\beta'\alpha_i' = 0 \in R$ for all i. Thus,

$$\sigma(a^{3s}t) = I + a^{3s}t \cdot f'(a^{3s}t) \left(\sum \alpha_i'\beta'\right) / a^{3s},$$

so we have $\sigma(a^{3s}t) = \tau_a$ upon setting

$$\begin{aligned}\tau &= I + tf'(a^{3s}t)\left(\sum \alpha_i'\beta'\right) \\ &= \prod \left(I + tf'(a^{3s}t)\,\alpha_i'\beta'\right) \in \mathbb{M}_n\big(R[t]\big).\end{aligned}$$

Since $n \geqslant 3$, each column vector $tf'(a^{3s}t)\alpha_i'$ has a zero coordinate, so (I.9.9)(2) implies that $\tau \in \mathrm{E}_n\big(R[t]\big)$. Combining this with $\tau(0) = I$, we see from (1.6) that $\tau \in \mathrm{E}_n\big(R[t], (t)\big)$, as desired. □

Corollary 1.8. *Let $n \geqslant 3$, $a \in R$, and $\sigma \in \mathrm{GL}_n\big(R[t], (t)\big)$ be such that $\sigma_a \in \mathrm{E}_n\big(R_a[t], (t)\big)$. Then there exists an integer k such that $\sigma(a^k t) \in \mathrm{E}_n\big(R[t], (t)\big)$.*

Proof. By (1.7), there exists $\tau \in \mathrm{E}_n\big(R[t], (t)\big)$ such that $\tau_a = \sigma_a(a^p t)$ for a suitable integer p. After replacing σ by $\sigma(a^p t)$ (which is harmless for the conclusion of the Corollary), we may assume that $\tau_a = \sigma_a$. Thus, upon writing

$$\sigma = I + t\sigma_1 + t^2\sigma_2 + \cdots \quad \text{and} \quad \tau = I + t\tau_1 + t^2\tau_2 + \cdots$$

where $\sigma_i,\ \tau_i \in \mathbb{M}_n(R)$, there is an integer k such that $a^k\sigma_i = a^k\tau_i$ for all i. (Note that only finitely many matrices are involved here.) It follows that

$$\sigma(a^k t) = I + \sum (a^k t)^i \sigma_i = I + \sum (a^k t)^i \tau_i = \tau(a^k t),$$

which lies in $\mathrm{E}_n\big(R[t], (t)\big)$, since τ does. □

The next proposition we shall prove is directly inspired by Quillen's results in (V.1.1) and (V.1.2).

Proposition 1.9. *Let $n \geqslant 3$, $a \in R$, and $\sigma \in \mathrm{GL}_n\big(R[t]\big)$ be such that $\sigma_a \in \mathrm{E}_n\big(R_a[t]\big)$. Then there exists an integer k such that, for two further indeterminates x, y (commuting with each other and with t), we have*

$$\sigma\big((x + a^k y)\,t\big)\,\sigma(xt)^{-1} \in \mathrm{E}_n\big(R[t, x, y],\ (y)\big). \tag{1.10}$$

In particular, if c, $d \in R$ are such that $c \equiv d$ (mod $a^k R$), then we have

$$\sigma(ct)\sigma(dt)^{-1} \in \mathrm{E}_n\big(R[t],\ (t)\big).$$

Proof. Consider the matrix

$$\sigma'(t, x, y) := \sigma\big((x + y)\,t\big)\,\sigma(xt)^{-1} \in \mathrm{GL}_n\big(R\,[t, x, y],\ (y)\big).$$

Since $\sigma_a \in \mathrm{E}_n\big(R_a[t]\big)$, we have $(\sigma')_a \in \mathrm{E}_n\big(R_a[t, x, y]\big)$, and so by (1.6) (applied to the ground ring $R_a\,[t, x]$),

$$(\sigma')_a \in \mathrm{E}_n\big(R_a[t, x, y],\ (y)\big).$$

By (1.8), there exists an integer k such that

$$\sigma'(t, x, a^k y) \in \mathrm{E}_n\big(R[t, x, y],\ (y)\big);$$

that is, (1.10) holds.

To prove the last statement in the Proposition, let $c = d + a^k e$, where k is as above. Specializing $x \mapsto d$ and $y \mapsto e$, we get from (1.10):

$$\sigma(ct)\,\sigma(dt)^{-1} \in \mathrm{E}_n\big(R[t]\big).$$

Since this matrix becomes I when $t = 0$, it lies in $\mathrm{GL}_n\big(R[t], (t)\big)$, and therefore in $\mathrm{E}_n\big(R[t],\ (t)\big)$ (again by (1.6)), as desired. □

Corollary 1.11. *Let $\sigma \in \mathrm{GL}_n\big(R[t],\ (t)\big)$ where $n \geqslant 3$, and let $a,\ b \in R$ be such that $aR + bR = R$. If $\sigma_a \in \mathrm{E}_n\big(R_a[t]\big)$ and $\sigma_b \in \mathrm{E}_n\big(R_b[t]\big)$, then $\sigma \in \mathrm{E}_n\big(R[t]\big)$.*

Proof. By (1.10), if we pick a sufficiently large integer k, then

$$\sigma(ct)\,\sigma(dt)^{-1} \in \mathrm{E}_n\big(R[t]\big)$$

whenever $c \equiv d \pmod{a^k R}$ or $c \equiv d \pmod{b^k R}$. Now $aR + bR = R$ implies that $a^k r + b^k s = 1$ for suitable elements $r,\ s \in R$. Writing

$$\sigma(t) = \sigma(t)\,\sigma(a^k rt)^{-1}\sigma(a^k rt)\,\sigma(0)^{-1},$$

we see that $\sigma(t) \in \mathrm{E}_n\big(R[t]\big)$ since $1 \equiv a^k r \pmod{b^k R}$ and $a^k r \equiv 0 \pmod{a^k R}$. (Note that $\sigma(0) = I$ here since $\sigma \in \mathrm{GL}_n\big(R[t], (t)\big)$.) □

We can now prove the following analogue of Quillen's Patching Theorem (V.1.6).

$\mathbf{GL_n}$-Patching Theorem 1.12. *Let $n \geqslant 3$ and $\sigma \in \mathrm{GL}_n\big(R[t], (t)\big)$. Then*

($\mathrm{A_n}$) *The Quillen set $\mathrm{Q}(\sigma) := \{a \in R :\ \sigma_a \in \mathrm{E}_n\big(R_a[t]\big)\}$ is an ideal in R.*
($\mathrm{B_n}$) *If $\sigma_{\mathfrak{m}} \in \mathrm{E}_n\big(R_{\mathfrak{m}}[t]\big)$ for every maximal ideal $\mathfrak{m}$ of R, then $\sigma \in \mathrm{E}_n\big(R[t]\big)$.*

Proof. ($\mathrm{A_n}$) Let $\mathfrak{A} = \mathrm{Q}(\sigma)$. Clearly, $a \in \mathfrak{A} \Rightarrow ra \in \mathfrak{A}$ for every $r \in R$. If $a, b \in \mathfrak{A}$, replace R by R_{a+b}, in which the pair $a,\ b$ is unimodular. By (1.11) (applied to the base ring R_{a+b}), we have $\sigma_{a+b} \in \mathrm{E}_n\big(R_{a+b}[t]\big)$. Therefore, we have $a + b \in \mathfrak{A}$, which completes the proof that $\mathfrak{A}$ is an ideal.

($\mathrm{B_n}$) Under the assumption in ($\mathrm{B_n}$), we would like to prove that $1 \in \mathfrak{A}$. If otherwise, there would exist a maximal ideal $\mathfrak{m} \supseteq \mathfrak{A}$. Since $\sigma_{\mathfrak{m}} \in \mathrm{E}_n\big(R_{\mathfrak{m}[t]}\big)$, there exists an element $a \in R\backslash\mathfrak{m}$ such that $\sigma_a \in \mathrm{E}_n\big(R_a\,[t]\big)$. But then $a \in \mathfrak{A}$, which contradicts $\mathfrak{A} \subseteq \mathfrak{m}$. □

Corollary 1.13. *Let $\sigma \in \mathrm{GL}_n\big(R[t]\big)$, where $n \geqslant 3$. If $\sigma_{\mathfrak{m}} \in \mathrm{GL}_n\big(R_{\mathfrak{m}}\big) \cdot \mathrm{E}_n\big(R_{\mathfrak{m}}[t]\big)$ for every maximal ideal $\mathfrak{m}$ of R, then $\sigma \in \mathrm{GL}_n(R) \cdot \mathrm{E}_n\big(R[t]\big)$.*

Proof. We are free to multiply σ from the left by any matrix in $\mathrm{GL}_n(R)$. Thus, after left-multiplying σ by $\sigma(0)^{-1}$, we may assume that $\sigma \in \mathrm{GL}_n\big(R[t], (t)\big)$. For any maximal ideal $\mathfrak{m} \subset R$, we have a factorization $\sigma_{\mathfrak{m}} = \alpha\beta$ where $\alpha \in \mathrm{GL}_n\big(R_{\mathfrak{m}}\big)$ and $\beta \in \mathrm{E}_n\big(R_{\mathfrak{m}}[t]\big)$. Evaluating this matrix at 0, we get $I = \alpha \cdot \beta(0)$. Thus,

$$\sigma_{\mathfrak{m}} = \beta(0)^{-1}\beta \in \mathrm{E}_n\big(R_{\mathfrak{m}}[t]\big).$$

It follows from (1.12) that $\sigma \in \mathrm{E}_n\big(R[t]\big)$. (Of course, the original σ has already been modified by a left factor from $\mathrm{GL}_n(R)$.) □

To conclude this section, we shall prove some more patching results for the groups GL_n and E_n that are of the same spirit as (V.1.3). These results hold for any commutative ring, but their proofs make use of the preceding results on the general linear groups of polynomial rings.

Proposition 1.14. *Let $a, b \in R$ be such that $aR + bR = R$. Then for any $\sigma \in \mathrm{E}_n(R_{ab})$ $(n \geqslant 3)$, there exist $\alpha \in \mathrm{E}_n(R_b)$ and $\beta \in \mathrm{E}_n(R_a)$ such that $\sigma = (\alpha)_a(\beta)_b$.*

Proof. To make the proof easier to follow, let us first record a commutator identity that will be used for the argument. We shall use the conjugation notation ${}^{x}y = xyx^{-1}$. Let $\sigma_i = \alpha_i\beta_i$ $(1 \leqslant i \leqslant m)$ in any group. We have

$$\sigma_1 \cdots \sigma_m = (\sigma_1 \cdots \sigma_{m-1})\,(\alpha_m\beta_m) = [{}^{\sigma_1\cdots\sigma_{m-1}}\alpha_m]\,(\sigma_1 \cdots \sigma_{m-1})\,\beta_m.$$

Repeating this process for $\sigma_1 \cdots \sigma_{m-1}$, etc., we obtain inductively the commutator identity

$$(1.15) \qquad \sigma_1 \cdots \sigma_m = [{}^{\gamma_m}(\alpha_m)]\,[{}^{\gamma_{m-1}}(\alpha_{m-1})] \cdots [{}^{\gamma_1}(\alpha_1)]\,\beta_1 \cdots \beta_m,$$

where $\gamma_k := \sigma_1 \cdots \sigma_{k-1}$ $(1 \leqslant k \leqslant m)$.

Consider any element $\sigma = \sigma_1 \cdots \sigma_m \in \mathrm{E}_n(R_{ab})$, where $\sigma_k = e_{i_k j_k}(c_k)(c_k \in R_{ab})$. The matrices $\gamma_k = \sigma_1 \cdots \sigma_{k-1}$ defined above are also in $\mathrm{E}_n(R_{ab})$, so we have

$$\gamma_k e_{i_k j_k}(c_k t)\gamma_k^{-1} \in \mathrm{E}_n\big(R_{ab}[t],\ (t)\big).$$

Viewing R_{ab} as $(R_b)_a$, we have by (1.7) a matrix $\tau_k = \tau_k(t) \in \mathrm{E}_n\big(R_b[t]\big)$ such that, for a suitable integer p,

$$(1.16) \qquad (\tau_k)_a = \gamma_k e_{i_k j_k}(c_k a^p t)\gamma_k^{-1} \quad (1 \leqslant k \leqslant m).$$

On the other hand, for another suitably chosen integer q, $c_k b^q$ "comes from" R_a for all k. Since $aR + bR = R$, we have $a^p r + b^q s = 1$ for some $r, s \in R$. Then

$$\sigma_k = e_{i_k j_k}(c_k a^p r + c_k b^q s) = \alpha_k\beta_k$$

where $\alpha_k = e_{i_k j_k}(c_k a^p r)$ and $\beta_k = e_{i_k j_k}(c_k b^q s)$. Now by (1.15),

$$\sigma = \prod_{k=m}^{1} (\gamma_k\alpha_k\gamma_k^{-1}) \cdot (\beta_1\,\beta_2 \cdots \beta_m).$$

Here, each $\beta_k \in \mathrm{E}_n(R_a)_b$ and, by specializing $t \mapsto r$ in (1.16), we see that each $\gamma_k\alpha_k\gamma_k^{-1} \in \mathrm{E}_n(R_b)_a$. □

To state the K-theoretic consequence of (1.14), it will be convenient to formally introduce the notion of unstable K_1-groups (which is already implicit in (III.3.8)). For a commutative ring R and for any integer $n \geqslant 3$, Suslin's Normality Theorem (I.9.14) guarantees that $\mathrm{E}_n(R)$ is normal in $\mathrm{GL}_n(R)$, so we can form the factor group

$$K_{1,n}(R) := \mathrm{GL}_n(R)/\mathrm{E}_n(R), \tag{1.17}$$

which may be called the n^{th} *unstable* K_1*-group* of R. We have a natural directed system of groups

$$K_{1,3}(R) \to \cdots \to K_{1,n}(R) \to K_{1,n+1}(R) \to \cdots,$$

whose direct limit is the abelian group $K_1(R)$. For each $n \geqslant 3$, $K_{1,n}$ is a covariant functor from the category of commutative rings to the category of groups.

With the above notations, we can now deduce the following nice consequence of (1.14).

Theorem 1.18. *Let a, b be non 0-divisors of R such that $aR + bR = R$, and that $R_a \cap R_b = R$ in the total ring of quotients of R. For $n \geqslant 3$ and $\alpha \in \mathrm{GL_n}(R_b)$, $\beta \in \mathrm{GL}_n(R_a)$, let $\bar{\alpha}$ and $\bar{\beta}$ denote the classes they represent in $K_{1,n}(R_b)$ and $K_{1,n}(R_a)$. Then, in the commutative diagram*

$$\begin{tikzcd} K_{1,n}(R) \arrow[r, "i"] \arrow[d, "j"'] & K_{1,n}(R_a) \arrow[d, "j'"] \\ K_{1,n}(R_b) \arrow[r, "i'"'] & K_{1,n}(R_{ab}) \end{tikzcd} \tag{1.19}$$

$i'(\bar{\alpha}) = j'(\bar{\beta})$ iff there exists $\gamma \in \mathrm{GL}_n(R)$ such that $\bar{\alpha} = j(\bar{\gamma})$ and $\bar{\beta} = i(\bar{\gamma})$.

Proof. The "if" part is a direct consequence of the commutativity of the diagram. For the "only if" part, assume that $i'(\bar{\alpha}) = j'(\bar{\beta})$. Then the matrix $\sigma = \alpha\beta^{-1} \in \mathrm{GL}_n(R_{ab})$ represents the 0-class in $K_{1,n}(R_{ab})$, so we have $\sigma \in \mathrm{E}_n(R_{ab})$. By (1.14), we have a decomposition $\sigma = (\alpha_0^{-1})_a(\beta_0)_b$, for suitable $\alpha_0 \in \mathrm{E}_n(R_b)$ and $\beta_0 \in \mathrm{E}_n(R_a)$. Let $\gamma := \alpha_0\alpha = \beta_0\beta$. Since $R_a \cap R_b = R$, we have

$$\gamma \in \mathrm{GL}_n(R_a) \cap \mathrm{GL}_n(R_b) = \mathrm{GL}_n(R).$$

Thus, $\bar{\gamma} \in K_{1,n}(R)$, with $j(\bar{\gamma}) = \overline{\alpha_0\alpha} = \bar{\alpha}$ and $i(\bar{\gamma}) = \overline{\beta_0\beta} = \bar{\beta}$, as desired. □

Remark 1.20. The result (1.18) comes close to saying that (1.19) is a pullback diagram. The only thing lacking is the *uniqueness* of the element $\bar{\gamma} \in K_{1,n}(R)$ mapping to the given $\bar{\alpha}$ and $\bar{\beta}$. If, in a given situation, we know that the map i or the map j is *injective*, then the uniqueness of $\bar{\gamma}$ is at hand, and (1.19) will be a pullback diagram indeed. In this connection, it is perhaps relevant to point out that, if we go to the stable K-groups, there is in general a Mayer-Vietoris long exact sequence

$$\cdots \to K_2(R_{ab}) \to K_1(R) \to K_1(R_a) \oplus K_1(R_b) \to K_1(R_{ab}) \to K_0(R) \to \cdots$$

due to Thomason (and Trobaugh), which holds as long as $aR + bR = R$ even without the assumption that a, b are non 0-divisors; see their paper in the Grothendieck Festschrift III, pp. 247–435, Birkhäuser, 1990.

§2. Patching Theorem for Elementary Group Action

In this section, we return to the study of linear group actions on the space of unimodular rows over a polynomial ring $R[t]$. For $n \geqslant 3$, we shall prove a Patching Theorem (2.3) for the action of the *elementary group* $\mathrm{E}_n(R[t])$ on such rows. This result was somehow *not* covered in Suslin's 1977 paper, and was given only some years later in [Rao: 1985]. We'll present Rao's result in this section, with the caveat that the proof to be given here covers only the case where R is an integral domain. A nice pay-off of this new Patching Theorem is another result of Rao on the elementary transformation of a unimodular polynomial vector to its "top coefficient row" (when this latter row happens to be unimodular). This application will be presented in (2.8) below.

One clarification is in order here. The principal goal of this chapter is to study the structure of $\mathrm{SL}_n\big(k[t_1, \ldots, t_m]\big)$ (over a field k). As far as this group is concerned, the results in this section are not needed. In fact, the later sections §§3–6 of this chapter will be completely independent of this section. Thus, readers interested in following the shortest route to the structure of $\mathrm{SL}_n\big(k[t_1, \ldots, t_m]\big)$ may also proceed directly to §§3–6 at this point, after reading the preliminary §1 above.

Throughout this section, R denotes a commutative ring. For an element $a \in R$, we write R_a as usual for the localization of R at the multiplicative set $\{a^i : i \geqslant 0\}$. In general, a subscript "a" shall mean taking the image of something upon applying the localization map at a. This notational convention will be used freely without further mention in the following.

Before we come to the E_n-Patching Theorem, we set the stage by stating two preparatory results. The first one is just a slight variant of Proposition 1.9. Note that, in (1.9), we started with a matrix α defined over $R[t]$. In the following alternative formulation, we start instead with an α defined only over $R_a[t]$. The proof of (1.9) essentially carries over without change.

Proposition 2.1. *Let* $n \geqslant 3$, $a \in R$, *and* $\alpha \in \mathrm{E}_n\big(R[t]_a\big)$. *Then there exists an integer* k *such that, for any* c, $d \in R$ *with* $c \equiv d$ (mod $a^k R$), *we have*

$$\alpha(ct)\alpha(dt)^{-1} \in \mathrm{E}_n\big(R[t],\ (t)\big)_a.$$

Proof. As in the proof of (1.9), for two new indeterminates x, y, we consider the matrix

$$\alpha'(t, x, y) := \alpha((x+y)t)\,\alpha(xt)^{-1} \in \mathrm{E}_n\big(R_a[t, x, y],\ (y)\big).$$

By (1.7), there exists an integer k such that

$$\alpha'(t, x, a^k y) = \alpha((x + a^k y)t)\,\alpha(xt)^{-1} \in \mathrm{E}_n\big(R[t, x, y],\ (y)\big)_a.$$

If $c = d + a^k e$ $(e \in R)$, specializing $x,\ y$ to $d,\ e \in R$ gives the desired conclusion, just as in the proof of (1.9). □

Next, we prove a useful fact on the relative general linear group $\mathrm{GL}_n(A, J)$ with respect to the localizations of a ring A at a pair of comaximal elements.

Proposition 2.2. *Let $s_1,\ s_2$ be two comaximal elements in a commutative ring A, and let J be an ideal of A. Let $\sigma_i \in \mathrm{GL}_n\big(A_{s_i},\ J_{s_i}\big)$ be such that*

$$(\sigma_1)_{s_2} = (\sigma_2)_{s_1} \in \mathrm{GL}_n\big(A_{s_1 s_2},\ J_{s_1 s_2}\big).$$

Then there exists a unique $\sigma \in \mathrm{GL}_n(A,\ J)$ such that $\sigma_{s_i} = \sigma_i$ for $i = 1, 2$.

Proof. In I.3.10, we have verified the pullback property for a commutative square with its four corners given by the rings $A,\ A_{s_1},\ A_{s_2}$ and $A_{s_1 s_2}$. The present proposition is simply a variant of this fact for the relative general linear groups. First of all, working with the individual entries of the matrices σ_i and σ_i^{-1} $(i = 1, 2)$, we get from I.3.10 unique matrices, say $\sigma,\ \tau \in \mathbb{M}_n(A)$, localizing to σ_i and σ_i^{-1}. Then $\sigma\,\tau - I_n$ localizes to the zero matrix at both s_1 and s_2, so $\sigma\,\tau = I_n$. This shows that $\sigma \in \mathrm{GL}_n(A)$. To check that $\sigma \in \mathrm{GL}_n(A,\ J)$, we pass from A to its factor ring A/J. By I.3.10, applied to the four rings

$$A/J,\ \ (A/J)_{s_1},\ \ (A/J)_{s_2},\ \text{ and } (A/J)_{s_1 s_2},$$

we see that $\overline{\sigma}$ localizes to I_n at both s_1 and s_2 (since $\sigma_i \in \mathrm{GL}_n(A_{s_i},\ J_{s_i})$). Therefore, $\overline{\sigma} = I_n$, which shows that $\sigma \in \mathrm{GL}_n(A,\ J)$, as desired. □

For $f,\ g \in \mathrm{Um}_n(A)$ over a ring A, recall that $f \sim_G g$ means that we can transform f to g by an element of the group $G \subseteq \mathrm{GL}_n(A)$. Using this notation for $A = R[t]$, we shall now prove the following local-global principle for unimodular rows over A *under the action of the elementary group* $\mathrm{E}_n(A)$ for $n \geqslant 3$. Note that the corresponding local-global principle for the general linear group $\mathrm{GL}_n(A)$ has been proved earlier in III.2.4 and III.2.5 for all n.

$\mathbf{E_n}$-Patching Theorem for Unimodular Rows 2.3. *Let $f(t) \in \mathrm{Um}_n\big(R[t]\big)$, where $n \geqslant 3$. Then*

(1) *the Quillen set $\mathrm{Q}(f) := \{\, s \in R \mid f(t) \sim_{\mathrm{E}_n(R_s[t])} f(0)\,\}$ is an ideal in R;*

(2) *if $f(t) \sim_{\mathrm{E}_n(R_{\mathfrak{m}}[t])} f(0)$ for all $\mathfrak{m} \in \mathrm{Max}\,R$, then $f(t) \sim_{\mathrm{E}_n(R[t])} f(0)$.*

Proof. As we have stated at the beginning of this section, we'll only prove this result under the assumption that R is an integral domain. This additional assumption makes the argument a little easier, at the very last step of the proof.

As usual, (1) $\Longrightarrow$ (2) since, once we know $\mathrm{Q}(f)$ is an ideal, the hypothesis in (2) will then imply that $\mathrm{Q}(f) = R$, which means exactly that $f(t) \sim_{\mathrm{E}_n(R[t])} f(0)$.

To check (1), we only need to show that

$$s_1,\ s_2 \in \mathrm{Q}(f) \Longrightarrow s_1 + s_2 \in \mathrm{Q}(f).$$

After inverting $s_1 + s_2$, we may assume that $s_1 + s_2 = 1$. Let $\sigma_i(t) \in \mathrm{E}_n\big(R_{s_i}[t]\big)$ be such that $f(t)\sigma_i(t) = f(0)$ for $i = 1, 2$. After replacing $\sigma_i(t)$ by $\sigma_i(t)\sigma_i(0)^{-1}$, we may assume that $\sigma_i(0) = I_n$, so that $\sigma_i(t) \in \mathrm{E}_n\big(R_{s_i}[t],\ (t)\big)$ (see (1.6)). Let

$$\alpha(t) := \sigma_2(t)^{-1}\sigma_1(t) \in \mathrm{E}_n\big(R_{s_1 s_2}[t]\big), \quad \text{with } \alpha(0) = I_n.$$

Applying (2.1) to the localizations $R_{s_i}[t] \to R_{s_1 s_2}[t]$ with the above α, we get a single integer k that works for both localizations. Fix an equation $s_1^k p + s_2^k q = 1$ $(p, q \in R)$ and let $c = s_2^k q$, with $1 - c = s_1^k p$. Then the conclusion of (2.1) guarantees that

$$\alpha(ct)\,\alpha(0\,t)^{-1} = \alpha(ct) \in \mathrm{E}_n\big(R_{s_1}[t],\ (t)\big)_{s_2}, \quad \alpha(t)\,\alpha(ct)^{-1} \in \mathrm{E}_n\big(R_{s_2}[t],\ (t)\big)_{s_1}.$$

Let $\beta_i \in \mathrm{E}_n(R_{s_i}[t],\ (t))$ be such that β_1 localizes to $\alpha(ct)$, and β_2 localizes to $\alpha(t)\,\alpha(ct)^{-1}$. Then

$$\beta_2\beta_1 = \alpha = \sigma_2^{-1}\sigma_1 \in \mathrm{E}_n\big(R_{s_1 s_2}[t]\big)$$

implies that $\sigma_1\beta_1^{-1} \in \mathrm{E}_n\big(R_{s_1}[t],\ (t)\big)$ and $\sigma_2\beta_2 \in \mathrm{E}_n\big(R_{s_2}[t],\ (t)\big)$ localize to the same element in $\mathrm{E}_n\big(R_{s_1 s_2}[t]\big)$. Thus, by (2.2), there exists $\sigma \in \mathrm{GL}_n\big(R[t],\ (t)\big)$ localizing (at s_1 and s_2, respectively) to these two matrices. Then, by (1.11) (which requires $n \geqslant 3$), $\sigma \in \mathrm{E}_n\big(R[t]\big)$. Calculating over $R_{s_1 s_2}[t]$, we have

$$\begin{aligned} f(t)\,\sigma(t) &= f(t)\,\sigma_1(t)\,\beta_1(t)^{-1} \\ &= f(0)\,\alpha(ct)^{-1} \\ &= f(0)\,\sigma_1(ct)^{-1}\,\sigma_2(ct) \\ &= f(ct)\,\sigma_2(ct) \\ &= f(0). \end{aligned}$$

Since R is an integral domain (so the localization $R[t] \to R_{s_1 s_2}[t]$ is an injection), the equation $f(t)\sigma(t) = f(0)$ must already hold over $R[t]$. This shows that the Quillen set $\mathrm{Q}(f)$ contains 1, as desired. $\square$

Remark 2.4. In the proof of Theorem 2.3 given in [Rao: 1985], after the matrices $\beta_i \in \mathrm{E}_n\big(R_{s_i}[t],\ (t)\big)$ were chosen (as above), it was assumed that they satisfied the equation $f(0)\,\beta_i(t) = f(0)$ over $R_{s_i}[t]$. These equations (for $i = 1, 2$) do hold over $R_{s_1 s_2}[t]$ (by a direct calculation similar to the one we gave), but it is not clear to me why they may be assumed to hold *over* $R_{s_i}[t]$. This is why we introduced the extra assumption that R is an integral domain in the above proof, which enabled us to consummate the argument without going through any additional steps.

Next, we present Rao's application of (2.3) to unimodular polynomial vectors with a unimodular leading coefficient row. To make this application more smooth, we first distill a part of the needed arguments into a general lemma.

Lemma 2.5. *Let* $T = \{t^i : i \geqslant 0\}$ *in a commutative ring* A*. Suppose a row vector* $h = (h_1, \ldots, h_n) \in A^n$ *is such that* (1) $h_T \in \mathrm{Um}_n(A_T)$*, and* (2) $\overline{h} \in \mathrm{Um}_n(A/tA)$*. Then* $h \in \mathrm{Um}_n(A)$.

Proof. If otherwise, there exists a maximal ideal M containing all h_i. Then $M_T = A_T$ by (1), so some power $t^i \in T \cap M$, which implies $t \in M$. But (2) yields an equation $h_1k_1 + \cdots + h_nk_n = 1 + ta$ (for suitable $a, k_1, \ldots, k_n \in A$). Transposition yields $1 \in M$, a contradiction. $\square$

Remark. Note that the general set-up of Lemma 2.5 is rather similar to that of Roberts' Theorem VI.4.1. However, the situation is simpler here, and the proof is also considerably easier.

Now let us recall a general construction in the theory of polynomials. For a given polynomial $g(t) = b_m t^m + \cdots + b_0 \in R[t]$ with leading coefficient $b_m \neq 0$, the *reciprocal polynomial* of $g(t)$ is defined to be

$$g^*(t) = t^m g(t^{-1}) = b_m + b_{m-1}t + \cdots + b_0 t^m \in R[t]. \tag{2.6}$$

Of course, if $g(t) = 0$, we take $g^*(t) = 0$ too. We caution the reader that, in general, $g^{**}(t) \neq g(t)$. For instance, if $g(t) = bt^2 + ct$ where $b,\ c \neq 0$, then $g^{**}(t) = bt + c \neq g(t)$.

For any polynomial vector $f = (f_1, \ldots, f_n)$ over $R[t]$, we define the "top coefficient row" $L(f)$ to be the row $(c_1, \ldots, c_n)$, where c_i is the leading coefficient of f_i. To facilitate our exposition, we introduce a tentative terminology: let us say that the polynomial vector f is *special* if $L(f) \in \mathrm{Um}_n(R)$; that is, if the leading coefficients of the f_i's generate the unit ideal in R. In general, $f(t) = (f_1, \ldots, f_n) \in \mathrm{Um}_n\big(R[t]\big)$ does not guarantee that f is special; nor does it guarantee that

$$f^*(t) = (f_1^*(t), \ldots, f_n^*(t)) \in \mathrm{Um}_n\big(R[t]\big).$$

However, these two conditions turn out to be equivalent, as the last part of the following lemma shows.

Lemma 2.7. *Let* $f = (f_1, \ldots, f_n) \in \mathrm{Um}_n\big(R\,[t]\big)$*. Then*

(A) f^* *is special.*

(B) $f^*(t)$ *is unimodular over the ring* $R\,[t, t^{-1}]$.

(C) f *is special iff* $f^* \in \mathrm{Um}_n\big(R\,[t]\big)$.

Proof. (A) Consider $f(0) := (a_1, \ldots, a_n) \in \mathrm{Um}_n(R)$. Say $a_1, \ldots, a_k$ are nonzero, and the other a_i's are zero. Then already $(a_1, \ldots, a_k) \in \mathrm{Um}_k(R)$. But $a_1, \ldots, a_k$ are among the coordinates of $L(f^*)$, which clearly implies that $L(f^*) \in \mathrm{Um}_n(R)$.

(B) If $\sum_i g_i(t) f_i(t) = 1$, where, say $\deg f_i = m_i$, then $\sum_i g_i(t^{-1}) f_i(t^{-1}) = 1$ leads to

$$1 = \sum_i [\, g_i(t)\, t^{-m_i}]\, t^{m_i} f_i(t^{-1}) = \sum_i [\, g_i(t)\, t^{-m_i}]\, f_i^*(t).$$

Since $g_i(t)\, t^{-m_i} \in R[t, t^{-1}]$ for all i, this proves (B).

(C) First assume $f^* \in \mathrm{Um}_n\big(R[t]\big)$. Then $f^*(0) \in \mathrm{Um}_n(R)$. Since $f^*(0) = L(f)$, this means f is special. Conversely, assume f is special; that is, $L(f) \in \mathrm{Um}_n(R)$. To show that $f^*(t)$ is unimodular over $A := R[t]$, it suffices to check the conditions (1), (2) in Lemma 2.5 for $h := f^*$. By (B), $f^*(t)$ is unimodular over $A_t = R[t, t^{-1}]$, so (1) holds. Next, over $A/tA \cong R$, $f^*(t)$ reduces to $L(f)$, which is unimodular over R, so (2) holds as well. □

We are now ready to prove the following result from [Rao: 1985] about *special* unimodular rows over a polynomial ring $R[t]$, which refines (III.2.1) in the case $n\ geqslant$ 3.

Theorem 2.8. (R. A. Rao) *Let* $f = (f_1, \ldots, f_n)$ *be a* special *unimodular row over* $A = R[t]$*, where* $n \geqslant 3$. *Then, for any* $a \in R$:

$$f(t) \sim_{\mathrm{E}_n(A)} f(a) \sim_{\mathrm{E}_n(R)} L(f). \tag{2.9}$$

Proof. *Step* 1. We claim that $f(t) \sim_{\mathrm{E}_n(A)} f(0)$. By (2.3), it suffices to check this over $R_{\mathfrak{m}}[t]$ for every $\mathfrak{m} \in \mathrm{Max}\, R$. Since $L(f) \in \mathrm{Um}_n(R)$, *some* f_i is unitary over $R_{\mathfrak{m}}[t]$. In this case, III.2.6 gives $f(t) \sim_{\mathrm{E}_n(R_{\mathfrak{m}}[t])} f(0)$, as desired.

Step 2. Given any $a \in R$, specializing $f(t) \sim_{\mathrm{E}_n(A)} f(0)$ by $t \mapsto a$ gives $f(a) \sim_{\mathrm{E}_n(R)} f(0)$. Thus, we get $f(t) \sim_{\mathrm{E}_n(A)} f(a)$.

Step 3. Now look at $f^*(t)$, which is special by (2.7)(A), and unimodular by (2.7)(C). Thus, by *Step* 1 above, $f^*(t) \sim_{\mathrm{E}_n(A)} f^*(0)$. Specializing this by $t \mapsto 1$, we get $f^*(1) \sim_{\mathrm{E}_n(R)} L(f)$; that is, $f(1) \sim_{\mathrm{E}_n(R)} L(f)$. In view of *Step* 2, this completes the proof. □

We note that the conclusion $f(t) \sim_{\mathrm{E}_n(R[t])} f(a)$ was also re-proved in [Sivatski: 2000]. An immediate consequence of (2.8) is the following nice statement.

Corollary 2.10. *Let* $f = (f_1, \ldots, f_n) \in \mathrm{Um}_n\big(R[t]\big)$*, where* $n \geqslant 3$. *If some* f_i *is unitary, then* f *is completable to a matrix in* $\mathrm{E}_n\big(R[t]\big)$.

Note that this result is a common generalization of III.2.6 and V.2.6. To wit, it generalizes III.2.6 from the case of a local ground ring to the case of any commutative ground ring, and it also shows that, in V.2.6, we may replace the general linear group by the smaller elementary group (in case $n \geqslant 3$).

§3. Mennicke Symbols

In this section, we record some general theorems on the representation of certain elements of $\mathrm{SL}_3(R)/\mathrm{E}_3(R)$ for *any commutative ring* R. Recall from (I.9.14) that for $n \geqslant 3$, $\mathrm{E}_n(R)$ is a *normal* subgroup of $\mathrm{GL}_n(R)$, so the factor group $\mathrm{SL}_n(R)/\mathrm{E}_n(R)$ makes sense for $n \geqslant 3$, and in particular for $n = 3$. Note that there is a natural group homomorphism

$$\mathrm{SL}_3(R)/\mathrm{E}_3(R) \to \mathrm{SL}(R)/\mathrm{E}(R) = SK_1(R),$$

so elements of $\mathrm{SL}_3(R)/\mathrm{E}_3(R)$ also give rise to elements of the group $SK_1(R)$ (defined in (I.7.6)). However, we shall be able to prove results for the group $\mathrm{SL}_3(R)/\mathrm{E}_3(R)$, without "stabilizing" it to $SK_1(R)$. *Throughout this section, R will always denote a commutative ring.*

Given any unimodular pair $(a, b) \in \mathrm{Um}_2(R)$, pick two elements $c, d \in R$ such that $ad - bc = 1$. Then

$$\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{SL}_3(R), \tag{3.1}$$

so it gives rise to an element in $\mathrm{SL}_3(R)/\mathrm{E}_3(R)$. The fundamental observation here is the following.

Lemma 3.2. *The class of the above matrix in* $\mathrm{SL}_3(R)/\mathrm{E}_3(R)$ *is independent of the choice of the elements* c, d. *Consequently, we may denote this class by the* "Mennicke symbol"

$$[a, b] \in \mathrm{SL}_3(R)/\mathrm{E}_3(R). \tag{3.3}$$

Proof. If $ad' - bc' = 1$ also, then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a & b \\ c' & d' \end{pmatrix}^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d' & -b \\ -c' & a \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ cd' - dc' & 1 \end{pmatrix} \in \mathrm{E}_2(R) \subseteq \mathrm{E}_3(R).$$

Therefore $\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} a & b & 0 \\ c' & d' & 0 \\ 0 & 0 & 1 \end{pmatrix}$ define the same element in the factor group $\mathrm{SL}_3(R)/\mathrm{E}_3(R)$.[(*)] □

Some of the striking properties of the Mennicke symbol $[a, b]$ are given in the following result.

[(*)]The argument here shows, indeed, that the unimodular row (a, b) determines uniquely an element $\mathrm{E}_2(R) \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in the right coset space $\mathrm{E}_2(R) \setminus \mathrm{SL}_2(R)$.

Proposition 3.4. *Let* $a, a', b, t \in R$.

(1) *If* $a \in \mathrm{U}(R)$*, then* $[a, b] = 1$.
(2) *If* (a, b), $(a', b) \in \mathrm{Um}_2(R)$*, then* $(aa', b) \in \mathrm{Um}_2(R)$*, and*

$$[aa', b] = [a, b]\,[a', b].$$

(3) *If* $(a, b) \in \mathrm{Um}_2(R)$*, then* $[b, a] = [a, b] = [a + bt, b]$.

Proof. (1) Since $a(a^{-1}) - b(0) = 1$, $[a, b]$ is the class of the matrix $M = \begin{pmatrix} a & b & 0 \\ 0 & a^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}$. But

$$\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix} = \begin{pmatrix} 1 & ba \\ 0 & 1 \end{pmatrix} \in \mathrm{E}_2(R),$$

and by Whitehead's Lemma, $\begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix} \in \mathrm{E}_2(R)$ also. Thus, $\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \in \mathrm{E}_2(R)$, and in particular, $M \in \mathrm{E}_3(R)$.

(2) Say $ad - bc = 1$, and $a'd' - bc' = 1$. By elementary transformations, we have

$$\begin{pmatrix} a' & b & 0 \\ c' & d' & 0 \\ 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} a' & 0 & -b \\ c' & 0 & -d' \\ 0 & 1 & 0 \end{pmatrix} \mapsto \begin{pmatrix} a' & 0 & -b \\ c' & 0 & -d' \\ 0 & 1 & a \end{pmatrix}$$

Left multiplying the last matrix by $\begin{pmatrix} a & 0 & b \\ 0 & 1 & 0 \\ c & 0 & d \end{pmatrix}$ (whose class is $[a, b]$), we see that $[a, b]\,[a', b]$ is represented by

$$\begin{pmatrix} aa' & b & 0 \\ c' & 0 & -d' \\ ca' & d & 1 \end{pmatrix} \mapsto \begin{pmatrix} aa' & b & 0 \\ * & * & 0 \\ ca' & d & 1 \end{pmatrix} \mapsto \begin{pmatrix} aa' & b & 0 \\ * & * & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This proves both conclusions of (2).

(3) If $ad - bc = 1$, then $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} -b & a \\ -d & c \end{pmatrix}$ by elementary transformations. Therefore, by (2) and (1):

$$[a, b] = [-b, a] = [b, a]\,[-1, a] = [b, a].$$

The last equation in (3) is clear from the elementary column transformation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a + bt & b \\ c + dt & d \end{pmatrix}. \qquad \square$$

With the calculus of the Mennicke symbols in place, we can now prove with relative ease the triviality of the SK_1-class of certain elements of $\mathrm{SL}_2(R)$, as the following example shows.

Example 3.5. For any elements x, y in a commutative ring R, we have seen (in (I.9.10)) that the Cohn matrix $C = \begin{pmatrix} 1+xy & x^2 \\ -y^2 & 1-xy \end{pmatrix}$ belongs to $\mathrm{E}_3(R)$ (under the embedding $\mathrm{SL}_2(R) \subseteq \mathrm{SL}_3(R)$). This can now be quickly reverified by a Mennicke symbol calculation:

$$[1+xy,\ x^2] = [1+xy,\ x]^2 = [1,\ x]^2 = 1.$$

In fact, more generally, for any $m, n \geqslant 0$ with $r = m+n-1 \geqslant 0$, we can say that

$$C_{n,m} = \begin{pmatrix} 1+xy & x^n y^m \\ (-1)^r x^m y^n & 1-xy+x^2y^2-\cdots+(-1)^r x^r y^r \end{pmatrix} \in \mathrm{E}_3(R),$$

since a similar Mennicke symbol calculation shows that $[1+xy, x^n y^m] = 1$. In particular, the SK_1-class of the generalized Cohn matrix $C_{n,m}$ is trivial. It is also of interest to determine for which pairs (n, m) is $C_{n,m} \in \mathrm{E}_2(R)$. In the case $R = k[x, y]$ (k a field), Tom Dorsey has pointed out that $C_{n,m} \in \mathrm{E}_2(R)$ iff $|n-m| \leqslant 1$. This can be easily verified by using Theorem (I.8.2). On the other hand, the example of the real coordinate ring of the 2-sphere (I.8.14) shows that, if $x^2+y^2 = 1$ in a commutative ring A, the Mennicke symbol $[x, y]$ need not represent the trivial class in $SK_1(A)$. Note that the matrix $M = \begin{pmatrix} x & y \\ -y & x \end{pmatrix}$ completing $(x,\ y)$ to an element of $\mathrm{SL}_2(A)$ is related to the Cohn matrix C in the following way:

$$M \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} M^{-1} = \begin{pmatrix} 1+xy & x^2 \\ -y^2 & 1-xy \end{pmatrix} = C.$$

Example 3.5′. It is perhaps also of interest to mention an example of a *non-trivial* Mennicke symbol of the form $[1+xy, x^2]$. Take the k-subalgebra $R \subseteq k[x, y]$ generated by the monomials x^2, y^2 and xy, where k is a field of characteristic $\neq 2$. Since $\begin{pmatrix} 1+xy & x^2 \\ -y^2 & 1-xy \end{pmatrix}$ is defined over R and has determinant 1, the Mennicke symbol $[1+xy, x^2] \in \mathrm{SL}_3(R)/\mathrm{E}_3(R)$ makes sense. V. Srinivas has shown that this symbol represents a nontrivial element of $SK_1(R)$; see his 1987 paper referenced in Chapter VIII. This is a significant example in that the normal "monomial algebra" R is known to have all f.g. projectives free (according to [Murthy-Pedrini: 1972]), but $SK_1(R) \neq \{1\}$. For more information on this, see (VIII.9).

In the next result, due to M.P. Murthy, we shall compute the Mennicke symbol in an important special case. Murthy's result is to be compared with (III.2.6). Recall that a polynomial $f(t) \in R[t]$ is said to be *unitary* if its leading coefficient is a unit in R.

Theorem 3.6. *Let* $(R, \mathfrak{m})$ *be a commutative local ring, and let* $(f, g) \in \mathrm{Um}_2\big(R[t]\big)$ *be such that* f *is unitary. Then* $[f, g] = 1$ *in* $\mathrm{SL}_3\big(R[t]\big)/\mathrm{E}_3\big(R[t]\big)$.

Proof. We shall induct on $n = \deg f$, the case $n = 0$ being trivial (since then $f \in \mathrm{U}(R)$, and (3.4)(1) applies). For $n > 0$, we may assume that $\deg g < n$ (since we can replace g by its remainder upon division by f, thanks to (3.4)(3)).

Case 1. $g(0) \in \mathrm{U}(R)$. Since

$$[f,\ g] = [f - g(0)^{-1} f(0) g,\ g],$$

we may assume that $f(0) = 0$; that is, $f = tf'$ for some unitary f' of degree $n - 1$. But then

$$[f,\ g] = [tf',\ g] = [t,\ g]\,[f',\ g],$$

where $[t, g] = [t,\ g(0)] = 1$ by (3.4), and $[f',\ g] = 1$ by the inductive hypothesis. Thus, $[f,\ g] = 1$ in this case.

Case 2. $g(0) \in \mathfrak{m}$. Fix an equation $hf - kg = 1$, where $h, k \in R[t]$. Let $k = qf + k'$, where $\deg(k') < n$. Now

$$1 = hf - (qf + k')g = h'f - k'g, \quad \text{where } h' = h - qg. \tag{3.7}$$

By the elementary transformation

$$\begin{pmatrix} f & g \\ k' & h' \end{pmatrix} \mapsto \begin{pmatrix} f + k' & g + h' \\ k' & h' \end{pmatrix},$$

we have $[f, g] = [f + k',\ g + h']$. Since $\deg k' < n$, $f + k'$ is still unitary of degree n. But from (3.6), $(h'(0), g(0)) \in \mathrm{Um}_2(R)$. Since $g(0) \in \mathfrak{m}$, we have $h'(0) \in \mathrm{U}(R)$, and hence $(g + h')(0) \in \mathrm{U}(R)$. By *Case 1*, $[f + k', g + h'] = 1$, so $[f, g] = 1$, as desired. □

The argument above is taken from [Gupta-Murthy: 1980], except that we have eliminated their use of the resultant of f and g in the proof there.

§4. Suslin's Stability Theorem

In this section, we shall, again, consider only commutative rings. For such a ring R, a unimodular row $(f, g) \in \mathrm{Um}_2\big(R[t]\big)$ gives rise to a Mennicke symbol

$$[f, g] \in \mathrm{SL}_3\big(R[t]\big)/\mathrm{E}_3\big(R[t]\big).$$

If $\mathfrak{m}$ is a maximal ideal of R, let we write $[f, g]_{\mathfrak{m}}$ for the image of $[f, g]$ in $\mathrm{SL}_3\big(R_{\mathfrak{m}}[t]\big)/\mathrm{E}_3\big(R_{\mathfrak{m}}[t]\big)$; that is, the Mennicke symbol of $(f, g) \in \mathrm{Um}_2\big(R_{\mathfrak{m}}[t]\big)$.

We shall now put together the results in the previous sections to prove Suslin's Stability Theorem for fields. The following result is a direct consequence of (1.13).

Proposition 4.1. *Let* $(f, g) \in \mathrm{Um}_2\big(R[t]\big)$*. If, for any maximal ideal* $\mathfrak{m} \subset R$*, there exists*

$$(a, b) \in \mathrm{Um}_2(R_\mathfrak{m}) \subseteq \mathrm{Um}_2\big(R_\mathfrak{m}[t]\big)$$

such that $[f, g]_\mathfrak{m} = [a, b]$*, then* $[f, g] = [f(0), g(0)]$.

Proof. Pick $h, k \in R[t]$ such that $fh - gk = 1$ and let

$$\sigma = \begin{pmatrix} f & g & 0 \\ k & h & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{SL}_3\big(R[t]\big). \tag{4.2}$$

The hypothesis on $[f, g]_\mathfrak{m}$ implies that

$$\sigma_\mathfrak{m} \in \mathrm{SL}_3(R_\mathfrak{m}) \cdot \mathrm{E}_3\big(R_\mathfrak{m}[t]\big),$$

for every maximal ideal $\mathfrak{m} \subset R$. Therefore, by (1.13),

$$\sigma = \alpha \cdot \beta, \quad \text{where} \quad \alpha \in \mathrm{GL}_3(R), \quad \text{and} \quad \beta \in \mathrm{E}_3\big(R[t]\big).$$

Evaluating this at 0, we have $\sigma(0) = \alpha \cdot \beta(0)$. Therefore,

$$\sigma = \sigma(0)\beta(0)^{-1}\beta \in \sigma(0) \cdot \mathrm{E}_3\big(R[t]\big),$$

which implies that $[f, g] = [f(0), g(0)]$ (in view of (4.2)). □

Corollary 4.3. *If* $(f, g) \in \mathrm{Um}_2\big(R[t]\big)$ *and* f *is unitary, then* $[f, g] = [f(0), g(0)]$.

Proof. For any maximal ideal $\mathfrak{m} \subset R$, we have $[f, g]_\mathfrak{m} = 1$ by (3.6). Thus, the local hypothesis in (4.1) is satisfied, and (4.1) gives $[f, g] = [f(0), g(0)]$. □

We shall now apply the above results to polynomial rings over fields. Corollary (4.3) provides the key to proving the triviality of Mennicke symbols over such polynomial rings.

Proposition 4.4. *Let* k *be any field and* $A = k[t_1, \ldots, t_m]$*. Then* $[f, g] = 1$ *for any* $(f, g) \in \mathrm{Um}_2(A)$.

Proof. We induct on m, the case $m = 0$ being trivial (as $\mathrm{SL}_r(k) = \mathrm{E}_r(k)$ for any r). Say $f \neq 0$. For $m \geqslant 1$, we may assume, after a change of variables, that f has a leading term $c\,t_m^s$ as a polynomial in t_m over $R = k[t_1, \ldots, t_{m-1}]$ where $c \in k^*$ (see (III.1.9)). Then by (4.3),

$$[f, g] = [f(t_1, \ldots, t_{m-1}, 0),\ g(t_1, \ldots, t_{m-1}, 0)].$$

By the inductive hypothesis, the RHS is trivial, so we are done! □

We are now finally in a position to present the following main result of Suslin on linear groups over a polynomial ring. Note that the bound $n \geqslant 3$ below is the best we can expect, in view of Cohn's example in (I.8).

Suslin's Stability Theorem (for fields) 4.5. *For a polynomial ring* $A = k[t_1, \ldots, t_m]$ *over any field* k*, we have* $\mathrm{SL}_n(A) = \mathrm{E}_n(A)$ *for* $n \geqslant 3$*. In particular,* $K_1(A) \cong k^*$*, and for all* $n \geqslant 3$*, the canonical map*

$$\varphi_n : K_{1,n}(A) \longrightarrow K_1(A) \cong k^* \tag{4.6}$$

(where $K_{1,n}(A) = \mathrm{GL}_n(A)/\mathrm{E}_n(A)$ *) is an isomorphism.*

Before we present the proof of (4.5), let us stress the point of view that this result is a (nonstable) K_1–analogue of Serre's Conjecture. Indeed, since

$$\mathrm{GL}_n(A) = \mathrm{U}(A) \cdot \mathrm{SL}_n(A) = k^* \cdot \mathrm{SL}_n(A)$$

(where $\mathrm{U}(A) = k^* = \mathrm{GL}_1(k)$ is viewed as embedded in $\mathrm{GL}_n(A)$ by the usual suspension process), the first conclusion of (4.5) amounts to

$$\mathrm{GL}_n(A) = \mathrm{GL}_1(k) \cdot \mathrm{E}_n(A) \quad \text{for } n \geqslant 3. \tag{4.7}$$

Thus, while Serre's Conjecture says that any f.g. projective A-module is extended from k (and hence free), (4.7) confirms that any invertible $n \times n$ matrix over A (with $n \geqslant 3$) is, up to a sequence of elementary column transformations, also "extended from k".

Having made the above remark, we now proceed to the following

Proof of (4.5). Once we have $\mathrm{SL}_n(A) = \mathrm{E}_n(A)$ for $n \geqslant 3$, we get $SK_1(A) = \{1\}$, $K_1(A) \cong \mathrm{U}(A) = k^*$, and the fact that φ_n is an isomorphism.

To prove $\mathrm{SL}_n(A) = \mathrm{E}_n(A)$, note that for $n \geqslant 3$, the argument in the first paragraph of the proof of (III.3.7) shows that, for any $\alpha \in \mathrm{GL}_n(A)$, there exist $\sigma, \sigma' \in \mathrm{E}_n(A)$ such that $\sigma'\alpha\sigma \in \mathrm{GL}_2(A) \subseteq \mathrm{GL}_n(A)$. Thus, if $\alpha \in \mathrm{SL}_n(A)$, we have $\alpha_1 := \sigma'\alpha\sigma \in \mathrm{SL}_2(A)$. The class of α_1 in $\mathrm{SL}_3(A)/\mathrm{E}_3(A)$ is a Mennicke symbol, so by (4.4), we have $\alpha_1 \in \mathrm{E}_3(A)$, which yields $\alpha \in \mathrm{E}_n(A)$. □

For another proof of (4.5) using a Horrocks-type argument (akin to the proof of Serre's Conjecture in (V.2.7)), see Remark (5.14) below. The same result (4.5) was shown to be true by Suslin for Laurent polynomial rings of the most general kind, namely,

$$A = k[t_1, \ldots, t_m,\ t_{m+1}^{\pm 1}, \ldots, t_{m+r}^{\pm 1}].$$

We refer the reader to the original source [Suslin: 1977a] for more details.

§5. K_1–Analogue of Horrocks' Theorem

In the paper [Suslin: 1977a], Suslin not only proved the Stability Theorem (for polynomial rings over noetherian rings) by Quillen's patching techniques, but also obtained a K_1–analogue on Horrocks' Theorem. Here again, the role of finitely generated projective modules over a polynomial ring $R[t]$ is replaced by that of matrices in $\mathrm{GL}_n(R[t])$ modulo $\mathrm{E}_n(R[t])$. For the results to work out, we need to assume, once more, that $n \geqslant 3$. Under this assumption, we can invoke the normality of $\mathrm{E}_n(R[t])$ in $\mathrm{GL}_n(R[t])$, which often makes a crucial difference. Throughout the following discussion, R continues to denote a commutative ring.

Our goal in this section is to state and to prove Suslin's K_1–analogue of Horrocks' Theorem. There are two main tools needed for this proof. The first is the GL_n-Patching Theorem (1.12), or more specifically, its corollary (1.13), which we have already at our disposal. The second is a key factorization theorem for the group $\mathrm{E}_n(R[t,t^{-1}])$ over a local ring R, which itself requires a rather lengthy proof. In order to present the proof of the K_1-Horrocks' Theorem without first going into another detour, we shall state here the result we need on $\mathrm{E}_n(R[t,t^{-1}])$, in a form that will be sufficient for our purposes.

Factorization Theorem 5.1. *Let $(R,\mathfrak{m})$ be a commutative local ring, and let $n \geqslant 3$. If $\gamma \in \mathrm{E}_n(R[t,t^{-1}])$ reduces to the identity matrix I_n modulo $\mathfrak{m}[t,t^{-1}]$, then there exists a factorization $\gamma = \gamma_+\gamma_-$ where $\gamma_+ \in \mathrm{E}_n(R[t])$ and $\gamma_- \in \mathrm{E}_n(R[t^{-1}])$.*

The proof of (5.1) will be given in §6, where we will delve more deeply into the structure of $\mathrm{E}_n(R[t,t^{-1}])$ for a local ring R. Indeed, we shall see in §6 that a more precise result is possible in (5.1), in that γ_+ and γ_- there can each be chosen to be congruent to I_n modulo $\mathfrak{m}[t]$ and $\mathfrak{m}[t^{-1}]$ respectively. This finer conclusion, however, will not be needed in this section.

Assuming the Factorization Theorem 5.1, we may now proceed to the result we want. Since the assumption $n \geqslant 3$ will be always in force, we'll use freely the normality of $\mathrm{E}_n(S)$ in $\mathrm{GL}_n(S)$ for any commutative ring S (see (I.9.14)). In particular the unstable K_1-groups $K_{1,n}(S) = \mathrm{GL}_n(S)/\mathrm{E}_n(S)$ will be meaningful throughout.

Suslin's K_1-Horrocks Theorem 5.2. *For any commutative ring R and any integer $n \geqslant 3$, we have a pullback diagram*

(5.3)
$$\begin{array}{ccc}
K_{1,n}(R) & \xrightarrow{i_+} & K_{1,n}(R[t]) \\
{\scriptstyle i_-}\big\downarrow & & \big\downarrow{\scriptstyle j_-} \\
K_{1,n}(R[t^{-1}]) & \xrightarrow{j_+} & K_{1,n}(R[t,\,t^{-1}])
\end{array}$$

where $i_\pm$, $j_\pm$ are the functorial maps for the unstable K_1-groups.(*)

Proof. For any $\delta \in \mathrm{GL}_n(R)$ ($n \geqslant 3$ fixed), we write $\bar{\delta}$ for the class of δ in $K_{1,n}(R) = \mathrm{GL}_n(R)/\mathrm{E}_n(R)$. The same notation will be used for the rings $R[t]$, $R[t^{-1}]$, and $R[t,t^{-1}]$. The statement that (5.3) is a pullback diagram means that, *for any* $\alpha \in$

(*)We have seen before a "similar-looking" commutative diagram in Th. (1.18). However, the diagram here is of a different nature, and the conclusions of (1.18) certainly do not apply. Nevertheless, the two diagrams will eventually be combined to prove something useful: for a sneak preview, see the diagram in (5.12) below.

$\mathrm{GL}_n\big(R[t]\big)$ *and* $\beta \in \mathrm{GL}_n\big(R[t^{-1}]\big)$ *with* $j_+(\bar{\beta}) = j_-(\bar{\alpha})$, *there exists a unique* $\bar{\delta} \in K_{1,n}(R)$ *such that* $i_+(\bar{\delta}) = \bar{\alpha}$ *and* $i_-(\bar{\delta}) = \beta$. Note that the uniqueness of $\bar{\delta}$ is already clear, since R is a retract of $R[t]$ and $R[t^{-1}]$, which implies that $i_\pm$ are *injective* maps. The task at hand is to prove the existence of $\bar{\delta}$.

The equation $j_+(\bar{\beta}) = j_-(\bar{\alpha})$ in $K_{1,n}\big(R[t,t^{-1}]\big)$ means that $\alpha\beta^{-1} \in \mathrm{E}_n\big(R[t,t^{-1}]\big)$. *It will suffice to prove that* $\alpha \in \mathrm{E}_n\big(R[t]\big)\cdot\delta$ *for some* $\delta \in \mathrm{GL}_n(R)$. For, by symmetry, this would also give $\beta \in \mathrm{E}_n\big(R[t^{-1}]\big)\cdot\delta_0$ for some $\delta_0 \in \mathrm{GL}_n(R)$. Since $\bar{\alpha}$ and $\bar{\beta}$ become equal in $K_{1,n}\big(R[t,t^{-1}]\big)$, $\bar{\delta}$ and $\bar{\delta}_0$ are also equal there. But

$$(5.4) \qquad j_-i_+ = j_+i_- : \ K_{1,n}(R) \to K_{1,n}\big(R[t,t^{-1}]\big)$$

is also injective since R is a retract of $R[t,t^{-1}]$. (A retraction from $R[t,t^{-1}]$ to R is given by specializing t and t^{-1} to 1.) Thus, $\delta_0 \in \mathrm{E}_n(R)\cdot\delta$, and $\bar{\delta} = \bar{\delta}_0$ in $K_{1,n}(R)$ maps to $\bar{\alpha}$ in $K_{1,n}\big(R[t]\big)$ and to $\bar{\beta}$ in $K_{1,n}\big(R[t^{-1}]\big)$, as desired.

To prove that $\alpha \in \mathrm{E}_n\big(R[t]\big)\cdot\mathrm{GL}_n(R) = \mathrm{GL}_n(R)\cdot\mathrm{E}_n\big(R[t]\big)$, we may invoke (1.13) (Corollary to the GL_n-Patching Theorem) to assume that R is *local*, say with maximal ideal $\mathfrak{m}$. Without loss of generality, we may also assume that $\alpha(0) = I_n$ (which can be achieved by replacing α by $\alpha\cdot\alpha(0)^{-1}$ and β by $\beta\cdot\alpha(0)^{-1}$). From $\alpha\beta^{-1} \in \mathrm{E}_n\big(R[t,t^{-1}]\big)$, we have

$$(5.5) \qquad \det(\alpha) = \det(\beta) \in R[t] \cap R[t^{-1}] = R,$$

so the new assumption $\alpha(0) = I_n$ forces $\det(\alpha) = \det(\beta) = 1$. Since $\mathrm{SL}_n = \mathrm{E}_n$ over the euclidean domains $(R/\mathfrak{m})[t]$ and $(R/\mathfrak{m})[t^{-1}]$ (by (I.5.4)), we may further modify α and β, modulo $\mathrm{E_n}\big(R[t]\big)$ and $\mathrm{E}_n\big(R[t^{-1}]\big)$ respectively, to assume that they are both congruent to I_n modulo $\mathfrak{m}$. Applying (5.1) to $\alpha\beta^{-1} \in \mathrm{E}_n\big(R[t,t^{-1}]\big)$, we get a factorization $\alpha\beta^{-1} = \gamma_+\gamma_-$, where $\gamma_+ \in \mathrm{E}_n\big(R[t]\big)$, and $\gamma_- \in \mathrm{E}_n\big(R[t^{-1}]\big)$. Thus,

$$\gamma_+^{-1}\alpha = \gamma_-\beta \in \mathrm{GL}_n\big(R[t]\big) \cap \mathrm{GL}_n\big(R[t^{-1}]\big) = \mathrm{GL}_n(R),$$

so $\alpha \in \gamma_+\cdot\mathrm{GL}_n(R) \subseteq \mathrm{E}_n\big(R[t]\big)\cdot\mathrm{GL}_n(R)$, as desired. □

We close this section by proving the following $K_{1,n}$–analogue of our earlier results on K_0.

Theorem 5.6. *Let* $f \in R[t]$ *be a monic polynomial over a commutative ring* R, *and let* $n \geqslant 3$. *Then the functorial map*

$$(5.7) \qquad h : \ K_{1,n}\big(R[t]\big) \to K_{1,n}\big(R[t,t^{-1},f^{-1}]\big)$$

is injective; in particular, so is the functorial map

$$(5.8) \qquad i_f : \ K_{1,n}\big(R[t]\big) \to K_{1,n}\big(R[t,f^{-1}]\big).$$

Proof. The injectivity of h will follow once we draw the correct commutative diagram of functorial maps. Let

$$f(t) = t^m + a_{m-1}t^{m-1} + \cdots + a_0, \quad \text{and}$$
$$(5.9) \qquad g(s) = 1 + a_{m-1}s + \cdots + a_0 s^m,$$

where $s := t^{-1}$, and $a_i \in R$. Then $f(t) = t^m g(s)$, and

$$(5.10) \qquad R[t, t^{-1}, f^{-1}] = R[s, s^{-1}, s^m g^{-1}] = R[s, s^{-1}, g^{-1}].$$

Note that s and g are non 0-divisors in $R[s]$, and (5.9) implies that they are comaximal there. Moreover, we have

$$(5.11) \qquad R[s, s^{-1}] \cap R[s, g^{-1}] = R[s].$$

Indeed, any nonzero element from the LHS has the form

$$w = (b_0 + b_1 s + \cdots)/s^d = (c_0 + c_1 s + \cdots)/g^e,$$

where $d \geqslant 0$, $e \geqslant 0$, $b_i, c_j \in R$, with $b_0 \neq 0$. Clearing the denominators and comparing constant terms (in s), we see that d must be zero, and so $w \in R[s]$.

Utilizing (5.10), we produce the following commutative diagram:

$$(5.12) \qquad \begin{tikzcd}
K_{1,n}R \arrow[r, "i_+"] \arrow[d, "i_-"'] & K_{1,n}(R[t]) \arrow[r, "i_f"] \arrow[d, "j_-"] \arrow[ddr, "h"] & K_{1,n}R[t, f^{-1}] \arrow[dd] \\
K_{1,n}(R[s]) \arrow[r, "j_+"] \arrow[d, "i_g"'] & K_{1,n}(R[s, s^{-1}]) \arrow[d, "j'"] & \\
K_{1,n}(R[s, g^{-1}]) \arrow[r] & K_{1,n}(R[s, s^{-1}, g^{-1}]) \arrow[r, phantom, "="] & K_{1,n}(R[t, t^{-1}, f^{-1}])
\end{tikzcd}$$

In one sentence, the idea of the proof is that *the injectivity of $i_g i_-$ implies the injectivity of $h = j' j_-$*, by the good properties of the two squares on the left-side! We give more details below.

The equation (5.11) for the comaximal pair s, g in $R[s]$ enables us to use (1.18) for the bottom square. Let $h(\bar{\alpha}) = 1$, where $\alpha \in \mathrm{GL}_n\big(R[t]\big)$. Then $j'(j_-(\bar{\alpha})) = 1$ implies the existence of a $\sigma \in \mathrm{GL}_n\big(R[s]\big)$ such that $j_+(\bar{\sigma}) = j_-(\bar{\alpha})$ and $i_g(\bar{\sigma}) = 1$. Next, by (5.2) applied to the top square, there exists a $\delta \in \mathrm{GL}_n(R)$ such that $i_+(\bar{\delta}) = \bar{\alpha}$ and $i_-(\bar{\delta}) = \bar{\sigma}$. In particular, $i_g i_-(\bar{\delta}) = i_g(\bar{\sigma}) = 1$. Now $i_g i_-$ is an *injective* map, since R is a retract of the ring $R[s, g^{-1}]$. (We can retract $R[s, g^{-1}]$ to R by $s \mapsto 0$ and $g \mapsto 1$.) Therefore, we have $\bar{\delta} = 1$ and $\bar{\alpha} = i_+(\bar{\delta}) = 1$, as desired. □

Remark 5.13. According to (5.2), the top square on the left of (5.12) is a pullback diagram. Once we have proved the injectivity of h and hence of i_f, we can apply the latter to deduce that j_+ is injective. With this extra information, we can now say that the *bottom* square on the left of (5.12) is also a pullback diagram (see Remark (1.20)), and consequently, so is the LHS rectangle composed of the two squares in (5.12).

Remark 5.14. It is worth pointing out (although it is no longer a surprise at this stage) that the injectivity of the map i_f in (5.8) can be used to give another proof of Suslin's Stability Theorem (4.5) for all fields k. Indeed, suppose we know the truth of (4.5) for $m-1$ variables, and let $\alpha \in \mathrm{SL}_n\big(k[t_1, \ldots, t_m]\big)$, where $n \geqslant 3$. Viewing α as in $\mathrm{SL}_n\big(k(t_1)[t_2, \ldots, t_m]\big)$, we'll have $\alpha \in \mathrm{E}_n\big(k(t_1)[t_2, \ldots, t_m]\big)$. Thus, for some monic polynomial $f \in k[t_1]$, we have

$$\alpha \in \mathrm{E}_n\big(k[t_2, \ldots, t_m]\,[t_1, f^{-1}]\big).$$

Applying the injectivity of i_f (with $R = k[t_2, \ldots, t_m]$), we conclude that $\alpha \in \mathrm{E}_n\big(k[t_1, \ldots, t_m]\big)$, as desired.

Remark 5.15. Besides Suslin's theorem (5.2), there are also other variants of K_1-statements of the Horrocks type, for instance, about completing unimodular rows to *elementary* matrices under a localization map $R[t] \to R[t]_h$, where $h \in R[t]$ is a monic polynomial. For an explicit statement of such a result, due to Ravi Rao, see VIII.5.10.

§6. Structure Theorem on $\mathrm{E}_n\big(R[t, t^{-1}]\big)$

In §4, Suslin's K_1-Horrocks' Theorem was proved under the assumption of the truth of the Factorization Theorem (5.1) for the group $\mathrm{E}_n\big(R[t, t^{-1}]\big)$ over a commutative local ring R. To complete our exposition, we must now try to prove the stated factorization theorem. This will be done in the present section, which will, in fact, give much more structural information on the group $\mathrm{E}_n\big(R[t, t^{-1}]\big)$ (in the local case). The first half of this section will be devoted to the formulation and proof of this structural result (6.17). After this, we'll return to supply the missing proof of (a somewhat stronger version of) the Factorization Theorem (5.1); see (6.24).

Throughout this section, R denotes a commutative ring. Regarding the integer n as fixed, we introduce the following notations for this section: let $M(R)$ denote the group of monomial matrices[(*)] in $\mathrm{SL}_n(R)$, and let

(6.1) $$\widetilde{T}(R) = \{(a_{ij}) \in \mathrm{SL}_n(R): \quad a_{ij} = 0 \quad \text{for } i > j\}$$

(6.2) $$T(R) = \{(a_{ij}) \in \widetilde{T}(R): \quad a_{ii} = 1 \quad \text{for all } i\}.$$

For any r $(1 \leqslant r \leqslant n)$, let $N_r(R)$ (resp. $N^r(R)$) denote the subgroup of $\mathrm{E}_n(R)$ consisting of matrices whose ith row (resp. ith column) is a unit times the ith unit vector. Whenever the intended ring R is clear from the context, we shall frequently shorten the notations $M(R)$, $\widetilde{T}(R)$, $T(R)$, $N_r, \ldots$ to M, $\widetilde{T}$, T and N_r, etc.

We start by recording a few basic facts about the groups introduced above.

(6.3) $M \subseteq \mathrm{E}_n(R)$.

[(*)] *A monomial matrix* is a square matrix having exactly one nonzero entry in each row and in each column.

(6.4) $T \subseteq \widetilde{T} \subseteq N^1 \cap N_n$.

(6.5) *If* $(R,\mathfrak{m})$ *is a commutative local ring with residue field* $k = R/\mathfrak{m}$, *then the natural* ("*bar*") *maps*

$$M(R) \to M(k), \quad T(R) \to T(k), \quad \textit{and} \quad N^r(R) \to N^r(k)$$

are all surjective.

Here, (6.4) is clear, and (6.3) follows easily from Whitehead's Lemma. The proof for (6.5) is based on the argument of "adjusting determinants". We'll demonstrate this in the case of N^r, and leave the two other (easier) cases to the reader. For ease of notations, let $r = n$. Any $\alpha_0 \in N^n(k)$ has the block form

$$\begin{pmatrix} \bar{\beta} & 0 \\ \bar{\gamma} & a \end{pmatrix}, \quad \text{where } \beta \in \mathrm{GL}_{n-1}(R),\ \gamma \in R^{n-1},\ \text{and } a \in k^*.$$

(We used here the fact that $\mathrm{GL}_{n-1}(R) \to \mathrm{GL}_{n-1}(k)$ is surjective!) Then, for $u := \det(\beta)^{-1} \in \mathrm{U}(R)$, the matrix

$$\alpha = \begin{pmatrix} \beta & 0 \\ \gamma & u \end{pmatrix} \in \mathrm{SL}_n(R) \tag{6.6}$$

lifts α_0, since $a = \det\big(\bar{\beta}\big)^{-1} = \bar{u} \in k$. Now $\mathrm{SL}_n(R) = \mathrm{E}_n(R)$ by (I.5.4), so we have $\alpha \in N^n(R)$, as desired.

Remark 6.7. Of course, the maps $N_r(R) \to N_r(k)$ and $\widetilde{T}(R) \to \widetilde{T}(k)$ are surjective too. However, we have chosen to include in (6.3) – (6.5) *only* those facts that will be needed in the future.

Next, we prove the following result for any field k.

Lemma 6.8. *For* $r \leqslant n$, *we have* $\mathrm{SL}_n(k) = N_r \cdot M \cdot T = N^r \cdot M \cdot T$.

Proof. It suffices to prove the first equality, as the second will follow from a similar argument. We shall assume here Bruhat's Decomposition Theorem for SL_n which says that $\mathrm{SL}_n(k) = T \cdot M \cdot T$ (see, e.g. [Milnor: 1971, (12.6)]). Since $T \subseteq N_n$, this gives $\mathrm{SL}_n(k) = N_n \cdot M \cdot T$. For $r < n$ and $\sigma \in \mathrm{SL}_n(k)$, let ω be the matrix obtained from I_n by changing the sign of its r th row and then transposing the r th and n th rows. Then $\omega \in M \subseteq \mathrm{SL}_n(k)$, so we have $\omega^{-1}\sigma \in T \cdot M \cdot T$. From this, it follows that

$$\sigma \in (\omega T \omega^{-1})(\omega M) \cdot T \subseteq N_r \cdot M \cdot T,$$

as desired. □

In the rest of this section, we shall let

$$\delta_r := \mathrm{diag}(1, \ldots, t, \ldots, 1) \in \mathrm{GL}_n\big(R[t,t^{-1}]\big), \tag{6.9}$$

where t occurs in the r th position $(1 \leqslant r \leqslant n)$.

Lemma 6.10. *Let* $\sigma \in \mathrm{E}_n\big(R[t]\big)$*, where* $n \geqslant 3$*. If* $\sigma \equiv I_n \ \big(\mathrm{mod}\ (t)\big)$*, then*

$$\delta_r\, \sigma\, \delta_r^{-1},\ \ \delta_r^{-1}\sigma\, \delta_r \in \mathrm{E}_n\big(R[t]\big) \quad \textit{for any } r \leqslant n.$$

Proof. Note that δ_r is *not* in $\mathrm{GL}_n\big(R[t]\big)$, so we cannot deduce $\delta_r\, \sigma\, \delta_r^{-1} \in \mathrm{E}_n\big(R[t]\big)$ from Suslin's Normality Theorem (I.9.14). The idea here is, however, that if $\sigma \equiv I_n\ (\mathrm{mod}\ (t))$, there will be "enough t" in σ to offset the fact that $\delta_r^{-1} \notin \mathbb{M}_n\big(R[t]\big)$. In fact, we shall show that a careful modification of the earlier proof for (I.9.14) will yield $\delta_r\, \sigma\, \delta_r^{-1}, \delta_r^{-1}\, \sigma\, \delta_r \in \mathrm{E}_n\big(R[t]\big)$, under the assumption that $\sigma \equiv I_n\ (\mathrm{mod}\ (t))$. Let

$$\sigma = \prod_{l=1}^{m} e_{i_l j_l}(a_l + t f_l) = \prod_{l=1}^{m} e_{i_l j_l}(a_l)\ \cdot\ e_{i_l j_l}(t f_l),$$

where $a_l \in R,\ f_l \in R[t]$, and $e_{ij}(g)$ denotes $I_n + g e_{ij}\ \ (i \neq j)$ (as in §1). Setting

$$\gamma_p = \prod_{l=1}^{p} e_{i_l j_l}(a_l) \in \mathrm{E}_n(R) \quad (p \leqslant m).$$

we have $\gamma_m = I_n$ (since $\sigma \equiv I_n$ (mod (t))), and

$$\sigma = \prod_{l=1}^{m} \gamma_l e_{i_l j_l}(t f_l)\gamma_l^{-1} = \prod_{l=1}^{m} (I_n + \alpha_{i_l}\ \cdot\ t f_l \cdot \beta_{j_l}), \tag{6.11}$$

where α_{i_l} is the i_l th column of γ_l, and β_{j_l} is the j_l th row of γ_l^{-1} (so that $\beta_{j_l}\alpha_{i_l} = 0$, as a result of $i_l \neq j_l$). In (5.11), we can then handle one factor at a time, so let us assume that $\sigma = I_n + \alpha \cdot t f \cdot \beta$, with $\alpha,\ \beta \in \mathrm{Um}_n(R)$, and $\beta\alpha = 0$. By (I.9.11), we can write $\alpha = \alpha_1 + \cdots + \alpha_N$, where each α_i has at most two nonzero coordinates (in R), with $\beta\alpha_i = 0$. As in the proof of (I.9.12), we are then reduced to the case where $\alpha = \alpha_i$; in particular, since $n \geqslant 3$, α has a zero coordinate. Now

$$\delta_r\, \sigma\, \delta_r^{-1} = \delta_r\ (I_n + \alpha(tf)\beta)\, \delta_r^{-1} = I_n + \alpha'\beta'\,, \tag{6.12}$$

with $\alpha' = \delta_r \alpha f$ and $\beta' = \beta t \delta_r^{-1}$ *both defined over* $R[t]$. Moreover,

$$\beta'\alpha' = (\beta t)(\alpha f) = 0$$

and α' has a zero coordinate. Therefore, by (I.9.9) (and (6.12)), $\delta_r\, \sigma\, \delta_r^{-1} \in \mathrm{E}_n\big(R[t]\big)$. Similarly, we can check that $\delta_r^{-1}\, \sigma\, \delta_r \in \mathrm{E}_n\big(R[t]\big)$. □

The next lemma is another conjugation result, but σ is now a matrix over R.

Lemma 6.13. *If* $\sigma \in N_r(R)$*, then* $\delta_r^{-1}\, \sigma\, \delta_r \in \mathrm{E}_n\big(R[t]\big)$*. If* $\sigma \in N^r(R)$*, then* $\delta_r\, \sigma\, \delta_r^{-1} \in \mathrm{E}_n\big(R[t]\big)$.

Proof. For ease of notations, let $r=n$. If $\sigma\in N_r(R)$, then $\sigma=\begin{pmatrix}\sigma' & v\\ 0 & a\end{pmatrix}$ for some $a\in\mathrm{U}(R)$, and

$$\delta_n^{-1}\,\sigma\,\delta_n=\begin{pmatrix}\sigma' & t\,v\\ 0 & a\end{pmatrix}=\begin{pmatrix}I_{n-1} & va^{-1}(t-1)\\ 0 & 1\end{pmatrix}\cdot\sigma.$$

This belongs to $\mathrm{E}_n\big(R[t]\big)$, as claimed. The case where $\sigma\in N^r(R)$ can be handled similarly. □

Corollary 6.14. *Let $\sigma\in\mathrm{E}_n\big(R[t]\big)$, where $n\geqslant 3$. If $\delta_r\,\sigma\,\delta_r^{-1}$ lies in $\mathrm{GL}_n\big(R[t]\big)$, then it lies in $\mathrm{E}_n\big(R[t]\big)$.*

Proof. Write σ in the form $\sigma_1\sigma_0$, where $\sigma_0=\sigma(0)\in\mathrm{E}_n(R)$ and $\sigma_1\in\mathrm{E}_n\big(R[t]\big)$ is $\equiv I_n$ (mod (t)). It is easy to check that the condition $\delta_r\,\sigma\,\delta_r^{-1}\in\mathrm{GL}_n\big(R[t]\big)$ amounts exactly to $\sigma_0\in N^r(R)$, so (6.13) implies $\delta_r\,\sigma_0\,\delta_r^{-1}\in\mathrm{E}_n\big(R[t]\big)$. On the other hand, (6.10) implies $\delta_r\,\sigma_1\,\delta_r^{-1}\in\mathrm{E}_n\big(R[t]\big)$, so we have $\delta_r\,\sigma\,\delta_r^{-1}\in\mathrm{E}_n\big(R[t]\big)$. □

Remark. From the proof given above, it should be clear to the reader that (6.14) is just a self-strengthening of (6.10). Of course, (6.14) also works for the conjugate $\delta_r^{-1}\,\sigma\,\delta_r$; but this will not be needed later.

For the rest of this section, we shall assume that $(R,\mathfrak{m})$ *is a commutative local ring, with residue field* $k=R/\mathfrak{m}$. In working with the rings $R[t]$, $R[t^{-1}]$, and $R[t,t^{-1}]$, we shall be dealing with the relative $\mathrm{GL_n}$-groups

$$\mathrm{GL}_n\big(R[t],\ \mathfrak{m}[t]\big),\quad \mathrm{GL}_n\big(R[t^{-1}],\ \mathfrak{m}[t^{-1}]\big),\quad\text{and}$$

$$\mathrm{GL}_n\big(R[t,t^{-1}],\ \mathfrak{m}[t,t^{-1}]\big).\tag{6.15}$$

To simplify the notations, we shall denote these groups more informally by

$$\mathrm{GL}_n\big(R[t],\ \mathfrak{m}\big),\quad \mathrm{GL}_n\big(R[t^{-1}],\ \mathfrak{m}\big),\quad\text{and}\quad \mathrm{GL}_n\big(R[t,\ t^{-1}],\ \mathfrak{m}\big),\tag{6.16}$$

and we shall use the same simplified notations for the relative elementary groups (introduced in (I.9)); that is, $\mathrm{E}_n\big(R[t,\ t^{-1}],\ \mathfrak{m}\big)$, etc.

Recalling the definition of $\widetilde{T}$ from (6.1), we can now prove the following main result for this section.

Structure Theorem 6.17. *For $n\geqslant 3$, we have*

$$\mathrm{E}_n\big(R[t,\ t^{-1}]\big)=\mathrm{E}_n\big(R[t]\big)\cdot\widetilde{T}\big(R[t,\ t^{-1}]\big)\cdot\mathrm{E}_n\big(R[t,t^{-1}],\ \mathfrak{m}\big).$$

Proof. Let V denote the set on the RHS. We proceed in a number of steps.

Step 1. For any $r\leqslant n$, $\delta_r\,V\,\delta_r^{-1}=V$. It suffices to prove that $\delta_r\,V\,\delta_r^{-1}\subseteq V$ (since $\delta_r^{-1}V\delta_r\subseteq V$ can be proved similarly). Certainly, $\widetilde{T}\big(R[t,t^{-1}]\big)$ is stabilized under conjugation by δ_r, and the same is true of $\mathrm{E}_n\big(R[t,t^{-1}],\mathfrak{m}\big)$ since it is a normal subgroup of $\mathrm{GL}_n\big(R[t,\ t^{-1}]\big)$ by (I.9.14). Therefore, it is enough for us to show that(*)

$$\text{(6.18)} \qquad \sigma \in \mathrm{E}_n\big(R[t]\big) \Longrightarrow \delta_r\, \sigma\, \delta_r^{-1} \in V.$$

Write again $\sigma = \sigma_1\sigma_0$ as in the proof of (6.14). By (6.10), $\delta_r\, \sigma_1\, \delta_r^{-1} \in \mathrm{E}_n\big(R[t]\big)$, so we are reduced to verifying that $\delta_r\, \sigma_0\, \delta_r^{-1} \in V$. Writing "bar" for the passage from R to $k = R/\mathfrak{m}$ and applying (6.8), we have a decomposition

$$\text{(6.19)} \qquad \bar{\sigma}_0 = \alpha_1\alpha_2\alpha_3, \quad \text{where} \quad \alpha_1 \in N^r(k), \quad \alpha_2 \in M(k), \quad \alpha_3 \in T(k).$$

By (6.5), there exist $\beta_1 \in N^r(R)$, $\beta_2 \in M(R)$, and $\beta_3 \in T(R)$ such that $\bar{\beta}_i = \alpha_i$ for all i. Thus, we can write $\sigma_0 = \beta_1\beta_2\beta_3\beta_4$, where $\beta_4 \in \mathrm{SL}_n\big(R,\ \mathfrak{m}\big)$. We certainly have $\delta_r\beta_3\delta_r^{-1} \in \widetilde{T}\big(R[t, t^{-1}]\big)$, and by (I.9.22)(3), $\beta_4 \in \mathrm{E}_n\big(R,\ \mathfrak{m}\big)$, so

$$\delta_r\beta_4\delta_r^{-1} \in \mathrm{E}_n\big(R[t, t^{-1}],\ \mathfrak{m}\big).$$

On the other hand, (6.13) gives $\delta_r\beta_1\delta_r^{-1} \in \mathrm{E}_n\big(R[t]\big)$. Therefore, we are done if we can show that

$$\text{(6.20)} \qquad \delta_r\beta_2\delta_r^{-1} \in \mathrm{E}_n\big(R[t]\big) \cdot \widetilde{T}\big(R[t,\ t^{-1}]\big).$$

But $\beta_2 \in M(R)$ implies that $\delta_r\beta_2\delta_r^{-1} \in M\big(R[t, t^{-1}]\big)$, so we can write it as $\pi\gamma$, where π is a permutation matrix, and γ is a diagonal matrix in $\mathrm{GL}_n\big(R[t,\ t^{-1}]\big)$. If π is an *even* permutation, we are done since then $\pi \in \mathrm{E}_n\big(\mathbb{Z}\big) \subseteq \mathrm{E}_n\big(R[t]\big)$, and this guarantees that $\gamma \in \widetilde{T}\big(R[t, t^{-1}]\big)$. If π is an *odd* permutation, we can finish similarly by taking $\varepsilon = \mathrm{diag}(-1, 1, \ldots, 1)$ and writing $\pi\gamma = (\pi\varepsilon)(\varepsilon\gamma)$.

Step 2. For $i \neq j$ and any $a \in R$, we have

$$\text{(6.21)} \qquad e_{ij}(at) \cdot V = V \quad \textit{and} \quad e_{ij}(at^{-1}) \cdot V = V.$$

The former is clear, since $e_{ij}(at) \in \mathrm{E}_n\big(R[t]\big)$. For the latter, we express $e_{ij}(at^{-1})$ as $\delta_i^{-1} e_{ij}(a)\delta_i$ and deduce from *Step* 1 that

$$\begin{aligned} e_{ij}(at^{-1}) \cdot V &= \delta_i^{-1} e_{ij}(a) \cdot \delta_i V \delta_i^{-1} \cdot \delta_i \\ &= \delta_i^{-1} e_{ij}(a) V \delta_i \\ &= \delta_i^{-1} V \delta_i = V. \end{aligned}$$

Step 3. In view of $e_{ij}(f+g) = e_{ij}(f)\, e_{ij}(g)$ and the relations

$$[e_{ij}(f),\ e_{jk}(g)] = e_{ik}(fg) \quad (i, j, k \text{ distinct})$$

(*) One should *not* assume that V is a group here. Rather, we rely on the normality of $\mathrm{E}_n\big(R[t, t^{-1}],\ \mathfrak{m}\big)$ to make this reduction.

in (I.7.2), the group $\mathrm{E}_n\big(R[t,t^{-1}]\big)$ is generated by the set

(6.22) $$S=\{e_{ij}(at),\ e_{ij}(at^{-1}):\ i\neq j,\ a\in R\}.$$

Let $V_0:=\{\tau\in\mathrm{E}_n\big(R[t,t^{-1}]\big):\ \tau V=V\}$, which is contained in V since $1\in V$. Clearly, V_0 is a subgroup of $\mathrm{E}_n\big(R[t,t^{-1}]\big)$, and *Step* 2 implies that $S\subseteq V_0$. Thus, $\mathrm{E}_n\big(R[t,t^{-1}]\big)=V_0\subseteq V_1$, as desired. □

Having proved the Structure Theorem 6.17, we now return to consider the yet outstanding Factorization Theorem 5.1. Our attention will be focused on the following three groups:

(6.23) $$\begin{cases} G=\mathrm{E}_n\big(R[t,t^{-1}]\big)\cap \mathrm{GL}_n\big(R[t,t^{-1}],\ \mathfrak{m}\big)\supseteq \mathrm{E}_n\big(R[t,\ t^{-1}],\ \mathfrak{m}\big),\\ G_+=\mathrm{E}_n\big(R[t]\big)\cap \mathrm{GL}_n\big(R[t],\ \mathfrak{m}\big)\supseteq \mathrm{E}_n\big(R[t],\ \mathfrak{m}\big),\\ G_-=\mathrm{E}_n\big(R[t^{-1}]\big)\cap \mathrm{GL}_n\big(R[t^{-1}],\ \mathfrak{m}\big)\supseteq \mathrm{E}_n\big(R[t^{-1}],\ \mathfrak{m}\big),\end{cases}$$

(Here, we are using the notational simplifications introduced in (6.16).) The Factorization Theorem stated earlier in (5.1) says that, for $n\geqslant 3$, any matrix in G can be written as $\gamma_+\gamma_-$, where $\gamma_+\in\mathrm{E}_n\big(R[t]\big)$ and $\gamma_-\in\mathrm{E}_n\big(R[t^{-1}]\big)$. We shall prove a more precise version of this, as follows.

Factorization Theorem 6.24. *For $n\geqslant 3$, we have $G=G_+\cdot G_-$.*

The proof of this is completed in a sequence of three lemmas about the groups in (6.23). *In each of these lemmas, we shall assume that $n\geqslant 3$.*

Lemma 6.25. *For $U:=G_+\cdot G_-$, we have*

$$\sigma\in\mathrm{E}_n\big(R[t,t^{-1}]\big)\implies \sigma U\sigma^{-1}=U.$$

Proof. Note that U is not known to be a group. However,

$$S:=\{\sigma\in\mathrm{E}_n\big(R[t,t^{-1}]\big):\ \sigma U\sigma^{-1}=U\}$$

is a subgroup of $\mathrm{E}_n\big(R[t,t^{-1}]\big)$. In order to show that $S=\mathrm{E}_n\big(R[t,t^{-1}]\big)$, it suffices to show that, for any $a\in R$, $e_{ij}(at)$, $e_{ij}(at^{-1})\in S$ (since $\mathrm{E}_n\big(R[t,t^{-1}]\big)$ is generated by these two kinds of matrices). In the following, we'll show that $e_{ij}(at^{-1})\in S$. Once this is known, we can replace t by t^{-1} to get

$$e_{ij}(at)\,(G_-G_+)\,e_{ij}(-at)=G_-G_+\,.$$

Taking inverses, we'll get

$$e_{ij}(at)\,(G_+G_-)\,e_{ij}(-at)=G_+G_-\,,$$

so $e_{ij}(at)\in S$ as well.

To see that $e_{ij}(at^{-1})\in S$, let us assume (for ease of notation) that $i=1$ and $j=n$. Consider an arbitrary element $\gamma_+\gamma_-\in U$, where $\gamma_+\in G_+$ and $\gamma_-\in G_-$.

After changing this product into $(\gamma_+ \cdot \gamma_+(0)^{-1})(\gamma_+(0)\gamma_-)$, we may assume that $\gamma_+ \equiv I_n \pmod{(t)}$. Let the first and last rows of γ_+ be

$$(1+tf_1, \ldots, tf_n) \quad \text{and} \quad (tg_1, \ldots, 1+tg_n) \quad \big(f_i,\ g_i \in R[t]\big).$$

Then the first row of $e_{1n}(at^{-1})\gamma_+$ is

$$(1+tf_1+ag_1,\ tf_2+ag_2,\ \ldots,\ tf_n+at^{-1}+g_n).$$

Let $b = g_1(0) \in \mathfrak{m}$ (since $\gamma_+ \equiv I_n \pmod{\mathfrak{m}}$); then $1+ab \in \mathrm{U}(R)$. To eliminate the only occurrence of t^{-1} in $e_{1n}(at^{-1})\gamma_+$, we multiply it from the right by $e_{1n}(-a(1+ab)^{-1}t^{-1})$ to get a matrix $\alpha \in \mathbb{M}_n\big(R[t]\big)$. Since $\det(\alpha) = \det(\gamma_+) = 1$, $\alpha \in \mathrm{GL}_n\big(R[t]\big)$. Modulo $\mathfrak{m}$, we have

$$\alpha \equiv e_{1n}(at^{-1}) \cdot \gamma_+ \cdot e_{1n}(-at^{-1}) \equiv I_n,$$

so in fact $\alpha \in \mathrm{GL}_n\big(R[t],\ \mathfrak{m}\big)$. Using the identity

$$e_{1n}(ct^{-1}) = \delta_n\, e_{1n}(c)\, \delta_n^{-1} \quad (\forall\, c \in R),$$

we can rewrite

$$\alpha = \delta_n\, [e_{1n}(a) \cdot (\delta_n^{-1}\gamma_+\delta_n) \cdot e_{1n}(-a(1+ab)^{-1})]\, \delta_n^{-1}. \tag{6.26}$$

By (6.10), $\delta_n^{-1}\gamma_+\delta_n$ belongs to $\mathrm{E}_n\big(R[t]\big)$, and hence so does the matrix in brackets in (6.26). Since $\alpha \in \mathrm{GL}_n\big(R[t]\big)$, an application of (6.14) now yields $\alpha \in \mathrm{E}_n\big(R[t]\big)$. Therefore, $\alpha \in G_+$, and we can decompose the conjugate $e_{1n}(at^{-1})\gamma_+\gamma_-e_{1n}(at^{-1})^{-1}$ into

$$\begin{aligned} &e_{1n}(at^{-1})\gamma_+e_{1n}\big(-a(1+ab)^{-1}t^{-1}\big)e_{1n}\big(a(1+ab)^{-1}t^{-1}\big)\gamma_-e_{1n}(-at^{-1}) \\ &\quad = \alpha \cdot e_{1n}\big(a(1+ab)^{-1}t^{-1}\big)\gamma_-e_{1n}(-at^{-1}) \in G_+G_-. \end{aligned}$$

This shows that $e_{1n}(at^{-1})Ue_{1n}(at^{-1})^{-1} \subseteq U$. Replacing a by $-a$, we have also $e_{1n}(at^{-1})^{-1}Ue_{1n}(at^{-1}) \subseteq U$, so $e_{1n}(at^{-1}) \in S$ (for every $a \in R$), as desired. $\square$

Next, we prove the following "approximation" to (6.24).

Lemma 6.27. *$G = G_+ \cdot E$, where $E = \mathrm{E}_n\big(R[t, t^{-1}],\ \mathfrak{m}\big)$.*

Proof. Let $\alpha \in G$; that is, $\alpha \in \mathrm{E}_n\big(R[t, t^{-1}]\big)$ with $\alpha \equiv I_n \pmod{\mathfrak{m}}$. By the Structure Theorem 6.17, we can factor α into $\alpha_1\alpha_2\alpha_3$, where

$$\alpha_1 \in \mathrm{E}_n\big(R[t]\big), \quad \alpha_2 \in \widetilde{T}\big(R[t, t^{-1}]\big), \quad \text{and} \quad \alpha_3 \in E.$$

To show that $\alpha \in G_+ \cdot E$, we may thus assume that $\alpha_3 = I_n$. Writing "bar" again for reduction modulo $\mathfrak{m}$ (with $\bar{R} = k$), we have

$$\bar{\alpha}_1 = \bar{\alpha}_2^{-1} \in \mathrm{E}_n\big(k[t]\big) \cap \widetilde{T}\big(k[t, t^{-1}]\big) \subseteq \widetilde{T}\big(k[t]\big). \tag{6.28}$$

From this, it is easy to see that $\bar{\alpha}_1$ lifts to a matrix $\beta \in \widetilde{T}\big(R[t]\big)$. Then $\beta\alpha_2 \equiv I_n$ (mod $\mathfrak{m}$), and

$$\beta\alpha_2 \in \widetilde{T}\big(R[t]\big) \cdot \widetilde{T}\big(R[t,t^{-1}]\big) = \widetilde{T}\big(R[t,t^{-1}]\big).$$

By (I.9.22)(1), we have $\beta\alpha_2 \in E$, and $\beta \in \mathrm{E}_n\big(R[t]\big)$. Since $\alpha_1\beta^{-1} \equiv I_n$ (mod $\mathfrak{m}$) also, it follows that $\alpha_1\beta^{-1} \in G_+$, and hence $\alpha = (\alpha_1\beta^{-1})(\beta\alpha_2) \in G_+ \cdot E$, as desired. □

We shall now complete the proof of (6.24) as follows.

Lemma 6.29. $E \subseteq G_+G_-$. (*In particular,* $G \subseteq G_+(G_+G_-) = G_+G_-$ *.*)

Proof. (Cf. Step 3 in the proof of (6.17).) For $U = G_+G_-$ (as in (6.25)),

$$U_0 := \{\tau \in E : \ \tau U = U\}$$

is a subgroup of E contained in U. For matrices

$$\alpha \in \mathrm{E}_n\big(R[t,t^{-1}]\big), \quad \beta \in \mathrm{E}_n\big(R[t],\ \mathfrak{m}\big), \quad \text{and} \quad \gamma \in \mathrm{E}_n\big(R[t^{-1}],\ \mathfrak{m}\big),$$

repeated applications of (6.25) give the equations

$$\begin{aligned} \alpha^{-1}\beta\alpha U = \alpha^{-1}\beta U\alpha = \alpha^{-1}U\alpha = U \quad &(\text{since } \beta \in G_+), \quad \text{and} \\ \alpha^{-1}\gamma\alpha U = \alpha^{-1}U\gamma\alpha = \alpha^{-1}U\alpha = U \quad &(\text{since } \gamma \in G_-). \end{aligned}$$

Therefore, $\alpha^{-1}\beta\alpha$, $\alpha^{-1}\gamma\alpha \in U_0$. Since E is generated by matrices of the form $\alpha^{-1}\beta\alpha$ and $\alpha^{-1}\gamma\alpha$, we have $E = U_0 \subseteq U$, as desired. □

Remark 6.30. Suslin has in fact proved that, *if a matrix in* $\mathrm{E}_n\big(R[t]\big)$ *is congruent to* I_n (mod $\mathfrak{m}$)*, then it belongs to* $\mathrm{E}_n\big(R[t],\ \mathfrak{m}\big)$. This fact (which *does not* follow from (1.5)!) implies that the inclusion signs in (6.23) are all equalities. This more refined information is, however, not needed in the proof of the Factorization Theorem (6.24) presented above.

Notes on Chapter VI

This chapter is mainly an exposition on [Suslin: 1977a], which is the source of both Suslin's Normality Theorem and his Stability Theorem. We have already covered the former in the preparatory Chapter I, so in this chapter, we focused our attention on the latter. The proofs in [Suslin: 1977a] are difficult to improve upon, but the division of the material into six sections here, each with a specific goal, hopefully makes the mathematics a bit easier to follow. In writing this exposition, we have consulted extensively the lecture notes of [Gupta-Murthy: 1980], which provided us a second reliable source (beside Suslin's original paper) for the key ideas of the proofs of Suslin's results.

The GL_n-Patching Theorem (1.12) is very much modeled upon Quillen's Patching Theorem (V.1.6). Though Suslin referred to this as "A Theorem of Quillen", it was he who first formulated and proved this result. The introduction of the functors $K_{1,n}$ $(n \geqslant 3)$ on commutative rings prepared the way to the Stability Theorem, and made possible the simple diagram-style formulation of Theorem (1.18). A quick coverage of the method of Mennicke symbols (highlighted by Murthy's Theorem (3.6) for such symbols over the polynomial ring of a local ring) paved the way to a full proof of Suslin's Stability Theorem (over a field k) in §4.

The material in §2 is not needed for the rest of the chapter, but it provides a natural continuation of the results in §1. We have included this material in the present chapter since it is expected to be useful in handling some of the still unsolved problems in the study of the elementary group action on unimodular polynomial vectors. For a statement of some of these problems, see VIII.5.

To complete the analogy with the case of projective modules, we also presented Suslin's K_1–analogue of Horrocks' Theorem. The formulation of this theorem in terms of a simple pullback diagram in (5.2) makes it particularly easy to understand and to remember. This pullback viewpoint also led to very natural proofs of the injectivity statements in Theorem (5.6).

A key fact for the proof of (5.2) is the Factorization Theorem (5.1) for $\mathrm{E}_n\big(R[t, t^{-1}]\big)$ $(n \geqslant 3)$ over a local ring R. This theorem may look "easy" at first sight, since $\mathrm{E}_n\big(R[t, t^{-1}]\big)$ is generated as a group by $\mathrm{E}_n\big(R[t]\big)$ and $\mathrm{E}_n\big(R[t^{-1}]\big)$. Unfortunately, it does not seem possible to translate this observation into a valid proof of (5.1). The proof is only accomplished in §6, after considerable work devoted to the proof of another theorem, (6.17), on a triple factorization of $\mathrm{E}_n\big(R[t, t^{-1}]\big)$. The work in §6 is both technical and subtle, but so far unavoidable. A quicker or more efficient treatment must await future efforts.

For more information on the K_1–analogue of Serre's Conjecture, and on the work done on related themes after 1977, see §9, §11, and §13 of Chapter VIII.

Chapter VII

The Quadratic Analogue of Serre's Conjecture

This chapter will be a short exposition on some results in the literature (up to about 1977) concerning the quadratic analogue of Serre's Conjecture. In this investigation, one considers f.g. projective modules P (say over $R = k[x_1, \ldots, x_d]$, k a field), equipped with, respectively, the following three types of structures, S: (1) *quadratic forms*, (2) *symmetric bilinear forms*, or (3) *symplectic forms*. In the spirit of Serre's Problem on projective modules over the polynomial ring R, the main question to be asked in this chapter is the following:

> *Is the object (P, S) over $k[x_1, \ldots, x_d]$ necessarily extended, in a suitable sense, from an object of a similar kind over k?*

For convenience of the exposition, we shall restrict our attention to cases (2) and (3), and skip the more difficult case of quadratic forms. All rings considered in this chapter will be assumed to be commutative (with an identity).

§1. Inner Product Spaces

Definition 1.1. We say that (P, B) is an *inner product space* (IPS) over a commutative ring R if $P \in \mathfrak{P}(R)$ (that is, P is a f.g. projective R-module) and $B : P \times P \to R$ is a symmetric bilinear form, satisfying the following "nonsingularity" condition:

(1.2) *For any $f \in P^* = \mathrm{Hom}_R(P, R)$, there exists a unique $m \in P$ such that $f = B(-\,, m)$.* (*In other words, $m \mapsto B(-\,, m)$ gives an isomorphism from P to P^*.*)

For this class of objects, we have an obvious notion of "isomorphism": we say that (P, B) is *isomorphic* (or *isometric*) *to* (P', B') if there exists an R-isomorphism $g : P \to P'$ such that $B'(g(m_1), g(m_2)) = B(m_1, m_2)$ for any $m_1, m_2 \in P$. If this is the case, we shall write $(P, B) \cong (P', B')$ (or just $P \cong P'$ if the forms involved are clear from the context).

From Def. (1.1), it follows that the underlying (f.g. projective) module P of an inner IPS structure must be *self-dual.* The converse is true too: if $P \in \mathfrak{P}$ is self-dual, we can fix an R-isomorphism $\varphi : P \to P^*$ and define an IPS structure on P by taking $B(m, m') = \varphi(m)(m')$ (for all $m, m' \in P$).

If (P, B) is an IPS over R and P is R-free, we shall say that (P, B) is a *free inner product space*. In this situation, it is customary to associate to B a symmetric matrix $(B(e_i, e_j))$, where $\{e_1, \ldots, e_n\}$ is an R-basis for P. With respect to this matrix, we can easily verify that the nonsingularity condition (1.2) is tantamount to $(B(e_i, e_j)) \in \mathrm{GL}_n(R)$. If we use a different basis $\{e'_1, \ldots, e'_n\}$ for P, we will get a new matrix $(B(e'_i, e'_j))$ congruent to the old one, that is,

$$(B(e'_i, e'_j)) = X^t \cdot (B(e_i, e_j)) \cdot X, \quad (t \text{ denotes "transpose"}),$$

where $X \in \mathrm{GL}_n(R)$ is the matrix expressing $\{e'_i\}$ in terms of $\{e_j\}$. Thus, the isometry class of a free IPS of rank n gives rise to a congruence class of symmetric matrices in $\mathrm{GL}_n(R)$ — and, of course, conversely.

If (P_1, B_1), (P_2, B_2) are IPS's over R, we shall define $B_1 \perp B_2$ to be the symmetric bilinear form given by

$$(B_1 \perp B_2)((m_1, m_2),\ (m'_1, m'_2)) = B_1(m_1, m'_1) + B_2(m_2, m'_2)$$

on $P_1 \oplus P_2$. This form clearly satisfies the nonsingularity condition (1.2), and the resulting IPS $(P_1 \oplus P_2, B_1 \perp B_2)$ shall be denoted by $P_1 \perp P_2$ (if B_1, B_2 are clear from the context). This is called the *orthogonal sum* of (P_1, B_1) and (P_2, B_2), and we shall say that (P_1, B_1), (P_2, B_2) are *orthogonal summands* of $P_1 \perp P_2$.

Lemma 1.3. *Let (P, B) be an IPS, and let M be an R-submodule of P such that $B|M \times M$ is nonsingular. Let $N = M^\perp = \{n \in P : B(n, M) = 0\}$. Then*

$$(M,\ B|M \times M), \quad (N,\ B|N \times N)$$

are IPS's over R, and $P = M \perp N$.

Proof. Clearly, $M \cap N = 0$. Let $x \in P$. By hypothesis, the linear functional $B(x, -)$ on M is of the form $B(m, -)$ for some $m \in M$. Thus, $B(x - m,\ M) = 0$, and so $x - m \in N$. This shows that $P = M \oplus N$. In particular, $M, N \in \mathfrak{P}(R)$, and the rest is clear. □

Definition 1.4. Let $P \in \mathfrak{P}(R)$. We shall define $\mathbb{H}(P)$ to be the IPS supported by $P \oplus P^*$, with form

$$B^P((m, f),\ (m', f')) = f(m') + f'(m) \quad (m, m' \in P;\ f, f' \in P^*).$$

(The verification of the nonsingularity condition (1.2) for B^P will be left to the reader.)

An IPS of the form $\mathbb{H}(P)$ will be called a *hyperbolic* IPS. The most basic special case of this is the *hyperbolic plane* $\mathbb{H}(R)$: this is a free IPS of rank 2, supported by $R \oplus R^* \cong R^2$, with form

$$((r,s),\ (r',s')) \longmapsto rs' + r's \quad (r,r',s,s' \in R).$$

With respect to the standard basis on R^2, this gives rise to the symmetric matrix $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Similarly, $\mathbb{H}(R^m) \cong \mathbb{H}(R) \perp \cdots \perp \mathbb{H}(R)$ (m times) gives rise to the symmetric matrix $\begin{pmatrix} J & & 0 \\ & \ddots & \\ 0 & & J \end{pmatrix}$. By a permutation of the standard basis, we see that this is congruent to $\begin{pmatrix} 0 & I_m \\ I_m & 0 \end{pmatrix}$.

Lemma 1.5. (1) *Every IPS is an orthogonal summand of a free IPS.*

(2) *If* 2 *is invertible in* R*, every IPS is an orthogonal summand of some* $\mathbb{H}(R^m)$.

Proof. Let (P, B) be a given IPS. Choose $Q \in \mathfrak{P}(R)$ such that $P \oplus Q$ is free, say $\cong R^n$. Consider the orthogonal sum $P \perp P \perp \mathbb{H}(Q)$. This is a *free* IPS since

$$\begin{aligned} P \oplus P \oplus (Q \oplus Q^*) &\cong P \oplus P^* \oplus (Q \oplus Q^*) \\ &\cong (P \oplus Q) \oplus (P \oplus Q)^* \\ &\cong R^n \oplus (R^n)^* \\ &\cong R^{2n}. \end{aligned}$$

Thus, (P, B) is an orthogonal summand of (R^{2n}, B_1) for the form B_1 on $P \perp P \perp \mathbb{H}(Q)$. This proves (1). To prove (2), it is sufficient to show that (if 2 is invertible) (R^{2n}, B_1) is an orthogonal summand of some $\mathbb{H}(R^m)$. Let $S = (B_1(e_i, e_j))$ be the symmetric (invertible) matrix associated with B_1. Then $(R^{4n}, B_1 \perp -B_1)$ has matrix $\begin{pmatrix} S & 0 \\ 0 & -S \end{pmatrix}$. We claim that this is congruent to $\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$. In fact, if $T = S^{-1}$, and $X = \begin{pmatrix} \frac{1}{2}T & I \\ \frac{1}{2}T & -I \end{pmatrix}$, we have

$$\begin{aligned} X^t \cdot \begin{pmatrix} S & 0 \\ 0 & -S \end{pmatrix} \cdot X &= \begin{pmatrix} \frac{1}{2}T & \frac{1}{2}T \\ I & -I \end{pmatrix} \begin{pmatrix} S & 0 \\ 0 & -S \end{pmatrix} \begin{pmatrix} \frac{1}{2}T & I \\ \frac{1}{2}T & -I \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2}I & -\frac{1}{2}I \\ S & S \end{pmatrix} \begin{pmatrix} \frac{1}{2}T & I \\ \frac{1}{2}T & -I \end{pmatrix} \\ &= \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}. \end{aligned}$$

This equation clearly implies that X is invertible; in particular, $(R^{4n}, B_1 \perp -B_1) \cong \mathbb{H}(R^{2n})$. $\square$

Let (P, B) be an IPS over R. If $f: R \to R'$ is a homomorphism of (commutative) rings, we can define, by "scalar extension", a new pair (P', B') over R', where $P' = R' \otimes_R P$, and B' is specified by

$$B'(r_1' \otimes m_1,\ r_2' \otimes m_2) = r_1' r_2' \cdot f(B(m_1, m_2)) \quad (r_i' \in R',\ m_i \in P).$$

It is routine to verify that (P', B') is an IPS over R'; we say that (P', B') is *extended from* (the IPS (P, B) over) R, and write $(P', B') = R' \otimes_R (P, B)$. More weakly, we say that an IPS (P_1', B_1') over R' is *stably extended* from R if there exist IPS's (P_2', B_2'), (P_3', B_3') extended from R such that

$$(1.6) \qquad (P_1', B_1') \perp (P_2', B_2') \cong (P_3', B_3').$$

We have, in particular, $P_1' \oplus P_2' \cong P_3'$ where P_2' and P_3' are extended from $\mathfrak{P}(R)$, so the module P_1' itself is stably extended from R.

Lemma 1.7. *Let $f: R \to R'$ be as above and (P_1', B_1') be an IPS over R'. Assume that* 2 *is invertible in R. Then (P_1', B_1') is stably extended from R iff there exists m such that $(P_1', B_1') \perp \mathbb{H}(R'^m)$ is extended from R.*

Proof. The "if" part is clear, since $\mathbb{H}(R'^m)$ is isometric to the extension of $\mathbb{H}(R^m)$. Conversely, suppose (1.6) holds, where, say, (P_i', B_i') is extended from (P_i, B_i), $i = 2, 3$. By (1.5)(2), there exists an IPS (P_4, B_4) over R such that $(P_2, B_2) \perp (P_4, B_4) \cong \mathbb{H}(R^m)$. "Adding" on the extension (P_4', B_4') of (P_4, B_4) to (1.6), we get

$$(P_1', B_1') \perp \mathbb{H}(R'^m) \cong (P_3', B_3') \perp (P_4', B_4'),$$

which is the extension of $(P_3, B_3) \perp (P_4, B_4)$. □

To pursue the quadratic analogue of Serre's Problem, we shall be interested in the notion of extension of IPS's with respect to the inclusion homomorphism $R \to R[t_l, \dots, t_d] = R'$. If an IPS (P, B) over R extends to (P', B') over R', we can recapture (P, B) from (P', B') by reducing modulo $(t_1, \dots, t_d)$ (i.e. further "extending" along $R' \to R$ defined by specializing all $t_i \mapsto 0$). Thus, if an IPS (P', B') over R' is extended, it must be extended from (P_0', B_0'), where the subscript 'zero' denotes reduction modulo $(t_1, \dots, t_d)$. The analogue of Serre's Conjecture for IPS's is, therefore, the following problem:

(1.8) *If k is a field, is every* IPS (P', B') *over* $k[t_l, \dots, t_d]$ *extended from the* IPS (P_0', B_0') *over* k?

Note that, by the Quillen-Suslin Theorem, $P' \in \mathfrak{P}(k[t_1, \dots, t_d])$ must be *free*, so P' is indeed extended from P_0'. This, in itself, does not guarantee the truth of (1.8), but it surely makes (1.8) into a very natural question.

Since the (P', B') above must be a *free* IPS, it amounts essentially to a symmetric matrix $S \in \mathrm{GL}_n(k[t_1, \dots, t_d])$. The invertibility of S amounts to the fact that $\det S \in k \setminus \{0\}$. In view of these observations, we can translate (1.8) into the following very concrete purely matrix theoretic problem:

$$(1.8') \qquad \begin{array}{l} \textit{If } k \textit{ is a field, and } S \textit{ is a symmetric matrix over } k[t_1, \dots, t_d] \\ \qquad \textit{with } \det S \in k \setminus \{0\}, \textit{ is } S \textit{ congruent over} \\ \qquad k[t_1, \dots, t_d] \textit{ to } S_0 = S(0, \dots, 0)? \end{array}$$

In contrast to the Quillen-Suslin solution of Serre's Conjecture, the answer to (1.8) and (1.8′), though not fully known at present, seems to depend on:

(1) *the nature of* k;

(2) *the number of variables* d; and, to some extent, also on

(3) *the rank of* P' (*i.e. the size of the matrix* S).

We shall now survey some of the results on the above problem; among these are both positive and negative results.

(1.9) *The answer is "no" (for any* d *and* rank $P' = 2$) *if* char $k = 2$. In fact, the symmetric matrix $S = \begin{pmatrix} t_1 & 1 \\ 1 & 0 \end{pmatrix} \in \mathrm{GL}_2\big(k[t_1, \dots, t_d]\big)$ provides an obvious counterexample. Indeed, if S is congruent over $k[t_1, \dots, t_d]$ to

$$S_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

the form B' must be "alternating" (i.e. $B(m, m) = 0$ for every $m \in k[t_1, \dots, t_d]^2$). This is certainly not the case since $B'(e_1, e_1) = t_1$ for the unit vector $e_1 = (1, 0)$ in $P' = k[t_1, \dots, t_d]^2$! If we take $\begin{pmatrix} S & & \\ & \ddots & \\ & & S \end{pmatrix}$, we can get counterexamples over $k[t_1, \dots, t_d]$ (char $k = 2$) for any *even* rank.

In view of the above, *we shall assume, in the following, that* char $k \neq 2$. We have then the following results.

(1.10) *The answer to* (1.8) *is "yes" if* rank $P' = 2$. (The case rank $P' = 1$ is, of course, a triviality.) This is proved in [Parimala: 1976c, Prop. 1.1], using the tools of Galois cohomology.

(1.11) *The answer to* (1.8) *is "yes" if* $d = 1$. This is a result of Harder; see Theorem 13.4.3 in [Knebusch: 1970].

(1.12) *The answer to* (1.8) *is "yes" if* char $k \neq 2$ *and all quaternion algebras over* k *split.* (In particular, (1.8) is true over any algebraically closed field and any finite field of characteristic not 2.) This result is due to [Raghunathan: 1978] (and independently to Kopeiko). Special cases of this result (mainly for $d \leqslant 3$) were first obtained in [Bass: 1977].

The simplest example of a field k with a nonsplit quaternion algebra is $k = \mathbb{R}$. Over this field, Hamilton's quaternion algebra is a division algebra. In this case, one has the following remarkable result of [Parimala: 1976b, 1978].

(1.13) *There exists a rank* 4 *IPS over* $\mathbb{R}[t_1, t_2]$ *that is not extended from* $\mathbb{R}$. *In fact, there exist infinitely many isometry classes of such rank* 4 *IPS*'s!

(1.14) Assume that $\operatorname{char} k \neq 2, 3$. If k has a quaternion division algebra D, Knus and Ojanguren (1977) showed that *there exist rank* 3 *and rank* 4 *inner product spaces* (P, B) *over* $k[t_1, t_2]$ *that are not extended from* k. Their construction was closely related to that of Parimala, both depending on the use of nonfree projective right ideals in $D[t_1, t_2]$ (which exist according to II.3.6). The result of Knus and Ojanguren shows that Raghunathan's extendibility result mentioned in (1.12) above is essentially the best possible. By a remark of A. Geramita, one can construct from this, by using orthogonal sums, IPS's of rank $3n$ (for any $n \geqslant 1$) over $\mathbb{R}[t_1, t_2]$ that are not extended from $\mathbb{R}$. (Better still: see Proposition 4.13 below.)

Nevertheless, one has the following theorem of [Karoubi: 1972].

(1.15) *Any IPS over* $k[t_1, \ldots, t_d]$ $(\operatorname{char} k \neq 2)$ *is stably extended from* k.

Also, an analogue of Quillen's Patching Theorem has been established for the category of inner product spaces over arbitrary commutative rings:

(1.16) *Let* R *be a commutative ring, and let* (P, B) *be an IPS over* $R[t]$. *Then* (P, B) *is extended from* R *iff* $(P_{\mathfrak{m}}, B_{\mathfrak{m}})$ *is extended from* $R_{\mathfrak{m}}$ *for any maximal ideal* $\mathfrak{m} \in \operatorname{Max} R$. (*Here,* $(P_{\mathfrak{m}}, B_{\mathfrak{m}})$ *denotes the extension of* (P, B) *along the localization homomorphism* $R[t] \to R_{\mathfrak{m}}[t]$.) This result was proved in [Bass-Connell-Wright: 1976].

Using the result (1.16), [Parimala: 1978] has shown, on the other hand, that:

(1.17) *The quadratic analogue of the Local Horrocks Theorem does not hold: namely, there exist commutative local rings* (*in fact discrete valuation rings*) R, *and an IPS over* $R[t]$, *which "extends to an IPS over* $\mathbb{P}^1_R$", *but is not extended from an IPS over* R. (Parimala showed later that a "modified" version of Horrocks' theorem does hold. For a statement of this version, see Item (7) in VIII.10.)

In the balance of this chapter, we shall prove Karoubi's Theorem in §2, Harder's Theorem in §3, and we shall present Parimala's counterexamples (1.13), (1.17) in §4. The other results we mentioned above require more background preparations, so we have to refer the reader to the original sources. In a final section, we shall show, however, that

(1.18) *Any "symplectic space" over* $k[t_1, \dots, t_d]$ *is hyperbolic, and is, in particular, extended from* k.

§2. Karoubi's Theorem

The fact (1.15) mentioned in the last section will be derived from a more general result, as follows.

Theorem 2.1. [Karoubi: 1972] *Let R be a commutative ring in which* 2 *is invertible, and let* (P, B) *be an IPS over* $R[t_1, \dots, t_d]$. *If P is stably extended from R, then* (P, B) *is also stably extended from R.*

If $R = k$, a field of characteristic not 2, this result clearly implies (1.15) since, for any IPS (P, B) over $k[t_1, \dots, t_d]$, the supporting module P (being projective) must be stably free over $k[t_1, \dots, t_d]$ by II.5.9,(*) and hence (P, B) is stably extended from k. More generally, the following statement can be deduced from (2.1).

Corollary 2.2. *Let R be a commutative ring in which* 2 *is invertible. Assume that R is regular or von Neumann regular. Then any IPS* (P, B) *over* $R[t_1, \dots, t_d]$ *is stably extended from* R.

Proof. If R is regular, $K_0 R \to K_0 R[t_1, \dots, t_d]$ is an isomorphism by Grothendieck's Theorem (II.5.8), so

$$[P] = [R[t_1, \dots, t_d] \otimes_R Q_1] - [R[t_1, \dots, t_d] \otimes_R Q_2]$$

for suitable $Q_i \in \mathfrak{P}(R)$. This implies that

$$P \oplus [R[t_1, \dots, t_d] \otimes_R (Q_2 \oplus R^m)] \cong R[t_1, \dots, t_d] \otimes_R (Q_1 \oplus R^m)$$

for suitable m, so P is stably extended from R. Thus, (2.1) yields the desired conclusion. If R is, instead, von Neumann regular, then (V.2.15) says that P is (even) extended from R. □

Proof of (2.1). *Step 1. It suffices to treat the case* $d = 1$. For, if this case is known, then the general case follows by induction on d. Indeed, let

$$S = R[t_1, \dots, t_d], \quad \text{and} \quad S' = R[t_1, \dots, t_{d-1}].$$

Using the 1-variable case, we know that (P, B) is stably extended from S'. By (1.7), there exist m and an IPS (P_1, B_1) over S' such that

(*)Of course, P is in fact *free* by the Quillen-Suslin Theorem. But we don't need this for the purposes of proving or applying (2.1).

$$(2.3) \qquad (P, B) \perp \mathbb{H}(S^m) \cong S \otimes_{S'} (P_1, B_1).$$

If we ignore the inner product structures for the time being, and reduce this equation modulo t_d, we see that P_1 is stably extended from R. By the inductive hypothesis, there exist an integer n, and an IPS (P_2, B_2) over R such that

$$(P_1, B_1) \perp \mathbb{H}(S'^n) \cong S' \otimes_R (P_2, B_2).$$

Tensoring this up to S, and adding on $\mathbb{H}(S^n)$ to (2.3), we obtain

$$(P, B) \perp \mathbb{H}(S^{m+n}) \cong S \otimes_R (P_2, B_2),$$

so (P, B) is stably extended from R.

We shall thus assume, in the following, that $S = R[t]$.

Step 2. It suffices to treat the case where (P, B) *is a free IPS.* In fact, if we assume the result for this case, then the general case can be deduced as follows. Let (P, B) be an IPS over $S = R[t]$, with P stably extended from R, say

$$P \oplus S^m \cong S \otimes_R Q, \quad \text{where } Q \in \mathfrak{P}(R).$$

Let subscript 'zero' denote reduction modulo t, so $P_0 \oplus R^m \cong Q$. Tensoring this back up to S, we get

$$(S \otimes_R P_0) \oplus S^m \cong S \otimes_R Q \cong P \oplus S^m.$$

Upon replacing (P, B) by $P \perp \mathbb{H}(S^m)$, we may thus assume that $P \cong S \otimes_R P_0$. Choose an IPS (P_1, B_1) over R such that $(P_0, B_0) \perp (P_1, B_1)$ is R-free (see (1.5)(1)). If we replace (P, B) by $(P, B) \perp S \otimes_R (P_1, B_1)$, the latter is a free IPS over S, and we need only show that *it* is stably extended from R.

Step 3. Consider now a *free* IPS (S^n, B). Let M be the symmetric matrix associated to B with respect to the standard basis on S^n. We can write (uniquely)

$$M = M_0 + M_1 \cdot t + \cdots + M_r \cdot t^r,$$

where each M_i is a symmetric $(n \times n)$ matrix over R. We wish to apply successive congruence transformations to M (over S); meanwhile, we allow ourselves the liberty of replacing M by $\begin{pmatrix} M & 0 & 0 \\ 0 & 0 & I_p \\ 0 & I_p & 0 \end{pmatrix}$, since this corresponds to adding $\mathbb{H}(S^p)$ to (P, B).

Let us see what happens if we "congruence" $\begin{pmatrix} M & 0 & 0 \\ 0 & 0 & I_n \\ 0 & I_n & 0 \end{pmatrix}$ by $X = \begin{pmatrix} I_n & \alpha & \beta \\ 0 & I_n & 0 \\ 0 & 0 & I_n \end{pmatrix}$ (α, β to be determined):

$$(2.4) \qquad X \cdot \begin{pmatrix} M & 0 & 0 \\ 0 & 0 & I_n \\ 0 & I_n & 0 \end{pmatrix} \cdot X^t = \begin{pmatrix} M' & \beta & \alpha \\ \beta^t & 0 & I_n \\ \alpha^t & I_n & 0 \end{pmatrix}$$

where $M' = M + \beta\alpha^t + \alpha\beta^t$. If $r > 1$, we can put $\alpha = \frac{1}{2}M_r \cdot t^{r-1}$, $\beta = -t \cdot I_n$, which will make

$$M' = M - M_r \cdot t^r = M_0 + M_1 \cdot t + \cdots + M_{r-1} \cdot t^{r-1},$$

and consequently the RHS of (2.4) will involve powers of t only up to t^{r-1}. By repeating this argument, we see that it is sufficient to treat the case where $M = M_0 + M_1 \cdot t$.

Step 4. Here, $M = M_0 + M_1 \cdot t = M_0 \cdot (I + N \cdot t)$, where $N = M_0^{-1} M_1$. Since M is invertible over $S = R[t]$, so is $I + N \cdot t$. The latter has formal inverse

$$I - N \cdot t + N^2 \cdot t^2 - N^3 \cdot t^3 + \cdots$$

over $R[[t]]$, but the inverse lies in $\mathbb{M}_n\big(R[t]\big)$. Consequently, N must be *nilpotent*. Now we invoke the following lemma.

Lemma 2.5. *If S is a commutative ring in which 2 is invertible, and Z is a nilpotent matrix over S, then there exists a polynomial $f \in S[x]$ such that $f(Z)^2 = I + Z$.*

Applying this to $Z = N \cdot t$, we can write $I + N \cdot t = f(N \cdot t)^2$ where $f \in S[x]$. Since M_0, M_1 are symmetric, we have

$$\begin{aligned} M_0\, N = M_1 = M_1^T = N^T M_0, &\quad \text{and hence} \\ M_0\, f(N \cdot t) = f(N^T \cdot t)\, M_0. & \end{aligned}$$

Here, we used "T" for the transpose since "t" is already used to denote the indeterminate. It follows that

$$f(N \cdot t)^T\, M_0\, f(N \cdot t) = M_0\, f(N \cdot t)^2 = M_0 \cdot (I + N \cdot t) = M.$$

Since $f(N \cdot t) \in \mathrm{GL}_n(S)$, this shows that M is congruent over S to M_0, as desired.

Step 5. It remains to prove Lemma 2.5, but this is just a special case of what we have done in the proof of III.4.11. If 2 is invertible in S, Newton's binomial formula

$$\sqrt{1+Z} = 1 + \frac{1}{2}Z - \frac{1}{8}Z^2 + \frac{1}{16}Z^3 - \frac{5}{128}Z^4 + \cdots$$

can be used for a nilpotent element Z in any S-algebra. (The power series in Z on the RHS above is defined over $\mathbb{Z}[1/2]$, as we have observed after III.4.12.) Alternatively, here is a cute little argument given in [Swan: 1975]. If $I + Z' = \big(I - \frac{1}{2}Z\big)^2 (I + Z)$, then $Z' = \frac{1}{4}Z^2(Z - 3I)$ has a *smaller* index of nilpotency than Z (if $Z \neq 0$), while any polynomial in Z' is a polynomial in Z (assuming 2 is invertible). Thus, (2.5) follows by induction on the index of nilpotency of Z. $\square$

§3. Harder Theorem: Easier Proof

In this section, we shall prove Harder's Theorem mentioned in (1.11). The proof of this result that appeared in [Knebusch: 1970] makes heavy use of the modern methods of algebraic geometry. There exists, however, an easier proof of the Harder result (never mind the verbal contradiction !) that was given in [Gerstein: 1973a, b]. We'll present Gerstein's proof here, which is based on the beautiful classical ideas of Hermite on the minima of integral quadratic forms.

For uniformity of treatment, we shall consider symmetric bilinear forms on f.g. projective modules over euclidean domains A, with a degree function ∂ satisfying certain conditions. To get results for rational function fields in one variable over a field k, we can simply specialize the results to the situation $A = k[t]$ (k a field), with $\partial =$ ordinary degree.

Note that if A is any euclidean domain (or even just a PID), then a f.g. A-module L is projective iff it is free, iff it is torsion-free. (Indeed, if the latter holds, we can embed L into a f.g. free module, and hence infer that L is free from (II.2.4).) Thus, for the purposes of considering symmetric bilinear forms over A, we can focus our attention on (f.g.) torsion-free A-modules.

We shall now fix the following notations for this section: A is a euclidean domain, with quotient field K, and ∂ is the degree function on $A\backslash\{0\}$. *We assume further, that ∂ satisfies the following two properties*:

(3.1) $$\partial(ab) = \partial(a) + \partial(b) \text{ for all } a, b \in A\backslash\{0\}.$$

(3.2) $$\partial(a+b) \leqslant \max(\partial(a),\ \partial(b)) \text{ for all } a, b, a+b \in A\backslash\{0\}.$$

Under (3.1) and (3.2), it is clear that a nonzero element u in A is a unit iff $\partial(u) = 0$. We can extend ∂ uniquely to a group homomorphism $\partial : \dot{K} = K\backslash\{0\} \to \mathbb{Z}$ by defining $\partial(a/b) = \partial(a) - \partial(b)$ for $a, b \in A\backslash\{0\}$. Note that *the quotient field K can have arbitrary characteristic, unless otherwise declared.*

Let (V, B) be a nonzero IPS (inner product space) over K. A f.g. A-submodule $L \subseteq V$ is said to be an *A-lattice* in V if L contains a K-basis of V. As we have already observed, L must be A-free since it is torsion-free, and it is easy to verify that an A-basis for L will automatically be a K-basis for V. An element (or "value") $a \in \dot{K}$ is said to be *represented by* V (resp. by L) if there exists a vector $v \in V$ (resp. $v \in L$) such that $B(v, v) = a$. We shall write $D_B(V)$ (resp. $D_B(L)$) for the set of values represented by V (resp. by L). These are subsets in $\dot{K}$; notice that we are not assuming that $D_B(L) \subseteq A$ (at least for the time being).

Lemma 3.3. *The set $\partial(D_B(L)) \subseteq \mathbb{Z}$ has a minimum. (We shall denote this minimum by* $\min_B L$*.)*

Proof. By the Well-Ordering Principle, it is sufficient to show that $\partial(D_B(L))$ is bounded from below. Let $v_1, \ldots, v_n \in L$ be an A-basis for L, and let $a \in A$ be a common denominator for all $B(v_i, v_j)$, say $B(v_i, v_j) = a_{ij}/a$ where $a_{ij} \in A$.

For any vector $v = \sum b_i v_i \in L$ ($b_i \in A$), we have $B(v, v) = \sum b_i b_j a_{ij}/a$, so if $B(v, v) \neq 0$, we have $\partial(a \cdot B(v, v)) \in \partial(A \backslash \{0\}) \geqslant 0$, whence $\partial(B(v, v)) \geqslant -\partial(a)$ by (3.1). □

Keeping the notations in the above proof, let us write $\det_B L = \det(B(v_i, v_j))$. If we use a different A-basis for L, say $\{v_i'\}$, the two matrices $(B(v_i, v_j))$ and $(B(v_i', v_j'))$ will be congruent by a matrix in $\mathrm{GL}_n(A)$, so we have

$$\det(B(v_i', v_j')) \in \det(B(v_i, v_j)) \cdot \mathrm{U}(A)^2,$$

where $\mathrm{U}(A)$ denotes the group of units in A. Thus, $\det_B L$ will be well-defined if we take it to be an element in $\dot{K}/\mathrm{U}(A)^2$. Since ∂ is trivial on $\mathrm{U}(A)$, it follows that $\partial(\det_B L)$ ($\in \mathbb{Z}$) is well-defined.

Definition 3.4. We say that L is an *integral lattice* (w.r.t. B) if $B(L \times L) \subseteq A$. In this case, we have clearly $\min_B L \geqslant 0$, and $\partial(\det_B L) \geqslant 0$.

We shall now prove the following result that provides, for a class of integral lattices, a certain inequality relating $\min_B L$ and $\partial(\det_B L)$. The arguments used for the proof are modeled upon Hermite's classical work on the minimum of integral quadratic forms.

Theorem 3.5. ([Gerstein: 1973a, b]) *Let A be a euclidean domain, with degree function ∂ satisfying* (3.1), (3.2). *Let (V, B) be an n-dimensional IPS over the quotient field K of A, and let L be an integral lattice in V. Assume that V is anisotropic (i.e. $0 \neq w \in V \Rightarrow B(w, w) \neq 0$).*(*) *Then*

$$0 \leqslant \min_B L \leqslant \frac{1}{n} \cdot \partial(\det_B L).$$

Proof. As before, let $\{v_i\}$ be an A-basis for L. Suppose the vector $v = \sum a_i v_i$ ($a_i \in A$) makes $\partial(B(v, v)) = \min_B L$. Then, clearly, $\gcd(a_1, \ldots, a_n) = 1$, i.e. $(a_1, \ldots, a_n)$ is unimodular. Since A is a Hermite ring (by (I.5.5) or (II.2.4)), $(a_1, \ldots, a_n)$ can be completed to an invertible matrix over A; in other words, v can be extended to an A-basis for L. Thus, we may assume that $v = v_1$, so $\partial(B(v_1, v_1)) = \min_B L$. To simplify the notations, let us write $a_{ij} = B(v_i, v_j) \in A$.

We shall now carry out an induction on n, the case $n = 1$ being trivial. For $n > 1$, let v_i' ($2 \leqslant i \leqslant n$) be the projection of v_i onto the orthogonal complement of v_1 in V; i.e. $v_i' = v_i - \frac{a_{i1}}{a_{11}} v_1$. Clearly, $\{v_1, v_2', \ldots, v_n'\}$ are K-linearly independent. Let

$$M = \bigoplus_{i \geqslant 2} A \cdot v_i', \quad \text{and} \quad L' = A \cdot v_1 \oplus M.$$

From the definition of v_i', we have $B(v_i', v_j') = b_{ij}/a_{11}$ for suitable $b_{ij} \in A$ ($i, j \geqslant 2$), so

(*)Clearly, we need only require this condition for $0 \neq w \in L$. Thus, "V anisotropic" is synonymous with "L anisotropic".

$$\det_B L' = \det \begin{pmatrix} a_{11} & 0 \\ 0 & \frac{1}{a_{11}} \cdot (b_{ij}) \end{pmatrix} = \frac{1}{a_{11}^{n-2}} \cdot \det(b_{ij}). \tag{3.6}$$

Let C be the restriction of $a_{11} \cdot B$ to $M \times M$. Then, w.r.t. C, M is an integral lattice in the anisotropic space $K \cdot M$, and (3.6) yields

$$\det_C M = a_{11}^{n-2} \det_B L' = a_{11}^{n-2} \det_B L.$$

(We have $\det_B L = \det_B L'$ since the matrix relating $\{v_1, \dots, v_n\}$ and $\{v_1, v_2', \dots, v_n'\}$ is upper triangular, with 1's on the diagonal.) By the inductive hypothesis, $\min_C M \leqslant \frac{1}{n-1} \partial(\det_C M)$, i.e.

$$\partial(a_{11}) + \min_B M \leqslant \frac{1}{n-1} [\, \partial(\det_B L) + (n-2) \cdot \partial(a_{11}) \,],$$
$$\min_B M \leqslant \frac{1}{n-1} [\, \partial(\det_B L) - \partial(a_{11}) \,].$$

We now claim that

$$\min_B L = \partial(a_{11}) \leqslant \min_B M. \tag{3.7}$$

If this is true, then $\partial(a_{11}) \leqslant \frac{1}{n-1} [\partial(\det_B L) - \partial(a_{11})]$ clearly yields the desired inequality $\partial(a_{11}) \leqslant \frac{1}{n} \partial(\det_B L)$. To prove our claim (3.7), consider a vector $w = \sum \lambda_i v_i$, $\lambda_i \in A$. Changing basis, we have

$$w = \left(\lambda_1 + \frac{a_{21}\lambda_2 + \cdots + a_{n1}\lambda_n}{a_{11}} \right) \cdot v_1 + \lambda_2 v_2' + \cdots + \lambda_n v_n'. \tag{3.8}$$

Now suppose $\lambda_2, \dots, \lambda_n \in A$ have been chosen so that the nonzero vector $w' = \lambda_2 v_2' + \cdots + \lambda_n v_n' \in M$ satisfies $\partial(B(w', w')) = \min_B M$. Next, we write

$$a_{21}\lambda_2 + \cdots + a_{n1}\lambda_n = a_{11} q + r,$$

where $q, r \in A$, and either $r = 0$ or $\partial(r) < \partial(a_{11})$. Then, choosing $\lambda_1 = -q$, the coefficient $\beta = r/a_{11}$ of v_1 in (3.8) is either 0 or satisfies $\partial(\beta) < 0$. Since $w \neq 0$, the anisotropicity hypothesis guarantees that

$$B(w, w) = \beta^2 a_{11} + B(w', w') \neq 0.$$

If $\beta = 0$, we have

$$\partial(a_{11}) \leqslant \partial(B(w, w)) = \partial(B(w', w')) = \min_B M,$$

as desired. Thus, we may suppose $\beta \neq 0$, $\partial(\beta) < 0$. Then, from (3.1) and (3.2), we get

$$\begin{aligned} \partial(a_{11}) \leqslant \partial(B(w, w)) &\leqslant \max\{\partial(\beta^2 a_{11}),\ \partial(B(w', w'))\} \\ &= \max\{\partial(a_{11}) + 2\,\partial(\beta),\ \min_B M\}. \end{aligned}$$

The latter maximum must be given by $\min_B M$, for if it was given by $\partial(a_{11}) + 2\partial(\beta)$, we would have been led to $0 \leqslant 2 \cdot \partial(\beta)$, a contradiction. This proves the outstanding claim (3.7). □

The arguments used above are variations of those of Hermite used in his classical work on the minimum of integral lattices over the rational integers, $\mathbb{Z}$. The latter is, of course, an euclidean domain, with degree function $\partial'(a) = |a|$ for $a \in \mathbb{Z}\backslash\{0\}$. This degree function, however, does not satisfy the axioms (3.1) and (3.2); rather, it satisfies $\partial'(ab) = \partial'(a)\,\partial'(b)$. If we exploit this property instead of (3.1), (3.2), and rework the above inductive procedure for an integral $\mathbb{Z}$-lattice L in (V, B) (= anisotropic IPS over $\mathbb{Q}$), the outcome will be the following upper bound for the minimum of L:

$$\min_B L = \min_{0 \neq v \in L} |B(v, v)| \leqslant \left(\frac{4}{3}\right)^{\frac{n-1}{2}} |\det_B L|^{\frac{1}{n}} .$$

This is, of course, the famous inequality of Hermite that has led to so many important results in the integral theory of quadratic forms. (See, e.g. [Newman: 1972].)

Coming back to Theorem 3.5, let us now record some of its nice corollaries.

Corollary 3.9. *Keep the notations and hypotheses in* (3.5). *If* $\partial(\det_B L) < n$, *then* L *represents some unit in* A.

Note that the integral lattice $(L, B \mid L \times L)$ will be an IPS over A iff $\det_B L \in \mathrm{U}(A)/\mathrm{U}(A)^2$, i.e. iff $\partial(\det_B L) = 0$. If this is the case, L is said to be an (integral) *unimodular lattice* in V.

Corollary 3.10. *Let* (A, ∂) *be as in* (3.5). *Then any anisotropic IPS, L, over* A *admits an orthogonal* A*-basis. (With respect to such a basis, the inner product structure on L will take on a diagonal form, with units of* A *down the diagonal.)*

Proof. Let (V, B) be the extension of L to the quotient field K of A. Then L is an (integral) unimodular lattice in V. By (3.9), there exists $v_1 \in L$ such that $B(v_1, v_1) \in \mathrm{U}(A)$. By (1.3), we have $L = A \cdot v_1 \perp L_1$, where $(L_1, B \mid L_1 \times L_1)$ is an (anisotropic) IPS over A, so the induction proceeds. □

We are now in a position to apply the above to $A = k[t]$, k a field. Note that $\partial =$ ordinary degree on $k[t]$ clearly satisfies the axioms (3.1) and (3.2). We may thus record the following consequence of (3.10).

Corollary 3.11. (Harder) *Let* k *be any field (of arbitrary characteristic). Then any anisotropic IPS, L, over* $k[t]$ *is extended from an IPS over* k.(*)

Proof. With respect to an orthogonal $k[t]$-basis on L, the inner product structure takes on the matrix diag $(f_1, \ldots, f_n)$, where $f_i \in \mathrm{U}(k[t]) = k\backslash\{0\}$. □

If we try to waive the hypothesis on *anisotropicity* in the above developments, the results (3.5) through (3.11) may no longer hold. For instance, let L be the free IPS

(*)This result can be extended to a much more general setting: we can take k to be a skew field, $k[t]$ a twisted polynomial ring with involution, and take L to be an anisotropic Hermitian space w.r.t. the involution. See [Djoković: 1975].

over $k[t]$ given by the symmetric matrix $\begin{pmatrix} t & 1 \\ 1 & 0 \end{pmatrix}$. If k has characteristic 2, L *cannot* have an orthogonal $k[t]$-basis, for otherwise L would be extended from k, which it is not, by (1.9). Thus, the results (3.5) through (3.11) fail to apply to the lattice L. Of course L here is *not* anisotropic, since $B(e_2, e_2) = 0$.

The above counterexample is peculiar to characteristic 2. Indeed, if char $k \neq 2$, we can check easily that

$$e_1 - \frac{t-1}{2} e_2 \quad \text{and} \quad e_1 - \frac{t+1}{2} e_2 \tag{3.12}$$

provide an orthogonal $k[t]$-basis for L $\left(\text{and consequently } \begin{pmatrix} t & 1 \\ 1 & 0 \end{pmatrix} \text{ is the extension of } \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right)$. This is not surprising in view of the following result, which says that we can bypass the anisotropicity assumption completely in case char $k \neq 2$.

Theorem 3.13. (Harder) *Let k be a field of characteristic not 2. Then any IPS, (L, B), over $k[t]$ has an orthogonal $k[t]$-basis (and is therefore extended from an IPS over k).*

Proof. In view of (3.11), we may assume that L admits an 'isotropic' vector $v \neq 0$ (such that $B(v, v) = 0$). Since $A = k[t]$ is a PID, we may "shrink" v suitably to assume that v can be extended to an A-basis for L. Then there exists a linear functional $L \to A$ that sends v to 1. By the nonsingularity of (L, B), we can find $u \in L$ with $B(u, v) = 1$. Let $L_1 = A\,u \oplus Av$. The restriction $B|L_1 \times L_1$ is nonsingular since it has matrix $\begin{pmatrix} s & 1 \\ 1 & 0 \end{pmatrix}$, where $s = B(u, u)$. By (1.3), we have an orthogonal decomposition $L = L_1 \perp L_2$ (where $L_2 = L_1^{\perp}$). Invoking an inductive hypothesis, we may assume that L_2 has an orthogonal A-basis, so it is sufficient to show that L_1 also has an orthogonal A-basis. Since 2 is invertible in A, this can be checked exactly as in (3.12): namely, the two vectors $u - \frac{s-1}{2}\, v$ and $u - \frac{s+1}{2}\, v$ provide an orthogonal A-basis for L_1. □

Remark 3.14. The above argument does show something even for char $k = 2$; namely, no matter what the characteristic is, an *arbitrary* IPS, (L, B), over $k[t]$ will always decompose into an orthogonal sum $L_0 \perp L_1 \perp \cdots \perp L_m$, where L_0 is extended from k, and all other L_i's have rank 2, with matrices of the type $S_i = \begin{pmatrix} s_i & 1 \\ 1 & 0 \end{pmatrix}$. In other words, the matrix of B will be congruent over $k[t]$ to

$$\begin{pmatrix} \varepsilon_1 & & & & & \\ & \ddots & & & 0 & \\ & & \varepsilon_r & & & \\ & & & S_1 & & \\ & 0 & & & \ddots & \\ & & & & & S_m \end{pmatrix},$$

where $\varepsilon_1, \ldots, \varepsilon_r \in k \backslash \{0\}$. If char $k \neq 2$, we may arrange that $m = 0$, by (3.13).

§4. Parimala's Counterexamples

In [Parimala: 1976b, 1978], a counter-example to the analog of Serre's Problem for inner product spaces was constructed. Her counterexample is a 4×4 invertible symmetric matrix S over $\mathbb{R}[x, y]$ that is not congruent to $S(0, 0)$ — hence the inner product structure induced by S on $\mathbb{R}[x, y]^4$ cannot be extended from $\mathbb{R}$. Parimala's 4×4 matrix over $\mathbb{R}[x, y]$ turns out to be a variation of a certain 2×2 matrix over $\mathbb{C}[x, y]$, so it is best for us to begin by describing a basic connection between the matrix groups $\mathrm{GL}_4(\mathbb{R}[x, y])$ and $\mathrm{GL}_2(\mathbb{C}[x, y])$.

More generally, let A be any commutative $\mathbb{R}$-algebra, and let $B = \mathbb{C} \otimes_{\mathbb{R}} A = A \oplus i\,A$. Any matrix $N \in \mathbb{M}_n(B)$ has a unique form $N = \alpha + i\beta$ where $\alpha, \beta \in \mathbb{M}_n(A)$, so we can associate to N the matrix $\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \in \mathbb{M}_{2n}(A)$. It is routine to check that the rule: $\alpha + i\beta \mapsto \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$ yields an injective ring homomorphism $\varepsilon : \mathbb{M}_n(B) \to \mathbb{M}_{2n}(A)$; in particular, ε induces an injective group homomorphism $\mathrm{GL}_n(B) \to \mathrm{GL}_{2n}(A)$.

Let "bar" denote complex conjugation on $\mathbb{M}_n(B)$. A matrix $N \in \mathbb{M}_n(B)$ is said to be *Hermitian* if $N = \overline{N}^{\,t}$. If $N_1, N_2 \in \mathbb{M}_n(B)$ are Hermitian, we shall say that they are *Hermitian congruent* if there exists $U \in \mathrm{GL}_n(B)$ such that $N_2 = U N_1 \overline{U}^{\,t}$. This is clearly an equivalence relation on Hermitian matrices.

Lemma 4.1. (1) *For any* $N \in \mathbb{M}_n(B)$, $\varepsilon\big(\overline{N}^{\,t}\big) = \varepsilon(N)^t$.

(2) N *is Hermitian iff* $\varepsilon(N)$ *is symmetric.*

(3) *Let* N_1, N_2 *be Hermitian. If* N_1, N_2 *are Hermitian congruent, then* $\varepsilon(N_1)$, $\varepsilon(N_2)$ *are congruent.*

(4) *For any* $N \in \mathbb{M}_n(B)$, *we have* $\det \varepsilon(N) = \det N \cdot \overline{\det N}$.

Proof. (1) If $N = \alpha + i\beta$, then $\overline{N}^{\,t} = \alpha^t - i\beta^t$, so

$$\varepsilon\big(\overline{N}^{\,t}\big) = \begin{pmatrix} \alpha^t & -\beta^t \\ \beta^t & \alpha^t \end{pmatrix} = \varepsilon(N)^t.$$

(2) and (3) follow immediately from (1). For (4), we use the fact that the determinant of a matrix is unchanged under block elementary transformations. Thus,

$$\begin{aligned} \det\,\varepsilon(N) &= \det \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} = \det \begin{pmatrix} \alpha + i\beta & \beta \\ -\beta + i\alpha & \alpha \end{pmatrix} \\ &= \det \begin{pmatrix} \alpha + i\beta & \beta \\ 0 & \alpha - i\beta \end{pmatrix} \\ &= \det\,(\alpha + i\beta)\det\,(\alpha - i\beta) \\ &= \det N \cdot \overline{\det N}. \quad \square \end{aligned}$$

In the following, we shall specialize to $A = \mathbb{R}\,[x, y]$, $B = \mathbb{C} \otimes_{\mathbb{R}} A = \mathbb{C}[x, y]$, and $n = 2$. A certain sequence of Hermitian matrices T_n $(n \geqslant 0)$ will be constructed in $\mathrm{GL}_2(\mathbb{C}[x, y])$. Upon application of the embedding ε, we shall arrive at a sequence of symmetric matrices $S_n = \varepsilon(T_n) \in \mathrm{GL}_4(\mathbb{R}[x, y])$. It will turn out that the S_n $(n \geqslant 1)$ are mutually non-congruent, and none of them is congruent to a (symmetric) matrix in $\mathrm{GL}_4(\mathbb{R})$. Thus, one obtains an infinite family of essentially different inner product structures on $\mathbb{R}[x, y]^4$, none of which can be extended from inner product structures on $\mathbb{R}^4$.

The construction of the Hermitian matrices T_n stems from [Parimala-Sridharan: 1975]; it depends on a certain classification of the rank 1 left projective modules over $H[x, y]$, where H denotes the division ring of Hamilton's real quaternions. Here, as in (II.3), the rank of a module $P \in \mathfrak{P}(H[x, y])$ is taken to be $\dim_H P/(x, y)P$. In [Parimala-Sridharan: 1975], it was shown that the rank 1 left projectives over $H[x, y]$ are in 1-1 correspondence with certain classes of matrices in $\mathrm{GL}_2(\mathbb{C}[x, y])$ under a suitable equivalence relation. The matrices T_n we want will arise, under this correspondence, from a certain sequence of rank 1 projective modules $P_n \in \mathfrak{P}(H[x, y])$.

The construction of the P_n's is directly inspired by [Ojanguren-Sridharan: 1971] (see (II.3)). In fact, let $f, g \in \mathbb{R}[x, y]$, and let $\phi : H[x, y]^2 \to H[x, y]$ be defined by

$$\phi(e_1) = f + i, \quad \phi(e_2) = g + j,$$

where $\{1, i, j, ij\}$ is the usual $\mathbb{R}$-basis for the quaternions. The map ϕ is onto, since

$$\phi\begin{pmatrix} g + j \\ -f - i \end{pmatrix} = (g + j)(f + i) - (f + i)(g + j) = ji - ij = -2ij$$

is invertible in H (and hence in $H[x, y]$). The left module $P_{f,g} = \ker \phi$ will therefore be stably free of rank 1 over $H[x, y]$, and, by the 1-1 correspondence of Parimala-Sridharan mentioned above, $P_{f,g}$ will give rise to a certain class of matrices in $\mathrm{GL}_2(\mathbb{C}[x, y])$. Using explicit calculations, Parimala and Sridharan have determined that

$$P_{f,g} \longleftrightarrow \text{"class" of } T_{f,g}, \text{ where}$$

$$T_{f.g} = \begin{pmatrix} 4 + g^2(1 + f^2) & fg(1 + g^2) + ig(1 + f^2g^2) \\ fg(1 + g^2) - ig(1 + f^2g^2) & 1 + f^2g^4 \end{pmatrix}.$$

We shall not present the detailed calculations that led to the matrices above. Instead, we shall take these matrices for granted, and proceed with the construction of Parimala's counterexamples.

Notice that $T_{f,g}$ is a Hermitian matrix with determinant 4 (by computation). We can specialize further by setting $f = x$, $g = y^n$ $(n \geqslant 0)$, and get

$$T_n = T_{x,y^n} = \begin{pmatrix} a_n & b_n + ic_n \\ b_n - ic_n & d_n \end{pmatrix} \text{ where}$$

(4.2)
$$\begin{cases} a_n = 4 + y^{2n}(1 + x^2), & b_n = xy^n(1 + y^{2n}), \\ c_n = y^n(1 + x^2y^{2n}), & d_n = 1 + x^2y^{4n}. \end{cases}$$

If we apply the earlier map ε: $\mathrm{GL}_2\big(\mathbb{C}[x, y]\big) \to \mathrm{GL}_4\big(\mathbb{R}[x, y]\big)$, we will arrive at Parimala's symmetric matrices:

$$\begin{aligned} S_n &= \varepsilon(T_n) = \varepsilon\left[\begin{pmatrix} a_n & b_n \\ b_n & d_n \end{pmatrix} + i\begin{pmatrix} 0 & c_n \\ -c_n & 0 \end{pmatrix}\right] \\ &= \begin{pmatrix} a_n & b_n & 0 & c_n \\ b_n & d_n & -c_n & 0 \\ 0 & -c_n & a_n & b_n \\ c_n & 0 & b_n & d_n \end{pmatrix} \in \mathrm{GL}_4\big(\mathbb{R}[x, y]\big). \end{aligned} \tag{4.3}$$

By (4.1)(4), this symmetric matrix has determinant $= 4^2 = 16$ so in fact it lies in $\mathrm{GL}_4\big(\mathbb{Z}[\frac{1}{2}][x, y]\big)$.

Theorem 4.4. (Parimala) *Let S_n be as defined above.*

(1) *The matrix S_0 is congruent over $\mathbb{R}[x, y]$ to the identity matrix.*

(2) *For $m \neq n$, S_m is not congruent to S_n.*

(3) *For $n \geqslant 1$, the inner product structure induced by S_n on $\mathbb{R}[x, y]^4$ is not extended from an inner product structure on $\mathbb{R}^4$.*

Proof. Note that $S_n(0, 0) = \operatorname{diag}(4, 1, 4, 1)$ is congruent to the identity. If the space $\big(\mathbb{R}[x, y]^4,\ S_n\big)$ is extended from $\mathbb{R}^4$, it must be the extension of $\big(\mathbb{R}^4,\ S_n(0, 0)\big)$, and hence congruent to the identity. Thus, (3) will follow if we can prove (1) and (2).

For (1), note that

$$\begin{aligned} T_0 &= \begin{pmatrix} 4 + (1 + x^2) & 2x + i(1 + x^2) \\ 2x - i(1 + x^2) & 1 + x^2 \end{pmatrix} \\ &= \begin{pmatrix} 2 + ix & i \\ x & 1 \end{pmatrix} \begin{pmatrix} 2 - ix & x \\ -i & 1 \end{pmatrix}, \end{aligned}$$

so T_0 is Hermitian congruent to I_2 over $\mathbb{C}[x, y]$. Applying (4.1)(3), we conclude that $\varepsilon(T_0) = S_0$ is congruent to I_4 over $\mathbb{R}[x, y]$. This being so, we may as well *redefine* S_0 to be I_4, to facilitate the notations.

To prove (2), *assume* that there exists $U = (u_{ij}) \in \mathrm{GL}_4\big(\mathbb{R}[x, y]\big)$ such that $S_m = U\, S_n\, U^t$, where $0 \leqslant m < n$. Equating the last diagonal entries of the two sides, we get an equation

$$\begin{aligned} d_m = {} & (u_{41}a_n + u_{42}b_n + u_{44}c_n)u_{41} + (u_{41}b_n + u_{42}d_n - u_{43}c_n)u_{42} \\ & + (-u_{42}c_n + u_{43}a_n + u_{44}b_n)u_{43} + (u_{41}c_n + u_{43}b_n + u_{44}d_n)u_{44}. \end{aligned}$$

Multiplying this by d_n and noting that $4 = \det T_n = a_n d_n - (b_n^2 + c_n^2)$, we get

$$\begin{aligned} d_m d_n &= (4 + b_n^2 + c_n^2)(u_{41}^2 + u_{43}^2) + d_n^2(u_{42}^2 + u_{44}^2) \\ &\qquad + 2b_n d_n(u_{41}u_{42} + u_{43}u_{44}) + 2c_n d_n(u_{41}u_{44} - u_{42}u_{43}) \\ &= (d_n u_{42} + b_n u_{41} - c_n u_{43})^2 + (d_n u_{44} + b_n u_{43} + c_n u_{41})^2 \\ &\qquad + 4(u_{41}^2 + u_{43}^2). \end{aligned}$$

Let us now invoke the following technical lemma:

Lemma 4.5. *Suppose* $0 \leqslant m < n$, *and* $n \neq 2m$. *Then there do not exist* $t, u, v, w \in \mathbb{R}[x, y]$ *such that*

$$d_m d_n = (d_n t + b_n u - c_n v)^2 + (d_n w + c_n u + b_n v)^2 + 4(u^2 + v^2). \tag{4.6}$$

If we can prove this lemma, we will have shown that S_m is not congruent to S_n whenever $0 \leqslant m < n$ and $n \neq 2m$. This still leaves the case $n = 2m$ uncovered. It turns out, fortunately, that we can deduce this case for free. Indeed, assume

$$S_m = U\, S_{2m}\, U^t \quad (m > 0,\ \ U \in \mathrm{GL}_4\big(\mathbb{R}[x, y]\big).$$

If we replace y by y^2 in this equation, it becomes $S_{2m} = U(x, y^2) S_{4m} U(x, y^2)^t$, which implies that S_{2m} is congruent to S_{4m}. This is impossible by what we said at the beginning of this paragraph.

It is thus sufficient to grind out the

Proof of (4.5). Assume, on the contrary, that t, u, v, w exist in $\mathbb{R}[x, y]$, satisfying (4.6). Note that the LHS has x-degree

$$\deg_x d_m d_n = \deg_x d_m(1 + x^2 y^{4n}) = \begin{cases} 2 \text{ if } m = 0, \\ 4 \text{ if } m \neq 0. \end{cases}$$

(Recall that we have redefined S_0 to be I_4.) Thus,

$$u,\ v,\ \text{and } d_n t + b_n u - c_n v,\ d_n w + c_n u + b_n v$$

must have x-degree $\leqslant 2$ (and in fact $\leqslant 1$ if $m = 0$).(*) Say:

$$\begin{cases} u = u_0 + u_1 x + u_2 x^2, \\ v = v_0 + v_1 x + v_2 x^2, \\ d_n t + b_n u - c_n v = p_0 + p_1 x + p_2 x^2, \\ d_n w + c_n u + b_n v = q_0 + q_1 x + q_2 x^2, \end{cases} \tag{4.7}$$

where $u_i, v_i, p_i, q_i \in \mathbb{R}[y]$, with the further condition that $u_2 = v_2 = p_2 = q_2 = 0$ if $m = 0$. Setting $x = 0$ in (4.6), we obtain

$$1 = p_0^2 + q_0^2 + 4u_0^2 + 4v_0^2,$$

so $p_0, q_0, u_0, v_0 \in \mathbb{R}$. If $m \neq 0$, comparing the coefficients of x^4 on both sides of (4.6), we get $p_2^2 + q_2^2 + 4u_2^2 + 4v_2^2 = y^{4(m+n)}$. This implies that

$$p_2 = p_2' y^{2(m+n)}, \quad q_2 = q_2' y^{2(m+n)},$$
$$u_2 = u_2' y^{2(m+n)}, \quad v_2 = v_2' y^{2(m+n)},$$

(*) Here, we use the fact that $\sum g_i^2 = 0$ $\big(g_i \in \mathbb{R}[y]\big) \Rightarrow$ all $g_i = 0$.

where $p_2', q_2', u_2', v_2' \in \mathbb{R}$, with the convention that these are zero if $m=0$. Substituting the first two equations of (4.7) into the last two, we have

(4.8) $d_n t + b_n(u_0+u_1x+u_2x^2) - c_n(v_0+v_1x+v_2x^2) = p_0+p_1x+p_2x^2,$

(4.9) $d_n w + c_n(u_0+u_1x+u_2x^2) + b_n(v_0+v_1x+v_2x^2) = q_0+q_1x+q_2x^2.$

Since $\deg_x b_n = 1$ and $\deg_x c_n = \deg_x d_n = 2$, these equations imply that $\deg_x t \leqslant 2$, and $\deg_x w \leqslant 2$; say

$$t = t_0 + t_1x + t_2x^2, \quad \text{and} \quad w = w_0 + w_1x + w_2x^2,$$

where $t_i, w_i \in \mathbb{R}[y]$. If we compare, on the two sides of (4.8), the coefficients of x^0, x^2 and x^4, we obtain the following three equations:

$$p_0 = t_0 - y^n v_0 \tag{4.10}$$

$$p_2' \, y^{2(m+n)} = p_2 = t_2 + t_0 y^{4n} + y^n(1+y^{2n})u_1 - v_2' y^{2m+3n} - v_0 y^{3n}, \tag{4.11}$$

$$0 = t_2 y^{4n} - v_2' y^{2m+5n} \ \left(\text{i.e. } t_2 = v_2' y^{2m+n}\right). \tag{4.12}$$

Using (4.10), (4.12) to eliminate t_0 and t_2, we can rewrite (4.11) in the form

$$\begin{aligned} p_2' \, y^{2(m+n)} = v_2' \, y^{2m+n} &+ (p_0 + y^n v_0) y^{4n} \\ &+ y^n(1+y^{2n})u_1 - v_2' \, y^{2m+3n} - v_0 y^{3n}. \end{aligned}$$

Reading this equation in $\mathbb{R}[y]/(1+y^{2n})$, we have

$$-p_2' \, y^{2m} \equiv v_2' \, y^{2m+n} + p_0 + y^n v_0 + v_2' \, y^{2m+n} + v_0 y^n \ \left(\text{mod } (1+y^{2n})\right),$$

or

$$p_0 + p_2' y^{2m} + 2v_0 y^n + 2v_2' \, y^{2m+n} \equiv 0 \quad \left(\text{mod } (1+y^{2n})\right). \tag{$*$}$$

where, as we recall, $p_0, p_2', v_0, v_2' \in \mathbb{R}$.

Case 1. Let $m \neq 0$. Since $0 < m < n \neq 2m$, the integers $0, 2m, n, 2m+n$ are distinct modulo $2n$, so the images of $1, y^{2m}, y^n$ and y^{2m+n} in $\mathbb{R}[y]/(1+y^{2n})$ are $\mathbb{R}$-linearly independent. Consequently, the congruence $(*)$ implies that $p_0 = p_2' = v_0 = v_2' = 0$.

Case 2. Let $m = 0$. Here, we have already $p_2' = v_2' = 0$, and $(*)$ clearly implies $p_0 = v_0 = 0$.

Thus, in either case, $p_0 = p_2' = v_0 = v_2' = 0$. A similar argument with respect to (4.9) will yield $q_0 = q_2' = u_0 = u_2' = 0$. But these conclusions are clearly in contradiction to $p_0^2 + q_0^2 + 4u_0^2 + 4v_0^2 = 1$. $\square$

If all we want is to produce an IPS over $\mathbb{R}[x, y]$ that is not extended from $\mathbb{R}$, there is a quick way to do so, by a remark of Knus and Parimala (see (3.9) in the paper [Knus-Parimala: 1980] listed in the references on Chapter VIII). Indeed, let $z = xy$. For $n = 1$, the four polynomials in (4.2) become

$$a_1 = 4 + y^2 + z^2, \quad b_1 = z(1 + y^2), \quad c_1 = y(1 + z^2), \quad \text{and} \quad d_1 = 1 + y^2z^2.$$

Thus, viewed as a symmetric matrix over the polynomial ring $B := \mathbb{R}[y, z]$, S_1 defines a quadratic form

$$q(\alpha, \beta, \gamma, \delta) = a_1(\alpha^2 + \gamma^2) + d_1(\beta^2 + \delta^2) + 2b_1(\alpha\beta + \gamma\delta) + 2c_1(\alpha\delta - \beta\gamma)$$

over B. Let us prove the following result, which, in particular, implies (along with (1.14)) that the polynomial ring $B = \mathbb{R}[y, z]$ has nonextended inner product spaces of *any* rank $\geqslant 3$. (Of course, the same conclusion will then follow for $\mathbb{R}[x_1, \ldots, x_n]$ for any $n \geqslant 2$.)

Proposition 4.13. *For any* $r \geqslant 0$, $T_r := \begin{pmatrix} S_1 & 0 \\ 0 & I_r \end{pmatrix}$, *viewed as a symmetric matrix over* $B = \mathbb{R}[y, z]$, *defines an IPS (of rank* $4 + r$*) that is not extended from* $\mathbb{R}$.

Proof. If this IPS is extended from $\mathbb{R}$, T_r would be congruent over B to I_{4+r} (since S_1 reduces to $\operatorname{diag}(4, 1, 4, 1)$ when $y = z = 0$). In particular, any "value" taken by the quadratic form $q(\alpha, \beta, \gamma, \delta)$ over B would be a sum of $4 + r$ squares in B. A quick calculation yields the following special value

$$(\dagger) \qquad f(y, z) := q(-1, z, 0, y) = 4 + y^4z^2 + y^2z^4 - 4y^2z^2.$$

We'll now get a contradiction by checking that $f(y, z)$ is *not* a sum of (any number of) squares in B. This is done by applying the following "term-inspection method" in [Choi-Lam: 1976, 1977] (listed in the references on Chapter VIII). Assume instead, that $f(y, z) = \sum_i f_i^2$, where $f_i \in B$. Each f_i is an $\mathbb{R}$-linear combination of the monomials

$$1,\ y,\ z,\ y^2,\ yz,\ z^2,\ y^3,\ y^2z,\ yz^2,\ \text{and}\ z^3.$$

Clearly, the terms y^3, z^3 cannot occur, since $f(y, z)$ does not have the terms y^6, z^6. It follows that the terms y^2, z^2 cannot occur, and consequently, the terms y, z cannot occur either. Thus, each f_i involves only the terms 1, yz, y^2z, and yz^2. But then the coefficient of y^2z^2 in $f(y, z)$ must be *non-negative* (being a sum of squares in $\mathbb{R}$). This contradicts $(\dagger)$! ◻

In retrospect, the method of proof for (4.4)–(4.5) was based essentially on the same "reality principle" as in the sums-of-squares argument above. Parimala was trying to prove more, and hence her proofs were longer. This point aside, there was a remarkable synergy between Parimala's paper and those of Choi and Lam, though this was not immediately realized when these papers were written.

Another interesting observation we can make on the proof of (4.4) is as follows. If we assume Parimala's result (1.10) (that any rank 2 IPS over $\mathbb{R}[t_1, \dots, t_d]$ is extended from $\mathbb{R}$), we can, in fact, strengthen the conclusion of (4.4). First a definition:

Definition. An IPS (P, B) over a commutative ring R is said to be *decomposable* if $(P, B) \cong (P_1, B_1) \perp (P_2, B_2)$ for suitable nonzero IPS's (P_i, B_i). Otherwise, (P, B) (with $P \neq 0$) is said to be *indecomposable*.

Supplement to (4.4). *The IPS's* $\big(\mathbb{R}[x, y]^4, S_n\big)$ *are indecomposable for all* $n \geqslant 1$.

Proof. Assume that $\big(\mathbb{R}[x, y]^4, S_n\big) \cong (P_1, B_1) \perp (P_2, B_2)$, where $P_1 \neq 0 \neq P_2$. If rank $P_1 =$ rank $P_2 = 2$, (P_1, B_1) and (P_2, B_2) will be extended from $\mathbb{R}$ by (1.10), and hence $(\mathbb{R}[x, y]^4, S_n)$ is also extended from $\mathbb{R}$, contradicting (4.4). Thus, we may assume that rank $P_1 = 3$, rank $P_2 = 1$. Now P_2 is free by (II.1.2), and P_1 is free by the Quillen-Suslin Theorem. Let A_1, A_2 be invertible symmetric matrices associated to (P_1, B_1), (P_2, B_2). Then, $A_2 = (d)$ where $d \in \mathbb{R} \backslash \{0\}$. The isometry

$$\big(\mathbb{R}[x, y]^4,\ S_n\big) \cong (P_1, B_1) \perp (P_2, B_2)$$

implies that we have a congruence relation

$$U \cdot S_n \cdot U^t = \begin{pmatrix} A_1 & 0 \\ 0 & d \end{pmatrix}, \quad \text{where } U \in \mathrm{GL}_4\big(\mathbb{R}[x, y]\big).$$

Proceeding as in the proof of (4.4), we get

$$\begin{aligned} d\, d_n &= d\big(1 + x^2 y^{4n}\big) \\ &= \big(d_n u_{42} + b_n u_{41} - c_n u_{43}\big)^2 + \big(d_n u_{44} + b_n u_{43} + c_n u_{41}\big)^2 + 4\big(u_{41}^2 + u_{43}^2\big). \end{aligned}$$

This implies that $d > 0$, and, dividing this equation by $\sqrt{d}$, we get a contradiction to (4.5) if $n = 1$ or $n \geqslant 3$. Thus, S_1 and S_n $(n \geqslant 3)$ are indecomposable; it follows that S_2 is *also* indecomposable, since $S_2(x, y^2) = S_4(x, y)$. □

To close this section, let us make some further observations about the matrices $S_n = \varepsilon(T_n)$ in relation to the Local Horrocks' Theorem. The discussion will be a bit haphazard since it will involve the notion of inner product spaces over schemes (both affine and non-affine) which we have not formally defined in the text. Anyway, let $R = \mathbb{R}[x]$, and consider

$$\mathbf{A}_R^1 = \operatorname{Spec} R[y] = \operatorname{Spec} \mathbb{R}[x, y] \subseteq \mathbb{P}_R^1.$$

We claim that *the IPS structure given by* S_n *(n fixed) over the affine scheme* $\mathbf{A}_R^1$ *can be extended to an IPS structure over* $\mathbb{P}_R^1$. The required extension to $\mathbb{P}_R^1$ may be constructed, as usual, by a "gluing process" of inner product structures. The key observation is the following.

Lemma 4.14. (Parimala) *There exists a monic polynomial* $f = f_n(y) \in R[y]$ *such that* S_n *is congruent to the identity matrix* I_4 *over* $R[y]_f$.

If we can prove this lemma, then, just as in the proof of (IV.2.3) (the "if" part), we can "glue" the inner product structure given by S_n over $\operatorname{Spec} R[y]$ with the trivial rank 4 inner product structure given by I_4 over $\operatorname{Spec} R[y^{-1}]\backslash V(f)$, to obtain an inner product structure over the projective line $\mathbb{P}^1_R$. (By (IV.2.4), the set $V(f)$ is closed in $\mathbb{P}^1_R$ if $f \in R[y]$ is monic.) This will establish the claim we made in the paragraph preceding (4.14).

Proof of (4.14). According to Parimala, if we set

$$U_n = \begin{pmatrix} xy^n - iy^n & -(2+y^{2n}) - ixy^{2n} \\ 1 - ixy^{2n} & -xy^{3n} + iy^n \end{pmatrix}, \tag{4.15}$$

then $U_n \overline{U}{}^t_n = (1+y^{2n}) \cdot T_n$ where T_n is defined via (4.2). Applying the embedding ε, we get $V_n V_n^t = (1+y^{2n}) \cdot S_n$ for $V_n = \varepsilon(U_n)$. Thus, if we set $f = 1 + y^{2n}$ and work in $R[y]_f$, S_n will be congruent to $f \cdot I_4$. But

$$f \cdot I_2 = \begin{pmatrix} 1 & y^n \\ -y^n & 1 \end{pmatrix} \begin{pmatrix} 1 & -y^n \\ y^n & 1 \end{pmatrix},$$

so $f \cdot I_4$ (and hence S_n) is congruent to I_4 over $R[y]_f$. $\square$

(I am not sure what could have led one to the form of U_n above. The explicit construction given in (4.15) is apparently based on secret calculations of Parimala.)

Recall that S_n ($n \geqslant 1$) is not extended from $\mathbb{R}$ by (4.4). By Harder's Theorem, we can say that S_n is not even extended from $R = \mathbb{R}[x]$. In affine scheme terminology, the inner product structure given by S_n on $\mathbf{A}^1_R$ is not extended from any inner product structure on $\operatorname{Spec} R$. Nevertheless, we have shown above that the structure S_n on $\mathbf{A}^1_R$ can be extended to an inner product structure on $\mathbb{P}^1_R$. This indicates that *the Affine Horrocks' Theorem fails to have an analogue in the framework of inner product structures.*

If we allow ourselves to use the patching theorem of Bass-Connell-Wright (1.16), we can further improve the above into a counterexample for the inner product analogue of *local* Horrocks. In fact, since $S_n \in \mathrm{GL}_n\big(R[y]\big)$ is not extended from R, (1.16) implies that $S_n \in \mathrm{GL}_n\big(R_\mathfrak{p}[y]\big)$ is not extended from $R_\mathfrak{p}$ for at least one prime ideal $\mathfrak{p} \in \operatorname{Spec} R$. By functoriality, the inner product structure given by S_n over $\mathbf{A}^1_{R_\mathfrak{p}}$ still extends to one over $\mathbb{P}^1_{R_\mathfrak{p}}$. Upon replacing R by $R_\mathfrak{p}$, we arrive thus at a "local" counterexample. Since $R = \mathbb{R}[x]$ is a PID, the counterexample base ring $R_\mathfrak{p}$ is even a discrete valuation ring.

Curiously enough, Parimala has shown that there is actually a *unique* choice for the prime ideal $\mathfrak{p} \subset R = \mathbb{R}[x]$ for the above construction, namely, $\mathfrak{p} = (1+x^2)$. This is clear from the following lemma, which essentially precludes all other primes from our consideration.

Lemma 4.16. *For any prime* $\mathfrak{p}' \subset R = \mathbb{R}[x]$ *not containing* $1+x^2$, S_n *is congruent to* I_4 *over* $R_{\mathfrak{p}'}[y]$.

Proof. We use here a construction similar in spirit to that of (4.15). For

$$U_n' = \begin{pmatrix} 2+2ix & -2xy^n + iy^n(1-x^2) \\ 2xy^n & (1-x^2y^{2n}) + ix(y^{2n}-1) \end{pmatrix},$$

we can check that $U_n' \overline{U_n'}^t = (1+x^2) \cdot T_n$, so application of the embedding ε shows that $V_n' V_n'^t = (1+x^2) \cdot S_n$, where $V_n' = \varepsilon(U_n')$. If we throw in an inverse for $1+x^2$, we can show exactly as in the proof of (4.14) that S_n will become congruent to the identity I_4. (Again, it does not seem easy to give a satisfactory account for the construction of the matrices U_n'.) □

§5. Symplectic Spaces and Self-Duality

In this section, we offer an introductory treatment on symplectic spaces, and give some applications of the theory of such spaces to the study of projective and stably free nodules. In particular, we'll (finally) be able to prove the outstanding result, stated in III.6.8, that stably free modules of rank 2 (over any commutative ring) are self-dual. The first part of our exposition here follows closely [Bass: 1969]; the rest of the section is based on [Krusemeyer: 1975, 1976], and suggestions of R. G. Swan.

Definition 5.0. We say that (P, B) is a *symplectic space* (or a *symplectic form*) over a commutative ring R if $P \in \mathfrak{P}(R)$, and $B : P \times P \to R$ is an *alternating* bilinear form satisfying the nonsingularity condition (1.2).(*)

Here, "alternating" means that $B(p, p) = 0$ for every $p \in P$. From this, we get

$$0 = B(p+q,\ p+q) = B(p, q) + B(q, p) \quad (p, q \in P),$$

so in particular, the bilinear form B must be skew-symmetric. If (P, B) is a *free* symplectic space, say with $P = R^n$, the invertible matrix associated to B with respect to any basis on P is then an *alternating matrix*; that is, a skew-symmetric matrix with zeros on the diagonal. Conversely, any alternating matrix in $\mathrm{GL}_n(R)$ defines a symplectic space structure on the free R-module R^n.

The notions of isometry, orthogonal sums (resp. orthogonal summands) and scalar extensions for symplectic spaces are completely analogous to those for inner product spaces, so we do not repeat them here. The formation of a *hyperbolic symplectic form*, however, needs a slight modification: $\mathbb{H}(P) = P \oplus P^*$ (for $P \in \mathfrak{P}(R)$) as a *symplectic space* shall have the form defined by

$$(5.1) \qquad B^P((p, f),\ (p', f')) = f'(p) - f(p') \quad (p,\ p' \in P;\ \ f, f' \in P^*).$$

(*) Since this condition implies the existence of an R-isomorphism from P to P^*, it follows (just as in the IPS case) that any $P \in \mathfrak{P}(R)$ supporting a symplectic structure (in the sense of (5.0)) is always self-dual.

The sign change we have put in now makes the form B^P *alternating*. In particular, the *symplectic hyperbolic plane* $\mathbb{H} := \mathbb{H}(R)$ (supported by the module $R \oplus R^* \cong R^2$) has the pairing

$$((r, s),\ (r', s')) \longmapsto s'r - sr' \quad (r, r', s, s' \in R),$$

and its corresponding matrix (in the natural basis $\{e_1, e_2\}$) is $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

In working with symplectic spaces, the first thing we have to realize is the proposition below, which is indeed special to the symplectic category — with no analogue in the category of inner product spaces. This result was actually already implicit in III.5.12, except that it was disguised there as a fact on determinants.

Proposition 5.2. *Let* (P, B) *be a symplectic space over a nonzero commutative ring* R*, where* $P \in \mathfrak{P}(R)$ *has constant rank* n*. Then* n *must be even.*

Proof. After a localization, we may assume that R is a local ring. Then $P \cong R^n$. Fixing a basis $\{e_1, \ldots, e_n\}$ on P, the matrix $\mathcal{B}$ associated with B is alternating, with $\det(\mathcal{B}) \in \mathrm{U}(R)$. If n is odd, we have $\det(\mathcal{B}) = 0$ by III.5.12: this is a contradiction. Thus, n must be even. □

We'll next state a couple of lemmas on symplectic spaces. The first one is the analogue of (1.3) for symplectic spaces. The earlier proof for (1.3) carries over without any change.

Lemma 5.3. *Let* (P, B) *be a symplectic space, and let* M *be an* R*-submodule of* P *such that* $B \mid M \times M$ *is nonsingular. Let* $N = M^{\perp} = \{p \in P : B(p, M) = 0\}$*. Then* $(M,\ B \mid M \times M)$*,* $(N,\ B \mid N \times N)$ *are symplectic spaces, and* $P = M \perp N$*.*

Lemma 5.4. *Let* (P, B) *be a symplectic space, and let* L *be a direct summand of* P*. Then* P *has a submodule* L_1 *such that* $P = L^{\perp} \oplus L_1$ *and* $B : L_1 \times L \to R$ *induces an isomorphism* $L_1 \to L^*$*.*

Proof. Writing $P = L \oplus X$, we use the identification $P^* = L^* \oplus X^*$. The map $p \mapsto B(p,\ -)$ gives an isomorphism $\varphi : P \to P^*$, which we write as

$$\varphi : L \oplus X \longrightarrow L^* \oplus X^*.$$

We see easily that $L^{\perp} = \varphi^{-1}(X^*)$, so we are done by choosing L_1 to be the submodule $\varphi^{-1}(L^*) = X^{\perp}$. □

Proposition 5.5. *Let* (P, B) *be a symplectic space, and let* L *be a direct summand of* P *that is totally isotropic, i.e.* $B(L, L) = 0$*. If we write* $P = L^{\perp} \oplus L_1$ *as in* (5.4)*, then* $L_1 \cong L^*$*, and* $M := L + L_1$ *is a nonsingular submodule* (*and hence* $P = M \perp M^{\perp}$ *by* (5.3))*.*

Proof. Note that $M = L + L_1$ is a direct sum, since $L \subseteq L^{\perp}$. To show that M is nonsingular, we check that the map $\varphi : M \to M^*$ defined via B is an isomorphism. Using the decompositions $M = L \oplus L_1$ and $M^* = L^* \oplus L_1^*$, we represent φ by a matrix $\Phi = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$. Now note the following:

(1) $\alpha : L \to L^*$ is the zero map, since $B(L, L) = 0$.

(2) By the choice of L_1, $\beta : L_1 \to L^*$ is an isomorphism.

(3) By skew-symmetry, $\gamma : L \to L_1^*$ is just $-\beta^*$ (where β^* denotes the dual of β), if we use the usual identification $L_1^{**} = L_1$. Thus, γ is also an isomorphism.

From (1), (2) and (3), it follows that Φ is an isomorphism — with inverse given by $\begin{pmatrix} -\gamma^{-1}\delta\beta^{-1} & \gamma^{-1} \\ \beta^{-1} & 0 \end{pmatrix}$. □

Remark. Professor Swan has pointed out that, in the above proof, the symplectic space (M, B) is in fact isometric to the hyperbolic space $\mathbb{H}(L)$. The verification of this would require a couple of additional steps in the proof. Since we do not need this information for the application of (5.5) in the rest of this section, we shall not present the extra arguments here.

We now come to the following nice result of Bass.

Theorem 5.6. *Let* P, $L \in \mathfrak{P}(R)$, *where* $\operatorname{rk} L = 1$.

(1) *If* $P \oplus L$ *admits a symplectic structure, then* $P \cong L^* \oplus Q$ *for some* Q.

(2) *If* $\operatorname{rk} P = 1$ *also,* $P \oplus L$ *admits a symplectic structure iff* $P \cong L^*$. *In particular,* $R \oplus L$ *admits a symplectic structure iff* $L \cong R$.

Proof. Clearly $(1) \Rightarrow (2)$, so it suffices to prove (1). Let B be a symplectic form on $P \oplus L$. For any prime ideal $\mathfrak{p} \subset R$, we have $L_{\mathfrak{p}} \cong R_{\mathfrak{p}}$, so $L_{\mathfrak{p}}$ is totally isotropic. Thus, for any x, $y \in L$, $B(x, y)$ localizes to zero at every prime. This gives $B(x, y) = 0$, so L itself is totally isotropic. By (5.5), we have

$$P \oplus L = (L \oplus L_1) \oplus (L \oplus L_1)^{\perp}$$

for some $L_1 \cong L^*$. Factoring out L, we get $P \cong L^* \oplus Q$ where $Q = (L \oplus L_1)^{\perp}$, as desired. □

Corollary 5.7. *If* $P \oplus L \cong R^{2n}$ *where* $\operatorname{rk} L = 1$, *then* $P \cong L^* \oplus Q$ *for some* Q.

Proof. This is a special case of (5.6)(1) since R^{2n} always admits a symplectic structure. [The symplectic hyperbolic space $\mathbb{H}(R^n)$ (defined in (5.1)) is supported by the module $R^n \oplus (R^n)^* \cong R^{2n}$.] □

In the case $L = R$, this Corollary says that *odd-rank stably free modules of type* 1 *has a direct summand isomorphic to* R. We have, of course, already proved this by the more straightforward method in I.4.12. The above was Bass's original proof using symplectic methods.

How large a supply of symplectic spaces are there? Of course the answer to this question depends on the ring R. In the result below, we'll show that the structure of symplectic spaces over many rings can be very simple.

Theorem 5.8. *Suppose that every nonzero symplectic space over R has a unimodular element. Then every symplectic space (P, B) over R is isometric to $\mathbb{H}(R^m)$ for some m.*

Proof. Fix a unimodular element $e \in P$. Since there exists a linear functional on P taking e to 1, the nonsingular condition on (P, B) implies that $B(e, e') = 1$ for some $e' \in P$. Now, whenever two vectors $e,\ e' \in P$ satisfy $B(e,\ e') = 1$, we say that $\{e,\ e'\}$ is a *symplectic pair.* Such a symplectic pair is always linearly independent over R. Indeed, if $re + se' = 0$, then

$$0 = B(re + se',\ e') = r, \quad \text{and} \quad 0 = B(re + se',\ e) = -s.$$

Thus, $M := Re + Re'$ with the induced symplectic form is isometric to the hyperbolic plane $\mathbb{H}$. Now (5.3) gives $P = M \perp M^{\perp} \cong \mathbb{H} \perp M^{\perp}$, so we are done by induction. (At the end of the induction, we will have obtained a "symplectic basis" $\{e_1, e'_1, \ldots, e_m, e'_m\}$ for P, with $B(e_i,\ e'_i) = 0$ and $B(e_i,\ e_j) = B(e_i,\ e'_j) = 0$ for $i \neq j$.) $\square$

Theorem 5.8 applies notably to the following situations:

(a) when every $P \in \mathfrak{P}(R)$ is free; e.g. R is a field, a PID, a semilocal ring with only trivial idempotents; or more generally,
(b) when every $P \in \mathfrak{P}(R)$ is $\cong$ (free) $\oplus$ (ideal); e.g. when R is a Dedekind domain.

In particular, from the Quillen-Suslin Theorem and (5.8), we deduce the following.

Corollary 5.9. *If k is a field, then any symplectic space over $R = k[t_1, \ldots, t_d]$ is isomorphic to $\mathbb{H}(R^n)$ for some n; in particular, the symplectic space is extended from k. (In matrix terms, this says that every invertible alternating matrix M over $k[t_1, \ldots, t_d]$ is congruent to $M(0, \ldots, 0)$.)*

Coming back to a general commutative ring R, note that the proof of (5.8) also yields the following.

Corollary 5.10. *Up to isometry, the only symplectic structure on R^2 is that of $\mathbb{H}$, the symplectic hyperbolic plane.*

How about (P, B) with $P \cong R^4$? We can start the proof of (5.8) by choosing $e = e_0 = (1, 0, 0, 0)$. The proof then shows that P is isometric to $\mathbb{H} \perp Q$, where $Q := M^{\perp}$ is a symplectic space. Here, Q is stably free (of rank 2 and type 1), but it may not have a unimodular element (e.g. Q may be indecomposable). If that is the case, then the proof of (5.8) cannot be continued. For a more concrete illustration of this, see (5.27) below, where (after choosing B suitably) Q here can be *any* stably free module of rank 2 and type 1 !

In view of (5.10) and the above paragraph, it is of interest to ask if one can "determine" all symplectic structures (if any) on a rank 2 projective module. The answer to this question is provided by the following result from [Bass: 1969].

Theorem 5.11. *Let* $P \in \mathfrak{P}(R)$ *be of rank* 2*. Then* P *admits a symplectic structure iff* $\Lambda^2(P) \cong R$*, where* $\Lambda^2(P)$ *denotes the second exterior power of* P*. In this case, the set of isometry classes of symplectic forms on* P *is in one-one correspondence with the elements of the group* $\mathrm{U}(R)/G$*, where* G *is the image of the determinant homomorphism*

$$\text{(5.12)} \qquad \det : \operatorname{Aut}(P) \longrightarrow \mathrm{U}(R)$$

(*to be defined in the proof below*).

Proof. For better understanding of the proof, it will be convenient to replace the condition $\Lambda^2(P) \cong R$ by the equivalent condition $S \cong R$, where $S := \left(\Lambda^2(P)\right)^*$. Also, since S is a f.g. projective module of rank 1 (by I.4.20), we can further replace $S \cong R$ by the equivalent condition that S be *cyclic* as an R-module.

Suppose P admits a symplectic form B. We can think of the alternating form $B : P \times P \to R$ as an element in the R-module S (which we'll continue to denote by B). We are done if we can show that $S = R \cdot B$. It is enough to show this at all localizations of R at prime ideals, so we may as well assume that R is local. In this case, $P \cong R^2$, and S is free on a singleton basis B_0 given by the symplectic hyperbolic structure on $P \cong R^2$, so we have $B = r \cdot B_0$ for some $r \in R$. Since B is *nonsingular*, r must be a unit. Thus, $S = R \cdot B_0 = R \cdot B$, as desired.

For the converse, we assume that S is free on a singleton basis $B \in S$. We can again think of B as an alternating form $B : P \times P \to R$. Locally at any prime ideal $\mathfrak{p}$, the form $B_{\mathfrak{p}}$ also generates $S_{\mathfrak{p}}$, so it must be a unit multiple of the symplectic hyperbolic form on $P_{\mathfrak{p}} \cong R_{\mathfrak{p}}^2$. This implies that B is nonsingular locally at each $\mathfrak{p}$, which then implies that B itself is *nonsingular.* Thus, B gives the symplectic structure we sought on P.

To prove the last part of (5.11), we first recall the definition of the determinant of an automorphism $\varphi \in \operatorname{Aut}(P)$. For this, we take any R-module Q such that $P \oplus Q \cong R^n$ (for some n), and define

$$\text{(5.13)} \qquad \det \varphi := \det\left(\varphi \oplus \mathrm{Id}_Q\right) \in \mathrm{U}(R).$$

This invariant is easy to work with, since locally (at any prime ideal) P is free, and $\det \varphi$ localizes to the usual determinant of an automorphism on a free module. (This remark, incidentally, also shows the well-definition of $\det \varphi$.)

Assuming $\Lambda^2(P)$ is free, fix a symplectic form B on P that freely generates $S = \left(\Lambda^2(P)\right)^*$. It is easy to check that

(5.14) $$B(\varphi(x),\ \varphi(y)) = (\det\varphi)B(x,\ y)$$

for any $\varphi \in \mathrm{Aut}(P)$. (This follows, for instance, by a local calculation.) Now, our work in the first part of the proof showed that the symplectic forms on P are given by uB with u ranging over $\mathrm{U}(R)$. In particular, $\mathrm{U}(R)$ acts transitively (by multiplication) on the isometry classes of such symplectic forms. We are done if we can show that the isotropy subgroup of the class of B under this action is given by $G := \det(\mathrm{Aut}(P))$. First, suppose $\varphi \in \mathrm{Aut}(P)$ is an isometry from $(P,\ uB)$ to $(P,\ B)$. Then

(5.15) $$uB(x,\ y) = B(\varphi(x),\ \varphi(y)) = (\det\varphi)\ B(x,\ y) \quad (\forall\, x,\ y \in P),$$

in view of (5.14). The same equation will also apply locally, where certainly there exist $x,\ y$ with $B(x,\ y) = 1$. It follows that $u = \det\varphi$ (locally, and hence globally); that is, $u \in G$. Conversely, if $u \in G$, say $u = \det\varphi$ for some $\varphi \in \mathrm{Aut}(P)$, then (5.14) shows that φ is an isometry from $(P,\ uB)$ to $(P,\ B)$. This completes the proof of (5.11). □

Remark 5.16. The factor group $\mathrm{U}(R)/G$ (which "enumerates" the isometry classes of symplectic forms on P if they exist) is a group of exponent 2, since the automorphism on P given by multiplication by $u \in \mathrm{U}(R)$ has determinant u^2.

Remark 5.17. Assume, again, that $\Lambda^2(P)$ is free, and fix a symplectic form B on P that freely generates $S = \left(\Lambda^2(P)\right)^*$. Let $e \in \Lambda^2(P)$ be the dual basis of the singleton basis B. For any $x,\ y \in P$, we have $x \wedge y = re$ for some $r \in R$. Applying the functional B on $\Lambda^2(P)$, we get $B(x,\ y) = r$. Therefore, for the singleton basis e in $\Lambda^2(P)$, the symplectic form B on P is described by the equation

$$x \wedge y = B(x,\ y)\, e \quad (\text{for all } x,\ y \in P).$$

This is just a general formula. In some cases, B can be constructed more explicitly. For instance, if P (of rank 2) is stably free of type 1, (5.27) below offers a construction of a symplectic structure on P using suitable 4×4 alternating matrices over R.

The first corollary of (5.11), implicit in [Bass: 1969], verifies our earlier claim III.6.8 — after a long postponement.

Corollary 5.18. *Every f.g. stably free R-module P of rank 2 admits a symplectic structure. In particular, $P \cong P^*$.*

Proof. By (I.4.21), $\Lambda^2(P) \cong R$. Thus, the desired conclusion follows from Bass's result (5.11). □

Corollary 5.19. *Every f.g. stably free R-module P of rank 3 and type 1 is self-dual.*

Proof. Of course, this is only a special case of III.6.7(1). But here is a proof of the result independently of III.6. By I.4.12(2), $P \cong R \oplus Q$ for some Q. Since Q is (clearly) stably free and has rank 2, we have $Q^* \cong Q$ by (5.18). It follows that

$$P^* \cong R^* \oplus Q^* \cong R \oplus Q \cong P.$$

(With this method of proof, however, our luck runs out starting with the case rank $P = 5$.) □

Corollary 5.20. *Let R be a* UFD*, and let $P \in \mathfrak{P}(R)$ be of rank* 2*. Then P admits a symplectic structure. In particular, $P \cong P^*$.*

Proof. By (I.4.20), $\Lambda^2(P)$ is projective of rank 1. Since R is a UFD, we have $\Lambda^2(P) \cong R$ by (II.1.3). Thus, again, the desired conclusion follows from (5.11). □

At this point, the following historical note is in order. Corollary 5.20 above, obtained by Bass in 1969, was directly inspired by an effort to prove Serre's Conjecture for three variables, still open at that time. Let Q be a f.g. projective module of rank r over $R = k[x, y, z]$, where k is a field. To prove that Q is free, it suffices to work in the cases where $r \leqslant 3$, by II.7.1. If $r = 1$, Q is free by II.1.3 (since R is a UFD). If $r = 3$, then Q is stably free of type 1 (again by II.7.1), and therefore $Q \cong P \oplus R$ (for some P) by I.4.12(2). Thus, Serre's Conjecture for $R = k[x, y, z]$ reduces to checking that any f.g. projective R-module P *of rank* 2 is free. Using the symplectic techniques in 1969, Bass was able to show, as in (5.20) above, that P is *self-dual.* While this did not quite do the job, it certainly lent strong positive evidence toward Serre's Conjecture for three variables. A few years after Bass's work, symplectic techniques fluorished in the hands of Suslin and Vaserstein, and led to the first proofs of Serre's Conjecture for 3, 4, and 5 variables (over arbitrary fields). For a summary of the developments on Serre's Conjecture for a "small" number of variables in the period 1969–1974, see the items (11)–(14) in the Introductory Survey at the beginning of this book.

To conclude this section, we shall prove a couple more results on the existence of symplectic structure on certain projective modules of higher (even) rank; namely, those of the form $P(b_1, \ldots, b_{2m-1})$ (solution space of a unimodular row $(b_1, \ldots, b_{2m-1})$). For this, we need two preliminary lemmas.

Lemma 5.21. *A f.g. projective module P (over a commutative ring R) admits a symplectic structure iff its dual module P^* does.*

Proof. Since $P^{**} \cong P$, it suffices to prove the "only if" part. Fixing a symplectic structure on P, we have an R-isomorphism $\varphi : P \to P^*$ with the property that $\varphi(p)(p) = 0$ for every $p \in P$. Let $\varepsilon : P \to P^{**}$ be the canonical isomorphism, and consider the R-isomorphism $\varepsilon\,\varphi^{-1} : P^* \to P^{**}$. For any $\lambda \in P^*$, write $\lambda = \varphi(p)$ (for a unique $p \in P$). Then

$$\big((\varepsilon\,\varphi^{-1})(\lambda)\big)(\lambda) = \varepsilon(p)(\lambda) = \lambda(p) = \varphi(p)(p) = 0.$$

Thus, the isomorphism $\varepsilon\,\varphi^{-1} : P^* \to P^{**}$ gives rise to a symplectic structure on the dual module P^*. □

Lemma 5.22. *If* $A \in \mathrm{GL}_r(R)$ *is an alternating matrix over a commutative ring* R*, then so is* A^{-1}.

Proof. This can be deduced from (5.21) by choosing $P = R^r$, and taking the symplectic structure on P that is defined by the invertible alternating matrix A. A more direct proof not using symplectic structures can be given as follows.

Assuming that $R \neq 0$, r must be even, by III.5.12. According to the classical adjoint formula for computing the inverse $C := A^{-1}$, a diagonal entry C_{ii} is, up to a unit, the determinant of the matrix obtained from A by deleting the i th row and the i th column. This matrix is alternating of odd size $(r-1) \times (r-1)$, so $C_{ii} = 0$ (again by III.5.12). On the other hand, $AC = I_r$ gives $I_r = C^t A^t = -C^t A$, so $-C^t = C$. This means that C is skew-symmetric, so it is an alternating matrix, as claimed. □

To formulate our last group of results for this section, we introduce the following useful definition:

> *A row* $(b_1, \ldots, b_n) \in R^n$ *is called* skew-completable *if* $(0, b_1, \ldots, b_n)$ *can be completed to an invertible alternating matrix in* $\mathbb{M}_{n+1}(R)$.

There are two necessary conditions for this to happen. First, by III.5.12, n must be odd (if $R \neq 0$), and second, by a determinant expansion (along the first row), $(b_1, \ldots, b_n) \in R^n$ must be unimodular. While these conditions are, in general, not sufficient, it turns out that the necessary and sufficient conditions for the skew-completability of $(b_1, \ldots, b_n)$ are closely tied to the existence of symplectic structures. The following result, due to Swan, may be viewed as a symplectic analogue of the fact that $(b_1, \ldots, b_n)$ is completable iff $P(b_1, \ldots, b_n)$ is free. The proof of this result we'll give below is an expanded (and somewhat revised) version of that given in [Krusemeyer: 1975].

Theorem 5.23. (Swan) *Let* $a_1 b_1 + \cdots + a_n b_n = 1$ *over a commutative ring* R*, where* n *is an odd integer. Then the following are equivalent:*

(1) *The module* $P(b_1, \ldots, b_n)$ *admits a symplectic structure.*

(2) *The module* $P(a_1, \ldots, a_n)$ *admits a symplectic structure.*

(3) *The row* $(b_1, \ldots, b_n)$ *is skew-completable.*

(4) *The row* $(a_1, \ldots, a_n)$ *is skew-completable.*

In general, for n *odd, if* $(b_1, \ldots, b_n)$ *is completable, then it is skew-completable; the converse of this does not hold.*

Proof. The first part of the last statement is clear as soon as we know that (1) $\Rightarrow$ (3). Indeed, if $(b_1, \ldots, b_n)$ is completable, we have $P(b_1, \ldots, b_n) \cong R^{n-1}$. Since $n-1$ is even, $P(b_1, \ldots, b_n)$ has then a (hyperbolic) symplectic structure, and hence $(b_1, \ldots, b_n)$ is skew-completable. The converse of this does not hold, as we'll see from (5.25) below that $(b_1, b_2, b_3) \in \mathrm{Um}_3(R)$ is *always skew-completable*, though, of course, it need not be completable.

Coming to the main part of the theorem, note that (1) $\Leftrightarrow$ (2) follows from (5.21) and (I.4.10)(1). Thus, the theorem boils down to proving (1) $\Leftrightarrow$ (4). Assume (1) and fix a symplectic form B_0 on $Q := P(b_1, \ldots, b_n)$. We write $\alpha = a_1e_1 + \cdots + a_ne_n \in R^n$ and

$$R^n = R \cdot \alpha \oplus Q, \quad R^{n+1} = R \cdot e_0 \oplus R^n = (R \cdot e_0 \oplus R \cdot \alpha) \oplus Q,$$

where $\{e_0, \ldots, e_n\}$ is the standard basis on R^{n+1}. On $R \cdot e_0 \oplus R \cdot \alpha$, we fix the hyperbolic symplectic form H with $H(\alpha, e_0) = 1$. Then $B := H \perp B_0$ is a symplectic form on R^{n+1}. Consider the sum $\sum_{i \geqslant 1} a_i B(e_i, e_j) = B(\alpha, e_j)$. This is 1 if $j = 0$, and 0 if $j \geqslant 1$ (since $B(\alpha, \alpha) = 0 = B(\alpha, Q)$ implies $B(\alpha, R^n) = 0$). Therefore, the row $(0, a_1, \ldots, a_n)$ has dot product 1 with the first column of the alternating matrix

$$A := \big(B(e_i, e_j)\big)_{i,j \geqslant 0} \in \mathrm{GL}_{n+1}(R),$$

and dot product 0 with its other columns. But of course, the only row (of length $n+1$) with such properties is the first row of A^{-1}. Therefore, $(0, a_1, \ldots, a_n)$ is the first row of A^{-1}, which, according to Lemma 5.22, is an (invertible) alternating matrix. This completes the proof of (1) $\Rightarrow$ (4).

To prove the converse, we need only reverse a part of the argument above. Start with a skew completion, say C, of the row $(0, a_1, \ldots, a_n)$, and let $A = C^{-1} \in \mathrm{GL}_{n+1}(R)$. By (5.22), A is alternating, so it defines a symplectic form B on R^{n+1}. As in the paragraph above, we get $B(\alpha, e_0) = 1$; that is, $\{\alpha, e_0\}$ is a symplectic pair. (Here, we also have $B(\alpha, R^n) = 0$, though we won't need this for the rest of the argument.) By the proof of (5.8), this gives $R^{n+1} = M \perp M^{\perp}$, where $M = R \cdot e_0 \oplus R \cdot \alpha \cong \mathbb{H}$. Since $Q \cong R^{n+1}/M \cong M^{\perp}$ and $M^{\perp}$ is a symplectic space, it follows that $Q = P(b_1, \ldots, b_n)$ admits a symplectic structure. □

Remark 5.24. Clearly, the equivalence of (1) and (3) implies that, if $(b_1, \ldots, b_n)$ and $(c_1, \ldots, c_n)$ are conjugate under the action of $\mathrm{GL}_n(R)$, then $(b_1, \ldots, b_n)$ is skew-completable iff $(c_1, \ldots, c_n)$ is. This fact can be independently checked by an easy matrix calculation, which we'll leave to the reader.

Proposition 5.25. *If $n = 1$ or 3, all of the conditions* (1)–(4) *in* (5.23) *hold. If R is a Hermite ring (e.g. a polynomial ring over a field), these conditions hold for all odd n. However, if R is not Hermite and $n \geqslant 5$, they need not hold.*

Proof. We proceed directly to the case $n = 3$ (since the case $n = 1$ is trivial). To simplify the notations, we replace (a_1, a_2, a_3) by (a, b, c), and (b_1, b_2, b_3) by (p, q, r), where $ap + bq + cr = 1$. For the matrix

$$(5.26) \qquad C = \begin{pmatrix} 0 & a & b & c \\ -a & 0 & r & -q \\ -b & -r & 0 & p \\ -c & q & -p & 0 \end{pmatrix}, \quad \text{we have } \det(C) = (ap + bq + cr)^2 = 1.$$

(This follows from (III.7.9), or from a direct determinant expansion.) Thus, the row $(0, a, b, c)$ can be completed even to an alternating matrix of determinant 1. In particular, (4) in (5.23) holds. (*Note.* The skew-completability of unimodular rows of length 1 and 3 may be thought of as the analogue of the completability of unimodular rows of length 1 and 2.)

Now assume R is a Hermite ring. In this case, any $(b_1, \ldots, b_n) \in \mathrm{Um}_n(R)$ is completable. So if n is odd, $(b_1, \ldots, b_n)$ is skew-completable too, by the last part of (5.23). We note in passing that, in the case where $R = k[x_1, \ldots, x_d]$ over a field k, this fact may be interpreted as a *second* symplectic analogue of the truth of Serre's Conjecture — the first one being Corollary 5.9.

Finally, let n be odd and $\geqslant 5$. In (III.6.12), we have constructed (over a suitable commutative ring R) some stably free modules of the form $P(b_1, \ldots, b_n)$ that are not self-dual. For such modules, obviously, (1) in (5.23) does not hold. (From this, it follows easily that, if S is the ring $\mathbb{Z}[x_1, y_1, \ldots, x_n, y_n]$ with the relation $\sum_i x_i y_i = 1$, then the "generic unimodular row" $(y_1, \ldots, y_n)$ is not skew-completable over S as long as $n \neq 1, 3$.) $\square$

Of course, we also know that (5.23)(1) holds for $n = 3$ by applying (5.18), since $P(p, q, r)$ is stably free of rank 2 whenever $(p, q, r) \in \mathrm{Um}_3(R)$. This amounts to a second proof of (the first part of) (5.25). But, instead of accepting this as a logically redundant second proof, we can turn the situation around, and parlay (5.26) into an *explicit construction* of a symplectic structure on the stably free module $Q := P(p, q, r)$. Actually, such a symplectic structure was already attained earlier via some matrix calculations given in Chapter III (see the discussions after III.6.9). Here, we construct a symplectic structure on Q from a slightly different perspective, following closely the idea of the proof of (5.23) (specialized to the case $n = 3$). Indeed, just as in (5.26), for

$$(5.27) \qquad A = \begin{pmatrix} 0 & -p & -q & -r \\ p & 0 & -c & b \\ q & c & 0 & -a \\ r & -b & a & 0 \end{pmatrix}, \quad \text{we have } \det(A) = (ap + bq + cr)^2 = 1.$$

By a small miracle,[(*)] $CA = I_4$, so $A = C^{-1}$. Now let B be the symplectic form defined by A on R^4 (with basis $\{e_0, \ldots, e_3\}$). Then, as in the proof of (4) $\Rightarrow$ (1) in (5.23), the vector $\alpha = ae_1 + be_2 + ce_3$ satisfies $B(\alpha, e_0) = 1$ and $B(\alpha, Q) = 0$. It turns out that $B(e_0, Q) = 0$ as well since, for $u = u_1 e_1 + u_2 e_2 + u_3 e_3 \in Q$, we have

$$B(e_0, u) = u_1 B(e_0, e_1) + u_2 B(e_0, e_2) + u_3 B(e_0, e_3) = -(u_1 p + u_2 q + u_3 r) = 0.$$

[(*)] This is, however, largely a "commutative miracle". If R is just a general ring, the equation $CA = I_4$ will hold if and only if (1) a, b, c pairwise commute, (2) p, q, r pairwise commute, (3) a commutes with q, r, (4) b commutes with p, r, (5) c commutes with p, q, and (6) $ap - pa = bq - qb = cr - rc$ are killed by 2. If we also want $AC = I_4$, then *all* elements involved must pairwise commute. This means we may as well assume that R itself is commutative!

Therefore, $Q = M^{\perp}$ for $M = R \cdot e_0 \oplus R \cdot \alpha$, so B induces a symplectic structure on $Q = P(p, q, r)$. Summarizing, we have the following.

Proposition 5.28. *In the above notations, the alternating matrix $A \in \mathrm{SL}_4(R)$ in (5.27) defines a symplectic structure B on R^4, and by restriction, B induces a symplectic structure B_0 on $Q = P(p, q, r) \subseteq R^3$. In particular, $Q \cong Q^*$, and we have*

$$(R^4,\ B) \cong \mathbb{H} \perp (Q,\ B_0). \tag{5.29}$$

Remark 5.30. The determinantal identity in (5.27) (found in many classical books on determinant theory) is no accident. In general, the determinant of an alternating matrix of even size is the square of its *Pfaffian,* and the Pfaffian for A is just $ap + bq + cr$. The Pfaffian is named after Johann Friedrich Pfaff (1765–1825), who, twelve years older than Carl Friedrich Gauss, was a teacher (and later a life-long friend) of Gauss at the University of Helmstädt. Apparently, the Pfaffian was discovered by Pfaff through his work on differential forms and partial differential equations.

As a fun exercise, we invite the reader to construct the cubic Pfaffian for an alternating matrix in $\mathbb{M}_6(R)$, in the full glory of its 15 terms in as many variables!

We come now to the last result in this section, which concerns again the relationship between the notions of *completability* and *skew-completability.* In the last part of (5.23), we have pointed out that, for n odd, completability of $(b_1 \dots, b_n)$ implies its skew-completability, though not conversely. It might seem that this single statement summarizes all one can say about the relationship between completability and skew-completability. But if one looks deeper, there is a surprise! The following theorem from [Krusemeyer: 1976] gives another rather intriguing relation between the two notions.

Theorem 5.31. (Krusemeyer) *If $(b_1, \dots, b_n)$ is skew-completable over a commutative ring R, then $(b_1^2, b_2, \dots, b_n)$ is completable. Stated equivalently in view of* (5.23)*, if $(b_1, \dots, b_n) \in \mathrm{Um}_n(R)$ and $P(b_1, \dots, b_n)$ has a symplectic structure, then $P(b_1^2, b_2, \dots, b_n)$ is free.*

Corollary 5.32. *If n is odd and $(b_1, \dots, b_n)$ is completable, then $(b_1^2, b_2, \dots, b_n)$ is also completable.*

Of course, the hypotheses in Theorem 5.31 can hold only when n is an *odd* integer (assuming $R \neq 0$). Since any $(a, b, c) \in \mathrm{Um}_3(R)$ is skew-completable (by (5.26)), it follows from (5.31) that *any* $(a^2, b, c) \in \mathrm{Um}_3(R)$ *is completable.* This is the theorem of Swan and Towber mentioned at the beginning of III.4. What we can say from this deduction is that the Swan-Towber theorem admits two different generalizations: Suslin's $n!$ theorem III.4.1 is one, and Krusemeyer's theorem (5.31) is the other.

Proof of (5.31). The proof of this theorem in [Krusemeyer: 1976] is computational in nature, and makes use of various nontrivial Pfaffian identities. Here we present a short and completely self-contained proof that is communicated to us by Professor Swan. Let $B \in \mathrm{GL}_{n+1}(R)$ be an alternating matrix with a first row $(0, b_1, \ldots, b_n)$ (where n must be odd, assuming $R \neq 0$). As in the proof of (5.23), we index the rows and columns of B by $0, 1, \ldots, n$. Let

$$(5.33) \qquad D := (I + b_n e_{n0})^t \, B \, (I + b_n e_{n0}),$$

where $\{e_{ij}\}$ denote the matrix units. Since $D = (d_{ij})$ is congruent to B, it is also alternating (and invertible). The cofactor of d_{nn} is zero (as in the proof of (5.22)). Therefore, we can replace $d_{nn} = 0$ by $d_{nn} = 1$ without changing the determinant of D. Now apply elementary row operations to the resulting (still invertible) matrix to reduce its last column to $(0, \ldots, 0, 1)^t$. Because of the initial modification of B to D, these operations bring us to an invertible matrix D' with first row $(b_n^2, b_1, \ldots, b_{n-1}, 0)$ and last column $(0, \ldots, 0, 1)^t$. Thus, deleting the last row and last column of D' leaves an invertible matrix completing $(b_n^2, b_1, \ldots, b_{n-1})$. Since, by (5.24), we could have started with $(b_2, \ldots, b_n, b_1)$, this proves (5.31). □

Note that the proof of (5.31) is completely constructive — as long as the skew completion B is given. The best illustration for this construction of an explicit completion of $(b_1^2, b_2, \ldots, b_n)$ is provided by the case where $n = 3$ (the Swan-Towber case). In this case, writing $(a, b, c) \in \mathrm{Um}_3(R)$ for (b_1, b_2, b_3), we fix a unimodularity equation $ap + bq + cr = 1$, and start instead with (b, c, a). As in the proof of (5.25), this row is skew-completable to the alternating matrix

$$B = \begin{pmatrix} 0 & b & c & a \\ -b & 0 & p & -r \\ -c & -p & 0 & q \\ -a & r & -q & 0 \end{pmatrix} \in \mathrm{SL}_4(R).$$

After the congruence transformation in the proof above, this becomes

$$D = \begin{pmatrix} 0 & b+ar & c-aq & a \\ -b-ar & 0 & p & -r \\ -c+aq & -p & 0 & q \\ -a & r & -q & 0 \end{pmatrix} \in \mathrm{SL}_4(R).$$

Now change the southeast corner of D to 1, and carry out the indicated row operations to bring the matrix to

$$(5.34) \qquad D' = \begin{pmatrix} a^2 & b & c & 0 \\ -b-2ar & r^2 & p-qr & 0 \\ -c+2aq & -p-qr & q^2 & 0 \\ -a & r & -q & 1 \end{pmatrix} \in \mathrm{SL}_4(R).$$

Deleting the last row and last column now yields the $\mathrm{SL}_3(R)$ matrix (first given in III.4.4) that completes the unimodular row (a^2, b, c).

Remark 5.35. Professor Swan has pointed out to us that the converse of Theorem 5.31 is false. In fact, let R be the ring used in (III.6.12); namely, R is the complex coordinate ring of the sphere S^{2n+1}. Let $z_0, \ldots, z_n \in R$ be defined as in the paragraph preceding (III.6.12), and let $b_j := z_j^{r_j}$ with $\prod_{j \geqslant 0} r_j = n!/2$. By Suslin's $n!$ theorem, the module $P(b_0^2, b_1, \ldots, b_n)$ is free. However, if $n \equiv 0 \pmod 4$, a topological argument shows that $P = P(b_0, b_1, \ldots, b_n)$ is not self-dual. In particular, no symplectic structure can exist on P. For more details on this example, see [Swan: 2006] listed in the references on Chapter VIII.

Notes on Chapter VII

The investigation of the quadratic analogue of Serre's Problem began shortly after the solution of the original Serre Conjecture, though some of the results in this chapter (e.g. those in §2 and §3) were already known since the early 1970s. Karoubi's Theorem (2.1) is part of a larger work on periodicity in Hermitian K-theory [Karoubi: 1972]; our exposition is based on that of [Bass: 1977, Sec. 5]. Harder's result ((3.11), (3.13)) apparently first appeared in [Knebusch: 1970]; the (so-called) "easier" proof given in the text follows [Gerstein: 1973a]. (Details of this proof, however, were suppressed in the published version [Gerstein: 1973b].) As was observed in the text, Gerstein's argument is modeled upon the proof of Hermite's classical inequality on the minima of positive definite integral quadratic forms. Hermite described his inequality and its applications to Jacobi in a letter dated August 6, 1845. The diagonalizability result (3.10), however, depends substantially on the two extra properties (3.1) and (3.2) of the degree function ∂, and *does not* apply to $\mathbb{Z}$ with the ordinary (absolute value) norm. It was known at least since 1873 (Korkine-Zolotareff)$^{(*)}$ that there exists an 8-dimensional, positive definite IPS over $\mathbb{Z}$ that is *not* diagonalizable – contrary to a misleading claim once made by Hermite.

The extendibility result (1.10) for rank 2 inner product spaces over $k[t_1, \ldots, t_d]$ (for a field k of characteristic not 2) was first proved by Parimala (1978), using Galois cohomology techniques. A few years later, Knus, Ojanguren and Parimala generalized this result to the case where k is any normal ring with $2 \in \mathrm{U}(k)$; see their 1982 paper listed in the references on Chapter VIII.

The difficulty inherent in the characteristic 2 case of the quadratic Serre Problem (Remark (1.9)) was pointed out at the beginning of [Bass: 1977], while the main body of Bass′s paper deals with fields k with char $k \neq 2$. It was shown that, if such k is separably closed, then any IPS (P, B) over $k[t_1, \ldots, t_d]$ is extended from k, provided either $d \leqslant 3$, or rank $P \leqslant 6$. This result inspired the work of [Raghunathan: 1978] (see also [Raghunathan-Ramanathan: 1984]), which investigates principal G-bundles over affine spaces for various affine algebraic groups G over k. In the case where G is the orthogonal group, Raghunathan obtained the result stated in (1.12).

The exposition in §4 follows closely [Parimala: 1978]; see also [Parimala: 1976a]. The only thing extra is the immediate observation (Supplement to (4.4)) that Parimala's rank 4 inner product spaces over $\mathbb{R}[x, y]$ are, in fact, *indecomposable*. The Knus-Ojanguren rank 3 counterexample (mentioned in (1.14)) is, of course, automatically indecomposable, in view of (1.10). Given hindsight, the construction of nonextended inner product spaces of any rank > 3 in (4.13) over $\mathbb{R}[x, y]$ (using the term inspection method for sums-of-squares) looks surprisingly easy.

Indecomposable inner product spaces of rank $\geqslant 3$ over $\mathbb{R}[t_1, \ldots, t_d]$ (with $d \geqslant 2$) are not easy to classify. However, further constructions of such indecompos-

$^{(*)}$According to the book of Milnor and Husemoller (*Symmetric Bilinear Forms*, Springer Verlag, 1973), the existence of the Korkine-Zolotareff form was proved non-constructively by H.J.S. Smith in 1867.

ables, and the determination of the structure of their respective orthogonal groups, should be both worthy of investigations. For more information on the extensive work done in this area after 1977, see §10 of Chapter VIII.

The symplectic analogue of Serre's Problem (for polynomial rings) turned out to be no deeper than Serre's Problem itself, as we saw in (5.10). On the other hand, the coverage of symplectic techniques in §5 gave us the chance to finally prove the fact (first stated in III.6.8) that rank 2 stably free modules (of any type) over any commutative ring are self-dual. The historical remarks following the proof of (5.20) showed that the material in that section is, curiously, as relevant (if not more relevant) to the *linear* Serre Problem as to the symplectic Serre Problem. The results in §5 are, for the most part, drawn from [Bass: 1969], [Bass: 1972b, (4.11.12)], [Bass: 1974, Prop. 2], and [Krusemeyer: 1975, 1976]. An apparent oversight in the formulation of Swan's result in [Krusemeyer: 1975] is corrected here in Theorem 5.23.

In §5, we developed the theory of symplectic structures only to the point of accounting for how such structures may arise on rank 2 projective modules (especially stably free modules of type 1), and relating the completability and skew-completability of unimodular rows. The next stage of development would be to define and to study the symplectic Witt group, following [Vaserstein-Suslin: 1976]. Excellent expositions on this theory of Vaserstein and Suslin can be found in [Bass: 1974], and [Swan: 1975]; see also [Bass: 1972b]. The class of alternating matrices has recently found some nice applications in the study of sectionable and presectionable sequences in commutative rings; for an exposition on this, see [Lam-Swan: 2006].

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Appendix: Complete Intersections and Serre's Conjecture

This Appendix is a somewhat expanded version of the "Appendix on Complete Intersections" that appeared in the original (1978) version of this book. The goal here is to offer some comments on the problem of complete intersections in algebraic geometry, and to explain some of its connections to Serre's Conjecture in the period 1955-1976 (when the Conjecture stood open). Since such discussions are still essential today in coming to a full understanding of Serre's Conjecture, I have chosen to preserve the Appendix in this version of the book, in spite of the fact that the main text of it was written in 1977.

Roughly speaking, the Problem of Complete Intersections is concerned with the characterization of ideals in polynomial rings that can be generated by the "right" number of elements. Geometrically, this translates into the problem of recognizing algebraic sets (in affine or projective space) that can be expressed as the intersection of the "right" number of hypersurfaces. In a broader sense, the Complete Intersection Problem is also concerned with the determination of the minimal number of generators needed for a polynomial ideal over a field, or the determination of the minimal number of hypersurfaces needed to cut out a given algebraic variety. In a still broader sense, similar problems can be considered where the polynomial ideals are replaced by arbitrary ideals in a commutative noetherian ring.

As we have mentioned in the Introduction section of this book, the important link between the Complete Intersection Problem and Serre's Problem on projective modules over polynomial rings was first pointed out in [Serre: 1960/61]. Historically, this link has provided a good part of the motivation for Serre's Conjecture, since Serre's seminar report has made it quite clear that an affirmative solution to his "conjecture" would have immediate applications to the Complete Intersection Problem. Thus, to complete our exposition on Serre's Problem, it is only fitting that we devote some space to the theme of Complete Intersections: this is the *raison d'être* for the present Appendix. For obvious reasons, our discussions here will be focused more on the relations between c.i. and Serre's Conjecture, than on c.i. itself. For a broader survey on c.i. with more complete technical details, the reader may consult Jack Ohm's article

[24], ca. 1980.(*) For pointers to the more recent literature on complete intersections, see VIII.3.

The style of our survey here will be rather informal. In particular, a few undefined terms will be used, and a basic knowledge of commutative and homological algebra will be assumed. By and large, everything we need from ring theory can be found in Kaplansky's basic text *Commutative Rings.* All rings considered in this Appendix will be assumed to be commutative.

For a f.g. module M over a (commutative) ring R, we shall write $\mu(M)$ for the minimal number of generators for M. In particular, we'll use this notation for f.g. ideals I in R. Since we shall be dealing (almost exclusively) with noetherian rings R, $\mu(I)$ is defined and always finite. *What can one say about $\mu(I)$ for the ideals (or at least the prime ideals) I in a given (noetherian) ring R?* If R is a principal ideal ring, by definition $\mu(I) \leqslant 1$ for all I. If R is a Dedekind domain, a well-known result in commutative algebra says that all $\mu(I) \leqslant 2$. An obvious question is, therefore: will the $\mu(I)$'s be bounded by a function of dim R (the Krull dimension of R)? This is, unfortunately, far from being the case. In [5], Cohen showed that if R is a noetherian integral domain with dim $R > 1$, there cannot exist a finite bound on $\mu(I)$ for all ideals. Even if we are willing to restrict to *prime* ideals, Sally and Vasconcelos [25] have pointed out that there are noetherian rings R of dimension $\leqslant 2$, for which

$$\{\mu(\mathfrak{p}) : \mathfrak{p} \in \operatorname{Spec} R\}$$

is unbounded. On the positive side, they showed, however, that if R is a finitely generated algebra over a field k, and dim $R \leqslant 2$ (e.g. $R = k[t_l, t_2]$), then $\{\mu(\mathfrak{p}) : \mathfrak{p} \in \operatorname{Spec} R\}$ will remain bounded.

For the purposes of this survey, we shall mainly be interested in the case of polynomial rings over a field k; that is, $R = k\,[t_1, \ldots, t_n]$. For these rings, we can hope to get more, and better, results, though the situation is still not without pitfalls. For instance, we still can't hope to get

$$\{\, \mu(\mathfrak{p}) : \ \mathfrak{p} \in \operatorname{Spec} k[t_1, \ldots, t_n]\,\}$$

bounded, if $n \geqslant 3$. In fact, there are classical examples of height 2 prime ideals in $\mathbb{C}\,[t_1, t_2, t_3]$, constructed by F. S. Macaulay, which require an *arbitrarily large* number of generators.(†) This kind of examples is certainly an eye-opener for anyone just getting into the study of the $\mu(\mathfrak{p})$ question. However, Macaulay's proof techniques "have fallen into desuetude these days" (in the words of J. S. Joel (MR 0466156)). As a sad result, his venerable construction has "undoubtedly confounded generations of graduate students" (in the words of Jack Ohm [24]). Fortunately, the full details of Macaulay's examples are now readily available to modern readers in Geyer [13], or in

(*) This Appendix has its own bibliography, which appears at the end. The numbered citations [l], [2], [3], ... etc. refer to the papers listed there.

(†) Examples of this nature also exist in the power series ring $\mathbb{C}\,[[t_1, t_2, t_3]]$, according to a construction of T. T. Moh in J. Math. Soc. Japan **26** (1974), 722–734.

Sathaye's notes of Abhyankar's lectures [2]. Macaulay's height 2 primes, however, correspond to curves in 3-space with a singular point at the origin, so one can perhaps blame their "pathology" on the presence of the singularity.

To avoid singularities, we can focus on primes $\mathfrak{p} \subset k[t_1, \ldots, t_n]$ such that the factor ring $k[t_l, \ldots, t_n]/\mathfrak{p}$ is a regular domain. This *does* turn out to make a big difference, at least as far as finding a uniform bound on $\mu(\mathfrak{p})$ is concerned. Indeed, the following basic result follows from Forster [12].

Forster's Theorem. *Let R be a regular ring of dimension n (e.g. $R = k[t_1, \ldots, t_n]$ for a field k). For any ideal $I \subseteq R$ such that R/I is regular, $\mu(I) \leqslant n+1$.*

Of course, the case $R = k[t_1, \ldots, t_n]$ is of special interest to the study of affine geometry. In this case, Forster speculated that the upper bound $n+1$ above can be further improved to n. In other words, we have the following

Forster's Conjecture. *If an ideal $I \subset R = k\,[t_1, \ldots, t_n]$ $(n > 0)$ is such that R/I is regular, then $\mu(I) \leqslant n$.*

Classically, we do know, for instance, that $\mu(\mathfrak{m}) = n$ for any maximal ideal $\mathfrak{m} \subset k[t_1, \ldots, t_n]$, and, of course, $\mathfrak{p}$ is principal if $\mathrm{ht}(\mathfrak{p}) = 1$. Thus, the first interesting case of Forster's Conjecture is when $n = 3$, and $\mathfrak{p}$ is a height 2 prime defining a nonsingular curve in 3-space. In this case, Murthy [21] and Abhyankar [1] have independently proved Forster's Conjecture; namely, that $\mu(\mathfrak{p}) \leqslant 3$. Murthy has further remarked that this is the best possible bound for $n = 3$, by constructing nonsingular curves in $\mathbf{A}^3$ whose prime ideals require three generators.

The above discussion has not yet touched upon the Complete Intersection Question proper. But we shall go into that direction now. First let us recall a basic commutative algebra definition given before in III.5.9. A sequence $b_1, \ldots, b_n$ in a commutative ring R is called a *regular sequence* (or alternatively, an *R-sequence*) if $\sum b_i R \neq R$ and for each i, x_i is not a 0-divisor in the factor ring $R/(b_1 R + \cdots + b_{i-1} R)$. *An ideal of R generated by a regular sequence is called a complete intersection ideal* (or a c.i. ideal for short). For the most part, we shall be interested in c.i. ideals in *noetherian* rings only, so let us assume that R is noetherian in the following.

In a noetherian ring R, the height of a proper ideal I (written $\mathrm{ht}\,(I)$) is defined to be the infimum of $\mathrm{ht}\,(\mathfrak{p})$, where $\mathfrak{p}$ ranges over the prime ideals of R containing I. It is well known that the length of any R-sequence in I is at most $\mathrm{ht}\,(I)$ (see Theorem 132 in Kaplansky's *Commutative Rings*). On the other hand, the Generalized Principal Ideal Theorem of Krull says that $\mathrm{ht}\,(I) \leqslant \mu(I)$. Thus, if I is a c.i. ideal generated by an R-sequence $x_1, \ldots, x_r$, then we have

$$\mu(I) \leqslant r \leqslant \mathrm{ht}\,(I) \leqslant \mu(I),$$

so all three numbers here are equal. In some books, the equation $\mathrm{ht}\,(I) = \mu(I)$ is taken to be the definition of a c.i. ideal. These are the ideals I for which there exist $x_1, \ldots, x_r$ generating I where $r = \mathrm{ht}\,(I)$, but there is no guarantee that such $x_1, \ldots, x_r$ can be chosen to be an R-sequence. Here, we prefer to use the stronger (and more standard) definition of c.i. ideals introduced in the last paragraph. If R is a Cohen-Macaulay ring, it can be shown that the two definitions are the same. As it turns out, most of the rings in which we study c.i. ideals (e.g. regular local rings and polynomial rings over fields) are Cohen-Macaulay. In such rings, the c.i. ideals are indeed those $I \subsetneq R$ for which $\mathrm{ht}\,(I) = \mu(I)$.

The notion of a c.i. ideal has a very useful "local" version too: we'll say that $I \subsetneq R$ is a *locally c.i. ideal* if the localization $I_{\mathfrak{m}}$ is a c.i. ideal in $R_{\mathfrak{m}}$ for every maximal ideal $\mathfrak{m} \supseteq I$. (Of course, c.i. ideals are locally c.i.) The notions of c.i. ideals and locally c.i. ideals have clearly a geometric content: in the case $R = k[t_1, \ldots, t_n]$, if I is a c.i. ideal of height r, then the algebraic set V of codimension r defined by I can be cut out by r hypersurfaces in the affine n-space. If I is, instead, a locally c.i. ideal, then V is "locally" cut out by r hypersurfaces at every point.

In [27], Serre studied the codimension 2 complete intersection problem. Stated in the context of polynomial rings, the problem is: *how does one recognize a height* 2 *complete intersection ideal in* $k[t_1, \ldots, t_n]$? To deal with this question, let us first look for necessary conditions. If $I \subseteq R = k[t_1, \ldots, t_n]$ is a c.i. ideal of height 2,, we claim that the following are satisfied:

(A) *I is locally a c.i. ideal*;
(B) *I is unmixed*;
(C) *I has homological (projective) dimension* $\leqslant 1$;
(D) $\mathrm{Ext}^1_R(I, R)$ *is a cyclic R-module.*

Here, (A) is obvious by localization (in any R), and (B) is the classical unmixedness theorem for the Cohen-Macaulay ring $R = k[t_1, \ldots, t_n]$. For (C), write $I = (f, g)$, where f, g form an R-sequence; that is, $f \neq 0$, and g is not a zero divisor mod (f). Let φ be the epimorphism $R^2 \longrightarrow I$ defined by $\varphi(1, 0) = f$ and $\varphi(0, 1) = g$. A simple calculation shows that $\ker \varphi = R \cdot (-g, f)$. In fact, if $\varphi(h_1, h_2) = 0$, then $h_1 f = -g h_2$. This implies that $h_2 = hf$ for some h, and so $(h_1, h_2) = h \cdot (-g, f)$. Thus, we get a free resolution of length 1:

$$(*) \qquad 0 \longrightarrow R \xrightarrow{\psi} R^2 \xrightarrow{\varphi} I \longrightarrow 0, \qquad \psi(1) = (-g, f),$$

which establishes (C). From $(*)$, (D) also follows, for, if we write down the Ext-exact sequence from $(*)$, we get

$$R \cong \mathrm{Hom}_R(R, R) \longrightarrow \mathrm{Ext}^1_R(I, R) \longrightarrow \mathrm{Ext}^1_R(R^2, R) = 0,$$

so $\mathrm{Ext}^1_R(I, R)$ is an epimorphic image of R.

Having now established the necessary conditions (A), (B), (C) and (D), we would naturally like to know if they are also sufficient for the height 2 ideal I to be c.i. *For the moment, let us only assume the truth of* (C) *and* (D) *(with I just any f.g. module over a noetherian ring R).* Let ξ be a generator for $\mathrm{Ext}^1_R(I, R)$, and let

$$(**) \qquad 0 \longrightarrow R \longrightarrow E \longrightarrow I \longrightarrow 0$$

represent the extension class given by ξ. The Ext-exact sequence gives

$$\mathrm{Hom}_R(R, R) \longrightarrow \mathrm{Ext}^1_R(I, R) \longrightarrow \mathrm{Ext}^1_R(E, R) \longrightarrow \mathrm{Ext}^1_R(R, R) = 0.$$

The connecting homomorphism takes 1_R to the generator ξ so it is *onto*. This yields $\mathrm{Ext}^1_R(E, R) = 0$. On the other hand,

$$\mathrm{pd}\,(E) \leqslant \max\,\{\mathrm{pd}(R),\ \mathrm{pd}(I)\}$$

(where "pd" denotes the projective dimension), so (C) implies that $\mathrm{pd}(E) \leqslant 1$. Let

$$0 \longrightarrow P \longrightarrow Q \longrightarrow E \longrightarrow 0$$

be a resolution of E by f.g. R-projective modules P and Q. Adding a projective module to P and Q if necessary, we may assume that $P \cong R^t$ for some t. But then

$$\mathrm{Ext}^1_R(E, P) \cong \bigoplus^t \mathrm{Ext}^1_R(E, R) = 0,$$

i.e. every extension of P by E splits. In particular, $E \oplus P \cong Q$, which implies that E is a (f.g.) projective R-module.

At this point, the relevance of Serre's Conjecture becomes evident. For, returning to the case where $0 \neq I \subseteq R = k[t_1, \ldots, t_n]$, *if we know that the projective R-module E must be free, then we have necessarily $E \cong R^2$,* and hence $\mu(I) \leqslant 2$ by $(**)$. This implies that I is c.i. since R is a Cohen-Macaulay ring.

In the above argument, only the conditions (C) and (D) were used, but not the more "natural" conditions (A) and (B). It turns out, though, that (A) *and* (B) *together imply* (C), for any height 2 ideal I. We won't prove this implication here, but, assuming it, we may recapitulate the main result as follows.

Theorem 1 (Serre). *Let I be a height 2 ideal in $R = k[t_1, \ldots, t_n]$. If I is a c.i. ideal, then* (A), (B), *and* (D) *are valid. Conversely, if* (A), (B), *and* (D) *hold, and every f.g. rank 2 R-projective is free, then I is a c.i. ideal.*

Of course, the freeness of any f.g. projective R-module is now known from the Quillen-Suslin Theorem, so we could have stated Theorem 1 in the form of a clear-cut criterion for a height 2 ideal in R to be a complete intersection. However, we have deliberately kept the freeness of projective modules over R as a "hypothesis" in the second part of the theorem so as to preserve the historical perspective of Serre's result.

Thanks to the work in Murthy [21], a more general formulation of Serre's argument above is available. This more general formulation may be called the Serre-Murthy theorem. It states that, if I is a f.g. module over a commutative noetherian ring R with projective dimension 1, then $\mu\big(\mathrm{Ext}^1_R(I, R)\big)$ (the number of generators for the R-module $\mathrm{Ext}^1_R(I, R)$) is given by the least integer t for which there exists an exact sequence

$$(***) \qquad 0 \longrightarrow R^t \longrightarrow P \longrightarrow I \longrightarrow 0$$

where P is a f.g. projective R-module. In particular, if (say) R is a domain, and f.g. projectives are free over R, then the P above is generated by $\operatorname{rank}(P) = \operatorname{rank}(I) + t$ elements. Under such assumptions, we'll then have the inequality

$$\mu(I) \leqslant \operatorname{rank}(I) + \mu\big(\operatorname{Ext}^1_R(I, R)\big).$$

For a nice exposition on the Serre-Murthy theorem, see §10 of Ohm [24]. Sequences of the type $(***)$ with the first term free have been given certain special names in algebraic geometry; they have been called, for instance, "Bourbaki sequences", or "$\mathcal{E}$-type resolutions" in the geometry literature.

Returning now to the setting of Serre's Theorem 1, let us try to say something more about the module $\operatorname{Ext}^1_R(I, R)$ there. To simplify matters further, let us assume that k is algebraically closed, and I is the prime ideal corresponding to a codimension 2 *smooth* variety in $\mathbf{A}^n_k$ (so the coordinate ring $S = k[t_1, \ldots, t_n]/I$ is regular). Under these conditions, (A), (B) both become automatic, and (D) can be translated into another condition. In fact, in the present case, the R-module $\operatorname{Ext}^1_R(I, R)$ will be naturally isomorphic to the exterior power $\Lambda^{n-2}\Omega_k(S)$, where $\Omega_k(S)$ denotes the module of k-differentials of S. Thus, we obtain the following result.

Corollary 2. *Let k be algebraically closed, and I be the prime ideal corresponding to a codimension 2 smooth variety V in $\mathbf{A}^n_k$. Let $S = k[t_l, \ldots, t_n]/I$ be the coordinate ring of V. If I is a c.i. ideal, then $\Lambda^{n-2}\Omega_k(S)$ is cyclic (as an S-module). Conversely, if $\Lambda^{n-2}\Omega_k(S)$ is cyclic (and every f.g. rank 2 R-projective is free), then I is a c.i. ideal.*

The most concrete case of the above Corollary is when $n = 3$, in which case $\Lambda^{n-2}\Omega_k(S)$ is just $\Omega_k(S)$. Classically, there is a large class of nonsingular curves V that are known to have cyclic modules of differentials, e.g. curves of genus 0 and 1 (rational curves and elliptic curves). Applying Corollary 2 to these curves (and assuming the Murthy-Towber theorem that f.g. projectives are free over $k[t_1, t_2, t_3]$), we get:

Corollary 3. *Let k be algebraically closed, and V be a nonsingular irreducible curve of genus 0 or 1 in $\mathbf{A}^3_k$. Then the prime ideal $I \subseteq k[t_1, t_2, t_3]$ corresponding to V can be generated by a regular sequence f, g.*

Let us now return to the situation of Corollary 2. One main advantage of this Corollary is that the "c.i." property of I is characterized essentially as an "intrinsic" condition on V, i.e. as a condition involving only the coordinate ring S of V. Thus, if I, $I' \subseteq k[t_1, \ldots, t_n]$ are height 2 primes defining isomorphic smooth varieties in $\mathbf{A}^n_k$, then I is c.i. iff I' is c.i. This situation has been generalized in the work of Mohan Kumar. The following, for instance, is one of the consequences of his results.

Theorem 4 ([19]). *Let k be algebraically closed, and I be a height 2 radical ideal in $k[t_1, \ldots, t_n]$. Then I is a c.i. ideal iff there exists a c.i. ideal $J \subseteq k[t_1, \ldots, t_m]$ such that*

$$k[t_l, \ldots, t_n]/I \cong k[t_1, \ldots, t_m]/J \quad \text{(as } k\text{-algebras).}$$

The point is that there is no smoothness assumption needed here, and that m is allowed to be different from n. The ideal I is still assumed to be of height 2, though I do not know if this is really necessary.

Another result of Mohan Kumar [19] dealt with the case of "sufficiently big" height: *if I is the ideal of a smooth variety V in $\mathbf{A}_k^n$ (k algebraically closed),* dim $V \leqslant \frac{n}{2} - 1$, *then I is c.i. iff the normal bundle of $V \subseteq \mathbf{A}_k^n$ is trivial.* This is motivated by similar characterizations of *analytic* complete intersections for analytic submanifolds of $\mathbb{C}^n$ found earlier by Forster and Ramspott. (The dimension bound of Forster-Ramspott is, however, dim $V \leqslant \frac{2}{3}(n-1)$.)

In an important subsequent work [20], Mohan Kumar gave an affirmative answer to a conjecture of Eisenbud and Evans from [8] (see also [7]) concerning $\mu(I)$ for ideals I in a polynomial ring $A\,[t]$. (For a detailed statement on the three Eisenbud-Evans Conjectures, see VIII.1.) From this, Mohan Kumar deduced the following result, where k is now again an arbitrary field.

Theorem 5. *If $I \subseteq k\,[t_1, \ldots, t_n]$ is locally c.i., then $\mu(I) \leqslant n$.*

This extends a result of Murthy [22, Theorem 2.2] from the case of height 2 to arbitrary height, and, in particular, establishes Forster's Conjecture mentioned earlier in this Appendix. (Note that, if an ideal $I \subseteq R = k[t_1, \ldots, t_n]$ is such that R/I is regular, then local algebra shows that I is locally c.i.) Mohan Kumar's proof of Theorem 5 made crucial use of the Quillen-Suslin Theorem on Serre's Conjecture. For more discussions on this (from the viewpoint of conormal bundles), see VIII.3.6. In the case where k is an infinite field, Theorem 5 was also proved independently by Sathaye [26].

For information on more work on the (Forster)-Eisenbud-Evans Conjecture(s) done after 1977/78, see VIII.1 and VIII.3.

In the remainder of this survey, we shall discuss the related problem of "set-theoretic" (and locally set-theoretic) complete intersections. Geometrically, if we are only interested in expressing a variety $V \subseteq \mathbf{A}_k^n$ as a *set-theoretic* intersection of, say, r hypersurfaces, we do not need to know that the ideal I of V can be generated by r elements. Instead, it will be sufficient to know that I is the radical of some ideal I' that can be generated by r elements. This observation leads us to the following two new definitions.

(1) An ideal I in a (commutative) ring R is said to be *set-theoretically generated by* the elements $f_1, \ldots, f_r \in R$ if

$$\text{rad}\,(I) = \text{rad}\,(f_1, \ldots, f_r);$$

(This definition is, of course, partly motivated by Hilbert's Nullstellensatz.) Note that, if such f_i 's exist, they may be taken to be in I.

(2) I is said to be a *set-theoretic complete intersection* if I can be set-theoretically generated by $\text{ht}(I)$ elements. In view of this new definitions, the "old" type of complete intersections introduced before is sometimes referred to as *ideal-theoretic* complete intersections.

Note that an ideal I can often be set-theoretically generated by fewer than $\mu(I)$ elements. For instance, if some power of I is a principal ideal fR, then I is set-theoretically generated by the single element f, while $\mu(I) > 1$ if I itself is nonprincipal. For instance, if R is a Dedekind domain with a torsion Picard group, then the observation made above implies that *every* ideal in R is a set-theoretic c.i., while $\mu(I) = 2$ for any ideal I that is not principal.

Coming back to the geometric context, we'll say that an algebraic set V in $\mathbf{A}_k^n$ is a set-theoretic c.i. if the ideal corresponding to V in $k[t_1, \ldots, t_n]$ is a set-theoretic c.i. The problem of recognizing such set-theoretic complete intersections is, again, a rather delicate matter. We'll sample below a couple of natural questions that one could ask. The first one, first raised by Murthy [22] concerning the relationship between "locally c.i." and "set-theoretic c.i.", turned out to be a fountainhead of ideas for later investigations in this area:

(†) *If $V \subseteq \mathbb{A}_k^n$ is locally a complete intersection (e.g. in case V is smooth), is V then a set-theoretic complete intersection*?

The motivation for this question stemmed from a result of Ferrand and Szpiro (see [11], [22], [29]), which states that the answer to (†) is 'yes' in the case $n = 3$. (This result of Ferrand and Szpiro made essential use of the freeness of projective modules over $k[x, y, z]$ over any field k, which was known by 1975.) For $n = 4$ and dim $V = 2$, Murthy also answered (†) in the affirmative if either

(1) k is algebraically closed and some multiple of the canonical line bundle on V is trivial, or
(2) k is the algebraic closure of a finite field.

In the case where V is a curve (of pure dimension 1), an affirmative answer to the Question (†) (for general n) has subsequently been provided by the work of Mohan Kumar [20: Cor. 5] (and in part by Boratyński [4]). We'll come back to discuss this matter in the continuation of this survey in VIII.3; see Theorem VIII.3.3

The second question on set-theoretic complete intersections is a notoriously difficult classical problem, which "simply" asks:

(‡) *Is every curve in $\mathbf{A}_k^n$ a set-theoretic c.i.? At least in the case $n = 3$?*

In spite of much impressive progress on complete intersections, this single old question seems to have remained unsolved. The main difficulty is known to be in the case of characteristic 0, since Cowsik and Nori [6] have proved the following wonderful result:

Cowsik-Nori Theorem. *If* char $k = p > 0$, *any curve in* $\mathbf{A}_k^n$ *is a set-theoretic complete intersection.*

In the above survey, we have restricted the complete intersection problem to only the *affine* case. This is, of course, because the affine case is directly tied to the study of projective modules over affine algebras (that is, finitely generated commutative algebras over a field), and is, therefore, relevant to the main theme of this book. If we move from affine spaces to projective spaces, the problem of complete intersections can still be fruitfully pursued. Though this is no longer directly relevant to Serre's Problem, we should mention a few important results here. The oldest one is perhaps that of Kronecker (stated without proof in [18], ca. 1882), to the effect that, for an algebraically closed field k, *any algebraic set in* $\mathbb{P}_k^n$ *or* $\mathbf{A}_k^n$ *is an intersection of* $n+1$ *hypersurfaces.*(*) In the affine case, this result can be stated more generally in the language of commutative algebra, as follows. (For this algebraic statement, the field k need not be algebraically closed.)

Theorem 6. *Let* I *be an ideal in a commutative noetherian ring* R *of Krull dimension* n *(for instance,* $R = k[x_1, \ldots, x_n]$ *for a field* k*), then* I *can be set-theoretically generated by* $n+1$ *elements; that is, there exist* $f_1, \ldots, f_{n+1} \in I$ *such that*

$$\operatorname{rad}(I) = \operatorname{rad}(f_1, \ldots, f_{n+1}).$$

(The projective case corresponds to a similar statement over positively graded noetherian rings, which we shall not state explicitly.) This classical result is sometimes referred to as the Kronecker-van der Waerden Theorem.

As it turned out, Kronecker's result is not the best! In Eisenbud-Evans [9] (see also Storch [28] in the affine case), it is shown that, if (say) k is algebraically closed, *any algebraic set in* $\mathbb{P}_k^n$ *or* $\mathbf{A}_k^n$ *is an intersection of* n *hypersurfaces.* Again, we'll state this result more generally in purely algebraic terms in the affine case (and skip the more technical algebraic statement in the projective case).

Theorem 7. *If* A *is a* d*-dimensional noetherian ring, then any ideal in* $A[t]$ *is set-theoretically generated by* $d+1$ *elements. In particular, if* $n > 0$*, any ideal in* $k[x_1, \ldots, x_n]$ *(for any field* k *) is set-theoretically generated by* n *elements.*

In the case $n = 3$, it was Kneser [17] who first showed that any projective curve in $\mathbb{P}_k^3$ is the intersection of three surfaces. The case of the union V of two skew lines shows that a curve in $\mathbb{P}_k^3$ need not be a set-theoretic c.i. (see, e.g. the next paragraph). However, as an algebraic set, V is not irreducible. Along with (‡), it seems to be also unknown if an *irreducible* curve in $\mathbb{P}_k^3$ must be a set-theoretic c.i. On the other hand, there do exist examples of *nonsingular* surfaces in $\mathbb{P}_k^4$ that are not set-theoretic c.i.'s: see Hartshorne [15]. Finally, in sharp contrast to the affine

(*) The assumption that k be algebraically closed is not absolutely essential for this statement (or for the similar statements to follow), and is certainly not needed in the affine case. However, the geometrical results for the projective case over non algebraically closed fields would require more careful assumptions on the existence of k-rational points, etc., so for simplicity we state such results only over algebraically closed fields here.

case, the problem of estimating the number of homogeneous generators needed for homogeneous ideals seems rather hopeless: Geyer [13] has produced examples of nonsingular irreducible curves in $\mathbb{P}^3_k$ whose homogeneous ideals require an arbitrarily large number of generators.

For more information on complete intersections in projective spaces, see Hartshorne's paper [16, §5]. We should also mention that there is a close relationship between set-theoretic complete intersections in projective spaces and the notion of connectedness. According to an early result of Hartshorne ([14], [15]), if V is a set-theoretic c.i. of positive dimension in $\mathbb{P}^n_k$ over an algebraically closed field k, then for any variety $W \subset V$ of codimension $\geqslant 2$, $V \setminus W$ is connected in the k-topology; in particular, V itself is connected. For a nice exposition on this basic connectedness theorem of Hartshorne, see (VI.4) in Kunz's book listed in the reference section of Chapter VIII.

The following list contains all articles referred to in this Appendix, plus a few other relevant ones. For a more extensive list of the literature up to about 1980, see the bibliography in [24]. Many more recent papers, survey articles and some new books on complete intersections are listed in the references to Chapter VIII. For some quick pointers to these newer sources, see VIII.3.

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Chapter VIII

New Developments (since 1977)

With certain conjectures in mathematics, the solution of the conjecture marked the end of a period of investigation, and workers in the field moved on to other directions of research. With other conjectures, the solution of a once-open problem would lead to analogues and extensions of the problem, and provide a fountainhead of ideas for further generalizations, or even formulation of significant new problems. Fortuitously, Serre's Conjecture is of the latter kind. In fact, the continuing research on problems related to Serre's Conjecture in the post Quillen-Suslin period (that is, 1977-present) turned out to be even more intensive and voluminous than the work done on the original conjecture between 1955 and 1976. In total, at least some six hundred papers have appeared after 1976 that, in one manner or another, impacted upon Serre's Problem — by way of refinements, generalizations, adaptations, variations, or applications. The surprising level, quality, quantity, and breadth of this ensuing work is, in my opinion, truly a tribute to the genius of Quillen, Suslin, and Serre.

In this chapter, I shall give a survey on some of the post-1976 developments mentioned above on the general theme of Serre's Problem. Unlike the earlier chapters, this one contains essentially no proofs. It is also hardly self-contained, since from time to time the survey touches upon topics that have not been discussed in detail before in our text. Writing in this relatively free style, we'll try to give as broad an overview as possible on the mathematics engendered by Serre's Problem from 1977 to the present. To help make sense out of the myriad developments that have taken place, we have made an effort at classifying such developments into a number of topical areas. Accordingly, this survey will consist of the following thirteen sections:

§1. $R[t_1, \ldots, t_n]$ for R Noetherian
§2. Projective Modules over Affine Algebras
§3. Complete Intersections
§4. Monomial Algebras and Discrete Hodge Algebras
§5. Unimodular Rows
§6. The Bass-Quillen Conjecture
§7. $R[t_1, \ldots, t_n]$ for R Non-Noetherian
§8. Noncommutative Polynomial Rings

It is hoped that the survey offered in this chapter will give an idea of what happened to Serre's Problem between 1977 and the present. In the bibliography at the end of the chapter, I will list all relevant papers I am aware of, although not all of these papers will be explicitly cited in the text.(*) I make no claim at completeness in this survey. On the contrary, I am sure there are many inadvertent omissions, and I apologize in advance to any author(s) whose pertinent papers I have failed to include. With this disclaimer, I hope the bibliography will nevertheless provide a handy record of the literature on Serre's Problem in the last 30 years, and convey a sense of the intensity and vibrancy of the recent research in this area.

Before we begin our survey, a quick word is also in order on the relationship between Serre's Problem and Algebraic K-Theory. These two topics originated from about the same time in the 1950s: nowadays, Serre's Problem is perhaps best viewed as a part of "nonstable" K_0-theory, with some possibilities of generalizations to K_1 (or even K_2). Thus, the work in algebraic K-theory in general is never too far removed from the interest of researchers working on Serre's Problem and its generalizations. Conversely, Serre's Problem is referred to or discussed in a considerable number of papers and books in algebraic K-theory. But Serre's Problem is primarily concerned with polynomial algebras and their variations, so it would be pointless to include, in our survey, papers on the algebraic K-theory of general rings. To make up for this omission, we'll just take this opportunity to mention a few of the standard books written on algebraic K-theory after 1970; e.g. [Milnor: 1971], [Silvester: 1981], [Rosenberg: 1994], [Inasaridze: 1995], [Srinivas: 1996a], and [Magurn: 2002]. See also Weibel's forthcoming book at the URL: http://math.rutgers.edu/~weibel/Kbook.html. Among the many surveys written on the subject of "classical" algebraic K-theory, the reader may consult [Bass: 1992], [Vaserstein-Suslin: 1974], [Vaserstein: 1976], [Suslin: 1982, 1984], and [Weibel: 1997]. For a "popular" introduction to Serre's Conjecture requiring no prior knowledge of higher algebra, see [Gustafson-Halmos-Zelmanowitz: 1978].

§1. $R[t_1, \ldots, t_n]$ for R Noetherian

Before and after Serre's Conjecture was proved by Quillen and Suslin in 1976, there was a lot of research work on projective modules over $R[t_1, \ldots, t_n]$ for various kinds of ground rings R. In this section, we survey some of the key results in this area in the case where R is *commutative noetherian* (and mostly of finite Krull dimension).

(*)For convenience, we have duplicated here a few references from the bibliography for Ch. I–VII – mainly for papers appearing in the period 1976–78. The overlap between the two bibliographies is otherwise negligible.

Parallel surveys on the commutative nonnoetherian case and the noncommutative case will be given later in §7 and §8. *Throughout this section, R denotes a commutative noetherian ground ring*, unless it is stated otherwise.

To facilitate our discussion, let us first recall some basic terminology. For a f.g. projective P over a ring A, an element $p \in P$ is said to be *unimodular* (see I.4) if $A \cdot p$ is a free direct summand of P with basis $\{p\}$ (see I.4). We say P is *cancellative* (or occasionally, "cancellable") if, for any f.g. projective A-modules P' and Q:

$$P \oplus Q \cong P' \oplus Q \Longrightarrow P \cong P'.$$

According to the inaugural results of Serre and Bass in projective module theory, if rank $P \geqslant d+1$ where $d = \dim(A)$, then P has a unimodular element, and P is cancellative. These facts are called, respectively, *Serre's Splitting Theorem*, and *Bass's Cancellation Theorem.* (We have already encountered the latter in V.4.8.) From the early 1960s and on, these two basic results have provided much of the paradigm for the research on projective modules (and their geometric analogues, vector bundles).

In the case where R is a field, the polynomial ring $A = R[t_1, \ldots, t_n]$ has Krull dimension n. Serre's original result on the existence of unimodular elements would only apply to (f.g.) projective A-modules of rank $\geqslant n+1$, and is thus far insufficient for proving Serre's Conjecture.

Partly prompted by Serre's Conjecture, Bass and Murthy [1967] proposed to study the "Serre dimension" of a ring A: the smallest integer s such that any f.g. projective A-module of rank $\geqslant s+1$ has a unimodular element. (A similar invariant was also considered in various subsequent papers, e.g. [Geramita-Roberts: 1970], and [Lissner-Moore: 1970].) Extending earlier results of Endô, Seshadri and Murthy, Bass and Murthy proved the following.

Theorem 1.1. *If R has dimension $\leqslant 1$ and $R/\mathrm{Nil}(R)$ has a finite normalization, and A denotes either the polynomial ring $R[t_1, \ldots, t_n]$ or the Laurent polynomial ring $R[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$, then A has Serre dimension $\leqslant 1$ if $n = 1$, or if $n \in \{1, 2\}$ and R is semilocal.*

From this, they deduced, for instance, that for any rank 1 abelian group π and any Dedekind domain R, the group ring $R\pi$ has Serre dimension $\leqslant 1$. Results of this nature led Bass and Murthy to ask the question: *is the Serre dimension of $R[T]$ bounded by* $\dim R$ *for a f.g. free abelian group or monoid T* (of any rank)? An affirmative answer to this for R a field and T a monoid would amount to a solution of Serre's Conjecture. The significance of the Bass-Murthy question is that it ushered in a new, if conjectural, paradigm, that the Serre dimension of $R[T]$ should be "controlled" by the (Krull) dimension of R, and not by the rank of the free (abelian) group or monoid T.

To unify the polynomial case and the Laurent polynomial case, one may consider more generally the ring

$$A = R[t_1^{\pm 1}, \ldots, t_m^{\pm 1}, s_1, \ldots, s_n] \quad (m, n \geqslant 0) \tag{1.2}$$

over a (commutative, noetherian) ground ring R. If $\dim R = d$, one can ask the following two questions.

Question 1.3. *Is* Serre-dim$(A) \leqslant d$*? (That is, does every f.g. projective* A*-module of rank* $\geqslant d+1$ *have a unimodular element?)*

Question 1.4. *Is every f.g. projective* A*-module of rank* $\geqslant d+1$ *cancellative?*

In the case where one of m, n is 0 (so A is a polynomial ring or a Laurent polynomial ring), (1.3) and (1.4) were explicitly raised as (a part of) Question XIV in Bass's survey article on "Classical" algebraic K-theory [Bass: 1972]. As mentioned above, Question (1.3) for $m+n=1$ was raised already in [Bass-Murthy: 1967], and answered affirmatively for a "majority" of ground rings R with Krull dimension $d=1$. And, in the case $d=1$, a "yes" answer for (1.3) will imply the same for (1.4) for projective modules of constant rank, since rank 1 projective modules are always cancellative (by an exterior power argument in the spirit of (I.4.11)). In the case $m=n=0$, of course, (1.3) and (1.4) were answered affirmatively by the original motivating results of Serre and Bass. An early work on the cancellation question (1.4) for $m=0$ appeared in [Swan: 1974].

In the case where P in (1.3) and (1.4) is *stably extended* from R, the "Big Rank Theorem" of Swan (V.4.7) implies that P is already extended from some (f.g. projective) R-module P_0, necessarily of rank $\geqslant d+1$. From the theorems of Serre and Bass, P_0 has a unimodular element, and is cancellative. From these and (V.4.7′), we check easily that P itself has a unimodular element and is cancellative. Thus, the answers to (1.3) and (1.4) are both "yes" if P happens to be stably extended. (This is always the case, for instance, if R is a regular ring, by (II.5.8).)

The questions (1.3) and (1.4) are closely related to the three "Eisenbud-Evans Conjectures" raised in [Eisenbud-Evans: 1973b]. Before we come to these conjectures, however, we must first go back to the seminal work done in [Eisenbud-Evans: 1973a]. In this paper, Eisenbud and Evans studied the notion of "basic elements" in f.g. modules that is due to Swan. Restricting ourselves to the commutative case (for convenience), an element m in a f.g. module M over a commutative ring A is called *basic* if $m \notin \mathfrak{p}M_{\mathfrak{p}}$ for any $\mathfrak{p} \in \mathrm{Spec}(A)$. (The notion of basic elements generalizes that of unimodular elements in projective modules.) In [Eisenbud-Evans: 1973a], the authors obtained existence results on basic elements in f.g. modules which implied at the same time Serre's splitting theorem, Bass's cancellation theorem, and the Forster-Swan theorem on the minimal number of generators of f.g. modules (to be stated below). Several nice expository accounts on the Eisenbud-Evans results on basic elements are now available; we recommend highly [Mandal: 1997, §4.1], and [Ischebeck-Rao: 2005, §9.4]. The latter reference contains especially some interesting schematic diagrams clarifying the relationship of the Eisenbud-Evans theorems to various other topics in commutative algebra and algebraic K-theory. For some further generalizations of the Eisenbud-Evans theorems, see [Bruns: 1976].

The Eisenbud-Evans Conjectures were prompted largely by the work in the basic elements paper mentioned above. In [Eisenbud-Evans: 1973b], however, these conjectures were formulated only in the case $A=R[s]$ (instead of the more general rings A of the type (1.2)). For the reader's easy reference, we include a complete statement of each of these conjectures below, for $A=R[s]$. For any f.g. module M over a ring, let us write $\mu(M)$ for the least number of elements that can generate M. The first Eisenbud-Evans Conjecture is essentially a generalization of (1.3) (in the case $A = R[s]$) from projective modules to f.g. A-modules:

EE$_1$: *If M is a f.g. module over $A = R[s]$ such that, for any $\mathfrak{p} \in \mathrm{Spec}(A)$, $\mu(M_{\mathfrak{p}}) \geqslant d+1$ (where $d = \dim R$), then M has a "basic" element.*

The second Eisenbud-Evans Conjecture is the restatement of a positive answer to (1.4) for polynomial rings in one variable:

EE$_2$: *A f.g. projective module over $A = R[s]$ of rank $\geqslant d+1$ (where $d = \dim R$) is cancellative.*

To motivate the third conjecture of Eisenbud and Evans, let us first state the classical result of Forster and Swan for $\mu(M)$ over an *arbitrary* commutative noetherian ring. For several different proofs of (and good perspectives on) this result, see [Ischebeck-Rao: 2005, §9.3].

Forster-Swan Theorem. *For a f.g. module M over any commutative noetherian ring A,*

$$\mu(M) \leqslant \max\{\,\mu(M_{\mathfrak{p}}) + \dim(A/\mathfrak{p}) : \ \mathfrak{p} \in \mathrm{Spec}(A)\,\}.$$

A good example illustrating the use of the Forster-Swan bound is the case of a nonzero ideal M over a Dedekind domain A. In this case, for any prime ideal $\mathfrak{p}$, we have $\mu(M_{\mathfrak{p}}) = 1$ since $A_{\mathfrak{p}}$ (being a discrete valuation ring) is a PID. It follows that, for $\mathfrak{p} \neq 0$, $\mu(M_{\mathfrak{p}}) + \dim(A/\mathfrak{p}) = 1$, while for $\mathfrak{p} = 0$, $\mu(M_{\mathfrak{p}}) + \dim(A/\mathfrak{p}) = 2$. The Forster-Swan Theorem thus gives the well-known result that $\mu(M) \leqslant 2$ for ideals M in a Dedekind domain A.

The point of the third Eisenbud-Evans Conjecture is that the Forster-Swan bound could be further improved if A is a polynomial ring $R[s]$:

EE$_3$: *For a f.g. module M over $A = R[s]$ where R is noetherian of dimension d, we have*

$$\mu(M) \leqslant \max\{\,\mu(M_{\mathfrak{p}}) + \dim(A/\mathfrak{p}) : \ \dim A/\mathfrak{p} < \dim A = d+1\,\}.$$

The improvement lies in the omission of the primes $\mathfrak{p} \subset A$ for which $\dim(A/\mathfrak{p}) = d+1$ in the computation of the maximum above. These are the primes in A lying at the bottom of the prime chains of maximum length in the polynomial ring $A = R[s]$.

All of the above questions and conjectures have now been settled affirmatively. However, the road to their full solutions was often via iterations and successive approximations. In the following, we'll give a summary of some of the key results. For an alternative survey, we refer the reader to [Bhatwadekar: 1999].

For $d = 1$, under the assumption that $R/\mathrm{Nil}(R)$ has finite normalization and $1/2 \in R$, Kang (1979) answered (1.3) affirmatively in the polynomial ring case ($m = 0$), and obtained partial positive answers in the case of general m, n. These results were later improved somewhat in [Roy: 1982], [Greither: 1982], and [Kang: 1984].

Sathaye (1978) proved that, in the case R is a domain, the third Eisenbud-Evans Conjecture (EE$_3$) is valid if it is valid for all ideals of $R[s]$. With this big reduction step, Sathaye proved (EE_3) for all affine domains R over infinite fields k. Using very similar ideas, Mohan Kumar (1978) made the same reduction step to the case of ideals in reduced (noetherian) polynomial rings. With the further help of the Quillen-Suslin theorem, Mohan Kumar obtained a full proof of (EE_3) for a general noetherian ring R. It is of historical interest to note that Sathaye's proof did not use the Quillen-Suslin Theorem, and that the papers of Sathaye and Mohan Kumar appeared back to back, by design, in the same issue of the *Inventiones Mathematicae* in 1978.

The proof of (EE_3) also led to an affirmation of "Forster's Conjecture", mentioned in our Appendix on Complete Intersections); more information on this will be given in (3.6) below. For a follow-up work of Mohan Kumar on the conjecture (EE_3), see [Mohan Kumar: 1981]. For some further sharpenings of the conjecture (EE_3) in the case of f.g. modules over a polynomial ring $A = k[s_1, \ldots, s_n]$ (where k is an infinite field), see [Lyubeznik: 1988a]. (The sharpening in this paper occurs in further limiting the prime ideals to those at which the module in question is not free.)

Only shortly after Sathaye and Mohan Kumar made their discoveries, B. Plumstead proved, in his Chicago thesis (1979), *all three* of the Eisenbud-Evans Conjectures (EE_1)–(EE_3)! However, it took several years before Plumstead was persuaded to write up his results for publication, in [Plumstead: 1983]. In this work, Plumstead also raised the Eisenbud-Evans Conjectures for *multivariate* polynomial rings $(A = R[s_1, \ldots, s_n])$. Plumstead's proofs of (EE_1)–(EE_3) featured the use of a number of interesting new techniques, including:

(1) the notion of "generalized dimension functions"; and

(2) the sheaf-theoretic patching of basic elements.

These techniques proved to be very useful in the subsequent work in this area. For an exposition on Plumstead's proofs, see [Mandal: 1997, §4.3]. (See also [Ischebeck-Rao: 2005, §9.6] for (EE_3).) Plumstead's work was later extended to the Laurent polynomial case $(A = R[t, t^{-1}])$ in [Mandal: 1982a, b; 1983]. For another extension of Plumstead's results and some applications, see [Lindel: 1984].

In the multivariate *polynomial* case, Bhatwadekar and Roy (1984) answered (1.3) affirmatively for all d-dimensional (noetherian) rings R. As a consequence of their work, any f.g. projective $R[s_1, \ldots, s_n]$-module of rank r can be generated by $r + d$ elements. Bhatwadekar and Roy also settled (1.4) positively for 2-dimensional normal rings R. Other partial results on (1.4) for polynomial rings appeared in [Bhatwadekar-Roy: 1987] and [Bhatwadekar: 1987], before a full affirmative answer was given in [Rao: 1988a]. In more recent work, Bhatwadekar (2001) proved the cancellativity of f.g. projectives of *all* ranks (or, most crucially, rank 2) when R is a normal 2-dimensional affine domain over a perfect field of characteristic $\neq 2$ and of cohomological dimension $\leqslant 1$.

As for the general *Laurent polynomial* case (with A as in (1.2)), Question (1.3) was answered affirmatively in [Bhatwadekar-Lindel-Rao: 1985]. Given this result, one can say the following about the extendibility of f.g. projective modules over A: to check that all such projectives P are extended from R, it would be sufficient to check the case where rank $P \leqslant d = \dim R$. In the case of polynomial rings, this was first observed in [Roitman: 1979] for $d = \dim R$, and earlier in [Roitman: 1977b] for $d = \text{gl.dim } R$ (the global dimension of R).

For the Laurent polynomial case, the cancellation problem in Question (1.4) proved to be more difficult. The extension of the Bhatwadekar-Lindel-Rao result to this case was only accomplished later in [Lindel: 1995]. In this paper (published posthumously with the assistance of A. Wiemers), Lindel gave a beautiful proof for the fact that, if P in (1.4) has rank $\geqslant \max\{2, d+1\}$ (over A as in (1.2)), then the "elementary group" (or the group of transvections on $P \oplus A$ acts transitively on its unimodular elements. This remarkable result, proved by Lindel *without* assuming the Quillen-Suslin Theorem, provided the last step for a full affirmative answer to the questions (1.3) and (1.4). For an exposition on some of Lindel's results, see [Mandal: 1997, §7.2], as well as [Ischebeck-Rao: Ch. 7].

Although the Serre dimension bound in (1.3) is generally the best, we need not be discouraged from seeking better bounds in more special situations. For instance, if R itself is a polynomial ring $R_0[x]$, then a general Laurent extension A of the type (1.2) over R is also one over R_0 (with one more polynomial variable). In this case, the Serre dimension of A is bounded by $\dim R_0 = \dim R - 1$, so A-projectives of rank $\geqslant d = \dim R$ will already have a unimodular element. More generally, one may consider the (noetherian) rings B between $R_0[x]$ and its total quotient ring; these rings are called *birational coverings* of $R_0[x]$. The question whether $B[s_1, \dots, s_n]$ has Serre dimension bounded by $\dim R_0$ was raised in [Bhatwadekar-Roy: 1983]. The answer is "yes" for $n = 0$ according to [Rao: 1982], where the corresponding cancellative result was also proved. For general n, the answer is known to remain "yes" as long as R_0 is *normal*, according to a result (the last one) proved in [Bhatwadekar-Lindel-Rao: 1985].

Questions of the type (1.3) and (1.4) can also be raised more generally with monoid algebras $A = R[M]$ where M is a commutative monoid "sufficiently resembling" $\mathbb{Z}^m \times \mathbb{N}^n$. More detailed information on such questions can be found in §§4–6.

The fact that the "K_0-stability range" of a Laurent Polynomial ring A (as in (1.2)) is influenced solely by $\dim R$ is a very useful paradigm. Certainly it has provideed the best and most general perspective for the truth of Serre's Conjecture so far. One step beyond, Suslin's stability theorem for the general linear group, mentioned in (III.3.8) (in the polynomial case) may be thought of as the K_1-manifestation of the same paradigm. But of course, the considerations of K_0 and K_1 stability are often intertwined. For instance, the proof in [Bhatwadekar-Lindel-Rao: 1985] for the existence of unimodular elements for projective modules over Laurent polynomial rings made ample use of Suslin's stability theorem for the general linear group over such rings.

Of course, there are other results in the literature on projective modules over $R[t_1, \dots, t_n]$ (for R noetherian) that we have not yet touched upon. Let us conclude by mentioning just one such result here, for the reason that, somewhat unusually, it has to do with the characteristic of the ground ring. This result, from [Roitman: 1986], states the following:

If R is a commutative d-dimensional noetherian ring of finite characteristic prime to $d\,!$, then for any n, any stably extended module in $\mathfrak{P}(R[t_1, \ldots, t_n])$ of rank $\geqslant d/2+1$ is extended from R.

This implies, in particular, that (for R as above) stably free $R[t_1, \ldots, t_n]$-modules of rank $\geqslant d/2+1$ are extended from R. In view of Grothendieck's theorem (II.5.8), Roitman's result affirms the following special case of the Bass-Quillen Conjecture:

If R is a (commutative) d-dimensional regular ring of finite characteristic prime to $d\,!$, then f.g. projective $R[t_1, \ldots, t_n]$-modules of rank $\geqslant d/2+1$ are extended from R.

These are interesting special results which happen to hold over noetherian polynomial rings of finite characteristic.

We remind the reader that the survey in this section was primarily focused on the case of commutative noetherian base rings. Discussions on the *non-noetherian* and *noncommutative* cases will be given later in §7 and §8 respectively.

§2. Projective Modules over Affine Algebras

By affine algebras, we mean as usual commutative finitely generated algebras over a field. Finitely generated projectives over such algebras correspond to vector bundles over affine varieties, and their study generalizes the study of finitely generated projectives over polynomial rings. As is to be expected, the study of projectives over affine algebras involves more algebro-geometric methods than in the case of polynomial rings. For such algebras, projectives are no longer free; the main focus of the research in this setting remains to be (as in §1) on the cancellation problem and the problem of existence of unimodular elements in projective modules.

In this section, we give a report on the work on (f.g.) projective modules over affine algebras. Not surprisingly, much of this work took place only *after* the solution of Serre's Conjecture by Quillen and Suslin in 1976. The best surveys for this line of research are the two excellent articles of Murthy (1995, 1999), where the reader can find complete statements (and often informative sketches of proofs) for the main results obtained in this area. Our report in this section is largely based on Murthy's, and is included here mainly to make this chapter entirely self-contained.

In the work on affine algebras, it was clear right from the start that the case of an algebraically closed ground field would be more amenable. For instance, the rank 2 projective module P corresponding to the tangent bundle of the real 2-sphere is nonfree and in fact indecomposable, but scalar extension from $\mathbb{R}$ to $\mathbb{C}$ makes P free. This and other similar examples suggest that $\mathbb{C}$ is easier to handle than $\mathbb{R}$, so it would certainly be a good idea to first focus on the case of affine algebras A over an *algebraically closed* ground field k. *These assumptions on k and on A will now be in place until further notice (and projective modules will be assumed to be f.g. over A).*

In view of the early theorems of Serre and Bass, not much more needs to be said about the case where A is 1-dimensional affine. In 1976, Murthy and Swan proved a cancellation theorem for rank 2 projective modules that is general enough to imply that, *in the case* $\dim A = 2$ (affine surfaces), *all projective modules are cancellative*. This work, [Murthy-Swan: 1976], appeared in the *Inventiones Mathematicae* just preceding Quillen's paper solving Serre's Conjecture. (The special issue of *Invent. Math.* in which these papers appeared was, appropriately enough, dedicated by that journal to J.-P. Serre.) In their paper, Murthy and Swan also proved that, *if* A *is 2-dimensional and regular, then all projectives over* A *have the form* "free $\oplus$ ideal" *iff all maximal ideals in* A *are generated by two elements*. These properties are shown to hold, for instance, in the case where A defines a ruled surface.

The next breakthrough came in [Suslin: 1977c]. In this paper, the Murthy-Swan cancellation theorem over surfaces was extended to the following: *if* $\dim A = n$, *then all projective* A*-modules* P *of rank* $\geqslant n$ *are cancellative*. In particular, *stably free* A*-modules* P *of rank* $\geqslant n$ *are all free*. (The crucial case for either statement is, of course, when $\text{rank}(P) = n$.) This line of research was continued in [Suslin: 1978b, c] and [Suslin: 1982b] over various other types of grounds fields. Subsequently, Suslin's rank n theorem was proved for finite ground fields in [Mohan Kumar-Murthy-Roy: 1988], and for C_1 ground fields of characteristic zero in [Bhatwadekar: 2003]. See also [Bhatwadekar: 2001] where it is proved that *every* f.g. projective module is cancellative over $A[x_1, \ldots, x_r]$ for normal affine surfaces A over certain perfect fields.

We shall say more about the case of general ground fields later; *for now, we continue to assume that* k *is an algebraically closed ground field.*

On analytical grounds (holomorphic vector bundles over Stein manifolds), it would appear that one could hope for much better results than Suslin's cancellation theorem. In fact, Suslin had written (albeit with reservation) in [Suslin: 1978b, p. 328]: "I suppose that the correct bound (for cancellativity) would be rank $P \geqslant (1+n)/2$." That this is *not* the case was later shown by [Mohan Kumar: 1985]. Using Chow group methods, Mohan Kumar constructed, for any prime p, a smooth, rational affine k-algebra A of dimension $n = p + 2$ admitting a rank p stably free module P that is not free. Such a projective module is *not* cancellative, and has rank $\geqslant (1+n)/2$ if $p \geqslant 3$. Since rank P in this example for $p = 3$ is just "2 away" from $n = \dim A$, the only "hope" left for general cancellativity of projectives over A (when k is algebraically closed) would be the case where rank $P = n - 1$.

This case has, however, remained largely mysterious. For $P = A^{n-1}$, the question would be whether stably free modules of rank $n-1$ is free (or, whether any unimodular vector of length n over A is completable to an invertible matrix). Even for $n = 3$, this special case of the cancellation question (at rank $n-1$) has remained unanswered. One positive case is when A has the form $R[x]$ where R is regular. In this case, by the main result in [Lindel: 1982] on the Bass-Quillen Conjecture, P (of rank $n-1$) must be extended from $P_0 = P/xP$ over R. Since R is (necessarily) an affine k-algebra of dimension $n-1$, Suslin's theorem guarantees the cancellativity of P_0, and this (easily) implies the cancellativity of P. But of course, the case where $A = R[x]$ is rather far from "typical."

Some results on the cancellativiity of projectives of rank $n-1$ were obtained in [Suslin: 1982b]. For instance, if A has the form $k[x_1, \ldots, x_n, f^{-1}]$ where $0 \neq f \in$

$k[x_1, \ldots, x_n]$, and char (k) is either 0 or $\geqslant n$, Suslin showed that all A-projectives of rank $\geqslant n-1$ are free. Some of Suslin's results were extended to other classes of affine algebras in the recent work [Murthy: 2002]. For instance, if $k = \overline{\mathbb{F}}_p$ where $p \geqslant n$, and A lies between an n-dimensional integral normal affine k-algebra B and its quotient field, Murthy showed that stably free A-modules P of rank $n-1$ are extended from B. Thus, if B is, for instance, $k[x_1, \ldots, x_n]$, then by the Quillen-Suslin Theorem P must be free. Murthy's results enabled him to produce various interesting classes of smooth rational affine n-folds over which all vector bundles of rank $\geqslant n-1$ are trivial. Explicit examples include the variety $\mathrm{SL}_r\big(\mathbb{C}\big)$ (where $n = r^2 - 1$), and the 3-folds

$$x + x^r y + z^s + w^t = 0 \qquad (r \geqslant 2,\ \gcd(s,t) = 1)$$

over $\mathbb{C}$ which appeared in the work of Koras and Russell (1997) on the linearization of $\mathbb{C}^*$-actions on the affine 3-space.

The theme of the cancellativity of projective modules of rank $n-1$ over *smooth* affine algebras of dimension n is recently revisited in a paper of R. Rao (to appear).

Coming back to the case of rank n, Mumford's early work on rational equivalence of 0-cycles on surfaces already showed that regular 2-dimensional affine algebras may have many indecomposable rank 2 projective modules. Explicit examples are the projectives defined by the unimodular row (x_1, x_2, x_3) over the smooth affine surfaces $x_1^m + x_2^m + x_3^m = 1$ for $m \geqslant 4$, as cited in [Murthy-Swan: 1976]. Later, ideas of Nori led to the construction of indecomposable projectives of rank n over, for instance, the affine n-folds defined by

$$x_1^m + \cdots + x_{n+1}^m = 1 \qquad \text{for } m \geqslant n+2;$$

see [Srinivas: 1993]. These examples of "top rank" indecomposable projectives showed that Serre's theorem on the existence of unimodular elements cannot be improved upon, even over smooth n-folds over algebraically closed fields.

Because of the above, the problem of finding the obstruction to the existence of unimodular elements in "top rank" projective modules over affine algebras took center stage. Here, again, topology provides the needed inspiration. In topology, an n-dimensional complex vector bundle over a CW-complex of dimension n splits off a trivial line bundle iff its top Chern class is zero. The task at hand for the algebraists is thus that of adapting this topological theorem into an algebraic setting.

In the theory of projective modules over a commutative ring A, Chern classes may be defined as follows. If P is a projective A-module of rank n, we define the top Chern class $\mathrm{C}_n(P)$ to be the element

$$\sum_{i=0}^{n} (-1)^i [\Lambda^i(P^*)] \in K_0(A),$$

where P^* denotes the A-dual $\mathrm{Hom}_A(P, A)$ of the module P, and Λ^i means taking the i th exterior power. If P admits a unimodular element, we may write $P \cong A \oplus Q$ for some projective module Q, in which case a quick calculation shows that $\mathrm{C}_n(P) =$

0. One may, therefore, ask if, conversely, $\mathrm{C}_n(P)=0$ for a projective module P of rank n (over an n-dimensional affine k-algebra A) would imply the existence of a unimodular element in P.

If $n \leqslant 2$, the answer is easily seen to be "yes". In the case $n=3$, the answer was shown to be also "yes" in [Mohan Kumar-Murthy: 1982] in the smooth case. In this paper, taking the Chern classes of vector bundles in the Chow groups, in the sense of Grothendieck, the authors showed that, *over smooth affine* 3*-folds, the stable isomorphism class of a vector bundle is determined by its Chern classes* (and there exist bundles with arbitrarily prescribed Chern classes). Thus, a rank 3 vector bundle P with top Chern class zero can be stably matched with some $Q \oplus T$ for the trivial line bundle T (by matching the first two Chern classes), and Suslin's cancellation theorem then gives $P \cong Q \oplus T$. For a quick exposé on the key ideas involved, see [Murthy-Mohan Kumar: 1984].

The work on the top Chern class in the case of algebraically closed ground fields culminated in [Murthy: 1994] (preceded by the announcement [Murthy: 1988]). The formulation of the main results in these papers depends on the use of certain subgroups $\{F^n K_0(A)\}$ in $K_0(A)$. Here, by the standard process of dévissage, we identify $K_0(A)$ with the Grothendieck group of f.g. A-modules of finite projective dimension. With this identification, $F^n K_0(A)$ is the subgroup of $K_0(A)$ generated by the class $[A/\mathfrak{m}]$ where $\mathfrak{m}$ ranges over all maximal ideals such that the localization $A_{\mathfrak{m}}$ is a regular local ring (of dimension n). Using the Riemann-Roch Theorem and earlier results from [Mohan Kumar: 1984] on complete intersection ideals of top height, Murthy (1994) proved that:

If $F^n K_0(A)$ has no $(n-1)!$-torsion, then for a f.g. projective A-module P of rank n, P has a unimodular element iff $\mathrm{C}_n(P)=0$.

In particular, this conclusion can be drawn if either char $(k)=0$, or $n=2$, or $n \geqslant 3$ and A is normal. In the case where A is a regular domain, Murthy further showed that

A local complete intersection ideal $I \subseteq A$ of height n is a complete intersection iff $[A/I]=0$ in $K_0(A)$.

For a sketch of the arguments used for the proof of these statements, see [Murthy: 1995], §§2–3.

Some of the results in [Murthy: 1994] are extended and applied in [Mandal: 1998] and [Mandal-Murthy: 1998]. The main theme of the latter paper is the study of the "projective generation" of an ideal I in an affine algebra A: if I can be expressed as an epimorphic image of a projective module P, we say that I is *projectively generated by* P. Note that, in the case where $I=A$, this corresponds to the existence of a unimodular element in P. Of course, the idea of "projective generation" comes, in part, from Serre's work on height 2 local complete intersection ideals in polynomial rings, which we have discussed in the "Appendix on Complete Intersections" preceding this chapter.

In [Mandal-Murthy: 1998], some Chern class criteria were given for a local complete intersection ideal I of height n to be projectively generated by a given P of rank n, and for an arbitrary ideal $I \subseteq A$ to be projectively generated by P given that I/I^2 is an epimorphic image of P. In the case where A is of the form $R[x]$ or $R[x, x^{-1}]$, some results on lifting surjections $P \to I/I^2$ to surjections $P \to I$ are obtained in [Datt Kumar-Mandal: 2002].

Two favorite themes in the work on projective generation are the (so-called) addition and subtraction principles. Generally speaking, given a projective module P and comaximal ideals I and J in a (say noetherian) ring A, the *addition principle* is an attempt to produce a surjection $P \longrightarrow\!\!\!\!\!\rightarrow I \cap J$ from a surjection $P \longrightarrow\!\!\!\!\!\rightarrow I$, and the *subtraction principle* is the reverse process. Principles of this nature are intended to be applied toward the study of the number of elements needed to generate an ideal in A. For instance, taking P to be free, we hope to generate $I \cap J$ by the same number of generators for I, and, taking I to be A, we hope to get a unimodular element in P by starting with a suitable surjection $P \longrightarrow\!\!\!\!\!\rightarrow J$. Of course, such "principles" do not hold in general, but in more specific circumstances, one hopes that they would apply.

One of the earliest investigations on addition and subtraction principles was carried out in the seminal work [Mohan Kumar: 1984]. For the free module $P = A^n$ over an n-dimensional affine algebra A (over an algebraically closed field k), Mohan Kumar proved addition and subtraction principles for a pair of comaximal local complete intersection ideals I, J of height n. From these principles, it follows that, if any two of the ideals I, J, and $I \cap J$ above are complete intersections, then so is the third. If, moreover, A is a regular affine domain and every maximal ideal of A is a complete intersection, then in fact every local complete intersection ideal of height n is a complete intersection.

The theme of additions and subtractions was revived in [Sridharan: 1995a, b]. In the first paper, using a homotopy result from [Mandal: 1992] (to be discussed below), Raja Sridharan formulated versions of Mohan Kumar's addition and subtraction principles for affine algebras A over *non algebraically closed* fields, and also partly for noetherian rings. For instance, let A be noetherian with $\dim A = n \geqslant 3$, and let P be a rank n projective A-module with trivial determinant (that is, $\Lambda^n P \cong A$). If P has a unimodular element, then P maps onto every height n ideal generated by n elements (addition principle). Conversely, if P maps onto some n-generated height n ideal $J \subseteq A$, then with suitable assumptions on J, P contains a unimodular element (subtraction principle). Analogous results in the 2-dimensional case appeared in the second paper [Sridharan: 1995b]. Later, generalizations to a pair of comaximal ideals I, J of maximum height in a noetherian ring A (with $\Lambda^n P \cong A$) were obtained in [Mandal-Sridharan: 1996], [Bhatwadekar-Sridharan: 1998a], and in [Maltenfort: 1999] (for more general P). For a "stable" version of the subtraction principle, see [Sridharan: 1998].

Much of the progress between 1995 and 2000 in the work on addition and subtraction principles was the result of M.V. Nori's important suggestion of defining an "Euler class" for the study of top rank projective modules over affine k-algebras where k is a *not necessarily algebraically closed* field. Let us now briefly describe this important development and its repercussions.

In topology, for a given space X of dimension n, the top (n^{th}) *Euler class* of a rank n real vector bundle B is the obstruction to the existence of a nonzero section for B. For instance, the nonvanishing of the second Euler class of the tangent bundle for the 2-sphere implies that S^2 is not parallelizable. Nori's program, which began to emerge in the early 1990s, was to define the notion of an "Euler class" for rank n projective modules P over an n-dimensional affine k-algebra (k a field), taking values in a suitable "Euler class group", such that the vanishing of this Euler class would provide a criterion for the existence of a unimodular element in P. In short, one hopes to "mimic" the standard topological theory in the study of algebraic vector bundles over affine varieties. Murthy's successful work (1994) on the top Chern class over algebraically closed fields certainly heightened one's desire to search for a "finer" Euler class invariant for algebraic vector bundles over more general ground fields k.

Using obstruction theory, Nori proved a certain homotopy principle for sections of real vector bundles over smooth manifolds. This work appeared as an appendix in [Mandal: 1992]. In this paper, Mandal studied an algebraic analogue of this homotopy principle for projective modules, likewise proposed by Nori. In somewhat informal terms, this algebraic analogue asked the following:

Let P be a projective module of rank n over a regular ring A of finite dimension, I be an ideal in $A[t]$, and $s: P \to I(0)$ be a given surjection, where $I(0) = \{f(0) : f \in I\}$. If $n \geqslant \dim A[t]/I + 2$ and $\varphi: P[t] \to I/I^2$ is a surjection such that $\varphi(0) \equiv s \pmod{I(0)^2}$, can φ be "lifted" to a surjection $\tilde{\varphi}: P[t] \to I$ such that $\tilde{\varphi}(0) = s$?

We shall refer to a positive answer to this question (mainly for smooth affine algebras A of dimension n) as Nori's Conjecture, with the understanding that some smoothness conditions may sometimes be required on $A[t]/I$.

Mandal (1992) proved Nori's Conjecture (without smooth conditions) in the important case where I contains a monic polynomial, and also in some cases where $I = I_0[t]$. (If I contains no monics, an example of Bhatwadekar, Mohan Kumar and Srinivas showed that Nori's Conjecture may fail if A is not regular: see [Bhatwadekar-Sridharan: 1998a].) Some variants of Mandal's result appeared in [Mandal-Sridharan: 1996], and the local case of Nori's Conjecture (with A regular) was largely settled in [Mandal-Varma: 1997]. This latter work was subsequently further extended in [Bhatwadekar-Keshari: 2003].

The definition of the Euler class group $E(A)$, due to Nori, appeared in print first in [Mandal-Sridharan: 1996]. Let A be a regular affine domain of dimension $n \geqslant 2$ over an infinite perfect field k. One forms a free abelian group G with basis given by formal pairs $(\mathfrak{m}, \omega_{\mathfrak{m}})$ where $\mathfrak{m}$ is a maximal ideal of A and $\omega_{\mathfrak{m}}$ is a generator of $\Lambda^n(\mathfrak{m}/\mathfrak{m}^2)$. If an ideal J is an intersection of distinct maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$ and ω_J is a generator of $\Lambda^n(J/J^2)$, then ω_J induces a generator ω_i of $\Lambda^n(\mathfrak{m}_i/\mathfrak{m}_i^2)$, so we can define the formal sum

$$(J, \omega_J) := \sum_i (\mathfrak{m}_i, \omega_i) \in G.$$

The "relations" for the Euler class group arise from "curves" in $\operatorname{Spec}\big(A[t]\big)$. Consider any local complete intersection ideal I of height n in $A[t]$ such that

$I/I^2 \cong (A[t]/I)^n$, and let $\omega(t)$ be any generator of $\Lambda^n(I/I^2)$. Further, assume that $I(0)$ and $I(1)$ are finite intersections of maximal ideals in A (where, as before, $I(a)$ means $\{f(a) : f \in I\}$). By definition, the Euler class group of A is $E(A) := G/H$, where H is the subgroup of G generated by all formal differences

$$(I(0),\ \omega(0)) - (I(1),\ \omega(1))$$

obtained from all valid choices of I and $\omega(t)$ above.

For any rank n projective A-module P with trivial determinant, let χ be a generator of $\Lambda^n P$. (One thinks of χ as an "orientation".) Take any surjection $\alpha : P \to J$ where J is an intersection of finitely many maximal ideals in A. This induces an isomorphism

$$\Lambda^n \bar{\alpha} : \Lambda^n(P/JP) \longrightarrow \Lambda^n(J/J^2),$$

so one can define the *Euler class* $e(P, \chi)$ to be the image in $E(A)$ of the element $(J,\ \Lambda^n \bar{\alpha}(\bar{\chi})) \in G$. That this image is independent of the choices of J and α is proved in the Appendix of [Sridharan: 1996].

Using the results of Mandal, Mandal-Varma, and Mandal-Sridharan (*loc. cit.*), Bhatwadekar and Sridharan (1998a) proved Nori's Conjecture in the case where $\dim A[t]/I = 1$ (with $\operatorname{rank}(P) = \dim(A) \geqslant 3$) over an infinite perfect field k. This enabled them to give a simpler description of the relation subgroup $H \subseteq G$ in the definition of the Euler class group $E(A) = G/H$. Combining this with a main result(*) in [Sridharan: 1998], they accomplished the goal of Nori's program, proving that:

> *A top rank projective A-module P with trivial determinant (and χ a generator of $\Lambda^n P$) has a unimodular element iff its Euler class $e(P, \chi)$ is zero in the Euler class group $E(A)$.*

The "simpler" description of the relation subgroup $H \subseteq G$ enables one to think of $E(A)$ as a group of "oriented" zero cycles modulo a certain "rational equivalence". Using this viewpoint, Bhatwadekar and Sridharan (1999) introduced a quotient group $E_0(A)$ of $E(A)$ obtained by "ignoring" the orientations. This group, in turn, maps onto the usual Chow group $\mathrm{CH}_0(A)$ of zero cycles on $\operatorname{Spec} A$ modulo rational equivalence. In the case where k is algebraically closed, the three groups $E(A)$, $E_0(A)$ and $\mathrm{CH}_0(A)$ are shown to be isomorphic, and, upon identifying these groups, the Euler class $e(P, \chi)$ essentially coincides with the top Chern class $\mathrm{C}_n(P)$.

Let us now come to the case of real affine algebras. Earlier work on bundles over real varieties has appeared in [Kong: 1977], [Barge-Ojanguren: 1987], [Swan: 1993], and [Sridharan: 1996]. For later work on real affine algebras, see [Ojanguren-Parimala: 1990] and [Keshari: 2004]. The main purpose of [Bhatwadekar-Sridharan: 1999] is to study $E(A)$ in the case $k = \mathbb{R}$. For smooth real varieties of dimension $\geqslant 2$, it is shown that $E_0(A) \cong \mathrm{CH}_0(A)$, and $E(A)$, $E_0(A)$ are described in terms of the compact connected components of the real locus $X(\mathbb{R})$. For instance, one consequence of this description is that, if n is even, then for a top rank projective

(*)This main result, incidentally, makes use of the solution to Serre's Conjecture.

module P with trivial determinant, $\mathrm{C}_n(P)=0$ iff P is *stably* isomorphic to $A\oplus P'$ for some P', and, in case $X(\mathbb{R})$ has no compact connected components, then (for n odd or even), $\mathrm{C}_n(P)=0$ iff $P\cong A\oplus P'$ for some P'.

The work on the Euler class group $E(A)$ has progressed further since 1999. For general noetherian rings A containing the rational field $\mathbb{Q}$, an "L-based" Euler class group was introduced in [Bhatwadekar-Sridharan: 2000] for any given $[L]\in \mathrm{Pic}(A)$ (an "orientation bundle") to handle the case of top rank projective modules P with $\Lambda^n P\cong L$. If $\chi:\Lambda^n P\to L$ is an isomorphism, an Euler class $e(P,\chi)$ is again defined in $E(A,L)$, whose vanishing is the precise obstruction to the existence of a unimodular element in P. (For affine algebras over infinite perfect fields, part of this work was also carried out in [Robertson: 2000].) In the general noetherian case, some of the arguments used in the geometric case are no longer available. Instead, one assumes that $\mathbb{Q}\subseteq A$ (or more precisely, $1/(n-1)!\in A$) in order to exploit a result of [Rao: 1988b] (and [Suslin: 1977b], see §5 below) to the effect that stably free $R[t]$-modules of rank d are free over a d-dimensional noetherian local ring R with $1/d!\in R$.

In their paper, Bhatwadekar and Sridharan (2000) showed that $E_0(A,L)$ (the L-based version of the E_0-group) is independent of the choice of the class $[L]\in \mathrm{Pic}(A)$, so that in fact

$$E_0(A,L)=E_0(A,A)=E_0(A).$$

Also, the kernel of the canonical epimorphism $E(A)\to E_0(A)$ is shown to be related to the orbit space of unimodular rows under the special linear group action. More general work relating Euler classes to the existence of unimodular elements in stably free modules can be found in [Bhatwadekar-Sridharan: 2002]. We shall come back to this matter in a later section (§5) devoted specifically to the work on unimodular rows and stably free modules over commutative rings.

In the case where B is a polynomial ring $A[t]$ where A is a (commutative) noetherian ring of dimension $n\geqslant 3$ containing the rationals, an Euler class group $E(B)=E^n(B)$ of ideals of codimension n is defined in [Das: 2003], and applied to the study of addition/subtraction principles, generation of height n ideals $I\subset B$, Quillen-Suslin theory relating $E(A)$, $E\big(A[t]\big)$, and $E\big(A\langle t\rangle\big)$, and the existence of unimodular elements in f.g. projective B-module P. If P has rank n and trivial determinant, with χ a generator of $\Lambda^n P$, it is shown that P has a unimodular element iff a certain Euler class $e(P,\chi)\in E^n(B)$ is zero. Further refinements and extensions of this work, sometimes with the additional assumption that $n=\dim R$ is even, can be found in the subsequent paper [Das-Sridharan: 2003].

§3. Complete Intersections

In the earlier "Appendix on Complete Intersections", we have given a brief summary of the work on complete intersections done up to about 1977. In this section, we pick up from where we left off in that Appendix. However, after Serre's Conjecture was solved in 1976, the research on complete intersections partly took its own course, and has remained relevant to Serre's Problem only through the general common theme of finding the minimal number of generators for polynomial ideals and modules. Since our main interest is on Serre's Problem and not complete intersections *per se*, we'll limit our report in this section mostly to those developments since 1977 that are related to some aspects of projective module theory. Other results on complete intersections will only be mentioned peripherally (toward the end of the section) without a detailed discussion. Also, many parts of the presentations in the two previous sections (for instance, those on the Forster-Swan theorem and the Eisenbud-Evans Conjecture (EE_3), the addition and subtraction principles, etc.) can all be rightfully classified as results on complete intersections. For obvious reasons, these results will not be repeated here.

Before we begin, let us also point out that, since 1977, several books and surveys relevant to the theme of complete intersections have appeared in the literature. Of these, we would especially like to mention, in chronological order, [Szpiro: 1979], [Ohm: 1980], [Kunz: 1985], [Mandal: 1997], and [Ischebeck-Rao: 2005]. Mohan Kumar's notes on Szpiro's lectures are specifically devoted to space curves, while Ohm's article is a broad survey on c.i. with an emphasis on homological and commutative algebra methods. The books of Kunz, Mandal and Ischebeck-Rao offer easily accessible coverage of many aspects of c.i., especially those that are directly linked to Serre's Problem. Many of the results mentioned in our earlier Appendix are now available (with proofs) from one or more of these books, including, for instance, the Kronecker-van der Waerden theorem with the Storch-Eisenbud-Evans refinement, the characteristic p Cowsik-Nori theorem, and the solution to Forster's Conjecture by Sathaye and Mohan Kumar, etc. (Incidentally, a large collection of concrete examples of local, set theoretic, and ideal theoretic complete intersections is available from (V.3.13) in [Kunz: 1985].) Novice readers will certainly be very well served to consult any of these fine expository accounts.

In our Appendix on Complete Intersections, we have mentioned Murthy's Question, which we recall here for the convenience of the reader:

(†) *If an affine variety $V \subseteq \mathbf{A}_k^n$ is locally a complete intersection (e.g. in case V is smooth), is V then a set-theoretic complete intersection?*

This question has been answered affirmatively in the case where V is a curve. (In particular, any nonsingular curve in $\mathbf{A}_k^n$ is a set-theoretic c.i.) Since the solution in this case (at least the one we'll explain) depends on each of the following:

(1) the Quillen-Suslin theorem,

(2) Suslin's $n!$ theorem, and

(3) Serre's method on the projective generation of ideals,

it provides a fine example of the synergy of Serre's Conjecture and complete intersection problems. For this reason, we'll give a synopsis of this work below. For uniformity, *all rings considered in this section will be assumed to be commutative.*

For a f.g. ideal I in a ring R, the R/I-module I/I^2 is often used as a tool for understanding the structure of I. In the literature, I/I^2 is sometimes called the *conormal module* or the *conormal bundle* (associated to I). If R happens to be a local ring and $I \neq R$, Nakayama's Lemma implies that $\mu(I) = \mu(I/I^2)$. (As usual, $\mu(M)$ denotes the least number of generators for a f.g. module M.) Without the local assumption, it is nevertheless true that

$$\mu(I/I^2) \leqslant \mu(I) \leqslant \mu(I/I^2) + 1, \tag{3.1}$$

and one has also $\operatorname{ht}(I) \leqslant \mu(I/I^2)$ in case R is noetherian (and $I \neq R$). Proofs for these classical facts can be found in [Ischebeck-Rao: 2005, (10.2.1)].

The conormal module I/I^2 responds well to the c.i. or locally c.i. properties of the f.g. ideal I. Indeed, if I is c.i., say generated by a regular sequence $x_1, \ldots, x_n$, then the R/I-module I/I^2 is free on the basis $\{\overline{x}_1, \ldots, \overline{x}_n\}$; and, if I is locally c.i., say of height r, then I/I^2 is a f.g. projective R/I-module of rank r. In the former case, in particular, we have $\mu(I) = \mu(I/I^2)$.

We start by stating some nice facts on the number of set-theoretic generators for an ideal. By a clever use of Suslin's $n!$ theorem (on the completability of a unimodular row of the form $\big(a_1, \ldots, a_{n-1}, a_n^{(n-1)!}\big)$), Boratyński (1978) proved the following remarkable result.

Theorem 3.2. *Let I be a proper f.g. ideal in a ring R. Then I contains a subideal J with $\operatorname{rad}(J) = \operatorname{rad}(I)$ such that J is a quotient of a f.g. projective R-module P of rank $\mu := \mu(I/I^2)$. In the case where f.g. projectives are free over R (e.g. R is a polynomial ring over a field or a PID), I can be set-theoretically generated by μ elements.*

The last part here follows since the additional assumption on R guarantees that $P \cong R^\mu$, so that $\mu(J) \leqslant \mu$, and $\operatorname{rad}(J) = \operatorname{rad}(I)$ implies that I and J are set-theoretically generated by the same number of elements. For the proof of (3.2), one writes $I = (f_1, \ldots, f_\mu) + I^2$, and takes

$$J = (f_1, \ldots, f_{\mu-1}) + I^{(\mu-1)!}.$$

Here, the choice of the factorial in the exponent is designed to enable the use of Suslin's $n!$ theorem over a suitable localization of R. For more details on this proof, see [Mandal: 1997, p. 77].

We can now answer Murthy's Question affirmatively for curves, via the following result on polynomial ideals.

Theorem 3.3. *Let $R = k[t_1, \ldots, t_n]$, where k is a field and $n \geqslant 3$. Then any locally c.i. ideal of height $n-1$ in R is a set-theoretic c.i.*

To prove this, one uses a construction from [Ferrand: 1975] (an exposition on which can be found in [Ischebeck-Rao: 2005, (10.3.3)]). According to this construction, if R is a noetherian ring of Krull dimension $n \geqslant 3$, and $I_0 \subseteq R$ is a *locally c.i.* ideal of height $n-1$, then I_0 has a locally c.i. subideal I with $\operatorname{rad}(I) = \operatorname{rad}(I_0)$

such that I/I^2 is free over R/I of rank $n-1$. In the case $R=k[t_1,\ldots,t_n]$, Theorem 3.2 above gives another subideal $J\subseteq I$ with $\operatorname{rad}(J)=\operatorname{rad}(I)$ such that J is generated by $\mu(I/I^2)=n-1$ elements. Thus, I_0 itself is set-theoretically generated by $n-1=\operatorname{ht} I_0$ elements, as claimed. This result is due to Ferrand and Szpiro in the case $n=3$, and to Mohan Kumar in general (Cor. 5 in [Mohan Kumar: 1978]). Mohan Kumar's proof, however, did not make use of Boratyński's result. In fact, his paper seems to have been written independently of [Boratyński: 1978].

Theorem 3.3 could also have been deduced from Ferrand's construction and another result of Boratyński (1978). This result, stated below, holds without any height assumption on I, but requires the triviality of the conormal bundle.

Theorem 3.4. *Let R be a noetherian ring over which all f.g. projectives are free. If $I\subsetneq R$ is a locally c.i. ideal such that I/I^2 is free as an R/I-module, then I is a set-theoretic c.i.*

In the case $R=k[t_1,\ldots,t_n]$ where k is a field, Boratyński (1982) showed that the above holds already if it is only assumed that the conormal bundle I/I^2 is a direct sum of torsion line bundles (that is, a direct sum of rank 1 projectives representing torsion elements in the Picard group of R/I).

The results above dealt only with *set-theoretic* c.i.'s, but we shall try to give now some information on the numbers $\mu(I)$ themselves. Recall from (3.1) that $\mu(I)$ is given by $\mu(I/I^2)$ or $1+\mu(I/I^2)$. In the literature, a (f.g.) ideal $I\subsetneq R$ is said to be *efficiently generated* if $\mu(I)=\mu(I/I^2)$. For instance, as we've mentioned before, a c.i. ideal is always efficiently generated — though not conversely. A well known general observation in the study of efficient generation is the following:

If $\mu(I/I^2)>\dim(R)$ in a noetherian ring R, then I is efficiently generated.

For a proof of this, see [Kunz: 1985, (V.5.17)].(*) The main effort in the study of $\mu(I)$ often lies in getting the conclusion of efficient generation under "weaker" (or less special) assumptions than $\mu(I/I^2)>\dim(R)$.

The following result on efficient generation of polynomial ideals, the first of its kind, is Theorem 5 in [Mohan Kumar: 1978]; its proof depends on the Quillen-Suslin Theorem.

Theorem 3.5. *For any field k, any ideal I in $R=k[t_1,\ldots,t_n]$ with*

$$\mu(I/I^2)\geqslant\dim(R/I)+2$$

is efficiently generated.

To illustrate the power of this theorem, let us derive from it the truth of Forster's Conjecture in the following form, which we had stated earlier as Theorem 5 in the Appendix on Complete Intersections.

(*) In fact, the weaker assumption $\mu(I/I^2)>\dim(R/\operatorname{rad} R)$ would have sufficed for this conclusion (by a standard Jacobson radical argument).

Theorem 3.6. *If* $I \subset k\,[t_1, \ldots, t_n]$ $(n > 0)$ *is locally c.i.* (k *a field), then* $\mu(I) \leqslant n$.

The fact that (3.5) can be used to prove (3.6) was a remark on p. 235 of [Mohan Kumar: 1978].(*) Let us give here a brief sketch. Using the Forster-Swan theorem (as stated in §1), we get $\mu := \mu(I/I^2) \leqslant n$ from the locally c.i. assumption on I. We can then apply (3.5) to deduce $\mu(I) \leqslant n$. The only time (3.5) doesn't apply is when $\mu < \dim\,(R/I) + 2$. In this case, (3.1) gives

$$\mu(I) \leqslant \mu + 1 \leqslant \dim\,(R/I) + 2.$$

This leaves only the cases $\dim\,(R/I) = n,\ n-1$ uncovered, but these cases can be handled easily.

For a nice and self-contained account of Forster's Conjecture and an exposition on related results, see Rabeya Basu's M. Phil. dissertation ([Basu: 2002]).

For infinite fields k, Lyubeznik (1986) proved that any ideal $I \subseteq k[t_1, \ldots, t_n]$ has an n-generated subideal J that is a "reduction" of I in the sense of Northcott and Rees; i.e. $JI^r = I^{r+1}$ for some r. This statement can be seen to contain (in the case k is infinite) *both* (3.6) and the Eisenbud-Evans Theorem that I can be set-theoretically generated by n elements.

Although Murthy's Question (†) has remained unanswered in general for $\dim V > 1$, and it is still unknown exactly which (or whether all) polynomial ideals $I \subset k[t_1, \ldots, t_n]$ are efficiently generated, the results (3.2)–(3.6) above provided a good basis for subsequent investigations on complete intersections. For some more work using the method of conormal modules or dealing with Murthy's Question, see [Vasconcelos: 1978], [Herzog: 1978], [Boratyński: 1979, 1980, 1982, 1984, 1986a, b], [Mandal: 1984, 1988], and [Mandal-Roy: 1987]. We would like to make a few remarks about the last three papers here since the results therein are directly related to, and in fact improve upon, the ones we have discussed so far in this section.

A relevant philosophical observation in this general area of study is that, if something can be proved for ideals in a polynomial ring $k[t_1, \ldots, t_n]$ $(n > 0)$ over a field k, there could very well be an analogue of the result provable for ideals *containing a monic polynomial* in a polynomial ring $A\,[t]$ over a (finite-dimensional) noetherian ring A. Conversely, a result of this latter type is immediately applicable to ideals in a multivariate polynomial ring $k[t_1, \ldots, t_n]$ $(n > 0)$ by letting $A = k[t_1, \ldots, t_{n-1}]$ after a suitable ("Nagata-type") change of variables.

The results we'll sketch below will all be in the spirit of extending the previous results of this section to the $A\,[t]$ setting. We'll follow the general order of (3.2), (3.3), and (3.5), even though their $A\,[t]$ analogues were not discovered in that particular order. The first one, from [Mandal-Roy: 1987], is a variant of Boratyński's Theorem 3.2. Here, R is no longer an arbitrary commutative ring, but one of the type $A\,[t]$ where A is a noetherian ring. However, again as in Theorem 3.5′, no assumptions are made on the projective modules over R or over A.

(*) In this paper, Forster's Conjecture followed directly from the Eisenbud-Evans Conjecture (EE_3), without recourse to Theorem 3.5. These were the "Two Conjectures" in the title of the paper.

Theorem 3.2′. *Let $R = A[t]$, where A is a noetherian ring. Then any ideal $I \subseteq R$ containing a monic polynomial in t is set-theoretically generated by $\mu(I/I^2)$ elements.*

The proof of this is not at all easy! Besides using the original arguments of Boratyński, Mandal and Roy also applied the Affine Horrocks' Theorem, a patching lemma due to Plumstead, plus a slight improvement of Suslin's $n!$ theorem. The details can be found in [Mandal-Roy: 1987]; for an exposition, see Theorem 6.2.6 in [Mandal: 1997].

The second result, due to Mandal (1988) and Lyubeznik (1992), generalizes (3.3) (Mohan Kumar's positive answer to Murthy's Question for curves) to the $A[t]$ setting.

Theorem 3.3′. *Let $R = A[t]$, where A is a noetherian ring of Krull dimension d, and let $I \subseteq R$ be an ideal containing a monic polynomial in t with* $\dim(R/I) \geqslant 1$. *If I is a locally c.i., then it is set-theoretically generated by* $d = \dim(R) - 1$ *elements.*

The proof of this makes use of (3.2′) above, in lieu of Boratyński's (3.2). In [Mandal: 1988], this theorem was proved under the additional assumption that R is Cohen-Macaulay. This assumption was subsequently shown to be unnecessary by Lyubeznik. The geometric content of (3.3′) is given by the following corollary, obtained by taking A to be $k[t_1, \ldots, t_{n-1}]$. (Here, $\dim(V)$ is arbitrary, while the special case $\dim(V) = 1$ recaptures (3.3).)

Corollary 3.7. *For any field k and any $n \geqslant 2$, every $V \subset \mathbf{A}_k^n$ that is a locally c.i. of constant positive dimension is set-theoretically definable by $n-1$ equations.*

It is also worth noting that we can get rid of the polynomial setting in (3.3′) to get a result for any noetherian ring.

Corollary 3.8. *Let J is a locally c.i. ideal in a noetherian ring A of dimension d. If* $\dim(A/J) \geqslant 1$, *then J can be set-theoretically generated by d elements. (In particular, a hypersurface in a smooth d-dimensional affine variety can be set-theoretically defined by d equations.)*

Proof. Let I be the ideal in $A[t]$ generated by J and t. Theorem 3.3′ implies that I is set-theoretically generated by d elements. Setting $t = 0$, we see that J is set-theoretically generated by d elements. □

The last statement in (3.8) was prompted by a question raised in [Forster: 1984] in generalization of Forster's Conjecture discussed earlier. The question is the following:

> *If an algebraic subset V of a d-dimensional affine variety X consists only of irreducible components of positive dimensions, is V always definable set-theoretically by d equations?*

Mandal's result answered this affirmatively in codimension 1 for smooth X (while the case of a curve on a smooth surface was settled earlier in [Boratyński: 1986c]). Later, Lyubeznik (1988b, 1992) answered Forster's question negatively if X is not smooth, and positively if X is smooth over an algebraically closed field.

Finally, we come to the $A[t]$-version of (3.5). The following result from [Mandal: 1984] is a very nice improvement of Mohan Kumar's theorem (3.5) on the efficient generation of polynomial ideals.

Theorem 3.5′. *Let $R = A\,[t]$ where A is a noetherian ring of finite Krull dimension, and let I be an ideal of R containing a monic polynomial in t, with*

$$\mu(I/I^2) \geqslant \dim(R/I) + 2. \tag{3.9}$$

Then I is efficiently generated.

Actually, Mohan Kumar's proof of (3.5) was also based on working with $A\,[t]$ (rather than just with $k[t_1, \ldots, t_n]$). Under the hypothesis (3.9), Mohan Kumar's argument gives the efficient generation of I *if one assumes that f.g. projectives are free over* A. The novelty of Mandal's result lies in the fact that efficient generation is proved *without* this assumption. In [Mandal: 1984], efficient generation is also established for ideals I in rings of the more general form

$$R = A\,[s_1^{\pm 1}, \ldots, s_m^{\pm 1}, t_1, \ldots, t_n\,] \quad (m,\ n \geqslant 0)$$

under the condition (3.9) and $\mathrm{ht}\,(I) > \dim\,(A)$. This height assumption on I is, of course, designed to "produce" the monic polynomial hypothesis in (3.5′) after a change of variables (using Suslin's Monic Polynomial Theorem (III.3.3)). For the case $m = 0$, see also the discussions in the introductory section of [Rao: 1985].

For other cases where the formula $\mu(I) = \mu(I/I^2)$ holds, see [Nashier: 1983a,b,c] and [Nashier: 1985], although some results in the former are superseded by those of Mandal stated above). We note in passing that the case of (positively) graded rings is actually easier: if $R = \bigoplus_{i \geqslant 0} R_i$ is a (commutative) noetherian graded ring where R_0 is a field, then for any *homogeneous* ideal $I \subseteq R$, the equation $\mu(I) = \mu(I/I^2)$ always holds, according to Exercise 12 in [Kunz: 1985, (V.5)].

There is also a line of investigation in the theory of complete intersections that is directly descended from the classical work of David Hilbert. For a field k, Hilbert's Nullstellensatz implies that any maximal ideal in $k[t_1, \ldots, t_n]$ is a complete intersection (and thus efficiently generated). This prompted a fairly large body of work studying the c.i. or efficient generation properties for maximal ideals in polynomial rings over a commutative base ring A, starting with [Geramita: 1973] and [Davis-Geramita: 1973, 1977]. The last paper contains the following remarkable result.

Theorem 3.10. *Let $R = A\,[t_1, \ldots, t_n]$, where $n > 0$ and A is a noetherian ring. A maximal ideal $\mathfrak{m} \subset R$ is efficiently generated in each of the following cases:*

(1) *The localization $R_{\mathfrak{m}}$ is* not *regular;*

(2) $\mathfrak{m} \cap A$ *is a maximal ideal of* A*;*

(3) A *is a Hilbert ring* (*) *(e.g. any affine algebra over a field);*

(4) A *is a regular ring, with* $\dim(A) \leqslant 1$*;*

(5) $n > 1$.

If A *is regular, and either* $\dim(A) \leqslant 1$ *or* $n > 1$*, any maximal ideal in* R *is a complete intersection.*

In (4), the regularity assumption is needed, for otherwise the 1-dimensional local ring $A = k[[t^2, t^3]]$ (for k a field) provides a counterexample. This ring A is, of course, not normal. But Bhatwadekar (1984b) has also constructed a normal (but not regular) local domain $A \supset \mathbb{C}$ of dimension 2 such that $A[t]$ has a maximal ideal $\mathfrak{m}$ of height 2 with $\mu(\mathfrak{m}/\mathfrak{m}^2) = 2$ but $\mu(\mathfrak{m}) = 3$. In particular, $\mathfrak{m}$ is *not* efficiently generated.

All of the above results did not cover the case where A is a regular ring of arbitrary dimension. Indeed. the last statement in Theorem 3.10 has led to the following:

Conjecture 3.11. *For any regular ring* A*, any maximal ideal* $\mathfrak{m}$ *in* $R = A[t_1, \ldots, t_n]$ *is a complete intersection.*

This Conjecture, in its full generality, has remained unproven as of this date. Of course, in view of Case (5) in (3.10), we need only deal with the crucial case $n = 1$. Thus, from here on, *let us assume that* $R = A[t]$*, where* A *is regular.* For such R, (3.11) has been proved in many cases, including the following:

(a) A is a regular local ring with $\dim(A) = 2$ [Geramita: 1973]. (Later, this case was further generalized. In [Estes-Matijevic: 1979], it was proved that the conclusion of (3.11) also holds if A is a noetherian Towber ring.)

(b) A is a local ring of an affine algebra over an infinite perfect field, or A is a power series ring over any field [Bhatwadekar: 1982].

(c) A contains $\mathbb{Q}$ [Bhatwadekar-Varma: 1990].

Perhaps not surprisingly, the problem of deciding if maximal ideals in $R = A[t]$ are c.i.'s is rather closely related to the Bass-Quillen Conjecture in V.3. (This shows once more that the study of complete intersections is never too far removed from the understanding of Serre's Problem.) If $\mathfrak{m} \subset R = A[t]$ is maximal where A is a noetherian ring containing $\mathbb{Q}$, Bhatwadekar (1984b) has shown that there is a f.g. projective R-module P of rank $\mu := \mu(\mathfrak{m}/\mathfrak{m}^2)$ that maps surjectively onto $\mathfrak{m}$. Now let us specialize to the case where A is a regular local ring. If the Bass-Quillen Conjecture holds, then P is free, and hence $\mu(\mathfrak{m}) \leqslant \mu$, which (easily) implies that $\mathfrak{m}$ is a complete intersection. Thus, if we assume some of the results on the Bass-Quillen Conjecture to be surveyed in §6, this would directly give the conclusions in (a), (b) and (c) above, in case A is a regular local ring containing $\mathbb{Q}$.

(*) A ring A is called *Hilbert* if every prime ideal in A is an intersection of maximal ideals.

For many other results on the efficient generation of maximal ideals $\mathfrak{m} \subset A[t]$, see (besides the papers already cited) [Boratyński-Davis-Geramita: 1978], [Campanella: 1980], [Kim: 1987], [Sato: 1990], and [Varma: 1992]. Instead of maximal ideals, one may also consider more generally 0-dimensional ideals (that is, ideals $I \subseteq R = A[t_1, \ldots, t_n]$ such that $\dim(R/I) = 0$). For an investigation on the efficient generation of such ideals I, see [Varma: 1990].

In the above, we have focused our survey only on those aspects of complete intersection theory that have a direct bearing on the theory of projective modules and the efficient generation of polynomial ideals. To tackle the other parts of the vast literature on complete intersections would perhaps take us too far afield. In the following, therefore, we shall content ourselves by mentioning only a few sample papers in some other directions of work in the c.i. theory.

• The classical problem, whether every curve in $\mathbf{A}_k^3$ is a set-theoretic c.i., has apparently remained unsolved. One can look at a local version of this problem, and ask the following purely algebraic question: *is every height* 2 *ideal in a regular local ring of dimension* 3 *generated up to radical by two elements?* Some positive cases of this were treated (in the spirit of addition and subtraction principles) in [Mohan Kumar: 1990], and later generalizations to higher dimensions were obtained in [Lyubeznik: 1992, (5.2)]. For other works dealing with complete intersections in local rings, see, e.g. [Faltings: 1981], [Huneke: 1985].

• For a study of the set of complete intersection points on affine varieties, see [Weibel: 1982, 1984].

• Papers studying set-theoretic c.i.'s using other techniques (such as Gröbner bases, Gorenstein rings or Cohen-Macaulay rings) include [Herzog: 1978], [Boratyński: 1984], [Valla: 1982, 1984], and [Robbiano-Valla: 1983, 1986].

• For determinantal varieties to be set-theoretic complete intersections, see [Valla: 1980], and an unpublished result of Hochster (stated as Exercise 30 on p. 316 of [Ischebeck-Rao: 2005]).

• The study of monomial curves is closely related to the theory of abelian semigroups and semigroup rings. Herzog (1970) showed that a curve C in $\mathbf{A}^3$ given parametrically by $t \mapsto (t^a, t^b, t^c)$ with $\gcd(a, b, c) = 1$ is c.i. iff the semigroup generated by a, b, c in $\mathbb{N}$ has a certain "symmetric" property. Later, Herzog showed that C is always a set-theoretic c.i. The following basic examples from pp. 137–140 of [Kunz: 1985] provide the best illustrations for this situation. For $(a, b, c) = (4, 5, 6)$, the ideal of C is $(x^3 - z^2,\ y^2 - xz)$, so it is a c.i. For $(a, b, c) = (3, 4, 5)$, the ideal I of C is generated by $x^3 - yz,\ y^2 - xz,\ z^2 - x^2y$ and not fewer elements, so it is not c.i. However,

$$\operatorname{rad}(I) = (x^4 - 2xyz + y^3,\ z^2 - xy), \tag{3.12}$$

so I is indeed a set-theoretic c.i. In the case $\gcd(a, b, c) = 1$, [Bresinsky: 1979a] re-proved Herzog's result constructively, giving "high school algebra" algorithms for finding equations of the type (3.12). *It seems to be still unknown, however, if monomial* *curves in* $\mathbf{A}^n$ *are set-theoretic complete intersections.* Some partial results can be found, e.g. in [Bresinsky: 1979b], [Eliahou: 1984, 1988], [Moh: 1984], and [Patil: 1990, 1994]. For related results in characteristic $p > 0$, see [Hartshorne: 1979] and [Ferrand: 1979]. Other papers dealing with c.i. issues on monomial curves include [Robbiano-Valla: 1983], and [Thoma: 1989a, b, 1991, 1995]. For a nice survey of known results, see [Mete: 2001].

• For alternative formulations of the proof of the Cowsik-Nori theorem in characteristic $p > 0$, see [Ferrand: 1979], and (2.5) in [Lyubeznik: 1989]. Many other important results on complete intersections (especially in characteristic p) appeared in [Lyubeznik: 1988b, 1992]. We shall only mention a couple of them here. The first one is the following "$(n-1)$-theorem" from [Lyubeznik: 1992, (5.9)], which is a very strong generalization of the Cowsik-Nori theorem.

Theorem 3.13. *If k is a field of characteristic $p > 0$, then every algebraic set $V \subseteq \mathbf{A}_k^n$ consisting only of irreducible components of positive dimensions is set-theoretically definable by $n-1$ equations.*

There was some progress toward Murthy's Question (†) too, in characteristic p. Here we have the following "$(n-2)$-theorem" from [Lyubeznik: 1992, (6.3)], which handles at least the case of surfaces in affine spaces of dimension $\geqslant 6$.

Theorem 3.14. *Let k be an algebraically closed field of characteristic $p > 0$, and $V \subset \mathbf{A}_k^n$ be locally c.i. of constant dimension d. If $2 \leqslant d \leqslant n-4$, then V is set-theoretically definable by $n-2$ equations. In particular, for $n \geqslant 6$, every locally c.i. surface in $\mathbf{A}_k^n$ is a set-theoretic c.i.*

In characteristic 2, Lyubeznik proved even an analogous "$(n-3)$-theorem", which is sufficient to settle Murthy's Question for 3-folds (with $n \geqslant 9$). We won't go into more details here, but should point out that [Lyubeznik: 1992] is a very rich source of results on complete intersections that are by no means limited to characteristic p.

• K-theory groups, Chow groups, and Euler class groups have all proved to be applicable to the study of complete intersections. We have already surveyed parts of this work in §2, so not much more needs to be said here. Suffice it to mention that, in the joint paper [Bloch-Murthy-Szpiro: 1989], Chow groups are used to get more positive answers to Murthy's Question (†) for certain types of smooth surfaces in $\mathbf{A}_k^4$ over algebraically closed fields k. For some extensions to $\mathbf{A}_k^n$ for $n \geqslant 5$, see [Mohan Kumar: 1989]. More recently, the work of Bloch, Murthy and Szpiro was also generalized to the setting of height 2 ideals containing monic polynomials in $A[t]$, where A is a 3-dimensional regular affine algebra over an algebraically closed field k; see [Mandal: 2001].

Beyond the above pointers to the literature, we should mention the sizable collection of conference articles, all devoted to the theme of complete intersections, in the Springer Lecture Notes volume edited by Greco and Strano (1984). Also, the

M.S.R.I. survey article of Lyubeznik (1989) contains extensive discussions on results and open problems on c.i., covering many topics (e.g. symbolic Rees algebras and local cohomological dimensions) that are not even touched upon in this section. This article also features a full bibliography on papers in the field written in the last three decades; we recommend it highly to the reader. For other surveys and historical/expository articles on the subject, see, e.g. [Kunz: 1979], [Forster: 1984], [Valla: 1984], and [Mohan Kumar: 2003].

In closing, let us point out that, in commutative algebra, the study of c.i. ideals is also closely linked to an important class of local rings that are called "complete intersections". A (commutative, noetherian) local ring R is called a complete intersection if there exists a regular local ring A and a c.i. ideal I in A such that $R \cong A/I$. In other words, R can be identified with a regular local ring modulo the ideal generated by a regular sequence. An equivalent definition, given by Grothendieck, is that *the completion of* R can be identified with a regular local ring modulo the ideal generated by a regular sequence. The following hierarchy for noetherian local rings is quite well known in this area of study::

(3.15) regular $\Longrightarrow$ complete intersection $\Longrightarrow$ Gorenstein $\Longrightarrow$ Cohen-Macaulay.

This gives three interesting and increasingly broader classes of generalizations of regular local rings. In studying a variety V locally at a point x, it is essential to work with the local ring $\mathcal{O}_{V,x}$ of V at x, and to know exactly which classes of local rings above (if any) $\mathcal{O}_{V,x}$ belongs to. This leads to a somewhat primitive classification of singularities at a point, in case $\mathcal{O}_{V,x}$ is not regular.

The fact that complete intersection local rings are Gorenstein is a basic result in commutative algebra. A proof for this can be found in [Kunz: 1985, (VI.3.22)]. We mentioned this relationship since it turned out to play a rather interesting role in Andrew Wiles's proof of Fermat's Last Theorem [Wiles: 1995]. In Chapter 3 of this famous paper, Wiles needed to know that certain Hecke algebras, which are known to be local Gorenstein rings, are indeed complete intersections. This step, which was required for fixing a gap in Wiles's proof of FLT first announced in June, 1993, was eventually accomplished jointly with Richard Taylor in [Taylor-Wiles: 1995], which provided the final piece of the argument needed to complete Wiles's proof of Fermat's Last Theorem.

For a related work on c.i. algebras, see [Lenstra: 1995]. In this paper, Lenstra sharpened Wiles's criterion (in the Appendix of [Wiles: 1995]) for a finite $\mathcal{O}$-free local Gorenstein algebra T over a complete discrete valuation ring $\mathcal{O}$ to be a complete intersection. In doing so, Lenstra was able to remove the Gorenstein assumption on T. For another follow-up work on finite complete intersection algebras, see [de Smit-Lenstra: 1997]. For abundant historical background and a wealth of general information on local rings that are complete intersections, see Chapter 2 of [Bruns-Herzog: 1998].

§4. Monomial Algebras and Discrete Hodge Algebras

In this section, we focus our attention on affine algebras (over a field k) of certain specific types, mainly those that are generated by monomials in $k[t_1, \dots, t_n]$, or those obtained from $k[t_1, \dots, t_n]$ by quotienting out a monomial ideal. Since these algebras bear a certain resemblance to the polynomial algebra $k[t_1, \dots, t_n]$, a natural gerneralization of Serre's Problem would be to study the (possible) freeness of the projective modules over them.

To survey the work in this area, we start with subalgebras of $k[t_1, \dots, t_n]$. In the case $n = 2$, the early work of [Murthy-Pedrini: 1972] showed that f.g. projective modules are indeed free over rings such as

$$A = k[x^n, xy, y^n] \quad \text{or} \quad A = k[x^n, x^{n-1}y, \dots, xy^{n-1}, y^n].$$

This result inspired the 1976 Chicago thesis of D.F. Anderson, which in part appeared in the publications [Anderson: 1978a, b, c]. In the first paper, it is shown that if k is algebraically closed and A is an affine normal subalgebra of $B = k[x, y]$ such that B is integral over A, then any f.g. projective A-module has the simple form (free) $\oplus$ (rank 1). (In particular, the only f.g. indecomposable projective A-modules are the invertible ideals.) The second paper, [Anderson: 1978b], is devoted to *monomial subalgebras* of $k[x, y]$, that is, affine subalgebras A of $k[x, y]$ that are generated by a certain set of monomials. If A is normal, Anderson proved that f.g. projective A-modules are free (in generalization of the Murthy-Pedrini examples). If, however, A is not assumed normal, then any f.g. projective A-module is *stably* isomorphic to (free) $\oplus$ (rank 1), but need not be free. (It will be stably free if $\operatorname{Pic}(A) = \{1\}$.)

Based on the results above, and the Quillen-Suslin Theorem, it was conjectured in two of the papers cited above that, for any normal monomial algebra $A \subseteq k[t_1, \dots, t_n]$ (*for any* n), any f.g. projective A-module is free. This later became known as *Anderson's Conjecture.* In retrospect, this conjecture was driven by algebraic geometry just as the original Serre Conjecture was. Affine toric varieties are natural generalizations of affine spaces, and normal monomial algebras $A \subseteq k[t_1, \dots, t_n]$ correspond to coordinate rings of affine toric varieties. Thus, in geometric terms, Anderson's Conjecture is the statement that algebraic vector bundles over such varieties are trivial.

In commutative algebra, a proper generalization of algebras generated by monomials is the class of *monoid algebras* $k[M]$, where M is a commutative monoid, and k is a commutative ring. Around the time when Anderson's work was published, the papers [Chouinard-Hardy-Shores: 1980] and [Chouinard: 1982a] studied conditions on k and M for $k[M]$ to be a Bézout ring or a K-Hermite ring (in the sense of I.4). Prompted by the work of Anderson, Chouinard (1982b) went on to study the structure of projective modules over $k[M]$ for Krull monoids M and noetherian rings k. In case M is a finitely generated Krull monoid with a torsion class group, Chouinard extended Quillen's Patching Theorem to $k[M]$. Using a suitable analogue of Horrocks' Theorem, he showed that, for M above, f.g. projectives over $k[M]$ are extended from k if k is a Dedekind domain. He then asked for the widest class of (commutative) monoids M for which projectives are free over $k[M]$ when k

is a field (or a PID). This was perhaps the first formal enunciation of Anderson's Conjecture in the more general framework of commutative monoid algebras.

David F. Anderson went on to become a prolific contributor to commutative algebra, but aside from a few subsequent papers on the Picard group (e.g. [Anderson: 1981, 1982, 1988, 1990]), he never returned to the theme of projective modules over monomial rings. His conjecture, on the other hand, caught the fancy of a young Georgian mathematician, I. Dzh. Gubeladze. Gubeladze worked on Anderson's Conjecture from 1979 to 1986. After some preliminary publications (some in the form of research announcements, starting with [Gubeladze: 1981]), Gubeladze proved Anderson's Conjecture in 1986 in a very strong form, working generally with commutative monoid algebras $k[M]$ over a ring k. The main result from [Gubeladze: 1988a] is as follows.

Gubeladze's Theorem 4.1. *For a commutative, torsionfree and cancellative monoid M, the following are equivalent:*

(1) *For any principal ideal domain k, any f.g. projective $k[M]$-module is free.*
(2) *For any field k, f.g. rank 1 projectives are free over $k[M]$; that is,* $\mathrm{Pic}(k[M]) = \{1\}$.
(3) *M is seminormal; that is, for any element a in the total quotient group of M, $a^2, a^3 \in M \Rightarrow a \in M$.*

Here, (1) $\Rightarrow$ (2) is trivial, and (2) $\Rightarrow$ (3) was known before (see, e.g. [Anderson-Anderson: 1982]). Thus, the gist of (4.1) is in (3) $\Rightarrow$ (1). Applying this implication to (semi) normal monomial algebras, we get immediately an affirmative answer to Anderson's Conjecture – even over any principal ideal domain k. Of course, the original form of Serre's Conjecture corresponds to the case where M is the free abelian monoid on the generators $t_1, \ldots, t_n$.

A detailed presentation of Gubeladze's original proof of (4.1) using polyhedral geometry can be found in Chapter VI, §1, of [Inasaridze: 1995]. But readers interested in Gubeladze's Theorem (4.1) will do well to consult also [Swan: 1992]. In this masterful exposition, Swan offered a reworking of Gubeladze's proof, replacing his geometric methods by purely algebraic ones, and added many new interesting results along the way.(*) We shall state only three of these refinements.

(4.2) (Higher-Dimensional Gubeladze). *If k is a regular domain of dimension d and M is any commutative, torsionfree, cancellative and seminormal monoid, then any f.g. projective $k[M]$-module of rank $> d$ is extended from k.*

This result is a direct generalization (in the regular case) of the "Big Rank Theorem" presented in (V.3.6) and (V.3.7) in our text.

(4.3) (Murthy's Problem). *If k is a Dedekind ring and M is any commutative, torsionfree and cancellative monoid, then any f.g. projective $k[M]$-module has the form (free) $\oplus$ (rank 1).*

Positive results on this problem were obtained in special cases in [Kang: 1979, 1984]. For polynomial rings and Laurent polynomial rings, the truth of this Conjecture

(*) A large part of Swan's exposition is also reproduced in (VI.§2) of [Inasaridze: 1995].

can also be deduced from [Bhatwadekar-Roy: 1984] and [Bhatwadekar-Lindel-Rao: 1985].

(4.4) (Lindel's Problem). *Let k be a PID and A be a commutative k-algebra with generators x_{ip} $(1 \leqslant i \leqslant m, 1 \leqslant p \leqslant n)$ and relations $x_{ip}x_{jq} = x_{jp}x_{iq}$ for all i, j, p, q. Then any f.g. projective A-module is free.*

Note that the defining relations on A are the results of setting to zero the 2×2 minors of the $m \times n$ matrix of the commuting indeterminates (x_{ip}). The algebras $A = S_{m,n}$ obtained in this fashion are called *Segre extensions* of k, since, in the case $m = 2$ (say), this is exactly the homogeneous coordinate ring of the image of the standard Segre embedding $\mathbb{P}^{n-1} \times \mathbb{P}^{n-1} \to \mathbb{P}^{n^2-1}$. (For instance, when $m = n = 2$, the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$ is given by

$$((\alpha, \beta), (\gamma, \delta)) \mapsto (\alpha\gamma, \alpha\delta, \beta\gamma, \beta\delta),$$

and the algebra $A = S_{2,2}$ is isomorphic to $k[x, y, z, w]$ with the single relation $wx = yz$.) See [Lindel: 1984] for one of the earliest works on projective modules over Segre extensions. It is of historical interest to note that Lindel had initially expressed skepticism on the possible truth of (4.4). As a concluding remark in [Lindel: 1984], he cautiously wrote: "Because we do not believe that this question has an affirmative answer, we are interested in calculating $K_0(S_{m,n})$."

We now turn our attention to quotient algebras $k[t_1, \ldots, t_n]/I$, where I is an ideal in $B = k[t_1, \ldots, t_n]$. Of course, if I is unrestricted, B/I is just a typical affine algebra over k. Let us, therefore, consider the case where I is a *monomial* ideal (i.e. an ideal generated by some monomials). The quotients B/I obtained in this way are called *discrete Hodge algebras* over k (where k can be any commutative ring).

The most important case of a discrete Hodge algebra is that of "face rings" in combinatorics. For any "vertex set"

$$V = \{v_1, \ldots, v_n\},$$

a (finite) *simplicial complex* Δ on V is a collection of subsets of V, called *faces*, such that any subset of a face is also a face. Any such Δ gives rise to a *face ring* (a.k.a. *Stanley-Reisner ring*) $k[t_1, \ldots, t_n]/I_\Delta$, where I_Δ is the ideal generated by all monomials $t_{i_1} \cdots t_{i_s}$ such that $\{v_{i_1}, \ldots, v_{i_s}\} \notin \Delta$. For instance, for $n = 3$, if the simplicial complex Δ has faces $\{v_1\}$, $\{v_2\}$, $\{v_3\}$, and $\{v_1, v_2\}$, then the non-faces are $\{v_1, v_3\}$, $\{v_2, v_3\}$ and $\{v_1, v_2, v_3\}$, and so the associated face ring is

$$\frac{k[t_1, t_2, t_3]}{(t_1t_3,\ t_2t_3,\ t_1t_2t_3)} = \frac{k[t_1, t_2, t_3]}{(t_1t_3,\ t_2t_3)}.$$

The face rings arising in this manner from simplicial complexes are essentially the *reduced* discrete Hodge algebras. Such rings are used extensively in combinatorics and polytope geometry.

(General Hodge algebras, or algebras with a "straightening law", are defined in [De Concini-Eisenbud-Procesi: 1982]. The nomenclature honors the work of

W.V.D. Hodge on Grassmann varieties and Schubert cycles. However, we shall only be concerned with *discrete* Hodge algebras (defined above) in this survey.)

The non-stable K-theory of discrete Hodge algebras was first investigated by T. Vorst. For projective modules, we have the following beautiful result from [Vorst: 1983]; see also [Lindel: 1984].

Theorem 4.5. *For a commutative noetherian ring k, the following are equivalent:*

(1) *k has the property* (E) *defined in* V.1.9*; that is, for any n, any f.g. projective $k[t_1, \ldots, t_n]$-module is extended from k.*
(2) *For any discrete Hodge k-algebra A, any f.g. projective A-module is extended from k.*

In particular, for any Dedekind ring k, any f.g. projective module over a discrete Hodge k-algebra is extended from k. This is thus yet another generalization of the Quillen-Suslin Theorem on Serre's Conjecture! For an exposition on Vorst's results on the theme of (4.5), see (VI.2) in [Inasaridze: 1995].

There is also a version of (4.5) in which the role of the polynomial algebras $k[t_1, \ldots, t_n]$ is replaced by that of the monoid algebras. We state only the following special case of it from [Gubeladze: 1988b] and [Swan: 1992].

Theorem 4.6. *Let k be a Dedekind ring, and M be any commutative, torsionfree, cancellative, and seminormal monoid. Then for any ideal $N \subsetneq M$, any f.g. projective module over $k[M]/k[N]$ is extended from k.*

Here, an *ideal* N of the commutative monoid M is a subset $N \subseteq M$ such that $N \cdot M \subseteq N$. This is equivalent to saying that the k-span $k[N]$ is an ideal in the monoid algebra $k[M]$ (so that the factor ring $k[M]/k[N]$ makes sense in the statement of (4.6)).

For a discrete Hodge algebra $A = B/I$ where I is a monomial ideal in $B = k[t_1, \ldots, t_n]$, the paper [Mandal: 1985] investigated the Eisenbud-Evans Conjectures for A (raised in §3 of [Vorst: 1985]). Mandal's results were entirely satisfactory except for certain bound assumptions on the ranks of the projective modules. These bound assumptions were improved later in the Münster thesis of A. Wiemers, who was a student of H. Lindel. Unfortunately, Lindel died in June, 1991 before the publication of his student's results in [Wiemers: 1992].

Wiemers' approach was based on the idea of "lifting" a projective module P from a discrete Hodge algebra to a polynomial ring, provided that P is large enough. To be precise, the result is as follows.

Theorem 4.7. *Let k be a commutative noetherian ring of Krull dimension d, and let $A = B/I$ be a discrete Hodge algebra as above. If P is a f.g. projective A-module such that* $\operatorname{rk}(P) \geqslant d + 1$, *or* $\operatorname{rk}(P) = d$ *and $d!$ is a unit in k, then there exists a f.g. projective B-module Q such that $P \cong Q/IQ$.*

Given this theorem, certain facts on A can be reduced to the case of the polynomial algebra $B = k[t_1, \ldots, t_n]$. For instance, Wiemers showed that, if $\operatorname{rk}(P) \geqslant d + 1$, then P has a direct summand $\cong A$, and P is cancellative in the category of f.g. projective A-modules. These results restored the "correct" bounds in part of Mandal's work on the Eisenbud-Evans Conjectures for discrete Hodge algebras. Wiemers also proved the surjectivity of the maps $K_0(B) \to K_0(A)$ and $\mathrm{Pic}(B) \to \mathrm{Pic}(A)$ in the above context without the noetherian assumption on k. For related work on Laurent polynomial rings, see [Wiemers: 1993].

In view of Vorst's results, it seems tempting to ask if (4.7) is true *without* any rank assumptions on P. To the best of my knowledge, this question has remained unanswered.

For other work on projective modules over subrings and quotient rings of $k[t_1, \ldots, t_n]$, see [Lissner-Moore: 1970], [Lissner-Lensted: 1978], and [Visweswaran: 1989].

§5. Unimodular Rows

Interest in the completability of unimodular rows (into invertible square matrices) started with Hermite, whose work in 1849 implied that any unimodular row of rational integers is completable. Steinitz's extension of this result to algebraic integers in 1911 led to the result that *any Dedekind domain is Hermite* (in the sense of I.4). [Kaplansky: 1949] defined a class of Hermite rings in a strong sense which, in the commutative case, are Hermite in our weak sense: see I.4.26. (We called such rings "K-Hermite" in the Appendix to I.4.) Later, building on the results of A.L. Foster in 1946, [Steger: 1966] defined a commutative ring R to be "ID" if every idempotent matrix over R is similar to a diagonal matrix. As we have mentioned before in I.4.7(6), Steger proved that such a ring R is also Hermite in our sense. In the case where R is connected (that is, without nontrivial idempotents), R is ID iff all f.g. projective R-modules are free. For this reason, [McDonald: 1984] renamed Steger's ID rings *projectively trivial*. For instance, all 0-dimensional and all semilocal commutative rings, as well as all Bézout domains. are projectively trivial. For much information on such projectively trivial rings (and other related rings, such as rings with many units, and the "local-global rings" of Estes and Guralnick), we refer the reader to Chapter 4 in McDonald's book. In (I.4.19), we have pointed out that, even though f.g. projectives are free over $A = k[x_1, \ldots, x_n]$ for a field k (Quillen-Suslin Theorem), the same need not be true over *localizations* of A; see also [Ischebeck-Ojanguren: 1998]. For additional references on rings over which f.g. projectives are free, see, e.g. [Tong: 1989] and [Cohn: 2003a, b].

Since $K_0\big(k[t_1, \ldots, t_m]\big) \cong \mathbb{Z}$ for any field k, it was known from the late 1950s that Serre's Conjecture is equivalent to the statement that the polynomial ring $R = k[t_1, \ldots, t_m]$ is Hermite; that is, any unimodular row $(f_1, \ldots, f_n) \in \mathrm{Um}_n(R)$ is completable. This interpretation of Serre's Conjecture, together with the works referred to in the previous paragraph, brought the problem of completing unimodular rows onto the center stage of general algebra. Even after Serre's Conjecture was fully solved, interest in the completability of unimodular rows over rings seemed to have continued unabated.

In III.4 of this book, we have included a detailed proof of Suslin's $n!$ Theorem. First published in [Suslin [1977b], this remarkable result marked the beginning of a new phase of research on the completion problem of unimodular rows. For an alternative proof for Suslin's Theorem, see [Roitman: 1985]. (Two earlier papers of Roitman (1977a, b), written around the time when Serre's Conjecture was solved, also contained useful information on the completion of unimodular rows over polynomial rings.) The length 3 case of Suslin's theorem was due to [Swan-Towber: 1975], which also contained the first treatment of the *converse* to the $n!$ theorem for general n. For more details on this, see III.4 and III.7.

Closely associated to the completion of unimodular rows to invertible matrices is the problem of completing a unimodular row $(0, b_1, \ldots, b_{2m-1})$ to an invertible *alternating* matrix. The latter (called the "skew-completability" of $(b_1, \ldots, b_{2m-1})$) turned out to be equivalent to having a symplectic structure on the stably free module defined by $(b_1, \ldots, b_{2m-1})$: this is Swan's theorem in (VII.5.23). Along with this, we have also proved Krusemeyer's theorem (VII.5.31), to the effect that the skew-completability of $(b_1, \ldots, b_{2m-1}) \in \mathrm{Um}_{2m-1}(R)$ implies the (usual) completability of the unimodular row $(b_1^2, b_2, \ldots, b_{2m-1})$. Both of these results were proved around the time when Serre's Conjecture was first solved, so we have included them in our main text in Chapter VII. Since

$$(5.1) \qquad \text{Pfaffian} \begin{pmatrix} 0 & a & b & c \\ -a & 0 & r & -q \\ -b & -r & 0 & p \\ -c & q & -p & 0 \end{pmatrix} = ap + bq + cr,$$

any row $(a, b, c) \in \mathrm{Um}_3(R)$ is *always* skew-completable to an alternating matrix in $\mathrm{SL}_4(R)$, upon taking an equation $ap + bq + cr = 1$. Thus, Krusemeyer's result gave another proof for the completability of (a^2, b, c) due to Swan-Towber, and his Pfaffian method (or Swan's proof of VII.5.31) led to the explicit completion given by the 3×3 northwest corner of the matrix in (VII.5.34). This particular completion, incidentally, was also arrived at independently in [Suslin: 1977b]. In fact, Suslin's proof of his $n!$ theorem in this paper is constructive, and leads to a completion of a "factorial" unimodular row

$$(a_0, a_1, a_2^2, \ldots, a_n^n)$$

(over any commutative ring) via the use of the Suslin matrices discussed in III.7 in our text.

The study of unimodular elements in stably free modules of the form $P(b_1, \ldots, b_n)$ can be partly extended to the case of non-unimodular sequences. We have already presented some material on the notion of *sectionable sequences* in III.5. For more results in this study, see [Lam-Swan: 2006], where alternating matrices again play a rather substantial role.

The following question of Nori offers a possible generalization of Suslin's $n!$ Theorem: if k is a field and

$$f_0, \ldots, f_n \in R = k[t_0, \ldots, t_n]$$

are such that

$$\text{rad}\,(f_0, \ldots, f_n) = (t_0, \ldots, t_n)$$

and the length of $R/(f_0, \ldots, f_n)$ is divisible by $n!$, is $(\varphi(f_0), \ldots, \varphi(f_n))$ completable in S for any k-algebra homomorphism $\varphi : R \to S$ such that $(\varphi(t_0), \ldots, \varphi(t_n))$ is unimodular? This is proved to be true in [Mohan Kumar: 1997] in the case where k is algebraically closed and the f_i 's are homogeneous polynomials.

A lot of work on unimodular rows is focused on the structure of the set of orbits of $\mathrm{Um}_n(R)$ under the action of the group $\mathrm{E}_n(R)$ for $n \geqslant 3$. Let us denote this orbit space by $\mathrm{W}_n(R)$, and *assume in the following that the ring R is commutative.* The earliest significant work on $\mathrm{W}_n(R)$ was done by Vaserstein in [Vaserstein-Suslin: 1976], where the following is shown:

Theorem 5.2. *If $\mathrm{E}_k(R)$ acts transitively on $\mathrm{Um}_k(R)$ for all $k \geqslant 4$, then $\mathrm{W}_3(R)$ is in 1-1 correspondence with the "elementary symplectic Witt group" $\mathrm{W}_E(R)$ of R.*

Here, the Witt group $\mathrm{W}_E(R)$ is defined to be the Grothendieck group of alternating matrices of Pfaffian 1 up to elementary stable equivalence. The one-one correspondence is induced by the "Vaserstein symbol", which (with only the commutativity assumption on R) associates to $(a, b, c) \in \mathrm{Um}_3(R)$ the class of the alternating matrix on the LHS of (5.1), where (p, q, r) is any triple such that $ap + bq + cr = 1$. Since $\mathrm{W}_E(R)$ is an abelian group, this one-one correspondence, if applicable, can then be used to produce an abelian group structure on $\mathrm{W}_3(R)$. In retrospect, the important special result (5.2) has no doubt planted the seed for the later work on finding possible abelian group structures on the higher orbit spaces $\mathrm{W}_n(R)$.

The transitivity assumption on $\mathrm{Um}_k(R)$ needed for Vaserstein's 1-1 correspondence in (5.2) is usually checked by dimension considerations. For instance, let R be a commutative ring such that $R/\mathrm{rad}(R)$ is noetherian of dimension $\leqslant 2$. Then (II.7.3$''$) implies that $\mathrm{E}_k(R)$ acts transitively on $\mathrm{Um}_k(R)$ for all $k \geqslant 4$, and so Vaserstein's 1-1 correspondence $\mathrm{W}_3(R) \leftrightarrow \mathrm{W}_E(R)$ applies to R. In particular, we could have taken R itself to be any noetherian ring of dimension $\leqslant 2$.

Instead of using the dimension, one could also have used Bass's notion of the stable range. Since we have not formally introduced this invariant so far, we take this opportunity to give its definition here. The *stable range* $\mathrm{sr}\,(R)$ of a ring R is the least integer r (if it exists) such that, for any $(a_1, \ldots, a_{r+1}) \in \mathrm{Um}_{r+1}(R)$, there exist $b_1, \ldots, b_r \in R$ such that

$$(a_1 + b_1 a_{r+1}, \ \ldots, \ a_r + b_r a_{r+1}) \in \mathrm{Um}_r(R).$$

If $\mathrm{sr}\,(R) = r$, it is easy to show that $\mathrm{E}_k(R)$ acts transitively on $\mathrm{Um}_k(R)$ for any $k > r$. Thus, Vaserstein's 1-1 correspondence in (5.2) applies whenever $\mathrm{sr}\,(R) \leqslant 3$.

By a classical result of Bass, *if R is a commutative noetherian ring of Krull dimension d, then* $\mathrm{sr}\,(R) \leqslant d + 1$. (This is proved by the same kind of "prime avoidance" argument used in the proof of II.7.3$'$.) In view of this result, a ring with stable range r is sometimes said to have *stable dimension* $r - 1$; for then the stable dimension of a noetherian ring R is bounded by $\dim R$. For the basic theory of the stable range of rings, see [Bass: 1968] and [Vaserstein: 1971].

While the ideas on $\mathrm{W}_3(R)$ in [Vaserstein-Suslin: 1976] did not lead directly to a general proof of Serre's Conjecture, they turned out to be seminal for the later study of unimodular rows after 1976. For a detailed exposition of the Vaserstein-Suslin results on $\mathrm{W}_3(R)$, see Bass's Bourbaki talk [Bass: 1974].

In studying group structures on the set $\mathrm{W}_n(R)$, the following basic topological example should be borne in mind. If $R = C(X, \mathbb{R})$ is the ring of continuous real-valued functions on a finite CW-complex X, then for $n \geqslant 2$, $\mathrm{W}_n(R)$ can be identified with $\pi^{n-1}(X)$, the set of homotopy classes of continuous mappings from X to the $(n-1)$-sphere S^{n-1} (see, e.g. [van der Kallen: 1983, 1989]). When $n = 2$ or 4, S^{n-1} is a topological group, so $\pi^{n-1}(X)$ has a group structure; otherwise, the cohomotopy set $\pi^{n-1}(X)$ has a group structure only when $\dim X \leqslant 2n - 4$, according to the classical work of Borsuk. Using this as a model, van der Kallen has obtained the following purely algebraic analogue: *if* $n \geqslant 3$ *and* $\mathrm{sr}\,(R) \leqslant 2n-3$, *then* $\mathrm{W}_n(R)$ *has the structure of an abelian group*. Thus, such a group structure on $\mathrm{W}_n(R)$ is available for a noetherian ring R of Krull dimension d as long as $n \geqslant \max\{3,\ (d+4)/2\}$. In fact, in this case, $\mathrm{W}_n(R)$ can be made into a right $\mathrm{GL}_n(R)$-module with the following action:

(5.3) $$[(a_1, \ldots, a_n)] * \gamma = [(a_1, \ldots, a_n)\,\gamma\,] - [\gamma_i] \in \mathrm{W}_n(R),$$

where γ_i is the i th row of $\gamma \in \mathrm{GL}_n(R)$. (Recall that the class $[\gamma_i]$ in $W_n(R)$ is *independent* of i, according to part III.6.1(2).) The main task here is that of proving that the "weak Mennicke symbol" on $\mathrm{Um}_n(R)$ defines a *bijection* from $\mathrm{W}_n(R)$ onto a target abelian group; see [van der Kallen: 1983, 1989, 2002]. For a study of Mennicke symbols and SK_1-stability for regular affine algebras, see [Rao-v.d. Kallen: 1994]. For more information on group structures on $\mathrm{W}_n(R)$ (using Witt groups and Suslin matrices), see the forthcoming work [Jose-Rao: 2005d].

In another direction, the work of [Vaserstein: 1986] and [Roitman: 1985] led to a family of unary operations $\{\psi_m : m \in \mathbb{Z}\}$ on the set $\mathrm{W}_n(R)$ for any (commutative) ring R, in case $n \geqslant 3$. These operations result from taking the m^{th} power of an entry in a unimodular row. To be more precise, if we continue to write $[u]$ for the $\mathrm{E}_n(R)$-orbit of $u \in \mathrm{Um}_n(R)$ (as in (5.3)), then for $n \geqslant 3$,

(5.4) $$\psi_m : \ [(a_1, \ldots, a_n)] \mapsto [(a_1^m, a_2, \ldots, a_n)]$$

turns out to be a *well-defined* self-map of $\mathrm{W}_n(R)$. [Here, when m is negative, the power a_1^m is taken modulo the ideal $a_2R + \cdots + a_nR$.] In particular,

$$\mathrm{im}\,(\psi_0) = [(1, 0, \ldots, 0)],$$

and ψ_1 is the identity map, while $\psi_m\psi_{m'} = \psi_{mm'}$ for all $m, m' \in \mathbb{Z}$. Also, since

$$[(a_1, a_2, \ldots, a_n)] = [(a_2, -a_1, \ldots, a_n)],$$

the well-definition of ψ_m implies, in particular, that

$$[(a_1^m, a_2, \ldots, a_n)] = [(a_1, a_2^m, \ldots, a_n)]$$

for all $(a_1, \dots, a_n) \in \mathrm{Um}_n(R)$. This may be viewed as a strong form of (III.4.9).

For different proofs for the well-definition of the operations ψ_m on $\mathrm{W}_n(R)$ $(n \geqslant 3)$, see [Roitman: 1985] and [Vaserstein: 1986].

The operation ψ_{-1} is already known from the earlier work [Vaserstein-Suslin: 1976]. As a self-map of $\mathrm{W}_n(R)$, ψ_{-1} has order $\leqslant 2$. Vaserstein noted that the well-definition of ψ_{-1} also implies the fact, mentioned above and proved before in III.6.1(2), that (for $n \geqslant 3$) any two rows of a matrix in $\mathrm{GL}_n(R)$ define the same element in $\mathrm{W}_n(R)$.

The operation ψ_2 is implicit in [Suslin: 1977b]. The use of ψ_2 does sometimes lead to new possibilities. For instance, Rao (1998c) has shown that, if $\mathrm{sr}\,(R) \leqslant 4$, then $\psi_2(\mathrm{W}_3(R))$ (consisting of classes of unimodular rows of the Swan-Towber form (a^2, b, c)) has an abelian group structure. For another work on the power operations ψ_i, see [Hinson: 1992].

If $A = R[t]$ where R is a commutative ring, and $f := (f_1, \dots, f_n) \in \mathrm{Um}_n(A)$ is "special" (in that the row of its leading coefficients, $(a_1, \dots, a_n)$, is unimodular over R), Rao (1985b) showed that

$$[(f_1, \dots, f_n)] = [(a_1, \dots, a_n)] \in \mathrm{W}_n(A), \quad \text{in case } n \geqslant 3.$$

From this, it follows that, again for $n \geqslant 3$:

$$(5.5) \qquad [(f_1, \dots, f_n)] = [f_1(r), \dots, f_n(r)] \in \mathrm{W_n}(A) \quad (\forall\, r \in R).$$

This latter result was also re-proved in [Sivatski: 2000]. For a detailed exposition on Rao's theorem, see VI.2.8.

In other works, [Hinson: 1991a] has considered a (pseudo) graph structure on $\mathrm{Um}_n(R)$ where two unimodular rows are connected by an edge if their inner product is 1. For any $\beta \in \mathrm{Um}_n(R)$ (where $n \geqslant 3$), Hinson showed that the $\mathrm{E}_n(R)$-orbit $[\beta]$ of β consists of unimodular rows connectible to β by paths of *even* length, while the path component of β is $[\beta] \cup [\gamma]$, where γ is any row with $\beta\gamma^t = 1$. These results are now included in III.6 of our text. Hinson has also studied the *diameter* of the graph structure on $\mathrm{Um}_n(R)$, showing that this diameter is 2 (say for all $n \geqslant 2$) iff R is a Boolean ring. In [Hinson: 1991a, b], the graph structure on $\mathrm{Um}_n(R)$ is further related to word lengths in the group $\mathrm{E}_n(R)$ with respect to the elementary matrices. Earlier papers on word lengths in $\mathrm{E}_n(R)$ and commutator subgroups of linear groups include [van der Kallen: 1982], [Dennis-Vaserstein: 1988, 1989], and [Sivatski-Stepanov: 1999].

We should also mention the paper [Gubeladze: 1992], in which the transitivity of $\mathrm{E}_n(R)$ on $\mathrm{Um}_n(R)$ is proved for monoid algebras $R = k[M]$, where k is a noetherian ring of dimension d and $M \subseteq \mathbb{Q}_+^r$ is an integral monoid extension (any $q \in \mathbb{Q}_+^r$ has an integral multiple in M), under Suslin's standard assumption $n \geqslant \max\{3, d+2\}$. This result also extends to monomial quotients of $k[M]$, thus covering, for instance, the case of discrete Hodge algebras. In the sequel [Gubeladze: 1993], the integrality condition for the monoid extension $M \subseteq \mathbb{Q}_+^r$ is significantly relaxed; in particular, no restriction is required for $r = 3$. (The case of general cancellative torsionfree monoids M has, however, remained untreated.) In [Gubeladze: 2001], descent results are obtained for the condition $[u]=[(1,0,\dots,0)]$ under a "subintegral" extension of commutative rings (again for $n \geqslant 3$). A related earlier paper is [Gubeladze: 1995b]. For a current work on effective computations in elementary matrices over a monoid ring with applications to the functor K_2, see [Gubeladze: 2006].

The problem of lifting unimodular rows from discrete Hodge algebras to polynomial algebras is more complicated. Some partial results can be found in [Wiemers: 1992]. More information on the completion of unimodular rows (especially from the viewpoint of computations) is given in §13. For work relating $\mathrm{W}_n(R)$ to Euler class groups, see [Bhatwadekar-Sridharan: 2000, 2002], and [van der Kallen: 2002].

There are still many unsolved problems involving unimodular rows. We take this opportunity to record some of them. The first of these, on unimodular polynomial vectors, is prompted by results such as VI.2.3(2).

(5.6) Local-Global Problem. *Let $f,\ g \in \mathrm{Um}_n\big(R[t]\big)$ be such that $[f(0)]=[g(0)] \in \mathrm{W}_n(R)$, where R is a commutative ring, and $n \geqslant 3$. If*

$$[f]=[g] \in \mathrm{W}_n\big(R_{\mathfrak{m}}[t]\big)$$

for all maximal ideals $\mathfrak{m} \subset R$, does the same equality hold in $\mathrm{W}_n\big(R[t]\big)$?

One significant case in which we can answer this question positively is when the unimodular vector f is "special" in the sense of VI.2; i.e. when the leading coefficients of the coordinates of f generate the unit ideal in R. In this case, by Rao's theorem (VI.2.8), $[f]=[f(0)] \in \mathrm{W}_n\big(R[t]\big)$. On the other hand, in $\mathrm{W}_n\big(R_{\mathfrak{m}}[t]\big)$, we have $[g]=[f]=[f(0)]=[g(0)]$. By the Patching Theorem VI.2.3 for the elementary group, it follows that

$$[g]=[g(0)]=[f(0)]=[f] \in \mathrm{W}_n\big(R[t]\big),$$

as desired. In particular, the answer to (5.6) is is "yes" if $f \in \mathrm{Um}_n(R)$. But the answer is unknown if $f(t)$ and $g(t)$ are *both* non-constant unimodular polynomial vectors.

The second problem is, in part, prompted by a very special case of (III.4.11) (and the Swan-Towber result on the completability of $(x^2, b, c) \in \mathrm{Um}_3(R)$). Consider a row $(1+a, b, c)$ over a commutative ring R, where a is nilpotent modulo the ideal $bR+cR$. Clearly, $(1+a, b, c) \in \mathrm{Um}_3(R)$. If $2 \in \mathrm{U}(R)$, then III.4.11 implies that $(1+a, b, c)$ is completable. *But what if* $2 \notin \mathrm{U}(R)$? For instance, one might begin by trying to deal with the "worst scenario" where $2=0 \in R$. This brings us to the following very concrete question:

(5.7) Murthy's $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$-Problem. *If R is a commutative ring of characteristic 2, and $a^n \in bR+cR$ for some $n \geqslant 2$, is the unimodular row $(1+a, b, c)$ always completable (to a matrix in $\mathrm{GL}_3(R)$) ?*

Although this question looks "simple", it has remained unanswered for more than 25 years. Even the case $n=2$ seems wide open!

The various "conjectures" (S_1), (S_2), (S_3), (H) (and their local versions) stated in (V.3), though obviously very strong, seemed to have so far survived the test of counterexamples! The one easiest to state in terms of unimodular rows is (H'), which says that for a (commutative) local ring R, any unimodular row over $R[t]$ is completable (that is, $R[t]$ is Hermite). As a third open problem, we remind our readers of what we have stated earlier in (V.3.8), which may be viewed as a more cautious form of (H'):

(5.8) Suslin's Problem $\mathbf{Su}(R)_n$. *Let R be a commutative local ring. If $n!$ is invertible in R, is every*

$$f = (f_0, f_1, \ldots, f_n) \in \mathrm{Um}_{n+1}\big(R[t]\big)$$

completable $\big($*to a matrix in* $\mathrm{GL}_{n+1}(R[t])\big)$?

Let us mention some key results obtained so far on $\mathrm{Su}(R)_n$, where, of course, we assume that $n \geqslant 2$. In (III.4.13), we have obtained a "yes" answer to (5.8) if all f_i 's have degree $\leqslant 1$. And of course, completability always holds if one of the f_i 's is unitary, by III.2.6. (In this case, the assumption on $n! \in \mathrm{U}(R)$ is unnecessary.) On an intuitive level, these are perhaps the best pieces of evidence for the truth of (5.8). If no restrictions are imposed on the f_i 's, the study of $\mathrm{Su}(R)_n$ is often coupled with the consideration of the Krull dimension of R (if it is finite). We'll give a summary of some known results in this direction, as follows.

In case R is noetherian local of dimension d, the answer to $\mathrm{Su}(R)_n$ is "yes" (again without the $n!$ assumption) if $n \geqslant d+1$, by the general Transitivity Theorem III.3.6. Therefore, the first interesting case to focus on is when $n = d$. In this case, Murthy observed that Suslin's Problem has an affirmative answer for $d = 2$. Murthy's argument, as presented in [Bhatwadekar-Roy: 1983, (2.7)], works in the multivariate case, and in particular verifies a special case of the "conjecture" (H').

Theorem 5.9. *For any noetherian local ring R with $\dim R \leqslant 2$ and $2 \in \mathrm{U}(R)$, the polynomial ring $R[t_1, \ldots, t_m]$ is Hermite.*

Here is a sketch of the proof. It suffices to show that, for $A := R[t_1, \ldots, t_m]$, $\mathrm{E}_k(A)$ acts transitively on $\mathrm{Um}_k(A)$ for all $k \geqslant 3$. If $k \geqslant 4$, this follows again from III.3.6. By (5.2), this would also follow for $k = 3$, if we can show that $\mathrm{W}_E(A) = 0$. Since $2 \in \mathrm{U}(R) \subseteq \mathrm{U}(A)$, Karoubi's theorem gives $\mathrm{W}_E(A) \cong \mathrm{W}_E(R) = 0$ (R being local), so we are done.

Incidentally, (5.9) can also be stated in the following "global" form, thanks to Quillen's Patching Theorem. Note that, here, there is no regularity assumption on R, but if R is regular, this recaptures V.3.3 in view of Grothendieck's Theorem II.5.8 !

Theorem 5.9$'$. *If R is noetherian ring with $\dim R \leqslant 2$ and $2 \in \mathrm{U}(R)$, then any stably extended f.g. projective $R[t_1, \ldots, t_m]$-module is extended from R.*

Coming back now to Suslin's Problem $\mathrm{Su}(R)_n$ for d-dimensional noetherian local rings R, the case where $n = d$ was settled by Rao *for any d*, in extension of Murthy's observation for $d = 2$. In [Rao: 1988b], it is shown that, if $d!$ is invertible in R, any $f \in \mathrm{Um}_{d+1}(R)$ can be elementarily transformed to a unimodular row of the form

$$(g_0^{d!},\ g_1,\ \ldots,\ g_d),$$

and is therefore completable, by the $d\,!$ theorem of Suslin.

Thus, the first open case of (5.8) is $\mathrm{Su}(R)_{d-1}$ for R (say noetherian) local of dimension d. Here, the first significant case $d=3$ has also been settled affirmatively: Rao (1991) showed that, *if R is noetherian local of dimension 3 with $1/2 \in \mathrm{U}(R)$, any $f \in \mathrm{Um}_3\big(R[t]\big)$ is completable.* Progress is being made toward proving this statement *in all dimensions* (in other words, proving $\mathrm{Su}(R)_2$ in full); see [Rao-Swan: 2006a]. For other information and earlier relevant work on Suslin's problem $\mathrm{Su}(R)_n$, see [Swan: 1978], [Roitman: 1986], and [Rao: 1987, 1988b].

If the answer to $\mathrm{Su}(R)_n$ is "yes" for all n, it will follow that, for any commutative local ring R containing the rational field $\mathbb{Q}$, $R[t]$ is Hermite. On the other hand, if the Bass-Quillen Conjecture is true, $\mathrm{Su}(R)_n$ would hold for all *regular* local rings, without any assumption on $n!$. For a survey on results on the Bass-Quillen Conjecture, see the next section. In comparison with the Bass-Quillen Conjecture, $\mathrm{Su}(R)_n$ deals with a broader class of rings: it is meaningful, and seems to have a reasonable chance of being true, for *arbitrary* commutative local rings. (To prove $\mathrm{Su}(R)_n$ true for all n, of course, it would be sufficient to work in the noetherian case, by the usual noetherian reduction argument via the Hilbert Basis Theorem.)

Before we conclude our discussions on Suslin's Problem, we should perhaps reiterate the fact that even Conjecture (H') (that $R[t]$ is Hermite for any commutative local ring R) has apparently remained open. For such R, Bhatwadekar and Rao (1983) have proved that *$R[t]$ is Hermite iff $R\langle t\rangle$ is Hermite*, where $R\langle t\rangle$ is the localization of $R[t]$ at the multiplicative set of monic polynomials (see IV.1). Thus, the open question here can also be phrased in the following equivalent form: *is every unimodular row completable over $R\langle t\rangle$ (for R commutative local)*?

The Conjecture (H') has an obvious analogue in the *noncommutative* case too, where R is allowed to be an arbitrary local ring. However, the noncommutative conjecture turned out to be false. Let $R = H[[x, y]]$ be the power series ring in x, y over the division ring H of the real quaternions, and let $A = R[t]$. The right A-epimorphism $A^2 \to A$ defined by

$$e_1 \mapsto xt + i, \quad \text{and} \quad e_2 \mapsto yt + j$$

has a stably free kernel P that is *not* free, according to [Parimala: 1981b]. (The argument for the non-freeness of P is similar to that used in the proof of Theorem II.3.5.) Thus, for the noncommutative local ring R, $R[t]$ fails to be right Hermite.

Coming back to the commutative case, it is pointed out in [Rao-Swan: 2006a] that, over a commutative local ring R, unimodular vectors over $R[t]$ need not be *elementarily* completable. In fact, an example is given for a vector in $\mathrm{Um}_3\big(R[t]\big)$ that is completable to a matrix in $\mathrm{GL}_3\big(R[t]\big)$, but not completable to one in $\mathrm{E}_3\big(R[t]\big)$. In this connection, let us also mention one result on completing a vector in $\mathrm{Um}_n\big(R[t]\big)$ to an elementary matrix. This result, in the spirit of a K_1-Horrocks' Theorem, has already been alluded to in our earlier remark (VI.5.15); it appeared as (1.1) in [Rao: 1987].

Theorem 5.10. *Let R be any commutative ring, and $h \in R[t]$ be any monic polynomial. For $n \geqslant 3$, if a vector $f = (f_1, \dots, f_n) \in \mathrm{Um}_n\big(R[t]\big)$ is completable to an elementary matrix over the localization $R[t]_h$, then it is already completable to an elementary matrix over $R[t]$.*

Note that, in the above, if we replace the group of elementary matrices by the group of invertible matrices (over both $R[t]$ and $R[t]_h$), the resulting statement (without the assumption $n \geqslant 3$) would be a special case of the Affine Horrocks' Theorem (IV.2.3). Not surprisingly, Rao's proof of (5.10) started with a reduction to the case where R is a (commutative) local ring. For another proof of Rao's result in the local case, see [Sridharan: 2004]. In this recent preprint, Raja Sridharan proved the local case of Rao's result by establishing a suitable analogue of Roberts' Theorem as formulated in (IV.4.1), with the freeness of projective modules replaced by the completability of unimodular rows to elementary matrices.

To the ultimate optimist, Rao's result on E_n-action is perhaps still not the best version one can hope for in the context of monic inversion in $R[t]$. Indeed, analogously to (5.6), one may pose the following

(5.11) Monic Inversion Problem. *Let $f,\ g \in \mathrm{Um}_n\big(R[t]\big)$, where R is a commutative ring, and $n \geqslant 3$. Let $h \in R[t]$ be a monic polynomial. If*

$$[f] = [g] \in \mathrm{W}_n\big(R[t]_h\big),$$

does the same equality hold in $\mathrm{W}_n\big(R[t]\big)$?

Rao's result provided an affirmative answer for this in the special case where $g \in \mathrm{Um}_n\big(R[t]\big)$ is the constant unit vector $(1, 0, \dots, 0)$. The answer to the general case in (5.11) seems still unknown. This problem is, in fact, partly motivated by the corresponding "linear" problem for projective modules over $R[t]$. We shall discuss this linear problem more fully in the next section in the context of the Bass-Quillen Conjecture; see Question 6.6 below.

Yet another fruitful direction of research on unimodular rows is the theory and applications of the Suslin Matrices. These matrices made their debut in [Suslin: 1977b] in connection with Suslin's $n!$ theorem (III.4.1); we have given a short exposé on their definition and elementary properties in (III.7). For a thorough survey on how Suslin's matrices are used in the study of unimodular rows over commutative rings R, see the notes of the ICTP lecture [Rao: 2004]. Some highlights are as follows.

• Using the Koszul complex associated with a unimodular sequence, Fossum, Foxby and Iversen (1978/79) have constructed a *Mennicke symbol*, denoted by "wt", from $\mathrm{Um}_{n+1}(R)$ $(n \geqslant 1)$ into the abelian group $K_1(R)$. The two defining properties for "wt" to be a Mennicke $(n+1)$-symbol are, respectively:

(MS_1) "wt" is constant on each $\mathrm{E}_{n+1}(R)$-orbit in $\mathrm{Um}_{n+1}(R)$; and

(MS_2) "wt" is multiplicative in each of its $n+1$ variables.

Suslin (1982c, Prop. 2.2) reinterpreted the Mennicke symbol "wt" in terms of Suslin matrices as follows. For $\alpha \in \mathrm{Um}_{n+1}(R)$, (5.12) $$\mathrm{wt}\,(\alpha) = [S_n(\alpha,\ \beta)] \in K_1(R),$$

for any vector β such that the inner product $\langle \alpha,\ \beta \rangle = 1$. Here, $S_n(\alpha,\ \beta) \in \mathbb{M}_{2^n}(R)$ is the Suslin matrix associated with the unimodular pair $\{\alpha,\ \beta\}$ (defined inductively in (III.7.1)). The inner product equation $\langle \alpha,\ \beta \rangle = 1$ guarantees that $S_n(\alpha,\ \beta) \in \mathrm{SL}_{2^n}(R)$ by (III.7.8)(3), so (5.12) above shows that $\mathrm{wt}\,(\alpha) \in SK_1(R)$. This SK_1-invariant for unimodular rows is quite significant. For instance, using it, Rao has given a purely algebraic proof for the observation of Nori and Swan (see III.6)) that a stably free module P *of even rank* defined by a unimodular row $\alpha = (a_0, \ldots, a_n)$ need not be self-dual. Rao took α to be the universal unimodular row over the generic ring A_n (say over the base ring $\mathbb{Z}$) defined in (III.7.11), and used Suslin's result $SK_1(A_n) \cong \mathbb{Z}$ to show that $P \not\cong P^*$ in case $n = \mathrm{rank}\ P \geqslant 4$ is even. The details of this calculation are given in the last part of the lecture notes [Rao: 2004]; see also [Rao-Swan: 2006b]. For more examples of stably free modules that are not self-dual, see [Swan: 2006].

• In (III.7.9), we have defined the *special unimodular vector group* $\mathrm{SUm}_n(R)$ that is generated by the Suslin matrices $S_n(\alpha,\ \beta)$ with $\langle \alpha,\ \beta \rangle = 1$. Using a new "fundamental property" of Suslin matrices, Jose and Rao (2005b, c) introduced a certain action of $\mathrm{SUm}_n(R)$ on the space of Suslin matrices $\Sigma_n(R)$ defined in (III.7.4). This action leads to a group homomorphism

$$\varphi : \mathrm{SUm}_n(R) \longrightarrow \mathrm{SO}_{2(n+1)}(R),$$

whose kernel turns out to be $\{uI_{2^n} : u^2 = 1\}$ (assuming that $2 \in \mathrm{U}(R)$). The group $\mathrm{SUm}_n(R)$ can thus be expected to (more or less) resemble the spin groups.

• The natural subgroup of $\mathrm{SUm}_n(R)$ generated by those Suslin matrices $S_n(\alpha,\ \beta)$ with α completable to a matrix in $\mathrm{E}_{n+1}(R)$ (and $\langle \alpha,\ \beta \rangle = 1$) is the *elementary unimodular vector group*, denoted by $\mathrm{EUm}_n(R)$. The basic calculus for this group is introduced in [Jose-Rao: 2005a]. In its sequel (2005b), analogues of both Quillen-Suslin's local-global theory and Horrocks' Theorem are developed for $\mathrm{EUm}_n\big(R[t]\big)$, based on the earlier work of [Suslin-Kopeiko: 1977] for the elementary orthogonal groups $\mathrm{EO}_{2n}\big(R[t]\big)$ over polynomial rings.

In relation to the last bullet item above, the paper [Basu-Khanna-Rao: 2005] contains also a quick survey on the local-global principle of Quillen-Suslin, with an eye to a uniform treatment of its applications to the cases of linear, orthogonal, and symplectic groups over a polynomial ring $R[t]$. The treatment here covers, in fact, the case when the base ring R is module-finite over its center.

In this section, we have focused mainly on unimodular rows over commutative rings since so much research has been done on this theme alone. But of course, unimodular rows can be generalized in (at least) two directions. We close by making a few comments on each of these generalizations.

(1) Unimodular rows are a special case of right invertible matrices M. Working over (nonzero) commutative rings R, say, we have $M \in \mathbb{M}_{r,n}(R)$ where $r \leqslant n$, with

$MN = I_r$ for some $N \in \mathbb{M}_{n,r}(R)$. (The "solution space" of M is a type r stably free module of rank $n-r$.) So far, not much of the rich calculus for unimodular rows has been developed for such right invertible matrices M. For a few papers containing information on the $M \in \mathbb{M}_{r,n}$ case, see [Lam: 1976], [Roitman: 1985], and [van der Kallen: 2002].

(2) Unimodular rows are also a special case of unimodular elements in projective modules (as defined in §1). The existence and behavior of such unimodular elements have always remained a topic of interest since Serre proved his famous splitting theorem in the Séminaire P. Dubreil on May 5, 1958. This interest is reflected, for instance, by the voluminous work on Questions (1.3) and (1.4) on modules over polynomial rings (and more generally over monoid rings). Much of the work on affine algebras and Euler class groups reported in §2 are also concerned with unimodular elements in projective modules over such algebras. On the topic of the transitivity of transvections on unimodular elements in projective modules, the posthumous work of [Lindel: 1995] is especially important; see §1. Some more references on the existence of unimodular elements (e.g. upon going back from $R\langle t\rangle$ to $R[t]$) will be given toward the end of §6.

§6. The Bass-Quillen Conjecture

After the solution of Serre's Conjecture in 1976, the place of this conjecture was simply taken over by the Bass-Quillen Conjecture, which we have presented in Chapter V. After more than 25 years, the most general case of this latter conjecture has remained unsolved. However, a lot of progress has been made. In this section, we shall give a quick survey on the main results obtained in this area. Before we begin, we should mention that a number of earlier surveys on the Bass-Quillen Conjecture have appeared in various places; see, e.g. [Bhatwadekar: 1984a, 1999] (in English), [Lemmer-Naudé: 1989] (in Afrikaans), and [Wang-Qin: 1994] (in Chinese), which our multilingual reader should also consult for the sake of completeness.

Using a slightly different notation from (V.3), let us write $\mathrm{BQ}(d)$ for the conjecture that, if R is a commutative regular ring of Krull dimension d, then R has the property (E) defined in V.1.9; that is, for any n, any $P \in \mathfrak{P}\big(R[t_1, \ldots, t_n]\big)$ is extended from R. [The earlier version (BQ_d) in (V.3) is simply the disjunction of the statements $\mathrm{BQ}(1), \ldots, \mathrm{BQ}(d)$.] As we have seen in V.3, it is enough to study the conjecture $\mathrm{BQ}(d)$ in one variable in the local case, where it says "simply":

$\mathrm{BQ}'(d)$: $R[t]$ *is Hermite for any regular local ring* $(R, \mathfrak{m})$ *of Krull dimension* d.

We know already from Ch. V that $\mathrm{BQ}(d)$ and $\mathrm{BQ}'(d)$ are true for $d = 1, 2$, and true for all d if R is a power series ring in d variables over a field k by the work of Mohan Kumar, Lindel and Lütkebohmert from 1977. At around the same time, Suslin (1978b, 1978c) obtained various positive results on (BQ_d) in the case of regular affine k-algebras (over a field k), showing, in particular, the truth of (BQ_3)

for all such algebras under the assumption char $(k) \neq 2$. The breakthrough came with Lindel (1980b, 1981), who proved the following.

Theorem 6.1. BQ(d) *holds for all commutative regular rings* R *of essentially finite type over a field* k*; that is,* $R = S^{-1}A$ *where* S *is a multiplicatively closed set in an affine* k*-algebra* A.

Lindel's original theorem assumed k to be a perfect field, but this assumption can be removed according to a remark of Mohan Kumar. Thus, the Bass-Quillen Conjecture is true in all dimensions *in the "geometric" case.* Lindel's proof depends on the Descent Lemma (V.5.11) and the technique of étale neighborhoods. (A local ring $(A', \mathfrak{m}')$ is said to be an *étale neighborhood* of a local subring $(A, \mathfrak{m})$ if A' is an étale extension of A and $A/\mathfrak{m} = A'/\mathfrak{m}'$. This amounts to saying that A' is faithfully flat and unramified over A.) Another ingredient of Lindel's arguments is the basic result V.1.11 (due to [Roitman: 1979]), to the effect that, if a commutative ring R has the property (E), then so does any localization of R. (As we have pointed out in V.1, this is a kind of converse to Quillen's Patching Theorem V.1.6.)

For a quick sketch of Lindel's proof using the above ideas, see Bhatwadekar's survey (1999). A more leisurely exposition is available from Mandal's Springer Lecture Notes volume (1997), §7.1. For further extensions of Lindel's results on étale neighborhoods, see, e.g. [Nashier: 1983], and [Dutta: 2000].

The next breakthrough on the Bass-Quillen Conjecture was made possible by the work of [Popescu: 1985, 1986]. In these important papers, Popescu obtained a main theorem on general Néron desingularization, showing that, under some conditions of separability, every regular morphism of Noetherian rings is a filtered inductive limit of finite type smooth morphisms. The original proof of this result was complicated, difficult to understand, and inaccessible to non-experts. However, subsequent expositions by other authors, especially [André: 1991], [Ogoma: 1994], and [Swan: 1998], have greatly clarified Popescu's work; see also [Popescu: 1990], and [Spivakovsky: 1999].

In a local setting, Popescu's result, in purely algebraic terms, states the following (Theorem 1.1 in [Swan: 1998]).

Theorem 6.2. *Let* R *be a regular local ring containing a perfect field* k*. For any affine* k*-subalgebra* $A \subseteq R$*, there exists a regular affine* k*-domain* B *and* k*-algebra homomorphisms* $A \xrightarrow{\alpha} B \xrightarrow{\beta} R$ *whose composite* $\beta\alpha$ *is the inclusion map of* A *into* R.

With this "desingularization" result, Popescu settled the equi-characteristic case of the Bass-Quillen Conjecture in all dimensions:

Theorem 6.3. [*Popescu: 1989*] BQ$'(d)$ *is true for any regular local ring* $(R, \mathfrak{m})$ (of *dimension* d) *containing a field* k.

The short proof runs as follows. We may assume in (6.3) that k is a prime (and hence perfect) field. Our job is to show that any unimodular sequence $f = (f_1(t), \ldots, f_n(t))$ over $R[t]$ is completable. Here, all f_i 's are defined over an affine k-subalgebra $A \subseteq R$. Picking $A \xrightarrow{\alpha} B \xrightarrow{\beta} R$ as in (6.2), we see that f can be "pullbacked" to $B[t]$, and hence to $B_{\mathfrak{p}}[t]$, for the prime ideal $\mathfrak{p} = \beta^{-1}(\mathfrak{m})$ in the *regular* affine k-domain B. By Lindel's result, such a pullback is completable over $B_{\mathfrak{p}}[t]$. Applying β, it follows that f is completable over $R[t]$, as desired!

Popescu's methods can be applied also to certain cases of mixed characteristic; that is, $\operatorname{char}(R) = 0$, but $\operatorname{char}(R/\mathfrak{m}) = p > 0$. In this case, the truth of $\mathrm{BQ}'(d)$ is proved in [Popescu: 1989] for $(R, \mathfrak{m})$ *excellent henselian*, as well as for $(R, \mathfrak{m})$ *unramified* $(p \notin \mathfrak{m}^2)$. These results, together with (6.3), are the strongest obtained on the Bass-Quillen Conjecture to this date.

Since $\mathrm{BQ}'(d)$ is about $R[t]$ being Hermite, Suslin's Problem (5.8) has clearly a direct bearing on it. If R is a noetherian local ring of dimension 3 with $6 \in \mathrm{U}(R)$, the work of Rao on Suslin's Problem already showed that $R[t]$ is Hermite, as we have explained in §5. In particular, we have:

Theorem 6.4. [Rao: 1988a, 1991] $\mathrm{BQ}'(3)$ *is true for a* (3-*dimensional*) *regular local ring* $(R, \mathfrak{m})$ *if* $\operatorname{char}(R/\mathfrak{m}) \neq 2, 3$. *In this case, all f.g. projectives over* $R[t]$ *are free.*

Rao (1987) also proved the truth of $\mathrm{BQ}'(4)$ in the case $\operatorname{char}(R) = p > 3$, though this is now a special case of the equi-characteristic result (6.3).

In (V.3), we have mentioned Quillen's Question (QQ_d), asking if all f.g. projectives are free over a ring of the type $R_s = R[s^{-1}]$, where $(R, \mathfrak{m})$ is a regular local ring of dimension $\leqslant d+1$, and s is a regular parameter (in the sense that $s \in \mathfrak{m} \setminus \mathfrak{m}^2$). Quillen posed this question in 1976, together with the observation that the truth of (QQ_d) would imply that of (BQ'_d) (see (V.3)). Although the converse of this is unknown, this connection between (BQ'_d) and (QQ_d) has generated a great deal of interest in the latter.

In his work on the cohomological characterization of the Brauer group, Gabber (1980) answered Quillen's question affirmatively for 3-dimensional regular local rings, thus completely settling (QQ_2). Gabber's proof involved the use of nonabelian cohomology. A simpler and more self-contained proof of Gabber's result is later given in [Swan: 1988]. See also [Bhatwadekar: 1984a] for a different treatment of Quillen's question in the case of unramified 3-dimensional regular local rings. In this paper, Bhatwadekar showed additionally that, if $(R, \mathfrak{m})$ is a regular local ring of dimension $d+1$, and R contains a field F such that the embedding $F \hookrightarrow R/\mathfrak{m}$ is a finite separable field extension, then for any $s \in \mathfrak{m} \setminus \mathfrak{m}^2$, any f.g. projective $R[s^{-1}]$-module of rank $\geqslant d$ is free. This generalizes a result of Horrocks for 3-dimensional regular local rings.

Building on Lindel's result, Bhatwadekar and Rao (1983) settled (QQ_d) in the geometric case, proving that if $(R, \mathfrak{m})$ is a regular local ring of an affine k-algebra (over a field k), then f.g. projectives are free over R_s for any regular parameter $s \in \mathfrak{m} \setminus \mathfrak{m}^2$. In fact, in case $R/\mathfrak{m}$ is infinite, they even proved that f.g. projectives over

(6.5) $$A = R_s[t_1, \ldots, t_n, t_{n+1}^{\pm 1}, \ldots, t_{n+m}^{\pm 1}]$$

are free. In a subsequent paper [Rao: 1985], these results are further generalized to the case where the element s is a product of r elements in a minimal generating set

$(s_1, \ldots, s_{d+1})$ of $\mathfrak{m}$. If $r \leqslant 2$, f.g. projectives are free over the ring A in (6.5), and for any $r \leqslant d+1$, f.g. projectives of rank $\geqslant [\frac{d+1}{2}]$ or $\geqslant r$ are free over A (assuming the infinitude of the residue field $R/\mathfrak{m}$).

After Popescu's desingularization theorem was available, it became clear that the Bhatwadekar-Rao result in the geometric case is sufficient to give an affirmative answer to Quillen's question (QQ_d) (in all dimensions) for any equicharacteristic regular local ring. Also, some cases of unramified regular local rings in mixed characteristics are treated in [Tessier: 1994].

Bhatwadekar and Rao (1983) have pointed out that, if a regular local ring R is localized at an element $s \in \mathfrak{m}^2$, the resulting ring R_s may have f.g. projective modules that are *not* free. An example for this, due to Samuel, is given by the power series ring

$$R = \mathbb{R}[[x_0, \ldots, x_n]], \quad n \neq 1, 3, \text{ or } 7.$$

If we localize this regular local ring at $s = x_0^2 + \cdots + x_n^2$, the stably free R_s-module defined by the unimodular row $(x_0, \ldots, x_n)$ turns out to be *nonfree*. For a proof of this, see [Swan: 1983], p. 439.

In [Bhatwadekar: 1999], several open questions related to (BQ_d) and (QQ_d) were mentioned. Let us record here just one of them that deals with $R\langle t\rangle$, the ring obtained by localizing $R[t]$ at the multiplicative set of monic polynomials.

Question 6.6. *For a commutative ring R and f.g. projective modules P, Q over $R[t]$, does*

$$R\langle t\rangle \otimes_{R[t]} P \cong R\langle t\rangle \otimes_{R[t]} Q \Longrightarrow P \cong Q? \tag{6.7}$$

If $P \in \mathfrak{P}^R\big(R[t]\big)$, we know the answer is "yes". Indeed, in this case, (V.2.2) implies that $Q \in \mathfrak{P}^R\big(R[t]\big)$ also, and (V.2.3) yields $P \cong Q$ over $R[t]$. If P is *not* assumed to be extended from R, the injectivity of the map $K_0R[t] \to K_0R\langle t\rangle$ (proved in (V.2.6)) implies that P, Q are stably isomorphic, but it is not clear if they must be isomorphic. (The fact that P, Q are stably isomorphic was first observed by Murthy and Pedrini (1972). For a direct proof of this using a lemma of Swan, see Thm. 3.7.5 in [Ischebeck-Rao: 2005].) In the case where R is local, some results are available. Here, Roy (1980) proved that:

If P', $Q \in \mathfrak{P}\big(R[t]\big)$ with $\operatorname{rk} P' < \operatorname{rk} Q$, *then P' embeds as a direct summand of Q iff $R\langle t\rangle \otimes_{R[t]} P'$ embeds as a direct summand of $R\langle t\rangle \otimes_{R[t]} Q$.*

This implies the truth of (6.7) if P has a rank 1 direct summand P_1. (Apply the above result to a direct complement P' of P_1.) Using Roy's result, Nashier (1983d) showed that, *for any $P \in \mathfrak{P}\big(R[t]\big)$* ($R$ local), *P and $R\langle t\rangle \otimes_{R[t]} P$ have the same minimal number of generators*. (This, in itself, is a generalization of Horrocks' Theorem (IV.2.1).) Nashier also observed that, in the situation of (6.6), since P and Q must already be stably isomorphic, the truth of the second Eisenbud-Evans Conjecture implies an affirmative answer for (6.7) in the case where R is noetherian of dimension $\leqslant 1$. For an analogue of Roy's result for Laurent polynomial rings, see [Mandal: 1991a].

Note that the truth of the implication (6.7) would imply that a module $P \in \mathfrak{P}(R[t])$ is cancellative if $R\langle t\rangle \otimes_{R[t]} P$ is cancellative. For, if $P \oplus Q' \cong Q \oplus Q'$ in $\mathfrak{P}(R[t])$, then applying cancellation in $\mathfrak{P}(R\langle t\rangle)$ gives

$$R\langle t\rangle \otimes_{R[t]} P \cong R\langle t\rangle \otimes_{R[t]} Q,$$

and (6.7) would yield $P \cong Q$. Indeed, a large part of the motivation behind Question (6.6) is the investigation of the extent to which the behavior of $P \in \mathfrak{P}(R[t])$ is "determined" by that of $R\langle t\rangle \otimes_{R[t]} P$.

A related question in a similar spirit is whether the existence of a unimodular element in $R\langle t\rangle \otimes_{R[t]} P$ would imply the same for P. An affirmative answer to this was provided in [Roitman: 1979] for R local. In the case of non-local noetherian rings R, the same was done in [Bhatwadekar-Roy: 1983], under the assumption that rank $P \geqslant \dim(R/\mathrm{rad}\, R) + 2$. A variant of this latter result was also given by Bhatwadekar and Roy for the Laurent polynomial ring $R[t, t^{-1}]$. Other cases with positive results were presented in [Bhatwadekar-Roy: 1983] (assuming that P/tP contains a unimodular element), [Bhatwadekar-Lindel-Rao: 1985], and [Bhatwadekar: 1989]. More recently, an affirmative answer was given in [Bhatwadekar-Sridharan: 2001] for all commutative noetherian rings of finite Krull dimension containing an infinite field.

In the earlier version of this book ("Serre's Conjecture"), the ring $R\langle t\rangle$ was denoted by $R(t)$, following Quillen's original paper solving Serre's Conjecture in 1976. In this new version of the book, I felt compelled to switch over to the notation $R\langle t\rangle$, which seems to be currently more standard in the commutative algebra literature. A lot of information is available on $R\langle t\rangle$; see, for instance, [Gilmer-Heinzer: 1978], [le Riche: 1980], [Brewer-Heinzer: 1980], and [Anderson-Anderson-Markanda: 1985]. (The last article contains a good bit of history on $R\langle t\rangle$, as well as on interesting notational matters.) The multivariate versions $R\langle t_1, \ldots, t_n\rangle$ can be defined inductively, and more recently, some authors have decided to call these the "Serre Conjecture rings"; see, e.g. [Cahen-Elkhayyari-Kabbaj: 1997], followed by [Yengui: 1999, 2000]. I am somewhat neutral about this terminology, but I do believe that Quillen's solution to Serre's Conjecture in 1976 (as exposited in Ch. V; see especially, (V.2), (V.3)) has helped make these rings rather popular!

The notation $R(t)$ was originally used by Nagata to denote the localization of $R[t]$ at the set of *primitive polynomials* (that is, polynomials whose coefficients generate the unit ideal in R). The multivariate version of this, $R(t_1, \ldots, t_n)$, is traditionally known as a *Nagata ring*. Much is known about these rings too, but for our purposes, the most important fact is perhaps that f.g. projectives of constant rank are free over $R(t_1, \ldots, t_n)$ (if $n \geqslant 1$). This fact is proved in [McDonald-Waterhouse: 1981] for the class of rings "with many units," which includes the Nagata rings. The same fact was proved independently by D.D. Anderson, and by [Ferrand: 1982].

Coming back to the Bass-Quillen Conjecture, let us mention in closing that strong results are also available for the analogues of this conjecture for *monoid algebras*. In fact, by combining the results of [Gubeladze: 1988a], [Popescu: 1985–86], and [Swan: 1992, 1998], it follows that, *for any commutative regular ring R containing a field and for any commutative, cancellative, torsionfree and seminormal monoid M*

without nontrivial units, any f.g. projective $R[M]$*-module is extended from* R. (Here, the "classical" Bass-Quillen Conjecture corresponds to the case where the monoid M is free, while Anderson's Conjecture corresponds to the case where R itself is a field.) Note that the assumption that M be without nontrivial units is necessary, since f.g. projectives over a *Laurent* polynomial ring $R[t, t^{-1}]$ may not be extended from a regular base ring R, as we have seen at the end of (V.4).

We close this section by mentioning another reference on the Bass-Quillen Conjecture. This conjecture, in the form that every f.g. projective is free over $R[t]$ when $(R, \mathfrak{m})$ is a commutative regular local ring, is known in the case where R is complete, as per the results in V.5. One natural way to tackle the Conjecture for a general regular local ring R would be, therefore, to apply some suitable form of faithfully flat descent on $\hat{R}/R$, where $\hat{R}$ denotes the completion of $(R, \mathfrak{m})$. Such an approach was studied in detail in [Le Bruyn: 1985], where some necessary and sufficient conditions were given for when descent may be carried out. These conditions involved the theory of maximal orders and Azumaya algebras. In a later work, Le Bruyn investigated the "type number" of maximal orders and the behavior of such numbers under polynomial extensions; a version of Quillen's Patching Theorem (V.1.6) in this context was obtained in [Le Bruyn: 1986].

§7. $R[t_1, \dots, t_n]$ for R Non-Noetherian

Almost all of the results reported in this text pertained to the case of polynomial rings over (commutative) noetherian coefficient rings. But clearly, the study of projective modules over $R[t_1, \dots, t_n]$ should not be limited to the noetherian case.

One of the earliest non-noetherian results was "hidden" in a research announcement published in 1971. In a short abstract in the AMS Notices, W.V. Vasconcelos and A. Simis stated that if R is a valuation domain, then any f.g. projective module over $R[t]$ is free [Vasconcelos-Simis: 1971]. Nothing was said about the multivariate case, and the detailed proof of the original univariate result remained unpublished. This is why I said the result was "hidden". I was unaware of it when I wrote my Springer Lectures Notes in 1977; otherwise I would certainly have included a report on it in the original text.

After the solution of Serre's Conjecture by Quillen and Suslin, work also began in the case of non-noetherian base rings. Brewer and Costa showed in 1978 that if N is the nilradical of a commutative ring R, then a f.g. projective $R[t_1, \dots, t_n]$-module P is extended from R provided that P/NP is extended from R/N. Using this result, they showed that a commutative 0-dimensional ring or a 1-dimensional arithmetical ring(*) R has the property (E); that is, for any n, any f.g. projective module over $R[t_1, \dots, t_n]$ is extended from R. (The 0-dimensional case of this is

(*) A commutative ring R is said to be *arithmetical* if, for any maximal ideal $\mathfrak{m} \subset R$, the localization $R_{\mathfrak{m}}$ is a valuation ring; that is, the ideals of $R_{\mathfrak{m}}$ form a chain. According to a theorem of C.U. Jensen, arithmetical rings are exactly those commutative rings whose ideal lattices satisfy the distributive laws.

now included in (V.2.13) in our text. See also Th. 2 in [Brumatti-Lequain: 1983].) In the same paper [Brewer-Costa: 1978], the authors also showed that any 1-dimensional (commutative) Prüfer domain R has the property (E). (In the case where R is Bézout, this means that any f.g. projective module over $R[t_1, \ldots, t_n]$ is free: this was proved at around the same time in [Maroscia: 1977].) Partial information was given for f.g. projectives over $R[t_1, \ldots, t_n]$ in case R is a 2-dimensional valuation domain. Like this author, Brewer and Costa in 1977 were not aware of the short research announcement [Vasconcelos-Simis: 1971]. The constructive aspects of the Maroscia-Brewer-Costa result on 1-dimensional Prüfer domains have also been explored: see the reference to the paper of Lombardi, Quitté and Yengui later in §12.

Independently of Brewer and Costa, Lequain and Simis (1980) showed that *any Prüfer domain has the property* (E). By Quillen's Patching Theorem (V.1.6), it suffices to prove that, over a valuation domain R, any $P \in \mathfrak{P}\big(R[t_1, \ldots, t_n]\big)$ is free. Thus, the task at hand is to prove the Vasconcelos-Simis result in the general *multivariate* case.

This is done in two steps. The first one is to reduce the consideration to the case of a *finite-dimensional* valuation domain. This step, due to Vasconcelos, is reasonably routine. The second step was inspired by the axiomatic version of Quillen's method developed in (V.3) of our text. However, there is one obvious obstruction. To apply Quillen Induction in the form (V.3.1), we have to work with a class of rings $\mathcal{F}$ that is closed with respect to the passage $R \mapsto R\langle t\rangle$, where $R\langle t\rangle$ denotes the localization of $R[t]$ at the multiplicative set of monic polynomials. If we let $\mathcal{F}$ be the class of Prüfer domains, this closure property unfortunately fails!

Lequain and Simis found a clever way around this difficulty by proving the following variation of the axiomatic Quillen Induction Theorem.

Second Induction Theorem 7.1. *Let $\mathcal{F}$ be a class of commutative rings satisfying the four properties below:*

(LS_0) *If $R \in \mathcal{F}$, then every nonmaximal prime ideal of R has finite height.*
(LS_1) $R \in \mathcal{F} \Rightarrow R\,[t]_{\mathfrak{p}[t]} \in \mathcal{F}$ *for any* $\mathfrak{p} \in \operatorname{Spec} R$.
(LS_2) $R \in \mathcal{F} \Rightarrow R_{\mathfrak{p}} \in \mathcal{F}$ *for any* $\mathfrak{p} \in \operatorname{Spec} R$.
(LS_3) $R \in \mathcal{F}$ *and* R *local* $\Rightarrow$ *any* $P \in \mathfrak{P}\big(R\,[t]\big)$ *is free.*

Then, any ring $R \in \mathcal{F}$ has the property (E).

The reader should carefully compare the four conditions above with the three conditions (Q_i) used in Quillen's Induction Theorem V.3.1. The condition (LS_0) here looks a bit technical, but it is automatically satisfied if the rings in $\mathcal{F}$ are all finite-dimensional. The condition (LS_2) is a slight strengthening of (Q_2). The condition

$$(Q_1) \qquad R \in \mathcal{F} \Longrightarrow R\,\langle t\rangle \in \mathcal{F},$$

is now absent, by design. Replacing it is (LS_1), which may be thought of as a kind of *weakening* of (Q_1). (Note that, in the presence of (LS_2), we have (Q_1) $\Rightarrow$ (LS_1), since $R\,[t]_{\mathfrak{p}[t]}$ is the localization of $R\,\langle t\rangle$ at a prime ideal.) Finally, the condition (Q_3) in (V.3.1) is unchanged; we have simply renamed it (LS_3).

The main idea of the proof of the Lequain-Simis Theorem is that *the new induction theorem in* (7.1) *applies well to the class* $\mathcal{F}$ *of valuation domains of finite dimension.* For this class of local rings, (LS_0) and (LS_2) are clearly satisfied, and (LS_1) can be checked by an argument involving the content of polynomials with coefficients from a valuation domain. Finally, (LS_3) is (essentially) the Vasconcelos-Simis result in 1-variable (which, of course, needs to be proved!).

Upon applying Theorem 7.1 to $\mathcal{F}$, therefore, one sees that f.g. projective modules over $R\,[t_1, \ldots, t_n]$ are free for any finite-dimensional valuation domain R. As we have pointed out before, this is sufficient for proving the theorem of Lequain and Simis. For a more detailed exposition on this result, including a careful proof of Th. 7.1, see Ch. VI (§4) of the recent monograph [Fontana-Huckaba-Papick: 1997].

The techniques and results of Lequain-Simis (including their Induction Theorem 7.1) were further utilized in [Brewer-Costa: 1979] in their study of the property (E) for integral domains with a Prüfer normalization. If such a domain R has the property that $\mathrm{Spec}\,(R_{\mathfrak{p}})$ is finite for every prime $\mathfrak{p}$, then R has the property (E) iff it is seminormal in the sense of II.5; that is, for any $a \in \mathrm{Q}(R)$ (quotient field of R), $a^2, a^3 \in R \Rightarrow a \in R$. In particular, *the property* (E) *holds for any* 1*-dimensional noetherian seminormal domain* R. This generalizes earlier results of [Endô: 1963] and [Bass-Murthy: 1967] (in the domain case).

The interesting notion of seminormality was originally motivated by the work of Andreotti, Bombieri, Norguet and Salmon on classification problems for algebraic curves. For a comprehensive discussion of the geometric issues involved, see [Davis: 1978]. For an arbitrary commutative ring R with integral closure $\bar{R}$ in its total ring of quotients $\mathrm{Q}(R)$, Traverso [1970] defined R to be *seminormal* if

$$R = \{\, x \in \bar{R} :\ x/1 \in R_{\mathfrak{p}} + \mathrm{rad}\,(\bar{R}_{\mathfrak{p}}) \text{ for every } \mathfrak{p} \in \mathrm{Spec}\, R \,\},$$

where $\mathrm{rad}\,(\bar{R}_{\mathfrak{p}})$ denotes the Jacobson radical of the ring $(\bar{R})_{\mathfrak{p}}$. For noetherian rings, this is a local property according to [Traverso: 1970]; see also [Vorst: 1979b]. *If* R *is a reduced noetherian ring with a finite normalization* (*i.e.* $\bar{R}$ *is a f.g.* R*-module*), *Traverso showed that, for any* $n \geqslant 1$:

$$j_* :\ \mathrm{Pic}(R) \to \mathrm{Pic}\big(R\,[t_1, \ldots, t_n]\big) \tag{7.2}$$

is an isomorphism iff R *is seminormal in the above sense.* (The fact that this holds in the case of Krull dimension $\leqslant 1$ was first proved in [Bass-Murthy: 1967].) Traverso also obtained other descriptions of seminormal rings — but always with a noetherian condition on them.

Later, Traverso's result for the Picard group was extended to *arbitrary* integral domains R simultaneously in [Brewer-Costa: 1979] and [Gilmer-Heitmann: 1980]. The latter authors observed that their results also worked for noetherian reduced rings, while Rush (1980) treated the case of reduced rings R with finitely many minimal primes; see also [Vorst: 1979b]. The relationships between Traverso's seminormality and the notion of "(2, 3)-closedness" in II.5.11(4) were elucidated in [Hamann: 1975], as well as in all of the papers listed above. But if R is, say, a general reduced ring, Traverso's original theorem on the Picard group can no longer be expected to hold.

Swan overcame this difficulty by *redefining* seminormality: in [Swan: 1980], a commutative ring R *is said to be seminormal if, for any* $b,\ c \in R$ *such that* $b^3 = c^2$, *there exists* $a \in R$ *such that* $b = a^2$ *and* $c = a^3$. Swan remarked that it is harmless to change over to this definition, since it agrees with that of Traverso in the case where $\mathrm{Q}(R)$ is a direct product of fields. (All previous theorems applied only in this case.)

One great advantage of Swan's definition is that it is a very simple "internal" first-order condition on the (commutative) ring R. Costa [1982] noted that Swan's definition *implies* the reducedness of R. His simple proof runs as follows. Suppose $d^2 = 0$ in a ring R seminormal in the above sense. Then $d^2 = 0 = d^3$ implies that $d = a^2 = a^3$ for some $a \in R$. Then $d = aa^2 = ad$, so $d = a(ad) = d^2 = 0$. The reducedness of R, in turn, implies that in Swan's definition, given $b^3 = c^2$, the element a there is in fact *unique*. For, if we also have $b = \bar{a}^2,\ c = \bar{a}^3$ for some $\bar{a} \in R$, then

$$(a - \bar{a})^3 = a^3 - 3\,a^2\,\bar{a} + 3\,a\,\bar{a}^2 - \bar{a}^3 = c - 3c + 3c - c = 0$$

implies that $a = \bar{a}$.

Swan's new definition of seminormality enabled him to give a characterization for the map j_* in (7.2) to be an isomorphism for *any* (commutative) ring R. His result is the following:

> *The map* j_* *is an isomorphism for some* $n \geqslant 1$ *iff it is an isomorphism for all* n, *iff the reduced ring* $R/\mathrm{Nil}\,(R)$ *is seminormal* (*in Swan's sense*), *where* $\mathrm{Nil}\,(R)$ *denotes the nil radical of* R.

From this statement, it follows without further proof, for instance, that the seminormality of R implies the seminormality of any polynomial ring $R[t_1, \ldots, t_n]$. For a direct arithmetic treatment of this (at least for domains and for noetherian rings) via the preservation of (2, 3)-root closure under polynomial extensions, see [Brewer-Costa-McCrimmon: 1979]. For a nice reworking of some of Swan's results from a "constructive" viewpoint, see [Coquand: 2005].

Prompted by some examples given in [Hamann: 1975], Swan was able to go even one step farther. For any rational prime p, call R *p-seminormal* if, for any $b, c, d \in R$ such that $b^3 = c^2$, $d^2 = p^2 b$, $d^3 = p^3 c$, there exists an element $a \in R$ such that $b = a^2$ and $c = a^3$. Then, *the kernel of*

$$\mathrm{Pic}\left(R\,[t_1, \ldots, t_n]\right) \longrightarrow \mathrm{Pic}\,(R)$$

is p*-torsion free iff* $R/\mathrm{Nil}\,(R)$ *is* p*-seminormal.*

For an excellent survey on the evolvement of the notion of seminormality with considerably more details than given here, we refer the reader to [Costa: 1982].

For other work dealing with the extendibility of rank 1 projectives over $R\,[t]$ and $R\,[t, t^{-1}]$ where R is a reduced ring, see [Ischebeck: 1977] and [Bhatwadekar-Varma: 1993]. For extendibility results where $R\,[t]$ is replaced by commutative $\mathbb{N}$-graded rings, see [Anderson: 1981, 1982].

The work described above is relevant to Serre's Problem since it determines completely those commutative rings R for which all rank 1 projectives for polynomial rings over R are extended (from R). For comparison, we should also mention the following interesting result from [Querré: 1980]: *a commutative domain* R *is normal iff all rank* 1 *reflexive* $R[t]$*-modules are extended from* R.

The ultimate question to ask would be, of course: *precisely what rings (or even just commutative domains) have the property* (E)? This is likely to be a very difficult question, as a full answer to it would have to include a solution to the Bass-Quillen Conjecture, as well as Gubeladze's solution to Anderson's Conjecture, as we have seen in the discussions in the Appendix to V.3. In this connection, let us mention a couple of non-noetherian results obtained by Chinese researchers. Huanyin Chen (1996a) proved the property (E) for any "w-inherent" ring; that is, a commutative locally coherent ring with weak global dimension $\leqslant 1$. More recently, Fanggui Wang proved the same result for any commutative ring of global dimension $\leqslant 2$. The proof of Wang's result depends on a judicious use of the Lequain-Simis induction method, and also on specific structural information on commutative local rings of global dimension 2. For more details, see [Wang: 2002].

As for the Eisenbud-Evans problems for efficient generation of modules, various results are also available in the non-noetherian case. We should especially mention the remarkable results in [Heitmann: 1984] on generating non-noetherian modules efficiently, based in part on the earlier work [Heitmann: 1976] on the generation of invertible ideals in non-noetherian domains (including Prüfer domains). In [Heitmann: 1984], non-noetherian versions of Serre's splitting, Bass's stable range theorem, Eisenbud-Evans's basic element theorem and the Forster-Swan theorem were obtained, using the new notions of j-spectrum and δ-dimension. For related works, see [Vasconcelos-Wiegand: 1978] and [Naudé: 1990]. Heitmann's work is recently further refined in [Coquand-Lombardi-Quitté: 2005], where a certain "Heitmann dimension" (a variant of Heitmann's δ–dimension) is introduced. Using this modified dimension, these authors studied the "elementary and constructive aspects" of Kronecker's theorem, Serre's splitting, Bass's stable range, cancellation theorem, etc. in the non-noetherian case, in extension of [Heitmann: 1984]. Parts of these results have already appeared in [Coquand: 2004] and [Coquand-Lombardi-Quitté: 2004]; see also the two related papers [Ducos: 2004a, b].

There are several papers that are more specific to the case of polynomial rings as well. The bounds on the number of module generators are, in general, weaker than those in the noetherian case. For instance, in [Brumatti: 1984], it is shown that, if R is a commutative ring of Krull dimension d, and $P \in \mathfrak{P}(R[t_1, \ldots, t_n])$ is locally generated by k elements, then P can be generated by $k(d+1)$ elements. (For a related result, see [Brumatti-Lequain: 1983].) A strong non-noetherian theorem later appeared in [Lemmer: 1994], where it is proved that, if P is a f.g. projective module of constant rank over $B = R[t_1, \ldots, t_n]$, then under suitable assumptions on the minimal prime spectra of certain quotients of R, P can be generated by $\mathrm{rk}(P) + \dim(B) - n$ elements. In the noetherian case, this retrieves the $\mathrm{rk}(P) + \dim(R)$ bound in [Bhatwadekar-Roy: 1984] mentioned in §2.

For other papers on the efficient generation of projective and stably free modules over $R[t_1, \ldots, t_n]$ in the non-noetherian case, see [Lemmer-Naudé: 1993], [Naudé: 1990], and [Naudé-Naudé: 1990], among others.

§8. Noncommutative Polynomial Rings

What can be said about (f.g.) projective modules over a polynomial ring A that is not commutative? Here, A can be of the form $R[t_1, \dots, t_n]$ where R is a noncommutative ring, or A can be an Ore extension, say $R[t, \sigma, \delta]$, where σ is an endomorphism of the ring R, and δ is a σ-derivation, that is, an additive map $R \to R$ with the property that $\delta(ab) = \sigma(a)\delta(b) + \delta(a)b$ for all $a, b \in R$. In the latter case, t need not commute with elements of R, but rather, satisfies the commutation rule

$$tr = \sigma(r)t + \delta(r) \quad \text{(for all } r \in R).$$

The most prominent examples among Ore extensions are the Weyl algebras.

The study of projective $k[t_1, \dots, t_n]$-modules for a noncommutative division ring k began with the work of Sharma, Ojanguren and Sridharan reported in (II.3). In the 1970s, the existence of nonfree stably free left/right ideals in such a polynomial ring $k[t_1, \dots, t_n]$ $(n \geqslant 2)$ was a new phenomenon. (In the commutative case, any f.g. stably free module of rank 1 is free by (I.4.11).) Such 1-sided ideals are studied in more detail in [Parimala-Sridharan: 1975], [Sridharan: 1981], and [Parimala: 1982], among others. Closely related to the above is the study of projective modules over quaternion algebras defined over commutative rings; see, e.g. [Parimala-Sridharan: 1976/77], and [Knus-Parimala: 1980].

For many more interesting examples of nonfree projective modules over polynomial (or affine) rings with noncommutative coefficients (constructed with the goal of studying quaternionic bundles over spheres), see [Swan: 1996].

The work mentioned in the opening paragraph of this section was utilized in [Parimala: 1978] to give the first examples of non-extended inner product spaces over $\mathbb{R}[x, y]$ (as we saw in VII.4). This work established for the first time the connections between projective 1-sided ideals in $H[x, y]$ (H denoting Hamilton's division ring of real quaternions) and the quadratic analogue of Serre's Conjecture. Later, these projective ideals are further linked to stable vector bundles over the projective plane $\mathbb{P}^2_{\mathbb{C}}$; see, e.g. [Knus-Parimala-Sridharan: 1981], and [Knus: 1982]. These works belong more properly to the study of the quadratic analogue of Serre's Conjecture, so we'll postpone their survey to §10 below.

For a division ring k finite-dimensional over its center, Knus and Ojanguren (1977) proved that any f.g. projective $k[t_1, t_2]$-module of rank > 1 is free. Slightly later, Suslin (1978a) proved the same result over $k[t_1, \dots, t_n]$ for any n. The strongest possible result was subsequently obtained in [Stafford: 1980], where it is shown that:

If k is a division ring with an infinite center, then any f.g. projective left module over $k[t_1, \dots, t_n]$ is either free or isomorphic to a left ideal.(*)

In a separate series of papers, Stafford has also investigated extensively the stable structure of noncommutative noetherian rings, and generalized the classical theorems

(*) To the best of my knowledge, however, the case where k has finite center has remained untreated.

of Serre (free summands), Bass (cancellation), and Forster-Swan (efficient generation) to this context. See, for instance, his papers [Stafford: 1977b, 1978a, 1981a, b, 1982].

The theme of the existence of nonfree stably free left ideals was also studied in [Stafford: 1985], where a general construction principle was given for such objects based on the use of two "strongly noncommuting" elements. This construction covers the known cases of $k[x, y]$, Weyl algebras, group rings of nonabelian, poly-(infinite cyclic) groups, and quite remarkably, applies to the enveloping algebra of any finite-dimensional nonabelian Lie algebra. For related work of a similar spirit, see [Artamonov: 1981] on group rings of solvable groups, and [Antoniou-Iyudu-Wisbauer: 2003] on the enveloping algebras of the RIT (relativistic internal time) Lie algebras.

There is only a small literature on the structure of projective modules over Ore extensions; see, e.g. [Archer-Hart: 1980] for some information on the case $R[t, \delta]$. However, a lot of information is available for Weyl algebras in [Stafford: 1977a, 1978b]. In the latter paper, Stafford showed that, for the Weyl algebra A_n over a field of characteristic 0, f.g. projective (left) A_n-modules are either free or isomorphic to a left ideal of A_n.

Some of the key tools used in Quillen's solution of Serre's Conjecture have been successfully generalized to the noncommutative setting. For instance, Quillen's Patching Theorem (V.1.6) was proved for $\mathbb{N}$-graded algebras in [Artamonov: 1983] (with a K_1-analogue in [Artamonov: 1995]); see also [Lindel: 1984]. And a Horrocks type theorem for noncommutative rings was obtained in [Artamonov: 1989a]. These additional tools eventually paved the way to various quantum generalizations of Serre's Conjecture: we refer the reader to §11 for more details.

A paper devoted to the study of simple modules and primitivity questions for $R = k[t_1, \ldots, t_n]$ over a division ring k is [Amitsur-Small: 1978]. Although this work did not involve projective modules over R, it is worth mentioning as one of the earliest papers dealing with multivariate polynomial rings over general division rings.

§9. K_1 (and Higher K_n) Analogues

One nice effect of Serre's Conjecture (and Serre's original theorem on projective modules) was the stimulus it gave to the investigation of K_1 (and possibly higher K_n's) in the direction of stable and nonstable behavior of such functors. This investigation began with Bass's seminal work in 1964 proving the surjective stability of K_1 and conjecturing its injective stability, in terms of the stable range of rings. A version of injective stability was later proved in the commutative case in [Bass-Milnor-Serre: 1967], and in the noncommutative case in [Bass: 1968]. This work on K_1 culminated in [Vaserstein: 1969], which contained the fullest and most satisfying form of injective stability for the general linear group, not only at the critical dimensions but in some sense in all dimensions. This work, incidentally, was also the origin of Vaserstein's Lemma, now included as (I.9.4) in our text.

Normality results on the elementary groups $\mathrm{E}_n(R)$ (and their relative versions $\mathrm{E}_n(R, J)$) were very much a part of the investigations on K_1-stability. Nevertheless, Suslin's proof (in [Suslin: 1977a]) of the normality of $\mathrm{E}_n(R, J)$ for $n \geqslant 3$ in the commutative case ((I.9.14) in our text) came as a total surprise for many. Analogues of Suslin's normality theorem were later obtained for the elementary orthogonal group $\mathrm{EO}_{2n}(R) \subseteq \mathrm{O}_{2n}(R)$ $(n \geqslant 3)$ in [Suslin-Kopeiko: 1977]; see [Kopeiko: 1983] for the failure of the result in the case $n = 2$. Similar results for the elementary symplectic group $\mathrm{Ep}_{2n}(R) \subseteq \mathrm{Sp}_{2n}(R)$ $(n \geqslant 3)$ appeared later in [Kopeiko: 1978], [Taddei: 1982], and in the general setting of Chevalley-Demazure group schemes over commutative rings in [Taddei: 1983]. In the special case where R is a Dedekind ring of the arithmetic type, Bass, Milnor and Serre have shown considerably earlier that $\mathrm{Ep}_{2n}(R)$ is normal in $\mathrm{Sp}_{2n}(R)$ for $n \geqslant 2$; see [Bass-Milnor-Serre: 1967, Thm. 12.4].

The classification of normal subgroups of $\mathrm{GL}_n(R)$ and $\mathrm{Sp}_{2n}(R)$ is closely tied to the study of $\mathrm{E}_n(R)$ and $\mathrm{Ep}_{2n}(R)$. This is, however, further removed from Serre's Conjecture, so (besides [Bass-Milnor-Serre: 1967]) we mention only [Bass: 1968, V.2], [Golubčik: 1981], [Vaserstein: 1981, 1989], and [Stepanov-Vavilov: 2000] as representative works in this area. (The last cited paper contains, especially, an extensive list of 88 papers dealing with the subgroup structure of classical groups over rings.) For more information in the commutative case, see [McDonald: 1984]. For normality issues and their relations to local-global principles for rings finite over their centers, see [Basu-Khanna-Rao: 2005]. There are also strong results on the nilpotency and solvability of the nonstable K_1-groups $\mathrm{SL}_n(R)/\mathrm{E}_n(R)$ and $\mathrm{GL}_n(R)/\mathrm{E}_n(R)$. In this direction, we have already briefly referred to the papers [Bak: 1991] and [Vaserstein: 1992] in a remark following the statement of Suslin's Normality Theorem in (I.9.14). For generalizations of such nonstable results to automorphism groups of projective modules with unimodular elements, see [Basu-Rao: 2005].

Suslin's Stability Theorem III.3.8 for polynomial rings (and Laurent polynomial rings) was perhaps as much a *tour de force* as his 1976 solution of Serre's Conjecture. The symplectic analogue of this result for $A = R[t_1, \ldots, t_m]$ (R a commutative noetherian ring) was obtained in [Kopeiko: 1978], who proved that the map

$$\mathrm{Sp}_{2n}(A)/\mathrm{Ep}_{2n}(A) \longrightarrow K_1\mathrm{Sp}(A)$$

is an isomorphism whenever $2n \geqslant \max\{4, \dim A + 2\}$. (Here, $K_1\mathrm{Sp}(A)$ denotes the *symplectic K_1-group* of A; that is, the factor group $\mathrm{Sp}(A)/\mathrm{Ep}(A)$.) In particular, if R is a field, then $\mathrm{Sp}_{2n}(A) = \mathrm{Ep}_{2n}(A)$ for $n \geqslant 2$. For subsequent work on the case where R is a regular ring, see [Kopeiko: 1992, 1995a]. Some results were also obtained for Laurent polynomial rings, in [Kopeiko: 1978, 1995b, 1999].

In [Grunewald-Mennicke-Vaserstein: 1991], the nonstable K_1-symplectic groups were considered over a polynomial ring $A = R[t_1, \ldots, t_m]$ where the (commutative) base ring R is not necessarily noetherian or regular. One assumes, instead, that R is *locally principal*; that is, the localizations of R at maximal ideals are principal ideal rings. (Of course, this implies that $\dim R = 1$.) Under this assumption, one gets

$$(9.0) \qquad \mathrm{Sp}_{2n}(A) = \mathrm{Sp}_{2n}(R) \cdot \mathrm{Ep}_{2n}(A) \text{ (for } n \geqslant 2), \quad \text{and} \quad K_1\mathrm{Sp}(A) = K_1\mathrm{Sp}(R).$$

As a byproduct of this work, it also follows that

$$(9.1) \qquad \mathrm{SL}_n(A) = \mathrm{SL}_n(R) \cdot \mathrm{E}_n(A) \text{ (for } n \geqslant 3), \quad \text{and} \quad SK_1(A) = SK_1(R).$$

(Of course, in the case where R is a field or a euclidean domain, these equations recapture Suslin's original stability results.) In the same paper of Grunewald et al, similar results were obtained for the Laurent polynomial ring $R\,[t, t^{-1}]$ over a locally principal ring R.

In case R is a regular ring of essentially finite type over a field, Lindel has proved (the Bass-Quillen Conjecture) that R has the property (E) defined in V.1.9; that is, f.g. projective modules over $A = R[t_1, \ldots, t_m]$ are extended from R. The K_1-analogue of this would ask if all invertible matrices over A are also "extended from R", or more precisely, if

$$\text{(9.2)} \qquad \mathrm{GL}_n(A) = \mathrm{GL}_n(R) \cdot \mathrm{E}_n(A) \ \ (\text{for } n \geqslant 3), \quad \text{and} \quad SK_1(A) = SK_1(R).$$

This was shown to be true by Vorst (1981), in generalization of Suslin's result in the case where R is a field. Vorst's proof is based on a reduction to the local case (by Quillen Patching for K_1), where Lindel's technique of étale neighborhoods (see §6) can be applied. This reduces the consideration to the case of power series rings over fields. Here, the proof is completed via the Weierstrass Preparation Theorem (as in the earlier work of Mohan Kumar and Lindel-Lütkebohmert), using an induction on $\dim R$, with the case of dimension 0 covered by Suslin's Stability Theorem for fields. In this proof, Vorst also used the following K_1-analogue of Roitman's basic localization result (V.1.11):

> *For a commutative ring R, if every invertible matrix over $R[t_1, \ldots, t_m]$ is "extended from R", then the same also holds for any localization $S^{-1}R$ of R.*

In view of the powerful Néron-Popescu desingularization theorem stated in (6.2), we can now infer that Vorst's result above also holds for any commutative regular ring R containing a field k. Just as in the case of projective modules, however, the value of these results for GL_n lies in their *nonstable* nature. As a matter of fact, in Quillen's general algebraic K-theory, one of the earliest results (due to Quillen) was that the higher K-groups have the basic property

$$\text{(9.3)} \qquad K_n(R) \cong K_n\big(R[t_1, \ldots, t_m]\big) \quad (\forall m)$$

for any left regular ring R, so the *stable* results are already well at hand for $n = 0, 1$.

Vorst's article [Vorst: 1982] offers an extensive survey of the work on the higher K-theory of polynomial extensions. We shall only mention a few key results here and refer the reader to [Vorst: 1979a,b, 1982] for more details. For any commutative ring R, let

$$\text{(9.4)} \qquad N_m K_n(R) = \ker\Big(K_n\big(R[t_1, \ldots, t_m]\big) \xrightarrow{\pi_m} K_n(R)\Big),$$

where π_m is induced by the R-algebra map $R[t_1, \ldots, t_m] \to R$ sending all t_i 's to 0. If R is a reduced ring, then $N_m K_n(R) = 0$ implies $N_m K_n(R_{\mathfrak{p}}) = 0$ for all prime ideals $\mathfrak{p}$ of R, and one has an injective natural map

(9.5) $$N_m K_n(R) \to \prod_{\mathfrak{p}} N_m K_n(R_{\mathfrak{p}}) \quad (\mathfrak{p} \in \operatorname{Spec} R)$$

giving a local-global principle for the elements of the group $N_m K_n(R)$.

A subtle problem in algebraic K-theory is to determine those (say commutative) rings R that are *K_n-regular*, in the sense that $N_m K_n(R) = 0$ for all m; or equivalently,

(9.6) $$K_n(R) \to K_n\big(R[t_1, \ldots, t_m]\big)$$

is an isomorphism for all m. By Quillen's theorem, *regular rings are K_n-regular,* so it is of interest to seek other classes of K_n-regular rings.

The right place to start is the case $n = 0$, which already turned out to be difficult. In the early 1970s, Murthy raised the question *whether all normal domains R are K_0-regular.* (See Remark 3.5 in [Murthy-Pedrini: 1972].) Bass reiterated this question in a somewhat more conservative form (Problem IV in [Bass: 1972]), asking *if all UFD's are K_0-regular.* (Murthy's question was also mentioned in II.5 in the earlier version of this book.) Since normality is preserved by polynomial extensions, Murthy's question boils down to asking if $N_1 K_0(R) = 0$ for normal domains R. Unfortunately, the answer turned out to be "No". Utilizing one of the "bad" noetherian rings constructed by Nagata (in his book *Local Rings*), Weibel (1980, Th. 2.7) produced the first example of a 2-dimensional local normal domain R such that $R[t]$ has a rank 2 indecomposable projective module that is not stably free. Thus, P cannot be stably extended from R, and we have $N_1 K_0(R) \neq 0$, showing that R is not K_0-regular. Weibel's normal domain R has other remarkable properties too. For instance, there are projective $R[t, t^{-1}, s]$-modules that are not even stably a direct sum of projective modules extended from $R[t, s]$, $R[t^{-1}, s]$, and $R[t, t^{-1}]$.

While Weibel's first example R appears to be "big" (because of the nature of its construction), descent methods can be used to "bring it down" to rings of a geometric nature; for instance, affine domains. To make a long story short, the following 3-dimensional normal domain

$$R = k[w, x, y, z]/(w^3 - x^2 - yz) \quad (k \text{ a field})$$

offered by Swan (in [Weibel: 1980, Thm. 3.6]) turned out to have also $N_1 K_0(R) \neq 0$, so it is not K_0-regular. This affine domain (with only a singularity at the origin) is in fact a UFD, so it provides a counterexample to Bass's Problem IV as well. Later, in studying normal quasiprojective varieties X over algebraically closed fields, Srinivas was even able to produce examples of X with $N_1 K_0(X)$ of *infinite* rank; see [Srinivas: 1986b].

While normal domains need not be K_0-regular, there are many examples of *non-normal* domains that *are* K_0-regular. For instance, if R is a 1-dimensional noetherian seminormal domain, then by the results of Brewer and Costa mentioned before in §7, R has the property (E); that is, all projectives over $R[t_1, \ldots, t_n]$ are extended from R. This, of course, implies that R is K_0-regular, though R need not be normal. A case in point is the elliptic curve C defined by $y^2 = x^3 + x^2$. Since the only singularity of C is an ordinary double point at the origin, it is well known that the coordinate ring of C is seminormal (albeit not normal). In fact, according to Brewer and Costa (1979), examples of nonnormal domains with the property (E) exist in any prescribed dimension.

Unfortunately, all of the above discussions leave us little clue as to how one might characterize the K_0-regular rings. More work is definitely needed here. In this connection, we should mention that Kotlarsky (1982) has investigated a notion called "T-regularity" ("T" for "Tor") which implies K_0-regularity, and is implied by regularity. In this way, he obtained a new proof of Grothendieck's theorem (II.5.8).

If R is a commutative ring of essentially finite type over a field, with dim $R \leqslant n$, it has been conjectured that R is regular iff it is K_{n+1}-regular. This is proved to be true for $n = 0$, 1 in [Vorst: 1979a], but is unknown for $n \geqslant 2$. (It is also not known if the assumption that R be of essentially finite type over a field is needed at all in this conjecture.)

As is customary in algebraic K-theory, we write $NK_n(R)$ for $N_1K_n(R)$. With the help of van der Kallen, Vorst (1979a) proved that, for any commutative ring R,

$$NK_{n+1}\big(R\,[t]\big) = 0 \Longrightarrow NK_n(R) = 0. \tag{9.7}$$

In particular, for such R, it follows readily that *K_{n+1}-regularity implies K_n-regularity*. The $n = 0$ case of this result was also proved in [Dayton-Weibel: 1980]: it answered in part Problem III raised in [Bass: 1972]. Another part of this problem asked if $NK_1(R) = 0$ implies $NK_0(R) = 0$.(*) This is shown to be true for 1-dimensional noetherian rings with finite normalization (see [Vorst: 1979b], extending [Pedrini: 1974, Thm. 8]), but has apparently remained unknown in general. As to whether $NK_0(R) = 0$ implies $NK_1(R) = 0$, Bass has given a nonreduced counterexample in [Bass: 1972, p. 13]. The existence of *reduced* counterexamples follows from the results in [Vorst: 1979b].

One more question worth mentioning on K_n-regularity is Problem VI_n in [Bass: 1972]. This one asked *if $NK_n(R) = 0$ for a commutative ring R would guarantee the K_n-regularity of R*. In the case $n = 0$, this question was first raised by Sharma and Strooker, and explicitly stated in a footnote on p. 116 of [Murthy-Pedrini: 1972]. We can think of this question as a *stable form* of our earlier Question V.1.16 (asking if, for commutative rings, the property (E_1) implies the property (E)). To the best of my knowledge, the answer to the Sharma-Strooker question is still unknown.

As we have seen in §4, it is sometimes possible to extend the work on polynomial algebras to discrete Hodge algebras, that is, quotients of polynomial algebras by ideals generated by monomials. Here, Vorst's result for projective modules quoted in (4.5) provides an important paradigm. The analogue of this result for the extension of SL_r-matrices (for $r \geqslant 3$) from a commutative base ring to its discrete Hodge algebras was obtained in the same paper [Vorst: 1983], where (4.5) first appeared. (The same holds for GL_r-matrices with $r \geqslant 3$, under the additional assumption that the discrete Hodge algebra is reduced.) Later, a generalization of Suslin's Stability Theorem (III.3.8) for K_1 was obtained from polynomial rings (over noetherian rings)

(*)This is *not* answered by (9.7), since (9.7) only says that $NK_1\big(R[t]\big) = 0$ implies $NK_0(R) = 0$.

to discrete Hodge algebras, with essentially the same dimension bounds as Suslin's in the polynomial ring case; see [Vorst: 1985].

Results on K_2-stability (for Milnor's functor K_2) were initiated in [Dennis: 1972]. For subsequent work, see [Stein: 1973, 1978], [Dennis-Stein: 1974], [Vaserstein: 1975], [van der Kallen: 1976], [Suslin-Tulenbaev: 1976], and [Kolster: 1982], etc. For a K_2-analogue of Suslin's Stability Theorem for polynomial rings (VI.4.5), see [Tulenbaev: 1982]. (The dimension bounds for K_2-stability are usually "1 larger" than the corresponding bounds used for K_1.) For stability for higher K-groups and relations to the homology stability of general linear groups, the reader can consult [Suslin: 1980] and [van der Kallen: 1980].

We close this section with some remarks on the more recent work on the higher K-theory of monoid algebras, due to J. Gubeladze and others.(*) This is related to the further study of Serre's Problem in the following way. The truth of Serre's Conjecture on $k[t_1, \ldots, t_n]$ (k a field) prompted Anderson's Conjecture on the freeness of projectives over normal monomial algebras, and Gubeladze's solution of Anderson's Conjecture showed, in particular, that for any regular ring R and any (commutative) cancellative, torsionfree and seminormal monoid M, the inclusion of R into the monoid algebra $R[M]$ induces an isomorphism in their K_0-groups. It turns out, curiously enough, that the situation changes radically when one passes to the next functor K_1. In his work on affine cones over projective curves, Srinivas (1987) showed that the group

$$SK_1\big(k[x, y, z]/(z^2 - xy)\big)$$

is *nontrivial* for any algebraically closed field k of characteristic $\neq 2$. The ring in question is isomorphic to $k[x^2, xy, y^2]$, which is the monoid ring of the seminormal (even normal) monoid generated by x^2, xy and y^2. From this example, we see that the Bass-Heller-Swan theorem on the K_1-regularity of regular rings *does not* extend directly to the setting of monomial algebras.

Going beyond this, Gubeladze [1995a] showed that, for any finitely generated seminormal monoid M sandwiched between $\mathbb{Z}_+^r$ and $\mathbb{Q}_+^r$, and for any (commutative) regular ring R, the inclusion $R \to R[M]$ induces an isomorphism in SK_1 iff M is free (i.e. isomorphic to $\mathbb{Z}_+^r$), iff $R[M]$ is K_1-regular. In the case where M is nonfree, explicit constructions were given for elements in $SK_1\big(R[M]\big)$ that are not extended from $SK_1(R)$. Thus, for "most" f.g. seminormal monoids M, one simply should not expect $K_1\big(R[M]\big)$ to be equal to $K_1(R)$.

Gubeladze pointed out that the above phenomenon is due to the lack of the general excision property of the Bass-Whitehead functor K_1. If, instead, the (cancellative,

(*) A large part of our discussions here is based on the more extensive survey of these matters in [Gubeladze: 1999].

torsionfree) monoid M is without nontrivial units and is p-divisible for some prime p (and hence seminormal), he has conjectured that

$$(\mathrm{NC})_i \qquad K_i(R) = K_i\big(R[M]\big) \textit{ for any regular ring } R \textit{ and any } i \geqslant 0.$$

Gubeladze called this the "Nilpotence Conjecture" since it can be easily reformulated into a statement on the (local) nilpotence of a certain natural "Frobenius action" of $\mathbb{N}$ on the quotient group $K_i(R[M])/K_i(R)$.

An excellent historical perspective on the Nilpotence Conjecture $(\mathrm{NC})_i$ is given in §1 in [Gubeladze: 2005]. Here is just a quick synopsis. $(\mathrm{NC})_0$ is essentially the stable version of Anderson's Conjecture (that is, the (stable) triviality of vector bundles on affine toric varieties), proved in [Gubeladze: 1988b]. $(\mathrm{NC})_1$ was proved in [Gubeladze: 1990], and $(\mathrm{NC})_2$ was proved in [Mushkudiani: 1995]. For $i \geqslant 2$, $(\mathrm{NC})_i$ was proved for p-divisible monoids intermediate between $\mathbb{Z}_+^r$ and $\mathbb{Q}_r^+$ in [Gubeladze: 1995b], where it was also shown that excision in algebraic K-theory holds for monoid rings for p-divisible monoids. Finally, in [Gubeladze: 2005], $(\mathrm{NC})_i$ is proved in the case R is a localization of a polynomial ring $k[t_1, \ldots, t_n]$, where k is any field of characteristic zero.

§10. Quadratic Analogues of Serre's Conjecture

In this area of investigation, the initial problem was to study the "quadratic form version" of Serre's Conjecture; that is, to ask if every quadratic space(*) over a polynomial ring $k[x_1, \ldots, x_n]$ (k being a field, say) is extended from k. The basic theorems of Karoubi and Harder presented in Chapter VII, §§2-3, were among the earliest results on this problem in the positive direction. With the affirmative solution of Serre's Conjecture secured by Quillen and Suslin in 1976, interest in its quadratic space analogue certainly heightened.

For k algebraically closed with $\operatorname{char} k \neq 2$, Bass proved (in [Bass: 1977]) that quadratic spaces over $k[x_1, \ldots, x_n]$ $(n \leqslant 3)$ are extended from k. Shortly thereafter, as we have mentioned in VII.1.12, Bass's result was extended in [Raghunathan: 1978] to all n, and to all fields k $(\operatorname{char} k \neq 2)$ whose quaternion algebras are matrix algebras. But at around the same time, Parimala's discovery of rank 4 *nonextended* quadratic spaces over $\mathbb{R}[x, y]$ (presented in detail in VII.4) substantially changed the direction of research on the quadratic analogue of Serre's Problem. From 1977 on, the main focus of this research essentially shifted to the construction, decomposition, and classification of nonextended quadratic spaces over $k[x_1, \ldots, x_n]$, with k often allowed to be a commutative noetherian ring.

In this section, in continuation of the Notes section on Chapter VII, we give a quick summary on some of the work done in this area after 1977. The topics to be surveyed are as follows.

(*) In this section, we shall use the term "quadratic spaces" or "quadratic forms" mostly as synonyms for the inner product spaces (IPS's) that were defined earlier in Chapter VII, §1. This is a fairly standard practice in the literature.

- Quadratic Spaces over Rings
- Extendibility Questions for Polynomial Rings
- Non-Extended and Indecomposable Quadratic Spaces
- Quadratic Bundles on Projective Spaces
- Base Rings of Dimension 1
- Local-Global Principles
- Horrocks' Theorem

We start with the first item, which provides a brief sketch of the background information on quadratic spaces over rings in general. The subsequent items will be more specific to polynomial rings.

(1) **Quadratic Spaces over Rings.** Just as in the linear case, the cancellation problem of quadratic spaces (with respect to orthogonal sums) is a main concern, starting with Witt's classical work over fields of characteristic $\neq 2$ in 1937. The earliest cancellation results for quadratic spaces over rings are due to A. Bak, H. Bass, A. Roy, and L. N. Vaserstein. For instance, Roy's version (for quadratic spaces, i.e. IPS in the sense of Chapter VII) states the following: *for a commutative noetherian ring* A *with* $2 \in \mathrm{U}(A)$, *any quadratic space* Q *over* A *with Witt index*$^{(*)}$ $\geqslant 1 + \dim(A)$ *is cancellative*; that is,

$$Q \perp Q'' \cong Q' \perp Q'' \Longrightarrow Q \cong Q'.$$

This result, from [Roy: 1968], is modelled upon Bass' Cancellation Theorem in the linear case. Quadratic analogues of Serre's Splitting Theorem are available later in [Bertuccioni: 1987].

In his study of stability properties of various functors on commutative rings modelled on Chevalley groups of root systems, M. Stein (1978) modified Bass's notion of stable range ("sr") into a new notion of *absolute stable rank* ("asr") for commutative rings that is more suitable for the study of the orthogonal groups. Ten years later, this notion was further extended to *noncommutative* rings in [Magurn-v.d. Kallen-Vaserstein: 1988]. In general, $\mathrm{sr}(A) \leqslant \mathrm{asr}(A)$ for any ring A; and, if R is commutative ring with $\mathrm{Max}(R)$ noetherian of finite dimension d, then any module-finite R-algebra A has $\mathrm{asr}(A) \leqslant d+1$. This refines Bass's well-known upper bound on $\mathrm{sr}(A)$. Using the absolute stable rank, Magurn, van der Kallen and Vaserstein obtained a Witt cancellation theorem for a very general class of quadratic spaces over rings with involutions encompassing all classical examples of quadratic forms. In [Stafford: 1990], an upper bound for the absolute stable rank is also given in terms of the Krull dimension of a right noetherian ring in the sense of Rentschler and Gabriel.

$^{(*)}$A quadratic space Q is said to have Witt index $\geqslant r$ if Q has an orthogonal summand $\mathbb{H}(P)$ where $\mathrm{rank}(P) \geqslant r$. Here, the hyperbolic quadratic space $\mathbb{H}(P)$ is as defined in VII.1.4.

(2) **Extendibility Questions for Polynomial Rings.** In studying the quadratic analogues of Serre's Problem, we are specifically interested in base rings of the form $A = R[x_1, \ldots, x_n]$, where R is (say) a commutative noetherian ring. Starting with the classical case, let us first assume R is a field k, of characteristic not 2. For $n \geqslant 2$, we have pointed out in VII.1.12 and VII.1.14 that

> *All IPS's over* $A = k[x_1, \ldots, x_n]$ *are extended from* k *iff all quaternion algebras over* k *split.*

If this is not the case, we may look for sufficient conditions for the extendibility of a *given* inner product A-space (P, B). An early result in this spirit is that of Ojanguren (1978), which states that (P, B) *is extended from* k *if* $(\overline{P}, \overline{B})$ *has Witt index* $\geqslant 1$, where $(\overline{P}, \overline{B})$ denotes the reduction of (P, B) modulo the ideal $(x_1, \ldots, x_n)$. Note that this result subsumes the "if" part of the displayed statement above, since the splitting of all quaternion algebras over k implies the Witt index condition on $(\overline{P}, \overline{B})$ if $\operatorname{rank}(P) \geqslant 3$ (and all rank 2 IPS's over A are extended by VII.1.10).

A more general extension theorem, valid for $A = R[x_1, \ldots, x_n]$ where R is a commutative d-dimensional noetherian ring (with $2 \in \mathrm{U}(R)$), is obtained in [Suslin-Kopeiko: 1977]. In this result, the Witt index lower bound is replaced by $1 + \dim(R)$, while the supporting module P is assumed to be stably extended from R. For an alternative treatment of this result, see [Bertuccioni: 1982]. For various quantitative improvements of the Suslin-Kopeiko theorem (for instance when $\dim R \leqslant 3$, or when R is a regular ring of essentially finite type over a field), see [Rao: 1984b, c].

The paper [Suslin-Kopeiko: 1977] also contained stability results on the orthogonal groups of hyperbolic spaces over a polynomial ring $R[x_1, \ldots, x_n]$. In the sequel [Kopeiko-Suslin: 1979], it was shown that, in case R is a field k, any isotropic quadratic space (P, B) over $k[x_1, \ldots, x_n]$ is cancellative. In particular, in view of Karoubi's theorem, (P, B) must be extended from k. For further generalizations of this result (when k is no longer a field), see (5b) below.

For a polynomial ring $A = R[t]$ where R is noetherian with $2 \in \mathrm{U}(R)$, Roy's cancellation theorem was improved in [Parimala: 1984] into the following: *any quadratic A-space with Witt index* $\geqslant \dim(A)$ *is cancellative.* The improvement here consists of replacing Roy's Witt index lower bound $1 + \dim(A)$ by $\dim(A)$. This is a significant improvement, as it corresponds, in the linear case, to Plumstead's solution of the second Eisenbud-Evans Conjecture (EE_2) discussed earlier in §1. Later, in [Mandal: 1986], Parimala's cancellation theorem was generalized to the case where A is a discrete Hodge algebra over R with $\dim A > \dim R$.

Another kind of extension question to consider is the behavior of quadratic spaces upon scalar extension from $k[x_1, \ldots, x_n]$ to $k(x_1, \ldots, x_n)$, where k is, say, a field of characteristic $\neq 2$. For some work in this direction, see [Kopeiko: 1991], which built on [Ojanguren: 1980].

(3) **Non-Extended and Indecomposable Quadratic Spaces.** As we have pointed out in VII.4, Parimala's construction of rank 4 nonextended IPS's over $A = k[x, y]$ was the result of working with certain nonfree projective right ideals in $H[x, y]$, where H is Hamilton's division ring of real quaternions. Note that $H[x, y]$ may

be viewed as a quaternion algebra over $\mathbb{R}[x, y]$, generated by i, j with the usual relations

$$i^2 = j^2 = -1, \quad \text{and} \quad ij = -ji.$$

More generally, if a field k (with char $k \neq 2, 3$) has a (generalized) quaternion algebra D, then $B := D[x, y]$ is a quaternion (and hence Azumaya) algebra over $A = k[x, y]$. According to Ojanguren and Sridharan, B has a nonfree projective right ideal P (see II.3), and the endomorphism ring $C = \mathrm{End}_B(P)$ is an Azumaya A-algebra not isomorphic to B. As a result of this, Knus and Ojanguren (1977) showed that B and C are also not isometric when they are viewed as quadratic spaces with quadratic structure given, respectively, by their Azumaya algebra reduced norms N_B and N_C. This implies that the rank 4 quadratic space (C, N_C) over $A = k[x, y]$ is not extended from k, nor is the rank 3 space given by the orthogonal complement of $A \cdot 1$ in (C, N_C).

The seminal work on nonextended spaces above was later developed into a classification of rank 3 quadratic spaces, and rank 4 quadratic spaces of trivial discriminant, *over any domain* A (with $2 \in \mathrm{U}(A)$) in terms of projective modules over Azumaya A-algebras of rank 4 in [Knus-Ojanguren-Sridharan: 1978]; see also [Knus: 1977] and [Knus-Parimala: 1980]. This work is further extended in [Parimala-Sridharan: 1983], where the existence of indecomposable rank 4 quadratic $k[x_1, \dots, x_n]$-spaces ($n \geqslant 2$) of discriminant $d \in \dot{k} \setminus \dot{k}^2$ was shown to be equivalent to the existence of involutorial quaternion algebras of the second kind over the quadratic extension $k(\sqrt{d})/k$.

The work above prompted the existence question of indecomposable quadratic spaces of rank > 4 over polynomial rings (already raised in the 1978 version of this book). For $\mathbb{R}[x, y]$, such spaces of rank 6 and rank $4n$ were constructed, respectively, in [Knus-Ojanguren-Parimala: 1982] and [Ojanguren-Parimala-Sridharan: 1983]. These results culminate in [Parimala: 1986], where it is shown that, for a field k of characteristic not 2 and for any integer $r \geqslant 3$, there exists an indecomposable quadratic space Q of rank r over $k[x, y]$ iff k admits an anisotropic quadratic form q of rank r. In fact, when q exists, Q may be chosen to be an indecomposable quadratic space over $k[x, y]$ which reduces to q modulo the ideal (x, y).

(4) **Quadratic Bundles on Projective Spaces.** Much of the work on quadratic spaces over $k[x, y]$ (a.k.a. quadratic bundles over the affine plane $\mathbf{A}_k^2$) is linked to the study of quadratic bundles over the projective space $\mathbb{P}_k^2$. The crucial fact here is the following result (Thm. 2.1 in [Knus-Parimala-Sridharan: 1981]), obtained partly in answer to a question of Knebusch:

> *Any quadratic bundle P on $\mathbf{A}_k^2$ extends to $\mathbb{P}_k^2$, and uniquely (up to isometry) if P is anisotropic.*(*)

An analogous fact is proved for Hermitian bundles. In the case $k = \mathbb{C}$, the Hermitian case applies particularly well to the study of projective (say, left) ideals in $H[x, y]$

(*)For a higher-dimensional analogue of this result, see [Quebbemann: 1989].

where H denotes the division ring of real quaternions. The early papers [Parimala-Sridharan: 1975] and [Parimala: 1976a] (referred to in VII.4) classified such left ideals P in terms of rank 2 positive definite Hermitian spaces of discriminant 1 over $\mathbb{C}[x, y]$. These restrict to rank 4 positive definite quadratic spaces over $\mathbb{R}[x, y]$, and in the mean time extend uniquely to rank 2 Hermitian bundles $\mathcal{E}_P$ over $\mathbb{P}^2_{\mathbb{C}}$. Knus, Parimala and Sridharan (1981) have further shown that $\mathcal{E}_P$ is a *stable bundle* over $\mathbb{P}^2_{\mathbb{C}}$ with Chern class $c_1 = 0$ and $c_2 =$ even. The classification of the P's can thus be related to the classification of certain rank 2 stable bundles on $\mathbb{P}^2_{\mathbb{C}}$ due to [Barth: 1977], [Hulek: 1980], and [Barth-Hulek: 1978]. A follow-up work in this area with the goal of classifying anisotropic quadratic spaces over $\mathbb{R}[x, y]$ using linear algebraic data is [Ojanguren-Parimala-Sridharan: 1984]. For surveys on some of the work above on bundles over $\mathbb{P}^2_{\mathbb{C}}$ with a good supply of explicit examples, see [Knus: 1982, 1984].

In connection with Hermitian bundles on $\mathbb{P}^2_{\mathbb{C}}$, we should also mention that the indecomposable quadratic spaces of rank $4n$ over $\mathbb{R}[x, y]$ referred to in the third paragraph of (3) were obtained as restrictions of suitable positive definite Hermitian bundles of rank $2n$ over $\mathbb{P}^2_{\mathbb{C}}$.

There is work on Hermitian spaces too, starting with Parimala's 1976 paper listed in the references on Chapters I-VII. For the construction of indecomposable Hermitian spaces over $\mathbb{R}[x, y]$, see [Parimala-Sinclair-Sridharan-Suresh: 1999].

In "Serre's Conjecture" (p. 204), I raised the question on the uniqueness of Krull-Schmidt decompositions of positive definite quadratic spaces over $\mathbb{R}[t_1, \dots, t_n]$. This was fully resolved for $n = 2$ by Knus, Ojanguren and Parimala (1982). For a projective scheme X over $\mathbb{R}$, they showed that a vector bundle over the complexification $X_{\mathbb{C}}$ has at most one positive definite Hermitian structure, and that such Hermitian bundles over $X_{\mathbb{C}}$ satisfy the classical Krull-Schmidt theorem for orthogonal decompositions. In view of the extension result mentioned in the opening paragraph of (4), this implies the Krull-Schmidt theorem (and hence also the Witt cancellation theorem) for both positive definite Hermitian spaces over $\mathbb{C}[x, y]$, and positive definite quadratic spaces over $\mathbb{R}[x, y]$.

(5) **Base Rings of Dimension** 1. Since the Bass-Quillen Conjecture is known to be true for a Dedekind base ring, there is certainly good incentive to study the extension problem for quadratic forms over $R[x_1, \dots, x_n]$ where R is a Dedekind domain, or more generally, a commutative ring of Krull dimension 1. However, we have already shown in VII.4 (after [Parimala: 1978]) that, for the discrete valuation ring $R = \mathbb{R}[t]_{(1+t^2)}$, there is a quadratic space (of rank 4 and discriminant 1) over $R[t]$ that is *not* extended from R. This means that the naive quadratic analogue of the Bass-Quillen Conjecture fails — even locally. However, it turned out that there are still some positive results, as follows.

(a) In the spirit of Harder's theorem, Parimala (1981a) showed that, if R is a complete discrete valuation ring with $2 \in \mathrm{U}(R)$, any quadratic space over $R[t]$ is extended from R. (Special cases of this result were first obtained in [Knus-Parimala: 1980].) It is the *completeness* on R that made the difference here, although, unfortunately, completeness no longer helps if we go to higher dimensions. It was shown in [Parimala: 1981b] that, even for the (complete, regular local) power series ring $R = \mathbb{R}[[x, y]]$, there is a quadratic space over $R[t]$ that is not extended from R. This, incidentally, implies the failure of the quadratic analogue of the results of Mohan Kumar and Lindel-Lütkebohmert presented in V.5.

(b) Back to the 1-dimensional case, it was shown in [Parimala: 1981a] that, for a Dedekind domain R with $2 \in \mathrm{U}(R)$, every *isotropic* quadratic space over $R[x_1, \ldots, x_n]$ is extended from R. Later, this result was extended to discrete Hodge algebras over R in [Mandal: 1986], and to the case of polynomial rings over regular rings of dimension 2 in [Parimala: 1982a].

Relaxing the Dedekind ring assumption in (b), let R be a 1-dimensional reduced noetherian ring with a finite normalization and $2 \in \mathrm{U}(R)$. In the addendum to the paper [Parimala-Sinclair: 1982], it is shown that any quadratic space over $R[x_1, \ldots, x_n]$ of rank $\geqslant 3$ which contains a hyperbolic space of rank 2 modulo $(x_1, \ldots, x_n)$ is extended from R. For more results in this direction, see [Sinclair: 1984].

In [Parimala: 1983a], results were obtained on decomposing a quadratic space (P, B) over a Laurent polynomial ring $R[t, t^{-1}]$ (R a Dedekind domain) into an orthogonal sum $(P_1, B_1) \perp t\,(P_2, B_2)$ where (P_i, B_i) are quadratic spaces over R. For a continuation of this work (for a ring R as in the last paragraph), see [Sinclair: 1985].

There is not much work on the case where the polynomial algebra $R[x_1, \ldots, x_n]$ is replaced by a monomial algebra. For the 1-dimensional monomial algebra $k[t^2, t^3]$ over a field k, some classification work on quadratic spaces is available from [Parimala-Sridharan: 1980]. In this paper, some indecomposable quadratic spaces of rank 3 and rank 4 are also constructed. This shows, incidentally, that the quadratic analogue of Anderson's Conjecture fails in general.

(6) **Local-Global Principles.** In VII.1.16, we have already mentioned that Quillen's Patching Theorem for extendibility was proved by Bass, Connell and Wright for quadratic spaces over $R[t]$ where R is any commutative ring. For the Laurent polynomial ring $R[t, t^{-1}]$, a corresponding patching theorem was proved, under suitable assumptions, in [Parimala-Sinclair: 1983]. More precisely, if R is a commutative domain with $2 \in \mathrm{U}(R)$, then an *anisotropic* quadratic space (P, B) over $R[t, t^{-1}]$ is extended from R iff $(P_{\mathfrak{p}}, B_{\mathfrak{p}})$ is extended from $R_{\mathfrak{p}}$ for every prime ideal $\mathfrak{p} \subset R$. This is a somewhat surprising result, since the corresponding local-global statement for the Laurent polynomial ring is *not* true in the linear case (according to Swan's example in V.4.14). The *anisotropicity* assumption on (P, B) is, however, essential for the result of Parimala and Sinclair.

In the context of number theory, there is another kind of local-global principle for quadratic spaces. For a global field k, let $\{k_v\}$ be the family of completions of k. In classical quadratic form theory, the Hasse-Minkowski theorem provides the strongest possible local-global principle that enables us to predict the behavior of a quadratic form q over k through the behavior of q over the completions k_v. In studying the extension problem for quadratic spaces over a polynomial ring $k[x_1, \ldots, x_n]$, it is, therefore, natural to also try a local-global approach with respect to completions.

For an IPS (P, B) of rank $\leqslant 4$ over $k[x_1, \ldots, x_n]$ (k a global field), Parimala and Sridharan (1982) showed that (P, B) is extended from k iff each

$$(P_v, B_v) := k_v \otimes_k (P, B)$$

(as an IPS over $k_v[x_1, \ldots, x_n]$) is extended from k_v. The same local-global principle, *without any rank assumptions on* (P, B), was later obtained in [Parimala: 1983b], with a proof using the classical Hasse-Minkowski theorem together with the method of nonabelian cohomology.

(7) **Horrocks' Theorem.** It was already pointed out, after the proof of VII.4.14, that the geometric form of the Affine Horrocks' Theorem does not hold in the quadratic setting; that is, a quadratic space over $\mathbf{A}^1_R$ extendible to $\mathbb{P}^1_R$ need not be extended from a quadratic space over R. However, a *modified* version of the Affine Horrocks' Theorem V.2.2 was shown to be true in [Parimala: 1981]. More precisely, let f be a monic polynomial in $R[t]$ where R is any commutative ring with $2 \in \mathrm{U}(R)$, and let (P, B) be a quadratic space over $R[t]$. *If* $(\overline{P}, \overline{B}) := R[t]/(t) \otimes_{R[t]} (P, B)$ *contains a hyperbolic plane, then* (P, B) *is extended from* R *iff* $R[t]_f \otimes_{R[t]} (P, B)$ *is extended from* R.

A much broader view than studying the quadratic analogue of Serre's Problem was offered in [Raghunathan: 1978], where the author investigated the extension problem for principal G-bundles over the affine space $\mathbf{A}^n_k$ for an affine algebraic group G. For $G = \mathrm{GL}_n(k)$ or $\mathrm{SL}_n(k)$, this corresponds to the classical "linear case" of Serre's Problem, while, for $G = \mathrm{O}_n(k)$ or $\mathrm{SO}_n(k)$, this corresponds to the "quadratic analogue" of Serre's Problem under consideration in this section. Various cases of affine algebraic groups G (and ground fields k) are found in which the extension problem has affirmative answers for all n; e.g. k is a global field and G is a split classical semisimple and simply connected group, to name just one example beyond the classical linear case. For some subsequent work in this area of study, see [Raghunathan-Ramanathan: 1984], and [Raghunathan: 1989].

§11. Quantum Versions of Serre's Conjecture

With the advent of the quantum group technology, it is not surprising that there are quantum versions of Serre's Conjecture as well.

We begin by defining certain "quantum polynomial algebras", which are in essence multiplicative analogues of the classical Weyl algebras. Let k be a field, and let $Q = (q_{ij})$ be an $n \times n$ matrix over k^* such that $q_{ii} = 1$ and $q_{ji} = q_{ij}^{-1}$ for all i, j. The k-algebra

$$A_Q = k\langle t_1, \ldots, t_n \rangle$$

generated by $t_1, \ldots, t_n$ with the relations $t_i t_j = q_{ij} t_j t_i$ (for all i, j) is called a *quantum polynomial algebra* in n variables (after V. A. Artamonov). This algebra made its first appearance in [McConnell-Petit: 1988], although attention in that paper was focused on the Laurent version

$$k\,\langle t_1, \ldots, t_n, t_1^{-1}, \ldots, t_n^{-1}\rangle.$$

For $n = 2$, this is a familiar algebra in the context of quantum mechanics; see also [Jategaonakar: 1984]. For general n, A_Q is a quantum analogue of the algebra of functions on the affine n-space.(†) It may be thought of as a crossed product algebra $k * G$ where G is the free abelian monoid generated by the t_j's. (In the Laurent polynomial case, we take G to be the free abelian group generated by the t_i 's.)

The quantum version of Serre's Problem is the study of the freeness property of f.g. projective modules P over the quantum polynomial ring A_Q. In a nutshell, Artamonov has proved that, if $\mathrm{rk}(P) \geqslant 2$, P must indeed be free under one of the following assumptions(*):

(1) *Each q_{ij} is a root of unity in k.*
(2) *The q_{ij} 's generate a free abelian group in k^* of maximal rank* (*namely*, $n(n-1)/2$).

Case (2) here was treated in [Artamonov: 1994], while Case (1) was deduced (in the Laurent polynomial case) from a general treatment of projective modules over crossed products in [Artamonov: 1995] (see also [Artamonov: 1989b]). The proof in Case (1) essentially follows the classical pattern: first compute that $K_0(A_Q) \cong \mathbb{Z}$, then prove a suitable cancellation theorem to arrive at the desired non-stable result. The main part of the proof has to do with the automorphisms of projective modules and Quillen patching for K_1.

The work above can still be generalized in several directions: k may be allowed to be a division ring, and, instead of insisting that the t_i 's commute with the elements $a \in k$, we may only require a twist law:

$$t_i a = \sigma_i(a) t_i \qquad \text{(for all } i, \text{ and all } a \in k),$$

where the σ_i 's are automorphisms of k over its center. (Of course, the parameters q_{ij} will have to satisfy some compatibility conditions with the σ_i 's to guarantee associativity.) Finally, we may introduce the t_i^{-1} 's for a given (possibly empty) subset of the i's. With these modifications taken into account, one arrives at a more general version of the quantum polynomial ring, which, for convenience, we still denote by A_Q. Under suitable independence conditions on the parameters $\{q_{ij}\}$, it is proved in [Artamonov: 1999] that all f.g. projective modules of rank $\geqslant 3$ over A_Q are free.

For surveys on the recent work on quantum polynomial rings, their projective modules, and their relations to crossed products, derivations, automorphisms and quantum groups, etc., see [Artamonov: 1997, 1998]. The former survey concludes with five open problems.

(†)Another quantum analogue of $k[t_1, \ldots, t_n]$ is given by the *quantized coordinate ring* of affine n-space, although this analogue will not concern us in this section.

(*)The rank 1 case is, in general, a true exception.

§12. Algorithmic Methods

The 80s and 90s of the last century witnessed truly remarkable advances in the development of algorithmic methods in mathematics as the result of the availability of increasingly powerful software packages in computations. In the area of commutative algebra and algebraic geometry, (where software packages such as *REDUCE*, *Maucaulay*, and *Singular* are popular), the theory and methods of Gröbner bases especially played an important role in this revolution, and brought fundamental changes to these classical subjects. It is thus entirely to be expected that the mathematics surrounding Serre's Problem would also be significantly impacted by the advent of these compelling modern tools.

In [Fitchas-Galligo: 1990], a new proof was given for a remarkable result of J. Kollár on an effective version of Hilbert's Nullstellensatz. This proof is valid in all characteristics, and is completely within the framework of commutative polynomial algebra. Using this effective Nullstellensatz, Fitchas and Galligo gave the first effective (quantitative) version of the Quillen-Suslin Theorem on Serre's Conjecture. For refinements and more discussions on this effective version from the viewpoint of degree bounds and computational complexity, see the subsequent papers [Caniglia et al: 1992] and [Fitchas: 1993].

At around the same time, Logar and Sturmfels [1992] took another step forward by viewing Serre's Conjecture as an algorithmic problem in solving polynomial linear systems, and giving a constructive proof of the Quillen-Suslin Theorem. These authors gave an algorithm for computing a free basis of a projective module over $A = \mathbb{C}[t_1, \ldots, t_n]$ if such a module is presented as the image, kernel, or cokernel of a module homomorphism $\varphi : A^m \to A^\ell$ given by an $\ell \times m$ polynomial matrix M. For instance, if φ is subjective (so $\ell \leqslant m$ and the maximal minors of M generate the unit ideal), an algorithm is given for finding a free basis of $\ker(\varphi)$, which amounts to completing M to an $m \times m$ invertible matrix over A. (Given the stable version of Serre's Conjecture, the special case $\ell = 1$ of this already amounts to an algorithmic proof of the Quillen-Suslin Theorem for $\mathbb{C}[t_1, \ldots, t_n]$.) Some steps of the algorithms constructed (including those used for the projectivity tests) can be carried out by Gröbner basis techniques. See also [Park-Woodburn: 1995]. For a nice introduction to some of these algorithmic methods and many explicit computational examples, see Chapter 5 in [Cox-Little-O'Shea: 2005], and [Miller-Sturmfels: 2005]. For other algorithmic work on completing unimodular rows over $k[t_1, \ldots, t_n]$ where k is a PID or Dedekind domain, see [Gago-Vargas: 2000, 2002].

A variant of the methods of Logar and Sturmfels appeared later in [Laubenbacher-Woodburn: 2000], which provides the construction of a locally free basis for a projective A-module at a maximal ideal of $A = \mathbb{C}[t_1, \ldots, t_n]$. In the case of affine monomial subalgebras $B \subseteq A$, algorithmic work is also available. For such B arising from seminormal monoids of monomials in $t_1, \ldots, t_n$, an algorithmic proof of Gubeladze's theorem on the freeness of projective modules over B is given in [Laubenbacher-Woodburn: 1997]. The construction of the algorithm follows closely Swan's algebraic formulation of the proof of Gubeladze's theorem in [Swan:

1992], but involves new ideas in devising algorithms for patching projective modules over Milnor and Karoubi squares.

The case of discrete Hodge algebras (quotients of polynomial algebras by monomial ideals) and bundles over affine toric varieties is treated in [Laubenbacher-Schlauch: 2000]. In this paper, an algorithm is found for determining the projectivity of a module over such Hodge algebras, and in the case such a (f.g.) module P is projective of constant rank, a method is proposed for finding a free basis for P, giving thus an algorithmic version of the theorem of [Vorst: 1983]. The projectivity test is based on the use of Fitting ideals, and, as is to be expected, the construction of the algorithm hearkens back to that for the polynomial algebra $A = \mathbb{C}[t_1, \ldots, t_n]$. Some of the results here are already implicit in [Laubenbacher-Woodburn: 1997], in view of the work in §14 of [Swan: 1992] on discrete Hodge algebras. Indeed, this latter alternative provides an algorithm for monomial quotients of arbitrary monoid algebras – rather than just polynomial algebras.

In an important development in 1995, Park and Woodburn published an algorithmic version of Suslin's Stability Theorem $\mathrm{SL}_r = \mathrm{E}_r$ $(r \geqslant 3)$ (see §9) for the polynomial algebra A; that is, given an $r \times r$ polynomial matrix M of determinant 1, an algorithm is found for factorizing M into a product of elementary matrices over A. The construction is based on induction on $r \geqslant 3$, and ultimately comes down to the factorization of 3×3 matrices of the form $\begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix}$, where $N \in \mathrm{SL}_2(A)$. (Here, Murthy's result (VI.3.6) plays an important role.) Suslin's Stability Theorem is not used in this work, so the algorithm represents, in particular, a new proof of Suslin's result. Crucial to finding this algorithm is a subalgorithm for reducing a unimodular row of polynomials $(f_1, \ldots, f_r)$ $(r \geqslant 3)$ to $(1, 0, \ldots, 0)$ by a sequence of elementary transformations. This, in turn, gives another algorithmic proof for the Quillen-Suslin Theorem. Again, Gröbner basis techniques may be used in the implementation of these algorithms.

The results of Park and Woodburn were obtained independently in their Ph.D. dissertations, respectively at the University of California, Berkeley, and the New Mexico State University ([Park: 1995], [Woodburn: 1994]), but the results were published jointly in [Park-Woodburn: 1995]. This work treats the case $A = k[t_1, \ldots, t_n]$ where k is a field. The case where k is a euclidean domain is treated subsequently in [Gago-Vargas: 2001].

As was already seen in (I.8), the case 2×2 matrices is a true exception to Suslin's Stability Theorem. This situation is elucidated algorithmically in the later work of Park. In [Park: 1999a], an algorithm is developed that, for a multivariate polynomial matrix $M \in \mathrm{SL}_2\big(R[t_1, \ldots, t_n]\big)$ over a euclidean domain R, determines precisely if M allows a factorization into 2×2 elementary matrices, and in case it does, finds explicitly such a factorization. (Some corrections may be necessary for Th. 2.1 in Park's paper.)

In the literature on linear groups, a commutative ring R is said to be a GE_2-ring if $\mathrm{SL}_2(R) = \mathrm{E}_2(R)$. The seminal paper on the subject of GE-rings in general is, of course, [Cohn: 1966]. For later work, we'll just mention a few references, not all of which are of an algorithmic nature. Discrete GE_2-subrings of the complex field

$\mathbb{C}$ were determined in [Dennis: 1975]; for some algorithmic methods on the GE_2-question for imaginary quadratic rings of the form $\mathbb{Z}[\sqrt{d}\,]$, see [Tuler: 1983]. In the interesting paper [Bachmuth-Mochizuki: 1982], it was shown that Laurent polynomial rings *in one variable* over any discrete valuation ring is a GE_2-ring. On the other hand, the equation $SL_2(R) = E_2(R)$ was shown to fail, rather "spectacularly", for Laurent polynomial rings R in two or more variables over any coefficient domain that is not a field. The proofs of these results depended on a subgroup theorem for amalgamated products due to Karrass and Solitar. Published at around the same time was the paper [Chu: 1982], in which Huah Chu obtained non-GE_2 results for certain commutative graded rings as well as for Laurent polynomial rings over commutative domains with a regular sequence of length 2. He also showed that a commutative polynomial ring $A[x]$ is GE_2 iff dim $R = 0$; for a related work, see [Costa: 1988]. Some algorithmic results on deciding whether a matrix of determinant 1 over a commutative polynomial ring $A[x]$ is elementary appeared in [Chu: 1984],

From the viewpoint of constructive mathematics, we should also mention the paper [Lombardi-Quitté: 2003], in which constructive versions of local-global principles in commutative algebra are used to deal with invertible polynomial matrices, and to give constructive proofs of the Quillen-Suslin Theorem and the Suslin Stability Theorem. A sequel to this paper, [Lombardi-Quitté-Yengui: 2005], deals with the constructive aspects of the Maroscia-Brewer-Costa generalization of the Quillen-Suslin Theorem to the case of Prüfer base domains of Krull dimension 1 discussed in §7. However, no constructive methods have yet been given for the more general Lequain-Simis result on the extendibility of f.g. projective modules over polynomial rings with a Prüfer base domain of arbitrary Krull dimension. Other papers of Coquand, Lombardi and Quitté on the constructive aspects of Serre's Splitting Theorem, Bass's Cancellation Theorem, Forster-Swan Theorem and the Heitmann dimension in the non-noetherian case have already been mentioned earlier in §7.

In the case of quantum polynomial algebras, algorithmic solutions to Serre's Conjecture were developed in detail in [Artamonov: 1996].

§13. Applications of Serre's Conjecture

Besides the natural connections to vector bundles, projective "polynomial modules" occur also in many branches of algebra, algebraic geometry, and combinatorics. The Quillen-Suslin solution to Serre's Conjecture affirming the freeness of such polynomial modules is sometimes a useful step in making progress on problems connected with polynomial modules in these areas, and has led to new questions for further investigations. It would be impossible to exhaust the literature on problems of this nature. In the following, we shall only try to give a sampling of the research work in this spirit done after 1977.

- In studying the Jacobian Problem, Stuart Wang considered a pair of polynomial algebras

$$R = k[f_1, \ldots, f_n] \subseteq S = k[x_1, \ldots, x_n] \tag{13.1}$$

such that S is separable over R, where k is field of characteristic 0. In Theorem 46 of [Wang: 1980], he collected many equivalent conditions for R to be equal to S in (13.1), including, e.g. S being integral over R, S being f.g. projective (and hence free) over R, and in the case where k is algebraically closed, the injectivity (resp. surjectivity) of the polynomial map

$$(f_1, \ldots, f_n) : k^n \longrightarrow k^n .$$

Wang showed that, in any case, S has projective dimension $\leqslant 1$ over R, so it is not far from being projective. This result led Wang to a characteristic-free version of the Jacobian Conjecture, where, under the Jacobian condition, the desired conclusion $R = S$ is replaced by S being f.g. free over R. This was, however, negated later by a counterexample of P. Nousianen. For a discussion of this, as well as an extensive survey on the classical Jacobian Problem, see the article of Bass-Connell-Wright [1982].

- In areas such as polytope geometry, approximation theory, and geometric design, spaces of piecewise polynomial functions (or *splines*) over polyhedral complexes arise naturally, providing interesting instances for the use of polynomial modules — over the ring of global polynomial functions. A very accessible introduction to multivariate polynomial spline functions can be found in Chapter 8 of [Cox-Little-O'Shea: 2005]; see also the article [Cox: 2005] for a nice perspective on the general use of polynomial modules in Applied Mathematics. The Quillen-Suslin Theorem (V.2.9) is useful for such investigations, since a polynomial module is guaranteed to have a free basis as soon as one knows that the module is projective (which is usually much easier to check, if it is true). For an initiation to the study of the freeness of spline modules and its relations to the face ring (or Stanley-Reisner ring) of the polyhedral complex in question, see [Billera: 1989] and [Billera-Rose: 1992]. The problem of *computing* free bases for spline modules was also hinted at in [Logar-Sturmfels: 1992]. For more recent work on freeness issues, see [Schenck-Stillman: 1997], [Deo-Mazumdar: 1998], and [Rose: 1996, 2004]. Gröbner basis techniques are often used extensively in the investigations in this direction; see, for instance, [Billera-Rose: 1989, 1991] and [Rose: 1995].

- If a prime ideal $\mathfrak{p} \subseteq k[t_1, \ldots, t_n]$ of dimension d is in a "Noether normal position", then the affine algebra $A = k[t_1, \ldots, t_n]/\mathfrak{p}$ is a f.g. module over $k[t_1, \ldots, t_d]$, and A is a Cohen-Macaulay ring iff it is a projective (and hence free) $k[t_1, \ldots, t_d]$-module. This, again, led to the interesting algorithmic problem of computing a free basis for A; see, e.g. [Logar: 1990].

- Serre's Conjecture has also found applications in group theory. Using the Quillen-Suslin Theorem and an extension of it by Artamonov [1978], Gupta, Gupta and Noskov [1994] obtained results on the metabelian inner rank of certain one-relator groups, and gave necessary and sufficient conditions for a subset of a free metabelian group to be part of a basis.

- Shortly after the solution of Serre's Conjecture, R.W. Richardson found interesting applications of the Quillen-Suslin Theorem to the study of algebraic transformation groups and invariant theory. Let $A(G)$ be the k-algebra of regular functions on a semisimple affine algebraic group over an algebraically closed field k of characteristic 0, and let $C(G)$ be the subalgebra of G-invariants of $A(G)$. Richardson proved that, if $C(G)$ is a polynomial algebra, then for each irreducible rational representation λ of G, the isotypic component $A(G)_\lambda$ is a f.g. flat (and hence projective) $C(G)$-module. It follows that each $A(G)_\lambda$ and $A(G)$ itself are free over $C(G)$. A second application of a variant of the Quillen-Suslin Theorem shows the existence of a G-stable vector space E_λ such that the product map

$$C(G) \otimes_k E_\lambda \longrightarrow A(G)_\lambda$$

is an isomorphism. Finally, the multiplicity of λ in the G-module E_λ is determined to be the dimension of the zero weight space of the representation λ, thus giving rather complete information on $A(G)_\lambda$. All of this constitute the main theorem in [Richardson: 1976, 1979]. The "variant" of the Quillen-Suslin Theorem alluded to above is the following, which is of independent interest.

(13.2) Theorem. *Let $R = k[t_1, \ldots, t_n]$ where k is a PID, and let A be a finite direct product of matrix algebras over k. If $\rho : A \to \mathbb{M}_r(R)$ is a k-algebra homomorphism, then there exists a matrix $g \in \mathrm{GL}_r(R)$ such that $g^{-1}\rho(a)g \in \mathbb{M}_r(K)$ for all $a \in A$.*

Richardson observed that, if k is a field of characteristic $\neq 2$, (13.2) is in fact equivalent to the Quillen-Suslin Theorem for the ground field k.

For further results of the same spirit for smooth G-varieties and their coordinate rings, see [Richardson: 1981].

- **Equivariant Serre Problem**: Let G be a reductive algebraic group over $\mathbb{C}$. In the equivariant K-theory, [Bass-Haboush: 1987] showed that G-vector bundles over a G-module B are stably trivial. In the case where G is *abelian*, it is proved in [Masuda-Moser-Jauslin-Petrie: 1996] that every G-vector bundle over B is in fact trivial; that is, isomorphic to $B \times F$ for some representation F of G. This result is in part based on Gubeladze's Theorem (4.1), which is applied to the ring of G-invariants (or the coordinate ring of the algebraic quotient $B//G$). Among various other works written on equivariant algebraic vector bundles over representation spaces for reductive groups (and their relations to linearization problems on algebraic group actions), we shall only cite a couple more papers: [Masuda-Petrie: 1995], and [Castella: 2000]. For some markedly different computations of equivariant vector bundles in cases other than representation spaces, see [Bell: 2001], where certain Russell-Koras threefolds under hyperbolic $\mathbb{C}^*$-actions are shown to have non stably-trivial equivariant vector bundles.

- Recently, M.P. Murthy has discovered a remarkable relationship between the Hermite property of a commutative polynomial ring $A = R[x_1, \ldots, x_m]$ and sets of ideal generators for the ideals $Ax_1 + \cdots + Ax_n$ in A $(n \leqslant m)$. A special case of Murthy's result states that, if the polynomial ring A above is Hermite, then for any set $\{f_1, \ldots, f_k\} \subseteq A$ generating the ideal $Ax_1 + \cdots + Ax_n$, there exists an invertible matrix $\alpha \in \mathrm{GL}_k(A)$ such that

$$(f_1, \dots , f_k)\,\alpha = (x_1, \dots , x_n, 0, \dots , 0);$$

in other words, $\mathrm{GL}_k(A)$ acts *transitively* on the set

$$\{\, (f_1, \dots , f_k) \in A^k : \; Af_1 + \cdots + Af_k = Ax_1 + \cdots + Ax_n \,\},$$

for any $n \leqslant m$ (and for any k). By the Quillen-Suslin Theorem, the polynomial ring $A = R[x_1, \dots , x_m]$ is Hermite when R is a field or a PID. Thus, Murthy's theorem implies that the above transitivity property holds for the ideal generators of $Ax_1 + \cdots + Ax_n$ $(n \leqslant m)$ in this case. More generally, it also holds when R is a regular local ring containing a field, or a commutative noetherian ring of Krull dimension 1 (by applying Popescu's Theorem (6.3), and the main result of [Bhatwadekar-Roy: 1984]). The proof of Murthy's result, based on a clever induction on m, is an extension of an argument in [Sathaye: 1978] on the Forster-Eisenbud-Evans Conjectures. Murthy also applied his new theorem to prove a strong cancellative property for the module $A^m/A \cdot (x_1, \dots , x_m)$; for more details, see [Murthy: 2003].

Next, we come to the applications of the Quillen-Suslin Theorem in other science fields. In the engineering and communication sciences, polynomial matrices are widely used in different areas including, for instance, circuit theory, control theory and mutlidimensional systems theory, digital filters, wavelet analysis, signal processing, and seismic data processing, etc. Since the Quillen-Suslin Theorem, in its matrix formulation, represents a fundamental breakthrough in our understanding of polynomial matrices, it is not surprising that the theorem would get added into the mathematical toolkit of research engineers in the above areas. We shall close by mentioning only a few relevant works in this connection.

For an introduction to the use of polynomials and polynomial matrices in circuits and multidimensional systems theory, we recommend the early textbooks [Kailath: 1980] and [Bose: 1982]. For a survey on the progress and directions in the field up to the mid-80s, see [Bose et al: 1985]. The authors of [Youla-Pickel: 1984] opined that "Some of the most impressive accomplishments in circuits and systems have been obtained by an in-depth, exploitation of the properties of elementary polynomial matrices.". The paper of Youla and Pickel contains a "tutorial account" (for the electrical engineering community) on the solution of Serre's Conjecture, using a minimum of abstract algebra, and specifically emphasizing the aspect of completing unimodular rectangular polynomial matrices to invertible (square) matrices from an engineering viewpoint. A predecessor of this paper is [Youla-Gnavi: 1979], which offers an independent proof of Seshadri's Theorem on Serre's Conjecture in two variables in the context of systems theory. The later paper [Sule: 1994] addresses the use of projective modules over commutative rings (in particular multivariate polynomial rings) in the treatment of feedback stabilization in multidimensional systems and signal processing problems. Factorization of polynomial matrices is also a major concern in the research on these topics; see, e.g. [Guiver-Bose: 1982], [Basu-Tan: 1989], [Fornasini-Valcher:

1997], [Lin: 1999a, b], [Charoenlarpnopparut-Bose: 1999], and [Bose: 2000], among others.

In a recent paper [Lin-Bose: 2001] (see also [Lin: 1999a] and [Lin-Bose: 2000]), several variants of Serre's Conjecture were proposed. For convenience, we'll call these variants the *Lin-Bose Conjectures.* These conjectures are mainly concerned with problems of completing and factoring multivariate polynomial matrices, motivated by considerations in multidimensional systems theory. To state the main Lin-Bose Conjecture, we proceed as follows.

Let $R = k[t_1, \dots, t_n]$ where k is a field, and let A be an $m \times \ell$ matrix over R, where $\ell < m$. We say that A satisfies the *Lin-Bose Condition* if the $\ell \times \ell$ ("maximal") minors of A, say $a_1, \dots, a_r$, happen to generate a *principal* ideal in R; that is, $a_1 R + \cdots + a_r R = dR$ for some $d \in R$. The following question on such a matrix A arises rather naturally:

Question (13.3). *If an $m \times \ell$ matrix A over $R = k[t_1, \dots, t_n]$ satisfies the Lin-Bose Condition, and $a_1 R + \cdots + a_r R = dR$ as above, can A be completed into an $m \times m$ matrix (over R) with determinant d?*

Note that this question has a negative answer if R is just an affine domain. For instance, if R is the coordinate ring of the real 2-sphere (that is, $R = \mathbb{R}[x, y, z]$ with $x^2 + y^2 + z^2 = 1$), then the column matrix $A = (x, y, z)^T$ satisfies the Lin-Bose Condition since $xR + yR + zR = R$. However, A cannot be completed over R to a 3×3 matrix with determinant 1, as we have shown in (I.4).

In the case where $R = k[t_1, \dots, t_n]$ (where k is a field), an affirmative answer to the question above was conjectured in [Lin-Bose: 2001]. Note that in the case $d = 1$, this amounts essentially to Serre's Conjecture (in the form of the freeness of stably free modules over R). The conjecture was also proved to be true for univariate and bivariate polynomial matrices $(n = 1, 2)$ in [Guiver-Bose: 1982]. For general n, the truth of that conjecture is easy to verify in case $\ell = 1$ or $\ell = m - 1$; see, e.g. [Gattazzo: 1991], [Lin: 1999a].

In [Lin-Bose: 2001], the authors stated a number of other conjectures on the polynomial matrix A, many of which they showed to be equivalent to (or at least implied by) the main Lin-Bose Conjecture mentioned above. For instance, keeping the notations and assumptions in the question above, an equivalent form of the Conjecture is that A can be factored in the form CD over R, where C is $m \times \ell$, D is $\ell \times \ell$, with $\det D = d$. Yet another equivalent form of the Conjecture is that there exists an equation $D = BA$, where B is $\ell \times m$, D is $\ell \times \ell$, with $\det D = d$.

Shortly after the appearance of the Lin-Bose Conjectures, they were proved (apparently in characteristic zero) by J.-F. Pommaret (2001), by a combination of methods from homological algebra and algebraic analysis. Other proofs, not using algebraic analysis, were given later in [Park: 2003] and [Wang-Feng: 2004]. Recently, yet another proof appeared in [Srinivas: 2004], in all characteristics. In this work, Srinivas pointed out that his proof, "depending on one's point of view, may be viewed as a consequence of a simple lemma in module theory (due to V. Trivedi), or by 'pure thought', as a consequence of the universal properties of blowing up, of the Grassmannian, and the Plücker embedding." The "simple lemma in module theory" Srinivas referred to is completely tractable. Appearing as Lemma 1 in Trivedi's work on the Bertini theorems [Trivedi: 1994], this result states the following:

If R is a commutative local domain and M is a f.g. torsionfree R-module of rank r, then $M \cong R^r$ iff there is a surjection from $\Lambda^r M$ onto R.

As for the "pure thought" part, Srinivas's use of the Plücker embedding of the Grassmannian led to the following somewhat more sophisticated module-theoretic statement:

Let E, F be f.g. projective modules over a commutative domain R such that $r := \text{rank}\, F < \text{rank}\, E$. If $f \in \text{Hom}_R(E, F)$ is such that the image of $\Lambda^r(f) : \Lambda^r E \to \Lambda^r F$ is an invertible R-module, then $f(E)$ is a projective R-module.

In a concluding remark on his paper, Srinivas pointed that his "blow-up" proof of the above statement gives a geometric meaning to the Lin-Bose Condition (the unimodularity of the "reduced minors" $a_1/d, \ldots, a_r/d$ of the matrix A) in Question (13.3).

At this point, we should perhaps mention also in passing a few works on linear systems from a somewhat different point of view. For linear systems over commutative rings, especially on questions of reachability, observability, pole assignability, stability and shifting, a good general introduction is available in [Brewer-Bunce-Van Vleck: 1986]. (The Quillen-Suslin Theorem is exploited in Theorem 3.11 there.) Among the many research articles written in this area, we shall only cite a few representative ones, such as [Sontag: 1976, 1981], [Bumby-Sontag-Sussmann-Vasconcelos: 1981], [Bumby: 1981], [Tannenbaum: 1982a, b], [Sharma: 1984, 1986, 1994], [Brewer-Naudé-Naudé: 1984], [Brewer-Heinzer-Lantz: 1985], [Brewer-Katz-Ullery: 1987a,b], [Brewer-Klingler-Minnaar: 1990], [Brewer-Klingler: 1991], and [Fagnani-Zampieri: 1997]. From a more analytic viewpoint, the paper [Oberst: 1990] establishes a categorical duality between multidimensional linear shift-invariant systems and f.g. modules over a multivariate polynomial ring.

Next we turn our attention to the use of Serre's Conjecture mathematics in the field of signal processing. Here, Suslin's Stability Theorem to the effect that, for k a field and $r \geqslant 3$, any matrix in $\text{SL}_r\big(k[t_1, \ldots, t_n]\big)$ is a product of elementary matrices has a concrete meaning. Using their realization algorithm for this problem, Park and Woodburn [1995] have effectively obtained a way of expressing certain multidimensional multi-channel filter banks as a cascade of simpler filter banks called "elementary ladder steps."

For the study of n-D multirate signal processing, Laurent polynomial rings come naturally into play, as the polyphase representation of an FIR (finite impulse response) filter bank gives rise to a rectangular matrix over a multivariate Laurent polynomial ring

$$R = k[t_1^{\pm 1}, \ldots, t_n^{\pm 1}] = k[t_1, \ldots, t_n, t_1^{-1}, \ldots, t_n^{-1}],$$

where, say $k = \mathbb{R}$. The solution of Serre's Conjecture, extended to the Laurent polynomial case, says that such a matrix has a left inverse iff it is unimodular (that is, its maximal minors are unimodular). Thus, the study of perfect reconstruction FIR filter banks corresponds to the study of unimodular rectangular matrices over R, and the design and synthesis of such invertible multirate systems corresponds to the mathematical problem of parametrizing all left inverses of the polyphase matrix. This algebraic problem can be solved in the univariate case by the Euclidean Algorithm, and in the general case by Gröbner basis techniques. For some work in this area in the last ten years, see [Kalker-Park-Vetterli: 1995], [Park-Kalker-Vetterli: 1997], and [Park: 1999b, 2004].

The decomposition problem of certain filter banks is considered in [Tolhuizen-Hollmann-Kalker: 1995], where it is noted that decomposing a causal biorthogonal multidimensional 2-band filter bank into elementary ladder steps is equivalent to finding an algorithm to determine if a matrix in $\mathrm{SL}_2\big(k[t_1, \ldots, t_n]\big)$ is a product of elementary matrices. (Recall that Suslin's Stability Theorem does not apply to 2×2 matrices over $k[t_1, \ldots, t_n]$.) As noted in §12, [Park: 1999a] accomplished the task of finding such a realization algorithm. In the Tolhuizen-Hollmann-Kalker paper (*loc. cit.*), the importance and difficulty of extending a realization algorithm to the Laurent polynomial case (for use for FIR filter banks) was mentioned. The matrix

$$A = \begin{pmatrix} 1+(x+y)(x-y) & (x+y)^2 \\ -(x-y)^2 & 1-(x+y)(x-y) \end{pmatrix} \in \mathrm{SL}_2\big(k[x, y]\big)$$

has a "leading form" matrix

$$\begin{pmatrix} x^2-y^2 & x^2+2xy+y^2 \\ -x^2+2xy-y^2 & -(x^2-y^2) \end{pmatrix},$$

so Thm. (I.8.2) implies that A is not "realizable" over $k[x, y]$. Park showed, however, that A is "realizable" over the Laurent polynomial ring $k[x^{\pm 1}, y^{\pm 1}]$, contrary to a conjecture of Tolhuizen, Hollmann, and Kalker. To the best of my knowledge, no realization algorithm is known yet for matrices in $\mathrm{SL}_2\big(k[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]\big)$.

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