Lecture 1: Overview

September 28, 2018

Let X be an algebraic curve over a finite field $\mathbf{F}_q$, and let K_X denote the field of rational functions on X. Fields of the form K_X are called *function fields*. In number theory, there is a close analogy between function fields and *number fields*: that is, fields which arise as finite extensions of $\mathbf{Q}$. Many arithmetic questions about number fields have analogues in the setting of function fields. These are typically much easier to answer, because they can be connected to algebraic geometry.

Example 1 (The Riemann Hypothesis)**.** Recall that the *Riemann zeta function* $\zeta(s)$ is a meromorphic function on $\mathbf{C}$ which is given, for $\mathrm{Re}(s) > 1$, by the formula

$$\zeta(s) = \prod_p \frac{1}{1 - p^{-s}} = \sum_{n>0} \frac{1}{n^s},$$

where the product is taken over all prime numbers p. The celebrated *Riemann hypothesis* asserts that $\zeta(s)$ vanishes only when $s \in \{-2, -4, -6, \ldots\}$ is negative even integer or when $\mathrm{Re}(s) = \frac{1}{2}$.

To every algebraic curve X over a finite field $\mathbf{F}_q$, one can associate an analogue ζ_X of the Riemann zeta function, which is a meromorphic function on $\mathbf{C}$ which is given for $\mathrm{Re}(s) > 1$ by

$$\zeta_X(s) = \prod_{x \in X} \frac{1}{1 - |\kappa(x)|^{-s}} = \sum_{D \subseteq X} \frac{1}{|\mathcal{O}_D|^s}$$

here $|\kappa(x)|$ denotes the cardinality of the residue field $\kappa(x)$ at the point x, D ranges over the collection of all effective divisors in X and $|\mathcal{O}_D|$ denotes the cardinality of the ring of regular functions on D. The *Riemann hypothesis for* X asserts that all zeroes of the function $\zeta_X(s)$ satisfy $\mathrm{Re}(s) = \frac{1}{2}$. This is equivalent to the more concrete assertion that the size of the set $X(\mathbf{F}_{q^n})$ of $\mathbf{F}_{q^n}$-valued points of X satisfies an estimate of the form

$$|X(\mathbf{F}_{q^n})| = q^n + O(q^{n/2}).$$

This was originally conjectured by Artin, proved by Hasse in the case of elliptic curves, and proved by Weil for algebraic curves in general (and later generalized to arbitrary algebraic varieties by Deligne). Weil's proof proceeds by studying intersection theory on the algebraic surface $X \times_{\mathrm{Spec}(\mathbf{F}_q)} X$ (in modern language, it is a consequence of the Hodge index theorem for $X \times_{\mathrm{Spec}(\mathbf{F}_q)} X$, together with some elementary linear algebra). By contrast, the classical Riemann hypothesis (for the usual Riemann zeta function $\zeta(s)$) remains an open question.

The algebraic curve associated to a function field has an analogue in the number field setting. To every number field K, one can associate the affine scheme $\mathrm{Spec}(\mathcal{O}_K)$, where $\mathcal{O}_K$ is the ring of algebraic integers in

K. There is a close analogy between schemes of this form and algebraic curves over finite fields:

Function Field Arithmetic	Number Field Arithmetic
Function Field K_X	Number Field $\mathbf{Q}$
Algebraic curve X	Affine Scheme $\mathrm{Spec}(\mathbf{Z})$
Closed points $x \in X$	Prime numbers p
Residue field $\kappa(x)$	Residue field $\mathbf{Z}/p\mathbf{Z}$

But there is a fundamental weakness in this analogy. The affine scheme $\mathrm{Spec}(\mathbf{Z})$ behaves very much like an algebraic curve over a finite field if we consider its points *as a topological space*. However, it behaves very differently when we think about its points *as a functor*. If X is a scheme and R is a commutative ring, we let $X(R) = \mathrm{Hom}(\mathrm{Spec}(R), X)$ denote the set of R-valued points of X. In the case where $X = \mathrm{Spec}(\mathbf{Z})$, this is just the set of ring homomorphisms from $\mathbf{Z}$ into R: that is, it always consists of exactly one element. It follows that the product $\mathrm{Spec}(\mathbf{Z}) \times \mathrm{Spec}(\mathbf{Z})$, formed in the category of schemes, is just $\mathrm{Spec}(\mathbf{Z})$. In particular, it is a very poor replacement for the algebraic surface $X \times_{\mathrm{Spec}(\mathbf{F}_q)} X$ of Example 1. This motivates the following:

Question 2. Can one replace the affine scheme $\mathrm{Spec}(\mathbf{Z})$ by some other mathematical object S, for which the product $S \times S$ is more interesting? (And, in particular, not isomorphic to S?)

Several years ago, Scholze proposed an answer to Question 2, at least after completing at some prime number p. To explain the idea, we need a brief digression.

Definition 3. Let K be a field. A *non-archimedean absolute value* on K is a map

$$| \, |_K : K \to \mathbf{R}_{\geq 0} \qquad x \mapsto |x|$$

which satisfies the following conditions:

$$|0|_K = 0 \qquad |1|_K = 1$$

$$|xy|_K = |x|_K |y|_K \qquad |x+y|_K \leq \sup\{|x|_K, |y|_K\}.$$

If K is equipped with a non-archimedean absolute value, then the subset

$$\mathcal{O}_K = \{x \in K : |x|_K \leq 1\}$$

is a subring of K, which we call the *valuation ring of* K. This is a local ring, whose unique maximal ideal is given by $\mathfrak{m}_K = \{x \in K : |x|_K < 1\}$. We let k denote the quotient field $\mathcal{O}_K / \mathfrak{m}_K$ and refer to it as the *residue field of* k.

We say that an absolute value $| \, |_K$ on K is *nontrivial* if $\mathcal{O}_K \neq K$: equivalently, if there exists an element $\pi \in K$ satisfying $0 < |\pi|_K < 1$. Any such element is called a *pseudo-uniformizer*. We say that K is *complete* with respect to a nontrivial absolute value $| \, |_K$ if it is complete with respect to the metric given by $d(x,y) = |x-y|_K$. Equivalently, K is complete if the valuation ring $\mathcal{O}_K$ is π-adically complete, with respect to any choice of pseudo-uniformizer $\pi \in K$.

A *completely valued field* is a field K equipped with a nontrivial non-archimedean absolute value $| \, |_K$ such that K is complete with respect to $| \, |_K$.

Construction 4 (Tilting)**.** Fix a prime number p. For any field K, we let $K^\flat$ denote the set given by the inverse limit of the system

$$\cdots \to K \xrightarrow{x \mapsto x^p} K \xrightarrow{x \mapsto x^p} K \xrightarrow{x \mapsto x^p} K.$$

In other words, $K^\flat$ is the collection of all sequences $\{x_n\}_{n \geq 0}$ in K which satisfy $x_n^p = x_{n-1}$ for $n > 0$.

The set $K^\flat$ comes equipped with an obvious multiplication law, given by

$$\{x_n\}_{n\geq 0} \cdot \{y_n\}_{n\geq 0} = \{x_n y_n\}_{n\geq 0}.$$

This is well-defined since the operation $x \mapsto x^p$ is multiplicative. One cannot define an addition law in the same way (unless K has characteristic p), because the map $x \mapsto x^p$ is not additive. Nevertheless, we have the following:

Proposition 5. *Let K be a completely valued field which is algebraically closed, and suppose that the residue field of K has characteristic p. Then $K^\flat$ can be equipped with the structure of a field of characteristic p, with multiplication defined as above and addition given by*

$$\{x_n\}_{n\geq 0} + \{y_n\}_{n\geq 0} = \{\lim_{m\to\infty}(x_{n+m} + y_{n+m})^{p^m}\}_{n\geq 0}.$$

We will refer to $K^\flat$ as the tilt of K.

We will prove Proposition 5 in the next lecture.

Notation 6. Let K be a completely valued field which is algebraically closed. For each $x = \{x_n\}_{n\geq 0} \in K^\flat$, we set $x^\sharp = x_0$, so that $x \mapsto x^\sharp$ defines a map of sets $\sharp : K^\flat \to K$. If K has characteristic p, then this map is an isomorphism. If K has characteristic zero, then it is multiplicative but not additive (note that it cannot be a ring homomorphism, since $K^\flat$ and K have different characteristics). Instead we have

$$(x+y)^\sharp = \lim_{m\to\infty}((x^{1/p^m})^\sharp + (y^{1/p^m})^\sharp)^{p^m}.$$

We define a map $|\,|_{K^\flat} : K^\flat \to \mathbf{R}_{\geq 0}$ by the formula $|x|_{K^\flat} = |x^\sharp|_K$. We will see in the next lecture that this is a non-archimedean absolute value on $K^\flat$, and endows $K^\flat$ with the structure of a completely valued field of characteristic p. We denote the valuation ring of $K^\flat$ by

$$\mathcal{O}_K^\flat = \{x \in K^\flat : |x^\sharp|_K \leq 1\} = \varprojlim(\cdots \to \mathcal{O}_K \xrightarrow{x\mapsto x^p} \mathcal{O}_K \xrightarrow{x\mapsto x^p} \mathcal{O}_K).$$

One can show that if K is algebraically closed, then $K^\flat$ is also algebraically closed. Consequently, tilting provides a construction

$$\begin{array}{c}\{\text{Algebraically closed completely valued fields of residue characteristic } p\} \\ \Big\downarrow \flat \\ \{\text{Algebraically closed completely valued fields of characteristic } p\}.\end{array}$$

Beware that this construction is not invertible.

Definition 7. Let C be an algebraically closed completely valued field of characteristic p. An *untilt* of C is a pair (K, ι), where K is an algebraically closed completely valued field with residue characteristic p and $\iota : C \simeq K^\flat$ is an isomorphism of fields which carries the valuation ring $\mathcal{O}_C$ to the valuation ring $\mathcal{O}_K^\flat$.

Example 8. Up to isomorphism, there is a unique untilt of C which has characteristic p (given by the pair (C, id_C)). However, we will see that there exist many other untilts (K, ι), where K has characteristic zero. In this case, one can think of K as a "characteristic zero incarnation" of the field C, which is an algebra over the field $\mathbf{Q}_p$ of p-adic rational numbers.

Let A be a commutative ring. Then the affine scheme $\mathrm{Spec}(A)$ can be understood in terms of its "functor of points," which assigns to another commutative ring R the set $\mathrm{Hom}(A, R)$ of all A-algebra structures on R. In the cases $A = \mathbf{Z}$ and $A = \mathbf{Q}$, this functor is not very interesting: every commutative ring admits a unique $\mathbf{Z}$-algebra structure, and at most one $\mathbf{Q}$-algebra structure.

Heuristic Idea 9 (Scholze)**.** Let C be an algebraically closed completely valued field of characteristic p. Then

$$\{\text{Untilts of } C\}/\simeq$$

is a good replacement for the set of C-valued points of $\mathrm{Spec}(\mathbf{Z})$. Similarly,

$$\{\text{Characteristic zero untilts of } C\}/\simeq$$

is a good replacement for the set of C-valued points of $\mathrm{Spec}(\mathbf{Q})$.

One virtue of this idea is that it allows us to make sense of products like $\mathrm{Spec}(\mathbf{Z}) \times \mathrm{Spec}(\mathbf{Z})$ in a nontrivial way. Rather than taking a product in the category of schemes (which gives nothing interesting), we instead apply Heuristic 9: if C is an algebraically closed completely valued field of characteristic p, then we should think of "C-valued points" of $\mathrm{Spec}(\mathbf{Z}) \times \mathrm{Spec}(\mathbf{Z})$ as corresponding to *pairs* of untilts of C. This is different from the interpretation that Heuristic 9 suggests for $\mathrm{Spec}(\mathbf{Z})$ itself: while C has only one $\mathbf{Z}$-algebra structure, it has many different untilts.

To exploit this idea effectively, we need to address the following:

Question 10. Let C be an algebraically closed completely valued field of characteristic p. What can we say about the collection of all untilts of C? How can they be classified?

We observed above that, up to isomorphism, C has only one untilt of characteristic p (namely, the field C itself). Let us therefore confine our attention to the classification of *characteristic zero* untilts of C. Note that since C is a perfect field of characteristic p, the construction $x \mapsto x^p$ induces an automorphism of C called the *Frobenius map*, which we denote by $\varphi_C : C \to C$. If (K, ι) is any untilt of C, then we can construct a family of untilts given by $\{(K, \varphi_C^n \circ \iota)\}_{n \in \mathbf{Z}}$. When K has characteristic zero, these untilts are pairwise non-isomorphic but should nevertheless be considered "the same" in some sense. Let us therefore rephrase Question 10 as follows:

Question 11. Let C be an algebraically closed completely valued field of characteristic p. What can one say about the quotient

$$\{\text{Isomorphism classes of characteristic zero untilts of } C\}/\varphi_C^{\mathbf{Z}}?$$

Question 11 has a very beautiful answer, which is the subject of this course.

Theorem 12. *Let C be an algebraically closed completely valued field of characteristic p. There exists a Dedekind scheme X equipped with a bijection*

$$\begin{array}{c} \{ \text{ Closed points } x \in X \ \} \\ \big\downarrow \wr \\ \{\text{Isomorphism classes of characteristic zero untilts of } C\}/\varphi_C^{\mathbf{Z}}. \end{array}$$

For each closed point $x \in X$, the corresponding untilt of C can be identified with the residue field $\kappa(x)$ of X.

The scheme X of Theorem 12 is called the *Fargues-Fontaine curve.* It has many amazing properties:

- There is a canonical isomorphism $\mathbf{Q}_p \simeq \mathrm{H}^0(X, \mathcal{O}_X)$. In particular, X can be regarded as a $\mathbf{Q}_p$-scheme.

The structural morphism $\pi : X \to \mathrm{Spec}(\mathbf{Q}_p)$ is not of finite type (for example, the residue fields of X at its closed points are not finite extensions of $\mathbf{Q}_p$). Nevertheless, X behaves in many respects like a *complete* algebraic curve of genus zero:

- For every rational function f on X, there is a degree formula

$$\sum_{x \in X} \deg_x(f) = 0.$$

  Note that if X were an algebraic curve, then this would imply that X is projective: that is, it does not have any "missing" points.

- Every line bundle $\mathscr{L}$ on X has a well-defined degree $\deg(\mathscr{L})$, any any two line bundles of the same degree are isomorphic (if X were an algebraic curve, this would say that the genus of X is equal to zero).

- The cohomology group $\mathrm{H}^1(X; \mathcal{O}_X)$ vanishes (if X were an algebraic curve, this would also be equivalent to saying that X has genus zero).

- Every vector bundle $\mathscr{E}$ on X has a canonical Harder-Narasimhan filtration (as if $\mathscr{E}$ were a vector bundle on an algebraic curve), which is non-canonically split (as if the curve were of genus zero).

- The fibers of the map $\pi : X \to \mathrm{Spec}(\mathbf{Q}_p)$ are simply connected. More precisely, pullback along π induces an equivalence

$$\{ \text{ Representations of } \mathrm{Gal}(\mathbf{Q}_p) \} \to \{ \text{ Local systems on } X \}$$

  (for various definitions of the right and left hand side).

Motivated by Heuristic 9, Fargues and Scholze have proposed a program for applying geometric ideas to the study of the local Langlands program. The philosophy of their approach can be roughly summarized by the slogan

$$\{\text{Local Langlands for } \mathbf{Q}_p\} \simeq \{\text{Geometric Langlands for the curve } X\}.$$

The Fargues-Fontaine curve X is also closely connected other ideas in arithmetic: p-adic Hodge theory, the classification of p-divisible groups, Banach-Colmez spaces, Tate local duality, ...

Lecture 2: Tilting

October 5, 2018

Let p be a prime number, which we regard as fixed throughout this lecture. In Lecture 1, we defined the *tilt* $K^\flat$ of an algebraically closed completely valued field K of residue characteristic p. In this lecture, we review the tilting construction in more detail, working in the more general setting of *perfectoid fields*.

Definition 1. A *perfectoid field* is a field K equipped with a nonarchimedean absolute value $| \, |_K : K \to \mathbf{R}_{\geq 0}$ satisfying the following axioms:

$(A1)$ The residue field $k = \mathcal{O}_K / \mathfrak{m}_K$ has characteristic p. Equivalently, the prime number p belongs to the maximal ideal $\mathfrak{m}_K$, so that $|p|_K < 1$.

$(A2)$ The field K is complete with respect to the absolute value $| \, |_K$.

$(A3)$ The Frobenius map $\varphi : \mathcal{O}_K / p \, \mathcal{O}_K \to \mathcal{O}_K / p \, \mathcal{O}_K$ is surjective. That is, for every element $x \in \mathcal{O}_K$, we can write $x = y^p + pz$ for some $y, z \in \mathcal{O}_K$.

$(A4)$ The maximal ideal $\mathfrak{m}_K$ is not generated by p. In other words, there exists some element $x \in K$ satisfying $|p|_K < |x|_K < 1$.

Remark 2. In the situation of Definition 1, choose $x \in K$ satisfying $|p|_K < |x|_K < 1$. Then $x \in \mathcal{O}_K$, so we can write $x = y^p + pz$ for some $y, z \in \mathcal{O}_K$. Since $|pz|_K \leq |p|_K < |x|_K$, we must have $|x|_K = |y^p|_K = |y|_K^p$. In particular, we have $|x|_K < |y|_K < 1$, so that $y \in \mathfrak{m}_K \setminus x \, \mathcal{O}_K$. It follows that the maximal ideal $\mathfrak{m}_K$ is not principal: that is, the valuation ring $\mathcal{O}_K$ is not a discrete valuation ring.

Remark 3. In the situation of Definition 1, suppose that K is characteristic p. In this case, axiom $(A1)$ is automatic, axiom $(A3)$ says that the field K is *perfect* (that is, every element of K has a pth root), and axiom $(A4)$ says that the absolute value $| \, |_K$ is nontrivial. In other words, a perfectoid field of characteirstic p is just a completely valued perfect field of characteristic p.

Example 4. Let K be a completely valued field of residue characteristic p. Suppose that every element $x \in K$ has a pth root (this condition is satisfied, for example, if K is algebraically closed). Then axioms $(A3)$ and $(A4)$ are satisfied, so K is a perfectoid field.

Example 5. For each $n > 0$, let $\mathbf{Z}[\zeta_{p^n}]$ denote ring obtained from $\mathbf{Z}$ by adjoining a primitive p^nth root of unity, given by the quotient $\mathbf{Z}[x]/(1 + x^{p^{n-1}} + x^{2p^{n-1}} + \cdots + x^{(p-1)p^{n-1}})$; equivalently $\mathbf{Z}[\zeta_{p^n}]$ can be described as the ring of integers in the number field $\mathbf{Q}(\zeta_{p^n})$.

Let $\mathbf{Z}_p^{\mathrm{cyc}}$ denote the p-adic completion of the union $\bigcup_{n>0} \mathbf{Z}[\zeta_{p^n}]$ and set $\mathbf{Q}_p^{\mathrm{cyc}} = \mathbf{Z}_p^{\mathrm{cyc}}[1/p]$. Then $K = \mathbf{Q}_p^{\mathrm{cyc}}$ is a perfectoid field with ring of integers $\mathcal{O}_K = \mathbf{Z}_p^{\mathrm{cyc}}$. Axiom $(A3)$ follows from the observation that the image of the Frobenius map

$$\varphi : \mathbf{Z}_p^{\mathrm{cyc}}/p\mathbf{Z}_p^{\mathrm{cyc}} \to \mathbf{Z}_p^{\mathrm{cyc}}/p\mathbf{Z}_p^{\mathrm{cyc}}$$

is a subgroup of $\mathbf{Z}_p^{\mathrm{cyc}}/p\mathbf{Z}_p^{\mathrm{cyc}} \simeq \bigcup_{n>0} \mathbf{F}_p[\zeta_{p^n}]$ which contains each of the roots of unity ζ_{p^n}, by virtue of the equation $\zeta_{p^n} = (\zeta_{p^{n+1}})^p$.

Note that the pth power map $\mathbf{Q}_p^{\mathrm{cyc}} \to \mathbf{Q}_p^{\mathrm{cyc}}$ is *not* surjective: for example, there is no element $x \in \mathbf{Q}_p^{\mathrm{cyc}}$ satisfying $x^p = p$.

As in the previous lecture, we let $K^\flat$ denote the inverse limit of the system

$$\cdots \to K \xrightarrow{x \mapsto x^p} K \xrightarrow{x \mapsto x^p} K,$$

whose elements can be identified with sequences $\vec{x} = \{x_0, x_1, \ldots \in K : x_n = x_{n+1}^p\}$. We regard $K^\flat$ as a monoid with respect to the obvious multiplication

$$\{x_n\}_{n \geq 0} \cdot \{y_n\}_{n \geq 0} = \{x_n \cdot y_n\}_{n \geq 0}.$$

When K is a perfectoid field, we can equip $K^\flat$ with a compatible addition law. To prove this, it is convenient to first work with the subset $\mathcal{O}_K^\flat \subseteq K^\flat$ consisting of those sequences $\{x_n\}_{n \geq 0}$ where each x_n belongs to $\mathcal{O}_K$ (note that if this condition is satisfied for any integer $n \geq 0$, then it is satisfied for all integers $n \geq 0$).

Proposition 6. *Let K be a completely valued field of residue characteristic p. Then canonical map $\mathcal{O}_K \to \mathcal{O}_K / p\, \mathcal{O}_K$ induces a bijection*

$$\mathcal{O}_K^\flat \to \varprojlim(\cdots \to \mathcal{O}_K / p\, \mathcal{O}_K \xrightarrow{x \mapsto x^p} \mathcal{O}_K / p\, \mathcal{O}_K)$$

Proof. Let us assume that K has characteristic zero (in characteristic p, there is nothing to prove). Our assumption that K is complete implies that $\mathcal{O}_K$ can be realized as the inverse limit $\varprojlim_n \mathcal{O}_K / p^n\, \mathcal{O}_K$. For each $n \geq 1$, let $Z(n)$ denote the limit of the inverse system of sets

$$\cdots \to \mathcal{O}_K / p^n\, \mathcal{O}_K \xrightarrow{x \mapsto x^p} \mathcal{O}_K / p^n\, \mathcal{O}_K \xrightarrow{x \mapsto x^p} \mathcal{O}_K / p^n\, \mathcal{O}_K \xrightarrow{x \mapsto x^p} \mathcal{O}_K / p^n\, \mathcal{O}_K .$$

Then $\mathcal{O}_K^\flat$ is the inverse limit $\varprojlim_n Z(n)$, and we wish to show that the projection map $\mathcal{O}_K^\flat \to Z(1)$ is a bijection. For this, it will suffice to show that each of the transition maps $Z(n) \to Z(n-1)$ is a bijection. In other words, it will suffice to show that the vertical maps in the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \mathcal{O}_K / p^n\, \mathcal{O}_K \arrow[r, "(\bullet)^p"] \arrow[d] & \mathcal{O}_K / p^n\, \mathcal{O}_K \arrow[r, "(\bullet)^p"] \arrow[d] & \mathcal{O}_K / p^n\, \mathcal{O}_K \arrow[d] \\
\cdots \arrow[r] & \mathcal{O}_K / p^{n-1}\, \mathcal{O}_K \arrow[r, "(\bullet)^p"] \arrow[ur, dashed] & \mathcal{O}_K / p^{n-1}\, \mathcal{O}_K \arrow[r, "(\bullet)^p"] \arrow[ur, dashed] & \mathcal{O}_K / p^{n-1}\, \mathcal{O}_K
\end{tikzcd}$$

induce an isomorphism after taking the inverse limit in the horizontal direction. For this, we note the existence (and uniqueness) of dotted arrows rendering the diagram commutative: this comes from the elementary observation that for $x, y \in \mathcal{O}_K$, we have

$$(x \equiv y \pmod{p^{n-1}}) \Rightarrow (x^p \equiv y^p \pmod{p^n}).$$

$\square$

Corollary 7. *Let K be a completely valued field of residue characteristic p. Then we can equip $\mathcal{O}_K^\flat$ with the structure of a commutative ring, where the multiplication is defined pointwise and the addition is uniquely determined by the requirement that*

$$\{x_n\}_{n \geq 0} + \{y_n\}_{n \geq 0} = \{z_n\}_{n \geq 0} \Rightarrow x_n + y_n \equiv z_n \pmod{p}.$$

Remark 8. In the situation of Corollary 7, we can describe the addition law on $\mathcal{O}_K^\flat$ more explicitly. Suppose we are given elements $\{x_n\}_{n \geq 0}$ and $\{y_n\}_{n \geq 0}$ in $\mathcal{O}_K^\flat$. Write $\{x_n\}_{n \geq 0} + \{y_n\}_{n \geq 0} = \{z_n\}_{n \geq 0}$, so that we have $x_m + y_m \equiv z_m \pmod{p}$ for each $n \geq 0$. Writing $z_m = x_m + y_m + pw$ for some $w \in \mathcal{O}_K$, we obtain

$$\begin{aligned}
z_0 &= z_m^{p^m} \\
&= (x_m + y_m + pw)^{p^m} \\
&= \sum_{i=0}^{p^m} \binom{p^m}{i} (pw)^i (x_m + y_m)^{p^m - i} \\
&\equiv (x_m + y_m)^{p^m} \pmod{p^m}.
\end{aligned}$$

It follows that z_0 is given concretely as the limit $\lim_{m\to\infty}(x_m + y_m)^{p^m}$. More generally, each z_n is given concretely as $\lim_{m\to\infty}(x_{n+m} + y_{n+m})^{p^m}$.

Note that, to prove Proposition 6, we do not need to assume that K is a perfectoid field: it is enough to assume axioms $(A1)$ and $(A2)$ of Definition 1. However, at this level of generality, the tilt $K^\flat$ might be "too small."

Exercise 9. Let $K = \mathbf{Q}_p$ be the field of p-adic rational numbers, equipped with the usual p-adic absolute value. Show that $K^\flat = \mathcal{O}_K^\flat$ is isomorphic to $\mathbf{F}_p$.

Our next goal is to show that, when K is a perfectoid field, the tilt $K^\flat$ is very large (Proposition 13).

Notation 10. Let K be a completely valued field of residue characteristic p and let $x = \{x_n\}_{n\geq 0}$ be an element of $K^\flat$. We set $x^\sharp = x_0 \in K$. The construction $x \mapsto x^\sharp$ then determines a multiplicative map $\sharp : K^\flat \to K$. For each $x \in K^\flat$, we define $|x|_{K^\flat} = |x^\sharp|_K$.

Example 11. Suppose that K is algebraically closed (or, more generally, that every element of K admits a pth root). Then the map $x \mapsto x^\sharp$ determines a surjection $K^\flat \to K$,

Example 12. Suppose that K is a perfect field of characteristic p. Then the map $\sharp : K^\flat \to K$ is bijective.

Proposition 13. *Let K be a perfectoid field. Then:*

(1) *For every element $x \in \mathcal{O}_K$, there exists an element $x' \in \mathcal{O}_K^\flat$ satisfying $x \equiv x'^\sharp \pmod p$.*

(2) *For every element $y \in K$, there exists an element $y' \in K^\flat$ satisfying $|y|_K = |y'|_{K^\flat}$.*

Proof. Assertion (1) follows from Proposition 6 together with the observation that, if K satisfies axiom $(A3)$, then the transition maps in the diagram

$$\cdots \to \mathcal{O}_K / p\,\mathcal{O}_K \xrightarrow{x\mapsto x^p} \mathcal{O}_K / p\,\mathcal{O}_K \xrightarrow{x\mapsto x^p} \mathcal{O}_K / p\,\mathcal{O}_K$$

are surjective.

To prove (2), we may assume without loss of generality we may assume that $y \neq 0$. Using axiom $(A4)$ of Definition 1, we can choose an element $x \in K$ with $|p|_K < |x|_K < 1$. Replacing x by an element which is congruent modulo p, we can assume that $x = x'^\sharp$ for some $x' \in K^\flat$ (by virtue of (1)). We are therefore free to modify y by multiplying it by a suitable power of x, and can therefore reduce to the case where $|x|_K \leq |y|_K < 1$. In this case, we have $|p|_K < |y|_K < 1$. Using part (1) again, we can choose $y' \in K^\flat$ with $y'^\sharp \equiv y \pmod p$, so that $|y|_K = |y'^\sharp|_K = |y'|_{K^\flat}$. $\square$

Exercise 14. Show that the converse of Proposition 13 is also true: if K is a completely valued field of residue characteristic p, then assertion (1) of Proposition 13 implies that K satisfies axiom $(A3)$ of Definition 1, and assertion (2) of Proposition 13 implies that K satisfies axiom $(A4)$ of Definition 1. In other words, the axioms for a perfectoid field are exactly what we need to guarantee that the tilt $K^\flat$ is "sufficiently large."

Using Proposition 13, we can choose an element π in $K^\flat$ such that $0 < |\pi|_{K^\flat} < 1$. For each $n \in \mathbf{Z}$, we have

$$\pi^{-n}\,\mathcal{O}_K^\flat = \{x \in K^\flat : |x|_{K^\flat} \leq |\pi|_{K^\flat}^{-n}\}$$

It follows that, as a set, we can identify $K^\flat$ with the direct limit

$$\mathcal{O}_K^\flat \xrightarrow{\pi} \mathcal{O}_K^\flat \xrightarrow{\pi} \mathcal{O}_K^\flat \xrightarrow{\cdots},$$

where the transition maps are given by multiplication by π. This proves the following:

Proposition 15. *Let K be a perfectoid field. Then the inclusion $\mathcal{O}_K^\flat \hookrightarrow K^\flat$ extends uniquely to a multiplicative bijection $\mathcal{O}_K^\flat[\pi^{-1}] \simeq K^\flat$. Consequently, there is a unique ring structure on $K^\flat$ which is compatible with its multiplication and which coincides, on $\mathcal{O}_K^\flat$, with the ring structure of Corollary 7.*

Exercise 16. Show that the addition law on $K^\flat$ is given in general by the formula

$$\{x_n\}_{n\geq 0} + \{y_n\}_{n\geq 0} = \{\lim_{m\to\infty}(x_{m+n}+y_{m+n})^{p^m}\}_{n\geq 0}$$

Theorem 17. *Let K be a perfectoid field. Then $K^\flat$, with the ring structure of Proposition 15 and the map $|\,|_{K^\flat} : K^\flat \to \mathbf{R}_{\geq 0}$, is a perfectoid field of characteristic p.*

Proof. Note that if $\{x_n\}_{n\geq 0}$ is nonzero element of $K^\flat$, then each x_n is a nonzero element of K; it follows that $\{x_n^{-1}\}_{n\geq 0}$ is also an element of $K^\flat$ which is a multiplicative inverse for $\{x_n\}_{n\geq 0}$. This proves that $K^\flat$ is a field. Proposition 6 realizes $\mathcal{O}_K^\flat$ as an inverse limit of copies of $\mathcal{O}_K/p\,\mathcal{O}_K$ (with transition maps given by the Frobenius). Since p vanishes in $\mathcal{O}_K/p\,\mathcal{O}_K$, it vanishes in $\mathcal{O}_K^\flat$ and therefore also in $K^\flat$: that is, $K^\flat$ is a field of characteristic p. We claim that $|\,|_{K^\flat}$ is a non-archimedean absolute value on $K^\flat$. The identities

$$|0|_{K^\flat} = 0 \qquad |1|_{K^\flat} = 1 \qquad |x\cdot y|_{K^\flat} = |x|_{K^\flat}\cdot|y|_{K^\flat}$$

are immediate from the definition. It will therefore suffice to show that for $x = \{x_n\}_{n\geq 0}$ and $y = \{y_n\}_{n\geq 0} \in K^\flat$, we have

$$|x+y|_{K^\flat} \leq \max(|x|_{K^\flat}, |y|_{K^\flat}).$$

Using the formula of Exercise 16, we are reduced to proving that

$$|(x_m+y_m)^{p^m}|_K \leq \max(|x_m|_K^{p^m}, |y_m|_K^{p^m}),$$

which follows (after extracting p^mth roots) from the analogous fact for the absolute value $|\,|_K$.

The field $K^\flat$ is perfect by construction: every element $(x_0, x_1, x_2, \ldots) \in K^\flat$ has a unique pth root, given by the shifted sequence $(x_1, x_2, x_3, \ldots) \in K^\flat$. Moreover, the absolute value on $K^\flat$ is nontrivial because it takes the same values as the absolute value on K (Proposition 13). We will complete the proof by showing that $K^\flat$ is complete. Let us assume that K has characteristic zero (if K has characteristic p, then the map $\sharp : K^\flat \to K$ is an isomorphism of valued fields and there is nothing to prove). Using Proposition 13, we can choose an element $\pi \in K^\flat$ satisfying $|\pi|_{K^\flat} = |p|_K$. We wish to show that the ring $\mathcal{O}_K^\flat$ is π-adically complete: that is, that it can be realized as the inverse limit of the system

$$\cdots \to \mathcal{O}_K^\flat/(\pi^{p^3}) \to \mathcal{O}_K^\flat/(\pi^{p^2}) \to \mathcal{O}_K^\flat/(\pi^p) \to \mathcal{O}_K^\flat/(\pi).$$

For each $m \geq 0$, the map of sets

$$\mathcal{O}_K^\flat \to \mathcal{O}_K \qquad (x = \{x_n\}_{n\geq 0}) \mapsto (x_m = (x^{1/p^m})^\sharp)$$

induces a ring homomorphism $\mathcal{O}_K^\flat \to \mathcal{O}_K/p\,\mathcal{O}_K$ which annihilates π^{p^m}, and therefore factors through a map $u_m : \mathcal{O}_K^\flat/(\pi^{p^m}) \to \mathcal{O}_K/p\,\mathcal{O}_K$. These maps fit into a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \mathcal{O}_K^\flat/(\pi^{p^2}) \arrow[r] \arrow[d, "u_2"] & \mathcal{O}_K^\flat/(\pi^p) \arrow[r] \arrow[d, "u_1"] & \mathcal{O}_K^\flat/(\pi) \arrow[d, "u_0"] \\
\cdots \arrow[r] & \mathcal{O}_K/p\,\mathcal{O}_K \arrow[r, "\varphi"] & \mathcal{O}_K/p\,\mathcal{O}_K \arrow[r, "\varphi"] & \mathcal{O}_K/p\,\mathcal{O}_K
\end{tikzcd}$$

where the inverse limit of the lower diagram agrees with $\mathcal{O}_K^\flat$ by virtue of Proposition 6. It will therefore suffice to show that each of the maps u_m is an isomorphism. This reduces immediately to the case $m = 0$, where it is a special case of Lemma 18 below. □

Lemma 18. *Let K be a perfectoid field and let $\pi \in K^\flat$ be a nonzero element satisfying $|p|_K \leq |\pi|_{K^\flat} < 1$. Then the map $\sharp : K^\flat \to K$ induces an isomorphism $\mathcal{O}_K^\flat/(\pi) \to \mathcal{O}_K/(\pi^\sharp)$.*

Proof. Surjectivity follows from Proposition 13. To prove injectivity, we note that if $x \in \mathcal{O}_K^\flat$ has the property that $x^\sharp \equiv 0 \pmod{\pi^\sharp}$, then $|x|_{K^\flat} = |x^\sharp|_K \leq |\pi^\sharp|_K = |\pi|_{K^\flat}$ so that x is divisibly by π in $\mathcal{O}_K^\flat$. □

Lecture 3: Untilting

October 5, 2018

In this lecture, we let $C^\flat$ denote a perfectoid field of characteristic p.

Warning 1. We will often use the superscript $^\flat$ to signal that an object under consideration "lives in characteristic p". In particular, declaring that $C^\flat$ is a perfectoid field of characteristic p is not meant to signal that $C^\flat$ is given as the tilt of a perfectoid field C. In fact, our emphasis is the opposite: we take the field $C^\flat$ as given, and would like to understand all possible *untilts* of $C^\flat$. Recall that an untilt of $C^\flat$ is defined to be a pair (K, ι), where K Is a perfectoid field and $\iota : C \simeq K^\flat$ is an (continuous) isomorphism.

Question 2. Let $C^\flat$ be a perfectoid field of characteristic p. How can one classify the untilts of $C^\flat$?

Remark 3. Let K be a perfectoid field of characteristic zero. Note that giving a continuous isomorphism $\iota : C^\flat \simeq K^\flat$ is equivalent to giving an isomorphism of valuation rings

$$\mathcal{O}_C^\flat \to \mathcal{O}_K^\flat = \varprojlim(\cdots \to \mathcal{O}_K / p\,\mathcal{O}_K \xrightarrow{\varphi} \mathcal{O}_K / p\,\mathcal{O}_K \xrightarrow{\varphi} \mathcal{O}_K / p\,\mathcal{O}_K).$$

We saw in the previous lecture that this induces an isomorphism of quotient rings $\iota_0 : \mathcal{O}_C^\flat / (\pi) \simeq \mathcal{O}_K / (p)$ for some element $\pi \in C^\flat$ satisfying $0 < |\pi|_{C^\flat} < 1$. Conversely, any such isomorphism ι_0 can be lifted to an isomorphism of valuation rings $\mathcal{O}_C^\flat \simeq \mathcal{O}_K^\flat$, since $\mathcal{O}_C^\flat$ is isomorphic to the inverse limit

$$\cdots \to \mathcal{O}_C^\flat / (\pi) \xrightarrow{\varphi} \mathcal{O}_C^\flat / (\pi) \xrightarrow{\varphi} \mathcal{O}_C^\flat / (\pi).$$

We may therefore rephrase Question 2 as follows: how can we classify perfectoid fields K of characteristic zero equipped with an isomorphism $\mathcal{O}_K / (p) \simeq \mathcal{O}_C^\flat / (\pi)$?

For an untilt (K, ι) of C, let us abuse notation by writing $\sharp : C^\flat \to K$ for the composite map $C^\flat \xrightarrow{\iota} K^\flat \xrightarrow{\sharp} K$. This map does not need to be surjective. However, it is not too far off. We saw in the previous lecture that every element $x \in \mathcal{O}_K$ is congruent modulo p to an element in the image of the map $\sharp$: that is, we can find an element $c_0 \in \mathcal{O}_C^\flat$ satisfying $x = c_0^\sharp + x'p$, for some $x' \in \mathcal{O}_K$. Applying the same argument to x', we obtain $x = c_0^\sharp + c_1^\sharp p + x'' p^2$, for some $x'' \in \mathcal{O}_K$. Iterating this argument, we obtain a description of x as an infinite sum

$$x = c_0^\sharp + c_1^\sharp p + c_2^\sharp p^2 + c_3^\sharp p^3 + \cdots,$$

for some sequence of elements $c_0, c_1, c_2, \ldots \mathcal{O}_C^\flat$; note that this infinite sum makes sense because the ring $\mathcal{O}_K$ is p-adically complete. The decomposition above is not at all unique: generally an element $x \in \mathcal{O}_K$ can be decomposed as a sum $\sum_{n \geq 0} c_n^\sharp p^n$ in many different ways. For example, if K is algebraically closed, then any element $x \in \mathcal{O}_K$ can be written in the form $c_0^\sharp$ by choosing a compatible sequence of p^nth roots of x; in characteristic zero, these pth roots are not unique.

One virtue of working with expressions like $\sum_{n \geq 0} c_n^\sharp p^n$ is that they make sense simultaneously in *every* untilt K of $C^\flat$. Moreover, it is possible to give work out formulas for adding and multiplying these expressions which are independent of the choice of K. To make this idea precise, it will be convenient to review the theory of Witt vectors.

Notation 4. Let R be a perfect ring of characteristic p: that is, a commutative ring such that $p = 0$ in R and every element $x \in R$ has a unique pth root. We let $W(R)$ denote the ring of Witt vectors of R. Then $W(R)$ is characterized up to (unique) isomorphism by the following properties:

(1) There is an isomorphism $W(R)/pW(R) \simeq R$.

(2) The element p is not a zero-divisor in $W(R)$.

(3) The ring $W(R)$ is p-adically complete.

Example 5. Let $R = \mathbf{F}_p$ be the finite field with p-elements. Then $W(R) \simeq \mathbf{Z}_p$ can be identified with the ring of p-adic integers.

Notation 6. For every element $x \in R$, we let $[x]$ denote its *Teichmüller representative* in $W(R)$. Then $[x]$ is uniquely determined by the following properties:

- The quotient map $W(R) \twoheadrightarrow W(R)/pW(R) \simeq R$ carries $[x]$ to x.
- The element $[x] \in W(R)$ admits a p^nth root, for every $n \geq 0$.

Concretely, one can construct the Teichmüller representative $[x]$ as the limit $\lim_{n\to\infty}(\overline{x^{1/p^n}})^{p^n}$, where $\overline{x^{1/p^n}}$ is any element of $W(R)$ representing the p^nth root $x^{1/p^n} \in R$. The construction of Teichmüller representatives determines a map

$$[\bullet] : R \to W(R)$$

which is multiplicative (that is, we have $[xy] = [x][y]$) but not additive.

Remark 7. Let R be a perfect ring of characteristic p and let x be an element of $W(R)$. Then x has some image $c_0 \in R$ under the quotient map $W(R) \twoheadrightarrow W(R)/pW(R) \simeq R$. The Teichmüller lift $[c_0]$ is then congruent to x modulo p, so we can write $x = [c_0] + x'p$ for some $x' \in W(R)$. Iterating this observation, we obtain an identity

$$x = [c_0] + [c_1]p + [c_2]p^2 + [c_3]p^3 + \cdots ,$$

called the *Teichmüller expansion* of x. Note that, in contrast to the situation before, this expansion is unique: if

$$\sum [c_n]p^n = \sum [c_n']p^n,$$

then an easy induction shows that $c_n = c_n'$ for each n.

Remark 8. Let R be a perfect ring of characteristic p. Then the ring of Witt vectors $W(R)$ can be characterized by a universal property:

($*$) For any p-adically complete ring A, reduction modulo p induces a bijection

$$\operatorname{Hom}(W(R), A) \to \operatorname{Hom}(R, A/pA).$$

In other words, every ring homomorphism $R \to A/pA$ can be lifted uniquely to a ring homomorphism $W(R) \to A$.

Let us now specialize to the situation of interest to us.

Construction 9. Let $C^\flat$ be a perfectoid field of characteristic p and let $\mathcal{O}_C^\flat$ be the valuation ring of $C^\flat$. Then $\mathcal{O}_C^\flat$ is a perfect ring of characteristic p. We let $\mathbf{A}_{\text{inf}}$ denote the ring of Witt vectors $W(\mathcal{O}_C^\flat)$.

Remark 10. The ring $\mathbf{A}_{\text{inf}}$ is one of Fontaine's *period rings*; it will play an essential role in this course.

Remark 11. Let $C^\flat$ be a perfectoid field of characteristic p and let (K, ι) be an untilt of $C^\flat$. The map $\sharp : \mathcal{O}_C^\flat \to \mathcal{O}_K$ is not a ring homomorphism (unless K has characteristic p). However, it induces a ring homomorphism $\mathcal{O}_C^\flat \to \mathcal{O}_K / p\, \mathcal{O}_K$. Since $\mathcal{O}_K$ is p-adically complete, the universal property of Remark 8 to lifts this to a ring homomorphism

$$\theta : \mathbf{A}_{\mathrm{inf}} = W(\mathcal{O}_C^\flat) \to \mathcal{O}_K .$$

Concretely, this map is given by the formula

$$\theta([c_0] + [c_1]p + [c_2]p^2 + \cdots) = c_0^\sharp + c_1^\sharp p + c_2^\sharp p^2 + \cdots .$$

From the discussion at the beginning of the lecture, we deduce that θ is surjective.

Remark 12. In the situation of Remark 11, the map θ is *local*: that is, an element $\sum [c_n] p^n$ of $\mathbf{A}_{\mathrm{inf}}$ is invertible if and only if its image $\sum c_n^\sharp p^n \in \mathcal{O}_K$ is invertible (in both cases, invertibility is equivalent to the requirement that $|c_0|_{C^\flat} = 1$).

Remark 13. Let $C^\flat$ be a perfectoid field of characteristic p and let K be a characteristic zero untilt of $C^\flat$. We saw in the previous lecture that it is possible to choose an element $\pi \in \mathcal{O}_C^\flat$ with $|\pi|_{C^\flat} = |p|_K$, and that the map $\sharp : C^\flat \to K$ induces an isomorphism of commutative rings $\mathcal{O}_C^\flat / \pi\, \mathcal{O}_C^\flat \simeq \mathcal{O}_K / p\, \mathcal{O}_K$. In other words, $\mathcal{O}_C^\flat$ and $\mathcal{O}_K$ have a common quotient ring. Remark 11 shows that both $\mathcal{O}_C^\flat$ and $\mathcal{O}_K$ can be realized as a quotient of the same ring $\mathbf{A}_{\mathrm{inf}} = W(\mathcal{O}_C^\flat)$: one gets $\mathcal{O}_C^\flat$ from $\mathbf{A}_{\mathrm{inf}}$ by reducing modulo p, and $\mathcal{O}_K$ by reducing modulo the kernel $\ker(\theta)$. These quotient maps fit into a commutative diagram

$$\begin{tikzcd} \mathbf{A}_{\mathrm{inf}} \arrow[r, "\theta"] \arrow[d] & \mathcal{O}_K \arrow[d] \\ \mathcal{O}_C^\flat \arrow[r, "\sharp"] & \mathcal{O}_K / p\, \mathcal{O}_K . \end{tikzcd}$$

In the situation of Remark 11, the map θ is never injective: in other words, it is always possible to write an element $x \in \mathcal{O}_K$ as a sum $\sum_{n \geq 0} c_n^\sharp p^n$ in multiple ways.

Example 14. Let $C^\flat$ be a perfectoid field of characteristic p and let K be an untilt of $C^\flat$. We saw in the previous lecture that there exists an element $\pi \in \mathcal{O}_C^\flat$ satisfying $|\pi|_{C^\flat} = |p|_K$ (if K has characteristic p, we just take $\pi = 0$). It follows that the elements p and $\pi^\sharp$ have the same absolute value in K, and therefore differ by multiplication by an invertible element $\overline{u} \in \mathcal{O}_K$. Write $\overline{u} = \theta(u)$ for $u \in \mathbf{A}_{\mathrm{inf}}$ (so that u is also invertible, by Remark 12). The identity $\pi^\sharp = \overline{u} p$ then implies that $[\pi] - up \in \mathbf{A}_{\mathrm{inf}}$ belongs to the kernel of θ.

Definition 15. Let $C^\flat$ be a perfectoid field of chacteristic p. We say that an element $\xi \in \mathbf{A}_{\mathrm{inf}}$ is *distinguished* if it has the form $[\pi] - up$, where $|\pi|_{C^\flat} < 1$ and u is an invertible element of $\mathbf{A}_{\mathrm{inf}}$. In other words, ξ is distinguished if its Teichmüller expansion

$$\xi = [c_0] + [c_1]p + [c_2]p^2 + \cdots$$

has the property that $|c_0|_{C^\flat} < 1$ and $|c_1|_{C^\flat} = 1$.

Example 14 shows that, for every untilt K of $C^\flat$, the kernel of the induced map $\theta : \mathbf{A}_{\mathrm{inf}} = W(\mathcal{O}_C^\flat) \to \mathcal{O}_K$ always contains a distinguished element ξ. We now establish a converse:

Proposition 16. *Let $C^\flat$ be a perfectoid field of characteristic p and let ξ be a distinguished element of $\mathbf{A}_{\mathrm{inf}} = W(\mathcal{O}_C^\flat)$. Then the quotient $\mathbf{A}_{\mathrm{inf}}/(\xi)$ can be identified with the valuation ring $\mathcal{O}_K$ in a perfectoid field K. Moreover, the canonical map*

$$\mathcal{O}_C^\flat = \mathbf{A}_{\mathrm{inf}}/(p) \to \mathbf{A}_{\mathrm{inf}}/(\xi, p) \simeq \mathcal{O}_K / (p)$$

exhibits K as an untilt of $C^\flat$ (see Remark 3).

Corollary 17. *Let $C^\flat$ be a perfectoid field of characteristic p, let K be an untilt of $C^\flat$, and let $\theta : \mathbf{A}_{\text{inf}} \to \mathcal{O}_K$ be as above. Then $\ker(\theta)$ is a principal ideal, generated by any choice of distinguished element $\xi \in \ker(\theta)$.*

Proof. It follows from Example 14 that $\ker(\theta)$ contains a distinguished element ξ, so that θ induces a surjection $\overline{\theta} : \mathbf{A}_{\text{inf}}/(\xi) \twoheadrightarrow \mathcal{O}_K$. Proposition 16 shows that we can identify $\mathbf{A}_{\text{inf}}/(\xi)$ with $\mathcal{O}_{K'}$ for some untilt K' of $C^\flat$. Since $\mathcal{O}_K$ is an integral domain, the kernel of θ is a prime ideal of $\mathcal{O}_{K'}$, and is therefore either (0) (in which case $\overline{\theta}$ is an isomorphism) or the maximal ideal $\mathfrak{m}_{K'}$ (which is impossible, since $\mathcal{O}_K$ is not a field). $\square$

Corollary 18. *Let $C^\flat$ be a perfectoid field of characteristic p. Then the construction*

$$\xi \mapsto \textit{ Fraction field of } \mathbf{A}_{\text{inf}}/(\xi)$$

induces a bijection

$$\{\textit{Distinguished elements of } \mathbf{A}_{\text{inf}}\}/\textit{multiplication by units} \simeq \{\textit{Untilts of } C^\flat\}/\textit{isomorphism.}$$

To prove Proposition 16, we will need the following purely algebraic fact, which we leave to the reader.

Exercise 19. Let R be a commutative ring (not necessarily Noetherian!) containing a pair of elements x and y. Suppose that:

- The element x is not a zero-divisor in R, and R is x-adically complete.
- The image of y is not a zero-divisor in R/xR, and R/xR is y-adically complete.

Show that:

- The element y is not a zero-divisor in R, and R is y-adically complete.
- The image of x is not a zero-divisor in R/yR, and R/yR is x-adically complete.

Proof of Proposition 16. Let ξ be a distinguished element of $\mathbf{A}_{\text{inf}}$, so that we can write $\xi = [\pi] - up$ for some $\pi \in \mathfrak{m}_C^\flat$ and some invertible element u in $\mathbf{A}_{\text{inf}}$. If $\pi = 0$, then $\mathbf{A}_{\text{inf}}/(\xi) \simeq \mathbf{A}_{\text{inf}}/(p) \simeq \mathcal{O}_C^\flat$ and we have nothing to prove. Let us therefore assume that π is not zero. Let $\mathcal{O}_K$ denote the quotient ring $\mathbf{A}_{\text{inf}}/\xi\mathbf{A}_{\text{inf}}$. (Beware that this notation is misleading, since we do not yet know that there is a valued field K having $\mathcal{O}_K$ as its valuation ring.) We then have a canonical map $\theta : \mathbf{A}_{\text{inf}} \twoheadrightarrow \mathcal{O}_K$ (with kernel generated by ξ); for each element $x \in \mathcal{O}_C^\flat$, we will denote $\theta([x])$ by $x^\sharp \in \mathcal{O}_K$.

We now apply Exercise 19 to the elements $x = p$ and $y = \xi$ of the ring $\mathbf{A}_{\text{inf}}$. The construction of $\mathbf{A}_{\text{inf}}$ as the ring of Witt vectors $W(\mathcal{O}_C^\flat)$ shows that $\mathbf{A}_{\text{inf}}$ is p-adically complete and p-torsion free. Moreover, the image of ξ in the quotient $\mathbf{A}_{\text{inf}}/p\mathbf{A}_{\text{inf}} \simeq \mathcal{O}_C^\flat$ is π, satisfying $0 < |\pi|_{C^\flat} < 1$. It follows that $\mathcal{O}_C^\flat$ is ξ-adically complete and ξ-torsion free. Applying Exercise 19, we deduce the following:

- The ring $\mathbf{A}_{\text{inf}}$ is ξ-adically complete and ξ-torsion free.
- The quotient ring $\mathcal{O}_K = \mathbf{A}_{\text{inf}}/(\xi)$ is p-adically complete and p-torsion free.

We next prove the following:

(a) For any element $y \in \mathcal{O}_K$, there exists an element $x \in \mathcal{O}_C^\flat$ such that y is a unit multiple of $x^\sharp$.

To prove (a), we may assume without loss of generality that $y \neq 0$. Since $\mathcal{O}_K$ is p-adically complete, we can write $y = p^n y'$ for some $y' \in \mathcal{O}_K$ which is not divisible by p. Replacing y by y', we may assume that y is not divisible by p. Since θ is surjective, we can choose $x \in \mathcal{O}_C^\flat$ such that $y \equiv x^\sharp \pmod p$. Then $x^\sharp$ is not divisible by p, so x is not divisible by π. We can therefore write $\pi = xx'$ for some $x' \in \mathfrak{m}_C^\flat$. We have $y = x^\sharp + \pi^\sharp w = x^\sharp(1 + x'^\sharp w)$ for some $w \in \mathcal{O}_K$. Since some power of x' is divisible by π in the ring $\mathcal{O}_C^\flat$, some power of $x'^\sharp$ is divisible by p in the ring $\mathcal{O}_K$. It follows that $1 + x'^\sharp w$ is an invertible element of $\mathcal{O}_K$ (with inverse given by the p-adically convergent sum $1 - x'^{\sharp}w + (x'^{\sharp}w)^2 - (x'^{\sharp}w)^3 + \cdots)$. This proves that y is a unit multiple of $x^{\sharp}$, as desired.

Note that the element x appearing in (a) is not uniquely determined: we are free to multiply it by any unit in $\mathcal{O}_C^{\flat}$. However, this is our only freedom:

(b) Let $x, x' \in \mathcal{O}_C^{\flat}$ be elements such that $x^{\sharp}$ is divisible by $x'^{\sharp}$ in $\mathcal{O}_K$. Then x is divisible by x' in $\mathcal{O}_C^{\flat}$: that is, we have $|x|_{C^{\flat}} \leq |x'|_{C^{\flat}}$.

Suppose otherwise. We then have $|x|_{C^{\flat}} > |x'|_{C^{\flat}}$, so we can write $x' = tx$ for some $t \in \mathfrak{m}_{C^{\flat}}$. Since x is not zero, it divides π^n for $n \gg 0$. Consequently, our assumption that $x^{\sharp}$ is a multiple of $x'^{\sharp}$ guarantees that $(\pi^n)^{\sharp}$ is a unit multiple of $(\pi^n t)^{\sharp}$ in $\mathcal{O}_K$. Since $\pi^{\sharp}$ is a unit multiple of p and $\mathcal{O}_K$ is p-torsion-free, it follows that $t^{\sharp}$ is a unit in $\mathcal{O}_K$. This is impossible, since the image of $t^{\sharp}$ is nilpotent in the ring $\mathcal{O}_K / p\, \mathcal{O}_K \simeq \mathcal{O}_C^{\flat} / \pi\, \mathcal{O}_C^{\flat}$.

We next claim:

(c) The ring $\mathcal{O}_K$ is an integral domain.

To prove (c), let y be any nonzero element of $\mathcal{O}_K$; we wish to show that y is not a zero divisor. By virtue of (a), we may assume that $y = x^{\sharp}$ for some nonzero element $x \in \mathcal{O}_C^{\flat}$. Then x divides π^n for some large n; we may therefore replace x by π^n. In this case, y is a unit multiple of p^n, which we have already seen is not a zero divisor in $\mathcal{O}_K$.

For each element $y \in \mathcal{O}_K$, let us define $|y|_K = |x|_{C^{\flat}}$, where x is any element of $\mathcal{O}_{C^{\flat}}$ satisfying $y = x^{\sharp} \cdot \text{unit}$. It follows from (a) and (b) that $|y|_K$ is well-defined. Moreover, we have obvious identities

$$|0|_K = 0 \qquad |1|_K = 1 \qquad |y \cdot z|_K = |y|_K \cdot |z|_K.$$

Moreover, it follows from (b) that for each $y, z \in \mathcal{O}_K$, we have $|y|_K \leq |z|_K$ if and only if y is divisible by z. This immediately implies that $|y + z|_K \leq \max(|y|_K, |z|_K)$. We can therefore extend $|\bullet|_K$ uniquely to a non-archimedean absolute value on the fraction field K of $\mathcal{O}_K$. Moreover, an element $\frac{y}{z}$ of K satisfies $|\frac{y}{z}|_K = \frac{|y|_K}{|z|_K} \leq 1$ if and only if $|y|_K \leq |z|_K$: that is, if and only if y is divisible by $z \in \mathcal{O}_K$. It follows that $\mathcal{O}_K$ is the valuation ring of K (with respect to the absolute value $|\bullet|_K$).

Note that $|p|_K = |\pi|_{C^{\flat}} < 1$, so that K has residue characteristic p. Moreover, since p is not a zero-divisor in $\mathcal{O}_K$, the field K has characteristic zero: that is, p is a pseudo-uniformizer of $\mathcal{O}_K$. Consequently, the assertion that $\mathcal{O}_K$ is p-adically complete guarantees that it is complete with respect to its absolute value. The maximal ideal $\mathfrak{m} \subseteq \mathcal{O}_K$ is not generated by p: for example, it contains the element $(\pi^{1/p})^{\sharp}$, which is not divisible by p. Finally, we note that the isomorphisms

$$\mathcal{O}_K / p\, \mathcal{O}_K \simeq \mathbf{A}_{\mathrm{inf}}/(\xi, p) \simeq (\mathbf{A}_{\mathrm{inf}}/p\mathbf{A}_{\mathrm{inf}})/(\xi) = \mathcal{O}_C^{\flat} / \pi\, \mathcal{O}_C^{\flat}$$

guarantee that the Frobenius map is surjective on $\mathcal{O}_K / p\, \mathcal{O}_K$. Consequently, K is a perfectoid field of characteristic zero which is an untilt of C (Remark 3). □

Lecture 4: Holomorphic Functions of the Variable p

October 5, 2018

Throughout this lecture, we let $C^\flat$ denote a perfectoid field of characteristic p and $\mathcal{O}_C^\flat$ its valuation ring. As in the previous lecture, we are interested in classifying *untilts* of $C^\flat$: that is, perfectoid fields K equipped with an isomorphism $\iota : C^\flat \simeq K^\flat$

Exercise 1. Let K be a field. Suppose we are given two non-archimedean absolute values $| \bullet |_K, | \bullet |'_K : K \to \mathbf{R}_{\geq 0}$. Show that the following conditions are equivalent:

(a) The absolute values $| \bullet |_K$ and $| \bullet |'_K$ determine the same topology on K (one given by the metric $d(x, y) = |x - y|_K$, the other by $d'(x, y) = |x - y|'_K$.

(b) The valuation rings $\mathcal{O}_K = \{x \in K : |x|_K \leq 1\}$ and $\mathcal{O}'_K = \{x \in K : |x|'_K \leq 1\}$ are the same.

(c) There exists a constant $\alpha > 0$ such that $|x|'_K = |x|_K^\alpha$ for all $x \in K$ (moreover, if the valuations $| \bullet |_K$ and $| \bullet |'_K$ are nontrivial, then α is unique).

In general, when speaking of a valued field K, we will assume only that the topology of K is given; the absolute value $| \bullet |_K$ itself is only well-defined up to a constant exponent. If K is of characteristic zero with residue characteristic p, then there is a canonical way to normalize the absolute value of K: we can demand that, on the subfield $\mathbf{Q} \subseteq K$, the absolute value $| \bullet |_K$ restricts to the usual p-adic absolute value (so that $|p|_K = \frac{1}{p}$). On the other hand, if K is given to us as an untilt of $C^\flat$ (and we have fixed a normalization for the absolute value on $C^\flat$), then there is another way to normalize the absolute value of K: we can equip K with the unique absolute value for which the isomorphism ι satisfies $|x|_{C^\flat} = |\iota(x)|_{K^\flat} = |\iota(x)^\sharp|_K$. In general, these normalizations are different. We can exploit this to construct an invariant:

Construction 2. Let (K, ι) be an untilt of $C^\flat$. We let $r(K, \iota)$ denote the real number given by $|p|_K$, where $| \bullet |_K$ has been normalized so that ι is compatible with absolute values.

Remark 3. The real number r depends on a choice of absolute value on $C^\flat$; modifying the absolute value by some exponent α has the effect of replacing r by r^α.

Note that we have $0 \leq r(K, \iota) < 1$. Moreover, up to isomorphism, there is only one untilt (K, ι) satisfying $r(K, \iota) = 0$: namely, the characteristic p untilt given by $K \simeq C^\flat$.

Heuristic Idea 4. The collection of all (isomorphism classes of) untilts of $C^\flat$ behaves in some respects like the unit disk $\{z \in \mathbf{C} : |z| < 1\}$ in the complex plane. Here the function $(K, \iota) \mapsto r(K, \iota) = |p|_K$ is analogous to the function $z \mapsto |z|$, and the characteristic p untilt $(C^\flat, \mathrm{id})$ is analogous to the complex number 0.

This heuristic is supported by some of the constructions of the previous lecture. Recall that $\mathbf{A}_{\mathrm{inf}}$ denotes the ring of Witt vectors $W(\mathcal{O}_C^\flat)$. Since $\mathcal{O}_C^\flat$ is a perfect ring of characteristic p, every element $x \in \mathbf{A}_{\mathrm{inf}}$ admits a unique Teichmüller expansion

$$x = [c_0] + [c_1]p + [c_2]p^2 + [c_3]p^3 + \cdots$$

It is useful to think of the elements of $\mathbf{A}_{\text{inf}}$ can be viewed as "power series in the variable p" which can be evaluated in any untilt K of $C^\flat$ by applying the homomorphism $\theta : \mathbf{A}_{\text{inf}} \twoheadrightarrow \mathcal{O}_K$ of the previous lecture, thereby producing an element

$$\theta(x) = c_0^\sharp + c_1^\sharp p + c_2^\sharp p^2 + \cdots \in \mathcal{O}_K,$$

just as a power series $f(z) = \sum_{n \geq 0} c_n z^n$ with complex coefficients satisfying $|c_n| \leq 1$ can be evaluated at any complex number z satisfying $|z| < 1$.

It will be useful to consider various enlargements of the ring $\mathbf{A}_{\text{inf}}$. Fix a quasi-uniformizer $\pi \in C^\flat$: that is, an element of $C^\flat$ satisfying $0 < |\pi|_{C^\flat} < 1$. For each untilt K of $C^\flat$, the image $\pi^\sharp$ is a quasi-uniformizer in K: that is, the map

$$\mathbf{A}_{\text{inf}} \to \mathcal{O}_K$$

extends to a map of localizations

$$\mathbf{A}_{\text{inf}}[\frac{1}{[\pi]}] \to \mathcal{O}_K[\frac{1}{\pi^\sharp}] = K.$$

Note that the localization on the left hand side is independent of π. Concretely, each element of the localization $\mathbf{A}_{\text{inf}}$ admits a unique Teichmüller expansion

$$\sum_{n \geq 0} [c_n] p^n,$$

where each c_n belongs to $C^\flat$ and the sequence $\{|c_n|\}_{n \geq 0}$ is bounded; here $\sum_{n \geq 0} [c_n] p^n$ is defined as the fraction

$$\frac{\sum_{n \geq 0} [c_n \pi^m] p^n}{[\pi^m]}$$

for m sufficiently large.

If we are interested in functions that we can evaluate only on *characteristic zero* untilts of $C^\flat$, there is a further enlargement we can make: for each characteristic zero untilt K of $C^\flat$, the map

$$\mathbf{A}_{\text{inf}}[\frac{1}{[\pi]}] \to K$$

factors through the localization $\mathbf{A}_{\text{inf}}[\frac{1}{[\pi]}, \frac{1}{p}]$. Once again, elements of this localization have a concrete description: they admit Teichmüller expansions

$$\sum_{-\infty < n < \infty} [c_n] p^n,$$

where we allow negative powers of p but require that $c_n = 0$ for $n \ll 0$ (and that the set $\{|c_n|\}$ is bounded).

We can extend Heuristic 4 as follows:

Perfectoid Geometry	Complex Analysis
$\{\mathrm{Untilts}(K,\iota)\}/\sim$	Open disk $\{z \in \mathbf{C} : \lvert z\rvert < 1\}$
Real number $r(K,\iota) = \lvert p\rvert_K$	Real number $\lvert z\rvert$
Prime number p	Coordinate function z
Period ring $\mathbf{A}_{\mathrm{inf}}$	Power series $\sum_{n\geq 0} c_n z^n$, $\lvert c_n\rvert \leq 1$
$\mathbf{A}_{\mathrm{inf}}[\frac{1}{p}]$	Laurent series $\sum_{n\geq -k} c_n z^n$, $\lvert c_n\rvert \leq 1$
$\mathbf{A}_{\mathrm{inf}}[\frac{1}{[\pi]}]$	Power series $\sum_{n\geq 0} c_n z^n$, $\lvert c_n\rvert$ bounded
$\mathbf{A}_{\mathrm{inf}}[\frac{1}{p}, \frac{1}{[\pi]}]$	Laurent series $\sum_{n\geq -k} c_n z^n$, $\lvert c_n\rvert$ bounded

The collections of power series that appear on the right hand side are a bit strange looking (in the setting of complex analysis). Note if $f(z) = \sum_{n\geq 0} c_n z^n$ is a power series with complex coefficients, then the condition that the coefficients c_n are bounded is sufficient, but not necessary, to guarantee that $f(z)$ converges in the entire unit disk $\{z \in \mathbf{C} : |z| < 1\}$. However, the collection of such power series is not closed under addition or multiplication. It is better to look instead at the collection of power series

$$\{\sum_{n\geq 0} c_n z^n : \limsup |c_n|^{1/n} \leq 1\},$$

which are exactly the Taylor expansions of holomorphic functions on $\{z \in \mathbf{C} : |z| < 1\}$. Also, the Laurent series $\sum_{n\geq -k} c_n z^n$ appearing on the right hand side are actually *meromorphic* at $z = 0$. This has a natural enlargement, where we consider the ring of *all* holomorphic functions on the punctured unit disk $\{z \in \mathbf{C} : 0 < |z| < 1\}$. Our goal over the next few lectures will be to study an analogue of this ring of holomorphic functions in perfectoid geometry, which we will denote by B. The ring B will be a certain enlargement of the ring $\mathbf{A}_{\mathrm{inf}}[\frac{1}{p}, \frac{1}{[\pi]}]$, whose elements can be viewed heuristically as "holomorphic functions of p." Before giving a definition of B, let us give an example of the sort of function that we would like it to contain.

Exercise 5. Let K be a completely valued field of characteristic zero. Show that, for every element $x \in \mathfrak{m}_K$, the power series

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

converges.

Construction 6. Let e be an element of $\mathcal{O}_C^\flat$ satisfying $|e-1|_{C^\flat} < 1$. Then, for each untilt K of $C^\flat$, the image $e^\sharp$ satisfies $|e^\sharp - 1|_K < 1$. If K has characteristic zero, then it follows from Exercise 5 that the logarithm

$$\log(e^\sharp) = \log(1 + (e^\sharp - 1)) = \sum_{n>0} (-1)^{n+1} \frac{(e^\sharp - 1)^n}{n}$$

is a well-defined element of K.

It is tempting to denote the function $(K,\iota) \mapsto (\log(e^\sharp) \in K)$ by $\log([e])$. However, the expression $\log([e])$ has no meaning in the ring of Witt vectors $\mathbf{A}_{\mathrm{inf}}$, or even in the localization $\mathbf{A}_{\mathrm{inf}}[1/p, \frac{1}{[\pi]}]$; it has an "essential singularity" at $p = 0$.

The definition of the ring B is somewhat complicated. It will be defined as a certain inverse limit of rings $B_{[a,b]}$, where a and b are (certain) real numbers satisfying $0 < a \leq b < 1$. The idea is that $B_{[a,b]}$ should consist of "holomorphic functions of p" which are defined on the strip $a \leq |p|_K \leq b$.

Construction 7. Let a and b be real numbers satisfying $0 < a \leq b < 1$, and suppose that a and b belong to the *value group* of the field $C^\flat$: that is, there exist elements $\pi_a, \pi_b \in C^\flat$ satisfying $|\pi_a|_{C^\flat} = a$ and $|\pi_b|_{C^\flat} = b$ (these elements then belong to the maximal ideal $\mathfrak{m}_C^\flat$, and are well-defined up to multiplication by units in $\mathcal{O}_C^\flat$).

We let $\mathbf{A}_{\inf}[\frac{[\pi_a]}{p}, \frac{p}{[\pi_b]}]$ denote the subring of $\mathbf{A}_{\inf}[\frac{1}{p}, \frac{1}{[\pi]}]$ generated by $\mathbf{A}_{\inf}$ together with the elements $\frac{[\pi_a]}{p}$ and $\frac{p}{[\pi_b]}$. Note that this subring depends only on the real numbers a and b, and not on the elements π_a and π_b.

Let $\widehat{\mathbf{A}_{\inf}[\frac{[\pi_a]}{p}, \frac{p}{[\pi_b]}]}$ denote the p-adic completion of $\mathbf{A}_{\inf}[\frac{[\pi_a]}{p}, \frac{p}{[\pi_b]}]$ in the usual algebraic sense: that is, the ring

$$\varprojlim_n \mathbf{A}_{\inf}[\frac{[\pi_a]}{p}, \frac{p}{[\pi_b]}]/p^n \mathbf{A}_{\inf}[\frac{[\pi_a]}{p}, \frac{p}{[\pi_b]}].$$

We define $B_{[a,b]} = \widehat{\mathbf{A}_{\inf}[\frac{[\pi_a]}{p}, \frac{p}{[\pi_b]}]}[\frac{1}{p}]$.

Remark 8. Note that since $\mathbf{A}_{\inf}[\frac{[\pi_a]}{p}, \frac{p}{[\pi_b]}]$ contains the element $x = \frac{p}{[\pi_b]}$ satisfying $x \cdot [\pi_b] = p$, the element $[\pi_b]$ becomes invertible in the ring $B_{[a,b]}$. Since π divides some power of π_b, the element $[\pi]$ also becomes invertible in $B_{[a,b]}$. Consequently, we can view $B_{[a,b]}$ as an algebra over $\mathbf{A}_{\inf}[\frac{1}{p}, \frac{1}{[\pi]}]$. In fact, $B_{[a,b]}$ can be viewed as a completion of $\mathbf{A}_{\inf}[\frac{1}{p}, \frac{1}{[\pi]}]$ with respect to a certain Banach norm; we will return to this viewpoint in the next lecture.

Remark 9. Let K be an untilt of $C^\flat$, and suppose that $a \leq |p|_K \leq b$ (so that, in particular, K has characteristic zero). It follows that $|\pi_a^\sharp|_K \leq |p|_K \leq |\pi_b^\sharp|_K$, so that $\frac{\pi_a^\sharp}{p}$ and $\frac{p}{\pi_b^\sharp}$ belong to $\mathcal{O}_K$. Consequently, the canonical map $\mathbf{A}_{\inf}[\frac{1}{p}, \frac{1}{[\pi]}] \to K$ carries the subring $\mathbf{A}_{\inf}[\frac{[\pi_a]}{p}, \frac{p}{[\pi_b]}]$ into $\mathcal{O}_K$. Since $\mathcal{O}_K$ is p-adically complete, this map extends over the completion $\widehat{\mathbf{A}_{\inf}[\frac{[\pi_a]}{p}, \frac{p}{[\pi_b]}]}$. Inverting p, we obtain a map

$$B_{[a,b]} = \widehat{\mathbf{A}_{\inf}[\tfrac{[\pi_a]}{p}, \tfrac{p}{[\pi_b]}]}[\frac{1}{p}] \to \mathcal{O}_K[\frac{1}{p}] = K.$$

These maps fit into a commutative diagram

$$\begin{tikzcd}
\mathbf{A}_{\inf} \arrow[r, hook] \arrow[dr, hook] & \mathbf{A}_{\inf}[\frac{[\pi_a]}{p}, \frac{p}{[\pi_b]}] \arrow[r] \arrow[d, hook] & \widehat{\mathbf{A}_{\inf}[\frac{[\pi_a]}{p}, \frac{p}{[\pi_b]}]} \arrow[r] \arrow[d, hook] & \mathcal{O}_K \arrow[d] \\
& \mathbf{A}_{\inf}[\frac{1}{p}, \frac{1}{[\pi]}] \arrow[r] & B_{[a,b]} \arrow[r] & K.
\end{tikzcd}$$

Remark 10. Suppose we are given another pair of real numbers $0 < a' \leq b' < 1$ belonging to the value group of $C^\flat$, so that we can write $a' = |\pi_{a'}|_{C^\flat}$ and $b' = |\pi_{b'}|_{C^\flat}$. Suppose that the closed interval $[a', b']$ is contained in $[a, b]$: that is, we have $a \leq a' \leq b' \leq b$. Then π_a is divisible by $\pi_{a'}$, and $\pi_{b'}$ is divisible by π_b. It follows that the ring $\mathbf{A}_{\inf}[\frac{[\pi_a]}{p}, \frac{p}{[\pi_b]}]$ is contained in $\mathbf{A}_{\inf}[\frac{[\pi_{a'}]}{p}, \frac{p}{[\pi_{b'}]}]$ (where we regard both as subrings of $\mathbf{A}_{\inf}[\frac{1}{p}, \frac{1}{[\pi]}]$). Passing to p-adic completions, we obtain a map

$$\widehat{\mathbf{A}_{\inf}[\tfrac{[\pi_a]}{p}, \tfrac{p}{[\pi_b]}]} \to \widehat{\mathbf{A}_{\inf}[\tfrac{[\pi_{a'}]}{p}, \tfrac{p}{[\pi_{b'}]}]}.$$

Inverting p, we get a ring homomorphism $B_{[a,b]} \to B_{[a',b']}$.

We can now define the object that we are interested in.

Definition 11. Let B denote the inverse limit $\varprojlim B_{[a,b]}$, taken over all closed intervals $[a,b] \subseteq (0,1)$ such that a and b belong to the value group of $C^\flat$.

Lecture 5: Norms

October 29, 2018

Our goal in this lecture is to describe another way of thinking about some of the rings appearing in the previous lecture. First, we review some topological algebra.

Definition 1. Let V be a topological vector space over $\mathbf{Q}_p$. We say that V is a *p-adic Banach space* if there exists an open $\mathbf{Z}_p$-submodule $V_0 \subseteq V$, which is closed under addition, such that V_0 is p-adically complete as an abelian group and satisfies $V = V_0[\frac{1}{p}]$.

Example 2. Fix a non-archimedean norm $| \bullet |_{\mathbf{Q}_p}$ on $\mathbf{Q}_p$, compatible with the usual topology. For example, we can take the usual p-adic norm, characterized by $|p|_{\mathbf{Q}_p} = \frac{1}{p}$; however, it will be convenient not to assume this.

Let V be a vector space over $\mathbf{Q}_p$. We define a *norm* on V to be a function $| \bullet |_V : V \to \mathbf{R}_{\geq 0}$ satisfying

$$|\lambda v|_V = |\lambda|_{\mathbf{Q}_p} \cdot |v|_V \qquad |v + w|_V \leq \max(|v|_V, |w|_V)$$

(this is sometimes called a *pre-norm*, with the term *norm* reserved for the case where $|v|_V = 0 \Rightarrow v = 0$).

Any norm on V equips V with the structure of a (pre)metric space, with metric $d(v, w) = |v - w|$. If V is separated and complete with respect to this metric, then it is a p-adic Banach space (take $V_0 = \{v \in V : |v|_V \leq 1\}$ to be the "unit ball" of V).

Remark 3. Every p-adic Banach space V can be obtained from the construction of Example 2. Let $V_0 \subseteq V$ be an open $\mathbf{Z}_p$-module which is p-adically complete. We can then define a map $| \bullet |_V : V \to \mathbf{R}_{\geq 0}$ by the formula

$$|v|_V = \inf\{|\lambda|_{\mathbf{Q}_p} : v \in \lambda \cdot V_0\};$$

this is a norm on V, having V_0 as the unit ball.

Example 4. Let V be a vector space over $\mathbf{Q}_p$ equipped with a pair of norms $| \bullet |_V$ and $| \bullet |'_V$ (possibly with respect to different choices of absolute value $| \bullet |_{\mathbf{Q}_p}$ and $| \bullet |'_{\mathbf{Q}_p}$. We can then regard V as a metric space with respect to the metric $d(v, w) = |v - w|_V + |v - w|'_V$. If V is complete with respect to this metric, then it is a p-adic Banach space (the intersection of unit balls $V_0 = \{v \in V : |v|_V \leq 1 \text{ and } |v|'_V \leq 1\}$ satisfies the requirements of Definition 1).

Alternatively, in the case $| \bullet |_{\mathbf{Q}_p} = | \bullet |'_{\mathbf{Q}_p}$ (which we can always arrange by raising to an appropriate power), we can equip V with the norm $v \mapsto |v|_V + |v|_{V'}$.

Example 5. Let K be any completely valued field of characteristic zero and residue characteristic p. Then K is a p-adic Banach space.

Example 6. Let V_0 be any abelian group which is p-adically complete and p-torsion free. Then V_0 has the structure of a module over the ring $\mathbf{Z}_p$, and the tensor product $V = \mathbf{Q}_p \otimes_{\mathbf{Z}_p} V_0 = V_0[\frac{1}{p}]$ can be regarded as a p-adic Banach space (by equipping it with the topology where the subsets $p^n V_0$ form a neighborhood basis of the identity).

Example 7. Let M be an abelian group which is p-torsion free. We can then apply the construction of Example 6 to the p-adic completion $\widehat{M} = \varprojlim M/p^n M$ to obtain a p-adic Banach space $\widehat{M}[\frac{1}{p}]$.

Example 8. Let V be a $\mathbf{Q}_p$-vector space equipped with a norm. Then the completion of V (as a metric space) is a $\mathbf{Q}_p$-Banach space.

Examples 7 and 8 are related. If V is a $\mathbf{Q}_p$-vector space equipped with a norm, then the unit ball $V_0 = \{v \in V : |v|_V \leq 1\}$ is a p-torsion free abelian group. The completion of V with respect to its norm can then be identified with $\widehat{V}_0[\frac{1}{p}]$, where $\widehat{V}_0$ is the p-adic completion of V_0.

Variant 9. *Suppose that V is equipped with a pair of norms $|\bullet|_V$ and $|\bullet|_{V'}$. Then the completion of V with respect to the metric $d(v,w) = |v-w|_V + |v-w|'_V$ is given by $\widehat{V}_0[\frac{1}{p}]$, where $V_0 = \{v \in V : |v|_V \leq 1, |v|'_V \leq 1\}$.*

Let us now turn to the example of interest to us. Fix a perfectoid field $C^\flat$, with valuation ring $\mathcal{O}_C^\flat$ and set $\mathbf{A}_{\text{inf}} = W(\mathcal{O}_C^\flat)$. Fix an element $\pi \in C^\flat$ satisfying $0 < |\pi|_{C^\flat} < 1$ and consider the localization $\mathbf{A}_{\text{inf}}[\frac{1}{p}, \frac{1}{[\pi]}]$. Every element of this ring admits a Teichmüller expansion

$$\sum_{n \gg -\infty} [c_n] p^n$$

where the coefficients $c_n \in C^\flat$ are bounded.

Definition 10. [Gauss Norms] Fix a real number $0 < \rho < 1$. For each element $f = \sum_{n \gg -\infty} [c_n] p^n \in \mathbf{A}_{\text{inf}}[\frac{1}{p}, \frac{1}{[\pi]}]$, we define

$$|f|_\rho = \sup\{|c_n|_{C^\flat} \cdot \rho^n\}.$$

Remark 11. In the situation of Definition 10, the real numbers $|c_n|_{C^\flat} \cdot \rho^n$ vanish for $n \ll 0$ and decay exponentially as $\rho \to \infty$. Consequently, the supremum $\sup\{|c_n|_{C^\flat} \cdot \rho^n\}$ is exists and is realized by finitely many values of n.

Notation 12. We let Y denote the set of all isomorphism classes of characteristic zero untilts (K, ι) of $C^\flat$. We will use the letter y to denote a typical point of Y, given by an untilt (K, ι) of $C^\flat$. For every such point y, we have a surjective ring homomorphism

$$\theta_y : \mathbf{A}_{\text{inf}} \to \mathcal{O}_K \qquad \sum_{n \geq 0} [c_n] p^n \mapsto \sum_{n \geq 0} c_n^\sharp p^n$$

which extends to a ring homomorphism $\mathbf{A}_{\text{inf}}[\frac{1}{p}, \frac{1}{[\pi]}] \to K$. We denote the value of this homomorphism on an element $f \in \mathbf{A}_{\text{inf}}[\frac{1}{p}, \frac{1}{[\pi]}]$ by $f(y) \in K$.

Given $0 < a \leq b < 1$, we let $Y_{[a,b]} \subseteq Y$ denote the subset consisting of those points $y = (K, \iota)$ satisfying $a \leq |p|_K \leq b$.

Remark 13. Let $y = (K, \iota)$ be a point of Y and let $\rho = |p|_K$. Then, for every element $f = \sum_{n \gg -\infty} [c_n] p^n \in \mathbf{A}_{\text{inf}}[\frac{1}{p}, \frac{1}{[\pi]}]$, we have

$$|f(y)|_K = |\sum_{n \gg -\infty} c_n^\sharp p^n|_K \leq \sup\{|c_n^\sharp|_K \cdot |p|_K^n\} = \sup\{|c_n|_{C^\flat} \cdot \rho^n\} = |f|_\rho.$$

Remark 14. Let $f = \sum_{n \gg -\infty} [c_n] p^n$ be an element of $\mathbf{A}_{\text{inf}}[\frac{1}{p}, \frac{1}{[\pi]}$. We will say that a real number $\rho \in (0, 1)$ is *generic for* f if the supremum $\sup\{|c_n|_{C^\flat} \rho^n\}$ is achieved exactly once. That is, ρ is generic for f if there is an integer n such that $|f|_\rho = |c_n|_{C^\flat} \rho^n$, and for all integers $m \neq n$ we have $|c_m|_{C^\flat} \rho^m < |f|_\rho$. In this case, if $y = (K, \iota)$ is a point of Y satisfying $|p|_K = \rho$, the inequality of Remark 13 can be replaced by an equality $|f|_\rho = |f(y)|_K$.

Exercise 15. Let f be an element of $\mathbf{A}_{\inf}[\frac{1}{p}, \frac{1}{[\pi]}]$. Show that the set

$$\{\rho \in (0,1) : \ \rho \text{ is not generic for } f\ \}$$

is a discrete subset of $(0,1)$. In other words, if ρ is not generic for f, then $\rho \pm \epsilon$ *will* be generic for f for all sufficiently small $\epsilon \neq 0$.

Proposition 16. *For each $0 < \rho < 1$, the map $|\bullet|_\rho : \mathbf{A}_{\inf}[\frac{1}{p}, \frac{1}{[\pi]}] \to \mathbf{R}_{\geq 0}$ is a norm (in the sense of Example 2), compatible with the norm on $\mathbf{Q}_p$ satisfying $|p|_{\mathbf{Q}_p} = \rho$.*

Proof. We first show that $|f+g|_\rho \leq \max(|f|_\rho, |g|_\rho)$. Write $f + g = \sum_{n \gg -\infty} [c_n] p^n$. Suppose that the following conditions are satisfied:

($*$) The real number ρ is generic for f and belongs to the value group of $C^\flat$.

In this case, we can choose a point $y = (K, \iota) \in Y$ satisfying $|p|_K = \rho$ (for example, by taking $\mathcal{O}_K = \mathbf{A}_{\inf}/([c] - p)$, where $c \in C^\flat$ is any element satisfying $|c|_{C^\flat} = \rho$). Remark 14 then gives

$$|f+g|_\rho = |(f+g)(y)|_K \leq \max(|f(y))|_K, |g(y)|_K) \leq \max(|f|_\rho, |g|_\rho).$$

It follows from Exercise 15 that the collection of real numbers ρ satisfying ($*$) is dense in $(0,1)$. Consequently, it follows by continuity that $|f+g|_\rho \leq \max(|f|_\rho, |g|_\rho)$ for all $\rho \in (0,1)$.

It follows for that each $f \in \mathbf{A}_{\inf}[\frac{1}{p}, \frac{1}{[\pi]}]$ and each integer n, we have $|nf|_\rho \leq |f|_\rho$. By a continuity argument, we conclude that $|\lambda f|_\rho \leq |f|_\rho$ for each $\lambda \in \mathbf{Z}_p$. If λ is an invertible element of $\mathbf{Z}_p$, then the same argument gives $|f|_\rho \leq |\lambda f|_\rho$, so that $|\lambda f|_\rho = |f|_\rho = |\lambda|_{\mathbf{Q}_p} \cdot |f|_\rho$. Since every nonzero element of $\mathbf{Q}_p$ factors as $p^n u$, where u is an invertible element of $\mathbf{Z}_p$, we are reduced to checking the identity $|\lambda f|_\rho = |\lambda|_{\mathbf{Q}_p} \cdot |f|_\rho$ in the case $\lambda = p$: that is, the identity $|pf|_\rho = \rho \cdot |f|_\rho$. This follows immediately from the definition. $\square$

Variant 17. *For every pair of elements $f, g \in \mathbf{A}_{\inf}[\frac{1}{p}, \frac{1}{[\pi]}]$, we have $|f \cdot g|_\rho = |f|_\rho \cdot |g|_\rho$.*

Proof. Assume first that the following condition is satisfied:

($*$) The element ρ is generic for f, g, and $f \cdot g$, and belongs to the value group of $C^\flat$.

As in the proof of Proposition 16, we can choose a point $y = (K, \iota) \in Y$ satisfying $|p|_K = \rho$. In this case, Remark 13 gives

$$|f \cdot g|_\rho = |(f \cdot g)(y)|_K = |f(y)|_K \cdot |g(y)|_K = |f|_\rho \cdot |g|_\rho.$$

We conclude by observing that the collection of real numbers $\rho \in (0,1)$ satisfying ($*$) is dense, so by continuity we have an equality $|fg|_\rho = |f|_\rho \cdot |g|_\rho$ for all $\rho \in (0,1)$. $\square$

Proposition 18. *Suppose that a and b belong to the value group of $C^\flat$, so that we can choose elements $\pi_a, \pi_b \in C^\flat$ satisfying $|\pi_a|_{C^\flat} = a$ and $|\pi_b|_{C^\flat} = b$. Then the intersection of unit balls*

$$V_0 = \{f \in \mathbf{A}_{\inf}[\frac{1}{p}, \frac{1}{[\pi]} : |f|_a \leq 1, |f|_b \leq 1\}$$

is the subring $\mathbf{A}_{\inf}[\frac{[\pi_a]}{p}, \frac{p}{[\pi_b]}]$ of the previous lecture.

Proof. It follows from Proposition 16 and Variant 17 that V_0 is a subring of $\mathbf{A}_{\inf}[\frac{1}{p}, \frac{1}{[\pi]}]$. This subring clearly contains $\mathbf{A}_{\inf}$: note that if $f = \sum_{n \geq 0} [c_n] p^n$ belongs to $\mathbf{A}_{\inf}$, then we automatically have

$$|f|_\rho = \sup_{n \geq 0} \{|c_n|_{C^\flat} \cdot \rho^n\} \leq 1$$

for any $0 < \rho < 1$. Moreover, it also contains $\frac{[\pi_a]}{p}$ and $\frac{p}{[\pi_b]}$, by virtue of the equalities

$$|\frac{[\pi_a]}{p}|_a = 1 \qquad |\frac{[\pi_a]}{p}|_b = \frac{a}{b} < 1$$

$$|\frac{p}{[\pi_b]}|_a = \frac{a}{b} < 1 \qquad |\frac{p}{[\pi_b]}|_b = 1.$$

This shows that $\mathbf{A}_{\inf}[\frac{[\pi_a]}{p}, \frac{p}{[\pi_b]}]$ is contained in V_0.

We now prove the reverse containment. Suppose that $f = \sum_{n \gg -\infty} [c_n] p^n$ satisfies $|f|_a \leq 1$ and $|f|_b \leq 1$; we wish to show that f belongs to $\mathbf{A}_{\inf}[\frac{[\pi_a]}{p}, \frac{p}{[\pi_b]}]$. By assumption, the absolute values $|c_n|_{C^\flat}$ are bounded above. We may therefore choose some integer $m \gg 0$ such that each product $\pi_b^m c_n$ belongs to $C^\flat$. We then have

$$f = (\sum_{n<m} [c_n] p^n) + (\sum_{n \geq 0} [c_{n+m} \pi_b^m] p^n)(\frac{p}{[\pi_b]})^m$$

where the second summand belongs to $\mathbf{A}_{\inf}[\frac{p}{[\pi_b]}]$ (and therefore also to the unit ball of $\mathbf{A}_{\inf}[\frac{[\pi_a]}{p}, \frac{p}{[\pi_b]}]$). Subtracting, we can reduce to the case where the Teichm"uller expansion of f is finite.

Our assumption that $|f|_a \leq 1$ and $|f|_b \leq 1$ guarantees that, for each integer n, we have

$$|c_n|_{C^\flat} \cdot a^n \leq 1 \qquad |c_n|_{C^\flat} \cdot b^n \leq 1,$$

so that $c_n \pi_a^n$ and $c_n \pi_b^n$ belong to $\mathcal{O}_C^\flat$. For $n \leq 0$, this implies that $[c_n] p^n = [c^n \pi_a^n](\frac{[\pi_a]}{p})^{-n}$ belongs to $\mathbf{A}_{\inf}[\frac{[\pi_a]}{p}]$. For $n \geq 0$, we instead learn that $[c_n] p^n = [c_n \pi_b^n](\frac{p}{[\pi_b]})^n$ belongs to $\mathbf{A}_{\inf}[\frac{p}{[\pi_b]}]$. It follows that f belongs to the ring $\mathbf{A}_{\inf}[\frac{[\pi_a]}{p}, \frac{p}{[\pi_b]}]$, as desired. $\square$

Corollary 19. *Suppose that a and b belong to the value group of $C^\flat$. Then the ring $B_{[a,b]}$ of the previous lecture can be identified with the completion of $\mathbf{A}_{\inf}[\frac{1}{p}, \frac{1}{[\pi]}]$ with respect to the pair of norms $| \bullet |_a$ and $| \bullet |_b$.*

We will henceforth use this Corollary to extend the definition of $B_{[a,b]}$ to the case where a and b do not necessarily belong to the value group of $C^\flat$). Note that if $y = (K, \iota) \in Y$ is an untilt satisfying $a \leq |p|_K \leq b$, then Remark 13 implies that the homomorphism

$$\mathbf{A}_{\inf}[\frac{1}{p}, \frac{1}{[\pi]}] \to K \qquad f \mapsto f(y)$$

admits a continuous extension $B_{[a,b]} \to K$, which we will also denote by $f \mapsto f(y)$.

Lecture 6: Definition of the Fargues-Fontaine Curve

October 29, 2018

Throughout this lecture, we fix a perfectoid field $C^\flat$ of characteristic p, with valuation ring $\mathcal{O}_C^\flat$. Fix an element $\pi \in C^\flat$ with $0 < |\pi|_{C^\flat} < 1$. We let $\mathbf{A}_{\inf}$ denote the ring of Witt vectors $W(\mathcal{O}_C^\flat)$. In the previous lecture, we defined the *Gauss norm* $|\bullet|_\rho : \mathbf{A}_{\inf}[\frac{1}{p}, \frac{1}{[\pi]}] \to \mathbf{R}_{\geq 0}$, for every real number $\rho \in (0,1)$. By definition, it is given by the formula $|\sum [c_n] p^n|_\rho = \sup\{|c_n|_{C^\flat} \cdot \rho^n\}$. For every pair of real numbers $0 < a \leq b < 1$, we let $B_{[a,b]}$ denote the completion of $\mathbf{A}_{\inf}[\frac{1}{p}, \frac{1}{[\pi]}]$ with respect to the pair of norms $|\bullet|_a$ and $|\bullet|_b$.

Exercise 1. Show that, for $0 < a \leq c \leq b < 1$, we have $|f|_c \leq \sup\{|f|_a, |f|_b\}$. Consequently, the completion of $\mathbf{A}_{\inf}[\frac{1}{p}, \frac{1}{[\pi]}]$ with respect to any finite collection of Gauss norms $|\bullet|_{\rho_0}, \cdots, |\bullet|_{\rho_n}$ is given $B_{[a,b]}$, where $a = \min\{\rho_i\}$ and $b = \max\{\rho_i\}$.

Recall that the ring B is defined as the inverse limit $\varprojlim B_{[a,b]}$, where $[a,b]$ ranges over the collection of all closed intervals contained in $(0,1)$. Equivalently, we can describe B as the completion of $\mathbf{A}_{\inf}[\frac{1}{p}, \frac{1}{[\pi]}]$ with respect to *all* of the Gauss norms $|\bullet|_\rho$ (for $0 < \rho < 1$). This inverse limit inherits a topology, and each of the norms $|\bullet|_\rho$ on $\mathbf{A}_{\inf}[\frac{1}{p}, \frac{1}{[\pi]}]$ admits a unique continuous extension to B (which we will also denote by $|\bullet|_\rho$). Moreover, a sequence $\{f_n\}_{n \geq 0}$ converges to $f \in B$ if and only if $\lim_{n \to \infty} |f - f_n|_\rho = 0$ for all $\rho \in (0,1)$. By virtue of Exercise 1, the collection of real numbers ρ which satisfies this condition is convex (so it suffices to check convergence for real numbers of the form $\frac{1}{N}$ and $\frac{N-1}{N}$, for example).

Warning 2. The ring B is a topological vector space over $\mathbf{Q}_p$, but it is *not* a p-adic Banach space: its topology cannot be defined by a single norm. It is instead an example of a *p-adic Frechet space*. However, it can still be regarded as a completion of $\mathbf{A}_{\inf}[\frac{1}{p}, \frac{1}{[\pi]}]$ in the following sense: every element $f \in B$ can be realized as the limit of a sequence $\{f_n\}$, where each f_n belongs to $\mathbf{A}_{\inf}[\frac{1}{p}, \frac{1}{[\pi]}]$. For example, we can take any sequence satisfying

$$|f - f_n|_{\frac{1}{n}} \leq \frac{1}{n} \qquad |f - f_n|_{1-\frac{1}{n}} \leq \frac{1}{n}$$

for $n > 1$.

Let us describe these completion a little bit more concretely. Let V be a $\mathbf{Q}_p$-vector space equipped with a non-archimedean norm $|\bullet|_V$. Suppose we are given a collection of vectors $\{v_i\}_{i \in I}$ in V with the property that, for every real number $\epsilon > 0$, we have $|v_i|_V < \epsilon$ for all but finitely many $i \in I$. In this case, the sum $\sum_{i \in I} v_i$ converges (absolutely) in the completion $\widehat{V}$ of V with respect to the norm $|\bullet|_V$.

Exercise 3. Let V be a $\mathbf{Q}_p$-vector space equipped with a norm $|\bullet|_V$, and suppose we are given a sequence of points $v_0, v_1, v_2, \ldots \in V$. Show that the following conditions are equivalent:

- The sequence $\{v_n\}_{n \geq 0}$ is a Cauchy sequence (with respect to the metric $d(v,w) = |v - w|_V$).
- $\lim_{n \to \infty} |v_n - v_{n-1}|_V = 0$.
- The sum $v_0 + \sum_{n > 0} (v_n - v_{n-1})$ is (absolutely) convergent in the completion $\widehat{V}$ of V.

If these conditions are satisfied, then the limit $\lim_{n\to\infty} v_n$ (in the completion $\widehat{V}$ of V) coincides with $v_0 + \sum_{n>0}(v_n - v_{n-1})$. Consequently, any element of $\widehat{V}$ can be written as an (absolutely convergent) sum of elements of V.

Variant 4. *Let $\widehat{V}$ be the completion of a $\mathbf{Q}_p$-vector space V is a $\mathbf{Q}_p$-vector space with respect to a pair of norms $|\bullet|_V$ and $|\bullet|_{V'}$. In this case, a sum $\sum_{i\in I} v_i$ converges in $\widehat{V}$ provided that* $\lim |v_i|_V = \lim |v_i|'_V = 0$.

Let us now specialize to the case of interest to us.

Example 5 (Teichmüller Expansions)**.** Suppose we are given a formal sum

$$\sum_{n\in\mathbf{Z}} [c_n] p^n,$$

where each c_n is an element of $C^\flat$. Then:

- The sum converges for the Gauss norm $|\bullet|_\rho$ if and only if

$$\lim_{n\to\infty} |c_n|_{C^\flat} \rho^n = 0 \qquad \lim_{n\to\infty} |c_{-n}|_{C^\flat} \rho^{-n} = 0.$$

- The sum converges in $B_{[a,b]}$ if and only if it converges for the Gauss norms $|\bullet|_a$ and $|\bullet|_b$. That is, if and only if we have

$$\lim_{n\to\infty} |c_n|_{C^\flat} b^n = 0 \qquad \lim_{n\to\infty} \frac{|c_{-n}|_{C^\flat}}{a^n} = 0.$$

- The sum converges in B if and only if it converges with respect to the Gauss norm $|\bullet|_\rho$ for *every* $\rho \in (0,1)$. This is equivalent ot the statement

$$\limsup_{n>0} |c_n|_{C^\flat}^{1/n} \leq 1 \qquad \lim_{n\to\infty} |c_{-n}|_{C^\flat}^{1/n} = 0.$$

Remark 6 (Complex-Analytic Analogue)**.** Let f be a holomorphic function defined on the punctured unit disk $D^\times = \{z \in \mathbf{C} : 0 < |z| < 1\}$. Then f admits a Laurent series expansion

$$f(z) = \sum c_n z^n,$$

where the coefficients c_n are complex numbers satisfying the conditions

$$\limsup_{n>0} |c_n|^{1/n} \leq 1 \qquad \lim_{n\to\infty} |c_{-n}|^{1/n} = 0.$$

Conversely, for any sequence $\{c_n\}_{n\in\mathbf{Z}}$ of complex numbers satisfying these conditions, the sum $\sum c_n z^n$ determines a holomorphic function on $D^\times$.

Warning 7. It follows from Example 5 that every collection $\{c_n\}_{n\in\mathbf{Z}}$ of elements of $C^\flat$ satisfying the conditions

$$\limsup_{n>0} |c_n|_{C^\flat}^{1/n} \leq 1 \qquad \lim_{n\to\infty} |c_{-n}|_{C^\flat}^{1/n} = 0$$

determines an element of the ring B, given by $\sum_{n\in\mathbf{Z}} [c_n] p^n$. However, it is not clear that every element of B can be represented in this way, or that such representations are unique when they exist.

Recall that $C^\flat$ is a perfect field of characteristic p, so the Frobenius map

$$\varphi : C^\flat \to C^\flat \qquad \varphi(c) = c^p$$

is an automorphism of $C^\flat$. This automorphism restricts to an automorphism of the valuation ring $\mathcal{O}_C^\flat$, and therefore induces an automorphism of the ring of Witt vectors $\mathbf{A}_{\mathrm{inf}} = W(\mathcal{O}_C^\flat)$. We will denote both of these automorphisms also by φ. Note that $\varphi([\pi]) = [\pi]^p$, so that inverting $[\pi]$ has the same effect as inverting $\varphi([\pi])$. Consequently, the Frobenius automorphism of $\mathbf{A}_{\mathrm{inf}}$ extends to an automorphism of $\mathbf{A}_{\mathrm{inf}}[\frac{1}{p}, \frac{1}{[\pi]}]$, which we will again denote by φ. On Teichmüller expansions, it is given by the formula

$$\varphi(\sum_{n \gg -\infty} [c_n] p^n) = \sum_{n \gg -\infty} [c_n^p] p^n.$$

In particular, we have

$$|\varphi(\sum_{n \gg -\infty} [c_n] p^n)|_\rho = \sup\{|c_n|_{C^\flat}^p \rho^n\} = (\sup\{|c_n|_{C^\flat} \rho^{n/p}\})^p = |\sum_{n \gg -\infty} [c_n] p^n|_{\rho^{1/p}}^p,$$

which we can write more simply as

$$|\varphi(f)|_{\rho^p} = (|f|_\rho)^p.$$

It follows that the automorphism φ of $\mathbf{A}_{\mathrm{inf}}[\frac{1}{p}, \frac{1}{[\pi]}]$ extends to an isomorphism $B_{[a,b]} \simeq B_{[a^p, b^p]}$. Passing to the inverse limit over all intervals $[a,b] \subseteq (0,1)$, we obtain an automorphism of the ring B, which we will (once again) denote by φ.

Notation 8. For every integer n, we let $B^{\varphi = p^n}$ denote the subset of B consisting of those elements x satisfying $\varphi(x) = p^n x$.

Note that if f belongs to $B^{\varphi=p^m}$ and g belongs to $B^{\varphi=p^n}$, then we have $\varphi(fg) = \varphi(f) \cdot \varphi(g) = (p^m f) \cdot (p^n g) = p^{n+m} fg$, so that fg belongs to $B^{\varphi = p^{n+m}}$. It follows that we can regard the sum

$$\bigoplus_{n \in \mathbf{Z}} B^{\varphi = p^n}$$

as a graded ring. We can now finally define the main object of study in this course:

Definition 9. The *Fargues-Fontaine curve* is the scheme $\mathrm{Proj}(\bigoplus_{n \geq 0} B^{\varphi = p^n})$.

To get a feeling for what is going on, let's try to write down some elements of the graded ring $\bigoplus_{n \in \mathbf{Z}} B^{\varphi = p^n}$. Suppose that f is an element of B which admits a convergent Teichmüller expansion

$$f = \sum_{n \in \mathbf{Z}} [c_n] p^n,$$

so that

$$\limsup_{n>0} |c_n|^{1/n} \leq 1 \qquad \lim_{n \to \infty} |c_{-n}|^{1/n} = 0.$$

In this case, the elements $p^k f$ and $\varphi(f)$ also admits convergent Teichmüller expansions

$$p^k f = \sum_{n \in \mathbf{Z}} [c_n] p^{n+k} = \sum_{n \in \mathbf{Z}} [c_{n-k}] p^n$$

$$\varphi(f) = \sum_{n \in \mathbf{Z}} [c_n^p] p^n.$$

Consequently, to satisfy the equation $\varphi(f) = p^k f$, it is sufficient (but perhaps not necessary) to have a termwise equality of Teichm"uller expansions $c_{n-k} = c_n^p$.

Example 10. Suppose that $k < 0$. Then, for each $n \in \mathbf{Z}$, the sequence

$$c_{n+k} = c_n^{1/p}, c_{n+2k} = c_n^{1/p^2}, c_{n+3k} = c_n^{1/p^3}, \cdots$$

is required to converge to zero. It follows that $c_n = 0$ for all n. In other words, there are no "obvious" nonzero elements of $B^{\varphi=p^k}$ for $k < 0$. (We will see in Lecture 11 that there are no nonzero elements at all: that is, the ring $\bigoplus_{n\in\mathbf{Z}} B^{\varphi=p^n}$ is nonnegatively graded.)

Example 11. Suppose that $k = 0$. In this case, for a Teichmüller expansion $\sum_{n\in\mathbf{Z}}[c_n]p^n$ to represent an element of $B^{\varphi=p^k}$, it is sufficient to have $c_n = c_n^p$ for all n: that is, each coefficient belongs to the subfield $\mathbf{F}_p \subseteq C^\flat$. In this case, the convergence condition on the coefficients c_n just demands that $c_n = 0$ for $n \ll 0$. These are exactly the Teichmüller expansions of elements of $\mathbf{Q}_p = W(\mathbf{F}_p)[\frac{1}{p}]$. We therefore obtain a map $\mathbf{Q}_p \to B^{\varphi=p^0}$. We will see later that this map is an isomorphism.

Example 12. Suppose that $k > 0$. In this case, the condition $c_{n-k} = c_n^p$ shows that the entire sequence is determined by a finite number of terms $c_0, c_1, \ldots, c_{k-1}$. Moreover, for the Teichmüller expansion to converge, each of these coefficients must belong to $\mathfrak{m}_C^\flat$. Via this procedure, we can write down a large number of elements of $B^{\varphi=p^k}$ (beware that it is not clear if these elements are distinct, or if all elements of $B^{\varphi=p^k}$ can be obtained in this way).

Example 13. In the case $k = 1$, we see that every element $c \in \mathfrak{m}_C^\flat$ determines an element of $B^{\varphi=p}$, given by the formula $\sum_{n\in\mathbf{Z}}[c^{1/p^n}]p^n$. We will study these elements in the next lecture.

Remark 14. Note that every element of the ring $\mathbf{A}_{\mathrm{inf}}[\frac{1}{p}, \frac{1}{[\pi]}]$ admits a *unique* Teichmüller expansion $\sum_{n\gg-\infty}[c_n]p^n$, and therefore belongs to $\mathbf{A}_{\mathrm{inf}}[\frac{1}{p}, \frac{1}{[\pi]}]^{\varphi=p^k}$ if and only if $c_{n-k} = c_n^p$ for all n. If $k \neq 0$, the vanishing of c_n for $n \ll 0$ implies the vanishing of c_n all n. In other words, the graded ring

$$\bigoplus_n \mathbf{A}_{\mathrm{inf}}[\frac{1}{p}, \frac{1}{[\pi]}]^{\varphi=p^n}$$

is just the field $\mathbf{Q}_p$. To obtain interesting elements of the ring $\bigoplus_{n\in\mathbf{Z}} B^{\varphi=p^n}$, it is important to complete the vector space $\mathbf{A}_{\mathrm{inf}}[\frac{1}{p}, \frac{1}{[\pi]}]$ by allowing "essential singularities at $p = 0$."

Lecture 7: The Artin-Hasse Exponential

October 12, 2018

Throughout this lecture, we fix a perfectoid field $C^\flat$ of characteristic p, with valuation ring $\mathcal{O}_C^\flat$. Fix an element $\pi \in C^\flat$ with $0 < |\pi|_{C^\flat} < 1$, and let B denote the completion of $\mathbf{A}_{\text{inf}}[\frac{1}{p}, \frac{1}{[\pi]}]$ with respect to the family of Gauss norms $|\bullet|_\rho$ for $0 < \rho < 1$. In the previous lecture, we showed that for each element $a \in \mathfrak{m}_C^\flat$, the infinite sum

$$\sum_{n\in\mathbf{Z}} \frac{[a^{p^n}]}{p^n}$$

converges to an element $x \in B$ satisfying $\varphi(x) = px$: that is, it is an element of the Frobenius eigenspace $B^{\varphi=p}$. We can now ask the following questions:

(1) Does every element of the eigenspace $B^{\varphi=p}$ have the form $\sum_{n\in\mathbf{Z}} \frac{[a^{p^n}]}{p^n}$ for some element $a \in \mathfrak{m}_C^\flat$? If so, is the element a uniquely determined?

(2) Note that $B^{\varphi=p}$ is a vector space over $\mathbf{Q}_p$. Is the collection of elements of the form $\sum_{n\in\mathbf{Z}} \frac{[a^{p^n}]}{p^n}$ closed under addition? If so, how do we add them?

Note that if every element of B were to admit a *unique* Teichmüller expansion, then the analysis of the previous lecture would give an affirmative answer to Question (1). We will eventually show that the answer to Question (1) is "yes" (even though we do not have existence and uniqueness results for Teichmüller expansions in general), but this will need to wait until we know a little bit more about the ring B. Our goal in this lecture is to show that, even without knowing the answer to (1), we can nevertheless answer Question (2) by describing the construction $a \mapsto \sum_{n\in\mathbf{Z}} \frac{[a^{p^n}]}{p^n}$ in a different way.

Exercise 1. Let A be an algebra over $\mathbf{Q}_p$ equipped with a norm $|\bullet|_A$ satisfying the condition

$$|x \cdot y|_A \leq |x|_A \cdot |y|_A.$$

Let $x \in A$ be an element satisfying $|x - 1|_A < 1$. Show that the infinite sum

$$\log(x) = \sum_{k>0} \frac{(-1)^{k+1}}{k}(x-1)^k$$

is a well-defined element of the completion $\widehat{A}$ (that is, the individual terms $\frac{(-1)^{k+1}}{k}(x-1)^k$ converge to zero as $k \to \infty$).

Assume that A is commutative, and let $y \in A$ be another element satisfying $|y - 1|_A < 1$. Show that xy satisfies $|xy - 1|_A < 1$ and $\log(xy) = \log(x) + \log(y)$ (in the completion $\widehat{A}$).

Example 2. Let x be an element of $C^\flat$ satisfying $|x - 1|_{C^\flat} < 1$. Note that $[x] - 1$ is an element of the ring $\mathbf{A}_{\text{inf}} = W(\mathcal{O}_C^\flat)$, and therefore admits a Teichmüller expansion

$$[x] - 1 = \sum_{n\geq 0} [c_n] p^n$$

where the coefficients c_n satisfy $|c_n|_{C^\flat} \leq 1$. Moreover, we have $c_0 = x - 1$, so that $|c_0|_{C^\flat} < 1$. For each real number $\rho \in (0,1)$, we have

$$|[x] - 1|_\rho = \sup\{|c_n|_{C^\flat}\rho^n\} \leq \max(|c_0|_{C^\flat}, \rho) < 1.$$

Applying Exercise 1, we conclude that the series

$$\log([x]) = \sum_{k>0} \frac{(-1)^{k+1}}{k}([x]-1)^k$$

converges with respect to the Gauss norm $|\bullet|_\rho$. Since ρ is arbitrary, it follows that $\log([x])$ is a well-defined element of the ring B. Moreover, if y is another element of $C^\flat$ satisfying $|y-1|_{C^\flat} < 1$, we have an identity

$$\log([xy]) = \log([x][y]) = \log([x]) + \log([y]).$$

Remark 3. For each $x \in 1 + \mathfrak{m}_C^\flat$, we have

$$\varphi(\log([x])) = \log(\varphi([x])) = \log([x^p]) = p\log([x]).$$

That is, $\log([x])$ actually belongs to the eigenspace $B^{\varphi=p} \subseteq B$.

We now have two explicit procedures for producing elements of the vector space $B^{\varphi=p}$: by forming Teichmüller expansions $\sum_{n\in\mathbf{Z}} \frac{[a^{p^n}]}{p^n}$ (which converge for elements $a \in \mathfrak{m}_C^\flat$ which are "close to zero"), and by forming logarithms $\log([x])$ (which converge for elements $x \in 1 + \mathfrak{m}_C^\flat$ which are "close to one"). We will show that these procedures produce the same elements.

Theorem 4. *There exists a commutative diagram of sets*

$$\begin{tikzcd} & 1+\mathfrak{m}_C^\flat \arrow[dr, "\log[\bullet]"] & \\ \mathfrak{m}_C^\flat \arrow[ur, "E"] \arrow[rr, "a\mapsto\sum\frac{[a^{p^n}]}{p^n}"] & & B^{\varphi=p}, \end{tikzcd}$$

where E is bijective.

Corollary 5. *The collection of elements of B of the form $\sum_{n\in\mathbf{Z}}\sum_{n\in\mathbf{Z}} \frac{[a^{p^n}]}{p^n}$ is closed under addition.*

Let's assume for a moment that Theorem 4 is true, and try to guess the nature of the function E. It follows from Theorem 4 that we can define an addition law $\bigoplus$ on the set $\mathfrak{m}_C^\flat$, having the property that

$$\sum_{n\in\mathbf{Z}} \frac{[(a \bigoplus b)^{p^n}]}{p^n} = (\sum_{n\in\mathbf{Z}} \frac{[a^{p^n}]}{p^n}) + (\sum_{n\in\mathbf{Z}} \frac{[b^{p^n}]}{p^n}).$$

Namely, we define $a \oplus b = E^{-1}(E(a)\cdot E(b))$, so that we have an identity $E(a\oplus b) = E(a)\cdot E(b)$. We can therefore think of E as something like an exponential map, which relates the modified addition law $\bigoplus$ on $\mathfrak{m}_C^\flat$ (related to the addition of Teichmüller expansions) to the usual multiplication on $1+\mathfrak{m}_C^\flat$.

Lemma 6. *Let $\exp(x) = \sum_{n\geq 0}\frac{x^n}{n!}$ be the power series for the exponential function, regarded as an element of $\mathbf{Q}[[x]]$. Then we have an identity of formal power series*

$$\exp(x) = \prod_{d>0}(\frac{1}{1-x^d})^{\frac{\mu(d)}{d}}.$$

Here μ denotes the Möbius function

$$\mu(d) = \begin{cases} (-1)^n & \text{if } d = p_1\cdots p_n \text{ for distinct primes } p_1, p_2, \ldots, p_n \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Taking the logarithm of the right hand side yields

$$\begin{aligned}
\log(\prod_{d>0}(\frac{1}{1-x^d})^{\frac{\mu(d)}{d}} &= \sum_{d>0}\log((\frac{1}{1-x^d})^{\frac{\mu(d)}{d}}) \\
&= \sum_{d>0}\frac{\mu(d)}{d}\log(\frac{1}{1-x^d}) \\
&= \sum_{d>0}\frac{\mu(d)}{d}\sum_{d'>0}\frac{x^{d'd}}{d'} \\
&= \sum_{n>0}\frac{x^n}{n}\sum_{d|n}\mu(d) \\
&= x.
\end{aligned}$$

where the final equality follows from the identity

$$\sum_{d|n}\mu(d) = \begin{cases} 1 & \text{if } n=1 \\ 0 & \text{otherwise.}\end{cases} \tag{1}$$

□

The exponential function $x \mapsto \exp(x)$ has good convergence properties over the real numbers (where the coefficients $\frac{1}{n!}$ are small), but much weaker convergence properties when working p-adically (where the coefficients $\frac{1}{n!}$ are large). However, we can get a function with better p-adic behavior by "leaving out" the problematic terms in the product decomposition of Lemma 6.

Definition 7. Fix a prime number p. The *Artin-Hasse exponential* $E(x)$ is the power series

$$E(x) = \prod_{(d,p)=1}(\frac{1}{1-x^d})^{\frac{\mu(d)}{d}},$$

where the product is taken over the collection of all positive integers d which are relatively prime to p.

Note that the coefficients of this power series are integral at p: that is, we can think of $E(x)$ as a power series with coefficients in the subring $\mathbf{Z}_{(p)} \subseteq \mathbf{Q}$, given by (in contrast with the usual exponential series $\exp(x)$).

Exercise 8. Show that, as a formal power series with rational coefficients, the Artin-Hasse exponential $E(x)$ is given by the formula

$$E(x) = \exp(x + \frac{x^p}{p} + \frac{x^{p^2}}{p^2} + \cdots).$$

(Hint: take the logarithm of both sides and argue as in Lemma 6).

Since the power series $E(x) = 1 + x + \text{ higher order terms }$ has coefficients in $\mathbf{Z}_{(p)}$, the construction $a \mapsto E(a)$ determines a bijection from $\mathfrak{m}_C^\flat$ to $1+\mathfrak{m}_C^\flat$. We claim that this bijection satisfies the requirements of Theorem 4. In other words, we claim that for each element $a \in \mathfrak{m}_C^\flat$, we have an equality

$$\sum_{n\in\mathbf{Z}}\frac{[a^{p^n}]}{p^n} = \log([E(a)]) = \log([\prod_{(d,p)=1}(\frac{1}{1-a^d})^{\frac{\mu(d)}{d}}])$$

in the ring B. We will establish this identity by manipulation of formal series, and leave it to the reader to justify that our manipulations are legal (that is, that all of the infinite sums and products that we consider are convergent with respect to each of the Gauss norms).

We first recall that for any element $y \in \mathcal{O}_C^\flat$, the Teichmüller representative $[y] \in W(\mathcal{O}_C^\flat)$ can be computed as the limit $\varinjlim_{k \to \infty} \widetilde{y}_k^{p^k}$, where $\widetilde{y}_k$ is any element of $W(\mathcal{O}_C^\flat)$ lying over $y^{p^{-k}}$. In particular, for $x \in \mathfrak{m}$, we have

$$[1-x] = \lim_{k \to \infty} (1 - [x^{p^{-k}}])^{p^k}$$

$$\begin{aligned}
\log \frac{1}{[1-x]} &= \lim_{k \to \infty} p^k \log\left(\frac{1}{1-[x^{p^{-k}}]}\right) \\
&= \lim_{k \to \infty} p^k \sum_{m>0} \frac{[x^{mp^{-k}}]}{m} \\
&= \sum_{\alpha \in \mathbf{Z}[1/p], \alpha > 0} \frac{[x^\alpha]}{\alpha}.
\end{aligned}$$

We now write

$$\begin{aligned}
\log\left[\prod_{(d,p)=1} (1-a^d)^{\frac{-\mu(d)}{d}}\right] &= \sum_{(d,p)=1} \frac{\mu(d)}{d} \log \frac{1}{[1-a^d]} \\
&= \sum_{(d,p)=1} \sum_{\alpha \in \mathbf{Z}[1/p], \alpha>0} \mu(d) \frac{[a^{d\alpha}]}{d\alpha} \\
&= \sum_{\beta \in \mathbf{Z}[1/p], \beta > 0} \sum_{d} \mu(d) \frac{[a^\beta]}{\beta},
\end{aligned}$$

where, in the final expression, we write $\beta = p^n k$ for $(k,p) = 1$ and d ranges over all divisors of k. It follows from Equation (1) that this inner sum vanishes for $k \neq 1$: that is, we can neglect all values of β which are not powers of p. Doing so, we obtain the expression

$$\sum_{n \in \mathbf{Z}} \frac{[a^{p^n}]}{p^n},$$

as desired.

Lecture 8: The Field B_{dR}

October 29, 2018

Throughout this lecture, we fix a perfectoid field $C^\flat$ of characteristic p, with valuation ring $\mathcal{O}_C^\flat$. Fix an element $\pi \in C^\flat$ with $0 < |\pi|_{C^\flat} < 1$, and let B denote the completion of $\mathbf{A}_{\mathrm{inf}}[\frac{1}{p}, \frac{1}{[\pi]}]$ with respect to the family of Gauss norms $|\bullet|_\rho$ for $0 < \rho < 1$. Recall our heuristic picture: B is an analogue of the ring of holomorphic functions on the punctured unit disk $D^\times = \{z \in \mathbf{C} : 0 < |z| < 1\}$. In the previous lecture, we studied some elements of the eigenspace $B^{\varphi = p}$ which can be described in two equivalent ways:

- As convergent Teichmüller expansions $\sum \frac{[a^{p^n}]}{p^n}$, where $|a|_{C^\flat} < 1$.
- As logarithms $\log([x])$, where $|x - 1|_{C^\flat} < 1$.

We now study the "vanishing loci" of these objects, viewed as functions on the "space" Y of (characteristic zero) untilts $y = (K, \iota)$ of $C^\flat$."

Construction 1. Let $\mathbf{Q}_p^{\mathrm{cyc}}$ denote the perfectoid field introduced in Lecture 2 (given by the completion of the union of cyclotomic extensions $\bigcup_{n>0} \mathbf{Q}_p[\zeta_{p^n}]$). We let ϵ denote the sequence of element of $\mathbf{Q}_p^{\mathrm{cyc}}$ given by $(1, \zeta_p, \zeta_{p^2}, \zeta_{p^3}, \cdots)$, which we regard as an element of the tilt $(\mathbf{Q}_p^{\mathrm{cyc}})^\flat$. By construction, we have $\epsilon^\sharp = 1$, so that $(\epsilon - 1)^\sharp \in p\mathbf{Z}_p^{\mathrm{cyc}}$. It follows that $\epsilon - 1$ is a pseudo-uniformizer of the tilt $(\mathbf{Q}_p^{\mathrm{cyc}})^\flat$: that is, we have

$$0 < |\epsilon - 1|_{(\mathbf{Q}_p^{\mathrm{cyc}})^\flat} < 1.$$

Let (K, ι) be an untilt of $C^\flat$ equipped with a (continuous) homomorphism $u : \mathbf{Q}_p^{\mathrm{cyc}} \hookrightarrow K$. Then the induced map $(\mathbf{Q}_p^{\mathrm{cyc}})^\flat \to K^\flat \simeq C^\flat$ carries ϵ to pseudo-uniformizer of the field $C^\flat$, which we will denote by $f(\epsilon)$.

Proposition 2. *The construction $(K, \iota, u) \mapsto u(\epsilon)$ induces a bijection*

$$\{\textit{Untilts } (K, \iota) \textit{ of } C^\flat \textit{ with an embedding } \mathbf{Q}_p^{\mathrm{cyc}} \hookrightarrow K\} \simeq \{x \in C^\flat : 0 < |x - 1|_{C^\flat} < 1\}.$$

Proof. Fix an element $x \in C^\flat$ satisfying $0 < |x - 1|_{C^\flat} < 1$. We wish to show that there is an essentially unique untilt K equipped with an embedding $u : \mathbf{Q}_p^{\mathrm{cyc}} \hookrightarrow K$ satisfying $(x^{1/p^n})^\sharp = f(\zeta_{p^n})$ for $n \geq 0$. Note that giving the embedding u is equivalent to choosing a compatible sequence of p^nth roots of unity in K, so we are just looking for an untilt K of $C^\flat$ having the property that $(x^{1/p})^\sharp$ is a primitive pth root of unity in K. This is equivalent to the requirement that $(x^{1/p})^\sharp$ satisfies the pth cyclotomic polynomial

$$1 + t + \cdots + t^{p-1} = 0;$$

that is, that the associated map $\theta : \mathbf{A}_{\mathrm{inf}} \to \mathcal{O}_K$ annihilates the element $\xi = 1 + [x^{1/p}] + \cdots + [x^{(p-1)/p}] \in \mathbf{A}_{\mathrm{inf}}$. Consequently, to prove the existence and uniqueness of K, it will suffice to show that ξ is a *distinguished* element of $\mathbf{A}_{\mathrm{inf}}$ (see Lecture 3). Writing $\xi = \sum_{n \geq 0} [c_n] p^n$, we wish to show that $|c_0|_{C^\flat} < 1$ and $|c_1|_{C^\flat} = 1$. Note that, since $x \equiv 1 \pmod{\mathfrak{m}_C^\flat}$ the image of ξ in the ring of Witt vectors $W(k) = W(\mathcal{O}_C^\flat / \mathfrak{m}_C^\flat)$ is p. We therefore have $|c_0|_{C^\flat} < 1$ and $|c_1 - 1|_{C^\flat} < 1$ (which is even more than we need). $\square$

Note that, if K is an untilt of $C^\flat$ for which there exists *one* embedding $u : \mathbf{Q}_p^{\mathrm{cyc}} \hookrightarrow K$, then we can always find many others: namely, we can precompose f with the automorphism of $\mathbf{Q}_p^{\mathrm{cyc}}$ given by $\zeta_{p^n} \mapsto \zeta_{p^n}^{\alpha}$ for any $\alpha \in \mathbf{Z}_p^\times$. Under the correspondence of Proposition 2, this corresponds to the action of $\mathbf{Z}_p^\times$ on $\{x \in C^\flat : 0 < |x-1|_{C^\flat} < 1\}$ by exponentiation. We therefore obtain the following:

Corollary 3. *There is a canonical bijection*

$$\begin{array}{c}\{\textit{Untilts } K \textit{ containing } p^n\textit{th roots of unity for all } n\}/\textit{isomorphism} \\ \downarrow\sim \\ \{x \in C^\flat : 0 < |x-1|_{C^\flat} < 1\}/\mathbf{Z}_p^\times.\end{array}$$

Moreover, replacing an untilt (K, ι) by $(K, \iota \circ \varphi_C)$ has the effect of replacing the corresponding element $x \in C^\flat$ by $x^{1/p}$. We therefore have:

Corollary 4. *There is a canonical bijection*

$$\begin{array}{c}\{\textit{Untilts } K \textit{ containing } p^n\textit{th roots of unity for all } n\}/\phi_{C^\flat}^{\mathbf{Z}} \\ \downarrow\sim \\ \{x \in C^\flat : 0 < |x-1|_{C^\flat} < 1\}/\,\mathbf{Q}_p^\times\,.\end{array}$$

The inverse bijection carries the equivalence class of an element $x \in C^\flat$ to the "vanishing locus" of $\log([x]) \in B$

The proof is based on the following:

Exercise 5. Let K be a field of characteristic zero which is complete with respect to non-archimedean absolute value having residue characteristic p, so that the logarithm $\log(y) \in K$ is defined for $y \in K$ satisfying $|y-1|_K < 1$. Show that the construction $y \mapsto \log(y)$ induces a bijection

$$\{y \in K : |y-1|_K < |p|_K^{1/(p-1)}\} \to \{z \in K : |z| < |p|_K^{1/(p-1)}$$

(hint: the inverse is given by the exponential map $z \mapsto \exp(z) = \sum \frac{z^n}{n!}$).

Remark 6. The constant $|p|_K^{1/(p-1)}$ appearing in Exercise 5 is the best possible. Note that if K contains a primitive pth root of unity ζ_p, then ζ_p satisfies $|\zeta_p - 1|_K = |p|_K^{1/(p-1)} < 1$, and we have

$$p\log(\zeta_p) = \log(\zeta_p^p) = \log(1) = 0,$$

so that $\log(\zeta_p) = 0 = \log(1)$. It follows that the logarithm map is not injective when restricted to the *closed* disk $\{y \in K : |y-1|_K \leq |p|_K^{1/(p-1)}\}$.

Proof of Corollary 4. Let x be an element of $C^\flat$ satisfying $0 < |x-1|_{C^\flat} < 1$. Then, for any untilt (K, ι) of $C^\flat$, the element $x^\sharp$ satisfies $|(x^{p^n})^\sharp - 1|_K < |p|_K^{1/(p-1)}$ for $n \gg 0$. Consequently, if $\log(x^\sharp) = 0$, then $\log((x^{p^n})^\sharp) = 0$ and therefore $(x^{p^n})^\sharp = 1$ by virtue of Exercise **??**. Choose n as small as possible, so that $(x^{p^{n-1}})^\sharp \neq 1$ (this is possible since we have assumed that $x \neq 1$). Composing ι with a suitable power of the Frobenius map $\varphi_{C^\flat}$, we can assume that $n = 0$: that is, we have $x^\sharp = 1$ but $(x^{1/p})^\sharp \neq 1$, so that (K, ι) is the untilt associated to the element x in the proof of Proposition 2. $\square$

Remark 7. One can show that when $C^\flat$ is algebraically closed, then every untilt of $C^\flat$ is also algebraically closed. It follows in this case that *every* Frobenius orbit of characteristic zero untilts of $C^\flat$ can be realized as the vanishing locus of some element of $B^{\varphi=p}$ having the form $\log([x])$; moreover, this element of $B^{\varphi=p}$ is unique up to the action of $\mathbf{Q}_p$.

Our next goal is to show that the functions of the form $\log([x])$ have *simple* zeros: that is, they do not vanish with multiplicity at any point of K. To make this idea precise, we need some auxiliary constructions.

Construction 8. Let $y = (K, \iota)$ be a characteristic zero untilt of $C^\flat$, so that we have a canonical surjection $\theta : \mathbf{A}_{\mathrm{inf}} \twoheadrightarrow \mathcal{O}_K$ whose kernel is a principal ideal (ξ). We saw in Lecture 3 that the ring $\mathbf{A}_{\mathrm{inf}}$ is ξ-adically complete: that is, it is isomorphic to the inverse limit of the tower

$$\cdots \to \mathbf{A}_{\mathrm{inf}}/(\xi^4) \twoheadrightarrow \mathbf{A}_{\mathrm{inf}}/(\xi^3) \twoheadrightarrow \mathbf{A}_{\mathrm{inf}}/(\xi^2) \twoheadrightarrow \mathbf{A}_{\mathrm{inf}}/(\xi) \simeq \mathcal{O}_K .$$

We let $B^+_{\mathrm{dR}} = B^+_{\mathrm{dR}}(y)$ denote the inverse limit of the diagram

$$\cdots \to (\mathbf{A}_{\mathrm{inf}}/(\xi^4))[\frac{1}{p}] \twoheadrightarrow (\mathbf{A}_{\mathrm{inf}}/(\xi^3))[\frac{1}{p}] \twoheadrightarrow \mathbf{A}_{\mathrm{inf}}/(\xi^2)[\frac{1}{p}] \twoheadrightarrow (\mathbf{A}_{\mathrm{inf}}/(\xi))[\frac{1}{p}] \simeq \mathcal{O}_K[\frac{1}{p}] = K.$$

Remark 9. The ring B^+_{dR} does not depend on the choice of generator ξ for the ideal $\ker(\theta)$: in each of the expressions above, we can replace the ideal (ξ^n) by $\ker(\theta)^n$. However, it does depend on the choice of untilt $y = (K, \iota)$.

Remark 10. In the situation of Construction 8, we can replace each of the localizations $(\mathbf{A}_{\mathrm{inf}}/(\xi^n))[\frac{1}{p}]$ by $(\mathbf{A}_{\mathrm{inf}}/(\xi^n))[\frac{1}{[\pi]}]$. Having made this replacement, the construction is sensible even when $K \simeq C^\flat$ has characteristic p. In this case, each quotient $\mathbf{A}_{\mathrm{inf}}(\xi^n)[\frac{1}{[\pi]}]$ can be identified with $W_n(\mathcal{O}^\flat_C)[\frac{1}{[\pi]}] \simeq W_n(\mathcal{O}^\flat_C[\frac{1}{\pi}]) \simeq W_n(C^\flat)$. Consequently, the "characteristic p" analogue of the ring B^+_{dR} is just the ring of Witt vectors $W(C^\flat)$.

Proposition 11. *In the situation of Construction 8, B^+_{dR} is a complete discrete valuation ring, and the element ξ is a uniformizer. In other words:*

(a) *The image of ξ in B^+_{dR} is not a zero-divisor.*

(b) *The ring B^+_{dR} is ξ-adically complete.*

(c) *The quotient $B^+_{\mathrm{dR}}/(\xi)$ is a field.*

Proof. We first prove (a). Let f be an element of B^+_{dR}, given by a system of elements $x_n \in (\mathbf{A}_{\mathrm{inf}}/(\xi^n))[\frac{1}{p}]$. We saw in Lecture 3 that ξ is not a zero divisor in $\mathbf{A}_{\mathrm{inf}}$ and p is not a zero-divisor in $\mathbf{A}_{\mathrm{inf}}/(\xi)$, and is therefore not a zero-divisor in $\mathbf{A}_{\mathrm{inf}}/(\xi^n)$ for all n. Consequently, we can view the quotient $\mathbf{A}_{\mathrm{inf}}/(\xi^n)$ as a subring of the localization $(\mathbf{A}_{\mathrm{inf}}/(\xi^n))[\frac{1}{p}]$. We can therefore choose some $k \gg 0$ (depending on n) such that $p^k x_n$ belongs to $\mathbf{A}_{\mathrm{inf}}/(\xi^n)$. If $\xi x = 0$, then each $p^k x_n$ is annihilated by ξ in $\mathbf{A}_{\mathrm{inf}}/(\xi^n)$, so we can write $p^k x_n = \xi^{n-1} y_n$ for some $y_n \in \mathbf{A}_{\mathrm{inf}}/(\xi^n)$. Reducing modulo (ξ^{n-1}), we conclude that $p^k x_{n-1} = 0$ in $\mathbf{A}_{\mathrm{inf}}/(\xi^{n-1})$, so that $x_{n-1} = 0$. Since n is arbitrary, it follows that $x = 0$.

Note that each of the projection maps $B^+_{\mathrm{dR}} \twoheadrightarrow (\mathbf{A}_{\mathrm{inf}}/(\xi^m))[\frac{1}{p}]$ annihilates (ξ^m) and therefore factors as a surjection $\rho : B^+_{\mathrm{dR}}/(\xi^m) \twoheadrightarrow (\mathbf{A}_{\mathrm{inf}}/(\xi^m))[\frac{1}{p}]$. We claim that ρ is an isomorphism: that is, if x is an element of B^+_{dR} whose image in $(\mathbf{A}_{\mathrm{inf}}/(\xi^m))[\frac{1}{p}]$ vanishes, then x is divisible by ξ^m. Write $x = \{x_n\}_{n \geq 0}$ as above. For each $n \geq m$, we can choose $k(n) \gg 0$ such that $p^{k(n)} x_n \in \mathbf{A}_{\mathrm{inf}}/(\xi^n)$. Then the image of $p^{k(n)} x_n$ in $\mathbf{A}_{\mathrm{inf}}/(\xi^m)$ vanishes, so we can write $p^{k(n)} x_n = \xi^m y_n$ for some $y_n \in \mathbf{A}_{\mathrm{inf}}/(\xi^{n-m})$. Then x is the product of ξ^m with the element of B^+_{dR} given by the sequence $\{\frac{y_n}{p^{k(n)}}\}_{n \geq m}$.

It follows from the preceding argument that B^+_{dR} can be identified with the limit $\varprojlim B^+_{\mathrm{dR}}/(\xi^m)$ and is therefore ξ-adically complete. Moreover, in the case $m = 1$ we obtain an isomorphism $B^+_{\mathrm{dR}}/(\xi) \simeq K$ which proves (c). $\square$

Remark 12. Since B^+_{dR} is a complete discrete valuation ring whose residue field $B^+_{\mathrm{dR}}/(\xi) \simeq K$ has characteristic zero, it is abstractly isomorphic to the formal power series ring $K[[\xi]]$. Beware that there is no canonical isomorphism of B^+_{dR} with $K[[\xi]]$.

Definition 13. We let B_{dR} denote the fraction field of the discrete valuation ring B^+_{dR} (that is, the ring $B^+_{\mathrm{dR}}[\frac{1}{\xi}]$).

Heuristically, if we think of the collection Y of all characteristic zero untilts $y = (K, \iota)$ of $C^\flat$ as as an analogue of the punctured unit disk $D^\times$ then $B^+_{\mathrm{dR}}(y)$ is the analogue is the analogue of the *completed local ring* $\widehat{\mathcal{O}}_{D^\times,y}$ at a point $y \in D^\times$ (which is isomorphic to the power series ring $\mathbf{C}[[t]]$, where t is any local coordinate of $D^\times$ at the point y (for example, we can take $t = z - y$).

Note that, for every characteristic zero untilt $y = (K, \iota)$ of $C^\flat$, we have a canonical map $\mathbf{A}_{\mathrm{inf}} \to B^+_{\mathrm{dR}}$. Moreover, the composite map $\mathbf{A}_{\mathrm{inf}} \to B^+_{\mathrm{dR}} \to B^+_{\mathrm{dR}}/(\xi) \simeq K$ carries p and $[\pi]$ to invertible elements $p, \pi^\sharp \in K$. Consequently, the images of p and $[\pi]$ in B^+_{dR} are invertible: that is, we have a map $\mathbf{A}_{\mathrm{inf}}[\frac{1}{p}, \frac{1}{[\pi]}] \to B^+_{\mathrm{dR}}$, which we will denote by $f \mapsto \widehat{f}_y$.

Proposition 14. *Let $0 < a \leq b < 1$ be real numbers satisfying $a \leq |p|_K \leq b$. Then the map $f \mapsto \widehat{f}_y$ admits a canonical extension to a ring homomorphism $B_{[a,b]} \to B^+_{\mathrm{dR}}$, which we will also denote by $f \mapsto \widehat{f}_y$.*

Note that Proposition 14 is consistent with our heuristics: if we think of $B_{[a,b]}$ as the ring of "holomorphic" functions on the untilts K satisfying $a \leq |p|_K \leq b$, then we should be able to evaluate elements of $B_{[a,b]}$ not only *at* the point K (to obtain an element of K itself), but also "infinitesimally close" to K (to obtain an element of B^+_{dR}).

Proof. Without loss of generality, we may assume that $a = |p|_K = b$. Moreover, we can assume that the pseudo-uniformizer $\pi \in C^\flat$ is chosen so that $|\pi|_{C^\flat} = |p|_K$. In this case, the ring $B_{[a,b]}$ is obtained from the subring $\mathbf{A}_{\mathrm{inf}}[\frac{[\pi]}{p}, \frac{p}{[\pi]}]$ by first p-adically completing and then inverting the prime number p. For each $n \geq 0$, $\overline{e}$ determines a ring homomorphism

$$\overline{e}_n : \mathbf{A}_{\mathrm{inf}}[\frac{[\pi]}{p}, \frac{p}{[\pi]}] \to B^+_{\mathrm{dR}}/(\xi^n) \simeq (\mathbf{A}_{\mathrm{inf}}/(\xi^n))[\frac{1}{p}].$$

We claim that there exists an integer $k \gg 0$ (depending on n) such that the image of $\overline{e}_n$ is contained in

$$p^{-k}(\mathbf{A}_{\mathrm{inf}}/(\xi^n)) \subseteq (\mathbf{A}_{\mathrm{inf}}/(\xi^n))[\frac{1}{p}].$$

Assuming this is true, we can use the fact that $p^{-k}(\mathbf{A}_{\mathrm{inf}}/(\xi^n))$ is p-adically complete to extend $\overline{e}_n$ to a map (of abelian groups) from the p-adic completion of $\mathbf{A}_{\mathrm{inf}}[\frac{[\pi]}{p}, \frac{p}{[\pi]}]$ to $p^{-k}(\mathbf{A}_{\mathrm{inf}}/(\xi^n))$. Inverting p, we then obtain a map (of commutative rings)

$$e_n : B_{[a,b]} \to B^+_{\mathrm{dR}}/(\xi^n) \simeq (\mathbf{A}_{\mathrm{inf}}/(\xi^n))[\frac{1}{p}].$$

These maps are compatible as n varies, and determine the desired homomorphism $B_{[a,b]} \to B^+_{\mathrm{dR}}$.

It remains to prove the existence of k. Define $f, g \in B^+_{\mathrm{dR}}/(\xi^n)$ by the formulae

$$f = \overline{e}_n(\frac{[\pi]}{p}) \qquad g = f^{-1} = \overline{e}_n(\frac{p}{[\pi]}).$$

Our assumption that $|\pi|_{C^\flat} = |p|_K$ guarantees that the images of f and g under the map $B_{\mathrm{dR}}^+/(\xi^n) \to B_{\mathrm{dR}}^+/(\xi) \simeq K$ belong to the valuation ring $\mathcal{O}_K$. Consequently, we can find elements $f', g' \in \mathbf{A}_{\mathrm{inf}}/(\xi^n)$ satisfying

$$f \equiv f' \mod \xi \qquad g \equiv g' \mod \xi.$$

We therefore have

$$f = f' + \frac{\xi}{p^c} f'' \qquad g = g' + \frac{\xi}{p^c} g''$$

for some other elements $f'', g'' \in \mathbf{A}_{\mathrm{inf}}/(\xi^n)$ and some integer $c \gg 0$. In this case, every power of f admits a binomial expansion

$$\begin{aligned} f^m &= (f' + \frac{\xi}{p^c} f'')^m \\ &= \sum_{i=0}^{m} \binom{m}{i} f'^{m-i} (\frac{\xi}{p^c} f'')^i \\ &= \sum_{i=0}^{n-1} \binom{m}{i} f'^{m-i} (\frac{\xi}{p^c} f'')^i \\ &\in p^{-nc}(\mathbf{A}_{\mathrm{inf}}/(\xi^n)), \end{aligned}$$

and similarly with g in place of f. $\square$

Lecture 9: Divisors

October 22, 2018

Throughout this lecture, we fix a perfectoid field $C^\flat$ of characteristic p, with valuation ring $\mathcal{O}_C^\flat$. Let Y denote the set of all isomorphism classes of characteristic zero untilts $K = (K, \iota)$ of $C^\flat$. For each $0 < a \leq b < 1$, we let $Y_{[a,b]} \subseteq Y$ denote the subset consisting of those untilts K satisfying $a \leq |p|_K \leq b$.

Recall that our heuristic is that Y behaves somewhat like a Riemann surface, with the ring $B_{[a,b]}$ (obtained by completing $A_{\text{inf}}[\frac{1}{p}, \frac{1}{[\pi]}]$ with respect to the Gauss norms $|\bullet|_a$ and $|\bullet|_b$) behaves like the ring of holomorphic functions on $B_{[a,b]}$.

For each characteristic zero untilt K of $C^\flat$, we let $B_{\text{dR}}^+(K)$ denote the discrete valuation ring constructed in the previous lecture (with residue field K). In Lecture 8, we proved that if $a \leq |p|_K \leq b$. then the canonical map $\theta : A_{\text{inf}} \twoheadrightarrow \mathcal{O}_K$ lifts to a map $e : B_{[a,b]} \to B_{\text{dR}}^+(K)$. For each $x \in B_{[a,b]}$, we let $\text{ord}_K(x) \in \mathbf{Z}_{\geq 0} \cup \{\infty\}$ denote the valuation of $e(x)$ in $B_{\text{dR}}^+(K)$ (so that $\text{ord}_K(x) = \infty$ if $e(x) = 0$, and otherwise $\text{ord}_K(x)$ is the unique integer n such that $(e(x))$ coincides with the nth power of the maximal ideal of $B_{\text{dR}}^+(K)$. We will refer to $\text{ord}_K(x)$ as the *order of vanishing* of x at the untilt K.

Our main objective over the next several lectures will be to prove the following:

Theorem 1. *Assume that $C^\flat$ is algebraically closed, and fix $0 < a \leq b < 1$. Then:*

(1) *Let x be a nonzero element of $B_{[a,b]}$. Then $\text{ord}_K(x) < \infty$ for each $K \in Y_{[a,b]}$. Moreover, there are only finitely many elements $K \in Y_{[a,b]}$ for which $\text{ord}_K(x) \neq 0$.*

The first part of (1) asserts that each of the maps $e : B_{[a,b]} \to B_{\text{dR}}^+(K)$ is injective. In particular, the ring $B_{[a,b]}$ is an integral domain.

(2) *Let x and y be nonzero elements of $B_{[a,b]}$. Then x is divisible by y if and only if $\text{ord}_K(x) \geq \text{ord}_K(y)$ for each $K \in Y_{[a,b]}$.*

For each nonzero element $x \in B_{[a,b]}$, we let $\text{Div}_{[a,b]}(x)$ denote the formal sum $\sum_{K \in Y_{[a,b]}} \text{ord}_K(x) \cdot K$, which we regard as an element of the free abelian group generated by the set $Y_{[a,b]}$. If x is a nonzero element of B, we let $\text{Div}(x)$ denote the formal sum $\sum_{K \in Y} \text{ord}_K(x) \cdot K$. Beware that this latter sum may have infinitely many terms; however, it has only finitely many summands lying in each $Y_{[a,b]}$.

Example 2. Let ξ be a distinguished element of $\mathbf{A}_{\text{inf}}$. We distinguish two cases:

(1) If ξ is a unit multiple of p, then it is invertible in the ring B. In this case, we have $\text{Div}(\xi) = 0$.

(2) If ξ is not a unit multiple of p, then there is a unique characteristic zero untilt K of $C^\flat$ such that the map $\theta : A_{\text{inf}} \twoheadrightarrow \mathcal{O}_K$ annihilates ξ. By construction, the image of ξ is a uniformizer of the discrete valuation ring B_{dR}^+. It follows that $\text{ord}_K(\xi) = 1$ and $\text{ord}_{K'}(\xi) = 0$ for $K' \neq K$, so the divisor $\text{Div}(\xi)$ is equal to K.

Example 3. Let x be an element of $C^\flat$ satisfying $0 < |x-1|_{C^\flat} < 1$. We saw in the previous lecture that there is precisely one Frobenius orbit of untilts on which $\log([x])$ vanishes. Moreover, one of the untilts K belonging to this locus is given by the vanishing locus of the distinguished element

$$\xi = 1 + [x^{1/p}] + \cdots + [x^{(p-1)/p}] = \frac{[x]-1}{[x^{1/p}]-1} \in \mathbf{A}_{\mathrm{inf}}.$$

Note that the image of $[x^{1/p}]$ in K is a primitive pth root of unity ζ_p, so that $\zeta_p - 1$ is invertible in K (though not in $\mathcal{O}_K$) and therefore $[x^{1/p}] - 1$ is invertible in $B^+_{\mathrm{dR}}(K)$. It follows that $[x]-1$ is a unit multiple of ξ in B^+_{dR}, and is therefore a uniformizer. The congruence

$$\log([x]) = \sum_{k>0} \frac{(-1)^{k+1}}{k}([x]-1)^k \equiv [x]-1 \pmod{([x]-1)^2}$$

shows that $\mathrm{ord}_K(\log([x])) = 1$. By symmetry, we conclude that $\log([x])$ vanishes to order exactly one at each point belonging to the Frobenius orbit of K: that is, we have

$$\mathrm{Div}(\log([x])) = \sum_{n\in\mathbf{Z}} \varphi^n(K).$$

For nonzero elements $x, y \in B$, we write $\mathrm{Div}(x) \geq \mathrm{Div}(y)$ if $\mathrm{ord}_K(x) \geq \mathrm{ord}_K(y)$ for each $K \in Y$. From Theorem 1, we immediately deduce the following:

Corollary 4. *Assume that $C^\flat$ is algebraically closed. The ring B is an integral domain. Moreover, if x and y are nonzero elements of B, then x is divisible by y if and only if* $\mathrm{Div}(x) \geq \mathrm{Div}(y)$.

We now state two further results we will prove later:

Theorem 5. *The canonical map $\mathbf{Q}_p \to B^{\varphi=1}$ is an isomorphism, and the vector space $B^{\varphi=p^n}$ vanishes when n is negative.*

Theorem 6. *Suppose that $C^\flat$ is algebraically closed. Then every untilt of $C^\flat$ is algebraically closed.*

Remark 7. The rough idea of Theorem 6 is easy to explain. If $C^\flat$ admits an untilt K which is not algebraically closed, then K admits some finite algebraic extension L. In this case, one would like to argue that L is also a perfectoid field and that $L^\flat$ is a finite algebraic extension of $K^\flat \simeq C^\flat$, contradicting our assumption that $C^\flat$ is algebraically closed. Fleshing out this argument requires some work, which we defer to a future lecture.

Let us collect some consequences.

Corollary 8. *Assume that $C^\flat$ is algebraically closed. Then every untilt K of $C^\flat$ belongs to the vanishing locus of* $\log([x])$*, for some element $x \in C^\flat$ satisfying $0 < |x-1|_{C^\flat} < 1$. Moreover, the map*

$$\psi : \{y \in C^\flat : |y-1|_{C^\flat} < 1\} \xrightarrow{\log(y^\sharp)} K$$

is surjective, whose kernel is generated by x (as a subspace of the $\mathbf{Q}_p$-vector space $\{y \in C^\flat : |y-1|_{C^\flat} < 1\}$).

Proof. To prove the first assertion, it will suffice (by results of the previous lecture) to show that every untilt K of $C^\flat$ contains a compatible system of p^nth roots of unity; this is immediate from Theorem 6. Note that if z is an element of K satisfying $|z|_K < |p|_K^{1/(p-1)}$, then the exponential $\exp(z)$ is well-defined. Since K is algebraically closed (Theorem 6), we can choose a compatible system of p^nth roots of unity of $\exp(z)$: that is, we can write $\exp(z) = y^\sharp$ for some $y \in C^\flat$. Some simple estimates yield $|y-1|_{C^\flat} = |\exp(z)-1|_K = |z|_K < 1$, so that $z = \log(y^\sharp)$. It follows that the image of ρ contains all *sufficiently small* elements of K. However, ρ is a map of $\mathbf{Q}_p$-vector spaces, and every element of K becomes sufficiently small after multiplying by a large power of p. This proves that ρ is surjective, and the kernel of ρ was described in the previous lecture. ∎

Corollary 9. *Assume that $C^\flat$ is algebraically closed. Then the map*

$$1 + \mathfrak{m}_C^\flat \xrightarrow{\log([x])} B^{\varphi=p}$$

is an isomorphism.

Proof. Injectivity is clear (since $\log([x])$ vanishes on a single Frobenius orbit of Y for $x \neq 1$). To prove surjectivity, we must show that every element $f \in B^{\varphi=p}$ has the form $\log([x])$ for some $x \in \mathfrak{m}_C^\flat + 1$. Assume that $f \neq 0$ (otherwise, we can take $x = 1$). We first claim that the divisor $\mathrm{Div}(f)$ is nonzero. Otherwise, Corollary 4 would imply that f is invertible. Then the inclusion $f \in B^{\varphi=p}$ guarantees $f^{-1} \in B^{\varphi=p^{-1}}$, so that $f^{-1} = 0$ by Theorem 5, which is clearly impossible.

Since $\mathrm{Div}(f)$ is nonzero, we can choose an untilt K of $C^\flat$ satisfying $\mathrm{ord}_K(f) \geq 1$. Since f belongs to $B^{\varphi=p}$, it follows that $\mathrm{ord}_{K'}(f) \geq 1$ for any $K' \in Y$ which belongs to the Frobenius orbit of K. Choose an element $x \in 1 + \mathfrak{m}_C^\flat$ such that $x \neq 1$ and $\log([x])$ vanishes at K. Then

$$\mathrm{Div}(\log([x])) = \sum_{n \in \mathbf{Z}} \varphi^n(K) \leq \mathrm{Div}(f).$$

Applying Corollary 4, we deduce that f is divisible by $\log([x])$: that is, we can write $f = \log([x]) \cdot g$. Since both f and $\log([x])$ belong to $B^{\varphi=p}$ (and B is an integral domain), it follows that g belongs to $B^{\varphi=1}$. Using Theorem 5, we conclude that $g \in \mathbf{Q}_p$ is a scalar. We can therefore arrange (after replacing x by a suitable scalar multiple in the $\mathbf{Q}_p$-vector space $1 + \mathfrak{m}_C^\flat$) that $g = 1$, so that $f = \log([x])$ as desired. $\square$

Lecture 10: Structure of the Fargues-Fontaine Curve

October 29, 2018

Throughout this lecture, we fix an *algebraically closed* perfectoid field $C^\flat$ of characteristic p, with valuation ring $\mathcal{O}_C^\flat$. Let Y denote the set of all isomorphism classes of characteristic zero untilts $y = (K, \iota)$ of $C^\flat$. To each nonzero element f of the ring B, we associate the "divisor"

$$\sum_{y \in Y} \mathrm{ord}_y(f) \cdot y$$

(which is generally an infinite sum, though "locally" finite). We recall three results from the previous lecture (which we have not yet proved):

Theorem 1. (1) *Every nonzero element $f \in B$, has finite order of vanishing $\mathrm{ord}_y(f)$ at each point $y \in Y$.*

(2) *Another nonzero element $g \in B$ is divisible by f if and only if $\mathrm{Div}(f) \leq \mathrm{Div}(g)$: that is, $\mathrm{ord}_y(f) \leq \mathrm{ord}_y(g)$ for each $y \in Y$.*

Theorem 2. *For $n < 0$, the eigenspace $B^{\varphi = p^n}$ vanishes.*

Theorem 3. *Every untilt of $C^\flat$ is algebraically closed.*

Let us now collect some consequences.

Corollary 4. *The ring $B^{\varphi=1}$ is a field.*

In fact, the field $B^{\varphi=1}$ can be identified with $\mathbf{Q}_p$; we stated this without proof in the previous lecture, but will not need it yet.

Proof of Corollary 4. Let f be a nonzero element of $B^{\varphi=1}$; we wish to prove that f is invertible in B (in which case it is clear that the inverse f^{-1} also belongs to $B^{\varphi=1}$. By virtue of Theorem 1, it will suffice to show that the divisor $\mathrm{Div}(f)$ vanishes. Since f is fixed by the Frobenius, the divisor $\mathrm{Div}(f)$ is likewise fixed by the Frobenius. Consequently, if $\mathrm{Div}(f) \neq 0$, then we can write $\mathrm{Div}(f) \geq \sum_{n \in \mathbf{Z}} \varphi^n(y)$ for some $y = (K, \iota) \in Y$. It follows from Theorem 3 that K contains a copy of $\mathbf{Q}_p^{\mathrm{cyc}}$, so that we can write $\sum_{n \in \mathbf{Z}} \varphi^n(y) = \log([\epsilon])$ for some $\epsilon \in 1 + \mathfrak{m}_C^\flat$. Applying Theorem 1, we can write $f = g \cdot \log([\epsilon])$. It follows that $g \in B^{\varphi = p^{-1}} = \{0\}$, contradicting our assumption that $f \neq 0$. $\square$

Corollary 5. *For $n \geq 0$, every nonzero element $f \in B^{\varphi = p^n}$ factors as a product $\lambda \log([\epsilon_1]) \cdots \log([\epsilon_n])$ for some $\lambda \in B^{\varphi=1}$ and $\epsilon_1, \ldots, \epsilon_n \in 1 + \mathfrak{m}_{C^\flat}$. Moreover, the factors are uniquely determined up reordering and multiplication by elements of $\mathbf{Q}_p^\times$.*

Proof. We prove existence by induction on n. If $n = 0$, there is nothing to prove. we will therefore assume that $n > 0$. Note that if $\mathrm{Div}(f) = 0$, then f is invertible (Theorem 1) and the inverse f^{-1} belongs to $B^{\varphi = p^{-n}}$, contradicting Theorem 2. As in the proof of Corollary 4, we learn that f is divisible by $\log([\epsilon])$ for some element $\epsilon \neq 1$ of $1+\mathfrak{m}_C^\flat$. Writing $f = g \cdot \log([\epsilon])$, we conclude that $g \in B^{\varphi=p^{n-1}}$. It follows from our inductive hypothesis that we can write $g = \lambda \log([\epsilon_1]) \cdots \log([\epsilon_{n-1}])$ for some $\lambda \in B^{\varphi=1}$ and $\epsilon_1, \ldots, \epsilon_n \in 1 + \mathfrak{m}_{C^\flat}$, so that $f = \lambda \log([\epsilon_1]) \cdots \log([\epsilon_{n-1}]) \cdot \log([\epsilon])$.

To prove uniqueness, it will suffice to show for $1 \neq \epsilon \in 1 + \mathfrak{m}_C^\flat$, the element $\log([\epsilon])$ is a *prime* element of the graded ring $\bigoplus_{n\geq 0} B^{\varphi=p^n}$: that is, if $\log([\epsilon])$ divides a product $f \cdot g$, then either $\log([\epsilon])$ divides f or $\log([\epsilon])$ divides g. Since $\log([\epsilon])$ is homogeneous, it suffices to check this in the case where f and g are homogeneous: that is, we may assume that $f \in B^{\varphi=p^m}$ and $g \in B^{\varphi=p^n}$. Choose a point $y \in Y$ belonging to the vanishing locus of $\log([\epsilon])$. Then either f or g must vanish at the point y; without loss of generality, we may assume that $f(y) = 0$. The equation $\varphi(f) = p^m f$ guarantees that the divisor $\mathrm{Div}(f)$ is Frobenius-invariant, so we must have $\mathrm{Div}(f) \geq \sum_{n \in \mathbf{Z}} \varphi^n(y) = \mathrm{Div}(\log([\epsilon]))$. Applying Theorem 1, we conclude that $\log([\epsilon])$ divides f. □

Let P denote the graded ring $\bigoplus_{n\geq 0} B^{\varphi=p^n}$. Recall that the *Fargues-Fontaine curve* X_{FF} is defined to be the scheme $\mathrm{Proj}(P)$. By definition, the points of X_{FF} (as a topological space) can be identified with homogeneous prime ideals $\mathfrak{p} \subseteq P$ which do not contain the "irrelevant" ideal $\bigoplus_{n>1} B^{\varphi=p^n}$. Let us give two examples of such ideals:

- It follows from Theorem 1 that B is an integral domain. Consequently, the graded ring P is also an integral domain, so the zero ideal $(0) \subseteq P$ is prime. This prime ideal corresponds to the *generic point* of the Fargues-Fontaine curve X_{FF}.
- Let $(K, \iota) \in Y$ be a characteristic zero untilt of $C^\flat$, and choose an element $\epsilon \in 1 + \mathfrak{m}_C^\flat$ such that $\epsilon \neq 1$ and $\log([\epsilon])$ vanishes at K. It follows from the proof Corollary 5 that the principal ideal $(\log([\epsilon]))$ is prime, and therefore corresponds to a point of the Fargues-Fontaine curve that we will denote by x_K. Note that multiplying $\log([\epsilon])$ by a unit in $\mathbf{Q}_p$ does not change the principal ideal $(\log([\epsilon]))$. Consequently, the point x_K depends only on the untilt K. Moreover, we have $x_K = x_{K'}$ if and only if K and K' belong to the same Frobenius orbit of Y.

We now show that these are the *only* points of the Fargues-Fontaine curve:

Proposition 6. *Let x be a point of the Fargues-Fontaine curve X_{FF} which is not the generic point. Then we have $x = x_K$ for some point $(K, \iota) \in Y$. Moreover, the residue field of X_{FF} at the point x can be identified with K.*

Proof. By construction, the scheme $X_{\mathrm{FF}} = \mathrm{Proj}(P)$ can be obtained by gluing together open affine subschemes of the form $P[\frac{1}{f}]^0 = B[\frac{1}{f}]^{\varphi=1}$, where f is a nonzero homogeneous element of P having positive degree. Let us suppose that x belongs to one of these open subschemes, and therefore corresponds to a nonzero prime ideal $\mathfrak{p} \subseteq B[\frac{1}{f}]^{\varphi=1}$. Choose an element of $\mathfrak{p}$ and write it as a fraction $\frac{g}{f^n}$ for some element $g \in B^{\varphi=p^n}$. It follows from Corollary 5 that, after scaling by a unit, we may assume that this element factors as a product $\frac{\log([\epsilon_1])}{f} \cdots \frac{\log([\epsilon_n])}{f}$. Since $\mathfrak{p}$ is prime, we may assume that it contains one of the factors, which we write as $\frac{\log([\epsilon])}{f}$. Let $y = (K, \iota) \in Y$ be a point at which $\log([\epsilon])$ vanishes. We claim that $x = x_K$, or equivalently that $\mathfrak{p}$ is generated by $\frac{\log([\epsilon])}{f}$. To prove this (and the last claim of Proposition 6), it will suffice to show that the principal ideal $(\frac{\log([\epsilon])}{f})$ is maximal, and that the quotient field

$$B[\frac{1}{f}]^{\varphi=1}/(\frac{\log([\epsilon])}{f})$$

can be identified with K. Since f does not vanish at K, we have a canonical ring homomorphism

$$\rho : B[\frac{1}{f}]^{\varphi=1} \subseteq B[\frac{1}{f}] \to K;$$

we claim that ρ is a surjection whose kernel is generated by $\frac{\log([\epsilon])}{f}$

To prove surjectivity, we note that ρ is already surjective when restricted to $\frac{1}{f}B^{\varphi=p}$, since every element of K has the form $\log(y^\sharp)$ for some $y \in 1 + \mathfrak{m}_C^\flat$ (see Lecture 9). To prove injectivity, we can use Corollary 5 to write every element of $B[\frac{1}{f}]^{\varphi=1}$ as a product $\lambda \frac{\log([\epsilon_1])}{f} \cdots \frac{\log([\epsilon_n])}{f}$. If this point belongs to $\ker(\rho)$, then some fraction $\frac{\log([\epsilon_i])}{f}$ must be annihilated by ρ. The desired result then follows from the observation that $\frac{\log([\epsilon_i])}{f}$ and $\frac{\log([\epsilon])}{f}$ differ by multiplication by some nonzero element of $\mathbf{Q}_p$. $\square$

Corollary 7. *The construction $y = (K, \iota) \mapsto x_K$ induces a bijection*

$$Y/\varphi^{\mathbf{Z}} \simeq \{\textit{Closed points of } X_{\mathrm{FF}}\}$$

Corollary 8. *The Fargues-Fontaine curve X_{FF} is a Dedekind scheme.*

Proof. By definition, we can cover X_{FF} by open affine subschemes of the form $R = B[\frac{1}{f}]^{\varphi=1}$. The proof of Proposition 6 shows that every nonzero prime ideal of R is a maximal ideal generated by a single element. In particular, every prime ideal of R is finitely generated so, by a theorem of Cohen, R is Noetherian. Since every nonzero prime ideal in R is maximal, it has Krull dimension 1. Moreover, since every maximal ideal of R is generated by a single element, the ring R is regular. It follows that R is a Dedekind ring, so that X_{FF} is a Dedekind scheme. $\square$

Lecture 11: Trivial Eigenspaces of the Frobenius

October 31, 2018

Throughout this lecture, we fix a perfectoid field $C^\flat$ of characteristic p. Our goal is to prove the following result, which was stated without proof in the previous lecture:

Theorem 1. *Let n be a negative integer. Then $B^{\varphi=p^n} = \{0\}$.*

Recall that Theorem 1 is consistent with the analysis of Lecture 6: a bi-infinite Teichmüller expansion which "obviously" belongs to the eigenspace $B^{\varphi=p^n}$ must be identically zero. For example, when $n = -1$, such a Teichmüller expansion would look like

$$\sum_{m \in \mathbf{Z}} p^m [c]^{p^m}$$

for some $c \in C^\flat$. Such a sum cannot converge for *any* of the Gauss norms $| \bullet |_\rho$ unless c is equal to zero.

This does not translate directly to a proof of Theorem 1, because we do not have existence and uniqueness of Teichmüller expansions for elements of the ring B. However, it suggests an approach to the problem: assuming the existence of a nonzero element $f \in B^{\varphi=p^n}$, we might hope to derive a contradiction (when n is negative) by studying the properties of the Gauss norms $|f|_\rho$. Here it will be useful not to focus on a particular value of ρ, but instead to study the function $\rho \mapsto |f|_\rho$ (where the element f is fixed). As we will see in a moment, the properties of this function are more apparent if we make a logarithmic change of variable.

Notation 2. Let K be a field equipped with a non-archimedean absolute value $| \bullet |_K$. For each element $x \in K$, we set $v(x) = -\log |x|_K$, which we regard as an element of $\mathbf{R} \cup \{\infty\}$. We refer to $v(x)$ as the *valuation of x*. Note that we have

$$\mathcal{O}_K = \{x \in K : v(x) \geq 0\} \qquad \mathfrak{m}_K = \{x \in K : v(x) > 0\} \qquad (v(x) = \infty) \Leftrightarrow (x = 0).$$

$$v(xy) = v(x) + v(y) \qquad v(x + y) \geq \min(v(x), v(y)).$$

Remark 3. Recall that, if K is a field equipped with a non-archimedean absolute value $| \bullet |_K$, then for any $\alpha > 0$, the absolute value $| \bullet |_K^\alpha$ defines the same topology on K. In other words, for many purposes, it is useful to regard the absolute value on K as only well-defined up to a constant exponent. Likewise, it is useful to regard the valuation v on K as only well-defined up to a constant factor. In the case of a discrete valuation, it is often convenient to normalize so that the map $v : K \to \mathbf{R} \cup \{\infty\}$ takes values in $\mathbf{Z} \cup \{\infty\}$. However, we are interested in perfectoid fields where such a normalization is impossible.

Notation 4. Let f be an element of $\mathbf{A}_{\inf}[\frac{1}{p}, \frac{1}{[\pi]}]$. Then, for every positive real number s, we define $v_s(f) \in \mathbf{R} \cup \{\infty\}$ by the formula

$$v_s(f) = -\log |f|_{\exp(-s)}.$$

More concretely, if f has a Teichmüller expansion $\sum_{n \gg -\infty} [c_n] p^n$, we have

$$|f|_\rho = \sup\{|c_n|_{C^\flat} \rho^n\} \qquad v_s(f) = \inf\{v(c_n) + ns\}.$$

In particular, we have $v_s(f) = \infty$ if and only if $f = 0$.

Note that, from the multiplicativity and ultrametric properties of the Gauss norms, we have

$$v_s(fg) = v_s(f) + v_s(g) \qquad v_s(f+g) \geq \min(v_s(f), v_s(g)).$$

Suppose that f is nonzero. Note that for a fixed positive real number s, the infimum $v_s(f) = \inf\{v(c_n) + ns\}$ is achieved for finitely many values of n, taken from some set $\{n_0 < n_1 < \cdots < n_k\}$. It follows that, for ϵ sufficiently small, we have

$$v_{s+\epsilon}(f) = v_s(f) + n_0\epsilon \qquad v_{s-\epsilon}(f) = v_s(f) - n_k\epsilon.$$

This proves the following:

Proposition 5. *Let f be a nonzero element of $\mathbf{A}_{\inf}[\frac{1}{p}, \frac{1}{[\pi]}]$. Then the construction $s \mapsto v_s(f)$ determines a function*

$$v_\bullet(f) : \mathbf{R}_{>0} \to \mathbf{R}$$

which is piecewise linear with integer slopes. Moreover, it is concave (that is, the slopes are decreasing).

Example 6. Let f be a distinguished element of $\mathbf{A}_{\inf}$, so that f admits a Teichmüller expansion $\sum_{n\geq 0}[c_n]p^n$ where $|c_0|_{C^\flat} < 1$ and $|c_1|_{C^\flat} = 1$. We then have

$$v_s(f) = \min(v(c_0) + 0 \cdot s, v(c_1) = 1 \cdot s) = \min(v(c_0), s).$$

Remark 7. Let f be a nonzero element of $\mathbf{A}_{\inf}[\frac{1}{p}, \frac{1}{[\pi]}]$. It follows from Proposition 5 that the function $v_\bullet(f)$ has well-defined left and right derivatives at each point $s \in \mathbf{R}_{>0}$ (though the left and right derivatives need not be the same). We will denote the values of these left and right derivatives (at a point $s \in \mathbf{R}_{>0}$) by $\partial_- v_s(f)$ and $\partial_+ v_s(f)$, respectively. Note that $\partial_- v_s(f)$ and $\partial_+ v_s(f)$ are integers satisfying $\partial_- v_s(f) \geq \partial_+ v_s(f)$.

Our next goal is to extend the definition of $v_s(f)$ to the case where f belongs to a suitable completion of $\mathbf{A}_{\inf}[\frac{1}{p}, \frac{1}{[\pi]}]$. Note that if we are given a sequence of elements $f_1, f_2, \ldots \in \mathbf{A}_{\inf}[\frac{1}{p}, \frac{1}{[\pi]}]$ which converges for the Gauss norm $| \bullet |_{\exp(-s)}$, then the sequence of Gauss norms $|f_i|_{\exp(-s)}$ must converge. In fact, we can do a bit better:

Proposition 8. *Let s be a positive real number and let $f_1, f_2, \ldots \in \mathbf{A}_{\inf}[\frac{1}{p}, \frac{1}{[\pi]}]$ be a Cauchy sequence for the Gauss norm $| \bullet |_{\exp(-s)}$ which does not converge to zero. Then the sequences*

$$v_s(f_i) \qquad \partial_- v_s(f_i) \qquad \partial_+ v_s(f_i)$$

are eventually constant.

Proof. Set $a = \lim_{i\to\infty} v_s(f_i) = -\log(\lim_{i\to\infty} |f_i|_{\exp(-s)})$. We can then choose an integer $n \gg 0$ such that, for $n' > n$, we have $v_s(f_{n'} - f_n) > a$. Then $v_s(f_n) = a$. It follows by continuity that, for any fixed $n' > n$, there exists a small interval $I = (s - \epsilon, s + \epsilon)$ (depending on n') such that $v_t(f_{n'} - f_n) > v_t(f_n)$ for $t \in I$. It then follows that $v_t(f_{n'}) = v_t(f_n)$ for $t \in I$, so that

$$v_s(f_{n'}) = v_s(f_n) \qquad \partial_- v_s(f_{n'}) = \partial_- v_s(f_n) \qquad \partial_+ v_s(f_{n'}) = \partial_+ v_s(f_n).$$

$\square$

Corollary 9. *Fix $0 < a \leq b < 1$. Let $f_1, f_2, \ldots \in \mathbf{A}_{\inf}[\frac{1}{p}, \frac{1}{[\pi]}]$ be a Cauchy sequence for the Gauss norms $|\bullet|_a$ and $|\bullet|_b$. Assume that the sequence $\{f_n\}$ does not converge to zero for both of the Gauss norms $|\bullet|_a$ and $|\bullet|_b$. Then the sequence of functions $\{s \mapsto v_s(f_n)\}$ is eventually constant on the closed interval $[-\log(b), -\log(a)]$.*

Proof. Without loss of generality we may assume that each f_n is nonzero. We also assume that the sequence $\{f_n\}$ does not converge to zero with respect to $|\bullet|_b$ (the proof in the other case is similar. Set $s = -\log(b)$. It follows from Proposition 8 that we can choose an integer $m \gg 0$ such that the sequence of real numbers $\{v_s(f_{m'})\}_{m' \geq m}$ takes some constant value $r \in \mathbf{R}$ and the sequence of integers $\{\partial_+ v_s(f_{m'})\}_{m' \geq m}$ takes some constant value $k \in \mathbf{Z}$. Since each of the functions $v_\bullet(f_{m'})$ is concave, it follows that $v_t(f_{m'}) \leq r + k(t-s)$ for $t \geq s$. In particular, each $v_t(f_{m'})$ is bounded above by $r' = \max(r, r + k\log(\frac{b}{a}))$ on the interval $[-\log(b), -\log(a)]$.

Choose $n \gg m$ such that, for $n' \geq n$, we have $|f_{n'} - f_n|_\rho < \exp(-r')$ for $\rho \in [a,b]$, or equivalently we have

$$v_t(f_{n'} - f_n) > r'$$

for $t \in [-\log(b), -\log(a)]$. Combining this with the inequality $v_t(f_n) \leq r'$, we conclude that $v_t(f_n) = v_t(f_{n'})$ for $t \in [-\log(b), -\log(a)]$. □

Note that, for $f \in B_{[a,b]}$, the Gauss norm $|f|_\rho$ is well-defined for $\rho \in [a,b]$. Consequently, we can define $v_s(f) = -\log|f|_{\exp(-s)}$ for $s \in [-\log(b), -\log(a)]$. Write f as the limit of a sequence $\{f_n\}$ in $\mathbf{A}_{\inf}[\frac{1}{p}, \frac{1}{[\pi]}]$ which converges for the norms $|\bullet|_a$ and $|\bullet|_b$. If $f \neq 0$, then the sequence $\{f_n\}$ cannot converge to zero with respect to both $|\bullet|_a$ and $|\bullet|_b$. Applying Corollary 9, we deduce that there exists an integer $n \gg 0$ such that $v_s(f) = v_s(f_n)$ for each $s \in [-\log(b), -\log(a)]$. Combining this observation with Proposition 5, we obtain the following:

Corollary 10. *Let f be a nonzero element of $B_{[a,b]}$. Then the construction $s \mapsto v_s(f)$ determines a function*

$$v_\bullet(f) : [-\log(b), -\log(a)] \to \mathbf{R}$$

which is piecewise linear with integer slopes. Moreover, it is concave (that is, the slopes are decreasing).

Remark 11 (The Hadamard Three-Circle Theorem)**.** Corollary 10 has a counterpart in complex analysis. Suppose we are given positive real numbers $a < b$, and let $f : \{z \in \mathbf{C} : a \leq |z| \leq b\} \to \mathbf{C}$ be a continuous function which is holomorphic on the interior of the annulus. Then the classical *Hadamard three-circle theorem* asserts that the function

$$s \mapsto \sup_{|z| = e^s} |f(z)|$$

is convex on the interval $[\log(a), \log(b)]$ (for closer analogue in the setting of complex analysis, see Lecture 13).

Corollary 12. *Let f be a nonzero element of B. Then the construction $s \mapsto v_s(f)$ determines a function*

$$v_\bullet(f) : \mathbf{R}_{>0} \to \mathbf{R}$$

which is piecewise linear with integer slopes. Moreover, it is concave (that is, the slopes are decreasing).

We now turn to the proof of Theorem 1. First, we recall that the Gauss norms satisfy the identities

$$|\varphi(f)|_{\rho^p} = |f|_\rho^p \qquad |p^n f|_\rho = \rho^n |f|_\rho.$$

We can rewrite these identities as

$$v_{ps}(\varphi(f)) = p v_s(f) \qquad v_s(p^n f) = ns + v_s(f).$$

Consequently, if f is a nonzero element of B satisfying $\varphi(f) = p^n f$, we have

$$p v_{s/p}(f) = v_s(\varphi(f)) = v_s(p^n f) = ns + v_s(f).$$

For $s > 0$, set $h(s) = \partial_+ v_s(f)$. Right-differentiating the preceding identity with respect to s, we obtain an equality

$$h(s/p) = n + h(s).$$

However, since $s \mapsto v_s(f)$ is concave, the function h must be nonincreasing. We therefore have

$$h(s) \leq h(s/p) = n + h(s)$$

so that $n \geq 0$.

Lecture 12: Detection of Zeroes

October 29, 2018

Throughout this lecture, we fix an algebraically closed perfectoid field $C^\flat$ of characteristic p. Then every untilt K of $C^\flat$ is algebraically closed, so the map $\sharp : C^\flat \to K$ is surjective (we will prove this later).

Notation 1. We let Y denote the set of isomorphism classes of untilts (K, ι) of $C^\flat$. We will use the letter y to denote a typical point of Y. For $0 < a \leq b < 1$, we let $Y_{[a,b]}$ denote the subset of Y consisting of those points $y = (K, \iota)$ satisfying $a \leq |p|_K \leq b$.

We use letters like f and g to denote typical elements of the ring $B_{[a,b]}$. For $y \in Y_{[a,b]}$, we let $f(y)$ denote the image of f under the ring homomorphism $B_{[a,b]} \to K$ constructed in Lecture 5. We also let $\mathrm{ord}_y(f)$ denote the order of vanishing of f at the point y (so that $\mathrm{ord}_y(f) > 0$ if and only if $f(y) = 0$).

Our goal, over the next few lectures, is to prove the following result which was promised in Lecture 10:

Theorem 2. *Let f be a nonzero element of $B_{[a,b]}$. Then:*

(1) *The divisor $\mathrm{Div}_{[a,b]}(f) = \sum_{y \in B_{[a,b]}} \mathrm{ord}_y(f) \cdot y$ is finite. In other words, the order of vanishing $\mathrm{ord}_y(f)$ is finite for all points $y \in B_{[a,b]}$, and vanishes for all but finitely many points of $B_{[a,b]}$.*

(2) *Let g be another element of $B_{[a,b]}$. If $\mathrm{Div}_{[a,b]}(g) \geq \mathrm{Div}_{[a,b]}(f)$ (that is, if $\mathrm{ord}_y(g) \geq \mathrm{ord}_y(f)$ for all $y \in B_{[a,b]}$), then g is (uniquely) divisible by f.*

Let us first dispense with the uniqueness.

Proposition 3. *The ring $B_{[a,b]}$ is an integral domain.*

Proof. Set $\alpha = -\log(a)$ and $\beta = -\log(b)$. For each $f \in B_{[a,b]}$, we have a function

$$[\beta, \alpha] \to \mathbf{R} \cup \{\infty\} \qquad s \mapsto v_s(f) = -\log |f|_{\exp(-s)}.$$

We saw in the previous lecture that if f is not zero, then $v_\bullet(f)$ is a piecewise linear concave function on the interval $[\beta, \alpha]$ (with integer slopes): in particular, it is everywhere finite. If g is also nonzero, then $v_\bullet(g)$ is also a piecewise linear concave function. It follows that the function $v_s(fg) = v_s(f) + v_s(g) < \infty$ for all $s \in [\beta, \alpha]$, so that fg is nonzero in $B_{[a,b]}$. $\square$

We now consider a special case of Theorem 2, where $f = \xi$ is a *distinguished* element of the ring $\mathbf{A}_{\mathrm{inf}}$. In this case, assertion (1) is clear and (2) reduces to the following:

Proposition 4. *Let ξ be a distinguished element of $\mathbf{A}_{\mathrm{inf}}$ which vanishes at a point $y \in Y_{[a,b]}$. If $g \in B_{[a,b]}$ also vanishes at y, then g is divisible by ξ (uniquely, by virtue of Proposition 3).*

Proof. Suppose first that g belongs to the ring $\mathbf{A}_{\mathrm{inf}}[\frac{1}{p}, \frac{1}{[\pi]}]$. Then we can write $g = \frac{g_0}{p^m [\pi]^n}$ for some $g_0 \in \mathbf{A}_{\mathrm{inf}}$ and some $m, n \geq 0$. Since $g(y) = 0$, it follows that $g_0(y) = 0$: that is, g_0 belongs to the kernel of the map

$\theta : \mathbf{A}_{\mathrm{inf}} \to \mathcal{O}_K$ determined by the untilt $y = (K, \iota)$. We saw in Lecture 3 that this kernel is generated by the distinguished element ξ. We can therefore write $g_0 = \xi \cdot h_0$ for some $h_0 \in \mathbf{A}_{\mathrm{inf}}$. It follows that $g = \xi \cdot h$, where $h = \frac{h_0}{p^m [\pi]^n}$.

We now treat the general case. Let g be any element of $B_{[a,b]}$. Then we can write g as the limit of a sequence $g_1, g_2, g_3, \ldots \in \mathbf{A}_{\mathrm{inf}}[\frac{1}{p}, \frac{1}{[\pi]}]$. It follows that $g(y)$ is the limit of the sequence $\{g_i(y)\}$ in the field K corresponding to the point $y = (K, \iota) \in Y_{[a,b]}$. For each $i > 0$, write $g_i(y) = c_i^\sharp$ for some element $c_i \in C^\flat$. If $g(y) = 0$, then the sequence $\{g_i(y)\}$ converges to zero in K. Since $|g_i(y)|_K = |c_i^\sharp|_K = |c_i|_{C^\flat}$, it follows that the sequence $\{c_i\}$ converges to zero in $C_\flat$, and therefore the sequence of Teichmüller representatives $\{[c_i]\}$ converges to zero with respect to the Gauss norms $| \bullet |_a$ and $| \bullet |_b$. It follows that the sequence $\{g_i - [c_i]\}$ also converges to g. Replacing each g_i by $g_i - [c_i]$, we can assume that each g_i vanishes at the point y. By the first part of the proof, we can write $g_i = \xi \cdot h_i$, for some (uniquely determined) $h_i \in \mathbf{A}_{\mathrm{inf}}[\frac{1}{p}, \frac{1}{[\pi]}]$. We will complete the proof by showing that $\{h_i\}$ is a Cauchy sequence (for the Gauss norms $| \bullet |_a$ and $| \bullet |_b$): then it converges to a unique element $h \in B_{[a,b]}$ which satisfies $g = \xi \cdot h$ by continuity. To prove this, we observe that

$$|g_i - g_j|_a = |\xi \cdot (h_i - h_j)|_a = |\xi|_a \cdot |h_i - h_j|_a$$

so that

$$|h_i - h_j|_a = \frac{|g_i - g_j|_a}{|\xi|_a} \to 0$$

as $i, j \to \infty$ (and similarly for the Gauss norm $| \bullet |_b$). □

Proposition 4 suggests a strategy for proving Theorem 2. Let f be a nonzero element of $B_{[a,b]}$, and let g be another element satisfying $\mathrm{Div}_{[a,b]}(g) \geq \mathrm{Div}_{[a,b]}(f)$. Suppose that $\mathrm{Div}_{[a,b]}(f)$ is not zero: that is, there is some point $y_1 \in Y_{[a,b]}$ such that $f(y_1) = 0$. Then we also have $g(y_1) = 0$. Let $\xi_1 \in \mathbf{A}_{\mathrm{inf}}$ be a distinguished element which vanishes at y_1. Applying Proposition 4, we can write $f = \xi_1 \cdot f_1$ and $g = \xi_1 \cdot g_1$ for some elements $f_1, g_1 \in B_{[a,b]}$. Then we have $\mathrm{Div}_{[a,b]}(g_1) \geq \mathrm{Div}_{[a,b]}(f_1)$. We can then repeat the argument: if $\mathrm{Div}_{[a,b]}(f_1) \neq 0$, we can find another distinguished element ξ_2 vanishing at a point $y_2 \in Y_{[a,b]}$ such that $f_1 = \xi_2 \cdot f_2$ and $g_1 = \xi_2 \cdot g_2$. Continuing in this way, we obtain sequences $\{f_n\}$, $\{g_n\}$, and $\{\xi_n\}$ satisfying

$$f = \xi_1 \cdot \xi_2 \cdots \xi_n \cdot f_n \qquad g = \xi_1 \cdot \xi_2 \cdots \xi_n \cdot g_n.$$

To prove Theorem 2, we must show the following:

(1′) This process eventually stops: that is, we eventually end up in a situation where $\mathrm{Div}_{[a,b]}(f_n) = 0$. In this case, we have $\mathrm{Div}_{[a,b]}(f) = y_1 + y_2 + \cdots + y_n$.

(2′) When the process stops, the element f_n is a unit (and therefore automatically divides g_n).

We can prove (1′) very easily from the results of the previous lecture. Let h be any nonzero element of the ring $B_{[a,b]}$, and write h as the limit of a sequence $\{h_i \in \mathbf{A}_{\mathrm{inf}}[\frac{1}{p}, \frac{1}{[\pi]}]\}$ which converges for the Gauss norms $|\bullet|_a$ and $| \bullet |_b$. In the previous lecture, we proved that the sequence of functions $\{v_\bullet(h_i)\}$ is eventually constant on the interval $[\beta, \alpha]$. Moreover, we can do a little better: the sequence of left derivatives $\partial_- v_\beta(h_i)$ and right derivatives $\partial_+ v_\alpha(h_i)$ are eventually constant, converging to integers $\partial_- v_\beta(h)$ and $\partial_+ v_\alpha(h)$, respectively.

Exercise 5. Check that the integers $\partial_- v_\beta(h)$ and $\partial_+ v_\alpha(h)$ depend only on h, and not on the choice of Cauchy sequence $\{h_i\}$ converging to h.

Since each of the functions $v_\bullet(h_i)$ is concave, we have $\partial_- v_\beta(h_i) \geq \partial_+ v_\alpha(h_i)$ for each i, and therefore $\partial_- v_\beta(h) \geq \partial_+ v_\beta(h)$.

Proposition 6. *Let f be a nonzero element of $B_{[a,b]}$, and set $N = \partial_- v_\beta(f) - \partial_+ v_\alpha(f) \geq 0$. Then the construction sketched above must terminative in $\leq N$ steps. That is, f cannot be divisible by a product $\xi_1 \cdot \xi_2 \cdot \cdots \cdot \xi_{N+1}$ of distinguished elements ξ_i vanishing at points $y_i \in Y_{[a,b]}$.*

Proof. We saw in the previous lecture that if ξ is a distinguished element of $\mathbf{A}_{\rm inf}$, then the function $s \mapsto v_s(\xi)$ is given by the formula

$$v_s(\xi) = \begin{cases} s & \text{if } s \leq v(\xi) \\ v(\xi) & \text{otherwise.} \end{cases}.$$

In particular, if the point at which ξ vanishes belongs to $Y_{[a,b]}$, then $v(\xi)$ belongs to the interval $[\beta, \alpha]$, and therefore

$$\partial_- v_\beta(\xi) = 1 \qquad \partial_+ v_\alpha(\xi) = 0.$$

In particular, if we can write $f = \xi_1 \cdot \xi_2 \cdots \xi_{N+1} \cdot f_{N+1}$, then we have

$$\begin{aligned} N &= \partial_- v_\beta(f) - \partial_+ v_\alpha(f) \\ &= (\sum_{i=1}^{N+1} \partial_- v_\beta(\xi_i) - \partial_+ v_\alpha(\xi_i)) + (\partial_- v_\beta(f_{N+1}) - \partial_+ v_\alpha(f_{N+1}) \\ &\geq (\sum_{i=1}^{N+1} \partial_- v_\beta(\xi_i) - \partial_+ v_\alpha(\xi_i)) \\ &= N+1 \end{aligned}$$

which is a contradiction. □

We will deduce Theorem 2 from the following:

Theorem 7. *Let f be a nonzero element of $B_{[a,b]}$. The following conditions are equivalent:*

- (a) *The element f is invertible in $B_{[a,b]}$.*
- (b) *The integer $N = \partial_- v_\beta(f) - \partial_+ v_\alpha(f)$ is equal to zero (in particular, the concave function $s \mapsto v_s(f)$ is linear on the interval $[\beta, \alpha]$).*
- (c) *The divisor $\mathrm{Div}_{[a,b]}(f)$ is equal to zero. That is, there is no point $y \in Y_{[a,b]}$ such that $f(y) = 0$.*

Note that the implication $(b) \Rightarrow (c)$ follows from Proposition 6. The implication $(a) \Rightarrow (b)$ is also clear: if f has an inverse $f^{-1} \in B_{[a,b]}$, then it is easy to see that

$$\partial_- v_\beta(f^{-1}) = -\partial_- v_\beta(f) \qquad \partial_+ v_\alpha(f^{-1}) = -\partial_+ v_\alpha(f)$$

so that

$$N = \partial_- v_\beta(f) - \partial_+ v_\alpha(f) = -(\partial_- v_\beta(f^{-1}) - \partial_+ v_\alpha(f^{-1}) \leq 0.$$

Over the next few lectures, we will show (using different arguments) that both of the implications $(a) \Rightarrow (b)$ and $(b) \Rightarrow (c)$ is reversible. Once that is done, it will follow that $(c) \Rightarrow (a)$: that is, a nonzero element $f \in B_{[a,b]}$ satisfying $\mathrm{Div}_{[a,b]}(f) = 0$ is invertible. This will complete the proof of $(2')$ and with it the proof of Theorem 2.

Lecture 13: Digression-Jensen's Formula

October 31, 2018

The constructions of the previous lecture(s) have an analogue in complex analysis.Let a and b be positive real numbers, and let f be a holomorphic function defined on the annulus $\{z \in \mathbf{C} : a < |z| < b\}$ which is not identically zero. Define a function

$$A(\bullet, f) : (\log(a), \log(b)) \to \mathbf{R} \qquad A(s, f) = \frac{1}{2\pi} \int_0^{2\pi} \log |f(\exp(s + i\theta))| d\theta.$$

In other words, $A(s, f)$ is the average value of $|\log f(z)|$ on the circle $\{z \in \mathbf{C} : |z| = \exp(s)\}$ of radius $\exp(s)$.

Exercise 1. Show that the average $A(s, f)$ is well-defined even when f vanishes at some points of the circle $\{z \in \mathbf{C} : |z| = \exp(s)\}$: that is, the function $\theta \mapsto \log f(\exp(s + i\theta))$ is always integrable, even when it fails to be well-defined at finitely many points. Moreover, if the holomorphic function f is fixed, then $s \mapsto A(s, f)$ is continuous.

Example 2. Let f be the holomorphic function given by $f(z) = z$. Then we have

$$A(s, f) = \frac{1}{2\pi} \int_0^{2\pi} \log | \exp(s + i\theta)| d\theta \frac{1}{2\pi} \int_0^{2\pi} \log(\exp(s)) d\theta = s.$$

Example 3. Suppose that $f(z) = \exp(g(z))$, for some holomorphic function g on the annulus $\{z \in \mathbf{C} : a < |z| < b\}$. We then have

$$A(s, f) = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{Re}(g(\exp(s + i\theta))) d\theta = \mathrm{Re}(\overline{g}(\exp(s))),$$

where $\overline{g}$ is the radially symmetric holomorphic function given by

$$\overline{g}(z) = \frac{1}{2\pi} \int_0^{2\pi} g(\exp(i\theta) z) d\theta.$$

Note that $\overline{g}$ is a holomorphic function which is constant on each circle $\{z \in \mathbf{C} : |z| = \exp(s)\}$ and is therefore constant. It follows that the function $s \mapsto A(s, f)$ is constant on the interval $(\log(a), \log(b))$.

Example 4. Suppose that the holomorphic function $f(z)$ has no zeroes in the the annulus $\{z \in \mathbf{C} : a < |z| < b\}$. Then f determines a continuous map

$$\{z \in \mathbf{C} : a < |z| < b\} \to \mathbf{C} \setminus \{0\}$$

which has a well-defined *winding number* $N \in \mathbf{Z}$ around the origin. This winding number vanishes if and only if f can be written globally as the exponential of another holomorphic function $g(z)$. We can always arrange this by multiplying f by a suitable power of z: that is, we can write $f(z) = z^N \exp(g(z))$ for some holomorphic function g on $\{z \in \mathbf{C} : a < |z| < b\}$. In this case, we have

$$A(s, f) = A(s, z^N) + A(s, \exp(g)) = Ns + c$$

for some real constant c, by virtue of Examples 2 and 3.

Let us now return to the case of a general holomorphic function $f : \{z \in \mathbf{C} : a < |z| < b\} \to \mathbf{C}$. It follows from Example 4 that the function $s \mapsto A(s, f)$ is piecewise linear with integer slopes: it is linear when restricted to any interval $I \subseteq (\log(a), \log(b))$ whose interior does not contain any point of the form $\log|z|$, where z is a zero of f.

The function $s \mapsto A(s, f)$ fails to be linear exactly when s has the form $\log|z|$, where z is a zero of f. It follows from the above reasoning that the difference $\partial_+ A(s, f) - \partial_- A(s, f)$ is given by the difference between the winding numbers of the functions

$$\theta \mapsto f(\exp(s + \epsilon + i\theta)) \qquad \theta \mapsto f(\exp(s - \epsilon + i\theta))$$

where ϵ is some small real number. This difference is equal to the number of zeros of f on the circle $\{z \in \mathbf{C} : |z| = \exp(s)\}$, counted with multiplicity. In particular, it is a nonnegative integer. Non-negativity implies that the piecewise-linear function $s \mapsto A(s, f)$ is actually convex.

For nonzero $g \in B_{[a,b]}$, the function $s \mapsto v_s(g)$ appearing in Theorem 6 can be regarded as an analogue of the function $s \mapsto A(s, f)$ in the setting of p-adic geometry. More accurately, it is an analogue of the concave function $s \mapsto -A(-s, f)$, where the sign is a matter of convention.

Example 5 (Jensen's Formula)**.** Let f be a holomorphic function defined on the open disk $\{z \in \mathbf{C} : |z| < b\}$, and assume for simplicity that $f(0) \neq 0$. For $t > 0$, set $h(t) = A(\log(t), f)$. Then, for sufficiently small t, the function $h(t)$ takes the constant value $\log|f(0)|$. We therefore have

$$h(t) = \log|f(0)| + \int_0^t h'(r)dr = \log|f(0)| + \int_0^t \frac{A'(\log(r), f)}{r} dr.$$

Here $A'(\log(r), f)$ is a well-defined integer provided that f does not have any zeroes on the circle $\{z \in \mathbf{C} : |z| = r\}$, given by the number $N(r, f)$ of zeroes of f on the disk $\{z \in \mathbf{C} : |z| < r\}$ (counted with multiplicity). We may rewrite the preceding equality as

$$\frac{1}{2\pi} \int_0^{2\pi} \log|f(t\exp(i\theta))| = \log|f(0)| + \int_0^t \frac{N(r, f)}{r} dr,$$

which is an identity known as *Jensen's formula*.

Let us now return to the case of interest to us. Fix a perfectoid field $C^\flat$ of characteristic p and real numbers $0 < a \leq b < 1$, and set $\alpha = -\log(a)$, $\beta = -\log(b)$. Recall that, to every nonzero element $f \in B_{[a,b]}$, we can associate a convex piecewise linear function

$$v_\bullet(f) : [\beta, \alpha] \to \mathbf{R} \qquad s \mapsto v_s(f) = -\log|f|_{\exp(-s)}.$$

Heuristically, one can think of the function $s \mapsto v_\bullet(f)$ as a p-adic analogue of the function $s \mapsto -A(-s, f)$ constructed above (here the insertion of signs is really just a matter of convention). We saw in the previous lecture that this function is piecewise linear with integer slopes, and that it is *concave*. Even better, $v_\bullet(f)$ can be promoted to a *germ* of a piecewise linear function on a neighborhood of the interval $[\beta, \alpha]$: in other words, it has well-defined left and right derivatives

$$\partial_- v_\beta(f) \qquad \partial_+ v_\alpha(f),$$

these are integers, given by $\partial_- v_\beta(f')$ and $\partial_+ v_\alpha(f')$ where $f' \in \mathbf{A}_{\inf}[\frac{1}{p}, \frac{1}{[\pi]}]$ is any sufficiently close approximation to f with respect to the Gauss norms $|\bullet|_a$ and $|\bullet|_b$. In this case, the function $s \mapsto v_s(f')$ is defined for all $s \in \mathbf{R}_{>0}$ (and is piecewise linear with integer slopes), so the difference

$$\partial_- v_\beta(f) - \partial_+ v_\alpha(f) = \partial_- v_\beta(f') - \partial_+ v_\alpha(f') \geq 0$$

is a nonnegative integer. Our goal is to show that, like in the complex-analytic world, this integer has a concrete interpretation: when $C^\flat$ is algebraically closed, it is equal to the degree of the divisor $\mathrm{Div}_{[a,b]}(f)$ (that is, the number of points of $y \in Y_{[a,b]}$ where the function f vanishes, each counted with multiplicity $\mathrm{ord}_y(f)$). In the last lecture, we reduced the proof of this to the following assertion:

Theorem 6. *Let f be a nonzero element of $B_{[a,b]}$. Then:*

(1) *If $\partial_- v_\beta(f) = \partial_+ v_\alpha(f)$, then f is invertible in $B_{[a,b]}$.*

(2) *If $C^\flat$ is algebraically closed and $\partial_- v_\beta(f) \neq \partial_+ v_\alpha(f)$, then there is a point $y \in Y_{[a,b]}$ such that $f(y) = 0$.*

Remark 7. We saw in the previous lecture that the converse of (1) and (2) hold.

Warning 8. Let f be as in Theorem 6. If $\partial_- v_\beta(f) = \partial_+ v_\alpha(f)$, then the function $s \mapsto v_s(f)$ is linear on the interval $[\beta, \alpha]$. However, the converse is false in general. Note that if $a < b$, then the concavity of $v_\bullet(f)$ yields inequalities

$$\partial_- v_\beta(f) \geq \partial_+ v_\beta(f) \geq \partial_- v_\alpha(f) \geq \partial_+ v_\alpha(f),$$

and that $v_\bullet(f)$ is linear on the interval $[\beta, \alpha]$ if and only if we have an equality $\partial_+ v_\beta(f) \geq \partial_- v_\alpha(f)$. In the case where $C^\flat$ is algebraically closed, this is equivalent to the requirement that f does not vanish at any point of

$$Y_{(a,b)} = \{y = (K, \iota) \in Y : a < |p|_K < b\}.$$

However, it is possible for f to satisfy this condition while vanishing at points $y = (K, \iota)$ satisfying $|p|_K = a$ or $|p|_K = b$, in which case one of the inequalities

$$\partial_- v_\beta(f) \geq \partial_+ v_\beta(f) \qquad \partial_- v_\alpha(f) \geq \partial_+ v_\alpha(f)$$

would be strict and f would not be invertible.

Let us now prove part (1) of Theorem 6 (we will prove (2) in a future lecture). Assume first that f belongs to the ring $\mathbf{A}_{\inf}[\frac{1}{p}, \frac{1}{[\pi]}]$, and therefore admits a unique Teichmüller expansion

$$f = \sum_{n \gg -\infty} [c_n] p^n$$

where the real numbers $|c_n|_{C^\flat}$ are bounded. In this case, the function $s \mapsto v_s(f)$ is defined for all $s > 0$, and given by the formula

$$v_s(f) = \inf_{n \in \mathbf{Z}} (v(c_n) + ns).$$

The equality $\partial_- v_\beta(f) = \partial_+ v_\alpha(f)$ is equivalent to the requirement that the function $s \mapsto v_s(f)$ is linear in a small neighborhood of the interval $[\beta, \alpha]$, and therefore coincides with the linear function

$$s \mapsto v(c_{n_0} s) + n_0 s = v_s([c_{n_0}] p^{n_0})$$

on that neighborhood; it follows that we have

$$v(c_n) + ns > v(c_{n_0}) + n_0 s$$

for $n_0 \neq n$ and $s \in [\beta, \alpha]$. Restated in terms of absolute values, we have

$$|c_{n_0}|_{C^\flat} \rho^{n_0} > |c_n|_{C^\flat} \rho^n$$

for all $n \neq n_0$ and $\rho \in [a, b]$.

We wish to show that in this case, the function f is invertible. Replacing f by the quotient $\frac{f}{[c_{n_0}] p^{n_0}}$, we can reduce to the case where $n_0 = 0$ and $[c_{n_0}] = 1$, so that our inequality can be rewritten as

$$1 > |c_n|_{C^\flat} \rho^n$$

for $n \neq 0$ and $\rho \in [a, b]$. Setting $\epsilon = \sum_{n \neq 0} [c_n] p^n$, we have $|\epsilon|_\rho = \sup\{|c_n| \rho^n\}_{n \neq 0} < 1$. It follows that ϵ is topologically nilpotent in the ring $B_{[a,b]}$, so that $f = 1 + \epsilon$ has an inverse given by the convergent sum

$$f^{-1} = (1+\epsilon)^{-1} = 1 - \epsilon + \epsilon^2 - \epsilon^3 + \cdots .$$

This completes the proof of part (1) in the special case where f belongs to $\mathbf{A}_{\mathrm{inf}}[\frac{1}{p}, \frac{1}{[\pi]}]$.

We now treat the general case. Let f be a nonzero element of $B_{[a,b]}$, which we write as the limit of a sequence

$$f_1, f_2, f_3, \cdots \in \mathbf{A}_{\mathrm{inf}}[\frac{1}{p}, \frac{1}{[\pi]}]$$

which is Cauchy for the Gauss norms $| \bullet |_\rho$ for $\rho \in [a, b]$. We proved in Lecture 11 that we have

$$\partial_- v_\beta(f) = \partial_- v_\beta(f_n) \qquad \partial_+ v_\alpha(f) = \partial_+ v_\alpha(f_n)$$

for $n \gg 0$. Passing to a subsequence, we may assume that these equalities hold for all n. In this case, our hypothesis $\partial_- v_\beta(f) = \partial_+ v_\alpha(f)$ guarantees that we also ahve $\partial_- v_\beta(f_n) = \partial_+ v_\alpha(f_n)$ for each n. By the special case treated above, this means that each f_n is invertible when regarded as an element of the completion $B_{[a,b]}$. Let us denote the inverse by f_n^{-1}. For each $\rho \in [a, b]$, we have

$$|f_m^{-1} - f_n^{-1}|_\rho = |\frac{f_n - f_m}{f_m \cdot f_n}|_\rho = \frac{|f_n - f_m|_\rho}{|f_m|_\rho \cdot |f_n|_\rho}.$$

As m and n tend to infinity, the numerator of this expression goes to zero (since the sequence $\{f_n\}$ is Cauchy with respect to the Gauss norm $| \bullet |_\rho$) and the denominator tends to $|f|_\rho^2$. It follows that the quantity $|f_m^{-1} - f_n^{-1}|_\rho$ tends to zero. That is, $\{f_n^{-1}\}$ is a Cauchy sequence with respect to each of the Gauss norms $| \bullet |_\rho$ for $\rho \in [a, b]$, and therefore converges to an element $g \in B_{[a,b]}$. It follows by continuity that $f \cdot g = 1$, so that f is invertible as desired.

Lecture 14: The Metric Structure of Y

November 2, 2018

Throughout this lecture, we fix an algebraically closed perfectoid field $C^\flat$ of characteristic p. Fix real numbers $0 < a \leq b < 1$ and set $\alpha = -\log(a)$, $\beta = -\log(b)$. Recall that our goal is to prove the following:

Theorem 1 (Existence of Roots, Version 1)**.** *Let f be a nonzero element of $B_{[a,b]}$, and suppose that $\partial_- v_\beta(f) \neq \partial_+ v_\alpha(f)$. Then there is a point $y \in Y_{[a,b]}$ such that $f(y) = 0$.*

Note that, in the situation of Theorem 1, we can find some $\rho \in [a, b]$ such that $\partial_- v_s(f) \neq \partial_+ v_s(f)$ for $s = -\log(\rho)$. To prove that Theorem 1 has a root in $Y_{[a,b]}$, it suffices to show that it has a root in $Y_{[\rho,\rho]}$. That is, we may assume without loss of generality that $a = \rho = b$. It will therefore suffice to prove the following special case of Theorem 1:

Theorem 2 (Existence of Roots, Version 2)**.** *Let f be a nonzero element of $B_{[\rho,\rho]}$ and set $s = -\log(\rho)$. If $\partial_- v_s(f) > \partial_+ v_s(f)$, then f vanishes at some point $y \in Y_{[\rho,\rho]}$.*

Note that, using the arguments of Lectures 12 and 13, Theorem 2 is equivalent to the following apparently stronger statement:

Corollary 3. *Let f be a nonzero element of $B_{[\rho,\rho]}$. Then f admits a factorization*

$$f = g \cdot \xi_1 \cdot \xi_2 \cdot \cdots \cdot \xi_n,$$

where each ξ_i is a distinguished element of $\mathbf{A}_{\text{inf}}$ vanishing at some point $y_i \in Y_{[\rho,\rho]}$, and g is an invertible element of $B_{[\rho,\rho]}$.

Here the hypothesis that $C^\flat$ is algebraically closed is essential: if $C^\flat$ is not algebraically closed, then the function $s \mapsto v_s(f)$ can fail to be differentiable due to "zeroes" coming from untilts of finite extension fields of $C^\flat$, rather than of $C^\flat$ itself. To say that $C^\flat$ is algebraically closed is to say that any *polynomial* equation in $C^\flat$ has a solution. To get from there to solving "analytic" equations like $f(y) = 0$, we will need to make some approximation arguments.

Notation 4. Let $\overline{Y}$ denote the set of isomorphism classes of untilts of $C^\flat$. We write $\overline{Y} = Y \cup \{0\}$, where Y is the set of isomorphism classes of characteristic zero untilts of $C^\flat$ and 0 denotes the isomorphism class of the characteristic p untilt (given by $C^\flat$ itself). For each point $y \in \overline{Y}$, we let ξ_y denote a distinguished element of $\mathbf{A}_{\text{inf}}$ which vanishes at y (so ξ_y is determined up to multiplication by a unit in $\mathbf{A}_{\text{inf}}$); for example, we can take $\xi_0 = p$.

For every pair of points $x, y \in \overline{Y}$, we let $d(x, y)$ denote the absolute value $|\xi_x(y)|_K$, where $y = (K, \iota)$. We will refer to $d(x, y)$ as the *distance from x to y*. Note that this quantity does not depend on the choice of distinguished element ξ_x: if ξ'_x is another distinguished element of $\mathbf{A}_{\text{inf}}$ vanishing at x, then $\xi_x(y)$ and $\xi'_x(y)$ differ by multiplication by a unit in $\mathcal{O}_K$, and therefore have the same absolute value in K.

Example 5. For $y = (K, \iota) \in \overline{Y}$, we have $d(0, y) = |p|_K$; this is the "distance from the origin" that we introduced earlier.

Proposition 6. *The function $d : \overline{Y} \times \overline{Y} \to \mathbf{R}_{\geq 0}$ is an ultrametric. That is, we have*

$$d(x,y) = 0 \Leftrightarrow x = y$$

$$d(x,y) = d(y,x)$$

$$d(x,z) \leq \max\{d(x,y), d(y,z)\}.$$

Proof. Note that $d(x,y) = 0$ if and only if the distinguished element ξ_x vanishes at y, which holds if and only if $x = y$.

Fix any pair of points $x, y \in \overline{Y}$, corresponding to untilts K_x and K_y of $C^\flat$. Since $C^\flat$ is algebraically closed, we can write $\xi_x(y) = c^\sharp$ for some $c \in C^\flat$. Then c belongs to the maximal ideal $\mathfrak{m}_C^\flat$, so that ξ_x and $\xi_x - [c]$ have the same image under the map

$$\mathbf{A}_{\text{inf}} = W(\mathcal{O}_C^\flat) \to W(\mathcal{O}_C^\flat / \mathfrak{m}_C^\flat) = W(k).$$

It follows that $\xi_x - [c]$ is also a distinguished element of $\mathbf{A}_{\text{inf}}$ which vanishes at the point y. We may therefore assume without loss of generality that $\xi_y = \xi_x - [c]$, so that

$$d(y,x) = |\xi_y(x)|_{K_x} = |\xi_x(x) - c^\sharp|_{K_x} = |c|_{C^\flat} = |c^\sharp|_{K_y} = |\xi_x(y)|_{C^\flat} = d(x,y);$$

here we write $c^\sharp$ both for the image of $[c]$ in K_x and its image in K_y.

To prove the third assertion, suppose we are given a point $z \in \overline{Y}$ corresponding to an untilt K_z. We then have

$$d(x,z) = |\xi_x(z)|_{K_z} = |\xi_y(z) + c^\sharp|_{K_z} \leq \max(|\xi_y(z)|_{K_z}, |c^\sharp|_{K_z}) = \max(d(y,z), d(x,y))$$

(where this time $c^\sharp$ denotes the image of $[c]$ in K_z). $\square$

Proposition 7. *The set $\overline{Y}$ is* complete *with respect to the metric $d(x,y)$.*

Proof. Suppose we are given a Cauchy sequence $y_0, y_1, y_2, \ldots \in \overline{Y}$. Let ξ_{y_0} be a distinguished element of $\mathbf{A}_{\text{inf}}$ which vanishes at y_0. Arguing as in the proof of Proposition 6, we can choose a sequence distinguished elements ξ_{y_n} vanishing at the points y_n, such that

$$\xi_{y_n} = \xi_{y_{n-1}} + [c_n],$$

where c_n is an element of $C^\flat$ satisfying $|c_n|_{C^\flat} = d(y_{n-1}, y_n)$.

Let $\pi \in C^\flat$ be a pseudo-uniformizer. Since the sequence $\{y_n\}$ is Cauchy, the sum

$$\sum_{n>0} [c_n]$$

converges with respect to the $[\pi]$-adic topology on $\mathbf{A}_{\text{inf}}$ (recall that $\mathbf{A}_{\text{inf}}$ is $[\pi]$-adically complete, since it is p-adically complete and p-torsion free and $\mathcal{O}_C^\flat = \mathbf{A}_{\text{inf}}/(p)$ is π-adically complete and π-torsion free). Set $\xi = \xi_{y_0} + \sum_{n>0}[c_n]$. Then ξ and ξ_{y_0} have the same image under the map

$$\mathbf{A}_{\text{inf}} = W(\mathcal{O}_C^\flat) \to W(\mathcal{O}_C^\flat / \mathfrak{m}_C^\flat) = W(k),$$

so ξ is a distinguished element of $\mathbf{A}_{\text{inf}}$ vanishing at some point $y \in \overline{Y}$. We then compute

$$d(y, y_m) = |\xi(y_m)|_{K_{y_m}} = |\xi_m(y_m) + \sum_{n>m} c_n^\sharp|_{K_{y_m}} \leq \max\{|c_n|_{C^\flat}\}_{n>m},$$

which tends to zero as $m \to \infty$. It follows that the Cauchy sequence $\{y_n\}$ converges to y. $\square$

Let us now return to the situation of Theorem 2. Fix $0 < \rho < 1$, and let f be a nonzero element of $B_{[\rho,\rho]}$. Recall that, if $y = (K, \iota)$ is a point of Y satisfying $d(0, y) = |p|_K = \rho$, then we have $|f(y)|_K \leq |f|_\rho$. In general, this inequality is strict. However, if f is an *invertible* element of $B_{[\rho,\rho]}$, then we also have

$$|(\frac{1}{f})(y)|_K \leq |\frac{1}{f}|_\rho = \frac{1}{|f|_\rho},$$

which implies that $|f(y)|_K = |f|_\rho$.

Let us say that an element f of $B_{[\rho,\rho]}$ is *good* if it satisfies the conclusion of Corollary 3: that is, if f admits a factorization

$$f = g \cdot \xi_1 \cdot \xi_2 \cdot \cdots \cdot \xi_n,$$

where g is an invertible element of $B_{[\rho,\rho]}$ and each ξ_i is a distinguished element vanishing at some point $y_i \in Y_{[a,b]}$. We then compute

$$\begin{aligned} |f(y)|_K &= |g(y)|_K \cdot |\xi_1(y)|_K \cdot \cdots \cdot |\xi_n(y)|_K \\ &= |g|_\rho \cdot \prod_{i=1}^{n} d(y_i, y) \\ &= \frac{|f|_\rho}{\prod_{i=1}^{n} |\xi_i|_\rho} \prod_{i=1}^{n} d(y_i, y) \\ &= |f|_\rho \prod_{i=1}^{n} \frac{d(y_i, y)}{\rho}. \end{aligned}$$

In other words, in particular, we see that the equality $|f(y)|_K = |f|_\rho$ holds in the *generic case* where y is at distance ρ from each of the zeroes y_i of the function f. However, we have a strict inequality whenever $d(y_i, y) < \rho$ for some i.

Proposition 8. *Let f be a good element of $B_{[\rho,\rho]}$ having n zeroes in $Y_{[\rho,\rho]}$ (counted with multiplicity), and let g be any nonzero element of $B_{[\rho,\rho]}$. Suppose that $|f - g|_\rho < |f|_\rho$. Then, for any point $y = (K, \iota) \in Y_{[\rho,\rho]}$ satisfying $g(y) = 0$, there exists a point $y' \in B_{[\rho,\rho]}$ satisfying $f(y') = 0$ and $d(y, y') < \rho(\frac{|f-g|_\rho}{|f|_\rho})^{1/n}$.*

Proof. Let $y_1, \ldots, y_n$ be the zeroes of f (counted with multiplicity). If $g(y) = 0$, we have

$$\begin{aligned} |f - g|_\rho &\geq |(f - g)(y)|_K \\ &= |f(y)|_K \\ &= |f|_\rho \prod_{i=1}^{n} \frac{d(y_i, y)}{\rho}. \end{aligned}$$

It follows that at least one of the factors $\frac{d(y_i,y)}{\rho}$ must be less than or equal to $(\frac{|f-g|_\rho}{|f|_\rho})^{1/n}$. □

Corollary 9. *Let f be a nonzero element of $B_{[\rho,\rho]}$ which is given as the limit of a Cauchy sequence $\{f_i\}$ with respect to the Gauss norm $| \bullet |_\rho$. Suppose that each f_i is good. If $\partial_- v_s(f) > \partial_+ v_s(f)$ for $s = -\log(\rho)$, then f vanishes at some point in $Y_{[\rho,\rho]}$.*

Proof. Passing to a subsequence, we may assume that

$$v_s(f) = v_s(f_i) \qquad \partial_- v_s(f) = \partial_- v_s(f_i) \qquad \partial_+ v_s(f) = \partial_+ v_s(f_i)$$

$$|f_{i+1} - f_i|_\rho < |f|_\rho$$

for all i. Set $n = \partial_- v_s(f) - \partial_+ v_s(f) > 0$. Then each f_i has exactly n zeroes in $Y_{[\rho,\rho]}$, counted with multiplicity. Applying Proposition 8, we can choose a sequence $\{y_i\}$ in $Y_{[\rho,\rho]}$ such that $f_i(y_i) = 0$ and

$$d(y_{i+1}, y_i) \leq \rho(\frac{|f_{i+1} - f_i|_\rho}{|f|_\rho})^{1/n}$$

It follows that the sequence $\{y_i\}$ is Cauchy and therefore converges to some point $y \in \overline{Y}$ (Proposition 7). We then have

$$|f_i(y)|_K \leq |f_i|_\rho \cdot \frac{d(y_i, y)}{\rho} = |f|_\rho \cdot \frac{d(y_i, y)}{\rho} \to 0$$

as $i \to \infty$, so $f(y) = \lim_{i\to\infty} f_i(y)$ vanishes in K. $\square$

Lecture 15: Zeroes of Primitive Elements

November 6, 2018

Throughout this lecture, we fix an algebraically closed perfectoid field $C^\flat$ of characteristic p. Our goal is to show if f is a nonzero element of the ring $B_{[\rho,\rho]}$ which is not invertible, then f vanishes at some point y lying on the "circle"

$$Y_{[\rho,\rho]} = \{y = (K, \iota) \in Y : |p|_K = \rho\}.$$

In the last lecture, we said that an element f of $B_{[\rho,\rho]}$ is *good* if we can write

$$f = g \cdot \xi_1 \cdot \xi_2 \cdot \dots \cdot \xi_n,$$

where g is invertible and each ξ_i is a distinguished element of $\mathbf{A}_{\text{inf}}$ vanishing at some point $y_i \in Y_{[\rho,\rho]}$. Moreover, we proved that if f is a nonzero, noninvertible element of $B_{[\rho,\rho]}$ which can be realized as the limit $\varinjlim_{i\to\infty} f_i$ (with respect to the Gauss norm $|\bullet|_\rho$ where each f_i is good, then f has a zero. Note that any element of $B_{[\rho,\rho]}$ can be approximated arbitrarily well by elements of the ring $\mathbf{A}_{\text{inf}}[\frac{1}{p}, \frac{1}{[\pi]}]$, which admit Teichmüller expansions

$$\sum_{n \gg -\infty} [c_n] p^n.$$

Moreover, every such sum can be approximated arbitrarily well by elements with *finite* Teichmüller expansions. It will therefore suffice to prove the following:

Proposition 1. *Let f be an element of $B_{[\rho,\rho]}$ which admits a finite Teichmüller expansion $\sum_{n=-N}^{N} [c_n] p^n$. Then f is good.*

It will be convenient to introduce a bit of terminology.

Definition 2. Let f be an element of $\mathbf{A}_{\text{inf}}$, given by a Teichmüller expansion $\sum_{n\geq 0} [c_n] p^n$. We say that f is *primitive* if $c_0 \neq 0$ and $|c_d|_{C^\flat} = 1$ for some integer d. If d is the smallest such integer, then we say that f is *primitive of degree d*.

Remark 3. Let f be an element of $\mathbf{A}_{\text{inf}}$. Then f is primitive if and only if it satisfies the following conditions:

- The element f is not divisible by p.
- The element f has nonzero image $\overline{f}$ under the map

  $$\mathbf{A}_{\text{inf}} = W(\mathcal{O}_C^\flat) \to W(\mathcal{O}_C^\flat / \mathfrak{m}_C^\flat) = W(k),$$

  where k denotes the residue field of C. In this case, the degree d of f is characterized by the equality of ideals $(p^d) = (\overline{f})$ in $W(k)$.

It follows that if $f = gh$ is primitive, then g and h are also primitive, with $\deg(f) = \deg(g) + \deg(h)$.

Remark 4. An element $f \in \mathbf{A}_{\text{inf}}$ is primitive of degree 1 if and only if it is distinguished and corresponds to a *characteristic zero* untilt of $C^\flat$.

Remark 5. Let f be an element of $B_{[\rho,\rho]}$ which admits a finite Teichmüller expansion $\sum_{n=-N}^{N}[c_n]p^n$. Then we can write $f = p^m \cdot [c] \cdot g$, where m is an integer, c is a nonzero element of $C^\flat$, and $g \in \mathbf{A}_{\inf}$ is primitive. Moreover, the integer m and the absolute value $|c|_{C^\flat}$ are uniquely determined.

Exercise 6. Let $f = \sum_{n \gg -\infty}[c_n]p^n$ be an element of $\mathbf{A}_{\inf}[\frac{1}{p}, \frac{1}{[\pi]}]$. Show that the following conditions are equivalent:

- The element f is nonzero and the supremum $\sup_n\{|c_n|_{C^\flat}\}$ is actually achieved for some integer n.
- The function $s \mapsto v_s(f)$ fails to be differentiable at only finitely many points.
- The element f factors as a product $p^m \cdot [c] \cdot g$, where $g \in \mathbf{A}_{\inf}$ is primitive.

Example 7. Let ϵ be an element of $1 + \mathfrak{m}_C^\flat$. Then the element $[\epsilon] - 1 \in \mathbf{A}_{\inf}$ is not primitive, since it has vanishing image in $W(k)$. More explicitly, the problem is that $[\epsilon]-1$ has infinitely many zeroes in Y: we have

$$\begin{aligned} [\epsilon] - 1 &= \frac{[\epsilon]-1}{[\epsilon^{1/p}]-1} \cdot ([\epsilon^{1/p}] - 1) \\ &= \frac{[\epsilon]-1}{[\epsilon^{1/p}]-1} \cdot \frac{[\epsilon^{1/p}]-1}{[\epsilon^{1/p^2}]-1} \cdot ([\epsilon^{1/p^2}] - 1) \\ &= \cdots \end{aligned}$$

We will deduce Proposition 1 from the following:

Proposition 8. *Let f be an element of $\mathbf{A}_{\inf}$ which is primitive of degree $d > 0$. Then f admits a factorization*

$$f = \xi_1 \cdot \xi_2 \cdot \ldots \cdot \xi_d,$$

where each ξ_i is a distinguished element vanishing at some point $y_i \in Y$.

Proof of Proposition 1 from Proposition 8. Let f be an element of $B_{[\rho,\rho]}$ which admits a finite Teichmüller expansion. By Remark 5, we can write $f = p^m \cdot [c] \cdot g$, where g is a primitive element of $\mathbf{A}_{\inf}$. Let d be the degree of g. If $d = 0$, then g is invertible in $\mathbf{A}_{\inf}$ and therefore f is invertible in $B_{[\rho,\rho]}$. Otherwise, we can use Proposition 8 to write $g = \xi_1 \cdot \ldots \cdot \xi_d$, where each ξ_i is a distinguished element of $\mathbf{A}_{\inf}$ vanishing at some point $y_i \in Y$. Rearranging the product if necessary, we may assume that $y_1, y_2, \cdots, y_m$ belong to the circle $Y_{[\rho,\rho]}$, and that $y_i \notin Y_{[\rho,\rho]}$ for $m < i \leq d$. Then ξ_i is invertible in $B_{[\rho,\rho]}$ for $m < i \leq d$ (this follows from Lecture 13, but is also easy to see directly). The factorization

$$f = (p^m \cdot [c] \cdot \prod_{i=m+1}^{d} \xi_i) \cdot \xi_1 \cdot \cdots \cdot \xi_m$$

now shows that f is good. $\square$

Let $f = \sum_{n \geq 0}[c_n]p^n$ be an element of $\mathbf{A}_{\inf}$ which is primitive of degree d, and consider the function

$$v_\bullet(f) : \mathbf{R}_{>0} \to \mathbf{R} \qquad s \mapsto v_s(f) = \inf_n (v(c_n) + ns).$$

Note that for $n \geq d$, we have $v(c_n) + ns > v(c_d) + ds = ds$, so we might as well only take the infimum only over the set $\{0, 1, \ldots, d\}$. For s sufficiently small, this infimum is realized when $n = d$ and we have $v_s(f) = ds$. When s is sufficiently large, the infimum is realized when $n = 0$ and we have $v_s(f) = v(c_0)$. It follows that, if $d \neq 0$, then there is some smallest positive real number s such that the function $v_\bullet(f)$ is not differentiable at s: that is, for which $\partial_- v_s(f) > \partial_+ v_s(f)$. Write $s = -\log(\lambda)$ for $\lambda \in (0,1)$. We will prove the following:

Proposition 9. *Let f be an element of $\mathbf{A}_{\mathrm{inf}}$ which is primitive of degree $d > 0$, and let $\lambda \in (0,1)$ be defined as above. Then there exists a point $y \in Y_{[\lambda,\lambda]}$ satisfying $f(y) = 0$.*

Proof of Proposition 8 from Proposition 9. Let f be an element of $\mathbf{A}_{\mathrm{inf}}$ which is primitive of degree d; we wish to show that f can be written as a product of d distinguished elements (corresponding to characteristic zero untilts of $C^\flat$). We proceed by induction on d; the case $d = 1$ is immediate from Remark 4. To carry out the inductive hypothesis, we observe that Propositino 9 guarantees that there is a point $y \in Y$ satisfying $f(y) = 0$, so that f factors as a product $f = g \cdot \xi$, where ξ is a distinguished element vanishing at y. Remark 3 then shows that g is primitive of degree $d-1$ and can therefore be factored as a product of $d-1$ distinguished elements by our inductive hypothesis. $\square$

Remark 10. To prove Proposition 8 from Proposition 9, the equality $d(y, 0) = \lambda$ is irrelevant. We include it in the statement of Proposition 9 to highlight that the overall strategy is a bit subtle. Let f be a nonzero element of $B_{[\rho,\rho]}$ which is not invertible. Then f is *a priori* defined *only* at points belonging to the circle $Y_{[\rho,\rho]}$, and we wish to show that there is a point $y \in Y_{[\rho,\rho]}$ satisfying $f(y) = 0$. To find this point, we are writing f as the limit $\varinjlim f_n$ of elements which admit finite Teichmüller expansions, and can therefore be evaluated at *any* point of Y. For $n \gg 0$, we expect that the functions f_n must also vanish at some point $y_n \in Y_{[\rho,\rho]}$, and the argument of Lecture 14 shows that we can choose these points so that the sequence $\{y_n\}$ converges to a point $y \in Y_{[\rho,\rho]}$ where f vanishes. However, Proposition 9 does not produce the points y_n directly. Each f_n vanishes at finitely many points of Y, some of which lie on the circle $Y_{[\rho,\rho]}$ and some of which do not. The proof of Proposition 9 will actually select a zero of f_n that is *furthest from the origin*. In order to find the desired zero lying on the circle $Y_{[\rho,\rho]}$, we actually need to apply Proposition 9 repeatedly (to primitive elements of $\mathbf{A}_{\mathrm{inf}}$ which are factors of $p^m \cdot [c] \cdot f_n$)

Let us now begin the proof of Proposition 9. Note that we have $v_s(f) = ds$ for $s \in (0, -\log(\lambda)]$, In particular, we have $v_{-\log(\lambda)}(f) = -d\log(\lambda)$, or equivalently $|f|_\lambda = \lambda^d$. It follows that for any point $y = (K_y, \iota)$ lying on the circle $Y_{[\lambda,\lambda]}$, we have $|f(y)|_{K_y} \leq |f|_\lambda = \lambda^d$. We saw in the previous lecture that if f is good (when regarded as an element of $B_{[\lambda,\lambda]}$), then the equality is strict if and only if y is "close" to a point where f vanishes: that is, if and only if there is a point $y' \in Y_{[\lambda,\lambda]}$ satisfying $d(y, y') < \lambda$ and $f(y') = 0$. Of course, we do not yet know that f is good (that's a special case of what we are trying to prove). But it suggests that if we can find a point $y \in Y_{[\lambda,\lambda]}$ with $|f(y)|_{K_y} < \lambda^d$, then we will be on the right track. We therefore begin by proving the following weaker version of Proposition 9 (we will complete the proof in the next lecture).

Lemma 11. *Let f be an element of $\mathbf{A}_{\mathrm{inf}}$ which is primitive of degree $d > 0$, and let $\lambda \in (0,1)$ be defined as above. Then there exists a point $y \in Y_{[\lambda,\lambda]}$ satisfying $|f(y)|_{K_y} \leq \lambda^{d+1}$.*

Proof. Write $f = \sum_{n \geq 0} [c_n] p^n$. Multiplying f by $[c_d^{-1}]$ if necessary, we may assume without loss of generality that $c_d = 1$. By our choice of λ, we have

$$|c_i|_{C^\flat} \lambda^i \leq \lambda^d$$
$$|c_i|_{C^\flat} \leq \lambda^{d-i}$$

and that equality holds for at least one value of i.

Consider the polynomial

$$F(x) = x^d + c_{d-1}x^{d-1} + \cdots + c_1 x + c_0 \in C^\flat[x].$$

Since $C^\flat$ is algebraically closed, this polynomial factors as a product of linear factors: that is, we can find elements $r_1, \ldots, r_d \in C^\flat$ satisfying $F(x) = (x - r_1)(x - r_2) \cdots (x - r_d)$. Let λ' denote the largest of the absolute values of these roots (we will see in a moment that $\lambda' = \lambda$). Without loss of generality, we may assume that

$$|r_i|_{C^\flat} = \lambda \text{ for } i = 1, \ldots, m \qquad |r_i|_{C^\flat} < \lambda \text{ for } i = m+1, m+2, \ldots, d.$$

Let $e_m(r_1, \ldots, r_d)$ denote the mth symmetric function of r_1 through r_d. Then

$$e_m(r_1, \ldots, r_d) = r_1 \cdots r_m + \text{ terms of absolute value } < \lambda'^m \ .$$

We therefore have

$$\lambda'^m = |e_m(r_1, \ldots, r_d)|_{C^\flat} = |c_{n-m}|_{C^\flat} \leq \lambda^m.$$

On the other hand, there is some integer $0 \leq i < d$ satisfying

$$\lambda^{d-i} = |c_i|_{C^\flat} = |e_{d-i}(r_1, \ldots, r_d)|_{C^\flat} \leq \lambda'^{d-i}.$$

Combining these, we obtain $\lambda = \lambda'$.

Set $r = r_1$, and note that $c_i = \pm e_{d-i}(r_1, \ldots, r_n)$ is divisible by r^{d-i} for $0 \leq i \leq d$. Set $\xi = p - [r]$. Then ξ is a distinguished element of $\mathbf{A}_{\mathrm{inf}}$ vanishing at a point $y \in Y$ satisfying $d(0, y) = |r|_{C^\flat} = \lambda' = \lambda$. Let K denote the corresponding untilt of $C^\flat$ and $\theta : \mathbf{A}_{\mathrm{inf}} \twoheadrightarrow \mathcal{O}_K$ associated quotient map. Then

$$\begin{aligned}
p^{-d} f(y) &= p^{-d}\theta(f) \\
&= \sum_{n \geq 0} c_n^\sharp p^{n-d} \\
&\equiv \sum_{i=0}^{d} (\frac{c_i}{r^{d-i}})^\sharp \pmod{p} \\
&\equiv (\sum_{i=0}^{d} \frac{c_i}{r^{d-i}})^\sharp \pmod{p} \\
&= (r^{-d} F(r))^\sharp \\
&= 0.
\end{aligned}$$

In other words, we have $f(y) \equiv 0 \pmod{p^{d+1}}$ in $\mathcal{O}_K$, which is equivalent to the desired inequality $|f(y)|_K \leq \lambda^{d+1}$. $\square$

Lecture 16: Converging to a Zero

November 8, 2018

Throughout this lecture, we fix an algebraically closed perfectoid field $C^\flat$ of characteristic p. Let f be an element of the ring $\mathbf{A}_{\text{inf}}$ which is primitive of degree d: that is, an element which admits a Teichmüller expansion $\sum_{n\geq 0}[c_n]p^n$ satisfying

$$c_0 \neq 0 \qquad |c_i|_{C^\flat} < 1 \text{ for i ¡ d} \qquad |c_d|_{C^\flat} = 1.$$

Assume that $d > 0$, and let $\lambda \in (0,1)$ be the largest element for which the function $s \mapsto v_s(f)$ fails to be differentiable at $-\log(\lambda)$; that is, λ satisfies

$$|c_i|\lambda^i \leq \lambda^d \text{ for all } i$$

$$|c_i|\lambda^i = \lambda^d \text{ for some } i < d$$

Our goal in this lecture is to complete the proof of the following result:

Proposition 1. *Then there exists a point $y \in Y$ satisfying $d(0,y) = \lambda$ and $f(y) = 0$.*

Note that we have $|f|_\lambda = \lambda^d$. Consequently, for each point $y \in Y$ satisfying $d(0,y) = \lambda$, we automatically have

$$|f(y)| \leq |f|_\lambda = \lambda^d.$$

Moreover, we expect the inequality to be strict if and only if y is "close" to a root of f. More precisely, if f factors as a product of distinguished elements of $\mathbf{A}_{\text{inf}}$ (which will follow once Proposition 1 has been proved), then we expect

$$|f(y)| = \lambda^d \cdot \prod \frac{d(y',y)}{\lambda},$$

where the product is taken over the collection of all y' satisfying $d(0,y') = \lambda$ and $f(y') = 0$ (counted with multiplicity!); here at most d factors appear. In particular, we should be able to choose at least one such point y' satisfying

$$\frac{d(y',y)}{\lambda} \leq (\frac{|f(y)|}{\lambda^d})^{1/d}.$$

We now show that this is the case.

Lemma 2. *Let y be a point of Y satisfying $d(0,y) = \lambda$, and suppose that $|f(y)| = \lambda^d \cdot \alpha$ for some $\alpha < 1$. Then there exists a point $y' \in Y$ satisfying $d(y,y') \leq \lambda \cdot \alpha^{1/d}$ and $f(y') \leq \lambda^{d+1} \cdot \alpha$.*

Proof of Proposition 1 from Lemma 2. We proved in Lecture 15 that there exists a point $y_1 \in Y$ satisfying $d(0,y_1) = \lambda$ and $|f(y_1)| \leq \lambda^{d+1}$. Applying Lemma 2, we can choose a point $y_2 \in Y$ satisfying $d(y_1,y_2) \leq \lambda^{1+\frac{1}{d}}$ and $|f(y_2)| \leq \lambda^{d+2}$. Note that we then also have $d(0,y_2) = \lambda$, so we can apply Lemma 2 again to choose a point $y_3 \in Y$ satisfying $d(y_2,y_3) \leq \lambda^{1+\frac{2}{d}}$ and $|f(y_3)| \leq \lambda^{d+3}$. Continuing in this way, we obtain a sequence of points $\{y_n\}$ on the circle $Y_{[\lambda,\lambda]}$ satisfying

$$d(y_n,y_{n+1}) \leq \lambda^{1+\frac{n}{d}} \qquad |f(y_n)| \leq \lambda^{d+n}.$$

The first inequality implies that the sequence $\{y_n\}$ is Cauchy, and therefore converges to a point $y \in Y_{[\lambda,\lambda]}$. The second inequality implies that $|f(y)| = \lim_{n\to\infty} |f(y_n)| = 0$, so that $f(y) = 0$. $\square$

Proof of Lemma 2. Fix a point $y = (K, \iota) \in Y$ satisfying $d(0, y) = \lambda$ and $|f(y)|_K \leq \lambda^d \cdot \alpha$. Let ξ be a distinguished element of $\mathbf{A}_{\mathrm{inf}}$ satisfying $\xi(y) = 0$. Since $\mathbf{A}_{\mathrm{inf}}$ is ξ-adically complete and every element of $\mathbf{A}_{\mathrm{inf}}/\xi$ belongs to the image of $\sharp : \mathcal{O}_C^\flat \to \mathcal{O}_K$, we can write f as a sum

$$\sum_{n\geq 0} [c_n]\xi^n$$

(beware that this representation is *not* unique, because the map $\sharp : \mathcal{O}_C^\flat \to \mathcal{O}_K$ is not bijective). Note that under the reduction map

$$\mathbf{A}_{\mathrm{inf}} = W(\mathcal{O}_C^\flat) \to W(\mathcal{O}_C^\flat/\mathfrak{m}_C^\flat) = W(k),$$

the image of ξ is a unit multiple of p (since ξ is distinguished) and the image of f is a unit multiple of p^d (since f is primitive of degree d). It follows that $|c_i|_{C^\flat} < 1$ for $i < d$ and that $|c_d|_{C^\flat} = 1$. Replacing f by $f/[c_d]$, we may assume without loss of generality that $c_d = 1$. Note that we have

$$|c_0|_{C^\flat} = |[c_0](y)|_K = |f(y)|_K = \lambda^d \cdot \alpha.$$

We will assume that $c_0 \neq 0$ (otherwise, we can take $y' = y$).

Consider the polynomial

$$F(x) = c_0 + c_1 x + \cdots + c_{d-1}x^{d-1} + x^d \in C_\flat[x].$$

Since $C^\flat$ is algebraically closed, we can factor $F(x)$ as a product of linear factors

$$F(x) = (x - r_1)(x - r_2)\cdots(x - r_d)$$

for some elements $r_1, r_2, \ldots, r_d \in C^\flat$. Choose $r \in \{r_1, \ldots, r_d\}$ so that the absolute value of r is as small as possible. Note that, for $0 \leq m \leq d$, we have $c_m = \pm e_{d-m}(r_1, \ldots, r_d)$, where e_{d-m} denotes the $(d-m)$th elementary symmetric polynomial. We therefore have

$$\begin{aligned}
|r|^m_{C^\flat}|c_m|_{C^\flat} &\leq |r|^m_{C^\flat} \sup_{J\subseteq\{1,\ldots,d\},|J|=d-m} \prod_{j\in J} |r_j|_{C^\flat} \\
&\leq \frac{\prod_{j=1}^d |r_j|_{C^\flat}}{|r|^m} \\
&= |c_0|_{C^\flat} \\
&= \lambda^d \cdot \alpha.
\end{aligned}$$

In the special case $m = d$, we have $|r|^d_{C^\flat} \leq \lambda^d \cdot \alpha$, or $|r|_{C^\flat} \leq \lambda \cdot \alpha^{1/d}$. Set $\xi' = \xi - [r]$. Then ξ' is also a distinguished element, vanishing at a point $y' \in Y$. We have

$$d(y, y') = |\xi'(y)|_K = |-[r](y)|_K = |r|_{C^\flat} \leq \lambda \cdot \alpha^{1/d}.$$

It follows that $d(y, y') < \lambda = d(0, y)$, so that we have $d(0, y') = \lambda$. Let K' be the untilt of $C^\flat$ corresponding to the point y' and let $\sharp : C^\flat \to K'$ be the usual map (given by $x^\sharp = [x](y')$). Then $\xi(y') = (\xi' + [r])(y') = r^\sharp$. We therefore have

$$\begin{aligned}
\frac{f(y')}{[c_0]} &= \sum_{n\geq 0} \frac{c_n^\sharp}{c_0^\sharp} \xi(y')^n \\
&= \sum_{n\geq 0} (\frac{c_n r^n}{c_0})^\sharp.
\end{aligned}$$

Note that the ration $\frac{c_n r^n}{c_0}$ belongs to $\mathcal{O}_{C^\flat}$ for $n \leq d$ (by virtue of the inequality $|c_n r^n|_{C^\flat} \leq |c_0|$ established above). For $n > d$, we have

$$|\frac{c_n r^n}{c_0}|_{C^\flat} \leq |\frac{r^n}{c_0}|_{C^\flat} \leq \frac{|r|^n_{C^\flat}}{\lambda^d \cdot \alpha} \leq \frac{\lambda^n \cdot \alpha^{n/d}}{\lambda^d \cdot \alpha} \leq \lambda = |p|_{K'}.$$

It follows that each $(\frac{c_n r^n}{c_0})^\sharp$ belongs to the valuation ring $\mathcal{O}^\flat_K$, and is divisible by p (in $\mathcal{O}^\flat_K$) when $n > d$. We therefore compute

$$\begin{aligned}
\frac{f(y')}{[c_0]} &= \sum_{n \geq 0} (\frac{c_n r^n}{c_0})^\sharp \\
&\equiv \sum_{n=0}^{d} (\frac{c_n r^n}{c_0})^\sharp \pmod p \\
&\equiv (\sum_{n=0}^{d} \frac{c_n r^n}{c_0})^\sharp \pmod p \\
&= \frac{F(r)^\sharp}{c_0^\sharp} \\
&= 0.
\end{aligned}$$

We therefore have

$$|f(y')|_{K'} \leq |[c_0]|_{K'} \cdot |p|_{K'} = \lambda^d \cdot \alpha \cdot \lambda = \lambda^{d+1} \cdot \alpha.$$

□

Lecture 17: Algebraic Closure of Untilts

November 9, 2018

Our goal in this lecture is to prove the following result, which we have used several times without proof:

Theorem 1. *Let K be a perfectoid field. If the tilt $K^\flat$ is algebraically closed, then K is algebraically closed.*

We will prove Theorem 1 using an approximation argument which is similar to (but much easier than) the strategy of the last two lectures. The key point is to prove the following:

Proposition 2. *Let K be a perfectoid field such that the tilt $K^\flat$ is algebraically closed, and let $f(x) = x^n + a_1 x^{n-1} + \cdots + a_n \in K[x]$ be a non-constant irreducible polynomial. Let y be an element of K. Then there exists an element $y' \in K$ satisfying*

$$|y - y'|_K \leq |f(y)|_K^{1/n} \qquad |f(y')|_K \leq |pf(y)|_K.$$

Proof of Theorem 1 from Proposition 2. Let K be a perfectoid field such that $K^\flat$ is algebraically closed. We assume that K has characteristic zero (otherwise there is nothing to prove). We wish to show that K is algebraically closed: that is, that every non-constant polynomial $f(x) \in K[x]$ has a root in K. Without loss of generality, we may assume that $f(x)$ is monic and irreducible of degree $n > 0$. Replacing $f(x)$ by $p^{nd} f(\frac{x}{p^d})$ for $d \gg 0$, we may assume that the coeffcients of f belong to $\mathcal{O}_K$. Setting $y_0 = 0$, it follows that $f(y_0) \in \mathcal{O}_K$, or equivalently that $|f(y_0)|_K \leq |p^0|_K$. Applying Proposition 2, we deduce that there exists $y_1 \in K$ satisfying $|y_0 - y_1|_K \leq |f(y_0)|_K^{1/n} \leq |p^0|_K^{1/n}$ and $|f(y_1)|_K \leq |pf(y_0)|_K \leq |p|_K$. Applying Proposition 2 to the element y_1, we obtain an element $y_2 \in \mathcal{O}_K$ satisfying $|y_1 - y_2|_K \leq |f(y_1)|_K^{1/n} \leq |p|_K^{1/n}$ and $|f(y_2)|_K \leq |pf(y_1)|_K \leq |p^2|_K$. Proceeding in this way, we obtain a sequence of elements $y_0 = 0, y_1, y_2, \ldots \in K$ satisfying

$$|y_m - y_{m+1}|_K \leq |p^m|_K^{1/n} \qquad |f(y_m)|_K \leq |p^m|_K.$$

It follows from the first inequality (and the completeness of K) that the sequence $\{y_m\}$ converges to an element $y \in K$. Then

$$|f(y)|_K = \lim_{m \to \infty} |f(y_m)|_K = 0,$$

so that y is a root of f. $\square$

For the proof of Proposition 2, we will use the following result from the theory of valued fields:

Theorem 3. *Let K be a field which is complete with respect to a non-archimedean absolute value $|\bullet|_K$, and let L be a finite extension field of K. Then $|\bullet|_K$ can be extended uniquely to an absolute value on the field $|\bullet|_L$.*

Remark 4. In the situation of Theorem 3, the absolute value $|\bullet|_L$ is given concretely by the formula

$$|x|_L = |N_{L/K}(x)|_K^{1/\deg(L/K)},$$

where $N_{L/K} : L \to K$ denotes the norm map and $\deg(L/K)$ denotes the degree of the field extension $K \hookrightarrow L$. To prove this, we are free to enlarge L and may thereby assume that L is a normal extension of K. In this case, we can write

$$N_{L/K}(x) = (\prod_{\gamma \in \mathrm{Gal}(L/K)} \gamma(x))^{d_0},$$

where d_0 is the inseparable degree of L over K. We therefore have

$$|N_{L/K}(x)|_K^{1/\deg(L/K)} = (\prod_{\gamma \in \mathrm{Gal}(L/K)} |\gamma(x)|_L)^{1/|\mathrm{Gal}(L/K)|}.$$

The desired identity then follows from formula $|x|_L = |\gamma(x)|_L$ for $\gamma \in \mathrm{Gal}(L/K)$ (by virtue of the uniqueness asserted in Theorem 3).

Warning 5. In the situation of Theorem 3, one cannot drop the assumption that K is complete. If K is not complete, then the norm $| \bullet |_K$ can generally be extended in many different ways to extension fields L over K, and the formula $|x|_L = |N_{L/K}(x)|_K^{1/\deg(L/K)}$ of Remark 4 need not define an absolute value on L.

Corollary 6. *Let K be a field which is complete with respect to a non-archimedean absolute value $| \bullet |_K$, and let $f(x) = x^n + a_1 x^{n-1} + \cdots + a_n$ be an irreducible polynomial with coefficients in K. If a_n belongs to $\mathcal{O}_K$, then each a_i belongs to $\mathcal{O}_K$.*

Proof. Let L be a finite normal extension of K over which the polynomial $f(x)$ factors as a product $f(x) = (x - r_1) \cdot (x - r_2) \cdots (x - r_n)$. Equip L with the absolute value $| \bullet |_L$ of Theorem 3. Since the roots r_i are conjugate by the action of the Galois group $\mathrm{Gal}(L/K)$, they must all have the same absolute value; that is, there exists a real number λ satisfying $|r_i|_L = \lambda$ for all i. Then $a_n = (-1)^n \prod_{i=1}^n r_i$. Consequently, if a_n belongs to $\mathcal{O}_K$, then each r_i belongs to $\mathcal{O}_L$. It follows that the polynomial

$$f(x) = \prod_{i=1}^n (x - r_i)$$

has coefficients in $\mathcal{O}_L$, so that each a_i belongs to $\mathcal{O}_L \cap K = \mathcal{O}_K$ as desired. □

Proof of Proposition 2. Let K be a perfectoid field such that the tilt $K^\flat$ is algebraically closed, and let $f(x) = x^n + a_1 x^{n-1} + \cdots + a_n \in K[x]$ be a non-constant irreducible polynomial. We wish to show that, for each element $y \in K$, we can find another point $y' \in K$ satisfying

$$|y - y'|_K \leq |f(y)|_K^{1/n} \qquad |f(y')|_K \leq |pf(y)|_K.$$

Replacing $f(x)$ by the polynomial $f(x+y)$, we can reduce to the case $y = 0$; in this case, we wish to show that there exists $y' \in K$ satisfying

$$|y'|_K \leq |f(0)|_K^{1/n} \qquad |f(y')|_K \leq |pf(0)|_K.$$

Let us assume that $f(0) \neq 0$ (otherwise, we can take $y' = 0$ and there is nothing to prove). Note that the value group of K is the same as the value group of $K^\flat$, and is therefore divisible (since $K^\flat$ is algebraically closed). We can therefore choose an element $c \in K$ satisfying $|c|_K = |f(0)|_K^{1/n}$. In this case, we can rewrite the inequalities above as

$$|\frac{y'}{c}|_K \leq 1 \qquad |\frac{1}{c^n} f(c \cdot \frac{y'}{c})|_K \leq |p|_K.$$

Replacing $f(x)$ by the monic polynomial $\frac{1}{c^n} f(cx)$ (and y' by $\frac{y'}{c}$), we can reduce to the case where $|f(0)|_K = 1$. In this case, we wish to show that there exists $y' \in K$ satisfying

$$|y'|_K \leq 1 \qquad |f(y')|_K \leq |p|_K.$$

Write $f(x) = x^n + a_1 x^{n-1} + \cdots + a_n$. Our assumption that $|f(0)|_K = 1$ guarantees that a_n belongs to $\mathcal{O}_K$. Applying Corollary 6, we see that each of the coefficients a_i belongs to $\mathcal{O}_K$. We can therefore choose elements $b_i \in \mathcal{O}_K^\flat$ satisfying $b_i^\sharp \equiv a_i \pmod{p}$. Set

$$g(x) = x^n + b_1 x^{n-1} + b_2 x^{n-2} + \cdots + b_n \in K^\flat[x].$$

Since $K^\flat$ is algebraically closed, the polynomial $g(x)$ factors as a product

$$g(x) = (x - r_1) \cdots (x - r_n)$$

for some $r_1, r_2, \ldots, r_n \in K^\flat$. Note that we have

$$|r_1|_{K^\flat} \cdots |r_n|_{K^\flat} = |(-1)^n b_n|_{K^\flat} \leq 1.$$

It follows that there must exist $r \in \{r_1, \ldots, r_n\}$ satisfying $|r|_{K^\flat} \leq 1$, so that r belongs to $\mathcal{O}_K^\flat$. Setting $y' = r^\sharp$, we have $|y'|_K = |r|_{K^\flat} \leq 1$, and

$$\begin{aligned}
f(y') &= y'^n + a_1 y'^{n-1} + \cdots + a_n \\
&\equiv y'^n + b_1^\sharp y'^{n-1} + \cdots + b_n^\sharp \pmod{p} \\
&= (r^\sharp)^n + b_1^\sharp (r^\sharp)^{n-1} + \cdots + b_n^\sharp \\
&\equiv (g(r))^\sharp \pmod{p} \\
&= 0
\end{aligned}$$

so that $|f(y')|_K \leq |p|_K$, as desired. $\square$

Lecture 18: Bounded and Meromorphic Functions of p

November 18, 2018

Throughout this lecture, we fix a perfectoid field $C^\flat$ of characteristic p. Our goal in this lecture is to give an "intrinsic" description of $\mathbf{A}_{\mathrm{inf}} = W(\mathcal{O}_C^\flat)$ as a subring of B: roughly speaking, it consists of "holomorphic" functions on Y whose value at any point $y \in Y$ belongs to the valuation ring $\mathcal{O}_{K_y}$ of the perfectoid field K_y corresponding to y.

Theorem 1. *Let f be a nonzero element of B. The following conditions are equivalent:*

(1) *For each $\rho \in (0,1)$, we have $|f|_\rho \leq 1$.*

(2) *The element f belongs to the subring $\mathbf{A}_{\mathrm{inf}} \subseteq B$.*

Corollary 2. *Let f be a nonzero element of B. Then:*

- *The element f belongs to the localization $\mathbf{A}_{\mathrm{inf}}[\frac{1}{p}]$ if and only if there exists an integer n such that $|f|_\rho \leq \rho^n$ for all $\rho \in (0,1)$.*
- *The element f belongs to the localization $\mathbf{A}_{\mathrm{inf}}[\frac{1}{[\pi]}]$ if and only if there exists a constant $C > 0$ satisfying $|f|_\rho \leq C$ for all $\rho \in (0,1)$.*
- *The element f belongs to the localization $\mathbf{A}_{\mathrm{inf}}[\frac{1}{p}, \frac{1}{[\pi]}]$ if and only if there exists a constant $C > 0$ and an integer n satisfying $|f|_\rho \leq C\rho^n$ for all $\rho \in (0,1)$.*

We will deduce Theorem 1 from the following weaker assertion:

Lemma 3. *Let f be an element of B. Suppose that there exists an integer m such that*

$$|f|_\rho \leq \rho^m$$

for all $0 < \rho < 1$. Then we can write

$$f = [c]p^m + g,$$

where $c \in \mathcal{O}_C^\flat$ and g satisfies an inequality of the form $|g|_\rho \leq \rho^{m+1}$.

Proof of Theorem 1 from Lemma 3. The implication (2) $\Rightarrow$ (1) is immediate. Conversely, suppose that (1) is satisfied, and set $f_0 = f$. Applying Lemma 3, we can write $f_0 = [c_0] + f_1$, where $[c_0] \in \mathcal{O}_C^\flat$, and f_1 satisfies $|f_1|_\rho \leq \rho$ for all $\rho \in (0,1)$. Applying Lemma 3 again, we can write $f_1 = [c_1]p + f_2$, where $c_1 \in \mathcal{O}_C^\flat$ and f_2 satisfies $|f_2|_\rho \leq \rho$ for all $\rho \in (0,1)$. Continuing in this way, we obtain a sequence of elements $f_0, f_1, f_2, \ldots \in B$ and $c_0, c_1, c_2, \ldots \in \mathcal{O}_C^\flat$ satisfying

$$f_0 = [c_0] + [c_1]p + \cdots + [c_{n-1}]p^{n-1} + f_n$$

$$|f_n|_\rho \leq \rho^n.$$

Note that the sequence $\{f_n\}_{n\geq 0}$ converges to zero with respect to the each of the Gauss norms $|\bullet|_\rho$. It follows that the infinite sum $\sum_{n\geq 0}[c_n]p^n$ converges in B to f, so that f belongs to $\mathbf{A}_{\mathrm{inf}}$ as desired. $\square$

Proof of Lemma 3. Replacing f by $\frac{f}{p^m}$, we can reduce to the case $m = 0$. In this case, we have an element $f \in B$ satisfying $|f|_\rho \leq 1$ for all $\rho \in (0,1)$; we wish to write $f = [c] + g$ for $c \in \mathcal{O}_C^\flat$, where g satisfies $|g|_\rho \leq \rho$ for all $\rho \in (0,1)$.

Choose a sequence $f_1, f_2, \ldots \in \mathbf{A}_{\inf}[\frac{1}{p}, \frac{1}{[\pi]}]$ which converges to f in B. Each f_i then admits a unique Teichmüller expansion

$$f_i = \sum_{n \gg -\infty} [c_{n,i}] p^n.$$

Set $f_i^+ = \sum_{n \geq 0} [c_{n,i}] p^n$. We claim that the sequence $f_1^+, f_2^+, \ldots$ also converges to f in B. To prove this, we must show that for each $\rho \in (0,1)$, we have

$$\lim_{i \to \infty} |f_i - f_i^+|_\rho = 0.$$

Let ϵ be a small positive real number. Then the sequence $f_1, f_2, \ldots$ converges to f with respect to the Gauss norm $| \bullet |_{\epsilon \cdot \rho}$. It follows that, for i sufficiently large (depending on ϵ), we have

$$|f_i|_{\epsilon \cdot \rho} = |f|_{\epsilon \cdot \rho} \leq 1.$$

For such i, we have

$$|c_{-n,i}|_{C^\flat} (\epsilon \rho)^{-n} \leq 1.$$

If n is positive, this gives

$$|c_{-n,i}|_{C^\flat} \rho^{-n} \leq \epsilon^n \leq \epsilon.$$

We therefore have

$$|f_i - f_i^+|_\rho = \sup_{n > 0} (|c_{-n,i}|_{C^\flat} \rho^{-n}) \leq \epsilon$$

for sufficiently large i.

Replacing the sequence $\{f_i\}$ with $\{f_i^+\}$, we may assume that each f_i admits a Teichmüller expansion of the form

$$f_i = \sum_{n \geq 0} [c_{n,i}] p^n.$$

Then, for every pair of indices i and j, the difference $f_i - f_j$ admits a Teichmüller expansion of the form $[c_{0,i} - c_{0,j}]$ + higher order terms. For any $\rho \in (0,1)$, we have

$$|f_i - f_j|_\rho \geq |c_{0,i} - c_{0,j}|_{C^\flat}.$$

Since the sequence $\{f_i\}$ is Cauchy with respect to the Gauss norm $| \bullet |_\rho$, it follows that $\{c_{0,i}\}$ is a Cauchy sequence in the field $C^\flat$. Since $C^\flat$ is complete, this Cauchy sequence converges to some element $c \in C^\flat$. Moreover, for $i \gg 0$, we have

$$|c_{0,i}|_{C^\flat} \leq |f_i|_\rho = |f|_\rho \leq 1,$$

so that $c_{0,i}$ belongs to $\mathcal{O}_C^\flat$ (for $i \gg 0$) and therefore $c \in \mathcal{O}_C^\flat$.

Exercise 4. Show that, if $\{c_i\}$ is a Cauchy sequence in $\mathcal{O}_C^\flat$ converging to a point $c \in \mathcal{O}_C^\flat$, then we have $[c] = \lim_{i \to \infty} [c_i]$ in the ring B.

For each i, set $g_i = f_i - [c_{0,i}] = \sum_{n > 0} [c_{n,i}] p^i$. Applying the exercise, we see that the limit $\lim_{i \to \infty} g_i$ exists and is given by

$$\lim_{i \to \infty} g_i = (\lim_{i \to \infty} f_i) - (\lim_{i \to \infty} [c_{0,i}]) = f - [c].$$

That is, we can write $f = [c] + g$, where $g = \lim_{i \to \infty} g_i$. We will complete the proof by showing that $|g|_\rho \leq \rho$ for all $\rho \in (0,1)$, or equivalently that $v_s(g) \geq s$ for all $s \in \mathbf{R}_{>0}$.

Let us assume that $g \neq 0$ (otherwise there is nothing to prove). Passing to a subsequence, we may then also assume that $g_i \neq 0$ for all i. Each g_i admits a Teichmüller expansion where only positive powers of p occur, so that the piecewise linear function $v_\bullet(g_i)$ has strictly positive slopes. When restricted to any compact interval $I \subseteq \mathbf{R}_{>0}$, the function $v_\bullet(g)$ agrees with $v_\bullet(g_i)$ for $i \gg 0$. It follows that the piecewise linear function $s \mapsto v_s(g)$ also has strictly positive (and integral) slopes. Suppose, for a contradiction, that there exists some $s > 0$ such that $v_s(g) < s$. Choose $0 < s' < s$ such that $v_s(g) - s + s' < 0$. Since the function $v_\bullet(g)$ is piecewise linear with slopes ≥ 1 everywhere, we have

$$v_{s'}(g) \leq v_s(g) - s + s' < 0.$$

Setting $\rho' = e^{-s'}$, we have $|g|_{\rho'} > 1$. Then

$$1 < |g|_{\rho'} = |f - [c]|_{\rho'} \leq \max(|f|_{\rho'}, |[c]|_{\rho'}) = \max(|f|_{\rho'}, |c|_{C^\flat}) \leq 1,$$

which is a contradiction. $\square$

From Theorem 1, it is easy to describe the invariant subring $B^{\varphi=1} \subseteq B$:

Theorem 5. *The unit map $\mathbf{Q}_p \to B^{\varphi=1}$ is an isomorphism.*

Lemma 6. *Let f be a nonzero element of $B^{\varphi=1}$. Then there exists an integer n such that $|f|_\rho = \rho^n$ for all $0 < \rho < 1$.*

Proof. Note that for $0 < \rho < 1$, we have

$$|f|_\rho^p = |\varphi(f)|_{\rho^p} = |f|_{\rho^p}.$$

In other words, the function $s \mapsto v_s(f)$ satisfies the identity $v_{ps}(f) = pv_s(f)$. Differentiating both sides (on the left) with respect to s and dividing by p, we obtain $\partial_- v_{ps}(f) = \partial_- v_s(f)$. Since the function $s \mapsto v_s(f)$ is concave, the function $s \mapsto \partial_- v_s(f)$ is nondecreasing; the above equality implies that it is constant. In other words, $s \mapsto v_s(f)$ is a linear function of s, which we can write as $v_s(f) = ns + r$ for some integer n and some real number r. The equality $v_{ps}(f) = pv_s(f)$ then implies that $r = 0$, so that $v_s(f) = ns$ for all $s > 0$ and therefore $|f|_\rho = \rho^n$ for all $0 < \rho < 1$. $\square$

Proof of Theorem 5. Let f be a nonzero element of $B^{\varphi=1}$. It follows from Lemma 6 and Corollary 2 that f belongs to the subring $\mathbf{A}_{\mathrm{inf}}[\frac{1}{p}] \subseteq B$. That is, f admits a unique Teichmüller expansion

$$f = \sum_{n \gg -\infty} [c_n] p^n,$$

where each c_n belongs to $\mathcal{O}_C^\flat$. We then have

$$\sum_{n \gg -\infty} [c_n] p^n = f = \varphi(f) = \sum_{n \gg -\infty} [c_n^p] p^n,$$

so that each coefficient c_n satisfies $c_n = c_n^p$ in the field $C^\flat$, and therefore belongs to the finite field $\mathbf{F}_p \subseteq C^\flat$. The equality $f = \sum_{n \gg -\infty} [c_n] p^n$ now shows that f belongs to $\mathbf{Q}_p = W(\mathbf{F}_p)[\frac{1}{p}]$, as desired. $\square$

Lecture 19: Line Bundles on the Fargues-Fontaine Curve and Their Cohomology

November 18, 2018

Throughout this lecture, we fix an algebraically closed perfectoid field $C^\flat$ of characteristic p. Let X denote the Fargues-Fontaine curve, given by

$$X = \mathrm{Proj}(\bigoplus_{n \geq 0} B^{\varphi = p^n}).$$

We have seen that X is a Dedekind scheme. Our first goal in this lecture is to describe the Picard group $\mathrm{Pic}(X)$ consisting of isomorphism classes of line bundles on X.

Construction 1. By general nonsense, every graded module M over the graded ring $\bigoplus_{n \geq 0} B^{\varphi = p^n}$ determines a quasi-coherent sheaf $\widetilde{M}$ on X. If $x \in X$ is a closed point given by the vanishing of a function $\log([\epsilon]) \in B^{\varphi = p}$ and $U = X - \{x\}$ is the complementary affine open set, then $\widetilde{M}(U)$ is the degree zero part of the graded module $M[\frac{1}{\log([\epsilon])}]$.

In particular, we can apply this construction to the graded module $M = \bigoplus_{n \in \mathbf{Z}} B^{\varphi = p^{n+1}}$ (obtained from $\bigoplus_{n \in \mathbf{Z}} B^{\varphi = p^n}$ by shifting the grading). This yields a line bundle $\mathcal{O}(1)$ on X, given on the affine subset above by the formula

$$\mathcal{O}(1)(U) = M[\frac{1}{\log([\epsilon])}]^0 = (B[\frac{1}{\log([\epsilon])}]^{\varphi = p}).$$

More generally, for any integer m, we can consider the line bundle $\mathcal{O}(m) = \mathcal{O}(1)^{\otimes m}$ on X, given by

$$\mathcal{O}(m)(U) = (B[\frac{1}{\log([\epsilon])}]^{\varphi = p^m}).$$

Remark 2. If M is any graded module over the ring $\bigoplus_{n \geq 0} B^{\varphi = p^n}$ and M^0 is the degree zero part of M, then we have a canonical map

$$M^0 \to \mathrm{H}^0(X; \widetilde{M}).$$

In particular, the line bundles $\mathcal{O}(m)$ are equipped with canonical maps

$$\mu : B^{\varphi = p^m} \to \mathrm{H}^0(X; \mathcal{O}(m)).$$

We now prove the following:

Theorem 3. *The construction $m \mapsto \mathcal{O}(m)$ induces an isomorphism of abelian groups $\rho : \mathbf{Z} \to \mathrm{Pic}(X)$.*

Let $\mathrm{Div}(X)$ denote the group of divisors on X: that is, the free abelian group generated by the set of closed points of X. There is a canonical map

$$\mathrm{Div}(X) \to \mathrm{Pic}(X),$$

which carries a closed point $x \in X$ to the line bundle $\mathcal{O}(x)$ given by the *inverse* of the ideal sheaf of X (which is an invertible sheaf, since X is a Dedekind scheme). On the other hand, we also have a homomorphism $\deg : \mathrm{Div}(X) \to \mathbf{Z}$, given by $\deg(x) = 1$ for each closed point $x \in X$.

Lemma 4. *The diagram*

$$\begin{tikzcd} & \mathrm{Div}(X) \arrow[dl, "\deg"'] \arrow[dr] & \\ \mathbf{Z} \arrow[rr, "\rho"] & & \mathrm{Pic}(X) \end{tikzcd}$$

is commutative.

Proof. It will suffice to show that, for every closed point $x \in X$, the line bundle $\mathcal{O}(x)$ is isomorphic to the line bundle of $\mathcal{O}(1)$ of Construction 1. Choose $\epsilon \in 1 + \mathfrak{m}_C^\flat$ such that $\log([\epsilon])$ vanishes at x. Under the canonical map

$$B^{\varphi = p} \to \mathrm{H}^0(X; \mathcal{O}(1)),$$

we can view $\log([\epsilon])$ as a global section of $\mathcal{O}(1)$ which vanishes to order 1 at the point x, and therefore extends to an isomorphism $\mathcal{O}(x) \simeq \mathcal{O}(1)$. $\square$

For any Dedekind scheme X, the canonical map $\mathrm{Div}(X) \to \mathrm{Pic}(X)$ is surjective. It follows from Lemma 4 that the map $\rho : \mathbf{Z} \to \mathrm{Pic}(X)$ is also surjective. To complete the proof of Theorem 3, it will suffice to show that ρ is injective: that is, that the line bundles $\mathcal{O}(m)$ are nontrivial whenever m is nonzero. This is a consequence of the following more precise assertion, which describes the cohomology of the line bundles $\mathcal{O}(m)$:

Theorem 5. (*a*) *For each integer m, the canonical map $\mu : B^{\varphi = p^m} \to \mathrm{H}^0(X; \mathcal{O}(m))$ is an isomorphism.*

(*b*) *For $m > 0$, the cohomology groups $\mathrm{H}^i(X; \mathcal{O}(m))$ vanish for $i > 0$.*

Corollary 6. *The canonical map $\mathbf{Q}_p \to \mathrm{H}^0(X, \mathcal{O}_X)$ is an isomorphism, and the cohomology group $\mathrm{H}^1(X, \mathcal{O}_X)$ vanishes.*

Remark 7. Since X is a Dedekind scheme, the cohomology groups $\mathrm{H}^i(X; \mathcal{O}(m))$ automatically vanish for $i > 1$. Beware, however, that the cohomology groups $\mathrm{H}^1(X; \mathcal{O}(m))$ do not vanish for $m < 0$.

Remark 8. Let Y be a (complete) algebraic curve over an algebraically closed field k. Then Y has genus zero if and only either of the equivalent conditions is satisfied;

- The Picard group $\mathrm{Pic}(Y)$ is isomorphic to $\mathbf{Z}$.
- The cohomology group $\mathrm{H}^1(Y; \mathcal{O}_Y)$ vanishes.

Theorems 3 and 5 assert that these properties hold for the Fargues-Fontaine curve X. This supports the heuristic that X is "like" an algebraic curve of genus 0.

Proof of Theorem 5. Let t be a nonzero element of $B^{\varphi = p}$ (so that we can write $t = \log([\epsilon])$ for some unique $\epsilon \in 1 + \mathfrak{m}_C^\flat$), so that t determines a global section of $\mathcal{O}(1)$ (which we will also denote by t). This section vanishes at a single point $x \in X$. For any integer m, we have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & B^{\varphi = p^m} \arrow[r, "t"] \arrow[d, "\mu"] & B^{\varphi = p^{m+1}} \arrow[r] \arrow[d, "\mu"] & B^{\varphi = p^{m+1}}/tB^{\varphi = p^m} \arrow[r] \arrow[d, "\nu"] & 0 \\ 0 \arrow[r] & \mathrm{H}^0(X, \mathcal{O}(m)) \arrow[r] & \mathrm{H}^0(X, \mathcal{O}(m+1)) \arrow[r] & \mathrm{H}^0(X, \mathcal{O}(m+1)/t\,\mathcal{O}(m)). & \end{tikzcd}$$

We first prove the following:

$(*)$ If $m \geq 0$, then the map $\nu : B^{\varphi=p^{m+1}}/tB^{\varphi=p^m} \to \mathrm{H}^0(X, \mathcal{O}(m+1)/t\,\mathcal{O}(m))$ is an isomorphism.

Recall that every nonzero element of $B^{\varphi=p^{m+1}}$ factors as a product $t_0 t_1 \cdots t_m$, where each t_i is a nonzero element of $B^{\varphi=p}$ vanishing at a single point $x_i \in X$. Such an element is annihilated by the composite map

$$B^{\varphi=p^{m+1}} \to B^{\varphi=p^{m+1}}/tB^{\varphi=p^m} \xrightarrow{\nu} \mathrm{H}^0(X, \mathcal{O}(m+1)/t\,\mathcal{O}(m))$$

if and only if the product $t_0 t_1 \cdots t_m$ vanishes at the point x: that is, if and only if $x = x_i$ for some i. In this case, t_i is a unit multiple of t, so that $t_0 \cdots t_m$ belongs to $tB^{\varphi=p^m}$. This proves that ν is injective.

To verify the surjectivity of ν, we can use the commutativity of the diagram

$$\begin{tikzcd}
B^{\varphi=p}/tB^{\varphi=1} \arrow[r, "t^m"] \arrow[d, "\nu"] & B^{\varphi=p^{m+1}}/tB^{\varphi=p^m} \arrow[d, "\nu"] \\
\mathrm{H}^0(X, \mathcal{O}(1)/t\,\mathcal{O}(0)) \arrow[r, "t^m", "\sim"'] & \mathrm{H}^0(X, \mathcal{O}(m+1)/t\,\mathcal{O}(m))
\end{tikzcd}$$

to reduce to the case $m = 0$. In this case, the assertion is equivalent to the surjectivity of the map

$$B^{\varphi=p} \simeq 1 + \mathfrak{m}_C^\flat \xrightarrow{\epsilon \mapsto \log(\epsilon^\sharp)} K_x$$

(where K_x is the untilt of $C^\flat$ corresponding to the point x), which was established in Lecture 10.

Since $\mathcal{O}(m+1)/t\,\mathcal{O}(m)$ is supported at a single point of X, its cohomology vanishes in degree 1. We therefore have a long exact sequence

$$0 \to \mathrm{H}^0(X, \mathcal{O}(m)) \to \mathrm{H}^0(X, \mathcal{O}(m+1)) \to u\mathrm{H}^0(X, \mathcal{O}(m+1)/t\,\mathcal{O}(m)) \to \mathrm{H}^1(X; \mathcal{O}(m)) \to \mathrm{H}^1(X; \mathcal{O}(m+1)) \to 0$$

It follows from $(*)$ that the map u is surjective for $m \geq 0$. Combining this observation with $(*)$ and the snake lemma, we deduce (again for $m \geq 0$) that multiplication by t induces isomorphisms

$$\ker(B^{\varphi=p^m} \to \mathrm{H}^0(X, \mathcal{O}(m))) \to \ker(B^{\varphi=p^{m+1}} \to \mathrm{H}^0(X, \mathcal{O}(m+1)))$$

$$\mathrm{coker}(B^{\varphi=p^m} \to \mathrm{H}^0(X, \mathcal{O}(m))) \to \mathrm{coker}(B^{\varphi=p^{m+1}} \to \mathrm{H}^0(X, \mathcal{O}(m+1)))$$
$$\mathrm{H}^1(X, \mathcal{O}(m)) \to \mathrm{H}^1(X, \mathcal{O}(m+1)).$$

Passing to the limit, we conclude that the kernel and cokernel of μ coincide with the kernel and cokernel of the canonical isomorphism

$$(B[\frac{1}{t}])^{\varphi=1} \to \varinjlim_{n \geq m} \mathrm{H}^0(X, \mathcal{O}(nx)) = \mathrm{H}^0(U, \mathcal{O}_U),$$

where $U = X - \{x\}$ is the affine open subset of X complementary to the vanishing locus of t. Similarly, the canonical map

$$\mathrm{H}^1(X, \mathcal{O}(m)) \to \varinjlim_{n \geq m} \mathrm{H}^1(X, \mathcal{O}(nx)) \simeq \mathrm{H}^1(U, \mathcal{O}_U) \simeq 0$$

is an isomorphism, so $\mathrm{H}^1(X, \mathcal{O}(m))$ vanishes. This proves Theorem 5 in the case $m \geq 0$.

To handle the case $m < 0$, it will suffice to show that the cohomology groups $\mathrm{H}^0(X, \mathcal{O}(-n)) = \mathrm{H}^0(X, \mathcal{O}(-nx))$ vanish for $n > 0$. To prove this, we observe that a nonzero element of $\mathrm{H}^0(X, \mathcal{O}(-nx))$ can be identified with a nonzero element of $\mathrm{H}^0(X, \mathcal{O}_X)$ vanishing to order n at the point x. However, the first part of the proof shows that $\mathrm{H}^0(X, \mathcal{O}_X) \simeq B^{\varphi=1} \simeq \mathbf{Q}_p$ is a field, so a nonzero element of $\mathrm{H}^0(X, \mathcal{O}_X)$ cannot vanish at any point of X. $\square$

Definition 9. We let $\deg : \mathrm{Pic}(X) \to \mathbf{Z}$ denote the inverse of the isomorphism $\mathbf{Z} \to \mathrm{Pic}(X)$ appearing in Theorem 3. If $\mathcal{L}$ is a line bundle on X, we will refer to $\deg(\mathcal{L})$ as the *degree* of $\mathcal{L}$.

More generally, if $\mathcal{E}$ is a vector bundle of rank r on X, we let $\deg(\mathcal{E})$ denote the *degree* of $\mathcal{E}$, defined by the formula

$$\deg(\mathcal{E}) = \deg(\bigwedge^r \mathcal{E}).$$

Note that a short exact sequence $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$ of vector bundles of ranks r', r, and r'' determines a canonical isomorphism

$$\bigwedge^r(\mathcal{E}) \simeq (\bigwedge^{r'} \mathcal{E}') \otimes (\bigwedge^{r''} \mathcal{E}''),$$

hence an equality $\deg(\mathcal{E}) = \deg(\mathcal{E}') + \deg(\mathcal{E}'')$.

For any vector bundle $\mathcal{E}$ on X, we define the *slope* of $\mathcal{E}$ by the formula

$$\mathrm{slope}(\mathcal{E}) = \frac{\deg(\mathcal{E})}{\mathrm{rank}(\mathcal{E})}.$$

Lecture 20: The Harder-Narasimhan Filtration

November 19, 2018

Throughout this lecture, we fix an algebraically closed perfectoid field $C^\flat$ of characteristic p. Let X denote the Fargues-Fontaine curve, given by

$$X = \mathrm{Proj}(\bigoplus_{n \geq 0} B^{\varphi = p^n}).$$

Our goal in this lecture is to show that every vector bundle $\mathcal{E}$ on X admits a canonical *Harder-Narasimhan filtration* (just as if X were an algebraic curve defined over a field).

We begin with some generalities. Recall that, if $\mathcal{E}$ is a nonzero vector bundle on X, the *slope* $\mathrm{slope}(\mathcal{E})$ is defined by the formula

$$\mathrm{slope}(\mathcal{E}) = \frac{\deg(\mathcal{E})}{\mathrm{rank}(\mathcal{E})}.$$

Exercise 1. Let $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$ be a short exact sequence of nonzero vector bundles on X, so that we have equalities

$$\deg(\mathcal{E}) = \deg(\mathcal{E}') + \deg(\mathcal{E}'') \qquad \mathrm{rank}(\mathcal{E}) = \mathrm{rank}(\mathcal{E}') + \mathrm{rank}(\mathcal{E}'').$$

Using this, show that:

- If $\mathrm{slope}(\mathcal{E}') = \mathrm{slope}(\mathcal{E}'')$, then $\mathrm{slope}(\mathcal{E}') = \mathrm{slope}(\mathcal{E}) = \mathrm{slope}(\mathcal{E}'')$.
- If $\mathrm{slope}(\mathcal{E}') < \mathrm{slope}(\mathcal{E}'')$, then $\mathrm{slope}(\mathcal{E}') < \mathrm{slope}(\mathcal{E}) < \mathrm{slope}(\mathcal{E}'')$.
- If $\mathrm{slope}(\mathcal{E}') > \mathrm{slope}(\mathcal{E})$, then $\mathrm{slope}(\mathcal{E}') > \mathrm{slope}(\mathcal{E}) > \mathrm{slope}(\mathcal{E}'')$.

Remark 2. Let $\mathcal{E}$ be a vector bundle on X and let $\mathcal{E}' \subsetneq \mathcal{E}$ be a subsheaf which is a vector bundle of the same rank (so that the quotient $\mathcal{E}'' = \mathcal{E} / \mathcal{E}'$ is a coherent sheaf with finite support on X). Then $\deg(\mathcal{E}') < \deg(\mathcal{E})$ and therefore $\mathrm{slope}(\mathcal{E}') < \mathrm{slope}(\mathcal{E})$. To prove this, we can replace $\mathcal{E}$ and $\mathcal{E}'$ by their top exterior powers and thereby reduce to the case where $\mathcal{E}$ and $\mathcal{E}'$ are line bundles, in which case the result is obvious (since there are no nonzero maps from $\mathcal{O}(m)$ to $\mathcal{O}(n)$ for $m > n$, and every nonzero map from $\mathcal{O}(n)$ to itself is an isomorphism). Note that this can be regarded as a degenerate version of Exercise 1, where we adopt the convention that $\mathrm{slope}(\mathcal{E}'') = \infty$.

Definition 3. Let $\mathcal{E}$ be a nonzero vector bundle on X and let λ be a rational number. We say that $\mathcal{E}$ is *semistable of slope* λ if $\mathrm{slope}(\mathcal{E}) = \lambda$ and, for every nonzero subbundle $\mathcal{E}' \subseteq \mathcal{E}$, we have $\mathrm{slope}(\mathcal{E}') \leq \lambda$. By convention, we say that the zero vector bundle is semistable of every slope.

Remark 4. Let $\mathcal{E}$ be a vector bundle on X which is semistable of slope λ and let $\mathcal{E}' \subseteq \mathcal{E}$ be a coherent subsheaf. Then $\mathcal{E}'$ is also a vector bundle, but not necessarily a vector subbundle (since the quotient $\mathcal{E} / \mathcal{E}'$ might not be a vector bundle). However, $\mathcal{E}'$ is always contained in a vector subbundle $\overline{\mathcal{E}}' \subseteq \mathcal{E}$ of the same rank. Using Remark 2 we obtain

$$\mathrm{slope}(\mathcal{E}') \leq \mathrm{slope}(\overline{\mathcal{E}}') \leq \lambda.$$

Moreover, the first inequality is strict if $\mathcal{E}'$ is not a subbundle of $\mathcal{E}$.

Proposition 5. *Let $\mathcal{E}$ be a vector bundle on X which is semistable of rank λ. For any surjection of vector bundles $\mathcal{E} \twoheadrightarrow \mathcal{E}''$, we have* $\mathrm{slope}(\mathcal{E}'') \geq \lambda$.

Proof. We have an exact sequence $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$, and the semistability of $\mathcal{E}$ gives $\mathrm{slope}(\mathcal{E}') \leq \mathrm{slope}(\mathcal{E}) = \lambda$. Applying Exercise 1, we see that $\mathrm{slope}(\mathcal{E}'') \geq \lambda$. □

Corollary 6. *Let $\mathcal{E}$ and $\mathcal{F}$ be semistable vector bundles of slopes λ and μ. If $\lambda > \mu$, then every map of vector bundles $f : \mathcal{E} \to \mathcal{F}$ vanishes.*

Proof. If $f \neq 0$, then the image $\mathrm{Im}(f)$ is a nonzero coherent subsheaf of $\mathcal{F}$, hence a vector bundle of rank > 0. Remark 4 and Proposition 5 then give

$$\lambda = \mathrm{slope}(\mathcal{E}) \leq \mathrm{slope}(\mathrm{Im}(f)) \leq \mathrm{slope}(\mathcal{F}) = \mu,$$

contradicting our assumption that $\lambda > \mu$. □

Proposition 7. *Let $f : \mathcal{E} \to \mathcal{F}$ be a map of vector bundles on $\mathcal{E}$ which are semistable of slope λ. Then* $\ker(f)$ *and* $\mathrm{coker}(f)$ *(formed in the category of coherent sheaf on X) are vector bundles.*

Proof. If $f = 0$, there is nothing to prove. Otherwise, we again have inequalities

$$\lambda = \mathrm{slope}(\mathcal{E}) \leq \mathrm{slope}(\mathrm{Im}(f)) \leq \mathrm{slope}(\mathcal{F}) = \lambda.$$

It follows that equality must hold in both cases, so that $\mathrm{Im}(f)$ has slope λ. Moreover, Remark 4 shows that it is a vector subbundle of $\mathcal{F}$, so that $\mathrm{coker}(f)$ is a vector bundle on X and we have exact sequences

$$0 \to \ker(f) \to \mathcal{E} \to \mathrm{Im}(f) \to 0$$

$$0 \to \mathrm{Im}(f) \to \mathcal{F} \to \mathrm{coker}(f) \to 0.$$

Applying Exercise 1, we conclude that $\ker(f)$ and $\mathrm{coker}(f)$ (if nonzero) also have slope λ. Every subbundle of $\ker(f)$ can also be regarded as a subbundle of $\mathcal{E}$, and therefore has slope $\leq \lambda$ by virtue of our assumption that $\mathcal{E}$ is semistable. This proves the $\ker(f)$ is semistable of slope λ. We claim that $\mathrm{coker}(f)$ is also semistable of slope λ. Assume otherwise: then there exists a subbundle $\overline{\mathcal{F}}' \subseteq \mathrm{coker}(f)$ of slope $> \lambda$. Let $\mathcal{F}'$ be the inverse image of $\overline{\mathcal{F}}'$ in $\mathcal{F}$, so that we have an exact sequence

$$0 \to \mathrm{Im}(f) \to \mathcal{F}' \to \overline{\mathcal{F}}' \to 0.$$

Applying Exercise 1, we deduce that $\mathrm{slope}(\mathcal{F}') > \lambda$, contradicting the semistability of $\mathcal{F}$. □

Proposition 8. *Let $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$ be an exact sequence of vector bundles on X. If $\mathcal{E}'$ and $\mathcal{E}''$ are semistable of slope λ, then so is $\mathcal{E}$.*

Proof. Exercise 1 shows that $\mathcal{E}$ has slope λ. Let $\mathcal{F} \subseteq \mathcal{E}$ be any vector subbundle. Let $\mathcal{F}' = \mathcal{F} \cap \mathcal{E}'$ and let $\mathcal{F}''$ be the image of $\mathcal{F}$ in $\mathcal{E}''$. Then $\mathcal{F}'$ and $\mathcal{F}''$ are vector bundles which can be regarded as subsheaves of $\mathcal{E}'$ and $\mathcal{E}''$, respectively, so Remark 4 implies that $\mathrm{slope}(\mathcal{F}'), \mathrm{slope}(\mathcal{F}'') \leq \lambda$. Using the exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0,$$

we deduce that $\mathrm{slope}(\mathcal{F}) \leq \lambda$. □

Corollary 9. *Let* $\mathrm{Coh}(X)$ *denote the category of coherent sheaves on X and let* $\mathrm{Vect}_\lambda(X) \subseteq \mathrm{Coh}(X)$ *denote the full subcategory whose objects are vector bundles on X which are semistable of slope* 0. *Then* $\mathrm{Vect}_\lambda(X)$ *is closed under kernels, cokernels, and extensions in* $\mathrm{Coh}(X)$. *In particular, it is an abelian category.*

Warning 10. The collection of *all* vector bundles on X does not form an abelian category (note that if $f : \mathcal{E} \to \mathcal{F}$ is a map of vector bundles, then in general the cokernel $\operatorname{coker}(f)$ in the category of coherent sheaves is not a vector bundle).

Definition 11. Let $\mathcal{E}$ be a vector bundle on X. We say that a filtration

$$0 = \mathcal{E}_0 \subsetneq \mathcal{E}_1 \subseteq \mathcal{E}_2 \subsetneq \cdots \subseteq \mathcal{E}_m = \mathcal{E}$$

is a *Harder-Narasimhan filtration* if the following conditions are satisfied:

- Each of the quotient vector bundles $\mathcal{E}_i \,/\, \mathcal{E}_{i-1}$ is semistable of some slope λ_i.
- The slopes λ_i are strictly decreasing: that is, we have $\lambda_1 > \lambda_2 > \cdots > \lambda_m$.

Theorem 12. *Let $\mathcal{E}$ be a vector bundle on X. Then $\mathcal{E}$ has a unique Harder-Narasimhan filtration.*

Let us first establish uniqueness. We will proceed by induction on the rank r of $\mathcal{E}$. Suppose that $\mathcal{E}$ is equipped with two Harder-Narasimhan filtrations

$$0 = \mathcal{E}_0 \subsetneq \mathcal{E}_1 \subseteq \mathcal{E}_2 \subsetneq \cdots \subseteq \mathcal{E}_m = \mathcal{E}$$

$$0 = \mathcal{E}'_0 \subsetneq \mathcal{E}'_1 \subseteq \mathcal{E}'_2 \subsetneq \cdots \subseteq \mathcal{E}'_n = \mathcal{E}\,.$$

where the successive quotients have slopes $\lambda_1 > \cdots > \lambda_m$ and $\lambda'_1 > \cdots > \lambda'_n$, respectively. We wish to show that these filtrations are the same. We will show that $\mathcal{E}_1 = \mathcal{E}'_1$; the desired result will then follow by applying the inductive hypothesis to the filtrations

$$0 = \mathcal{E}_1 \,/\, \mathcal{E}_1 \subseteq \mathcal{E}_2 \,/\, \mathcal{E}_1 \subsetneq \cdots \subseteq \mathcal{E}_m \,/\, \mathcal{E}_1 = \mathcal{E} \,/\, \mathcal{E}_1$$

$$0 = \mathcal{E}'_1 \,/\, \mathcal{E}'_1 \subseteq \mathcal{E}'_2 \,/\, \mathcal{E}'_1 \subsetneq \cdots \subseteq \mathcal{E}'_n \,/\, \mathcal{E}'_1 = \mathcal{E} \,/\, \mathcal{E}'_1\,.$$

We first claim that $\lambda_1 = \lambda'_1$. Suppose otherwise. Then we may assume without loss of generality that $\lambda_1 > \lambda'_1$. It follows that $\lambda_1 > \lambda'_i$ for $1 \leq i \leq n$. Applying Corollary 6, we conclude that $\operatorname{Hom}(\mathcal{E}_1, \mathcal{E}'_i \,/\, \mathcal{E}'_{i-1}) = 0$. Since $\mathcal{E}$ admits a finite filtration whose successive quotients are $\mathcal{E}'_i \,/\, \mathcal{E}'_{i-1}$, it follows that $\operatorname{Hom}(\mathcal{E}_1, \mathcal{E}) = 0$. This is a contradiction, since the inclusion map $\mathcal{E}_1 \hookrightarrow \mathcal{E}$ is a nonzero element of $\operatorname{Hom}(\mathcal{E}_1, \mathcal{E})$.

The equality $\lambda_1 = \lambda'_1$ guarantees that we have a strict inequality $\lambda_1 > \lambda'_i$ for $i > 1$. As above, we conclude that $\operatorname{Hom}(\mathcal{E}_1, \mathcal{E}'_i \,/\, \mathcal{E}'_{i-1}) = 0$. Since the quotient bundle $\mathcal{E} \,/\, \mathcal{E}'_1$ admits a finite filtration whose successive quotients have the form $\mathcal{E}'_i \,/\, \mathcal{E}'_{i-1}$ with $i > 1$, it follows that $\operatorname{Hom}(\mathcal{E}_1, \mathcal{E} \,/\, \mathcal{E}'_1)$ vanishes. In particular, the composite map

$$\mathcal{E}_1 \hookrightarrow \mathcal{E} \twoheadrightarrow \mathcal{E} \,/\, \mathcal{E}'_1$$

must be zero, so we must have $\mathcal{E}_1 \subseteq \mathcal{E}'_1$. Applying the same argument with the roles of $\mathcal{E}_1$ and $\mathcal{E}'_1$ reversed, we deduce that $\mathcal{E}'_1 \subseteq \mathcal{E}_1$. We therefore have equality $\mathcal{E}_1 = \mathcal{E}'_1$, which (together with our inductive hypothesis) proves the uniqueness part of Theorem 12. To prove existence, we need the following:

Lemma 13. *Let $\mathcal{E}$ be a vector bundle on X. Then there exists an integer $N(\mathcal{E})$ with the following property: for every coherent subsheaf $\mathcal{F} \subseteq \mathcal{E}$, we have $\deg(\mathcal{F}) \leq N(\mathcal{E})$.*

Proof. We proceed by induction on the rank of $\mathcal{E}$. Note that if $\mathcal{E}$ is a line bundle, then every subsheaf $\mathcal{F} \subseteq \mathcal{E}$ is either a line bundle of smaller degree or zero; we can therefore take $N(\mathcal{E}) = \max(\deg(\mathcal{E}), 0)$. To handle the general case, we observe that if $\mathcal{E}$ has rank > 1 then we can choose an exact sequence of vector bundles

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0,$$

where $\mathcal{E}'$ and $\mathcal{E}''$ have smaller rank (for example, we can take $\mathcal{E}'$ to be the line subbundle of $\mathcal{E}$ determined by any rational section of $\mathcal{E}$). If $\mathcal{F}$ is a coherent subsheaf of $\mathcal{E}$, then $\mathcal{F}$ fits into an exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

where $\mathcal{F}' = \mathcal{F} \cap \mathcal{E}'$ and $\mathcal{F}''$ is a subsheaf of $\mathcal{E}''$. We then have

$$\deg(\mathcal{F}) = \deg(\mathcal{F}') + \deg(\mathcal{F}'') \leq N(\mathcal{E}') + N(\mathcal{E}''),$$

so setting $N(\mathcal{E}) = N(\mathcal{E}') + N(\mathcal{E}'')$ satisfies the requirements of Lemma 13. □

Proof of Theorem 12. Let $\mathcal{E}$ be a vector bundle on X; we wish to show that $\mathcal{E}$ admits a Harder-Narasimhan filtration. We proceed by induction on the rank $\mathrm{rank}(\mathcal{E})$. Let S be the collection of all rational numbers of the form $\mathrm{slope}(\mathcal{E}')$, where $\mathcal{E}' \subseteq \mathcal{E}$ is a nonzero subbundle. It follows from Lemma 13 that S has a largest element. Let λ denote the largest element of S. Then there exists a nonzero subbundle $\mathcal{E}' \subseteq \mathcal{E}$ of slope λ. Choose such a subbundle whose rank is as large as possible. Note that $\mathcal{E}'$ is semistable of slope λ: it cannot admit a subbundle of larger slope, because that would contradict the maximality of λ.

Set $\mathcal{E}'' = \mathcal{E} / \mathcal{E}'$. Then $\mathcal{E}''$ is a vector bundle whose rank is smaller than $\mathcal{E}$. It follows from our inductive hypothesis that $\mathcal{E}''$ admits a Harder-Narasimhan filtration

$$0 = \mathcal{E}''_0 \subsetneq \mathcal{E}''_1 \subseteq \cdots \subsetneq \mathcal{E}''_m = \mathcal{E}'',$$

so that the slopes $\lambda_i = \mathrm{slope}(\mathcal{E}''_i / \mathcal{E}''_{i-1})$ form a decreasing sequence $\lambda_1 > \lambda_2 > \lambda_3 > \cdots > \lambda_m$. For $0 \leq i \leq m$, let $\overline{\mathcal{E}}''_i \subseteq \mathcal{E}$ denote the inverse image of $\mathcal{E}''_i$, so that $\overline{\mathcal{E}}''_0 = \mathcal{E}'$. We will complete the proof by showing that

$$0 \subsetneq \mathcal{E}' = \overline{\mathcal{E}}_0 \subsetneq \overline{\mathcal{E}}''_1 \subsetneq \cdots \subsetneq \overline{\mathcal{E}}''_m = \mathcal{E}$$

is a Harder-Narasimhan filtration of $\mathcal{E}$. By construction, the successive quotients of this filtration are given by $\mathcal{E}'$ and $\mathcal{E}''_i / \mathcal{E}''_i$, which are semistable of slopes λ and λ_i, respectively. It will therefore suffice to show that we have inequalities $\lambda > \lambda_1 > \lambda_2 > \cdots > \lambda_m$. Assume, for a contradiction, that this fails: that is, we have $\lambda \leq \lambda_1$. We have an exact sequence

$$0 \to \mathcal{E}' \to \overline{\mathcal{E}}''_1 \to \mathcal{E}''_1 \to 0,$$

satisfying $\mathrm{slope}(\mathcal{E}') = \lambda$ and $\mathrm{slope}(\mathcal{E}''_1) = \lambda_1$. Applying Exercise 1, we deduce that $\mathrm{slope}(\overline{\mathcal{E}}''_1) \geq \lambda$. This is impossible: we cannot have $\mathrm{slope}(\overline{\mathcal{E}}''_1) > \lambda$ (since λ was chosen to be the largest element of S), and we cannot have $\mathrm{slope}(\overline{\mathcal{E}}''_1) = \lambda$ (since $\mathcal{E}'$ was chosen to be maximal among subbundles of $\mathcal{E}$ having slope λ). □

Lecture 21: Covers of the Fargues-Fontaine Curve

November 21, 2018

Throughout this lecture, we fix an algebraically closed perfectoid field $C^\flat$ of characteristic p. Let X denote the Fargues-Fontaine curve, given by

$$X = \mathrm{Proj}(\bigoplus_{n \geq 0} B^{\varphi = p^n}).$$

Recall that if $\mathcal{E}$ is a nonzero vector bundle on X, the slope of $\mathcal{E}$ is defined by the formula

$$\mathrm{slope}(\mathcal{E}) = \frac{\deg(\mathcal{E})}{\mathrm{rank}(\mathcal{E})}.$$

We say that $\mathcal{E}$ is *semistable* if every nonzero subbundle $\mathcal{E}' \subseteq \mathcal{E}$ satisfies $\mathrm{slope}(\mathcal{E}') \leq \mathrm{slope}(\mathcal{E})$. Our first goal in this lecture is to prove the following:

Proposition 1. *Let $\lambda = \frac{d}{n}$ be a rational number (where d and n are integers with $n > 0$). Then there exists a semistable vector bundle $\mathcal{E}$ on X having degree d and rank n (hence slope $\lambda = \frac{d}{n}$).*

Remark 2. The vector bundle $\mathcal{E}$ appearing in the statement of Proposition 1 is unique up to isomorphism. We will return to this point in a future lecture.

In the case $n = 1$, Proposition 1 is trivial. Note that every line bundle $\mathcal{L}$ on X is automatically semistable (since the only nonzero subbundle of $\mathcal{L}$ is $\mathcal{L}$ itself), so Proposition 1 merely asserts that for every integer d, there exists a line bundle of degree d. Here we are exploiting the fact that line bundles on X are easy to make: every divisor $D \subseteq X$ determines a line bundle $\mathcal{O}_X(D)$ on X. To produce vector bundles, we will need to work harder. One possible strategy is to look for a map of schemes

$$\pi : \widetilde{X} \to X$$

which is finite and flat of degree n. In that case, for any line bundle $\mathcal{L}$ on $\widetilde{X}$, the direct image $\pi_*(\mathcal{L})$ will be a vector bundle of rank n on X. There is an obvious source of examples to consider.

Construction 3. Let E be a finite extension of the field $\mathbf{Q}_p$ having degree n. We let X_E denote the fiber product $X \times_{\mathrm{Spec}(\mathbf{Q}_p)} \mathrm{Spec}(E)$.

Let us enumerate some easy properties of this construction.

- Since E is a finite extension field of $\mathbf{Q}_p$, the map $\mathrm{Spec}(E) \to \mathrm{Spec}(\mathbf{Q}_p)$ is finite étale of degree n (in particular, it is finite flat of degree n). It follows that the projection map $\pi : X_E \to X$ is finite étale of degree n. In particular, X_E is also a Dedekind scheme.

- Since the unit map $\mathbf{Q}_p \to \mathrm{H}^0(X, \mathcal{O}_X)$ is an isomorphism, it follows that the unit map $E \to \mathrm{H}^0(X_E, \mathcal{O}_{X_E})$ is also an isomorphism. In particular, $\mathrm{H}^0(X_E, \mathcal{O}_{X_E})$ is a field, so the scheme X_E is connected.

- Let $U \subseteq X$ be a nonempty affine open subset. Then $X - U$ can be given as the vanishing locus of a homogeneous element $t \in \bigoplus_{m \geq 0} B^{\varphi = p^m}$ (which is nonzero of positive degree). In this case, we have seen that U can be described as the spectrum of the ring $B[\frac{1}{t}]^{\varphi=1}$. Setting $U_E = U \times_{\mathrm{Spec}(\mathbf{Q}_p)} E$, it follows that U_E can be described as the spectrum of the ring

$$B[\frac{1}{t}]^{\varphi=1} \otimes_{\mathbf{Q}_p} E \simeq (B[\frac{1}{t}] \otimes_{\mathbf{Q}_p} E)^{\varphi=1}.$$

Here we extend the Frobenius automorphism $\varphi : B \to B$ to the tensor product $B \otimes_{\mathbf{Q}_p} E$ by letting it act trivially on the second factor.

- Let $x \in X$ be a closed point corresponding to an untilt K of $C^\flat$. Then the fiber product $X_E \times_X \{x\}$ can be identified with the spectrum of the tensor product $E \otimes_{\mathbf{Q}_p} K$. Since K is algebraically closed, this tensor product just factors as a Cartesian product of n copies of K. That is, every closed point of X has exactly n points of X_E lying over it, each of which has the same residue field.

- Let Y denote the set of isomorphism classes of characteristic zero untilts (K, ι) of $C^\flat$. Recall that the set of closed points of X can be identified with the quotient $Y/\varphi^{\mathbf{Z}}$. Using the same reasoning, we see that the collection of closed points of the curve X_E can be identified with the quotient $Y_E/\varphi^{\mathbf{Z}}$; here Y_E denotes the set of isomorphism classes of triples (K, ι, u), where K is a perfectoid field of characteristic zero, $\iota : C^\flat \simeq K^\flat$ is an isomorphism, and $u : E \to K$ is a map of $\mathbf{Q}_p$-algebras (note in this situation, K is an algebraically closed extension field of $\mathbf{Q}_p$, so there are exactly n choices for the embedding u).

Example 4. Suppose that E is the unramified degree n extension of $\mathbf{Q}_p$, given by $E = W(\mathbf{F}_{p^n})[\frac{1}{p}]$. If K is an untilt of $C^\flat$, then the following data are equivalent:

- $\mathbf{Q}_p$-algebra maps $e : E \to K$.
- $\mathbf{Z}_p$-algebra maps $W(\mathbf{F}_{p^n}) \to \mathcal{O}_K$.
- $\mathbf{F}_p$-algebra maps $\mathbf{F}_{p^n} \to \mathcal{O}_K / p\, \mathcal{O}_K$.
- $\mathbf{F}_p$-algebra maps $\mathbf{F}_{p^n} \to \mathcal{O}_C^\flat / \pi$, where $\pi \in C^\flat$ satisfies $|\pi|_{C^\flat} = |p|_K$.
- $\mathbf{F}_p$-algebra maps $\mathbf{F}_{p^n} \to \mathcal{O}_C^\flat$ (since $\mathbf{F}_p$ is étale over $\mathbf{F}_p$).
- $\mathbf{F}_p$-algebra maps $\mathbf{F}_{p^n} \to C^\flat$.

We therefore obtain a bijection $Y_E \simeq Y \times \mathrm{Hom}_{\mathbf{F}_p}(\mathbf{F}_{p^n}, C^\flat)$; here $\mathrm{Hom}_{\mathbf{F}_p}(\mathbf{F}_{p^n}, C^\flat$ denotes the set of $\mathbf{F}_p$-algebra maps from $\mathbf{F}_{p^n}$ into $C^\flat$. This has exactly n elements, which are cyclically permuted by the action of the Frobenius map $\varphi_C^\flat$. It follows that in this case, we have canonical bijections

$$\begin{aligned} \text{Closed points of } X_E &\simeq Y_E/\varphi^{\mathbf{Z}} \\ &\simeq (Y \times \mathrm{Hom}_{\mathbf{F}_p}(\mathbf{F}_{p^n}, C^\flat))/\varphi^{\mathbf{Z}} \\ &\simeq Y/\varphi^{n\mathbf{Z}}. \end{aligned}$$

In the situation of Example 4, the description of the set of closed points of X_E has a counterpart at the level of functions. Let $U \subseteq X$ be a nonempty affine open subset and let $U_E \subseteq X_E$ be its inverse image in X_E, so that we can write

$$U_E = \mathrm{Spec}((B[\frac{1}{t}] \otimes_{\mathbf{Q}_p} E)^{\varphi=1}).$$

In the case where E is unramified over $\mathbf{Q}_p$, this description can be simplified. Note that each embedding of $u : \mathbf{F}_{p^n} \hookrightarrow C^\flat$ factors through $\mathcal{O}_C^\flat$, and therefore induces a map

$$W(\mathbf{F}_{p^n}) \to W(\mathcal{O}_C^\flat) = \mathbf{A}_{\inf} \to B \to B[\frac{1}{t}],$$

which extends to a $\mathbf{Q}_p$-algebra map $\overline{u} : E \to B[\frac{1}{t}]$. Tensoring $\overline{u}$ with the identity map on $B[\frac{1}{t}]$, we obtain a map

$$q_u : B[\frac{1}{t}] \otimes_{\mathbf{Q}_p} E \to B[\frac{1}{t}].$$

Exercise 5. Show that the maps q_u induce an isomorphism

$$B[\frac{1}{t}] \otimes_{\mathbf{Q}_p} E \simeq \prod_{u : \mathbf{F}_{p^n} \hookrightarrow C^\flat} B[\frac{1}{t}].$$

This doesn't require knowing much about the situation: you can replace $B[\frac{1}{t}]$ by any $\mathbf{Q}_p$-algebra R which admits a map $E \to R$.

The Frobenius automorphism of $B[\frac{1}{t}]$ extends uniquely to an E-linear automorphism of $B[\frac{1}{t}] \otimes_{\mathbf{Q}_p} E$. Under the isomorphism

$$B[\frac{1}{t}] \otimes_{\mathbf{Q}_p} E \simeq \prod_{u : \mathbf{F}_{p^n} \hookrightarrow C^\flat} B[\frac{1}{t}],$$

this automorphism cyclically permutes the factors. Concretely, if we identify an element of the right hand side with an n-tuple $(f_0, f_1, \ldots, f_{n-1})$ of elements of $B[\frac{1}{t}]$, then we have $\varphi(f_0, f_1, \ldots, f_{n-1}) = (\varphi(f_{n-1}), \varphi(f_0), \varphi(f_1), \ldots, \varphi(f_{n-2}))$. It follows that $(f_1, f_2, \ldots, f_n)$ is invariant under the Frobenius if and only if we have $f_i = \varphi(f_{i-1})$ for $0 < i < n$ and $f_0 = \varphi(f_{n-1})$. The first equation guarantees that $f_i = \varphi^i(f_0)$ for $0 < i < n$, so that we can rewrite the second as $\varphi^n(f_0) = f_0$. This proves the following:

Proposition 6. *Let E be the unramified degree n extension of $\mathbf{Q}_p$ and let $U \subsetneq X$ be the vanishing locus of a homogeneous element $t \in \bigoplus B^{\varphi = p^m}$. Then we have*

$$U_E = \mathrm{Spec}(B[\frac{1}{t}]^{\varphi^n = 1}).$$

Let us now return to the case where E is *any* finite extension field of $\mathbf{Q}_p$. Let $\pi : X_E \to X$ be the projection map. Note that if $\mathcal{E}$ is any vector bundle of rank r on X_E, then $\pi_* \mathcal{E}$ is a vector bundle of rank nr on X. Moreover, this construction induces an equivalence of categories

$$\{\text{Vector bundles on } X_E\} \simeq \{\text{Vector bundles on } X \text{ with an action of the field } E\}.$$

Every vector bundle $\mathcal{E}$ on X_E has a well-defined degree, defined by the formula $\deg(\mathcal{E}) = \deg(\pi_* \mathcal{E})$. If $\mathcal{E}$ is not zero, we can define the slope $\mathrm{slope}(\mathcal{E})$ by the formula $\mathrm{slope}(\mathcal{E}) = \frac{\deg(\mathcal{E})}{\mathrm{rank}(\mathcal{E})} = \frac{1}{n}\mathrm{slope}(\pi_* \mathcal{E})$.

Proposition 7. *Let $\mathcal{E}$ be a nonzero vector bundle on X_E and let λ be a rational number. The following conditions are equivalent:*

- (a) *The vector bundle $\mathcal{E}$ is semistable of slope λ. That is,* $\mathrm{slope}(\mathcal{E}) = \lambda$ *and, for every nonzero subbundle $\mathcal{E}' \subseteq \mathcal{E}$, we have* $\mathrm{slope}(\mathcal{E}') \leq \lambda$.
- (b) *The direct image $\pi_* \mathcal{E} \in \mathrm{Vect}(X)$ is semistable of slope $\frac{\lambda}{n}$, in the sense of the previous lecture.*

Proof. We have already observed that $\mathcal{E}$ has slope λ if and only if $\pi_* \mathcal{E}$ has slope $\frac{\lambda}{n}$. The implication $(b) \Rightarrow (a)$ is now clear: note that $\pi_* \mathcal{E}$ is semistable and $\mathcal{E}'$ is a nonzero subbundle of $\mathcal{E}$, then $\pi_* \mathcal{E}'$ is a nonzero subbundle of $\pi_* \mathcal{E}$, and therefore satisfies

$$\deg(\mathcal{E}') = n \deg(\pi_* \mathcal{E}') \leq n \deg(\pi_* \mathcal{E}) = \deg(\mathcal{E}).$$

Conversely, suppose that $\mathcal{E}$ is semistable of slope λ.

Assume that $\mathcal{F} = \pi_* \mathcal{E}$ is not semistable; we will see that this leads to a contradiction. Let

$$0 = \mathcal{F}_0 \subsetneq \mathcal{F}_1 \subsetneq \mathcal{F}_2 \subsetneq \cdots \subsetneq \mathcal{F}_m = \mathcal{F}$$

be the Harder-Narasimhan filtration of $\mathcal{F}$, with slopes $\lambda_i = \mathrm{slope}(\mathcal{F}_i / \mathcal{F}_{i-1})$ satisfying $\lambda_1 > \lambda_2 > \cdots > \lambda_m$. Then we must have $\lambda_1 > \mathrm{slope}(\mathcal{F}) = \frac{\lambda}{n}$. Note that, for any nonzero element $x \in E$, multiplication by x induces an automorphism of $\mathcal{F} = \pi_* \mathcal{E}$ which automatically preserves the Harder-Narasimhan filtration. It follows that the action of E on $\mathcal{F}$ preserves the subbundle $\mathcal{F}_1$, so that we can write $\mathcal{F}_1 = \pi_* \mathcal{E}'$ for some subbundle $\mathcal{E}' \subseteq \mathcal{E}$. We then have $\mathrm{slope}(\mathcal{E}') = n\lambda_1 > \lambda$, contradicting the semistability of $\mathcal{E}$. $\square$

Proof of Proposition 1. Let $\lambda = \frac{d}{n}$ be a rational number; we wish to show that there exists a semistable vector bundle on X having rank n and degree d. Let E be a finite extension of $\mathbf{Q}_p$ of degree n, and let $\mathcal{L}$ be a line bundle of degree d on X_E (for example, we can take $\mathcal{L} = \mathcal{O}_{X_E}(D)$, where D is a divisor of degree d on X_E). Then $\mathcal{L}$ is automatically semistable as a vector bundle on X_E (since it has rank 1). Applying Proposition 7, we see that $\pi_* calL$ is a semistable vector bundle on X, which evidently has rank n and degree d. $\square$

Our next goal is to better understand the vector bundle produced by our proof of Proposition 1. *A priori*, it depends on a few choices: the finite extension $E \supseteq \mathbf{Q}_p$, and the choice of line bundle $\mathcal{L}$ on X_E. But it turns out that both of these choices are irrelevant. In the second case, this is because of the following:

Theorem 8. *Let E be a finite extension of $\mathbf{Q}_p$. Then the degree map $\deg : \mathrm{Pic}(X_E) \to \mathbf{Z}$ is an isomorphism.*

In Lecture 19, we proved Theorem 8 in the special case $E = \mathbf{Q}_p$. The point was that for every pair of closed points $x, x' \in X$, the divisor $x - x'$ is linearly equivalent to zero: that is, we can find a rational function f on X having a simple zero at x and a simple pole at x'. More precisely, we can take

$$f = \frac{\log([\epsilon])}{\log([\epsilon'])} \in B[\frac{1}{\log([\epsilon'])}]^{\varphi=1},$$

where $\epsilon, \epsilon' \in 1 + \mathfrak{m}_C^\flat$ are chosen so that the vanishing loci of $\log([\epsilon])$ and $\log([\epsilon'])$ in Y are exactly the Frobenius orbits corresponding to the points x and x', respectively. The function f is Frobenius-invariant, but individually the numerator and denominator are not: they both belong to the eigenspace $B^{\varphi=p}$. To prove Theorem 8 in general, we need to do something analogous for the cover X_E. We will return to this in the next lecture.

Lecture 22: Line Bundles on Covers

November 23, 2018

Throughout this lecture, we fix an algebraically closed perfectoid field $C^\flat$ of characteristic p. Let X denote the Fargues-Fontaine curve, given by

$$X = \mathrm{Proj}(\bigoplus_{n \geq 0} B^{\varphi = p^n}).$$

In the previous lecture, we explained a strategy for producing semistable vector bundles of any given slope $\lambda = \frac{m}{n}$ on X:

- First, choose a finite degree n extension E of $\mathbf{Q}_p$.
- Then, choose a line bundle $\mathcal{L}$ of degree m on the fiber product $X_E = X \times_{\mathrm{Spec}(\mathbf{Q}_p)} \mathrm{Spec}(E)$.

The direct image of $\mathcal{L}$ along the map $X_E \to X$ is then a semistable vector bundle of degree m and rank n. This vector bundle is *a priori* dependent on the choice of extension E and line bundle $\mathcal{L}$. But it turns out not to matter: up to isomorphism, the resulting vector bundle depends only on the integers d and n. First, independence of $\mathcal{L}$ is a consequence of the following:

Theorem 1. *Let E be a finite extension of $\mathbf{Q}_p$. Then the degree map* $\deg : \mathrm{Pic}(X_E) \to \mathbf{Z}$ *is an isomorphism.*

To prove Theorem 1, it will suffice to show that, for every pair of closed points $x, x' \in X_E$, we have $\mathcal{O}_{X_E}(x) = \mathcal{O}_{X_E}(x')$. We have already seen that this is true when $E = \mathbf{Q}_p$. Essentially, we proved this by observing that $\mathcal{O}_X(x)$ and $\mathcal{O}_X(x')$ can be identified with another line bundle $\mathcal{O}(1)$, whose definition did not depend on a choice of point of X. We would like to show that something similar happens for the scheme X_E.

Notation 2. For the remainder of this lecture, we fix a finite extension field E of $\mathbf{Q}_p$ of degree n. Then the inclusion $\mathbf{Q}_p \hookrightarrow E$ admits an essentially unique factorization as

$$\mathbf{Q}_p \hookrightarrow E_0 \hookrightarrow E,$$

where E_0 is an unramified extension of $\mathbf{Q}_p$ having some degree d (so that $E_0 \simeq W(\mathbf{F}_{p^d})[\frac{1}{p}]$ and E is a totally ramified extension of E_0 having some degree e; we then have $n = d \cdot e$. We let $\mathcal{O}_E$ denote the ring of integers of E, and $\pi \in \mathcal{O}_E$ a choice of uniformizer.

Exercise 3. Choose an embedding $\mathbf{F}_{p^d} \hookrightarrow \mathcal{O}_C^\flat$, which extends to a map $W(\mathbf{F}_{p^d}) \to W(\mathcal{O}_C^\flat) = \mathbf{A}_{\mathrm{inf}} \to B$ and therefore a map $E_0 \to B$, whose image is stable under the dth power of the Frobenius map. Let t be any homogeneous element of the graded ring $\bigoplus_{n \geq 0} B^{\varphi = p^n}$. Show that the canonical map

$$B[\frac{1}{t}] \otimes_{\mathbf{Q}_p} E \to B[\frac{1}{t}] \otimes_{E_0} E$$

induces an isomorphism

$$(B[\frac{1}{t}] \otimes_{\mathbf{Q}_p} E)^{\varphi=1} \simeq (B[\frac{1}{t}] \otimes_{E_0} E)^{\varphi^d=1}.$$

In the special case $E_0 = E$, this recovers the isomorphism $(B[\frac{1}{t}] \otimes_{\mathbf{Q}_p} E)^{\varphi=1} \simeq B[\frac{1}{t}]^{\varphi^n=1}$ of the previous lecture.

It follows that, if $U \subseteq X$ is the complement of the vanishing locus of t and we set $U_E = U \times_{\mathrm{Spec}(\mathbf{Q}_p)} E$, then (when t has positive degree) we can write $U_E = \mathrm{Spec}((B[\frac{1}{t}] \otimes_{E_0} E)^{\varphi^d=1})$.

Construction 4. We attempt to construct a line bundle $\mathcal{O}_{X_E}(1)$ on X_E as follows:

- To each affine open subset $U \subseteq X$ as above (given by the complement of the vanishing locus of t), we set
$$\mathcal{O}_{X_E}(1)(U_E) = (B[\frac{1}{t}] \otimes_{E_0} E)^{\varphi^d=\pi}.$$

To simultaneously show that this construction "works" and prove Theorem 1, it will suffice to show that $\mathcal{O}_{X_E}(1)$ agrees with the line bundle $\mathcal{O}_{X_E}(x)$, for any choice of point $x \in X_E$. In other words, we need to find a section of $\mathcal{O}_{X_E}(1)$ which vanishes exactly at the point x. First, we need some terminology.

Notation 5. In the previous lecture, we let Y_E denote the set of isomorphism classes of triples (K, ι, u), where (K, ι) is an untilt of $C^\flat$ and $u : E \to K$ is an embedding of fields. Let $Y_E^\circ \subseteq Y_E$ denote the subset consisting of those triples where $u|_{E_0}$ is given by the composite map $E_0 \to B \to K$ (corresponding to the embedding $\mathbf{F}_{p^d} \hookrightarrow C^\flat$ that we have chosen). Then Y_E° is not stable under the Frobenius, but is stable under its dth power; moreover, the inclusion $Y_E^\circ \hookrightarrow Y_E$ induces a bijection

$$Y_E^\circ / \varphi^{d\mathbf{Z}} \simeq Y_E / \varphi^{\mathbf{Z}}.$$

Recall that, for each point $y = (K, \iota)$ of Y, we have an evaluation map

$$B \to K \qquad f \mapsto f(y).$$

If we promote y to a point $\overline{y} = (K, \iota, e)$ of Y_E°, then this evaluation map admits an E-linear extension

$$B \otimes_{E_0} E \to K,$$

which we will denote by $f \mapsto f(\overline{y})$.

In fact, we can do a little bit better. Recall that K can be identified with the residue field of a discrete valuation ring $B_{\mathrm{dR}}^+(y)$ (with uniformizer we denote by ξ) and that the homomorphism $B \to K$ lifts to a map

$$B \to B_{\mathrm{dR}}^+(y) \qquad f \mapsto \widehat{f}_y.$$

In particular, this allows us to view $B_{\mathrm{dR}}^+(y)$ as an algebra over the field $E_0 \subseteq B$. Since E is a separable extension field of E_0, the E_0-algebra map

$$E \xrightarrow{u} K \simeq B_{\mathrm{dR}}^+(y)/(\xi)$$

lifts uniquely to a homomorphism $E \to B_{\mathrm{dR}}^+(y)$. Amalgamating, we obtain a homomorphism

$$B \otimes_{E_0} E \to B_{\mathrm{dR}}^+(y),$$

which we will denote by $f \mapsto \widehat{f}_{\overline{y}}$. In particular, this allows us to define an *order of vanishing* $\mathrm{ord}_{\overline{y}}(f) \in \mathbf{Z}_{\geq 0} \cup \{\infty\}$ for each $f \in B \otimes_{E_0} E$: namely, the supremum of those integers m such that $\widehat{f}_{\overline{y}}$ is divisible by ξ^m.

We will deduce Theorem 1 from the following result, which we prove in the next lecture:

Theorem 6. *Let x be a closed point of X_E, corresponding to a subset $S \subseteq Y_E^{\circ}$ which is an orbit for the action of $\varphi^{d\mathbf{Z}}$. Then there exists an element $f \in (B \otimes_{E_0} E)^{\varphi^d = \pi}$ satisfying*

$$\mathrm{ord}_{\overline{y}}(f) = \begin{cases} 1 & \text{if } \overline{y} \in S \\ 0 & \text{otherwise.} \end{cases}$$

Example 7. In the case $E = \mathbf{Q}_p$ and $\pi = p$, we can choose $f \in B^{\varphi = p}$ to be an element of the form $\log([\epsilon])$ for $\epsilon \in 1 + \mathfrak{m}_C^{\flat}$.

Proof of Theorem 1 from Theorem 6. Assuming Theorem 6, we show that for each closed point $x \in X_E$, the line bundle $\mathcal{O}_{X_E}(x)$ on X_E is isomorphic to the presheaf $\mathcal{O}_{X_E}(1)$ described in Construction 4: this will show both that $\mathcal{O}_{X_E}(1)$ extends to a line bundle and that $\mathcal{O}_{X_E}(x)$ is independent of x. Choose $f \in (B \otimes_{E_0} E)^{\varphi^d = \pi}$ satisfying the conclusion of Theorem 6. We will show that, for every affine open subset $U \subseteq X$ (complementary to the vanishing locus of some homogeneous element $t \in \bigoplus B^{\varphi = p^m}$), multiplication by f induces an isomorphism

$$\mathcal{O}_{X_E}(x)(U_E) \to \mathcal{O}_{X_E}(1)(U_E) = (B[\frac{1}{t}] \otimes_{E_0} E)^{\varphi^d = \pi}.$$

Note that $B \otimes_{E_0} E$ is a finite flat ring extension of B (of degree e). Let $N(f) \in B$ denote the norm of f along this ring extension (that is, the determinant of the B-module homomorphism of $B \otimes_{E_0} E$ given by multiplication by f). Note that, for each point $y \in Y$, we have $\widehat{N(f)}_y = \prod \widehat{f}_{\overline{y}}$, where the product is taken over the set of all preimages of y in Y_E°. It follows that the vanishing locus of $N(f)$ is given by a single orbit of $\varphi^{d\mathbf{Z}}$ on Y (and that $N(f)$ has simple zeros at each point where it vanishes). Then the product $N(f)\varphi(N(f))\varphi^2(N(f)) \cdots \varphi^{d-1}(N(f)) \in B$ vanishes on a single $\varphi^{\mathbf{Z}}$-orbit of Y (again with simple zeros), and can therefore be written as a product $u \log([\epsilon])$ where u is an invertible element of B and $\epsilon \in 1 + \mathfrak{m}_C^{\flat}$. Here $\log([\epsilon])$ vanishes at a single point of X, which can be identified with the image of x under the projection map $X_E \to X$. Note that since f divides the norm $N(f)$, it divides the product $N(f)\varphi(N(f))\varphi^2(N(f)) \cdots \varphi^{d-1}(N(f)) = u \log([\epsilon])$, and therefore also divides $\log([\epsilon])$.

We now distinguish two cases:

- Suppose that x does not belong to U_E. Then $\log([\epsilon])$ is a divisor of t, so f is a divisor of t and is therefore invertible in the ring $B[\frac{1}{t}] \otimes_{E_0} E$. In this case, multiplication by f induces an isomorphism of
$$\mathcal{O}_{X_E}(x)(U_E) = (B \otimes_{E_0} E)^{\varphi^d = 1} \xrightarrow{f} (B \otimes_{E_0} E)^{\varphi^d = \pi},$$
with inverse given by multiplication by $\frac{1}{f}$.
- Suppose that x belongs to U_E. Choose some other point $x' \in X_E$ which does not belong to U_E, and let $f' \in (B \otimes_{E_0} E)^{\varphi = \pi}$ satisfy the conclusion of Theorem 6 for the point x'. The preceding argument shows that f' is invertible in $B[\frac{1}{t}] \otimes_{E_0} E$. It follows that the ratio $\frac{f}{f'}$ is a well-defined element of $(B[\frac{1}{t}] \otimes_{E_0} E)^{\varphi = 1}$, which we can identify with a regular function on U_E with a simple zero at the point x. Consequently, multiplication by $\frac{f'}{f}$ induces an isomorphism $\mathcal{O}_{X_E}(U_E) \to \mathcal{O}_{X_E}(x)(U_E)$. It will therefore suffice to show that the composite map
$$\mathcal{O}_{X_E}(U_E) \xrightarrow{\frac{f'}{f}} \mathcal{O}_{X_E}(x)(U_E) \xrightarrow{f} (B[\frac{1}{t}] \otimes_{E_0} E)^{\varphi^d = \pi}$$
is an isomorphism. In other words, we may replace x by x' and thereby reduce to the case treated above.

$\square$

Lecture 23-Formal Groups

November 27, 2018

Throughout this lecture, we fix an algebraically closed perfectoid field $C^\flat$ of characteristic p. Let X denote the Fargues-Fontaine curve, given by

$$X = \mathrm{Proj}(\bigoplus_{m \geq 0} B^{\varphi = p^m}).$$

Let E be a finite extension of $\mathbf{Q}_p$, so that $\mathbf{Q}_p \subseteq E_0 \subseteq E$ where $\mathbf{Q}_p \hookrightarrow E_0$ is an unramified extension of degree d and $E_0 \hookrightarrow E$ is a totally ramified extension of degree e. Let $\mathcal{O}_E$ denote the ring of integers of E and let $\pi \in \mathcal{O}_E$ be a uniformizer. Our goal in this lecture is to explain the following result, which was stated without proof in the last lecture:

Theorem 1. *Let x be a closed point of X_E, corresponding to a subset $S \subseteq Y_E^\circ$ which is an orbit for the action of $\varphi^{d\mathbf{Z}}$. Then there exists an element $f \in (B \otimes_{E_0} E)^{\varphi^d = \pi}$ satisfying*

$$\mathrm{ord}_{\overline{y}}(f) = \begin{cases} 1 & \text{if } \overline{y} \in S \\ 0 & \text{otherwise.} \end{cases}$$

In the case $E = \mathbf{Q}_p$, we can take $\pi = p$ and choose f be an element of the form $\log([\epsilon])$, for some $\epsilon \in 1 + \mathfrak{m}_C^\flat$. We would like to do something analogous for a general extension E of $\mathbf{Q}_p$. Note that the power series

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

has a special feature: for any completely valued field K of characteristic zero, $x \mapsto \log(x)$ induces a group homomorphism from the open unit disk $1 + \mathfrak{m}_K$ (regarded as an abelian group with respect to multiplication) to K (regarded as a group under addition). To generalize this, it is useful to examine other group structures on the open unit disk.

Definition 2. Let R be a commutative ring. A *formal group law* over R is a power series $F(u,v) \in R[[u,v]]$ satisfying the identities

$$F(0,u) = u \qquad F(u,v) = F(v,u) \qquad F(u, F(v,w)) = F(F(u,v),w).$$

Let F be a formal group law over R, and let A be a commutative R-algebra. Then, for any pair of nilpotent elements $u, v \in A$, we can evaluate the power series F on the elements u and v to obtain a new nilpotent element $F(u,v) \in A$. It follows from the above identities that the induced map

$$F : \{\text{Nilpotent elements of } A\} \times \{\text{Nilpotent elements of } A\} \to \{\text{Nilpotent elements of } A\}$$

is commutative and associative, and has a unit given by the zero element $0 \in A$. It therefore equips the set $\{\text{Nilpotent elements of } A\}$ with the structure of a commutative monoid, which we will denote by $\widehat{\mathbf{G}}(A)$. The construction $A \mapsto \widehat{\mathbf{G}}(A)$ determines a functor from commutative R-algebras to commutative monoids. We will refer to $\widehat{\mathbf{G}}$ as the *formal group associated to F*.

Exercise 3. Let F be a formal group law over a commutative ring R. Show that, for every commutative R-algebra A, the commutative monoid $\mathbf{G}(A)$ is a group.

Example 4 (The Formal Additive Group)**.** The power series $F(u,v) = u + v$ satisfies the requirements of Remark **??**, and therefore gives rise to a formal group law. It corresponds to the functor

$$\widehat{\mathbf{G}}_a : \{\text{Commutative } R\text{-algebras}\} \to \{\text{Abelian groups}\},$$

where $\widehat{\mathbf{G}}_a(A)$ is the collection of nilpotent elements of A, regarded as a group under addition. We refer to $\widehat{\mathbf{G}}_a$ as the *formal additive group.*

Example 5 (The Formal Multiplicative Group)**.** The power series $F(u,v) = u+v+uv$ satisfies the requirements of Remark **??**, and therefore gives rise to a formal group law. It corresponds to the functor

$$\widehat{\mathbf{G}}_m : \{\text{Commutative } R\text{-algebras}\} \to \{\text{Abelian groups}\},$$

where $\widehat{\mathbf{G}}_m(A)$ can be identified with the set $\{x \in A : \ x-1 \text{ is nilpotent}\}$, regarded as a group under multiplication. We refer to $\widehat{\mathbf{G}}_m$ as the *formal multiplicative group.*

Note that the formal additive group and the formal multiplicative group are different in general, but are isomorphic whenever the ground ring R has characteristic zero: in that case, we have natural isomorphisms

$$\widehat{\mathbf{G}}_m(A) \simeq \widehat{\mathbf{G}}_a(A)$$

$$1 + x \mapsto \log(1+x) = x - \frac{x^2}{2} + \cdots .$$

This is a general phenomenon: over a ring of characteristic zero, all formal groups look like the formal additive group $\widehat{\mathbf{G}}_a$.

Remark 6. Let F be a formal group law over a commutative ring R and let $\widehat{\mathbf{G}}$ be the associated formal group. If we ignore the group structure on $\widehat{\mathbf{G}}$ (that is, if we think of $\widehat{\mathbf{G}}$ as a set-valued functor on the category of commutative R-algebras), then it is isomorphic to the formal affine line $\widehat{\mathbf{A}}^1 \simeq \operatorname{Spf}(R[[t]])$.

Suppose that F and F' are formal group laws, defining formal groups $\widehat{\mathbf{G}}$ and $\widehat{\mathbf{G}}'$, respectively. A *homomorphism of formal groups* from $\widehat{\mathbf{G}}$ and $\widehat{\mathbf{G}}'$ is a natural transformation of group-valued functors $H : \widehat{\mathbf{G}} \to \widehat{\mathbf{G}}'$. Ignoring the group structures, we can view H as a map of formal schemes from $\operatorname{Spf}(R[[t]])$ to itself which preserves the "origin" given by the vanishing locus of t. It follows that H is induced by a map of power series rings

$$R[[t]] \to R[[t]] \qquad t \mapsto h(t),$$

where $h(t) \in R[[t]]$ is a power series with vanishing constant term. The condition that this power series induces a group homomorphism can be written concretely as the formula

$$h(F(u,v)) = F'(h(u), h(v)).$$

We say that H is an *isomorphism of formal groups* if it is an isomorphism of group valued-functors, or equivalently if the power series $h(t) = c_1 t + c_2 t^2 + c_3 t^3 + \cdots$ has $c_1 \in R^{\times}$.

Definition 7. Let F be a formal group law over a commutative ring R, with associated formal group $\widehat{\mathbf{G}}$. A *logarithm* for F is an isomorphism of formal groups $\widehat{\mathbf{G}} \to \widehat{\mathbf{G}}_a$, given by a power series $h(t) = c_1 t + c_2 t^2 + \cdots$ which satisfies the identity

$$h(F(u,v)) = h(u) + h(v).$$

We further assume that $c_1 = 1$ (this is a normalization condition: it can always be arranged by multiplying the power series h by a scalar).

Proposition 8. *Let R be a commutative ring which contains the field $\mathbf{Q}$ of rational numbers. Then every formal group law $F(u,v)$ has a unique logarithm.*

Proof. Let us first prove uniqueness. Write $F(u,v) = \sum c_{i,j} u^i v^j$, so that $c_{1,0} = 1$. Set

$$f(v) = \frac{\partial F}{\partial u}(0,v) = \sum c_{1,j} v^j = 1 + c_{1,1} v + c_{1,2} v^2 + \cdots .$$

Let h be a logarithm for F, so that h satisfies the identity

$$h(F(u,v)) = h(u) + h(v).$$

Differentiating with respect to u and setting $u = 0$

$$f(v) h'(v) = \frac{\partial F}{\partial u}(0,v) h'(F(0,v)) = \frac{\partial h(F(u,v))}{\partial u}|_{u=0} = h'(0) = 1.$$

It follows that h must satisfy $h(0) = 0$ and $h'(v) = \frac{1}{f(v)}$, and is therefore uniquely determined.

To prove existence, we define h to be the unique power series satisfying the preceding identities, given formally by the formula

$$h(t) = \int_0^t \frac{1}{f(v)} dv.$$

We claim that h has the desired property: that is, that we have

$$h(F(u,v)) = h(u) + h(v).$$

Since $h(0) = 0$, this identity holds after setting $u = 0$. It will therefore suffice to check that it holds after differentiating with respect to u: that is, that we have an identity of power series

$$\frac{\partial F}{\partial u}(u,v) h'(F(u,v)) = h'(u)$$

or equivalently

$$\frac{\partial F}{\partial u}(u,v) f(u) = f(F(u,v)).$$

This follows from the associativity identity

$$F(F(t,u),v) = F(t, F(u,v))$$

by differentiating with respect to t and then setting $t = 0$. □

Construction 9. Let R be a commutative ring and let F be a formal group law over R. The associated formal group $\widehat{\mathbf{G}}$ might not be isomorphic to the formal additive group $\widehat{\mathbf{G}}_a$. However, after extending scalars to the ring $R_{\mathbf{Q}} = R \otimes \mathbf{Q}$ and applying Proposition 8, we deduce that there exists a unique power series $\log_F(t)$ with coefficients in $R_{\mathbf{Q}}$ such that $\log_F(t) \equiv t \pmod{t^2}$ and

$$\log_F(F(u,v)) = \log_F(u) + \log_F(v).$$

We refer to $\log_F(t)$ as the *logarithm of F*.

Remark 10. Let us assume that R is torsion-free, so that R can be identified with a subring of $R_{\mathbf{Q}}$. In general, the coefficients of the power series $\log_F$ do not belong to $R \subseteq R_{\mathbf{Q}}$: that is, they involve denominators. Inspecting the proof of Proposition 8, we see that these denominators arise from the process of integration. That is, if we set $f(v) = \frac{\partial F}{\partial u}(0,v)$, then $f(v) = 1$ so we can write $\frac{1}{f(v)} = 1 + a_1 v + a_2 v^2 + a_3 v^3 + \cdots$. We then have

$$\log_F(t) = t + \frac{a_1}{2} t^2 + \frac{a_2}{3} t^3 + \cdots .$$

In particular, the denominators occuring in the power series $\log_F(t)$ are "no worse" than the denominators which occur in the power series expansion for the usual logarithm: the coefficient of t^n belongs to $\frac{1}{n} R$.

Let R be a commutative ring. The formal additive group $\widehat{\mathbf{G}}_a$ has some additional structure: for every commutative R-algebra A, the set

$$\widehat{\mathbf{G}}_a(A) = \{\text{Nilpotent elements of } A\}$$

is not just an abelian group under addition, but an R-module. Given a formal group law F over R with associated formal group $\widehat{\mathbf{G}}$, we can attempt to use the logarithm $\log_F$ to transport this structure to $\widehat{\mathbf{G}}$.

Construction 11. Let R be a commutative ring and let F be a formal group over R. For each element $\lambda \in R$, let $[\lambda](t) \in R_{\mathbf{Q}}[[t]]$ be the power series given by the formula

$$\log_F^{-1}(\lambda \log_F(t)).$$

Exercise 12. Let R be a commutative ring and let F be a formal group over R. Show that the construction $(\lambda \in R) \mapsto ([\lambda](t) \in R_{\mathbf{Q}}[[t]])$ has the following properties:

- We have $[\lambda](t) \equiv \lambda t \pmod{t^2}$.
- For $\lambda, \mu \in R$, we have $[\lambda\mu](t) = [\lambda]([\mu](t))$. Similarly, we have $[1](t) = t$.
- For $\lambda, \mu \in R$, we have $[\lambda + \mu](t) = F([\lambda](t), [\mu](t))$. Similarly, we have $[0](t) = 0$.

Hint: we might as well replace R by $R_{\mathbf{Q}}$.

Exercise 13. Let R be a torsion-free commutative ring, let F be a formal group law over R, and let $\widehat{\mathbf{G}}$ be the associated formal group. Set

$$R_0 = \{\lambda \in R : [\lambda](t) \text{ has coefficients in } R\}.$$

Show that R_0 is a subring of R (combine Exercises 12 and 3).

Moreover, show that for any commutative algebra R-algebra A (which might not be torsion-free!), the abelian group $\widehat{\mathbf{G}}(A)$ has the structure of a module over R_0, where multiplication by $\lambda \in R_0$ is implemented by the map

$$(x \in \widehat{\mathbf{G}}(A) = \{\text{Nilpotent elements of } A\}) \mapsto ([\lambda](x) \in \widehat{\mathbf{G}}(A) = \{\text{Nilpotent elements of } A\}$$

In the situation of Exercise 13, we will say that the formal group $\widehat{\mathbf{G}}$ is a *formal R_0-module.* (When R is torsion-free, this is simply a *property* of a formal group over R, not an additional structure.)

As a tool for proving Theorem 1, we cite the following classical result:

Theorem 14 (Lubin-Tate)**.** *Let E be a finite extension of $\mathbf{Q}_p$, let $q = p^d$ be the order of the residue field of E, and let $\pi \in \mathcal{O}_E$ be a uniformizer. Let $f(t) \in \mathcal{O}_E[[t]]$ be any power series satisfying*

$$f(t) \equiv \pi t \pmod{t^2}$$

$$f(t) \equiv t^q \pmod{\pi}.$$

Then there is a unique formal group law $F(u,v) \in \mathcal{O}_E[[u,v]]$ satisfying

$$\pi \log_F(t) = \log_F(f(t))$$

(that is, so that $f(t)$ coincides with the power series $[\pi](t)$ of Construction 11). We will denote the associated formal group by $\widehat{\mathbf{G}}_{\mathrm{LT}}$ and refer to it as the Lubin-Tate formal group of E. Moreover, $\widehat{\mathbf{G}}_{\mathrm{LT}}$ is a formal $\mathcal{O}_E$-module (that is, the power series $[\lambda](t)$ has coefficients in $\mathcal{O}_E$ for each $\lambda \in \mathcal{O}_E$).

Remark 15. One can show that the formal group $\widehat{\mathbf{G}}_{\mathrm{LT}}$ depends only on E, and not on the choice of power series $f(t)$. However, we will not need this for our applications: it will be enough to fix any choice of power series $f(t)$ satisfying the congruences above, such as $f(t) = \pi t + t^q$.

Example 16. In the case $E = \mathbf{Q}_p$ and $\pi = p$, the multiplicative formal group law $F(u, v) = u + v + uv$ satisfies the requirements of Theorem 14 for the power series

$$f(t) = (1+t)^p - 1 = pt + \cdots + pt^{p-1} + t^p.$$

Lecture 24-Lubin-Tate Formal Groups

November 30, 2018

Throughout this lecture, we fix an algebraically closed perfectoid field $C^\flat$ of characteristic p. Let E be a finite extension of $\mathbf{Q}_p$, so that $\mathbf{Q}_p \subseteq E_0 \subseteq E$ where $\mathbf{Q}_p \hookrightarrow E_0$ is an unramified extension of degree d and $E_0 \hookrightarrow E$ is a totally ramified extension of degree e. Set $q = p^d$. Let $\mathcal{O}_E$ denote the ring of integers of E and let $\pi \in \mathcal{O}_E$ be a uniformizer. Let $F(u, v)$ be the Lubin-Tate formal group law associated to the polynomial $f(t) = \pi t + t^q$ and let $\mathbf{G}_{\mathrm{LT}}$ denote the associated formal group. That is, F is the unique formal $\mathcal{O}_E$-module satisfying

$$[\pi](t) = \pi t + t^q.$$

In the previous lecture, we viewed the formal group $\mathbf{G}_{\mathrm{LT}}$ as a functor

$$\{\text{Commutative } \mathcal{O}_E\text{-algebras}\} \to \{\text{Abelian groups}\}$$

given by $\mathbf{G}_{\mathrm{LT}}(A) = \{$ Nilpotent elements of $A\}$ (with group structure given by $(u, v) \mapsto F(u, v)$). In this lecture, it will be convenient to consider a more general construction.

Notation 1. Let A be an $\mathcal{O}_E$-algebra which is complete with respect to an ideal I, which we view as a *topological* commutative ring by endowing it with the I-adic topology. We then define

$$\begin{aligned} \mathbf{G}_{\mathrm{LT}}(A) &= \varprojlim \mathbf{G}_{\mathrm{LT}}(A/I^n) \\ &= \{\text{Topologically nilpotent elements } x \in A\}. \end{aligned}$$

Beware that there is some ambiguity in our notation: the definition of $\mathbf{G}_{\mathrm{LT}}(A)$ depends on whether we view A as a discrete $\mathcal{O}_E$-algebra (in which case it consists only of nilpotent elements of A) or as a topological $\mathcal{O}_E$-algebra (in which case it consists of the topologically nilpotent elements of A; in particular, it includes the ideal I). Hopefully our usage will be clear in context.

Variant 2. *Let A be a $\mathcal{O}_E$-algebra which is complete with respect to an ideal I. We define $\widetilde{\mathbf{G}}_{\mathrm{LT}}(A)$ by the formula*

$$\widetilde{\mathbf{G}}_{\mathrm{LT}}(A) = \varprojlim(\cdots \xrightarrow{p} \mathbf{G}_{\mathrm{LT}}(A) \xrightarrow{p} \mathbf{G}_{\mathrm{LT}}(A).$$

We refer to the functor $A \mapsto \widetilde{\mathbf{G}}_{\mathrm{LT}}(A)$ as the universal cover of the Lubin-Tate formal group $\mathbf{G}_{\mathrm{LT}}$.

Remark 3. In the situation of Variant 2, the universal cover $\widetilde{\mathbf{G}}_{\mathrm{LT}}$ can also be defined as the inverse limit of the tower

$$\widetilde{\mathbf{G}}_{\mathrm{LT}}(A) = \varprojlim(\cdots \xrightarrow{\pi} \mathbf{G}_{\mathrm{LT}}(A) \xrightarrow{\pi} \mathbf{G}_{\mathrm{LT}}(A)$$

(since π^e is a unit multiple of p in the ring $\mathcal{O}_E$.

Example 4. Let K be an algebraically closed field containing E which is complete with respect to an absolute value $|\bullet|_K$ compatible with the valuation on $\mathcal{O}_E$. Then, as a set, we can identify $\mathbf{G}_{\mathrm{LT}}(\mathcal{O}_K)$ with the maximal ideal $\mathfrak{m}_K$ of the valuation ring $\mathcal{O}_K$. Under this identification, multiplication by π is given by $t \mapsto \pi t + t^q$. Since K is algebraically closed, this map is surjection, and every element of $\mathfrak{m}_K$ has exactly q

preimages. In other words, the map $\mathbf{G}_{\mathrm{LT}}(\mathcal{O}_K) \xrightarrow{\pi} \mathbf{G}_{\mathrm{LT}}(\mathcal{O}_K)$ is a surjection of $\mathcal{O}_E$-modules whose kernel has order q, and must therefore be isomorphic to the residue field $\mathcal{O}_E/(\pi) \simeq \mathbf{F}_q$ as a module over $\mathcal{O}_E$. It follows that the canonical map $\widetilde{\mathbf{G}}_{\mathrm{LT}}(\mathcal{O}_K) \to \mathbf{G}_{\mathrm{LT}}(\mathcal{O}_K)$ is surjective.

We can be a bit more precise: for each $n \geq 0$, the kernel of the map $\pi^n : \mathbf{G}_{\mathrm{LT}}(\mathcal{O}_K) \to \mathbf{G}_{\mathrm{LT}}(\mathcal{O}_K)$ has order q^n, but contains only q elements which are annihilated by π. It follows that this kernel must be isomorphic to the quotient $\mathcal{O}_E/(\pi^n)$ as a module over $\mathcal{O}_E$. Passing to the inverse limit, we deduce that the kernel of the surjection

$$\widetilde{\mathbf{G}}_{\mathrm{LT}}(\mathcal{O}_K) \twoheadrightarrow \mathbf{G}_{\mathrm{LT}}(\mathcal{O}_K)$$

is free of rank 1 as a $\mathcal{O}_E$-module.

Example 5. In the situation of Notation 1, suppose that $\pi \in \mathcal{O}_E$ vanishes in A (so that A has characteristic p, since π^e is a unit multiple of p). Then the polynomial $[\pi](t) = \pi t + t^q$ induces the Frobenius map

$$\mathbf{G}_{\mathrm{LT}}(A) \xrightarrow{x \mapsto x^q} \mathbf{G}_{\mathrm{LT}}(A).$$

Consequently, if A is a *perfect* $\mathcal{O}_E/(\pi)$-algebra, multiplication by π induces a bijection from $\mathbf{G}_{\mathrm{LT}}(A)$ to itself. It follows that multiplication by p also induces a bijection of $\mathbf{G}_{\mathrm{LT}}(A)$ with itself (again, because p is a unit multiple of π^e). In particular, the projection map $\widetilde{\mathbf{G}}_{\mathrm{LT}}(A) \to \mathbf{G}_{\mathrm{LT}}(A)$ is a bijection.

Example 6. In the situation of Notation 1, assume that the ideal I contains p (this will be the case in all the situations we care about). In this case, the canonical map

$$\widetilde{\mathbf{G}}_{\mathrm{LT}}(A) \to \widetilde{\mathbf{G}}_{\mathrm{LT}}(A/I)$$

is bijective. To prove this, it will suffice to show that each of the maps

$$\widetilde{\mathbf{G}}_{\mathrm{LT}}(A/I^{n+1}) \to \widetilde{\mathbf{G}}_{\mathrm{LT}}(A/I^n)$$

is an isomorphism. This follows from the observation that for $u, v \in I^n$, we have $F(u,v) \equiv u + v \pmod{I^{2n}}$, so that we have a short exact sequence of abelian groups

$$0 \to I^n/I^{n+1} \to \mathbf{G}_{\mathrm{LT}}(A/I^{n+1}) \to \mathbf{G}_{\mathrm{LT}}(A/I^n) \to 0$$

where the first group is annihilated by p; we therefore have a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \mathbf{G}_{\mathrm{LT}}(A/I^{n+1}) \arrow[r, "p"] \arrow[d] & \mathbf{G}_{\mathrm{LT}}(A/I^{n+1}) \arrow[r, "p"] \arrow[d] & \mathbf{G}_{\mathrm{LT}}(A/I^{n+1}) \arrow[d] \\
\cdots \arrow[r] & \mathbf{G}_{\mathrm{LT}}(A/I^n) \arrow[r, "p"] \arrow[ur, dashed] & \mathbf{G}_{\mathrm{LT}}(A/I^n) \arrow[r, "p"] \arrow[ur, dashed] & \mathbf{G}_{\mathrm{LT}}(A/I^n),
\end{tikzcd}$$

where the existence of the dotted arrows guarantees that the vertical maps induce an isomorphism after passing to the limit.

Remark 7. In the situation of Example 6, we do not need to divide out by the entire ideal I; if J is an ideal contained in I, then the map

$$\widetilde{\mathbf{G}}_{\mathrm{LT}}(A) \to \widetilde{\mathbf{G}}_{\mathrm{LT}}(A/J)$$

is an isomorphism (since both sides are isomorphic to $\widetilde{\mathbf{G}}_{\mathrm{LT}}(A/I)$).

Example 8. Consider the $\mathcal{O}_E$-algebra $\mathbf{A}_{\mathrm{inf}} \otimes_{\mathcal{O}_{E_0}} \mathcal{O}_E$. Note that the inclusion $E_0 \hookrightarrow E$ induces an isomorphism $\mathcal{O}_{E_0}/(p) \simeq \mathcal{O}_E/(\pi)$, and therefore also an isomorphism

$$\mathcal{O}_{C^\flat} \simeq \mathbf{A}_{\mathrm{inf}}/(p) \simeq (\mathbf{A}_{\mathrm{inf}} \otimes_{\mathcal{O}_{E_0}} \mathcal{O}_E)/(\pi).$$

We have a commutative diagram of abelian groups

$$\begin{tikzcd}
\widetilde{\mathbf{G}}_{\mathrm{LT}}(\mathbf{A}_{\mathrm{inf}} \otimes_{\mathcal{O}_{E_0}} \mathcal{O}_E) \arrow[r, "\sim"] \arrow[d] & \widetilde{\mathbf{G}}_{\mathrm{LT}}(\mathcal{O}_C^\flat) \arrow[d, "\sim"] \\
\mathbf{G}_{\mathrm{LT}}(\mathbf{A}_{\mathrm{inf}} \otimes_{\mathcal{O}_{E_0}} \mathcal{O}_E) \arrow[r] & \mathbf{G}_{\mathrm{LT}}(\mathcal{O}_C^\flat).
\end{tikzcd}$$

Here the upper horizontal map is an isomorphism by Remark 7 and the right vertical map is an isomorphism by Example 5. It follows that the reduction map

$$\mathbf{G}_{\mathrm{LT}}(\mathbf{A}_{\mathrm{inf}} \otimes_{\mathcal{O}_{E_0}} \mathcal{O}_E) \to \mathbf{G}_{\mathrm{LT}}(\mathcal{O}_C^\flat)$$

is a surjection: in fact, it has a canonical section, given by the composition

$$\mathbf{G}_{\mathrm{LT}}(\mathcal{O}_C^\flat) \xleftarrow{\sim} \widetilde{\mathbf{G}}_{\mathrm{LT}}(\mathcal{O}_C^\flat) \xleftarrow{\sim} \widetilde{\mathbf{G}}_{\mathrm{LT}}(\mathbf{A}_{\mathrm{inf}} \otimes_{\mathcal{O}_{E_0}} \mathcal{O}_E) \to \mathbf{G}_{\mathrm{LT}}(\mathbf{A}_{\mathrm{inf}} \otimes_{\mathcal{O}_{E_0}} \mathcal{O}_E).$$

In the case where $E = \mathbf{Q}_p$ and $\mathbf{G}_{\mathrm{LT}}$ is the formal multiplicative group, this is the Teichmüller section

$$(x \in 1 + \mathfrak{m}_C^\flat) \mapsto ([x] \in 1 + \mathfrak{m}_{\mathbf{A}_{\mathrm{inf}}}).$$

Example 9. Suppose we are given a point of Y_E°, corresponding to an untilt (K, ι) of $C^\flat$ equipped with an E_0-algebra map $E \to K$. We then have a canonical surjection $\mathbf{A}_{\mathrm{inf}} \to \mathcal{O}_K$, which induces a commutative diagram

$$\begin{tikzcd}
\widetilde{\mathbf{G}}_{\mathrm{LT}}(\mathbf{A}_{\mathrm{inf}} \otimes_{\mathcal{O}_{E_0}} \mathcal{O}_E) \arrow[r, "\sim"] \arrow[d] & \widetilde{\mathbf{G}}_{\mathrm{LT}}(\mathcal{O}_K) \arrow[d] \\
\mathbf{G}_{\mathrm{LT}}(\mathbf{A}_{\mathrm{inf}} \otimes_{\mathcal{O}_{E_0}} \mathcal{O}_E) \arrow[r] & \mathbf{G}_{\mathrm{LT}}(\mathcal{O}_K).
\end{tikzcd}$$

Here the top horizontal map is an isomorphism (by Remark 7 again) and the right vertical map is surjective (Example 4). We therefore obtain a surjection

$$\mathbf{G}_{\mathrm{LT}}(\mathcal{O}_C^\flat) \to \mathbf{G}_{\mathrm{LT}}(\mathcal{O}_K),$$

whose kernel is free of rank 1 as a $\mathcal{O}_E$-module.

In the special case where $E = \mathbf{Q}_p$ and $\mathbf{G}_{\mathrm{LT}}$ is the formal multiplicative group, this reduces to the map

$$1 + \mathfrak{m}_C^\flat \to 1 + \mathfrak{m}_K \qquad x \mapsto x^\sharp.$$

Construction 10. Let $\log_F$ denote the logarithm for the formal group law F, which we regard as a power series

$$\log_F(t) = t + \frac{c_2}{2}t^2 + \frac{c_3}{3}t^3 + \cdots \in E[[t]].$$

We observed in the previous lecture that the coefficients c_n belong to $\mathcal{O}_E$. Let x be any element belonging to the maximal ideal of the local ring $\mathbf{A}_{\mathrm{inf}} \otimes_{\mathcal{O}_{E_0}} \mathcal{O}_E$. We claim that the power series

$$\log_F(x) = x + \frac{c_2}{2}x^2 + \frac{c_3}{3}x^3 + \cdots$$

converges in the ring $B \otimes_{E_0} E = B + \pi B + \cdots + \pi^{e-1}B$. To prove this, we observe that each product $c_n x^n$ can be written uniquely as a sum

$$c_n x^n = a_{n,0} + a_{n,1}\pi + \cdots + a_{n,e-1}\pi^{e-1},$$

where each coefficient $a_{n,i}$ belongs to $\mathbf{A}_{\inf}$; we claim that the elements $\frac{1}{n} a_{n,i}$ converge to 0 (as $n \to \infty$) with respect to each of the Gauss norms $|\bullet|_\rho$ on $\mathbf{A}_{\inf}$. To prove this, write $x = x_0 + \pi y$, where x_0 belongs to the maximal ideal of $\mathbf{A}_{\inf}$. Then, for $n \geq e(m+1)$, the element $c_n x^n$ belongs to the ideal generated by the elements

$$x_0^{em}, p x_0^{e(m-1)}, \cdots, p^{m-1} x_0^e, p^m \in \mathbf{A}_{\inf}.$$

It follows that each coefficient of $c_n x^n$ (when written as a sum of powers of π) has Gauss norm

$$\leq \max(|x_0^{em}|_\rho, |p x_0^{e(m-1)}|_\rho, \cdots, |p^m|_\rho) = \max |x_0|_\rho^{em}, \rho^m.$$

These norms decay exponentially as $n \to \infty$, while the Gauss norms $|\frac{1}{n}|_\rho$ grow linearly in n.

Construction 10 supplies a canonical map $\mathcal{O}_E$-linear map

$$\mathbf{G}_{\mathrm{LT}}(\mathbf{A}_{\inf} \otimes_{\mathcal{O}_{E_0}} \mathcal{O}_E) \xrightarrow{\log_F} B \otimes_{E_0} E,$$

which is also equivariant with respect to the Frobenius endomorphism φ^d. Composing this map with the "Teichmüller section" of Example 8, we obtain a homomorphism of $\mathcal{O}_E$-modules

$$\mathbf{G}_{\mathrm{LT}}(\mathcal{O}_C^\flat) \to B \otimes_{E_0} E$$

which is again equivariant with respect to the Frobenius φ^d. Since $\mathcal{O}_C^\flat$ is an algebra over $\mathcal{O}_E/(\pi)$, it follows from the construction of the Lubin-Tate formal group that the Frobenius map φ^d coincides with multiplication by π on the $\mathcal{O}_E$-module $\mathbf{G}_{\mathrm{LT}}(\mathcal{O}_C^\flat)$. It follows that the preceding construction gives a map

$$\log_F : \mathbf{G}_{\mathrm{LT}}(\mathcal{O}_C^\flat) \to (B \otimes_{E_0} E)^{\varphi^d = \pi}.$$

This proves a weak form of the result promised in Lecture 22:

Theorem 11. *Let x be a closed point of X_E, corresponding to a subset $S \subseteq Y_E^\circ$ which is an orbit for the action of $\varphi^{d\mathbf{Z}}$. Then there exists a nonzero element $f \in (B \otimes_{E_0} E)^{\varphi^d = \pi}$ vanishing on S.*

Proof. Choose a point y of S, corresponding to an untilt (K, ι) of $C^\flat$ equipped with an E_0-algebra map $E \to K$. Then Example 9 implies that the natural map

$$\mathbf{G}_{\mathrm{LT}}(\mathcal{O}_C^\flat) \twoheadrightarrow \mathbf{G}_{\mathrm{LT}}(\mathcal{O}_K)$$

is a surjection, whose kernel is a free $\mathcal{O}_E$-module of rank 1. Let u be a generator of the kernel. We then take $f = \log_F(u) \in (B \otimes_{E_0} E)^{\varphi^d = \pi}$. By construction, f vanishes at the point y (and therefore on the entire orbit S). $\square$

To completely fulfill our promise from Lecture 22, we need to show that the function f of Theorem 11 vanishes simply at each point of the orbit S, and does not vanish on any other point of Y_E°. This is actually automatic: we will prove this in the next lecture.

Lecture 25-Functions on Y_E°

December 2, 2018

Throughout this lecture, we fix an algebraically closed perfectoid field $C^\flat$ of characteristic p. Let E be a finite extension of $\mathbf{Q}_p$, so that $\mathbf{Q}_p \subseteq E_0 \subseteq E$ where $\mathbf{Q}_p \hookrightarrow E_0$ is an unramified extension of degree d and $E_0 \hookrightarrow E$ is a totally ramified extension of degree e. We let $\mathbf{F}_q$ denote the residue field of E (so $q = p^d$) and π a uniformizer of the ring of integers $\mathcal{O}_E$. Fix an embedding $\mathbf{F}_q \hookrightarrow C^\flat$, which determines a map $\mathcal{O}_{E_0} \to \mathbf{A}_{\mathrm{inf}}$, hence $E_0 \to B$. Our first goal is to prove the following:

Proposition 1. *The ring $B \otimes_{E_0} E$ is an integral domain.*

Proof. We already know that B is an integral domain; let $K(B)$ denote its fraction field. We first note that B is integrally closed in $K(B)$. Given elements $f, g \in B$ such that $g \neq 0$ and $\frac{f}{g}$ is integral over B, it follows that for each point $y \in Y$ the image of $\frac{f}{g}$ in the field $B_{\mathrm{dR}}(y)$ is integral over $B^+_{\mathrm{dR}}(y)$. We therefore have $\frac{f}{g} \in B^+_{\mathrm{dR}}(y)$: that is, $\mathrm{ord}_y(f) \geq \mathrm{ord}_y(g)$. It follows that f is divisible by g in the ring B, so that $\frac{f}{g}$ belongs to B¿

Then $B \otimes_{E_0} E$ embeds in $K(B) \otimes_{E_0} E$. It will therefore suffice to show that the tensor product $K(B) \otimes_{E_0} E$ is a field. Enlarging E if necessary, we may assume that it is a Galois extension of E_0. Note that $K(B) \otimes_{E_0} E$ is an étale $K(B)$-algebra, so it is automatically a finite product of fields $K_1 \times \cdots \times K_m$. Then the subring $E \times \cdots \times E \subseteq K_1 \times \cdots \times K_m = K(B) \otimes_{E_0} E$ is invariant under the action of $\mathrm{Gal}(E/E_0)$, and can therefore be written as $L \otimes_{E_0} E$ for some E_0-subalgebra $L \subseteq K(B)$. Then L is an integral domain, so it is a field extension of E_0 (of degree m). We will complete the proof by showing that $L \otimes_{E_0} E$ is a field: that is, L and E are linearly disjoint over E_0. Since E is a totally ramified extension of E_0, it will suffice to show that L is an unramified extension of E_0. Let e' be the ramification degree of L over E_0 and let $f \in \mathcal{O}_L$ be a uniformizer. Then f is integral over $\mathcal{O}_{E_0} \subseteq B$, and therefore belongs to B (since B is integrally closed in $K(B)$). Note that, for each point $y \in Y$, we can identify $f(y)$ with the image of f under a continuous map of valued fields $L \to K_y$ (where K_y is the untilt of $C^\flat$ corresponding to y), and therefore have $|f(y)|_{K_y} = |p|_{K_y}^{1/e'}$. It follows that $|f|_\rho = \rho^{1/e'}$ for each $\rho \in (0,1)$: that is, we have $v_s(f) = \frac{s}{e'}$ for $s \in \mathbf{R}_{>0}$. Since the function $v_\bullet(f)$ is piecewise linear with *integer* slopes, we must have $e' = 1$ as desired. $\square$

Corollary 2. *Let f be a nonzero element of $B \otimes_{E_0} E$ and let $N_{E/E_0}(f)$ denote its norm (along the finite flat map $\mathrm{Spec}(B) \to \mathrm{Spec}(B \otimes_{E_0} E)$). Then $N_{E/E_0}(f) \neq 0$.*

Corollary 3. *Let f and g be elements of $B \otimes_{E_0} E$, where $g \neq 0$. The following conditions are equivalent:*

(1) *The element f is divisible by g.*

(2) *For each point $\widetilde{y} \in Y_E^\circ$, we have $\mathrm{ord}_{\widetilde{y}}(f) \geq \mathrm{ord}_{\widetilde{y}}(g)$.*

Proof. The implication (1) $\Rightarrow$ (2) is trivial. For the converse, assume that (2) is satisfied; we wish to show that g divides f. Write $N_{E/E_0}(g) = g \cdot h$. Replacing f by $f \cdot h$ and g by $g \cdot h = N_{E/E_0}(g)$, we can reduce to the case where g belongs to B. We can write f uniquely as a sum $f_0 + f_1\pi + \cdots + f_{e-1}\pi^{e-1}$, where each f_i belongs to B. It will therefore suffice to show that each f_i is divisible by g: that is, that we have $\mathrm{ord}_y(f_i) \geq \mathrm{ord}_y(g)$ for each point $y \in Y$. Let ξ be a uniformizer of the valuation ring $B^+_{\mathrm{dR}}(y)$ and set $n = \mathrm{ord}_y(g)$. Our hypothesis that $\mathrm{ord}_{\widetilde{y}}(f) \geq \mathrm{ord}_{\widetilde{y}}(g) = n$ for each point $\widetilde{y} \in Y^\circ_E$ lying over Y guarantees that the image of f vanishes in

$$\prod_{\widetilde{y} \to y} B^+_{\mathrm{dR}}(y)/(\xi^n) = (B^+_{\mathrm{dR}}(y)/(\xi^n)) \otimes_{E_0} E \simeq B^+_{\mathrm{dR}}(y)/(\xi^n) + \pi B^+_{\mathrm{dR}}(y)/(\xi^n) + \cdots + \pi^{e-1} B^+_{\mathrm{dR}}(y)/(\xi^n),$$

so that we have $\mathrm{ord}_y(f_i) \geq n$ for each i, as desired. □

Remark 4. Let f be a nonzero element of $B \otimes_{E_0} E$. Then the vanishing locus of f (as a subset of Y°_E) has at most countably many points. To see this, we observe that $N_{E/E_0}(f)$ is a nonzero element of B, and therefore vanishes on at most finitely many points of each annulus $Y_{[a,b]}$, and therefore at most countably many points of Y.

Corollary 5. *Let f be a nonzero element of $(B \otimes_{E_0} E)^{\varphi^d = \pi}$. Then the vanishing locus of f (as a subset of Y°_E) consists of a single orbit of $\varphi^{d\mathbf{Z}}$. Moreover, f has a simple zero at each point of this vanishing locus.*

Proof. Set $\pi' = N_{E/E_0}(\pi) \in \mathcal{O}_{E_0}$ and $f' = N_{E/E_0}(f)$, so that f belongs to $B^{\varphi^d = \pi'}$. Then the vanishing locus of f' is the image in Y of the vanishing locus of f, and the order of vanishing of f' at each point $y \in Y$ is the sum of the order of vanishing of f at each point of Y°_E lying over y. It will therefore suffice to show that f' vanishes on a single $\varphi^{d\mathbf{Z}}$-orbit (and with multiplicity 1 at each point of this orbit).

For each $\rho \in (0,1)$, we have

$$\rho^{p^d} \cdot |f'|_{\rho^{p^d}} = |\pi' f'|_{\rho^{p^d}} = |f'^{\varphi^d}|_{\rho^{p^d}} = |f'|_\rho^{p^d}$$

or, setting $s = -\log(\rho)$,

$$p^d s + v_{p^d s}(f') = p^d v_s(f).$$

Differentiating with respect to s and dividing by p^d, this gives

$$1 + \partial_- v_{p^d s}(f') = \partial_- v_s(f')$$

which implies that f' has exactly one zero (counted with multiplicity) on the half-open annulus $Y_{(\rho^p, \rho]} = \{y \in Y : \rho^p < d(y,0) \leq \rho\}$, Since the vanishing locus of f' is a union of $\varphi^{d\mathbf{Z}}$ orbits, it must consist of a single such orbit (with multiplicity 1). □

Let $\mathbf{G}_{\mathrm{LT}}$ denote the Lubin-Tate group of the previous lecture, and let $\sigma : \mathbf{G}_{\mathrm{LT}}(\mathcal{O}^\flat_C) \to \mathbf{G}(\mathbf{A}_{\mathrm{inf}} \otimes_{\mathcal{O}_{E_0}} \mathcal{O}_E)$ be the section of the projection map constructed in the previous lecture (which reduces to the usual Teichmüller section in the case $E = \mathbf{Q}_p$ and $\mathbf{G}_{\mathrm{LT}} = \widehat{\mathbf{G}}_m$).

Proposition 6. *The map*

$$\log_F(\sigma(\bullet)) : \mathbf{G}_{\mathrm{LT}}(\mathcal{O}^\flat_C) \to (B \otimes_{E_0} E)^{\varphi^d = \pi}$$

is an isomorphism.

Proof. We first prove injectivity. Let $\widetilde{y}$ be any point of Y°_E having image $y \in Y$ and let K_y denote the corresponding untilt of $C^\flat$, so that $\widetilde{y}$ equips $\mathcal{O}_{K_y}$ with the structure of a $\mathcal{O}_E$-algebra. Note that, on a sufficiently small ball around the origin in K_y, the construction $t \mapsto \log_F(t)$ is bijective (this happens already for the open ball of radius $|p|^{1/(p-1)}_{K_y}$). For any $t \in \mathbf{G}_{\mathrm{LT}}(K_y)$, the image $[\pi^m](t)$ will belong to this ball for $m \gg 0$. It follows that

$$(\log_F(t) = 0) \Leftrightarrow (\log_F([\pi^m](t)) = 0) \Leftrightarrow ([\pi^m](t) = 0 \text{ for } m \gg 0).$$

Let u be any nonzero element of $\mathbf{G}_{\mathrm{LT}}(\mathcal{O}_C^\flat)$. The above reasoning shows that $\log_F(\sigma(u))$ vanishes at a point $\widetilde{y} \in Y_E^\circ$ if and only if $[\pi^m]\sigma(u) = \sigma(\varphi^{md}(u))$ vanishes at $\widetilde{y}$ for some $m \gg 0$. Since each $\sigma(\varphi^{md}(u))$ is a nonzero element of

$$\mathbf{G}_{\mathrm{LT}}(\mathbf{A}_{\mathrm{inf}} \otimes_{\mathcal{O}_{E_0}} \mathcal{O}_E) \simeq \mathfrak{m}_{\mathbf{A}_{\mathrm{inf}} \otimes_{\mathcal{O}_{E_0}} \mathcal{O}_E},$$

there are at most countably many points $\widetilde{y} \in Y_E^\circ$ which satisfy this condition (Remark 4). It follows that $\log_F(\sigma(u))$ cannot be identically zero.

We now prove surjectivity. Let f be a nonzero element of $(B \otimes_{E_0} E)^{\varphi^d = \pi}$. Then the vanishing locus of f consists of a single $\varphi^{d\mathbf{Z}}$ orbit on Y_E° (Corollary 5). In Lecture 24, we constructed a nonzero element $u \in \mathbf{G}_{\mathrm{LT}}(\mathcal{O}_C^\flat)$ such that $\log_F(\sigma(u))$ vanishes on the same locus. Since $\log_F(\sigma(u))$ is also nonzero (by the first part of the proof), it follows that $\lambda \log_F(\sigma(u)) = f$, where λ is a unit in the ring $B \otimes_{E_0} E$. It then follows that λ is a nonzero element of

$$(B \otimes_{E_0} E)^{\varphi^d = 1} \simeq (B \otimes_{\mathbf{Q}_p} E)^{\varphi = 1} \simeq B^{\varphi = 1} \otimes_{\mathbf{Q}_p} E \simeq E.$$

Replacing u by λu (using the structure of $\mathbf{G}_{\mathrm{LT}}(\mathcal{O}_C^\flat)$ as a vector space over E), we can arrange that $\log_F(\sigma(u)) = f$. $\square$

It follows from the preceding discussion that we have canonical bijections

$$\begin{aligned} \{\text{Closed points of } X_E\} &\simeq \{\varphi^{d\mathbf{Z}}\text{-orbits on } Y_E^\circ\} \\ &\simeq ((B \otimes_{E_0} E)^{\varphi^d = \pi} - \{0\})/E^\times \\ &\simeq (\mathbf{G}_{\mathrm{LT}}(\mathcal{O}_C^\flat) - \{0\})/E^\times. \end{aligned}$$

Lecture 26-Isocrystals

December 6, 2018

Throughout this lecture, we fix an algebraically closed perfectoid field $C^\flat$ of characteristic p and let

$$X = \mathrm{Proj}(\bigoplus_{n \geq 0} B^{\varphi = p^n})$$

be the Fargues-Fontaine curve. In Lecture 21, we explained how to construct a semistable vector bundle $\mathcal{E}$ on X of any rank $n > 0$ and any degree m (hence of any rational slope $\frac{m}{n}$). Namely, one can choose a degree n extension $E \supset \mathbf{Q}_p$ and a line bundle $\mathcal{L}$ of degree m on the curve X_E; we can then take $\mathcal{E} = \rho_* \mathcal{L}$, where $\rho : X_E \to X$ is the projection map. Over the last few lectures, we proved that this construction is independent of the choice of $\mathcal{L}$ (since a line bundle on X_E is determined up to isomorphism by its degree). Our next goal is to show that it is also independent of E.

As in the previous lecture, let us write $\mathbf{Q}_p \subseteq E_0 \subseteq E$ where E_0 is an unramified extension of $\mathbf{Q}_p$ of degree d and E is a totally ramified extension of E_0 having degree e (so that $n = d \cdot e$). Then $E_0 = W(\mathbf{F}_{p^d})[\frac{1}{p}]$, and we assume that we have fixed an embedding $\mathbf{F}_{p^d} \hookrightarrow C^\flat$. Let $\pi \in \mathcal{O}_E$ be a uniformizer. Let $U \subseteq X$ be an affine open subset given by the complement of the vanishing locus of some homogeneous element $t \in \bigoplus_{n>0} B^{\varphi = p^n}$. Then the vector bundle $\mathcal{E}$ constructed above can be given by the formula

$$\mathcal{E}(U) = (B[\frac{1}{t}] \otimes_{E_0} E)^{\varphi^d = \pi^m}.$$

We now describe a variant of this construction.

Definition 1. Let k be a perfect field of characteristic p, let $W(k)$ denote the ring of Witt vectors of k, and set $K = W(k)[\frac{1}{p}]$ be its fraction field. Then the Frobenius automorphism of k induces an automorphism of K, which we will denote by φ_K.

An *isocrystal* (over k) is a finite-dimensional vector space V over K equipped with a Frobenius-semilinear automorphism: that is, an isomorphism of abelian groups $\varphi_V : V \to V$ satisfying $\varphi_V(\lambda v) = \varphi_K(\lambda)\varphi_V(v)$ for $\lambda \in K$ and $v \in V$.

Remark 2. Our terminology is not standard; many authors use the term *F-isocrystal* or *Frobenius isocrystal* to refer to the notion of isocrystal that we just defined.

Example 3. Let X be a smooth projective algebraic variety over a perfect field k. Then the (rationalized) *crystalline cohomology groups* $\mathrm{H}^m_{\mathrm{crys}}(X; W(k))[\frac{1}{p}]$ have a Frobenius semilinear automorphism induced by the absolute Frobenius map $\varphi : X \to X$, and can therefore be regarded as isocrystals over k.

Example 4. Let k be a perfect field and let $T(u, v) \in k[[u, v]]$ be a formal group law over k which is not isomorphic to the additive group. Then the associated formal group $\mathbf{G}_T$ is determined (up to isomorphism) by its *Dieudonné module* $\mathbf{D}(\mathbf{G}_F)$: this is a free $W(k)$-module of finite rank equipped with a Frobenius semilinear endomorphism F and an inverse-Frobenius semilinear endomorphism V satisfying $FV = VF = p$. The rationalized Dieudonné module $\mathbf{D}(\mathbf{G}_F)[\frac{1}{p}]$ is then an isocrystal over k (the Frobenius endomorphism F has inverse given by $\frac{V}{p}$).

Example 5. Let m and n be relatively prime integers, with $n > 0$, and let $V_{\frac{m}{n}} = K^n$. Then we can equip V with the structure of an isocrystal by defining

$$\varphi_{V_{\frac{m}{n}}}(x_1, x_2, \ldots, x_n) = (\varphi_K(x_2), \varphi_K(x_3), \cdots, \varphi_K(x_n), p^m \varphi_K(x_1)).$$

This isocrystal is characterized by a universal property: giving a map from V into another isocrystal W (which is K-linear and Frobenius equivariant) is equivalent to giving an element of the eigenspace $W^{\varphi^n = p^m}$.

Theorem 6 (Dieudonné-Manin Classification)**.** *Let k be an algebraically closed field of characteristic p. Then:*

- *The category of isocrystals over k is semisimple. That is, every isocrystal over k can be written as a direct sum of simple objects.*
- *The simple isocrystals over k are exactly those of the form $V_{\frac{m}{n}}$, where m and n are relatively prime integer with $n > 0$.*

Construction 7. Let $k = \overline{\mathbf{F}}_p$ be the algebraic closure of $\mathbf{F}_p$ in the field $C^\flat$. Then the inclusion $\overline{\mathbf{F}}_p \hookrightarrow C^\flat$ extends to a map $W(\overline{\mathbf{F}}_p) \to \mathbf{A}_{\mathrm{inf}}$, hence to a map

$$K = W(\overline{\mathbf{F}}_p)[\frac{1}{p}] \to B.$$

Let V be an isocrystal over K. We let $\mathcal{E}_V$ denote the quasi-coherent sheaf on $X = \mathrm{Proj}(\bigoplus_{n \geq 0} B^{\varphi = p^n})$ associated to the graded module

$$\bigoplus_{n \geq 0} \mathrm{Hom}_K(V, B)^{\varphi = p^n}.$$

In other words, if U is an affine open subset of X given by the complement of the vanishing locus of some homogeneous element $t \in \bigoplus_{n>0} B^{\varphi = p^n}$, then we have

$$\mathcal{E}_V(U) = \{ \ \phi\text{-equivariant } K\text{-linear maps } V \to B[\frac{1}{t}]\}.$$

In the special case where $V = V_{\frac{m}{n}}$ is the isocrystal of Example 5, we will denote the quasi-coherent sheaf $\mathcal{E}_V$ by $\mathcal{O}(\frac{m}{n})$.

Example 8. Fix relatively prime integers m and n with $n > 0$. Let $U \subset X$ be the affine open subset given by the vanishing locus of some homogeneous element $t \in \bigoplus_{n>0} B^{\varphi = p^n}$. We then have

$$\begin{aligned} \mathcal{O}(\frac{m}{n})(U) &\simeq (B[\frac{1}{t}])^{\varphi^n = p^m} \\ &= (\rho_* \mathcal{O}_{X_E}(m))(U) \end{aligned}$$

where E is the *unramified* extension of $\mathbf{Q}_p$ of degree n and $\rho : X_E \to X$ is the projection map. It follows that $\mathcal{O}(\frac{m}{n})$ is a semistable vector bundle of degree m and rank n.

Remark 9. It follows from Example 8 and the Dieudonné-Manin classification that, for every isocrystal V over $\overline{\mathbf{F}}_p$, the quasi-coherent sheaf $\mathcal{E}_V$ of Construction 7 is a vector bundle on X (whose rank is equal to the dimension of V as a vector space over K).

Example 10. Let E be a totally ramified extension of $\mathbf{Q}_p$ with uniformizer $\pi \in \mathcal{O}_E$, let $E^\vee$ denote the dual of E as a $\mathbf{Q}_p$-vector space (which we can identify with E via the trace pairing), and regard $V = E^\vee \otimes_{\mathbf{Q}_p} K$ as an isocrystal via the formula

$$\varphi_V(x \otimes y) = \pi^m x \otimes \varphi_K(y).$$

Unwinding the definitions, we have

$$\mathcal{E}_V(U) = \mathrm{Hom}_K(V, B[\frac{1}{t}])^{\varphi=1} = (E \otimes_{\mathbf{Q}_p} B[\frac{1}{t}])^{\varphi=\pi^m} = \rho_* \mathcal{O}_{X_E}(m)$$

where $\rho : X_E \to X$ is the projection map.

Exercise 11. Let E be a finite extension of $\mathbf{Q}_p$ (not necessarily unramified or totally ramified), let $\rho : X_E \to X$ be the projection map, and consider the vector bundle $\rho_* \mathcal{O}_{X_E}(m)$ (which is semistable of rank n and degree m).

- Show that $\rho_* \mathcal{O}_{X_E}(m)$ can be written as $\mathcal{E}_V$, where V is a suitable isocrystal over $\overline{\mathbf{F}}_p$ (hint: take $V = E^\vee \otimes_{\mathbf{Q}_p} K$, endowed with a suitable Frobenius action which depends on m).
- If m is relatively prime to n, show that V is isomorphic to $V_{\frac{m}{n}}$ (hint: use the Dieudonné-Manin classification).
- Conclude that if m and n are relatively prime, then $\rho_* \mathcal{O}_{X_E}(m)$ is isomorphic to the vector bundle $\mathcal{O}(\frac{m}{n})$ of Construction 7.

We can now state the classification theorem for semistable vector bundles on X in a more precise form.

Definition 12. Let μ be a rational number, which we write as $\mu = \frac{m}{n}$ where m and n are relatively prime and $n > 0$. We say that an isocrystal V over $\overline{\mathbf{F}}_p$ is *isoclinic of slope* μ if it is isomorphic to a direct sum of copies of the isocrystal $V_{\frac{m}{n}}$ of Example 5.

Example 13. An isocrystal over $\overline{\mathbf{F}}_p$ is isoclinic of slope 0 if and only if it is isomorphic to a sum of copies of K (with the usual Frobenius action). In this case, the vector bundle $\mathcal{E}_V$ is a sum of copies of $\mathcal{O}_X$: that is, it is a trivial vector bundle on X.

Remark 14. By the Dieudonné-Manin classification, every isocrystal V over $\overline{\mathbf{F}}_p$ splits uniquely as a direct sum of isoclinic isocrystals (of different slopes).

Theorem 15. (1) *For every vector bundle $\mathcal{E}$ on X, the Harder-Narasimhan filtration of $\mathcal{E}$ splits: that is, $\mathcal{E}$ can be written (non-uniquely) as a sum of semistable vector bundles.*

(2) *For every rational number μ, the construction*

$$V \mapsto \mathcal{E}_V$$

induces an equivalence of categories

$$\{\textit{Isoclinic isocrystals of slope } \mu\}^{\mathrm{op}} \to \{\textit{Semistable vector bundles on } X \textit{ of slope } \mu\}.$$

Corollary 16. *Every vector bundle $\mathcal{E}$ on X can be obtained by applying Construction 7 to some isocrystal V over $\overline{\mathbf{F}}_p$.*

Warning 17. The category of vector bundles on X is not equivalent to the category of isocrystals over $\overline{\mathbf{F}_p}$. The construction $V \mapsto \mathcal{E}_V$ is fully faithful when restricted to isoclinic isocrystals of some fixed slope μ, but is not fully faithful in general. For example, let (K, φ_K) denote the field K regarded as an isocrystal via its usual Frobenius automorphism, and let $(K, p\varphi_K)$ denote the field K regarded as an isocrystal via the map $x \mapsto \frac{\varphi_K(x)}{p}$. Then

$$\mathcal{E}_{(K,\varphi_K)} \simeq \mathcal{O}_X \qquad \mathcal{E}_{(K,p\varphi_K)} \simeq \mathcal{O}_X(1).$$

There are no maps from $(K, p\varphi_K)$ to (K, φ_K) in the category of isocrystals, but there are plenty of maps from $\mathcal{O}_X$ to $\mathcal{O}_X(1)$ in the category of vector bundles on X.

In the next lecture, we will use the following consequence of Theorem 15.

Corollary 18. *Let $\mathcal{E}$ be a vector bundle on X which is semistable of slope 0. Then $\mathcal{E}$ is trivial (that is, it is a sum of copies of $\mathcal{O}_X$).*

Lecture 27-Some Applications

December 7, 2018

In this final lecture, we describe some consequences of the classification of vector bundles on the Fargues-Fontaine curve. Recall that we have seen that the Fargues-Fontaine curve X behaves in many respects like an algebraic curve of genus 0. We now give one more heuristic piece of evidence for this.

Theorem 1. *The Fargues-Fontaine curve X is geometrically simply connected: that is, the projection map $X \to \mathrm{Spec}(\mathbf{Q}_p)$ induces an isomorphism of étale fundamental groups $\pi_1(X) \to \pi_1(\mathrm{Spec}(\mathbf{Q}_p)) = \mathrm{Gal}(\mathbf{Q}_p)$. Equivalently, pullback along the projection map*

$$X \to \mathrm{Spec}(\mathbf{Q}_p)$$

induces an equivalence of categories

$$\{ \text{ Étale covers of } \mathrm{Spec}(\mathbf{Q}_p)\} \to \{ \text{ Étale covers of } X\}.$$

We will need the following:

Lemma 2. *Let $\mathcal{E}$ and $\mathcal{E}'$ be vector bundles on X which are semistable of slopes μ and μ', respectively. Then the tensor product $\mathcal{E} \otimes \mathcal{E}'$ is semistable (of slope $\mu + \mu'$).*

Proof. By the classification, we may assume without loss of generality that $\mathcal{E} = \rho_* \mathcal{O}_{X_E}(d)$, where $\rho : X_E \to X$ is the projection map for some finite extension E of $\mathbf{Q}_p$. Then $\mathcal{E} \otimes \mathcal{E}' = (\rho_* \mathcal{O}_{X_E}(d))\, \mathcal{E}' = \rho_*(\mathcal{O}_{X_E}(d) \otimes \rho^* \mathcal{E}')$. Since the functors ρ_* and ρ^* preserve semistability, we are reduced to proving that the tensor product functor $\mathcal{O}_{X_E}(d) \otimes \bullet$ preserves semistability, which is immediate from the definitions. $\square$

Proof of Theorem 1. Let $\rho : \widetilde{X} \to X$ be a finite étale cover; we wish to show that $\widetilde{X}$ can be written uniquely as $X \times_{\mathrm{Spec}(\mathbf{Q}_p)} \mathrm{Spec}(E)$, where E is an étale $\mathbf{Q}_p$-algebra (that is, a product of finite extensions of $\mathbf{Q}_p$). Set $\mathcal{A} = \rho_* \mathcal{O}_{\widetilde{X}}$ and take $E = \mathrm{H}^0(X, \mathcal{A})$ (so that $E = \mathrm{H}^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}})$ is an algebra over $\mathbf{Q}_p$). To complete the proof, it will suffice to show the canonical map $E \otimes_{\mathbf{Q}_p} \mathcal{O}_X \to \mathcal{A}$ is an isomorphism (this forces $\widetilde{X} \simeq X \times_{\mathrm{Spec}(\mathbf{Q}_0)} \mathrm{Spec}(E)$, which in turn forces E to be an étale algebra over $\mathbf{Q}_p$). Equivalently, we wish to show that the vector bundle $\mathcal{A}$ is trivial.

By virtue of the classification, it will suffice to show that $\mathcal{A}$ is semistable of slope 0. We first observe that, since ρ is a finite étale morphism, the trace pairing $\mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \xrightarrow{\mathrm{tr}} \mathcal{O}_X$ is nondegenerate, so the vector bundle $\mathcal{A}$ is isomorphic to its dual $\mathcal{A}^\vee$. Since $\deg(\mathcal{A}) = -\deg(\mathcal{A}^\vee)$, it follows that $\deg(\mathcal{A}) = 0$ and therefore that $\mathcal{A}$ has slope zero. Assume, for a contradiction, that $\mathcal{A}$ is not semistable. Let $\mathcal{A}' \subseteq \mathcal{A}$ be the first step of the Harder-Narasimhan filtration of $\mathcal{A}$. Then $\mathcal{A}'$ is a semistable vector bundle of slope $\mu > 0$. Moreover, for every semistable vector bundle $\mathcal{E}$ of slope $> \mu$, every map $\mathcal{E} \to \mathcal{A}$ is zero. By Lemma 2, the tensor product $\mathcal{A}' \otimes \mathcal{A}'$ is semistable of slope $\mu + \mu > \mu$. It follows that the multiplication map

$$\mathcal{A}' \otimes \mathcal{A}' \hookrightarrow \mathcal{A} \otimes \mathcal{A} \xrightarrow{m} \mathcal{A}$$

must be the zero map. Let $U \subseteq X$ be an affine open subset over which the vector bundle $\mathcal{A}'$ has a nonzero section s. Then s can be viewed as a nonzero regular function on the scheme $\widetilde{X} \times_X U$ satisfying $s^2 = 0$. This is impossible, since the scheme $\widetilde{X} \times_X U$ is reduced (it is even a disjoint union of Dedekind schemes). □

Corollary 3. (1) *Pullback along the projection map $u : X \to \mathrm{Spec}(\mathbf{Q}_p)$ induces an equivalence of categories*

$$\{\textit{Finite abelian groups with a continuous action of } \mathrm{Gal}(\mathbf{Q}_p)\} \to \{\textit{Étale local systems on } X\}$$

(2) *If M is a finite abelian group with a continuous action of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$, then the induced map*

$$\mathrm{H}^*(\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p); M) \to \mathrm{H}^*_{\mathrm{et}}(X, u^*M)$$

is an isomorphism for $* = 0, 1$.

Proof. Assertion (1) is immediate from Theorem 1. To prove (2), it will suffice to establish the following more general assertion:

(2′) Let E be a finite extension of $\mathbf{Q}_p$, let $u : X_E \to \mathrm{Spec}(E)$ be the projection map, and let M be a finite abelian group with a continuous action of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/E)$, then the induced map

$$\mathrm{H}^*(\mathrm{Gal}(\overline{\mathbf{Q}}_p/E); M) \to \mathrm{H}^*_{\mathrm{et}}(X_E, u^*M)$$

is an isomorphism for $* = 0, 1$.

By a descent argument, to prove (2′) for a Galois module M, we are free to enlarge the field E. We may therefore assume without loss of generality that the Galois group $\mathrm{Gal}(\overline{\mathbf{Q}}_p/E)$ acts trivially on M. Then both cohomology groups are isomorphic to M when $* = 0$, and when $* = 1$ we are reduced to proving that u induces an isomorphism

$$\mathrm{Hom}(\pi_1(E), M) \to \mathrm{Hom}(\pi_1(X), M)$$

which follows from Theorem 1. □

Let C be a smooth proper algebraic curve defined over an algebraically closed field F of characteristic zero. Then the étale cohomology of C satisfies Poincaré duality. More precisely, for every positive integer n, there is a *trace map*

$$e_C : \mathrm{H}^2_{\mathrm{et}}(C; \mu_n) \xrightarrow{\sim} \mathbf{Z}/n\mathbf{Z}$$

with the following property: for every constructible locally constant sheaf of $(\mathbf{Z}/n\mathbf{Z})$-modules $\mathcal{F}$ on C having dual $\mathcal{G} = \mathrm{Hom}(\mathcal{F}, \mu_n)$, the pairing

$$\mathrm{H}^*_{\mathrm{et}}(C; \mathcal{F}) \times \mathrm{H}^{2-*}_{\mathrm{et}}(C; \mathcal{G}) \to \mathrm{H}^2_{\mathrm{et}}(C; \mu_n) \xrightarrow{e_C} \mathbf{Z}/n\mathbf{Z}$$

is perfect.

Tate proved that the absolute Galois group $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ is a (profinite) *Poincare duality* group of dimension 2. More precisely, there are isomorphisms $e_{\mathbf{Q}_p} : \mathrm{H}^2_{\mathrm{et}}(\mathrm{Spec}\,\mathbf{Q}_p; \mu_n) \xrightarrow{\sim} \mathbf{Z}/n\mathbf{Z}$ having the same property. In this sense, the scheme $\mathrm{Spec}\,\mathbf{Q}_p$ behaves like a smooth proper curve over an algebraically closed field. However, there are other respects in which it behaves differently. Note that on any $\mathbf{Z}[\frac{1}{n}]$-scheme Z, we have an exact sequence of étale sheaves

$$0 \to \mu_n \to \mathbf{G}_m \xrightarrow{n} \mathbf{G}_m \to 0.$$

This gives rise to a long exact sequence

$$\mathrm{Pic}(Z) \xrightarrow{n} \mathrm{Pic}(Z) \to \mathrm{H}^2_{\mathrm{et}}(Z; \mu_n) \to \mathrm{Br}(Z) \xrightarrow{n} \mathrm{Br}(Z)$$

(here $\mathrm{Br}(Z)$ denotes the *cohomological Brauer group* $\mathrm{H}^2_{\mathrm{et}}(Z; \mathbf{G}_m)$).

The Fargues-Fontaine curve provides a "geometric" way of thinking about Tate duality. Note that the projection map $X \to \mathrm{Spec}(\mathbf{Q}_p)$ induces a map of long exact sequences

$$\begin{tikzcd}
\mathrm{Pic}(\mathbf{Q}_p) \arrow[r, "n"] \arrow[d] & \mathrm{Pic}(\mathbf{Q}_p) \arrow[r] \arrow[d] & \mathrm{H}^2_{\mathrm{et}}(\mathrm{Spec}\,\mathbf{Q}_p; \mu_n) \arrow[r] \arrow[d] & \mathrm{Br}(\mathbf{Q}_p) \arrow[r, "n"] \arrow[d] & \mathrm{Br}(\mathbf{Q}_p) \arrow[d] \\
\mathrm{Pic}(X) \arrow[r, "n"] & \mathrm{Pic}(X) \arrow[r] & \mathrm{H}^2_{\mathrm{et}}(X; \mu_n) \arrow[r] & \mathrm{Br}(X) \arrow[r, "n"] & \mathrm{Br}(X).
\end{tikzcd}$$

One can show that the cohomology groups $\mathrm{H}^2_{\mathrm{et}}(\mathrm{Spec}(\mathbf{Q}_p); \mu_n)$ and $\mathrm{H}^2_{\mathrm{et}}(X; \mu_n)$ are both (canonically) isomorphic to $\mathbf{Z}/n\mathbf{Z}$, and that the middle vertical map is an isomorphism (combined with a descent argument, this implies that the comparison map of Corollary 3 is also an isomorphism on cohomology in degree 2). However, the "origin" of this $\mathbf{Z}/n\mathbf{Z}$ can be understood differently for $\mathrm{Spec}(\mathbf{Q}_p)$ and for X. The top line induces a short exact sequence

$$0 \to \mathrm{Pic}(\mathbf{Q}_p)/n\,\mathrm{Pic}(\mathbf{Q}_p) \to \mathrm{H}^2_{\mathrm{et}}(\mathrm{Spec}\,\mathbf{Q}_p; \mu_n) \to \mathrm{Br}(\mathbf{Q}_p)[n] \to 0$$

where the first term vanishes (since $\mathbf{Q}_p$ is a field) and the third term is isomorphic to $\mathbf{Z}/n\mathbf{Z}$ (since the Brauer group $\mathrm{Br}(\mathbf{Q}_p)$ is isomorphic to $\mathbf{Q}/\mathbf{Z}$ by the theory of the Hasse invariant). On the other hand, the lower line gives a short exact sequence

$$0 \to \mathrm{Pic}(X)/n\,\mathrm{Pic}(X) \to \mathrm{H}^2_{\mathrm{et}}(X; \mu_n) \to \mathrm{Br}(X)[n] \to 0$$

where the first term is isomorphic to $\mathbf{Z}/n\mathbf{Z}$ (since $\mathrm{Pic}(X) \simeq \mathbf{Z}$) and the third term vanishes by virtue of the following:

Theorem 4. *The Brauer group* $\mathrm{Br}(X)$ *is zero.*

Sketch. Let x be an element of the Brauer group of $\mathrm{Br}(X)$. Then x can be represented by an Azumaya algebra $\mathcal{A}$ on X, which we can regard as a vector bundle of rank n^2 on X for some $n > 0$. Let us assume that n has been chosen as small as possible. As in the proof of Theorem 1, the vector bundle $\mathcal{A}$ is isomorphic to its dual and therefore has slope 0. Assume that $\mathcal{A}$ is not semistable, and let $\mathcal{E} \subseteq \mathcal{A}$ be the first step of the Harder-Narasimhan stratification of $\mathcal{A}$. Let $\mathcal{I} \subseteq \mathcal{A}$ be the image of the multiplication map $\mathcal{A} \otimes \mathcal{E} \to \mathcal{A}$. Then $\mathcal{I}$ is a left ideal of $\mathcal{A}$, hence a left $\mathcal{A}$-module. It follows that $\mathcal{I}$ must have rank kn for some $k > 0$, and a local calculation shows that $k < n$. Let $\mathcal{B}$ denote the sheaf of endomorphisms of $\mathcal{I}$ as a left $\mathcal{A}$-module. One can then argue that $\mathcal{B}$ is an Azumaya algebra of rank k^2 and that the opposite algebra $\mathcal{B}^{\mathrm{op}}$ is Morita equivalent to $\mathcal{A}$ (via the bimodule $\mathcal{I}$), and therefore also represents the same Brauer class x. This contradicts the minimality of n. We may therefore assume that $\mathcal{A}$ is semistable of slope 0, and therefore of the form $A \otimes_{\mathbf{Q}_p} \mathcal{O}_X$ for some Azumaya algebra A over $\mathbf{Q}_p$. By minimality, it follows that A is a central division algebra over $\mathbf{Q}_p$; let $\mu = \frac{m}{n} \in \mathbf{Q}/\mathbf{Z} \simeq \mathrm{Br}(\mathbf{Q}_p)$ be its Hasse invariant. Then we can identify A with the algebra of endomorphisms of the isocrysal $V_{\frac{m}{n}}$ defined in the previous lecture. By functoriality, the algebra A acts (on the right) on the associated vector bundle $\mathcal{E}_{V_{\frac{m}{n}}} = \mathcal{O}(\frac{m}{n})$ on X: that is, we can regard $\mathcal{O}(\frac{m}{n})$ as a right $\mathcal{A} = A \otimes_{\mathbf{Q}_p} \mathcal{O}$-module. This induces an isomorphism $\mathcal{A} \simeq \mathrm{End}(\mathcal{O}(\frac{m}{n}))$, which shows that the Brauer class of $\mathcal{A}$ vanishes. $\square$