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To Jasmine

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Preface

The theory of algebraic spaces and stacks has its origins in the study of moduli spaces in algebraic geometry. It is closely related to the problem of constructing quotients of varieties by equivalence relations or group actions, and the basic definitions are natural outgrowths of this point of view. The foundations of the theory were introduced by Deligne and Mumford in their fundamental paper on the moduli space of curves [**23**] and by Artin building on his work on algebraic approximations [**9**]. Though it has taken some time, algebraic spaces and stacks are now a standard part of the modern algebraic geometers toolkit and are used throughout the subject.

This book is an introduction to algebraic spaces and stacks intended for a reader familiar with basic algebraic geometry (for example Hartshorne's book [**41**]). We do not strive for an exhaustive treatment. Rather we aim to give the reader enough of the theory to pursue research in areas that use algebraic spaces and stacks, and to proceed on to more advanced topics through other sources. Numerous exercises are included at the end of each chapter, ranging from routine verifications to more challenging further developments of the theory.

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Introduction

A basic theme in algebraic geometry is to classify various geometric structures using moduli spaces or parameter spaces, and the study of such classification problems leads naturally to the notions of algebraic spaces and stacks. Just as nonreduced or nonseparated schemes are important even if one is only motivated by the study of smooth projective varieties, so algebraic spaces and stacks are natural extensions of the notion of scheme.

An important classical situation where one encounters algebraic spaces and stacks is when constructing quotients of schemes by group actions. If X is a scheme and G is a finite group acting freely on X then the quotient X/G of X by the G-action always exists as an algebraic space, but this quotient may not exist as a scheme (see 5.3.2 for an example). More generally, one can construct algebraic spaces as quotients of schemes by free actions of algebraic groups. Loosely speaking, an algebraic space is a geometric object obtained by gluing together schemes using the étale topology rather than the Zariski topology. An algebraic space can always be realized as the quotient of a scheme X by an equivalence relation $\Gamma \hookrightarrow X \times X$ such that the two projections $\Gamma \to X$ are étale. Of course a given algebraic space can have many different presentations as a quotient, and care has to be taken in defining the proper category in which the quotient can be taken.

Algebraic stacks enter into the study of group actions on schemes when the action is no longer free. If G is a group, say finite for simplicity, acting on a scheme X the equivalence relation $\Gamma \subset X \times X$ given by declaring points in the same orbit equivalent, need no longer be an étale equivalence relation. In topology it is well-known how to form quotients by nonfree actions. Namely, one considers a contractible space EG with free G-action and then defines the quotient $[X/G]$ to be the quotient of $X \times EG$ by the diagonal action. This quotient has many advantages compared to the naive quotient. For example, the projection map $X \times EG \to [X/G]$ is a principal G-bundle. In algebraic geometry there is no analogue of the space EG, but still one can form the stack quotient $[X/G]$ for a group acting on a scheme X and this quotient enjoys many of the good properties of the corresponding quotient in topology. This stack quotient $[X/G]$ is no longer a space but something more general. For instance, if k is an algebraically closed field then the k-points of the stack is not a set but rather a category; namely, the category whose objects are the k-points of X and for which a morphism $x \to x'$ is given by an element $g \in G$ such that $gx = x'$.

Not all algebraic stacks are quotients of schemes by group actions, though many important algebraic stacks can be realized in this way. To get a sense of the general definition, recall that the Yoneda imbedding identifies the category of schemes with a subcategory of the category of functors

$$(\text{schemes})^{\text{op}} \to (\text{Sets}).$$

Under this identification a scheme X corresponds to the functor h_X sending a scheme T to the set $\text{Hom}(T, X)$ of morphisms of schemes $T \to X$. A functor is called *representable* if it is isomorphic to h_X for some scheme X. From this point of view, algebraic stacks can be viewed as the natural generalization of schemes obtained by replacing the category of sets by groupoids (categories in which all morphisms are isomorphisms). So an algebraic stack can be viewed as a functor

$$(\text{schemes})^{\text{op}} \to (\text{Groupoids})$$

satisfying certain conditions generalizing those characterizing schemes among all functors from schemes to sets. To make this precise one encounters a number of technical difficulties including the fact that groupoids do not form a category but a 2-category (functors between categories do not form a set but rather a category). Once these are overcome, however, one can generalize most of standard algebraic geometry to algebraic stacks. This includes the theory of quasi-coherent and coherent sheaves, cohomology (both cohomology of quasi-coherent sheaves as well as étale and other theories), intersection theory and so on. In fact, the more general setting of algebraic stacks also allows some constructions that are impossible in the ordinary category of schemes (see for example the root stack construction in section 10.3).

Most functors

$$M : (\text{schemes})^{\text{op}} \to (\text{Sets})$$

occurring in moduli theory can naturally be lifted to functors (ignoring the technical issues just mentioned)

$$\mathscr{M} : (\text{schemes})^{\text{op}} \to (\text{Groupoids}).$$

Typically the functor M sends a scheme T to the set of isomorphism classes of certain geometric objects over T (for example families of curves of a given genus g, vector bundles on a fixed variety pulled back to T, etc.). The lifting $\mathscr{M}$ is obtained by sending T to the groupoid whose objects are the geometric objects in question over T and whose morphisms are isomorphisms between them. In this way M is obtained from $\mathscr{M}$ by sending a scheme T to the set of isomorphism classes in the groupoid $\mathscr{M}(T)$.

A *fine moduli space* for the moduli problem is a scheme X and an isomorphism of functors $h_X \simeq M$. As we illustrate below, for many moduli problems the presence of nontrivial automorphisms prevents the moduli problem from having a fine moduli space. In this situation one classically looks for a *coarse moduli space*, which is a scheme X with a morphism of functors $M \to h_X$ that is universal for morphisms from M to representable functors and such that for any algebraically closed field k the induced map $M(\text{Spec}(k)) \to h_X(\text{Spec}(k)) = X(k)$ is a bijection. Thus, for example, if we wish to classify curves of genus g over a field k a coarse moduli space X would, in particular, have its points in 1-1 correspondence with isomorphism classes of such curves. It would also be a fine moduli space if there exists a family of curves $C_X \to X$ over X such that for any k-scheme T and family of curves Y/T of genus g there exists a unique morphism $T \to X$ such that Y is the pullback of C_X. In this case we say also "X classifies families of curves of genus g."

This definition also gives a definition of coarse moduli space for stacks: A coarse moduli space for a stack $\mathscr{M}$ is a coarse moduli space for the corresponding functor of isomorphism classes. As we discuss in this book, the Keel-Mori theorem gives some natural conditions under which an algebraic stack has a coarse moduli space. The typical situation in moduli theory is that one does not have a fine moduli space but the stack $\mathscr{M}$ is a nice algebraic stack admitting a coarse moduli space. In this situation, the stack $\mathscr{M}$ is often a better object to work with than the coarse moduli space. For example, the algebraic stack $\mathscr{M}$ could be smooth and the coarse space singular; also one has a so-called universal family over $\mathscr{M}$ but not over the coarse space. It is often advantageous to study the geometry of $\mathscr{M}$ directly without passing to the coarse moduli space.

Rather than delving into too many technicalities about the exact definitions of algebraic stacks here in the introduction, let us illustrate some of these ideas with the moduli of elliptic curves. For simplicity we work here over $\mathbb{C}$; a more complete discussion can be found in the main text (in particular Chapter 13, §13.1; see also [**42**, §26] for another point of view). We learned the following example from [**39**, Chapter 2].

Recall (for example from [**41**, Chapter IV, §4]) that an *elliptic curve* over $\mathbb{C}$ is a pair (E, e), where E is a smooth projective genus 1 curve over $\mathbb{C}$, and $e \in E$ is a closed point (we often denote such an elliptic curve simply by E, omitting the marked point from the notation). Any such curve can be described as the zero locus in $\mathbb{P}^2$ of a homogeneous polynomial of the form

$$Y^2Z - X(X-Z)(X-\lambda Z),$$

where $\lambda \in \mathbb{C} - \{0,1\}$ and the point e is given by $[0:1:0]$. This polynomial in fact defines a family

$$\mathscr{E} \hookrightarrow \mathbb{P}^2 \times (\mathbb{A}^1 - \{0,1\}),$$

over the punctured affine line, in which every elliptic curve is isomorphic to a fiber over some point $\lambda \in \mathbb{A}^1 - \{0,1\}$. The line $\mathbb{A}^1 - \{0,1\}$ can therefore be thought of as a parameter space for elliptic curves. However, the representation of an elliptic curve as a fiber is not unique and depends on a choice of equation for the curve as a subscheme of $\mathbb{P}^2$. In fact, there is an action of the symmetric group S_3 on $\mathbb{A}^1 - \{0,1\}$ generated by the automorphisms

$$\lambda \mapsto 1/\lambda, \quad \lambda \mapsto \frac{1}{1-\lambda},$$

and one checks that two points $\lambda, \lambda' \in \mathbb{C} - \{0,1\}$ define isomorphic elliptic curves if and only if they lie in the same S_3-orbit. Therefore, if we want to parametrize abstract elliptic curves without a projective imbedding, we should consider the quotient of $\mathbb{A}^1 - \{0,1\}$ by this S_3-action. In the category of schemes this quotient is given by taking the spectrum of the S_3-invariants in the ring $\mathbb{C}[\lambda]_{\lambda(\lambda-1)}$ which it turns out is isomorphic to the polynomial ring $\mathbb{C}[j]$ in one variable j given by the formula

$$j = 2^8 \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda-1)^2}.$$

This implies that there is a bijection between isomorphism classes of elliptic curves over $\mathbb{C}$ and complex numbers $j \in \mathbb{C}$. The affine line $\mathbb{A}^1_j$ is not, however, a fine moduli space for moduli of elliptic curves. There does not exist a family of elliptic curves $\mathscr{E} \to \mathbb{A}^1_j$ such that for any $\mathbb{C}$-scheme T and family of elliptic curves E/T

there exists a unique morphism $\rho_E : T \to \mathbb{A}^1_j$ such that E is isomorphic to the pullback along ρ_E of $\mathscr{E}$ (see 13.1.1 for the precise definition of an elliptic curve over a scheme S).

This can be seen explicitly as follows. Consider the family of elliptic curves $\mathscr{E}_t$ defined over the t-line $\mathbb{A}^1_t - \{0\}$ given by the equation

$$Y^2Z = X^3 - tZ^3.$$

The j-invariant of every fiber (including the generic fiber) is 0, so the corresponding map $\mathbb{A}^1_t - \{0\} \to \mathbb{A}^1_j$ is a constant map. The elliptic curve E_0 given by $Y^2Z = X^3 - Z^3$ also has j-invariant 0, so if the j-line was a fine moduli space for elliptic curves one would expect that the family $\mathscr{E}_t$ was isomorphic to $E_0 \times (\mathbb{A}^1_t - \{0\})$. This is not the case, however. Indeed, over the function field $\mathbb{C}(t)$ these two curves are not isomorphic, but they do become isomorphic over the field extension $\mathbb{C}(t^{1/6})$ (this is exercise 4.D in the text; the reader may wish to work out this exercise now).

This example not only shows that the j-line $\mathbb{A}^1_j$ is not a fine moduli space for moduli of elliptic curves, but in fact that the functor

$$M_{1,1} : (\mathbb{C}\text{-schemes})^{\mathrm{op}} \to \mathrm{Sets}$$

sending a scheme S to the set of isomorphism classes of elliptic curves over S is not representable. Indeed, the preceding discussion implies that $M_{1,1}$ fails one of the basic properties of representable functors; so-called *étale descent*. If U is a $\mathbb{C}$-scheme and

$$F : (\mathbb{C}\text{-schemes})^{\mathrm{op}} \to \mathrm{Sets}$$

is the functor sending a scheme S to the set of morphisms $S \to U$, then for any field extension $K \hookrightarrow L$ the map

$$F(\mathrm{Spec}(K)) \to F(\mathrm{Spec}(L))$$

is injective. As exercise 4.D demonstrates, the failure of this condition for $M_{1,1}$ arises from the fact that elliptic curves have nontrivial automorphisms. On the other hand, there is an algebraic stack $\mathscr{M}_{1,1}$ which associates to a scheme S the groupoid whose objects are elliptic curves over S with isomorphisms between them. This stack $\mathscr{M}_{1,1}$ is a so-called Deligne-Mumford stack, it is smooth, there is a universal elliptic curve $\mathscr{E} \to \mathscr{M}_{1,1}$ over it, and so on. Furthermore, the stack $\mathscr{M}_{1,1}$ admits a coarse moduli space which in this case is the natural map $\mathscr{M}_{1,1} \to \mathbb{A}^1_j$ associating to an elliptic curve its j-invariant.

Outline of the book:

A proper treatment of the subject requires a fair amount of foundational work. Our general philosophy is that covering a subset of the theory with all the details included is preferable to a cursory exposition that covers more topics. A considerable portion of the book is therefore devoted to foundational topics such as Grothendieck topologies, descent, and fibered categories. While perhaps a bit 'dry', these subjects are so central to the theory that any student wishing to study the subject needs a thorough understanding of these topics.

Chapter 1 collects together various facts from "standard algebraic geometry" that, nonetheless, may not be part of a standard course on the subject. The reader may skip this chapter referring back as needed later. In addition to technical background, this chapter also contains a discussion of schemes from a more functor-theoretic point of view than the standard treatment. This serves as a lead-in to the definitions of algebraic spaces and stacks, and it may be helpful for the reader to consider this functor-theoretic point of view of schemes first.

As mentioned above, schemes can be viewed via the Yoneda imbedding as functors satisfying various properties with respect to the Zariski topology reflecting the fact that they can be covered by affines. The notion of algebraic space is obtained from this point of view by replacing the Zariski topology by the étale topology. For this to make sense we need some foundational material on Grothendieck topologies and sites, and we develop this material in Chapter 2. In this chapter we also discuss some technical points about simplicial topoi, which provide a basic technical tool (in particular for cohomological descent).

One way to phrase the notion of stack is to say that a stack is a sheaf taking values in groupoids (categories in which all morphisms are isomorphisms), instead of sets. Making this definition precise, however, is nontrivial and one has to tackle several 2-categorical issues. Chapter 3 is devoted to the basic definitions and results concerning fibered categories, which provide a solution to these 2-categorical issues.

Chapter 4 discusses descent, which provides the categorical version of the sheaf condition for presheaves. Of basic importance here is faithfully flat descent, which enables one to work with bigger topologies, such as the fppf or étale topologies, in the algebro-geometric context. In this chapter we also discuss some basic results about torsors and principal homogeneous spaces, which play an important role in many examples, and introduce the condition for a fibered category to be a stack.

We then proceed to the development of the theory of algebraic spaces in Chapter 5. The main difference between our treatment and that in other standard sources (for example Knutson's book [**45**]) is that we make no assumptions on the diagonal in the definition of algebraic space. This we feel is the most natural approach though it entails a little extra care in the development of the theory.

To develop the theory of algebraic spaces beyond the basic definitions we need some results about quotients of schemes by finite flat equivalence relations. We discuss these in Chapter 6. This is a very classical topic which is useful both for algebraic spaces but also for the discussion of coarse moduli spaces in later chapters. In Chapter 6 we show in particular that reasonable algebraic spaces can be stratified by schemes.

In Chapter 7 we turn to the study of quasi-coherent sheaves on algebraic spaces. We discuss Stein factorization, Chow's lemma, and finiteness of cohomology. At this point we have generalized most of the standard scheme theory to algebraic spaces, and now turn our attention to algebraic stacks.

In Chapter 8 we introduce the basic definitions for algebraic stacks, as well as several examples. We also discuss Deligne-Mumford stacks, which have many special properties not enjoyed by general algebraic stacks.

Chapter 9 is devoted to the study of quasi-coherent sheaves on algebraic stacks. We have chosen to work with the lisse-étale site as in [**49**], though there are persuasive arguments for using bigger topologies such as the fppf topology. In particular, working with the lisse-étale site introduces some complications when considering pullbacks of quasi-coherent sheaves, and we discuss how to handle these in Chapter 9.

In Chapter 10 we discuss several basic constructions and examples. The notion of a proper morphism of stacks is introduced, and among other examples we discuss root stacks and a stack-theoretic version of the usual Proj construction.

We then turn to the Keel-Mori theorem and the construction of coarse moduli spaces in Chapter 11. Briefly a coarse moduli space for an algebraic stack $\mathscr{X}$ is a morphism $\pi : \mathscr{X} \to X$ to an algebraic space that is universal for such morphisms and such that for an algebraically closed field k the map π induces a bijection between isomorphism classes of the groupoid $\mathscr{X}(k)$ and $X(k)$. A basic example is the moduli stack $\mathscr{M}_{1,1}$ of elliptic curves discussed above, where the j-invariant defines a map $\mathscr{M}_{1,1} \to \mathbb{A}^1$, which in fact is a coarse moduli space.

The last two chapters discuss applications of the theory.

In Chapter 12 we discuss gerbes and their connection with Azumaya algebras and cohomology classes. A gerbe bound by a smooth abelian group scheme $\boldsymbol{\mu}$ over a scheme X is a special kind of algebraic stack over X. The isomorphism classes of such gerbes are given by $H^2(X, \boldsymbol{\mu})$. For $\boldsymbol{\mu} = \mathbb{G}_m$ we get a bijection between $\mathbb{G}_m$-gerbes over X and the group $H^2(X, \mathbb{G}_m)$, which is closely related to the Brauer group of X. Some questions about Brauer groups can be reformulated to algebro-geometric questions about the associated $\mathbb{G}_m$-gerbes, and we discuss some of this in Chapter 12.

In Chapter 13 we discus various moduli stacks of curves. In some sense this brings the book back to the original paper of Deligne and Mumford [**23**]. We discuss moduli of elliptic curves, the Deligne-Mumford compactification $\overline{\mathscr{M}}_g$ of the moduli stack of curves of genus $g \geq 2$, and moduli of stable maps. In this chapter we use some facts from curve theory that may not be so familiar to the beginning reader (for example, the stable reduction theorem), but we summarize the basic results used. The aim here is to synthesize the theory of stacks with classical curve theory.

Finally, there is a glossary where we summarize some of the category theory used in the text.

Nothing in this book is original, and all the results herein can be found in some form in the existing literature. We indicate our main sources in each chapter introduction, and to help the reader connect with the literature we also provide precise references for the main theorems. The basic treatments for the material presented in this book are [**9**], [**23**], [**36**], [**45**], [**49**], and [**71**]. One major topic that we do not cover in this book is Artin's method for proving representability by an algebraic space or stack using deformation theory. The original papers of Artin [**4**], [**5**], [**6**], [**7**], and [**9**] remain the definitive references on this topic.

CHAPTER 1

Summary of background material

In this chapter we review some of the basic facts and definitions that we will need later. The reader can browse this chapter to start, referring back only when needed.

1.1. Flatness

For a more detailed discussion of the notion of flatness see [**2**, Chapter V].

1.1.1. Let R be a ring. Recall that an R-module M is called *flat* if the functor

$$(-)\otimes_R M : \mathrm{Mod}_R \to \mathrm{Mod}_R$$

is an exact functor, where Mod_R denotes the abelian category of R-modules. The module M is called *faithfully flat* if M is flat, and if for any two R-modules N and N' the natural map

$$\mathrm{Hom}_R(N, N') \to \mathrm{Hom}_R(N\otimes_R M, N'\otimes_R M)$$

is injective.

Proposition 1.1.2. *Let M be an R-module. The following are equivalent:*

(i) *M is faithfully flat.*

(ii) *M is flat and for any R-module N' the map*

$$N' \to \mathrm{Hom}_R(M, N'\otimes M), \quad y\mapsto (m\mapsto y\otimes m) \tag{1.1.2.1}$$

is injective.

(iii) *A sequence of R-modules*

$$N'\to N\to N'' \tag{1.1.2.2}$$

is exact if and only if the sequence

$$N'\otimes_R M\to N\otimes_R M\to N''\otimes_R M \tag{1.1.2.3}$$

is exact.

(iv) *A morphism of R-modules $N'\to N$ is injective if and only if the morphism $N'\otimes_R M\to N\otimes_R M$ is injective.*

(v) *M is flat and if $N\otimes_R M = 0$ for some R-module N, then $N = 0$.*

(vi) *M is flat and for every maximal ideal $\mathfrak{m}\subset R$ we have $M/\mathfrak{m}M\neq 0$.*

Proof. First let us show that (i) is equivalent to (ii). If $F\to N$ is a surjective morphism of R-modules, then for any R-module N' we have a commutative square

$$\begin{tikzcd}
\mathrm{Hom}_R(N, N') \arrow[r] \arrow[d, hook] & \mathrm{Hom}_R(N\otimes M, N'\otimes M) \arrow[d, hook] \\
\mathrm{Hom}_R(F, N') \arrow[r] & \mathrm{Hom}_R(F\otimes M, N'\otimes M),
\end{tikzcd}$$

where the vertical maps are injective. In particular, choosing $F = \bigoplus_{i\in I} R$ to be a free module, in which case we have natural isomorphisms

$$\mathrm{Hom}_R(F, N') \simeq \prod_{i\in I} N', \quad \mathrm{Hom}_R(F\otimes M, N'\otimes M) \simeq \prod_{i\in I} \mathrm{Hom}_R(M, N'\otimes M),$$

we see that an R-module M is faithfully flat if and only if for any R-module N' the map

$$N' \to \mathrm{Hom}_R(M, N'\otimes M), \quad y \mapsto (m \mapsto y\otimes m)$$

is injective, thereby showing the equivalence of (i) and (ii).

Next, let us show that (ii) implies (iv). If M is flat and $N' \to N$ is injective, then $N'\otimes M \to N\otimes M$ is also injective by the flatness assumption on M. So to show that (ii) implies (iv) it suffices to show that if (ii) holds and $N' \to N$ is a morphism of R-modules such that $N'\otimes M \to N\otimes M$ is an injection, then $N' \to N$ is also an inclusion. For this, note that we obtain a commutative diagram

$$\begin{tikzcd} N' \arrow[r, hook] \arrow[d] & \mathrm{Hom}_R(M, N'\otimes M) \arrow[d] \\ N \arrow[r, hook] & \mathrm{Hom}_R(M, N\otimes M), \end{tikzcd} \tag{1.1.2.4}$$

where the horizontal arrows are inclusions by (ii). If $N'\otimes M \to N\otimes M$ is an inclusion, then so is the right vertical arrow in the diagram, from which it follows that $N' \to N$ is also injective.

Statement (v) follows from (iv) applied to the map $N \to 0$.

Statement (v) implies (vi) by taking $N = R/\mathfrak{m}$.

Also (v) is equivalent to (iii). Indeed (iii) is equivalent to the statement that M is flat and for any sequence (1.1.2.2) for which (1.1.2.3) is exact the sequence (1.1.2.2) is exact. Now observe that if M is flat and (1.1.2.2) is a sequence of R-modules then setting

$$H := \mathrm{Ker}(N \to N'')/\mathrm{Im}(N' \to N)$$

we have

$$H\otimes M := \mathrm{Ker}(N\otimes M \to N''\otimes M)/\mathrm{Im}(N'\otimes M \to N\otimes M).$$

From this it follows that (v) implies (iii), and the converse direction is immediate (consider the sequence $0 \to N \to 0$).

To prove the proposition we are therefore reduced to showing that (vi) implies (ii). For this we show that a counterexample to (ii) yields a counterexample for (vi). So suppose N' is an R-module and $x \in N'$ is a nonzero element mapping to zero under the map (1.1.2.1). Denote by $L \subset N'$ the submodule generated by x, and let $\mathfrak{a} \subset R$ be the kernel of the map $R \to N'$ sending $f \in R$ to $f\cdot x$, so we have $R/\mathfrak{a} \simeq L$. If M is flat, then the map

$$L\otimes M \to N'\otimes M$$

is an inclusion, from which we deduce that the map (1.1.2.1) for L,

$$L \to \mathrm{Hom}_R(M, L\otimes M),$$

is the zero map. The image of $x \in L$ under this map is via the isomorphism $L \simeq R/\mathfrak{a}$ identified with the projection map

$$M \to M\otimes R/\mathfrak{a} \simeq M/\mathfrak{a}M,$$

which implies that $M/\mathfrak{a}M = 0$. If $\mathfrak{m}$ is a maximal ideal containing $\mathfrak{a}$ we then get that $M/\mathfrak{m}M = 0$ thereby obtaining a counterexample to (vi). $\square$

DEFINITION 1.1.3. A morphism of schemes $f : X \to Y$ is *flat* if for every point $x \in X$ the map

$$\mathscr{O}_{Y,f(x)} \to \mathscr{O}_{X,x}$$

is flat. The morphism f is *faithfully flat* if f is flat and surjective.

REMARK 1.1.4. If $R \to R'$ is a ring homomorphism, then

$$\mathrm{Spec}(R') \to \mathrm{Spec}(R)$$

is flat (resp. faithfully flat) if and only if R' is flat (resp. faithfully flat) as an R-module (this is exercise 1.A).

PROPOSITION 1.1.5. *Let $f : X \to Y$ be a flat morphism of locally noetherian schemes that is locally of finite type. Then for any open subset $U \subset X$ the image $f(U) \subset Y$ is open.*

PROOF. See [**26**, IV.2.4.6]. $\square$

COROLLARY 1.1.6. *Let $f : X \to Y$ be a faithfully flat morphism of locally noetherian schemes that is locally of finite type, and let $Y = \bigcup_i U_i$ be an open covering, with each U_i affine. Then for each i, there is a Zariski covering $f^{-1}(U_i) = \bigcup_j V_{ij}$ with V_{ij} quasi-compact and $f(V_{ij}) = U_i$.*

PROOF. It suffices to exhibit for every $x \in f^{-1}(U_i)$ a quasi-compact open neighborhood $x \in V \subset f^{-1}(U_i)$ such that $f(V) = U_i$. For this start with an affine neighborhood $x \in V_0 \subset f^{-1}(U_i)$, and then take $V = V_0 \cup V_1 \cup \cdots V_r$, where the $V_j \subset f^{-1}(U_i)$ are affine open subsets such that the open images $f(V_i)$ cover the quasi-compact U_i. $\square$

REMARK 1.1.7. The condition 'flat morphism of locally noetherian schemes that is locally of finite type' in 1.1.5 and 1.1.6 can be replaced by the more general condition 'flat morphism of schemes that is locally of finite presentation' discussed in the next section.

1.2. Morphisms locally of finite presentation

The basic reference for the material in this section is [**26**, IV §8].

1.2.1. If A is a ring and M is an A-module, then M is called of *finite presentation* if there exists an exact sequence

$$A^r \to A^s \to M \to 0$$

for some integers r and s. Note that in the case when A is noetherian, this is equivalent to M being finitely generated (as the kernel of any surjection $A^s \to M$ is automatically finitely generated), but in general M being of finite presentation is a stronger condition than being finitely generated.

1.2.2. If $A \to B$ is a ring homomorphism, then we say that B is of *finite presentation over* A, (or that B is a *finitely presented A-algebra*) if there exists a surjection

$$\pi : A[X_1, \ldots, X_s] \to B$$

with kernel $\mathrm{Ker}(\pi)$ a finitely generated ideal in $A[X_1, \dots, X_s]$. If A is noetherian this is equivalent to B being a finitely generated A-algebra, but in general B being of finite presentation is a stronger condition than being finitely generated.

1.2.3. These notions generalize to schemes as follows.

Let X be a scheme. A quasi-coherent sheaf $\mathscr{F}$ on X is called *locally finitely presented* if for every affine open subset $\mathrm{Spec}(B) \subset X$ the module $\Gamma(\mathrm{Spec}(B), \mathscr{F})$ is a finitely presented B-module.

Note that if X is locally noetherian then a quasi-coherent sheaf is locally finitely presented if and only if it is coherent.

A morphism of schemes $f : X \to Y$ is called *locally of finite presentation* if for every affine open $\mathrm{Spec}(A) \subset Y$ and affine open $\mathrm{Spec}(B) \subset f^{-1}(\mathrm{Spec}(A))$, the A-algebra B is of finite presentation over A.

A morphism $f : X \to Y$ is said to be of *finite presentation* (or a *finitely presented morphism*) if f is locally of finite presentation and quasi-compact and quasi-separated (recall that by definition a morphism of schemes $f : X \to Y$ is quasi-separated if the diagonal morphism is quasi-compact).

In the case when Y is noetherian, the morphism f is locally of finite presentation if and only if f is locally of finite type, and finitely presented if and only if of finite type.

Example 1.2.4 ([**67**, Tag 01KH]). The notion of a quasi-separated morphism usually is not studied extensively in a first year course in algebraic geometry, as any reasonable morphism of schemes that one might encounter in such a course will be quasi-separated. However, in the study of algebraic spaces and stacks properties of the diagonal morphism play a central role and notions such as quasi-separated morphisms occur naturally. Here is an example of a morphism of schemes $f : X \to Y$ which is quasi-compact and locally of finite presentation but not quasi-separated.

Let k be a field and let $Y = \mathrm{Spec}(k[x_1, x_2, \dots,])$ be the spectrum of the polynomial ring on infinitely many variables x_i $(i \geq 1)$. Let $z \in Y$ denote the closed point obtained by setting all variables $x_i = 0$, and let $P_i \in Y$ $(i \geq 1)$ be the closed point obtained by setting $x_i = 1$ and all other variables equal to 0. Let $U \subset Y$ denote the complement of z. Then U is not quasi-compact. Indeed, U is covered by the basic open subsets $D(x_i)$ and each P_i is contained in a unique such basic open, whence no finite subset of the $D(x_i)$ can cover U. Let X be the scheme obtained by taking two copies of Y and gluing them along the open subset U with the identity morphism, and let $X_1, X_2 \subset X$ be the inclusions of the two copies of Y. Let $f : X \to Y$ be the morphism which restricts to the identity on each X_i. Then f is quasi-compact and locally of finite presentation, but not quasi-separated. Indeed, we have $X_1 \times_Y X_2 \simeq Y$, and the square

$$\begin{tikzcd} U \arrow[r] \arrow[d] & X_1 \times_Y X_2 \arrow[d] \\ X \arrow[r, "\Delta"] & X \times_Y X \end{tikzcd}$$

is cartesian.

1.2.5. The utility of these notions is that they often allow one to reduce questions over arbitrary base schemes to questions over base schemes of finite type over $\mathbb{Z}$.

To explain, consider a partially ordered set $(I, \geq)$ which we assume to be filtering (that is for every $\lambda, \mu \in I$ there exists $\tau \in I$ such that $\tau \geq \lambda$ and $\tau \geq \mu$). We often think of I as a category with objects the elements of I and

$$\mathrm{Hom}(\lambda, \mu) = \begin{cases} \{*\} & \text{if } \lambda \geq \mu, \\ \emptyset & \text{otherwise.} \end{cases}$$

Fix a scheme B. A *projective system of B-schemes indexed by I* is a functor

$$S_\bullet : I \to (B\text{-schemes}).$$

More concretely, for every $\lambda \in I$ we specify a B-scheme S_λ and if $\lambda \geq \mu$ then we specify a morphism over B,

$$\theta_{\lambda,\mu} : S_\lambda \to S_\mu,$$

and these transition maps have to be compatible for triples $\tau \geq \lambda \geq \mu$.

We say that a projective system $S_\bullet$ of B-schemes indexed by I has *affine transition maps* if for every $\lambda \geq \mu$ the map $\theta_{\lambda,\mu}$ is an affine morphism.

EXAMPLE 1.2.6. Suppose $B = \mathrm{Spec}(\mathbb{Z})$ and A is a ring. Let I be the partially ordered set of finitely generated subrings of A, and for $\lambda \in I$ let A_λ be the corresponding subring. We then get a projective system of affine schemes setting $S_\lambda = \mathrm{Spec}(A_\lambda)$.

LEMMA 1.2.7. *Let $S_\bullet$ be a projective system of B-schemes indexed by I with affine transition maps. Then the inverse limit $\varprojlim_{\lambda \in I} S_\lambda$ exists in the category of B-schemes, and for every $\lambda \in I$ the map $\varprojlim_{\lambda \in I} S_\lambda \to S_\lambda$ is affine.*

PROOF. Fix any $\lambda \in I$ and for $\mu \geq \lambda$ let $\mathscr{A}_\mu$ denote the quasi-coherent sheaf of algebras on S_λ corresponding to the affine morphism $S_\mu \to S_\lambda$. Setting

$$\mathscr{A} := \varinjlim_{\mu \geq \lambda} \mathscr{A}_\mu$$

we get a scheme $S := \underline{\mathrm{Spec}}_{S_\lambda}(\mathscr{A})$ equipped with a morphism $\rho_\mu : S \to S_\mu$ over S_λ for each $\mu \geq \lambda$. By the universal property of the relative spectrum of a sheaf of algebras [**26**, II.1.2.7] the scheme S represents $\varprojlim_\lambda S_\lambda$. $\square$

The following is a useful characterization of a morphism being locally of finite presentation in terms of its functor of points.

PROPOSITION 1.2.8. *A morphism of schemes $f : X \to Y$ is locally of finite presentation if and only if for every projective system of Y-schemes $\{S_\lambda\}_{\lambda \in I}$ with each S_λ an affine scheme, the natural map*

$$\varinjlim_\lambda \mathrm{Hom}_Y(S_\lambda, X) \to \mathrm{Hom}_Y(\varprojlim S_\lambda, X)$$

is a bijection.

PROOF. See [**26**, IV.8.14.2]. $\square$

EXAMPLE 1.2.9. The assumption that f is locally of finite presentation is necessary as the following simple example shows. Let $Y = \mathrm{Spec}(k)$ be the spectrum of a field, and let X be the spectrum of a k-algebra A which is not finitely generated over k. Let I be the partially ordered set of finite subsets of A, and for $\lambda \in I$ let $A_\lambda \subset A$ be the k-subalgebra generated by the elements of λ. Let $S_\lambda = \mathrm{Spec}(A_\lambda)$.

Then $X = \varprojlim_\lambda S_\lambda$, and the identity map $X = \varprojlim_\lambda S_\lambda \to X$ gives an element of $\mathrm{Hom}_Y(\varprojlim S_\lambda, X)$ not in the image of $\varinjlim_\lambda \mathrm{Hom}_Y(S_\lambda, X)$.

1.2.10. Fix a scheme B and let $S_\bullet$ be a projective system of B-schemes indexed by a partially ordered set I. A *family of quasi-coherent sheaves* on $S_\bullet$ is a collection of data $\{(\mathscr{F}_\lambda)_{\lambda\in I}, u_{\lambda\mu}\}$ consisting of a quasi-coherent sheaf $\mathscr{F}_\lambda$ on S_λ for each $\lambda \in I$, and for every $\lambda \geq \mu$ an isomorphism $u_{\lambda\mu} : \theta^*_{\lambda,\mu}\mathscr{F}_\mu \to \mathscr{F}_\lambda$ such that the natural cocycle condition holds for triples $\tau \geq \lambda \geq \mu$.

Assume now that the transition maps in $S_\bullet$ are all affine so the projective limit $S := \varprojlim_{\lambda\in I} S_\lambda$ exists, and let $\rho_\lambda : S \to S_\lambda$ denote the projection. For a family $\{(\mathscr{F}_\lambda)_\lambda, u_{\lambda\mu})\}$ set $\mathscr{F} := \rho^*_\lambda\mathscr{F}_\lambda$ for any choice of λ. The quasi-coherent sheaf $\mathscr{F}$ on S is up to canonical isomorphism independent of the choice of λ.

Next consider two families $\{(\mathscr{F}_\lambda)_\lambda, u_{\lambda\mu})\}$ and $\{(\mathscr{G}_\lambda)_\lambda, v_{\lambda\mu})\}$. For every λ we have a natural map

$$\rho^*_\lambda : \mathrm{Hom}_{S_\lambda}(\mathscr{F}_\lambda, \mathscr{G}_\lambda) \to \mathrm{Hom}_S(\mathscr{F}, \mathscr{G})$$

which is compatible with the transition maps so we get a map

$$\rho^* : \varinjlim_\lambda \mathrm{Hom}_{S_\lambda}(\mathscr{F}_\lambda, \mathscr{G}_\lambda) \to \mathrm{Hom}_S(\mathscr{F}, \mathscr{G}). \tag{1.2.10.1}$$

Theorem 1.2.11. *(i) Assume that there exists λ such that S_λ is quasi-compact and quasi-separated and $\mathscr{F}_\lambda$ is locally finitely presented. Then the map* (1.2.10.1) *is an isomorphism.*

(ii) Assume there exists λ such that S_λ is quasi-compact and quasi-separated. Then for any locally finitely presented quasi-coherent sheaf $\mathscr{F}$ on S, there exists μ and a finitely presented quasi-coherent sheaf $\mathscr{G}_\mu$ on S_μ such that $\mathscr{F} \simeq \rho^*_\mu\mathscr{G}_\mu$.

Proof. See [**26**, IV.8.5.2]. □

Many properties of sheaves are also stable under passing to the limit:

Theorem 1.2.12. *Let $S_\bullet$ be a projective system of B-schemes with affine transition maps indexed by a partially ordered set I, and let S denote $\varprojlim_\lambda S_\lambda$. Assume that for some λ the scheme S_λ is quasi-compact. Let $\{(\mathscr{F}_\lambda)_\lambda, u_{\lambda\mu})\}$ be a family of quasi-coherent sheaves on $S_\bullet$ with pullback $\mathscr{F}$ on S.*

(i) Assume that $\mathscr{F}_\lambda$ is finitely presented for some λ and let n be an integer. Then $\mathscr{F}$ is locally free of rank n if and only if there exists μ such that $\mathscr{F}_\mu$ is locally free of rank n.

(ii) Fix $\lambda \in I$ and let

$$\mathscr{F}_\lambda \to \mathscr{G}_\lambda \to \mathscr{H}_\lambda \to 0$$

be a sequence of quasi-coherent sheaves on S_λ. Assume that the sheaves $\mathscr{F}_\lambda$ and $\mathscr{G}_\lambda$ are of finite type and that $\mathscr{H}_\lambda$ is finitely presented. Then the sequence on S,

$$\mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0,$$

obtained by pullback to S is exact if and only if for some $\mu \geq \lambda$ the pullback of the sequence to S_μ is exact.

Proof. See [**26**, IV 8.5.5 and 8.5.6]. □

1.2.13. Similarly many properties of morphisms behave well with respect to passing to the limit. Fix as before a projective system $S_\bullet$ of B-schemes indexed by a partially ordered set I, and assume that there exists an element $\lambda_0 \in I$ such that $\lambda \geq \lambda_0$ for all $\lambda \in I$. Let S denote $\varprojlim_\lambda S_\lambda$.

For a scheme X_{λ_0} over S_{λ_0}, denote by $X_\lambda := X_{\lambda_0} \times_{S_{\lambda_0}} S_\lambda$, and let X denote $S \times_{S_{\lambda_0}} X_{\lambda_0}$. For $\lambda \geq \mu$ the map $S_\lambda \to S_\mu$ induces a morphism $X_\lambda \to X_\mu$.

If Y_{λ_0} is a second scheme over S_{λ_0}, then base change induces for every $\lambda \geq \mu$ a map

$$\mathrm{Hom}_{S_\mu}(X_\mu, Y_\mu) \to \mathrm{Hom}_{S_\lambda}(X_\lambda, Y_\lambda),$$

and by passing to the limit a map

$$e : \varinjlim_{\lambda} \mathrm{Hom}_{S_\lambda}(X_\lambda, Y_\lambda) \to \mathrm{Hom}_S(X, Y). \tag{1.2.13.1}$$

THEOREM 1.2.14. *(i) Suppose X_{λ_0} is quasi-compact and that Y_{λ_0} is locally of finite presentation over S_{λ_0}. Then the map* (1.2.13.1) *is an isomorphism.*

(ii) Assume S_{λ_0} is quasi-compact and quasi-separated. Then for any finitely presented S-scheme X there exists $\lambda \in I$ and a scheme X_λ over S_λ such that $X \simeq X_\lambda \times_{S_\lambda} S$.

PROOF. See [**26**, IV 8.8.2]. □

THEOREM 1.2.15. *Assume S_{λ_0} is quasi-compact and quasi-separated, and that X_{λ_0} and Y_{λ_0} are finitely presented S_{λ_0}-schemes. Let $f_{\lambda_0} : X_{\lambda_0} \to Y_{\lambda_0}$ be a morphism, and let P be one of the following properties of morphisms: an isomorphism, a monomorphism, an imbedding, a closed imbedding, an open imbedding, separated, surjective, radicial, affine, quasi-affine, finite, quasi-finite, proper, projective, quasi-projective.*

Then the base change $f : X \to Y$ has property P if and only if there exists $\lambda \in I$ such that the morphism $f_\lambda : X_\lambda \to Y_\lambda$ has property P.

PROOF. See [**26**, IV 8.10.5]. □

1.3. Étale and smooth morphisms

The basic reference for the development of the theory of smooth and étale morphisms, as presented in this section, is [**26**, IV, §17].

DEFINITION 1.3.1. Let $f : X \to Y$ be a morphism of schemes. We call f *formally smooth* (resp. *formally unramified*, *formally étale*) if for every affine Y-scheme $Y' \to Y$ and every closed imbedding $Y'_0 \hookrightarrow Y'$ defined by a nilpotent ideal, the map

$$\mathrm{Hom}_Y(Y', X) \to \mathrm{Hom}_Y(Y'_0, X) \tag{1.3.1.1}$$

is surjective (resp. injective, bijective). If f is also locally of finite presentation then f is called *smooth* (resp. *unramified*, *étale*).

REMARK 1.3.2. If $Y'_0 \hookrightarrow Y'$ is defined by an ideal $I \subset \mathscr{O}_{Y'}$ with $I^n = 0$ for some n, then setting $Y'_i \subset Y'$ equal to the closed subscheme defined by I^{i+1} we get a sequence of closed imbeddings of Y-schemes

$$Y'_0 \hookrightarrow Y'_1 \hookrightarrow \cdots \hookrightarrow Y'_{n-1} = Y',$$

with Y'_i defined in Y'_{i+1} by a square-zero ideal. This in turn gives a factorization of (1.3.1.1) as

$$\mathrm{Hom}_Y(Y', X) \to \mathrm{Hom}_Y(Y'_{n-2}, X) \to \cdots \to \mathrm{Hom}_Y(Y'_1, X) \to \mathrm{Hom}_Y(Y'_0, X).$$

From this it follows that a morphism $f : X \to Y$ is formally smooth (resp. formally unramified, formally étale) if and only if for every closed imbedding of Y-schemes $Y_0' \hookrightarrow Y'$ defined by a square-zero ideal the map (1.3.1.1) is surjective (resp. injective, bijective).

1.3.3. Recall that if S is a set and G is a group acting on S, then S is called a *G-torsor* if S is nonempty and the action of G is simply transitive.

One of the key basic properties of differentials is the following. For any commutative diagram of solid arrows of schemes

(1.3.3.1)
$$\begin{tikzcd} Y_0' \arrow[r, "y_0"] \arrow[d, hook, "i"] & X \arrow[d, "f"] \\ Y' \arrow[r, "a"] \arrow[ur, dashed] & Y, \end{tikzcd}$$

where i is a square-zero closed imbedding with ideal $\mathscr{I} \subset \mathscr{O}_{Y'}$, the set of dotted arrows filling in the diagram is either empty or a torsor under the group

$$\mathrm{Hom}_{Y_0'}(y_0^*\Omega^1_{X/Y}, \mathscr{I}).$$

This can be seen as follows (we leave some of the details as exercise 1.I). Let $\Delta : X \to X \times_Y X$ be the diagonal morphism, and consider the surjection of sheaves of rings on X:

$$\Delta^{-1}\mathscr{O}_{X\times_Y X} \to \mathscr{O}_X.$$

If J denotes the kernel of this map, then by definition [**41**, definition on page 175] we have $\Omega^1_{X/Y} \simeq J \otimes_{\Delta^{-1}\mathscr{O}_{X\times_Y X}} \mathscr{O}_X$ (which we henceforth abbreviate J/J^2) . Given two maps $\rho_1, \rho_2 : Y' \to X$ filling in (1.3.3.1) we obtain a commutative diagram

$$\begin{tikzcd} Y_0' \arrow[r, hook, "i"] \arrow[d, "y_0"] & Y' \arrow[d, "\rho := \rho_1 \times \rho_2"] \\ X \arrow[r, "\Delta"] & X \times_Y X. \end{tikzcd}$$

From this we obtain a commutative diagram of sheaves of rings on $|Y_0|$:

$$\begin{tikzcd} y_0^{-1}\Delta^{-1}\mathscr{O}_{X\times_Y X} \arrow[r] \arrow[d, "\rho"] & y_0^{-1}\mathscr{O}_X \arrow[d, "y_0"] \\ \mathscr{O}_{Y'} \arrow[r] & \mathscr{O}_{Y_0'}. \end{tikzcd}$$

Since the kernel of $\mathscr{O}_{Y'} \to \mathscr{O}_{Y_0'}$ is square-zero, the morphism ρ factors through the pullback to Y_0 of

$$\mathscr{P}^1_{X/Y} := \Delta^{-1}\mathscr{O}_{X\times_Y X}/J^2.$$

The pullback $p_1^* : \mathscr{O}_X \to \mathscr{P}^1_{X/Y}$ coming from the first projection $X \times_Y X \to X$ defines a section of the map $\mathscr{P}^1_{X/Y} \to \mathscr{O}_X$ so this gives a decomposition

$$\mathscr{P}^1_{X/Y} \simeq \mathscr{O}_X \oplus J/J^2.$$

The map $y_0^{-1}\mathscr{P}^1_{X/Y} \to \mathscr{O}_{Y'}$ therefore induces a map $y_0^{-1}J/J^2 \to \mathscr{I}$. We define

$$\rho_1 - \rho_2 \in \mathrm{Hom}_{Y_0'}(y_0^*\Omega^1_{X/Y}, \mathscr{I})$$

to be this map. By exercise 1.I this construction defines a torsorial action of $\mathrm{Hom}_{Y_0'}(y_0^*\Omega^1_{X/Y}, \mathscr{I})$ on the set of morphisms $Y' \to X$ filling in (1.3.3.1) if this set is nonempty.

PROPOSITION 1.3.4. *(i) If $f : X \to Y$ is smooth (resp. unramified, étale), and $g : Y' \to Y$ is any morphism then the base change $f' : X \times_Y Y' \to Y'$ is smooth (resp. unramified, étale).*

(ii) A composition of smooth (resp. unramified, étale) morphisms is smooth (resp. unramified, étale).

(iii) Suppose given a composition of morphisms locally of finite presentation

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

with gf and g smooth, and such that the map $f^*\Omega^1_{Y/Z} \to \Omega^1_{X/Z}$ is an isomorphism. Then f is étale.

PROOF. These follow immediately from the definitions and the discussion in 1.3.3, and are left as exercise 1.J. □

PROPOSITION 1.3.5. *A morphism of schemes $f : X \to Y$ is smooth if and only if it is locally of finite presentation and the following condition holds: For every commutative diagram of schemes*

(1.3.5.1)
$$\begin{tikzcd} Y_0' \arrow[r, "y_0"] \arrow[d, hook, "i"] & X \arrow[d, "f"] \\ Y' \arrow[r, "a"] & Y, \end{tikzcd}$$

where i is a closed imbedding defined by a square-zero ideal, there exists an open covering $Y' = \bigcup_i U_i'$ and morphisms $\rho_i : U_i' \to X$ over Y such that the restriction of ρ_i to $U_{i,0}' := U_i \times_{Y'} Y_0'$ is equal to the restriction of y_0.

PROOF. The 'only if' direction is immediate.

For the 'if' direction, fix a diagram (1.3.5.1) with Y' affine and assume given a covering $Y' = \bigcup_i U_i$ with each U_i also affine and morphisms $\rho_i : U_i \to X$ as in the proposition. We need to show that we can change our choices of ρ_i so that they agree on the overlaps $U_{ij} := U_i \cap U_j$.

For this, define elements

$$\epsilon_{ij} := \rho_i|_{U_{ij}} - \rho_j|_{U_{ij}} \in \mathrm{Hom}_{U_{ij}}(y_0^*\Omega^1_{X/Y}, \mathscr{J}),$$

where $\mathscr{J} \subset \mathscr{O}_{Y'}$ is the ideal of $Y_0' \subset Y'$. In other words, let ϵ_{ij} be the unique class which under the torsor action sends $\rho_j|_{U_{ij}}$ to $\rho_i|_{U_{ij}}$. For three indices i, j, k it follows immediately from the definition that on U_{ijk} we have

$$\epsilon_{ij} + \epsilon_{jk} = \epsilon_{ik}$$

so the elements ϵ_{ij} define a Čech cocycle with coefficients in the quasi-coherent sheaf $\mathscr{H}om(y_0^*\Omega^1_{X/Y}, \mathscr{J})$. Since Y_0' is affine we have

$$H^1(Y_0', \mathscr{H}om(y_0^*\Omega^1_{X/Y}, \mathscr{J})) = 0,$$

so this implies that there exists elements $\lambda_i \in \mathscr{H}om_{U_i}(y_0^*\Omega^1_{X/Y}|_{U_i}, \mathscr{J})$ such that

$$\epsilon_{ij} = \lambda_j|_{U_{ij}} - \lambda_i|_{U_{ij}}.$$

Setting $\rho_i' := \lambda_i * \rho_i$ we see that the maps ρ_i' agree on the overlaps U_{ij} and therefore glue together to give a global map $\rho : Y' \to X$ over Y filling in (1.3.5.1). □

PROPOSITION 1.3.6. *(i) Let $f : X \to Y$ be a smooth morphism of schemes. Then the sheaf $\Omega^1_{X/Y}$ is a locally free sheaf of finite rank on X.*

(ii) If $f : X \to Y$ is étale, then $\Omega^1_{X/Y} = 0$.

(iii) If $g : X \to Y$ is a smooth morphism, and $i : Z \hookrightarrow X$ is a locally finitely presented closed imbedding, then the composition $f := ig : Z \to Y$ is smooth if and only if the sequence

$$0 \to i^*\mathscr{I}_Z \to i^*\Omega^1_{X/Y} \to \Omega^1_{Z/Y} \to 0 \tag{1.3.6.1}$$

is exact and locally split, where $\mathscr{I}_Z$ denotes the ideal sheaf of Z in X.

PROOF. To prove (i), we may work locally on X and Y so we may assume that X and Y are affine, say

$$X = \mathrm{Spec}(B), \quad Y = \mathrm{Spec}(A).$$

We then need to show that the B-module $\Omega^1_{B/A}$ is projective.

For a B-module M, let $B[M]$ denote the B-algebra with underlying B-module the direct sum $B \oplus M$, and the multiplication given by

$$(b, m) \cdot (b', m') := (bb', bm' + b'm).$$

There is a surjection $\pi_M : B[M] \to B$ given by $(b, m) \mapsto b$, whose kernel is a square-zero ideal.

Consider a morphism of A-algebras $f : B \to B[M]$ such that the composite map

$$B \xrightarrow{f} B[M] \xrightarrow{\pi_M} B$$

is the identity map. We can then write $f(b) = (b, \partial(b))$ for a function $\partial : B \to M$. From the relations

$$(b+b', \partial(b)+\partial(b')) = (b, \partial(b))+(b', \partial(b')) = f(b)+f(b') = f(b+b') = (b+b', \partial(b+b')),$$

$$(bb', b\partial(b') + b'\partial(b)) = (b, \partial(b))(b', \partial(b')) = f(b)f(b') = f(bb') = (bb', \partial(bb')),$$

and

$$(a, \partial(a)) = f(a) = (a, 0), \quad a \in A,$$

we conclude that ∂ is an A-derivation $B \to M$. Conversely, given such a derivation ∂, we obtain a morphism of A-algebras $B \to B[M]$ which composes with π_M to the identity.

Suppose we are now given a diagram of B-modules

$$\begin{tikzcd} & \Omega^1_{B/A} \arrow[d, "\partial"] \\ M \arrow[r, two heads, "t"] & N, \end{tikzcd}$$

where t is surjective. If $f_0 : B \to B[N]$ denotes the map corresponding to ∂, then we obtain a commutative diagram of solid arrows

$$\begin{tikzcd} B[N] & B \arrow[l, "f_0"'] \arrow[dl, dashed] \\ B[M] \arrow[u, two heads, "t"] & A, \arrow[l] \arrow[u] \end{tikzcd}$$

where we also write $t : B[M] \to B[N]$ for the surjection defined by t. By the preceding discussion giving a dotted arrow as in the diagram is equivalent to giving a lifting of ∂,

$$\tilde{\partial} : \Omega^1_{B/A} \to M.$$

Now if we assume that $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is smooth, then such a lifting exists, and therefore $\Omega^1_{B/A}$ is projective implying (i).

From this discussion (ii) also follows, for if $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is étale, then the preceding discussion implies that for any B-module M we have

$$\mathrm{Hom}_B(\Omega^1_{B/A}, M) = 0,$$

which implies that $\Omega^1_{B/A} = 0$.

Finally, let us prove (iii). We always have an exact sequence

$$i^*\mathscr{I}_Z \to i^*\Omega^1_{X/Y} \to \Omega^1_{Z/Y} \to 0,$$

so it is equivalent to show that $Z \to Y$ is smooth if and only if the map $i^*\mathscr{I}_Z \to i^*\Omega^1_{X/Y}$ is injective and locally split. To prove this we may work locally on X and Y, so as above we may assume that $X = \mathrm{Spec}(B)$ and $Y = \mathrm{Spec}(A)$ are affine. Let $I \subset X$ be the ideal defining Z, and let R denote B/I so we have $Z = \mathrm{Spec}(R)$. We have to show that the map induced by $d : B \to \Omega^1_{B/A}$,

$$\bar{d} : I/I^2 \to \Omega^1_{B/A} \otimes_B R,$$

is injective and locally split if and only if $Z \to Y$ is smooth. Now observe that if $\bar{d}$ is injective and locally split, then $\Omega^1_{R/A}$ is a projective R-module and therefore in fact the map $\bar{d}$ is split over R. It therefore suffices to show that $Z \to Y$ is smooth if and only if $\bar{d}$ identifies I/I^2 with a direct summand of $\Omega^1_{B/A} \otimes_B R$.

The condition that I/I^2 is a direct summand is equivalent to the condition that for any R-module N the map

$$\mathrm{Hom}_R(\Omega^1_{B/A} \otimes R, N) \to \mathrm{Hom}_R(I/I^2, N) \tag{1.3.6.2}$$

is surjective (consider $N = I/I^2$). This map can be interpreted as follows. We have

$$\mathrm{Hom}_R(\Omega^1_{B/A} \otimes R, N) \simeq \mathrm{Hom}_B(\Omega^1_{B/A}, N),$$

so this set is in bijection with the set of A-algebra maps

$$B \to B[N]$$

which compose with the projection to B to the identity. Since N is an R-module, giving such a map is equivalent to giving an A-algebra map

$$f : B \to R[N]$$

whose composition with the projection to R is the quotient map. Now given such a map we get an induced map

$$I \to \mathrm{Ker}(R[N] \to R) \simeq N.$$

This map factors through I/I^2, and sending f to the resulting map $I/I^2 \to N$ is the map (1.3.6.2).

Now suppose that $Z \to Y$ is smooth. Let $j : \widetilde{Z} \hookrightarrow X$ denote the closed subscheme of X defined by I^2. We then have a closed imbedding $Z \hookrightarrow \widetilde{Z}$ defined by the square-zero ideal I/I^2. Now consider the commutative diagram of solid arrows

$$\begin{tikzcd} Z \arrow[r, equal] \arrow[d, hook] & Z \arrow[d] \\ \widetilde{Z} \arrow[r] \arrow[ur, dashed] & Y. \end{tikzcd}$$

Since $Z \to Y$ is smooth there exists a dotted arrow as in the diagram. Choosing one such morphism we get a homomorphism

$$R \to B/I^2$$

lifting the projection $B/I^2 \to B/I = R$. This homomorphism defines an isomorphism $R[I/I^2] \simeq B/I^2$. We therefore get a map

$$B \to R[I/I^2]$$

whose composition with $R[I/I^2] \to R$ is the quotient map, and such that the composite map

$$I \hookrightarrow B \to R[I/I^2]$$

is equal to the projection $I \to I/I^2$ followed by the inclusion $I/I^2 \hookrightarrow R[I/I^2]$.

This map corresponds (1.3.6.2) to a morphism $\Omega^1_{B/A} \otimes R \to I/I^2$ whose composition with $\bar{d}$ is the identity. This shows that if $Z \to Y$ is smooth then $\bar{d}$ is injective and locally split.

For the converse suppose $\bar{d}$ is injective and split, and consider a commutative diagram of solid arrows

$$\begin{tikzcd} \mathrm{Spec}(C_0) \arrow[r, "h_0"] \arrow[d] & Z \arrow[d] \\ \mathrm{Spec}(C) \arrow[r] \arrow[ur, dashed] & Y, \end{tikzcd}$$

where $C \to C_0$ is surjective with square-zero kernel J.

Since $X \to Y$ is smooth, there exists a morphism $\tilde{h} : \mathrm{Spec}(C) \to X$ such that the composition

$$\mathrm{Spec}(C_0) \hookrightarrow \mathrm{Spec}(C) \to X$$

is the composition of h_0 with $Z \hookrightarrow X$. We have to show that we can find $\tilde{h}$ that factors through Z. The map $\tilde{h}$ defines a map

$$I \hookrightarrow B \to C$$

which gives a map $\kappa : I/I^2 \to J$. For the map $\tilde{h}$ to factor through Z it suffices that this map is zero. Now if this map is not zero, then choose an element in

$$\mathrm{Hom}_R(\Omega^1_{B/A} \otimes R, J)$$

mapping to κ under (1.3.6.2). Changing our choice of $\tilde{h}$ by the negative of this map κ (under the torsorial action discussed in 1.3.3), we obtain a map $\tilde{h}$ which factors through Z. $\square$

1.3.7. Part (iii) of the proposition gives a useful way to describe smooth and étale morphisms locally.

Let

$$f : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$$

be a smooth morphism of affine schemes, and let $x \in \mathrm{Spec}(B)$ be a point. Assume that $\Omega^1_{B/A}$ is free of finite rank r (we can always arrange this by shrinking on $\mathrm{Spec}(B)$). Choose a surjection

$$A[x_1, \dots, x_n] \to B$$

and let I denote the kernel. The sequence

$$0 \longrightarrow I/I^2 \xrightarrow{\bar{d}} \Omega^1_{A[x_1,\dots,x_n]/A} \otimes B \longrightarrow \Omega^1_{B/A} \longrightarrow 0 \tag{1.3.7.1}$$

is exact by 1.3.6 (iii), so after possibly shrinking some more to a neighborhood of x there exists $n-r$ elements $f_1, \dots, f_{n-r} \in I$ such that $f_1, \dots, f_{n-r}$ map to a basis for I/I^2. We have

$$\bar{d}(f_j) = \sum_{i=1}^n \frac{\partial f_j}{\partial x_i} dx_i. \tag{1.3.7.2}$$

Since the classes of the f_j form a basis for I/I^2 and the differentials dx_i form a basis for $\Omega^1_{A[x_1,\dots,x_n]/A}$, the condition that $\bar{d}$ is a split injection is equivalent to the condition that the $(n-r) \times (n-r)$ minors of the matrix

$$(\partial f_j/\partial x_i) \in \mathbb{M}_{n \times (n-r)}(B) \tag{1.3.7.3}$$

generate the unit ideal in B. By Nakayama's lemma, we can therefore find an element $g \in A[x_1, \dots, x_r]$ which is invertible at x such that the $(n-r) \times (n-r)$ minors of the matrix $(\partial f_j/\partial x_i)$ generate the unit ideal in $A[x_1, \dots, x_r][1/g]$ and such that the ideal in $A[x_1, \dots, x_r][1/g]$ generated by I is equal to the ideal generated by the elements $f_1, \dots, f_{n-r}$. We have shown the 'only if' direction of the following:

PROPOSITION 1.3.8. *Let $f : X \to Y$ be a morphism locally of finite presentation. Then f is smooth if and only if for every point $x \in X$ there exists affine neighborhoods*

$$x \in \mathrm{Spec}(B) \subset X, \quad f(x) \in \mathrm{Spec}(A) \subset Y$$

with $f(\mathrm{Spec}(B)) \subset \mathrm{Spec}(A)$, and such that

$$B \simeq (A[x_1, \dots, x_n]/(f_1, \dots, f_s))[1/g] \tag{1.3.8.1}$$

for some $f_1, \dots, f_s, g \in A[x_1, \dots, x_n]$ and $s \leq n$ for which the $s \times s$-minors of the matrix (1.3.7.3) *generate the unit ideal in B. The morphism f is étale if and only if for any point $x \in X$ we can find an affine neighborhood* (1.3.8.1) *as above with $n = s$.*

PROOF. For the converse suppose we have an isomorphism (1.3.8.1). The condition on the minors of (1.3.7.3) insures that the sequence (1.3.7.1) is exact, and locally split. By 1.3.6 (iii) this implies that $f : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is smooth. Finally, note that from the local description (1.3.8.1), the rank of $\Omega^1_{X/Y}$ is equal to

$n - s$, so $\Omega^1_{X/Y} = 0$ if and only if for each presentation (1.3.8.1) we have $n = s$. This implies the statement describing étale morphisms. □

1.3.9. Suppose now that $f : X \to Y$ is a smooth morphism and let $x \in X$ be a point. We know that the sheaf $\Omega^1_{X/Y}$ is locally free of finite rank. It follows that there exists a neighborhood $x \in U \subset X$ and sections $f_1, \ldots, f_n \in \Gamma(U, \mathscr{O}_U)$ such that the differentials $df_1, \ldots, df_n$ form a basis for $\Omega^1_{X/Y}|_U$. Let

$$\boldsymbol{f} : U \to \mathbb{A}^n_Y$$

be the morphism defined by $f_1, \ldots, f_n$. Since $\boldsymbol{f}^*\Omega^1_{\mathbb{A}^n_Y/Y} \to \Omega^1_{X/Y}|_U$ is an isomorphism by construction, it follows from (1.3.4 (iii)) that $\boldsymbol{f}$ is étale.

Corollary 1.3.10. *Let $f : X \to Y$ be a smooth morphism, and let $x \in X$ be a point. Then there exists an étale morphism $\pi : Y' \to Y$ with image containing $f(x)$ and a morphism $s : Y' \to X$ such that $f \circ s = \pi$.*

Proof. After shrinking on X we may assume that there exists an étale morphism $\boldsymbol{f} : X \to \mathbb{A}^n_Y$ for some n. Let Y' denote the fiber product of the diagram

$$\begin{tikzcd} & X \arrow[d, "\boldsymbol{f}"] \\ Y \arrow[r, "0"] & \mathbb{A}^n_Y. \end{tikzcd}$$

Then Y' fits into a diagram

$$\begin{tikzcd} Y' \arrow[r, "s"] \arrow[d] & X \\ Y & \end{tikzcd}$$

as desired. □

Corollary 1.3.11. *Let A be a ring, and let $f \in A[X]$ be a monic polynomial with derivative f'. Then the finite ring homomorphism*

$$A \to A[X]/(f)$$

is étale if and only if f' maps to a unit in $A[X]/(f)$.

Proof. By 1.3.6 (ii) applied to the closed imbedding

$$\mathrm{Spec}(A[X]/(f)) \hookrightarrow \mathrm{Spec}(A[X])$$

over $\mathrm{Spec}(A)$, it follows that $\mathrm{Spec}(A[X]/(f)) \to \mathrm{Spec}(A)$ is étale if and only if the map

$$\cdot f' : A[X]/(f) \to A[X]/(f)$$

is an isomorphism. □

Example 1.3.12. If K is a field and $L = K[X]/(f)$ is a field extension defined by an irreducible polynomial f, then L/K is étale if and only if $f' \neq 0$.

1.3.13. An important consequence of 1.3.8, which we now explain, is the so-called invariance of the étale site under infinitesimal thickenings. Let $i : S_0 \hookrightarrow S$ be a closed imbedding defined by a nilpotent ideal. Let $\mathrm{Et}(S)$ (resp. $\mathrm{Et}(S_0)$) denote the category whose objects are étale S-schemes (resp. étale S_0-schemes) and whose morphisms are S-morphisms (resp. S_0-morphisms). There is a functor

$$F : \mathrm{Et}(S) \to \mathrm{Et}(S_0), \quad (Z \to S) \mapsto (Z \times_S S_0 \to S_0). \tag{1.3.13.1}$$

THEOREM 1.3.14 ([**36**, Exposé I, 8.3]). *The functor F is an equivalence of categories.*

PROOF. The full faithfulness of F can be seen as follows. Suppose $Z_1 \to S$ and $Z_2 \to S$ are two étale morphisms. Let $Z_{j,0}$ $(j = 1, 2)$ denote the reduction to S_0 of Z_j, and suppose given an isomorphism

$$\sigma_0 : Z_{1,0} \to Z_{2,0}.$$

We then obtain a commutative diagram of solid arrows:

$$\begin{tikzcd} Z_{1,0} \arrow[r, "\sigma_0"] \arrow[d] & Z_{2,0} \arrow[r, hook] & Z_2 \arrow[d] \\ Z_1 \arrow[urr, dashed] \arrow[rr] & & S. \end{tikzcd}$$

Now since $Z_2 \to S$ is étale and $Z_{1,0} \hookrightarrow Z_1$ is defined by a nilpotent ideal, there exists locally on Z_1 a unique dotted arrow filling in the diagram. By the uniqueness these locally defined morphisms agree on overlaps and therefore there exists globally on Z_1 a unique dotted arrow $\sigma : Z_1 \to Z_2$ filling in the diagram. By the same argument applied to σ_0^{-1} there exists a unique morphism $\tau : Z_2 \to Z_1$ reducing to σ_0^{-1}. The morphisms $\tau\circ\sigma$ and $\sigma\circ\tau$ must be the identities as they reduce to identity morphisms over S_0.

It remains to show the essential surjectivity. For this observe first that if $Z \to S$ is an étale morphism with reduction $Z_0 \to S_0$, then $Z_0 \hookrightarrow Z$ is defined by a nilpotent ideal so the map on topological spaces $|Z_0| \to |Z|$ is an isomorphism. Therefore any open subset $U_0 \subset Z_0$ lifts uniquely to an open subset $U \subset Z$.

Now fix an étale morphism $Z_0 \to S_0$. We then construct a lifting $Z \to S$ of Z_0 as follows. First note that the problem of finding a lifting of Z_0 is local on Z_0 in the following sense. If $Z_0 = \bigcup_{i\in I} U_0^{(i)}$ is an open covering and for each i we are given an étale lifting $U^{(i)} \to S$ of $U_0^{(i)}$, then the $U^{(i)}$ glue together to give a lifting Z of Z_0. Indeed, for any two indices $i, j \in I$ the open subset $U_0^{(ij)} := U_0^{(i)} \cap U_0^{(j)}$ lifts by the preceding remark to unique open subsets

$$U^{(ij)} \subset U^{(i)}, \quad U^{(ji)} \subset U^{(j)}.$$

Moreover, by the uniqueness of liftings already shown, there is a unique isomorphism $\varphi_{ij} : U^{(ij)} \to U^{(ji)}$ reducing to the identity on $U_0^{(ij)}$. Furthermore, the uniqueness of liftings of morphisms implies that the assumption in the gluing lemma [**41**, Chapter II, exercise 2.12] are satisfied, so we get the desired lifting Z by gluing together the $U^{(i)}$ along the $U^{(ij)}$.

It remains to construct local liftings. By 1.3.8, we may therefore assume that $S = \mathrm{Spec}(A)$ for some ring A, $S_0 = \mathrm{Spec}(A_0)$ with $A_0 = A/I$ for some nilpotent ideal I, and $Z_0 = \mathrm{Spec}(B_0)$, where

$$B_0 = (A_0[X_1, \ldots, X_n]/(f_1, \ldots, f_n))[1/g],$$

for polynomials $f_1, \ldots, f_n, g \in A_0[X_1, \ldots, X_n]$ such that the determinant

$$\det(\partial f_i/\partial_{X_j})$$

is invertible in B_0. We then get a lifting of B_0 simply by lifting the polynomials $f_1, \dots, f_n, g$ to polynomials $\tilde{f}_1 \dots, \tilde{f}_n, \tilde{g} \in A[X_1, \dots, X_n]$ and setting

$$B := (A[X_1, \dots, X_n]/(\tilde{f}_1, \dots, \tilde{f}_n))_{\tilde{g}},$$

which is an étale A-algebra by 1.3.8. □

1.3.15. Let X be a scheme. A *geometric point of* X is a morphism $\bar{x} : \mathrm{Spec}(k) \to X$, where k is a separably closed field. An *étale neighborhood* of a geometric point $\bar{x}$ is a commutative diagram

$$\begin{tikzcd} & U \arrow[d, "b"] \\ \mathrm{Spec}(k) \arrow[ur, "u"] \arrow[r, "\bar{x}"] & X, \end{tikzcd}$$

where the morphism b is étale. Given two étale neighborhoods $(i = 1, 2)$,

$$\begin{tikzcd} & U_i \arrow[d, "b_i"] \\ \mathrm{Spec}(k) \arrow[ur, "u_i"] \arrow[r, "\bar{x}"] & X, \end{tikzcd}$$

a morphism between them is an X-morphism $g : U_1 \to U_2$ such that $u_2 = g \circ u_1$ and $b_1 = b_2 \circ g$. We therefore get a category $I_{\bar{x}}$ of étale neighborhoods of $\bar{x}$, which is filtering.

Define

$$\mathscr{O}_{X,\bar{x}} := \varinjlim_{(U,u) \in I_{\bar{x}}} \Gamma(U, \mathscr{O}_U).$$

This as an $\mathscr{O}_{X,x}$-algebra, called the *strict henselization of* X *at* $\bar{x}$.

PROPOSITION 1.3.16. *The ring* $\mathscr{O}_{X,\bar{x}}$ *is a local henselian ring with separably closed residue field.*

PROOF. There is a natural map

$$\mathscr{O}_{X,\bar{x}} \to k$$

induced by the maps

$$u^* : \Gamma(U, \mathscr{O}_U) \to k$$

by passing to the limit. The kernel of this map is a maximal ideal. Indeed, if $f \in \mathscr{O}_{X,\bar{x}}$ is an element whose image in k is nonzero, then we can find an object $(U, u) \in I_{\bar{x}}$ and an element $f_U \in \Gamma(U, \mathscr{O}_U)$ mapping to f in $\mathscr{O}_{X,\bar{x}}$. Let $V \subset U$ be the open subset where f_U is invertible. Since f_U maps to a nonzero element in k, the map u factors through a morphism $v : \mathrm{Spec}(k) \to V$ and (V, v) is also an étale neighborhood of $\bar{x}$. It follows that $f \in \mathscr{O}_{X,\bar{x}}$ is a unit, which implies that the image of $\mathscr{O}_{X,\bar{x}}$ in k is a field.

To show that $\mathscr{O}_{X,\bar{x}}$ is Henselian with separably closed residue field, it suffices to show that if $F \in \mathscr{O}_{X,\bar{x}}[T]$ is a polynomial and $b_0 \in k$ is a root of F in k such that $F'(b_0) \neq 0$, then b_0 is the image of a root of F in $\mathscr{O}_{X,\bar{x}}$ (for this characterization of Henselian rings see, for example, [**28**, Chapter 7]). For this let $(U, u) \in I_{\bar{x}}$ be an étale neighborhood such that the finitely many coefficients of F are in the image of $\Gamma(U, \mathscr{O}_U)$, and let $F_U \in \Gamma(U, \mathscr{O}_U)[T]$ be a polynomial mapping to F. Then

$$Z := \underline{\mathrm{Spec}}_U(\mathscr{O}_U[T]/F_U)_{F'_U}$$

is an étale U-scheme and the root b_0 defines a morphism $z : \mathrm{Spec}(k) \to Z$ over u, making (Z, z) an étale neighborhood of $\bar{x}$. The image of $T \in \Gamma(Z, \mathscr{O}_Z)$ in $\mathscr{O}_{X,\bar{x}}$ is then a root of F mapping to b_0 as desired.

$\square$

COROLLARY 1.3.17. *Let $f : Y \to X$ be a finite morphism of schemes, and let $\bar{x} : \mathrm{Spec}(k) \to X$ be a geometric point. Then the map on sets of connected components*

$$\pi_0(Y \times_X \mathrm{Spec}(\mathscr{O}_{X,\bar{x}})) \to \pi_0(Y \times_{X,\bar{x}} \mathrm{Spec}(k))$$

is a bijection.

PROOF. This follows from 1.3.16 and [**28**, 7.5]. $\square$

1.4. Schemes as functors

1.4.1. While not strictly necessary for the rest of the book, we review in this section how to characterize schemes as functors with certain properties (this characterization can for example be found in [**29**, VI-14]). This point of view is the starting point for the subsequent definitions of algebraic spaces and stacks. The basic idea is to start with the notion of affine scheme, and iteratively define more general objects. We are not advocating this as a preferred alternative to the usual definition of scheme, but the results of this section serve as a warmup for subsequent constructions. The end result is that for an affine base scheme S we have a hierarchy of categories of geometric objects

$$\begin{array}{c} (\text{affine } S\text{-schemes}) \\ \big\downarrow \\ (\text{separated } S\text{-schemes}) \\ \big\downarrow \\ (S\text{-schemes}) \\ \big\downarrow \\ (S\text{-algebraic spaces}) \\ \big\downarrow \\ (\text{algebraic stacks over } S). \end{array}$$

1.4.2. Let $\mathscr{C}$ be a category. Recall the Yoneda imbedding A.2.2

$$h : \mathscr{C} \to \mathrm{Fun}(\mathscr{C}^{\mathrm{op}}, \mathrm{Set})$$

which sends an object $X \in \mathscr{C}$ to the functor h_X given by

$$Y \mapsto \mathrm{Hom}_{\mathscr{C}}(Y, X).$$

DEFINITION 1.4.3. (i) A functor $F : \mathscr{C}^{\mathrm{op}} \to \mathrm{Set}$ is *representable* if $F \simeq h_X$ for some $C \in \mathscr{C}$.

(ii) For two functors $F, G : \mathscr{C}^{\mathrm{op}} \to \mathrm{Set}$ a morphism of functors $f : F \to G$ is called *relatively representable* if for every $X \in \mathscr{C}$, and for every $g : h_X \to G$, the fiber product $h_X \times_G F : \mathscr{C}^{\mathrm{op}} \to \mathrm{Set}$ is representable.

REMARK 1.4.4. By Yoneda's lemma, for any object $X \in \mathscr{C}$ the map of sets

$$(\text{morphisms of functors } g : h_X \to G) \to G(X), \quad g \mapsto g(\mathrm{id}_X : X \to X)$$

is a bijection, so we will often think of $g : h_X \to G$ as an element of $G(X)$.

REMARK 1.4.5. For $X \in \mathscr{C}$ and $F : \mathscr{C}^{\mathrm{op}} \to \mathrm{Set}$ we often abusively denote a morphism of functors $h_X \to F$ simply by $X \to F$, identifying X with its image under the Yoneda imbedding.

For the rest of this section we take $\mathscr{C} = \mathrm{Aff}_S$ to be the category of affine schemes over some fixed affine scheme S. Any S-scheme X defines a functor

$$h_X : \mathscr{C}^{\mathrm{op}} \to \mathrm{Set}, \quad \mathrm{Spec}(A) \mapsto \mathrm{Hom}_S(\mathrm{Spec}(A), X).$$

In the case when X is an affine S-scheme this is just the functor associated to X by the Yoneda imbedding.

DEFINITION 1.4.6. A morphism of functors $f : F \to G$ is an *affine open (resp. closed) imbedding* if the following conditions hold:

(i) f is relatively representable.
(ii) For all $X \in \mathrm{Aff}_S$ and $g : h_X \to G$, the map $F \times_G h_X \to h_X$ is an open (resp. closed) imbedding.

REMARK 1.4.7. Note that here we are working with the category of affine S-schemes, so the condition in (i) that f is relatively representable means that for every affine S-scheme X and $g : h_X \to G$ the fiber product $F \times_G h_X$ is isomorphic to h_Y for an affine S-scheme Y.

REMARK 1.4.8. Note that in (ii), both the functors $F \times_G h_X$ and h_X are representable functors, so it makes sense to say that a morphism between them is an open or closed imbedding.

DEFINITION 1.4.9. A *big Zariski sheaf* on Aff_S is a functor

$$F : \mathrm{Aff}_S^{\mathrm{op}} \to \mathrm{Set}$$

such that for any $U \in \mathrm{Aff}_S$ and open covering $\{U_i\}_{i \in I}$ of U by affine open subschemes $U_i \hookrightarrow U$, the sequence of sets

$$F(U) \to \prod_{i \in I} F(U_i) \rightrightarrows \prod_{(i,j) \in I \times I} F(U_i \cap U_j)$$

is exact.

A morphism of big Zariski sheaves

$$g : F \to G$$

is called *surjective* if for any $U \in \mathrm{Aff}_S$ and $u \in G(U)$ there exists an open covering $U = \bigcup_{i \in I} U_i$ of U such that $u|_{U_i} \in G(U_i)$ is in the image of $F(U_i)$ for every $i \in I$.

REMARK 1.4.10. A map of Zariski sheaves $g : F \to G$ is surjective in the above sense, if and only if g is an epimorphism in the category of big Zariski sheaves on Aff_S (this is exercise 1.K).

PROPOSITION 1.4.11. *A functor $F : \mathrm{Aff}_S^{\mathrm{op}} \to \mathrm{Set}$ is representable by a separated S-scheme if and only if the following hold:*

(i) *F is a big Zariski sheaf.*
(ii) *The diagonal morphism $\Delta : F \to F \times F$ is an affine closed imbedding.*

(iii) *There exists a family of objects* $\{X_i\}$ *in* Aff_S *and morphisms* $\pi_i : h_{X_i} \to F$ *which are affine open imbeddings and such that the map of Zariski sheaves* $\coprod_i h_{X_i} \to F$ *is surjective.*

Moreover, the functor

(1.4.11.1)
$$h_- : (\textit{separated } S\textit{-schemes}) \to (\textit{functors } F : \mathrm{Aff}_S^{\mathrm{op}} \to \mathrm{Set} \textit{ satisfying (i)-(iii)})$$

is an equivalence of categories.

Proof. We start by showing that if X is a separated S-scheme and F denotes h_X, then F satisfies conditions (i)–(iii).

For (i), let $U \in \mathrm{Aff}_S$ be an object and $U = \bigcup_i U_i$ an open covering. Then statement (i) is equivalent to the statement that giving a morphism of schemes

$$f : U \to X$$

is equivalent to giving a collection of morphisms

$$\{f_i : U_i \to X\}$$

such that for every $i, j \in I$ the restriction $f_i|_{U_i \cap U_j}$ and $f_j|_{U_i \cap U_j}$ are equal, which is immediate.

For (ii), note that the Yoneda imbedding commutes with products, so $h_X \times h_X \cong h_{X \times X}$ and the diagonal morphism

$$\Delta : h_X \to h_X \times h_X$$

is identified with the morphism

$$h_X \to h_{X \times_S X}$$

induced by the diagonal morphism of schemes

$$X \to X \times_S X,$$

which is a closed imbedding since X/S is separated.

To verify property (iii), let $X = \bigcup X_i$ be an open covering by affines, and let

$$\pi_i : h_{X_i} \to h_X$$

be the map induced by the inclusion $X_i \hookrightarrow X$. For any $T \in \mathrm{Aff}_S$ and morphism $h_T \to h_X$ corresponding to a morphism of schemes $f : T \to X$ the fiber product

$$h_T \times_{h_X} h_{X_i}$$

is represented by $f^{-1}(X_i)$ since the Yoneda imbedding commutes with fiber products. Since X/S is separated and S is affine, $f^{-1}(X_i)$ is an affine open subset of T, and therefore π_i is an affine open imbedding.

The statement that the map

$$\coprod_i h_{X_i} \to F$$

is a surjective map of big Zariski sheaves amounts to the statement that given a morphism of schemes $f : T \to X$ with $T \in \mathrm{Aff}_S$ there exists an open covering $T = \bigcup_{i \in I} T_i$ such that $f|_{T_i}$ factors through X_i for some i. This is clear as we can just take $T_i = f^{-1}(X_i)$.

Next, suppose given a functor F satisfying (i)–(iii), and choose morphisms $\pi_i : h_{X_i} \to F$ as in (iii). For every i and j, consider the fiber product of the diagram

$$\begin{tikzcd} & h_{X_i} \arrow[d, "\pi_i"] \\ h_{X_j} \arrow[r, "\pi_j"] & F. \end{tikzcd}$$

Since the maps π_i are representable by open imbeddings this fiber product is representable by an affine scheme V_{ij}, and we get open imbeddings

$$\begin{tikzcd} V_{ij} \arrow[r, hook] \arrow[d, hook] & X_i \\ X_j. & \end{tikzcd}$$

The isomorphism of functors

$$h_{X_j} \times_F h_{X_i} \to h_{X_i} \times_F h_{X_j}$$

obtained by switching the factors induces an isomorphism

$$\varphi_{ij} : V_{ij} \to V_{ji}.$$

This enables us to glue to the schemes X_i together along the subschemes V_{ij}. Indeed, it follows immediately from the construction that the following two conditions hold, so we can apply [**41**, Chapter II, exercise 2.12]:

(1) For any two indices i and j we have $\varphi_{ij} = \varphi_{ji}^{-1}$.
(2) For any three indices i, j, k we have

$$\varphi_{ij}(V_{ij} \cap V_{ik}) = V_{ji} \cap V_{jk},$$

and $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$ on $V_{ij} \cap V_{ik}$.

Let X denote the scheme obtained by gluing, and let G denote $\coprod_i h_{X_i}$. Notice that the fiber product

$$G \times_{h_X} G$$

is isomorphic to

$$\coprod_{i,j} h_{V_{ij}}.$$

This implies that the surjection of Zariski sheaves

$$G \twoheadrightarrow F$$

factors through an isomorphism $h_X \simeq F$.

To also prove the last statement of 1.4.11, it remains to see that the functor (1.4.11.1) is fully faithful. Let X and Y be two separated schemes, and let h_X and h_Y denote the corresponding functors on Aff_S. Choose an open covering $X = \bigcup_i U_i$ with each U_i affine, and let U_{ij} denote the intersection $U_i \cap U_j$ which is again an affine scheme since X is S-separated. We then have a coequalizer diagram of big Zariski sheaves

$$\coprod_{i,j} h_{U_{ij}} \rightrightarrows \coprod_i h_{U_i} \to h_X$$

and therefore we have

$$\mathrm{Hom}(h_X, h_Y) = \mathrm{Eq}(\prod_i h_Y(U_i) \rightrightarrows \prod_{i,j} h_Y(U_{ij})).$$

Since the map

$$\mathrm{Hom}(X, Y) \to \mathrm{Eq}(\prod_i h_Y(U_i) \rightrightarrows \prod_{i,j} h_Y(U_{ij}))$$

is an isomorphism, we obtain the full faithfulness. □

1.4.12. Now that we have the notion of a separated scheme defined purely functorially we can repeat the argument to get an alternate definition of scheme defined purely in terms of functors.

Let

$$F, G : \mathrm{Aff}_S^{\mathrm{op}} \to \mathrm{Set}$$

be two functors. We say that a morphism of functors $\epsilon : F \to G$ is *representable by separated schemes* if for any $U \in \mathrm{Aff}_S$ and morphism $h_U \to G$ the fiber product

$$F \times_G h_U \to h_U$$

is a separated U-scheme (which is a notion defined purely functorially using 1.4.11). Repeating the above argument one then obtains the following:

Proposition 1.4.13. *A functor* $F : \mathrm{Aff}_S^{\mathrm{op}} \to \mathrm{Set}$ *is representable by a* S*-scheme if and only if the following hold:*

(i)' *F is a big Zariski sheaf.*
(ii)' *The diagonal morphism $\Delta : F \to F \times F$ is representable by separated schemes.*
(iii)' *There exists a family of objects $\{X_i\}$ in* Aff_S *and morphisms $\pi_i : h_{X_i} \to F$ which are open imbeddings and such that the map of Zariski sheaves $\coprod_i h_{X_i} \to F$ is surjective.*

Moreover, the functor

$$h_- : (S\textit{-schemes}) \to (\textit{functors } F : \mathrm{Aff}_S^{\mathrm{op}} \to \textit{Set satisfying (i)'–(iii)'})$$

is an equivalence of categories.

Proof. Omitted. □

Example 1.4.14. To get used to the above point of view, let us verify the conditions (i)–(iii) for projective space (over $\mathbb{Z}$).

Let Aff denote the category of affine schemes, and let

$$\mathscr{P}^n : \mathrm{Aff}^{\mathrm{op}} \to \mathrm{Set}$$

denote the functor which sends an affine scheme $\mathrm{Spec}(R)$ to the set of isomorphism classes of surjections of R-modules

$$R^{n+1} \xrightarrow{\pi} L,$$

where L is a projective R-module of rank 1. Equivalently, $\mathscr{P}^n(\mathrm{Spec}(R))$ is the set of isomorphism classes of surjections of sheaves

$$\mathscr{O}_{\mathrm{Spec}(R)}^{n+1} \twoheadrightarrow \mathscr{L},$$

with $\mathscr{L}$ locally free of rank 1.

Condition (i) for this functor is immediate (this essentially amounts to the statement that sheaves and morphisms between them can be constructed locally).

To verify condition (ii), let $\mathrm{Spec}(R)$ be an affine scheme, and suppose given two elements of $\mathscr{P}^n(\mathrm{Spec}(R))$ that

$$\mathscr{O}^{n+1}_{\mathrm{Spec}(R)} \xrightarrow{\pi_j} \mathscr{L}_j, \quad j = 1, 2.$$

The fiber product W of the diagram

$$\begin{tikzcd} & \mathrm{Spec}(R) \arrow[d] \\ \mathscr{P}^n \arrow[r, "\Delta"] & \mathscr{P}^n \times \mathscr{P}^n \end{tikzcd}$$

is the functor on the category of affine R-schemes which to any such scheme $f : \mathrm{Spec}(B) \to \mathrm{Spec}(R)$ associates the unital set if

$$\mathrm{Ker}(f^*\pi_1) = \mathrm{Ker}(f^*\pi_2),$$

and the empty set otherwise. Let $K_1 \subset \mathscr{O}^{n+1}_{\mathrm{Spec}(R)}$ denote the locally free sheaf $\mathrm{Ker}(\pi_1)$, and let $\gamma : K_1 \to \mathscr{L}_2$ denote the composite morphism

$$K_1 \hookrightarrow \mathscr{O}^{n+1}_{\mathrm{Spec}(R)} \xrightarrow{\pi_2} \mathscr{L}_2.$$

Let $K_1^\vee$ denote the dual of K_1 so we can also think of γ as a global section of $K_1^\vee \otimes \mathscr{L}_2$. The functor W is then the functor which to $f : \mathrm{Spec}(B) \to \mathrm{Spec}(R)$ associates the unital set if $f^*\gamma = 0$, and the empty set otherwise. Now the condition that a section of a vector bundle is zero is representable by a closed subscheme (locally if we trivialize the vector bundle and write the section as a vector, then the representing closed subscheme is the closed subscheme defined by the entries of the vector). This verifies (ii).

Finally, let us verify (iii). For $i = 0, \ldots, n$ let $\sigma_i : \mathscr{U}_i \subset \mathscr{P}^n$ denote the subfunctor whose value on an affine scheme $\mathrm{Spec}(R)$ is the set of quotients

$$\pi : \mathscr{O}^{n+1}_{\mathrm{Spec}(R)} \to \mathscr{L} \tag{1.4.14.1}$$

such that if $j_i : \mathscr{O}_{\mathrm{Spec}(R)} \hookrightarrow \mathscr{O}^{n+1}_{\mathrm{Spec}(R)}$ is the inclusion of the i-th component, then the composite morphism

$$\mathscr{O}_{\mathrm{Spec}(R)} \xhookrightarrow{j_i} \mathscr{O}^{n+1}_{\mathrm{Spec}(R)} \xrightarrow{\pi} \mathscr{L} \tag{1.4.14.2}$$

is surjective. Note that if if this is the case then this composite map is actually an isomorphism. Giving an element (1.4.14.1) of $\mathscr{U}_i(\mathrm{Spec}(R))$ is therefore equivalent to giving elements

$$x_s \in R, \quad s = 0, \ldots, n, \quad s \neq i,$$

the map π being the map

$$\mathscr{O}^{n+1}_{\mathrm{Spec}(R)} \to \mathscr{O}_{\mathrm{Spec}(R)}$$

whose composition with $j_s : \mathscr{O}_{\mathrm{Spec}(R)} \hookrightarrow \mathscr{O}^{n+1}_{\mathrm{Spec}(R)}$ is equal to multiplication by x_s if $s \neq i$ and the identity if $s = i$. In particular, we find that $\mathscr{U}_i$ is represented by $\mathbb{A}^n$.

The map σ_i is representable by open imbeddings. Indeed, given any object (1.4.14.1), the fiber product of the resulting diagram

$$\begin{tikzcd} & \mathrm{Spec}(R) \arrow[d] \\ \mathscr{U}_i \arrow[r, hook] & \mathscr{P}^n \end{tikzcd}$$

is represented by the complement in $\mathrm{Spec}(R)$ of the zero locus of the section of $\mathscr{L}$ defined by the composition (1.4.14.2). It is also clear that the map

$$\coprod_{i=0}^{n} \mathscr{U}_i \to \mathscr{P}^n$$

is a surjective map of big Zariski sheaves and therefore (iii) holds. This is the functorial interpretation of the standard open covering of $\mathbb{P}^n$.

1.5. Hilbert and Quot schemes

The basic reference for this section is [**35**, part IV].

1.5.1. Let $f : X \to S$ be a finitely presented separated morphism of schemes, let L be a relatively ample invertible sheaf on X, let $P \in \mathbb{Q}[z]$ be a polynomial, and let F be a quasi-coherent locally finitely presented sheaf on X.

Remark 1.5.2. If S is noetherian, then f finitely presented is equivalent to f being of finite type, and F quasi-coherent and locally finitely presented is equivalent to F being coherent.

1.5.3. Define

$$\underline{\mathrm{Quot}}^P(F/X/S) : (\mathrm{Sch}/S)^{\mathrm{op}} \to \mathrm{Set}$$

to be the functor which to any $S' \to S$ associates the set of isomorphism classes of quotients of quasi-coherent sheaves

$$F_{S'} \twoheadrightarrow G,$$

where $F_{S'}$ denotes the pullback of F to $X_{S'} := X \times_S S'$, and G is a locally finitely presented quasi-coherent sheaf on $X_{S'}$ whose support is proper over S' and such that for every point $s' \in S'$ the Hilbert polynomial of G restricted to the fiber of $X_{S'}$ over s' is equal to P.

Theorem 1.5.4. *The functor $\underline{\mathrm{Quot}}^P(F/X/S)$ is representable by a scheme quasi-projective over S (projective if X/S is proper).*

Proof. See [**35**, part IV, Théorème 3.2]. □

Remark 1.5.5. If $F = \mathscr{O}_X$, then $\underline{\mathrm{Quot}}^P(F/X/S)$ is called the *Hilbert scheme* of X/S and will be denoted $\underline{\mathrm{Hilb}}^P_{X/S}$.

Theorem 1.5.4 has many important consequences. Here we just mention a few which will be needed later. The following is a special case of [**65**, 5.2.2].

Theorem 1.5.6. *Let $f : X \to S$ be a quasi-projective morphism between noetherian schemes with S reduced. Then there exists a projective birational morphism $S' \to S$ which is an isomorphism over a dense open subset of S such that the strict transform of X in $X_{S'} := X \times_S S'$ is flat over S'.*

We will deduce this from the following variant result.

THEOREM 1.5.7. *Let $f : X \to S$ be a projective morphism with S integral and noetherian, and let $U \subset S$ be the maximal open subset of S over which f is flat. Then U is dense in S, and there exists a blowup $S' \to S$ with center in $S - U$ such that the strict transform of X in $X_{S'}$ is flat over S'.*

THEOREM 1.5.7 IMPLIES 1.5.6. Replacing S by the disjoint union of its irreducible components with the reduced structure, we may assume that S is integral.

Since f is quasi-projective, we can find a commutative diagram

$$\begin{tikzcd} X \arrow[r, hook] \arrow[dr, "f"'] & \overline{X} \arrow[d, "\bar{f}"] \\ & S, \end{tikzcd}$$

where j is a dense open imbedding and $\bar{f} : \overline{X} \to S$ is projective. Now observe that if $S' \to S$ is a blowup with nowhere dense center such that the strict transform of $\overline{X}$ in $\overline{X}_{S'}$ is flat over S', then the strict transform of X in $X_{S'}$ is also flat over S'. To prove 1.5.6 it therefore suffices to consider the case when f is projective and S is integral. $\square$

PROOF OF THEOREM 1.5.7. By [**26**, IV.11.1.1] (or exercise 9.4 in Chapter III of [**41**]), the set of points in X where f fails to be flat is closed, and since f is proper the image $Z \subset S$ of this set under f is also closed and $Z = S - U$. Since S is integral, f is flat over the generic point of S, so U is nonempty and dense in S.

Fix a relatively ample line bundle on X. For any point $u \in U$ we can consider the Hilbert polynomial P_u of the fiber X_u, and since f is flat over U all these Hilbert polynomials are equal to a fixed polynomial $P \in \mathbb{Q}[z]$, since Hilbert polynomials are constant in flat families.

Consider the Hilbert scheme

$$\underline{\mathrm{Hilb}}^P_{X/S} \to S.$$

Over U we have a section (corresponding to X_U/U)

$$s : U \to \underline{\mathrm{Hilb}}^P_{X/S}.$$

Let S' denote the scheme-theoretic closure of $s(U)$. Then $S' \to S$ is a projective morphism, whence a blowup by [**26**, III.2.3.5] (or [**41**, II.7.17]). Moreover, over S' we have a closed imbedding

$$Z \hookrightarrow X_{S'}$$

such that Z/S' is flat with Hilbert polynomial P, and such that the restriction of Z to U is X_U. It follows that Z is the strict transform of X, which proves the theorem. $\square$

1.6. Exercises

EXERCISE 1.A. Let $R \to R'$ be a ring homomorphism. Show that the morphism of schemes

$$\mathrm{Spec}(R') \to \mathrm{Spec}(R)$$

is flat (resp. faithfully flat) in the sense of 1.1.3 if and only if R' is flat (resp. faithfully flat) as an R-module.

EXERCISE 1.B. Let S be a scheme and let $\mathscr{O}$ be the functor on the category of S-schemes which sends X/S to $\Gamma(X, \mathscr{O}_X)$. Show that $\mathscr{O}$ is represented by $\mathbb{A}^1_S$.

EXERCISE 1.C. (a) Let $n \geq 1$ be an integer and let

$$GL_n : (\mathrm{Sch})^{\mathrm{op}} \to \mathrm{Set}$$

be the functor sending a scheme Y to the set $GL_n(\Gamma(Y, \mathscr{O}_Y))$. Prove that GL_n is representable by an affine scheme.

(b) Let X represent the functor GL_n. Prove that the group structure on $GL_n(\Gamma(Y, \mathscr{O}_Y))$ induces morphisms

$$m : X \times X \to X, \quad i : X \to X, \quad e : \mathrm{Spec}(\mathbb{Z}) \to X$$

such that the following diagrams commute:

(i)

$$\begin{tikzcd} \mathrm{Spec}(\mathbb{Z}) \times X \arrow[r, "e\times\mathrm{id}"] \arrow[dr, equal] & X \times X \arrow[d, "m"] \\ & X \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X \times \mathrm{Spec}(\mathbb{Z}) \arrow[r, "\mathrm{id}\times e"] \arrow[dr, equal] & X \times X \arrow[d, "m"] \\ & X. \end{tikzcd}$$

(ii)

$$\begin{tikzcd} X \times X \times X \arrow[r, "m\times\mathrm{id}"] \arrow[d, "\mathrm{id}\times m"] & X \times X \arrow[d, "m"] \\ X \times X \arrow[r, "m"] & X. \end{tikzcd}$$

(iii)

$$\begin{tikzcd} X \arrow[r, "i\times\mathrm{id}"] \arrow[d] & X \times X \arrow[d, "m"] \\ \mathrm{Spec}(\mathbb{Z}) \arrow[r, "e"] & X \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X \arrow[r, "\mathrm{id}\times i"] \arrow[d] & X \times X \arrow[d, "m"] \\ \mathrm{Spec}(\mathbb{Z}) \arrow[r, "e"] & X. \end{tikzcd}$$

EXERCISE 1.D. (a) Let

$$\mathbb{A}^n - \{0\} : (\mathrm{Sch})^{\mathrm{op}} \to \mathrm{Set}$$

be the functor sending a scheme Y to the set of n-tuples $(y_1, \ldots, y_n)$ of sections $y_i \in \Gamma(Y, \mathscr{O}_Y)$ such that for every point $y \in Y$ the images of the y_i in $k(y)$ are not all zero. Show that $\mathbb{A}^n - \{0\}$ is representable.

(b) Let

$$(\mathbb{A}^n - \{0\})/\mathbb{G}_m : (\mathrm{Sch})^{\mathrm{op}} \to \mathrm{Set}$$

be the functor sending a scheme Y to the quotient of the set $(\mathbb{A}^n - \{0\})(Y)$ by the equivalence relation

$$(y_1, \ldots, y_r) \sim (y'_1, \ldots, y'_r)$$

if there exists a unit $u \in \Gamma(Y, \mathscr{O}^*_Y)$ such that $y_j = uy'_j$ for all j. Show that $(\mathbb{A}^n - \{0\})/\mathbb{G}_m$ is not representable.

EXERCISE 1.E. Let (Top) be the category of topological spaces with morphisms being continuous maps. Let

$$F : (\mathrm{Top})^{\mathrm{op}} \to \mathrm{Set}$$

be the functor sending a topological space S to the collection $F(S)$ of all its open subsets.

(a) Endow $\{0,1\}$ with the coarsest topology in which the subset $\{0\} \subset \{0,1\}$ is closed. Show that the open subsets in this topology are $\emptyset$, $\{1\}$, and $\{0,1\}$.

(b) Show that $\{0,1\}$ with the above topology represents F.

(c) Let (HausTop) $\subset$ (Top) denote the full subcategory of Hausdorff topological spaces. Show that the restriction of F,

$$F|_{(\text{HausTop})} : (\text{HausTop})^{\text{op}} \to \text{Set},$$

is not representable.

For further discussion of this and other related examples see [**71**, 2.1.3]

Exercise 1.F. Let S be a noetherian scheme and $f : X \to S$ a finite type morphism of schemes. Let F be a coherent sheaf on X, which is locally free over a dense open subset $U \subset X$. Show that there exists a projective birational morphism $\pi : X' \to X$ and a locally free sheaf E on X' of finite rank such that for some dense open subset $U' \subset X'$ we have $E|_{U'} \simeq \pi^* F|_{U'}$.

Exercise 1.G. Let k be a field and let K/k be a finite extension of fields. Show that $\text{Spec}(K) \to \text{Spec}(k)$ is étale if and only if K/k is separable.

Exercise 1.H. Let R be a ring and let $F \in R[X_0, \ldots, X_n]$ be a homogeneous polynomial of degree d. Let

$$Z(F) : (R\text{-schemes})^{\text{op}} \to \text{Set}$$

be the functor sending an R-scheme T to the set of isomorphism classes of quotients $\pi : \mathscr{O}_T^{n+1} \to L$, where L is an invertible $\mathscr{O}_T$-module and such that the image of F, viewed as an element of $\text{Sym}^d_R(R^{n+1})$, under the map

$$\text{Sym}^d_R(R^{n+1}) \longrightarrow \text{Sym}^d(\mathscr{O}_T^{n+1}) \xrightarrow{\ \pi\ } L^{\otimes d}$$

is zero. Verify the conditions in 1.4.11 for the functor $Z(F)$ thereby showing it is representable, and show that the representing scheme is isomorphic to

$$\text{Proj}(R[X_0, \ldots, X_n]/(F)).$$

Exercise 1.I. Verify that the construction in paragraph 1.3.3 defines a simply transitive action of $\text{Hom}_{Y_0}(y_0^* \Omega^1_{X/Y}, \mathscr{I})$ on the set of dotted arrows as in diagram (1.3.3.1), assuming that at least one such arrow exists.

Exercise 1.J. Prove Proposition 1.3.4.

Exercise 1.K. Verify that a morphism $f : F \to G$ of big Zariski sheaves on a scheme S is surjective in the sense of 1.3.4 if and only if it is an epimorphism in the category of big Zariski sheaves on Aff_S. Furthermore, show that if $f : F \to G$ is a surjective morphism of big Zariski sheaves, then G is the coequalizer in the category of big Zariski sheaves of the diagram

$$F \times_G F \underset{\text{pr}_2}{\overset{\text{pr}_1}{\rightrightarrows}} F$$

Exercise 1.L. Let $f : X \to S$ be a projective morphism of schemes, and let L and M be two invertible sheaves on X. Let

$$I : (S\text{-schemes})^{\text{op}} \to \text{Set}$$

be the functor which to any S-scheme S' associates the set of isomorphisms $L_{S'} \to M_{S'}$ of invertible sheaves on $X_{S'} := X \times_S S'$. Show that I is representable by a quasi-projective S-scheme. Hint: Let $U \to X$ be the complement of the zero section of the total space $\mathbb{V}(L^\vee \otimes M)$ of the invertible sheaf $L^\vee \otimes M$ and note that giving an isomorphism of invertible sheaves $L \to M$ is equivalent to giving a closed subscheme $\Gamma \subset U$ such that the projection to X is an isomorphism. Now consider the Hilbert scheme of U.

Exercise 1.M. Let B be a ring. Show that if M is a finitely presented projective B-module, then the associated quasi-coherent sheaf $\widetilde{M}$ on $\mathrm{Spec}(B)$ is a locally free sheaf of finite rank.

CHAPTER 2

Grothendieck topologies and sites

In this chapter we introduce the basic definitions and results concerning Grothendieck topologies and sites. The main examples that we will need later are the étale and flat site of a scheme, but for technical reasons we will also have occasion to consider other sites (for example, the lisse-étale site, or the étale site of a simplicial scheme).

One of the key ideas of Grothendieck is that in order to develop the standard topological machinery of sheaves one does not need to restrict attention only to topological spaces. Rather the theory can be developed in a more general setting of a category with certain collections of maps which are called coverings. Such a category with coverings is called a site. The collection of open subsets of a topological space, with morphisms being inclusions, is one example of a site, but there are many other important examples which arise in algebraic geometry. There is also the notion of topos which by definition is a category equivalent to the category of sheaves of sets on some site. The important point here is that different sites can have equivalent categories of sheaves, and the topos is the more important invariant. For example, the category of abelian sheaves on a site can be defined purely in terms of the topos and therefore cohomology depends only on the topos and not the defining site. We will make use of this in various places where choosing different representative sites for the same topos is helpful in proofs.

After developing the basic material on sites and topoi, we discuss cohomology and simplicial techniques. The discussion of simplicial topoi is rather technical but plays a crucial role in what follows. In particular, the spectral sequence of a covering (see paragraph 2.4.25) is one of the key tools in generalizing various cohomological results from schemes to algebraic spaces and stacks. The material on simplicial topoi is a generalization of the more familiar theory of Čech cohomology.

All of this material, and of course much more, can be found in [**10**]. Other good introductions are [**3**] (in particular Theorem 4.14 in Chapter II) and [**71**]. With an eye towards the lisse-étale site of a stack studied later, we pay particular attention in this chapter to issues surrounding sites in which finite projective limits are not representable. For more details about Čech cohomology and cohomology of presheaves see [**10**, V] (see also [**67**, Tag 01FQ]). The material on cohomological descent is developed in even greater generality in [**10**, Exposé Vbis] (see also [**22**]).

2.1. Sites

2.1.1. The notion of a sheaf on a site is a generalization of the notion of a sheaf on a topological space. Before stating the definition of a site, let us recall for motivation the definition of a sheaf on a topological space X. For such a space, let $\mathrm{Op}(X)$ denote the collection of open subsets of X. We can view $\mathrm{Op}(X)$ as a category in which for two open sets $U, V \subset X$ there exists a unique morphism

$U \to V$ if $U \subset V$ and no morphisms $U \to V$ otherwise. Then a presheaf on X is simply a functor

$$F : \mathrm{Op}(X)^{\mathrm{op}} \to \mathrm{Set}.$$

Moreover, F is a sheaf if and only if for every $U \in \mathrm{Op}(X)$ and covering $U = \bigcup_i U_i$ the sequence

$$F(U) \to \prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \cap U_j)$$

is an equalizer diagram (see for example A.3.2). The key observation is that the notion of a sheaf on X depends only on the category $\mathrm{Op}(X)$ and the distinguished collections of maps $\{U_i \to U\}$ which are coverings.

This point of view leads to the following definition:

Definition 2.1.2. Let C be a category. A *Grothendieck topology* on C consists of a set $\mathrm{Cov}(X)$ of collections of morphisms $\{X_i \to X\}_{i\in I}$ for every object $X \in C$ such that the following hold:

(i) If $V \to X$ is an isomorphism, then $\{V \to X\} \in \mathrm{Cov}(X)$.

(ii) If $\{X_i \to X\}_{i\in I} \in \mathrm{Cov}(X)$ and $Y \to X$ is any arrow in C, then the fiber products $X_i \times_X Y$ exist in C and the collection

$$\{X_i \times_X Y \to Y\}_{i\in I}$$

is in $\mathrm{Cov}(Y)$.

(iii) If $\{X_i \to X\}_{i\in I} \in \mathrm{Cov}(X)$, and if for every $i \in I$ we are given $\{V_{ij} \to X_i\}_{j\in J_i} \in \mathrm{Cov}(X_i)$, then the collection of compositions

$$\{V_{ij} \to X_i \to X\}_{i\in I, j\in J_i}$$

is in $\mathrm{Cov}(X)$.

Remark 2.1.3. We call the collections $\{X_i \to X\}_{i\in I}$ in $\mathrm{Cov}(X)$ the *coverings of* X.

Remark 2.1.4. The above notion of Grothendieck topology is called a 'pre-topology' in [**10**].

Definition 2.1.5. A category with a Grothendieck topology is called a *site*.

Here are several examples of sites (the reader should verify that the above axioms hold in each case).

Example 2.1.6 (Classical topology). Let X be a topological space, and as above let $\mathrm{Op}(X)$ be the category whose objects are open subsets of X and for which $\mathrm{Hom}(U, V)$ is a singleton if $U \subset V$ and the empty set otherwise. For $U \in \mathrm{Op}(X)$ we define $\mathrm{Cov}(U)$ to be the collections $\{U_i \to U\}_{i\in I}$ for which $U = \bigcup_i U_i$.

In particular, if X is a scheme, then the usual Zariski topology defines a site, referred to as the *small Zariski site* (or simply *Zariski site* if no confusion occurs).

Example 2.1.7 (Big classical topology). Let C be the category of topological spaces with morphisms continuous maps. For a topological space X define $\mathrm{Cov}(X)$ to be the collections of families $\{X_i \to X\}_{i\in I}$ for which each $X_i \to X$ is an open imbedding and $X = \bigcup_{i\in I} X_i$.

Example 2.1.8 (Big Zariski site of a scheme). Let X be a scheme, and let C be the category of X-schemes. For $(U \to X) \in C$ define $\mathrm{Cov}(U)$ to be the set of collections of X-morphisms $\{U_i \to U\}_{i\in I}$ for which each $U_i \to U$ is an open imbedding and $U = \bigcup_{i\in I} U_i$.

EXAMPLE 2.1.9 (Localized site). Let C be a site, and let $X \in C$ be an object. Define C/X to be the category whose objects are arrows $Y \to X$ in C and whose morphisms are commutative triangles

$$\begin{tikzcd} Y' \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\ & X. & \end{tikzcd}$$

Define $\mathrm{Cov}(Y \to X)$ to be the set of collections of X-morphisms $\{Y_i \to Y\}_{i \in I}$ for which the underlying collection of morphisms $\{Y_i \to Y\}_{i \in Y}$ in C, obtained by forgetting the morphisms to X, is in $\mathrm{Cov}(Y)$.

EXAMPLE 2.1.10 (Diagrams). Let Δ be a category and let C be a site. Let

$$F : \Delta^{\mathrm{op}} \to C$$

be a functor. Define C_F as follows.

The objects of C_F are pairs $(\delta, X \to F(\delta))$, where $\delta \in \Delta$ and $X \to F(\delta)$ is a morphism in C. A morphism

$$(\delta', X' \to F(\delta')) \to (\delta, X \to F(\delta))$$

is a pair (f, f^b), where $f : \delta \to \delta'$ is a morphism in Δ and $f^b : X' \to X$ is a morphism in C such that the diagram

$$\begin{tikzcd} X' \arrow[r, "f^b"] \arrow[d] & X \arrow[d] \\ F(\delta') \arrow[r, "F(f)"] & F(\delta) \end{tikzcd}$$

commutes. For $(\delta, X \to F(\delta)) \in C_F$ the set of coverings of this object is the set of collections

$$\{(\delta_i, X_i) \stackrel{(f_i, f_i^b)}{\to} (\delta, X)\}_{i \in I}$$

such that each $f_i : \delta \to \delta_i$ is an isomorphism and the collection $\{f_i^b : X_i \to X\}_{i \in I}$ is in $\mathrm{Cov}(C/F(\delta))$.

REMARK 2.1.11. An important special case of the preceding example, which we will consider in section 2.4, is when Δ is the category of finite ordered sets with order preserving maps.

EXAMPLE 2.1.12 (Small étale site). Let X be a scheme. Define $\mathrm{Et}(X)$ to be the full subcategory of the category of X-schemes whose objects are étale morphisms $U \to X$. A collection of morphisms $\{U_i \to U\}_{i \in I}$ is in $\mathrm{Cov}(U)$ if the map

$$\coprod_{i \in I} U_i \to U$$

is surjective. Note that by 1.3.4 (iii) all morphisms in $\mathrm{Et}(X)$ are étale.

EXAMPLE 2.1.13 (Big étale site). Let X be a scheme and let (Sch/X) be the category of X-schemes. We put a Grothendieck topology on (Sch/X) by declaring that $\mathrm{Cov}(U)$ is the set of collections $\{U_i \to U\}_{i \in I}$ of X-morphisms for which each morphism $U_i \to U$ is étale and the map

$$\coprod_{i \in I} U_i \to U$$

is surjective.

Example 2.1.14 (fppf site). Let X be a scheme, and let (Sch/X) be the category of X-schemes. Define a Grothendieck topology on this category by declaring that a collection of morphisms $\{U_i \to U\}_{i\in I}$ is a covering if each morphism $U_i \to U$ is flat and locally of finite presentation and the map

$$\coprod_{i\in I} U_i \to U$$

is surjective.

Example 2.1.15 (Lisse-étale site [**49**, §12]). Let X be a scheme, and let $\text{Lis-Et}(X)$ be the full subcategory of the category of X-schemes whose objects are smooth morphisms $U \to X$. A collection of maps $\{U_i \to U\}_{i\in I}$ in $\text{Lis-Et}(X)$ is a covering if each map $U_i \to U$ is étale and the map

$$\coprod_{i\in I} U_i \to U$$

is surjective.

Example 2.1.16 (Smooth site). Let X be a scheme and define C to be the full subcategory of the category of X-schemes whose objects are smooth morphisms $U \to X$, as in the previous example. Define a collection of maps $\{U_i \to U\}_{i\in I}$ to be a covering if each map $U_i \to U$ is smooth and the map $\coprod_{i\in I} U_i \to U$ is surjective.

2.2. Presheaves and sheaves

Definition 2.2.1. A *presheaf* on a category C is a functor

$$F : C^{\mathrm{op}} \to \mathrm{Set}.$$

We usually write $\widehat{C}$ for the category of presheaves on C.

Definition 2.2.2. Let C be a category with a Grothendieck topology.

(i) A presheaf F on C is called *separated* if for every $U \in C$ and $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}(U)$ the map $F(U) \to \prod_{i\in I} F(U_i)$ is injective.

(ii) A presheaf F on C is called a *sheaf* if for every object $U \in C$ and covering $\{U_i \to U\}_{i\in I}$ the sequence

$$F(U) \to \prod_{i\in I} F(U_i) \rightrightarrows \prod_{i,j\in I} F(U_i \times_U U_j) \tag{2.2.2.1}$$

is exact, where the two maps on the right are induced by the two projections $U_i \times_U U_j \to U_i$ and $U_i \times_U U_j \to U_j$ (note also that these fiber products exist by the axioms for a Grothendieck topology).

We sometimes write $C^\sim$ for the category of sheaves on C.

Remark 2.2.3. To say that the sequence (2.2.2.1) is exact means that the map

$$F(U) \to \prod_{i\in I} F(U_i)$$

identifies $F(U)$ with the equalizer of the two maps

$$\prod_{i\in I} F(U_i) \rightrightarrows \prod_{i,j\in I} F(U_i \times_U U_j).$$

THEOREM 2.2.4. *Let C be a site. The inclusion*

$$(\textit{sheaves on } C) \hookrightarrow (\textit{presheaves on } C)$$

has a left adjoint $F \mapsto F^a$.

PROOF. We follow the argument from [**71**, proof of 2.64]. In fact, we will show that the two inclusions

$$(\text{separated presheaves on } C) \hookrightarrow (\text{presheaves on } C) \tag{2.2.4.1}$$

and

$$(\text{sheaves on } C) \hookrightarrow (\text{separated presheaves on } C) \tag{2.2.4.2}$$

have left adjoints. Taking the composition of these two left adjoints we obtain the desired functor $F \mapsto F^a$.

To construct a left adjoint to (2.2.4.1), let F be a presheaf on C, and define F^s to be the quotient of F which to any $U \in C$ associates $F(U)/\sim$, where two sections $a, b \in F(U)$ are equivalent if there exists a covering $\{U_i \to U\}_{i \in I} \in \mathrm{Cov}(U)$ such that a and b have the same image under

$$F(U) \to \prod_{i \in I} F(U_i).$$

Note that if $V \to U$ is a morphism in C, then we have $\{U_i \times_U V \to V\}_{i \in I} \in \mathrm{Cov}(V)$ and a commutative diagram:

$$\begin{tikzcd} F(U) \arrow[r] \arrow[d] & \prod_{i \in I} F(U_i) \arrow[d] \\ F(V) \arrow[r] & \prod_{i \in I} F(U_i \times_U V). \end{tikzcd}$$

From this it follows that the composition

$$F(U) \to F(V) \to F(V)/\sim$$

factors through $F(U)/\sim$, so F^s is a presheaf. It is also clear from the construction that if G is a separated presheaf then any map $F \to G$ factors (necessarily uniquely since $F \to F^s$ is surjective) through F^s. This completes the construction of the left adjoint $F \mapsto F^s$ to (2.2.4.1).

For the left adjoint to (2.2.4.2), let F be a separated presheaf on C, and define F^a to be the presheaf which associates to any $U \in C$ the set of pairs

$$(\{U_i \to U\}_{i \in I}, \{a_i\}),$$

consisting of a covering $\{U_i \to U\}_{i \in I}$ of U and an element

$$\{a_i\} \in \mathrm{Eq}(\prod_{i \in I} F(U_i) \rightrightarrows \prod_{i,j \in I} F(U_i \times_U U_j)),$$

of the equalizer, modulo the equivalence relation

$$(\{U_i \to U\}_{i \in I}, \{a_i\}) \sim (\{V_j \to U\}_{j \in J}, \{b_j\})$$

if a_i and b_j have the same image in $F(U_i \times_U V_j)$ for all $i \in I$ and $j \in J$. We leave to the reader the verification that F^a is a sheaf, and that the natural map

$$F \to F^a$$

sending $a \in F(U)$ to $(\{\mathrm{id} : U \to U\}, \{a\}) \in F^a(U)$ is universal for maps to sheaves. $\square$

Remark 2.2.5. The left adjoint $F \mapsto F^a$ is called *sheafification*, and if F is a presheaf we call F^a the *sheaf associated to* F.

Remark 2.2.6. As in the classical case of topological spaces we can talk about sheaves of groups, rings, modules over a ring, etc. on a site.

Remark 2.2.7. Note that by the construction the sheafification functor commutes with finite products.

Definition 2.2.8. A *topos* is a category T equivalent to the category of sheaves of sets on a site.

Usually the topos of a site is more important than the site itself. Working with the topos rather than the site itself adds some flexibility, as different sites can induce equivalent topoi, and it is often convenient to consider different sites defining the same topos.

Remark 2.2.9. For a scheme X we write X_{et} (resp. X_{ET}, $X_{\text{lis-ét}}$, X_{fppf}) for the topos of sheaves on the small étale site (resp. big étale site, lisse-étale site, fppf-site) of X.

Example 2.2.10. Define $\mathrm{Et}^{\mathrm{aff}}(X)$ to be the full subcategory of $\mathrm{Et}(X)$ consisting of étale morphisms $U \to X$ with U affine. Define a collection of morphisms $\{U_i \to U\}_{i \in I}$ in $\mathrm{Et}^{\mathrm{aff}}(X)$ to be a covering if it is a covering in the étale site $\mathrm{Et}(X)$ (see 2.1.12). Then $\mathrm{Et}^{\mathrm{aff}}(X)$ is a site whose associated topos is equivalent to the topos of sheaves on $\mathrm{Et}(X)$. This follows from exercise 2.H.

Similarly the topos of sheaves on the big étale site (2.1.13), the flat site (2.1.14), the lisse-étale site (2.1.15), or the smooth site (2.1.16) can be defined using analogous sites of affine X-schemes.

Remark 2.2.11. The notions of sheaves of abelian groups, rings, etc. can be defined purely in terms of the associated topos T. For example, a sheaf of abelian groups is specified by a collection of data (A, m, e), where A is a sheaf of sets, and

$$m : A \times A \to A, \quad e : \{*\} \to A$$

are morphisms of sheaves of sets such that the following hold:

(i) (Associativity). The diagram

$$\begin{tikzcd} A \times A \times A \arrow[r, "1\times m"] \arrow[d, "m\times 1"] & A \times A \arrow[d, "m"] \\ A \times A \arrow[r, "m"] & A \end{tikzcd}$$

commutes.

(ii) (Commutativity). The diagram

$$\begin{tikzcd} A \times A \arrow[dd, "\text{flip}"] \arrow[dr, "m"] & \\ & A \\ A \times A \arrow[ur, "m"'] & \end{tikzcd}$$

commutes, where the flip map $A \times A \to A \times A$ is the map interchanging the factors.

(iii) (Existence of inverses). If Γ denotes the fiber product of the diagram

$$\begin{tikzcd} & A \times A \arrow[d, "m"] \\ \{*\} \arrow[r, "e"] & A, \end{tikzcd}$$

then the composite map

$$\Gamma \longrightarrow A \times A \xrightarrow{\mathrm{pr}_1} A$$

is an isomorphism.

(iv) (Identity) The composite map

$$A \xrightarrow{\mathrm{id}_A \times e} A \times A \xrightarrow{m} A$$

is the identity map on A.

In what follows we will therefore talk about abelian groups, rings, etc. in an arbitrary topos T. If we fix a site C with associated topos T, then these notions get identified with the usual notions of sheaves of abelian groups, rings, etc.

Remark 2.2.12. Axiom (iii) can be replaced by the following condition:

(iii)' There exists a morphism $i : A \to A$ such that the diagram

(2.2.12.1)
$$\begin{tikzcd} A \arrow[r, "(i,\mathrm{id})"] \arrow[d] & A \times A \arrow[d, "m"] \\ \{*\} \arrow[r, "e"] & A \end{tikzcd}$$

commutes.

Indeed given axioms (i), (ii), (iii)', and (iv) and a morphism $(f, g) : P \to A \times A$ such that $m \circ (f, g) = e$, the following diagram commutes:

$$\begin{tikzcd}
P \arrow[rr, "if"] \arrow[d, "{(f,g)}"] \arrow[dd, bend right=60, "g"'] & & A \arrow[d, "\mathrm{id}\times e"'] \arrow[dd, bend left=60, "\mathrm{id}"] \\
A \times A \arrow[r, "(i\times \mathrm{id})\times \mathrm{id}"] \arrow[d, "\mathrm{pr}_2"] & A \times A \times A \arrow[r, "\mathrm{id}\times m"] \arrow[d, "m\times \mathrm{id}"] & A \times A \arrow[d, "m"'] \\
A \arrow[r, "e\times \mathrm{id}"] \arrow[rr, bend right=30, "\mathrm{id}"'] & A \times A \arrow[r, "m"] & A.
\end{tikzcd}$$

It follows that $g = if$ and that the diagram

$$\begin{tikzcd}
P \arrow[d, "f"] \arrow[dr, "{(f,g)}"] & \\
A \arrow[r, "{(\mathrm{id},i)}"] & A \times A
\end{tikzcd}$$

commutes, so (2.2.12.1) is cartesian and (iii) also holds. Thus the axioms for a group can be phrased without assuming the existence of fiber products.

2.2.13. Recall (for example from A.3.1) that if C and I are categories and

$$F : I \to C$$

is a functor, then the limit

$$\varprojlim F : C^{\mathrm{op}} \to \mathrm{Set}$$

is the functor sending $X \in C$ to the set of natural transformations of functors

$$k_X \to F,$$

where k_X is the constant functor sending every object of I to X and every morphism in I to the identity. Similarly the colimit

$$\varinjlim F : C \to \mathrm{Set}$$

is defined to be the functor sending $X \in C$ to the set of natural transformations of functors

$$F \to k_X.$$

The following basic lemma, taken from [**10**, I, 2.3], gives a useful way to prove that a category has finite projective limits.

LEMMA 2.2.14. *Let C be a category. Then the following are equivalent:*

(i) *For any small category I and functor $F : I \to C$, the limit $\varprojlim F$ is representable.*
(ii) *Products and equalizers in C are representable.*
(iii) *Products and fiber products are representable.*

PROOF. Since products, equalizers, and fiber products are all limits (see for example A.3.2, A.3.3, A.3.4), we have $(i) \implies (ii)$ and $(i) \implies (iii)$.

For $(ii) \implies (i)$, let $F : I \to C$ be a functor, and let $\mathrm{Ar}(I)$ be the set of arrows in I. Then there are two maps:

$$p_1, p_2 : \prod_{i \in \mathrm{Ob}(I)} F(i) \to \prod_{u \in \mathrm{Ar}(I)} F(\mathrm{target}(u)).$$

The map p_1 sends an element $(a_i)_{i \in \mathrm{Ob}(I)} \in \prod_{i \in \mathrm{Ob}(I)} F(i)$ to the element of $\prod_{u \in \mathrm{Ar}(I)} F(\mathrm{target}(u))$ whose u-component is the element $a_{\mathrm{target}(u)}$, and the map p_2 sends such an element to the element with u-component the image of $a_{\mathrm{source}(u)}$ under the map

$$F(u) : F(\mathrm{source}(u)) \to F(\mathrm{target}(u)).$$

Granting (ii) we obtain that $\varprojlim F$ is the equalizer of the two maps p_1 and p_2, and hence is representable.

Finally $(iii) \implies (ii)$ follows from the fact that the equalizer of two maps

$$u, v : X \to Y$$

can be identified with the fiber product of the diagram

$$\begin{tikzcd} & X \arrow[d, "{(u,v)}"] \\ Y \arrow[r, "\Delta"] & Y \times Y. \end{tikzcd}$$

$\square$

EXAMPLE 2.2.15. If I is the empty category then there exists a unique functor $F : I \to C$, and $\varprojlim F$ is the functor sending $X \in C$ to the unital set $\{*\}$. In particular, $\varprojlim F$ is representable if and only if C has a final object.

PROPOSITION 2.2.16. *Let T be a topos, and let $F : I \to T$ be a functor with I small. Then the limit $\varprojlim F$ is representable.*

PROOF. By the previous lemma, it suffices to show that products and equalizers are representable. For this let C be a site such that T is equivalent to the topos of sheaves on C. If $\{F_i\}_{i \in I}$ is a set of sheaves on C, the product of the F_i is representable by the sheaf

$$(X \in C) \mapsto \prod_{i \in I} F_i(X).$$

Furthermore, if $u, v : F \to G$ are two maps of sheaves, then the equalizer of u and v is the sheaf sending $X \in C$ to the equalizer of the two maps

$$F(X) \rightrightarrows G(X),$$

defined by u and v. $\square$

DEFINITION 2.2.17. Let T and T' be topoi. A *morphism of topoi* $f : T \to T'$ is an adjoint pair (in the sense of A.5.1)

$$f = (f^*, f_*, \varphi),$$

where

$$f_* : T \to T', \quad f^* : T' \to T$$

are functors and φ is an isomorphism

$$\varphi : \mathrm{Hom}_T(f^*(-), -) \to \mathrm{Hom}_{T'}(-, f_*(-))$$

of bifunctors

$$T'^{\mathrm{op}} \times T \to \mathrm{Set},$$

and such that f^* commutes with finite limits.

There are variants of the above definitions considering also a sheaf of rings.

DEFINITION 2.2.18. A *ringed topos* is a pair (T, Λ), where T is a topos and Λ is a ring object in T. A *morphism of ringed topoi*

$$(T, \Lambda) \to (T', \Lambda')$$

is a pair $(f, f^\sharp)$ consisting of a morphism of topoi $f : T \to T'$ and a morphism of rings in T'

$$f^\sharp : \Lambda' \to f_*\Lambda.$$

REMARK 2.2.19. Since f^* is left adjoint to f_*, giving the morphism $f^\sharp$ is equivalent to giving a morphism $f^*\Lambda' \to \Lambda$ which we somewhat abusively also denote by $f^\sharp$.

Let C and C' be sites with associated topoi T and T', and let $f : C' \to C$ be a functor.

DEFINITION 2.2.20. The functor f is *continuous* if for every $X \in C'$ and $\{X_i \to X\}_{i\in I} \in \mathrm{Cov}(X)$, the family $\{f(X_i) \to f(X)\}_{i\in I}$ is in $\mathrm{Cov}(f(X))$, and if f commutes with fiber products when they exist in C'.

EXAMPLE 2.2.21. Let $f : X \to Y$ be a continuous map of topological spaces, and consider the resulting classical topologies $\mathrm{Op}(X)$ and $\mathrm{Op}(Y)$ as in 2.1.6. Then pullback defines a functor

$$f^{-1} : \mathrm{Op}(Y) \to \mathrm{Op}(X)$$

which is a continuous morphism of sites in the sense of 2.2.20.

2.2.22. In many cases (though not always), a continuous functor between sites induces a morphism of topoi, as we now explain.

For a continuous functor $f : C' \to C$ define

$$f_* : T \to T'$$

to be the functor sending a sheaf $F \in T$ to the sheaf which to an object $X \in C'$ associates

$$(f_*F)(X) := F(f(X)).$$

That f_*F is a sheaf follows from observing that if $\{X_i \to X\}$ is a covering of X in C', then there is a commutative diagram

$$\begin{tikzcd}
(f_*F)(X) \arrow[r] \arrow[d, equal] & \prod_{i\in I}(f_*F)(X_i) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, equal] & \prod_{i,j}(f_*F)(X_i \times_X X_j) \arrow[d, "\simeq"] \\
F(f(X)) \arrow[r] & \prod_i F(f(X_i)) \arrow[r, shift left] \arrow[r, shift right] & \prod_{i,j} F(f(X_i) \times_{f(X)} f(X_j)),
\end{tikzcd}$$

where the bottom row is exact since F is a sheaf and f is continuous, and where the right vertical identification is given by the canonical isomorphism (again using that f is continuous)

$$f(X_i \times_X X_j) \simeq f(X_i) \times_{f(X)} f(X_j).$$

REMARK 2.2.23. Note that for any categories C' and C and functor $f : C' \to C$, the preceding construction defines a functor on presheaf categories

$$f_* : \widehat{C} \to \widehat{C}'.$$

REMARK 2.2.24. In most situations (though there are non-pathological examples where this is not the case; for example, the crystalline site of a scheme in positive characteristic is not functorial [**15**]) morphisms of topoi that we will consider arise from natural morphisms of sites.

REMARK 2.2.25. Note that a continuous morphism of sites $f : C' \to C$ is in the opposite direction to the intuition from topological spaces. That is, a continuous map of topological spaces $X \to Y$ induces a morphism of sites $\mathrm{Op}(Y) \to \mathrm{Op}(X)$, which in turn defines a morphism of topoi $\mathrm{Op}(X)^\sim \to \mathrm{Op}(Y)^\sim$.

PROPOSITION 2.2.26. *(i) Let $f : C' \to C$ be a functor between small categories. Then the functor*

$$f_* : \widehat{C} \to \widehat{C}'$$

has a left adjoint $\hat{f}^*$.

(ii) If $f : C' \to C$ is a continuous morphism of sites with C and C' small, then the induced functor between topoi $f_* : T \to T'$ has a left adjoint

$$f^* : T' \to T.$$

PROOF. Note that (ii) follows from (i), for granting (i) we get a left adjoint to $f_* : T \to T'$ by sending a sheaf $F' \in T'$ to the sheaf associated to the presheaf $\hat{f}^* F'$.

To prove (i), define for an object $U \in C$ the category I_U to be the category whose objects are pairs (U', ρ), where $U' \in C'$ and $\rho : U \to f(U')$ is a morphism in C, and for which a morphism

$$(U', \rho) \to (V', \epsilon)$$

is a morphism $g : U' \to V'$ such that the diagram

$$\begin{tikzcd} & f(U') \arrow[dd, "f(g)"] \\ U \arrow[ur, "\rho"] \arrow[dr, "\epsilon"'] & \\ & f(V') \end{tikzcd}$$

commutes.

We define

$$(\hat{f}^* F)(U) := \varinjlim_{(U', \rho) \in I_U^{op}} F(U').$$

Note that if $h : V \to U$ is a morphism in C, then there is an induced functor

$$h^* : I_U \to I_V, \quad (U', \rho) \mapsto (U', \rho \circ h),$$

which induces a map

$$(\hat{f}^* F)(U) \to (\hat{f}^* F)(V)$$

making $\hat{f}^* F$ a presheaf on C.

There is a natural transformation

$$\hat{f}^* f_* \to \mathrm{id}_{\widehat{C}},$$

given by the natural maps

$$(\hat{f}^* f_* F)(U) = \varinjlim_{(U',\rho)\in I_U^{op}} F(f(U')) \xrightarrow{\rho^*} F(U),$$

where $F \in \widehat{C}$ and $U \in C$.

If $G \in \widehat{C}'$ and $F \in \widehat{C}$, we therefore get a map

$$\mathrm{Hom}_{\widehat{C}'}(F, f_* G) \to \mathrm{Hom}_{\widehat{C}}(\hat{f}^* F, G), \tag{2.2.26.1}$$

by sending a morphism $h : F \to f_* G$ to the composition

$$\hat{f}^* F \xrightarrow{\hat{f}^*(h)} \hat{f}^* f_* G \xrightarrow{\hat{f}^* f_* \to \mathrm{id}} G.$$

This transformation is clearly functorial in both F and G.

Note that for any $U' \in C'$, there is a canonical object $(U', \mathrm{id}_{f(U')})$ of $I_{f(U')}$, and evaluation on this object defines a morphism

$$F(U') \to (f_* \hat{f}^* F)(U')$$

for any $F \in \widehat{C}'$. This map is functorial in U' so we get a morphism of presheaves $F \to f_* \hat{f}^* F$. Given a map $h : \hat{f}^* F \to G$ (where $G \in \widehat{C}$), we therefore get a map

$$F \to f_* \hat{f}^* F \xrightarrow{f_* h} f_* G,$$

and this defines a map

$$\mathrm{Hom}_{\widehat{C}}(\hat{f}^* F, G) \to \mathrm{Hom}_{\widehat{C}'}(F, f_* G).$$

We leave to the reader the verification that this defines an inverse to (2.2.26.1), and realizes $\hat{f}^*$ as a left adjoint to f_*. □

REMARK 2.2.27. Let $X' \in C'$ be an object and let $h_{X'}$ be the sheaf associated to the corresponding representable presheaf. Then $f^* h_{X'}$ is the sheaf $h_{f(X')}$ associated to the presheaf represented by $f(X')$. This follows from noting that by the Yoneda lemma we have

$$\mathrm{Hom}_{T'}(h_{X'}, G) \simeq G(X')$$

for any sheaf $G \in T'$. We therefore have

$$\mathrm{Hom}_T(f^* h_{X'}, G) \simeq \mathrm{Hom}_{T'}(h_{X'}, f_* G) = (f_* G)(X') = G(f(X')) \simeq \mathrm{Hom}_{T'}(h_{f(X')}, G),$$

which again by the Yoneda lemma gives an isomorphism $f^* h_{X'} \simeq h_{f(X')}$.

EXAMPLE 2.2.28. Notice that in the case of a continuous map of topological spaces $f : X \to Y$ with the classical topology, the colimit over the categories I_U in the preceding proof reduces to the usual colimit in the definition of pullback of presheaves [**41**, p. 65].

REMARK 2.2.29. As in the case of sheaves on a topological space, there can be some notational confusion when working with sheaves of modules. In this setting the pullback functor f^* is often denoted f^{-1} and pullback of sheaves of modules is denoted by f^*.

REMARK 2.2.30. In general, the functor f^* need not commute with finite limits. An example of particular importance to the theory of stacks is the following due to Behrend [**13**, 4.42].

Let k be a field and let $X = \mathbb{A}^1_k$ be the affine line over k, with coordinate t (so $X = \mathrm{Spec}(k[t])$). Let

$$i : Y := \mathrm{Spec}(k) \hookrightarrow X$$

be the inclusion defined by setting $t = 0$. There is a continuous morphism of sites

$$\text{Lis-Et}(X) \to \text{Lis-Et}(Y)$$

sending a smooth X-scheme $Z \to X$ to the fiber product $Z \times_X Y \to Y$. Let $\mathscr{O}_X$ (resp. $\mathscr{O}_Y$) denote the presheaf on Lis-Et(X) (resp. Lis-Et(Y)) (see 2.1.15) associating to any $U \to X$ (resp. $U \to Y$) the usual global sections of the structure sheaf of U (in the Zariski topology). This presheaf is represented by $\mathbb{A}^1_X$ (resp. $\mathbb{A}^1_Y$) and, in fact, we will see later that it is a sheaf (see 4.1.2). It follows from this and 2.2.27 that $i^*\mathscr{O}_X = \mathscr{O}_Y$. On the other hand, we have a global section $t \in \Gamma(\text{Lis-Et}(X), \mathscr{O}_X)$, which defines an injective map

$$\cdot t : \mathscr{O}_X \to \mathscr{O}_X.$$

Since $i^*t = 0$, the pullback of this map to Lis-Et(Y) is the zero map. It follows that i^* does not preserve inclusions, and hence does not commute with finite limits (since kernels are a limit).

PROPOSITION 2.2.31. *Let $f : C' \to C$ be a continuous morphism of sites with C and C' small, and assume that finite limits in C' are representable and that f commutes with all finite limits in C'. Then f^* commutes with finite limits, and hence f defines a morphism of topoi (which we again denote by the same letter)*

$$f : T \to T'.$$

PROOF. Since the functor sending a presheaf to its associated sheaf commutes with finite limits (as noted in 2.2.7), to prove the proposition it suffices to show that the functor on presheaf categories

$$\hat{f}^* : \widehat{C}' \to \widehat{C}$$

commutes with finite limits.

Recall from the proof of 2.2.26 that for an object $U \in C$ and $F \in \widehat{C}'$ we have

$$\hat{f}^*F(U) = \varinjlim_{(U',\rho)\in I_U^{\mathrm{op}}} F(U'),$$

where I_U is the category of pairs $(U', \rho : U \to f(U'))$. To prove the proposition it suffices to show that for all $U \in C$ the category I_U^{op} is filtered, as filtered colimits commute with finite limits (see A.3.7).

It is clear that I_U is nonempty.

To verify condition A.3.6 (ii) let $(U_1', \rho_1), (U_2', \rho_2)$ be two objects of I_U. Let U_3' denote $U_1' \times U_2'$, which exists since C' has finite limits, and let ρ_3 denote the map

$$(\rho_1, \rho_2) : U \to f(U_3') \simeq f(U_1') \times f(U_2'),$$

where the last isomorphism uses the fact that f commutes with finite limits. We then get a diagram in I_U

$$\begin{tikzcd} & (U'_1, \rho_1) \\ (U'_3, \rho_3) \arrow[ur] \arrow[dr] & \\ & (U'_2, \rho_2) \end{tikzcd}$$

verifying A.3.6 (ii) for I_U^{op}.

For A.3.6 (iii) let $(U'_1, \rho_1), (U'_2, \rho_2) \in I_U$ be two objects and let $a, b : (U'_1, \rho_1) \to (U'_2, \rho_2)$ be two morphisms. Let U_3 be the equalizer of the two maps $U'_1 \to U'_2$, which exists by our assumptions on C'. Then since f commutes with finite limits $f(U_3)$ is the equalizer of the two induced maps $f(U'_1) \to f_(U'_2)$. This implies that ρ_1 factors uniquely through a morphism $\rho_3 : U \to f(U_3)$ so we get a diagram

$$(U_3, \rho_3) \longrightarrow (U_1, \rho_1) \rightrightarrows (U_2, \rho_2),$$

which gives A.3.6 (iii) for I_U^{op}. $\square$

Remark 2.2.32. The assumptions in 2.2.31 imply, in particular, that C' has a final object and that f takes this final object to a final object in C (see Example 2.2.15).

Example 2.2.33. Let X be a scheme, and let $\mathrm{Et}(X)$ be the small étale site of X consisting of étale X-schemes (see 2.1.12). Then all finite limits in $\mathrm{Et}(X)$ exist (since finite products and equalizers are representable). It follows that if C is a site with associated topos T, then any continuous functor

$$f : \mathrm{Et}(X) \to C$$

which commutes with finite products and equalizers, induces a morphism of topoi

$$f : T \to X_{\mathrm{et}}.$$

For example, we can take C to be the lisse-étale site of X (see 2.1.15) and f the natural inclusion $\mathrm{Et}(X) \hookrightarrow \text{Lis-Et}(X)$, or C the big fppf site (see 2.1.14) again with f the inclusion.

Remark 2.2.34. The induced morphism of topoi

$$\epsilon : X_{\text{lis-et}} \to X_{\mathrm{et}}$$

has the property that ϵ_* is exact.

2.2.35. If $f : C' \to C$ is a continuous morphism of sites, then as observed above the pullback functor $f^* : T' \to T$ on the associated topoi need not preserve finite limits, and in particular may not even commute with finite products. This implies, for example, that if $F' \in T'$ is a sheaf of groups, then f^*F' does not have a natural group structure.

Proposition 2.2.36. *Assume that finite products in C' are representable, and that f commutes with finite products. Then the pullback functors $\hat{f}^* : \widehat{C}' \to \widehat{C}$ and $f^* : T' \to T$ commute with finite products.*

PROOF. By Remark 2.2.7 sheafification commutes with finite products, so it suffices to prove that $\hat{f}^*$ commutes with finite products. By induction it further suffices to show that if $F_1', F_2' \in \widehat{C}'$ are two presheaves then the natural map

$$\hat{f}^*(F_1' \times F_2') \to (\hat{f}_1^* F_1') \times (\hat{f}_2^* F_2')$$

is an isomorphism. For this in turn it suffices to show that for any $U \in C$ the map on sections

$$\hat{f}^*(F_1' \times F_2')(U) \to (\hat{f}_1^* F_1')(U) \times (\hat{f}_2^* F_2')(U)$$

is an isomorphism. Let I_U denote the category defined in the proof of 2.2.26. Then by definition

$$\hat{f}^*(F_1' \times F_2')(U) = \varinjlim_{(U',\rho)\in I_U^{\mathrm{op}}} F_1'(f(U')) \times F_2'(f(U')),$$

and

$$(\hat{f}_1^* F_1')(U) \times (\hat{f}_2^* F_2')(U) = (\varinjlim_{(U',\rho)\in I_U^{\mathrm{op}}} F_1'(f(U'))) \times (\varinjlim_{(U',\rho)\in I_U^{\mathrm{op}}} F_2'(f(U'))),$$

and we need to show that the map

$$\varinjlim_{(U',\rho)\in I_U^{\mathrm{op}}} F_1'(f(U')) \times F_2'(f(U')) \to (\varinjlim_{(U',\rho)\in I_U^{\mathrm{op}}} F_1'(f(U'))) \times (\varinjlim_{(U',\rho)\in I_U^{\mathrm{op}}} F_2'(f(U')))$$

is an isomorphism.

LEMMA 2.2.37. *Finite products in I_U are representable.*

PROOF. It suffices to show that the product of two elements (U_1', ρ_1) and (U_2', ρ_2) are representable. This product is representable by $U_1' \times U_2'$, which exists by the assumptions on C', with the map

$$U \xrightarrow{\rho_1 \times \rho_2} f(U_1') \times f(U_2') \longrightarrow f(U_1' \times U_2'),$$

where the second morphism is the inverse of the natural map $f(U_1' \times U_2') \to f(U_1') \times f(U_2')$, which is an isomorphism by our assumptions on f. □

It follows that the category I_U^{op} has finite coproducts. This combined with the following general lemma implies 2.2.36. □

LEMMA 2.2.38. *Let C be a category with finite coproducts, and let $F_1, \ldots, F_r : C \to \mathrm{Set}$ be functors. Then the natural map*

$$\varinjlim_C (F_1 \times \cdots \times F_r) \to \prod_{i=1}^r \varinjlim_C (F_i)$$

is an isomorphism.

PROOF. By induction it suffices to consider the case when $r = 2$. In this case there is a natural identification between

$$(\varinjlim_C F_1) \times (\varinjlim_C F_2)$$

and $\varinjlim_{C\times C}(F_1 \times F_2)$, where $F_1 \times F_2 : C \times C \to \mathrm{Set}$ is the functor sending $(U, V) \in C \times C$ to $F_1(U) \times F_2(U)$. To avoid confusion we write $F_1 \times F_2$ for this functor on

$C \times C$ and $F_1 \tilde{\times} F_2$ for the functor on C sending $U \in C$ to $F_1(U) \times F_2(U)$. The natural map

$$\varinjlim_C (F_1 \tilde{\times} F_2) \to \varinjlim_C (F_1) \times \varinjlim_C (F_2) \simeq \varinjlim_{C\times C} (F_1 \times F_2) \tag{2.2.38.1}$$

is induced by the diagonal functor $\Delta : C \to C \times C$. Note that this map is surjective. Indeed for any two objects $U, V \in C$ and elements $u \in F_1(U)$ and $v \in F_2(V)$ the corresponding element $[u \times v] \in \varinjlim_C (F_1) \times \varinjlim_C (F_2)$ is equal to the class of the element $\mathrm{pr}_{1*}(u) \times \mathrm{pr}_{2*}(v)$ in $F_1(U \coprod V) \times F_2(U \coprod V)$, where $\mathrm{pr}_1 : U \to U \coprod V$ and $\mathrm{pr}_2 : V \to U \coprod V$ are the natural maps.

There is also a functor $\pi : C \times C \to C$ sending (U, V) to $U \coprod V$ (which exists by assumption). Moreover, there is a natural transformation of functors

$$F_1 \times F_2 \to (F_1 \tilde{\times} F_2) \circ \pi$$

which on an object $(U, V) \in C \times C$ is given by the map

$$\mathrm{pr}_{1*} \times \mathrm{pr}_{2*} : F_1(U) \times F_2(V) \to F_1(U \coprod V) \times F_2(U \coprod V).$$

The functor π induces a morphism

$$\epsilon : \varinjlim_{C\times C} (F_1 \times F_2) \to \varinjlim_C (F_1 \tilde{\times} F_2).$$

To prove that (2.2.38.1) is also injective it suffices to show that the composition of ϵ with (2.2.38.1) is the identity. For this note that by definition this composition is the map

$$\varinjlim_C (F_1 \tilde{\times} F_2) \to \varinjlim_C (F_1 \tilde{\times} F_2)$$

sending the class of a pair $(a, b) \in F_1(U) \times F_2(U)$ to the class of $(\mathrm{pr}_{1*}(a), \mathrm{pr}_{2*}(b)) \in F_1(U \coprod U) \times F_2(U \coprod U)$. Let $\gamma : U \coprod U \to U$ be the natural map. Then the class of $(\mathrm{pr}_{1*}(a), \mathrm{pr}_{2*}(b))$ in $\varinjlim_C (F_1 \tilde{\times} F_2)$ equals the class of $(\gamma_* \mathrm{pr}_{1*}(a), \gamma_* \mathrm{pr}_{2*}(b)) = (a, b)$. □

Remark 2.2.39. All the sites we will consider in this book will have finite products, and all the continuous morphisms of sites which we encounter will commute with finite products. Thus while in general we will not get morphisms of topoi from such morphisms of sites, we at least have well-defined pullback and pushforward operations on the categories of sheaves of abelian groups, rings, modules, etc.

Remark 2.2.40. Lemma 2.2.38 also appears in the paper [**68**, 1.8].

2.3. Cohomology of sheaves

2.3.1. Let T be a topos, and let Λ be a ring in T. Denote by Mod_Λ the category of Λ-modules in T.

Theorem 2.3.2. *The category* Mod_Λ *is an abelian category with enough injectives.*

That Mod_Λ is an abelian category we leave as exercise 2.J. As for the existence of enough injectives, we give a proof of this theorem in a special case, which suffices for everything that follows. The interested reader can see [**67**, Tag 01DQ] for a proof in general.

2.3.3. Let pt denote the topos of sheaves on the one-point space. A *point* of the topos T is a morphism of topoi

$$x : \mathrm{pt} \to T.$$

We say that the topos T *has enough points* if there exists a set of points

$$\{x_i : \mathrm{pt} \to T\}_{i \in I}$$

of T such that the induced functor

$$T \to \mathrm{Set}^I, \quad F \mapsto \{x_i^* F\} \tag{2.3.3.1}$$

is faithful.

Proof of Theorem 2.3.2 in the case of enough points. We now prove 2.3.2 in the case when the topos T has enough points. Fix a collection of points $\{x_i : \mathrm{pt} \to T\}_{i \in I}$ such that the functor (2.3.3.1) is faithful. For $F \in T$ and $i \in I$ let F_i denote $x_i^* F$. In particular, we have a ring (in the usual sense) Λ_i for each $i \in I$. If $F \in \mathrm{Mod}_\Lambda$, then each F_i is a Λ_i-module. Choose for each $i \in I$, an injective Λ_i-module I_i, and an inclusion $F_i \hookrightarrow I_i$. This inclusion defines by adjunction a morphism

$$p_i : F \to x_{i*} I_i,$$

and so taking the product over I we get a map

$$p : F \to I := \prod_{i \in I} x_{i*} I_i.$$

Since x_{i*} has an exact left adjoint, the sheaf $x_{i*} I_i$ is injective, and since the product of injectives is also injective the sheaf I is an injective Λ-module. Finally, p is an inclusion. For this it suffices to note that for every $i \in I$ the map $F_i \to I_i$ is an inclusion. □

2.3.4. We have a functor

$$\Gamma(T, -) : \mathrm{Mod}_\Lambda \to \mathrm{Ab}$$

from the category of Λ-modules Mod_Λ to the category of abelian groups, obtained by

$$\mathrm{Hom}_{\mathrm{Mod}_\Lambda}(\Lambda, F).$$

This is a left exact functor. We denote the resulting right derived functors by

$$H^i(T, -) : \mathrm{Mod}_\Lambda \to \mathrm{Ab}.$$

Remark 2.3.5. It follows immediately from 7.5.1 that one has injective resolutions of sheaves of Λ-modules. One can also construct flat resolutions in this very general setting; see [**67**, Tags 03EW and 06YL].

For technical reasons it is sometimes useful to consider resolutions by acyclic sheaves rather than injective sheaves. Let C be a site with associated topos T.

Definition 2.3.6. A sheaf of Λ-modules $F \in \mathrm{Mod}_\Lambda$ is *C-acyclic* if for every object $X \in C$ we have

$$H^i(C/X, F) = 0$$

for all $i > 0$, where C/X denotes the localized site defined in 2.1.9, and we abusively also write F for its pullback to C/X (see also exercise 2.D). If $C = T$ with the canonical topology (see exercise 2.N), then we call a C-acyclic sheaf *flasque*.

REMARK 2.3.7. Note that $H^i(C/X, F)$ is the i-th derived functor of the functor

$$\mathrm{Mod}_\Lambda \to \mathrm{Ab}, \quad F \mapsto \mathrm{Hom}_T(h_X, F).$$

2.3.8. Let PMod_Λ denote the category of presheaves of Λ-modules on C. Then PMod_Λ is also an abelian category with enough injectives (in fact equal to the category of sheaves on C endowed with the topology in which the only coverings are collections of isomorphisms).

Viewing a sheaf as a presheaf we get a functor

$$u : \mathrm{Mod}_\Lambda \to \mathrm{PMod}_\Lambda$$

which has an exact left adjoint given by sheafification. In particular u takes injectives to injectives.

2.3.9. Let $\mathscr{X} := \{X_i \to X\}_{i\in I}$ be a covering in C and let $F \in \mathrm{PMod}_\Lambda$ be a presheaf of Λ-modules. For a set of elements $\underline{i} = (i_0, \dots, i_r) \in I^{r+1}$ let $X_{\underline{i}}$ denote the fiber product

$$X_{\underline{i}} := X_{i_0} \times_X X_{i_1} \cdots \times_X X_{i_r}.$$

We define the *Čech cohomology complex* of F, denoted $C^\bullet(\mathscr{X}, F)$, as follows. Set

$$C^r(\mathscr{X}, F) := \prod_{\underline{i}\in I^{r+1}} F(X_{\underline{i}}),$$

so an element $f \in C^r(\mathscr{X}, F)$ consists of sections $f_{\underline{i}} \in F(X_{\underline{i}})$ for each $\underline{i} \in I^{r+1}$. The differential

$$d_r : C^r(\mathscr{X}, F) \to C^{r+1}(\mathscr{X}, F)$$

sends an element $f \in C^r(\mathscr{X}, F)$ to the element of $C^{r+1}(\mathscr{X}, F)$ given by

$$(df)_{i_0,\dots,i_{r+1}} := \sum_{j=0}^{r+1} (-1)^j f_{i_0,\dots,\hat{i}_j,\dots,i_r}.$$

We leave to the reader the verification that this indeed defines a complex. The *Čech cohomology groups of* F, denoted $\check{H}^i(\mathscr{X}, F)$, are defined by

$$\check{H}^i(\mathscr{X}, F) := H^i(C^\bullet(\mathscr{X}, F)).$$

2.3.10. For $\underline{i} \in I^{r+1}$ define

$$j_{\underline{i}!}\Lambda_{X_{\underline{i}}}$$

to be the sheaf associated to the presheaf of Λ-modules on C/X sending $Y \to X$ to

$$\bigoplus_{a:Y\to X_{\underline{i}}} \Lambda(Y),$$

where the direct sum is over X-morphisms $Y \to X_{\underline{i}}$. For any $F \in \mathrm{PMod}_\Lambda$ we have a canonical isomorphism

$$\mathrm{Hom}_{\mathrm{PMod}_\Lambda}(j_{\underline{i}!}\Lambda_{X_{\underline{i}}}, F) \simeq F(X_{\underline{i}}),$$

and therefore if Z_r denotes the sheaf

$$\bigoplus_{\underline{i}\in I^{r+1}} j_{\underline{i}!}\Lambda_{X_{\underline{i}}},$$

then

$$\mathrm{Hom}_{\mathrm{PMod}_\Lambda}(Z_r, F) \simeq C^r(\mathscr{X}, F).$$

By the Yoneda lemma it follows that there are differentials

$$\partial_r : Z_r \to Z_{r-1}$$

with $\partial_{r-1} \circ \partial_r = 0$ for all r inducing the differentials $d_r : C^r(\mathscr{X}, F) \to C^{r+1}(\mathscr{X}, F)$. The differential ∂_r can be described explicitly as the map obtained by summing the maps

$$j_{\underline{i}!}\Lambda_{X_{\underline{i}}} \to \bigoplus_{\underline{s}} j_{\underline{s}!}\Lambda_{X_{\underline{s}}}$$

induced by summing over $e = 0, \dots, r$ the natural maps

$$j_{\underline{i}!}\Lambda_{X_{\underline{i}}} \to j_{i_0 \dots \hat{i}_e \dots i_r!}\Lambda_{X_{i_0 \dots \hat{i}_e \dots i_r}}$$

multiplied by $(-1)^{r-e}$.

Note also that there is a natural map

$$Z_\bullet \to \Lambda|_{C/X}. \tag{2.3.10.1}$$

LEMMA 2.3.11. *For any $Y \in C/X$ we have*

$$H_p(Z_\bullet(Y)) = 0$$

for $p > 0$.

PROOF. If there does not exist any X-morphism $Y \to X_i$ for any i then the complex $Z_\bullet(Y)$ is the zero complex and there is nothing to prove. So assume that there exists at least one such morphism and fix one

$$a_0 : Y \to X_\alpha$$

for some $\alpha \in I$. Define maps

$$h : Z_r(Y) \to Z_{r+1}(Y)$$

as follows. We have

$$Z_r(Y) = \bigoplus_{\underline{i} \in I} \bigoplus_{a: Y \to X_{\underline{i}}} \Lambda(Y)$$

and similarly for $Z_{r+1}(Y)$. So an element of $Z_r(Y)$ (resp. $Z_{r+1}(Y)$) is a collection of elements $(\gamma_{(\underline{i},a)})_{(\underline{i},a)}$ of elements of $\Lambda(Y)$, almost all zero, indexed by the set of pairs $(\underline{i}, a)$, where $\underline{i} \in I^{r+1}$ and $a : Y \to X_{\underline{i}}$ is an X-morphism. Given such a collection define $h((\gamma_{(\underline{i},a)})_{(\underline{i},a)})$ to be the collection of elements $(\gamma'_{(\underline{i}',b)})_{(\underline{i}',b)}$ where $\gamma'_{(\underline{i}',b)} = 0$ unless $\underline{i}' = (i_0 \dots i_r \alpha)$ and b is of the form

$$b = (a, a_0) : Y \to X_{i_0 \cdots i_r} \times_X X_\alpha,$$

in which case we define

$$\gamma'_{(\underline{i}',b)} := \gamma_{(\underline{i},a)}.$$

To prove the lemma it suffices to show that

$$\partial h + h\partial = \mathrm{id}, \tag{2.3.11.1}$$

which we leave to the reader. □

2.3.12. For an exact sequence

$$0 \to F'' \to F \to F' \to 0$$

in PMod_Λ the resulting sequence of complexes

$$0 \to C^\bullet(\mathscr{X}, F'') \to C^\bullet(\mathscr{X}, F) \to C^\bullet(\mathscr{X}, F') \to 0$$

is also exact (note that here it is essential to work with presheaves rather than sheaves). Taking cohomology we therefore get a long exact sequence

$$\cdots \to \check{H}^{i-1}(\mathscr{X}, F') \to \check{H}^i(\mathscr{X}, F'') \to \check{H}^i(\mathscr{X}, F) \to \check{H}^i(\mathscr{X}, F') \\ \to \check{H}^{i+1}(\mathscr{X}, F'') \to \cdots .$$

In this way the functors $\{\check{H}^i(\mathscr{X}, -)\}_{i\geq 0}$ on the category PMod_Λ are given the structure of a δ-functor.

Proposition 2.3.13. *The functor*

$$\check{H}^i(\mathscr{X}, -) : \mathrm{PMod}_\Lambda \to \mathrm{Ab}$$

is the i-th right derived functor of the functor $\check{H}^0(\mathscr{X}, -)$.

Proof. By [**41**, Chapter III, 1.4 and 1.3A] it suffices to show that $\check{H}^i(\mathscr{X}, I) = 0$ for $i > 0$ and I injective in PMod_Λ. This follows from noting that

$$\check{H}^i(\mathscr{X}, I) = H^i(\mathrm{Hom}_{\mathrm{PMod}}(Z_\bullet, I)),$$

and the functor $\mathrm{Hom}_{\mathrm{PMod}}(-, I)$ is exact since I is injective. $\square$

2.3.14. For $F \in \mathrm{Mod}_\Lambda$ and $i \geq 0$ let $\underline{\mathscr{H}}^i(F) \in \mathrm{PMod}_\Lambda$ denote the presheaf sending $X \in C$ to

$$H^i(C/X, F).$$

Then $\underline{\mathscr{H}}^i(F)$ is the i-th derived functor of the inclusion functor

$$\mathrm{Mod}_\Lambda \hookrightarrow \mathrm{PMod}_\Lambda.$$

From the spectral sequence of a composite of functors [**48**, XX, 9.6] applied to the composite

$$\begin{tikzcd} \mathrm{Mod}_\Lambda \arrow[r] \arrow[rr, bend left, "H^0(C/X,-)"] & \mathrm{PMod}_\Lambda \arrow[r, "\check{H}^0(\mathscr{X},-)"] & \mathrm{Ab} \end{tikzcd}$$

and using 2.3.13 we get a spectral sequence

(2.3.14.1) $$E_2^{st} = \check{H}^s(\mathscr{X}, \underline{\mathscr{H}}^t(F)) \implies H^{s+t}(C/X, F)$$

for any $F \in \mathrm{Mod}_\Lambda$.

Proposition 2.3.15. *A sheaf $F \in \mathrm{Mod}_\Lambda$ is C-acyclic if and only if for all coverings $\mathscr{X} = \{X_i \to X\}_{i\in I}$ in C we have $\check{H}^i(\mathscr{X}, F) = 0$ for $i > 0$.*

Proof. If F is C-acyclic, then $\underline{\mathscr{H}}^i(F) = 0$ for $i \neq 0$ and the spectral sequence (2.3.14.1) gives an isomorphism

$$\check{H}^i(\mathscr{X}, F) \simeq H^i(C/X, F).$$

This implies the "only if" direction.

Conversely suppose F is not C-acyclic and let i_0 be the smallest integer ≥ 1 for which there exists an object $X \in C$ with $H^{i_0}(C/X, F) \neq 0$. Fix such an object X and let $\alpha \in H^{i_0}(C/X, F)$ be a nonzero class. Let $\{X_i \to X\}_{i\in I}$ be a covering such that the image of α in each $H^{i_0}(C/X_i, F)$ is zero. This implies that the image of α in

$$\check{H}^0(\mathscr{X}, \underline{\mathscr{H}}^{i_0}(F))$$

is zero and therefore from the spectral sequence (2.3.14.1) we see that α is induced by a nonzero class in $\check{H}^{i_0}(\mathscr{X}, F)$. This implies the "if" direction. $\square$

COROLLARY 2.3.16. *A sheaf $F \in \mathrm{Mod}_\Lambda$ is C-acyclic if and only if the underlying sheaf of abelian groups is a C-acyclic object of* $\mathrm{Mod}_{\mathbb{Z}}$.

PROOF. This follows from 2.3.15 and the observation that the formation of the complex $C^\bullet(\mathscr{X}, F)$ depends only on the abelian group structure on F and not the Λ-module structure. $\square$

2.3.17. If $F \in \mathrm{Mod}_\Lambda$ is an object and $F \to I^\bullet$ is an injective resolution, then by 2.3.16 the complex of abelian groups underlying $I^\bullet$ is a flasque resolution of the sheaf of abelian groups underlying F. By [**41**, Chapter III, 1.2A] it follows that $\Gamma(T, I^\bullet)$ also computes the cohomology of F viewed as a sheaf of abelian groups. In other words, the cohomology groups do not depend on the Λ-module structure on F.

2.4. Simplicial topoi

2.4.1. The standard way to compute the cohomology of a stack is using a covering as we will see in Chapter 9, §2. In this section we discuss a general framework for dealing with cohomology of coverings. The contents of this section should be viewed as a generalization and unification of Čech cohomology as normally presented in a first year graduate course on algebraic geometry (and slightly generalized in the preceding section) and the Mayer-Vietoris sequence in topology.

2.4.2. Let Δ denote the category whose objects are finite nonempty totally ordered sets and whose morphisms are order preserving maps. For an integer $n \geq 0$ let $[n] \in \Delta$ denote the set $\{0, 1, \dots, n\}$ with the standard ordering. Any object of Δ is isomorphic to $[n]$ for some n, so we will often think of Δ as the category whose objects are $\{[n]\}_{n\geq 0}$ and whose morphisms are the order preserving maps.

If $\mathscr{C}$ is a category, then a *simplicial object in $\mathscr{C}$* is a functor

$$X_\bullet : \Delta^{\mathrm{op}} \to \mathscr{C}.$$

For such a functor we write X_n for $X_\bullet([n])$. We can also consider functors $Y_\bullet : \Delta \to \mathscr{C}$, which are called *cosimplicial objects in $\mathscr{C}$*.

2.4.3. Let T be a topos. Then the category T^Δ of cosimplicial objects in T is also a topos. In fact we can write down a site whose associated topos is T^Δ as follows. Let C be a site with topos T, and assume that C has a final object (this can always be arranged for example using exercise 2.D). Let $F : \Delta^{\mathrm{op}} \to C$ be the constant functor sending all objects of Δ^{op} to the final object of C. Let C_F be the resulting category of Δ-diagrams in C defined as in 2.1.10. Recall that the objects of C_F are pairs (S, U), where $S \in \Delta$ and $U \in C$. A morphism

$$(S, U) \to (S', U')$$

is a pair (δ, f), where $\delta : S' \to S$ is a morphism in Δ and $f : U \to U'$ is a morphism in C. For composable morphisms

$$(S, U) \xrightarrow{(\delta, f)} (S', U') \xrightarrow{(\epsilon, g)} (S'', U'')$$

the composition is defined to be the map

$$(\delta\epsilon, gf) : (S, U) \to (S'', U'').$$

A collection of maps

$$\{(\delta_i, f_i) : (S_i, U_i) \to (S, U)\}$$

is defined to be a covering if each map δ_i is an isomorphism and the maps $\{f_i : U_i \to U\}$ form a covering in C. In other words, as a category C_F is just $\Delta^{\mathrm{op}} \times C$, and so the category of presheaves

$$F_\bullet : C_F^{\mathrm{op}} = (\Delta^{\mathrm{op}} \times C)^{\mathrm{op}} \to \mathrm{Set}$$

is equivalent to the category of functors from Δ to the category of presheaves on C. Now by the definition of the topology on C_F the condition that such a presheaf $F_\bullet$ be a sheaf is equivalent to the condition that for every $S \in \Delta$ the composite functor

$$C_F \simeq S \times C^{\mathrm{op}} \hookrightarrow (\Delta^{\mathrm{op}} \times C)^{\mathrm{op}} \xrightarrow{F_\bullet} \mathrm{Set}$$

is a sheaf. This shows that T^Δ is equivalent to the category of sheaves on C_F.

Remark 2.4.4. More generally the category of functors from a given category to a topos T is a topos.

2.4.5. This can be generalized as follows. For a simplicial object $F_\bullet$ in T we define a topos $T/F_\bullet$ as follows. For each $[n] \in \Delta$ we can consider the localized topos T/F_n. For a morphism $\delta : [n] \to [m]$ we have a morphism of topoi

$$\delta : T/F_m \to T/F_n,$$

defined as in exercise 2.F. The category $T/F_\bullet$ is defined to be the category of systems $\{(G_n, \epsilon_n, G(\delta))\}_{n\in\mathbb{N}}$ consisting of an object $\epsilon_n : G_n \to F_n$ in T/F_n for each n, and for every morphism $\delta : [n] \to [m]$ in Δ map

$$G(\delta) : G_n \to \delta_* G_m$$

in T/F_n such that for a composition

$$[n] \xrightarrow{\delta} [m] \xrightarrow{\epsilon} [k]$$

the map

$$G_k \xrightarrow{G(\epsilon)} \epsilon_* G_m \xrightarrow{\epsilon_* G(\delta)} \epsilon_* \delta_* G_n \simeq (\epsilon\delta)_* G_n$$

is equal to $G(\epsilon\delta)$. A morphism $\{(G_n, \epsilon_n, G(\delta))\}_n \to \{(G'_n, \epsilon_n, G'(\delta))\}_n$ in $T/F_\bullet$ is a collection of maps $\{h_n : G_n \to G'_n\}_{n\in\mathbb{N}}$ in T/F_n such that for any morphism $\delta : [n] \to [m]$ in Δ the diagram

$$\begin{tikzcd} G_n \arrow[r, "G(\delta)"] \arrow[d, "h_n"] & \delta_* G_m \arrow[d, "\delta_* h_m"] \\ G'_n \arrow[r, "G'(\delta)"] & \delta_* G'_m \end{tikzcd}$$

commutes.

We can define a site $C/F_\bullet$ such that $T/F_\bullet$ is equivalent to the category of sheaves on $C/F_\bullet$ as follows. The objects of $C/F_\bullet$ are triples $(n, U, u \in F_n(U))$, where $n \in \mathbb{N}$ is a natural number, $U \in C$ is an object, and $u \in F_n(U)$ is a section. A morphism $(n, U, u) \to (m, V, v)$ is a pair (δ, f), where $\delta : [m] \to [n]$ is a morphism in Δ and $f : U \to V$ is a morphism in C such that the image of v under the map $f^* : F_m(V) \to F_m(U)$ is equal to the image of u under the map $\delta^* : F_n(U) \to F_m(U)$. A collection of morphisms $\{(\delta_i, f_i) : (n_i, U_i, u_i) \to (n, U, u)\}$ is a covering in $C/F_\bullet$ if $n_i = n$ for all i, each δ_i is the identity map, and the collection $\{f_i : U_i \to U\}$ is a covering in C. We leave it as exercise 2.I that $C/F_\bullet$ is a site with associated topos $T/F_\bullet$.

REMARK 2.4.6. For an object $\{(G_n, \epsilon_n, G(\delta))\}_{n\in\mathbb{N}}$ of $T/F_\bullet$ and morphism $\delta : [n] \to [m]$ in Δ, we can by adjunction also view $G(\delta)$ as a morphism

$$\delta^* G_n = G_n \times_{F_n,\delta} F_m \to G_m$$

over F_m.

REMARK 2.4.7. In the case when $F_\bullet$ is the constant punctual sheaf, the topos $T/F_\bullet$ is the category of cosimplicial objects in T.

EXAMPLE 2.4.8. Let T be the category of sheaves of Λ-modules on a topological space X, and let $\{U_i\}_{i\in I}$ be an open covering of X. We then get a simplicial topological space $Y_\bullet$ which in degree n is equal to $\coprod_{i_0,\dots,i_n}(U_{i_0} \cap \cdots \cap U_{i_n})$ and with the transition maps induced by the inclusions (this is the coskeleton of the morphism $\coprod_{i\in I} U_i \to X$ in the sense of 2.4.14 below). Let $F_n \in T$ denote the sheaf on X represented by Y_n, so we get a simplicial object $F_\bullet \in T^{\Delta^{\mathrm{op}}}$. The category $T/F_\bullet$ can in this case be described as follows. For a system $\{(G_n, \epsilon_n, G(\delta))\}_{n\in\mathbb{N}}$ as above, the sheaf G_n is a sheaf on $\coprod_{i_0,\dots,i_n}(U_{i_0} \cap \cdots \cap U_{i_n})$, and hence corresponds to a collection of sheaves $\{G_n^{(i_0,\dots,i_n)}\}$, where $G_n^{(i_0,\dots,i_n)}$ is a sheaf on $U_{i_0} \cap \cdots \cap U_{i_n}$. For a morphism $\delta : [n] \to [m]$ the corresponding map $G(\delta)$ is given by a collection of maps

$$G_n^{(i_{\delta(0)}, i_{\delta(1)},\dots,i_{\delta(n)})}|_{U_{i_0}\cap\cdots U_{i_m}} \to G_m^{(i_0,\dots,i_m)},$$

compatible with compositions.

EXAMPLE 2.4.9. Let $e \in T$ be the final object, and let $\underline{e}$ denote the constant simplicial object taking value e. Then $T/\underline{e}$ is equivalent to the category T^Δ.

2.4.10. If $f : F_\bullet \to F'_\bullet$ is a morphism of simplicial objects of T then there is an induced morphism of topoi

$$(f^*, f_*) : T/F_\bullet \to T/F'_\bullet$$

defined as follows. For $[n] \in \Delta$ let $(f_n^*, f_{n*}) : T/F_n \to T/F'_n$ be the natural morphism of topoi defined in exercise 2.F. For an object $(G_\bullet, \epsilon_\bullet, \{G(\delta)\}) \in T/F_\bullet$ define $f_*(G_\bullet, \epsilon_\bullet, \{G(\delta)\}) \in T/F'_\bullet$ to be the object which in degree n has sheaf $f_{n*}(\epsilon_n : G_n \to F_n)$ and with transition map given for $\delta : [n] \to [m]$ by the map

$$f_{n*}G_n \xrightarrow{G(\delta)} f_{n*}\delta_* G_m \simeq \delta_* f_{m*} G_m,$$

where the last isomorphism is induced by the commutative diagram of topoi

$$\begin{tikzcd} T/F_m \arrow[r, "\delta"] \arrow[d, "f_m"] & T/F_n \arrow[d, "f_n"] \\ T/F'_m \arrow[r, "\delta"] & T/F'_n. \end{tikzcd}$$

The pullback functor f^* sends a system $\{(\epsilon_n : G_n \to F'_n, \{G(\delta)\})\}$ to the system which in degree n is given by $f_n^* G_n \in T/F_n$ and sends a morphism $\delta : [n] \to [m]$ to the morphism

$$f_n^* G_n \xrightarrow{f_n^* G(\delta)} f_n^* \delta_* G_m \longrightarrow \delta_* f_m^* G_m,$$

where the second morphism is the natural adjunction map. Note that f^* commutes with finite projective limits since each f_n^* commutes with finite projective limits.

In particular, taking $F'_\bullet = \underline{e}$ as in 2.4.9 we get a morphism of topoi $T/F_\bullet \to T^\Delta$.

2.4.11. There is a morphism of topoi

$$e : T/F_\bullet \to T.$$

To define this, note first that there is a morphism of topoi $\gamma : T^\Delta \to T$. For $H \in T$ we define $\gamma^* H$ to be the constant cosimplicial object sending every $[n] \in \Delta$ to H and all morphisms to identity maps. Then functor γ_* sends a cosimplicial object $G_\bullet \in T^\Delta$ to the equalizer of the two maps

$$\delta_0^*, \delta_1^* : G_0 \to G_1.$$

The morphism e is defined to be the composition of γ with the natural morphism $T/F_\bullet \to T^\Delta$.

The pullback functor e^* can be described as follows. For $[n] \in \Delta$ let $e_n : T/F_n \to T$ be the localization morphism. Then e^* sends $H \in T$ to the system which in degree n is given by $e_n^* H$, and with transition morphisms given by sending $\delta : [n] \to [m]$ to the natural map

$$e_n^* H \longrightarrow \delta_* \delta^* e_n^* H \xrightarrow{\simeq} \delta_* e_m^* H,$$

where the first map is the adjunction map, and the second is induced by the natural isomorphism $\delta^* e_n^* \simeq e_m^*$.

The functor e_* sends a system $(\epsilon_n : G_n \to F_n, \{G(\delta)\})$ to the equalizer of the two maps $e_{0*} G_0 \to e_{1*} G_1$ induced by the two inclusions $[0] \hookrightarrow [1]$ in Δ.

2.4.12. If $F_\bullet \in T^{\Delta^{\mathrm{op}}}$ and $F'_\bullet \in T'^{\Delta^{\mathrm{op}}}$ and $\lambda : f^* F_\bullet \to F'_\bullet$ is a morphism in $T'^{,\Delta^{\mathrm{op}}}$, then there is an induced morphism of topoi

$$f : T'/F'_\bullet \to T/F_\bullet.$$

This morphism is most easily defined on the level of sites. Let C (resp. C') be a site with associated topos T (resp. T') and assume given a continuous morphism of sites $g : C \to C'$ inducing the morphism of topoi f (by exercise 2.N we can always find such sites). We then get a continuous morphism of sites

$$\tilde{g} : C/F_\bullet \to C'/F'_\bullet.$$

This functor sends a triple $(n, U, u \in F_n(U))$ to the triple $(n, g(U), g^*(u))$, where $g^*(u)$ is the image of u under the map

$$F_n(U) \xrightarrow{\lambda} (f_* F'_n)(U) = F'_n(g(U)).$$

The functor f^* sends a system $\{(\epsilon_n : G_n \to F_n)\}$ to the system which in degree n is given by $f^* \epsilon_n : f^* G_n \to f^* F_n \to F'_n$ and with transition morphisms given by sending $\delta : [n] \to [m]$ to the morphism

$$f^* G_n \xrightarrow{f^* G(\delta)} f^* \delta_* G_m \longrightarrow \delta_* f^* G_m,$$

where the second morphism is the adjunction map.

Note also that the diagram

$$\begin{tikzcd} T'/F'_\bullet \arrow[r, "f"] \arrow[d, "\epsilon'"] & T/F_\bullet \arrow[d, "\epsilon"] \\ T' \arrow[r, "f"] & T \end{tikzcd}$$

commutes.

2.4.13. The most common choice of the simplicial object $F_\bullet$ in the above is the following. Let $e \in T$ denote the final object in T, and let $X \to e$ be a morphism in T which is a covering with respect to the canonical topology (see exercise 2.N). We then get a simplicial object $X_\bullet$ in T by sending $[n] \in \Delta$ to the $(n+1)$-fold product in T:

$$X_n := \overbrace{X \times X \cdots \times X}^{n+1}.$$

If $\delta : [m] \to [n]$ is a morphism in Δ we define the corresponding map

$$\delta_* : X_n \to X_m$$

to be the map whose i-th component is the map

$$\mathrm{pr}_{\delta(i)} : X_n \to X.$$

We can then consider the simplicial topos $T/X_\bullet$ (2.4.5), and the natural augmentation (2.4.11)

$$e : T/X_\bullet \to T.$$

Remark 2.4.14. The simplicial object $X_\bullet$ is called the *coskeleton of* $X \to e$. The construction makes sense in any category with finite fiber products, and is often applied as follows. Let C be a category with finite fiber products, and let $f : X \to Y$ be a morphism in C. Via f we can then view X as an object of C/Y and we can apply the construction to X with the morphism to the final object of C/Y. The result is a simplicial object $X_\bullet$ in C/Y which in level n is given by the $(n+1)$-fold fiber product of X with itself over Y. This simplicial object in C is called the *coskeleton of* $f : X \to Y$.

2.4.15. Let Λ denote a ring in T, and let $\Lambda_\bullet \in T/X_\bullet$ denote the ring $e^*\Lambda$. We then have an exact functor

$$e^* : \mathrm{Mod}_\Lambda \to \mathrm{Mod}_{\Lambda_\bullet}.$$

Composing this exact functor with the δ-functor obtained by taking cohomology in $\mathrm{Mod}_{\Lambda_\bullet}$ we obtain another δ-functor sending $F \in \mathrm{Mod}_\Lambda$ to $H^i(T/X_\bullet, e^*F)$. Since the δ-functor $H^i(T, -)$ is universal, there is for any $F \in \mathrm{Mod}_\Lambda$ a natural map

$$a^i_F : H^i(T, F) \to H^i(T/X_\bullet, e^*F),$$

which for $i = 0$ is the map induced by the adjunction map $F \to e_*e^*F$.

Proposition 2.4.16. *For every* $F \in \mathrm{Mod}_\Lambda$ *and* $i \geq 0$*, the map* a^i_F *is an isomorphism.*

2.4.17. Before starting the proof of 2.4.16, let us make some general observations. If $F_\bullet \in \mathrm{Mod}_{\Lambda_\bullet}$ is a $\Lambda_\bullet$-module in $T/X_\bullet$, then we can form the *Čech complex* *of* F, denoted $\mathscr{C}^\bullet(F_\bullet)$, as follows. This is a complex in Mod_Λ concentrated in degrees ≥ 0, and with degree n-term

$$\mathscr{C}^n(F_\bullet) := e_{n*}F_n,$$

where $e_n : T/X_n \to T$ is the projection. The differential

$$d : \mathscr{C}^n(F_\bullet) \to \mathscr{C}^{n+1}(F_\bullet)$$

is obtained by taking the alternating sum of the maps

$$\partial_i : e_{n*}F_n \to e_{n+1*}F_{n+1}$$

induced by the unique injective order preserving map $[n] \to [n+1]$ whose image does not contain i. Just as in the case of usual Čech cohomology [**41**, Chapter III, §4], we have $d^2 = 0$ so we have a complex $\mathscr{C}^\bullet(F_\bullet)$ in T.

Observe that for $F \in \mathrm{Mod}_\Lambda$, there is a natural map

$$F \to \mathscr{C}^\bullet(e^*F). \tag{2.4.17.1}$$

LEMMA 2.4.18. *For every $F \in \mathrm{Mod}_\Lambda$ the map* (2.4.17.1) *is a quasi-isomorphism.*

PROOF. Since (2.4.17.1) is a morphism of complexes of sheaves and X is a covering of the final object of T, to check that it is a quasi-isomorphism it suffices to verify that it becomes a quasi-isomorphism after pulling back to T/X. Now the restriction to T/X of $\mathscr{C}^\bullet(e^*F)$ is simply the Čech complex associated to the pullback of F and the covering $\mathrm{pr}_2 : X \times X \to X$. This reduces us to the case when there is a section $s : e \to X$ of the covering $X \to e$. In this case we obtain a homotopy

$$\{h_n : \mathscr{C}^n(e^*F) \to \mathscr{C}^{n-1}(e^*F)\}$$

as follows. Let $h_n : \mathscr{C}^n(e^*F) \to \mathscr{C}^{n-1}(e^*F)$ denote the map induced by pullback from the map

$$s_n := s \times \mathrm{id} : X_{n-1} \simeq e \times X_{n-1} \to X \times X_{n-1} = X_n.$$

For $0 \leq i \neq n$ let $\gamma_i^n : X_n \to X_{n-1}$ be the map induced by the inclusion $[n-1] \hookrightarrow [n]$ whose image does not contain i. Then for $i \geq 1$ the diagram

$$\begin{tikzcd} X_n \arrow[r, "s_{n+1}"] \arrow[d, "\gamma_i^n"] & X_{n+1} \arrow[d, "\gamma_i^{n+1}"] \\ X_{n-1} \arrow[r, "s_n"] & X_n \end{tikzcd}$$

commutes. From this it follows that for $i \geq 1$ we have $h_{n+1}\partial_i = \partial_{i-1}h_n$ (maps $\mathscr{C}^n(e^*F) \to \mathscr{C}^n(e^*F)$), and from the definition $h_{n+1}\partial_0 = \mathrm{id}$. This implies that

$$\mathrm{id}_{\mathscr{C}^n(F_\bullet)} = h_{n+1}d - dh_n,$$

so the $\{h_n\}$ give the desired homotopy. □

EXAMPLE 2.4.19. In the setting of 2.4.8, the Čech complex $\mathscr{C}^\bullet(e^*F)$ associated to a sheaf F on X can be described as follows. For a multi-index $\underline{i} = (i_0, \dots, i_n)$ of elements of I, let $j_{\underline{i}} : U_{i_0} \cap \cdots \cap U_{i_n} \hookrightarrow X$ be the inclusion, and let $F_{\underline{i}}$ denote the restriction of F to $U_{i_0} \cap \cdots \cap U_{i_n}$. Then

$$\mathscr{C}^n(e^*F) = \prod_{\underline{i}=(i_0,\dots,i_n)} j_{\underline{i}*}F_{\underline{i}},$$

and the transition maps are induced by the inclusions. Note that this differs from the complex $\mathscr{C}(\mathfrak{U}, \mathscr{F})$ in [**41**, III, §4], whose definition requires the choice of a total ordering on I. However, the two complexes are quasi-isomorphic by [**72**, Theorem 8.3.8].

2.4.20. Next let us describe more explicitly the injective objects in $\mathrm{Mod}_{\Lambda_\bullet}$. For every $[n] \in \Delta$, there is a restriction functor

$$e_n'^* : T/X_\bullet \to T/X_n, \quad F_\bullet \mapsto F_n.$$

This functor is exact, and has a right adjoint e'_{n*} which sends a sheaf $G \in T/X_n$ to the sheaf

$$[m] \mapsto \prod_{\rho:[m]\to[n]} \rho_* G$$

with the natural transition maps. Here the product is taken in the category T/X_m, and $\rho_* G$ denotes the pushforward of G under the morphism of topoi $T/X_n \to T/X_m$ induced by $X(\rho) : X_n \to X_m$. For a morphism $\delta : [m'] \to [m]$ in Δ defining a morphism of topoi $\delta : T/X_m \to T/X_{m'}$ the transition map

$$(e_{n*}G)_{m'} \to \delta_*(e_{n*}G)_m$$

is defined to be the composition of the map

$$\prod_{\rho':[m']\to[n]} \rho'_* G \to \prod_{\rho:[m]\to[n]} \delta_* \rho_* G$$

induced by the map of sets

$$\{\rho : [m] \to [n]\} \to \{\rho' : [m'] \to [n]\}, \quad \rho \mapsto \rho \circ \delta$$

and the natural isomorphism (note that δ_* admits a left adjoint)

$$\prod_{\rho:[m]\to[n]} \delta_* \rho_* G \simeq \delta_* \prod_{\rho:[m]\to[n]} \rho_* G.$$

We therefore have a morphism of topoi

$$e'_n : T/X_n \to T/X_\bullet,$$

such that the composition with the augmentation $e : T/X_\bullet \to T$ is the projection $e_n : T/X_n \to T$. These morphisms of topoi e'_n also induce adjoint funtors

$$e_n'^* : \mathrm{Mod}_{\Lambda_\bullet} \to \mathrm{Mod}_{\Lambda_n}, \quad e'_{n*} : \mathrm{Mod}_{\Lambda_n} \to \mathrm{Mod}_{\Lambda_\bullet}.$$

Since $e_n'^*$ is exact the functor e'_{n*} takes injectives to injectives.

Now if $J_\bullet \in \mathrm{Mod}_{\Lambda_\bullet}$ is an injective object, choose for each n an inclusion $J_n \hookrightarrow I_n$ of J_n into an injective object of Mod_{Λ_n}. The resulting map

$$J_\bullet \to \prod_n e'_{n*} I_n$$

is then an injective morphism of injective objects in $\mathrm{Mod}_{\Lambda_\bullet}$. Since any monomorphism of injective objects is split it follows that $J_\bullet$ is a direct summand of an injective sheaf of the form $\prod_n e'_{n*} I_n$.

LEMMA 2.4.21. *Let $J_\bullet \in \mathrm{Mod}_{\Lambda_\bullet}$ be an injective object. Then for any m the sheaf $e_m'^* J_\bullet \in T/X_m$ is acyclic for e_{m*}.*

PROOF. Since a direct summand of a sheaf acyclic for e_{m*} is also acyclic, it suffices to consider the case when $J_\bullet = \prod_n e'_{n*}I_n$ for $I_n \in \mathrm{Mod}_{\Lambda_n}$ injective. Since a product of acyclic sheaves is also acyclic it further suffices to show that for an injective sheaf $I_n \in \mathrm{Mod}_{\Lambda_n}$ the sheaf

$$e'^*_m e'_{n*}I_n = \prod_{\rho:[m]\to[n]} \rho_* I_n$$

is acyclic. The result then follows from exercise 2.L which shows that the sheaves $\rho_* I_n$ are flasque and therefore acyclic for e_{m*}. $\square$

LEMMA 2.4.22. *Let $J_\bullet \in \mathrm{Mod}_{\Lambda_\bullet}$ be an injective object. Then the natural map $e_*J_\bullet \to \mathscr{C}^\bullet(J_\bullet)$ is a quasi-isomorphism.*

PROOF. Since $J_\bullet$ is a direct summand of an injective sheaf of the form $\prod_n e'_{n*}I_n$, it suffices to consider the case when $J_\bullet = \prod_n e'_{n*}I_n$. Furthermore, the formation of the Čech complex commutes with products so it suffices to consider when $J_\bullet = e'_{n*}I_n$ for an injective sheaf I_n in Mod_{Λ_n}.

Let $e_n : T/X_n \to T$ be the projection. Then from the definition of e_{n*} and $\mathscr{C}^\bullet(-)$, we have

$$\mathscr{C}^\bullet(e'_{n*}I_n) = e_{n*}I \otimes_{\mathbb{Z}} \mathscr{R}^\bullet,$$

where $\mathscr{R}^\bullet$ is the complex of finitely generated free abelian groups given by

$$\mathscr{R}^m = \prod_{[m]\to[n]} \mathbb{Z},$$

and the natural transition maps. In other words, $\mathscr{R}^\bullet$ is the usual Čech complex of the constant sheaf $\mathbb{Z}$ on the 1-point space with respect to the covering given by taking $n+1$-copies of the single nonempty open set. In particular, the complex $\mathscr{R}^\bullet$ is quasi-isomorphic to $\mathbb{Z}$, and therefore $\mathscr{C}^\bullet(e'_{n*}I_n)$ is quasi-isomorphic to $e_{n*}I_n$. $\square$

LEMMA 2.4.23. *Let $F_\bullet \in \mathrm{Mod}_{\Lambda_\bullet}$ be a $\Lambda_\bullet$-module such that for every n and $i > 0$ we have $R^i e_{n*}F_n = 0$. Then for every i the sheaf $R^i e_* F_\bullet$ is isomorphic to $\mathscr{H}^i(\mathscr{C}^\bullet(F_\bullet))$.*

PROOF. Let $F_\bullet \to J^\bullet_\bullet$ be an injective resolution in $\mathrm{Mod}_{\Lambda_\bullet}$. Taking the Čech complex of each $J^n_\bullet$ we obtain a bicomplex of sheaves of Λ-modules $\mathscr{C}(J^\bullet_\bullet)$, and maps

$$\begin{array}{ccc} e_*J^\bullet_\bullet & \xrightarrow{a} & \mathscr{C}(J^\bullet_\bullet) \\ & & \uparrow{\scriptstyle b} \\ & & \mathscr{C}^\bullet(F_\bullet). \end{array}$$

The map a is induced by the maps

$$e_*J^n_\bullet \to \mathscr{C}^\bullet(J^n_\bullet)$$

which are quasi-isomorphisms by 2.4.22. The map b is induced by the maps

$$e_{n*}F_n \to e_{n*}J^\bullet_n$$

which are quasi-isomorphisms by our assumptions on $F_\bullet$. It follows that both a and b induce quasi-isomorphisms with the total complex of $\mathscr{C}(J^\bullet_\bullet)$, and therefore $e_*J^\bullet_\bullet$ and $\mathscr{C}^\bullet(F_\bullet)$ are quasi-isomorphic. $\square$

Proof of Proposition 2.4.16. Note that for $F \in \mathrm{Mod}_\Lambda$ we have

$$H^0(T/X_\bullet, e^*F) = \mathrm{Ker}(\delta_0^* - \delta_1^* : H^0(X_0, F) \to H^0(X_1, F)) \simeq H^0(X, F).$$

By [**41**, Chapter III, Theorem 1.3A] it therefore suffices to show that the δ-functor

$$\{H^i(T/X_\bullet, e^*(-))\}$$

is effaceable. That is we have to show that every object $F \in \mathrm{Mod}_\Lambda$ admits an imbedding $F \hookrightarrow J$, where

$$H^i(T/X_\bullet, e^*J) = 0$$

for all $i > 0$. In fact this is true for any injective $J \in \mathrm{Mod}_\Lambda$. For such a J, the sheaf e^*J has the property that its restriction to each T/X_n is acyclic for e_{n*}, and therefore by 2.4.23 we have $R^ie_*e^*J \simeq \mathscr{H}^i(\mathscr{C}^\bullet(e^*J))$, and these vanish for $i > 0$ by 2.4.18. Therefore we get

$$H^i(T/X_\bullet, e^*J) \simeq H^i(T, J) = 0$$

for $i > 0$. □

Remark 2.4.24. The reader familiar with derived categories will note that the proof shows that for any object F of the bounded below derived category of Λ-modules $D^+(T, \Lambda)$ the map

$$F \to Re_*e^*F$$

in $D^+(T, \Lambda)$ is an isomorphism.

2.4.25. The spectral sequence of a covering.

2.4.26. One very useful consequence of 2.4.16 is the following. Let $F_\bullet \in \mathrm{Mod}_{\Lambda_\bullet}$ be a $\Lambda_\bullet$-module in $T/X_\bullet$ and choose an injective resolution

$$F_\bullet \to I_\bullet^\bullet$$

in the category $\Lambda_\bullet$-modules in $T/X_\bullet$. Taking global sections over each of the T/X_n we then get a bicomplex

$$\begin{tikzcd}
 & \vdots & \vdots & \\
\cdots \arrow[r] & \Gamma(T/X_n, I_n^m) \arrow[r] \arrow[u] & \Gamma(T/X_n, I_n^{m+1}) \arrow[r] \arrow[u] & \cdots \\
\cdots \arrow[r] & \Gamma(T/X_{n-1}, I_{n-1}^m) \arrow[r] \arrow[u] & \Gamma(T/X_{n-1}, I_{n-1}^{m+1}) \arrow[r] \arrow[u] & \cdots \\
 & \vdots \arrow[u] & \vdots \arrow[u] &
\end{tikzcd}$$

whose associated total complex computes the cohomology $H^*(T/X_\bullet, F_\bullet)$ by 2.4.22. Moreover, for a fixed integer n, the row

$$\cdots \to \Gamma(T/X_n, I_n^m) \to \Gamma(T/X_n, I_n^{m+1}) \to \cdots$$

computes the cohomology $H^*(T/X_n, F_n)$. The spectral sequence of a double complex [**48**, p. 819] therefore gives a spectral sequence

$$E_1^{pq} = H^q(T/X_p, F_p) \implies H^{p+q}(T/X_\bullet, F_\bullet).$$

In particular, if $F_\bullet = e^*F$ for some sheaf $F \in \mathrm{Mod}_\Lambda$ we get a spectral sequence
$$E_1^{pq} = H^q(T/X_p, e_p^*F) \implies H^{p+q}(T, F),$$
which we refer to as the *spectral sequence of the covering.*

2.5. Exercises

Exercise 2.A. Let G be a group, and define BG to be the category with one object $*$ and with
$$\mathrm{Hom}(*, *) = G.$$

(a) Show that if F is a presheaf on BG then $F(*)$ admits a natural action of G.

(b) Show that the induced functor
$$(\text{presheaves on } BG) \to (\text{sets with } G\text{-action}), \quad F \mapsto F(*)$$
is an equivalence of categories.

Exercise 2.B. For a scheme X, let $\mathrm{Fet}(X)$ be the category whose objects are finite étale morphisms $U \to X$ and whose morphisms are X-morphisms. Define a family $\{U_i \to U\}$ of morphisms in $\mathrm{Fet}(X)$ to be a covering if the map
$$\coprod_i U_i \to U$$
is surjective.

(a) Prove that $\mathrm{Fet}(X)$ with the above topology is a site.

(b) Let k be a field, and fix an algebraic closure $k \hookrightarrow \Omega$. Let G denote the absolute Galois group of k. Let X denote $\mathrm{Spec}(k)$, and for a sheaf F on $\mathrm{Fet}(X)$ define
$$F_\Omega := \varinjlim_{k \subset k' \subset \Omega} F(\mathrm{Spec}(k'))$$
where the colimit is taken over finite separable extension $k \subset k'$ inside Ω. Show that F_Ω has a natural action of G such that for any $f \in F_\Omega$ the subgroup
$$\{g \in G | g(f) = f\} \subset G$$
is of finite index (in general, if G acts on a set S then the action is called *continuous* if for any $s \in S$ the subgroup of elements $g \in G$ with $g(s) = s$ is of finite index).

(c) With notation as in (b), show that the induced functor
$$(\text{sheaves on } \mathrm{Fet}(X)) \to (\text{sets with continuous } G\text{-action}), \quad F \mapsto F_\Omega$$
is an equivalence of categories.

(d) (requires experience with étale morphisms) Let X be a normal connected scheme, and fix an algebraic closure $k(X) \hookrightarrow \Omega$. Let $k(X) \subset K \subset \Omega$ be the compositum of all finite extensions $k(X) \subset L \subset \Omega$ such that the normalization of X in L is étale over X. Let G denote the Galois group of K over $k(X)$. Show that the category of sheaves on $\mathrm{Fet}(X)$ is naturally equivalent to the category of sets with continuous action of G.

Exercise 2.C. A topological space X is called *sober* if every irreducible closed subset has a unique generic point (for example X could be the underlying topological space of a scheme). For a topological space let $Op(X)$ denote the site of open subsets of X, and let X_{cl} denote the topos of sheaves on this site.

Let pt denote the topos associated to the one-pointed space (so pt is the category of sets).

(a) Let $f : X \to Y$ be a continuous map of topological spaces. Show that the functor

$$f^{-1} : Op(Y) \to Op(X), \quad U \mapsto f^{-1}(U)$$

is a continuous functor and induces a morphism of topoi $f : X_{cl} \to Y_{cl}$.

(b) Show that every point of $x \in X$ defines a point $j_x : \mathrm{pt} \to X_{cl}$ of the topos X_{cl} for which the pullback functor j_x^* sends a sheaf F to its stalk F_x at x.

(c) Let X be a sober topological space, and let $j : \mathrm{pt} \to X_{cl}$ be a point of X_{cl}. Let $U \subset X$ be the union of those open sets $V \subset X$ for which j^*h_V is the empty set. Show that U is open and that the complement Z is irreducible. Let $\eta \in Z$ be the generic point, and show that $j = j_\eta$.

(d) Let X be any topological space and let Y be a sober topological space. Show that any morphism of topoi

$$f : X_{cl} \to Y_{cl}$$

is induced by a unique continuous map $X \to Y$.

Exercise 2.D. Let C be a site with associated topos T, and let $G \in T$ be an object. Define T/G to be the category whose objects are morphisms $F \to G$ of sheaves on C and whose morphisms are morphisms over G. Also define C/G to be the category whose objects are pairs (X', x'), where $X' \in C$ is an object and $x' \in G(X')$ (equivalently a morphism $x' : h_{X'} \to G$ of presheaves on C). A morphism $(X'', x'') \to (X', x')$ in C/G is a morphism $h : X'' \to X'$ such that $h^*x' = x''$ in $G(X'')$. Note that in the case when $G = h_X$ for an object $X \in C$ with h_X a sheaf, the category C/G is simply the category whose objects are morphisms $X' \to X$ in C and whose morphisms are X-morphisms, and in this case we often write C/X instead of C/h_X.

(a) Define a family of morphisms $\{(X'_i, x'_i) \to (X', x')\}$ in C/G to be a covering if $\{X'_i \to X'\}$ is a covering in C. Show that this defines a Grothendieck topology on C/G.

(b) Show that the category of sheaves on C/G is equivalent to T/G. In particular, T/G is a topos (called the *localized topos*).

(c) Define

$$j^* : T \to T/G$$

by sending F to $F \times G$ with the projection to G. Show that this functor j^* commutes with finite limits and has a right adjoint j_* given by the formula

$$j_*H(Y) = \mathrm{Hom}_{T/G}(h_Y^a \times G, H).$$

In particular, there is a morphism of topoi

$$j : T/G \to T.$$

(d) Let Λ be a sheaf of rings in T, and let Λ_G denote the sheaf of rings $j^*\Lambda$ in T/G. Show that the functor j^* sends injective Λ-modules in T to injective Λ_G-modules in T/G.

Exercise 2.E. Verify the identity (2.3.11.1).

EXERCISE 2.F. Let T be a topos and let $f : F \to G$ be a morphism in T. Show that there is a morphism of topoi

$$(f^*, f_*) : T/F \to T/G$$

in which f^* sends $(H \to G)$ to $\mathrm{pr}_2 : H \times_G F \to F$ and f_* is characterized by the property that for any $H \to G$ in T/G and $M \to F$ in T/F we have

$$\mathrm{Hom}_{T/G}(H, f_*M) = \mathrm{Hom}_{T/F}(H \times_G F, M).$$

EXERCISE 2.G. Let Δ denote the simplicial category 2.4.2.

(a) Show that for each n and $i = 0, \ldots, n$ there exists a unique surjective map

$$s_i : [n+1] \to [n]$$

with two elements mapping to i, and a unique injective map

$$\partial_i : [n-1] \to [n]$$

whose image does not contain i.

(b) Show that these maps satisfy the following identities:

$$s_j s_i = s_i s_{j-1}, \quad i < j,$$

$$\partial_j \partial_i = \partial_i \partial_{j-1}, \quad i \leq j,$$

and

$$\partial_j s_i = \begin{cases} s_i \partial_{j-1} & \text{if } i < j, \\ \text{identity} & \text{if } i = j \text{ or } i = j+1, \\ s_{i-1} \partial_j & \text{if } i > j+1. \end{cases}$$

(c) Let $\mathscr{C}$ be a category, and suppose given for every $n \geq 0$ an object X_n of $\mathscr{C}$, and for every $i = 0, \ldots, n$ maps

$$f_i : X_n \to X_{n+1}, \quad g_i : X_n \to X_{n-1}$$

such that the identities

$$f_i f_j = f_{j-1} f_i, \quad i < j,$$

$$g_i g_j = g_{j-1} g_i, \quad i \leq j,$$

and

$$f_i g_j = \begin{cases} g_{j-1} f_i & \text{if } i < j, \\ \text{identity} & \text{if } i = j \text{ or } i = j+1, \\ g_j f_{i-1} & \text{if } i > j+1. \end{cases}$$

Show that there exists a unique simplicial object

$$X_\bullet : \Delta^{\mathrm{op}} \to \mathscr{C}$$

such that $X([n]) = X_n$ and such that

$$X_\bullet(s_i) = f_i, \quad X_\bullet(\partial_i) = g_i.$$

EXERCISE 2.H. Let C be a site, and let $C' \subset C$ be a full subcategory such that the following hold:

(i) For every $U \in C$ there exists a covering $\{U_i \to U\}_{i \in I}$ of U with $U_i \in C'$ for every i.

(ii) If $\{U_i \to U\}$ is a covering of an object $U \in C'$ with $U_i \in C'$ for all i, then for any morphism $V \to U$ in C' the fiber products $V \times_U U_i$ are in C'.

Show that there is a Grothendieck topology on C' in which a collection of morphisms $\{U_i \to U\}$ in C' is a covering if and only if it is a covering in C. Furthermore, show that the topos defined by C' with this topology is equivalent to the topos defined by C.

Exercise 2.I. Show that $C/F_\bullet$, defined in 2.4.5, is a site with associated topos isomorphic to $T/F_\bullet$.

Exercise 2.J. Let T be a topos and Λ a ring in T. Show that the category of left Λ-modules in T is an abelian category.

Exercise 2.K. Let $(f, f^\sharp) : (T', \Lambda') \to (T, \Lambda)$ be a morphism of ringed topoi. Show that if M' is a Λ'-module in T' then f_*M' has a natural structure of a Λ-module in T, and that we get a functor

$$f_* : \mathrm{Mod}_{(T',\Lambda')} \to \mathrm{Mod}_{(T,\Lambda)}.$$

Further show that this functor has a left adjoint

$$\mathrm{Mod}_{(T,\Lambda)} \to \mathrm{Mod}_{(T',\Lambda')}.$$

As in the case of ringed spaces, when considering ringed topoi we usually write f^{-1} for the pullback functor for sheaves of sets, and f^* for the pullback on modules.

Exercise 2.L. Let $(f, f^\sharp) : (T', \Lambda') \to (T, \Lambda)$ be a morphism of ringed topoi. Show that the induced functor

$$f_* : \mathrm{Mod}_{(T',\Lambda')} \to \mathrm{Mod}_{(T,\Lambda)}$$

takes flasque sheaves to flasque sheaves and that for $F \in \mathrm{Mod}_{(T',\Lambda')}$ flasque we have $R^i f_* F = 0$ for $i > 0$.

Exercise 2.M. Let (T, Λ) be a ringed topos, and let $M \in \mathrm{Mod}_\Lambda$ be a Λ-module. Let $EXT(\Lambda, M)$ be the category whose objects are short exact sequences in Mod_Λ,

$$0 \to M \to E \to \Lambda \to 0,$$

and whose morphisms are the commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d, "\sigma"] & \Lambda \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M \arrow[r] & E' \arrow[r] & \Lambda \arrow[r] & 0.
\end{tikzcd}$$

Show that the set of isomorphism classes in $EXT(\Lambda, M)$ is canonically in bijection with $H^1(T, M)$.

Exercise 2.N. Let C be a category with fiber products. Define a collection of morphisms $\{X_i \to X\}$ to be a covering if for every morphism $Y \to X$ and $Z \in C$ the sequence

$$\mathrm{Hom}_C(Y, Z) \to \prod_i \mathrm{Hom}_C(X_i \times_X Y, Z) \rightrightarrows \prod_{i,j} \mathrm{Hom}_C(X_i \times_X X_j \times_X Y, Z)$$

is exact.

(i) Show that this defines a Grothendieck topology on C (called the *canonical topology*) such that every representable presheaf on C is a sheaf.

(ii) Show that if T is a topos, then the functor sending $F \in T$ to the functor

$$T^{\mathrm{op}} \to \mathrm{Set}, \quad G \mapsto \mathrm{Hom}_T(G, F)$$

defines an equivalence between T and the topos of sheaves on T endowed with the canonical topology.

(iii) Let $f : T' \to T$ be a morphism of topoi. Show that the pullback functor $f^* : T \to T'$ is continuous with respect to the canonical topologies on T and T' and that the morphism of topoi $T' \to T$ induced by this morphism of sites and the identifications in (ii) is equal to the original morphism of topoi f.

Exercise 2.O. Let $\Gamma : C \to D$ be a functor which admits a right adjoint.

(i) Show that if $F : I \to C$ is a functor such that $\varprojlim F$ exists in C, then the limit over the composition

$$I \xrightarrow{F} C \xrightarrow{\Gamma} D$$

exists in D, and is equal to $\Gamma(\varprojlim F)$.

(ii) Use (i) to give another proof of Proposition 2.2.16.

CHAPTER 3

Fibered categories

One of the main problems when defining the notion of stack is how to deal with fibered products, pullbacks, and other familiar constructions without making any choices. For example, if $f : X \to Y$ is a continuous map of topological spaces and F is a sheaf of abelian groups on Y, then the pullback sheaf f^*F on X is only well defined up to canonical isomorphism. Namely the sheaf f^*F together with the adjunction map $F \to f_*f^*F$ is an initial object in the category of pairs (G, ϵ), where G is an abelian sheaf on X and $\epsilon : F \to f_*G$ is a morphism of abelian sheaves on Y. One could make a construction defining f^* canonically using the espace étale of a sheaf, but then one does not get associativity of the pullback. Usually when dealing with schemes this kind of ambiguity in definitions does not pose much of a problem, but when considering stacks more care must be taken. There are various approaches for dealing with this. Here we will follow the original approach of Grothendieck via fibered categories [**35**, Exposé 190].

In the classical study of moduli problems, one considers a functor

$$F : (\text{schemes})^{\text{op}} \to \text{Set}$$

and ask if this functor is representable. As discussed in the introduction, the presence of automorphisms often prevents this from being the case. For example, the functor $M_{1,1}$ sending a scheme S to the set of isomorphism classes of elliptic curves over S is not representable. The key insight which leads to the theory of stacks is that the problem arises from passing to isomorphism classes, and rather we should consider also the automorphisms. This leads one to consider the idea of a functor $\mathscr{M}_{1,1}$ sending a scheme S to the category whose objects are elliptic curves over S and whose morphisms are isomorphisms of elliptic curves over S. To make this idea precise, however, one needs to wrestle with problems of higher category theory. The machinery of fibered categories is one way to handle this. The basic point is to consider instead of a functor the category $\mathscr{M}_{1,1}$ whose objects are pairs $(S, (E, e))$ consisting of a scheme S and an elliptic curve over S, and whose morphisms $(S', (E', e')) \to (S, (E, e))$ are pairs (f, g) consisting of morphisms $f : S' \to S$ and $g : E' \to E$ such that the induced morphism $E' \to E \times_S S'$ is an isomorphism of elliptic curves over S' (note that this last statement is independent of any choice of fiber product $E \times_S S'$). There is a functor

$$p : \mathscr{M}_{1,1} \to (\text{schemes})$$

sending $(S, (E, e))$ to S. This is a basic example of a fibered category. Note that for any fixed scheme S the category $\mathscr{M}_{1,1}(S)$ (the "fiber") whose objects are objects of $\mathscr{M}_{1,1}$ with image under p equal to S (strict equality, not just isomorphisms) and whose morphisms are morphisms projecting to the identity on S is simply the category of elliptic curves over S and isomorphisms between them.

In this chapter we develop the basic formalism of fibered categories. Particular attention is paid to categories fibered in groupoids which are the most important for the purposes of the theory of stacks.

Our exposition here follows closely that in [**71**], where many more details and extensions can be found. The original source for this material is [**36**, VI].

3.1. Definition of fibered category and basic properties

Let C be a category.

Definition 3.1.1. A *category over C* is a pair (F, p), where F is a category and $p : F \to C$ is a functor.

A morphism $\phi : u \to v$ in F is called *cartesian* if for any other object $w \in F$ with a morphism $\psi : w \to v$ and factorization

$$p(w) \xrightarrow{h} p(u) \xrightarrow{p(\phi)} p(v)$$

of $p(\psi)$, there exists a unique morphism $\lambda : w \to u$ such that $\phi \circ \lambda = \psi$ and $p(\lambda) = h$. In a picture:

$$\begin{tikzcd}
w \arrow[r, dashed] \arrow[rr, bend left, "\psi"] \arrow[d, mapsto] & u \arrow[r, "\phi"] \arrow[d, mapsto] & v \arrow[d, mapsto] \\
p(w) \arrow[r, "h"] & p(u) \arrow[r, "p(\phi)"] & p(v).
\end{tikzcd}$$

If $\phi : u \to v$ is a cartesian morphism, then the object u is called a *pullback* of v along $p(\phi)$. Note that if $\phi : u \to v$ and $\phi' : u' \to v$ are two pullbacks of v along $p(\phi)$, then there exists a unique isomorphism $\lambda : u \to u'$ with $p(\lambda) = \mathrm{id}_{p(u)}$ such that $\phi' \circ \lambda = \phi$.

3.1.2. For a category $p : F \to C$ over C and object $U \in C$, we write $F(U)$ for the category whose objects are objects $u \in F$ such that $p(u) = U$ (actual equality in C) and whose morphisms are morphisms $f : u' \to u$ in F such that $p(f) = \mathrm{id}_U$.

Definition 3.1.3. (i) A *fibered category over C* is a category $p : F \to C$ over C such that for every morphism $f : U \to V$ in C and $v \in F(V)$ there exists a cartesian morphism $\phi : u \to v$ such that $p(\phi) = f$ (so in particular $u \in F(U)$).

(ii) If

$$p_F : F \to C, \quad p_G : G \to C$$

are fibered categories, then a morphism of fibered categories $F \to G$ is a functor $g : F \to G$ such that the following hold:

(1) $p_G \circ g = p_F$ (equality of functors, *not* isomorphism).
(2) g sends cartesian morphisms in F to cartesian morphisms in G.

(iii) If

$$g, g' : F \to G$$

are morphisms of fibered categories, then a *base preserving natural transformation* $\alpha : g \to g'$ is a natural transformation of functors such that for every $u \in F$ the morphism $\alpha_u : g(u) \to g'(u)$ in G projects to the identity morphism in C (so α_u is a morphism in $G(p_F(u))$). We denote by

$$HOM_C(F, G)$$

the category whose objects are morphisms of fibered categories $F \to G$ and whose morphisms are base preserving natural transformations.

REMARK 3.1.4. For a fixed category C, the collection of categories fibered over C is an example of a 2-*category*. Roughly, a 2-category is a generalization of the notion of a category in which for any two objects A, B the morphisms between them form a category, instead of a set. Since we will not use this higher category theory language in this book, however, we do not discuss this notion further.

EXAMPLE 3.1.5. A basic example to illustrate the utility of fibered categories is the following. Let $f : X \to Y$ be a morphism of schemes. For any morphism $t : T \to Y$ one can then consider the fiber product $T \times_Y X$. This fiber product, however, is only unique up to unique isomorphism and so when one considers 'the fiber product $T \times_Y X$' there is, implicitly, a choice being made. While usually not a concern in this setting, this kind of choice becomes a nontrivial technical obstacle in the development of the theory of stacks, and fibered categories are used to address this problem. In this example of fiber products, fibered categories enter as follows.

Let C be the category of Y-schemes, and let F be the category whose objects are collections of data

$$(t : T \to Y, P, a, b),$$

where $t : T \to Y$ is a Y-scheme, P is a scheme, and $a : P \to T$ and $b : P \to X$ are morphisms such that the square

$$\begin{tikzcd} P \arrow[r, "b"] \arrow[d, "a"'] & X \arrow[d, "f"] \\ T \arrow[r, "t"] & Y \end{tikzcd}$$

is cartesian (note that the statement that a square is cartesian is a property of the diagram and does not require any choices). A morphism

$$(t' : T' \to Y, P', a', b') \to (t : T \to Y, P, a, b)$$

in F is a pair of morphisms (α, β), where $\alpha : T' \to T$ is a Y-morphism and $\beta : P' \to P$ is a morphism such that $a \circ \beta = \alpha \circ a'$ and $b \circ \beta = b'$.

There is a functor $p : F \to C$ sending $(t : T \to Y, P, a, b)$ to $t : T \to Y$. For $t : T \to Y$ in C the category $F(t : T \to Y)$ is the category of cartesian diagrams

$$\begin{tikzcd} P \arrow[r] \arrow[d] & X \arrow[d] \\ T \arrow[r] & Y. \end{tikzcd}$$

Any two objects of $F(T)$ are uniquely isomorphic, and the choice of an object of this category is equivalent to the choice of a fiber product $T \times_Y X$. The axioms for a fibered category in this example amount to the existence of fiber products in the category of schemes.

LEMMA 3.1.6. *If $p : F \to C$ is a fibered category, then every morphism $\psi : w \to v$ can be factored as*

$$w \xrightarrow{\lambda} u \xrightarrow{\phi} v,$$

where ϕ is a cartesian morphism and λ is a morphism in $F(p(w))$.

PROOF. By the definition of fibered category, there exists a cartesian morphism $\phi : u \to v$ over $p(\psi) : p(w) \to p(v)$, and by the universal property of a cartesian morphism the morphism ψ factors uniquely through a morphism $\lambda : w \to u$ projecting to the identity. □

3.1.7. Note that if $g : F \to G$ is a morphism of fibered categories over C, then for every object $U \in C$ we obtain a functor

$$g_U : F(U) \to G(U).$$

LEMMA 3.1.8. *Let $g : F \to G$ be a morphism of fibered categories over C. Then g is fully faithful (as a functor between the categories F and G) if and only if for every object $U \in C$ the functor*

$$g_U : F(U) \to G(U)$$

is fully faithful.

PROOF. The "only if" direction is immediate.

For the "if" direction let $x, y \in F$ be two objects. We then have a commutative diagram:

$$\begin{tikzcd} \mathrm{Hom}_F(x,y) \arrow[rr, "g"] \arrow[dr, "p_F"'] & & \mathrm{Hom}_G(g(x), g(y)) \arrow[dl, "p_G"] \\ & \mathrm{Hom}_C(p_F(x), p_F(y)). & \end{tikzcd}$$

Fix a morphism

$$h : p_F(x) \to p_F(y)$$

in C. Then to prove the full faithfulness, it suffices to show that the map g induces a bijection on the fibers over the element

$$h \in \mathrm{Hom}_C(p_F(x), p_F(y)).$$

Let

$$\tilde{h} : y' \to y$$

be a cartesian arrow over h. Then by the universal property of a cartesian arrow we have bijections

$$\mathrm{Hom}_{F(p_F(x))}(x, y') \to \{\text{morphisms } x \to y \text{ mapping to } h\}$$

$$(z : x \to y') \mapsto (\tilde{h} \circ z : x \to y),$$

and

$$\mathrm{Hom}_{G(p_F(x))}(g(x), g(y')) \to \{\text{morphisms } g(x) \to g(y) \text{ mapping to } h\},$$

$$(z : g(x) \to g(y')) \mapsto (g(\tilde{h}) \circ z : g(x) \to g(y)).$$

From this we obtain the lemma as the functor g induces a bijection

$$\mathrm{Hom}_{F(p_F(x))}(x, y') \to \mathrm{Hom}_{G(p_F(x))}(g(x), g(y'))$$

by assumption. □

DEFINITION 3.1.9. A morphism $g : F \to G$ of fibered categories over C is an *equivalence* if there exists a morphism of fibered categories $h : G \to F$ and base preserving isomorphisms

$$h \circ g \simeq \mathrm{id}_F, \quad g \circ h \simeq \mathrm{id}_G.$$

Proposition 3.1.10. *A morphism of fibered categories $g : F \to G$ over C is an equivalence if and only if for every $U \in C$ the functor*

$$g_U : F(U) \to G(U)$$

is an equivalence. Equivalently, if and only if for every $U \in C$ the functor g_U is fully faithful and essentially surjective.

Proof. For every $y \in G$ choose an object $h(y) \in F(p_G(y))$ and an isomorphism

$$\alpha_y : y \to g_{p_G(y)}(h(y))$$

in $G(p_G(y))$. This is possible since $g_{p_G(y)}$ is an equivalence. If $\phi : y \to y'$ is an arrow in G, then there exists by 3.1.8 a unique morphism

$$h(\phi) : h(y) \to h(y')$$

in F such that the diagram

$$\begin{tikzcd}
y \arrow[r, "\phi"] \arrow[d, "\alpha_y"] & y' \arrow[d, "\alpha_{y'}"] \\
g(h(y)) \arrow[r, "g(h(\phi))"] & g(h(y'))
\end{tikzcd}$$

commutes. In this way we get a functor

$$h : G \to F,$$

and the isomorphisms α_y define an isomorphism

$$\alpha : \mathrm{id}_G \to g \circ h.$$

Next we check that h takes cartesian arrows to cartesian arrows. So suppose $y \to y'$ is a cartesian arrow in G, and consider the diagram

$$\begin{tikzcd}
w \arrow[r, dashed] \arrow[rr, bend left, "\psi"] \arrow[d, mapsto] & h(y) \arrow[r] \arrow[d, mapsto] & h(y') \arrow[d, mapsto] \\
p_F(w) \arrow[r] & p_F(h(y)) \arrow[r] & p_F(h(y')).
\end{tikzcd}$$

Since the functor g is fully faithful, to show that there exists a unique dotted arrow filling in the diagram, it suffices to do so after applying the functor g and the isomorphism α. That is, it suffices to find a unique dotted arrow filling in the diagram

$$\begin{tikzcd}
g(w) \arrow[r, dashed] \arrow[rr, bend left, "g(\psi)"] \arrow[d, mapsto] & y \arrow[r] \arrow[d, mapsto] & y' \arrow[d, mapsto] \\
p_G(g(w)) \arrow[r] & p_G(y) \arrow[r] & p_G(y').
\end{tikzcd}$$

Since $y \to y'$ was assumed cartesian, there exists a unique such arrow, and therefore h takes cartesian arrows to cartesian arrows.

Finally, we construct an isomorphism

$$\beta : \mathrm{id}_F \to h \circ g.$$

For this, note that for any $x \in F$ we have an isomorphism

$$\alpha_{g(x)} : g(x) \to g(h(g(x))).$$

Since g is fully faithful, there exists a unique isomorphism

$$\beta_x : x \to h(g(x))$$

projecting to the identity in C and such that $g(\beta_x) = \alpha_{g(x)}$. We leave to the reader the verification that these maps β_x define an isomorphism of functors as desired. □

3.2. The 2-Yoneda lemma

The classical Yoneda lemma embeds a category into its associated category of functors to sets. This point of view is crucial for the development of algebraic spaces, which are by definition certain functors on the category of schemes. For the theory of stacks, we also need a version of the Yoneda lemma for fibered categories, which we explain in this section.

3.2.1. Let

$$p_F : F \to C$$

be a fibered category over a category C. Let $X \in C$ be an object, and let C/X be the localized category of objects over X. The category C/X is a fibered category over C with functor $C/X \to C$ sending $(Y \to X) \in C/X$ to Y (see exercise 3.A). There is a functor

$$\xi : HOM_C((C/X), F) \to F(X)$$

sending a morphism of fibered categories

$$g : C/X \to F$$

to $g(\mathrm{id} : X \to X)$.

Proposition 3.2.2 (2-Yoneda lemma [**71**, 3.6.2]). *The functor ξ is an equivalence of categories.*

Proof. We define a quasi-inverse

$$\eta : F(X) \to HOM_C((C/X), F) \tag{3.2.2.1}$$

as follows. Let $x \in F(X)$ be an object, and for every arrow $\phi : Y \to X$ choose a pullback $\phi^* x \in F(Y)$ of x. Define

$$\eta_x : (C/X) \to F$$

by

$$(\phi : Y \to X) \mapsto \phi^* x.$$

Given a morphism

$$\begin{tikzcd} Y' \arrow[rr, "\epsilon"] \arrow[dr, "\phi'"'] & & Y \arrow[dl, "\phi"] \\ & X & \end{tikzcd}$$

in C/X, define

$$\eta_x(\epsilon) : \phi'^* x \to \phi^* x$$

to be the unique dotted arrow filling in the diagram

$$\begin{tikzcd} \phi'^*x \arrow[r, dashed] \arrow[rr, bend left] & \phi^*x \arrow[r] & x \\ Y' \arrow[r, "\epsilon"] & Y \arrow[r, "\phi"] & X. \end{tikzcd}$$

We leave to the reader the verification that this definition of $\eta_x(\epsilon)$ is compatible with compositions so η_x is a morphism of fibered categories.

Notice also that given a morphism $f : x' \to x$ in $F(X)$ we get a morphism of functors

$$\eta_f : \eta_{x'} \to \eta_x.$$

Indeed, for any morphism $\phi : Y \to X$ and pullbacks ϕ^*x' and ϕ^*x of x' and x, respectively, we define $\eta_f(\phi)$ to be the unique (since ϕ^*x is a pullback) morphism filling in the square

$$\begin{tikzcd} \phi^*x' \arrow[r, "\eta_f(\phi)"] \arrow[d] & \phi^*x \arrow[d] \\ x' \arrow[r, "f"] & x. \end{tikzcd}$$

In this way we obtain a morphism of functors η_f, and in turn a functor η as in (3.2.2.1).

The composite functor

$$F(X) \xrightarrow{\eta} HOM_C((C/X), F) \xrightarrow{\xi} F(X)$$

sends $x \in F(X)$ to $\mathrm{id}_X^* x$, which is canonically isomorphic to x. This canonical isomorphism defines an isomorphism of functors

$$\mathrm{id}_{F(X)} \simeq \xi \circ \eta.$$

Next consider the composition

$$HOM_C((C/X), F) \xrightarrow{\xi} F(X) \xrightarrow{\eta} HOM_C((C/X), F).$$

This composition sends $f : (C/X) \to F$ to the functor $C/X \to F$ which to any $\phi : Y \to X$ associates $\phi^* f(\mathrm{id}_X)$. This functor is canonically isomorphic to f. To see this it suffices to show that $f(\phi)$ is a pullback of $f(\mathrm{id}_X)$. For this note that id_X is a final object in C/X so there exists a unique cartesian arrow $\phi \to \mathrm{id}_X$ in C/X. Since f takes cartesian arrows to cartesian arrows by assumption this implies that $f(\phi) \to f(\mathrm{id}_X)$ is also a cartesian arrow.

We therefore also get that $\mathrm{id}_{HOM_C((C/X),F)} \simeq \eta \circ \xi$. □

Corollary 3.2.3. *Let $X, Y \in C$ be two objects. Then the functor*

$$HOM_C((C/X), (C/Y)) \to \mathrm{Hom}_C(X, Y), \quad f \mapsto f(\mathrm{id}_X)$$

is an equivalence of categories.

Proof. Take $F = (C/Y)$ in 3.2.2. □

Remark 3.2.4. If X is an object of a category C and $F \to C$ is a fibered category, we will often write

$$X \to F$$

for an object of $F(X)$. This is justified by 3.2.3, which shows that we can also think of objects of $F(X)$ as morphisms of fibered categories $C/X \to F$.

More generally we will view presheaves on C as fibered categories as follows.

Definition 3.2.5. A *category fibered in sets over C* is a fibered category

$$p : F \to C$$

such that for every $U \in C$ the only morphisms in $F(U)$ are the identity morphisms (that is, $F(U)$ is a set).

Lemma 3.2.6. *Let $q : G \to C$ be a fibered category, and let $p : F \to C$ be a category fibered in sets over C. Then*

$$HOM_C(G, F)$$

is a set.

Proof. If $f, g : G \to F$ are morphisms of fibered categories and $\alpha : f \to g$ is a morphism in $HOM_C(G, F)$, then for every $x \in G$ over $X \in C$ we get a morphism

$$\alpha_x : f(x) \to g(x)$$

in $F(X)$. Since F is fibered in sets this arrow must be the identity morphism so we must have $f(x) = g(x)$ for all $x \in G$. This implies that $f = g$ (note that verifying that f and g agree on morphisms is immediate since F is fibered in sets). $\square$

In particular, we can speak of the category of categories fibered in sets over C. Note also that for a category fibered in sets $p : F \to C$ we have for any morphism $g : V \to U$ in C a well-defined pullback map

$$g^* : F(U) \to F(V)$$

compatible with composition.

3.2.7. Conversely, let

$$F : C^{\mathrm{op}} \to \mathrm{Set}$$

be a functor. Then F has an associated fibered category $\mathcal{F}$ defined as follows. The objects of $\mathcal{F}$ are pairs (U, x), where $U \in C$ and $x \in F(U)$. A morphism

$$(U', x') \to (U, x)$$

is a morphism $g : U' \to U$ in C such that $g^* x = x'$ in $F(U')$. The verification that $\mathcal{F}$ is a fibered category is exercise 3.B.

Proposition 3.2.8. *The functor*

$$\Gamma : (\text{presheaves on } C) \to (\text{categories fibered in sets over } C)$$

sending a presheaf F to the fibered category $\mathcal{F}$ defined above is an equivalence of categories.

PROOF. Given a category fibered in sets $p : \mathcal{F} \to C$, define

$$F : C^{\mathrm{op}} \to \mathrm{Set}$$

by sending $U \in C$ to the set

$$\mathcal{F}(U) \simeq HOM_C((C/U), \mathcal{F})$$

and a morphism $g : V \to U$ to $g^* : \mathcal{F}(U) \to \mathcal{F}(V)$.

Then $\Gamma(F)$ is the fibered category whose fiber over $U \in C$ is $\mathcal{F}(U)$ and whose morphisms are defined by the maps g^*. In particular there is a natural map

$$\Gamma(F) \to \mathcal{F}$$

which is an equivalence since it induces an equivalence in the fiber over any $U \in C$.

It is also clear that if we apply the above construction with $\mathcal{F} = \Gamma(F)$ for some presheaf F, then we simply recover the presheaf F. $\square$

REMARK 3.2.9. The preceding proposition enables us to also view presheaves as fibered categories, and we will often not distinguish in the notation between a functor F and the corresponding fibered category.

3.3. Splittings of fibered categories

Though fibered categories are crucial for dealing with the categorical issues that arise in the theory of stacks, it is often convenient to think of a fibered category $p : F \to C$ as the collection of categories $F(U)$ ($U \in C$) together with pullback functors $F(U) \to F(V)$ for morphisms $V \to U$. The key notion that puts this on solid footing is the notion of a splitting of a fibered category which we explain in this section.

3.3.1. Let $p : F \to C$ be a fibered category. A *splitting of* p is a subcategory $K \subset F$ such that the following hold:

(i) Every arrow in K is cartesian.
(ii) For every morphism $f : U \to V$ in C and $v \in F(V)$ there exists a unique arrow $u \to v$ in K over f.
(iii) If $u \in F(U)$ is an object over some $U \in C$, then $\mathrm{id}_u : u \to u$ is in K.

We refer to the pair (F, K) as a *split fibered category*

THEOREM 3.3.2 ([**71**, 3.45]). *Let $p : F \to C$ be a fibered category. Then there exists a split fibered category $(\widetilde{F}, K)$ over C and an equivalence of fibered categories $\widetilde{F} \to F$.*

PROOF. We sketch the construction of $\widetilde{F}$ leaving the verification that it works to the reader.

Define the category $\widetilde{F}$ as follows. The objects of $\widetilde{F}$ are pairs (U, u), where $U \in C$ is an object and $u : C/U \to F$ is a morphism of fibered categories. A morphism

$$(V, v) \to (U, u)$$

in $\widetilde{F}$ is a pair (g, α), where $g : V \to U$ is a morphism in C and $\alpha : v \to u \circ g$ is an isomorphism in $HOM_C(C/V, F)$ between the two morphisms

$$(C/V) \xrightarrow{g} (C/U) \xrightarrow{u} F$$

and

$$v : (C/V) \to F.$$

The composition of arrows

$$(W, w) \xrightarrow{(g',\alpha')} (V, v) \xrightarrow{(g,\alpha)} (U, u)$$

is given by the composite morphism $g \circ g' : W \to U$ and the isomorphism of functors

$$w \xrightarrow{\alpha'} v \circ g' \xrightarrow{\alpha \circ g'} u \circ g \circ g'.$$

There is a projection functor

$$\tilde{p} : \widetilde{F} \to C, \quad (U, u) \mapsto U$$

which makes $\widetilde{F}$ a fibered category over C. There is also a functor

$$\widetilde{F} \to F, \quad (U, u) \mapsto u(\mathrm{id} : U \to U)$$

which is an equivalence of fibered categories by the 2-Yoneda Lemma 3.2.2.

Finally, we have a splitting $K \subset \widetilde{F}$ given by the subcategory whose objects are the same as the objects of $\widetilde{F}$, but whose only morphisms are those of the form

$$(V, u \circ g) \xrightarrow{(g,\mathrm{id})} (U, u)$$

for $g : V \to U$ a morphism in C. □

3.4. Categories fibered in groupoids

3.4.1. Recall that a *groupoid* is a category in which all morphisms are isomorphisms. A *category fibered in groupoids over a category* C is a fibered category $p : F \to C$ such that for every $U \in C$ the category $F(U)$ is a groupoid. Morphisms of categories fibered in groupoids are defined to be morphisms of fibered categories.

REMARK 3.4.2. In the theory of stacks, we usually only consider categories fibered in groupoids. This is natural from the point of view of moduli, where one is usually only interested in classifying certain objects and isomorphisms between them.

PROPOSITION 3.4.3. *Let F and F' be categories fibered in groupoids over C. Then the category*

$$HOM_C(F, F')$$

of morphisms of categories fibered in groupoids $F \to F'$ is a groupoid.

PROOF. We have to show that if $f, g : F \to F'$ are two morphisms of fibred categories and $\xi : f \to g$ is a morphism, then for any $x \in F$ the map

$$\xi_x : f(x) \to g(x)$$

is an isomorphism. This is clear for if $X \in C$ denotes the image of x in C, then ξ_x is, by definition of base-preserving natural transformation, a morphism in $F'(X)$, which is a groupoid. □

3.4.4. One important construction of categories fibered in groupoids is the following. Let C be a category with finite fiber products. Following [**49**, 2.4.3] a *groupoid in* C is a collection of data,

$$(X_0, X_1, s, t, \epsilon, i, m),$$

as follows:

(i) X_0 and X_1 are objects of C.
(ii)

$$s : X_1 \to X_0, t : X_1 \to X_0, \epsilon : X_0 \to X_1, i : X_1 \to X_1, m : X_1 \times_{s,X_0,t} X_1 \to X_1$$

are morphisms in C (we usually refer to s as the 'source', t as the 'target', ϵ as the 'identity', i as the 'inverse', and m as the 'composition' morphism).

This data is required to satisfy the following conditions:

A.

$$s \circ \epsilon = t \circ \epsilon = \mathrm{id}, s \circ i = t, t \circ i = s, s \circ m = s \circ \mathrm{pr}_2, t \circ m = t \circ \mathrm{pr}_1.$$

B. (Associativity) The two compositions

$$\begin{tikzcd} X_1 \times_{s,X_0,t} X_1 \times_{s,X_0,t} X_1 \arrow[r, shift left, "m\times \mathrm{id}"] \arrow[r, shift right, "\mathrm{id}\times m"'] & X_1 \times_{s,X_0,t} X_1 \arrow[r, "m"] & X_1 \end{tikzcd}$$

are equal.

C. (Identity) The two compositions

$$\begin{tikzcd} & X_1 \times_{s,X_0} X_0 \arrow[dr, "\epsilon\times \mathrm{id}"] & & \\ X_1 \arrow[ur, equal] \arrow[dr, equal] & & X_1 \times_{s,X_0,t} X_1 \arrow[r, "m"] & X_1 \\ & X_0 \times_{X_0,t} X_1 \arrow[ur, "\mathrm{id}\times\epsilon"'] & & \end{tikzcd}$$

are equal to id_{X_1}.

D. (Inverse) The diagrams

$$\begin{tikzcd} X_1 \arrow[r, "i\times \mathrm{id}"] \arrow[d, "s"] & X_1 \times_{s,X_0,t} X_1 \arrow[d, "m"] \\ X_0 \arrow[r, "\epsilon"] & X_1 \end{tikzcd} \qquad \begin{tikzcd} X_1 \arrow[r, "\mathrm{id}\times i"] \arrow[d, "t"] & X_1 \times_{s,X_0,t} X_1 \arrow[d, "m"] \\ X_0 \arrow[r, "\epsilon"] & X_1 \end{tikzcd}$$

commute.

3.4.5. For every $U \in C$, consider the category $\{X_0(U)/X_1(U)\}$ whose objects are elements $u \in X_0(U)$, and for which a morphism $u \to u'$ is an element $\xi \in X_1(U)$ for which $s(\xi) = u$ and $t(\xi) = u'$. Given a composition

$$u'' \xrightarrow{\eta} u' \xrightarrow{\xi} u$$

we define $\xi \circ \eta$ to be the image under m of the element

$$(\xi, \eta) \in X_1(U) \times_{s,X_0(U),t} X_1(U).$$

The above axioms imply that $\{X_0(U)/X_1(U)\}$ is a category and in fact a groupoid (the inverse of $\xi \in X_1(U)$ is given by $i(\xi)$).

Note also that for any morphism $f : V \to U$ there is a functor

$$f^* : \{X_0(U)/X_1(U)\} \to \{X_0(V)/X_1(V)\}$$

induced by the pullback maps

$$f^* : X_0(U) \to X_0(V), \quad f^* : X_1(U) \to X_1(V),$$

and for a composition

$$W \xrightarrow{g} V \xrightarrow{f} U$$

the two functors

$$g^* \circ f^*, (fg)^* : \{X_0(U)/X_1(U)\} \to \{X_0(W)/X_1(W)\}$$

are equal (not just isomorphic).

Define a fibered category

$$p : \{X_0/X_1\} \to C$$

with objects pairs (U, u), where $U \in C$ and $u \in \{X_0(U)/X_1(U)\}$. A morphism

$$(V, v) \to (U, u)$$

in $\{X_0/X_1\}$ is a pair (f, α), where $f : V \to U$ is a morphism in C and $\alpha : v \to f^*u$ is an isomorphism in $\{X_0(V)/X_1(V)\}$.

The composition of arrows

$$(W, w) \xrightarrow{(g,\beta)} (V, v) \xrightarrow{(f,\alpha)} (U, u)$$

is defined to be $(f \circ g, g^*(\alpha) \circ \beta)$. The functor p sends (U, u) to U. In this way we obtain a fibered category over C whose fiber over $U \in C$ is the groupoid $\{X_0(U)/X_1(U)\}$.

3.4.6. Let C be a category as above and $p : F \to C$ a category fibered in groupoids. For any object $X \in C$ we then also get a category fibered in groupoids:

$$p_{/X} : F_{/X} \to (C/X).$$

This category $F_{/X}$ has objects $(y, f : p(y) \to X)$, where $y \in F$ and f is a morphism in C. A morphism

$$(y', f' : p(y') \to X) \to (y, f : p(y) \to X)$$

is a morphism $g : y' \to y$ in F such that the triangle

$$\begin{tikzcd} p(y') \arrow[rr, "p(g)"] \arrow[dr, "f'"'] & & p(y) \arrow[dl, "f"] \\ & X & \end{tikzcd}$$

commutes. The functor $p_{/X}$ sends $(y, f : p(y) \to X)$ to the object $f : p(y) \to X$ in C/X. Notice that for any morphism $f : Y \to X$, the fiber

$$F_{/X}(f : Y \to X)$$

is canonically equivalent to $F(Y)$, and in particular is a groupoid.

3.4.7. Let $X \in C$ be an object and let $x, x' \in F(X)$ be two objects in the fiber over X. We can then define a presheaf

$$\underline{\mathrm{Isom}}(x, x') : (C/X)^{\mathrm{op}} \to \mathrm{Set}$$

as follows.

For any morphism $f : Y \to X$, choose pullbacks f^*x and f^*x', and set

$$\underline{\mathrm{Isom}}(x, x')(f : Y \to X) := \mathrm{Isom}_{F(Y)}(f^*x, f^*x').$$

For a composition

$$Z \xrightarrow{g} Y \xrightarrow{f} X$$

the pullback $(fg)^*x$ (resp. $(fg)^*x'$) is a pullback along g of f^*x (resp. f^*x'), and therefore there is a canonical map

$$g^* : \underline{\mathrm{Isom}}(x, x')(f : Y \to X) \to \underline{\mathrm{Isom}}(x, x')(fg : Z \to X)$$

compatible with composition.

In particular, if $x = x'$ we get a presheaf of groups:

$$\underline{\mathrm{Aut}}_x : (C/X)^{\mathrm{op}} \to \mathrm{Groups}.$$

Remark 3.4.8. Up to canonical isomorphism the presheaf $\underline{\mathrm{Isom}}(x, x')$ is independent of the choice of pullbacks.

3.4.9. Fiber products.

3.4.10. Let us first consider fiber products of groupoids. Let

$$\begin{tikzcd} & \mathscr{G}_1 \arrow[d, "f"] \\ \mathscr{G}_2 \arrow[r, "g"] & \mathscr{G} \end{tikzcd}$$

be a diagram of groupoids. We define a new groupoid

$$\mathscr{G}_1 \times_{\mathscr{G}} \mathscr{G}_2$$

as follows. The objects of $\mathscr{G}_1 \times_{\mathscr{G}} \mathscr{G}_2$ are triples (x, y, σ), where $x \in \mathscr{G}_1$ and $y \in \mathscr{G}_2$ are objects and

$$\sigma : f(x) \to g(y)$$

is an isomorphism in $\mathscr{G}$. A morphism

$$(x', y', \sigma') \to (x, y, \sigma)$$

is a pair of isomorphisms $a : x' \to x$ and $b : y' \to y$ such that the diagram

$$\begin{tikzcd} f(x') \arrow[r, "\sigma'"] \arrow[d, "f(a)"] & g(y') \arrow[d, "g(b)"] \\ f(x) \arrow[r, "\sigma"] & g(y) \end{tikzcd}$$

commutes. There are functors

$$p_j : \mathscr{G}_1 \times_{\mathscr{G}} \mathscr{G}_2 \to \mathscr{G}_j \quad j = 1, 2$$

and a natural isomorphism of functors

$$\Sigma : f \circ p_1 \to g \circ p_2.$$

The category $\mathscr{G}_1 \times_{\mathscr{G}} \mathscr{G}_2$ together with the functors p_1, p_2, and the isomorphism Σ have the following universal property. Suppose $\mathscr{H}$ is another groupoid and that

$$\alpha : \mathscr{H} \to \mathscr{G}_1, \beta : \mathscr{H} \to \mathscr{G}_2, \quad \gamma : f \circ \alpha \to g \circ \beta$$

are two functors and γ is an isomorphism of functors. Then there exists a collection of data

$$(h : \mathscr{H} \to \mathscr{G}_1 \times_{\mathscr{G}} \mathscr{G}_2, \lambda_1, \lambda_2)$$

where h is a functor,

$$\lambda_1 : \alpha \to p_1 \circ h, \quad \lambda_2 : \beta \to p_2 \circ h$$

are isomorphisms of functors, and the diagram

$$\begin{tikzcd} f \circ \alpha \arrow[r, "f(\lambda_1)"] \arrow[d, "\gamma"] & f \circ p_1 \circ h \arrow[d, "\Sigma\circ h"] \\ g \circ \beta \arrow[r, "g(\lambda_2)"] & g \circ p_2 \circ h \end{tikzcd}$$

commutes. The data

$$(h, \lambda_1, \lambda_2)$$

is unique up to unique isomorphism.

REMARK 3.4.11. The fiber product $\mathscr{G}_1 \times_{\mathscr{G}} \mathscr{G}_2$ is sometimes called the 2-*categorical fiber product*. In this book the above is the only notion of fiber product of groupoids so we omit the reference to 2-categories.

3.4.12. Let C be a category and let

$$\begin{tikzcd} & F_1 \arrow[d, "c"] \\ F_2 \arrow[r, "d"] & F_3 \end{tikzcd}$$

be a diagram of categories fibered in groupoids over C.

Consider a category G fibered in groupoids over C, morphisms of fibered categories

$$\alpha : G \to F_1, \quad \beta : G \to F_2$$

and an isomorphism $\gamma : c \circ \alpha \to d \circ \beta$ of morphisms of fibered categories $G \to F_3$. Giving the data (α, β, γ) is equivalent to giving an object of

$$HOM_C(G, F_1) \times_{HOM_C(G,F_3)} HOM_C(G, F_2).$$

Such data defines for any other category fibered in groupoids H a morphism of groupoids

$$HOM_C(H, G) \to HOM_C(H, F_1) \times_{HOM_C(H,F_3)} HOM_C(H, F_2), \tag{3.4.12.1}$$

$$(h : H \to G) \mapsto (\alpha \circ h, \beta \circ h, \gamma \circ h).$$

PROPOSITION 3.4.13. *(i) There exists a collection of data $(G, \alpha, \beta, \gamma)$ as above, such that for every category fibered in groupoids H over C the map* (3.4.12.1) *is an isomorphism.*

(ii) If $(G', \alpha', \beta', \gamma')$ is another collection of data as in (i), then there exists a triple (F, u, v), where $F : G \to G'$ is an equivalence of fibered categories, $u : \alpha \to$

$\alpha' \circ F$ and $v : \beta \to \beta' \circ F$ are isomorphisms of morphisms of fibered categories, and the following diagram commutes:

$$\begin{tikzcd} c \circ \alpha \arrow[r, "c \circ u"] \arrow[d, "\gamma"] & c \circ \alpha' \circ F \arrow[d, "\gamma'"] \\ d \circ \beta \arrow[r, "d \circ v"] & d \circ \beta' \circ F. \end{tikzcd}$$

Moreover, if (F', u', v') is a second such triple, then there exists a unique isomorphism $\sigma : F' \to F$ such that the diagrams

$$\begin{tikzcd} \alpha \arrow[r, "u'"] \arrow[dr, "u"'] & \alpha' \circ F' \arrow[d, "\sigma"] \\ & \alpha' \circ F \end{tikzcd}$$

and

$$\begin{tikzcd} \beta \arrow[r, "v'"] \arrow[dr, "v"'] & \beta' \circ F' \arrow[d, "\sigma"] \\ & \beta' \circ F \end{tikzcd}$$

commute.

Remark 3.4.14. We usually write $F_1 \times_{F_3} F_2$ for the fibered category in (i) and refer to it as the *fiber product of* F_1 *and* F_2 *over* F_3. As in the case of schemes, this is an abuse of language, and really the data of $F_1 \times_{F_3} F_2$ together with the morphisms to F_1 and F_2 and the natural transformation is the fiber product, though we usually suppress it from the notation. In the case when C is the punctual category the fiber product construction of 3.4.13 reduces to that of 3.4.10.

Proof of Proposition 3.4.13. Let

$$p_i : F_i \to C$$

be the given functors to C ($i = 1, 2, 3$), and define G to be the category of triples (x_1, x_2, σ), where $x_i \in F_i$ are objects such that $p_1(x_1) = p_2(x_2)$, and $\sigma : c(x_1) \to d(x_2)$ is an isomorphism in $F_3(p_1(x_1)) = F_3(p_2(x_2))$. A morphism

$$(x_1', x_2', \sigma') \to (x_1, x_2, \sigma)$$

is a pair of morphisms $f_i : x_i' \to x_i$ in F_i ($i = 1, 2$) such that $p_1(f_1) = p_2(f_2)$ and such that the diagram

$$\begin{tikzcd} c(x_1') \arrow[r, "c(f_1)"] \arrow[d, "\sigma'"] & c(x_1) \arrow[d, "\sigma"] \\ d(x_2') \arrow[r, "d(f_2)"] & d(x_2) \end{tikzcd}$$

commutes. Let $\alpha : G \to F_1$ be the functor sending (x_1, x_2, σ) to x_1, and let β be the functor sending (x_1, x_2, σ) to x_2. The isomorphisms σ defines an isomorphism $\gamma : c \circ \alpha \to d \circ \beta$.

Now let H be another category fibered in groupoids, and let

$$\xi_1 : H \to F_1, \quad \xi_2 : H \to F_2$$

be two morphisms with an isomorphism $\delta : c \circ \xi_1 \to d \circ \xi_2$. Define

$$\xi : H \to G$$

by sending an object $y \in H$ to

$$(\xi_1(y), \xi_2(y), \delta_y : c(\xi_1(y)) \to d(\xi_2(y))).$$

Then we have (equality of functors)

$$\alpha \circ \xi = \xi_1, \quad \beta \circ \xi = \xi_2.$$

In this way we obtain a functor

$$F : HOM_C(H, F_1) \times_{HOM_C(H,F_3)} HOM_C(H, F_2) \to HOM_C(H, G) \tag{3.4.14.1}$$

whose composition with the functor 3.4.12.1 is equal to the identity functor.

For a morphism of fibered categories $\xi : H \to G$, set

$$\xi_1 := \alpha \circ \xi, \quad \xi_2 := \beta \circ \xi,$$

and let $\sigma : c \circ \xi_1 \to d \circ \xi_2$ be the isomorphism defined by $\gamma : c \circ \alpha \to d \circ \beta$. Then by definition we have for any object $y \in H$,

$$\xi(y) = (\xi_1(y), \xi_2(y), \sigma_y : c(\xi_1(y)) \to d(\xi_2(y))),$$

so ξ is in the essential image of (3.4.14.1). This proves (i).

For (ii), consider the equivalence

$$\Xi : HOM_C(G, G') \to HOM_C(G, F_1) \times_{HOM_C(G,F_3)} HOM_C(G, F_2).$$

The category of triples (F, u, v), where $F : G \to G'$ is a morphism of fibered categories, and

$$u : \alpha \to \alpha' \circ F, \quad v : \beta \to \beta' \circ F$$

are isomorphisms such that the diagram commutes is equivalent to the category of pairs (F, λ), where $F : G \to G'$ is a morphism of fibered categories and $\lambda : \Xi(F) \to (\alpha, \beta, \gamma)$ is an isomorphism in

$$HOM_C(G, F_1) \times_{HOM_C(G,F_3)} HOM_C(G, F_2).$$

Since Ξ is an equivalence of categories it follows that there exists such a triple and it is unique up to unique isomorphism. Furthermore, applying the same argument with G and G' interchanged, we get a morphism $F' : G' \to G$ such that the composition $F \circ F'$ (resp. $F' \circ F$) is isomorphic to the identity functor on G' (resp. G), by the uniqueness. □

3.5. Exercises

EXERCISE 3.A. Let C be a category and X be an object of C. Define C/X to be the category whose objects are morphisms $Y \to X$ and whose morphisms are given by commutative triangles

$$\begin{tikzcd} Y' \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\ & X. & \end{tikzcd}$$

Let $p : C/X \to C$ be the functor sending $Y \to X$ to Y. Show that $p : C/X \to C$ is a fibered category.

Exercise 3.B. Let C be a category, and let $F : C^{\text{op}} \to \text{Set}$ be a functor. Let

$$p : \mathcal{F} \to C, \quad (U, x) \mapsto U$$

be the category over C associated to F as in 3.2.7. Show that $\mathcal{F}$ is a fibered category over C.

Exercise 3.C. For a group G let C_G denote the category with one object and whose morphisms are given by elements of G. Let $p : G \to H$ be a group homomorphism, and write also $p : C_G \to C_H$ for the corresponding functor.

(i) Show that $p : C_G \to C_H$ is a fibered category if and only if p is surjective.

(ii) In the case when p is surjective, show that giving a splitting of p is equivalent to giving a subgroup $K \subset G$ such that the composition $K \to G \to H$ is bijective.

(iii) Using (ii), give an explicit example of a fibered category which does not admit a splitting.

For further discussion of this example see [**71**, Example 3.14].

Exercise 3.D. Let $p : F \to C$ be a fibered category. Show that p is fibered in groupoids if and only if every morphism in F is cartesian. Deduce from this that every base preserving functor from a fibered category to a category fibered in groupoids is a morphism of fibered categories.

Exercise 3.E. Let C be a category with finite fiber products, and suppose given three morphisms

$$s, t : X_1 \to X_0, \quad m : X_1 \times_{s, X_0, t} X_1 \to X_1$$

such that the following hold:

(i) The square

$$\begin{tikzcd} X_1 \times_{s,X_0,t} X_1 \arrow[r, "m"] \arrow[d, "\mathrm{pr}_2"] & X_1 \arrow[d, "s"] \\ X_1 \arrow[r, "s"] & X_0 \end{tikzcd}$$

is cartesian.

(ii) The square

$$\begin{tikzcd} X_1 \times_{s,X_0,t} X_1 \arrow[r, "m"] \arrow[d, "\mathrm{pr}_1"] & X_1 \arrow[d, "t"] \\ X_1 \arrow[r, "t"] & X_0 \end{tikzcd}$$

is cartesian.

(iii) The square

$$\begin{tikzcd} X_1 \times_{s,X_0,t} X_1 \times_{s,X_0,t} X_1 \arrow[r, "m \times \mathrm{id}_{X_1}"] \arrow[d, "\mathrm{id}_{X_1} \times m"] & X_1 \times_{s,X_0,t} X_1 \arrow[d, "m"] \\ X_1 \times_{s,X_0,t} X_1 \arrow[r, "m"] & X_1 \end{tikzcd}$$

commutes.

(iv) There exists a morphism $\epsilon : X_0 \to X_1$ such that $s\epsilon = t\epsilon = \mathrm{id}_{X_0}$.

Show that there exists a unique groupoid in C with s and t as the source and target morphisms and m as the composition morphism. These are the axioms for a groupoid given in [**10**, V].

Exercise 3.F. Let F be a category fibered in groupoids over C, and let $x \in F$ and $y \in F$ be two objects lying over $X \in C$ and $Y \in C$ respectively. Assume that $X \times Y$ exists in C, and consider the fiber product of the diagram

$$\begin{tikzcd} & X \arrow[d, "x"] \\ Y \arrow[r, "y"] & F. \end{tikzcd}$$

Show that the fiber product of this diagram is isomorphic to the fibered category associated to the functor

$$\underline{\mathrm{Isom}}(x, y)$$

over $C/X \times Y$.

Exercise 3.G. Let S be a scheme, and let G/S be a group scheme which acts on an S-scheme X. Define the *action groupoid* associated to this G-action as follows (notation as in 3.4.4):

(i) $X_0 = X$, $X_1 = X \times_S G$.
(ii) The map $s : X_1 \to X_0$ is the first projection $X \times_S G \to X$ and the map t is the action map.
(iii) The map $\epsilon : X_0 \to X_1$ is the map induced by the identity section $S \to G$.
(iv) The map $i : X_1 \to X_1$ is the map which is the action map on the first factor and the map $g \mapsto g^{-1}$ on G.
(v) The map

$$X \times_S G \times_S G \simeq X_1 \times_{s, X_0, t} X_1 \to X_1$$

is the map induced by the composition law $G \times_S G \to G$.

Show that with these definitions, we get a groupoid in S-schemes. In this setting, we often write $\{X/G\}$ instead of $\{X_0/X_1\}$.

CHAPTER 4

Descent and the stack condition

Descent theory is the natural generalization to sites of the more familiar gluing properties that one encounters in the study of schemes. For example, if X is a scheme and $\mathscr{U} = \{U_i\}_{i \in I}$ is an open covering, then one can consider the category of collections of data $\{(F_i)_{i\in I}, (\sigma_{ij})_{i,j\in I}\}$ consisting of a quasi-coherent sheaf F_i on U_i for each i and for every $i, j \in I$ an isomorphism $\sigma_{ij} : F_i|_{U_i\cap U_j} \to F_j|_{U_i\cap U_j}$ such that $\sigma_{ii} = \mathrm{id}_{F_i}$ and for every $i, j, k \in I$ the diagram

$$\begin{tikzcd} F_i|_{U_{ijk}} \arrow[r, "\sigma_{ij}|_{U_{ijk}}"] \arrow[rr, bend left, "\sigma_{ik}|_{U_{ijk}}"] & F_j|_{U_{ijk}} \arrow[r, "\sigma_{jk}|_{U_{ijk}}"] & F_k|_{U_{ijk}} \end{tikzcd}$$

over $U_{ijk} := U_i \cap U_j \cap U_k$ commutes. The standard gluing exercise (for example [**41**, Chapter II, exercise 1.22]) shows that this category is equivalent to the category of quasi-coherent sheaves on X.

The most basic result in descent theory, faithfully flat descent for quasi-coherent sheaves, is a generalization of this gluing result replacing the Zariski topology with the fppf (or fpqc) topologies. In this chapter we discuss this result and some of its many consequences. In particular, it implies that representable functors are sheaves on the category of schemes with the fppf topology (perhaps over some base), and in particular are also sheaves for the étale topology. It follows that the Yoneda imbedding identifies the category of schemes with a full subcategory of the category of sheaves for the étale topology. This imbedding of the category of schemes leads to the notion of algebraic space, which is by definition a sheaf satisfying certain conditions on the category of schemes with the étale topology.

It is convenient to reformulate the various descent problems in terms of fibered categories. This leads to the notion of a stack which is loosely a fibered category whose objects, as well as morphisms between them, can be glued from local objects. The notion of stack is a 2-categorical generalization of the notion of sheaf, and just as an algebraic space is defined as a sheaf which locally is a scheme we will subsequently (in Chapter 8) define an algebraic stack to be a stack which locally is an algebraic space.

The sources for this chapter are [**35**], [**36**], and [**71**]. For further discussion of the various examples of effective descent morphisms discussed here see [**36**, VIII]. The problem of representing torsors by principal homogenous spaces is, in general, subtle (but see [**52**, III, §4] for some additional cases and references). In the last section we discuss the notion of stack originally introduced in [**34**].

4.1. Faithfully flat descent

Many of the arguments in this section we learned from the excellent exposition [**71**] (see also [**52**]).

4.1.1. If $X \to Y$ is a morphism of schemes, then we get a functor of points

$$h_X : (\text{schemes over } Y)^{\text{op}} \to \text{Set}, \quad (Z \to Y) \mapsto \{Y\text{-morphisms } Z \to X\}.$$

The main result of this section is the following theorem 4.1.2.

THEOREM 4.1.2 ([**36**, VIII, 5.2]). *For any morphism of schemes $X \to Y$, the functor h_X is a sheaf in the fppf topology (and therefore also in the étale topology) on the category of Y-schemes.*

REMARK 4.1.3. Note that it suffices to consider the absolute case when $Y = \text{Spec}(\mathbb{Z})$.

REMARK 4.1.4. Theorem 4.1.2 is also true with the fppf topology replaced by the finer fpqc topology (see for example [**71**, 2.55]). However, for later purposes the fppf topology suffices so we do not include this more general statement whose proof requires some additional foundational work.

The proof of Theorem 4.1.2 occupies the remainder of this section.
We follow the argument of [**71**, Proof of 2.55].

4.1.5. Let $f : A \to B$ be a ring homomorphism, and let M be an A-module. Denote by M_B the B-module $M \otimes_A B$ and let $M_{B\otimes_A B}$ denote the module $M \otimes_A (B \otimes_A B)$. Write also $f : M \to M_B$ for the map induced by f, and let

$$p_1, p_2 : M_B \to M_{B\otimes_A B}$$

be the two maps induced by the two maps

$$B \underset{b\mapsto 1\otimes b}{\overset{b\mapsto b\otimes 1}{\rightrightarrows}} B \otimes_A B.$$

PROPOSITION 4.1.6. *If $f : A \to B$ is faithfully flat and M is an A-module, then the sequence*

$$M \xrightarrow{f} M_B \underset{p_2}{\overset{p_1}{\rightrightarrows}} M_{B\otimes_A B}$$

is exact.

PROOF. Since B is faithfully flat over A, exactness of the sequence in question is equivalent to exactness of the sequence obtained by tensoring over A with B:

$$M_B \xrightarrow{f'} M_{B\otimes_A B} \underset{p_2'}{\overset{p_1'}{\rightrightarrows}} M_{B\otimes_A B\otimes_A B} \tag{4.1.6.1}$$

Here the map

$$f' : M_B \to M_{B\otimes_A B}$$

is the map

$$m \otimes b \mapsto m \otimes 1 \otimes b,$$

and the map p_1' (resp. p_2') is the map given by

$$m \otimes b_1 \otimes b_2 \mapsto m \otimes b_1 \otimes 1 \otimes b_2 \quad (\text{resp. } m \otimes b_1 \otimes b_2 \mapsto m \otimes 1 \otimes b_1 \otimes b_2).$$

The map f' is injective as the multiplication map $B \otimes_A B \to B$ induces a map $\eta : M_{B\otimes_A B} \to M_B$ such that $\eta \circ f' = \mathrm{id}_{M_B}$.

Now suppose $\alpha \in M_{B\otimes_A B}$ satisfies $p_1'(\alpha) = p_2'(\alpha)$. Consider the map

$$\tau : M_{B\otimes_A B\otimes_A B} \to M_{B\otimes_A B}$$

induced by the map

$$B \otimes_A B \otimes_A B \to B \otimes_A B, \quad b_1 \otimes b_2 \otimes b_3 \mapsto b_1 \otimes (b_2 b_3).$$

We have $\tau \circ p_1' = \mathrm{id}$ and $\tau \circ p_2' = f' \circ \eta$. We therefore get that

$$\alpha = \tau \circ p_1'(\alpha) = \tau \circ p_2'(\alpha) = f'\eta(\alpha)$$

which implies the exactness in the middle of (4.1.6.1). □

Remark 4.1.7. This argument is a variant of the argument proving 2.4.18.

Corollary 4.1.8. *If $V \to U$ is a faithfully flat map of affine schemes, and X is an affine scheme, then the sequence $h_X(U) \to h_X(V) \rightrightarrows h_X(V \times_U V)$ is exact.*

Proof. Let $U = \mathrm{Spec}(A)$, $V = \mathrm{Spec}(B)$, and $X = \mathrm{Spec}(R)$. By Proposition 4.1.6, we have the exact sequence $A \to B \rightrightarrows B \otimes_A B$, and we wish to show exactness of the sequence $\mathrm{Hom}(R, A) \to \mathrm{Hom}(R, B) \rightrightarrows \mathrm{Hom}(R, B \otimes_A B)$.

Since A injects into B, two maps from R to A which agree in B are the same. If $f : R \to B$ satisfies $1 \otimes f(r) = f(r) \otimes 1$, then $f(r)$ lies in A, so f is obtained from a map $R \to A$. □

Lemma 4.1.9. *Let $F : (schemes)^{op} \to Set$ be a presheaf which is a big Zariski sheaf. Then F is a sheaf for the fppf topology if and only if for every faithfully flat morphism $V \to U$ locally of finite presentation the sequence*

$$F(U) \to F(V) \rightrightarrows F(V \times_U V)$$

is exact.

Proof. For a general cover $\{U_i \to U\}$ let V denote $\coprod_i U_i$. Then we have a commutative diagram

$$\begin{tikzcd} F(U) \arrow[r] \arrow[d, equal] & F(V) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\wr"] & F(V \times_U V) \arrow[d, "\wr"] \\ F(U) \arrow[r] & \prod_i F(U_i) \arrow[r, shift left] \arrow[r, shift right] & \prod_{i,j} F(U_i \times_U U_j) \end{tikzcd}$$

where the vertical maps are isomorphisms since F is a Zariski sheaf. By assumption the top row is exact, whence the bottom row is exact also. □

Lemma 4.1.10. *Let $F : (schemes)^{op} \to Set$ be a presheaf satisfying the following conditions.*

(i) *F is a sheaf in the big Zariski topology.*
(ii) *If $V \to U$ is a fppf morphism of affine schemes, then the sequence $F(U) \to F(V) \rightrightarrows F(V \times_U V)$ is exact.*

Then F is a sheaf for the fppf topology.

Remark 4.1.11. The following proof also works if you work over some base scheme X.

PROOF OF LEMMA 4.1.10. Let F be a presheaf satisfying conditions (i) and (ii) in the lemma. By 4.1.9 it suffices to show that for every faithfully flat morphism $V \to U$ locally of finite presentation the sequence

$$F(U) \to F(V) \rightrightarrows F(V \times_U V) \tag{4.1.11.1}$$

is exact.

Let us start by showing that the map $F(U) \to F(V)$ is injective. For this, let $U = \bigcup_i U_i$ be an open covering by affines, and for each i choose an affine covering $V_i = \bigcup_j V_{ij}$. We then get a commutative diagram

$$\begin{tikzcd} F(U) \arrow[r] \arrow[d] & F(V) \arrow[d] \\ \prod_i F(U_i) \arrow[r] & \prod_{i,j} F(V_{ij}), \end{tikzcd} \tag{4.1.11.2}$$

where the vertical arrows are injections since F is a Zariski sheaf. Furthermore, since the image of each V_{ij} in U_i is open and U_i is quasi-compact, for each i there exists a finite number of indices $j_1, \ldots, j_r$ such that $\{V_{ij_s} \to U_i\}_{s=1}^r$ is an fppf covering of U_i, and by property (ii) the map

$$F(U_i) \to \prod_{s=1}^r F(V_{ij_s}) = F(\coprod_{s=1}^r V_{ij_s})$$

is injective. Therefore the bottom horizontal arrow in (4.1.11.2) is injective, which implies that $F(U) \to F(V)$ is also injective.

To prove exactness in the middle of (4.1.11.1), we reduce first to the case when U is affine. For this let $U = \bigcup_i U_i$ be a covering of U by affine open subsets, and let $V_i \subset V$ be the preimage of U_i. We then get a commutative diagram

$$\begin{tikzcd} F(U) \arrow[r] \arrow[d, "a"] & F(V) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "b"] & F(V \times_U V) \arrow[d] \\ \prod_i F(U_i) \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & \prod_i F(V_i) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & \prod_i F(V_i \times_{U_i} V_i) \\ \prod_{i,j} F(U_i \cap U_j) \arrow[r, "c"] & \prod_{i,j} F(V_i \cap V_j). & \end{tikzcd}$$

Here the first two columns are exact, and the maps a, b, and c are injective (c is injective since $V_i \cap V_j \to U_i \cap U_j$ is an fppf covering). A diagram chase now shows that if the sequences

$$F(U_i) \to F(V_i) \rightrightarrows F(V_i \times_{U_i} V_i)$$

are exact for all i, then the top sequence in the diagram is exact. Considering each of the U_i, we are therefore reduced to the case when U is affine.

Assuming that U is affine, we next reduce to the case when V is also quasi-compact. By 1.1.6 (see also 1.1.7), there exists an open covering $V = \bigcup_j V_j$, where each V_j is quasi-compact and the maps $V_j \to U$ are surjective. Let

$$x \in \mathrm{Eq}(F(V) \rightrightarrows F(V \times_U V))$$

be an element in the equalizer. By restriction, this defines for each j an element

$$x_j \in \mathrm{Eq}(F(V_j) \rightrightarrows F(V_j \times_U V_j)).$$

Assuming the case when V is quasi-compact, this element x_j is in the image of a unique element $y_j \in F(U)$. All these elements y_j must be equal, because for every i and j the projection $V_i \times_U V_j \to U$ is an fppf cover, and since F is separated the induced map

$$F(U) \to F(V_i \times_U V_j)$$

is injective. Since both y_i and y_j map to the image of x in $F(V_i \times_U V_j)$ this implies that $y_i = y_j$ for every pair of indices i and j. Let y denote the common value of the y_j. Then the image of y in $F(V)$ is equal to x since the map

$$F(V) \to \prod_j F(V_j)$$

is injective, and both x and y map to the element $(x_j) \in \prod_j F(V_j)$. This therefore reduces the proof to the case when U is affine and V is quasi-compact.

To handle this case, let $V = \bigcup_j V_j$ be a finite covering of V by affines. Then the map $\coprod_j V_j \to V$ is an fppf covering and $\coprod_j V_j$ is affine. Consider the commutative diagram

$$\begin{tikzcd}
F(U) \arrow[r] \arrow[d, equal] & F(V) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, hook] & F(V \times_U V) \arrow[d, hook] \\
F(U) \arrow[r] & F(\coprod_j V_j) \arrow[r, shift left] \arrow[r, shift right] & F((\coprod_j V_j) \times_U (\coprod_j V_j)).
\end{tikzcd}$$

By assumption (ii) the bottom row is exact, and since the middle vertical arrow is injective this implies that the top row is also exact in the middle. $\square$

Proof of Theorem 4.1.2. Note first that it suffices to consider the case when $Y = \mathrm{Spec}(\mathbb{Z})$, in which case the statement of the theorem is that for any scheme X, the functor

$$h_X : (\text{schemes})^{op} \to \text{Set}$$

is a sheaf for the fppf topology.

In the case when X is affine, the result follows from 4.1.10 and 4.1.8.

We reduce the general case to this special case. Let X be any scheme, and write $X = \bigcup X_i$ as a union of open affine subschemes. By Lemma 4.1.10, it is enough to consider an fppf covering of the form $t : V \to U$, where U and V are affine.

We have to check the exactness of

$$h_X(U) \xrightarrow{\alpha} h_X(V) \rightrightarrows h_X(V \times_U V).$$

For the injectivity, let f and g be two elements of $h_X(U)$ such that $\alpha(f) = \alpha(g)$. Then we have $V \xrightarrow{t} U \underset{g}{\overset{f}{\rightrightarrows}} X$ with $ft = gt$. In particular, the maps of sets must agree; since t is surjective, f and g must be set-theoretically equal. Now consider $U_i = f^{-1}(X_i) = g^{-1}(X_i)$. By the affine case, $f|_{U_i} = g|_{U_i}$ scheme-theoretically. Therefore, we get $f = g$.

Next we check exactness in the middle. For a scheme T write $|T|$ for the underlying topological space. Let $f \in h_X(V)$ be an element such that the two morphisms

$$fp_1, fp_2 : V \times_U V \to X$$

are equal.

Looking at the underlying topological spaces we get a commutative diagram

$$\begin{tikzcd}
{|V \times_U V|} \arrow[r, dashed] \arrow[rr, bend left, "|p_1|"] \arrow[rr, bend right, "|p_2|"'] & {|V| \times_{|U|} |V|} \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & {|V|} \arrow[r, "|f|"] \arrow[dr, "|t|"'] & {|X|} \\
& & & {|U|,} \arrow[u, dotted, "h"']
\end{tikzcd}$$

where the dashed arrow exists by the universal property of $|V| \times_{|U|} |V|$.

Lemma 4.1.12. *(i) The map*

$$|V \times_U V| \to |V| \times_{|U|} |V|$$

is a surjection.

(ii) A subset $R \subset |U|$ is open if and only if the preimage $|t|^{-1}(R) \subset |V|$ is open.

(iii) The topological space $|U|$ is the coequalizer in the category of topological spaces of the diagram

$$|V \times_U V| \rightrightarrows |V|.$$

Proof. For (i) let $x, x' \in V$ be two points with the same image $\bar{x} \in U$. We then get a commutative diagram

$$\begin{tikzcd}
\mathrm{Spec}(k(x)) \arrow[drr, "x"] \arrow[ddr] & & \\
& \mathrm{Spec}(k(x')) \arrow[r, "x'"] \arrow[d] & V \arrow[d] \\
& \mathrm{Spec}(k(\bar{x})) \arrow[r, "\bar{x}"] & U,
\end{tikzcd}$$

which induces a morphism

$$\mathrm{Spec}(k(x) \otimes_{k(\bar{x})} k(x')) \to V \times_U V.$$

Then any point of $\mathrm{Spec}(k(x) \otimes_{k(\bar{x})} k(x'))$ maps to a point of $V \times_U V$ mapping to $(x, x') \in |V| \times_{|U|} |V|$.

Statement (ii) follows from 1.1.5 (and 1.1.7), and statement (iii) follows from (i) and (ii). $\square$

It follows that $|f|$ descends to a continuous map

$$h : |U| \to |X|.$$

Let $V_i = f^{-1}(X_i)$ and $U_i = h^{-1}(X_i)$. Then $V_i \to U_i$ are fppf coverings. By the affine case, we have (unique) morphisms of schemes $h_i : U_i \to X_i$ so that $f|_{V_i} = h_i \circ t|_{V_i}$. Covering the intersections $X_i \cap X_j$ by affines and using the uniqueness, we have that the h_i agree on intersections $U_i \cap U_j$. Therefore, we get a morphism of schemes $h : U \to X$ so that $f = h \circ t$. $\square$

4.2. Generalities on descent

4.2.1. Let C be a category with finite fiber products (the case we will be interested in is when C is the category of schemes), and let

$$p : F \to C$$

be a fibered category. Choose for each morphism $f : X \to Y$ a pullback functor

$$f^* : F(Y) \to F(X).$$

For a morphism $f : X \to Y$ define a category

$$F(X \xrightarrow{f} Y)$$

as follows. The objects of $F(X \xrightarrow{f} Y)$ are pairs (E, σ), where $E \in F(X)$ is an object and

$$\sigma : \mathrm{pr}_1^* E \to \mathrm{pr}_2^* E$$

is an isomorphism in $F(X \times_Y X)$ such that in $F(X \times_Y X \times_Y X)$ the following diagram commutes

$$\begin{tikzcd} \mathrm{pr}_{12}^*\mathrm{pr}_1^*E \arrow[r, "\mathrm{pr}_{12}^*\sigma"] \arrow[d, equal] & \mathrm{pr}_{12}^*\mathrm{pr}_2^*E \arrow[r, equal] & \mathrm{pr}_{23}^*\mathrm{pr}_1^*E \arrow[d, "\mathrm{pr}_{23}^*\sigma"] \\ \mathrm{pr}_{13}^*\mathrm{pr}_1^*E \arrow[r, "\mathrm{pr}_{13}^*\sigma"] & \mathrm{pr}_{13}^*\mathrm{pr}_2^*E \arrow[r, equal] & \mathrm{pr}_{23}^*\mathrm{pr}_2^*E. \end{tikzcd} \tag{4.2.1.1}$$

Here we abusively write '=' for the canonical isomorphisms and

$$\mathrm{pr}_{12}, \mathrm{pr}_{23}, \mathrm{pr}_{13} : X \times_Y X \times_Y X \to X \times_Y X, \quad \mathrm{pr}_1, \mathrm{pr}_2 : X \times_Y X \to X$$

for the various projections.

A morphism

$$(E', \sigma') \to (E, \sigma)$$

in $F(X \xrightarrow{f} Y)$ is a morphism $g : E' \to E$ in $F(X)$ such that the square

$$\begin{tikzcd} \mathrm{pr}_1^*E' \arrow[r, "\mathrm{pr}_1^*g"] \arrow[d, "\sigma'"] & \mathrm{pr}_1^*E \arrow[d, "\sigma"] \\ \mathrm{pr}_2^*E' \arrow[r, "\mathrm{pr}_2^*g"] & \mathrm{pr}_2^*E \end{tikzcd}$$

commutes.

For an object $(E, \sigma) \in F(X \xrightarrow{f} Y)$ we refer to the isomorphism σ as *descent data* for the object E.

REMARK 4.2.2. We often write simply $F(X \to Y)$ for $F(X \xrightarrow{f} Y)$ if no confusion seems likely to arise.

4.2.3. There is a functor

$$\epsilon : F(Y) \to F(X \xrightarrow{f} Y). \tag{4.2.3.1}$$

Namely, given an object $E_0 \in F(Y)$ the two pullbacks

$$\mathrm{pr}_1^* f^* E_0, \ \mathrm{pr}_2^* f^* E_0$$

are both pullbacks of E_0 along the projection

$$X \times_Y X \to Y,$$

and therefore there is a canonical isomorphism

$$\sigma_{\mathrm{can}} : \mathrm{pr}_1^* f^* E_0 \to \mathrm{pr}_2^* f^* E_0.$$

The functor ϵ is defined by sending E_0 to $(f^* E_0, \sigma_{\mathrm{can}})$.

4.2.4. More generally if $\{X_i \to Y\}_{i \in I}$ is a set of morphisms in C, we define $F(\{X_i \to Y\})$ to be the category of collections of data $(\{E_i\}_{i \in I}, \{\sigma_{ij}\}_{i,j \in I})$, where $E_i \in F(X_i)$ and for each $i, j \in I$,

$$\sigma_{ij} : \mathrm{pr}_1^* E_i \to \mathrm{pr}_2^* E_j,$$

is an isomorphism in $F(X_i \times_Y X_j)$, such that for any three indices $i, j, k \in I$ the following diagram in $F(X_i \times_Y X_j \times_Y X_k)$ commutes:

$$\begin{tikzcd}
\mathrm{pr}_{12}^*\mathrm{pr}_1^* E_i \arrow[r, "\mathrm{pr}_{12}^*\sigma_{ij}"] \arrow[d, equal] & \mathrm{pr}_{12}^*\mathrm{pr}_2^* E_j \arrow[r, equal] & \mathrm{pr}_{23}^*\mathrm{pr}_1^* E_j \arrow[d, "\mathrm{pr}_{23}^*\sigma_{jk}"] \\
\mathrm{pr}_{13}^*\mathrm{pr}_1^* E_i \arrow[r, "\mathrm{pr}_{13}^*\sigma_{ik}"] & \mathrm{pr}_{13}^*\mathrm{pr}_2^* E_k \arrow[r, equal] & \mathrm{pr}_{23}^*\mathrm{pr}_2^* E_k.
\end{tikzcd}$$

As in the case of a single morphism, we refer to the set of isomorphisms $\{\sigma_{ij}\}$ as *descent data* on the $\{E_i\}_{i \in I}$, and there is a natural functor

$$\epsilon : F(Y) \to F(\{X_i \to Y\}). \tag{4.2.4.1}$$

DEFINITION 4.2.5. The collection of morphisms $\{X_i \to Y\}$ is of *effective descent for* F if the functor (4.2.4.1) is an equivalence of categories.

DEFINITION 4.2.6. For an object $(\{E_i\}, \{\sigma_{ij}\}) \in F(\{X_i \to Y\})$, we say that the descent data $\{\sigma_{ij}\}$ is *effective* if $(\{E_i\}, \{\sigma_{ij}\})$ is in the essential image of (4.2.3.1).

As the following lemma shows (generalizing 4.1.9), the most important case to consider is that of a single morphism $X \to Y$:

LEMMA 4.2.7. *Assume that coproducts exist in C, and that the formation of coproducts commutes with fiber products when they exist. Assume further that for any set of objects $\{X_i\}_{i \in I}$ in C the natural functor*

$$F(\coprod_i X_i) \to \prod_i F(X_i)$$

is an equivalence of categories.

Let $\{X_i \to Y\}$ be a set of morphisms in C, and let $Q := \coprod_i X_i$ denote the coproduct. Then

$$F(Y) \to F(\{X_i \to Y\})$$

is an equivalence if and only if

$$F(Y) \to F(Q \to Y)$$

is an equivalence.

PROOF. We leave this as exercise 4.B. □

REMARK 4.2.8. If $f : X \to Y$ is a morphism such that (4.2.3.1) is an equivalence, we often simply say that f is an *effective descent morphism for F*, or simply an *effective descent morphism* if the reference to F is clear.

In the rest of this section we discuss various important examples of effective descent morphisms.

4.2.9. When f admits a section. Let $p : F \to C$ be a fibered category as above, and let $f : X \to Y$ be a morphism in C.

PROPOSITION 4.2.10. *If there exists a section $s : Y \to X$ of f then f is an effective descent morphism.*

PROOF. Fix a section $s : Y \to X$ of f, and let

$$\eta : F(X \to Y) \to F(Y)$$

be the functor sending (E, σ) to s^*E. We claim that η is a quasi-inverse to ϵ.

It is immediate that the composite functor

$$F(Y) \xrightarrow{\epsilon} F(X \to Y) \xrightarrow{\eta} F(Y)$$

is isomorphic to the identity functor.

To show that the composite functor

$$F(X \to Y) \xrightarrow{\eta} F(Y) \xrightarrow{\epsilon} F(X \to Y)$$

is isomorphic to the identity, note first that for any $(E, \sigma) \in F(X \to Y)$ there is a canonical isomorphism

$$\rho_{(E,\sigma)} : E \to f^*s^*E$$

defined as the composition

$$E \simeq (\mathrm{id}, sf)^*\mathrm{pr}_1^*E \xrightarrow{(\mathrm{id},sf)^*\sigma} (\mathrm{id}, sf)^*\mathrm{pr}_2^*E \simeq f^*s^*E,$$

where

$$(\mathrm{id}, sf) : X \to X \times_Y X.$$

Notice also that by pulling back the diagram (4.2.1.1) along the morphism

$$(\mathrm{id}_{X\times_Y X}, sf\mathrm{pr}_2) : X \times_Y X \to (X \times_Y X) \times_Y X$$

we get a commutative diagram

$$\begin{tikzcd}
\mathrm{pr}_2^*f^*s^*E & \mathrm{pr}_1^*f^*s^*E \arrow[l, "\sigma_{\mathrm{can}}"'] & \mathrm{pr}_1^*E \arrow[l, "\mathrm{pr}_1^*\rho_{(E,\sigma)}"'] \arrow[d, equal] \\
\mathrm{pr}_2^*E \arrow[u, "\mathrm{pr}_2^*\rho_{(E,\sigma)}"'] \arrow[r, equal] & \mathrm{pr}_2^*E & \mathrm{pr}_1^*E. \arrow[l, "\sigma"']
\end{tikzcd}$$

This implies that $\rho_{(E,\sigma)}$ defines an isomorphism

$$(E, \sigma) \simeq (\epsilon \circ \eta)(E, \sigma)$$

thereby proving the proposition. □

4.2.11. Descent for sheaves in a site. The following example is a generalization of the standard fact that sheaves on a topological space can be constructed locally (see for example [**41**, Chapter II, exercise 1.22]).

Let C be a site in which finite limits are representable.

For an object $X \in C$ let C/X denote the site of objects over X and let $(C/X)^{\smallsmile}$ denote the associated topos, as in exercise 2.D. For a morphism $f : X \to Y$ in C write also f for the induced morphism of topoi

$$(C/X)^{\smallsmile} \to (C/Y)^{\smallsmile}.$$

The functor f^* in this morphism of topoi sends a sheaf F on C/Y to the sheaf on C/X given by

$$(W \to X) \mapsto F(W \to X \to Y).$$

Notice in particular that for a composition

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

we have an equality of functors (not just canonical isomorphism)

$$f^* \circ g^* = (g \circ f)^*.$$

Define Sh to be the category whose objects are pairs (X, E), where $X \in C$ and $E \in (C/X)^{\smallsmile}$, and for which a morphism

$$(X, E) \to (Y, F)$$

is a pair (f, ϵ), where

$$f : X \to Y$$

is a morphism in C and $\epsilon : E \to f^*F$ is a morphism in $(C/X)^{\smallsmile}$. The composition of morphisms

$$(X, E) \xrightarrow{(f,\epsilon)} (Y, F) \xrightarrow{(g,\rho)} (Z, G)$$

is defined to be the composition

$$g \circ f : X \to Z$$

together with the morphism

$$E \xrightarrow{\epsilon} f^*F \xrightarrow{f^*\rho} f^*g^*G = (g \circ f)^*G.$$

Let

$$p : \mathrm{Sh} \to C$$

be the functor sending (X, E) to X. Then p makes Sh a fibered category over C with fiber over $X \in C$ the category $\mathrm{Sh}(X) = (C/X)^{\smallsmile}$.

THEOREM 4.2.12. *Any covering $f : X \to Y$ in C is an effective descent morphism for* Sh.

PROOF. The category

$$\mathrm{Sh}(X \to Y)$$

is the category of pairs (E, σ), where E is a sheaf on C/X and $\sigma : \mathrm{pr}_1^*E \to \mathrm{pr}_2^*E$ is an isomorphism in $(C/X \times_Y X)^{\smallsmile}$ satisfying the cocycle condition over $X \times_Y X \times_Y X$. As before let

$$\epsilon : \mathrm{Sh}(Y) \to \mathrm{Sh}(X \to Y), \quad E \mapsto (f^*E, \sigma_{\mathrm{can}})$$

be the natural functor which we must show is an equivalence of categories.

Let $g : X \times_Y X \to Y$ be the projection. For $(E, \sigma) \in \mathrm{Sh}(X \to Y)$, we get two maps

$$\mathrm{pr}_1^*, \sigma^{-1} \circ \mathrm{pr}_2^* : f_*E \to g_*\mathrm{pr}_1^*E,$$

where (somewhat abusively) we write also $\mathrm{pr}_i^* : f_*E \to g_*\mathrm{pr}_i^*E$ for the map induced by applying f_* to the adjunction map $E \to \mathrm{pr}_{i*}\mathrm{pr}_i^*E$ and the isomorphism $f_*\mathrm{pr}_{i*} \simeq g_*$. Define

$$\eta : \mathrm{Sh}(X \to Y) \to \mathrm{Sh}(Y) \tag{4.2.12.1}$$

by sending (E, σ) to the equalizer of these two maps. Note that there are natural morphisms of functors

$$\mathrm{id}_{\mathrm{Sh}(Y)} \to \eta \circ \epsilon, \quad \epsilon \circ \eta \to \mathrm{id}_{\mathrm{Sh}(X \to Y)}$$

which we claim are isomorphisms.

That

$$\mathrm{id}_{\mathrm{Sh}(Y)} \to \eta \circ \epsilon$$

is an isomorphism follows immediately from the definition of a sheaf.

Now suppose given $(E, \sigma) \in \mathrm{Sh}(X \to Y)$, and let $F \in \mathrm{Sh}(Y)$ denote $\eta(E, \sigma)$. There is a natural map

$$F \to f_*E$$

whose adjoint $\rho : (f^*F, \sigma_{\mathrm{can}}) \to (E, \sigma)$ we need to show is an isomorphism. For this we can work locally on Y. Namely, if $Y' \to Y$ is a covering and

$$f' : X' \to Y'$$

denotes $X \times_Y Y'$, then we have a commutative diagram

$$\begin{tikzcd} \mathrm{Sh}(X \to Y) \arrow[r, "\text{restriction}"] \arrow[d, "\eta"] & \mathrm{Sh}(X' \to Y') \arrow[d, "\eta'"] \\ \mathrm{Sh}(Y) \arrow[r, "\text{restriction}"] & \mathrm{Sh}(Y'), \end{tikzcd}$$

where η' denotes the functor defined by replacing $X \to Y$ with $X' \to Y'$ in the above. Moreover, to show that $f^*F \to E$ is an isomorphism it suffices to show that it is an isomorphism after restricting to X'. It therefore suffices to prove the result after making a base change $Y' \to Y$ along a covering of Y.

In particular, taking $Y' = X$ we are then reduced to the case when the covering $X \to Y$ admits a section. In this case we already know by 4.2.10 that ϵ is an equivalence of categories which implies that η is also an equivalence since we already know that $\eta \circ \epsilon = \mathrm{id}$. $\square$

Corollary 4.2.13. *Let X and Y be schemes over a scheme S, and let $S' \to S$ be an fppf cover. Set*

$$S'' := S' \times_S S', X' := X \times_S S', X'' := X \times_S S'', Y' := Y \times_S S', Y'' := Y \times_S S'',$$

and let

$$\mathrm{pr}_1, \mathrm{pr}_2 : S'' \to S'$$

be the two projections. If $f' : X' \to Y'$ is an S'-morphism such that the two morphisms

$$\mathrm{pr}_1^*f', \mathrm{pr}_2^*f' : X'' \to Y''$$

are equal, then f' is induced by a unique morphism $f : X \to Y$.

Proof. Let h_X and h_Y denote the fppf sheaves represented by X and Y respectively, and let C denote the site of S-schemes with the fppf topology. Giving a morphisms f' such that $\mathrm{pr}_1^* f' = \mathrm{pr}_2^* f'$ is equivalent to giving a morphism

$$(h_{X'}, \sigma_{\mathrm{can}}) \to (h_{Y'}, \sigma_{\mathrm{can}})$$

in $\mathrm{Sh}(S' \to S)$. By 4.2.12 such a morphism is induced by a unique morphism $h_X \to h_Y$, which in turn by the Yoneda lemma corresponds to a morphism $f : X \to Y$. □

4.2.14. There is a variant of the above discussion for sheaves of modules. Namely suppose $\mathscr{O}$ is a sheaf of rings on the site C. Then for any $X \in C$ we get by restriction a sheaf of rings $\mathscr{O}_X$ on C/X, and for any morphism $f : X \to Y$ in C the morphism of topoi

$$f : (C/X)^{\sim} \to (C/Y)^{\sim}$$

extends to a morphism of ringed topoi

$$((C/X)^{\sim}, \mathscr{O}_X) \to ((C/Y)^{\sim}, \mathscr{O}_Y)$$

which we again denote simply by f.

For $X \in C$ let Mod_X denote the category of sheaves of $\mathscr{O}_X$-modules in $(C/X)^{\sim}$. Note that for a morphism $f : X \to Y$ the pullback functor

$$f^* : \mathrm{Mod}_Y \to \mathrm{Mod}_X$$

is simply the restriction functor.

Let MOD be the category whose objects are pairs (X, E), where $X \in C$ and $E \in \mathrm{Mod}_X$. A morphism

$$(X, E) \to (Y, F)$$

is a pair (f, ϵ), where $f : X \to Y$ is a morphism in C and $\epsilon : E \to f^*F$ is a morphism in Mod_X.

The projection

$$p : MOD \to C, \quad (X, E) \mapsto X$$

makes MOD a fibered category over C with fiber over $X \in C$ the category Mod_X. Repeating the proof of 4.2.12, replacing sets by sheaves of $\mathscr{O}$-modules, then yields the following:

Theorem 4.2.15. *Let $X \to Y$ be a covering in C. Then the functor*

$$\epsilon : \mathrm{Mod}_Y \to MOD(X \to Y)$$

is an equivalence of categories.

Proof. Proof omitted. □

4.3. Descent for quasi-coherent sheaves

4.3.1. Let S be a scheme, and let C be the category of S-schemes which we view as a site with the fppf topology. There is a presheaf of rings $\mathscr{O}$ on C given by sending an S-scheme T to $\Gamma(T, \mathscr{O}_T)$. This presheaf $\mathscr{O}$ is in fact a sheaf as it is represented by $\mathbb{A}^1_S$ (by exercise 1.B).

4.3.2. Let $\mathrm{Qcoh}(S)$ denote the category of quasi-coherent sheaves on S. For a quasi-coherent sheaf F on S, there is an associated presheaf F_{big} of $\mathscr{O}$-modules on C defined by sending an S-scheme $f : T \to S$ to

$$\Gamma(T, f^*F),$$

where f^*F denotes the quasi-coherent sheaf on T obtained by pullback (so f^*F is a Zariski sheaf on T). Note that this definition depends on the choice of pullback functors f^*, but up to canonical isomorphism the sheaf F_{big} is independent of the choice of pullbacks.

LEMMA 4.3.3. *For any quasi-coherent sheaf F on S, the presheaf F_{big} is a sheaf.*

PROOF. We apply Lemma 4.1.10.

Clearly F_{big} is a big Zariski sheaf, so we only need to check the sheaf condition for an fppf cover of the form $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ over S, with $A \to B$ a faithfully flat ring extension. Let M denote the A-module corresponding to the pullback of F to $\mathrm{Spec}(A)$. Then the sheaf condition for F is equivalent to the exactness of the sequence

$$0 \to M \to M \otimes_A B \rightrightarrows M \otimes_A (B \otimes_A B)$$

which is 4.1.6. □

4.3.4. If G is a sheaf of $\mathscr{O}$-modules on C, then for any S-scheme $f : T \to S$ we get a sheaf G_T of $\mathscr{O}_T$-modules on the small Zariski site of T by restricting G. In particular, we get a sheaf of $\mathscr{O}_S$-modules G_S. Observe that for a quasi-coherent sheaf F on S we have

$$F \simeq (F_{\mathrm{big}})_S.$$

LEMMA 4.3.5. *The map induced by restriction*

$$\mathrm{Hom}_{\mathscr{O}}(F_{\mathrm{big}}, G) \to \mathrm{Hom}_{\mathscr{O}_S}(F, G_S)$$

is an isomorphism.

PROOF. The assertion is Zariski local on S, so we may assume that S is affine. In this case we can write F as a cokernel

$$F_1 \to F_2 \to F \to 0$$

where the F_i are (possibly infinite) direct sums of copies of $\mathscr{O}_S$. Since the functor sending a quasi-coherent sheaf F to F_{big} is right exact we therefore get a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_{\mathscr{O}}(F_{\mathrm{big}}, G) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathscr{O}}(F_{2,\mathrm{big}}, G) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathscr{O}}(F_{1,\mathrm{big}}, G) \arrow[d] \\
0 \arrow[r] & \mathrm{Hom}_{\mathscr{O}_S}(F, G_S) \arrow[r] & \mathrm{Hom}_{\mathscr{O}_S}(F_2, G_S) \arrow[r] & \mathrm{Hom}_{\mathscr{O}_S}(F_1, G_S).
\end{tikzcd}$$

From this it follows that it suffices to prove the lemma in the case when F is a direct sum of copies of $\mathscr{O}_S$, and for this in turn it suffices to consider the case when $F = \mathscr{O}_S$ where the result amounts to the statement that the restriction map

$$\mathrm{Hom}_{\mathscr{O}}(\mathscr{O}, G) \to \Gamma(S, G_S)$$

is an isomorphism, which follows from the fact that S is the final object on C. □

DEFINITION 4.3.6. A *big quasi-coherent sheaf on* S is a sheaf of $\mathscr{O}$-modules F on C such that the following hold:

(1) For every S-scheme T, the sheaf F_T on the small Zariski site of T is quasi-coherent.
(2) For every morphism $g : T' \to T$ in C the natural map $g^*F_T \to F_{T'}$ is an isomorphism.

In what follows we denote the category of big quasi-coherent sheaves on S by $\mathrm{Qcoh}(S_{\mathrm{fppf}})$, and the usual category of quasi-coherent sheaves on the small Zariski site of S by $\mathrm{Qcoh}(S_{\mathrm{Zar}})$.

REMARK 4.3.7. If $F \in \mathrm{Qcoh}(S_{\mathrm{Zar}})$ is a quasi-coherent sheaf on S, then the associated big sheaf F_{big} is a big quasi-coherent sheaf on S.

PROPOSITION 4.3.8. *The functor*

$$\mathrm{Qcoh}(S_{\mathrm{Zar}}) \to \mathrm{Qcoh}(S_{\mathrm{fppf}}), \quad F \mapsto F_{\mathrm{big}} \tag{4.3.8.1}$$

is an equivalence of categories, with quasi-inverse sending $G \in \mathrm{Qcoh}(S_{\mathrm{fppf}})$ *to* G_S.

PROOF. This is immediate from the definition of a big quasi-coherent sheaf. □

REMARK 4.3.9. If

$$\epsilon : (S_{\mathrm{fppf}}, \mathscr{O}_{S_{\mathrm{fppf}}}) \to (S_{\mathrm{Zar}}, \mathscr{O}_{S_{\mathrm{Zar}}})$$

denotes the natural morphism of ringed topoi, then the functor (4.3.8.1) is the pullback ϵ^* and the inverse functor $G \mapsto G_S$ is the functor ϵ_*.

REMARK 4.3.10. Proposition 4.3.8 implies, in particular, that the category $\mathrm{Qcoh}(S_{\mathrm{fppf}})$ is abelian. However, the inclusion functor

$$\mathrm{Qcoh}(S_{\mathrm{fppf}}) \subset (\mathscr{O}\text{-modules on } C)$$

is not exact. For example, let $i : S' \hookrightarrow S$ be a closed subscheme, and let $\mathscr{J} \subset \mathscr{O}_S$ be the corresponding quasi-coherent sheaf of ideals. Then the kernel of the map

$$\mathscr{O} \to (i_*\mathscr{O}_{S'})_{\mathrm{big}}$$

in the category of $\mathscr{O}$-modules is the sheaf which to any S-scheme T associates the global sections of the kernel of the map

$$\mathscr{O}_T \to i_{T*}\mathscr{O}_{T\times_S S'},$$

where $i_T : T \times_S S' \hookrightarrow T$ is the closed imbedding obtained by base change. In general this kernel is a quotient of the pullback of $\mathscr{J}$, but if T is not flat over S these two may not be equal.

4.3.11. Define QCOH to be the category whose objects are pairs (T, E) where T is a scheme and $E \in \mathrm{Qcoh}(T_{\mathrm{Zar}})$ is a quasi-coherent sheaf. A morphism $(T', E') \to (T, E)$ is a pair (f, ϵ), where $f : T' \to T$ is a morphism of schemes and $E' \to f^*E$ is a morphism of quasi-coherent sheaves on T'.

There is a functor

$$p : \mathrm{QCOH} \to (\text{schemes})$$

which makes QCOH a fibered category over the category of schemes. The fiber over a scheme T is the category $\mathrm{Qcoh}(T_{\mathrm{Zar}})$.

THEOREM 4.3.12 ([**36**, VIII, 1.1]). *Let* $f : X \to Y$ *be an fppf covering of schemes. Then* f *is an effective descent morphism for* QCOH.

Proof. Consider first the case when f is quasi-compact and quasi-separated. In this case the functor

$$\eta : \mathrm{Sh}(X \to Y) \to \mathrm{Sh}(Y)$$

in (4.2.12.1) takes $\mathrm{QCOH}(X \to Y)$ to $\mathrm{QCOH}(Y)$ and therefore defines a quasi-inverse to the functor

$$\epsilon : \mathrm{QCOH}(Y) \to \mathrm{QCOH}(X \to Y).$$

Note that this argument does not directly give the result in general as the functor in (4.2.12.1) does not preserve quasi-coherence without the quasi-compact and quasi-separated assumption. However, we can reduce the general case to the quasi-compact and quasi-separated case as follows.

Let $(E, \sigma) \in \mathrm{QCOH}(X \to Y)$ be an object. We show that it is in the essential image of ϵ as follows.

First choose Zariski coverings

$$Y = \bigcup_i Y_i, \quad f^{-1}(Y_i) = \bigcup_j X_{ij},$$

with each Y_i affine and the X_{ij} quasi-compact with $f(X_{ij}) = Y_i$ for each j (such a covering exists by 1.1.6).

For each i and j, we then have a quasi-compact and quasi-separated cover $X_{ij} \to Y_i$, and therefore by the preceding case the restriction of (E, σ) to X_{ij} is induced by pullback from a unique object $F_{ij} \in \mathrm{QCOH}(Y)$. For any two indices j and j' we have a commutative diagram

$$\begin{tikzcd}
 & \mathrm{QCOH}(Y_i) \arrow[d] \arrow[dl, bend right] \arrow[dr, bend left] & \\
 & \mathrm{QCOH}((X_{ij} \coprod X_{ij'}) \to Y_i) \arrow[dl] \arrow[dr] & \\
\mathrm{QCOH}(X_{ij} \to Y_i) & & \mathrm{QCOH}(X_{ij'} \to Y_i)
\end{tikzcd}$$

which defines a unique isomorphism $\sigma_{ijj'} : F_{ij} \to F_{ij'}$ (loosely speaking, the sheaf F_{ij} is independent of j). It follows that in fact we get an equivalence of categories

$$\epsilon_i : \mathrm{QCOH}(Y_i) \to \mathrm{QCOH}(f^{-1}(Y_i) \to Y_i). \tag{4.3.12.1}$$

Now by the usual Zariski gluing of sheaves, the category $\mathrm{QCOH}(X \to Y)$ is equivalent to the category of data

$$(\{(E_i, \sigma_i)\}, \{\alpha_{ii'}\}),$$

as follows:

(1) Each $(E_i, \sigma_i) \in \mathrm{QCOH}(f^{-1}(Y_i) \to Y_i)$ is an object.
(2) For any two indices i, i'

$$\alpha_{ii'} : (E_i, \sigma_i)|_{Y_i \cap Y_{i'}} \to (E_{i'}, \sigma_{i'})|_{Y_i \cap Y_{i'}}$$

is an isomorphism in $\mathrm{QCOH}(f^{-1}(Y_i \cap Y_{i'}) \to Y_i \cap Y_{i'})$.
(3) These isomorphisms $\alpha_{ii'}$ are required to satisfy the usual cocycle condition over triple intersections.

Using the equivalences of categories (4.3.12.1) this data is equivalent to collections of data $(\{E_i\}, \{\beta_{ii'}\})$ as follows:

(1) Each E_i is a quasi-coherent sheaf on Y_i.
(2) For any two indices i and i'
$$\beta_{ii'} : E_i \to E_{i'}$$
is an isomorphism in $\mathrm{QCOH}(Y_i \cap Y_{i'})$.
(3) These isomorphisms are required to satisfy the cocycle condition on triple overlaps.

On the other hand, this last category is by the standard gluing of sheaves [**41**, Chapter II, exercise 1.22] equivalent to the category $\mathrm{QCOH}(Y)$. From this the theorem follows. $\square$

4.3.13. There is also a variant of 4.3.8 for étale sheaves. Let S be a scheme, and let
$$\eta : (S_{\mathrm{et}}, \mathscr{O}_{S_{\mathrm{et}}}) \to (S_{\mathrm{Zar}}, \mathscr{O}_{S_{\mathrm{Zar}}})$$
be the natural morphism of topoi, where S_{et} denotes the small étale topos of S. If F is a quasi-coherent sheaf on the small Zariski site of S, then the pullback η^*F is the sheaf (by 4.3.3) which to any étale morphism $g : U \to S$ associates $\Gamma(U_{\mathrm{Zar}}, g_{\mathrm{Zar}}^* F)$, where g_{Zar} denotes the morphism of small Zariski topoi associated to g.

Definition 4.3.14. A sheaf of $\mathscr{O}_{S_{\mathrm{et}}}$-modules E on S_{et} is *quasi-coherent* if $E \simeq \eta^*F$ for some quasi-coherent sheaf F on the Zariski site of S.

Proposition 4.3.15. *(i) The pullback functor η^* defines an equivalence of categories between the usual category of quasi-coherent sheaves in the Zariski topos of S, and the category of quasi-coherent sheaves in the étale topos of S.*

(ii) A sheaf of $\mathscr{O}_{S_{\mathrm{et}}}$-modules E on S_{et} is quasi-coherent if and only if there exists an étale covering $\{S_i \to S\}_{i \in I}$ such that the restriction of E to each $S_{i,\mathrm{et}}$ is quasi-coherent.

Proof. Note first that if F and F' are quasi-coherent sheaves on the Zariski site of S, then since $F' \simeq \eta_*\eta^*F'$ the pullback map
$$\mathrm{Hom}_{S_{\mathrm{Zar}}}(F, F') \to \mathrm{Hom}_{S_{\mathrm{et}}}(\eta^*F, \eta^*F') \simeq \mathrm{Hom}_{S_{\mathrm{Zar}}}(F, \eta_*\eta^*F')$$
is an isomorphism. This implies that the functor
$$\eta^* : \mathrm{Qcoh}(S_{\mathrm{Zar}}) \to \mathrm{Qcoh}(S_{\mathrm{et}})$$
is fully faithful, and therefore an equivalence giving (i).

For statement (ii), let E be of $\mathscr{O}_{S_{\mathrm{et}}}$-modules whose restriction to a covering $\{S_i \to S\}_{i \in I}$ is quasi-coherent. Let E_i be the restriction of E to $S_{i,\mathrm{et}}$ and let $\lambda_{ij} : \mathrm{pr}_1^* E_i \to \mathrm{pr}_2^* E_j$ be the isomorphism on $S_i \times_S S_j$ giving the descent data for E. By assumption there exists a quasi-coherent F_i on $S_{i,Zar}$ with an isomorphism $\rho_i : \eta_i^* F_i \simeq E_i$, and by (i) the pair (F_i, ρ_i) is unique up to unique isomorphism. Further, (i) implies that the descent isomorphisms λ_{ij} are induced by unique isomorphisms $\delta_{ij} : \mathrm{pr}_1^* F_i \to \mathrm{pr}_2^* F_j$ on the Zariski site of $S_i \times_S S_j$, and that these isomorphisms δ_{ij} satisfy the cocycle condition over the fiber products $S_i \times_S S_j \times_S S_k$. By effectivity of descent for quasi-coherent sheaves 4.3.12 we therefore get a quasi-coherent sheaf F on S_{Zar} corresponding to the descent data $(\{F_i\}, \{\delta_{ij}\})$. The pullback η^*F is isomorphic to E, because the corresponding étale sheaves with descent data with respect to $\{S_i \to S\}$ are isomorphic. $\square$

Remark 4.3.16. Similarly, if S is locally noetherian we can define a notion of coherent sheaf on S_{et} and the same proof shows that the category of étale coherent sheaves is equivalent to the category of coherent sheaves in the Zariski topology.

4.4. Examples

4.4.1. Descent for closed subschemes.

4.4.2. Let Y be a scheme, let $f : X \to Y$ be an fppf covering so we get the usual diagram

$$\begin{tikzcd} X \times_Y X \arrow[r, shift left, "\mathrm{pr}_1"] \arrow[r, shift right, "\mathrm{pr}_2"'] \arrow[rr, bend left, "g"] & X \arrow[r, "f"] & Y. \end{tikzcd}$$

For a closed subscheme $Z \hookrightarrow X$, let $\mathrm{pr}_i^* Z \hookrightarrow X \times_Y X$ denote the closed subscheme

$$Z \times_{X, \mathrm{pr}_i} (X \times_Y X).$$

Proposition 4.4.3. *The map*

$$\begin{array}{c} \{\text{closed subschemes } W \hookrightarrow Y\} \\ \downarrow \\ \{\text{closed subschemes } Z \hookrightarrow X \text{ such that } \mathrm{pr}_1^* Z = \mathrm{pr}_2^* Z\} \end{array}$$

sending a closed subscheme $W \subset Y$ to $f^{-1}(W) \subset X$ is a bijection.

Proof. Using the correspondence between closed subschemes and quasi-coherent sheaves of ideals, it suffices to show that the pullback map

$$\begin{array}{c} \{\text{quasi-coherent sheaves of ideals } \mathscr{J} \subset \mathscr{O}_Y\} \\ \downarrow \\ \{\text{quasi-coherent sheaves of ideals } \mathscr{I} \subset \mathscr{O}_X \text{ such that } \mathrm{pr}_1^* \mathscr{I} = \mathrm{pr}_2^* \mathscr{I}\} \end{array}$$

is a bijection, which follows from 4.3.12. $\square$

4.4.4. Descent for open imbeddings.

4.4.5. Let Op be the category whose objects are pairs $(X, U \subset X)$, where X is a scheme and U is an open subset of X. A morphism

$$(X', U' \subset X') \to (X, U)$$

is a morphism of schemes $f : X' \to X$ such that $U' \subset f^{-1}(U)$.

The functor

$$p : \mathrm{Op} \to (\text{schemes}), \quad (X, U \subset X) \mapsto X$$

makes Op a fibered category over the category of schemes.

Proposition 4.4.6. *Any fppf covering $f : S' \to S$ is an effective descent morphism for* Op.

PROOF. Let S'' denote $S' \times_S S'$. We have to show that if $U' \subset S'$ is an open subset such that $\mathrm{pr}_1^{-1}(U') = \mathrm{pr}_2^{-1}(U')$ in S'', then $U' = f^{-1}(U)$ for a unique open subset $U \subset S$.

The uniqueness is clear since $S' \to S$ is surjective.

For the existence, recall (see 1.1.5) that f is an open mapping so $U := f(U')$ is an open subset of S. We have $U' \subset f^{-1}(U)$ without assuming the descent condition. Now if $x \in f^{-1}(U)$ is a point not in U', then there exists a point $y \in U'$ such that x and y have the same image in U. Such a pair (x, y) defines a point of S'' which is in $\mathrm{pr}_2^{-1}(U)$ but not in $\mathrm{pr}_1^{-1}(U)$ contradicting our assumptions. Therefore we must have $U' = f^{-1}(U)$ as desired. $\square$

4.4.7. Descent for affine morphisms.

4.4.8. Let Aff denote the category whose objects are affine morphisms of schemes $f : X \to Y$, and whose morphisms

$$(f' : X' \to Y') \to (f : X \to Y)$$

are commutative squares:

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r] & Y. \end{tikzcd}$$

Let

$$p : \mathrm{Aff} \to (\text{schemes})$$

be the functor sending $f : X \to Y$ to Y. This functor makes Aff a fibered category over the category of schemes.

PROPOSITION 4.4.9. *Let $s : S' \to S$ be an fppf covering. Then s is an effective descent morphism for* Aff.

PROOF. We can rephrase the proposition as a statement about quasi-coherent sheaves of algebras. Namely, let Alg denote the category whose objects are pairs $(Y, \mathscr{A})$, where Y is a scheme and $\mathscr{A}$ is a quasi-coherent sheaf of $\mathscr{O}_Y$-algebras on Y. A morphism

$$(Y', \mathscr{A}') \to (Y, \mathscr{A})$$

is a pair (h, ϵ), where $h : Y' \to Y$ is a morphism of schemes and $\epsilon : h^*\mathscr{A} \to \mathscr{A}'$ is a morphism of quasi-coherent sheaves of $\mathscr{O}_{Y'}$-algebras.

Let

$$q : \mathrm{Alg} \to (\text{schemes})$$

be the functor sending $(Y, \mathscr{A})$ to Y. Then Alg is a fibered category over the category of schemes, and the functor

$$\mathrm{Alg} \to \mathrm{Aff}, \quad (Y, \mathscr{A}) \mapsto (\mathrm{Spec}_Y(\mathscr{A}) \to Y)$$

defines an equivalence of fibered categories. It therefore suffices to show that the fppf covering s is an effective descent morphism for Alg. This follows from 4.3.12, and the observation that the equivalence in loc. cit. is compatible with tensor products and therefore also algebra objects. $\square$

4.4.10. Descent for polarized schemes.

4.4.11. Let $\mathscr{P}ol$ be the category whose objects are pairs $(f : X \to Y, L)$, where f is a proper and flat morphism of schemes and L is a relatively ample invertible sheaf on X. A morphism

$$(f' : X' \to Y', L') \to (f : X \to Y, L)$$

is a triple (a, b, ϵ), where

$$a : Y' \to Y, \quad b : X' \to X$$

are morphisms of schemes such that the square

$$\begin{tikzcd} X' \arrow[r, "b"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r, "a"] & Y \end{tikzcd}$$

is cartesian, and $\epsilon : b^*L \to L'$ is an isomorphism of invertible sheaves on X'.

There is a functor

$$p : \mathscr{P}ol \to (\text{schemes})$$

sending $(f : X \to Y, L)$ to Y which makes $\mathscr{P}ol$ a fibered category over the category of schemes.

Proposition 4.4.12. *Any fppf covering $S' \to S$ is an effective descent morphism for $\mathscr{P}ol$.*

Proof. By an argument similar to the argument proving 4.3.12, which we leave to the reader, it suffices to consider the case when S is affine and $S' \to S$ is quasi-compact and quasi-separated.

The category $\mathscr{P}ol(S' \to S)$ is the category of data

$$(f' : X' \to S', L', \sigma, \epsilon),$$

where $X' \to S'$ is a proper and flat S'-scheme, L' is a relatively ample invertible sheaf on X', and

$$\sigma : \mathrm{pr}_1^* X' \to \mathrm{pr}_2^* X'$$

is an isomorphism of schemes over $S'' := S' \times_S S'$, and $\epsilon : \mathrm{pr}_1^* L' \to \sigma^* \mathrm{pr}_2^* L'$ is an isomorphism of invertible sheaves on $\mathrm{pr}_1^* X'$. Furthermore, the morphisms (σ, ϵ) satisfy the cocycle condition over $S''' := S' \times_S S' \times_S S'$.

Since S' is quasi-compact there exists by [**26**, III.7.9.10] an integer N such that

$$E' := f'_* L'^{\otimes N}$$

is a locally free sheaf on S', the map

$$f'^* E' = f'^* f'_* L'^{\otimes N} \to L'^{\otimes N}$$

is surjective, and the resulting morphism

$$X' \to \mathbb{P}(E')$$

is a closed imbedding.

Since the two projections $\mathrm{pr}_i : S'' \to S'$ are flat we have

$$\mathrm{pr}_i^* E' \simeq f''_{i*} \mathrm{pr}_i^* L'^{\otimes N}$$

where

$$f''_i : \mathrm{pr}_i^* X' \to S''$$

is the pullback of f'. In particular, the isomorphisms (σ, ϵ) define an isomorphism

$$\sigma_{E'} : \mathrm{pr}_1^* E' \to \mathrm{pr}_2^* E'$$

satisfying the cocycle condition over S'''. By descent for locally free sheaves (exercise 4.C) the pair $(E', \sigma_{E'})$ is induced by a locally free sheaf E on S.

Now the scheme X' defines a closed subscheme of

$$\mathbb{P}(E) \times_S S'$$

such that the two pullbacks to $\mathbb{P}(E) \times_S S''$ are equal. By descent for closed subschemes 4.4.3 we see that $X' \hookrightarrow \mathbb{P}(E')$ is obtained by pullback from a unique closed subscheme $X \hookrightarrow \mathbb{P}(E)$. Finally the descent of L' is obtained simply by setting L on X to be the pullback of $\mathscr{O}(1)$ on $\mathbb{P}(E)$.

This shows that every object of $\mathscr{P}ol(S' \to S)$ is in the essential image of $\mathscr{P}ol(S)$. The full faithfulness is shown similarly. If

$$(f_1 : X_1 \to S, L_1), \quad (f_2 : X_2 \to S, L_2)$$

are two objects over S and

$$\sigma' : (X_1', L_1') \to (X_2', L_2')$$

is an isomorphism over S' defining an isomorphism in $\mathscr{P}ol(S' \to S)$, then choose an integer N such that the sheaves

$$E_i := f_{i*} L_i^{\otimes N}$$

are locally free sheaves and we have closed imbeddings

$$X_i \hookrightarrow \mathbb{P}(E_i)$$

over S. The isomorphism σ' then defines an isomorphism $E_{1,S'} \to E_{2,S'}$ over S' which satisfies the cocycle condition over S'' and therefore defines an isomorphism

$$\sigma : E_1 \to E_2.$$

The resulting isomorphism $\mathbb{P}(E_1) \to \mathbb{P}(E_2)$ takes X_1 to X_2 and therefore defines an isomorphism

$$\sigma : (X_1, L_1) \to (X_2, L_2)$$

inducing σ'. It is also clear from the construction that σ is unique. □

EXAMPLE 4.4.13. Let g be a nonnegative integer not equal to 1, and let $\mathscr{M}_g$ be the category whose objects are proper flat morphisms $f : C \to S$ such that for every point $s \in S$ the fiber C_s is a geometrically connected proper smooth curve of genus g. Note that these conditions imply that the morphism f is in fact smooth. A morphism

$$(C' \to S') \to (C \to S)$$

in $\mathscr{M}_g$ is a cartesian square

$$\begin{tikzcd} C' \arrow[r] \arrow[d] & C \arrow[d] \\ S' \arrow[r] & S. \end{tikzcd}$$

There is a functor

$$p : \mathscr{M}_g \to (\text{schemes})$$

making $\mathscr{M}_g$ a fibered category over the category of schemes.

There is a morphism of fibered categories

$$\mathscr{M}_g \to \mathscr{P}ol$$

which if $g \geq 2$ sends $C \to S$ to $(C \to S, \Omega^1_{C/S})$, and if $g = 0$ sends $C \to S$ to $(C \to S, \Omega^{1,\otimes -1}_{C/S})$.

For an fppf covering $S' \to S$, we then get a commutative diagram

$$\begin{tikzcd} \mathscr{M}_g(S) \arrow[r] \arrow[d] & \mathscr{P}ol(S) \arrow[d] \\ \mathscr{M}_g(S' \to S) \arrow[r] & \mathscr{P}ol(S' \to S), \end{tikzcd}$$

where the right vertical arrow is an equivalence. This implies that every object of $\mathscr{M}_g(S' \to S)$ is in the essential image of $\mathscr{M}_g(S)$. That the functor

$$\mathscr{M}_g(S) \to \mathscr{M}_g(S' \to S)$$

is fully faithful follows from 4.2.13.

REMARK 4.4.14. The stack fibered category $\mathscr{M}_g$ will be discussed at length in what follows; in particular, in Chapter 13.

4.4.15. Descent for quasi-affine morphisms.

4.4.16. Recall that a morphism of schemes $f : X \to Y$ is called *quasi-affine* if there exists a factorization

$$\begin{tikzcd} X \arrow[r, hook, "j"] \arrow[dr, "f"'] & W \arrow[d, "g"] \\ & Y, \end{tikzcd}$$

where j is a quasi-compact open imbedding, and g is an affine morphism.

Let QAff be the category whose objects are quasi-affine morphisms $f : X \to Y$, and whose morphisms are commutative squares

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r] & Y. \end{tikzcd}$$

Let

$$p : \mathrm{QAff} \to (\text{schemes})$$

be the functor sending $f : X \to Y$ to Y. This functor makes QAff a fibered category over the category of schemes.

PROPOSITION 4.4.17. *Any fppf covering $S' \to S$ is an effective descent morphism for* QAff.

PROOF. The key point here is that a quasi-affine morphism $f : X \to Y$ has a canonical factorization

$$X \xrightarrow{j} Z \xrightarrow{g} Y, \tag{4.4.17.1}$$

where j is an open imbedding and g is affine. Namely, let $\mathscr{A}$ denote the quasi-coherent (since f is quasi-compact and quasi-separated) sheaf of algebras $f_*\mathscr{O}_X$, and set $Z = \mathrm{Spec}_Y(\mathscr{A})$. There is a canonical map $X \to Z$, and as explained in [**26**, proof of IV, 18.12.12] (see also [**67**, Tag 01SM]) this map is an open imbedding. Note also that the construction of this factorization of f commutes with flat base change on Y.

Consider now the functor

$$\mathrm{QAff}(S) \to \mathrm{QAff}(S' \to S) \tag{4.4.17.2}$$

which we need to show is an equivalence. The fact that schemes are fppf sheaves implies that this functor is fully faithful. Indeed, suppose given two quasi-affine morphisms $X_i \to S$ $(i = 1, 2)$. For any S-scheme $T \to S$, giving a T-morphism $X_{1,T} \to X_{2,T}$ is equivalent to giving an element in $h_{X_2}(X_{1,T})$, where

$$h_{X_2} : (\mathrm{Sch}/S)^{\mathrm{op}} \to \mathrm{Set}$$

is the functor represented by X_2. In particular, we have an exact sequence

$$\mathrm{Hom}_S(X_1, X_2) \to \mathrm{Hom}_{S'}(X_{1,S'}, X_{2,S'}) \rightrightarrows \mathrm{Hom}_{S'\times_S S'}(X_{1,S'\times_S S'}, X_{2,S'\times_S S'}),$$

and similarly with X_1 and X_2 interchanged. From this the full faithfulness of (4.4.17.2) follows.

So it remains to show that any pair $(f' : X' \to S', \sigma)$ consisting of a quasi-affine morphism f' and an isomorphism

$$\sigma : \mathrm{pr}_1^* X' \to \mathrm{pr}_2^* X'$$

over $S'' := S' \times_S S'$ satisfying the cocycle condition is induced by a quasi-affine morphism $X \to S$.

Let

$$X' \longrightarrow Z' \longrightarrow S'$$

be the factorization (4.4.17.1) for f'. Since the formation of this factorization commutes with flat base change, σ induces a unique isomorphism

$$\tilde{\sigma} : \mathrm{pr}_1^* Z' \to \mathrm{pr}_2^* Z'$$

such that the diagram of S''-schemes

$$\begin{tikzcd} \mathrm{pr}_1^* X' \arrow[r, "\sigma"] \arrow[d] & \mathrm{pr}_2^* X' \arrow[d] \\ \mathrm{pr}_1^* Z' \arrow[r, "\tilde{\sigma}"] & \mathrm{pr}_2^* Z' \end{tikzcd}$$

commutes. Similarly, the isomorphism $\tilde{\sigma}$ satisfies the cocycle condition. By descent for affine morphisms 4.4.9, the pair $(Z', \tilde{\sigma})$ is therefore induced by an affine morphism $Z \to S$. Now note that X' defines an object of $\mathrm{Op}(Z' \to Z)$ (notation as in 4.4.6), and is therefore the pullback of a unique open subset $X \hookrightarrow Z$. This open imbedding is quasi-compact since its pullback to Z' is quasi-compact, and therefore the composition $X \to Z \to S$ defines the desired object of $\mathrm{QAff}(S)$. $\square$

4.5. Application: Torsors and principal homogenous spaces

4.5.1. Let C be a site and let $\boldsymbol{\mu}$ be a sheaf of groups on C (not necessarily abelian). A *$\boldsymbol{\mu}$-torsor on C* is a sheaf $\mathscr{P}$ on C together with a left action ρ of $\boldsymbol{\mu}$ on $\mathscr{P}$, such that the following conditions hold:

(T1) For every $X \in C$ there exists a covering $\{X_i \to X\}$ such that $\mathscr{P}(X_i) \neq \emptyset$ for all i.

(T2) The map

$$\boldsymbol{\mu} \times \mathscr{P} \to \mathscr{P} \times \mathscr{P}, \quad (g, p) \mapsto (p, gp)$$

is an isomorphism.

Note that the second condition is equivalent to saying that if $\mathscr{P}(X)$ is nonempty, then the action of $\boldsymbol{\mu}(X)$ on $\mathscr{P}(X)$ is simply transitive. We say that a torsor $(\mathscr{P}, \rho)$ is *trivial* if $\mathscr{P}$ has a global section. In this case if we fix a global section p, then we have an isomorphism

$$\boldsymbol{\mu} \to \mathscr{P}, \quad g \mapsto gp$$

which identifies $\mathscr{P}$ with $\boldsymbol{\mu}$ and the action ρ with left-translation on $\boldsymbol{\mu}$.

A morphism of $\boldsymbol{\mu}$-torsors $(\mathscr{P}, \rho) \to (\mathscr{P}', \rho')$ is a morphism of sheaves $f : \mathscr{P} \to \mathscr{P}'$ such that the diagram

$$\begin{tikzcd}
\boldsymbol{\mu} \times \mathscr{P} \arrow[r, "\mathrm{id}_{\boldsymbol{\mu}} \times f"] \arrow[d, "\rho"] & \boldsymbol{\mu} \times \mathscr{P}' \arrow[d, "\rho'"] \\
\mathscr{P} \arrow[r, "f"] & \mathscr{P}'
\end{tikzcd}$$

commutes.

REMARK 4.5.2. The notion of a $\boldsymbol{\mu}$-torsor on C depends only on the topos $C^{\sim}$ and not on the underlying site. For this reason we will sometimes also speak of a torsor in a topos.

4.5.3. The notion of torsor is closely related to the more classical notion of a principal bundle. To explain this, fix a scheme X and consider the site Sch/X of schemes over X with the fppf topology. Assume $\boldsymbol{\mu}$ is representable by a flat locally finitely presented X-group scheme G.

DEFINITION 4.5.4. A *principal G-bundle* over X is a pair $(\pi : P \to X, \rho)$, where π is a flat, locally finitely presented, surjective morphism of schemes, and

$$\rho : G \times_X P \to P$$

is a morphism such that the following axioms hold:

(i) The diagram

$$\begin{tikzcd}
G \times_X G \times_X P \arrow[r, "\mathrm{id}_G \times \rho"] \arrow[d, "m \times \mathrm{id}_P"] & G \times_X P \arrow[d, "\rho"] \\
G \times_X P \arrow[r, "\rho"] & P
\end{tikzcd}$$

commutes, where $m : G \times_X G \to G$ is the map defining the group law on G.

(ii) If $e : X \to G$ is the identity section, then the composition

$$P \xrightarrow{(e\pi, \mathrm{id}_P)} G \times_X P \xrightarrow{\rho} P$$

is the identity map on P.

(iii) The map

$$(\rho, \mathrm{pr}_2) : G \times_X P \to P \times_X P$$

is an isomorphism.

A morphism of principal G-bundles $(P, \rho) \to (P', \rho')$ is a morphism of X-schemes $f : P \to P'$ such that the diagram

$$\begin{tikzcd} G \times_X P \arrow[r, "\mathrm{id}_G \times f"] \arrow[d, "\rho"] & G \times_X P' \arrow[d, "\rho'"] \\ P \arrow[r, "f"] & P' \end{tikzcd}$$

commutes.

4.5.5. For a principal G-bundle (P, ρ) over X, we get a $\boldsymbol{\mu}$-torsor $(\mathscr{P}, \rho)$, by letting $\mathscr{P}$ be the sheaf on Sch/X represented by P, with action induced by the action ρ. Conditions (i) and (ii) in the definition of a principal G-bundle are then equivalent to the induced map

$$\boldsymbol{\mu} \times \mathscr{P} \to \mathscr{P}$$

being an action, and condition (iii) in the definition of a principal G-bundle is equivalent to condition (T2) for $(\mathscr{P}, \rho)$. Furthermore, since $P \to X$ is flat, locally finitely presented, and surjective there exists fppf locally on X a section, which implies that there exists an fppf cover $\{X_i \to X\}$ with $\mathscr{P}(X_i) \neq \emptyset$ for all i.

This construction in fact defines a fully faithful (by Yoneda's lemma) functor

$$\text{(principal } G\text{-bundles on } X) \to (\boldsymbol{\mu}\text{-torsors on } X). \tag{4.5.5.1}$$

Proposition 4.5.6. *If the structure morphism $G \to X$ is affine, then* (4.5.5.1) *is an equivalence of categories.*

Proof. Since we already know that (4.5.5.1) is fully faithful, it suffices to show that if $(\mathscr{P}, \rho)$ is a torsor then $\mathscr{P}$ is representable by a flat, locally finitely presented X-scheme P. Indeed as already discussed above, the action of $\boldsymbol{\mu}$ on $\mathscr{P}$ then corresponds under the Yoneda imbedding to an action of G on P satisfying (i) and (ii), condition (T2) is equivalent to condition (iii) in the definition of a principal G-bundle, and condition (T1) implies that $P \to X$ is surjective.

Choose an fppf covering $\{X_i \to X\}$ with $\mathscr{P}(X_i) \neq \emptyset$ for each i. Then the restriction of $\mathscr{P}$ to X_i is representable by a scheme P_i affine over X_i, namely (non-canonically) $G \times_X X_i$. The descent data for $\mathscr{P}$ defines descent data for the P_i, and since each P_i is affine over X_i this descent data is effective by 4.4.9. The resulting affine X-scheme $P \to X$ then represents $\mathscr{P}$. $\square$

Remark 4.5.7. In the case when G/X is smooth, any principal G-bundle $P \to X$ is also smooth since this can be verified over an fppf covering $\{X_i \to X\}$ where P is trivial. It follows that $P \to X$ admits a section étale locally on X. From this, one obtains that the category of principal G-bundles in the case when G/X is smooth is also equivalent to the category of $\boldsymbol{\mu}$-torsors on Sch/X with the étale topology.

Example 4.5.8 ([**52**, p. 125]). Let X be a scheme and let n be an integer invertible on X. Let $\boldsymbol{\mu}_n$ denote the group scheme with $\boldsymbol{\mu}_n(S) = \{f \in \mathscr{O}_S^* | f^n = 1\}$ (so if $\Gamma(X, \mathscr{O}_X)$ contains a primitive n-th root of unity then $\boldsymbol{\mu}_n$ is isomorphic to $\mathbb{Z}/(n)$). In this case the category of $\boldsymbol{\mu}_n$-torsors $\mathrm{Tors}(\boldsymbol{\mu}_n)$ on the étale site of X can be described as follows.

Let Σ_n denote the category of pairs (L, σ), where L is an invertible sheaf on X and $\sigma : L^{\otimes n} \to \mathscr{O}_X$ is a trivialization of the n-th power of L. Note that here we can consider L as either a sheaf in the Zariski or the étale topology by 4.3.15. A morphism $(L, \sigma) \to (L', \sigma')$ in Σ_n is an isomorphism $\rho : L \to L'$ of line bundles such that the diagram

$$\begin{tikzcd} L^{\otimes n} \arrow[dd, "\rho^{\otimes n}"'] \arrow[dr, "\sigma"] & \\ & \mathscr{O}_X \\ L'^{\otimes n} \arrow[ur, "\sigma'"'] & \end{tikzcd}$$

commutes. There is a functor

$$F : \Sigma_n \to \mathrm{Tors}(\boldsymbol{\mu}_n)$$

defined as follows. For a pair $(L, \sigma) \in \Sigma_n$ let $\mathscr{P}_{(L,\sigma)}$ denote the sheaf on the étale site of X associating to any $U \to X$ the set of trivializations $\lambda : \mathscr{O}_U \to L|_U$ such that the composite map

$$\mathscr{O}_U \xrightarrow{\lambda^{\otimes n}} L^{\otimes n}|_U \xrightarrow{\sigma} \mathscr{O}_U$$

is the identity map. There is an action of $\boldsymbol{\mu}_n(U)$ on $\mathscr{P}_{(L,\sigma)}(U)$ for which $\zeta \in \boldsymbol{\mu}_n(U)$ sends λ to the trivialization $\zeta \cdot \lambda$, and this defines a simply transitive action of $\boldsymbol{\mu}_n$ on $\mathscr{P}_{(L,\sigma)}$ making $\mathscr{P}_{(L,\sigma)}$ a $\boldsymbol{\mu}_n$-torsor. This construction defines the functor F.

Note that it is essential that we work with the étale topology here. For if $(L, \sigma) \in \Sigma_n$ it may not be possible to find a trivialization $\lambda \in \mathscr{P}_{(L,\sigma)}$ Zariski locally, but we can always find such a trivialization étale locally. Indeed, after replacing X by a Zariski cover we can choose a trivialization $\tau : \mathscr{O}_X \to L$. Now the composite map

$$\mathscr{O}_X \xrightarrow{\tau^{\otimes n}} L^{\otimes n} \xrightarrow{\sigma} \mathscr{O}_X$$

is given by multiplication by some $f \in \mathscr{O}_X^*$. Étale locally on X we can find an n-th root g of f (in fact on the étale cover $\underline{\mathrm{Spec}}_X(\mathscr{O}_X[T]/(T^n - f)) \to X$) and replacing τ by $g^{-1}\tau$ we get an element of $\mathscr{P}_{(L,\sigma)}$.

There is also a functor $G : \mathrm{Tors}(\boldsymbol{\mu}_n) \to \Sigma_n$ defined in the following way. If $\mathscr{P}$ is a $\boldsymbol{\mu}_n$-torsor, let $L_{\mathscr{P}}$ be the line bundle corresponding to (see exercise 4.G) the $\mathscr{O}_X^*$-torsor $T_{\mathscr{P}}$ whose value over an étale $U \to X$ is the set of maps of sheaves of sets $\mathscr{P}|_U \to \mathscr{O}_U^*$ on U which commute with the action of $\boldsymbol{\mu}_n$, where $\boldsymbol{\mu}_n$ acts on $\mathscr{O}_U^*$ by multiplication. The $\mathscr{O}_X^*$-torsor structure is induced by the multiplication action of $\mathscr{O}_X^*$ on itself. Note that if $\mathscr{P}_U$ is trivial and $p \in \mathscr{P}(U)$ is a section, then we get a trivialization of $T_{\mathscr{P}}$ from the unique map $\mathscr{P}|_U \to \mathscr{O}_U^*$ sending p to 1 and compatible with the $\boldsymbol{\mu}_n$-action. If p' is another trivialization obtained by applying an element $\zeta \in \boldsymbol{\mu}_n$ to p, then the corresponding trivializations of $T_{\mathscr{P}}$ differ by the action of $\zeta \in \boldsymbol{\mu}_n$. It follows that there is a canonical trivialization $\sigma_{\mathscr{P}}$ of $L_{\mathscr{P}}^{\otimes n}$, and the association $\mathscr{P} \mapsto (L_{\mathscr{P}}, \sigma_{\mathscr{P}})$ defines the functor G. We leave to the reader the verification that F and G are quasi-inverses.

Example 4.5.9. A special case of the preceding example is when $X = \mathrm{Spec}(k)$ is the spectrum of a field of characteristic prime to n. In this case, all line bundles on X are trivial, so the category Σ_n in the preceding paragraph is equivalent to the category whose objects are pairs $\sigma \in k^*$ and in which a morphism $\sigma \to \sigma'$ is given by an element $\lambda \in k^*$ such that $\sigma' = \lambda^n \sigma$. In particular, the isomorphism classes of objects in Σ_n, and hence also the isomorphism classes of $\boldsymbol{\mu}_n$-torsors, are in bijection with $k^*/k^{*,n}$.

4.6. Stacks

Let C be a site.

DEFINITION 4.6.1. A category fibered in groupoids $p : F \to C$ is a *stack* if for every object $X \in C$ and covering $\{X_i \to X\}_{i \in I}$, the functor

$$F(X) \to F(\{X_i \to X\}_{i \in I})$$

is an equivalence of categories.

The following Proposition relates this definition to the original one in [**34**, II.1.2.1] (see also [**23**, 4.1]).

PROPOSITION 4.6.2. *A category fibered in groupoids $p : F \to C$ is a stack if and only if the following two conditions hold:*

(i) *For any $X \in C$ and objects $x, y \in F(X)$, the presheaf $\underline{\mathrm{Isom}}(x, y)$ on C/X defined in* 3.4.7 *is a sheaf.*
(ii) *For any covering $\{X_i \to X\}$ of an object $X \in C$, any descent data with respect to this covering is effective.*

PROOF. We leave this as exercise 4.F. $\square$

DEFINITION 4.6.3. A category fibered in groupoids $p : F \to C$ is called a *prestack* if condition (i) in 4.6.2 holds.

PROPOSITION 4.6.4. *Let*

$$\begin{array}{ccc} & & F_1 \\ & & \downarrow{\scriptstyle c} \\ F_2 & \xrightarrow{d} & F_3 \end{array}$$

be a diagram of stacks fibered in groupoids over C. Then the fiber product $F_1 \times_{F_3} F_2$ is also a stack fibered in groupoids.

PROOF. This follows from noting that for any covering $\{X_i \to X\}$ in C the maps

$$(F_1 \times_{F_3} F_2)(X) \to F_1(X) \times_{F_3(X)} F_2(X)$$

and

$$(F_1 \times_{F_3} F_2)(\{X_i \to X\}) \to F_1(\{X_i \to X\}) \times_{F_3(\{X_i \to X\})} F_2(\{X_i \to X\})$$

are equivalences of groupoids. $\square$

Just as one can associate a sheaf to any presheaf, one can associate a stack to any category fibered in groupoids (see also [**34**, II, 2.1.3] and [**49**, 3.2]).

THEOREM 4.6.5. *Let $p : F \to C$ be a category fibered in groupoids. Then there exists a stack F^a over C and a morphism of fibered categories $q : F \to F^a$ such that for any stack G over C the induced functor*

$$\mathrm{HOM}_C(F^a, G) \to \mathrm{HOM}_C(F, G)$$

is an equivalence of categories.

REMARK 4.6.6. As in the case of a sheaf associated to a presheaf, the pair (F^a, q) is unique in the following sense. Given a second such pair (F'^a, q') for F we get an equivalence

$$\mathrm{HOM}_C(F^a, F'^a) \to \mathrm{HOM}_C(F, F'^a)$$

and therefore a pair (g, ϵ), unique up to unique isomorphism, where $g : F^a \to F'^a$ is a morphism of functors and $\epsilon : g \circ q \simeq q'$ is an isomorphism of morphisms of fibered categories. Similarly we get a unique pair (h, η), where $h : F'^a \to F^a$ is a morphism of fibered categories and $\eta : h \circ q' \simeq q$ is an isomorphism. The composition $h \circ g : F^a \to F^a$ is then a morphism of fibered categories equipped with an isomorphism $h \circ g \circ q \simeq h \circ q' \simeq q$. It follows that there exists a unique isomorphism $h \circ g \simeq \mathrm{id}_{F^a}$ inducing this isomorphism $h \circ g \circ q \simeq q$, and similarly for $g \circ h$. It follows that g is an equivalence of fibered categories. In other words, (F^a, q) is unique up to an equivalence that is unique up to unique isomorphism.

PROOF OF THEOREM 4.6.5. We sketch here the proof of the theorem, leaving some verifications to the reader. For a complete proof see [**67**, Tag 02ZP].

The proof proceeds in two steps. First we construct a morphism $F \to F'$ which is universal for morphisms to prestacks, and then a morphism $F' \to F^a$ which is universal for morphisms to stacks.

The prestack F' is defined as follows. The objects of F' are the same as the objects of F but for two objects $x, y \in F'$ lying over $X, Y \in C$ we define the set of morphisms $x \to y$ in F' to be the set of pairs $(f : X \to Y, \varphi)$, where f is a morphism in C and $\varphi \in \Gamma(X, \underline{\mathrm{Isom}}(x, f^*y)^a)$, where $\underline{\mathrm{Isom}}(x, f^*y)^a$ denotes the sheaf associated to the presheaf $\underline{\mathrm{Isom}}(x, f^*y)$.

For three objects $x, y, z \in F'$ lying over $X, Y, Z \in C$, the composition of two morphisms

$$(f, \varphi) : x \to y, \quad (g, \psi) : y \to z$$

is defined to be the composition $gf : X \to Z$ together with the section of $\underline{\mathrm{Isom}}(x, (gf)^*z)$ provided by the image of $(\varphi, f^*\psi)$ under the map

$$\underline{\mathrm{Isom}}(x, f^*y)^a \times \underline{\mathrm{Isom}}(f^*y, f^*g^*z)^a \to \underline{\mathrm{Isom}}(x, f^*g^*z)^a$$

obtained by sheafifying the composition map for F.

In this way we obtain a prestack F', together with a natural map $\epsilon_F : F \to F'$ which is the identity on objects and induced by the maps

$$\underline{\mathrm{Isom}}(x, f^*y) \to \underline{\mathrm{Isom}}(x, f^*y)^a$$

on morphisms. Observe that if F is already a prestack, then the morphism ϵ_F is an equivalence of fibered categories.

In addition, for any other fibered category H with associated prestack H' (given by the preceding construction applied to H), and morphism ρ, we get by sheafifying the induced maps on $\underline{\mathrm{Isom}}$-presheaves a morphism of prestacks

$$\rho' : F' \to H'.$$

Moreover, in the case when H is already a prestack, the association $\rho \mapsto \rho'$ defines an equivalence of categories

$$HOM(F', H) \to HOM(F, H).$$

To complete the proof of the theorem, we now consider the construction of a stack from a prestack F. Define the objects of F^a over an object $X \in C$ to be collections of data

$$(\{X_i \to X\}_{i\in I}, (\{x_i\}, \varphi_{ij})),$$

where $\{X_i \to X\}_{i\in I}$ is a covering in C, and $(\{x_i\}, \varphi_{ij})$ is an object of $F(\{X_i \to X\})$. A morphism

$$(\{Y_s \to Y\}_{s\in S}, (\{y_s\}, \psi_{st})) \to (\{X_i \to X\}_{i\in I}, (\{x_i\}, \varphi_{ij})),$$

is a pair (f, ρ), where $f : Y \to X$ is a morphism in C, and $\rho : (\{y_s\}, \psi_{st})\tilde{} \to f^*(\{x_i\}, \varphi_{ij})\tilde{}$ is a morphism between the induced objects of

$$F(\{Y_s \times_Y (Y \times_X X_i)\}_{i,s} \to Y).$$

There is a morphism of fibered categories $F \to F^a$ sending an object $x \in F(X)$ to $(\{\mathrm{id} : X \to X\}, (x, \mathrm{id}_x)) \in F^a(X)$, and if F is already a stack then this is an isomorphism of fibered categories, and if $F \to H$ is a morphism from F to a stack H, then the induced functor

$$HOM(F^a, H) \to HOM(F, H)$$

is an equivalence of categories. It remains to verify that F^a is a stack, which we leave to the reader. □

EXAMPLE 4.6.7. Let S be a scheme, and let $(X_0, X_1, s, t, \epsilon, i, m)$ be a groupoid in the category of S-schemes (notation as in 3.4.4). By the construction in 3.4.5 we then get a fibered category $\{X_0/X_1\}$, and we can consider the associated stack with respect to the étale topology on Sch/S. We will denote the stack by $[X_0/X_1]$. In particular, if X is an S-scheme, and G is an S-group scheme acting on X then we can consider the associated action groupoid (see exercise 3.G) to get a stack which we denote by $[X/G]$.

REMARK 4.6.8. The stack F^a in 4.6.5 is sometimes called the *stackification* of F.

EXAMPLE 4.6.9. Let C be a site and let $f : A \to B$ be a morphism of sheaves of abelian groups on C. Then the sheaf A acts on B by $a * b := f(a) + b$. We can therefore form the quotient $F := \{A/B\}$. For $X \in C$ and two objects $b_1, b_2 \in B(X)$, the functor

$$\mathrm{Isom}_F(b_1, b_2) : C/X \to \mathrm{Set}$$

sends $Y \to X$ to the set of elements $a \in A(Y)$ such that $f(a) = b_2 - b_1$. In particular this is a sheaf so F is a prestack. The condition for F to be a stack is more subtle. Let $X \in C$ be an object and let $\{X_i \to X\}_{i\in I}$ be a covering. Suppose given $b_i \in B(X_i)$ and for every $i, j \in I$ elements $a_{ij} \in A(X_{ij})$ such that $f(a_{ij}) = b_j|_{X_{ij}} - b|_{X_{ij}}$, where X_{ij} denotes $X_i \times_X X_j$, and such that for three indices i, j, k we have

$$a_{ij}|_{X_{ijk}} + a_{jk}|_{X_{ijk}} = a_{ik}|_{X_{ijk}},$$

where $X_{ijk} := X_i \times_X X_j \times_X X_k$. The descent condition is then equivalent to the condition that there exist elements $\alpha_i \in A(X_i)$ such that for every i, j we have $a_{ij} = \alpha_i|_{X_{ij}} - \alpha_j|_{X_{ij}}$. For then the sections $b_i - f(\alpha_i) \in B(X_i)$ agree on the X_{ij} and therefore descend to a section $b \in B(X)$ which gives the effectivity of descent.

4.7. Exercises

Exercise 4.A. Let S be a scheme. If F is a sheaf with respect to the fppf topology on the category of S-schemes, then for any S-scheme T we get a functor

$$F_T : (\text{open subsets of } T)^{\text{op}} \to \text{Set}.$$

(a) Show that F_T is a sheaf on the small Zariski site of T.

(b) Show that for any morphism of S-scheme $f : T' \to T$, there is an induced morphism of sheaves

$$\theta_f : f^*F_T \to F_{T'}$$

on the small Zariski site of T', and that for a composition

$$T'' \xrightarrow{g} T' \xrightarrow{f} T$$

the diagram

$$\begin{tikzcd} (fg)^*F_T \arrow[r, "\theta_f"] \arrow[rr, bend left, "\theta_{fg}"] & g^*F_{T'} \arrow[r, "\theta_g"] & F_{T''} \end{tikzcd}$$

commutes.

(c) Let $\mathscr{C}$ denote the category of systems $\{\{F_T\}, \{\theta_f\}\}$ consisting of a small Zariski sheaf F_T on each S-scheme T, and for every S-morphisms $f : T' \to T$ a morphism $\theta_f : f^*F_T \to F_{T'}$, which are compatible with composition as in (b). Show that the the constructions in (a) and (b) define a fully faithful functor

$$(\text{Sch}/S)_{\text{fppf}} \to \mathscr{C}.$$

(d) Show that the essential image of the functor in $\mathscr{C}$ is the subcategory of systems $\{\{F_T\}, \{\theta_f\}\}$ satisfying the condition that for every fppf covering $\{f_i : T_i \to T\}$ in (Sch/S) the sequence

$$F_T(T) \xrightarrow{\prod \theta_{f_i}} \prod_i F_{T_i}(T_i) \underset{\theta_{\text{pr}_2}}{\overset{\theta_{\text{pr}_1}}{\rightrightarrows}} \prod_{i,j} F_{T_i \times_T T_j}(T_i \times_T T_j)$$

is exact.

Exercise 4.B. Prove Lemma 4.2.7.

Exercise 4.C. (a) Let

$$p : \mathscr{V}ec \to (\text{schemes})$$

be the fibered category whose objects are pairs (S, E), where S is a scheme and E is a locally free sheaf of finite rank on S, and where a morphism $(S', E') \to (S, E)$ is a pair (g, g^b), where $g : S' \to S$ is a morphism of schemes and $g^b : g^*E \to E'$ is an isomorphism of vector bundles on S'. The functor p sends a pair (S, E) to S. Show that an fppf covering is an effective descent morphism for $\mathscr{V}ec$.

(b) Let

$$p : \mathscr{F} \to (\text{schemes})$$

be the fibered category whose objects are quadruples (S, E, F, g), where S is a scheme, E and F are quasi-coherent sheaves on S, and $g : E \to F$ is a morphism of quasi-coherent sheaves on S. A morphism

$$(S', E', F', g') \to (S, E, F, g)$$

is defined to be a triple (h, s^b, t^b), where $h : S' \to S$ is a morphism of schemes, and $s^b : h^*E \to E'$ and $t^b : h^*F \to F'$ are isomorphisms of quasi-coherent sheaves such that the square

$$\begin{tikzcd} h^*E \arrow[r, "s^b"] \arrow[d, "h^*g"] & E' \arrow[d, "g'"] \\ h^*F \arrow[r, "t^b"] & F' \end{tikzcd}$$

commutes. Show that an fppf covering is an effective descent morphism for $\mathscr{F}$.

Exercise 4.D. Following up on the example from the introduction, let $\mathscr{E}_t$ (resp. E_0) be the elliptic curve over $\mathbb{C}(t)$ defined by the equation

$$Y^2Z = X^3 - tZ^3 \quad (\text{resp. } Y^2Z = X^3 - Z^3).$$

Let $\mathbb{C}(t) \hookrightarrow \Omega$ be an algebraic closure and fix a sixth root $t^{1/6} \in \Omega$ of t in Ω. Let I denote the set of isomorphisms of elliptic curves over Ω

$$\alpha : \mathscr{E}_{t,\Omega} \to E_{0,\Omega}.$$

(a) Show that the set I has a simply transitive action of the automorphism group of E_0, which is isomorphic to $\mathbb{Z}/(6)$.

(b) Define an action of the Galois group G of Ω over $\mathbb{C}(t)$ on I as follows. An element $g \in G$ acts on both $\mathscr{E}_{t,\Omega}$ and $E_{0,\Omega}$ through its natural action on Ω, and the action of g on I is defined by sending an isomorphism α to the composition

$$\mathscr{E}_{t,\Omega} \xrightarrow{g^{-1}} \mathscr{E}_{t,\Omega} \xrightarrow{\alpha} E_{0,\Omega} \xrightarrow{g} E_{0,\Omega}.$$

Show that this action commutes with the action of $\mathbb{Z}/(6)$ in (a) and that it can be described explicitly as follows. Let $\alpha_0 \in I$ be the isomorphism given by

$$(x, y) \mapsto (t^{-1/3}x, t^{-1/2}y).$$

Let $\chi : G \to \mu_6$ be the character on G given by sending $g \in G$ to the 6-th root of unity ζ characterized by $g * t^{1/6} = \zeta t^{1/6}$. Then $g \in G$ acts on I by sending α_0 to the isomorphism given by

$$(x, y) \mapsto (\chi(g)^{-2}t^{-1/3}x, \chi(g)^{-3}t^{-1/2}y).$$

(c) Show that this G-action on I has no fixed point thereby proving that $\mathscr{E}_{t,\mathbb{C}(t)}$ and E_0 are not isomorphic.

Exercise 4.E. Let $p : \mathscr{M}_{1,1} \to$ (schemes) be the fibered category with objects triples $(S, f : E \to S, e : S \to E)$ as follows:

(i) S is a scheme;

(ii) $f : E \to S$ is a proper flat morphism all of whose geometric fibers are smooth genus 1 curves.

(iii) $e : S \to E$ is a section of f.

We refer to such a triple as an *elliptic curve over S*. A morphism

$$(S', f' : E' \to S', e') \to (S, f : E \to S, e)$$

is a cartesian diagram

$$\begin{tikzcd} E' \arrow[r, "\tilde{g}"] \arrow[d, "f'"] & E \arrow[d, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

such that $e \circ g = \tilde{g} \circ e'$. The functor p sends a triple $(S, f : E \to S, e)$ to S. Show that any fppf covering is an effective descent morphism for $\mathscr{M}_{1,1}$.

Exercise 4.F. Fill in the proof of Proposition 4.6.2.

Exercise 4.G. Let C be a site and let $\mathscr{O}$ be a sheaf of commutative rings on C. Let $\mathscr{O}^*$ denote the sheaf which to any object U of C associates the invertible elements in the ring $\mathscr{O}(U)$. Just as in the case of a scheme with the Zariski topology, define an *invertible sheaf* on C to be a sheaf of $\mathscr{O}$-modules L such that for any object $U \in C$ there exists a covering $\{U_i \to U\}$ such that the restriction of L to the localized site C/U_i is isomorphic to the restriction of $\mathscr{O}$ (viewed as an $\mathscr{O}$-module).

(a) Let L be an invertible sheaf on C. Define

$$P_L : C^{\mathrm{op}} \to \mathrm{Set}$$

by sending $U \in C$ to the set of isomorphisms of $\mathscr{O}|_{C/U}$-modules $\mathscr{O}|_{C/U} \to L|_{C/U}$. Show that there is a natural action of $\mathscr{O}^*$ on P_L making P_L a torsor under $\mathscr{O}^*$.

(b) Show that the construction in (a) defines an equivalence of categories

$$(\text{invertible } \mathscr{O}\text{-modules on } C) \to (\mathscr{O}^*\text{-torsors on } C),$$

where on the left we consider only isomorphisms.

Exercise 4.H. Let S be a scheme, and let $\boldsymbol{\mu}$ be a sheaf of groups on the big étale site of S. Let $B\boldsymbol{\mu}$ be the fibered category over the category of S-schemes whose objects are pairs (T, P), where T is an S-scheme and P is a $\boldsymbol{\mu}|_{\mathrm{Sch}/T}$-torsor over the big étale site of T. A morphism $(T', P') \to (T, P)$ in $B\boldsymbol{\mu}$ is defined to be a pair (f, f^b), where $f : T' \to T$ is an S-morphism and $f^b : P' \to P|_{\mathrm{Sch}/T'}$ is an isomorphism of $\boldsymbol{\mu}|_{\mathrm{Sch}/T'}$-torsors.

(a) Show that $B\boldsymbol{\mu}$ is a stack with respect to the étale topology on Sch/S.

(b) If $\boldsymbol{\mu}$ is the sheaf associated to a constant group G, show that $B\boldsymbol{\mu}$ is equivalent to $[S/G]$, where G acts trivially on S.

Exercise 4.I. Let C be a site and let $f : F \to F'$ be a morphism of stacks over C. Show that f is an isomorphism if and only if the following two conditions hold:

(i) For every $X \in C$ the functor $F(X) \to F'(X)$ is fully faithful.

(ii) For every $X \in C$ and $x \in F'(X)$ there exists a covering $\{X_i \to X\}$ such that the restriction $x_i \in F'(X_i)$ of x is in the essential image of $F(X_i)$.

CHAPTER 5

Algebraic spaces

In this chapter we introduce algebraic spaces and establish some of their basic properties. By definition an algebraic space over a scheme S is a sheaf on the big étale site of the category of S-schemes, which satisfies certain properties similar to the properties needed to define the notion of scheme from the notion of affine scheme in 1.4.13.

One way to think about algebraic spaces is using étale equivalence relations. If X is a scheme over a base S, an equivalence relation on X is a monomorphism $R \hookrightarrow X \times_S X$ such that for every S-scheme T the corresponding subset of T-valued points $R(T) \subset X(T) \times X(T)$ is an equivalence relation. An étale equivalence relation is an equivalence relation R for which the two projections $R \to X$ are étale. For such an equivalence relation one can form the sheaf X/R on the big étale site by sheafifying the presheaf sending T to the quotient of $X(T)$ by the equivalence relation $R(T)$. The sheaf X/R is an algebraic space, and every algebraic space can be described in this way. Of course, an algebraic space can be written in many different ways as the quotient of a scheme by an étale equivalence relation, just as a scheme can be described in many different ways as a gluing of affine schemes.

While enjoying some advantages especially in the context of examples, this definition using equivalence relations is often difficult to work with in practice and it is useful to have a more global definition of algebraic space. The key observation is that if F is a sheaf of the form X/R for some étale equivalence relation R on a scheme X, then the diagram of sheaves

$$\begin{tikzcd} R \arrow[r] \arrow[d] & X \times X \arrow[d] \\ F \arrow[r, "\Delta_F"] & F \times F \end{tikzcd}$$

is cartesian, where the map $X \to F$ is the projection and Δ_F is the diagonal morphism. This implies (with some work) that for any scheme Y and morphism $Y \to F \times F$ (where we abusively also write Y for the corresponding étale sheaf) the fiber product $Y \times_{F \times F, \Delta_F} F$ is a scheme. This condition, combined with the existence of a covering by a scheme, leads to a presentation-independent definition of algebraic space (see 5.1.10). The connection with étale equivalence relations is discussed in section 5.2.

In addition to developing some of the basic foundational material on algebraic spaces in this chapter, we also discuss a number of examples.

We have chosen to develop the theory working over a fixed base scheme S. This is for aesthetic reasons and the reader can take $S = \mathrm{Spec}(\mathbb{Z})$ throughout. A general formalism (see Remark 5.1.16) gives an equivalence of categories between algebraic spaces over S in the sense of Definition 5.1.10 and the category of pairs

(X, f), where X is an algebraic space over $\mathbb{Z}$ and $f : X \to S$ is a morphism of algebraic spaces. The original sources for the material presented in this chapter are Knutson's book on algebraic spaces [**45**] and Artin's articles [**5, 8**]

5.1. Properties of sheaves and definition of algebraic space

In order to speak about various properties of algebraic spaces and morphisms between them, we first discuss some formal language which enables us to speak about many geometric properties of schemes and morphisms between them in terms of the corresponding sheaves on the big Zariski site.

DEFINITION 5.1.1 ([**45**, Chapter I, 1.5]). Let C be a site, and assume that representable presheaves are sheaves. A class of objects $S \subseteq C$ is *stable* if for every covering $\{U_i \to U\}$, the object U is in S if and only if $U_i \in S$ for each i. We call a property P of objects of C *stable* if the class of objects satisfying P is stable.

EXAMPLE 5.1.2. Let C be the category of schemes with the Zariski topology. Then the following properties are stable: locally noetherian, reduced, normal, regular.

If S is a scheme and we take C to be the category of S-schemes with the Zariski topology then the following properties are stable: locally of finite presentation over S, locally of finite type over S, being a local complete intersection over S.

DEFINITION 5.1.3 ([**45**, Chapter I, 1.6, 1.7, and 1.13]). Let C be a site.

(i) A *closed subcategory* of C is a subcategory $D \subseteq C$ such that the following hold:

(1) D contains all isomorphisms, and
(2) for all cartesian diagrams in C,

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r] & Y \end{tikzcd}$$

for which the morphism f is in D, we have $f' \in D$.

(ii) A closed subcategory $D \subseteq C$ is *stable* if for all $f : X \to Y$ in C and every covering $\{Y_i \to Y\}$, the morphism f is in D if and only if all the maps $f_i : X \times_Y Y_i \to Y_i$ are in D.

(iii) A stable closed subcategory $D \subseteq C$ is *local on domain* if for all $f : X \to Y$ in C and all coverings $\{X_i \xrightarrow{x_i} X\}$, the morphism f is in D if and only if the compositions $f \circ x_i$ are in D.

(iv) If P is a property of morphisms in C satisfied by isomorphisms and closed under composition, let D_P be the subcategory of C with the same objects as the objects in C, but whose morphisms are morphisms satisfying P. We say that P is *stable* (resp. *local on domain)* if the subcategory $D_P \subseteq C$ is stable (resp. local on domain).

PROPOSITION 5.1.4. *Let S be a scheme, and let C be the category of S-schemes* Sch$/S$ *with the étale topology.*

(i) The following properties of morphisms in C are stable: proper, separated, surjective, quasi-compact.

(ii) The following properties of morphisms in C are stable and local on domain: locally of finite type, locally of finite presentation, flat, étale, universally open, locally quasi-finite, smooth.

PROOF. This is left as exercise 5.A. □

DEFINITION 5.1.5. Let S be a scheme and let $f : F \to G$ be a morphism of sheaves on Sch/S with the étale topology.

(i) f is *representable by schemes* if for every S-scheme T and morphism $T \to G$ the fiber product $F \times_G T$ is a scheme.

(ii) Let P be a stable property of morphisms of schemes. If f is representable by schemes, we say that f *has property* P if for every S-scheme T the morphism of schemes $\mathrm{pr}_2 : F \times_G T \to T$ has property P.

REMARK 5.1.6. In the above definition (and for most of what follows), we somewhat abusively write T both for the scheme and the corresponding sheaf h_T on Sch/S. Similarly, we say that '$F \times_G T$ is a scheme' when properly we should say that '$F \times_G T$ is representable'. For further discussion of these abuses see section 1.4.

EXAMPLE 5.1.7. If F and G are representable sheaves, then any morphism $f : F \to G$ is representable by schemes. Indeed, if $F = h_X$ and $G = h_Y$ so f is induced by a morphism of schemes $X \to Y$, then for any S-schemes T and morphism $g : T \to G$ corresponding to a morphism of schemes $T \to Y$, we have $F \times_G T \simeq h_{T \times_Y X}$.

LEMMA 5.1.8. *Let S be a scheme, and let $f : X \to Y$ be a morphism of S-schemes. If P is a stable property of morphisms of schemes, then f has P if and only if the induced morphism of sheaves $h_f : h_X \to h_Y$ has P in the sense of* 5.1.5 *(ii).*

PROOF. Indeed, as remarked above, if T is an S-scheme, and $g : T \to Y$ is a morphism of schemes with corresponding morphism of functors $h_g : h_T \to h_Y$, then we have

$$h_T \times_{h_Y} h_X \simeq h_{X \times_Y T},$$

and this isomorphism is compatible with the projections to T. It follows that h_f has property P in the sense of 5.1.5 (ii) if and only if for every morphism of schemes $g : T \to Y$ the projection

$$\mathrm{pr}_2 : X \times_Y T \to T$$

has property P. By the definition of a stable property of morphisms, this is equivalent to f having property P. □

LEMMA 5.1.9. *Let S be a scheme and let F be a sheaf on* Sch/S *with the étale topology. Suppose that the diagonal morphism $\Delta : F \to F \times F$ is representable by schemes. Then if T is a scheme any morphism $f : T \to F$ is representable by schemes.*

PROOF. Indeed, if T and T' are schemes, the fiber product of the diagram

$$\begin{tikzcd} & T \arrow[d, "f"] \\ T' \arrow[r, "g"] & F \end{tikzcd}$$

is isomorphic to the fiber product of the diagram

$$\begin{tikzcd} & T \times_S T' \arrow[d, "f\times g"] \\ F \arrow[r, "\Delta"] & F \times F, \end{tikzcd}$$

and since Δ is representable by schemes this fiber product is a scheme. □

DEFINITION 5.1.10. Let S be a scheme. An *algebraic space* over S is a functor $X : (\mathrm{Sch}/S)^{op} \to \mathrm{Set}$ such that the following hold:

(i) X is a sheaf with respect to the big étale topology.
(ii) $\Delta : X \to X \times_S X$ is representable by schemes.
(iii) There exists an S-scheme $U \to S$ and a surjective étale morphism $U \to X$.

Morphisms of algebraic spaces over S are morphisms of functors.

REMARK 5.1.11. Note that condition (iii) makes sense in light of (ii) and 5.1.9.

EXAMPLE 5.1.12. Schemes over S are algebraic spaces over S.

REMARK 5.1.13. In [**45**], the additional condition that Δ is quasi-compact is included in the definition of an algebraic space. This is because in this case the diagonal is quasi-finite and separated, whence quasi-affine. This makes various descent arguments easier. Here we use the more general definition of algebraic space, as can be found in [**65**].

5.1.14. Let $g : S' \to S$ be a morphism of schemes and let AS/S (resp. AS/S') denote the category of algebraic spaces over S (resp. S'). Let $\mathscr{C}$ denote the category whose objects are pairs (X, f), where X is an algebraic space over S and $f : X \to S'$ is a morphism of algebraic spaces over S. A morphism $(X', f') \to (X, f)$ in $\mathscr{C}$ is a commutative diagram of AS/S:

$$\begin{tikzcd} X' \arrow[rr] \arrow[dr, "f'"'] & & X \arrow[dl, "f"] \\ & S'. & \end{tikzcd}$$

For an algebraic space Y over S' let Y_S denote the functor on $(\mathrm{Sch}/S)^{\mathrm{op}}$ sending an S-scheme T to the set of pairs (ϵ, y), where $\epsilon : T \to S'$ is an S-morphism and $y \in Y(\epsilon : T \to S')$ is an element. There is a natural morphism of functors $f_Y : Y_S \to S'$ sending (ϵ, y) to ϵ.

PROPOSITION 5.1.15. *(i) For an algebraic space Y/S' the functor Y_S is an algebraic space over S.*

(ii) The induced functor

$$\mathrm{AS}/S' \to \mathscr{C}, \quad Y \mapsto (Y_S, f_Y)$$

is an equivalence of categories.

PROOF. To see that Y_S is an étale sheaf, let $\{T_i \to T\}_{i\in I}$ be an étale covering of an S-scheme T, and let $\{(\epsilon_i, y_i)\} \in \prod_i Y_S(T_i)$ be an element in the equalizer of

$$\prod_i Y_S(T_i) \rightrightarrows \prod_{i,j} Y_S(T_i \times_T T_j).$$

We must show that this collection is induced by a unique element $(\epsilon, y) \in Y_S(T)$. For this, note first that the ϵ_i define an element of the equalizer of the two maps

$$\prod_i S'(T_i) \rightrightarrows \prod_{i,j} S'(T_i \times_S T_j)$$

and therefore are induced by a unique morphism $\epsilon : T \to S'$. Viewing T and the T_i as S'-schemes via this map the elements y_i are then given by an element of the equalizer of the two maps

$$\prod_i Y(T_i) \rightrightarrows \prod_{i,j} Y(T_i \times_T T_j),$$

and therefore are induced by a unique element $y \in Y(T)$. The resulting element $(\epsilon, y) \in Y_S(T)$ is then the unique element inducing the collection $\{(\epsilon_i, y_i)\}$.

To see that the diagonal of Y_S is representable, note that the fiber product P of the diagram

$$\begin{tikzcd} & Y_S \times_S Y_S \arrow[d, "f_Y \times f_Y"] \\ S' \arrow[r, "\Delta_{S'/S}"] & S' \times_S S' \end{tikzcd}$$

is isomorphic as a functor on the category (Sch/S') to $Y \times_{S'} Y$. Therefore for an S-scheme T and map $g : T \to Y_S \times_S Y_S$, the fiber product

$$T \times_{Y_S \times_S Y_S, \Delta_{Y_S}} Y_S$$

is given by first taking the fiber product $T' := T \times_{S' \times_S S', \Delta_{S'/S}} S'$, which is an S'-scheme, and then forming the fiber product $T' \times_{Y \times_{S'} Y, \Delta_Y} Y$. In particular, the diagonal of Y_S is representable.

Finally, to get an étale cover of Y_S let $U \to Y$ be an étale surjection over S' corresponding to an element $u \in Y(U)$. Let $g : U \to S'$ be the structure morphism. Then the pair (g, u) defines an element of $Y_S(U)$ or equivalently a morphism $u_S : U \to Y_S$ over S. By the preceding argument for the diagonal, for any S-scheme T and object $(\epsilon, y) \in Y_S(T)$ the fiber product $U \times_{Y_S} T$ is given by $U \times_{Y,y} T$ which implies that u_S is an étale surjection. This completes the proof of (i).

For (ii) note first that we recover an algebraic space Y/S' from (Y_S, f_Y) as the functor associating to a S'-scheme T the set of morphisms of functors $T \to Y_S$ over S'. Conversely, given an object $(X, f) \in \mathscr{C}$, let Y be the functor on (Sch/S') whose value on $T \to S'$ is the set of morphisms $X \to Y$ over S'. We claim that this is an algebraic space. That Y is a sheaf for the étale topology follows from the fact that we have descent for étale sheaves 4.2.12. For an S'-scheme T and morphism $T \to Y \times_{S'} Y$ the fiber product $T \times_{Y \times_{S'} Y, \Delta} Y$ is given by

$$(T \times_{X \times_S X, \Delta_X} X) \times_{S' \times_S S', \Delta_{S'}} S',$$

and in particular is a scheme. Finally, if $U \to X$ is an étale covering with U a scheme, then the composite map $U \to X \to S'$ gives U the structure of an S'-scheme and the induced map $U \to Y$ is also an étale surjection. In this way we obtain a functor $\mathscr{C} \to \mathrm{AS}/S'$ which is inverse to the functor in (ii). □

REMARK 5.1.16. In particular one can develop the theory of algebraic spaces as an absolute theory taking $S = \mathrm{Spec}(\mathbb{Z})$ and then obtain the notion of algebraic space over a scheme S as a pair (X, f), where X is an algebraic space (over $\mathbb{Z}$) and $f : X \to S$ is a morphism of algebraic spaces.

5.2. Algebraic spaces as sheaf quotients

DEFINITION 5.2.1. Fix a base scheme S. An *étale equivalence relation on a S-scheme X* is a monomorphism of schemes

$$R \hookrightarrow X \times_S X$$

such that the following hold:

(i) For every S-scheme T the T-points

$$R(T) \subset X(T) \times X(T)$$

is an equivalence relation on $X(T)$.

(ii) The two maps

$$s, t : R \to X$$

induced by the two projections from $X \times_S X$ are étale.

REMARK 5.2.2. In the above we can take $S = \mathrm{Spec}(\mathbb{Z})$, in which case we get an absolute notion of an étale equivalence relation on a scheme.

REMARK 5.2.3. Note that since R is an equivalence relation we have an inclusion $X \hookrightarrow R$ induced by the diagonal $\Delta_X : X \to X \times_S X$.

5.2.4. Taking the quotient by R we get a presheaf

$$(\mathrm{Sch}/S)^{\mathrm{op}} \to \mathrm{Set}, \quad T \mapsto X(T)/R(T).$$

We write X/R for the associated sheaf with respect to the étale topology on $(\mathrm{Sch}/S)^{\mathrm{op}}$.

PROPOSITION 5.2.5. *(i) X/R is an algebraic space.*

(ii) If Y is an algebraic space over S, and $X \to Y$ is an étale surjective morphism with X a scheme, then

$$R := X \times_Y X$$

is a scheme and the inclusion

$$R \hookrightarrow X \times_S X$$

is an étale equivalence relation. Moreover, the natural map

$$X/R \to Y$$

is an isomorphism.

The proof will be in several steps (see also [**45**, II, 1.3], [**65**, §5.7]).

5.2.6. To prove (i), let Y denote the sheaf X/R. To show that Y is an algebraic space, the key point is to show that the diagonal

$$\Delta_Y : Y \to Y \times_S Y$$

is representable. Indeed, once this is shown, then by 5.1.9 we find that for any diagram

$$\begin{tikzcd} & V \arrow[d] \\ U \arrow[r] & Y \end{tikzcd}$$

the fiber product $U \times_Y V$ is a scheme. In particular, the projection $X \to Y$ is representable by schemes. We claim that $X \to Y$ is the étale surjection required in 5.1.10 (iii). Indeed, if $T \to Y$ is a morphism with T a scheme, then to verify that $X \times_Y T \to T$ is étale surjective we can replace T by an étale cover. Since $X \to Y$ is in any case an epimorphism of étale sheaves, to verify that $X \to Y$ is étale surjective it suffices to consider morphisms $T \to Y$ which factor through X. Now if we have a factorization $T \to X \to Y$ then we get a commutative diagram with cartesian squares (using exercise 5.D)

$$\begin{tikzcd} T \times_Y X \arrow[r] \arrow[d] & R \arrow[r, "t"] \arrow[d, "s"] & X \arrow[d] \\ T \arrow[r] & X \arrow[r] & Y, \end{tikzcd}$$

and since s is étale surjective this implies that $T \times_Y X \to T$ is étale surjective [**26**, IV, 17.3.3 (iii)].

5.2.7. Let $j : U \hookrightarrow X$ be an open subscheme, and let R_U denote the fiber product of the diagram

$$\begin{tikzcd} & U \times_S U \arrow[d] \\ R \arrow[r, hook] & X \times_S X. \end{tikzcd}$$

Observe that R_U is an étale equivalence relation on U, and there is an induced map

$$\bar{j} : U/R_U \to Y.$$

Lemma 5.2.8. *The morphism $\bar{j}$ is representable by open imbeddings.*

Proof. It is clear from the definitions that $\bar{j}$ is a monomorphism. For an S-scheme T, a morphism

$$f : T \to Y$$

is contained in the T-points of U/R_U if and only if étale locally on T the morphism f factors through the open subset $U \subset X$. This condition on f is representable by an open subset of T. Indeed, this can be verified étale locally on T, so we may assume that there exists a morphism

$$\tilde{f} : T \to X$$

lifting f. In this case the required open subset is

$$\tilde{f}^{-1}(s(t^{-1}(U))),$$

where as before

$$s, t : R \to X$$

are the two projections. □

5.2.9. To see that Δ_Y is representable, let

$$f : W \to Y \times_S Y$$

be a morphism, and let

$$F := Y \times_{\Delta_Y, Y \times_S Y} W$$

be the fiber product. To prove that F is a scheme, we may work Zariski locally on S, so we may assume that S is an affine scheme.

We may also work Zariski locally on W, so we may also assume that W is an affine scheme.

5.2.10. In this case we can prove that F is a scheme as follows.

Since $X \to Y$ is a surjective morphism of sheaves, there exists an étale covering $W' \to W$ such that the composite map

$$W' \to W \to Y \times_S Y$$

factors through $X \times_S X$, and we may even assume that W' is an affine scheme since W is quasi-compact. Now observe that since the square

$$\begin{tikzcd} R \arrow[r] \arrow[d] & X \times_S X \arrow[d] \\ Y \arrow[r, "\Delta"] & Y \times_S Y \end{tikzcd}$$

is cartesian, if F' denotes $F \times_W W'$, then we have

$$F' \simeq Y \times_{\Delta_Y, Y \times_S Y} W' \simeq R \times_{X \times_S X} W'.$$

In particular, F' is a scheme and $F' \to W'$ is a monomorphism, which implies that F' is a separated S-scheme (since W' is a separated S-scheme).

Since the square

$$\begin{tikzcd} F' \arrow[r] \arrow[d] & W' \arrow[d] \\ F \arrow[r] & W \end{tikzcd}$$

is cartesian, the morphism $F' \to F$ is representable and étale surjective. Therefore F is the sheaf quotient of F' by the equivalence relation

$$R' := F' \times_F F' \subset F' \times_S F'.$$

Since $F \to W$ is a monomorphism, for either of the two projections

$$p_i : W' \times_W W' \to W'$$

the square

$$\begin{tikzcd} F' \times_F F' \arrow[r] \arrow[d, "p_i"] & W' \times_W W' \arrow[d, "p_i"] \\ F' \arrow[r] & W' \end{tikzcd}$$

is cartesian. This implies that R' is a scheme and that the two projections

$$p_i : F' \times_F F' \to F'$$

are quasi-compact and étale.

Lemma 5.2.11. *Let $U' \subset F'$ be a quasi-compact open subset, and let $F_{U'}$ denote the quotient $U'/R'_{U'}$. Let $\bar{j} : F_{U'} \to F$ be the natural representable open imbedding, and let $F'_{U'} \subset F'$ denote the fiber product of the diagram*

$$\begin{tikzcd} & F' \arrow[d] \\ F_{U'} \arrow[r, "\bar{j}"] & F. \end{tikzcd}$$

Then $F'_{U'}$ is a quasi-compact open subset of F' containing U'.

PROOF. Indeed, we have

$$F'_{U'} = p_1(p_2^{-1}(U')),$$

where as above we write $p_i : F' \times_F F' \to F'$ $(i = 1, 2)$ for two projections. Since the maps p_i are quasi-compact and U' was assumed quasi-compact we obtain the result. $\square$

5.2.12. Since we can also write

$$F'_{U'} = F_{U'} \times_W W',$$

we are then reduced to proving that $F_{U'}$ is a scheme, for every quasi-compact open subset $U' \subset F'$.

For this notice that since $F'_{U'}$ is quasi-compact, the morphism $F'_{U'} \to W'$ is quasi-affine by Zariski's main theorem [**26**, IV, 18.12.13]. That $F_{U'}$ is a scheme then follows from 4.4.17 which gives that the descent data on $F'_{U'}$ is effective. This completes the proof of statement (i) in 5.2.5.

For (ii) note that since the square

$$\begin{tikzcd} R \arrow[r] \arrow[d] & X \times_S X \arrow[d] \\ Y \arrow[r, "\Delta_Y"] & Y \times_S Y \end{tikzcd}$$

is cartesian, R is a scheme. The remaining statements in (ii) are immediate. This completes the proof of 5.2.5. $\square$

5.3. Examples of algebraic spaces

EXAMPLE 5.3.1. Let X be a scheme, and let G be a discrete group acting on X. Write

$$\rho : G \times X \to X$$

for the map giving the action. The action of G on X is called *free* if the map

$$j : G \times X \to X \times X, \quad (g, x) \mapsto (x, \rho(g, x)) \tag{5.3.1.1}$$

is a monomorphism. In this case, let X/G denote the sheaf (with respect to the étale topology) associated to the presheaf which to any scheme T associates the quotient

$$X(T)/G$$

of $X(T)$ by the G-action. Then X/G is the quotient of X by the étale equivalence relation (5.3.1.1), and therefore X/G is an algebraic space.

Note that in this example we do not require the group G to be finite.

EXAMPLE 5.3.2. Continuing with the preceding example, one might ask if the quotient X/G is in fact always a scheme. The following example of Hironaka shows that this is not the case (this is an extension of the example in [**41**, Appendix B, Example 3.4.1] of a scheme that is not quasi-projective and is also considered in [**45**, Chapter 0, Example 3]).

Let $\mathbb{P}$ denote $\mathbb{P}^3_{\mathbb{C}}$ with coordinates Y_0, Y_1, Y_2, Y_3, and define curves C_1 and C_2 by the equations

$$C_1 : Y_0Y_1 + Y_1Y_2 + Y_2Y_0 = Y_3 = 0,$$

$$C_2 : Y_0Y_1 + Y_1Y_3 + Y_3Y_0 = Y_2 = 0.$$

The intersection $C_1 \cap C_2$ consists of two points

$$P_1 = [1 : 0 : 0 : 0], \quad P_2 = [0 : 1 : 0 : 0].$$

For $i = 1, 2$, let $\mathbb{P}_i$ be the scheme obtained by first blowing up C_i and then blowing up the strict transform of C_{3-i}, and let $U_i \subset \mathbb{P}_i$ be the preimage of $\mathbb{P} - P_{3-i}$. Let Z be the scheme obtained by gluing U_1 and U_2 along their common open subset (the preimage of $\mathbb{P} - \{P_1, P_2\}$).

There is an involution

$$\sigma : Z \to Z$$

induced by the automorphism of $\mathbb{P}$ given by

$$Y_0 \mapsto Y_1, \quad Y_1 \mapsto Y_0, \quad Y_2 \mapsto Y_3, \quad Y_3 \mapsto Y_2.$$

The automorphism σ of Z has fixed points. Let $Z' \subset Z$ denote the open subset where σ acts freely. By definition, $Z - Z'$ is the inverse image of the diagonal under the graph morphism

$$Z \to Z \times Z, \quad x \mapsto (x, \sigma(x)).$$

Then σ acts freely on Z'. We claim that the quotient

$$Q := Z'/\sigma$$

is not a scheme.

To see this suppose to the contrary that Q is a scheme. Since Z' is smooth and the action of σ is free, the scheme Q is also smooth. As in [**41**, Appendix B, Example 3.4.1], the preimage of P_i in Z consists of two lines $l_i + m_i$ such that

$$l_1 + m_2$$

is algebraically equivalent to zero. On the other hand, since σ interchanges l_1 and m_2 their image in Q is an irreducible curve $t \subset Q$. Choose a general surface $W \subset U$ which meets but does not contain t, and let $\overline{W} \subset Z$ be the closure in Z of the preimage of W in Z'. Then $\overline{W}$ meets l_1 and m_2 properly so the intersection number

$$\overline{W}.(l_1 + m_2)$$

is strictly positive. This is impossible as $l_1 + m_2$ is algebraically equivalent to 0.

Example 5.3.3 ([**45**, Chapter 0, Example 0]). Let k be a field and consider the scheme

$$U := \mathrm{Spec}(k[s, t]/(st))$$

obtained by gluing two copies of the affine line along the origin. Let $U' \subset U$ be the open subset obtained by deleting the origin. Set

$$R := U \coprod U'.$$

Consider the two maps

$$\pi_1, \pi_2 : R \to U$$

defined as follows. The restriction of both π_1 and π_2 to U is the identity map. However, on U' we define π_1 to be the natural inclusion, but π_2 to be the map which switches the two components. Then the resulting map

$$\pi_1 \times \pi_2 : R \to U \times U$$

is an étale equivalence relation. Let $F = U/R$ be the resulting algebraic space.

We claim that F is not a scheme. Indeed, the map

$$s + t : U \to \mathbb{A}^1 \tag{5.3.3.1}$$

is universal in the category of ringed spaces for maps from U which factor through F (exercise 6.J). However, the induced map $F \to \mathbb{A}^1$ in the category of algebraic spaces is not an isomorphism. This can be seen for example by noting that the map (5.3.3.1) is not étale.

EXAMPLE 5.3.4 ([**67**, Tag 02Z0]). Let k be a field of characteristic 0 and let $\mathbb{Z}$ act on $\mathbb{A}^1_k$ by the usual translation action. This is a free action and the quotient $X := \mathbb{A}^1_k/\mathbb{Z}$ is an algebraic space which is not a scheme. To see this suppose to the contrary that X is a scheme. Then X has a surjective étale covering by $\mathbb{A}^1_k$ and therefore X is smooth and connected, and for any affine open $U \subset X$ the sections $\mathscr{O}_X(U)$ are included into the $\mathbb{Z}$-invariants of the rational function field $k(x)$, where $n \in \mathbb{Z}$ acts by $x \mapsto x + n$. The $\mathbb{Z}$-invariants in $k(x)$ are just the constant functions since a nonconstant $\mathbb{Z}$-invariant function $f \in k(x)$ would have infinitely many zeros and poles. It follows that the coordinate ring of any nonempty open subset of X is k implying that $X = \mathrm{Spec}(k)$. This contradicts the fact that the map $\mathbb{A}^1_k \to X$ is étale, and therefore X does not contain any nonempty open subspace which is a scheme.

REMARK 5.3.5. Note that $\mathbb{A}^1_k/\mathbb{Z}$ is naturally a group object in the category of algebraic spaces. It is known that quasi-separated group algebraic spaces over a field are necessarily schemes [**4**, 4.1].

EXAMPLE 5.3.6. Similarly, let $\mathbb{Z}^2$ act on $\mathbb{A}^1_{\mathbb{C}}$ by the imbedding $\mathbb{Z}^2 \hookrightarrow \mathbb{C}$ sending (a, b) to $a + ib$ and the standard translation action of $\mathbb{C}$ on $\mathbb{A}^1_{\mathbb{C}}$. Then the quotient $\mathbb{A}^1_{\mathbb{C}}/\mathbb{Z}^2$ is an algebraic space which is not a scheme by the same argument. On the other hand, the quotient of $\mathbb{C}$, with the analytic topology, by this action of $\mathbb{Z}^2$ is an elliptic curve.

5.4. Basic properties of algebraic spaces

Throughout this section we work over a base scheme S.

DEFINITION 5.4.1. Let P be a property of schemes which is stable in the étale topology (terminology as in 5.1.1). Then an algebraic space X *has property* P if there exists an étale surjection $U \to X$ where U is a scheme with property P.

EXAMPLE 5.4.2. For example, we can talk about algebraic spaces being locally noetherian, reduced, regular, purely n-dimensional, normal, etc.

5.4.3. If $f : X \to Y$ is a morphism of algebraic spaces which is representable by schemes and P is a property of morphisms of schemes which is stable in the étale topology then by 5.1.5 (ii) we can talk about f having property P. By definition the morphism f has property P if there is an étale cover $V \to Y$ such that $V \times_Y X \to V$ has property P.

EXAMPLE 5.4.4. For example P could be the property of being proper, dominant, quasi-compact, etc.

Other examples of such properties are the following:

DEFINITION 5.4.5. A morphism $f : X \to Y$ of algebraic spaces is an *imbedding* (resp. *open imbedding*, *closed imbedding*) if f is representable by schemes and is an imbedding (resp. open imbedding, closed imbedding) in the sense of 5.4.3.

PROPOSITION 5.4.6. *The full subcategory of* $(\mathrm{Sch}/S)_{\mathrm{ET}}$ *whose objects are algebraic spaces is closed under finite limits.*

PROOF. As in 2.2.14 (note that products in our case are fiber products over S), it suffices to show that the category of algebraic spaces is closed under fiber products. So consider the diagram

$$\begin{tikzcd} & X_2 \arrow[d] \\ X_1 \arrow[r] & X_3, \end{tikzcd}$$

and let F be the fiber product in the category of sheaves. We show that F is an algebraic space as follows.

Consider first the case when $X_3 = S$. To see that the diagonal of F is representable by schemes, let U be a scheme and let

$$U \to F \times_S F \tag{5.4.6.1}$$

be a morphism. Denote by P the fiber product of the diagram

$$\begin{tikzcd} & U \arrow[d] \\ F \arrow[r, "\Delta_F"] & F \times F. \end{tikzcd}$$

Let

$$\sigma : F \times_S F = (X_1 \times_S X_2) \times_S (X_1 \times_S X_2) \to (X_1 \times_S X_1) \times_S (X_2 \times_S X_2)$$

be the isomorphism switching the two middle factors, and let

$$F_1 : U \to X_1 \times_S X_1, \quad F_2 : U \to X_2 \times_S X_2$$

be the two morphisms corresponding via σ to (5.4.6.1). Then P is isomorphic to the fiber product of the diagram

$$\begin{tikzcd} & U \arrow[d, "(F_1 \times F_2) \circ \Delta_U"] \\ X_1 \times_S X_2 \arrow[r, "\Delta_{X_1} \times \Delta_{X_2}"] & (X_1 \times_S X_1) \times_S (X_2 \times_S X_2). \end{tikzcd}$$

We can expand this diagram to a bigger commutative diagram with cartesian squares

$$\begin{tikzcd} P \arrow[r] \arrow[d] & U \arrow[d, "\Delta_U"] \\ R \arrow[r] \arrow[d] & U \times_S U \arrow[d, "F_1 \times F_2"] \\ X_1 \times_S X_2 \arrow[r, "\Delta_{X_1} \times \Delta_{X_2}"] & (X_1 \times_S X_1) \times_S (X_2 \times_S X_2), \end{tikzcd}$$

where R is defined to be the fiber product of the bottom square. Since

$$R \simeq (X_1 \times_{\Delta_{X_1}, X_1 \times_S X_1} U) \times_S (X_2 \times_{\Delta_{X_2}, X_2 \times_S X_2} U)$$

and Δ_{X_1} and Δ_{X_2} are representable by schemes, R is a scheme. Therefore P is isomorphic to a fiber product of schemes (the top square), whence P is also a scheme.

To see that F admits an étale cover (still in the case $X_3 = S$), let

$$U_1 \to X_1, \quad U_2 \to X_2$$

be étale covers with U_1 and U_2 schemes. The the product

$$U_1 \times_S U_2 \to X_1 \times_S X_2 = F$$

is an étale cover of F by a scheme, so F is an algebraic space.

For the case of general X_3, consider again a morphism

$$U \to F \times_S F$$

from a scheme U, which defines by composition with the natural map

$$F \to X_1 \times_S X_2$$

a morphism

$$U \to (X_1 \times_S X_2) \times_S (X_1 \times_S X_2).$$

By the case when $X_3 = S$, the fiber product R of the diagram

$$\begin{tikzcd} & U \arrow[d] \\ (X_1 \times_S X_2) \arrow[r] & (X_1 \times_S X_2) \times_S (X_1 \times_S X_2) \end{tikzcd}$$

is a scheme. Now the fiber product P of the diagram

$$\begin{tikzcd} & U \arrow[d] \\ F \arrow[r, "\Delta_F"] & F \times_S F \end{tikzcd}$$

is isomorphic to the fiber product of the diagram

$$\begin{tikzcd} & R \arrow[d] \\ X_3 \arrow[r, "\Delta_{X_3}"] & X_3 \times_S X_3. \end{tikzcd}$$

Since Δ_{X_3} is representable by schemes this implies that P is a scheme.

To get an étale cover of $X_1 \times_{X_3} X_2$, note that the morphism

$$X_1 \times_{X_3} X_2 \to X_1 \times_S X_2$$

is representable by schemes, as the diagram

$$\begin{tikzcd} X_1 \times_{X_3} X_2 \arrow[r] \arrow[d] & X_1 \times_S X_2 \arrow[d] \\ X_3 \arrow[r, "\Delta_{X_3}"] & X_3 \times_S X_3 \end{tikzcd}$$

is cartesian and Δ_{X_3} is representable by schemes. In particular, if $V \to X_1 \times_S X_2$ is an étale cover by a scheme and we form the cartesian square

$$\begin{tikzcd} W \arrow[r] \arrow[d] & V \arrow[d] \\ X_1 \times_{X_3} X_2 \arrow[r] & X_1 \times_S X_2 \end{tikzcd}$$

then $W \to X_1 \times_{X_3} X_2$ is an étale cover by a scheme. □

In particular, we can use the diagonal morphism to define various separation properties of morphisms:

Definition 5.4.7. A morphism $f : X \to Y$ of algebraic spaces over S is *quasi-separated* (resp. *locally separated*, *separated*) if the diagonal morphism

$$\Delta_{X/Y} : X \to X \times_Y X$$

is quasi-compact (resp. an imbedding, a closed imbedding).

An algebraic space X over S is *quasi-separated* (resp. *locally separated*, *separated*) if the structure morphism $X \to S$ is quasi-separated (resp. locally separated, separated).

Remark 5.4.8. Note that the definition makes sense as the diagonal morphism $\Delta_{X/Y} : X \to X \times_Y X$ is representable by schemes. Indeed, if T is a scheme and $T \to X \times_Y X$ is a morphism, then since $X \times_Y X \to X \times_S X$ is a monomorphism we have an isomorphism

$$X \times_{\Delta_{X/Y}, X \times_Y X} T \simeq X \times_{\Delta_X, X \times_S X} T,$$

and $\Delta_X : X \to X \times_S X$ is representable by assumption.

Example 5.4.9 ([**67**, Tag 02Z7]). Let $S = \mathrm{Spec}(\mathbb{Q})$, and let X be the quotient of $\mathbb{A}^1_S$ by the action of $\mathbb{Z}$ given by $n * x := x + n$ (note that this is a free action). The equivalence relation on $\mathbb{A}^1_S$ corresponding to this action is the union over all $n \in \mathbb{Z}$ of the graphs

$$\mathbb{A}^1_S \hookrightarrow \mathbb{A}^1_S \times_S \mathbb{A}^1_S, \quad x \mapsto (x, x+n).$$

We therefore have a cartesian diagram

$$\begin{tikzcd} \coprod_{n \in \mathbb{Z}} \mathbb{A}^1_S \arrow[r] \arrow[d] & \mathbb{A}^1_S \times_S \mathbb{A}^1_S \arrow[d] \\ X \arrow[r, "\Delta_X"] & X \times_S X, \end{tikzcd}$$

which implies in particular that Δ_X is not quasi-compact, so X is not quasi-separated.

Example 5.4.10. The algebraic space F in 5.3.3 is locally separated but not separated.

Definition 5.4.11. Let P be a property of morphisms of schemes which is stable and local on domain in the étale topology, and let $f : X \to Y$ be a morphism of algebraic spaces. Then f *has* P if there exists étale covers $v : V \to Y$ and $u : U \to X$ such that the projection

$$U \times_Y V \to V$$

has property P (note that $U \times_Y V$ is a scheme by 5.1.9).

EXAMPLE 5.4.12. We can talk about a morphism of algebraic spaces being étale, flat, smooth, surjective, locally of finite type, etc.

REMARK 5.4.13. Let $f : X \to Y$ be a morphism of algebraic spaces representable by schemes, and let P be a property of morphisms of schemes which is local on target. If f has property P then for every étale covering $V \to Y$ the morphism

$$X \times_Y V \to V$$

has property P.

Similarly, if P is a property of morphisms of schemes which is stable and local on domain, and if $f : X \to Y$ is a morphism of algebraic spaces with property P, then for any commutative diagram

$$\begin{tikzcd} U \arrow[r, "u"] & X \times_Y V \arrow[r, "\pi"] \arrow[d] & V \arrow[d, "v"] \\ & X \arrow[r, "f"] & Y, \end{tikzcd}$$

where u and v are étale surjections from schemes, the composite morphism $\pi \circ u$ has property P.

This implies in particular that if $f : X \to Y$ is representable by schemes and P is stable and local on domain, then the definitions in 5.4.3 and 5.4.11 produce the same notion. The verifications of these assertions is left as exercise 5.F.

EXAMPLE 5.4.14. Let k be a field, and let $f : X \to \mathrm{Spec}(k)$ be an étale morphism from an algebraic space X. We claim that in this case X is a scheme isomorphic to a disjoint union of spectra of separable field extensions of k.

For this it suffices to show that if L/k is a finite Galois extension with group G, and if $s : \mathrm{Spec}(L) \to X$ is a morphism over k, then there exists a subgroup $H \subset G$ and a decomposition $X = \mathrm{Spec}(L^H) \coprod X'$ in the category of algebraic spaces, such that the map s is induced by the natural map $\mathrm{Spec}(L) \to \mathrm{Spec}(L^H)$. So fix such a morphism s, and consider the fiber product $X_L := X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(L)$. The morphism s induces a section $s_L : \mathrm{Spec}(L) \to X_L$ which is an open and closed imbedding since this can be verified after base change along an étale surjection $U \to X_L$ with U a scheme. Therefore $X_L = \mathrm{Spec}(L) \coprod Y$ for some algebraic space Y. In fact, if $H \subset G$ is the subgroup of elements for which the diagram

$$\begin{tikzcd} \mathrm{Spec}(L) \arrow[r, "h"] \arrow[d, "s_L"] & \mathrm{Spec}(L) \arrow[d, "s_L"] \\ X_L \arrow[r, "\mathrm{id} \times h"] & X_L \end{tikzcd}$$

commutes, then we get a G-equivariant decomposition

$$X_L = (\coprod_{G/H} \mathrm{Spec}(L)) \coprod X''$$

and hence by taking the quotient by G a decomposition $X = \mathrm{Spec}(L^H) \coprod X'$.

We will discuss global properties of algebraic spaces further after more theory is developed. Three useful notions that we can define already, however, are the following.

DEFINITION 5.4.15. An algebraic space X over a scheme S is called *quasi-compact* if there exists an étale covering $U \to X$ with U a quasi-compact scheme. The algebraic space X is called *noetherian* if X is locally noetherian and quasi-compact.

DEFINITION 5.4.16. Let $f : X \to Y$ be a morphism of algebraic spaces over a scheme S. We say that f *is quasi-compact* if for any quasi-compact scheme Y' and morphism $Y' \to Y$ the algebraic space $X' := X \times_Y Y'$ is quasi-compact.

DEFINITION 5.4.17. A morphism $f : X \to Y$ of algebraic spaces is *of finite type* if it is quasi-compact and locally of finite type in the sense of 5.4.11.

REMARK 5.4.18. If $f : X \to Y$ is a morphism of algebraic spaces representable by schemes, then the definition in 5.4.16 agrees with the definition in 5.4.3. The only nontrivial aspect of verifying this is to show that if f is quasi-compact in the sense of 5.4.3, then for any morphism $Y' \to Y$, with Y' a quasi-compact scheme, the scheme $X \times_Y Y'$ is quasi-compact. To verify this note that since Y' is quasi-compact we can replace Y' by a quasi-compact étale cover, so we may assume that $Y' \to Y$ factors through an étale surjection $U \to Y$ with U a scheme for which $X \times_Y U \to U$ is quasi-compact. Since the base change of a quasi-compact morphism is again quasi-compact this implies that $X \times_Y Y' \to Y'$ is a quasi-compact morphism, whence $X \times_Y Y'$ is quasi-compact.

5.5. Algebraic spaces are fppf sheaves

5.5.1. Though we primarily use the étale topology in this book, schemes are in fact sheaves for the fppf topology. By the following result the same is true for algebraic spaces. We learned this result from [**49**, A.4], where it is shown in fact that algebraic spaces are sheaves for the finer fpqc topology.

THEOREM 5.5.2. *Let S be a scheme, and let X/S be an algebraic space over S with quasi-compact diagonal. Then X is a sheaf with respect to the fppf-topology on* (Sch/S).

PROOF. Let $\overline{X}$ denote the sheaf with respect to the fppf topology associated to the presheaf X, and let $q : X \to \overline{X}$ be the natural map. We check that q is bijective.

If U is a scheme and $x_1, x_2 \in X(U)$ are two sections, then the condition that $x_1 = x_2$ is represented by the fiber product Z of the diagram

$$\begin{tikzcd} & U \arrow[d, "(x_1,x_2)"] \\ X \arrow[r, "\Delta"] & X \times_S X, \end{tikzcd}$$

which is a scheme, since X is an algebraic space. Since schemes define sheaves with respect to the fppf topology by 4.1.2 it follows that X is a separated presheaf, and therefore q is injective.

To see the surjectivity, let U be a scheme and let $s \in \overline{X}(U)$ be a section. By the injectivity of q already shown, to verify that s is in the image of q it suffices to consider the case when U is quasi-compact. Furthermore, if we write $X = \bigcup_i X_i$ as a union of open subspaces X_i, then

$$X(U) = \varinjlim X_i(U), \quad \overline{X}(U) = \varinjlim \overline{X}_i(U),$$

where $\overline{X}_i$ is the fppf-sheaf associated to X_i. We may therefore also assume that X is quasi-compact.

Let $X_0 \to X$ be an étale surjection with X_0 a quasi-compact scheme. To prove that s is étale locally in the image of q, it suffices to show that the fppf-sheaf

$$X_0 \times_{\overline{X},s} U$$

is representable by a scheme étale and quasi-compact over U. By fppf-descent theory for étale and quasi-compact morphisms (which are quasi-affine), we may replace U by an fppf cover and may therefore assume that s lifts to a section $\tilde{s} \in X(U)$. But by the injectivity of q we then have

$$X_0 \times_{\overline{X},s} U = X_0 \times_{X,\tilde{s}} U$$

which is quasi-compact over U since X_0 is quasi-compact, which implies that $X_0 \times_S U$ is quasi-compact over U, and $X_0 \times_{X,\tilde{s}} U$ is quasi-compact over $X_0 \times_S U$ since it is the fiber product of

$$\begin{tikzcd} & X_0 \times_S U \arrow[d] \\ X \arrow[r, "\Delta"] & X \times_S X, \end{tikzcd}$$

and the diagonal of X is quasi-compact by assumption. □

5.6. Exercises

Exercise 5.A. Prove proposition 5.1.4.

Exercise 5.B. Prove that the class of étale morphisms of schemes is the smallest class of morphisms in Sch which (i) includes all étale maps of affine schemes, and (ii) is stable and local on domain in the Zariski topology.

Exercise 5.C. Let S be a scheme and let X be a sheaf on (Sch/S) with the étale topology. Let U be a scheme and let $\pi : U \to X$ be a morphism representable by schemes.

(i) Show that if π is étale, then π is surjective if and only if π is an epimorphism in the category of sheaves on (Sch/S).

(ii) Give an example to show that if π is not étale, then π surjective does not necessarily imply that π is an epimorphism in the category of sheaves.

Exercise 5.D. Let C be a site, and let X and R be sheaves on C. Let $s \times t : R \hookrightarrow X \times X$ be an inclusion such that for every $U \in C$ the subset $R(U) \subset X(U) \times X(U)$ is an equivalence relation on $X(U)$. Let X/R be the sheaf on C associated to the presheaf $U \mapsto X(U)/R(U)$. Show that the diagram of sheaves

$$\begin{tikzcd} R \arrow[r, "t"] \arrow[d, "s"'] & X \arrow[d] \\ X \arrow[r] & X/R \end{tikzcd}$$

is cartesian.

EXERCISE 5.E. If P is a stable property of objects (resp. stable property of morphisms, stable local on domain property of morphisms) in the étale topology on the category of schemes, then P is a stable property of objects (resp. stable property of morphisms, stable local on domain property of morphisms) in the étale topology in the category of algebraic spaces.

EXERCISE 5.F. Verify the assertions in Remark 5.4.13.

EXERCISE 5.G. Fix a base scheme S, and let Y/S be an algebraic space. Let F be a sheaf on (Sch/S) with the étale topology equipped with a morphism of sheaves $g : F \to Y$. Show that if there exists an étale surjective morphism $U \to Y$, with U a scheme, such that the fiber product $F \times_Y U$ is an algebraic space, then F is an algebraic space.

EXERCISE 5.H. Let $f : S \to T$ be a morphism of schemes. Let $X \to S$ be an algebraic space over S. Show that by composing with the morphism f we can also view X as an algebraic space over T.

CHAPTER 6

Invariants and quotients

In this chapter we discuss some classical invariant theory (essentially the contents of [**25**, Exposé V]). The results of this section will be generalized later when we discuss the Keel-Mori theorem and coarse moduli spaces (Chapter 11), but in order to develop further the theory of algebraic spaces we need the results included in this chapter.

A useful example for the reader to keep in mind is that of a finite group G acting on an affine scheme $\mathrm{Spec}(A)$. In this case the coarse moduli space of the stack quotient $[\mathrm{Spec}(A)/G]$ (to be defined in Chapter 11) is the affine scheme $\mathrm{Spec}(A^G)$, where A^G denotes the subring of G-invariant elements in A. The basic issue is then to understand the relationship between $\mathrm{Spec}(A^G)$ and the G-orbits in $\mathrm{Spec}(A)$, and other properties of the morphism $\mathrm{Spec}(A) \to \mathrm{Spec}(A^G)$. For example, if k is an algebraically closed field, then the results of this section imply that the set of k–points of $\mathrm{Spec}(A^G)$ are in bijection with G-orbits of k-points of $\mathrm{Spec}(A)$.

We work in a slightly more general setting than that of a finite group action; namely, in the setting of finite flat groupoids. The case of finite group actions, however, captures most of the essential points.

In the last two sections we discuss various consequences for algebraic spaces of the results about quotients explained in the earlier sections. In particular we show that an algebraic space X, satisfying some mild assumptions, admits a dense open subspace $W \subset X$ which is a scheme. The basic idea is that if we present X by an étale equivalence relation R on a scheme U, then we can find a dense open subscheme $V \subset U$ such that the restriction of R to V is a finite étale equivalence relation and the results of this section apply to give a quotient V/R_V which is a scheme and a dense open in X.

Our discussion of finite flat groupoids follows closely [**25**, V], and the material on topological properties of algebraic spaces can be found in [**45**].

6.1. Review of some commutative algebra

The constructions in the next section are rather delicate and depend in a crucial way on some background material in commutative algebra. For the convenience of the reader we review the necessary commutative algebra ingredients in this section.

Recall first that if $\phi : A \to B$ is a ring homomorphism, then B is said to be *integral over* A if for every $b \in B$ there exists a monic polynomial $P \in \phi(A)[X]$ such that $P(b) = 0$ in B. We also sometimes say that ϕ is an *integral ring homomorphism.* The A-algebra B is called an *integral extension of* A if ϕ is also injective.

EXAMPLE 6.1.1. If $\phi : A \to B$ is a finite A-algebra, then ϕ is an integral ring homorphism.

The key result about integral ring homomorphisms is the following:

THEOREM 6.1.2 (Going Up Theorem). *Let $A \to B$ be an integral extension of A. Fix integers $s < r$ and let*

$$\mathfrak{p}_1 \subset \mathfrak{p}_2 \subset \cdots \mathfrak{p}_r \subset A, \quad \mathfrak{q}_1 \subset \mathfrak{q}_2 \subset \cdots \mathfrak{q}_s \subset B$$

be chains of prime ideals such that for every $i \leq s$ we have $\mathfrak{p}_i = \mathfrak{q}_i \cap A$. Then there exists a chain of prime ideals containing $\mathfrak{q}_s$,

$$\mathfrak{q}_{s+1} \subset \cdots \mathfrak{q}_r \subset B,$$

such that $\mathfrak{q}_i \cap A = \mathfrak{p}_i$ for all $i \leq r$.

PROOF. See [**11**, 5.11]. □

COROLLARY 6.1.3. *If $\phi : A \to B$ is an integral ring homomorphism, then the map of schemes*

$$\mathrm{Spec}(B) \to \mathrm{Spec}(A)$$

is a closed map. If B is an integral extension of A then this map is also surjective.

PROOF. See [**11**, Chapter 5, Exercise 1]. □

COROLLARY 6.1.4. *Let $\phi : A \to B$ be an integral extension. If $\mathfrak{p}_1, \mathfrak{p}_2 \subset B$ are two distinct prime ideals with $\mathfrak{p}_1 \cap A = \mathfrak{p}_2 \cap A$, then neither $\mathfrak{p}_1 \subset \mathfrak{p}_2$ nor $\mathfrak{p}_2 \subset \mathfrak{p}_1$.*

PROOF. See [**11**, 5.9]. □

6.1.5. If A is a ring and P is a finitely presented projective A-module, then the corresponding quasi-coherent sheaf $\widetilde{P}$ on $\mathrm{Spec}(A)$ is a locally free sheaf of finite rank. We define the *rank of* P to be the corresponding locally constant integer-valued function on $\mathrm{Spec}(A)$. If P has constant rank r and $P^\vee$ denotes the dual module $\mathrm{Hom}_A(P, A)$, then

$$\mathrm{End}_A(P) \simeq P^\vee \otimes_A P$$

is a projective module of rank r^2, and $\bigwedge^r P$ is a projective A-module of rank 1. In particular, the map induced by scalar multiplication

$$A \to \mathrm{End}_A(\bigwedge^r P)$$

is an isomorphism.

If $\alpha : P \to P$ is an A-linear map, then the induced map $\bigwedge^r P \to \bigwedge^r P$ is given by scalar multiplication by an element $\det(\alpha) \in A$, which we call the *determinant of* α. Using this one can also define the characteristic polynomial of α. Namely, consider the polynomial ring $A[T]$ in one variable and the module $P \otimes_A A[T]$ which is a projective $A[T]$-module of rank r. We then have an endomorphism

$$T \cdot \mathrm{id}_{P \otimes A[T]} - \alpha \otimes 1 \in \mathrm{End}_{A[T]}(P \otimes_A A[T]).$$

The determinant of this endomorphism is an element $P(T, \alpha) \in A[T]$ called the *characteristic polynomial of* α. In the case when P is a free module, these notions agree with the more standard ones (exercise 6.A), and if we write out

$$P(T, \alpha) = T^r - \sigma_1 T^{r-1} + \sigma_2 T^{r-2} - \cdots + (-1)^r \sigma_r,$$

then the constant term $\sigma_r \in A$ is equal to the determinant of α.

6.1.6. Now consider a finite flat ring extension $A \to B$ of constant rank r. This means that $A \to B$ is an injective homomorphism and that B is a projective module when viewed as an A-module of constant rank r. For any element $b \in B$, multiplication by b on B is an A-linear endomorphism of B, and therefore we can consider the determinant of this map which is an element $\mathrm{Nm}(b) \in A$, called the *norm of* b, as well as the characteristic polynomial $P(T, b) \in A[T]$.

LEMMA 6.1.7. *Let $b \in B$ be an element for which $\mathrm{Nm}(b)$ is an element of a prime $\mathfrak{p} \subset A$. Then there exists a prime $\mathfrak{q} \subset B$ containing b with $\mathfrak{q} \cap A = \mathfrak{p}$.*

PROOF. Replacing A by $A/\mathfrak{p}$ and B by $B/\mathfrak{p}B$, we may assume that A is an integral domain and that $\mathfrak{p}$ is the zero ideal. Let K be the field of fractions. Then there is a bijection between prime ideals of B whose intersection with A is the zero ideal and prime ideals in $B \otimes_A K$, so we may further assume that A is a field. In this case we need to show that if $\mathrm{Nm}(b) = 0$, then b is not a unit (whence contained in a prime). This is clear, for if b is a unit then multiplication by b is an automorphism of B whence has nonzero determinant. $\square$

Finally, let us recall the following result which will be used in the next section:

THEOREM 6.1.8 (Prime avoidance). *Let A be a ring, $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ a collection of prime ideals of A. Let $I \subset A$ be an ideal not contained in any of the $\mathfrak{p}_i$. Then there exists an element $x \in I$ not contained in any of the $\mathfrak{p}_i$.*

PROOF. See [**28**, Lemma 3.3]. $\square$

6.2. Quotients by finite flat groupoids

6.2.1. Consider a groupoid in schemes

$$s, t : X_1 \rightrightarrows X_0, \quad m : X_1 \times_{s, X_0, t} X_1 \to X_1.$$

A map of algebraic spaces $f : X_0 \to T$ is called *invariant* if the two compositions fs and ft are equal.

The following is the main result of this chapter:

THEOREM 6.2.2 ([**25**, V, 4.1]). *Assume that s and t are finite and flat, and that for any point $x \in X_0$ the set $s(t^{-1}(x))$ is contained in an affine open subset of X_0. Then there exists an invariant morphism $\pi : X_0 \to Y$ to a scheme Y, which is universal for invariant morphisms to schemes in the sense that for any other invariant morphism $g : X_0 \to Z$ to a scheme Z there exists a unique morphism $f : Y \to Z$ such that $g = f\pi$.*

EXAMPLE 6.2.3. An important situation which captures many of the key ideas of the proof is the case of an affine scheme $X_0 = \mathrm{Spec}(A)$ with action of a finite group G (so X_1 in this case is $X_0 \times G$ with s and t given by the projection and the action respectively). In this case the proof will show that $Y = \mathrm{Spec}(A^G)$, where A^G is the invariant subring of A under the induced action of G on A.

REMARK 6.2.4. It is a very interesting problem to try to generalize Theorem 6.2.2 to the case of equivalence relations which are not flat. An excellent paper in this direction is Kollár's [**46**].

REMARK 6.2.5. Example 5.3.2 shows that the condition that the sets $s(t^{-1}(x))$ are contained in an affine cannot be omitted.

The proof of Theorem 6.2.2 will be in several steps.

We follow the argument and notation of [**25**, Exposé V, Proof of Théorème 4.1] with minor modifications.

6.2.6. Consider first the case when X_0 (and hence also X_1) is affine, and the finite flat morphisms s and t have constant rank n. Write $X_0 = \mathrm{Spec}(A_0)$ and $X_1 = \mathrm{Spec}(A_1)$. Let

$$\delta_0 : A_0 \to A_1 \quad (\text{resp. } \delta_1 : A_0 \to A_1)$$

denote the map defining s (resp. t), and let $A \subset A_0$ denote the equalizer of the two maps

$$\delta_0, \delta_1 : A_0 \to A_1,$$

so we have an invariant map

$$\pi : \mathrm{Spec}(A_0) \to \mathrm{Spec}(A).$$

We claim that this morphism is universal for invariant morphisms to schemes.

Remark 6.2.7. The proof will show that the morphism π is even universal for invariant morphisms to locally ringed spaces.

6.2.8. Write

$$A_2 := A_1 \otimes_{\delta_0, A_0, \delta_1} A_1 \quad \text{and} \quad R' = \mathrm{Spec}(A_2),$$

and let

$$\delta_0' : A_1 \to A_2 \quad (\text{resp. } \delta_1' : A_1 \to A_2)$$

be the map sending $a \in A_1$ to $a \otimes 1$ (resp. $1 \otimes a$). Note that we have

$$R' \simeq X_1 \times_{s, X_0, t} X_1$$

so we also have the composition map

$$R' \to X_1$$

which corresponds to a map

$$\delta_2' : A_1 \to A_2.$$

These maps fit into a commutative diagram:

$$\begin{tikzcd} A_0 \arrow[r, "\delta_0"] & A_1 \arrow[r, shift left, "\delta_1'"] \arrow[r, shift right, "\delta_0'"'] & A_2 \\ & A_0 \arrow[u, "\delta_1"] \arrow[r, shift left, "\delta_1"] \arrow[r, shift right, "\delta_0"'] & A_1. \arrow[u, "\delta_2'"'] \end{tikzcd} \tag{6.2.8.1}$$

Lemma 6.2.9. *Let $a \in A_0$ be an element, and let*

$$P_{\delta_1}(T, \delta_0(a)) = T^n - \sigma_1 T^{n-1} + \cdots + (-1)^n \sigma_n$$

be the characteristic polynomial of multiplication by $\delta_0(a)$ on A_1 viewed as an A_0-module via δ_1. Then $\sigma_i \in A$ for all i, and

$$a^n - \sigma_1 a^{n-1} + \cdots + (-1)^n \sigma_n = 0.$$

PROOF. Since the squares on the right of (6.2.8.1) are cocartesian and $\delta_0'\delta_0 = \delta_1'\delta_0$ we have

$$\delta_0(P_{\delta_1}(T, \delta_0(a))) = P_{\delta_2'}(T, \delta_0'\delta_0(a)) = P_{\delta_2'}(T, \delta_1'\delta_0(a)) = \delta_1(P_{\delta_1}(T, \delta_0(a))),$$

and therefore $P_{\delta_1}(T, \delta_0(a))$ is a polynomial with coefficients in A. By the Cayley-Hamilton theorem we have

$$\delta_0(a)^n - \delta_1(\sigma_1)\delta_0(a)^{n-1} + \cdots + (-1)^n\delta_1(\sigma_n) = 0.$$

It follows that

$$\delta_0(a^n - \sigma_1 a^{n-1} + \cdots + (-1)^n\sigma_n) = 0,$$

and, therefore, also that

$$a^n - \sigma_1 a^{n-1} + \cdots + (-1)^n\sigma_n = 0.$$

□

COROLLARY 6.2.10. *The ring A_0 is integral over A.*

PROOF. In fact the proof of (6.2.9) exhibits for every $a \in A_0$ an integral equation for a over A. □

LEMMA 6.2.11. *Let $x, y \in X_0$ be two points with $\pi(x) = \pi(y)$. Then there exists a point $z \in X_1$ such that $s(z) = x$ and $t(z) = y$.*

PROOF. Let $\mathfrak{p}$ (resp. $\mathfrak{q}$) denote the prime ideal of A_0 corresponding to x (resp. y), and suppose by contradiction that $\mathfrak{p}$ is not equal to $\delta_0^{-1}(\mathfrak{r})$ for any prime $\mathfrak{r} \subset A_1$ with $\delta_1^{-1}(\mathfrak{r}) = \mathfrak{q}$. Then by 6.1.4 the prime $\mathfrak{p}$ is not contained in $\delta_0^{-1}(\mathfrak{r})$ for any such prime $\mathfrak{r} \subset A_1$, and therefore by prime avoidance (6.1.8) there exists an element $a \in \mathfrak{p}$ not contained in $\delta_0^{-1}(\mathfrak{r})$ for any $\mathfrak{r}$ with $\delta_1^{-1}(\mathfrak{r}) = \mathfrak{q}$.

Let $\sigma \in A_0$ denote the norm of $\delta_0(a) : A_1 \to A_1$ with A_1 viewed as an A_0-module via δ_1. Then σ is an element of A by 6.2.9, and in fact $\sigma \in \mathfrak{p} \cap A$ since by 6.2.9 we can write σ as a linear combination of positive powers of a.

On the other hand, since $\delta_0(a)$ is not contained in $\mathfrak{r}$ for any prime $\mathfrak{r} \subset A_1$ with $\delta_1^{-1}(\mathfrak{r}) = \mathfrak{q}$, the element σ is not in $\mathfrak{q}$ (by 6.1.7). This is a contradiction since

$$A \cap \mathfrak{p} = A \cap \mathfrak{q}.$$

□

6.2.12. Since the map $|\mathrm{Spec}(A_0)| \to |\mathrm{Spec}(A)|$ is a closed surjective map of topological spaces (by 6.1.3), it follows that the topological space $|\mathrm{Spec}(A)|$ is the quotient in the category of topological spaces of the space $|\mathrm{Spec}(A_0)|$ by the equivalence relation given by the maps s and t. Furthermore, if

$$g : \mathrm{Spec}(A_1) \to \mathrm{Spec}(A)$$

denotes the projection, then the sequence of sheaves on $\mathrm{Spec}(A)$,

$$0 \to \mathscr{O}_{\mathrm{Spec}(A)} \to \pi_*\mathscr{O}_{\mathrm{Spec}(A_0)} \rightrightarrows g_*\mathscr{O}_{\mathrm{Spec}(A_1)},$$

is exact, as this can be verified after taking global sections, where the exactness is by definition of A.

It follows that π is universal for invariant maps to schemes (even invariant maps to locally ringed spaces).

6.2.13. Before proceeding further, it is useful to make some more refined observations about the ring A.

First of all, if $A \to B$ is a flat morphism, then we can consider the base change of π to a morphism

$$X_0 \times_{\mathrm{Spec}(A)} \mathrm{Spec}(B) \to \mathrm{Spec}(B) \tag{6.2.13.1}$$

which is an invariant morphism for the groupoid

$$X_1 \times_{\mathrm{Spec}(A)} \mathrm{Spec}(B) \rightrightarrows X_0 \times_{\mathrm{Spec}(A)} \mathrm{Spec}(B)$$

obtained from our original groupoid $X_1 \rightrightarrows X_0$ by base change to $\mathrm{Spec}(B)$. Since $A \to B$ is flat, the map

$$B \to \mathrm{Eq}(A_0 \otimes_A B \rightrightarrows A_1 \otimes_A B)$$

is an isomorphism, so (6.2.13.1) is also universal for invariant maps to schemes.

Second, in the case when

$$X_1 \to X_0 \times X_0$$

is an imbedding, so X_1 is a finite flat equivalence relation on X_0, we can say more about the quotient $\mathrm{Spec}(A)$. Namely, we claim that $A \to A_0$ is finite and flat, and that the natural map

$$A_0 \otimes_A A_0 \xrightarrow{\delta_0 \otimes \delta_1} A_1 \tag{6.2.13.2}$$

is an isomorphism. This can be seen as follows.

Since the formation of A commutes with flat base change, it suffices to prove the second claim in the case when A is a local ring with infinite residue field. In this case, A_0 is semilocal (i.e. has only finitely many maximal ideals), being finite over the local A. Also the map (6.2.13.2) is surjective. Indeed, viewing this as a map of A_0-modules via the map

$$A_0 \to A_0 \otimes_A A_0, \quad a \mapsto 1 \otimes a,$$

it suffices to show that the map is surjective modulo a maximal ideal q of A_0. This is indeed the case as modulo q this is a map to a finite A_0/q-algebra which induces a monomorphism on the spectra.

It follows that there exist elements $a_1, \ldots, a_n \in A_0$ such that $\delta_0(a_1), \ldots, \delta_0(a_n)$ form a basis for A_1 viewed as an A_0-module via δ_1. To prove our claim now note that the induced map

$$\underline{a} : A \otimes_{\mathbb{Z}} \mathbb{Z}^n \to A_0$$

is an isomorphism as it is identified with the map on equalizers obtained from the commutative diagram of cocartesian squares

$$\begin{tikzcd}
A_0 \otimes_{\mathbb{Z}} \mathbb{Z}^n \arrow[r, shift left, "\delta_1 \otimes \mathrm{id}"] \arrow[r, shift right, "\delta_0 \otimes \mathrm{id}"'] \arrow[d, "\delta_0(\underline{a})"] & A_1 \otimes_{\mathbb{Z}} \mathbb{Z}^n \arrow[d, "\delta_1'\delta_0(\underline{a}) = \delta_0'\delta_0(\underline{a})"] \\
A_1 \arrow[r, shift left, "\delta_1'"] \arrow[r, shift right, "\delta_0'"'] & A_2,
\end{tikzcd}$$

whose vertical morphisms are isomorphisms by construction.

Corollary 6.2.14. *Let X be an algebraic space, and suppose there exists a finite flat surjection $p : Y \to X$ of constant rank with Y an affine scheme. Then X is also an affine scheme.*

PROOF. Indeed, let $X_1 \subset Y \times Y$ denote $Y \times_X Y$. Then X_1 is a finite flat equivalence relation on Y of constant rank, and so we have a universal invariant morphism

$$\pi : Y \to Y'$$

with Y' an affine scheme. Write

$$Y = \mathrm{Spec}(A_0), \quad Y' = \mathrm{Spec}(A), \quad X_1 = \mathrm{Spec}(A_1).$$

We claim that there exists a unique morphism

$$\rho : Y' \to X$$

such that $\rho \circ \pi = p$.

To see this let $\mathrm{Spec}(B) \to X$ be an étale morphism from an affine scheme, and let Y_B denote the base change of Y to $\mathrm{Spec}(B)$. Since p is finite the scheme Y_B is affine, say

$$Y_B = \mathrm{Spec}(B_0).$$

Also the base change $X_{1,B}$ of X_1 is an affine scheme, say

$$X_{1,B} = \mathrm{Spec}(B_1).$$

We then get a commutative diagram of solid arrows

$$\begin{tikzcd} B \arrow[r] & B_0 \arrow[r, shift left] \arrow[r, shift right] & B_1 \\ A \arrow[r] \arrow[u, dashed, "h"] & A_0 \arrow[r, shift left] \arrow[r, shift right] \arrow[u] & A_1 \arrow[u] \end{tikzcd}$$

where the horizontal rows are exact. This implies that there exists a dotted arrow h as in the diagram. Also the induced map

$$A_0 \otimes_A B \to B_0$$

is an isomorphism. Indeed, tensoring this map of B-algebras with B_0 we get the map

$$(A_0 \otimes_A A_0) \otimes_{A_0} B_0 \simeq (A_0 \otimes_A B) \otimes_B B_0 \to B_0 \otimes_B B_0$$

which is an isomorphism since it can be identified with the map

$$A_1 \otimes_{A_0} B_0 \to B_1.$$

In particular, the map

$$\mathrm{Spec}(B) \to \mathrm{Spec}(A)$$

is an étale morphism as this can be verified after making a faithfully flat extension. We conclude that there exists an étale surjection

$$\mathrm{Spec}(A') \to \mathrm{Spec}(A)$$

such that the composition

$$\mathrm{Spec}(A') \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A_0) \to \mathrm{Spec}(A_0) \to X$$

factors through $\mathrm{Spec}(A')$, and that this factorization is unique. We therefore obtain the morphism ρ.

We claim that ρ is an isomorphism. To see this we may work locally on X, and may therefore assume that X is an affine scheme in which case the result is immediate. $\square$

Remark 6.2.15. The statement of 6.2.14 remains true if p is only assumed finite and surjective. This is a special case of [**46**, Corollary 48] and the theory of Chevalley-Kleiman schemes. This is a deeper result, and we will only need 6.2.14 in what follows.

To complete the proof of Theorem 6.2.2, we now explain how to reduce the theorem to the case when X_0 is affine and s and t have constant rank.

6.2.16. For an integer n, let $X_0^{(n)} \subset X_0$ be the largest open subset over which the projection

$$s : X_1 \to X_0$$

has rank n (note that $X_0^{(n)} \subset X_0$ is also closed). Then

$$s^{-1}(X_0^{(n)}) = t^{-1}(X_0^{(n)}).$$

Indeed, since we have a groupoid in schemes, if R' denotes the fiber product

$$R' := X_1 \times_{s,X_0,t} X_1 \to X_1, \tag{6.2.16.1}$$

then the square

$$\begin{tikzcd} R' \arrow[r, "m"] \arrow[d, "\mathrm{pr}_1"] & X_1 \arrow[d, "s"] \\ X_1 \arrow[r, "s"] & X_0 \end{tikzcd}$$

is cartesian. Since by definition we also have a cartesian square

$$\begin{tikzcd} R' \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] & X_1 \arrow[d, "s"] \\ X_1 \arrow[r, "t"] & X_0 \end{tikzcd}$$

we see that both $s^{-1}(X_0^{(n)})$ and $t^{-1}(X_0^{(n)})$ are equal to the maximal open subset of X_1 over which the morphism

$$\mathrm{pr}_1 : R' \to X_1$$

has rank n. It follows that if $X_1^{(n)} \subset X_1$ denotes the inverse image of $s^{-1}(X_0^{(n)})$, then we get a subgroupoid

$$(X_1^{(n)} \rightrightarrows X_0^{(n)}) \subset (X_1 \rightrightarrows X_0)$$

and to give an invariant map $X_0 \to T$ to a scheme T is equivalent to giving an invariant map

$$X_0^{(n)} \to T$$

for each n. It therefore suffices to prove the theorem for the groupoids

$$X_1^{(n)} \rightrightarrows X_0^{(n)}$$

so we may assume that the maps s and t have constant rank n, for some integer n. We make this assumption for the rest of the proof.

Remark 6.2.17. A similar argument as the preceding one can be found in [**62**].

6.2.18. We say that a subset $F \subset X_0$ is *invariant* if for every $x \in F$ and $y \in X_1$ such that $t(y) = x$ we have $s(y) \in F$.

LEMMA 6.2.19. *Let $F \subset X_0$ be a subset. Then $F^{\mathrm{inv}} := s(t^{-1}(F))$ is invariant.*

PROOF. Let $x' \in F^{\mathrm{inv}}$ be a point, and let $z \in X_1$ be a point such that $t(z) = x'$. Let $r \in X_1$ be a point such that $s(r) = x'$ and $t(r) = x \in F$. Then if $q \in X_1$ is the image of (z, r) under the composition map

$$X_1 \times_{t,X_0,s} X_1 \to X_1,$$

then

$$s(q) = s(z), \quad t(q) = t(z).$$

Since $t(z) = x' \in F^{\mathrm{inv}}$, this implies that $s(z)$ is in $s(t^{-1}(F^{\mathrm{inv}}))$ as desired. $\square$

6.2.20. If $W \subset X_0$ is an open subset with complement F, let W' denote $X_0 - F^{\mathrm{inv}}$. The open subset W' is called the *maximal saturated open subset of* W. Note that since $F \subset F^{\mathrm{inv}}$ we have $W' \subset W$. Furthermore, if $Z \subset W$ is an invariant subset, then $Z \subset W'$ since Z^c is also invariant and contains F, whence $F^{\mathrm{inv}} \subset Z^c$.

LEMMA 6.2.21. *For any point $x \in X_0$, there exists an invariant affine open subset $x \in W \cap X_0$ containing x.*

PROOF. Choose an affine open subset V of X_0 containing $s(t^{-1}(x))$ (this is possible by assumption), and let V' be the maximal saturated open subset of V. Since $s(t^{-1}(x))$ is an invariant subset we have $s(t^{-1}(x)) \subset V'$. By prime avoidance, there exists an element $f \in \Gamma(V, \mathscr{O}_V)$ such that $D(f) \subset V'$ and $s(t^{-1}(x)) \subset D(f)$. Let W be the maximal saturated open subset of $D(f)$, so again we have $s(t^{-1}(x)) \subset W$ and W is invariant.

We claim that W is affine. Set $Z(f) := V' - D(f)$. Then $t^{-1}(Z(f))$ is the set of points of $s^{-1}(V') = t^{-1}(V')$ where t^*f is zero. Therefore $s(t^{-1}(Z(f))$ is the zero locus of the norm $\mathrm{Nm}_s(t^*f)$ in the s-direction of t^*f. It follows that W is equal to $D(f \cdot \mathrm{Nm}_s(t^*f))$. $\square$

6.2.22. Now choose an open covering $X_0 = \bigcup W_i$ of X_0 by invariant affine open subsets. Then by the affine case, each of the quotients W_i/R_{W_i} is an affine scheme, and the W_i/R_{W_i} form an open covering of X. This completes the proof of 6.2.2. $\square$

6.3. Topological properties of algebraic spaces

Recall from 5.4.7 that a morphism of algebraic spaces $X \to S$ is called *quasi-separated* if the diagonal morphism $X \to X \times_S X$ is quasi-compact. We sometimes also refer to such an X/S as a *quasi-separated algebraic space over* S.

LEMMA 6.3.1. *Let X be a quasi-separated algebraic space over a scheme S such that there exists an epimorphism $g : \mathrm{Spec}(K) \to X$ with K a field. Then X is also the spectrum of a field.*

PROOF. Let $\pi : U \to X$ be an étale surjective morphism with U a scheme. After replacing K by a finite separable field extension, we may assume that g factors through a morphism $g_U : \mathrm{Spec}(K) \to U$. Let $V \subset U$ be a quasi-compact connected open subset containing the image point of g_U. Then $V \to X$ is again an epimorphism, whence we may assume that U is quasi-compact. The fiber product

$U \times_X U$ is then also quasi-compact being the fiber product of the diagram

$$\begin{tikzcd} & U \arrow[d, "\pi\times\pi"] \\ X \arrow[r, "\Delta"] & X \times_S X, \end{tikzcd}$$

where Δ is quasi-compact by assumption. Since $U \times_X \mathrm{Spec}(K)$ is the fiber product of the diagram

$$\begin{tikzcd} U \times_X U \arrow[d, "\mathrm{pr}_2"] & \\ U & \mathrm{Spec}(K) \arrow[l, "g_U"'] \end{tikzcd}$$

it follows that $U \times_X \mathrm{Spec}(K)$ is quasi-compact and étale over $\mathrm{Spec}(K)$. Therefore $U \times_X \mathrm{Spec}(K)$ is a finite disjoint union of field extensions of K. Since the first projection $\mathrm{pr}_1 : U \times_X \mathrm{Spec}(K) \to U$ is an epimorphism this implies that U is also a finite disjoint union of spectra of fields. Since U was further assumed connected this implies that U is the spectrum of a field. This reduces us to the case when $g : \mathrm{Spec}(K) \to X$ is étale surjective.

In this case, the scheme $R := \mathrm{Spec}(K) \times_X \mathrm{Spec}(K)$ is quasi-compact and étale over $\mathrm{Spec}(K)$, and therefore R also isomorphic to a finite disjoint union of separable field extensions of K. From this and 6.2.14 it follows that if L denotes the equalizer of the two maps

$$\mathrm{pr}_1^*, \mathrm{pr}_2^* : K \to \Gamma(R, \mathscr{O}_R)$$

then $X \simeq \mathrm{Spec}(L)$. □

Example 6.3.2. The assumption that the diagonal of X is quasi-compact in 6.3.1 is necessary. Fix a field k of characteristic 0, and let $U = \mathrm{Spec}(k(x))$ be the spectrum of the rational function field in 1 variable over k. There is an action of $\mathbb{Z}$ on U in which $n \in \mathbb{Z}$ acts by sending x to $x+n$. This is a free action. Indeed, if R is a k-algebra, then any homomorphism $\varphi : k(x) \to R$ is injective since $k(x)$ is a field, which implies that the action of $\mathbb{Z}$ on the R-valued points $U(R)$ is free for all R. Let X be the algebraic space obtained by taking the quotient of U by this action. Then X admits an étale surjective morphism $U \to X$ from a field, but we claim that X is not a scheme, let alone the spectrum of a field. To see this suppose to the contrary that X is a scheme. Then the underlying topological space of X must be a single point (since it admits a surjective morphism from a point) whence X is affine, say $X = \mathrm{Spec}(A)$. Since $U \to X$ is étale and surjective, we get an inclusion $A \hookrightarrow k(x)^{\mathbb{Z}}$ from A into the $\mathbb{Z}$-invariants of $k(x)$. As in 5.3.4 this implies that $A = k$. This is a contradiction for then we would have $\mathrm{Spec}(k(x) \otimes_k k(x)) \simeq \coprod_{n\in\mathbb{Z}} \mathrm{Spec}(k(x))$ which is false (the left side is quasi-compact, whereas the right is not).

6.3.3. Let S be a scheme and X/S an algebraic space. Associated to X is a topological space $|X|$ defined as follows.

A *point* of X is a monomorphism

$$\iota : \mathrm{Spec}(k) \to X$$

with k a field. We define an equivalence relation on the set of points by declaring two points

$$\iota_j : \mathrm{Spec}(k_j) \to X, \quad j = 1, 2$$

equivalent if there exists an an isomorphism $\sigma : \mathrm{Spec}(k_1) \to \mathrm{Spec}(k_2)$ such that $\iota_1 = \iota_2 \circ \sigma$.

Let $|X|$ denote the set of equivalence class of points of X. If $Y \subset X$ is a closed subspace, then $|Y|$ is naturally a subset of $|X|$. We define a topology on $|X|$ by declaring that a subset $Q \subset |X|$ is closed if there exists a closed subspace $Y \subset X$ with $|Y| = Q$.

PROPOSITION 6.3.4. *Assume that X/S is quasi-separated, and let $f : \mathrm{Spec}(k) \to X$ be a morphism of algebraic spaces with k a field. Then there exists a point $\iota : \mathrm{Spec}(k') \to X$ and a factorization*

$$\mathrm{Spec}(k) \xrightarrow{g} \mathrm{Spec}(k') \xrightarrow{\iota} X$$

of f.

PROOF. Let $Y \to X$ be an étale surjective morphism with Y a scheme, and set

$$Z := \mathrm{Spec}(k) \times_{f,X} Y.$$

The projection $Z \to \mathrm{Spec}(k)$ is étale, which implies that

$$Z = \coprod_{z \in Z} \mathrm{Spec}(k(z)).$$

Let $h : Z \to Y$ be the second projection, and set

$$T := \coprod_{z \in Z} \mathrm{Spec}(k(h(z))),$$

where $k(h(z))$ denotes the residue field of the image of z in Y. Let R denote $Z \times_{\mathrm{Spec}(k)} Z$, so R is an étale equivalence relation on Z.

Consider the morphism

$$\pi : Z \times_{\mathrm{Spec}(k)} Z \to T \times_X T.$$

We have

$$T \times_X T = \coprod_{z, z' \in Z} \mathrm{Spec}(k(h(z))) \times_X \mathrm{Spec}(k(h(z'))),$$

and $\mathrm{Spec}(k(h(z))) \times_X \mathrm{Spec}(k(h(z')))$ is a finite étale $\mathrm{Spec}(k(h(z))$-scheme via the first projection (and similarly the second projection is also finite étale). In particular, each term $\mathrm{Spec}(k(h(z))) \times_X \mathrm{Spec}(k(h(z')))$ is a finite disjoint union of spectra of fields. It follows that the scheme-theoretic image $\overline{R} \subset T \times_X T$ of R under π is an étale equivalence relation on T. By construction we have a factorization

$$\begin{tikzcd} \mathrm{Spec}(k) \arrow[r, "g"] \arrow[rr, bend left, "f"] & T/\overline{R} \arrow[r, "\iota"] & X \end{tikzcd}$$

of f. Also note that the square

$$\begin{tikzcd} \overline{R} \arrow[r] \arrow[d] & Y \times_X Y \arrow[d] \\ T \arrow[r] & Y \end{tikzcd}$$

is cartesian, so $T/\overline{R} \to X$ is a monomorphism. Finally, note that the map $g : \mathrm{Spec}(k) \to T/\overline{R}$ is an epimorphism, from which it follows that $T/\overline{R}$ is the spectrum of a field, by 6.3.1. ☐

6.3.5. It follows that the topological space $|X|$ associated to an algebraic space is functorial in X. Namely, if $f : X \to Y$ is a morphism of quasi-separated algebraic spaces over a base S, define

$$|f| : |X| \to |Y|$$

to be the map sending a point $\iota : \mathrm{Spec}(k) \to X$ to the point of Y associated to the composition $f\iota : \mathrm{Spec}(k) \to Y$. This is a continuous map of topological spaces as for a closed subspace $Z \subset Y$ we have $|f|^{-1}(|Z|) = |f^{-1}(Z)|$.

DEFINITION 6.3.6. Let S be a scheme. A morphism of quasi-separated algebraic spaces $f : X \to Y$ over S is *closed* if the map on topological spaces $|f| : |X| \to |Y|$ is a closed map. A morphism $f : X \to Y$ is *universally closed* if for every morphism of algebraic spaces $Y' \to Y$, with Y'/S also quasi-separated, the base change $X \times_Y Y' \to Y'$ is a closed map. A morphism of algebraic spaces is *proper* if it is separated, of finite type, and universally closed.

REMARK 6.3.7. If $f : X \to Y$ is a representable morphism of algebraic spaces, then by 5.1.5 we have another possible definition of what it means for f to be proper. By exercise 6.G this earlier definition agrees with the one obtained from 6.3.6.

REMARK 6.3.8. Just as in the case of schemes, most properties of proper morphisms of algebraic spaces will be deduced from the corresponding properties for projective morphisms of schemes. The key ingredient for this is Chow's lemma which is proven in 7.4.1.

6.4. Schematic open subspaces of algebraic spaces

THEOREM 6.4.1 ([**45**, II.6.6]). *Let S be a scheme and let X/S be a quasi-separated algebraic space over S. Then there exists a scheme V and a dense open imbedding $V \hookrightarrow X$.*

PROOF. Cover S by open affines $S = \bigcup_i S_i$ and let X_i denote the restriction of X to S_i. If the theorem is true in the case when S is affine, then we can find for each i a dense open $V_i \subset X_i$ with V_i a scheme. The union $\bigcup_i V_i \subset X_i$ is then a dense open subscheme of X. It therefore suffices to consider the case when S is affine.

Choose an étale cover $U \to X$, and let $R := U \times_X U$ be the étale equivalence relation on U defining X. For any open subset $W \subset U$, we can restrict the equivalence relation R to an equivalence relation R_W on W. Namely set

$$R_W := R \times_{(U \times_S U)} (W \times_S W).$$

Then R_W is an étale equivalence relation on W and the map

$$W/R_W \to X$$

is an open imbedding. Covering U by quasi-compact opens, it follows that it suffices to prove the result in the case when X admits an étale covering $U \to X$ with U quasi-compact.

For an integer n, let $W^{(n)} \subset U$ be the largest open subset over which the first projection

$$s : R \to U$$

has rank n. By the same argument as in 6.2.16 this subspace descends to an open subspace $X^{(n)} \subset X$, and the union of the $X^{(n)}$'s is dense in X. This then further reduces us to the case when X admits a finite étale surjection from a quasi-compact scheme which is a dense open in an affine scheme. Note that for such a scheme any finite set of points is contained in an affine, and therefore by 6.2.2 the space X is in fact a scheme in this case. □

6.5. Exercises

EXERCISE 6.A. Show that in the case when A is a field and P is a finite dimensional vector space over A, the notions of determinant and characteristic polynomial defined in 6.1.5 agree with the standard notions in linear algebra.

EXERCISE 6.B. Let k be a field of characteristic $p > 0$, and set $R = k[\epsilon, t]/(\epsilon^2)$. Let $\mathbb{Z}/(p)$ act on R over $k[\epsilon]/\epsilon^2$ by $t \mapsto t + \epsilon$. Show that the natural map on invariant rings

$$(R^{\mathbb{Z}/(p)}) \otimes_{k[\epsilon]/\epsilon^2, \epsilon \mapsto 0} k \to (R \otimes_{k[\epsilon]/\epsilon^2, \epsilon \mapsto 0} k)^{\mathbb{Z}/(p)}$$

is neither injective nor surjective.

EXERCISE 6.C. With notation and assumptions as in 6.2.6, assume further that A_0 and A_1 are finitely generated algebras over a noetherian ring R, and that the maps giving the groupoid structure are morphisms over $\operatorname{Spec}(R)$. Show that then the ring of invariants A of the groupoid is finitely generated over R. Deduce that if in Theorem 6.2.2 we further assume that X_0 and X_1 are locally of finite type over a noetherian base ring R and s, t, and m are morphisms over $\operatorname{Spec}(R)$, then the scheme Y in 6.2.2 is also of finite type over R.

EXERCISE 6.D. Show by example that if R is a ring and G is a finite group acting on R, then the map $R^G \to R$ need not be finite.

EXERCISE 6.E. Let k be a field and X/k a projective k-scheme. Let G be a finite group acting on X over k. Show that there exists a finite-dimensional G-representation V over k and a G-invariant imbedding $X \hookrightarrow \mathbb{P}(V)$, where the action of G on $\mathbb{P}(V)$ is induced by the action of G on V. Deduce from this that if k is infinite, then any finite set of points of X is contained in a G-invariant affine open subset of X.

Now show that if X/k is a quasi-projective scheme, and G is a finite group acting on X, then there exists a universal G-invariant morphism $\pi : X \to \overline{X}$ to a scheme $\overline{X}$. Moreover, the morphism π is finite and surjective, and for every point $\bar{x} \in \overline{X}$ the group G acts transitively on $\pi^{-1}(\bar{x})$.

EXERCISE 6.F. For actions of positive–dimensional groups, the ring of invariants is not as well-behaved. For example, let k be an algebraically closed field and consider the action of the multiplicative group $\mathbb{G}_m$ on $\mathbb{A}^2_k$ given by

$$u * (a, b) := (ua, ub).$$

(a) Show that the ring of invariants of $k[X, Y]$ under this action is equal to k.

(b) Exhibit a localization A of $k[X, Y]$ such that the $\mathbb{G}_m$-action induces a $\mathbb{G}_m$-action on A for which $\operatorname{Spec}(A^{\mathbb{G}_m})$ is one-dimensional.

Exercise 6.G. Let S be a scheme, and let X/S be a quasi-separated algebraic space, and let $\pi : U \to X$ be an étale surjective morphism with U/S a scheme. Write also $\pi : |U| \to |X|$ for the induced morphism of topological spaces.

(i) Show that a subset $S \subset |X|$ is closed if and only if its preimage $\pi^{-1}(S) \subset |U|$ is closed.
(ii) Show that a finite type separated morphism of quasi-separated algebraic spaces $f : X \to Y$ over S is proper if and only if for every morphism $Y' \to Y$ with Y' a scheme, the base change map $X \times_Y Y' \to Y'$ is a closed map.
(iii) Conclude from this that for representable morphisms of algebraic spaces the two notions of properness obtained from 5.1.5 and 6.3.6 agree.

Exercise 6.H. Let S be a scheme. Prove the following:

(i) If $f : X \to Y$ and $g : Y \to Z$ are proper morphisms of algebraic spaces quasi-separated over S, then $gf : X \to Z$ is a proper morphism.
(ii) If

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "g"'] & & Y \arrow[dl, "h"] \\ & Z & \end{tikzcd}$$

is a commutative diagram of algebraic spaces quasi-separated over S with g proper and h separated, then f is proper.

Exercise 6.I. Let F be the algebraic space described in 5.3.3. Show that the natural map $\pi : F \to \mathbb{A}^1_k$ induces a homeomorphism $|F| \to |\mathbb{A}^1_k|$.

Exercise 6.J. With notation as in 5.3.3, show that the diagram

$$R \rightrightarrows U \to \mathbb{A}^1_k$$

is a coequalizer diagram in the category of ringed spaces.

Exercise 6.K. Let k be an algebraically closed field, and let G/k be a finite type separated algebraic space over k equipped with the structure of a group object in the category of algebraic spaces over k [**69**, 7.6.3]. Show that G is a scheme.

Exercise 6.L. Let X be a noetherian algebraic space quasi-separated over a scheme S. Show that the topological space $|X|$ is a noetherian topological space.

CHAPTER 7

Quasi-coherent sheaves on algebraic spaces

Just as in the theory of schemes, quasi-coherent sheaves on algebraic spaces play a fundamental role in the development of the theory. The main difference from the scheme-theoretic setting is that for algebraic spaces we have to define the notion of quasi-coherent sheaf using the étale topology instead of the Zariski topology. This switch in topology is justified by our earlier results on descent 4.3.15. Once the basic definitions are established, we discuss various applications such as the correspondence between affine morphisms and quasi-coherent sheaves of algebras, Stein factorization, scheme-theoretic image, etc. We also discuss a theorem showing that an algebraic space is a scheme if and only if its maximal reduced subspace is a scheme, and finiteness of cohomology for coherent sheaves on proper algebraic spaces.

The source for the material in this chapter is [**45**, Chapter III].

7.1. The category of quasi-coherent sheaves

Definition 7.1.1. Let X be an algebraic space. The *small étale site of* X is the site whose underlying category has objects étale morphisms of algebraic spaces $Y \to X$ and whose morphisms are X-morphisms. A collection of morphisms

$$\{Y_i \to Y\}_{i \in I}$$

in this category is a covering if the map

$$\coprod_{i \in I} Y_i \to Y$$

is surjective. We denote by $\mathrm{Et}(X)$ the étale site of X, and by X_{et} the associated topos.

Remark 7.1.2. We can also consider the *big étale site* of X which is the site whose underlying category is the category of algebraic spaces over X, and whose coverings are given by étale coverings as above.

7.1.3. Notice that any object $Y \to X$ of $\mathrm{Et}(X)$ admits a covering $\{Y_i \to Y\}$ with the Y_i schemes. Let $\mathrm{Et}'(X) \subset \mathrm{Et}(X)$ denote the full subcategory of objects $Y \to X$ with Y a scheme, and view $\mathrm{Et}'(X)$ as a site with the induced topology. Let $X_{\mathrm{et}'}$ denote the associated topos. The inclusion

$$\mathrm{Et}'(X) \hookrightarrow \mathrm{Et}(X)$$

defines by 2.2.31 a morphism of topoi

$$X_{\mathrm{et}} \to X_{\mathrm{et}'}$$

which by exercise 2.H is an equivalence.

Notice that if X is a scheme then $\mathrm{Et}'(X)$ is the usual étale site of X as in 2.1.12.

This is useful as it is sometimes easier to define sheaves on the category $\mathrm{Et}'(X)$. For example, there is a sheaf of rings $\mathscr{O}_X \in X_{\mathrm{et}}$, called the *structure sheaf*, obtained from the sheaf on $\mathrm{Et}'(X)$ which to any $Y \to X$ with Y a scheme associates the ring

$$\Gamma(Y_{\mathrm{zar}}, \mathscr{O}_Y).$$

7.1.4. For any object $Y \to X$ of $\mathrm{Et}(X)$, the localized category

$$\mathrm{Et}(X)|_{Y \to X}$$

is naturally equivalent to $\mathrm{Et}(Y)$. In particular, if $Y \to X$ is an object of $\mathrm{Et}'(X)$, then for any sheaf F on $\mathrm{Et}(X)$ its restriction to the localized category $\mathrm{Et}(X)_{Y \to X}$ is naturally viewed as a sheaf F_Y in the étale topos of Y.

This enables us to describe sheaves on X in more concrete terms using a covering. Namely, let $U \to X$ be an étale surjection with U a scheme, and consider the associated étale equivalence relation

$$R \rightrightarrows U$$

defining X. Set

$$R' := U \times_X U \times_X U$$

so we have three projections

$$\mathrm{pr}_{12}, \mathrm{pr}_{23}, \mathrm{pr}_{13} : R' \to R.$$

Let $(R \rightrightarrows U)_{\mathrm{et}}$ denote the category of pairs (F_U, ϵ), where F_U is an étale sheaf on U and

$$\epsilon : s^* F_U \to t^* F_U$$

is an isomorphism of sheaves on R, such that the diagram in the étale topos of R',

$$\begin{tikzcd}
\mathrm{pr}_{12}^* s^* F_U \arrow[r, "\mathrm{pr}_{12}^*\epsilon"] \arrow[d, "\simeq"] & \mathrm{pr}_{12}^* t^* F_U \arrow[r, "\simeq"] & \mathrm{pr}_{23}^* s^* F_U \arrow[r, "\mathrm{pr}_{23}^*\epsilon"] & \mathrm{pr}_{23}^* t^* F_U \arrow[d, "\simeq"] \\
\mathrm{pr}_{13}^* s^* F_U \arrow[rrr, "\mathrm{pr}_{13}^*\epsilon"] & & & \mathrm{pr}_{13}^* t^* F_U,
\end{tikzcd}$$

commutes. There is a natural functor

$$X_{\mathrm{et}} \to (R \rightrightarrows U)_{\mathrm{et}}$$

which is an equivalence by 4.2.12.

Notice that $(R \rightrightarrows U)_{\mathrm{et}}$ only involves the étale sites of schemes, and can be defined without the theory of algebraic spaces.

Similarly the category of $\mathscr{O}_X$-modules in X_{et} is equivalent to the category of pairs (M_U, ϵ), where M_U is a sheaf of $\mathscr{O}_U$-modules in U_{et}, and $\epsilon : s^* M_U \to t^* M_U$ is an isomorphism over R satisfying the cocycle condition over R'.

Definition 7.1.5. An $\mathscr{O}_X$-module M in X_{et} is *quasi-coherent* if there exists an étale surjection $U \to X$ with U a scheme such that M_U is a quasi-coherent sheaf on U.

Similarly, if X is locally noetherian, then an $\mathscr{O}_X$-module M is called *coherent* if there exists an étale surjection $U \to X$ with U a scheme such that M_U is a coherent sheaf on U.

REMARK 7.1.6. In the case when X is a scheme, these notions are equivalent to the usual notions of quasi-coherent and coherent sheaves in the Zariski topology by 4.3.15.

LEMMA 7.1.7. *Let X be an algebraic space.*

(i) An $\mathscr{O}_X$-module M on X is quasi-coherent if and only if for every étale morphism $V \to X$, with V a scheme, the sheaf M_V is a quasi-coherent sheaf on V.

(ii) If X is locally noetherian, then an $\mathscr{O}_X$-module M is coherent if and only if for every étale morphism $V \to X$ with V a scheme the sheaf M_V is a coherent sheaf on V.

PROOF. The proofs of both statements are very similar, so we prove (i) leaving the proof of (ii) to the reader.

The 'if' direction is immediate, so we need to show that if M is a quasi-coherent sheaf on X, then for every étale morphism $V \to X$ with V a scheme the sheaf M_V is a quasi-coherent sheaf on V. Let $U \to X$ be as in Definition 7.1.5, and consider the fiber product

$$\begin{tikzcd} V \times_X U \arrow[r, "p_2"] \arrow[d, "p_1"] & U \arrow[d] \\ V \arrow[r] & X. \end{tikzcd}$$

We have

$$p_1^* M_V \simeq p_2^* M_U,$$

and therefore $p_1^* M_V$ is a quasi-coherent sheaf on $V \times_X U$. Since p_1 is an étale surjection, it follows from this and 4.3.12 that M_V is also a quasi-coherent sheaf. $\square$

7.1.8. If $f : X \to Y$ is a morphism of algebraic spaces, then there is an induced morphism of ringed topoi

$$f : X_{\text{et}} \to Y_{\text{et}}.$$

As in the case of schemes this is induced by the functor

$$\text{Et}(Y) \to \text{Et}(X), \quad (U \to Y) \mapsto (U \times_Y X \to X),$$

which is a continuous morphism of sites.

In particular, if M is an $\mathscr{O}_Y$-module, then we define (see exercise 2.K for the notation)

$$f^* M := f^{-1} M \otimes_{f^{-1}\mathscr{O}_Y} \mathscr{O}_X,$$

and if N is an $\mathscr{O}_X$-module, then we view $f_* N$ as an $\mathscr{O}_Y$-module via the map

$$\mathscr{O}_Y \to f_* \mathscr{O}_X.$$

PROPOSITION 7.1.9. *Let $f : X \to Y$ be a morphism of algebraic spaces.*

(i) If M is quasi-coherent on Y, then $f^* M$ is quasi-coherent on X.

(ii) If f is quasi-compact and quasi-separated, then for any quasi-coherent sheaf N on X the sheaf $f_* N$ is quasi-coherent on Y.

PROOF. For (i), let $U \to Y$ be an étale surjection with U a scheme, and let

$$V \to U \times_Y X$$

be an étale surjection with V a scheme, so we have the commutative diagram

$$\begin{tikzcd} V \arrow[r, "g"] \arrow[d] & U \arrow[d] \\ X \arrow[r, "f"] & Y, \end{tikzcd}$$

where the vertical arrows are étale surjections, and V and U are schemes. Then we have

$$(f^*M)_V = g^*M_U,$$

which reduces (i) to the usual case of schemes.

For (ii), note that the assertion is étale local on Y so we may assume that Y is an affine scheme. Choose an étale surjection

$$U \to X$$

with U an affine scheme (this is possible since f is quasi-compact) and set

$$R := U \times_X U.$$

Since f is quasi-separated, R is a quasi-compact scheme. Let

$$g : U \to Y, \quad h : R \to Y$$

be the maps defined by projecting to X and composing with f. Then

$$f_*N \simeq \mathrm{Ker}(g_*N_U \rightrightarrows h_*N_R),$$

which reduces the proof to the case of schemes, where the result is [**26**, I, 9.2.1]. $\square$

REMARK 7.1.10. By a similar argument one can show that the formation of f_*F commutes with flat base change on Y as in the case of schemes [**41**, III, 9.3].

PROPOSITION 7.1.11. *Let S be a quasi-compact scheme and let $f : X \to S$ be a quasi-compact and quasi-separated morphism of algebraic spaces with X locally noetherian. Then any quasi-coherent sheaf F is the union of its coherent subsheaves.*

PROOF. Let $u : U \to X$ be an étale surjection with U a finite disjoint union of affine schemes for which the structure morphism to S factors through an affine open in S.

First note that u is quasi-compact and quasi-separated. To see that the diagonal

$$\Delta_{U/X} : U \to U \times_X U$$

is quasi-compact, note that this is an open imbedding as it is a section of the étale morphism

$$\mathrm{pr}_1 : U \times_X U \to U.$$

Now the scheme $U \times_X U$ is quasi-compact (and hence noetherian) being the fiber product of the diagram

$$\begin{tikzcd} & U \times_S U \arrow[d] \\ X \arrow[r, "\Delta_X"] & X \times_S X, \end{tikzcd}$$

and $U \times_S U$ is a finite disjoint union of affine schemes by our choice of U, and Δ_X is quasi-compact. It follows that $\Delta_{U/X}$ is quasi-compact, and hence u is quasi-separated.

That the morphism $u : U \to X$ is quasi-compact follows from noting that for any affine scheme V and morphism $v : V \to X$ the scheme $V \times_X U$ is the fiber product of the diagram

$$\begin{array}{ccc} & & V \times_S U \\ & & \downarrow \\ X & \xrightarrow{\Delta_X} & X \times_S X \end{array}$$

and hence the projection $V \times_X U \to V$ is quasi-compact since Δ_X is quasi-compact.

Now let F be a quasi-coherent sheaf on X, and let F_U be its restriction to U so we have an inclusion

$$F \hookrightarrow u_* F_U.$$

Write F_U as the union of coherent subsheaves

$$F_U = \bigcup_{i \in I} M_i,$$

and let $F_i \subset F$ be the intersection in $u_* F_U$ of F and $u_* M_i$. The sheaf F_i is a quasi-coherent subsheaf of F being the kernel of the composite morphism

$$F \hookrightarrow u_* F_U \to u_*(F_U / M_i),$$

and we have $F = \bigcup_i F_i$. Finally, note that the sheaves F_i are coherent as the inclusion $F_{i,U} \hookrightarrow F_U$ obtained by restricting $F_i \hookrightarrow F$ to U factors through M_i. $\square$

7.2. Affine morphisms and Stein factorization

Definition 7.2.1. A morphism of algebraic spaces $f : X \to Y$ is *affine* if it is representable by schemes and satisfies the condition in 5.4.3 with P having the property of being an affine morphism.

7.2.2. Let X be an algebraic space. If $\mathscr{A}$ is a quasi-coherent sheaf of $\mathscr{O}_X$-algebras on X, define

$$\mathrm{Spec}_X(\mathscr{A}) : (\mathrm{Sch})^{\mathrm{op}} \to \mathrm{Set}$$

to be the functor which to any scheme T associates the set of pairs

$$(f : T \to X, \epsilon : f^* \mathscr{A} \to \mathscr{O}_T),$$

where f is a morphism and ϵ is a homomorphism of $\mathscr{O}_T$-algebras.

Proposition 7.2.3. *(i)* $\mathrm{Spec}_X(\mathscr{A})$ *is an algebraic space, and the structure morphism*

$$\pi : \mathrm{Spec}_X(\mathscr{A}) \to X, \quad (f, \epsilon) \mapsto f \tag{7.2.3.1}$$

is an affine morphism.

(ii) The functor

$$\mathrm{Spec}_X(-) : (\textit{quasi-coherent sheaves of algebras on } X)^{op} \to (\textit{affine morphisms } \pi : Y \to X)$$

is an equivalence of categories.

Proof. First we show that $\mathrm{Spec}_X(\mathscr{A})$ is an algebraic space. That $\mathrm{Spec}_X(\mathscr{A})$ is a sheaf for the étale topology follows from descent for morphisms of quasi-coherent sheaves; see 4.3.12.

To see that the diagonal of $\mathrm{Spec}_X(\mathscr{A})$ is representable, let T be a scheme and let

$$(f, \epsilon), (f', \epsilon') \in \mathrm{Spec}_X(\mathscr{A})(T)$$

be two sections over T. Let

$$\Sigma : (T\text{-schemes})^{\mathrm{op}} \to \mathrm{Set}$$

be the functor which to any morphism $g : T' \to T$ associates the unital set if the two compositions $f \circ g$ and $f' \circ g$ are equal and the diagram

$$\begin{tikzcd} g^*f^*\mathscr{A} \arrow[dd, equal] \arrow[dr, "g^*\epsilon"] & \\ & \mathscr{O}_{T'} \\ g^*f'^*\mathscr{A} \arrow[ur, "g^*\epsilon'"'] & \end{tikzcd}$$

commutes, and otherwise associates to g the empty set. We have to show that Σ is representable by a subscheme of T.

For this let Σ' be the functor which to any T-scheme $g : T' \to T$ associates the unital set if $f \circ g = f' \circ g$, and the empty set otherwise. Then Σ' is representable by a subscheme of T, since we have a cartesian diagram

$$\begin{tikzcd} \Sigma' \arrow[r] \arrow[d] & T \arrow[d] \\ X \arrow[r, "\Delta"] & X \times X. \end{tikzcd}$$

It therefore suffices to consider the case when $f = f'$.

In this case, consider the two maps

$$\epsilon, \epsilon' : f^*\mathscr{A} \to \mathscr{O}_T.$$

Let $\mathscr{J} \subset \mathscr{O}_T$ be the ideal given by the image of the difference

$$\epsilon - \epsilon' : f^*\mathscr{A} \to \mathscr{O}_T.$$

Then Σ is represented by the closed subscheme defined by $\mathscr{J}$. This completes the proof that the diagonal of $\mathrm{Spec}_X(\mathscr{A})$ is representable.

To see that $\mathrm{Spec}_X(\mathscr{A})$ is an algebraic space, it suffices to note that if $U \to X$ is an étale surjection with U a scheme, then we have a cartesian diagram of sheaves

$$\begin{tikzcd} \mathrm{Spec}_U(\mathscr{A}_U) \arrow[r] \arrow[d] & \mathrm{Spec}_X(\mathscr{A}) \arrow[d] \\ U \arrow[r] & X, \end{tikzcd}$$

where $\mathscr{A}_U$ denotes the restriction of $\mathscr{A}$ to U. In particular, since $\mathrm{Spec}_U(\mathscr{A}_U)$ is a scheme (by the case of schemes [**26**, II, 1.2.6 and 1.3.1]), this shows that $\mathrm{Spec}_X(\mathscr{A})$ admits an étale cover by a scheme, and hence is an algebraic space, and also this shows that the morphism (7.2.3.1) is affine. This completes the proof of (i).

As for (ii), note that there is a functor

$$G : (\text{affine morphisms } \pi : Y \to X) \to (\text{quasi-coherent sheaves of algebras on } X)^{\text{op}}$$

sending $\pi : Y \to X$ to $\pi_* \mathscr{O}_Y$. Notice that for an affine morphism $\pi : Y \to X$ (or for that matter any quasi-compact and quasi-separated morphism) and quasi-coherent sheaf of algebras $\mathscr{A}$ on X we have

$$\mathrm{Hom}_X(Y, \mathrm{Spec}_X(\mathscr{A})) \simeq \mathrm{Hom}_{\mathscr{O}_Y}(\pi^* \mathscr{A}, \mathscr{O}_Y) \simeq \mathrm{Hom}_{\mathscr{O}_X}(\mathscr{A}, \pi_* \mathscr{O}_Y),$$

where on the right we consider morphisms of quasi-coherent sheaves of algebras. In particular, $\mathrm{Spec}_X(-)$ is right adjoint to G. We claim that the adjunction maps

$$G \circ \mathrm{Spec}_X(-) \to \mathrm{id}, \quad \mathrm{id} \to \mathrm{Spec}_X(-) \circ G$$

are isomorphisms. This can be verified étale locally on X, so it suffices to consider the case when X is a scheme where the result is [**26**, II, 1.2.6 and 1.3.1]. This completes the proof of the proposition. □

Using the functor $\mathrm{Spec}_X(-)$ we can generalize several basic constructions to algebraic spaces:

7.2.4. Scheme-theoretic image. Let $f : X \to Y$ be a quasi-compact and quasi-separated morphism of algebraic spaces. The *scheme-theoretic image of* f is defined to be the relative spectrum

$$\mathrm{Spec}_Y(\mathscr{O}_Y / K),$$

where

$$K := \mathrm{Ker}(\mathscr{O}_Y \to f_* \mathscr{O}_X).$$

If f is a quasi-compact imbedding then the scheme theoretic image is also called the *scheme-theoretic closure.*

7.2.5. Maximal reduced subspace. If X is an algebraic space, let $\mathscr{N}_X \subset \mathscr{O}_X$ be the subsheaf of locally nilpotent sections of $\mathscr{O}_X$. This is a quasi-coherent sheaf as this can be verified étale locally on X. Define the *maximal reduced subspace of* X, denoted X_{red}, to be the space

$$X_{\text{red}} := \mathrm{Spec}_X(\mathscr{O}_X / \mathscr{N}_X).$$

Then X_{red} is a reduced algebraic space, and comes equipped with a closed imbedding

$$i : X_{\text{red}} \hookrightarrow X.$$

7.2.6. Closed subspaces and quasi-coherent sheaves of ideals. Just as in the case of schemes [**41**, p. 85], if X is an algebraic space we can consider the set of isomorphism classes of closed imbeddings of algebraic spaces

$$i : Z \hookrightarrow X.$$

We call such an isomorphism class a *closed subspace of* X. We have a map

$$F : \{\text{closed subspaces } i : Z \hookrightarrow X\} \to \{\text{quasi-coherent sheaves of ideals } \mathscr{J} \subset \mathscr{O}_X\}$$

sending

$$(i : Z \hookrightarrow X) \mapsto \mathrm{Ker}(\mathscr{O}_X \to i_* \mathscr{O}_Z).$$

This map is a bijection with inverse

$$\mathscr{J} \mapsto (\mathrm{Spec}_X(\mathscr{O}_X / \mathscr{J}) \hookrightarrow X).$$

7.2.7. Support of coherent sheaf. If X is a locally noetherian algebraic space, and M is a coherent sheaf on X, then we define the *support of* M to be the closed subspace

$$\mathrm{Supp}(M) \subset X$$

corresponding to the quasi-coherent sheaf of ideals $\mathscr{I}$ which to any étale $U \to X$ associates the set of sections $f \in \mathscr{O}_X(U)$ for which the multiplication map

$$\cdot f : M_U \to M_U$$

is zero. Note that to check that $\mathscr{I}$ is quasi-coherent we may work étale locally on X, so it suffices to consider the case when X is an affine scheme, where the result is [**41**, II, exercise 5.6].

7.2.8. Stein factorization. A quasi-compact and quasi-separated morphism of algebraic spaces $f : X \to Y$ is called *Stein* if the map

$$\mathscr{O}_Y \to f_*\mathscr{O}_X$$

is an isomorphism. In general, if $f : X \to Y$ is quasi-compact and quasi-separated then we have a factorization of f,

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[rr, bend left, "f"] & X' \arrow[r, "b"] & Y, \end{tikzcd}$$

where a is Stein and b is affine. This factorization is obtained by setting

$$X' := \mathrm{Spec}_Y(f_*\mathscr{O}_X),$$

and is called the *Stein factorization of* f. The morphism b is the map induced by the adjunction map

$$f^*f_*\mathscr{O}_X \to \mathscr{O}_X.$$

One particular application of the Stein factorization that we will use is Theorem 7.2.10. Note that the property that a morphism of schemes $f : X \to Y$ is locally quasi-finite is étale local on source and target, and therefore we can also speak of a morphism of algebraic spaces being locally quasi-finite.

We say that a morphism of algebraic spaces $f : X \to Y$ is *quasi-finite* if it is locally quasi-finite and of finite type.

Remark 7.2.9. The definition of a quasi-finite morphism in [**41**, Chapter II, exercise 3.5] does not include the finite type assumption. Here we follow [**26**, II.6.2.3], where the finite type condition is included.

Theorem 7.2.10 (see also [**49**, A.2] and [**67**, Tag 082J]). *Let $f : X \to Y$ be a separated and quasi-finite morphism between algebraic spaces. Let $\mathscr{A}$ denote the quasi-coherent sheaf of $\mathscr{O}_Y$-algebras $f_*\mathscr{O}_X$, and let*

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[rr, bend left, "f"] & Z := \mathrm{Spec}_Y(\mathscr{A}) \arrow[r, "h"] & Y \end{tikzcd}$$

be the Stein factorization of f. Then the morphism g is an open imbedding. In particular, f is quasi-affine.

Proof. In the case of schemes this is [**26**, Chapter IV, 18.12.12]. Note first that if $w : Y' \to Y$ is a flat morphism of algebraic spaces and $f' : X' := X \times_Y Y' \to Y'$ is the base change of f to Y', then the natural map $w^*\mathscr{A} \to f'_*\mathscr{O}_{X'}$ is an isomorphism (since pushforward of a quasi-coherent sheaf along a quasi-compact and quasi-separated morphism commutes with flat base change). It follows that if $Z' := Z \times_Y Y'$ then the resulting factorization of f',

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[rr, bend left, "f'"] & Z' \arrow[r, "h'"] & Y', \end{tikzcd}$$

is the Stein factorization of f'. It therefore suffices to prove the theorem after base changing to an étale cover of Y. We may therefore assume that Y is an affine scheme.

In this case, since the morphism f is quasi-compact, there exists an étale surjection $U \to X$ with U an affine scheme. The resulting projection $U \to Y$ is then a quasi-finite separated morphism locally of finite type between schemes.

We now show that the morphism $g : X \to Z$ is étale.

For this it suffices to show that for every point $y \in Y$, there exists an étale morphism $Y' \to Y$ and a point $y' \in Y'$ over y such that the induced morphism

$$U \times_Y Y' \to Z \times_Y Y'$$

is étale at every point of $U \times_Y Y'$ lying over y'.

By [**26**, Chapter IV, 18.12.3] there exists an étale morphism of affine schemes $Y' \to Y$ and a point $y' \in Y'$ lying over y such that the Y'-scheme $q' : U' := U \times_Y Y' \to Y'$ is equal to a disjoint union $U' = U'_1 \coprod U'_2$ with $U'_1 \to Y'$ a finite morphism and $U'_2 \cap q^{-1}(y') = \emptyset$.

Let $X'_1 \subset X' := X \times_Y Y'$ denote the open subspace given by the equivalence relation

$$U'_1 \times_{X'} U'_1 \rightrightarrows U'_1.$$

Lemma 7.2.11. *The algebraic space X'_1 is a scheme.*

Proof. Note that U'_1 is affine being finite over Y'. It therefore suffices to show that the two projections $U'_1 \times_{X'} U'_1 \to U'_1$ are finite, since the quotient of an affine scheme by a finite étale equivalence relation is the scheme 6.2.2. Since U'_1 is finite over Y', each of the projections $U'_1 \times_{Y'} U'_1 \to U'_1$ are finite, so it suffices to show that the natural map $U'_1 \times_{X'} U'_1 \to U'_1 \times_{Y'} U'_1$ is a closed imbedding. This is true because the diagram

$$\begin{tikzcd} U'_1 \times_{X'} U'_1 \arrow[r] \arrow[d] & U'_1 \times_{Y'} U'_1 \arrow[d] \\ X' \arrow[r, "\Delta"] & X' \times_{Y'} X' \end{tikzcd}$$

is cartesian and Δ is a closed imbedding since $X' \to Y'$ is separated. □

Lemma 7.2.12. *The map $X'_1 \hookrightarrow X'$ is an open and closed imbedding.*

Proof. By construction the inclusion $X'_1 \hookrightarrow X'$ is an open imbedding. It therefore suffices to show that for any Y'-scheme T and morphism $T \to X'$ the open subscheme $X_1' \times_{X'} T \subset T$ is also closed. This is true because it is the image of $U_1' \times_{X'} T \to T$ and this morphism is finite as it factors as

$$U_1' \times_{X'} T \xrightarrow{a} U_1' \times_{Y'} T \xrightarrow{b} T$$

where b is finite since $U_1' \to Y'$ is finite, and a is a closed imbedding since the diagram

$$\begin{tikzcd} U_1' \times_{X'} T \arrow[r] \arrow[d] & U_1 \times_{Y'} T \arrow[d] \\ X' \arrow[r] & X' \times_{Y'} X' \end{tikzcd}$$

is cartesian. □

It follows that $X' = X_1' \coprod X_2'$ where the fiber product $X_2' \times_{Y'} y' = \emptyset$. It also follows that if

$$X_i' \to Z_i' \to Y'$$

is the Stein factorization of $X_i' \to Y'$ then $Z' = Z_1' \coprod Z_2'$.

Since $X' \times_{Y'} y' \subset X_1'$ it therefore suffices to show that $X_1' \to Z_1'$ is an open imbedding, which is immediate. This completes the proof that $X \to Z$ is étale.

Lemma 7.2.13. *The morphism $X \to Z$ is separated, quasi-finite, and of finite presentation.*

Proof. That the diagonal of $X \to Z$ is closed follows from considering the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[rr, bend left, "a"] & X \times_Z X \arrow[r, "b"] \arrow[d] & X \times_Y X \arrow[d] \\ & Z \arrow[r, "c"] & Z \times_Y Z, \end{tikzcd}$$

where a is a closed imbedding since $X \to Y$ is separated, and c and hence also b are closed imbeddings since Z is affine (and hence separated) over Y.

For any point $z \in Z$ with image $h(z) \in Y$ the fiber product $U \times_Z z$ is a subscheme of $U \times_Y h(z)$. Since $X \to Y$ is quasi-finite, the topological space of the scheme $U \times_Y h(z)$ is a finite set with the discrete topology. It follows that the topological space $U \times_Z z$ is also a finite set with the discrete topology.

Since $X \to Z$ is étale it is a morphism locally of finite presentation. It is also a quasi-compact morphism since we have an étale surjection $U \to X$ with U quasi-compact. □

Now consider any point $z \in Z$ in the image of X (which is also the image of the composition $U \to X \to Z$). By the above argument, there exists an étale morphism $Z' \to Z$ and a point $z' \in Z'$ such that $X' := X \times_Z Z'$ is equal to a disjoint union $X' = X_1' \coprod X_2'$ with X_1' a scheme finite over Z'. If we further assume that Z' is connected then in fact $X' = X_1'$ since $X' \to Z'$ is Stein. But then the morphism $X' \to Z'$ is a finite Stein morphism, hence an isomorphism.

Lemma 7.2.14. *The map $g : X \to Z$ is a monomorphism.*

PROOF. By the above argument there exists a faithfully flat morphism $Z' \to Z$ such that the projection $X \times_Z Z' \to Z'$ is an isomorphism.

Now suppose $a, b : T \to X$ are two morphisms from a scheme T such that $ga = gb$. We need to show that the natural map

$$T \times_{a\times b, X\times_Z X, \Delta} X \to T$$

is an isomorphism. This can be verified after making the faithfully flat base change $Z' \to Z$, where the result is clear. □

So we now know that $g : X \to Z$ is an étale monomorphism. Let $W \subset Z$ be the open image of g. Then W is a scheme, and admits a presentation as an algebraic space as

$$U \times_W U \rightrightarrows U.$$

It therefore suffices to show that the natural map

$$U \times_X U \to U \times_W U = U \times_Z U$$

is an isomorphism. This is clear because $X \to Z$ is a monomorphism. This completes the proof of the theorem. □

EXAMPLE 7.2.15. Some care has to be taken in using the factorization in 7.2.10, as Z may not enjoy all the good properties of X and Y. For example, if $f : X \to Y$ is a quasi-finite morphism with Y affine and noetherian, the ring $H^0(X, \mathscr{O}_X)$ need not be noetherian. The following example is due to Osserman, de Jong, Conrad, and Vakil.

Let k be a field and let E/k be an elliptic curve. Let N be a degree 0 nontorsion invertible sheaf on E, and let P be an invertible sheaf on E of degree ≥ 3. Then

$$H^0(E, N^{\otimes m} \otimes P^{\otimes n}) \neq 0$$

if and only if one of the following holds:

(i) $n > 0$,
(ii) $m = n = 0$.

Set

$$R := \bigoplus_{m,n\geq 0} H^0(E, N^{\otimes m} \otimes P^{\otimes n}),$$

and let $\mathfrak{p} \subset R$ be the maximal ideal. Then $\mathfrak{p}$ is not finitely generated, so R is not noetherian.

Let Z be the total space of the bundle $N^\wedge \bigoplus P^\wedge$ so

$$Z = \mathrm{Spec}_E(\mathrm{Sym}^\bullet_E(N \oplus P)).$$

Then

$$\Gamma(Z, \mathscr{O}_Z) = R.$$

Let $T \to E$ be the $\mathbb{G}_m$-torsor corresponding to $P^\wedge$, so

$$T = \mathrm{Spec}_E(\bigoplus_{n\in\mathbb{Z}} P^{\otimes n}),$$

where the algebra structure is given by the natural isomorphisms

$$P^{\otimes n} \otimes P^{\otimes m} \to P^{\otimes n+m}.$$

Let $\mathbb{V}(N)$ (resp. $\mathbb{V}(P)$) denote the E-scheme

$$\mathbb{V}(N) = \mathrm{Spec}_E(\mathrm{Sym}^\bullet N) \quad (\text{resp. } \mathbb{V}(P) = \mathrm{Spec}_E(\mathrm{Sym}^\bullet P)),$$

so we have a cartesian diagram

$$\begin{tikzcd} Z \arrow[r, "a"] \arrow[d, "b"] & \mathbb{V}(P) \arrow[d] \\ \mathbb{V}(N) \arrow[r] & E, \end{tikzcd}$$

and let

$$j : T \hookrightarrow \mathbb{V}(P)$$

be the natural closed imbedding. Set

$$X := a^{-1}(T) \subset E$$

so

$$X \simeq \mathbb{V}(N) \times_E T \simeq \mathrm{Spec}_E(\bigoplus_{m \in \mathbb{N}, n \in \mathbb{Z}} N^{\otimes m} \otimes P^{\otimes n}).$$

Then again we have

$$\Gamma(X, \mathscr{O}_X) = R.$$

We claim that X is quasi-affine. Let

$$A = \mathrm{Spec}(\bigoplus_{n \geq 0} \Gamma(E, P^{\otimes n})),$$

so

$$T = A - \{0\}.$$

Let M be any extension to a coherent sheaf on A of the pullback of N to T. Then X is an open subset of

$$\mathrm{Spec}_A(\mathrm{Sym}^\bullet M)$$

which is an affine scheme.

7.3. Nilpotent thickenings of schemes

THEOREM 7.3.1 ([**45**, III.3.6]). *Let X be an algebraic space, and suppose there exists a nilpotent quasi-coherent sheaf of ideals $\mathscr{J} \subset \mathscr{O}_X$ such that*

$$X' := \mathrm{Spec}_X(\mathscr{O}_X/\mathscr{J})$$

is a scheme. Then X is also a scheme.

PROOF. Note first that any open subset $U' \subset X'$ lifts uniquely to an open subspace $U \subset X$. Indeed, if $V \to X$ is an étale surjective morphism, then giving an open subspace of X is equivalent to giving an open subspace $W \subset V$ such that the preimages under the two projections $V \times_X V \to V$ agree, and similarly open subsets of X' are given by open subsets of $V' := V \times_X X'$ whose two preimages in $V' \times_{X'} V'$ agree. From this and the fact that the maps of spaces

$$|V'| \to |V|, \quad |V' \times_{X'} V'| \to |V \times_X V|$$

are homeomorphisms, we get the claim.

We can therefore cover X' by open subspaces $U \subset X$ such that $X' \times_X U$ is an affine scheme. It suffices to prove that each U is also a scheme, which reduces the proof to the case when X' is an affine scheme.

Furthermore, if $\mathscr{J}^n = 0$ we have a sequence of closed imbeddings

$$X' \hookrightarrow \mathrm{Spec}_X(\mathscr{O}_X/\mathscr{J}^2) \hookrightarrow \mathrm{Spec}_X(\mathscr{O}_X/\mathscr{J}^3) \hookrightarrow \cdots \hookrightarrow \mathrm{Spec}_X(\mathscr{O}_X/\mathscr{J}^n) = X.$$

We may therefore assume that $\mathscr{J}^2 = 0$. In this case, the sheaf $\mathscr{J}$ is isomorphic to $i_* i^* \mathscr{J}$ where $i : X' \hookrightarrow X$ is the inclusion. Since X' is affine we therefore have an exact sequence

$$0 \to \Gamma(X, \mathscr{J}) \to \Gamma(X, \mathscr{O}_X) \to \Gamma(X', \mathscr{O}_{X'}) \to 0.$$

Let A denote $\Gamma(X, \mathscr{O}_X)$, and let A' denote $\Gamma(X', \mathscr{O}_{X'})$, so we have a morphism

$$\rho : X \to \mathrm{Spec}(A)$$

which we claim is an isomorphism. If $V \to X$ is an étale morphism with V a scheme, then $V' := V \times_X X'$ is an étale X'-scheme, which since $A \to A'$ is surjective with nilpotent kernel lifts uniquely to an étale A-scheme $V_A \to \mathrm{Spec}(A)$. Moreover, we have a commutative diagram of A-schemes

$$\begin{tikzcd} V' \arrow[r] \arrow[d] & V \arrow[dl, dashed] \\ V_A & \end{tikzcd}$$

which since V_A is étale over $\mathrm{Spec}(A)$ implies that there exists a unique morphism $V \to V_A \times_{\mathrm{Spec}(A)} X$ over X. This morphism is an isomorphism. Indeed, to verify this we may make an étale base change $Y \to X$ with Y a scheme where it follows from the fact that the morphism becomes an isomorphism over $Y \times_X X'$.

In fact the map $V \to V_A$ is an isomorphism. To verify this we may assume that V is affine, in which case it follows from the fact that we have a commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & \Gamma(V', \mathscr{J}) \arrow[r] \arrow[d, equal] & \Gamma(V_A, \mathscr{O}_{V_A}) \arrow[r] \arrow[d] & \Gamma(V', \mathscr{O}_{V'}) \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \Gamma(V', \mathscr{J}) \arrow[r] & \Gamma(V, \mathscr{O}_V) \arrow[r] & \Gamma(V', \mathscr{O}_{V'}) \arrow[r] & 0. \end{tikzcd}$$

It follows that if $V \to X$ is étale then the composition $V \to X \to \mathrm{Spec}(A)$ is étale, and $V \times_X V \to V \times_{\mathrm{Spec}(A)} V$ is an isomorphism. In particular, if $V \to X$ is an étale surjection, and $R = V \times_X V$ the resulting étale equivalence relation then we get that $\mathrm{Spec}(A) = V/R$ whence ρ is an isomorphism. □

Remark 7.3.2. More generally, Theorem 7.3.1 holds when $\mathscr{I}$ is just a nil-ideal. This is shown in [**21**, 3.1.12].

7.4. Chow's lemma for algebraic spaces

Theorem 7.4.1 ([**45**, IV.3.1]). *Let S be a noetherian scheme, and let $f : X \to S$ be a separated morphism of finite type with X a reduced algebraic space. Then there exists a proper birational morphism $X' \to X$ with X' a quasi-projective S-scheme.*

Proof. Since $X \to S$ is separated the diagonal of X over S is quasi-compact. By 6.4.1 we can therefore find a dense open subspace $j : U \subset X$ with U a scheme.

Lemma 7.4.2. *Let $Y \to S$ be a finite type separated morphism of schemes. There exists a dense open subset $V \subset Y$, such that V is a finite disjoint union of affine open subschemes $V_i \subset Y$ with each V_i mapping to an affine open subset in S. In particular there exists a dense open subset $V \subset Y$ and an imbedding $V \hookrightarrow \mathbb{P}^n_S$ over S for some n.*

PROOF. Let $Y = Y_1 \cup \cdots \cup Y_r$ be the irreducible components of Y (which exists since Y is noetherian being of finite type over the noetherian S) and let $Y_i' \subset Y_i$ denote the complement in Y_i of the intersections $Y_i \cap Y_j$ for $j \neq i$. Then if Y' denotes the disjoint union of the Y_i' we have a dense open imbedding $Y' \hookrightarrow Y$. It therefore suffices to prove the lemma for Y', and for this in turn it suffices to prove the result for each Y_i'. This reduces the proof to the case when Y is irreducible. In this case let $W \subset S$ be an affine open subset meeting the image of Y and set $V := f^{-1}(W)$. $\square$

Applying 7.4.2 to the morphism $U \to S$, we see that after shrinking on U we may assume that U is a quasi-projective S-scheme. Choose an imbedding

$$i : U \hookrightarrow \mathbb{P}^n_S,$$

and let

$$X' \hookrightarrow X \times_S \mathbb{P}^n_S$$

be the scheme-theoretic closure of

$$j \times i : U \to X \times_S \mathbb{P}^n_S.$$

Also, let

$$\overline{U} \subset \mathbb{P}^n_S$$

be the scheme-theoretic closure of $i(U)$. We then have a morphism

$$\pi : X' \to \overline{U},$$

and the morphism $X' \to X$ factors through $\mathbb{P}^n_X$ and restricts to an isomorphism over U. It therefore suffices to prove the theorem for X', so we may assume that there exists a morphism

$$\pi : X \to P$$

from X to a reduced projective S-scheme P which is an isomorphism over a dense open subset of P.

Let $U \to X$ be an étale covering by a quasi-compact separated scheme. By Chow's lemma [**26**, II, 5.6.1] there exists a proper birational morphism $U' \to U$ such that the composition $U' \to P$ is quasi-projective. By 1.5.6, there exists a blowup $P' \to P$ along a nowhere dense closed subscheme of P such that the strict transform V' of U' is flat over P' and, in particular, V' is quasi-finite over P'. This implies that the strict transform $Y \to P'$ of X is also quasi-finite, since the map $V' \to Y$ is surjective. By [**26**, IV, 18.12.12], the morphism $Y \to P'$ is quasi-affine and, in particular, Y is a quasi-projective scheme. The projection $Y \to X$ is then the desired proper birational morphism. $\square$

7.5. Finiteness of cohomology

Let X be an algebraic space. The category Mod_X of $\mathscr{O}_X$-module on the étale site of X is an abelian category with enough injectives. For a morphism of algebraic spaces $f : X \to Y$ we therefore have derived functors

$$R^q f_* : \mathrm{Mod}_X \to \mathrm{Mod}_Y.$$

THEOREM 7.5.1. *Let $f : X \to Y$ be a proper morphism of locally noetherian algebraic spaces, and let F be a coherent sheaf on X. Then $R^q f_* F$ is a coherent sheaf on Y for all q.*

The proof follows [**45**, IV.4.1] and [**26**, III.3.2.1].

LEMMA 7.5.2 (Dévissage). *Let S be an affine scheme and let $f : X \to S$ be a quasi-compact and quasi-separated morphism of algebraic spaces with X locally noetherian. Let $\mathscr{A}$ denote the category of coherent sheaves on X. Let $\mathscr{A}' \subset \mathscr{A}$ be a full subcategory such that the following hold:*

(i) *The zero sheaf is in $\mathscr{A}'$.*

(ii) *For every exact sequence in $\mathscr{A}$,*

$$0 \to A' \to A \to A'' \to 0,$$

if two of A, A', and A'' are in $\mathscr{A}'$ then so is the third.

(iii) *For every integral closed subspace $Z \hookrightarrow X$ there exists a sheaf $G \in \mathscr{A}'$ with $Supp(G) = Z$.*

(iv) *If A and A' are in $\mathscr{A}$ and $A \oplus A' \in \mathscr{A}'$ then $A, A' \in \mathscr{A}'$.*

Then $\mathscr{A}' = \mathscr{A}$.

PROOF. By noetherian induction applied to the topological space $|X|$ (which is noetherian by exercise 6.L), we may assume that any coherent sheaf supported on a strictly smaller closed subspace of X is in $\mathscr{A}'$. It therefore suffices to prove that for any coherent sheaf F there exists a morphism

$$u : F \to G \tag{7.5.2.1}$$

with $G \in \mathscr{A}'$ and the kernel and cokernel of u supported on a strictly smaller closed subspace of X.

By 6.4.1 there exists a dense open subspace $j : U \hookrightarrow X$ with U a scheme. Let $Z_U \subset U$ be an irreducible component of U with the reduced subscheme structure, and let $i : Z \subset X$ be its scheme-theoretic closure. Then Z is an integral closed subspace of X, and so by assumption there exists a coherent sheaf $G \in \mathscr{A}'$ whose support is Z.

To construct the morphism (7.5.2.1), by composing with the adjunction map

$$F \to i_* i^* F,$$

we can replace F by $i_* i^* F$, and so it suffices to consider the case when $Z = X$.

Now if X is integral, there exists a dense open subspace (even subscheme) $v : V \hookrightarrow X$ such that $F|_V$ is isomorphic to a finite direct sum of copies of $\mathscr{O}_V$. In particular, we can find an isomorphism

$$F^t|_V \to G^r|_V$$

for some r and t. By assumption (iv) we may replace F by F^t, and therefore we may assume that we have an isomorphism $h : F|_V \to G^r|_V$. This induces a map

$$F \to v_* G^r|_V.$$

Since $v_*(G^r)|_V$ is equal to the union of its coherent subsheaves there exists a coherent subsheaf $M \subset v_* G^r|_V$ containing the image of G^r and such that the the map $F \to v_* G^r|_V$ factors through M. Since $G^r \in \mathscr{A}'$ and the kernel and cokernel of the map

$$G^r \to M$$

are supported in strictly smaller closed subspaces, we also have $M \in \mathscr{A}'$, so the map

$$F \to M$$

is the desired map. □

7.5.3. Turning to the proof of 7.5.1, note first of all that the assertion is étale local on Y, so we may assume that $Y = \mathrm{Spec}(R)$ for some ring R. In this case it suffices to show that for any coherent sheaf F on X the following two conditions hold:

A. The R-module $H^q(X, F)$ is finitely generated.
B. For any étale ring homomorphism $R \to R'$ the map

$$H^q(X, F) \otimes_R R' \to H^q(X_{R'}, F)$$

is an isomorphism, where $X_{R'}$ denotes the base change of X to $\mathrm{Spec}(R')$.

Let $\mathscr{A}$ denote the category of coherent sheaves on X, and let $\mathscr{A}' \subset \mathscr{A}$ denote the full subcategory of coherent sheaves for which conditions A and B hold. We show that the $\mathscr{A}'$ satisfies the conditions in 7.5.2, thereby showing that $\mathscr{A}' = \mathscr{A}$.

By noetherian induction we may assume that any coherent sheaf supported on a strictly smaller closed subspace of X is in $\mathscr{A}'$.

Condition (i) in 7.5.2 is immediate.

For condition (ii), note that given an exact sequence in $\mathscr{A}$,

$$0 \to A' \to A \to A'' \to 0,$$

we get a long exact sequence

$$\cdots \to H^q(X, A') \to H^q(X, A) \to H^q(X, A'') \to H^{q+1}(X, A') \to \cdots,$$

and for any ring homomorphism $R \to R'$ we get a morphism of long exact sequences

$$\begin{tikzcd}
\cdots \arrow[r] & H^q(X, A') \otimes R' \arrow[r] \arrow[d] & H^q(X, A) \otimes R' \arrow[r] \arrow[d] & H^q(X, A'') \otimes R' \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & H^q(X_{R'}, A') \arrow[r] & H^q(X_{R'}, A) \arrow[r] & H^q(X_{R'}, A'') \arrow[r] & \cdots .
\end{tikzcd}$$

From this and the five lemma it follows that condition (ii) in 7.5.2 also holds.

To verify condition (iii), choose a proper birational morphism $\pi : Y \to X$ with Y a projective R-scheme and consider the coherent (since π is projective) sheaf $\pi_* \mathscr{O}_Y$. We have a spectral sequence of R-modules

$$E_2^{pq} = H^p(X, R^q \pi_* \mathscr{O}_Y) \implies H^{p+q}(Y, \mathscr{O}_Y),$$

and for every $q > 0$ the coherent sheaf $R^q \pi_* \mathscr{O}_Y$ is supported on a strictly smaller closed subspace of X and is therefore in $\mathscr{A}'$. Also, each of the $H^{p+q}(Y, \mathscr{O}_Y)$ are finitely generated R-modules and commute with étale base change $R \to R'$. We claim that these facts imply that the $H^p(X, G)$ also are finitely generated and commute with étale base change.

To see this note first of all that all the E_r^{pq} are finitely generated for $q > 0$ as E_r^{pq} is a subquotient of E_{r-1}^{pq} and hence finitely generated if E_{r-1}^{pq} is finitely generated. Moreover, since E_r^{p0} $(r \geq 3)$ is the kernel of a map

$$E_{r-1}^{p0} \to E_{r-1}^{p+r-1, p+r-2}$$

we see that E_r^{p0} is finitely generated if and only if E_{r-1}^{p0} is finitely generated. We conclude that $H^p(X, G)$ is finitely generated if and only if E_∞^{p0} is finitely generated, and this in turn is equivalent to $H^p(Y, \mathscr{O}_Y)$ being finitely generated since this R-module admits a finite filtration all of whose graded pieces are finitely generated except possibly for the term E_∞^{p0}.

By a similar argument the formation of $H^p(X,G)$ commutes with étale base change if and only if the E_∞^{p0}-terms commute with étale base change, and this in turn is equivalent to the formation of $H^p(Y,\mathscr{O}_Y)$ commuting with étale base change.

This completes the proof of 7.5.1. $\square$

7.6. Exercises

Exercise 7.A. Let T be a topos and $A \to B$ a morphism of rings in T. For a B-module M, an *A-linear derivation* $B \to M$ is a map of A-modules $\partial : B \to M$ such that the diagram

$$\begin{tikzcd} B \times B \arrow[r, "\text{multiplication}"] \arrow[d, "\rho"] & B \arrow[d, "\partial"] \\ (B \times M) \times (B \times M) \arrow[r, "\gamma"] & M \end{tikzcd}$$

commutes, where ρ is the map given on local sections by

$$(b, b') \mapsto (b, \partial(b')) \times (b', \partial(b))$$

and γ is the map given on local sections by

$$(b, m) \times (b', m') \mapsto bm + b'm'.$$

(a) For any $X \in T$ let $\mathscr{D}er_A(B, M)(X)$ denote the $B(X)$-module of $A|_{T|_X}$-linear derivations $B|_{T|_X} \to M|_{T|_X}$ in the topos $T|_X$. Show that $\mathscr{D}er_A(B,M)$ defines a B-module (which we also denote by $\mathscr{D}er_A(B,M)$) in T.

(b) Show that there exists a unique B-module $\Omega^1_{B/A}$ with an A-linear derivation $d : B \to \Omega^1_{B/A}$ such that for any B-module M the map

$$\mathscr{H}om_B(\Omega^1_{B/A}, M) \to \mathscr{D}er_A(B, M), \quad \varphi \mapsto \varphi \circ d$$

is an isomorphism.

(c) Let $f : X \to Y$ be a morphism of schemes. Taking T to be the Zariski topos of X, $A = f^{-1}\mathscr{O}_Y$, $B = \mathscr{O}_X$, show that $\Omega^1_{\mathscr{O}_X/f^{-1}(\mathscr{O}_Y)}$, defined as in (b), is isomorphic to the usual differentials of the morphism f as defined in [**41**, II, §8].

(d) Repeat part (c) with the Zariski topology replaced by the étale topology.

(e) Let $f : X \to Y$ be a morphism of algebraic spaces. Let T denote the étale topos of X, $A = f^{-1}\mathscr{O}_Y$, $B = \mathscr{O}_X$, and let $\Omega^1_{X/Y}$ denote the sheaf of $\mathscr{O}_X$-modules $\Omega^1_{\mathscr{O}_X/f^{-1}\mathscr{O}_Y}$. Show that $\Omega^1_{X/Y}$ is a quasi-coherent sheaf on X.

(f) Let $f : X \to Y$ be a morphism of algebraic spaces. Show that for any commutative diagram of algebraic spaces

$$\begin{tikzcd} T_0 \arrow[r, "x"] \arrow[d, hook] & X \arrow[d] \\ T \arrow[r, "y"] \arrow[ur, dashed] & Y, \end{tikzcd}$$

if there exists a dotted arrow filling in the diagram then the set of such dotted arrows form a torsor under $\mathrm{Hom}_{\mathscr{O}_{T_0}}(x^*\Omega^1_{X/Y}, J)$.

Exercise 7.B. Continuing with the preceding exercise, we verify in this exercise some more basic properties of differentials.

(a) Let

$$\begin{tikzcd} X' \arrow[r, "g"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r, "h"] & Y \end{tikzcd}$$

be a commutative diagram of algebraic spaces. Show that there is a natural map $g^*\Omega^1_{X/Y} \to \Omega^1_{X'/Y'}$ which is an isomorphism if the square is cartesian.

(b) Show that for a diagram of algebraic spaces

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

there is an exact sequence

$$f^*\Omega^1_{Y/Z} \to \Omega^1_{X/Z} \longrightarrow \Omega^1_{X/Y} \longrightarrow 0.$$

generalizing the usual sequence for schemes [**41**, II, 8.11].

(c) Let

$$\begin{tikzcd} X' \arrow[r, "g"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r, "h"] & Y \end{tikzcd}$$

be a cartesian square of algebraic spaces. Show that the map

$$f'^*\Omega^1_{Y'/Y} \oplus g^*\Omega^1_{X/Y} \to \Omega^1_{X'/Y}$$

is an isomorphism.

(d) Let $f : X \to Y$ be a quasi-separated morphism of algebraic spaces. Show that the kernel J of the morphism induced by the diagonal

$$\mathscr{O}_{X \times_Y X} \to \Delta_* \mathscr{O}_X$$

is a quasi-coherent sheaf on $X \times_Y X$ and that $\Omega^1_{X/Y} \simeq \Delta^* J$.

Exercise 7.C. Let X be an algebraic space, and let $v : V \to X$ be an étale morphism.

(a) Show that the restriction functor

$$v^* : (\text{Sheaves of } \mathscr{O}_X\text{-modules in } X_{\text{et}}) \to (\text{Sheaves of } \mathscr{O}_V\text{-modules in } V_{\text{et}})$$

has a left adjoint $v_!$.

(b) Using the construction in (a) give an example of an algebraic space X and a sheaf of $\mathscr{O}_X$-modules F such that F is not quasi-coherent but its restriction to the Zariski site of X is a quasi-coherent sheaf.

Exercise 7.D. Let k be a field and let X/k be a separated smooth algebraic space of dimension 1. Show that X is a scheme (i.e., for smooth curves we get no added generality by considering algebraic spaces).

Exercise 7.E. Let S be a noetherian scheme, and let X/S be a separated algebraic space of finite type. Show that there exists an integer N such that for any coherent sheaf F on X we have $H^i(X, F) = 0$ for $i > N$.

CHAPTER 8

Algebraic stacks: Definitions and basic properties

Having developed the theory of algebraic spaces, we now turn to algebraic stacks. From the point of view of groupoids, an algebraic stack is a stack for the étale topology of the form $[X_0/X_1]$ (notation as in 4.6.7) for a groupoid in algebraic spaces $s, t : X_1 \rightrightarrows X_0$ with s and t smooth. It is preferable, however, to have a definition that does not involve a groupoid presentation and we formulate such an (equivalent) definition. An algebraic stack which can be described by a groupoid presentation with s and t étale is called a *Deligne-Mumford stack*. In this chapter we discuss the basic definitions for algebraic stacks. In addition, we prove the very useful characterization of Deligne-Mumford stacks as algebraic stacks whose objects have no infinitesimal automorphisms (Theorem 8.3.3). We also discuss the stack $\mathscr{M}_g$ classifying genus g curves, which will be discussed in further detail in Chapter 13.

Throughout this chapter S denotes a fixed base scheme and we work over the category of S-schemes with the étale topology. By *stack* we mean a stack in the sense of 4.6.1 over the category of S-schemes with the étale topology. As in the case of algebraic spaces 5.1.16, the reader who so desires can take $S = \mathrm{Spec}(\mathbb{Z})$ throughout to define the absolute notion of algebraic stack, and then consider pairs (X, f) where X is an algebraic stack and $f : X \to S$ is a morphism to get the notion of an algebraic stack over S, instead of introducing the notion of an algebraic stack over S as in Definition 8.1.4.

The sources for the material in this chapter are [**9**], [**23**], and [**49**].

8.1. Definition of algebraic stack and fiber products

Definition 8.1.1. A morphism of stacks $f : \mathscr{X} \to \mathscr{Y}$ is *representable* if for every scheme U and morphism $y : U \to \mathscr{Y}$ the fiber product

$$\mathscr{X} \times_{\mathscr{Y}, y} U$$

is an algebraic space.

Remark 8.1.2. Note the distinction between a representable morphism and a morphism representable by schemes 5.1.5. The terminology is justified by the following lemma.

Lemma 8.1.3. *If a morphism of stacks $f : \mathscr{X} \to \mathscr{Y}$ is representable, then for every algebraic space V and morphism $y : V \to \mathscr{Y}$ the fiber product $V \times_{\mathscr{Y}} \mathscr{X}$ is an algebraic space.*

Proof. Indeed, the fiber product $V \times_{\mathscr{Y}} \mathscr{X}$ is equivalent to a sheaf since the condition that the objects of this stack have no nontrivial automorphisms can be verified after making an étale base change $U \to V$ with U a scheme. The result therefore follows from exercise 5.G applied to $V \times_{\mathscr{Y}} \mathscr{X} \to V$. □

DEFINITION 8.1.4. A stack $\mathscr{X}/S$ is an *algebraic stack* if the following hold:

(i) The diagonal
$$\Delta : \mathscr{X} \to \mathscr{X} \times_S \mathscr{X}$$
is representable.

(ii) There exists a smooth surjective morphism $\pi : X \to \mathscr{X}$ with X a scheme.

A *morphism of algebraic stacks* $f : \mathscr{X} \to \mathscr{Y}$ is a morphism of stacks.

REMARK 8.1.5. Being an algebraic stack is a property of fibered categories over the category of S-schemes. Morphisms of algebraic stacks are defined to be morphisms of fibered categories. In particular, for two algebraic stacks $\mathscr{X}$ and $\mathscr{Y}$ over S we have a category of morphisms $HOM_S(\mathscr{X}, \mathscr{Y})$.

REMARK 8.1.6. Note that condition (i) implies that every morphism $t : T \to \mathscr{X}$, with T a scheme, is representable. Indeed, if $u : U \to \mathscr{X}$ is another morphism from a scheme, then the fiber product
$$U \times_{u,\mathscr{X},t} T$$
is isomorphic to the fiber product of the diagram
$$\begin{tikzcd} & U \times_S T \arrow[d, "u\times t"] \\ \mathscr{X} \arrow[r, "\Delta"] & \mathscr{X} \times_S \mathscr{X} \end{tikzcd}$$
and therefore the fiber product is an algebraic space by (i). It therefore makes sense to talk about a smooth surjective morphism $X \to \mathscr{X}$ as in (ii).

REMARK 8.1.7. At times in the literature what we call an algebraic stack here is referred to as an *Artin stack.*

LEMMA 8.1.8. *Let $\mathscr{X}/S$ be a stack over S. The diagonal $\Delta : \mathscr{X} \to \mathscr{X} \times_S \mathscr{X}$ is representable if and only if for every S-scheme U and two objects $u_1, u_2 \in \mathscr{X}(U)$ the sheaf $\underline{\mathrm{Isom}}(u_1, u_2)$ on* (Sch$/U$) *defined in* 3.4.7 *is an algebraic space.*

PROOF. This is immediate from the definition and noting that we have a cartesian square
$$\begin{tikzcd} \underline{\mathrm{Isom}}(u_1, u_2) \arrow[r] \arrow[d] & U \arrow[d, "u_1\times u_2"] \\ \mathscr{X} \arrow[r] & \mathscr{X} \times_S \mathscr{X}. \end{tikzcd}$$
□

REMARK 8.1.9. Let $\mathscr{X}/S$ be a stack over S and let X/S be an algebraic space. For two morphisms $u_1, u_2 : X \to \mathscr{X}$ we define the sheaf $\underline{\mathrm{Isom}}(u_1, u_2)$ on (Sch$/X$) as in 3.4.7. It follows from exercise 5.G that this sheaf $\underline{\mathrm{Isom}}(u_1, u_2)$ is an algebraic space if and only if there exists an étale covering $U \to X$ with U a scheme such that the pullback of $\underline{\mathrm{Isom}}(u_1, u_2)$ to U is an algebraic space. From this and 8.1.8 it follows that the diagonal $\Delta : \mathscr{X} \to \mathscr{X} \times_S \mathscr{X}$ is representable if and only if for every algebraic space X and morphisms $u_1, u_2 : X \to \mathscr{X}$ the sheaf $\underline{\mathrm{Isom}}(u_1, u_2)$ is an algebraic space.

PROPOSITION 8.1.10. *Let $\mathscr{X}/S$ be an algebraic stack. Then for any diagram*

$$\begin{tikzcd} & X \arrow[d, "x"] \\ Y \arrow[r, "y"] & \mathscr{X}, \end{tikzcd}$$

with X and Y algebraic spaces, the fiber product $X \times_{\mathscr{X}} Y$ is an algebraic space. In particular, any morphism $x : X \to \mathscr{X}$ from an algebraic space X to $\mathscr{X}$ is representable.

PROOF. Indeed, the fiber product $X \times_{\mathscr{X}} Y$ is isomorphic to the sheaf $\underline{\mathrm{Isom}}(\mathrm{pr}_1^* x, \mathrm{pr}_2^* y)$ over $X \times_S Y$, which is an algebraic space by Remark 8.1.9. □

REMARK 8.1.11. Condition (ii) in 8.1.4 can be replaced with the condition that there exists a smooth surjective morphism $\pi : X \to \mathscr{X}$ with X an algebraic space (this makes sense by 8.1.10).

EXAMPLE 8.1.12. Let X be an algebraic space and let G/S be a smooth group scheme which acts on X. Define $[X/G]$ to be the stack whose objects are triples $(T, \mathscr{P}, \pi)$, where

(i) T is an S-scheme.
(ii) $\mathscr{P}$ is a $G_T := G \times_S T$-torsor on the big étale site of T (see section 4.5).
(iii) $\pi : \mathscr{P} \to X \times_S T$ is a G_T-equivariant morphism of sheaves on (Sch/T).

Here we write G_T and $X \times_S T$ both for the schemes over T and the corresponding representable sheaves on (Sch/T). A morphism

$$(T', \mathscr{P}', \pi') \to (T, \mathscr{P}, \pi)$$

is a pair (f, f^b), where $f : T' \to T$ is an S-morphism of schemes and $f^b : \mathscr{P}' \to f^*\mathscr{P}$ is an isomorphism of $G_{T'}$-torsors on (Sch/T') such that the induced diagram

$$\begin{tikzcd} \mathscr{P}' \arrow[rr, "f^b"] \arrow[dr, "\pi'"'] & & f^*\mathscr{P} \arrow[dl, "f^*\pi"] \\ & X \times_S T' & \end{tikzcd}$$

commutes. Descent for sheaves 4.2.12 implies that $[X/G]$ is a stack. In fact, $[X/G]$ is an algebraic stack as can be seen as follows.

To verify the representability of the diagonal of $[X/G]$ (using 8.1.8), let T be an S-scheme and let $(\mathscr{P}_i, \pi_i)$ $(i = 1, 2)$ be two objects of $[X/G]$ over T. Let $I := \underline{\mathrm{Isom}}((\mathscr{P}_1, \pi_1), (\mathscr{P}_2, \pi_2))$ denote the sheaf on (Sch/T) which to any T'/T associates the set of isomorphisms of $G_{T'}$-torsors $\mathscr{P}_1|_{T'} \to \mathscr{P}_2|_{T'}$ compatible with the maps to X. To verify that this sheaf is an algebraic space, we may by exercise 5.G replace T by an étale covering and hence may assume that $\mathscr{P}_1$ and $\mathscr{P}_2$ are both trivial torsors. Fix isomorphisms of torsors $\sigma_i : \mathscr{P}_i \to G_T$, so that the maps π_i are given by morphisms of schemes

$$\rho_i : G_T \to X_T.$$

In this case the sheaf I is identified with the sheaf which to any T'/T associates the set of elements $g \in G(T')$ such that the induced diagram

$$\begin{tikzcd} G_{T'} \arrow[rr, "m_g"] \arrow[dr, "\rho_1"'] & & G_{T'} \arrow[dl, "\rho_2"] \\ & X_T & \end{tikzcd}$$

commutes, where m_g denotes right multiplication by g. The commutativity of this triangle is equivalent to the condition that $\rho_1(e) = \rho_2(g)$, where $e \in G_T(T)$ is the identity section. Therefore I is identified with the fiber product of the diagram

$$\begin{tikzcd} & G_T \arrow[d, "\rho_1(e)\times\rho_2"] \\ X_T \arrow[r, "\Delta"] & X_T \times_T X_T, \end{tikzcd}$$

which implies that I is a scheme in the case when both $\mathscr{P}_1$ and $\mathscr{P}_2$ are trivial, and therefore that I is an algebraic space in general.

To get a smooth covering of $[X/G]$, consider the map $q : X \to [X/G]$ defined by the pair (G_X, ρ), where G_X denotes the trivial torsor over X and the map $\rho : G_X = G \times X \to X$ is the action map of G on X. For any object $(T, \mathscr{P}, \pi)$ of $[X/G]$, the fiber product of the diagram

$$\begin{tikzcd} & X \arrow[d, "(G_X,\rho)"] \\ T \arrow[r, "(\mathscr{P},\pi)"] & {[X/G]} \end{tikzcd}$$

is isomorphic to $\mathscr{P}$, which étale locally on T is a smooth group scheme over T. This implies that q is a smooth surjection.

Notice that if G is an affine group scheme, then by 4.5.6 the category $[X/G](T)$ is equivalent to the groupoid of diagrams

$$\begin{tikzcd} P \arrow[r, "\pi"] \arrow[d] & X \\ T, & \end{tikzcd}$$

where $P \to T$ is a principal G-bundle and π is a G-equivariant map.

Remark 8.1.13. With notation as in the preceding example, if U is an algebraic space over S then the groupoid of morphisms $U \to [X/G]$ is again equivalent to the groupoid of pairs $(\mathscr{P}, \pi)$, where $\mathscr{P}$ is a G-torsor on the big étale site of U (see 7.1.2) and $\pi : \mathscr{P} \to X \times_S U$ is a G-equivariant morphism of sheaves.

Definition 8.1.14. If G/S is a smooth group scheme, the *classifying stack of* G is the stack quotient $[S/G]$ where G acts trivially on S. It is denoted BG (or B_SG if we wish to make the reference to S explicit).

8.1.15. Consider a diagram of algebraic S-stacks

$$\begin{tikzcd} & \mathscr{X} \arrow[d, "c"] \\ \mathscr{Y} \arrow[r, "d"] & \mathscr{Z}, \end{tikzcd}$$

and let $\mathscr{W}$ be the fiber product $\mathscr{Y} \times_{\mathscr{Z}} \mathscr{X}$, a priori just a fibered category over S.

Proposition 8.1.16. *The fibered category $\mathscr{W}$ is an algebraic stack.*

Proof. That $\mathscr{W}$ is a stack follows from 4.6.4.

To prove that $\mathscr{W}$ is algebraic, first we verify that the diagonal of $\mathscr{W}$ is representable. For this let T be an S-scheme and let

$$A := (x, y, \sigma), \quad A' := (x', y', \sigma') \in \mathscr{W}(T)$$

be two objects, where $x, x' \in \mathscr{X}(T)$, $y, y' \in \mathscr{Y}(T)$, and $\sigma : c(x) \to d(y)$ and $\sigma' : c(x') \to d(y')$ are isomorphisms in $\mathscr{Z}(T)$. Let P denote the fiber product of the diagram

$$\begin{tikzcd} & T \arrow[d, "A\times A'"] \\ \mathscr{W} \arrow[r, "\Delta"] & \mathscr{W} \times \mathscr{W}. \end{tikzcd}$$

The fibered category P is equivalent to the fibered category associated to the functor on T-schemes which to any such scheme $T' \to T$ associates the set of pairs of isomorphisms

$$\iota_1 : x|_{T'} \to x'|_{T'}, \quad \iota_2 : y|_{T'} \to y'|_{T'}$$

in $\mathscr{X}(T')$ and $\mathscr{Y}(T')$, respectively, such that the induced diagram in $\mathscr{Z}(T')$,

$$\begin{tikzcd} c(x|_{T'}) \arrow[r, "\iota_1"] \arrow[d, "\sigma"] & c(x'|_{T'}) \arrow[d, "\sigma'"] \\ d(y|_{T'}) \arrow[r, "\iota_2"] & d(y'|_{T'}) \end{tikzcd}$$

commutes.

Let $P_{\mathscr{X}}$ denote the fiber product of the diagram

$$\begin{tikzcd} & T \arrow[d, "x\times x'"] \\ \mathscr{X} \arrow[r, "\Delta"] & \mathscr{X} \times \mathscr{X}, \end{tikzcd}$$

and let $P_{\mathscr{Y}}$ denote the fiber product of the diagram

$$\begin{tikzcd} & T \arrow[d, "y\times y'"] \\ \mathscr{Y} \arrow[r, "\Delta"] & \mathscr{Y} \times \mathscr{Y}. \end{tikzcd}$$

Also consider $P_{\mathscr{Z}}$ defined to be the fiber product of the diagram

$$\begin{tikzcd} & T \arrow[d, "c(x)\times d(y')"] \\ \mathscr{Z} \arrow[r, "\Delta"] & \mathscr{Z}\times\mathscr{Z}, \end{tikzcd}$$

so $P_{\mathscr{Z}}$ classifies isomorphisms $c(x) \to d(y')$. We have a map

$$F : P_{\mathscr{X}} \times P_{\mathscr{Y}} \to P_{\mathscr{Z}} \times P_{\mathscr{Z}}$$

sending a pair $(\iota_1, \iota_2) \in P_{\mathscr{X}} \times P_{\mathscr{Y}}$ to the pair of isomorphisms $(d(\iota_2) \circ \sigma, \sigma' \circ c(\iota_1))$, and P is isomorphic to the fiber product of the diagram

$$\begin{tikzcd} & P_{\mathscr{X}}\times P_{\mathscr{Y}} \arrow[d, "F"] \\ P_{\mathscr{Z}} \arrow[r, "\Delta"] & P_{\mathscr{Z}}\times P_{\mathscr{Z}}. \end{tikzcd}$$

Since $P_{\mathscr{X}}$, $P_{\mathscr{Y}}$, and $P_{\mathscr{Z}}$ are all algebraic spaces it follows that P is also an algebraic space.

Next we show that $\mathscr{X} \times_{\mathscr{Z}} \mathscr{Y}$ has a smooth cover by a scheme.

For this consider first the case when $c : \mathscr{X} \to \mathscr{Z}$ is representable, and let $Y \to \mathscr{Y}$ be a smooth surjection with Y a scheme. Let Y' denote $Y \times_{\mathscr{Z}} \mathscr{X}$, and consider the commutative diagram

$$\begin{tikzcd} Y' \arrow[r, "q"] \arrow[d] & \mathscr{Y}\times_{\mathscr{Z}}\mathscr{X} \arrow[r] \arrow[d] & \mathscr{X} \arrow[d] \\ Y \arrow[r] & \mathscr{Y} \arrow[r] & \mathscr{Z}, \end{tikzcd}$$

where all the squares are cartesian, and Y' is an algebraic space since c is representable. Let $p : Y'' \to Y'$ be an étale surjective map with Y'' a scheme. Then the induced map

$$Y'' \to \mathscr{Y} \times_{\mathscr{Z}} \mathscr{X}$$

is a smooth surjection. Indeed, for any morphism $t : T \to \mathscr{Y} \times_{\mathscr{Z}} \mathscr{X}$ from a scheme T, we have

$$Y' \times_{\mathscr{Y}\times_{\mathscr{Z}}\mathscr{X}} T \simeq Y \times_{\mathscr{Y},\mathrm{prot}} T,$$

so the second projection

$$Y' \times_{\mathscr{Y}\times_{\mathscr{Z}}\mathscr{X}} T \to T$$

is a smooth surjective map of algebraic spaces, and therefore the composition

$$Y'' \times_{\mathscr{Y}\times_{\mathscr{Z}}\mathscr{X}} T \simeq Y'' \times_{Y'} (Y' \times_{\mathscr{Y}\times_{\mathscr{Z}}\mathscr{X}} T) \to T$$

is the composition of an étale surjection with a smooth surjection, whence a smooth surjection. This completes the proof of the proposition in the case when one of c or d is representable.

For the general case, first choose a smooth surjection $Y \to \mathscr{Y}$ with Y a scheme, and consider the induced map

$$Y \times_{\mathscr{Z}} \mathscr{X} \to \mathscr{Y} \times_{\mathscr{Z}} \mathscr{X}.$$

By the representable case, the fibered category $Y \times_{\mathscr{Z}} \mathscr{X}$ is an algebraic stack, and therefore there exists a smooth surjection $q : Y' \to Y \times_{\mathscr{Z}} \mathscr{X}$ with Y' a scheme. Then repeating the argument in the representable case we find that the composition

$$Y' \to Y \times_{\mathscr{Z}} \mathscr{X} \to \mathscr{Y} \times_{\mathscr{Z}} \mathscr{X}$$

is a smooth surjection. □

Definition 8.1.17. Let $\mathscr{X}/S$ be an algebraic stack. The *inertia stack*, denoted $\mathscr{I}_{\mathscr{X}}$, of $\mathscr{X}$ is the fiber product of the diagram

$$\begin{tikzcd} & \mathscr{X} \arrow[d, "\Delta"] \\ \mathscr{X} \arrow[r, "\Delta"] & \mathscr{X} \times_S \mathscr{X}. \end{tikzcd}$$

8.1.18. The inertia stack of an algebraic stack $\mathscr{X}$ can be described more explicitly as follows. Its objects are pairs (x, g), where $x \in \mathscr{X}$ is an object of $\mathscr{X}$ lying over a scheme T, and g is an automorphism of x in $\mathscr{X}(T)$. A morphism

$$(x', g') \to (x, g)$$

in $\mathscr{I}_{\mathscr{X}}$ is a morphism $f : x' \to x$ in $\mathscr{X}$ such that the diagram

$$\begin{tikzcd} x' \arrow[r, "f"] \arrow[d, "g'"] & x \arrow[d, "g"] \\ x' \arrow[r, "f"] & x \end{tikzcd}$$

commutes. The projection $p : \mathscr{I}_{\mathscr{X}} \to \mathscr{X}$ sends a pair (x, g) to x.

Note also that for any scheme T and object $x \in \mathscr{X}(T)$ the fiber product of the diagram

$$\begin{tikzcd} & T \arrow[d, "t"] \\ \mathscr{I}_{\mathscr{X}} \arrow[r] & \mathscr{X} \end{tikzcd}$$

is the functor $\underline{\mathrm{Aut}}_t$ on the category of T-schemes (which is an algebraic space). It is often useful to think of the inertia stack $\mathscr{I}_{\mathscr{X}}$ as a "relative group space" over $\mathscr{X}$.

8.2. Properties of algebraic stacks and morphisms between them

Definition 8.2.1. Let P be a property of S-schemes which is stable in the smooth topology (see 5.1.1 and 2.1.16 for the terminology). We say that an algebraic stack $\mathscr{X}/S$ *has property* P if there exists a smooth surjective morphism $\pi : X \to \mathscr{X}$ with X a scheme having property P.

Remark 8.2.2. Note that for such a property P, it makes sense to talk about an algebraic space over S having property P by 5.4.1.

Example 8.2.3. Examples of such properties P are locally noetherian, regular, locally of finite type over S, locally of finite presentation over S.

LEMMA 8.2.4. *Let P be a property of schemes which is stable with respect to the smooth topology, and let $\mathscr{X}/S$ be an algebraic stack having property P. Then for any smooth morphism $y : Y \to \mathscr{X}$ from an algebraic space Y, the space Y has property P.*

PROOF. Let $\pi : X \to \mathscr{X}$ be a smooth surjective morphism from a scheme X with property P. We then obtain the commutative diagram

$$\begin{tikzcd} Y \times_{\mathscr{X}} X \arrow[r, "b"] \arrow[d, "a"] & X \arrow[d, "\pi"] \\ Y \arrow[r, "y"] & \mathscr{X}, \end{tikzcd}$$

where a is a smooth surjective morphism, and b is smooth. Since X has property P, it follows that $X \times_{\mathscr{X}} Y$ also has property P, and since a is smooth and surjective this implies that Y also has property P. □

8.2.5. To define properties of morphisms of algebraic stacks, it is convenient to introduce the following terminology. Let $f : \mathscr{X} \to \mathscr{Y}$ be a morphism of algebraic stacks over S. A *chart for f* is the commutative diagram

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[dr, "q"] \arrow[rr, bend left, "h"] & \mathscr{X}' \arrow[r, "f'"] \arrow[d, "p'"] & Y \arrow[d, "p"] \\ & \mathscr{X} \arrow[r, "f"] & \mathscr{Y}, \end{tikzcd} \tag{8.2.5.1}$$

where X and Y are algebraic spaces, the square is cartesian, and g and p are smooth and surjective. If we further require that X and Y are schemes, then we call this a *a chart for f by schemes.*

DEFINITION 8.2.6. Let P be a property of morphisms of schemes which is stable and local on domain with respect to the smooth topology (see 5.1.3). We say that a morphism $f : \mathscr{X} \to \mathscr{Y}$ *has property P* if there exists a chart for f by schemes such that the morphism h (8.2.5.1) has property P.

EXAMPLE 8.2.7. For example, P could be the property of being smooth, locally of finite presentation, surjective.

PROPOSITION 8.2.8. *Let P be a property of morphisms of schemes which is stable and local on domain with respect to the smooth topology. Then a morphism of algebraic stacks $f : \mathscr{X} \to \mathscr{Y}$ has property P if and only if for every chart for f* (8.2.5.1) *the morphism h has property P.*

PROOF. Fix a chart (8.2.5.1), and let $Y' \to Y$ be a smooth surjective morphism of algebraic spaces. We then get the commutative diagram

$$\begin{tikzcd}
X \times_Y Y' \arrow[r, "g'"] \arrow[d] \arrow[rr, bend left, "h'"] & \mathscr{X} \times_{\mathscr{Y}} Y' \arrow[r] \arrow[d] & Y' \arrow[d] \\
X \arrow[r, "g"] \arrow[dr] & \mathscr{X}' \arrow[r] \arrow[d] & Y \arrow[d] \\
 & \mathscr{X} \arrow[r, "f"] & \mathscr{Y},
\end{tikzcd}$$

where the big outside diagram

$$\begin{tikzcd}
X \times_Y Y' \arrow[r] \arrow[dr] & \mathscr{X} \times_{\mathscr{Y}} Y' \arrow[r] \arrow[d] & Y' \arrow[d] \\
 & \mathscr{X} \arrow[r, "f"] & \mathscr{Y}
\end{tikzcd}$$

is also a chart for f. Since the projection $X \times_Y Y' \to X$ is smooth and surjective, we see that the morphism h in (8.2.5.1) has property P if and only if the morphism

$$h' : X \times_Y Y' \to Y'$$

has property P.

Now fix two charts

$$\begin{tikzcd}
X_j \arrow[r, "g_j"] \arrow[dr, "q"'] \arrow[rr, bend left, "h_j"] & \mathscr{X}'_j \arrow[r, "f'"] \arrow[d, "p'_j"] & Y_j \arrow[d, "p_j"] \\
 & \mathscr{X} \arrow[r, "f"] & \mathscr{Y},
\end{tikzcd}$$

for $j = 1, 2$. We show that h_1 has property P if and only if h_2 has property P.

Set $Y' = Y_1 \times_{\mathscr{Y}} Y_2$. Applying the previous discussion to the two maps

$$\mathrm{pr}_j : Y' \to Y_j, \quad j = 1, 2$$

we see that it suffices to consider the case when $Y_1 = Y_2$ and $p_1 = p_2$ (in which case we write just $\mathscr{X}'$ for $\mathscr{X} \times_{\mathscr{Y}} Y_1 = \mathscr{X} \times_{\mathscr{Y}} Y_2$). So in this case we have the diagram

$$\begin{tikzcd}
 & X_2 \arrow[ddr, "g_2"] \arrow[ddrr, "h_2"] & & \\
X_1 \arrow[drr, "g_1"'] \arrow[drrr, "h_1"] & & & \\
 & & \mathscr{X}' \arrow[r] \arrow[d] & Y \arrow[d] \\
 & & \mathscr{X} \arrow[r, "f"] & \mathscr{Y},
\end{tikzcd}$$

where g_1 and g_2 are smooth surjections. Let X' denote $X_1 \times_{\mathscr{X}'} X_2$. Since the property P is local in the smooth topology on the source, the morphism h_1 (resp. h_2) has property P if and only if the composite morphism

$$h_1 \circ \mathrm{pr}_1 = h_2 \circ \mathrm{pr}_2 : X' \to Y$$

has property P, and therefore h_1 has property P if and only if h_2 does. $\square$

DEFINITION 8.2.9. Let P be a property of morphisms of algebraic spaces which is stable with respect to the smooth topology on the category of algebraic spaces over S. We say that a representable morphism of algebraic stacks $f : \mathscr{X} \to \mathscr{Y}$ *has property* P if for every morphism $Y \to \mathscr{Y}$ with Y an algebraic space, the morphism of algebraic spaces (since f is representable)

$$\mathscr{X} \times_{\mathscr{Y}} Y \to Y$$

has property P.

EXAMPLE 8.2.10. For example, we can talk about a representable morphism of algebraic stacks being étale, smooth of relative dimension d, separated, proper, affine, finite, unramified, a closed imbedding, an open imbedding, an imbedding.

8.2.11. In particular, if $f : \mathscr{X} \to \mathscr{Y}$ is a morphism of algebraic stacks over S, then the diagonal morphism

$$\Delta_{\mathscr{X}/\mathscr{Y}} : \mathscr{X} \to \mathscr{X} \times_{\mathscr{Y}} \mathscr{X}$$

is representable, and we can make the following definition.

DEFINITION 8.2.12. Let $f : \mathscr{X} \to \mathscr{Y}$ be a morphism of algebraic stacks over S, and let

$$\Delta_{\mathscr{X}/\mathscr{Y}} : \mathscr{X} \to \mathscr{X} \times_{\mathscr{Y}} \mathscr{X}$$

be the diagonal morphism. We say that:

(i) f is *quasi-separated* if the diagonal $\Delta_{\mathscr{X}/\mathscr{Y}}$ is quasi-compact and quasi-separated.

(ii) f is *separated* if the diagonal $\Delta_{\mathscr{X}/\mathscr{Y}}$ is proper.

If $\mathscr{Y} = S$ and f is the structure morphism, then we simply say that $\mathscr{X}$ is *quasi-separated* (resp. *separated*).

EXAMPLE 8.2.13. There exist morphisms of algebraic stacks $f : \mathscr{X} \to \mathscr{Y}$ for which the diagonal is quasi-compact but not quasi-separated. For example, let k be a field of characteristic 0 and let G/k be a group algebraic space which is quasi-compact but not quasi-separated (for example the quotient $\mathbb{A}^1_k/\mathbb{Z}$ discussed in 5.3.4). Then the diagonal of BG over k is quasi-compact but not quasi-separated.

8.3. Deligne-Mumford stacks

DEFINITION 8.3.1. An algebraic stack $\mathscr{X}/S$ is called *Deligne-Mumford* if there exists a étale surjection $X \to \mathscr{X}$ with X a scheme.

8.3.2. Recall (Definition 1.3.1) that a morphism of schemes $g : Z \to W$ is called *formally unramified* if for every closed imbedding $S_0 \hookrightarrow S$ of affine schemes defined by a nilpotent ideal the map on scheme-valued points

$$Z(S) \to Z(S_0) \times_{W(S_0)} W(S)$$

is injective. By exercise 8.C, this is equivalent to the condition that $\Omega^1_{Z/W} = 0$. In particular, the property of being formally unramified is stable and local on a domain with respect to the étale topology on the category of schemes, and stable with respect to the smooth topology. It therefore makes sense to talk about a representable morphism of stacks being formally unramified.

THEOREM 8.3.3 ([**23**, 4.21] and [**49**, 8.1]). *Let $\mathscr{X}/S$ be an algebraic stack. Then $\mathscr{X}$ is Deligne-Mumford if and only if the diagonal*

$$\Delta : \mathscr{X} \to \mathscr{X} \times_S \mathscr{X}$$

is formally unramified.

REMARK 8.3.4. This result should loosely be interpreted as saying that a stack $\mathscr{X}$ is Deligne-Mumford if and only if the objects of $\mathscr{X}$ admit no infinitesimal automorphisms.

Precisely, let $\mathscr{X}/S$ be an algebraic stack and assume that the diagonal $\Delta : \mathscr{X} \to \mathscr{X} \times_S \mathscr{X}$ is of finite presentation. Then the diagonal Δ is formally unramified if and only if for every algebraically closed field k and object $x \in \mathscr{X}(k)$, the automorphism group scheme $\underline{\mathrm{Aut}}_x$ is a reduced finite k-group scheme (i.e. a finite group). To see the 'only if' direction, observe that if Δ is formally unramified then the finite type k-group scheme $\underline{\mathrm{Aut}}_x$ is formally unramified over k, being the fiber product of the diagram

$$\begin{tikzcd} & \mathrm{Spec}(k) \arrow[d, "(x,x)"] \\ \mathscr{X} \arrow[r, "\Delta"] & \mathscr{X} \times_S \mathscr{X}. \end{tikzcd}$$

Since a finite type formally unramified k-scheme is a finite disjoint union of copies of k by [**26**, IV.17.4.1 (d')], this implies that $\underline{\mathrm{Aut}}_x$ is isomorphic to the k-group scheme associated to a finite group.

Conversely, let $U \to \mathscr{X} \times_S \mathscr{X}$ be any morphism from a scheme corresponding to two objects $u_1, u_2 \in \mathscr{X}(U)$, and let $P \to U$ denote the fiber product of the diagram

$$\begin{tikzcd} & U \arrow[d] \\ \mathscr{X} \arrow[r, "\Delta"] & \mathscr{X} \times_S \mathscr{X}. \end{tikzcd}$$

By [**26**, IV.17.4.1], the morphism $P \to U$ is formally unramified if and only if for every point $z \in U$ the fiber $P_z \to \mathrm{Spec}(k(z))$ is formally unramified, and by descent this is in turn equivalent to the base change $P_\Omega \to \mathrm{Spec}(\Omega)$ being formally unramified, where $k(z) \hookrightarrow \Omega$ is an algebraic closure. Now the scheme P_Ω is either empty, or isomorphic to $\underline{\mathrm{Aut}}_x$, where $x \in \mathscr{X}(\Omega)$ is the pullback of u_1. Therefore if each of the automorphism group schemes $\underline{\mathrm{Aut}}_x$ are formally unramified this implies that the diagonal is also formally unramfied.

The following argument is due to Laumon and Moret-Bailly [**49**, proof of 8.1], and we follow their notation.

PROOF OF THEOREM 8.3.3. First we show that if $\mathscr{X}$ is Deligne-Mumford then the diagonal is formally unramified. For this choose an étale surjection $\pi : X \to \mathscr{X}$

with X a scheme, and consider the cartesian square

$$\begin{tikzcd} X \times_{\mathscr{X}} X \arrow[r, "b"] \arrow[d, "a"] & X \times_S X \arrow[d] \\ \mathscr{X} \arrow[r, "\Delta"] & \mathscr{X} \times_S \mathscr{X}, \end{tikzcd}$$

where the map a is an étale surjection. The composition of the map b with the first projection $\mathrm{pr}_1 : X \times_S X \to X$ is étale since we have the cartesian square

$$\begin{tikzcd} X \times_{\mathscr{X}} X \arrow[r, "\mathrm{pr}_1 \circ b"] \arrow[d, "\mathrm{pr}_1 \circ b"'] & X \arrow[d] \\ X \arrow[r] & \mathscr{X}. \end{tikzcd}$$

It follows that the map b is unramified (see exercise 8.D), which implies that $\Delta_{\mathscr{X}}$ is formally unramified.

Next we show that if the diagonal is formally unramified, then $\mathscr{X}$ is Deligne-Mumford.

Let k be an algebraically closed field and let $y : \mathrm{Spec}(k) \to \mathscr{X}$ be a morphism. Fix a smooth morphism $P : X \to \mathscr{X}$ with X an affine scheme such that

$$X_y := X \times_{\mathscr{X},y} \mathrm{Spec}(k)$$

is nonempty, and fix in addition the following data:

(1) A point $x' \in X_y(k)$ (note that X_y is only an algebraic space but since k is assumed algebraically closed such a point exists). Let k_0 denote the residue field of the image of x' in S.
(2) Let k_0^s be a separable closure of k_0.

We show that after possibly replacing S by an étale cover and shrinking on X there exists a closed subscheme $W \subset X$ étale over $\mathscr{X}$ and with W_y nonempty.

Let Z denote the fiber product of the diagram

$$\begin{tikzcd} & X \arrow[d, "P"] \\ X \arrow[r, "P"] & \mathscr{X} \end{tikzcd}$$

so we have cartesian squares

$$\begin{tikzcd} Z \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_2"] & X \arrow[d, "P"] \\ X \arrow[r, "P"] & \mathscr{X} \end{tikzcd} \qquad \begin{tikzcd} Z \arrow[r] \arrow[d] & X \times_S X \arrow[d] \\ \mathscr{X} \arrow[r, "\Delta"] & \mathscr{X} \times_S \mathscr{X}. \end{tikzcd}$$

We define a sheaf $\Omega^1_{X/\mathscr{X}}$ by descent theory as follows. Consider the commutative diagram with cartesian squares

$$\begin{tikzcd} Z \times_{\mathrm{pr}_1, X, \mathrm{pr}_1} Z \arrow[r, shift left, "q_1"] \arrow[r, shift right, "q_2"'] \arrow[d] & Z \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_2"] & X \arrow[d, "P"] \\ X \times_{\mathscr{X}} X \arrow[r, shift left] \arrow[r, shift right] & X \arrow[r, "P"] & \mathscr{X}, \end{tikzcd}$$

where

$$q_1, q_2 : Z \times_{\mathrm{pr}_1, X, \mathrm{pr}_1} Z \to Z$$

are the two projections. Let $\Omega^1_{(2)}$ denote the relative differentials of the map $\mathrm{pr}_2 : Z \to X$. We have canonical isomorphisms

$$q_1^* \Omega^1_{(2)} \simeq \Omega^1_{(Z \times_{\mathrm{pr}_1, X, \mathrm{pr}_1} Z)/(X \times_{\mathscr{X}} X)} \simeq q_2^* \Omega^1_{(2)},$$

which defines an isomorphism

$$\epsilon : q_1^* \Omega^1_{(2)} \to q_2^* \Omega^1_{(2)}.$$

This isomorphism satisfies the cocycle condition on the triple product (we leave this verification to the reader)

$$Z \times_{\mathrm{pr}_1, X, \mathrm{pr}_1} Z \times_{\mathrm{pr}_1, X, \mathrm{pr}_1} Z$$

and therefore we obtain an object (notation as in 4.3.11)

$$(\Omega^1_{(2)}, \epsilon) \in \mathrm{QCOH}(\mathrm{pr}_1 : Z \to X).$$

Let $\Omega^1_{X/\mathscr{X}}$ be the corresponding quasi-coherent sheaf on X (using 4.3.12). This sheaf is locally free of finite rank as its pullback $\mathrm{pr}_1^* \Omega^1_{X/\mathscr{X}} \simeq \Omega^1_{(2)}$ has this property.

Note also that there is a natural map

$$\Omega^1_{X/S} \to \Omega^1_{X/\mathscr{X}}. \tag{8.3.4.1}$$

This map is surjective, as can be seen as follows. Since the diagonal of $\mathscr{X}$ is unramified, which implies that the map $Z \to X \times_S X$ is unramified, we find that the sum of the two pullback maps

$$\mathrm{pr}_1^* \Omega^1_{X/S} \oplus \mathrm{pr}_2^* \Omega^1_{X/S} \to \Omega^1_{Z/S} \tag{8.3.4.2}$$

is surjective. From the composition

$$Z \xrightarrow{\mathrm{pr}_2} X \longrightarrow S$$

we see that the sequence

$$\mathrm{pr}_2^* \Omega^1_{X/S} \to \Omega^1_{Z/S} \to \Omega^1_{(2)} \to 0$$

is exact, and therefore the composite map

$$\mathrm{pr}_1^* \Omega^1_{X/S} \hookrightarrow \mathrm{pr}_1^* \Omega^1_{X/S} \oplus \mathrm{pr}_2^* \Omega^1_{X/S} \longrightarrow \Omega^1_{(2)}$$

is also surjective. Since this is the pullback of the map (8.3.4.1) to Z we conclude that (8.3.4.1) is also surjective.

Write also $d : \mathscr{O}_X \to \Omega^1_{X/\mathscr{X}}$ for the composite map

$$\mathscr{O}_X \xrightarrow{d} \Omega^1_{X/S} \longrightarrow \Omega^1_{X/\mathscr{X}}.$$

Since $\Omega^1_{X/\mathscr{X}}$ is locally free of finite rank, and the image of d in $\Omega^1_{X/S}$ locally generates $\Omega^1_{X/S}$, we can find an étale neighborhood U of our point x' and sections $f_1, \dots, f_r \in \Gamma(U, \mathscr{O}_U)$ such that the sections $df_1, \dots, df_r \in \Omega^1_{X/\mathscr{X}}$ form a basis over U. Replacing X by this neighborhood we may assume that we have such sections f_j globally.

Let

$$\boldsymbol{f} : X \to \mathscr{X} \times_S \mathbb{A}^r_S$$

be the map defined by the $\{f_j\}$. This morphism fits into a commutative diagram:

$$\begin{tikzcd} X \arrow[dd, "P"'] \arrow[dr, "\boldsymbol{f}"] & \\ & \mathscr{X} \times_S \mathbb{A}^r_S \arrow[dl] \\ \mathscr{X}. & \end{tikzcd}$$

The morphism $\boldsymbol{f}$ is étale in a neighborhood of x' as this is true after making any base change $T \to \mathscr{X}$ by 1.3.4 (iii). After shrinking further around x' we may therefore assume that $\boldsymbol{f}$ is étale.

Let

$$\boldsymbol{f}_y : X_y \to \mathbb{A}^r_k$$

denote the base change of $\boldsymbol{f}$ along $y : \mathrm{Spec}(k) \to \mathscr{X}$. This morphism is étale, being the base change of an étale morphism, and therefore has open image. Let $F \in k[T_1, \ldots, T_r]$ be a nonzero polynomial with corresponding open set $D(F)$ contained in the image of X_y. Since k_0^s is infinite, there exists $a_1, \ldots, a_r \in k_0^s$ such that $F(a_1, \ldots, a_r) \neq 0$. Let

$$Q \in \mathbb{A}^r_{k_0}$$

be the image of the point

$$(a_1, \ldots, a_r) \in \mathbb{A}^r_{k_0^s}.$$

The point Q is then a closed point of $\mathbb{A}^r_{k_0}$ which is in the image of X_y and whose residue field is étale over k_0. By exercise 8.F, there exists an étale neighborhood $S' \to S$ of the image of x' in S and a subscheme $E \subset \mathbb{A}^r_{S'}$ étale over S', and containing Q. Let $W \subset X \times_S S'$ be the inverse image under $\boldsymbol{f}$ of

$$\mathscr{X} \times_S E \subset \mathscr{X} \times_{S'} \mathbb{A}^r_{S'}.$$

Then W_y is nonempty, and the map $W \to \mathscr{X}$ is étale being the composition of two étale morphisms

$$W \longrightarrow \mathscr{X} \times_S E \longrightarrow \mathscr{X}.$$

This completes the proof of Theorem 8.3.3. □

COROLLARY 8.3.5. *Let $\mathscr{X}/S$ be an algebraic stack, such that for every S-scheme U and object $x \in \mathscr{X}(U)$ the automorphism group of x is trivial. Then $\mathscr{X}$ is an algebraic space.*

PROOF. Indeed, in this case the diagonal morphism $\mathscr{X} \to \mathscr{X} \times_S \mathscr{X}$ is representable by monomorphisms. In particular, $\mathscr{X}$ is a Deligne-Mumford stack, so admits an étale surjection $U \to \mathscr{X}$ with U a scheme, and if R denotes $U \times_{\mathscr{X}} U$ then $R \to U \times_S U$ is an étale equivalence relation on U such that $\mathscr{X} = U/R$. □

REMARK 8.3.6. In [**20**, 2.2.5] a variant version of 8.3.5 is given showing that in order to verify that an algebraic stack is an algebraic space it suffices to show that the automorphism group functors of geometric points of $\mathscr{X}$ are trivial.

8.4. Examples

8.4.1. Group quotients. Let X be a scheme over a base scheme S, and let G/S be a smooth group scheme which acts on X over S. Let $[X/G]$ be the algebraic stack defined in Example 8.1.12.

In this setting, the condition that the diagonal of $[X/G]$ is unramified can be described in terms of the group action as follows. Let

$$\bar{s} : \mathrm{Spec}(k) \to S$$

be a morphism with k algebraically closed, and let $G_{\bar{s}}$ denote the pullback of G along $\bar{s}$, so $G_{\bar{s}}$ is a smooth group scheme over k. Consider an object $(\mathscr{P}, \pi)$ in $[X/G]$ over k. Since k is algebraically closed the torsor $\mathscr{P}$ is trivial. Fix a trivialization $p \in \mathscr{P}(k)$ and let $x \in X(k)$ denote the image $\pi(p)$. We then have a map

$$\lambda : G_{\bar{s}} \to X_{\bar{s}}, \quad g \mapsto g \cdot x,$$

where $X_{\bar{s}}$ denotes $X \times_{S,\bar{s}} \mathrm{Spec}(k)$. The fiber product G_x of the diagram

$$\begin{tikzcd} & \mathrm{Spec}(k) \arrow[d, "x"] \\ G_{\bar{s}} \arrow[r, "\lambda"] & X_{\bar{s}} \end{tikzcd}$$

is isomorphic to the automorphism group scheme $\underline{\mathrm{Aut}}_{(\mathscr{P},\pi)}$.

Note that for different trivializations of $\mathscr{P}$ the resulting subgroup schemes of $G_{\bar{s}}$ are conjugate. For a point $t \in [X/G](k)$ it therefore makes sense to talk about the stabilizer group scheme $G_t \subset G_{\bar{s}}$. It is well defined up to conjugation by elements of $G_{\bar{s}}(k)$.

Corollary 8.4.2. *The stack $[X/G]$ is Deligne-Mumford if and only if for every point $\bar{s} : \mathrm{Spec}(k) \to S$ as above and $t \in [X/G](k)$, the stabilizer group scheme $G_t \subset G_{\bar{s}}$ is étale over $\bar{s}$.*

Proof. For a field L and morphism $z : \mathrm{Spec}(L) \to [X/G]$ with corresponding automorphism group scheme $\underline{\mathrm{Aut}}_z$, the L-scheme $\underline{\mathrm{Aut}}_z$ is unramified over L if and only if the base change of $\underline{\mathrm{Aut}}_z$ to an algebraic closure $\overline{L}$ is étale over $\overline{L}$ by [**26**, IV, 17.6.2]. Therefore $[X/G]$ is Deligne-Mumford if and only if for any algebraically closed field k and morphism t as above the group scheme G_t is étale as desired. □

8.4.3. $\mathscr{M}_g$.

8.4.4. Fix an integer $g \geq 2$, and let $\mathscr{M}_g$ be the fibered category over the category of schemes whose objects are pairs $(S, f : C \to S)$, where S is a scheme and f is a proper smooth morphism such that every geometric fiber of s is a connected genus g curve. A morphism

$$(S', f' : C' \to S') \to (S, f : C \to S)$$

in $\mathscr{M}_g$ is a cartesian square

$$\begin{tikzcd} C' \arrow[r, "\tilde{g}"] \arrow[d, "f'"] & C \arrow[d, "f"] \\ S' \arrow[r, "g"] & S. \end{tikzcd}$$

The functor sending $(S, f : C \to S)$ to S makes $\mathscr{M}_g$ a fibered category over the category of schemes. In this subsection we prove the following theorem.

THEOREM 8.4.5. *The fibered category $\mathscr{M}_g$ is a Deligne-Mumford stack.*

The proof occupies the remainder of this subsection.

By 4.4.13 we already know that $\mathscr{M}_g$ is a stack for the étale topology.

LEMMA 8.4.6. *Let $(S, f : C \to S)$ be an object of $\mathscr{M}_g$, and let $L_{C/S}$ denote the invertible sheaf $(\Omega^1_{C/S})^{\otimes 3}$.*

(i) The sheaf $f_*L_{C/S}$ is a locally free sheaf of rank $5g - 5$ on S.

(ii) The map $f^*f_*L_{C/S} \to L_{C/S}$ is surjective, and the resulting S-map

$$C \to \mathbb{P}(f_*L_{C/S})$$

is a closed imbedding.

(iii) For any morphism $g : S' \to S$, the natural map

$$g^*f_*L_{C/S} \to f'_*L_{C'/S'}$$

is an isomorphism, where $f' : C' \to S'$ denotes the base change of f to S'.

PROOF. For (i) and (iii), by cohomology and base change [**41**, III, 12.11], it suffices to show that if $S = \mathrm{Spec}(k)$ is the spectrum of an algebraically closed field k, and C/k is a smooth proper curve of genus g, then $H^0(C, L_{C/S})$ has dimension $5g-5$ and the higher cohomology groups $H^i(C, L_{C/S})$ are zero for $i > 0$. This latter statement is immediate as the only possibly nonzero group is $H^1(C, L_{C/S})$ (since C is a curve), and by Serre duality this group is dual to $H^0(C, (\Omega^1_{C/S})^{\otimes -2})$ which is zero since $(\Omega^1_{C/S})^{-2}$ has negative degree (because we assumed $g \geq 2$). Similarly, Riemann-Roch gives that

$$h^0(C, L_{C/S}) = 3\mathrm{deg}(\Omega^1_{C/S}) + 1 - g = 3(2g - 2) + 1 - g = 5g - 5.$$

Finally to prove (ii) it suffices in light of (iii) to verify it in the case when $S = \mathrm{Spec}(k)$ is the spectrum of an algebraically closed field. Statement (ii) therefore follows from the fact that $L_{C/S}$ is very ample in this case. $\square$

8.4.7. Let $\widetilde{M}_g$ denote the functor on the category of schemes which to any scheme S associates the isomorphism classes of pairs

$$(f : C \to S, \sigma : \mathscr{O}_S^{5g-5} \simeq f_*L_{C/S}),$$

where $(S, f : C \to S) \in \mathscr{M}_g(S)$. An isomorphism

$$(f' : C' \to S, \sigma' : \mathscr{O}_S^{5g-5} \simeq f'_*L_{C'/S}) \to (f : C \to S, \sigma : \mathscr{O}_S^{5g-5} \simeq f_*L_{C/S})$$

is given by an isomorphism of curves $\alpha : C' \to C$ such that the resulting diagram

$$\begin{tikzcd} & f'_*L_{C'/S} \arrow[dd, "\alpha"] \\ \mathscr{O}_S^{5g-5} \arrow[ur, "\sigma'"] \arrow[dr, "\sigma"'] & \\ & f_*L_{C/S} \end{tikzcd}$$

commutes.

8.4.8. Given any element $(f : C \to S, \sigma) \in \widetilde{M}_g(S)$, we get an imbedding
$$\iota : C \hookrightarrow \mathbb{P}^{5g-6}_S$$
from the composition
$$C \hookrightarrow \mathbb{P}(f_* L_{C/S}) \xrightarrow{\mathbb{P}\sigma} \mathbb{P}^{5g-6}.$$
By the construction of this imbedding ι, we have an isomorphism
$$\tau : \iota^* \mathscr{O}_{\mathbb{P}^{5g-6}}(1) \to L_{C/S} \tag{8.4.8.1}$$
such that the isomorphism σ is given by the composition
$$\mathscr{O}_S^{5g-5} \xrightarrow{c} f_* \iota^* \mathscr{O}_{\mathbb{P}^{5g-6}}(1) \xrightarrow{\tau} f_* L_{C/S}. \tag{8.4.8.2}$$
Here the map labelled c is obtained by pullback from the tautological map $\mathscr{O}^{5g-5}_{\mathbb{P}^{5g-6}} \to \mathscr{O}_{\mathbb{P}^{5g-6}}(1)$.

Note also that the Hilbert polynomial of the imbedding ι is by Riemann-Roch equal to
$$(6g-6)T + 1 - g.$$
Let $\mathscr{H}'$ denote the Hilbert scheme of closed subschemes of $\mathbb{P}^{5g-6}$ with Hilbert polynomial $(6g-6)T+1-g$. So $\mathscr{H}'$ is a projective scheme and we have a universal closed subscheme
$$\begin{tikzcd} \mathscr{Z} \arrow[r, hook] \arrow[dr, "g"'] & \mathscr{H}' \times \mathbb{P}^{5g-6} \arrow[d] \\ & \mathscr{H}' \end{tikzcd}$$
where g is flat. Let $\mathscr{W} \subset \mathscr{Z}$ denote the closed subset where g is not smooth. Since g is proper the image $g(\mathscr{W}) \subset \mathscr{H}'$ is a closed set. Let $\mathscr{H} \subset \mathscr{H}'$ denote the open complement. By the fiberwise criterion for smoothness [**26**, IV.17.8.2], the open subset $\mathscr{H}$ represents the functor on $\mathscr{H}'$-schemes, which to any $T \to \mathscr{H}'$ associates the unital set if $\mathscr{Z}_T \to T$ is smooth, and the empty set otherwise. Since the Hilbert polynomial of the fibers of $\mathscr{Z}_{\mathscr{H}} \to \mathscr{H}$ is of degree 1, the fibers of $\mathscr{Z}_{\mathscr{H}} \to \mathscr{H}$ are smooth curves. Let $H \subset \mathscr{H}$ denote the open and closed subset over which all the fibers have genus g (the fact that this is open and closed follows, for example, from [**41**, III, 9.9]), so we have an object
$$f : C \to H$$
of $\mathscr{M}_g(H)$, and a map of functors
$$\overline{F} : \widetilde{M}_g \to H.$$
Let $i : C \hookrightarrow \mathbb{P}^{5g-6}_H$ be the given projective imbedding, and let $L'_{C/H}$ be the line bundle $i^* \mathscr{O}_{\mathbb{P}^{5g-6}_H}(1)$.

For any geometric point $\bar{s} \to H$ the restriction $L'_{C_{\bar{s}}}$ of $L'_{C/H}$ to the fiber $C_{\bar{s}}$ is a very ample invertible sheaf with Hilbert polynomial $(6g-6)T+1-g$. It follows from this and Riemann-Roch that the degree of $L'_{C_{\bar{s}}}$ is $6g-6$. From this and Riemann-Roch we conclude that the dimension of $H^0(C_{\bar{s}}, L'_{C_{\bar{s}}})$ is equal to $5g-5$ and that the higher cohomology groups $H^i(C_{\bar{s}}, L'_{C_{\bar{s}}})$ are 0 for $i > 0$. From this and cohomology and base change it follows that the sheaf $f_* L'_{C/H}$ is locally free of rank $5g-5$ and that its formation commutes with arbitrary base change on H.

8.4.9. Let $\widetilde{H}$ denote the functor on H-schemes which to any such scheme $g : T \to H$ associates the set of isomorphisms

$$\lambda : g^* f_* L_{C/H} \to g^* f_* L'_{C/H}.$$

The functor $\widetilde{H}$ is representable by an affine smooth H-scheme. Indeed, the verification of this can be done locally around any point $h \in H$. Fix an affine neighborhood

$$h \in \mathrm{Spec}(A) \subset H$$

such that $f_* L_{C/H}$ and $f_* L'_{C/H}$ are trivial. Choosing trivializations of these vector bundles, the functor $\widetilde{H}$ restricted to the category of schemes over $\mathrm{Spec}(A)$ becomes identified with the functor which to any scheme $T \to \mathrm{Spec}(A)$ associates an element of

$$GL_{5g-5}(\Gamma(T), \mathscr{O}_T).$$

This functor is representable by an affine $\mathrm{Spec}(A)$-scheme by exercise 1.C.

8.4.10. The functor $\overline{F}$ lifts to a morphism of functors

$$F : \widetilde{M_g} \to \widetilde{H}$$

by sending (C, σ) over some scheme S to the closed imbedding $\iota : C \hookrightarrow \mathbb{P}^{5g-6}_S$ together with the isomorphism (8.4.8.1).

8.4.11. We now show that the morphism of functors F identifies $\widetilde{M_g}$ with a closed subscheme of $\widetilde{H}$. Over $\widetilde{H}$ we have the universal curve

$$\begin{tikzcd} C \arrow[r, hook, "\iota"] \arrow[dr, "f"'] & \mathbb{P}^{5g-6}_{\widetilde{H}} \arrow[d] \\ & \widetilde{H} \end{tikzcd}$$

and the tautological isomorphism

$$\lambda^u : f_* L_{C/\widetilde{H}} \to f_* L'_{C/\widetilde{H}}.$$

This diagram induces for every morphism $T \to \widetilde{H}$, a diagram of solid arrows on the base change C_T

$$\begin{tikzcd} f^* f_* L_{C_T/T} \arrow[r, "\lambda^u"] \arrow[d, two heads] & f^* f_* L'_{C_T/T} \arrow[d, two heads] \\ L_{C_T/T} \arrow[r, dashed] & L'_{C_T/T}. \end{tikzcd}$$

The functor $\widetilde{M_g}$ is in this way identified with the subfunctor of $\widetilde{H}$ which to any morphism $T \to \widetilde{H}$ associates the unital set if there exists a dotted arrow filling in this diagram, and the empty set otherwise.

8.4.12. This condition is represented by a closed subscheme of $\widetilde{H}$. Indeed, let K denote the kernel of the map

$$f^* f_* L_{C/\widetilde{H}} \to L_{C/\widetilde{H}}.$$

Then a dotted arrow filling in the diagram exists over some $T \to \widetilde{H}$ if and only if the composite map

$$K_T \hookrightarrow f^*f_*L_{C_T/T} \xrightarrow{\lambda} f^*f_*L'_{C_T/T} \twoheadrightarrow L'_{C_T/T} \tag{8.4.12.1}$$

is zero. Let r be an integer so that the map

$$f^*f_*(K \otimes L'^{\otimes r}_{C/\widetilde{H}}) \to K \otimes L'^{\otimes r}_{C/\widetilde{H}}$$

is surjective, and such that $f_*(K \otimes L'^{\otimes r}_{C/\widetilde{H}})$ is a locally free sheaf on $\widetilde{H}$ whose formation commutes with arbitrary base change on $\widetilde{H}$. Then the condition that the composite map (8.4.12.1) is zero over some $T \to \widetilde{H}$ is equivalent to the condition that the map of vector bundles on $\widetilde{H}$,

$$f_*(K \otimes L'^{\otimes r}_{C/\widetilde{H}}) \to f_*L'^{\otimes r+1}_{C/\widetilde{H}},$$

pulls back to the zero bundle on T. By exercise 8.G, this condition is represented by a closed subscheme of $\widetilde{H}$. This concludes the proof that $\widetilde{M}_g$ is representable by a quasi-projective scheme.

8.4.13. There is an action of $G := GL_{5g-5}$ on $\widetilde{M}_g$ given on S-points (where S is a scheme) by

$$g * (C/S, \sigma) \mapsto (C/S, \sigma \circ g), \quad g \in G(S).$$

Consider the map

$$\pi : \widetilde{M}_g \to \mathscr{M}_g, \quad (C/S, \sigma) \mapsto (S, C).$$

For any scheme S and morphism $f : S \to \mathscr{M}_g$ corresponding to a curve C/S, the fiber product

$$\widetilde{M}_g \times_{\mathscr{M}_g} S$$

is the G_S-torsor of isomorphisms $\sigma : \mathscr{O}_S^{5g-5} \to f_*L_{C/S}$. It follows that we have an isomorphism $\mathscr{M}_g \simeq [\widetilde{M}_g/G]$. In particular, $\mathscr{M}_g$ is an algebraic stack.

8.4.14. It remains to see that $\mathscr{M}_g$ is a Deligne-Mumford stack. For this it suffices by 8.4.2 to show that for any algebraically closed field k and smooth genus g curve C/k, the group scheme $\underline{\mathrm{Aut}}_k(C)$ is reduced. For this it suffices in turn to show that if $A' \to A$ is a surjective morphism of k-algebras with square-zero kernel I, then the map

$$\underline{\mathrm{Aut}}_k(C)(A') \to \underline{\mathrm{Aut}}_k(C)(A)$$

is injective. This is clear for $\bar{\alpha} : C_A \to C_A$ is an automorphism, then the set of liftings of $\bar{\alpha}$ to $C_{A'}$ are given by the set of dotted arrows filling in the diagram

$$\begin{tikzcd} C_A \arrow[r, "\bar{\alpha}"] \arrow[d, hook] & C_A \arrow[r, hook] & C_{A'} \arrow[d] \\ C_{A'} \arrow[rru, dashed] \arrow[rr] & & \mathrm{Spec}(A'). \end{tikzcd}$$

By the universal property of differentials 1.3.3, the set of such dotted arrows form a torsor under

$$\mathrm{Hom}(\bar{\alpha}^*\Omega^1_{C_A/A}, I \otimes_A \mathscr{O}_{C_A}) \simeq H^0(C_A, \bar{\alpha}^*T_{C_A/A}, I \otimes_A \mathscr{O}_C),$$

which is zero since it is zero in every fiber (since the tangent bundle has negative degree because $g \geq 2$).

This completes the proof of Theorem 8.4.5. □

REMARK 8.4.15. One can also consider fibered categories $\mathscr{M}_g$ for $g = 0, 1$. Using the fact that on $\mathbb{P}^1$ the negative of the canonical bundle is ample, one can show using a method similar to the above that $\mathscr{M}_0 \simeq BPGL_2$, the classifying stack of PGL_2. For $\mathscr{M}_1$ the problem is more subtle. There are examples (see [**64**, XIII, 3.1 (b)]) which show that descent fails if one defines $\mathscr{M}_1$ as classifying proper smooth morphisms $f : C \to S$ of schemes all of whose geometric fibers are connected genus 1 curves. However, if one defines $\mathscr{M}_1$ as classifying proper smooth morphisms of algebraic spaces such that all geometric fibers are connected curves of genus 1, then $\mathscr{M}_1$ is a stack and in fact algebraic. For such a morphism $f : C \to S$ the functor $\underline{\mathrm{Pic}}^0_S$ of degree 0 line bundles is an elliptic curve and one obtains a morphism $\mathscr{M}_1 \to \mathscr{M}_{1,1}$, to the stack $\mathscr{M}_{1,1}$ classifying elliptic curves (see Chapter 13). This morphism realizes $\mathscr{M}_1$ as the classifying stack $B\mathscr{E}$ of the universal elliptic curve $\mathscr{E} \to \mathscr{M}_{1,1}$.

REMARK 8.4.16. We discuss the Deligne-Mumford compactification $\overline{\mathscr{M}}_g$ as well as other stacks of curves in Chapter 13.

8.5. Exercises

EXERCISE 8.A. Let $\mathscr{X}$ be the stack quotient of $\mathbb{A}^3$ by the action of $\mathbb{Z}/(2)$ given by the involution

$$(a, b, c) \mapsto (a, -b, -c).$$

Describe explicitly the inertia stack of $\mathscr{X}$.

EXERCISE 8.B. Let k be a field of characteristic 0, and let X/k be a smooth k-scheme with action of a finite group G. Show that the inertia stack of the quotient $[X/G]$ is smooth over k.

EXERCISE 8.C. Show that a morphism of schemes $g : Z \to W$ is formally unramified if and only if $\Omega^1_{Z/W} = 0$.

EXERCISE 8.D. Let

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

be morphisms of algebraic spaces. Show that if gf is formally unramified then f is formally unramified.

EXERCISE 8.E. Let S be a scheme, and let $\mathscr{A} = \bigoplus_{d \geq 0} \mathscr{A}_d$ be a quasi-coherent sheaf of graded algebras. Let C denote the relative spectrum $\underline{\mathrm{Spec}}_S(\mathscr{A})$. Let $Z \hookrightarrow C$ be the closed subscheme defined by the sheaf of ideals

$$\bigoplus_{d \geq 1} \mathscr{A}_d \subset \mathscr{A},$$

and let $U \subset C$ be the complement of Z. There is an action of $\mathbb{G}_m$ on C induced by the grading on $\mathscr{A}$. Namely, for any S-scheme $g : T \to S$ and $u \in \Gamma(T, \mathscr{O}_T^*)$ we get a homomorphism

$$\rho_u : g^*\mathscr{A} \to g^*\mathscr{A}$$

given by multiplication by u^d on $g^*\mathscr{A}_d$.

(i) Show that the $\mathbb{G}_m$-action on C induces a $\mathbb{G}_m$-action on U.

(ii) Show that the stack quotient $[U/\mathbb{G}_m]$ is isomorphic to the scheme

$$\underline{\mathrm{Proj}}_S(\mathscr{A})$$

defined as in [**41**, p. 160].

Exercise 8.F. Let $f : A \to S$ be a smooth morphism of schemes. For an integer d, let $H_d \to S$ be the Hilbert scheme associating to any S-scheme T the set of closed subschemes $E \subset A_T$ for which the projection $E \to T$ is finite flat of rank d.

(i) Show that there exists an open subscheme $U_d \subset H_d$ such that a morphism $T \to H_d$ corresponding to $E \subset A_T$ factors through U_d if and only if $E \to T$ is étale.

(ii) Show that the projection $U_d \to S$ is smooth.

(iii) Deduce that if $s \in S$ is a point, and $E_s \subset A_s$ is a closed subscheme étale over $\mathrm{Spec}(k(s))$ of rank d, then there exists an étale neighborhood $(S', s') \to (S, s)$ of s, a closed subscheme $E' \subset A_{S'}$ finite étale over S' such that the fiber $E'_{s'} \subset A_{s'}$ is equal to the pullback of E_s.

Exercise 8.G. Let H be a scheme and let $f : E \to E'$ be a morphism of locally free sheaves on H of finite rank. Let

$$F : (\mathrm{Sch}/H)^{\mathrm{op}} \to \mathrm{Set}$$

be the functor sending $g : T \to H$ to the unital set if $g^*(f) : g^*E \to g^*E'$ is the zero map, and the empty set otherwise. Show that F is represented by a closed subscheme of H.

Exercise 8.H. Let p be a prime, let S be a scheme over the finite field $\mathbb{F}_p$, and let $\mathscr{X}/\mathbb{F}_p$ be an algebraic stack. For an $\mathbb{F}_p$-scheme T, let $F_T : T \to T$ be the Frobenius morphism on T. Show that the functors

$$F_T^* : \mathscr{X}(T) \to \mathscr{X}(T),$$

defined after choosing a splitting of $\mathscr{X}$, define a morphism of stacks $F_{\mathscr{X}} : \mathscr{X} \to \mathscr{X}$, which in the case when $\mathscr{X}$ is a scheme agrees with the usual Frobenius morphism on X.

Exercise 8.I. Let S be a scheme and let $\mathscr{X}/S$ be an algebraic stack. For a morphism $f : \mathscr{X} \to Y$ from $\mathscr{X}$ to an S-scheme Y, define the *image* of f to be the subset of Y which is the image of the composite morphism

$$U \xrightarrow{u} \mathscr{X} \xrightarrow{f} Y,$$

where $u : U \to \mathscr{X}$ is a smooth surjective morphism with U a scheme.

(i) Show that this definition of image of f is independent of the choice of $u : U \to \mathscr{X}$.

(ii) Show that in the case when $\mathscr{X}$ is an algebraic space, the image of f as defined here agrees with the image of the morphism of topological spaces $|\mathscr{X}| \to |Y|$ defined in 6.3.5.

Exercise 8.J. Let $f : X \to S$ be a proper flat morphism of algebraic spaces which étale locally on S is projective. Let $\mathscr{V}ec_{X/S}$ be the fibered category over the category of S-schemes whose fiber over $T \to S$ is the groupoid of locally free sheaves of finite rank on $X_T := X \times_S T$. Prove that $\mathscr{V}ec_{X/S}$ is an algebraic stack as follows.

(i) Show that $\mathscr{V}ec_{X/S}$ is a stack with respect to the étale topology, and reduce the proof to the case when X/S is projective.

(ii) In this case, fix a relatively ample invertible sheaf L on X, and define $\mathscr{V}ec^{r,n}_{X/S} \subset \mathscr{V}ec_{X/S}$ to be the substack whose fiber over $T \to S$ is the groupoid of locally free sheaves E on X_T of rank r such that for every geometric point $\bar{t} \to T$ the restriction $E_{\bar{t}}$ of E to the fiber $X_{\bar{t}}$ satisfies the following two conditions:

(a) For every $m \geq n$ the natural map $H^0(X_{\bar{t}}, E_{\bar{t}} \otimes L_{\bar{t}}^{\otimes m}) \otimes L_{\bar{t}}^{\otimes -m} \to E_{\bar{t}}$ is surjective.

(b) For every $m \geq n$ we have $H^i(X_{\bar{t}}, E_{\bar{t}} \otimes L_{\bar{t}}^{\otimes m}) = 0$ for $i > 0$.

Show that $\mathscr{V}ec^{r,n}_{X/S} \hookrightarrow \mathscr{V}ec_{X/S}$ is representable by open immersions.

(iii) Show that for an object E of $\mathscr{V}ec^{r,n}_{X/S}$ over $T \to S$, the sheaf $f_{T*}(E \otimes L^{\otimes n})$ is a locally free sheaf on T whose formation commutes with arbitrary base change $T' \to T$, where $f_T : X_T \to T$ is the base change of f. Using this, define a suitable stack $\widetilde{\mathscr{V}ec}^{r,n}_{X/S}$ classifying pairs (E, σ), where E is an object of $\mathscr{V}ec^{r,n}_{X/S}$ and σ is a basis for $f_{T*}(E \otimes L^{\otimes n})$, and show that $\widetilde{\mathscr{V}ec}^{r,n}_{X/S}$ is a scheme.

(iv) Deduce from the representability of $\widetilde{\mathscr{V}ec}^{r,n}_{X/S}$ that $\mathscr{V}ec^{r,n}_{X/S}$ is an algebraic stack.

CHAPTER 9

Quasi-coherent sheaves on algebraic stacks

Our treatment of quasi-coherent sheaves on algebraic stacks follows the book of Laumon and Moret-Bailly [**49**]. In particular, we define quasi-coherent sheaves as certain sheaves on the lisse-étale site. One can develop the theory of quasi-coherent sheaves on algebraic stacks using several different topologies. The lisse-étale topology is one choice, but one can also consider the big Zariski, big étale, fppf, or fpqc topologies. The small étale topology can also be used for Deligne-Mumford stacks, but for general Artin stacks it does not give a good theory. The big topologies have the advantage that they are almost tautologically functorial. On the other hand, they have the disadvantage that the inclusion of the category of quasi-coherent sheaves into the category of all $\mathscr{O}$-modules is not exact. This last property does hold for the lisse-étale topology, but the lisse-étale topos on the other hand is not functorial which presents some technical difficulties (in particular, the definition of the pullback of a quasi-coherent sheaf requires some care). For a morphism of algebraic stacks $f : \mathscr{X} \to \mathscr{Y}$ there is an induced morphism of lisse-étale sites, but this morphism does not induce a morphism of topoi.

Throughout this chapter we work over a fixed base scheme S.

9.1. The lisse-étale site

Let $\mathscr{X}/S$ be an algebraic stack.

Definition 9.1.1. An *$\mathscr{X}$-space* is a pair (T, t), where T is an algebraic space over S and $t : T \to \mathscr{X}$ is a morphism over S. A morphism of $\mathscr{X}$-spaces

$$(T', t') \to (T, t)$$

is a pair (f, f^b), where $f : T' \to T$ is an S-morphism of algebraic spaces and $f^b : t' \to t \circ f$ is an isomorphism of functors $T' \to \mathscr{X}$.

For the diagram

$$(T'', t'') \xrightarrow{(g,g^b)} (T', t') \xrightarrow{(f,f^b)} (T, t)$$

the composition $(T'', t'') \to (T, t)$ is defined to be the morphism $f \circ g : T'' \to T$ together with the isomorphism of functors

$$t'' \xrightarrow{g^b} t' \circ g \xrightarrow{g(f^b)} t \circ f \circ g \simeq t \circ (f \circ g).$$

We denote the category of $\mathscr{X}$-spaces by $\mathrm{AS}/\mathscr{X}$ (here AS stands for algebraic spaces).

Define the category of *$\mathscr{X}$-schemes*, denoted $\mathrm{Sch}/\mathscr{X}$, to be the full subcategory of $\mathrm{AS}/\mathscr{X}$ consisting of pairs (T, t) where T is a scheme.

REMARK 9.1.2. If $(f, f^b) : (T', t') \to (T, t)$ is a morphism of $\mathscr{X}$-spaces we often think of f^b as an isomorphism $t' \to t \circ f$ in $\mathscr{X}(T')$ (using the 2-Yoneda lemma).

REMARK 9.1.3. If $\mathscr{X}$ is an algebraic space, then the category of $\mathscr{X}$-spaces is the usual category of algebraic spaces over $\mathscr{X}$.

REMARK 9.1.4. If $f : \mathscr{Y} \to \mathscr{X}$ and $f' : \mathscr{Y}' \to \mathscr{X}$ are morphisms of algebraic stacks, then an *$\mathscr{X}$-morphism* $\mathscr{Y} \to \mathscr{Y}'$ is a pair (g, σ), where $g : \mathscr{Y} \to \mathscr{Y}'$ is a morphism of stacks and $\sigma : f \to f' \circ g$ is an isomorphism of morphisms $\mathscr{Y} \to \mathscr{X}$. The collection of $\mathscr{X}$-morphisms $\mathscr{Y} \to \mathscr{Y}'$ forms a category $\mathrm{HOM}_{\mathscr{X}}(\mathscr{Y}, \mathscr{Y}')$. A morphism $(g, \sigma) \to (g', \sigma')$ is an isomorphism $\lambda : g \to g'$ such that the induced diagram

$$\begin{tikzcd} f \arrow[r, "\sigma"] \arrow[dr, "\sigma'"'] & f' \circ g \arrow[d, "\lambda"] \\ & f' \circ g' \end{tikzcd}$$

commutes. If f and f' are representable morphisms, then such an isomorphism λ is unique if it exists, so $\mathrm{HOM}_{\mathscr{X}}(\mathscr{Y}, \mathscr{Y}')$ is equivalent to a set. We can therefore define a category $\mathrm{RS}/\mathscr{X}$ of *relative spaces over $\mathscr{X}$* whose objects are representable morphisms of algebraic stacks $\mathscr{Y} \to \mathscr{X}$ and whose morphisms are isomorphism classes of $\mathscr{X}$-morphisms.

There is an inclusion $\mathrm{AS}/\mathscr{X} \hookrightarrow \mathrm{RS}/\mathscr{X}$.

9.1.5. Note that if $f : \mathscr{X} \to \mathscr{Y}$ is a morphism of algebraic stacks, then there is a natural functor

$$\mathrm{AS}/\mathscr{X} \to \mathrm{AS}/\mathscr{Y}, \quad (T, t) \mapsto (T, f \circ t),$$

which restricts to a functor

$$\mathrm{Sch}/\mathscr{X} \to \mathrm{Sch}/\mathscr{Y}.$$

DEFINITION 9.1.6. The *lisse-étale site of $\mathscr{X}$* is the full subcategory

$$\text{Lis-Ét}(\mathscr{X}) \subset \mathrm{Sch}/\mathscr{X}$$

consisting of pairs (T, t), where $t : T \to \mathscr{X}$ is a smooth morphism. A collection of maps

$$\{(f_i, f_i^b) : (T_i, t_i) \to (T, t)\}$$

is defined to be a covering if the underlying collection of morphisms of schemes $\{f_i : T_i \to T\}$ is an étale covering. We write $\mathscr{X}_{\text{lis-ét}}$ for the resulting topos.

REMARK 9.1.7. We could also define the lisse-étale site using $\mathrm{AS}/\mathscr{X}$ instead of $\mathrm{Sch}/\mathscr{X}$. The associated topoi are canonically equivalent (exercise 2.H), and for technical reasons we will at times prefer working with the slightly larger site obtained using $\mathrm{AS}/\mathscr{X}$. Similarly we could use the full subcategory of $\mathrm{RS}/\mathscr{X}$ whose objects are smooth representable morphisms $\mathscr{Y} \to \mathscr{X}$ for which $\mathscr{Y}$ admits an étale cover by an algebraic space (or equivalently by a scheme).

REMARK 9.1.8. For any object $(T, t) \in \text{Lis-Ét}(\mathscr{X})$, there is a an inclusion

$$\text{Ét}(T) \hookrightarrow \text{Lis-Ét}(\mathscr{X}), \quad (h : T' \to T) \mapsto (T', t \circ h),$$

where $\text{Ét}(T)$ denotes the étale site of T. Observe that if

$$F : \text{Lis-Ét}(\mathscr{X}) \to \mathrm{Set}$$

is a functor, then F is a sheaf if and only if for every $(T,t) \in \text{Lis-Ét}(\mathscr{X})$ the restriction of F to $\text{Ét}(T)$ is a sheaf.

EXAMPLE 9.1.9 (The structure sheaf). Define $\mathscr{O}_{\mathscr{X}}$ to be the sheaf on $\text{Lis-Ét}(\mathscr{X})$ which to any (T,t) associates $\Gamma(T, \mathscr{O}_T)$.

9.1.10. The category of sheaves on $\text{Lis-Ét}(\mathscr{X})$ can also be described as follows. Let $\mathscr{C}$ denote the category of data $(\{F_{(T,t)}\}, \{\rho_{(f,f^b)}\})$ as follows:

(1) For each $(T,t) \in \text{Lis-Ét}(\mathscr{X})$ an étale sheaf of sets $F_{(T,t)}$ on T.
(2) For each morphism $(f, f^b) : (T', t') \to (T,t)$ in $\text{Lis-Ét}(\mathscr{X})$ a morphism of sheaves
$$\rho_{(f,f^b)} : f^{-1}F_{(T,t)} \to F_{(T',t')}.$$

This data is further required to satisfy the following conditions:

(i) For any composition in $\text{Lis-Ét}(\mathscr{X})$,
$$(T'', t'') \xrightarrow{(g,g^b)} (T', t') \xrightarrow{(f,f^b)} (T,t),$$
the diagram
$$\begin{tikzcd} g^{-1}f^{-1}F_{(T,t)} \arrow[r, "g^{-1}\rho_{(f,f^b)}"] \arrow[d, "\simeq"] & g^{-1}F_{(T',t')} \arrow[d, "\rho_{(g,g^b)}"] \\ (fg)^{-1}F_{(T,t)} \arrow[r, "\rho_{(f,f^b)\circ(g,g^b)}"] & F_{(T'',t'')} \end{tikzcd}$$
commutes.
(ii) If $(f, f^b) : (T', t') \to (T,t)$ is a morphism with f étale, then the map $\rho_{(f,f^b)}$ is an isomorphism.

A morphism
$$(\{F_{(T,t)}\}, \{\rho_{(f,f^b)}\}) \to (\{G_{(T,t)}\}, \{\lambda_{(f,f^b)}\})$$
in $\mathscr{C}$ is a collection of morphisms $\gamma_{(T,t)} : F_{(T,t)} \to G_{(T,t)}$, one for each (T,t), such that for any morphism $(f, f^b) : (T', t') \to (T,t)$ the diagram
$$\begin{tikzcd} f^{-1}F_{(T,t)} \arrow[r, "f^{-1}\gamma_{(T,t)}"] \arrow[d, "\rho_{(f,f^b)}"] & f^{-1}G_{(T,t)} \arrow[d, "\lambda_{(f,f^b)}"] \\ F_{(T',t')} \arrow[r, "\gamma_{(T',t')}"] & G_{(T',t')} \end{tikzcd}$$
commutes.

9.1.11. There is a functor
$$h : \mathscr{C} \to \mathscr{X}_{\text{lis-ét}} \tag{9.1.11.1}$$
which sends $(\{F_{(T,t)}\}, \{\rho_{(f,f^b)}\})$ to the functor
$$\text{Lis-Ét}(\mathscr{X})^{op} \to \text{Set}, \quad (T,t) \mapsto \Gamma(T, F_{(T,t)}). \tag{9.1.11.2}$$
The functor structure is given by associating to a morphism $(f, f^b) : (T', t') \to (T,t)$ the map
$$\Gamma(T, F_{(T,t)}) \to \Gamma(T', F_{(T',t')})$$

induced by $\rho_{(f,f^b)}$. Note that the functor (9.1.11.2) is a sheaf by axiom (ii) for the system $(\{F_{(T,t)}\}, \{\rho_{(f,f^b)}\})$.

PROPOSITION 9.1.12. *The functor* (9.1.11.1) *is an equivalence of categories.*

PROOF. An inverse to (9.1.11.1) is obtained by associating to a sheaf F the system with $F_{(T,t)}$ the restriction of F to the étale site of T, and transition maps those induced by the sheaf structure on F. $\square$

REMARK 9.1.13. In what follows, if F is a sheaf on Lis-Ét$(\mathscr{X})$ and (T,t) is an object of this category, we write $F_{(T,t)}$ for the restriction of F to the étale site of T.

DEFINITION 9.1.14. (i) Let Λ be a sheaf of rings on Lis-Ét$(\mathscr{X})$. A sheaf F of Λ-modules in $\mathscr{X}_{\text{lis-ét}}$ is *cartesian* if for every morphism $(f,f^b) : (T',t') \to (T,t)$ the map of $\Lambda_{(T',t')}$-modules

$$f^*F_{(T,t)} := f^{-1}F_{(T,t)} \otimes_{f^{-1}\Lambda_{(T,t)}} \Lambda_{(T',t')} \to F_{(T',t')}$$

is an isomorphism.

(ii) A sheaf of $\mathscr{O}_{\mathscr{X}}$-modules F is *quasi-coherent* if F is cartesian and for every object (T,t) the sheaf $F_{(T,t)}$ is a quasi-coherent sheaf on T.

(iii) If $\mathscr{X}$ is locally noetherian (which implies that for any $(T,t) \in$ Lis-Ét$(\mathscr{X})$ the scheme T is locally noetherian), then a quasi-coherent sheaf F on $\mathscr{X}$ is called *coherent* if each $F_{(T,t)}$ is coherent.

We denote the category of quasi-coherent sheaves on an algebraic stack $\mathscr{X}$ by QCoh$(\mathscr{X})$, or QCoh$(\mathscr{X}_{\text{lis-ét}})$ if we want to make explicit the reference to the lisse-étale site.

PROPOSITION 9.1.15. *Let $\mathscr{X}$ be an algebraic stack (resp. locally noetherian algebraic stack) and $x : X \to \mathscr{X}$ a smooth surjection with X a scheme. Then a cartesian sheaf of $\mathscr{O}_{\mathscr{X}}$-modules F is quasi-coherent (resp. coherent) if and only if the sheaf $F_{(X,x)}$ on X is quasi-coherent (resp. coherent).*

PROOF. Let $(Z,z) \in$ Lis-Ét$(\mathscr{X})$ be another object, and form the cartesian square

$$\begin{tikzcd} P \arrow[r, "q"] \arrow[d, "p"] & X \arrow[d, "x"] \\ Z \arrow[r, "z"] & \mathscr{X}. \end{tikzcd}$$

Since the morphism p is smooth, there exists an étale surjection $\pi : Z' \to Z$ and a morphism $s : Z' \to P$ such that $p \circ s = \pi$ (see for example [**67**, Tag 05JU]). To verify that $F_{(Z,z)}$ is quasi-coherent (resp. coherent) it suffices to verify that $\pi^*F_{(Z,z)} = F_{(Z',z\circ\pi)}$ is quasi-coherent. This reduces us to verifying that $F_{(Z,z)}$ is quasi-coherent (resp. coherent) in the case when the morphism z factors as

$$Z \xrightarrow{g} X \xrightarrow{x} \mathscr{X}.$$

But in this case we have

$$F_{(Z,z)} \simeq g^*F_{(X,x)}$$

since F is cartesian, which implies the proposition. $\square$

9.1.16. If $\mathscr{X}/S$ is a Deligne-Mumford stack over the base scheme S, we can also consider the étale site of $\mathscr{X}$, denoted $\text{Ét}(\mathscr{X})$. This is the full subcategory of $\text{Lis-Ét}(\mathscr{X})$ consisting of étale morphisms $X \to \mathscr{X}$ with X an algebraic space. Coverings in $\text{Ét}(\mathscr{X})$ are the same as coverings in $\text{Lis-Ét}(\mathscr{X})$. We write $\mathscr{X}_{\text{ét}}$ for the associated topos, and $\mathscr{O}_{\mathscr{X}_{\text{ét}}}$ for the restriction of the structure sheaf $\mathscr{O}_{\mathscr{X}}$ in $\mathscr{X}_{\text{lis-ét}}$ to $\mathscr{X}_{\text{ét}}$.

A sheaf $\mathscr{M}$ of $\mathscr{O}_{\mathscr{X}_{\text{ét}}}$-modules is called *quasi-coherent* if its restriction $\mathscr{M}_X$ to the étale site of any object $X \to \mathscr{X}$ of $\text{Ét}(\mathscr{X})$ is quasi-coherent.

9.1.17. The inclusion

$$\text{Ét}(\mathscr{X}) \hookrightarrow \text{Lis-Ét}(\mathscr{X})$$

induces a morphism of ringed topoi

$$r : (\mathscr{X}_{\text{lis-ét}}, \mathscr{O}_{\mathscr{X}}) \to (\mathscr{X}_{\text{ét}}, \mathscr{O}_{\mathscr{X}_{\text{ét}}}).$$

To see this let $\text{Ét}(\text{RS}/\mathscr{X})$ denote the full subcategory of the category of relative spaces over $\mathscr{X}$ consisting of étale representable morphisms $\mathscr{Y} \to \mathscr{X}$. Endowed with the étale topology the category $\text{Ét}(\text{RS}/\mathscr{X})$ also defines the étale topos of $\mathscr{X}$. The advantage of the site $\text{Ét}(\text{RS}/\mathscr{X})$ is that all finite projective limits exist in $\text{Ét}(\text{RS}/\mathscr{X})$, whereas the category $\text{Ét}(\mathscr{X})$ has finite nonempty projective limits but not a final object. Similarly define $\text{Lis-Ét}(\text{RS}/\mathscr{X})$ to be the category of smooth representable morphisms $\mathscr{Y} \to \mathscr{X}$ for which $\mathscr{Y}$ admits an étale cover by a scheme and coverings are étale surjective families of morphisms. The inclusion

$$\text{Ét}(\text{RS}/\mathscr{X}) \hookrightarrow \text{Lis-Ét}(\text{RS}/\mathscr{X})$$

then induces by 2.2.31 the desired morphism of ringed topoi r.

Note also that the restriction functor r_* is exact.

PROPOSITION 9.1.18. *Let $\mathscr{X}$ be a Deligne-Mumford stack. Then the restriction functor*

$$r_* : (\text{quasi-coherent sheaves on } \mathscr{X}_{\text{lis-ét}}) \to (\text{quasi-coherent sheaves on } \mathscr{X}_{\text{ét}})$$

is an equivalence of categories, with quasi-inverse r^*.

PROOF. Let us first observe that if M is a quasi-coherent sheaf on $\mathscr{X}_{\text{ét}}$, then r^*M is quasi-coherent on $\mathscr{X}_{\text{lis-ét}}$. If $W \to \mathscr{X}$ is an étale morphism with W a scheme, then the localized topos $\mathscr{X}_{\text{lis-ét}}|_W$ is isomorphic to $W_{\text{lis-ét}}$ since $W \to \mathscr{X}$ is étale. It therefore suffices to consider the case when W is an affine scheme. In this case r^*M is the sheaf on $\text{Lis-Ét}(W)$ which to any smooth morphism $f : U \to W$ associates $\Gamma(U_{\text{ét}}, f^*M)$, where f^*M denotes the pullback of M to $U_{\text{ét}}$. From this it follows that r^*M is quasi-coherent, and this explicit description of r^* also implies that the adjunction map $M \to r_*r^*M$ is an isomorphism.

It remains to show that if $\mathscr{M}$ is a quasi-coherent sheaf on $\mathscr{X}_{\text{lis-ét}}$, then the adjunction map

$$r^*r_*\mathscr{M} \to \mathscr{M}$$

is an isomorphism. For this note that if $f : U \to \mathscr{X}$ is a smooth morphism, then étale locally on U there exists a factorization $g : U \to V$ of f through an étale morphism $V \to \mathscr{X}$. In this case the map

$$(r^*r_*\mathscr{M})_U \to \mathscr{M}_U$$

is identified with the map

$$g^*\mathscr{M}_V \to \mathscr{M}_U$$

which is an isomorphism since $\mathscr{M}$ is cartesian. □

EXAMPLE 9.1.19. Let S be a scheme and let G be a finite group. Let BG denote the associated classifying stack over S (see 8.1.14). The étale topos of BG can be described more explicitly in the following way. Define the *G-equivariant étale site of S*, denoted $G\text{-Ét}(S)$, as follows. The objects of $G\text{-Ét}(S)$ are simply étale morphisms of schemes $T \to S$. A morphism $T'/S \to T/S$ in $G\text{-Ét}(S)$ is a pair (f,g), where $f : T' \to T$ is an S-morphism and $g \in G$ is an element of the group. The composition of a diagram

$$T'' \xrightarrow{(f',g')} T' \xrightarrow{(f,g)} T$$

in $G\text{-Ét}(S)$ is defined to be $(f \circ f', gg')$. A collection of morphisms $\{(f_i,g_i) : T_i \to T\}_{i\in I}$ in $G\text{-Ét}(S)$ is defined to be a covering if the maps $\{f_i : T_i \to T\}$ form an étale covering of T. There is a functor

$$Y : G\text{-Ét}(S) \to \text{Ét}(BG).$$

This functor sends an étale S-scheme T/S to the morphism $T \to BG$ defined by the trivial G-torsor P_T^0 over T. A morphism $(f,g) : T' \to T$ in $G\text{-Ét}(S)$ is sent to the morphism over BG given by $f : T' \to T$ and the morphism of trivial G-torsors given by right multiplication by g. Note that this functor is fully faithful, and every object of $\text{Ét}(BG)$ admits a covering by objects in the essential image. It follows in particular that Y induces an equivalence on the associated topoi (exercise 2.H).

Write $\mathscr{O}_{G-S}$ for the sheaf of rings on $G\text{-Ét}(S)$ which to any étale T/S associates $\Gamma(T,\mathscr{O}_T)$, and for any morphism $(f,g) : T' \to T$ associates $f^* : \Gamma(T,\mathscr{O}_T) \to \Gamma(T',\mathscr{O}_{T'})$. This sheaf of rings corresponds under the equivalence of topoi with $BG_{\text{ét}}$ to the structure sheaf.

Now sheaves of $\mathscr{O}_{G-S}$-modules have the following concrete description. We have a functor

$$\text{Ét}(S) \to G\text{-Ét}(S)$$

which sends an étale morphism $T \to S$ to itself, and a morphism $f : T' \to T$ to (f, id). By restriction a sheaf F of $\mathscr{O}_{G-S}$-modules defines a sheaf of $\mathscr{O}_S$-modules $\underline{F}$ on $\text{Ét}(S)$. This sheaf $\underline{F}$ has the additional structure of maps

$$t_g : \underline{F} \to \underline{F}, \quad g \in G$$

corresponding to the automorphisms $(\text{id}_T, g) : T \to T$ of any object $T/S \in G\text{-Ét}(S)$. The compatibility with composition is equivalent to the condition that for $g, g' \in G$ we have $t_{g'} \circ t_g = t_{gg'}$. In other words, we have a left action of G on $\underline{F}$. Conversely, given a sheaf of $\mathscr{O}_S$-modules $\underline{F}$ on $\text{Ét}(S)$ and a left action t of G on $\underline{F}$, we get a sheaf F on $G\text{-Ét}(S)$ by sending T/S to $\underline{F}(T)$ and a morphism

$$(f,g) : T' \to T$$

to the composition of $f^* : \Gamma(T,\underline{F}) \to \Gamma(T',\underline{F})$ with $t_g : \Gamma(T',\underline{F}) \to \Gamma(T',\underline{F})$. Moreover, these constructions give an equivalence between the category of $\mathscr{O}_{G-S}$-modules and the category of $\mathscr{O}_S$-modules with a left action of G. In this way we get an equivalence of categories

$$(\mathscr{O}_{BG_{\text{ét}}}\text{-modules on Ét}(BG)) \simeq (\mathscr{O}_S\text{-modules with left } G\text{-action on Ét}(S)).$$

It follows from the construction that this equivalence induces an equivalence of categories

(quasi-coherent sheaves on BG)$\simeq$(quasi-coherent sheaves on S with left G-action).

9.2. Comparison with simplicial sheaves and the étale topos

9.2.1. Let S be a scheme and $\mathscr{X}$ an algebraic stack over S. Let $p : X \to \mathscr{X}$ be a smooth surjection with X an algebraic space. We then obtain a simplicial algebraic space $X_\bullet$, the *coskeleton of* $p : X \to \mathscr{X}$, as follows. As in 2.4.2, let Δ denote the standard simplicial category and consider the functor sending $[n] \in \Delta$ to the $(n+1)$-fold fiber product

$$X_n := \overbrace{X \times_{\mathscr{X}} X \cdots \times_{\mathscr{X}} X}^{n+1}.$$

Given a morphism $\delta : [n] \to [m]$ let

$$\delta^* : X_m \to X_n$$

be the map given by

$$\mathrm{pr}_{\delta(0)} \times \mathrm{pr}_{\delta(1)} \times \cdots \times \mathrm{pr}_{\delta(n)} : X_m \to X \times_{\mathscr{X}} X \cdots \times_{\mathscr{X}} X,$$

where $\mathrm{pr}_i : X_m \to X$ is the i-th projection. Note that there is an augmentation,

$$e : X_\bullet \to \mathscr{X},$$

and each of the maps $e_n : X_n \to \mathscr{X}$ are smooth and surjective.

9.2.2. The sheaf on Lis-Ét$(\mathscr{X})$ defined by the object $p : X \to \mathscr{X}$ is a covering of the final object of $\mathscr{X}_{\text{lis-ét}}$, and so we can consider the localized topos

$$\mathscr{X}_{\text{lis-ét}}|_{X_\bullet} \to \mathscr{X}_{\text{lis-ét}},$$

as in 2.4.11.

Observe that for $[n] \in \Delta$, the localized topos $\mathscr{X}_{\text{lis-ét}}|_{X_n}$ is *not* equivalent to the topos $X_{n,\text{lis-ét}}$. The topos $\mathscr{X}_{\text{lis-ét}}|_{X_n}$ is the topos associated to the site whose objects are morphisms of algebraic spaces $g : U \to X_n$ such that the composition $U \to X_n \to \mathscr{X}$ is smooth. However, the morphism g need not be smooth itself.

Nonetheless, there is a morphism of topoi

$$a : \mathscr{X}_{\text{lis-ét}}|_{X_\bullet} \to X_{\bullet,\text{ét}},$$

where $X_{\bullet,\text{et}}$ is defined as in [**49**, 12.4]. Recall that $X_{\bullet,\text{ét}}$ is the category of systems $F_\bullet = (F_n, F(\delta))$ consisting of a sheaf F_n on $X_{n,\text{ét}}$ for every n and for every morphism $\delta : [n] \to [m]$ a morphism

$$F(\delta) : \delta^{-1}F_n \to F_m,$$

where we also write $\delta : X_m \to X_n$ for the map coming from the simplicial structure. These maps are further required to satisfy a compatibility with composition (see 2.4.5).

The functor a_* sends a system of sheaves $F_\bullet$ in $\mathscr{X}_{\text{lis-ét}}|_{X_\bullet}$ to the system of sheaves whose restriction to X_n is given by the restriction of F_n to the étale site of X_n. The functor a^{-1} sends a sheaf $G_\bullet$ on $X_{\bullet,\text{ét}}$ to the object in $\mathscr{X}_{\text{lis-ét}}|_{X_\bullet}$ whose restriction to $\mathscr{X}_{\text{lis-ét}}|_{X_n}$ is the pullback of G_n along the natural morphism of topoi

$$\mathscr{X}_{\text{lis-ét}}|_{X_n} \to X_{n,\text{ét}},$$

together with the transition maps induced by the transition maps of $G_\bullet$. If we view $\mathscr{X}_{\text{lis-ét}}|_{X_\bullet}$ as ringed by the restriction of the structure sheaf $\mathscr{O}_{\mathscr{X}}$, and $X_{\text{ét}}$ as ringed by the structure sheaf $\mathscr{O}_{X_{\text{ét}}}$, then a is even a morphism of ringed topoi. Note also that the functor a_* is exact.

9.2.3. We now have a diagram

$$\begin{tikzcd} \mathscr{X}_{\text{lis-ét}}|_{X_\bullet} \arrow[r, "a"] \arrow[d, "e"] & X_{\bullet,\text{ét}} \\ \mathscr{X}_{\text{lis-ét}}, & \end{tikzcd}$$

which induces for every quasi-coherent sheaf F on $\mathscr{X}$ a diagram

$$\begin{tikzcd} H^i(\mathscr{X}_{\text{lis-ét}}|_{X_\bullet}, e^{-1}F) & H^i(X_{\bullet,\text{ét}}, F_\bullet) \arrow[l, "a^*"'] \\ H^i(\mathscr{X}_{\text{lis-ét}}, F), \arrow[u, "e^*"] & \end{tikzcd}$$

where we write $F_\bullet$ for the sheaf $a_*e^{-1}F$. This sheaf $F_\bullet$ has a more concrete description: It is the sheaf on $X_{\bullet,\text{ét}}$, which in degree n is simply the restriction F_n of F to the étale site of X_n and whose transition maps are those induced by the transition maps of F (see 2.4.11). By 2.4.16 the map e^* is an isomorphism, and the map a^* is an isomorphism since a_* is exact.

Corollary 9.2.4. *For every i and $F \in \mathrm{Qcoh}(\mathscr{X})$, we have a natural isomorphism $H^i(\mathscr{X}_{\text{lis-ét}}, F) \simeq H^i(X_{\bullet,\text{ét}}, F_\bullet)$. In particular, by* 2.4.25 *there is a spectral sequence*

$$E_1^{pq} = H^p(X_q, F_q) \implies H^{p+q}(\mathscr{X}, F).$$

Proof. This follows from the discussion above. □

9.2.5. The above results can be relativized as follows. Let

$$f : \mathscr{X} \to \mathscr{Y}$$

be a morphism of algebraic stacks over a base scheme S. We can then define a functor

$$f_* : \mathrm{Mod}_{\mathscr{O}_{\mathscr{X}}} \to \mathrm{Mod}_{\mathscr{O}_{\mathscr{Y}}}$$

by sending an $\mathscr{O}_{\mathscr{X}}$-module F in $\mathscr{X}_{\text{lis-ét}}$ to the sheaf on Lis-Ét$(\mathscr{Y})$ which to any object $U \to \mathscr{Y}$ associates

$$H^0(\mathscr{X}_{U,\text{lis-ét}}, F_{\mathscr{X}_U}),$$

where $\mathscr{X}_U$ denotes the fiber product $\mathscr{X} \times_{\mathscr{Y}} U$ and $F_{\mathscr{X}_U}$ denotes the restriction of F to the lisse-étale site of the smooth $\mathscr{X}$-stack $\mathscr{X}_U$. The functor f_* is left exact, so we can form the derived functors

$$R^if_* : \mathrm{Mod}_{\mathscr{O}_{\mathscr{X}}} \to \mathrm{Mod}_{\mathscr{O}_{\mathscr{Y}}}.$$

Remark 9.2.6. Note that this functor f_* is not induced by a morphism of topoi from $\mathscr{X}_{\text{lis-ét}}$ to $\mathscr{Y}_{\text{lis-ét}}$, so a little more work is needed to show that the functors R^if_* have good properties. The basic strategy is to reduce everything to questions about morphisms of simplicial algebraic spaces, where we can use the functorial étale topos.

9.2.7. Fix an object $g : U \to \mathscr{Y}$ of Lis-Ét($\mathscr{Y}$), and let Lis-Ét($\mathscr{X}|_U$) denote the site whose objects are triples $(h : V \to \mathscr{X}, f_V, \alpha)$, where $V \to \mathscr{X}$ is an object of Lis-Ét($\mathscr{X}$), $f_V : V \to U$ is a morphism of algebraic spaces, and $\alpha : g \circ f_V \simeq f \circ h$ is an isomorphism in $\mathscr{Y}(V)$. A morphism

$$(h' : V' \to \mathscr{X}, f_{V'}, \alpha') \to (h : V \to \mathscr{X}, f_V, \alpha)$$

is a morphism $(a, a^b) : (V' \to \mathscr{X}) \to (V \to \mathscr{X})$ in Lis-Ét($\mathscr{X}$) such that the diagram

$$\begin{tikzcd} V' \arrow[rr, "a"] \arrow[dr] & & V \arrow[dl, "f_V"'] \\ & U & \end{tikzcd}$$

commutes, and such that the resulting diagram

$$\begin{tikzcd} g \circ f_{V'} \arrow[r, equal] \arrow[d, "\alpha'"] & g \circ f_V \circ a \arrow[d, "\alpha"] \\ f \circ h' \arrow[r, "a^b"] & f \circ h \circ a \end{tikzcd}$$

commutes. A covering in Lis-Ét($\mathscr{X}|_U$) is a collection of morphisms

$$\{(h_i : V_i \to \mathscr{X}, f_{V_i}, \alpha_i) \to (h : V \to \mathscr{X}, f_V, \alpha)\}_{i \in I}$$

such that

$$\{(h_i : V_i \to \mathscr{X}) \to (h : V \to \mathscr{X})\}_{i \in I}$$

is a covering in Lis-Ét($\mathscr{X}$). The resulting topos we denote by $\mathscr{X}_{\text{lis-ét}}|_U$. Note that if f is representable, then $\mathscr{X} \times_{\mathscr{Y}} U \to \mathscr{X}$ is an object of Lis-Ét($\mathscr{X}$), and the topos $\mathscr{X}_{\text{lis-ét}}|_U$ is the localized topos $\mathscr{X}_{\text{lis-ét}}|_{\mathscr{X} \times_{\mathscr{Y}} U}$.

More generally, let $f^{-1}U$ denote the sheaf on Lis-Ét($\mathscr{X}$) which to any $V \to \mathscr{X}$ denotes the set of $\mathscr{Y}$-morphisms $V \to U$. Then $\mathscr{X}_{\text{lis-ét}}|_U$ is equivalent to the localized topos $\mathscr{X}_{\text{lis-ét}}|_{f^{-1}U}$.

9.2.8. For a sheaf F of $\mathscr{O}_{\mathscr{X}}$-modules on $\mathscr{X}_{\text{lis-ét}}$, there is a natural map

$$\Gamma(\mathscr{X}_{U,\text{lis-ét}}, F_{\mathscr{X}_U}) \to \Gamma(\mathscr{X}_{\text{lis-ét}}|_U, F).$$

This map is in fact an isomorphism. Indeed, if $W \to \mathscr{X}_U$ is a smooth surjective morphism with W as a scheme, and W' denotes $W \times_{\mathscr{X}_U} W$, then W defines a covering of the final object in both $\mathscr{X}_{\text{lis-ét}}|_U$ and $\mathscr{X}_{U,\text{lis-ét}}$ and so the global sections of F in either $\mathscr{X}_{U,\text{lis-ét}}$ or $\mathscr{X}_{\text{lis-ét}}|_U$ are both identified with

$$\text{Eq}(\Gamma(W, F_W) \rightrightarrows \Gamma(W', F_{W'})).$$

We can therefore also think of the functor f_* as the functor which sends a sheaf F of $\mathscr{O}_{\mathscr{X}}$-modules to the sheaf associated to the presheaf sending $(U \to \mathscr{Y}) \in$ Lis-Ét($\mathscr{Y}$) to

$$\Gamma(\mathscr{X}_{\text{lis-ét}}|_U, F).$$

Now if F is a sheaf of $\mathscr{O}_{\mathscr{X}}$-modules, and $F \to I^\bullet$ is an injective resolution, then the restriction of this resolution to the localized topos $\mathscr{X}_{\text{lis-ét}}|_U$ is again an injective resolution of the restriction of F (see for example exercise 2.D), and therefore we find that the sheaf $R^i f_* F$ is the sheaf on Lis-Ét($\mathscr{Y}$) associated to the presheaf which sends $U \to \mathscr{Y}$ to

$$H^i(\mathscr{X}_{\text{lis-ét}}|_U, F).$$

9.2.9. Next we calculate the groups $H^i(\mathscr{X}_{\text{lis-ét}}|_U, F)$. For this let $W \to \mathscr{X}_U$ be a smooth surjective morphism with W a scheme, and let $W_\bullet$ denote the coskeleton (see 2.4.14). For ease of notation write $\widetilde{\mathscr{X}_\bullet}$ for the simplicial topos

$$(\mathscr{X}_{\text{lis-ét}}|_U)|_{W_\bullet},$$

and let $\pi : \widetilde{\mathscr{X}_\bullet} \to \mathscr{X}_{\text{lis-ét}}|_U$ denote the natural augmentation.

There is also a morphism of simplicial topoi

$$\rho : \widetilde{\mathscr{X}_\bullet} \to W_{\bullet,\text{ét}}$$

and the functor ρ_* is exact. We conclude that for any F on $\mathscr{X}_{\text{lis-ét}}$ we have

$$H^i(\mathscr{X}_{\text{lis-ét}}|_U, F) \simeq H^i(\widetilde{\mathscr{X}_\bullet}, F) \simeq H^i(W_{\bullet,\text{ét}}, F_{W_\bullet}).$$

On the other hand, the group $H^i(W_{\bullet,\text{ét}}, F_{W_\bullet})$ is by 9.2.4 also isomorphic to $H^i(\mathscr{X}_{U,\text{lis-ét}}, F_{\mathscr{X}_U})$.

Conclusion 9.2.10. The sheaf R^if_*F is the sheaf associated to the presheaf which to any $(U \to \mathscr{Y}) \in \text{Lis-Ét}(\mathscr{Y})$ associates $H^i(\mathscr{X}_U, F_{\mathscr{X}_U})$.

Proposition 9.2.11. *Let $f : \mathscr{X} \to \mathscr{Y}$ be a quasi-compact and quasi-separated morphism of algebraic stacks, and let F be a quasi-coherent sheaf on $\mathscr{X}$. Then R^if_*F is a quasi-coherent sheaf on $\mathscr{Y}$ for every i.*

Proof. The sheaf R^if_*F is the sheaf associated to the presheaf which to any $(U \to \mathscr{Y}) \in \text{Lis-Ét}(\mathscr{Y})$ associates $H^i(\mathscr{X}_U, F_{\mathscr{X}_U})$. To prove the proposition it suffices (using exercise 9.G) to show that for any morphism

$$(\text{Spec}(R') \to \mathscr{Y}) \to (\text{Spec}(R) \to \mathscr{Y})$$

between affine objects of Lis-Ét$(\mathscr{Y})$ with $\text{Spec}(R') \to \text{Spec}(R)$ smooth, the map

$$H^i(\mathscr{X}_{\text{Spec}(R)}, F_{\mathscr{X}_{\text{Spec}(R)}}) \otimes_R R' \to H^i(\mathscr{X}_{\text{Spec}(R')}, F_{\mathscr{X}_{\text{Spec}(R')}}) \tag{9.2.11.1}$$

is an isomorphism. For this note that since f is quasi-compact the stack $\mathscr{X}_{\text{Spec}(R)}$ is quasi-compact. Let $W \to \mathscr{X}_{\text{Spec}(R)}$ be a smooth surjective morphism with $\mathscr{X}$ a quasi-compact scheme, and let $W_\bullet$ denote the coskeleton. Since f is quasi-separated, each W_n is a quasi-compact and quasi-separated R-scheme. Let W' denote $W \times_{\text{Spec}(R)} \text{Spec}(R')$ and let $W'_\bullet$ denote the coskeleton of $W' \to \mathscr{X}_{\text{Spec}(R')}$. We then get a commutative diagram:

$$\begin{tikzcd} W'_\bullet \arrow[r, "e'"] \arrow[d] & \mathscr{X}_{\text{Spec}(R')} \arrow[r] \arrow[d] & \text{Spec}(R') \arrow[d] \\ W_\bullet \arrow[r, "e"] & \mathscr{X}_{\text{Spec}(R)} \arrow[r] & \text{Spec}(R). \end{tikzcd}$$

This diagram induces a morphism of spectral sequences

$$\begin{array}{ccc} E_1^{pq} = H^p(W_q, F_{W_q}) & \Longrightarrow & H^{p+q}(\mathscr{X}_{\text{Spec}(R)}, F_{\mathscr{X}_{\text{Spec}(R)}}) \\ & & \downarrow \\ E_1^{pq} = H^p(W'_q, F_{W'_q}) & \Longrightarrow & H^{p+q}(\mathscr{X}_{\text{Spec}(R')}, F_{\mathscr{X}_{\text{Spec}(R')}}). \end{array}$$

Since $R \to R'$ is flat, to prove that (9.2.11.1) is exact it suffices to show that the maps on E_1-terms

$$H^p(W_q, F_{W_q}) \otimes_R R' \to H^p(W'_q, F_{W'_q})$$

is an isomorphism. This reduces the proof to the case of an algebraic space. Repeating this argument with an étale covering of W_q we are further reduced to the case of schemes, where the result is [**26**, III.1.4.10]. □

9.2.12. It will be useful to have another characterization of quasi-coherent sheaves on an algebraic stack $\mathscr{X}$ in terms of a covering. Let $\pi : X \to \mathscr{X}$ be a smooth surjection with X an algebraic space, and let $e : X_\bullet \to \mathscr{X}$ be the coskeleton with étale topos $X_{\bullet,\text{ét}}$. Let $\mathscr{O}_{X_\bullet}$ denote the sheaf of rings in $X_{\bullet,\text{ét}}$ given by the structure sheaves $\mathscr{O}_{X_n}$ with the natural transition maps.

We say that a sheaf $F_\bullet$ of $\mathscr{O}_{X_\bullet}$-modules is *quasi-coherent* if each F_n is quasi-coherent and if each of the morphisms

$$F(\delta) : \delta^* F_n \to F_m \tag{9.2.12.1}$$

is an isomorphism.

A sheaf of $\mathscr{O}_{\mathscr{X}}$-modules $\mathscr{M}$ in $\mathscr{X}_{\text{lis-ét}}$ defines by restriction a sheaf $\mathscr{M}_{X_\bullet}$ of $\mathscr{O}_{X_\bullet}$-modules in $X_{\bullet,\text{ét}}$.

Proposition 9.2.13. *The association $\mathscr{M} \mapsto \mathscr{M}_{X_\bullet}$ defines an equivalence of categories between* $\mathrm{Qcoh}(\mathscr{X})$ *and the category of quasi-coherent $\mathscr{O}_{X_\bullet}$-modules.*

Proof. If $\mathscr{M}$ is a quasi-coherent sheaf on $\mathscr{X}$ it is clear from the definitions that $\mathscr{M}_{X_\bullet}$ is a quasi-coherent sheaf on $X_\bullet$.

An inverse to this functor can be defined as follows. For a smooth morphism $U \to \mathscr{X}$, let $X_{U,\bullet}$ denote the simplicial algebraic space obtained by base change to U, so we have a commutative diagram:

$$\begin{tikzcd} X_{U,\bullet} \arrow[r, "g_\bullet"] \arrow[d, "p"] & X_\bullet \arrow[d] \\ U \arrow[r] & \mathscr{X}. \end{tikzcd}$$

Given a quasi-coherent sheaf $\mathscr{N}_\bullet$ on the étale site of $X_\bullet$, let $\mathscr{N}_{U\bullet}$ be the pullback of $\mathscr{N}_\bullet$ to $X_{U,\bullet}$, and set

$$\gamma(\mathscr{N}_\bullet)(U) := \mathrm{Eq}(\Gamma(X_{U0}, \mathscr{N}_{U0}) \rightrightarrows \Gamma(X_{U1}, \mathscr{N}_{U1})).$$

Then the association

$$U \mapsto \gamma(\mathscr{N}_\bullet)(U)$$

defines a quasi-coherent sheaf $\gamma(\mathscr{N}_\bullet)$ on $\mathscr{X}$. Indeed, by descent theory for quasi-coherent sheaves, there exists a unique quasi-coherent sheaf $\gamma(\mathscr{N}_\bullet)_U$ on the étale site of U whose pullback to $X_{U_\bullet}$ is $\mathscr{N}_{U,\bullet}$, and $\gamma(\mathscr{N}_\bullet)(U)$ is simply the global sections of this sheaf $\gamma(\mathscr{N}_\bullet)_U$. From this we also get that for any morphism $\rho : U' \to U$ in the lisse-étale site of $\mathscr{X}$, the pullback map

$$\rho^* \gamma(\mathscr{N}_\bullet)_U \to \gamma(\mathscr{N}_\bullet)_{U'}$$

is an isomorphism, since this map pulls back to an isomorphism on $X_{U'\bullet}$. Therefore $\gamma(\mathscr{N}_\bullet)$ is a quasi-coherent sheaf on $\mathscr{X}$ and we obtain a functor,

$$\gamma : \mathrm{Qcoh}(\mathscr{O}_{X_\bullet}) \to \mathrm{Qcoh}(\mathscr{X}).$$

We leave to the reader the verification that this defines the desired inverse functor. □

REMARK 9.2.14. One can also consider the category of sheaves $F_\bullet$ of $\mathscr{O}_{X_\bullet}$-modules which have the property that for all n the sheaf F_n is a quasi-coherent sheaf on X_n but for which the maps (9.2.12.1) are not necessarily isomorphisms. Let us call such a sheaf a *locally quasi-coherent sheaf on* $X_\bullet$. Then the functor γ described in the proof of 9.2.13 extends to a functor

$$\gamma : (\text{locally quasi-coherent sheaves on } X_\bullet) \to \mathrm{Qcoh}(\mathscr{X})$$

which is right adjoint to the functor

$$\mathrm{Qcoh}(\mathscr{X}) \to (\text{quasi-coherent sheaves on } X_\bullet)$$
$$\hookrightarrow (\text{locally quasi-coherent sheaves on } X_\bullet).$$

9.2.15. In the case of Deligne-Mumford stacks we can also consider quasi-coherent sheaves on the étale site as in 9.1.16. Let $f : \mathscr{X} \to \mathscr{Y}$ be a morphism of Deligne-Mumford stacks over a base scheme S. Then f induces a morphism of ringed topoi

$$f^{\mathrm{et}} : (\mathscr{X}_{\mathrm{et}}, \mathscr{O}_{\mathscr{X}_{\mathrm{et}}}) \to (\mathscr{Y}_{\mathrm{et}}, \mathscr{O}_{\mathscr{Y}_{\mathrm{et}}}),$$

and for a quasi-coherent sheaf $\mathscr{F}_{\mathrm{et}}$ on the étale site of $\mathscr{X}$, we can form the derived pushforwards $R^i f^{\mathrm{et}}_* \mathscr{F}_{\mathrm{et}}$, which are sheaves of $\mathscr{O}_{\mathscr{Y}_{\mathrm{et}}}$-modules on $\mathscr{Y}_{\mathrm{et}}$, by taking the usual derived functors of the left exact pushforward functor f^{et}_*.

PROPOSITION 9.2.16. *Let $\mathscr{F}$ denote the quasi-coherent sheaf on the lisse-étale site of $\mathscr{X}$ associated to $\mathscr{F}_{\mathrm{et}}$ as in* 9.1.18. *The sheaf $R^i f^{\mathrm{et}}_* \mathscr{F}_{\mathrm{et}}$ is isomorphic to the restriction of $R^i f_* \mathscr{F}$ to the étale site of $\mathscr{Y}$. In particular, if f is quasi-compact and quasi-separated, then the sheaves $R^i f^{\mathrm{et}}_* \mathscr{F}_{\mathrm{et}}$ are quasi-coherent on $\mathscr{Y}$ (by* 9.2.11*).*

PROOF. The inclusion $\mathrm{Ét}(\mathscr{X}) \subset \text{Lis-Ét}(\mathscr{X})$ (resp. $\mathrm{Ét}(\mathscr{Y}) \subset \text{Lis-Ét}(\mathscr{Y})$) induces a morphism of ringed topoi

$$r_{\mathscr{X}} : (\mathscr{X}_{\text{lis-ét}}, \mathscr{O}_{\mathscr{X}_{\text{lis-ét}}}) \to (\mathscr{X}_{\mathrm{et}}, \mathscr{O}_{\mathscr{X}_{\mathrm{et}}}) \ \ (\text{resp. } (\mathscr{Y}_{\text{lis-ét}}, \mathscr{O}_{\mathscr{Y}_{\text{lis-ét}}}) \to (\mathscr{Y}_{\mathrm{et}}, \mathscr{O}_{\mathscr{Y}_{\mathrm{et}}})).$$

The pullback functor

$$r^*_{\mathscr{X}} : \mathrm{Mod}_{\mathscr{O}_{\mathscr{X}_{\mathrm{et}}}} \to \mathrm{Mod}_{\mathscr{O}_{\mathscr{X}_{\text{lis-ét}}}} \ \ (\text{resp. } r^*_{\mathscr{Y}} : \mathrm{Mod}_{\mathscr{O}_{\mathscr{Y}_{\mathrm{et}}}} \to \mathrm{Mod}_{\mathscr{O}_{\mathscr{Y}_{\text{lis-ét}}}})$$

is exact, and therefore the functor $r_{\mathscr{X}*}$ (resp. $r_{\mathscr{Y}*}$) is exact and takes injectives to injectives. Let $\mathscr{F}$ denote $r^*_{\mathscr{X}} \mathscr{F}_{\mathrm{et}}$, so $\mathscr{F}_{\mathrm{et}} = r_{\mathscr{X}*} \mathscr{F}$, and choose an injective resolution $\mathscr{F} \to I^\bullet$ in $\mathrm{Mod}_{\mathscr{O}_{\mathscr{X}_{\text{lis-ét}}}}$. Then $R^i f_* \mathscr{F}$ is represented by $\mathscr{H}^i(f_* I^\bullet)$, and since $r_{\mathscr{Y}*}$ is exact we get

$$r_{\mathscr{Y}*} R^i f_* \mathscr{F} \simeq \mathscr{H}^i(r_{\mathscr{Y}*} f_* I^\bullet).$$

Using the isomorphism $r_{\mathscr{Y}*} \circ f_* \simeq f^{\mathrm{et}}_* \circ r_{\mathscr{X}*}$ we obtain an isomorphism

$$r_{\mathscr{Y}*} R^i f_* \mathscr{F} \simeq \mathscr{H}^i(f^{\mathrm{et}}_* r_{\mathscr{X}*} I^\bullet).$$

Since $r_{\mathscr{X}*}$ is exact and takes injectives to injectives the complex $r_{\mathscr{X}*} I^\bullet$ is an injective resolution of $\mathscr{F}_{\mathrm{et}}$ and so we obtain the desired isomorphism $r_{\mathscr{Y}*} R^i f_* \mathscr{F} \simeq R^i f^{\mathrm{et}}_* \mathscr{F}_{\mathrm{et}}$. □

Similarly:

PROPOSITION 9.2.17. *With notation as in* 9.2.16, *for any $i \geq 0$ there is a natural isomorphism $H^i(\mathscr{X}_{\mathrm{et}}, \mathscr{F}_{\mathrm{et}}) \simeq H^i(\mathscr{X}_{\text{lis-ét}}, \mathscr{F})$.*

PROOF. Indeed, choose an injective resolution $\mathscr{F} \to I^\bullet$ as in the proof of 9.2.16, so that $r_{\mathscr{X}*}I^\bullet$ is an injective resolution of $\mathscr{F}_{\text{et}}$. Then $H^i(\mathscr{X}_{\text{et}}, \mathscr{F}_{\text{et}})$ is computed by the complex $\Gamma(\mathscr{X}_{\text{et}}, r_{\mathscr{X}*}I^\bullet)$ which is canonically isomorphic to the complex $\Gamma(\mathscr{X}_{\text{lis-ét}}, I^\bullet)$ which computes $H^i(\mathscr{X}_{\text{lis-ét}}, \mathscr{F})$. □

REMARK 9.2.18. With notation as in the proof of 9.2.16, note that since the restriction functor $r_{\mathscr{X}*} : \text{Mod}_{\mathscr{O}_{\mathscr{X}_{\text{lis-ét}}}} \to \text{Mod}_{\mathscr{O}_{\mathscr{X}_{\text{et}}}}$ is exact and takes injectives to injectives, the collection of functors

$$\{R^i f_*^{\text{et}} \circ r_{\mathscr{X}*}(-)\}_i : \text{Mod}_{\mathscr{O}_{\mathscr{X}_{\text{lis-ét}}}} \to \text{Mod}_{\mathscr{O}_{\mathscr{Y}_{\text{et}}}}$$

form a universal δ-functor. The collection of functors $\{r_{\mathscr{Y}*}R^i f_*(-)\}$ is also a δ-functor, and the isomorphism in 9.2.16 is induced by the natural isomorphism $f_*^{\text{et}} \circ r_{\mathscr{X}*} \simeq r_{\mathscr{Y}*} \circ f_*$ and the universal property of $\{R^i f_*^{\text{et}} \circ r_{\mathscr{X}*}(-)\}_i$. In particular, the isomorphism in 9.2.16 is compatible with the long exact sequences associated to short exact sequences of sheaves. Similarly, the isomorphism in 9.2.17 is δ-functorial.

9.3. Pulling back quasi-coherent sheaves

9.3.1. Let $f : \mathscr{X} \to \mathscr{Y}$ be a morphism of algebraic stacks. Because such a morphism does not induce a morphism of lisse-étale topoi, it takes a little more work to define the pullback along f of a quasi-coherent sheaf on $\mathscr{Y}$.

9.3.2. Fix a smooth surjection $Y \to \mathscr{Y}$ with Y an algebraic space, let $\mathscr{X}_Y$ denote the fiber product $\mathscr{X} \times_{\mathscr{Y}} Y$, and also fix a smooth surjection $X \to \mathscr{X}_Y$ with X an algebraic space so we have a commutative diagram:

(9.3.2.1)
$$\begin{tikzcd} X \arrow[r] \arrow[dr] & \mathscr{X}_Y \arrow[r] \arrow[d] & Y \arrow[d] \\ & \mathscr{X} \arrow[r] & \mathscr{Y}. \end{tikzcd}$$

Let $e_X : X_\bullet \to \mathscr{X}$ (resp. $e_Y : Y_\bullet \to \mathscr{Y}$) denote the coskeletons and let $f_\bullet : X_\bullet \to Y_\bullet$ denote the morphism of simplicial algebraic spaces induced by the map $X \to Y$ so we have a commutative diagram:

$$\begin{tikzcd} X_\bullet \arrow[r, "f_\bullet"] \arrow[d, "e_X"] & Y_\bullet \arrow[d, "e_Y"] \\ \mathscr{X} \arrow[r, "f"] & \mathscr{Y}. \end{tikzcd}$$

By 9.2.13 the restriction functors

$$r_{\mathscr{X}} : \text{Qcoh}(\mathscr{X}_{\text{lis-ét}}) \to \text{Qcoh}(X_{\bullet,\text{ét}}), \quad r_{\mathscr{Y}} : \text{Qcoh}(\mathscr{Y}_{\text{lis-ét}}) \to \text{Qcoh}(Y_{\bullet,\text{ét}})$$

are equivalences of categories. Now the morphism $f_\bullet$ does induce a morphism of ringed topoi

$$f_{\bullet,\text{ét}} : (X_{\bullet,\text{ét}}, \mathscr{O}_{X_\bullet}) \to (Y_{\bullet,\text{ét}}, \mathscr{O}_{Y_\bullet})$$

and so we can consider the pullback functor $f_\bullet^*$.

LEMMA 9.3.3. *Let $M_\bullet$ be a quasi-coherent sheaf on $Y_{\bullet,\text{ét}}$. Then $f_\bullet^* M_\bullet$ is a quasi-coherent sheaf on $X_{\bullet,\text{ét}}$.*

Proof. Indeed, the restriction of $f_\bullet^* M_\bullet$ to $X_{n,\text{ét}}$ is just the pullback $f_n^* M_n$ which is quasi-coherent by 7.1.9. Furthermore, for any morphism $\delta : [n] \to [m]$ inducing morphisms $X(\delta) : X_m \to X_n$ and $Y(\delta) : Y_m \to Y_n$ the map

$$X(\delta)^* f_n^* M_n \to f_m^* M_m$$

is via the isomorphism $X(\delta)^* f_n^* \simeq f_m^* Y(\delta)^*$ identified with the pullback along f_m of the isomorphism $Y(\delta)^* M_n \to M_m$. $\square$

9.3.4. Fix an inverse equivalence

$$r_{\mathscr{X}}^{-1} : \mathrm{Qcoh}(X_{\bullet,\text{ét}}) \to \mathrm{Qcoh}(\mathscr{X}_{\text{lis-ét}})$$

and define

$$f^* : \mathrm{Qcoh}(\mathscr{Y}_{\text{lis-ét}}) \to \mathrm{Qcoh}(\mathscr{X}_{\text{lis-ét}})$$

to be the composite functor

$$\mathrm{Qcoh}(\mathscr{Y}_{\text{lis-ét}}) \xrightarrow{r_{\mathscr{Y}}} \mathrm{Qcoh}(Y_{\bullet,\text{ét}}) \xrightarrow{f_\bullet^*} \mathrm{Qcoh}(X_{\bullet,\text{ét}}) \xrightarrow{r_{\mathscr{X}}^{-1}} \mathrm{Qcoh}(\mathscr{X}_{\text{lis-ét}}).$$

9.3.5. The functor f^* is up to canonical isomorphism of functors independent of the choice of the diagram (9.3.2.1). To see this consider a second choice of diagram,

$$\begin{tikzcd} X' \arrow[r] \arrow[dr] & \mathscr{X}_{Y'} \arrow[r] \arrow[d] & Y' \arrow[d] \\ & \mathscr{X} \arrow[r, "f"] & \mathscr{Y}, \end{tikzcd}$$

and set $Y'' := Y \times_{\mathscr{Y}} Y'$ and $X'' := X \times_{\mathscr{X}} X'$. We then have

$$\mathscr{X}_{Y''} \simeq \mathscr{X}_Y \times_{\mathscr{X}} \mathscr{X}_{Y'}$$

and another diagram

$$\begin{tikzcd} X'' \arrow[r] \arrow[dr] & \mathscr{X}_{Y''} \arrow[r] \arrow[d] & Y'' \arrow[d] \\ & \mathscr{X} \arrow[r] & \mathscr{Y}. \end{tikzcd} \tag{9.3.5.1}$$

It suffices to show that (9.3.2.1) and (9.3.5.1) define canonically isomorphic pullback functors. For this note that we have the commutative diagram

$$\begin{tikzcd} X'' \arrow[rr, "f''"] \arrow[d] \arrow[ddr] & & Y'' \arrow[d] \arrow[ddr] & \\ X \arrow[rr, "f"] \arrow[dr] & & Y \arrow[dr] & \\ & \mathscr{X} \arrow[rr] & & \mathscr{Y}, \end{tikzcd}$$

which induces a commutative diagram

$$\begin{tikzcd}
\mathrm{Qcoh}(X_{\bullet,\text{ét}}) & & \mathrm{Qcoh}(Y_{\bullet,\text{ét}}) \arrow[ll, "f''^*_\bullet"'] & \\
\mathrm{Qcoh}(X_{\bullet,\text{ét}}) \arrow[u] & & \mathrm{Qcoh}(Y_{\bullet,\text{ét}}) \arrow[ll, "f^*_\bullet"'] \arrow[u] & \\
& \mathrm{Qcoh}(\mathscr{X}) \arrow[ul] \arrow[uul] & & \mathrm{Qcoh}(\mathscr{Y}). \arrow[ul] \arrow[uul]
\end{tikzcd}$$

Since all the vertical arrows in this diagram are equivalences this induces an isomorphism between the pullback functors defined by (9.3.5.1) and (9.3.2.1).

PROPOSITION 9.3.6. *Let $f : \mathscr{X} \to \mathscr{Y}$ be a quasi-compact and quasi-separated morphism of algebraic stacks. Then the functor $f^* : \mathrm{Qcoh}(\mathscr{Y}) \to \mathrm{Qcoh}(\mathscr{X})$ is left adjoint to the functor $f_* : \mathrm{Qcoh}(\mathscr{X}) \to \mathrm{Qcoh}(\mathscr{Y})$.*

PROOF. Observe that for a quasi-coherent sheaf $F_\bullet$ on $X_\bullet$, the pushforward $f_{\bullet *}F_\bullet$ on $Y_\bullet$ need not by cartesian, but it is locally quasi-coherent in the sense of 9.2.14, and we have an isomorphism of functors,

$$f_* \circ \gamma_{\mathscr{X}} \simeq \gamma_{\mathscr{Y}} \circ f_{\bullet *} : (\text{quasi-coherent sheaves on } X_\bullet) \to \mathrm{Qcoh}(\mathscr{Y}),$$

where $\gamma_{\mathscr{X}}$ and $\gamma_{\mathscr{Y}}$ are as in 9.2.14. From this it follows that for any quasi-coherent sheaf $\mathscr{M}$ on $\mathscr{X}$ and quasi-coherent sheaf $\mathscr{N}$ on $\mathscr{Y}$ we have

$$\begin{aligned}
\mathrm{Hom}_{\mathscr{X}}(f^*\mathscr{N}, \mathscr{M}) &\simeq \mathrm{Hom}_{X_\bullet}(f_\bullet^*\mathscr{N}_{Y_\bullet}, \mathscr{M}_{X_\bullet}) \\
&\simeq \mathrm{Hom}_{Y_\bullet}(\mathscr{N}_{Y_\bullet}, f_{\bullet *}\mathscr{M}_{X_\bullet}) \\
&\simeq \mathrm{Hom}_{\mathscr{Y}}(\mathscr{N}, \gamma_{\mathscr{Y}} f_{\bullet *}\mathscr{M}_{X_\bullet}) \\
&\simeq \mathrm{Hom}_{\mathscr{Y}}(\mathscr{N}, f_*\mathscr{M}),
\end{aligned}$$

which is the desired adjunction. □

9.4. Exercises

EXERCISE 9.A. Verify that the lisse-étale site of a stack defined in 9.1.6 satisfies the axioms of a site.

EXERCISE 9.B. Let $\mathscr{X}$ be an algebraic stack, and let Λ be a sheaf of rings in $\mathscr{X}_{\text{lis-ét}}$. Show that a sheaf of Λ-modules F is cartesian if and only if for every morphism $(f, f^b) : (T', t') \to (T, t)$ with $f : T' \to T$ smooth, the map $f^*F_T \to F_{T'}$ is an isomorphism.

EXERCISE 9.C. Let $f : \mathscr{X} \to \mathscr{Y}$ be a morphism of algebraic stacks, and let $W \to \mathscr{X}$ be a smooth surjection with W a scheme. Let $W_\bullet$ denote the associated simplical algebraic space, and for each n let $f_n : W_n \to \mathscr{Y}$ be the projection. Show that there is a spectral sequence in the category of sheaves of $\mathscr{O}_{\mathscr{Y}}$-modules in $\mathscr{Y}_{\text{lis-ét}}$:

$$E_1^{pq} = R^p f_{q*}(F|_{W_{n,\text{lis-ét}}}) \implies R^{p+q} f_* F.$$

EXERCISE 9.D. Let G be a finite group, and let k be a field. Show that if $\mathscr{V}$ is a quasi-coherent sheaf on BG with associated G-representation V, then

$$H^i(BG_{\text{lis-ét}}, \mathscr{V}) \simeq H^i(G, V),$$

where $H^i(G, V)$ denotes the group cohomology of the representation V. Using this exhibit a noetherian algebraic stack $\mathscr{X}$ and a coherent sheaf $\mathscr{V}$ such that $H^i(\mathscr{X}, \mathscr{V})$ is nonzero for all $i \geq 0$.

EXERCISE 9.E. Let $X = \mathrm{Spec}(A)$ be an affine scheme, and let G be a finite group acting on X. Denote by $\mathscr{X}$ the stack quotient $[X/G]$. Let $\mathscr{A}$ denote the ring whose underlying group is the free A-module on generators e_g $(g \in G)$ and with multiplication defined by

$$(ae_g)(a'e_{g'}) := ag(a')e_{gg'}, \quad a, a' \in A,$$

and where we write $g(a')$ for the image of a' under the given action $g : A \to A$. Show that the category of quasi-coherent sheaves on $\mathscr{X}$ is equivalent to the category of left $\mathscr{A}$-modules.

EXERCISE 9.F. Let $\mathscr{X}$ be an algebraic stack. Define the big fppf site of $\mathscr{X}$ to be $\mathrm{AS}/\mathscr{X}$ with the topology given by declaring that a collection of morphisms

$$\{(T_i, t_i) \to (T, t)\}_{i \in I}$$

is a covering if the morphisms of algebraic spaces $\{T_i \to T\}_{i \in I}$ is an fppf covering in the usual sense. There is a sheaf of rings $\mathscr{O}$ given by

$$\mathscr{O}(T, t) := \Gamma(T, \mathscr{O}_T).$$

Define a *fppf-quasi-coherent sheaf* to be a sheaf of $\mathscr{O}$-modules $\mathscr{M}$ such that the following hold:

(1) For any $\mathscr{X}$-space (T, t), the restriction $\mathscr{M}_{(T,t)}$ of $\mathscr{M}$ to the étale site of T is quasi-coherent.
(2) For any morphism of $\mathscr{X}$-spaces

$$(f, f^b) : (T', t') \to (T, t)$$

the natural map $f^*\mathscr{M}_{(T,t)} \to \mathscr{M}_{(T',t')}$ is an isomorphism.

(i) Give an example to show that the kernel (in the category of sheaves of $\mathscr{O}$-modules) of a morphism of fppf-quasi-coherent sheaves need not be fppf-quasi-coherent.

(ii) Show that the category of fppf-quasi-coherent sheaves on $\mathscr{X}$ is equivalent to the category of quasi-coherent sheaves on $\mathscr{X}$ as defined in 9.1.14.

(iii) Deduce that the category of fppf-quasi-coherent sheaves is abelian but that the inclusion into sheaves of $\mathscr{O}$-modules is not exact.

EXERCISE 9.G. Let $\mathscr{X}$ be an algebraic stack over a scheme S, and let $\mathscr{F}$ be a sheaf of $\mathscr{O}_{\mathscr{X}}$-modules in $\mathscr{X}_{\text{lis-ét}}$. Suppose that the following two conditions hold:

(i) For every smooth morphism $\mathrm{Spec}(R) \to \mathscr{X}$ from an affine scheme, the restriction of $\mathscr{F}$ to the étale site of $\mathrm{Spec}(R)$ is a quasi-coherent sheaf.
(ii) For every smooth morphism $\mathrm{Spec}(R') \to \mathrm{Spec}(R)$ of smooth affine $\mathscr{X}$-schemes, the induced pullback map

$$\mathscr{F}(\mathrm{Spec}(R)) \otimes_R R' \to \mathscr{F}(\mathrm{Spec}(R'))$$

is an isomorphism.

Show that $\mathscr{F}$ is a quasi-coherent sheaf on $\mathscr{X}$.

Exercise 9.H. Let S be a scheme, G/S a smooth group scheme, and X/S an algebraic space over S with the action of G. Let

$$\rho : X \times_S G \to X$$

be the map giving the action of G on X, and let $\mathrm{pr} : X \times_S G \to X$ be the first projection. Define the category of *G-equivariant quasi-coherent sheaves on X*, denoted $\mathrm{Qcoh}^G(X)$, to be the category of pairs $(\mathscr{F}, \sigma)$, where $\mathscr{F}$ is a quasi-coherent sheaf on X and $\sigma : \rho^*\mathscr{F} \to \mathrm{pr}^*\mathscr{F}$ is an isomorphism such that for any S-scheme T and points $g, g' \in G(T)$ the induced diagram on $X_T = X \times_S T$,

$$\begin{tikzcd} \rho_g^*\rho_{g'}^*\mathscr{F}_T \arrow[r, "\rho_g^*\sigma_{g'}"] \arrow[d, "\simeq"] & \rho_g^*\mathscr{F}_T \arrow[d, "\sigma_g"] \\ \rho_{g'g}^*\mathscr{F}_T \arrow[r, "\sigma_{g'g}"] & \mathscr{F}_T \end{tikzcd},$$

commutes, where $\rho_g : X_T \to X_T$ is the map induced by the action, and $\mathscr{F}_T$ denotes the pullback of $\mathscr{F}$ to X_T.

(i) Show that the category $\mathrm{Qcoh}^G(X)$ is equivalent to the category $\mathrm{Qcoh}([X/G])$ of quasi-coherent sheaves on the stack quotient $[X/G]$.

(ii) In the case when $X = S$ with the trivial action of G, show that for an integer n the data of a map σ making $(\mathscr{O}_S^n, \sigma)$ a G-equivariant sheaf on S is equivalent to giving a morphism of S-group schemes $G \to GL_{n,S}$.

Exercise 9.I. Let k be a field and G/k a smooth group scheme. Let $\mathrm{Lie}(G)$ denote the Lie algebra of G which is a G-representation under the adjoint action. Let $\mathscr{L}ie$ denote the corresponding quasi-coherent sheaf on BG. Show that $\mathscr{L}ie$ is isomorphic to the sheaf which to any pair (T, P) consisting of a k-scheme T and a G-torsor P/T associates the G-invariant elements of $\Gamma(P, \Omega^1_{P/T})$.

Exercise 9.J. Let S be a scheme and let $\mathscr{X}/S$ be an algebraic stack. Let $\mathscr{I} \subset \mathscr{O}_{\mathscr{X}}$ be the sub-presheaf which to any smooth morphism $U \to \mathscr{X}$ with U a scheme associates the set of locally nilpotent elements of the ring $\Gamma(U, \mathscr{O}_U)$. Show that $\mathscr{I}$ is a quasi-coherent sheaf on $\mathscr{X}$.

Exercise 9.K. Let k be an algebraically closed field and G/k a smooth connected group scheme over k. Show that if $U \to BG$ is a representable étale morphism of algebraic stacks of finite type then U is isomorphic to a finite disjoint union of copies of BG. In particular, the small étale site of BG is trivial.

CHAPTER 10

Basic geometric properties and constructions for stacks

In this chapter we discuss the notions of proper and separated morphisms, and present three basic constructions of stacks: relative Spec, relative Proj, and the root stack construction. The principal issue in generalizing notions for schemes and algebraic spaces to algebraic stacks is that for a stack the diagonal morphism is not an immersion. For example, if G is a group with associated classifying stack BG over a field k, and if $x : \mathrm{Spec}(k) \to BG$ is the morphism corresponding to the trivial torsor, then the fiber product of the diagram

$$\begin{tikzcd} & \mathrm{Spec}(k) \arrow[d, "x \times x"] \\ BG \arrow[r, "\Delta"] & BG \times BG \end{tikzcd}$$

is isomorphic to G.

The sources for this chapter are [**23**] and [**49**, §7 and 14], and for the root stack construction [**17**].

10.1. Proper morphisms

Definition 10.1.1. A morphism $f : \mathscr{Z} \to \mathscr{X}$ is an *imbedding* (resp. *open imbedding*, *closed imbedding*) if it is representable and satisfies 8.2.9 with P the property of being an imbedding (resp. open imbedding, closed imbedding).

A *closed substack* of an algebraic stack $\mathscr{X}$ is defined to be an equivalence class of closed imbeddings $\mathscr{Z} \to \mathscr{X}$, where two closed imbeddings $f_i : \mathscr{Z}_i \to \mathscr{X}$ $(i = 1, 2)$ are declared equivalent if there exists a pair (g, σ) consisting of a morphism $g : \mathscr{Z}_1 \to \mathscr{Z}_2$ and an isomorphism $\sigma : f_2 \circ g \simeq f_1$.

Remark 10.1.2. Note that since the closed imbeddings are representable morphisms, a pair (g, σ) as in the definition of closed substack is unique up to unique isomorphism.

Definition 10.1.3. A morphism $f : \mathscr{X} \to Y$ from an algebraic stack $\mathscr{X}$ to a scheme Y is *closed* if for every closed substack $\mathscr{Z} \subset \mathscr{X}$ the image of $\mathscr{Z}$ in Y is closed, where the image of f is defined as in exercise 8.I.

A morphism of algebraic stacks $f : \mathscr{X} \to \mathscr{Y}$ is *universally closed* if for every morphism $Y \to \mathscr{Y}$ with Y a scheme the morphism $\mathscr{X} \times_{\mathscr{Y}} Y \to Y$ is closed.

Proposition 10.1.4. *Let $f : \mathscr{X} \to \mathscr{Y}$ be a representable separated morphism of finite type. Then f is universally closed if and only if f is proper in the sense of 8.2.9.*

PROOF. If $Y \to \mathscr{Y}$ is a morphism with Y a scheme, then the image of the induced morphism of algebraic spaces

$$\mathscr{X} \times_{\mathscr{Y}} Y \to Y$$

defined as in exercise 8.I (i) agrees with the notion of image defined in 6.3.5 by exercise 8.I (ii). From this the proposition follows. □

DEFINITION 10.1.5. (i) A morphism of algebraic stacks $f : \mathscr{X} \to \mathscr{Y}$ is *separated* if the diagonal $\mathscr{X} \to \mathscr{X} \times_{\mathscr{Y}} \mathscr{X}$ is proper (note that the diagonal is representable).

(ii) A morphism of algebraic stacks $f : \mathscr{X} \to \mathscr{Y}$ is *proper* if it is separated, of finite type, and universally closed.

PROPOSITION 10.1.6. *(i) If*

$$\begin{tikzcd} \mathscr{X}' \arrow[r] \arrow[d, "f'"] & \mathscr{X} \arrow[d, "f"] \\ \mathscr{Y}' \arrow[r] & \mathscr{Y} \end{tikzcd}$$

is a cartesian diagram of algebraic stacks and f is proper, then f' is also proper.

(ii) If $f : \mathscr{X} \to \mathscr{Y}$ is a morphism of algebraic stacks such that there exists a smooth surjection $Y \to \mathscr{Y}$ with Y a scheme for which the base change $\mathscr{X}_Y \to Y$ is proper, then f is proper.

(iii) For a composition of morphisms of algebraic stacks

$$\mathscr{X} \xrightarrow{f} \mathscr{Y} \xrightarrow{g} \mathscr{Z}, \tag{10.1.6.1}$$

if f and g are proper, then so is gf.

(iv) For a composition (10.1.6.1) *if gf is proper and g is separated, then f is proper.*

(v) For a composition (10.1.6.1) *if gf is proper and f is surjective, then g is proper.*

PROOF. The proofs of these statements are immediate from the definition as in the classical case [**26**, II, 5.4.2 and 5.4.3]. □

EXAMPLE 10.1.7. Let k be a field and A/k an abelian variety. Then the classifying stack BA is proper over Spec(k). Indeed, if $\pi : \mathrm{Spec}(k) \to BA$ is the standard projection, then

$$\mathrm{Spec}(k) \times_{BA} \mathrm{Spec}(k) \simeq A$$

whence the diagonal of BA is proper so BA is separated and π is a proper surjection. That BA is proper over Spec(k) therefore follows from (v).

REMARK 10.1.8. In practice one often proves that a stack is proper using a generalization of the valuative criterion for properness for schemes. This is discussed in section 11.5.

10.2. Relative Spec and Proj

10.2.1. Let $\mathscr{X}$ be an algebraic stack over a scheme S, and let $\mathscr{A}$ denote a quasi-coherent sheaf of algebras on $\mathscr{X}$. Define a stack $\underline{\mathrm{Spec}}_{\mathscr{X}}(\mathscr{A})$ as follows. The objects of $\underline{\mathrm{Spec}}_{\mathscr{X}}(\mathscr{A})$ are triples

$$(T, x, \rho),$$

where T is an S-scheme, $x \in \mathscr{X}(T)$ is an object, and $\rho : x^*\mathscr{A} \to \mathscr{O}_T$ is a morphism of sheaves of algebras on T. A morphism

$$(T', x', \rho') \to (T, x, \rho)$$

is a pair (g, g^b), where $g : T' \to T$ is a morphism of S-schemes and $g^b : x' \to x$ is a morphism in $\mathscr{X}$ over g such that the diagram

$$\begin{tikzcd} x'^*\mathscr{A} \arrow[rr, "g^b"] \arrow[dr, "\rho'"'] & & g^*x^*\mathscr{A} \arrow[dl, "g^*\rho"] \\ & \mathscr{O}_{T'} & \end{tikzcd}$$

commutes. Since we have descent for morphisms of quasi-coherent sheaves, the fibered category $\underline{\mathrm{Spec}}_{\mathscr{X}}(\mathscr{A})$ is a stack with respect to the étale topology.

There is a morphism of stacks:

$$\pi : \underline{\mathrm{Spec}}_{\mathscr{X}}(\mathscr{A}) \to \mathscr{X}, \quad (T, x, \rho) \mapsto (T, x).$$

For an algebraic space T and object $x \in \mathscr{X}(T)$, the fiber product of the diagram

$$\begin{tikzcd} & T \arrow[d, "x"] \\ \underline{\mathrm{Spec}}_{\mathscr{X}}(\mathscr{A}) \arrow[r] & \mathscr{X} \end{tikzcd}$$

is represented by the usual relative spectrum $\underline{\mathrm{Spec}}_T(x^*\mathscr{A})$ defined as in 7.2.2. By the following 10.2.2 it follows that $\underline{\mathrm{Spec}}_{\mathscr{X}}(\mathscr{A})$ is an algebraic stack.

PROPOSITION 10.2.2. *Let S be a scheme and let $f : \mathscr{Y} \to \mathscr{X}$ be a morphism of (not necessarily algebraic) stacks over S. Suppose that for every S-scheme T and morphism $x : T \to \mathscr{X}$ the fiber product $\mathscr{Y} \times_{\mathscr{X}} T$ is an algebraic space. If $\mathscr{X}$ is an algebraic stack, then $\mathscr{Y}$ is also an algebraic stack.*

PROOF. To see that the diagonal of $\mathscr{Y}$ is representable note that we have the commutative diagram

$$\begin{tikzcd} \mathscr{Y} \arrow[r, "\Delta_{\mathscr{Y}/\mathscr{X}}"] \arrow[rr, bend left, "\Delta_{\mathscr{Y}}"] \arrow[dr, "f"'] & \mathscr{Y} \times_{\mathscr{X}} \mathscr{Y} \arrow[r, "b"] \arrow[d] & \mathscr{Y} \times_S \mathscr{Y} \arrow[d] \\ & \mathscr{X} \arrow[r, "\Delta_{\mathscr{X}}"] & \mathscr{X} \times_S \mathscr{X}, \end{tikzcd}$$

where the square is cartesian. Since $\Delta_{\mathscr{X}}$ is representable, the morphism labelled b is also representable. To prove that $\Delta_{\mathscr{Y}}$ is representable it therefore suffices to show that the diagonal $\Delta_{\mathscr{Y}/\mathscr{X}}$ of $\mathscr{Y}$ over $\mathscr{X}$ is representable. For this let T be a scheme and $x : T \to \mathscr{X}$ a morphism, and suppose given two morphisms $\alpha, \beta : T \to \mathscr{Y}$ over x. The fiber product of the resulting diagram

$$\begin{tikzcd} & T \arrow[d, "\alpha\times\beta"] \\ \mathscr{Y} \arrow[r, "\Delta_{\mathscr{Y}/\mathscr{X}}"] & \mathscr{Y} \times_{\mathscr{X}} \mathscr{Y} \end{tikzcd}$$

is then isomorphic to the fiber product of the diagram

$$\begin{tikzcd} & \mathscr{Y}_T \arrow[d, "\alpha\times\beta"] \\ \mathscr{Y}_T \arrow[r, "\Delta"] & \mathscr{Y}_T \times_T \mathscr{Y}_T, \end{tikzcd}$$

where $\mathscr{Y}_T$ denotes $\mathscr{Y} \times_{\mathscr{X},x} T$. Since $\mathscr{Y}_T$ is assumed to be an algebraic space it follows that $\Delta_{\mathscr{Y}/\mathscr{X}}$ is representable.

To get a smooth covering of $\mathscr{Y}$, let $X \to \mathscr{X}$ be a smooth surjection with X a scheme, and then choose a smooth surjection $Y \to \mathscr{Y} \times_{\mathscr{X}} X$ with Y a scheme (recall that we are assuming that $\mathscr{Y} \times_{\mathscr{X}} X$ is an algebraic space). The composite morphism $Y \to \mathscr{Y} \times_{\mathscr{X}} X \to \mathscr{Y}$ is then the desired smooth surjection. □

10.2.3. Let $\mathscr{X}$ be an algebraic stack over a scheme S. Using 8.2.9 we define a morphism of algebraic stacks $\pi : \mathscr{Y} \to \mathscr{X}$ to be *affine* if for every morphism $x : X \to \mathscr{X}$ with X an algebraic space, the fiber product $\mathscr{Y} \times_{\mathscr{X}} X \to X$ is affine over X (so in particular π is representable).

If $\pi : \mathscr{Y} \to \mathscr{X}$ and $\pi' : \mathscr{Y}' \to \mathscr{X}$ are two affine morphisms, then the category $HOM_{\mathscr{X}}(\mathscr{Y}, \mathscr{Y}')$ of morphisms $\mathscr{Y} \to \mathscr{Y}'$ over $\mathscr{X}$ is equivalent to a set. It follows that it makes sense to talk about the category of affine morphisms $\pi : \mathscr{Y} \to \mathscr{X}$.

THEOREM 10.2.4. *The* $\underline{\mathrm{Spec}}_{\mathscr{X}}(-)$ *construction defines an equivalence of categories*

$$(\textit{quasi-coherent sheaves of algebras on } \mathscr{X})^{\mathrm{op}} \to (\textit{affine morphisms } \pi : \mathscr{Y} \to \mathscr{X}),$$

$$\mathscr{A} \mapsto \underline{\mathrm{Spec}}_{\mathscr{X}}(\mathscr{A}).$$

An inverse functor is given by sending an affine morphism $\pi : \mathscr{Y} \to \mathscr{X}$ *to the sheaf of* $\mathscr{O}_{\mathscr{X}}$*-algebras* $\pi_*\mathscr{O}_{\mathscr{Y}}$.

PROOF. This is left as exercise 10.D. □

10.2.5. Similarly we can define relative Proj. As before let S be a scheme and $\mathscr{X}/S$ an algebraic stack. Let

$$\mathscr{A} = \bigoplus_{d\geq 0} \mathscr{A}_d$$

be a quasi-coherent sheaf of graded algebras on $\mathscr{X}$. Let $\underline{\mathrm{Proj}}_{\mathscr{X}}(\mathscr{A})$ denote the following fibered category. The objects of $\underline{\mathrm{Proj}}_{\mathscr{X}}(\mathscr{A})$ are triples (T, x, ρ), where T is an S-scheme, $x : T \to \mathscr{X}$ is a morphism, and $\rho : T \to \mathrm{Proj}_T(x^*\mathscr{A})$ is a section of the T-scheme $\mathrm{Proj}_T(x^*\mathscr{A})$. A morphism

$$(T', x', \rho') \to (T, x, \rho)$$

is a morphism $\tilde{g} : x' \to x$ in $\mathscr{X}$ over a morphism of schemes $g : T' \to T$ such that the diagram

$$\begin{tikzcd} T' \arrow[r, "g"] \arrow[d, "\rho'"] & T \arrow[d, "\rho"] \\ \mathrm{Proj}_{T'}(x'^*\mathscr{A}) \arrow[r, "\tilde{g}"] & \mathrm{Proj}_T(x^*\mathscr{A}) \end{tikzcd}$$

commutes.

Then $\underline{\mathrm{Proj}}_{\mathscr{X}}(\mathscr{A})$ is a stack with respect to the étale topology, and for any morphism $x : T \to \mathscr{X}$ from a scheme T the fiber product $T \times_{x,\mathscr{X}} \underline{\mathrm{Proj}}_{\mathscr{X}}(\mathscr{A})$ is isomorphic to $\mathrm{Proj}_T(x^*\mathscr{A})$. By 10.2.2 it follows that $\underline{\mathrm{Proj}}_{\mathscr{X}}(\mathscr{A})$ is also an algebraic stack.

EXAMPLE 10.2.6. Let $g \geq 2$ be an integer and consider the algebraic stack $\mathscr{M}_g$ classifying curves of genus g as in 8.4.3. Let $\pi : \mathscr{C} \to \mathscr{M}_g$ be the following stack over $\mathscr{M}_g$. The objects of $\mathscr{C}$ are triples $(T, g : C \to T, s)$, where T is a scheme, $g : C \to T$ is a genus g curve over T (an object of $\mathscr{M}_g(T)$), and $s : T \to C$ is a section of g. A morphism

$$(T', g' : C' \to T', s') \to (T, g : C \to T, s)$$

is a cartesian diagram

$$\begin{tikzcd} C' \arrow[r, "\tilde{h}"] \arrow[d, "g'"] & C \arrow[d, "g"] \\ T' \arrow[r, "h"] & T, \end{tikzcd}$$

such that $\tilde{h} \circ s' = s \circ h$. For a scheme T and morphism $T \to \mathscr{M}_g$ corresponding to a curve $C \to T$, the fiber product

$$T \times_{\mathscr{M}_g} \mathscr{C}$$

is represented by C. Therefore by 10.2.2 the stack $\mathscr{C}$ is also algebraic. It is usually referred to as the *universal curve* over $\mathscr{M}_g$.

On the stack $\mathscr{C}$ there is a coherent sheaf $\Omega^1_{\mathscr{C}/\mathscr{M}_g}$. This is the quasi-coherent sheaf which to any morphism

$$T \to \mathscr{C}$$

corresponding to a curve $C \to T$ with section $s : T \to C$ associates $\Gamma(T, s^*\Omega^1_{C/T})$. We can then consider the quasi-coherent sheaf of graded algebras on $\mathscr{M}_g$,

$$\mathscr{A} = \bigoplus_{d \geq 0} \pi_*(\Omega^{1,\otimes d}_{\mathscr{C}/\mathscr{M}_g}),$$

and $\mathscr{C} \simeq \underline{\mathrm{Proj}}_{\mathscr{M}_g}(\mathscr{A})$.

10.2.7. There is a variant of the $\underline{\mathrm{Proj}}_{\mathscr{X}}(-)$-construction which is often useful. As before let $\mathscr{X}$ be an algebraic stack and let $\mathscr{A} = \bigoplus_{d \geq 0} \mathscr{A}_d$ be a graded algebra. We can then define the *stacky Proj* of $\mathscr{A}$ as follows. Let $\underline{\mathscr{P}\mathrm{roj}}_{\mathscr{X}}(\mathscr{A})$ denote the following fibered category. The objects of $\underline{\mathscr{P}\mathrm{roj}}_{\mathscr{X}}(\mathscr{A})$ are collections of data

$$(T, x, P \to T, \rho),$$

where T is a scheme, $x \in \mathscr{X}(T)$ is an object, $P \to T$ is a $\mathbb{G}_m$-torsor, and $\rho : P \to \mathbb{V}(x^*\mathscr{A})^\circ$ is a $\mathbb{G}_m$-equivariant morphism over T. Here we write $\mathbb{V}(x^*\mathscr{A})^\circ$ for the complement in $\underline{\mathrm{Spec}}_T(x^*\mathscr{A})$ of the closed subscheme defined by the ideal $\bigoplus_{d \geq 1} x^*\mathscr{A}_d$, which inherits a $\mathbb{G}_m$-action from the $\mathbb{G}_m$-action on $\underline{\mathrm{Spec}}_T(x^*\mathscr{A})$. A morphism

$$(T', x', P' \to T', \rho') \to (T, x, P \to T, \rho)$$

is defined to be a pair (g, h), where $g : x' \to x$ is a morphism in $\mathscr{X}$ over a morphism $\bar{g} : T' \to T$ and $h : P' \to P$ is a $\mathbb{G}_m$-equivariant morphism over $\bar{g}$ such that the diagram

$$\begin{tikzcd} P' \arrow[r, "h"] \arrow[d, "\rho'"] & P \arrow[d, "\rho"] \\ \underline{\mathrm{Spec}}_{T'}(x'^*\mathscr{A}) \arrow[r] & \underline{\mathrm{Spec}}_T(x^*\mathscr{A}) \end{tikzcd}$$

commutes, where the bottom horizontal arrow is the morphism induced by g.

There is a morphism

$$\underline{\mathscr{P}\mathrm{roj}}_{\mathscr{X}}(\mathscr{A}) \to \mathscr{X}, \quad (T, x, P \to T, \rho) \mapsto x.$$

EXAMPLE 10.2.8. Suppose $\mathscr{X}$ is an affine scheme, say $\mathscr{X} = \mathrm{Spec}(R)$. Then $\mathscr{A}$ corresponds to a graded R-algebra $A = \bigoplus_{d \geq 0} A_d$. The stack $\underline{\mathscr{P}\mathrm{roj}}_{\mathscr{X}}(\mathscr{A})$ is isomorphic to the stack quotient $[\mathbb{V}(A)^\circ/\mathbb{G}_m]$, where $\mathbb{V}(A)^\circ$ denotes the complement of the vertex in $\mathrm{Spec}(A)$. In this context we usually write simply $\mathscr{P}\mathrm{roj}(A)$ for $\underline{\mathscr{P}\mathrm{roj}}_{\mathscr{X}}(\mathscr{A})$. If A is generated in degree 1 then $\mathscr{P}\mathrm{roj}(A)$ is isomorphic to the usual $\mathrm{Proj}(A)$, but if A has generators in higher degrees then these may be different. For example, consider the polynomial ring in one variable $R[t]$ but with grading given by declaring the degree of t to be 2. Then $\mu_2 \subset \mathbb{G}_m$ acts trivially on $R[t]$ and therefore every point of $\mathscr{P}roj(R[t])$ has nontrivial stabilizer group scheme.

PROPOSITION 10.2.9. *With notation as in* 10.2.7, *the fibered category* $\underline{\mathscr{P}roj}_{\mathscr{X}}(\mathscr{A})$ *is an algebraic stack.*

PROOF. Note that $\underline{\mathscr{P}\mathrm{roj}}_{\mathscr{X}}(\mathscr{A})$ is clearly a stack with respect to the étale topology (since this is true for $\mathscr{X}$), and for any morphism $x : T \to \mathscr{X}$ with T a scheme, the fiber product

$$T \times_{\mathscr{X}} \underline{\mathscr{P}\mathrm{roj}}_{\mathscr{X}}(\mathscr{A})$$

is isomorphic to the stack quotient

$$[\mathbb{V}(x^*\mathscr{A})^\circ/\mathbb{G}_m]$$

which is an algebraic stack. The proposition therefore follows from exercise 10.A. $\square$

EXAMPLE 10.2.10. Let $\mathscr{X}$ be an algebraic stack. Let $\mathscr{I} \subset \mathscr{O}_{\mathscr{X}}$ be the subsheaf which to any smooth $U \to \mathscr{X}$ associates the subset of $\Gamma(U, \mathscr{O}_U)$ consisting of locally nilpotent sections. Then $\mathscr{I}$ is a quasi-coherent sheaf (exercise 9.J), and therefore the quotient $\mathscr{A} := \mathscr{O}_{\mathscr{X}}/\mathscr{I}$ is a quasi-coherent sheaf of algebras. The relative spectrum

$$\mathscr{X}_{\mathrm{red}} := \underline{\mathrm{Spec}}_{\mathscr{X}}(\mathscr{A}) \to \mathscr{X}$$

is called the *maximal reduced closed substack of* $\mathscr{X}$. The morphism $\mathscr{X}_{\mathrm{red}} \to \mathscr{X}$ is a closed imbedding, and if $\mathscr{Y}$ is any other reduced algebraic stack and $f : \mathscr{Y} \to \mathscr{X}$ is a morphism, then f factors uniquely through $\mathscr{X}$.

10.3. Root stacks

10.3.1. Let X be a scheme. A technical problem one often encounters in algebraic geometry concerns taking roots of divisors. Specifically given an integer n and an effective Cartier divisor $D \subset X$, it is sometimes of interest to find another effective Cartier divisor E such that $nE = D$. This is of course not possible in general. The best one can hope for is to find a morphism $f : Y \to X$ such that f^*D makes sense on Y and such that there exists E on Y such that $nE = f^*D$. The root stack construction described in this section is an attempt at finding a universal such (Y, E).

Of course divisors do not pull back along arbitrary morphisms, so we work with the following more flexible object:

Definition 10.3.2. Let X be a scheme. A *generalized effective Cartier divisor on X* is a pair (L, ρ), where L is an invertible sheaf on X and $\rho : L \to \mathscr{O}_X$ is a morphism of $\mathscr{O}_X$-modules. If (L', ρ') and (L, ρ) are two generalized effective Cartier divisors, then an isomorphism between is an isomorphism of line bundles $\sigma : L' \to L$ such that the diagram

$$\begin{tikzcd} L' \arrow[rr, "\sigma"] \arrow[dr, "\rho'"'] & & L \arrow[dl, "\rho"] \\ & \mathscr{O}_X & \end{tikzcd}$$

commutes.

Example 10.3.3. Let $D \subset X$ be an effective Cartier divisor. Then the ideal sheaf $\mathscr{I}_D$ together with the inclusion $j_D : \mathscr{I}_D \hookrightarrow \mathscr{O}_X$ is a generalized effective Cartier divisor on X. Note that if $D, D' \subset X$ are two effective Cartier divisors, then $(\mathscr{I}_D, j_D)$ and $(\mathscr{I}_{D'}, j_{D'})$ are isomorphic if and only if $D = D'$ in which case the isomorphism is unique.

Example 10.3.4. If L is an invertible sheaf on X, then we can take for ρ the zero map $L \to \mathscr{O}_X$ and the pair $(L, 0)$ is a generalized effective Cartier divisor.

10.3.5. If (L, ρ) and (L', ρ') are two generalized effective Cartier divisors, then we can form the product

$$(L, \rho) \cdot (L', \rho') := (L \otimes L', \rho \otimes \rho'),$$

where we write $\rho \otimes \rho'$ for the map

$$L \otimes L' \xrightarrow{\rho \otimes \rho'} \mathscr{O}_X \otimes_{\mathscr{O}_X} \mathscr{O}_X \simeq \mathscr{O}_X.$$

For a nonnegative integer n and a generalized effective Cartier divisor (L, ρ), we write $(L^{\otimes n}, \rho^{\otimes n})$ for the n-fold product of (L, ρ) with itself.

Let $\mathcal{D}iv^+(X)$ denote the set of isomorphism classes of generalized effective Cartier divisors. Then the product construction defines on $\mathcal{D}iv^+(X)$ the structure of a commutative monoid with unit.

10.3.6. The main advantage of working with generalized effective Cartier divisors is that they pull back easily. Namely, if $g : Y \to X$ is a morphism of schemes and (L, ρ) is a generalized effective Cartier divisor on X, then g^*L with the map $g^*\rho : g^*L \to g^*\mathscr{O}_X = \mathscr{O}_Y$ is a generalized effective Cartier divisor on Y.

Consider the fibered category $\mathscr{D}$ over the category of schemes, whose objects are pairs $(T, (L, \rho))$, where T is a scheme and (L, ρ) is a generalized effective Cartier divisor on T. A morphism $(T', (L', \rho')) \to (T, (L, \rho))$ is a pair (g, g^b) consisting of a morphism of schemes $g : T' \to T$ and an isomorphism $g^b : (L', \rho') \to (g^*L, g^*\rho)$.

Note that since invertible sheaves and morphisms between them satisfy effective descent, the fibered category $\mathscr{D}$ is a stack.

Proposition 10.3.7. *The stack $\mathscr{D}$ is isomorphic to the stack quotient $[\mathbb{A}^1/\mathbb{G}_m]$ of $\mathbb{A}^1$ by the standard multiplication action of $\mathbb{G}_m$. In particular, $\mathscr{D}$ is an algebraic stack.*

Proof. Consider the prestack $\{\mathbb{A}^1/\mathbb{G}_m\}$ whose objects are pairs $(T, f \in \Gamma(T, \mathscr{O}_T))$, where T is a scheme, and whose morphisms

$$(T', f' \in \Gamma(T', \mathscr{O}_{T'})) \to (T, f \in \Gamma(T, \mathscr{O}_T))$$

is a pair (g, u), where $g : T' \to T$ is a morphism and $u \in \Gamma(T', \mathscr{O}_{T'}^*)$ is a unit such that $f' = u \cdot g^\sharp(f)$ in $\Gamma(T', \mathscr{O}_{T'})$. We have a morphism of fibered categories

$$\{\mathbb{A}^1/\mathbb{G}_m\} \to \mathscr{D}$$

sending (T, f) to $(T, (\mathscr{O}_T, \cdot f))$ and a morphism $(g, u) : (T', f') \to (T, f)$ to the morphism

$$(T', (\mathscr{O}_{T'}, \cdot f')) \to (T, (\mathscr{O}_T, \cdot f))$$

given by g and multiplication by u on $\mathscr{O}_{T'}$. Note that this morphism is fully faithful and every object of $\mathscr{D}$ is locally in the image. Therefore this morphism of fibered categories induces an isomorphism

$$[\mathbb{A}^1/\mathbb{G}_m] \to \mathscr{D}.$$

□

10.3.8. The morphisms

$$\mathbb{A}^1 \to \mathbb{A}^1, \quad t \mapsto t^n, \quad \mathbb{G}_m \to \mathbb{G}_m, u \mapsto u^n$$

define a morphism of stacks

$$p_n : [\mathbb{A}^1/\mathbb{G}_m] \to [\mathbb{A}^1/\mathbb{G}_m].$$

Under the identification with $\mathscr{D}$ in 10.3.7, this morphism corresponds to the morphism of stacks

$$\mathscr{D} \to \mathscr{D}, \quad (T, (L, \rho)) \mapsto (T, (L^{\otimes n}, \rho^{\otimes n})).$$

10.3.9. Fix a generalized effective Cartier divisor (L, ρ) and an integer $n \geq 1$. Let $\mathscr{X}_n$ be the fibered category over the category of schemes, whose objects are triples $(f : T \to X, (M, \lambda), \sigma)$, where $f : T \to X$ is an X-scheme, (M, λ) is a generalized effective divisor on T, and

$$\sigma : (M^{\otimes n}, \lambda) \to (f^*L, f^*\rho)$$

is an isomorphism of generalized effective Cartier divisors on T. A morphism

$$(f' : T' \to X, (M', \lambda'), \sigma') \to (f : T \to X, (M, \lambda), \sigma)$$

is a pair (h, h^b), where $h : T' \to T$ is an X-morphism and $h^b : (M', \lambda') \to (h^*M, h^*\lambda)$ is an isomorphism of generalized effective Cartier divisors on T' such that the diagram

$$\begin{tikzcd} M'^{\otimes n} \arrow[rr, "h^{b\otimes n}"] \arrow[dr, "\lambda'"'] & & h^*M^{\otimes n} \arrow[dl, "h^*\lambda"] \\ & f'^*L \simeq h^*f^*L & \end{tikzcd}$$

commutes. The fibered category $\mathscr{X}_n$ is called the *n-th root stack associated to* (L, ρ). Let

$$\pi_n : \mathscr{X}_n \to X$$

be the morphism defined by sending $(f : T \to X, (M, \rho), \sigma)$ to $f : T \to X$.

THEOREM 10.3.10. *(i) The fibered category $\mathscr{X}_n$ is an algebraic stack with finite diagonal.*

(ii) If $L = \mathscr{O}_X$ and ρ is given by an element $f \in \Gamma(X, \mathscr{O}_X)$, then $\mathscr{X}_n$ is isomorphic to the stack quotient of

$$\mathrm{Spec}_X(\mathscr{O}_X[T]/(T^n - f))$$

by the action of $\boldsymbol{\mu}_n$ given by $\zeta * T = \zeta T$.

(iii) The map π_n is an isomorphism over the open subset $U \subset X$ where ρ is an isomorphism.

(iv) If n is invertible on X, then $\mathscr{X}_n$ is a Deligne-Mumford stack.

PROOF. Notice that $\mathscr{X}_n$ (as a fibered category over X) is isomorphic to the fiber product of the diagram

$$\begin{tikzcd} & \mathscr{D} \arrow[d, "p_n"] \\ X \arrow[r, "(L,\rho)"] & \mathscr{D}. \end{tikzcd}$$

This implies in particular statement (i).

Furthermore, statement (iii) follows from (ii) so it remains to describe $\mathscr{X}_n$ in the case when $L = \mathscr{O}_X$.

In this case let $f \in \Gamma(X, \mathscr{O}_X)$ be the element defining the map $\rho : \mathscr{O}_X \to \mathscr{O}_X$, so the map

$$f : X \to \mathbb{A}^1$$

is a lifting of the map $X \to [\mathbb{A}^1/\mathbb{G}_m]$ defined by (L, ρ). By exercise 10.F, the fiber product of the diagram

$$\begin{tikzcd} & {[\mathbb{A}^1/\mathbb{G}_m]} \arrow[d, "p_n"] \\ \mathbb{A}^1 \arrow[r] & {[\mathbb{A}^1/\mathbb{G}_m]} \end{tikzcd}$$

is isomorphic to $[\mathbb{A}^1/\boldsymbol{\mu}_n]$ with map to $\mathbb{A}^1$ given by the map $\mathbb{A}^1 \to \mathbb{A}^1$ sending z to z^n. From this statement (ii) and (iv) follow. $\square$

REMARK 10.3.11. The root stack was considered in [**17**] where it was applied to study stable map spaces.

10.4. Exercises

Exercise 10.A. Generalize Proposition 10.2.2 as follows. Let S be a scheme and let $f : \mathscr{Y} \to \mathscr{X}$ be a morphism of stacks over S. Suppose that for every scheme T and morphism $x : T \to \mathscr{X}$ the fiber product $\mathscr{Y} \times_{\mathscr{X}} T$ is an algebraic stack. Show that if $\mathscr{X}$ is an algebraic stack, then $\mathscr{Y}$ is an algebraic stack as well.

Exercise 10.B. Verify the details of example (10.2.10):

(i) Show that the presheaf $\mathscr{I} \subset \mathscr{O}_{\mathscr{X}}$ of locally nilpotent sections is a quasi-coherent subsheaf.

(ii) Show that the $\mathscr{X}_{\mathrm{red}} \to \mathscr{X}$ is a closed imbedding, and that it is universal for morphisms from reduced stacks to $\mathscr{X}$.

Exercise 10.C. Prove Proposition 10.2.4.

Exercise 10.D. Let X be an integral scheme, and let $\mathrm{Ca}^+(X)$ denote the monoid of effective Cartier divisors on X. Show that $\mathcal{D}iv^+(X)$ (defined in 10.3.5) is isomorphic to the disjoint union $\mathrm{Ca}^+(X) \coprod \mathrm{Pic}(X)$ with monoid structure given

$$D \cdot L := L(-D) \in \mathrm{Pic}(X)$$

for $D \in \mathrm{Ca}^+(X)$ and $L \in \mathrm{Pic}(X)$, and the usual monoid structures on $\mathrm{Ca}^+(X)$ and $\mathrm{Pic}(X)$.

Exercise 10.E. Let S be a scheme and let G/S be a flat finite type group scheme over S. Let X/S be an algebraic space with action of G, and let $[X/G]$ be the fibered category over the category of S-schemes whose fiber over an S-scheme T is the groupoid of diagrams

$$\begin{tikzcd} P \arrow[r] \arrow[d] & X \\ T, & \end{tikzcd}$$

where $P \to T$ is a principal G_T-bundle (see Definition 4.5.4), and $P \to X$ is a G-equivariant map. Show that $[X/G]$ is an algebraic stack (note that in the case when G/S is smooth this is 8.1.12).

Exercise 10.F. Let S be a scheme and $f : G \to H$ a homomorphism of smooth group schemes over S. Let X be an algebraic space over S with action ρ_X of G, let Y be an algebraic space over S with action ρ_Y of H, and let $g : X \to Y$ be a morphism of algebraic spaces over S compatible with the actions in the sense that the diagram

$$\begin{tikzcd} G \times_S X \arrow[r, "\rho_X"] \arrow[d, "f \times g"] & X \arrow[d, "g"] \\ H \times_S Y \arrow[r, "\rho_Y"] & Y \end{tikzcd}$$

commutes.

(i) Show that g induces a morphism of algebraic stacks $\bar{g} : [X/G] \to [Y/H]$.

(ii) Show that if $\pi_Y : Y \to [Y/H]$ is the canonical projection, then the fiber product of the diagram

$$\begin{tikzcd} & {[X/G]} \arrow[d, "\bar{g}"] \\ Y \arrow[r, "\pi_Y"] & {[Y/H]} \end{tikzcd} \tag{10.4.0.1}$$

is isomorphic to the stack quotient of $X \times_S H$ by the diagonal action of G.

(iii) Deduce from this that in the case when f is surjective, and the kernel K of f is flat over S, the fiber product of the diagram (10.4.0.1) is isomorphic to $[X/K]$.

EXERCISE 10.G. For an algebraic stack $\mathscr{X}$, let $\mathrm{Pic}(\mathscr{X})$ denote the set of isomorphism classes of locally free quasi-coherent sheaves of rank 1 on $\mathscr{X}$.

(i) Show that tensor product defines a structure of an abelian group on $\mathrm{Pic}(\mathscr{X})$, called the Picard group.

(ii) Let k be a field and let n be an integer. Let $\boldsymbol{\mu}_n$ denote the group scheme of n-th roots of unity over k which acts by multiplication on $\mathbb{A}^1_k$. Calculate the Picard group of the stack quotient $[\mathbb{A}^1_k/\boldsymbol{\mu}_n]$.

(iii) Let k be a field and consider the action of $\mathbb{G}_m^n$ on $\mathbb{P}^n_k$ given by

$$(u_1, \ldots, u_n) * [a_0 : \cdots : a_n] := [a_0 : u_1 a_1 : \cdots : u_n a_n].$$

Calculate the Picard group of the quotient $[\mathbb{P}^n_k/\mathbb{G}_m^n]$ (the motivated reader may wish to generalize this example to the case of a smooth toric variety and the stack quotient by the torus action).

EXERCISE 10.H. Let k be a field. Show that any isomorphism of stacks $\sigma : [\mathbb{A}^1_k/\mathbb{G}_m] \to [\mathbb{A}^1_k/\mathbb{G}_m]$ is uniquely isomorphic to the identity functor.

CHAPTER 11

Coarse moduli spaces

In this section we prove the so-called Keel-Mori theorem on the existence of coarse moduli spaces for algebraic stacks with finite diagonal. Aside from being interesting in its own right, it has many important consequences. In particular, using the local structure of Deligne-Mumford stacks over their coarse moduli spaces (Theorem 11.3.1) one can often reduce proofs of statements for Deligne-Mumford stacks to the corresponding statements for algebraic spaces and stacks obtained by taking the quotient of a scheme by a finite group action.

Our treatment of the Keel-Mori theorem follows closely [**19**].

One consequence of the Keel-Mori theorem (which can also be proven by other methods) is Chow's lemma for Deligne-Mumford stacks. We discuss this in section 11.4, and use it to deduce the valuative criterion for properness as well as finiteness of cohomology for coherent sheaves on proper Deligne-Mumford stacks. This material can be found in [**23**] and [**49**]. Theorems 11.3.1 and the notion of tame stack can be found in [**1**, 2.2.3 and 2.3.1].

11.1. Basics on coarse moduli spaces

DEFINITION 11.1.1. Let S be a scheme and $\mathscr{X}/S$ an algebraic stack over S. A *coarse moduli space* for $\mathscr{X}$ is a morphism $\pi : \mathscr{X} \to X$ to an algebraic space over S such that:

(i) π is initial for maps to algebraic spaces over S. That is, if $g : \mathscr{X} \to Z$ is a morphism from $\mathscr{X}$ to an algebraic space Z, then there exists a unique morphism $f : X \to Z$ such that $g = f \circ \pi$.

(ii) For every algebraically closed field k the map $|\mathscr{X}(k)| \to X(k)$ is bijective, where $|\mathscr{X}(k)|$ denotes the set of isomorphism classes in $\mathscr{X}(k)$.

The main result on coarse moduli spaces is the following, whose proof we give in the following section.

THEOREM 11.1.2 ([**19, 43**]). *Assume S is locally noetherian and that $\mathscr{X}$ is an algebraic stack locally of finite presentation over S with finite diagonal. Then there exists a coarse moduli space $\pi : \mathscr{X} \to X$. In addition:*

(i) *X/S is locally of finite type, and if $\mathscr{X}/S$ is separated, then X/S is also separated.*

(ii) *π is proper, and the map $\mathscr{O}_X \to \pi_*\mathscr{O}_{\mathscr{X}}$ is an isomorphism.*

(iii) *If $X' \to X$ is a flat morphism of algebraic spaces, then $\pi' : \mathscr{X}' := \mathscr{X} \times_X X' \to X'$ is a coarse moduli space for $\mathscr{X}'$.*

REMARK 11.1.3. The theorem also holds without the noetherian hypothesis (except for statement (i)), but we only give the proof in the noetherian setting. For a statement and proof in the more general setting see [**19**].

EXAMPLE 11.1.4. Let X be a locally noetherian scheme and let (L, ρ) be a generalized effective Cartier divisor on X. Let $\pi_n : \mathscr{X}_n \to X$ be the associated n-th root stack (see 10.3.9). Then $\pi_n : \mathscr{X}_n \to X$ is a coarse moduli space. Indeed, let $q : \mathscr{X}_n \to Y$ be a coarse moduli space, and note that by the universal property there exists a unique morphism $\delta : Y \to X$ compatible with the projections from $\mathscr{X}_n$. Since π_n is proper and q is proper and surjective, the map δ is also proper. Since δ is also quasi-finite this implies that δ is finite.

Now to check that this map δ is an isomorphism we may work étale locally on X (using property (iii) of 11.1.2) so it suffices to consider the case when $L = \mathscr{O}_X$ and ρ is given by a global section $f \in \Gamma(X, \mathscr{O}_X)$. In this case, we have

$$\mathscr{X}_n = [\mathrm{Spec}_X(\mathscr{O}_X[T]/(T^n - f)/\boldsymbol{\mu}_n)]$$

The coarse moduli space is then given by the relative spectrum of the $\boldsymbol{\mu}_n$-invariants in $\mathscr{O}_X[T]/(T^n - f)$ since the map $\mathscr{O}_Y \to \pi_* \mathscr{O}_{\mathscr{X}_n}$ is an isomorphism. Since the $\boldsymbol{\mu}_n$-invariants of $\mathscr{O}_X[T]/(T^n - f)$ are equal to $\mathscr{O}_X$ this implies that δ is an isomorphism.

11.2. Proof of the main theorem

In this section we prove 11.1.2 following the argument and notation of [**19**].

11.2.1. Note that the properness of π and condition (ii) in the definition of coarse moduli space implies that if the theorem is true then π induces a bijection between open substacks of $\mathscr{X}$ and open subsets of X.

This implies that to prove the theorem we may work Zariski locally on $\mathscr{X}$. If $\mathscr{X} = \bigcup \mathscr{X}_i$ is an open cover of $\mathscr{X}$ and each $\mathscr{X}_i$ has a coarse moduli space $\mathscr{X}_i \to X_i$ satisfying the properties in 11.1.2, then the intersections $\mathscr{X}_{ij}$ define open subsets $X_{ij} \subset X_i$ and $X_{ij} \subset X_j$ which define gluing data for a morphism $\pi : \mathscr{X} \to X$. One verifies immediately that this is also a coarse moduli space.

From this it follows that we may assume that S is an affine noetherian scheme.

Special case 1: There exists a finite flat surjection $\mathrm{Spec}(A_1) \to \mathscr{X}$**.**

Set $\mathrm{Spec}(A_2) := \mathrm{Spec}(A_1) \times_{\mathscr{X}} \mathrm{Spec}(A_1)$, and let $A_0 := \mathrm{Eq}(A_1 \rightrightarrows A_2)$. We then showed in section 6.2:

(a) The morphism $A_0 \to A_1$ is finite and integral (this implies that A_0 is of finite type over S by exercise 6.C);
(b) The underlying topological space of $\mathrm{Spec}(A_0)$ is the quotient of the topological space of $\mathrm{Spec}(A_1)$ by the equivalence relation defined by $\mathrm{Spec}(A_2)$;
(c) If $\pi : \mathscr{X} \to \mathrm{Spec}(A_0)$ is the projection, then π is universal for maps to schemes. Applying this to $\mathbb{A}^1$ we get that the map $\mathcal{O}_{\mathrm{Spec}(A_0)} \to \pi_* \mathcal{O}_{\mathscr{X}}$ is an isomorphism.

We claim that $\pi : \mathscr{X} \to \mathrm{Spec}(A_0)$ is a coarse moduli space. Note that (i)-(iii) in Theorem 11.1.2 clearly hold, and also the bijectivity on geometric points is clear (in light of the already proven properties of the quotient). It therefore suffices to show that π is also universal for maps to algebraic spaces.

Let Y be an algebraic space. We then need to show that the sequence

$$Y(A_0) \to Y(A_1) \rightrightarrows Y(A_2) \tag{11.2.1.1}$$

is an equalizer diagram.

Injectivity of $Y(A_0) \to Y(A_1)$.

Let $U \to Y$ be an étale surjection with U a scheme, and let $\eta_1, \eta_2 \in Y(A_0)$ be two elements mapping to the same element in $Y(A_1)$. Let $R \subset U \times U$ denote $U \times_Y U$. To show that η_1 and η_2 are equal, we may work étale locally on $\mathrm{Spec}(A_0)$. We may therefore assume that there exist liftings $\tilde{\eta}_i \in U(A_0)$ of the η_i. Let $\gamma \in R(A_1)$ be an element whose image in $U(A_1) \times U(A_1)$ is equal to $(\tilde{\eta}_1|_{A_1}, \tilde{\eta}_2|_{A_1})$. The element γ is in the equalizer of the two maps $R(A_1) \rightrightarrows R(A_2)$ since $(\tilde{\eta}_1|_{A_1}, \tilde{\eta}_2|_{A_1})$ is in the equalizer of the two maps

$$U(A_1) \times U(A_1) \rightrightarrows U(A_2) \times U(A_2).$$

Thus γ defines an element of $R(A_0)$ mapping to $(\tilde{\eta}_1, \tilde{\eta}_2)$.

Exactness in the middle.

Let $\eta \in Y(A_1)$ be an element in the equalizer, and let $\bar{\eta} : \mathscr{X} \to Y$ be the induced map.

Consider the diagram

$$\begin{tikzcd} \mathrm{Spec}(A_2) \arrow[r, shift left] \arrow[r, shift right] & \mathrm{Spec}(A_1) \arrow[r] \arrow[rr, bend left, "\eta"] & \mathrm{Spec}(A_0) \arrow[r, dashed] & Y. \end{tikzcd}$$

As above, we may further assume that Y is quasi-compact. Let $U \to Y$ be a quasi-compact étale surjection with U a scheme, and let V denote $\mathrm{Spec}(A_1) \times_Y U$. Since the two pullbacks of η to $\mathrm{Spec}(A_2)$ agree we get two projections,

$$R := \mathrm{Spec}(A_2) \times_Y U \rightrightarrows U,$$

which define a finite locally free groupoid in quasi-affine schemes. Let $V \to Q$ be the corresponding quotient in the category of ringed spaces, which exists by 6.2.2, so there is a morphism $Q \to \mathrm{Spec}(A_0)$ which is surjective. By the universal property of the quotient in the category of ringed spaces there is also a morphism $Q \to Y$.

Let $\mathscr{X}_U$ (resp. $\mathscr{X}_{U \times_Y U}$) denote the fiber product $\mathscr{X} \times_{\bar{\eta}, Y} U$ (resp. $\mathscr{X} \times_{\bar{\eta}, Y} (U \times_Y U)$).

To prove exactness in the middle it suffices to show that $Q \to \mathrm{Spec}(A_0)$ is étale. For if $\mathscr{X}_{U \times_Y U} \to Q'$ denotes the universal map to a scheme (which exists again by 6.2.2), then for either of the projections p_i the diagram

$$\begin{tikzcd} \mathscr{X}_{U \times_Y U} \arrow[r] \arrow[d, "p_i"] & Q' \arrow[d, "p_i"] \\ \mathscr{X}_U \arrow[r] & Q \end{tikzcd}$$

is cartesian, so either of the projections $Q' \to Q$ is étale. It follows that if $Q \to \mathrm{Spec}(A_0)$ is étale, then the natural map

$$Q' \to Q \times_{\mathrm{Spec}(A_0)} Q$$

is also étale and induces a bijection on k-valued points, for any algebraically closed field k. By [**36**, I.5.7] it follows that $Q' \to Q \times_{\mathrm{Spec}(A_0)} Q$ is an isomorphism. Then $Q \times_{\mathrm{Spec}(A_0)} Q \subset Q \times Q$ is the étale equivalence relation defining $\mathrm{Spec}(A_0)$, and by the universal property of coarse moduli space the two maps $Q \times_{\mathrm{Spec}(A_0)} Q \to Y$ are equal.

The formation of the space Q is compatible with flat base change on $\mathrm{Spec}(A_0)$. Looking at the completion of $\mathrm{Spec}(A_0)$ at a point, we can therefore assume that A_0 is a complete noetherian local ring with algebraically closed residue field. In this case $\mathrm{Spec}(A_1) = \coprod_s \mathrm{Spec}(A_1^s)$, where each A_1^s is also complete and local. Then $\mathrm{Spec}(A_1^s) \to \mathscr{X}$ is also surjective, so we may also assume that A_1 is also complete and local, and further that $A_0 \to A_1$ induces an isomorphism on residue fields (this is where we need to make a flat base change on $\mathrm{Spec}(A_0)$). Let k be the residue field of A_1, and let

$$\begin{tikzcd} & \mathrm{Spec}(\mathscr{O}_{Y,\bar{\eta}}) \arrow[d] \\ \mathrm{Spec}(k) \arrow[ur] \arrow[r, "\eta"] & Y \end{tikzcd}$$

be the strict henselization of Y at η. The ring A_2 is a product of complete local rings A_2^r such that the maps $A_1 \to A_2^r$ induce isomorphisms on residue fields. We therefore have a commutative diagram:

$$\begin{tikzcd} \coprod \mathrm{Spec}(k) \arrow[r, "\mathrm{id}"] \arrow[d] & \mathrm{Spec}(k) \arrow[r] \arrow[d] & \mathrm{Spec}(\mathscr{O}_{Y,\bar{\eta}}) \arrow[d] \\ \coprod \mathrm{Spec}(A_2^r) \arrow[r, shift left] \arrow[r, shift right] & \mathrm{Spec}(A_1) \arrow[r] \arrow[ur] & Y. \end{tikzcd}$$

By the universal property of strict henselization the map $\mathrm{Spec}(A_1) \to \mathrm{Spec}(\mathscr{O}_{Y,\bar{\eta}})$ descends to a morphism $\mathrm{Spec}(A_0) \to \mathrm{Spec}(\mathscr{O}_{Y,\bar{\eta}})$. Therefore to prove that $Q \to \mathrm{Spec}(A_0)$ is étale we are reduced to the case when η is induced by a morphism $\mathrm{Spec}(A_0) \to Y$. In this case one sees from the construction that $Q = \mathrm{Spec}(A_0) \times_Y U$.

Special case 2: There exists a finite flat surjection $U \to \mathscr{X}$ with U a scheme such that for every $x \in U$ the finite set $s(t^{-1}(x))$ is contained in an affine open subscheme.

Let $\mathscr{X}^{(n)} \subset \mathscr{X}$ denote the open and closed substack of morphisms $T \to \mathscr{X}$ such that the fiber product $U \times_{\mathscr{X}} T \to T$ is finite of rank n. Then $\mathscr{X} = \coprod \mathscr{X}^{(n)}$ so it suffices to consider the case when $U \to \mathscr{X}$ has constant rank n.

We show that there exists an open covering $\mathscr{X} = \bigcup \mathscr{X}_i$ such that $\mathscr{X}_i$ admits a finite flat surjection $U_i \to \mathscr{X}_i$ with U_i affine.

This is essentially the same argument as we used in 6.2.22. Let us review the argument.

Let R denote $U \times_{\mathscr{X}} U$. Recall from 6.2.18 that a subset $F \subset U$ is called *invariant* if for every $x \in F$ and $y \in R$ such that $t(y) = x$ we have $s(y) \in F$. If $F \subset I$ is a subset then by 6.2.19 $F^{\mathrm{inv}} := s(t^{-1}(F))$ is invariant. Recall also 6.2.20 that if $W \subset U$ is an open subset with complement F, then $W' := U - F^{\mathrm{inv}}$ is called the *maximal invariant open subset of* W.

Note that if we have inclusions $Z_1 \subset Z_2 \subset U$ of subsets of U and Z_2 is invariant then $Z_1^{\mathrm{inv}} \subset Z_2$. Furthermore, if $Z \subset W$ is an invariant subset then $Z \subset W'$ (since Z^c is also invariant and contains F whence $F^{\mathrm{inv}} \subset Z^c$).

Let $V \subset U$ be any affine open subset containing $s(t^{-1}(x))$ (such an open subset exists by assumption), and let $V' \subset V$ be the saturation. Since $s(t^{-1}(x))$ is an invariant subset of V we also have $s(t^{-1}(x)) \subset V'$. By prime avoidance, there exists an element $f \in \Gamma(V, \mathcal{O}_V)$ such that the open set $D(f)$ contains $s(t^{-1}(x))$ and

$D(f) \subset V'$. Then the saturation $U := D(f)'$ also contains $s(t^{-1}(f))$, and by the argument of 6.2.21 U is affine.

Special case 3: $\mathscr{X}$ admits a quasi-finite flat surjection $U \to \mathscr{X}$ with U a quasi-projective S-scheme.

THEOREM 11.2.2. *Let $\mathscr{X}/S$ be an algebraic stack of finite presentation with finite diagonal, and assume there exists a quasi-finite flat surjection $U \to \mathscr{X}$ with U a quasi-projective S-scheme. Then there exists an algebraic stack $\mathscr{W}/S$ and a surjective, separated, and étale morphism $\mathscr{W} \to \mathscr{X}$ which is representable by schemes, and a closed imbedding $Z \subset U \times_{\mathscr{X}} \mathscr{W}$ such that $Z \to \mathscr{W}$ is a finite flat surjection. Furthermore we can arrange for the following to hold:*

(i) For any quasi-compact open substack $\mathscr{W}' \subset \mathscr{W}$ the preimage $Z' \subset Z$ is quasi-projective over S;

(ii) For every algebraically closed field k and $w \in \mathscr{W}(k)$ with image $x \in \mathscr{X}(k)$, the map on automorphism groups (not automorphism sheaves!)

$$\mathrm{Aut}(w) \to \mathrm{Aut}(x)$$

is bijective (note that this map is automatically injective since $\mathscr{W} \to \mathscr{X}$ is representable).

PROOF. Let $\mathscr{H}$ be the stack over S whose objects are triples (T, x, Z), where T is a scheme, $x \in \mathscr{X}(T)$, and $Z \subset U \times_{\mathscr{X},x} T$ is a closed subscheme finite and flat over T. Let $p : \mathscr{H} \to \mathscr{X}$ be the projection $(T, x, Z) \mapsto (T, x)$. By the theory of the Hilbert scheme, p is representable by schemes and hence $\mathscr{H}$ is also an algebraic stack by 10.2.2.

Let $\mathscr{W}' \subset \mathscr{H}$ be the maximal open substack of $\mathscr{H}$ étale over $\mathscr{X}$. Let $I_{\mathscr{X}}$ denote the inertia stack of $\mathscr{X}$ defined as the fiber product of

$$\begin{tikzcd} & \mathscr{X} \arrow[d, "\Delta"] \\ \mathscr{X} \arrow[r, "\Delta"] & \mathscr{X} \times_S \mathscr{X}, \end{tikzcd}$$

and define $I_{\mathscr{W}'}$ similarly. Let $I'_{\mathscr{X}}$ denote the fiber product $I_{\mathscr{X}} \times_{\mathscr{X}} \mathscr{W}'$ so we have a morphism $j : I_{\mathscr{W}'} \to I'_{\mathscr{X}}$.

LEMMA 11.2.3. *The morphism $j : I_{\mathscr{W}'} \to I'_{\mathscr{X}}$ is an open and closed imbedding.*

PROOF. Since $\mathscr{W}'$ and $\mathscr{X}$ have finite diagonals, both $I_{\mathscr{W}'}$ and $I'_{\mathscr{W}'}$ are finite over $\mathscr{W}'$. Therefore the map j is proper. The map j is also a monomorphism since $\mathscr{W}' \to \mathscr{X}$ is representable. To prove the lemma it suffices to show that j is also formally étale. This follows from consideration of the commutative diagram

$$\begin{tikzcd} I_{\mathscr{W}'} \arrow[r, "j"] \arrow[d] & I'_{\mathscr{X}} \arrow[r] \arrow[d] & \mathscr{W}' \arrow[d, "\Delta"] \\ \mathscr{W}' \arrow[r, "a"] \arrow[dr, "b"'] & \mathscr{K} \arrow[r] \arrow[d, "c"] & \mathscr{W}' \times_S \mathscr{W}' \arrow[d] \\ & \mathscr{X} \arrow[r] & \mathscr{X} \times \mathscr{X}, \end{tikzcd}$$

where a is étale since c and b are étale, whence j is also étale. $\square$

Let $I''_{\mathscr{X}} \subset I'_{\mathscr{X}}$ be the complement of $I_{\mathscr{W}'}$, and let $\mathscr{W} \subset \mathscr{W}'$ be the complement of the (closed since $I'_{\mathscr{X}}$ is proper over $\mathscr{W}$) image of $I''_{\mathscr{X}}$ in $\mathscr{W}'$. We claim that $\mathscr{W}$ together with the tautological subscheme $Z \subset U \times_{\mathscr{X}} \mathscr{W}$ has the desired properties.

By construction $Z \to \mathscr{W}$ is finite and flat. Furthermore, if $\mathscr{U} \subset \mathscr{W}$ is a quasi-compact open substack, then Z' is a closed subspace of the étale, separated and quasi-compact U-space $U \times_{\mathscr{X}} \mathscr{U}$ whence quasi-affine over U. This implies that Z' is also quasi-projective. Finally, property (ii) follows from the definition of $\mathscr{W}$.

The only thing that needs to be verified then, is that $\mathscr{W} \to \mathscr{X}$ is surjective. For this let k be an algebraically closed field and $x \in \mathscr{X}(k)$ an object together with a lifting $u \in U(k)$. Let $w \in \mathscr{H}(k)$ be the point over x corresponding to the closed subscheme of $U \times_{\mathscr{X},x} \operatorname{Spec}(k)$ given by the whole fiber. Then clearly $\operatorname{Aut}(w) \to \operatorname{Aut}(x)$ is bijective so it suffices to show that $\mathscr{H} \to \mathscr{X}$ is étale at w. Let $\widehat{U}$ denote the spectrum of the completion $\widehat{\mathcal{O}}_{U,u}$. Since $U \times_{\mathscr{X}} \widehat{U}$ is quasi-finite and flat over $\widehat{U}$ we can write

$$U \times_{\mathscr{X}} \widehat{U} = P \coprod Q,$$

where P is finite over $\widehat{U}$ and Q has empty fiber over the closed point of $\widehat{U}$. From this it follows that

$$\mathscr{H} \times_{\mathscr{X}} \widehat{U} \simeq \operatorname{Hilb}_{P/\widehat{U}} \coprod \text{(something with empty fiber over a closed point)}.$$

The Hilbert scheme $\operatorname{Hilb}_{P/\widehat{U}}$ has one connected component isomorphic to $\widehat{U}$ given by the whole scheme P, and our point w is the closed point of this component. $\square$

PROPOSITION 11.2.4. *Let $\mathscr{Y}' \to \mathscr{Y}$ be a representable, separated, étale, and quasi-compact morphism of finite type algebraic stacks over S. Assume there exists a finite flat covering $U \to \mathscr{Y}$ with U a quasi-projective S-scheme. Assume that for any algebraically closed field k and $y' \in \mathscr{Y}(k)$ with image $y \in \mathscr{Y}(k)$ the map on groups*

$$\operatorname{Aut}_{\mathscr{Y}'(k)}(y') \to \operatorname{Aut}_{\mathscr{Y}(k)}(y)$$

is bijective. Then $\mathscr{Y}'$ admits a finite flat covering $U' \to \mathscr{Y}'$ with U' a quasi-projective S-scheme, and the map on coarse moduli spaces (which exist by special case 2) $Y' \to Y$ is étale.

PROOF. Set $U' := U \times_{\mathscr{Y}} \mathscr{Y}'$. Then the algebraic space U' is separated, quasi-compact, and quasi-finite over the quasi-projective S-scheme U, and therefore U' is also a quasi-projective S-scheme.

The morphism $Y' \to Y$ is locally of finite type, since both Y and Y' are locally of finite type over S. To prove that $Y' \to Y$ is étale, it therefore suffices to show that for an element $y' \in \mathscr{Y}'(k)$ with image $y \in \mathscr{Y}(k)$ the map on strict henselizations $\mathscr{O}_{Y,\bar{y}} \to \mathscr{O}_{Y',\bar{y}'}$ is an isomorphism (where we abusively write also y' and y for the images in Y' and Y respectively). Since the formation of coarse moduli space commutes with flat base change on Y, we may therefore assume that Y is equal to the spectrum of a strictly henselian local ring. Since U is finite over Y we can also assume that U is the spectrum of a strictly henselian local ring (let $u \in U(k)$ be a lifting of y), and that $U \to Y$ induces an isomorphism on residue fields. If R denotes $U \times_{\mathscr{Y}} U$, then R is finite over the strictly henselian local U and therefore also the spectrum of a finite product of strictly henselian local rings. In this case

$$\mathscr{O}_{Y,\bar{y}} = \operatorname{Eq}(\mathscr{O}_{U,\bar{u}} \rightrightarrows \prod_{\xi \in R(k), s(\xi)=u=t(\xi)} \mathscr{O}_{R,\bar{\xi}}).$$

The ring $\mathscr{O}_{Y',\bar{y}'}$ can be described similarly. Let $u' \in U'(k)$ be a point over y'. Then $\mathscr{O}_{Y',\bar{y}'}$ is equal to the equalizer of the two maps

$$\mathscr{O}_{U',\bar{u}'} \rightrightarrows \prod_{\xi' \in R'(k), s(\xi')=t(\xi')=u'} \mathcal{O}_{R',\xi'},$$

where $R' := U' \times_{\mathscr{Y}'} U'$ and s and t are the two projections to U'. Since the morphisms $U' \to U$ and $R' \to R$ are étale, the map $\mathscr{O}_{U,\bar{u}} \to \mathscr{O}_{U',\bar{u}'}$ is an isomorphism, and for every $\xi' \in R'(k)$ mapping to $\xi \in R(k)$ the map $\mathscr{O}_{R,\bar{\xi}} \to \mathscr{O}_{R',\bar{\xi}'}$ is an isomorphism. Thus to prove the theorem, it is enough to show that the natural map

$$\{\xi' \in R'(k) | s(\xi') = t(\xi')\} \to \{\xi \in R(k) | s(\xi) = t(\xi) = u\}$$

is a bijection. This is clear because the left side is canonically identified with $\mathrm{Aut}_{\mathscr{Y}'(k)}(y')$ and the right side is canonically identified with $\mathrm{Aut}_{\mathscr{Y}(k)}(y)$. □

Returning to the proof of 11.1.2 in the present case, choose data $U \to \mathscr{X}$, $\pi : \mathscr{W} \to \mathscr{X}$, and $Z \subset U \times_{\mathscr{X}} \mathscr{W}$ as in 11.2.2. Since $\mathscr{X}$ is quasi-compact we may further assume that $\mathscr{W}$ is also quasi-compact, in which case Z is quasi-projective over S by 11.2.2 (i).

Since $\mathscr{W}$ admits a finite flat covering by a quasi-projective S-scheme there exists a coarse moduli space $\mathscr{W} \to W$ for $\mathscr{W}$. Let $\mathscr{R}$ denote $\mathscr{W} \times_{\mathscr{X}} \mathscr{W}$.

LEMMA 11.2.5. *(i) Either projection $\mathscr{R} \to \mathscr{W}$ satisfies the assumptions of Proposition* 11.2.4*, so there exists a coarse moduli space $\mathscr{R} \to R$ such that the two projections $R \rightrightarrows W$ are both étale;*

(ii) The map $R \to W \times_S W$ is a closed imbedding and defines an étale equivalence relation on W.

PROOF. For (i), let $(w_1, w_2, \iota) \in \mathscr{R}(k)$ be a point corresponding to two elements $w_i \in \mathscr{W}(k)$ together with an isomorphism $\iota : \pi(w_1) \to \pi(w_2)$ in $\mathscr{X}(k)$. An automorphism of such an object in $\mathscr{R}(k)$ consists of two automorphisms $\alpha_i : w_i \to w_i$ in $\mathscr{W}(k)$ such that the diagram in $\mathscr{X}(k)$,

$$\begin{tikzcd} \pi(w_1) \arrow[r, "\iota"] \arrow[d, "\pi(\alpha_1)"'] & \pi(w_2) \arrow[d, "\pi(\alpha_2)"] \\ \pi(w_1) \arrow[r, "\iota"] & \pi(w_2), \end{tikzcd}$$

commutes. From this description of automorphisms in $\mathscr{R}(k)$ and the fact that the maps

$$\mathrm{Aut}_{\mathscr{W}(k)}(w_i) \to \mathrm{Aut}_{\mathscr{X}(k)}(\pi(w_i))$$

are bijective statement (i) follows.

The map $\Delta : R \to W \times_S W$ is unramified since either of the two projections $R \to W$ are étale. To show that Δ is an imbedding it therefore suffices (by [EGA, IV.17.2.6]) to show that for any algebraically closed field k the map

$$R(k) \to W(k) \times W(k)$$

is injective. This in turn is equivalent to the injectivity of the map

$$|\mathscr{R}(k)| \to |\mathscr{W}(k)| \times |\mathscr{W}(k)|$$

which again follows from the fact that $\pi : \mathscr{W} \to \mathscr{X}$ induces bijections on automorphism groups of k-valued points.

To prove that Δ is a closed imbedding, it suffices to show that Δ is proper. For this, note that there is the commutative diagram

$$\begin{tikzcd} Z \times_{\mathscr{X}} Z \arrow[r, "b"] \arrow[d, "a"'] & R \arrow[d, "\Delta"] \\ Z \times_S Z \arrow[r, "c"] & W \times_S W, \end{tikzcd}$$

where a is finite (being the base change of the diagonal $\mathscr{X} \to \mathscr{X} \times_S \mathscr{X}$ along the morphism $Z \times_S Z \to \mathscr{X} \times_S \mathscr{X}$), and b and c are proper and surjective.

It remains to see that $R \subset W \times_S W$ is an equivalence relation. That R is invariant under the "flip" map $W \times_S W \to W \times_S W$ follows from observing that the flip map

$$\text{"flip}'' : \mathscr{W} \times_{\mathscr{X}} \mathscr{W} \to \mathscr{W} \times_{\mathscr{X}} \mathscr{W}$$

induces by the universal property of coarse moduli space a map "flip" : $R \to R$ compatible with the flip map on $W \times_S W$.

To verify the associativity, it suffices to show that there exists a dotted arrow filling in the diagram

$$\begin{tikzcd} \mathscr{R} \times_{t,\mathscr{W},s} \mathscr{R} \arrow[r, "m"] \arrow[d] & \mathscr{R} \arrow[d] \\ R \times_{t,W,s} R \arrow[r, dashed] & R, \end{tikzcd}$$

where m is the composition morphism. For this in turn it suffices to show that the projection $\mathscr{R} \times_{t,\mathscr{W},s} \mathscr{R} \to R \times_{t,W,s} R$ is a coarse moduli space for $\mathscr{R} \times_{t,\mathscr{W},s} \mathscr{R}$.

For this, note the morphism $\mathscr{R} \times_{t,\mathscr{W},s} \mathscr{R} \to \mathscr{W}$ satisfies the assumption of Proposition 11.2.4, and therefore $\mathscr{R} \times_{t,\mathscr{W},s} \mathscr{R}$ has a coarse moduli space R' étale over W. Since R is also étale over W it follows that the map $R' \to R \times_W R$ is étale. To verify that this map is an isomorphism, it then suffices by [EGA, IV.17.2.6] to show that for any algebraically closed field the map

$$R'(k) \to R(k) \times_{W(k)} R(k)$$

is bijective. This again follows from the fact that either projection $\mathscr{R} \to \mathscr{W}$ induces a bijection on automorphism groups. $\square$

Let X denote the separated algebraic space W/R. The composite map $\mathscr{W} \to W \to X$ then induces a unique morphism $q : \mathscr{X} \to X$. We claim that this is a coarse moduli space for $\mathscr{X}$.

The bijectivity of $|\mathscr{X}(k)| \to X(k) = W(k)/R(k)$ is immediate from the corresponding bijectivity for $\mathscr{W}$ and $\mathscr{R}$.

Let Y be an algebraic space and $f : \mathscr{X} \to Y$ a morphism. The composite morphism $\mathscr{W} \to \mathscr{X} \to Y$ defines by the universal property of $\mathscr{W} \to W$ a unique morphism $W \to Y$. Furthermore the two compositions

$$R \rightrightarrows W \to Y$$

are equal since the two morphisms $\mathscr{R} \rightrightarrows Y$ are equal. We therefore obtain a unique map $\bar{f} : X \to Y$. One verifies immediately that the composition $\mathscr{X} \to X \to Y$ is equal to f. The uniqueness of $\bar{f}$ follows from the uniqueness of the map $W \to Y$.

Finally, property (i) in 11.1.2 follows from the above discussion, and property (iii) follows from the corresponding property for of $\mathscr{W} \to W$ and $\mathscr{R} \to R$. The properness of $\pi : \mathscr{X} \to X$ can be seen by noting that the square

$$\begin{tikzcd} \mathscr{W} \arrow[r] \arrow[d] & W \arrow[d] \\ \mathscr{X} \arrow[r] & X \end{tikzcd}$$

is cartesian. Indeed, if $\mathscr{W}' := \mathscr{X} \times_X W$, then the map $\mathscr{W} \to \mathscr{W}'$ is representable and étale, and induces a bijection $|\mathscr{W}(k)| \to |\mathscr{W}'(k)|$ for any algebraically closed field k. Since both projections $\mathscr{W} \to \mathscr{X}$ and $\mathscr{W}' \to \mathscr{X}$ induce bijections on automorphism groups of geometric points, one sees from this that for any morphism $\mathrm{Spec}(k) \to \mathscr{W}'$ the fiber product $\mathscr{W} \times_{\mathscr{W}'} \mathrm{Spec}(k)$ projects isomorphically onto $\mathrm{Spec}(k)$. From this and [EGA, IV.17.2.6] it follows that for any scheme T and morphism $T \to \mathscr{W}'$ the morphism $\mathscr{W} \times_{\mathscr{W}'} T \to T$ is an isomorphism, whence $\mathscr{W} \to \mathscr{W}'$ is an isomorphism.

General case.

THEOREM 11.2.6. *Let $\mathscr{X}$ be an algebraic stack of finite type over (our noetherian) S, with finite diagonal. Then there is a covering of $\mathscr{X} = \bigcup \mathscr{X}_i$ by open substacks such that each $\mathscr{X}_i$ admits a quasi-finite, flat, covering $U_i \to \mathscr{X}_i$ with U_i a quasi-projective S-scheme.*

PROOF. The key point is the following lemma from [**25**] (the following argument can also be found in [**56**, Proof 2.11]). □

LEMMA 11.2.7. *Let*

$$\begin{tikzcd} Y \arrow[r, "u"] \arrow[d, "v"'] & X \\ Z & \end{tikzcd}$$

be a diagram of finite type S-schemes with X quasi-projective over S. Let $z \in Z$ be a point closed in the fiber over its image in S and such that v is flat at the points $v^{-1}(z)$. Then if $v^{-1}(z)$ is nonempty, there exists a closed subscheme $F \subset X$ such that $u(u^{-1}(F) \cap v^{-1}(z))$ is finite nonempty and such that the restriction of v to $u^{-1}(F)$ is flat at the points of $v^{-1}(z)$.

PROOF. See [**25**, V.7.2]. □

We use this as follows. After replacing $\mathscr{X}$ by an open covering we may assume that there exists a smooth surjection $U \to \mathscr{X}$ with U a quasi-projective S-scheme. Let $s, t : R \rightrightarrows U$ be the resulting groupoid presentation of $\mathscr{X}$. Since the diagonal of $\mathscr{X}$ is finite R is finite over U via either projection, so R is also a quasi-projective S-scheme. Let $x \in \mathscr{X}(k)$ be a geometric point closed in the fiber over its image in S, and let $u \in U(k)$ be a closed point in the fiber over x. By the above lemma applied to

$$\begin{tikzcd} R \arrow[r, "s"] \arrow[d, "t"'] & U \\ U & \end{tikzcd}$$

we get a closed subscheme $W \subset U$ such that the composite map

$$U \times_{\mathscr{X}} W = R \times_{s,U} W \longrightarrow U$$

defined by $t : R \to U$ is flat over u, and the set $s(s^{-1}(W) \cap t^{-1}(u))$ is finite and nonempty. The first of these statements implies that $W \to \mathscr{X}$ is flat over x, and hence after shrinking on W we may assume that the morphism $W \to \mathscr{X}$ is flat. Then the second of these statements implies that $W \times_{\mathscr{X}} x$ is finite, and therefore (using [**26**, IV.13.1.3]) we can arrange that $W \to \mathscr{X}$ is also quasi-finite. □

This completes the proof of Theorem 11.1.2. □

11.3. Applications of the local structure of coarse moduli spaces

The following result makes the connection between Deligne-Mumford stacks and the notion of orbifold.

THEOREM 11.3.1. *Let S be a locally noetherian scheme and let $\mathscr{X}/S$ be a Deligne-Mumford stack locally of finite type and with finite diagonal. Let $\pi : \mathscr{X} \to X$ be its coarse moduli space. Let $\tilde{x} \to \mathscr{X}$ be a geometric point with image $\bar{x} \to X$ in X. Let $G_{\tilde{x}}$ be the automorphism group of $\tilde{x}$, which is a finite group since $\mathscr{X}$ is Deligne-Mumford. Then there exists an étale neighborhood $U \to X$ of $\bar{x}$ and a finite U-scheme $V \to U$ with action of $G_{\tilde{x}}$ such that*

$$\mathscr{X} \times_X U \simeq [V/G_{\tilde{x}}].$$

PROOF. We may assume that there exists an étale surjection $V \to \mathscr{X}$ such that $V \to X$ is quasi-finite. Let $V_{(\bar{x})}$ denote the fiber product

$$\mathrm{Spec}(\mathscr{O}_{X,\bar{x}}) \times_X V.$$

By [**26**, IV, 18.12.3], we have

$$V_{(\bar{x})} \simeq V'_{(\bar{x})} \coprod W,$$

where $V'_{(\bar{x})} \to \mathrm{Spec}(\mathscr{O}_{X,\bar{x}})$ is finite, and the image of W in $\mathrm{Spec}(\mathscr{O}_{X,\bar{x}})$ does not contain the closed point. Further, the scheme $V'_{(\bar{x})}$ decomposes into a disjoint union of spectra of strictly henselian local rings. By a standard limit argument, such decompositions exist after replacing X by an étale neighborhood, and replacing X by such a neighborhood we may assume that $V \to X$ is finite and that $V_{(\bar{x})}$ is connected. Consider the fiber product

$$Z := V_{(\bar{x})} \times_{\mathscr{X}_{(\bar{x})}} V_{(\bar{x})},$$

where $\mathscr{X}_{(\bar{x})}$ denotes the base change $\mathscr{X} \times_X \mathrm{Spec}(\mathscr{O}_{X,\bar{x}})$. Each of the two projections

$$p_1, p_2 : Z \to V_{(\bar{x})}$$

are finite étale, so via the first projection each connected component of Z is identified with $V_{(\bar{x})}$. Furthermore, the base change of Z to the closed point of $V_{(\bar{x})}$ is canonically isomorphic to the stabilizer group $G_{\tilde{x}}$. We therefore get an isomorphism

$$Z \simeq V_{(\bar{x})} \times G_{\tilde{x}}$$

such that the first projection $Z \to V_{(\bar{x})}$ is projection onto the first factor. The second projection $p_2 : Z \to V_{(\bar{x})}$ then induces a morphism

$$\rho : V_{(\bar{x})} \times G_{\tilde{x}} \to V_{(\bar{x})}.$$

We claim that this map is an action of $G_{\tilde{x}}$ on $V_{(\bar{x})}$ and that $\mathscr{X}_{(\bar{x})} \simeq [V_{(\bar{x})}/G_{\tilde{x}}]$.

That ρ is a group action can be seen as follows. Consider $Z' := V_{(\bar{x})} \times_{\mathscr{X}_{(\bar{x})}} V_{(\bar{x})} \times_{\mathscr{X}_{(\bar{x})}} V_{(\bar{x})}$. Again, Z' being finite and étale over the strictly henselian $V_{(x)}$, is isomorphic via the first projection to a disjoint union of copies of $V_{(\bar{x})}$, and by passing to the closed fiber the connected components are in bijection with $G_{\tilde{x}} \times G_{\tilde{x}}$. Consider the three projections

$$\mathrm{pr}_{12}, \mathrm{pr}_{23}, \mathrm{pr}_{13} : Z' \to Z$$

Viewing these maps as morphisms

$$V_{(\bar{x})} \times G_{\tilde{x}} \times G_{\tilde{x}} \to V_{(\bar{x})} \times G_{\tilde{x}},$$

it follows that pr_{12} (resp. pr_{23}, pr_{13}) sends a scheme-valued point (v, g_1, g_2) to

$$(\rho(g_1, v), g_2) \quad (\text{resp. } (\rho(g_2, v), g_1), (v, g_1 \cdot g_2)).$$

The equality

$$\mathrm{pr}_2 \circ \mathrm{pr}_{13} = \mathrm{pr}_2 \circ \mathrm{pr}_{23}$$

then implies that

$$\rho(g_1, \rho(g_2, v)) = \rho(g_1 g_2, v),$$

which says that ρ is an action. This also shows that in fact the groupoid in schemes

$$Z \rightrightarrows V_{(x)}$$

arising from the cover $V_{(\bar{x})} \to \mathscr{X}_{(\bar{x})}$ is isomorphic to the groupoid in schemes arising from $V_{(\bar{x})}$ with its $G_{\tilde{x}}$-action. From this it follows that $\mathscr{X}_{(\bar{x})} \simeq [V_{(\bar{x})}/G_{\tilde{x}}]$.

Finally, we get 11.3.1 by 'spreading out'. Namely, giving an action of the group $G_{\tilde{x}}$ is equivalent to giving a finite number of maps $V \to V$, so after replacing X by an étale neighborhood of $\bar{x}$ we may assume given a global action of $G_{\tilde{x}}$ on V. Now consider the two X-schemes

$$V \times G_{\tilde{x}}, \quad V \times_{\mathscr{X}} V.$$

Both are finite X-schemes, so after replacing X by another étale neighborhood we may assume given an isomorphism between them inducing the previously constructed isomorphism over $\mathrm{Spec}(\mathscr{O}_{X,\bar{x}})$. After further shrinking around $\bar{x}$ we can then ensure that the isomorphism is compatible with the finitely many groupoid maps, and then we have $\mathscr{X} \simeq [V/G]$. □

Definition 11.3.2. Let $\mathscr{X}$ be a Deligne-Mumford stack separated and of finite type over a scheme S. We say that $\mathscr{X}$ is *tame* if for every geometric point $\tilde{x}$: $\mathrm{Spec}(k) \to \mathscr{X}$ the automorphism group $G_{\tilde{x}}$ has order invertible in k.

Example 11.3.3. If S is a $\mathbb{Q}$-scheme, then every separated Deligne-Mumford stack of finite type over S is tame.

Proposition 11.3.4. *Let $\mathscr{X}/S$ be a Deligne-Mumford stack locally of finite type over a locally noetherian scheme S and with finite diagonal, and let $\pi : \mathscr{X} \to X$ be its coarse moduli space. If $\mathscr{X}$ is tame, then the functor*

$$\pi_* : (\text{quasi-coherent sheaves on } \mathscr{X}) \to (\text{quasi-coherent sheaves on } X)$$

is exact.

PROOF. The assertion is étale local on the coarse space X, so we may assume that $X = \mathrm{Spec}(A)$ is affine, and that $\mathscr{X} = [\mathrm{Spec}(B)/G]$, where B is a finite A-algebra and G is a finite group of order invertible in A which acts on B. In this case by exercise 9.H the category of quasi-coherent sheaves on $\mathscr{X}$ is equivalent to the category of B-modules with a G-action compatible with the G-action on B. The functor π_* sends such a B-module M to the invariants M^G. The result therefore follows from noting that taking invariants under G is an exact functor since the order of G is invertible in A. $\square$

11.3.5. Let $\mathscr{X}$ be a separated Deligne-Mumford stack of finite type over a locally noetherian scheme S, and let $\pi : \mathscr{X} \to X$ be its coarse moduli space. If $S' \to S$ is a morphism then we get by base change a morphism $\pi_{S'} : \mathscr{X}_{S'} \to X_{S'}$, where $\mathscr{X}_{S'}$ and $X_{S'}$ are the base changes to S' of $\mathscr{X}$ and X, respectively. The map $\pi_{S'} : \mathscr{X}_{S'} \to X_{S'}$ need not in general be a coarse moduli space. An explicit example is provided by exercise 6.B.

THEOREM 11.3.6. *With notation as in* 11.3.5*, let $q : \mathscr{X}_{S'} \to Y$ be the coarse moduli space of $\mathscr{X}_{S'}$, and let $p : Y \to X_{S'}$ be the map induced by the universal property of the coarse space Y. Then p is a universal homeomorphism. If the map $S' \to S$ is flat or if the stack $\mathscr{X}$ is tame, then p is an isomorphism.*

PROOF. The assertions are étale local on the coarse space X, so by 11.3.1 we may assume that $X = \mathrm{Spec}(A)$, and $\mathscr{X} = [\mathrm{Spec}(B)/G]$, where B is a finite A-algebra and G is a finite group acting on B over A. Since X is the coarse moduli space we have $A = B^G$. Now let $A \to A'$ be a ring homomorphism. We then need to show that the map of algebras

$$A' \to (B \otimes_A A')^G$$

induces a universal homeomorphism on spectra. By property (ii) in the definition of coarse moduli spaces, the map

$$\mathrm{Spec}((B \otimes_A A')^G) \to \mathrm{Spec}(A')$$

induces a bijection on geometric points, and since the map is also finite (since $(B \otimes_A A')^G$ is a subalgebra of the finite A'-algebra $B \otimes_A A'$) it follows that the map is a universal homeomorphism [**26**, IV.18.12.11].

Now if $A \to A'$ is flat, then indeed we have $(B \otimes_A A')^G$ equal to A' since the ring of invariants is the kernel of the map

$$B \to \prod_{g \in G} B$$

given by the product of the actions, and tensoring with A' is exact. Also, if $\mathscr{X}$ is tame, then we can choose G to have order invertible in A. In this case the ring of invariants is the image of the map

$$\frac{1}{|G|} \sum_{g \in G} g : B \to B,$$

and so formation of the invariants commutes with base change also in this case. $\square$

11.4. Chow's lemma for Deligne-Mumford stacks and applications

The main result of this section is the following:

THEOREM 11.4.1 ([**23**, 4.12], [**70**, 2.6], and [**49**, 16.6.1]). *Let S be a noetherian scheme, and let $\mathscr{X}/S$ be a Deligne-Mumford stack of finite type over S and with finite diagonal. Then there exists a proper surjective morphism $X' \to \mathscr{X}$, which is finite over a dense open substack of $\mathscr{X}$, such that the composition*

$$X' \to \mathscr{X} \to S$$

is projective (so in particular X' is a scheme).

PROOF. Let $\pi : \mathscr{X} \to X$ be the coarse moduli space of $\mathscr{X}$. By Chow's lemma for algebraic spaces (see 7.4.1), there exists a proper surjective generically finite morphism $W \to X$ with W a quasi-projective S-scheme. The coarse moduli space of $\mathscr{X}_W := \mathscr{X} \times_X W$ is then also a quasi-projective scheme being quasi-finite over W by 11.3.6. Replacing $\mathscr{X}$ by $\mathscr{X}_W$ we may therefore assume that the coarse space X is a quasi-projective S-scheme.

Similarly, by base changing to the disjoint union of the normalizations of the irreducible components of X (with their reduced structure) we may assume that X is irreducible and normal. Replacing $\mathscr{X}$ by its maximal reduced closed substack we may also assume that $\mathscr{X}$ is reduced. Let $w : W \to \mathscr{X}$ be an étale surjection with W a scheme. Then W is also reduced.

The morphism $W \to X$ need not be generically étale. However, we can arrange this by making a base change. Indeed, if $\eta \in X$ is the generic point, then by 11.3.1 there exists a finite extension $k(\eta) \to L$ such that the pullback to $\mathrm{Spec}(L)$ of the generic fiber $\mathscr{X}_\eta := \mathscr{X} \times_X \mathrm{Spec}(k(\eta))$ is isomorphic to $[V/G]$, where V is a finite L-scheme and G is a finite group acting on V. After further replacing L by an extension we may even assume that V has an L-point. It follows that by replacing X by the normalization X' of X in a finite field extension $k(\eta) \subset L$, and replacing $\mathscr{X}$ by the maximal reduced closed substack of $\mathscr{X} \times_X X'$, we may assume that $\mathscr{X} \to X$ is étale over a dense open subscheme of X. In this case the morphism $W \to X$ is also generically étale. We proceed assuming this additional property of X.

By Zariski's main theorem [**26**, III.4.4.3], we can find a factorization

$$\begin{tikzcd} W \arrow[r, hook, "j"] \arrow[dr] & \overline{W} \arrow[d, "p"] \\ & X, \end{tikzcd}$$

where j is a dense open imbedding and p is finite and generically étale. Let $X' \to X$ be the normalization of X in a suitable Galois extension of $k(\eta)$ containing all the function fields of the generic points of $\overline{W}$, and let G be the Galois group of X' over X. We can then find a morphism

$$\pi : X' \to \overline{W}$$

over X. Let $U \subset X'$ be the preimage of W, so we have

$$\begin{tikzcd} U \arrow[r, hook, "j'"] \arrow[d, "u"] & X' \arrow[ddl, "G"] \\ \mathscr{X} \arrow[d] & \\ X. & \end{tikzcd}$$

We have

$$\bigcup_{g \in G} g(U) = X'.$$

Indeed, since X is normal the scheme X is the coarse space of the stack $[X'/G]$ and so by 11.1.1 (ii) the geometric points of X are in bijection with the G-orbits of geometric points of X'. Since $U \to X$ is surjective it follows that the $g(U)$ cover X'.

We can therefore find a finite morphism $\pi : X' \to X$ with X' irreducible and normal, an open covering $X' = \bigcup_{i=1}^r V_i$ and elements $v_i \in \mathscr{X}(V_i)$. We now need to glue together the v_i to get a global object $v \in \mathscr{X}(X')$. This will require blowing up on X'.

First let us show that there exists a dense open subset $V \subset X'$ contained in $\bigcap_i V_i$ such the restrictions $v_i|_V \in \mathscr{X}(V)$ are all isomorphic. For this let $\lambda \in V$ be the generic point, which maps to the generic point $\eta \in X$. Over the algebraic closure of $k(\lambda)$, all the elements v_i become isomorphic by the property 11.1.1 (ii) of coarse moduli space. It follows that there exists a finite extension of $k(\lambda)$ over which the v_i become isomorphic. Replacing X' by the normalization of X' in such an extension we then get that the v_i are isomorphic over $k(\lambda)$. From this it follows that there also exists a dense open subset $V \subset X'$ over which they are isomorphic.

We may therefore further assume that there exists a dense open subset $V \subset \bigcap_i V_i$, an object $v \in \mathscr{X}(V)$ and isomorphisms $\epsilon_i : v \to v_i|_V$. Let Y denote

$$\coprod_i V_i,$$

let $\pi : Y \to X'$ be the étale surjection induced by the inclusions $V_i \subset X'$, and let $y \in \mathscr{X}(Y)$ denote the object obtained from the v_i. Consider the two objects $p_1^* y, p_2^* y \in \mathscr{X}(Y \times_{X'} Y)$, and the resulting algebraic space

$$I := \underline{\mathrm{Isom}}(p_1^* y, p_2^* y) \to Y \times_{X'} Y.$$

The space I is proper over $Y \times_{X'} Y$ since the diagonal of $\mathscr{X}$ is proper. If we denote by $A \subset Y \times_{X'} Y$ the preimage of $V \subset X'$, then we have a section

$$\begin{tikzcd} & I \arrow[d] \\ A \arrow[ur, "s"] \arrow[r, hook] & Y \times_{X'} Y. \end{tikzcd}$$

Let $\Gamma \subset I$ be the scheme-theoretic closure of $s(A)$. Since I is proper over $Y \times_{X'} Y$ the map $\Gamma \to Y \times_{X'} Y$ is proper and surjective. Since $Y \times_{X'} Y$ is étale over the irreducible X', every irreducible component of Γ dominates X'. We can therefore find a blowup $B \to X'$ such that the strict transform Q of Γ is flat over B. It follows that Q is also flat over the base change of $Y \times_{X'} Y$ to B and therefore isomorphic to this base change. We conclude that after replacing X' by a blowup, we can find an isomorphism $\sigma : \mathrm{pr}_1^* y \to \mathrm{pr}_2^* y$ over $Y \times_{X'} Y$ inducing the given isomorphism over A.

This isomorphism in fact satisfies the cocycle condition on $Y^3 := Y \times_{X'} Y \times_{X'} Y$. Indeed, the automorphism

$$\mathrm{pr}_{13}^*(\sigma) \circ \mathrm{pr}_{23}^*(\sigma)^{-1} \circ \mathrm{pr}_{13}^*(\sigma)^{-1} : \mathrm{pr}_3^*(y) \to \mathrm{pr}_3^*(y)$$

over Y^3 defines a map

$$Y^3 \to \underline{\mathrm{Aut}}_y$$

over X', whose restriction to a dense open subscheme of Y^3 maps to the identity. Since $\underline{\mathrm{Aut}}_y$ is a separated X'-space and Y^3 is reduced being étale over X', this implies that the map is the constant mapping to the identity implying that the cocycle condition holds over Y^3. By descent we therefore get an object $x' \in \mathscr{X}(X')$ defining a surjective proper morphism $X' \to \mathscr{X}$. $\square$

Remark 11.4.2. Theorem 11.4.1 remains true with $\mathscr{X}/S$ only assumed to be a separated Artin stack of finite type, but a different argument is needed to prove this (see [**60**]).

11.5. The valuative criterion for properness

Theorem 11.5.1 (Valuative criterion for properness ([**23**, 4.19] and [**49**, 7.12])). *Let $f : \mathscr{X} \to \mathscr{Y}$ be a separated finite type morphism of locally noetherian algebraic stacks over a base scheme S. Then f is proper if and only if for every discrete valuation ring R with field of fractions K and object θ in the groupoid*

$$\mathscr{X}(K) \times_{\mathscr{Y}(K)} \mathscr{Y}(R)$$

there exists a finite field extension $K \hookrightarrow K'$ such that the restriction

$$\theta' \in \mathscr{X}(K') \times_{\mathscr{Y}(K')} \mathscr{Y}(R'),$$

where R' is the normalization of R in K', is in the essential image of the functor

$$\mathscr{X}(R') \to \mathscr{X}(K') \times_{\mathscr{Y}(K')} \mathscr{Y}(R').$$

Diagramatically:

$$\begin{tikzcd}
\mathrm{Spec}(K') \arrow[r] \arrow[d, hook] & \mathrm{Spec}(K) \arrow[r] \arrow[d, hook] & \mathscr{X} \arrow[d, "f"] \\
\mathrm{Spec}(R') \arrow[r] \arrow[urr, dashed] & \mathrm{Spec}(R) \arrow[r] & \mathscr{Y}.
\end{tikzcd}$$

Remark 11.5.2. In the proof we will use Chow's lemma in its general form 11.4.2. If one further assumes that for any morphism $T \to \mathscr{Y}$, with T a scheme, the fiber product $\mathscr{X} \times_{\mathscr{Y}} T$ is a Deligne-Mumford stack with finite diagonal then the argument holds with the weaker version of Chow's lemma, 11.4.1.

Proof. Let us first show the 'only if' direction. So suppose f is proper and consider an object $\theta \in \mathscr{X}(K) \times_{\mathscr{Y}(K)} \mathscr{Y}(R)$. The object θ corresponds to objects $x_K \in \mathscr{X}(K)$, $y \in \mathscr{Y}(R)$ with image $y_K \in \mathscr{Y}(K)$, and an isomorphism $\sigma_K : f(x_K) \to y_K$ in $\mathscr{Y}(K)$. We have to show that after possibly replacing K by a finite extension and R by the normalization in such an extension, there exists an object $x \in \mathscr{X}(R)$ and isomorphisms

$$\lambda : x_K \to x|_K, \quad \sigma : f(x) \to y$$

such that σ_K is equal to the composition

$$f(x_K) \xrightarrow{f(\lambda)} x|_K \xrightarrow{\sigma} y|_K.$$

Let $\mathscr{X}_R$ denote the base change $\mathscr{X} \times_{\mathscr{Y},y} \mathrm{Spec}(R)$. Then θ corresponds to an object of $\mathscr{X}_R(K)$, and giving the extension (x, λ, σ) is equivalent to finding an object of $\mathscr{X}(R)$ with the given restriction $\mathscr{X}_R(K)$. We may therefore assume that $\mathscr{Y} = \mathrm{Spec}(R)$ and $\mathscr{X} = \mathscr{X}_R$.

Let G/K denote the automorphism group space of $x : \mathrm{Spec}(K) \to \mathscr{X}$ so we have a morphism $s : BG \to \mathscr{X}_K$.

LEMMA 11.5.3. *The map s is a closed imbedding.*

PROOF. The substack $s(BG)$ can be characterized as follows: It is the substack of $\mathscr{X}_K$ whose objects over a K-scheme T are the objects $z \in \mathscr{X}(T)$ such that there exists a faithfully flat surjection $T' \to T$ such that $z|_{T'} \simeq x|_{T'}$. Therefore if T is a scheme and $w : T \to \mathscr{X}$ is any morphism, the fiber product of the diagram

$$\begin{tikzcd} & T \arrow[d, "w"] \\ BG \arrow[r, "s"] & \mathscr{X} \end{tikzcd}$$

is a subscheme $C \subset T$ whose points are exactly the image of the proper (since f is separated) map

$$\underline{\mathrm{Isom}}(x|_T, w) \to T.$$

It follows that C is closed in T whence the lemma. $\square$

Let $\mathscr{Z} \hookrightarrow \mathscr{X}$ be the scheme-theoretic closure of s. Then the map $\mathscr{Z} \to \mathrm{Spec}(R)$ is surjective and flat (since R is a discrete valuation ring). We can therefore find a finite type flat R-scheme U with nonempty closed fiber, and a morphism $u : U \to \mathscr{Z}$. By the following lemma, we can therefore find a finite extension $R \to R'$ and a morphism $\mathrm{Spec}(R') \to \mathscr{Z}$. The generic point $\mathrm{Spec}(K') \to BG$ may not be isomorphic to $x|_{K'}$ but after making a finite extension of K' this will be the case. Replacing R' be the normalization in such a finite extension we therefore obtain the desired $\mathrm{Spec}(R') \to \mathscr{X}$.

LEMMA 11.5.4. *Let R be a discrete valuation ring, and let A be a finite type flat R-algebra such that the closed fiber of* $\mathrm{Spec}(A) \to \mathrm{Spec}(R)$ *is nonempty. Then there exists a finite extension of discrete valuation rings $R \to R'$ and a morphism* $\mathrm{Spec}(R') \to \mathrm{Spec}(A)$ *over* $\mathrm{Spec}(R)$.

PROOF. This is left as exercise 11.C. $\square$

Next we show the 'if' direction. To prove that f is proper, it suffices by 10.1.6 (ii) to show that there exists a smooth covering $\{Y_i \to \mathscr{Y}\}$ such that each of the base changes $\mathscr{X}_i \to Y_i$ are proper. It therefore suffices to consider the case when $\mathscr{Y}$ is an affine scheme. In this case by Chow's lemma there exists a proper representable surjective $p : P \to \mathscr{X}$ with P a scheme. By 10.1.6 (v) it suffices to show that the morphism $P \to \mathscr{Y}$ is proper. Now observe that the valuative criterion for $\mathscr{X} \to \mathscr{Y}$ and the valuative criterion for $P \to \mathscr{X}$ (which holds since this morphism is proper and representable) implies that the valuative criterion for $P \to \mathscr{Y}$ also holds. This implies that $P \to \mathscr{Y}$ is proper by the scheme-version of the theorem [**41**, Chapter II, 4.7]. □

11.6. Finiteness of cohomology

THEOREM 11.6.1 ([**49**, 15.6]). *Let $f : \mathscr{X} \to \mathscr{Y}$ be a proper morphism between finite type Deligne-Mumford stacks over a locally noetherian base scheme S, and let $\mathscr{F}$ be a coherent sheaf on $\mathscr{X}$.*

(i) For any $q \geq 0$ the sheaf $R^q f_* \mathscr{F}$ is coherent on $\mathscr{Y}$.

(ii) If $\mathscr{Y}$ is the spectrum of a noetherian ring A, then for every q the cohomology group $H^q(\mathscr{X}, \mathscr{F})$ is a finitely generated A-module.

REMARK 11.6.2. In the statement of 11.6.1 we can consider $\mathscr{F}$ and the sheaves $R^q f_* \mathscr{F}$ either in the étale or lisse-étale topology. By 9.2.16 the corresponding two versions of 11.6.1 are equivalent. For the remainder of this section we work with the étale topology.

PROOF OF THEOREM 11.6.1. If $\mathrm{Spec}(A) \to \mathscr{Y}$ is an étale morphism from an affine scheme and $\mathscr{X}_A$ denotes the fiber product $\mathscr{X} \times_{\mathscr{Y}} \mathrm{Spec}(A)$, then by 9.2.16 the value of $R^q f_* \mathscr{F}$ on $\mathrm{Spec}(A)$ is equal to $H^q(\mathscr{X}_A, \mathscr{F}_A)$, where $\mathscr{F}_A$ denotes the restriction of $\mathscr{F}$ to the étale site of $\mathscr{X}_A$. It therefore suffices to prove statement (ii) in the theorem.

For this, let $\pi : \mathscr{X} \to X$ be the coarse moduli space of $\mathscr{X}/A$, so we have a factorization

(11.6.2.1)
$$\begin{tikzcd} \mathscr{X} \arrow[r, "\pi"] \arrow[rr, bend left, "f"] & X \arrow[r, "\bar{f}"] & \mathrm{Spec}(A). \end{tikzcd}$$

From this we get a factorization of the global section functor

$$\begin{tikzcd} \mathrm{Mod}_{\mathscr{O}_{\mathscr{X}}} \arrow[r, "\pi_*"] \arrow[rr, bend left, "\Gamma(\mathscr{X}, -)"] & \mathrm{Mod}_{\mathscr{O}_X} \arrow[r, "\Gamma(X, -)"] & \mathrm{Mod}_A, \end{tikzcd}$$

and so the spectral sequence of a composition of functors gives us a spectral sequence (the Leray spectral sequence)

(11.6.2.2)
$$E_2^{pq} = H^p(X, R^q \pi_* \mathscr{F}) \implies H^{p+q}(\mathscr{X}, \mathscr{F}).$$

In particular, each $H^n(\mathscr{X}, \mathscr{F})$ admits a finite filtration whose associated graded pieces are subquotients of an A-module of the form $H^p(X, R^q \pi_* \mathscr{F})$. From this and finiteness of cohomology for proper morphisms of algebraic spaces (see 7.5.1), it therefore suffices to show that for every q the sheaf $R^q \pi_* \mathscr{F}$ is coherent on X.

This can be verified étale locally on X, so by 11.3.1 we may assume that $X = \mathrm{Spec}(R)$ is affine and that $\mathscr{X} = [\mathrm{Spec}(B)/G]$ for some finite B-algebra with action of a finite group G such that the map $R \to B$ identifies R with the G-invariants B^G. As in the beginning of the proof, in this setting to prove that $R^q \pi_* \mathscr{F}$ is coherent it suffices to show that the cohomology $H^q([\mathrm{Spec}(B)/G], \mathscr{F})$ is a finitely generated A-module. This can be seen as follows.

Let $U = \mathrm{Spec}(B)$ and let $s : U \to \mathscr{X}$ be the quotient map, which is an étale surjective map. Let $U_\bullet$ denote the associated simplicial scheme as in 9.2.1, and let $\mathscr{F}_\bullet$ denote the restriction of $\mathscr{F}$ to $U_\bullet$. By 9.2.4 there is a spectral sequence of R-modules

$$E_1^{pq} = H^q(U_p, \mathscr{F}_p) \implies H^{p+q}(\mathscr{X}, \mathscr{F}).$$

It therefore suffices to show that each $H^q(U_p, \mathscr{F}_p)$ is a finitely generated R-module. This is immediate for

$$U_p \simeq U \times G^p,$$

and so, in particular, is finite over $\mathrm{Spec}(R)$. □

11.6.3. In the case of a finite group quotient $\mathscr{X} = [\mathrm{Spec}(B)/G]$ as in the end of the proof, the argument in fact shows more. With notation as in the proof, since each U_n is finite over $\mathrm{Spec}(R)$, we have $H^q(U_p, \mathscr{F}_p) = 0$ for $q > 0$. Therefore $H^q(\mathscr{X}, \mathscr{F}) = H^q(E_1^{\bullet 0})$. Let $M = \Gamma(U, \mathscr{F})$ be the B-module with semilinear G-action corresponding to $\mathscr{F}$ as in exercise 9.E. Then

$$\Gamma(U_p, \mathscr{F}) = \mathrm{Fun}(G^n, M),$$

the set of functions $G^n \to M$. This is the same complex as the standard complex computing the group cohomology of M viewed as a G-representation over R. We therefore obtain an isomorphism

$$H^q(\mathscr{X}, \mathscr{F}) \simeq H^q(G, M), \tag{11.6.3.1}$$

where the right side denotes group cohomology. This isomorphism can be obtained geometrically as follows. Let

$$\epsilon : [\mathrm{Spec}(B)/G] \to \mathrm{Spec}(R) \times BG$$

be the map defined by the projection to $\mathrm{Spec}(R)$ and the natural map $[\mathrm{Spec}(B)/G] \to BG$. By 9.1.19 the sheaf $\epsilon_* \mathscr{F}$ corresponds to a G-representation over R, which in fact is the G-representation M. The isomorphism (11.6.3.1) is induced by the natural map

$$H^q(\mathrm{Spec}(R) \times BG, \epsilon_* \mathscr{F}) \to H^q([\mathrm{Spec}(B)/G], \mathscr{F}).$$

Note also that the construction of the isomorphism (11.6.3.1) also applies to quasi-coherent $\mathscr{F}$.

Example 11.6.4. This implies, in particular, that for a noetherian Deligne-Mumford stack $\mathscr{X}$ and coherent sheaf $\mathscr{F}$, the groups $H^q(\mathscr{X}, \mathscr{F})$ may be nonzero for infinitely many q. For example, let p be a prime and consider the classifying stack $B(\mathbb{Z}/(p))$ over $\mathrm{Spec}(\mathbb{F}_p)$. Then for all $q \geq 0$,

$$H^q(B(\mathbb{Z}/(p)), \mathscr{O}_{B(\mathbb{Z}/(p))}) \simeq H^q(\mathbb{Z}/(p), \mathbb{F}_p) \simeq \mathbb{F}_p.$$

Theorem 11.6.5. *Let $f : \mathscr{X} \to \mathscr{Y}$ be a proper morphism of finite type Deligne-Mumford stacks over S, and assume that S is quasi-compact. For a geometric point $\bar{x} \to \mathscr{X}$ let $G_{\bar{x}}$ (resp. $H_{f(\bar{x})}$) denote the stabilizer group of $\bar{x}$ (resp. $f(\bar{x})$), and let $K_{\bar{x}}$ denote the kernel of the natural map $G_{\bar{x}} \to H_{f(\bar{x})}$. If for every geometric point $\bar{x}$ the order of the group $K_{\bar{x}}$ is invertible in the field $k(\bar{x})$, then there exists an integer n_0 such that for any quasi-coherent sheaf $\mathscr{F}$ on $\mathscr{X}$ we have $R^q f_* \mathscr{F} = 0$ for $q > n_0$.*

PROOF. Let $\mathrm{Spec}(A) \to \mathscr{Y}$ be an étale morphism, and let $\mathscr{X}_A$ denote the fiber product $\mathscr{X} \times_{\mathscr{Y}} \mathrm{Spec}(A)$. If $\bar{x}_A : \mathrm{Spec}(\Omega) \to \mathscr{X}_A$ is a geometric point and $\bar{x} : \mathrm{Spec}(\Omega) \to \mathscr{X}$ denotes the composition of $\bar{x}_A$ with the projection $\mathscr{X}_A \to \mathscr{X}$, then we have

$$K_{\bar{x}} \simeq \mathrm{Aut}_{\mathscr{X}_A}(\bar{x}_A).$$

Using this observation and the fact that $\mathscr{Y}$ is quasi-compact (being of finite type over the quasi-compact S), it follows as in the proof of 11.6.1 that it suffices to consider the case when $\mathscr{Y} = \mathrm{Spec}(A)$ is an affine scheme, and further in this case it suffices to show that there exists an integer n_0 such that for any quasi-coherent sheaf $\mathscr{F}$ on $\mathscr{X}$ we have $H^q(\mathscr{X}, \mathscr{F}) = 0$ for $q > n_0$.

Consider again the factorization (11.6.2.1) and the resulting spectral sequence (11.6.2.2). By finiteness of cohomology for algebraic spaces (exercise 7.E), it suffices to show that there exists an integer m_0 such that $R^q\pi_*\mathscr{F} = 0$ for $q > m_0$. This again is a local question on the coarse space X, and therefore using 11.3.1 we may assume that $X = \mathrm{Spec}(R)$ and $\mathscr{X} = [\mathrm{Spec}(B)/G]$, where B is a finite R-algebra and G is a finite group, whose order we can arrange to be of order invertible in R (by the assumptions on the stabilizer groups). In this case for any G-representation M over R we have $H^q(G, M) = 0$ for $i > 0$, which implies that $R^q\pi_*\mathscr{F} = 0$ for $q > 0$ for any quasi-coherent sheaf $\mathscr{F}$ on $\mathscr{X}$. □

REMARK 11.6.6. In [**30**] and [**60**, 1.2] more general finiteness results for proper morphisms of Artin stacks are obtained.

11.7. Exercises

EXERCISE 11.A. (a) Let k be a field and $k \hookrightarrow \bar{k}$ an algebraic closure. Show that if F is a representable functor then the map

$$F(k) \to F(\bar{k})$$

is injective.

(b) Let $D > 1$ be an integer. Show that the two elliptic curves over $\mathbb{Q}$

$$E_1 : y^2 = x^3 + x, \quad E_2 : y^2 = x^3 + Dx$$

are isomorphic over $\overline{\mathbb{Q}}$ but not over $\mathbb{Q}$ if D is not a fourth power.

(c) Let M denote the functor

$$(\mathrm{schemes})^{\mathrm{op}} \to \mathrm{Set}$$

sending a scheme S to the set of isomorphism classes of elliptic curves over S. Deduce from (a) and (b) that M is not representable.

EXERCISE 11.B. Let $A = \bigoplus_{d \geq 0} A_d$ be a finitely generated graded ring, and consider the associated projective stack $\mathscr{P}roj(A)$. Show that the coarse moduli space of $\mathscr{P}roj(A)$ is the projective scheme $\mathrm{Proj}(A)$.

EXERCISE 11.C. Prove Lemma 11.5.4.

EXERCISE 11.D. Let k be a field of characteristic not equal to 2 or 3, and let U_λ denote $\mathbb{A}^1_k - \{0, 1\}$ with coordinate λ so

$$U_\lambda = \mathrm{Spec}(k[\lambda]_{\lambda(\lambda-1)}).$$

(i) Show that the two automorphisms of U_λ defined by

$$\lambda \mapsto 1/\lambda, \quad \lambda \mapsto \frac{1}{1-\lambda}$$

defines an action of the symmetric group S_3 on U_λ.

(ii) Let $j \in \Gamma(U_\lambda, \mathscr{O}_{U_\lambda})$ denote the element

$$j := 2^8 \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda - 1)^2)}.$$

Show that j is invariant under the S_3-action and defines an isomorphism

$$k[j] \simeq \Gamma(U_\lambda, \mathscr{O}_{U_\lambda})^{S_3}.$$

Exercise 11.E. Let S be a noetherian scheme, and let $\mathscr{X}/S$ be a separated Deligne-Mumford stack of finite type with coarse moduli space X/S. Show that $\mathscr{X}$ is proper over S if and only if X is proper over S.

Exercise 11.F. Let k be an algebraically closed field and let $\mathscr{X}/k$ be a Deligne-Mumford stack with the property that there exists an étale cover $U \to \mathscr{X}$ such that for every point $u \in U(k)$ the complete local ring $\widehat{\mathscr{O}}_{U,u}$ is isomorphic to either $k[[t]]$ or $k[[x,y]]/(xy)$. Show that if X is the coarse moduli space of $\mathscr{X}$ then for every point $x \in X(k)$ the completion $\widehat{\mathscr{O}}_{X,x}$ of the strict henselization of X at x is isomorphic to either $k[[t]]$ or $k[[x,y]]/(xy)$.

Exercise 11.G. Let k be an algebraically closed field and let $X \subset \mathbb{A}^2_k$ be the union of the two axes defined by $xy = 0$. Define an action of $\mathbb{Z}/(2)$ on X by

$$(x, y) \mapsto (y, x)$$

What is the coarse moduli space of the quotient $[X/\mathbb{Z}/(2)]$?

Exercise 11.H. Let k be a field and let $\mathscr{X}$ be a smooth Deligne-Mumford stack over k, with coarse moduli space X.

(i) Let $X' \subset X$ be an open subspace of X whose complement has codimension ≥ 2, and let $\mathscr{X}' \subset \mathscr{X}$ be the preimage of X'. Show that the restriction functor

$$\mathrm{Pic}(\mathscr{X}) \to \mathrm{Pic}(\mathscr{X}')$$

is an isomorphism.

(ii) Assuming that X is a scheme (or generalize the theory of Weil divisors to algebraic spaces), show that X is regular in codimension 1.

(iii) Assuming that X is a scheme, and that $\mathscr{X} \to X$ is an isomorphism away from a codimension 2 subset, show that the Weil class group $\mathrm{Cl}(X)$ is isomorphic to the Picard group $\mathrm{Pic}(\mathscr{X})$.

(iv) Use this to prove directly (i.e. without the calculation in [**41**, Chapter II, example 6.5.2]) that the class group of the cone

$$\mathrm{Spec}(k[x,y,z]/(xy - z^2))$$

in characteristic $\neq 2$ is equal to $\mathbb{Z}/(2)$ (hint: note that the cone is the coarse moduli space of $[\mathbb{A}^2/\mathbb{Z}/(2)]$ where the action is given by $(x,y) \mapsto (-x,-y)$).

Exercise 11.I. Let k be a field and let $\mathscr{X}/k$ be a smooth Deligne-Mumford stack over k with finite diagonal. Let $\pi : \mathscr{X} \to X$ be the coarse moduli space, and assume that there exists a dense open subspace $U \subset X$ with complement of codimension ≥ 2 such that π restricts to an isomorphism over U.

(i) Show that if $\sigma : \mathscr{X} \to \mathscr{X}$ is an isomorphism, then there are no automorphisms of σ. Deduce that the set of isomorphism classes of isomorphisms $\mathscr{X} \to \mathscr{X}$ is a group $\mathrm{Aut}(\mathscr{X})$.

(ii) Show that the natural map $\mathrm{Aut}(\mathscr{X}) \to \mathrm{Aut}(X)$ is an isomorphism of groups.

CHAPTER 12

Gerbes

If X is a scheme and $\boldsymbol{\mu}$ is a sheaf of abelian groups on the étale site of X, then the first cohomology group $H^1(X, \boldsymbol{\mu})$ is isomorphic to the group of $\boldsymbol{\mu}$ torsors on X. For example, if $\boldsymbol{\mu}$ is $\mathbb{G}_m$, then we recover the standard isomorphism between $H^1(X, \mathbb{G}_m)$ and the Picard group of X.

The group $H^2(X, \boldsymbol{\mu})$ also has a geometric interpretation. It is in bijection with the set of $\boldsymbol{\mu}$-gerbes as we discuss in this chapter. Loosely a $\boldsymbol{\mu}$-gerbe is a stack over X which is a twisted form of the classifying stack $B\boldsymbol{\mu}$. This interpretation of $H^2(X, \boldsymbol{\mu})$ in terms of gerbes can be a powerful tool as it enables one to bring geometric techniques to bear on questions about $H^2(X, \boldsymbol{\mu})$. To illustrate this we discuss in section 12.3 the connection between gerbes, Azumaya algebras, and twisted sheaves.

A complete treatment of the theory of gerbes can be found in Giraud's book [**34**]. Our approach to twisted sheaves as quasi-coherent sheaves on gerbes is based on Lieblich's approach in [**50**]. See also Căldăraru's thesis [**18**]. For basics on Azumaya algebras and Brauer-Severi schemes see [**37**, IV, V, VI]. For more about bands, discussed only briefly here in exercise 12.G, see [**34**, IV].

12.1. Torsors and H^1

12.1.1. Though strictly not necessary for our discussion of gerbes, we review in this section the relationship between torsors and cohomology.

Let T be a topos and let $\boldsymbol{\mu}$ be a sheaf of abelian groups in T. We establish in this section a bijection between the set of isomorphism classes of $\boldsymbol{\mu}$-torsors and $H^1(T, \boldsymbol{\mu})$.

Fix a site C whose topos is equivalent to T, and such that C has a final object.

12.1.2. If $\mathscr{P}$ and $\mathscr{P}'$ are two $\boldsymbol{\mu}$-torsors, we define a new $\boldsymbol{\mu}$-torsor $\mathscr{P} \wedge \mathscr{P}'$ as the quotient of $\mathscr{P} \times \mathscr{P}'$ by the action of $\boldsymbol{\mu}$ given by

$$g * (p, p') := (gp, g^{-1}p').$$

The action of $\boldsymbol{\mu}$ on $\mathscr{P} \times \mathscr{P}'$ given by

$$h * (p, p') = (hp, p')$$

then descends to $\mathscr{P} \wedge \mathscr{P}'$ making $\mathscr{P} \wedge \mathscr{P}'$ a $\boldsymbol{\mu}$-torsor.

12.1.3. Define $EXT(\mathbb{Z}, \boldsymbol{\mu})$ to be the category whose objects are short exact sequences of sheaves of abelian groups on C,

$$0 \longrightarrow \boldsymbol{\mu} \longrightarrow \mathscr{E} \xrightarrow{\pi} \mathbb{Z} \longrightarrow 0, \qquad (12.1.3.1)$$

where we write simply $\mathbb{Z}$ for the constant sheaf associated to $\mathbb{Z}$. Morphisms in $EXT(\mathbb{Z}, \boldsymbol{\mu})$ are given by commutative diagrams:

$$\begin{tikzcd} 0 \arrow[r] & \boldsymbol{\mu} \arrow[r] \arrow[d, equal] & \mathscr{E}' \arrow[r] \arrow[d, "\rho"] & \mathbb{Z} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \boldsymbol{\mu} \arrow[r] & \mathscr{E} \arrow[r] & \mathbb{Z} \arrow[r] & 0. \end{tikzcd}$$

Note that any morphism in $EXT(\mathbb{Z}, \boldsymbol{\mu})$ is necessarily an isomorphism, and that the automorphism group of any object is canonically isomorphic to $\Gamma(C, \boldsymbol{\mu})$.

Let $\mathrm{Tors}(\boldsymbol{\mu})$ denote the category of $\boldsymbol{\mu}$-torsors in T. There is a functor

$$EXT(\mathbb{Z}, \boldsymbol{\mu}) \to \mathrm{Tors}(\boldsymbol{\mu}) \tag{12.1.3.2}$$

which sends a short exact sequence (12.1.3.1) to the μ-torsor $\pi^{-1}(1)$ of liftings of the global section $1 \in \Gamma(C, \mathbb{Z})$ to $\mathscr{E}$.

Proposition 12.1.4. *The functor* (12.1.3.2) *is an equivalence of categories.*

Proof. First we show that every object $\mathscr{P}$ in $\mathrm{Tors}(\boldsymbol{\mu})$ is in the essential image of (12.1.3.2).

Let $X \to e$ be a covering of the final object of C such that $\mathscr{P}(X)$ is nonempty, and fix an element $x \in \mathscr{P}(X)$. Let X' denote $X \times X$ and let

$$p_1, p_2 : X' \to X$$

be the two projections. Since $\mathscr{P}$ is a torsor, there exists a unique element $\alpha \in \boldsymbol{\mu}(X')$ such that

$$\alpha \cdot p_1^* x = p_2^* x.$$

If X'' denotes the triple product $X \times X \times X$, with projections

$$q_{12}, q_{23}, q_{13} : X'' \to X',$$

then we have

$$q_{13}^*(\alpha) = q_{12}^*\alpha + q_{23}^*\alpha \tag{12.1.4.1}$$

in $\boldsymbol{\mu}(X'')$. Indeed, we have

$$q_{23}^* p_2^* x = q_{23}^*(\alpha \cdot p_1^* x) = (q_{23}^*\alpha) \cdot q_{12}^*(p_2^*\alpha) = (q_{23}^*\alpha) q_{12}^*(\alpha p_1^* x) = (q_{23}^*\alpha + q_{12}^*\alpha)(q_{12}^* p_1^* x).$$

On the other hand, we also have

$$q_{23}^* p_2^* x = q_{13}^* p_2^* x = q_{13}^*(\alpha \cdot p_1^* x) = (q_{13}^*\alpha) \cdot q_{13}^* p_1^* x.$$

Since $q_{13}^* p_1^* x = q_{12}^* p_1^* x$ this implies the equality (12.1.4.1).

Using the element $\alpha \in \boldsymbol{\mu}(X')$ we can associate to $\mathscr{P}$ an extension

$$0 \to \boldsymbol{\mu} \to \mathscr{E}_{\mathscr{P}} \to \mathbb{Z} \to 0$$

as follows. We will define this extension by defining an extension over C/X with descent data relative to $X \to e$. Over X, we take the trivial extension $\boldsymbol{\mu} \oplus \mathbb{Z}$, and for the isomorphism σ over X' we take the map

$$\boldsymbol{\mu}|_{X'} \oplus \mathbb{Z} \to \boldsymbol{\mu}|_{X'} \oplus \mathbb{Z}, \quad (\zeta, n) \mapsto (\zeta + n\alpha, n).$$

The equality (12.1.4.1) implies that the resulting isomorphism σ satisfies the cocycle condition on X''. We define $\mathscr{E}_{\mathscr{P}}$ to be the extension obtained by descent 4.2.12. It follows from the construction that the torsor of liftings of $1 \in \mathbb{Z}$ is isomorphic to the original torsor.

To complete the proof we show that the functor (12.1.3.2) is fully faithful. Observe that the automorphism group of any object in $EXT(\mathbb{Z}, \boldsymbol{\mu})$ is isomorphic to $\Gamma(C, \boldsymbol{\mu})$, and therefore for two extensions $\mathscr{E}_j$ ($j = 1, 2$) with associated torsors $\mathscr{P}_j$ the map

$$\mathrm{Hom}_{EXT(\mathbb{Z},\boldsymbol{\mu})}(\mathscr{E}_1, \mathscr{E}_2) \to \mathrm{Hom}_{\mathscr{T}ors(\boldsymbol{\mu})}(\mathscr{P}_1, \mathscr{P}_2)$$

is bijective if the source is nonempty. On the other hand, if $\sigma : \mathscr{P}_1 \to \mathscr{P}_2$ is an isomorphism of torsors, then there exists a unique isomorphism of extensions $\tilde{\sigma} : \mathscr{E}_1 \to \mathscr{E}_2$ inducing σ. Indeed, by the uniqueness and the fact that morphisms of sheaves can be constructed locally, it suffices to show that such an isomorphism $\tilde{\sigma}$ after passing to a covering of the final object e. This reduces the existence of $\tilde{\sigma}$ to the case when $\mathscr{P}_1$ and $\mathscr{P}_2$ are trivial torsors, where the result is immediate. $\square$

Corollary 12.1.5. *There is a natural bijection*

$$H^1(C, \boldsymbol{\mu}) \simeq \{\textit{isomorphism classes of } \boldsymbol{\mu}\textit{-torsors}\}.$$

Proof. This follows from 12.1.4 by passing to isomorphism classes on both sides, and noting that the isomorphism classes of extensions of $\mathbb{Z}$ by $\boldsymbol{\mu}$ are in bijection with $H^1(C, \boldsymbol{\mu})$ (see, for example, exercise 2.M). $\square$

Corollary 12.1.6. *Let $\boldsymbol{\mu}$ be an injective sheaf of abelian groups on C. Then any $\boldsymbol{\mu}$-torsor is trivial.*

Proof. Indeed, by 12.1.4 the set of isomorphism classes of $\boldsymbol{\mu}$-torsors is in bijection with $H^1(C, \boldsymbol{\mu})$ which is 0 since $\boldsymbol{\mu}$ is injective. $\square$

12.1.7. If $f : \boldsymbol{\mu} \to \boldsymbol{\mu}'$ is a morphism of sheaves of groups, and if $\mathscr{P}$ is a $\boldsymbol{\mu}$-torsor, then we can define a $\boldsymbol{\mu}'$-torsor $f_*\mathscr{P}$ as follows. As a sheaf, $f_*\mathscr{P}$ is the quotient of the product $\boldsymbol{\mu}' \times \mathscr{P}$ by the action of $\boldsymbol{\mu}$ given on sections by

$$u * (u', p) := (u' f(u)^{-1}, up).$$

The left translation action of $\boldsymbol{\mu}'$ on $\boldsymbol{\mu}' \times \mathscr{P}$ given by

$$v * (u', p) := (vu', p)$$

commutes with the action of $\boldsymbol{\mu}$, and therefore defines a $\boldsymbol{\mu}'$-action on $f_*\mathscr{P}$. This action makes $f_*\mathscr{P}$ a $\boldsymbol{\mu}'$-torsor. Indeed, this can be verified locally in the case when $\mathscr{P}$ is trivial, in which case the claim is immediate.

In this way we obtain a functor

$$f_* : \mathrm{Tors}(\boldsymbol{\mu}) \to \mathrm{Tors}(\boldsymbol{\mu}')$$

which upon passing to isomorphism classes and using the identification in 12.1.5 gives a map

$$f_* : H^1(C, \boldsymbol{\mu}) \to H^1(C, \boldsymbol{\mu}'). \tag{12.1.7.1}$$

As verified in exercise 12.A, this map agrees on the usual map on cohomology defined by the morphism f.

Remark 12.1.8. In the case when f is the constant map sending all elements of $\boldsymbol{\mu}$ to the identity element e' in $\boldsymbol{\mu}'$, for any $\boldsymbol{\mu}$-torsor $\mathscr{P}$ the $\boldsymbol{\mu}'$-torsor $f_*\mathscr{P}$ is canonically trivial. Indeed in this case the subsheaf

$$\{e'\} \times \mathscr{P} \subset \boldsymbol{\mu}' \times \mathscr{P}$$

is invariant under the action of $\boldsymbol{\mu}$, and therefore defines a global section $* \in \Gamma(C, f_*\mathscr{P})$ upon passing to the quotient by $\boldsymbol{\mu}$.

12.2. Generalities on gerbes

12.2.1. Let C be a site and let $\boldsymbol{\mu}$ be a sheaf of abelian groups on C.

Let $p : F \to C$ be a stack over C. Recall from 3.4.7 that for any object $x \in F$ over $X := p(x)$ we have a sheaf $\underline{\mathrm{Aut}}_x$ over C/X.

Definition 12.2.2. A *$\boldsymbol{\mu}$-gerbe over C* is a stack $p : F \to C$ together with an isomorphism of sheaves of groups

$$\iota_x : \boldsymbol{\mu}|_{C/p(x)} \to \underline{\mathrm{Aut}}_x$$

for every object $x \in F$ such that the following conditions hold:

(G1) For any object $Y \in C$ there exists a covering $\{Y_i \to Y\}_{i \in I}$ such that $F(Y_i)$ is nonempty for every i.

(G2) For any two objects $y, y' \in F(Y)$ over the same object $Y \in C$ there exists a covering $\{f_i : Y_i \to Y\}_{i \in I}$ such that the pullbacks $f_i^* y$ and $f_i^* y'$ are isomorphic in $F(Y_i)$ for all $i \in I$.

(G3) For every object $Y \in C$ and isomorphism $\sigma : y \to y'$ in $F(Y)$ the resulting diagram

$$\begin{tikzcd} & \boldsymbol{\mu} \arrow[dl, "\iota_y"'] \arrow[dr, "\iota_{y'}"] & \\ \underline{\mathrm{Aut}}_y \arrow[rr, "\sigma"] & & \underline{\mathrm{Aut}}_{y'} \end{tikzcd}$$

commutes.

A *morphism of $\boldsymbol{\mu}$-gerbes*

$$(F', \{\iota_{x'}\}) \to (F, \{\iota_x\})$$

is a morphism of stacks $f : F' \to F$ such that for every object $x' \in F'$ the diagram

$$\begin{tikzcd} & \boldsymbol{\mu} \arrow[dl, "\iota_{x'}"'] \arrow[dr, "\iota_{f(x')}"] & \\ \underline{\mathrm{Aut}}_{x'} \arrow[rr, "f_*"] & & \underline{\mathrm{Aut}}_{f(x')} \end{tikzcd}$$

commutes.

Remark 12.2.3. Observe that properties (G1) and (G2) imply that if $X \in C$ is an object and $x, x' \in F(X)$ are two objects of the fiber, the sheaf

$$P_{x,x'} := \underline{\mathrm{Isom}}_{C/X}(x, x')$$

on C/X is a torsor under $\boldsymbol{\mu}|_{C/X}$ acting via ι_x and the natural action of $\underline{\mathrm{Aut}}_x$.

Lemma 12.2.4. *Any morphism of $\boldsymbol{\mu}$-gerbes*

$$f : (F', \{\iota_{x'}\}) \to (F, \{\iota_x\})$$

is an isomorphism.

Proof. By exercise 4.I, it suffices to show that f is fully faithful, and that every object of F is locally in the image of F'. This second statement follows immediately from properties (G1) and (G2) in the definition of a gerbe. As for the full faithfulness, observe that if $x, x' \in F'(X)$ are objects over some $X \in C$, then the map

$$\underline{\mathrm{Isom}}(x, x') \to \underline{\mathrm{Isom}}(f(x), f(x'))$$

is a morphism of $\boldsymbol{\mu}$-torsors on C/X and therefore an isomorphism. □

Just as $\boldsymbol{\mu}$-torsors are classified by $H^1(C, \boldsymbol{\mu})$, we now explain how $\boldsymbol{\mu}$-gerbes are classified by $H^2(C, \boldsymbol{\mu})$.

12.2.5. First consider the following construction of $\boldsymbol{\mu}$-gerbes. Let

$$1 \longrightarrow \boldsymbol{\mu} \xrightarrow{a} G \xrightarrow{b} Q \longrightarrow 1$$

be an exact sequence of sheaves of groups (with G and Q possibly nonabelian). Given a Q-torsor $\mathscr{P}$, let $\mathscr{G}_{\mathscr{P}}$ be the fibered category over C whose objects are triples $(X, \widetilde{\mathscr{P}}, \epsilon)$, where $X \in C$, $\widetilde{\mathscr{P}}$ is a $G|_{C/X}$-torsor on C/X, and

$$\epsilon : b_*\widetilde{\mathscr{P}} \to \mathscr{P}|_{C/X}$$

is an isomorphism of $Q|_{C/X}$-torsors. A morphism

$$(X', \widetilde{\mathscr{P}}', \epsilon') \to (X, \widetilde{\mathscr{P}}, \epsilon)$$

in $\mathscr{G}_{\mathscr{P}}$ is a pair (f, f^b), where $f : X' \to X$ is a morphism in C, and $f^b : f^*\widetilde{\mathscr{P}} \to \widetilde{\mathscr{P}}'$ is an isomorphism of $G|_{C/X'}$-torsors such that the diagram

$$\begin{tikzcd}
b_*f^*\widetilde{\mathscr{P}} \arrow[r, "b_*f^b"] \arrow[d, "\simeq"] & b_*\widetilde{\mathscr{P}}' \arrow[dr, "\epsilon'"] & \\
f^*b_*\widetilde{\mathscr{P}} \arrow[r, "f^*\epsilon"] & f^*\mathscr{P}|_{C/X} \arrow[r, "\simeq"] & \mathscr{P}|_{C/X'}
\end{tikzcd}$$

commutes.

There is a functor

$$p : \mathscr{G}_{\mathscr{P}} \to C, \quad (X, \widetilde{\mathscr{P}}, \epsilon) \mapsto X$$

making $\mathscr{G}_{\mathscr{P}}$ a category fibered in groupoids over C. The fiber over $X \in C$ is the groupoid of pairs $(\widetilde{\mathscr{P}}, \epsilon)$, where $\widetilde{\mathscr{P}}$ is a $G|_{C/X}$-torsor on C/X and $\epsilon : b_*\widetilde{\mathscr{P}} \to \mathscr{P}|_{C/X}$ is an isomorphism of $Q|_{C/X}$-torsors.

For any object $x = (X, \widetilde{\mathscr{P}}, \epsilon) \in \mathscr{G}_{\mathscr{P}}$, the automorphism presheaf $\underline{\mathrm{Aut}}_x$ is the presheaf of groups which to any $g : X' \to X$ associates the group of automorphisms of the $G|_{C/X'}$-torsor $\widetilde{\mathscr{P}}|_{C/X'}$ whose pushout along b_* is the identity automorphism. This group of automorphisms is precisely $\boldsymbol{\mu}(X')$, so we have a canonical isomorphism

$$\iota_x : \boldsymbol{\mu}|_{C/X} \to \underline{\mathrm{Aut}}_x.$$

PROPOSITION 12.2.6. *(i) The fibered category $\mathscr{G}_{\mathscr{P}}$ with the isomorphisms ι_x defined above is a $\boldsymbol{\mu}$-gerbe.*

(ii) If $\mathscr{P}$ and $\mathscr{P}'$ are isomorphic torsors then the $\boldsymbol{\mu}$-gerbes $\mathscr{G}_{\mathscr{P}}$ and $\mathscr{G}_{\mathscr{P}'}$ are isomorphic.

Proof. First we prove (i). That $\mathscr{G}_{\mathscr{P}}$ is a stack follows immediately from the effectivity of descent for sheaves and morphisms of sheaves 4.2.12.

If $X \in C$ is any object, then there exists a covering $\{X_i \to X\}_{i \in I}$ such that $\mathscr{P}|_{C/X_i}$ is trivial. In this case the trivial torsor $G|_{C/X_i}$ gives an object of $\mathscr{G}_{\mathscr{P}}(X_i)$ showing that condition (G1) in the definition of a $\boldsymbol{\mu}$-gerbe holds.

Next let us verify condition (G2) in the definition of a $\boldsymbol{\mu}$-gerbe. If $y = (X, \widetilde{\mathscr{P}}, \epsilon)$ and $y' = (X, \widetilde{\mathscr{P}}', \epsilon')$ are two objects over the same X, then after replacing X by a covering we may assume that $\widetilde{\mathscr{P}}$ and $\widetilde{\mathscr{P}}'$ are trivial torsors. Fix an isomorphism $\sigma : \widetilde{\mathscr{P}} \to \widetilde{\mathscr{P}}'$. The resulting diagram

$$\begin{tikzcd} b_*\widetilde{\mathscr{P}} \arrow[r, "\epsilon"] \arrow[d, "b_*\sigma"] & \mathscr{P}|_{C/X} \\ b_*\widetilde{\mathscr{P}} \arrow[ur, "\epsilon'"'] & \end{tikzcd}$$

may not commute, but its failure to commute is given by a section of $Q(X)$. After replacing X by a covering this section lifts to a section of G, and modifying our choice of σ by such a lifting we obtain an isomorphism $y \to y'$ in $\mathscr{G}_{\mathscr{P}}(X)$.

Finally, condition (G3) in the definition of a gerbe is immediate from the construction of the isomorphisms ι_x. This completes the proof of (i).

For statement (ii), note that if $\lambda : \mathscr{P} \to \mathscr{P}'$ is an isomorphism of Q-torsors, then the resulting morphism

$$\mathscr{G}_{\mathscr{P}} \to \mathscr{G}_{\mathscr{P}'}, \quad (X, \widetilde{\mathscr{P}}, \epsilon) \mapsto (X, \widetilde{\mathscr{P}}, \lambda \circ \epsilon)$$

is an isomorphism of μ-gerbes. $\square$

12.2.7. In particular, to any cohomology class $\epsilon \in H^2(C, \boldsymbol{\mu})$ we can associate a $\boldsymbol{\mu}$-gerbe $\mathscr{G}_\epsilon$ as follows.

Choose an inclusion of sheaves of abelian groups on C,

$$i : \boldsymbol{\mu} \hookrightarrow I,$$

with I injective, and let K be the quotient $I/\boldsymbol{\mu}$, so we have a short exact sequence of sheaves of abelian groups

$$0 \to \boldsymbol{\mu} \to I \to K \to 0.$$

Since I is injective the boundary map

$$\partial : H^1(C, K) \to H^2(C, \boldsymbol{\mu})$$

is an isomorphism. Therefore the preceding construction, which associates to any element of $H^1(C, K)$ a $\boldsymbol{\mu}$-gerbe, also gives for each element $\epsilon \in H^2(C, \boldsymbol{\mu})$ a gerbe $\mathscr{G}_\epsilon$.

A priori this depends on the choice of inclusion $i : \boldsymbol{\mu} \hookrightarrow I$ into an injective object. That it does not depend on this choice can be seen as follows. If $i' : \boldsymbol{\mu} \hookrightarrow I'$ is a second inclusion into an injective with cokernel K', then choosing a morphism $\rho : I \to I'$ such that $i' = \rho \circ i$ we get the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \boldsymbol{\mu} \arrow[r, "i"] \arrow[d, equal] & I \arrow[r] \arrow[d, "\rho"] & K \arrow[r] \arrow[d, "\bar{\rho}"] & 0 \\ 0 \arrow[r] & \boldsymbol{\mu} \arrow[r, "i'"] & I' \arrow[r] & K' \arrow[r] & 0 \end{tikzcd}$$

which induces the commutative diagram

$$\begin{tikzcd} H^1(C,K) \arrow[dd, "\bar{\rho}"'] \arrow[dr, "\partial"] & \\ & H^2(C,\boldsymbol{\mu}) \\ H^1(C,K'). \arrow[ur, "\partial'"'] & \end{tikzcd}$$

It therefore suffices to show that if $\gamma \in H^1(C,K)$ is an element with image $\gamma' \in H^1(C,K')$ under $\bar{\rho}$, then the $\boldsymbol{\mu}$-gerbes $\mathscr{G}_\gamma$ and $\mathscr{G}_{\gamma'}$ are isomorphic. Let $\mathscr{P}$ denote the K-torsor corresponding to γ, and let $\mathscr{P}'$ denote the pushout torsor $\bar{\rho}_*\mathscr{P}$. Given an object $(X,\widetilde{\mathscr{P}},\epsilon) \in \mathscr{G}_{\mathscr{P}}(X)$, the pushout $(X,\rho_*\widetilde{\mathscr{P}},\bar{\rho}_*\epsilon)$ is an object of $\mathscr{G}_{\mathscr{P}'}(X)$, and in this way we get a morphism of $\boldsymbol{\mu}$-gerbes

$$\mathscr{G}_{\mathscr{P}} \to \mathscr{G}_{\mathscr{P}'}$$

which is necessarily an isomorphism by 12.2.4.

Theorem 12.2.8 ([**34**, IV, 3.4.2]). *The induced set map*

$$H^2(C,\boldsymbol{\mu}) \to \{\textit{isomorphism classes of } \boldsymbol{\mu}\textit{-gerbes}\} \tag{12.2.8.1}$$

is a bijection.

Proof. Using exercises 2.N and 12.B, it suffices to consider the case when the site C has a final object $e \in C$, which we assume for the rest of the proof.

Fix an inclusion $\iota : \boldsymbol{\mu} \hookrightarrow I$, where I is injective, and let K denote the quotient $I/\boldsymbol{\mu}$. We then have an isomorphism $H^1(C,K) \simeq H^2(C,\boldsymbol{\mu})$. As above, for a K-torsor $\mathscr{P}$ we write $\mathscr{G}_{\mathscr{P}}$ for the corresponding $\boldsymbol{\mu}$-gerbe. Also let $b : I \to K$ be the projection.

Given a $\boldsymbol{\mu}$-gerbe $\mathscr{G}$, we construct a K-torsor $\mathscr{P}$ such that $\mathscr{G} \simeq \mathscr{G}_{\mathscr{P}}$ as follows.

Lemma 12.2.9. *If $\boldsymbol{\mu}$ is an injective sheaf of abelian groups, then any $\boldsymbol{\mu}$-gerbe $\mathscr{G}$ is trivial.*

Proof. First note that if $X \in C$ and $x, x' \in \mathscr{G}(X)$ are two objects, then x and x' are isomorphic. Indeed, the sheaf $\underline{\mathrm{Isom}}(x,x')$ of isomorphisms $x \to x'$ is a torsor under $\boldsymbol{\mu}$ and therefore trivial by 12.1.6.

Second, if X is an object and $\{g_i : X_i \to X\}$ is a covering such that there exists objects $x_i \in \mathscr{G}(X_i)$ for every i, we can consider the functor $\mathscr{E}$ on C/X which to any $Y \to X$ associates the set of collections $(y,\{\lambda_i\})$, where $y \in \mathscr{G}(Y)$ and $\lambda_i : y|_{X_i\times_X Y} \to x_i|_{X_i\times_X Y}$ is an isomorphism in $\mathscr{G}(X_i \times_X Y)$. Two collections $(y,\{\lambda_i\})$ and $(y',\{\lambda_i'\})$ are isomorphic if there exists an isomorphism (necessarily unique)

$$\rho : y \to y'$$

such that $\lambda_i' \circ \rho = \lambda_i$ for all i. Then $\mathscr{E}$ is a sheaf on C/X, and we will show that $\Gamma(C/X,\mathscr{E})$ is nonempty, so that, in particular, $\mathscr{G}(X)$ is nonempty.

Let $\widetilde{\boldsymbol{\mu}}$ denote the sheaf on C/X,

$$\prod_i g_{i*}\boldsymbol{\mu}|_{C/X_i},$$

and let $\mathscr{R}$ denote the quotient of $\widetilde{\boldsymbol{\mu}}$ by the natural inclusion $\boldsymbol{\mu} \hookrightarrow \widetilde{\boldsymbol{\mu}}$. Since $\boldsymbol{\mu}$ and $\widetilde{\boldsymbol{\mu}}$ are injective, the sheaf $\mathscr{R}$ is also injective. There is an action of $\widetilde{\boldsymbol{\mu}}$ in which a section $\{\alpha_i\} \in \widetilde{\boldsymbol{\mu}}$ sends $(y, \{\lambda_i\})$ to $(y, \{\alpha_i \circ \lambda_i\})$. The subsheaf $\boldsymbol{\mu} \hookrightarrow \widetilde{\boldsymbol{\mu}}$ acts trivially on $\mathscr{E}$, so in fact we obtain an action of $\mathscr{R}$ on $\mathscr{E}$. This action makes $\mathscr{E}$ a torsor under $\mathscr{R}$, since any two objects of $\mathscr{G}$ over the same object of C are isomorphic. Since $\mathscr{R}$ is injective it follows that $\mathscr{E}$ is the trivial torsor. This proves that $\mathscr{G}(X)$ is nonempty for all $X \in C$.

In particular, the fiber of $\mathscr{G}$ over the final object of C is nonempty, implying that $\mathscr{G}$ is trivial. □

Let $\iota_*\mathscr{G}$ denote the I-gerbe obtained from $\mathscr{G}$ by pushout (see exercise 12.F). By the lemma, this gerbe is trivial. Fix a trivialization $\eta \in \iota_*\mathscr{G}$, and define $\mathscr{P}$ on C to be the presheaf which to any $X \in C$ associates the set of isomorphism classes of pairs (x, λ), where $x \in \mathscr{G}(X)$ and $\lambda : \iota_* x \simeq \eta|_X$ is an isomorphism in $\iota_*\mathscr{G}(X)$. There is an action of I on $\mathscr{P}$ given by a section $\gamma \in I(X)$ sending (x, λ) to $(x, \gamma \circ \lambda)$. Note that, if $\gamma \in \boldsymbol{\mu}(X)$, then (x, λ) and $(x, \gamma \circ \lambda)$ are isomorphic so this defines an action of K on $\mathscr{P}$, which makes $\mathscr{P}$ a K-torsor. We claim that $\mathscr{G} \simeq \mathscr{G}_{\mathscr{P}}$. For this it suffices to define a morphism of $\boldsymbol{\mu}$-gerbes $\mathscr{G} \to \mathscr{G}_{\mathscr{P}}$. Given an object $y \in \mathscr{G}(X)$ for some object $X \in C$, consider the I-torsor $\widetilde{\mathscr{P}}$ of isomorphisms between $\iota_* y$ and η. There is a natural map

$$\widetilde{\mathscr{P}} \to \mathscr{P}|_{C/X}$$

sending an isomorphism $\lambda : \iota_* y \to \eta$ to (y, λ), and this induces an isomorphism between the pushout to K of $\widetilde{\mathscr{P}}$ and $\mathscr{P}|_{C/X}$. In this way we obtain the desired morphism $\mathscr{G} \to \mathscr{G}_{\mathscr{P}}$, thereby proving the surjectivity of the map (12.2.8.1).

Finally, we prove the injectivity of (12.2.8.1). Let $\mathscr{P}$ and $\mathscr{P}'$ be two K-torsors, and suppose $\sigma : \mathscr{G}_{\mathscr{P}} \to \mathscr{G}_{\mathscr{P}'}$ is an isomorphism. We show that $\mathscr{P} \simeq \mathscr{P}'$. For this let $C/\mathscr{P}$ denote the localized site (exercise 2.H), and let $\mathscr{P}'|_{C/\mathscr{P}}$ denote the restriction of $\mathscr{P}'$ to this site. Giving a morphism of $\boldsymbol{\mu}$-torsors $\mathscr{P} \to \mathscr{P}'$ is equivalent to giving a trivialization of $\mathscr{P}'|_{C/\mathscr{P}}$. This we do as follows. Given $(X, p) \in C/\mathscr{P}$, we obtain an object $(\widetilde{\mathscr{P}}_0, \epsilon_p) \in \mathscr{G}_{\mathscr{P}}(X)$ by taking $\widetilde{\mathscr{P}}_0$ to be the trivial I-torsor, and ϵ_p the trivialization of $\mathscr{P}|_{C/X}$ defined by p. Applying σ to the object $(\widetilde{\mathscr{P}}_0, \epsilon_p)$ we obtain an object $(\sigma(\widetilde{\mathscr{P}}_0), \sigma(\epsilon))$ of $\mathscr{G}_{\mathscr{P}'}$. Now observe that since I is injective, and hence also $I|_{C/\mathscr{P}}$ is injective (exercise 2.H (d)), the torsor $\sigma(\widetilde{\mathscr{P}}_0)$ is also trivial. Choosing one such trivialization we also obtain a trivialization of $\mathscr{P}'|_{C/\mathscr{P}}$. □

12.3. Gerbes and twisted sheaves

12.3.1. Let S be a scheme and let $\mathbb{G}_m$ denote the sheaf on the big étale site of S which to any S-scheme T associates $\Gamma(T, \mathscr{O}_T^*)$. Any $\mathbb{G}_m$-gerbe over S is algebraic by exercise 12.E.

Let $\mathscr{G}$ be a $\mathbb{G}_m$-gerbe, and let E be a locally free sheaf on $\mathscr{G}$ of finite rank. For any field k and morphism $x : \mathrm{Spec}(k) \to \mathscr{G}$, we get an action of $\underline{\mathrm{Aut}}_x \simeq \mathbb{G}_{m,k}$ on the k-vector space $E(x) \simeq x^* E$. By exercise 12.I, any representation V of $\mathbb{G}_{m,k}$ has a unique decomposition

$$V \simeq \bigoplus_{i \in \mathbb{Z}} V_i,$$

where $\mathbb{G}_{m,k}$ acts on V_i by

$$u * v = u^i v.$$

The set of integers i for which V_i is nonzero is called the *weights* of the representation V.

DEFINITION 12.3.2. Let n be an integer. A locally free sheaf of finite rank E on $\mathscr{G}$ is called an *n-twisted sheaf* if for every field k and morphism $x : \mathrm{Spec}(k) \to \mathscr{G}$ we have

$$E(x) = E(x)_n.$$

LEMMA 12.3.3. *Let $\pi : \mathscr{G} \to S$ be a $\mathbb{G}_m$-gerbe over a scheme S.*

(i) If E is an n-twisted sheaf and F is an m-twisted sheaf, then $E \otimes F$ is an $n + m$-twisted sheaf.

(ii) If M is a locally free sheaf of finite rank on S, then $\pi^* M$ is a 0-twisted sheaf on $\mathscr{G}$, and the induced functor

(12.3.3.1)

$$\pi^* : (\textit{locally free sheaves of finite rank on } S) \to (0\textit{-twisted sheaves on } \mathscr{G})$$

is an equivalence of categories.

(iii) If E is an n-twisted locally free sheaf on $\mathscr{G}$, then the dual $E^\vee$ is a $(-n)$-twisted sheaf.

PROOF. Statements (i) and (iii) follow immediately from the corresponding property for tensor products of representations of $\mathbb{G}_m$. It is also clear that if M is a locally free sheaf of finite rank on S, then $\pi^* M$ is a 0-twisted sheaf on $\mathscr{G}$. It remains to show that the functor (12.3.3.1) defines an equivalence of categories. For this it suffices to show that if M is a locally free sheaf on S, then the adjunction map

$$M \to \pi_* \pi^* M \tag{12.3.3.2}$$

is an isomorphism, and that if $\mathscr{M}$ is a 0-twisted sheaf on $\mathscr{G}$, then the adjunction map

$$\pi^* \pi_* \mathscr{M} \to \mathscr{M} \tag{12.3.3.3}$$

is an isomorphism. Both of these assertions are étale local on S, so we may assume that the gerbe $\mathscr{G}$ is trivial, and hence isomorphic to $B\mathbb{G}_m$.

A quasi-coherent sheaf $\mathscr{V}$ on $B\mathbb{G}_m$ corresponds as in exercise 9.H to a quasi-coherent sheaf V on S together with an action of $\mathbb{G}_m$. Such an action corresponds in turn to a $\mathbb{Z}$-grading

$$V \simeq \bigoplus_{i \in \mathbb{Z}} V_i,$$

again by exercise 12.I. The functor π_* associates to the quasi-coherent sheaf $\mathscr{V}$ to 0-part V_0. From this it follows that the two adjunction maps (12.3.3.2) and (12.3.3.3) are isomorphisms. $\square$

12.3.4. Note that the equivalence in 12.3.3.1 is compatible with tensor products on both sides. In particular, if E is an n-twisted sheaf on the gerbe $\mathscr{G}$ then the sheaf of noncommutative algebras $\mathscr{E}nd(E) = E^\vee \otimes E$ is 0-twisted and therefore isomorphic to $\pi^* \mathscr{A}_E$ for a unique locally free sheaf of algebras $\mathscr{A}_E$ on S. This construction provides an important connection between quasi-coherent sheaves on gerbes and the study of Brauer groups as we now explain.

12.3.5. If S is a scheme, an *Azumaya algebra* is a quasi-coherent sheaf of $\mathscr{O}_S$-algebras $\mathscr{A}$ étale locally on S isomorphic to $\mathscr{E}nd(E)$ for some locally free sheaf E of finite rank. Note that for such a sheaf of algebras $\mathscr{A}$ the ring $\mathscr{O}_S$ is isomorphic to the center of $\mathscr{A}$ by exercise 12.K (i). Notice that if E is a locally free sheaf on S and L is a line bundle, then there is a canonical isomorphism of algebras

$$\mathscr{E}nd(E) \simeq \mathscr{E}nd(E \otimes L),$$

so the locally free sheaf E is not unique. In fact, given the Azumaya algebra $\mathscr{A}$, let $\mathscr{G}_{\mathscr{A}}$ denote the fibered category over the category of S-schemes whose objects are triples

$$(f : T \to S, E, \sigma),$$

where f is a morphism of schemes, E is a locally free sheaf on T and

$$\sigma : \mathscr{E}nd(E) \simeq f^*\mathscr{A}$$

is an isomorphism of $\mathscr{O}_T$-algbras. A morphism

$$(f' : T' \to S, E', \sigma') \to (f : T \to S, E, \sigma)$$

in $\mathscr{G}_{\mathscr{A}}$ is a pair (g, g^b), where $g : T' \to T$ is an S-morphism, and $g^b : g^*E \to E'$ is an isomorphism of vector bundles on T' such that the diagram

$$\begin{tikzcd}
\mathscr{E}nd(g^*E) \arrow[rr, "g^b"] \arrow[dr, "\sigma"'] & & \mathscr{E}nd(E') \arrow[dl, "\sigma'"] \\
& f'^*\mathscr{A} &
\end{tikzcd}$$

commutes. Note that for any object $(f : T \to S, E, \sigma)$ there is a natural map

$$\mathbb{G}_m(T) \to \mathrm{Aut}(f : T \to S, E, \sigma), \quad u \mapsto (\mathrm{id}, u).$$

PROPOSITION 12.3.6. *The stack $\mathscr{G}_{\mathscr{A}}$ is a $\mathbb{G}_m$-gerbe.*

PROOF. To verify that $\mathscr{G}_{\mathscr{A}}$ is a $\mathbb{G}_m$-gerbe, we may work étale locally on S, so it suffices to consider the case when $\mathscr{A}$ is isomorphic to $\mathscr{E}nd(\mathscr{O}_S^r)$ for some integer r.

Now if $(f : T \to S, E, \sigma)$ is any object, we can étale locally on T trivialize E in which case σ is given by an automorphism

$$\sigma : \mathscr{E}nd(\mathscr{O}_T^r) \to \mathscr{E}nd(\mathscr{O}_T^r).$$

By exercise 12.K (v), there exists after further localizing on T an element $g \in \mathscr{E}nd(\mathscr{O}_T^r)$ such that σ is conjugation by g. This element g defines an isomorphism

$$(f : T \to S, \mathscr{O}_T^r, \sigma) \to (f : T \to S, \mathscr{O}_T^r, \mathrm{id}),$$

and therefore any two objects of $\mathscr{G}_{\mathscr{A}}$ are locally isomorphic. Furthermore, the automorphism sheaf of any object is isomorphic to $\mathbb{G}_m$ acting via scalar multiplication. Again, this is a local question and therefore reduces to exercise 12.K (ii). □

12.3.7. There is a refinement of the stack $\mathscr{G}_{\mathscr{A}}$ which is often useful. Assume that the rank of $\mathscr{A}$ is invertible in S. Let $\mathscr{G}'_{\mathscr{A}}$ denote the stack whose objects are quadruples

$$(f : T \to S, E, \sigma, \lambda),$$

where $(f : T \to S, E, \sigma)$ is an object of $\mathscr{G}_{\mathscr{A}}$, and $\lambda : \mathscr{O}_T \to \bigwedge^r E$ is a trivialization of the determinant line bundle of E (the rank r of E is equal to the square root of the rank of $\mathscr{A}$). A morphism

$$(f' : T' \to S, E', \sigma', \lambda') \to (f : T \to S, E, \sigma, \lambda)$$

is a morphism $(g, g^b) : (T', E', \sigma') \to (T, E, \sigma)$ in $\mathscr{G}_{\mathscr{A}}$ such that the induced diagram

$$\begin{tikzcd} \mathscr{O}_T \arrow[r, "\lambda'"] \arrow[dr, "g^*\lambda"'] & \bigwedge^r E' \arrow[d, "g^b"] \\ & \bigwedge^r g^*E \end{tikzcd}$$

commutes. For any object $(T \to S, E, \sigma, \lambda)$ the subgroup $\mu_r \subset \mathbb{G}_m$ of r-th roots of unity preserves λ. Just as in the proof of 12.3.6, one shows that $\mathscr{G}'_{\mathscr{A}}$ is a μ_r-gerbe, and the pushout along the inclusion $\mu_r \hookrightarrow \mathbb{G}_m$ as in exercise 12.F is isomorphic to $\mathscr{G}_{\mathscr{A}}$.

12.3.8. By 12.2.8, the $\mathbb{G}_m$-gerbes over S are classified by $H^2(S, \mathbb{G}_m)$. The class $[\mathscr{G}_{\mathscr{A}}] \in H^2(S, \mathbb{G}_m)$ can be described as follows. Consider the exact sequence of sheaves of groups for the étale topology

$$1 \to \mathbb{G}_m \to GL_r \to PGL_r \to 1. \tag{12.3.8.1}$$

Let $P_{\mathscr{A}}$ denote the functor on the category of S-schemes which to any $f : T \to S$ associates the set of isomorphisms of $\mathscr{O}_T$-algebras

$$\sigma : M_r(\mathscr{O}_T) \to f^*\mathscr{A}.$$

Since any automorphism of $M_r(\mathscr{O}_T)$ is locally inner (by exercise 12.K (v)), and conjugation by an element of $GL_r(\mathscr{O}_T)$ is trivial if and only if the element is a scalar (by exercise 12.K (ii)), the functor $P_{\mathscr{A}}$ is a torsor under PGL_r. In particular, $P_{\mathscr{A}}$ is representable by a scheme over S.

Lemma 12.3.9. *The class $[\mathscr{G}_{\mathscr{A}}] \in H^2(S, \mathbb{G}_m)$ of the gerbe $\mathscr{G}_{\mathscr{A}}$ is equal to the image under the boundary map*

$$\partial : H^1(S, PGL_r) \to H^2(S, \mathbb{G}_m)$$

of the class $[P_{\mathscr{A}}] \in H^1(S, PGL_r)$ of the torsor $P_{\mathscr{A}}$.

Proof. By definition of the boundary map ∂, the class $\partial[P_{\mathscr{A}}]$ corresponds to the gerbe $\mathscr{G}'$ whose objects are triples $(g : T \to S, P, \delta)$, where g is a morphism of schemes, P is a GL_r-torsor over T, and $\delta : P \wedge^{GL_r} PGL_r \to g^*P_{\mathscr{A}}$ is an isomorphism of PGL_r-torsors over T. Here $P \wedge^{GL_r} PGL_r$ denotes the pushout along $GL_r \to PGL_r$ of the torsor P to a PGL_r-torsor. A morphism

$$(g' : T' \to SP', \delta') \to (g : T \to S, P, \delta)$$

is a pair (h, h^b), where $h : T' \to T$ is a morphism of S-schemes, and $h^b : h^*P \to P'$ is a morphism of GL_r-torsors such that the diagram

$$\begin{tikzcd} h^*(P \wedge^{GL_r} PGL_r) \arrow[dd, "h^b"] \arrow[dr, "g^*\delta"] & \\ & g'^*P_{\mathscr{A}} \\ P' \wedge^{GL_r} PGL_r \arrow[ur, "\delta'"'] & \end{tikzcd}$$

commutes. Note that the inclusion $\mathbb{G}_m \hookrightarrow GL_r$ defines an action of $\mathbb{G}_m$ on any object of $\mathscr{G}'$ and this makes $\mathscr{G}'$ a $\mathbb{G}_m$-gerbe. To prove the lemma it suffices to define a morphism of $\mathbb{G}_m$-gerbes $\mathscr{G}_{\mathscr{A}} \to \mathscr{G}'$ since any morphism of $\mathbb{G}_m$-gerbes is automatically an isomorphism by 12.2.4. We take the map obtained by sending $(g : T \to S, E, \sigma)$ to the GL_r-torsor

$$\underline{\mathrm{Isom}}(\mathscr{O}_T^r, E)$$

together with the isomorphism

$$\underline{\mathrm{Isom}}(\mathscr{O}_T^r, E) \wedge^{GL_r} PGL_r \to P_{\mathscr{A}}$$

induced by σ. □

12.3.10. Let $\pi : \mathscr{G}_{\mathscr{A}} \to S$ be the projection. On $\mathscr{G}_{\mathscr{A}}$ there is a tautological locally free sheaf $\mathscr{E}$ together with an isomorphism

$$\sigma : \mathscr{E}nd(\mathscr{E}) \to \pi^*\mathscr{A}$$

of algebras over $\mathscr{G}_{\mathscr{A}}$. Note that this implies that we can recover $\mathscr{A}$ from the pair $(\mathscr{G}_{\mathscr{A}}, \mathscr{E})$. Namely,

$$\mathscr{A} = \pi_*\mathscr{E}nd(\mathscr{E}).$$

Also $\mathscr{E}$ is a 1-twisted sheaf on $\mathscr{G}_{\mathscr{A}}$. There is a converse to this construction:

PROPOSITION 12.3.11. *Let $\pi : \mathscr{G} \to S$ be a $\mathbb{G}_m$-gerbe, and suppose there exists a 1-twisted locally free sheaf $\mathscr{E}$ on $\mathscr{G}$ of rank $r > 0$. Let $\mathscr{A}$ denote $\pi_*\mathscr{E}nd(\mathscr{E})$. Then $\mathscr{A}$ is an Azumaya algebra on S, the adjunction map $\pi^*\mathscr{A} \to \mathscr{E}nd(\mathscr{E})$ is an isomorphism, and $\mathscr{G} \simeq \mathscr{G}_{\mathscr{A}}$.*

PROOF. To prove that $\mathscr{A}$ is an Azumaya algebra and that $\pi^*\mathscr{A} \to \mathscr{E}nd(\mathscr{E})$ is an isomorphism, we may work étale locally on S, and it therefore suffices to consider the case when $\mathscr{G} = B\mathbb{G}_m$. In this case, $\mathscr{E}$ is isomorphic to $\pi^*E \otimes \chi$, where E is a locally free sheaf on S and χ denotes the line bundle on $B\mathbb{G}_m$ corresponding to the standard representation of $\mathbb{G}_m$. In this case we have $\mathscr{A} \simeq \mathscr{E}nd(E)$ and the result is immediate.

To show that $\mathscr{G}$ is isomorphic to $\mathscr{G}_{\mathscr{A}}$, note that the data

$$(\mathscr{E}, \pi^*\mathscr{E} \simeq \mathscr{E}nd(\mathscr{E}))$$

defines a morphism of $\mathbb{G}_m$-gerbes $\mathscr{G} \to \mathscr{G}_{\mathscr{A}}$ which by 12.2.4 must be an isomorphism. □

REMARK 12.3.12. Proposition 12.3.11 provides the link between the study of twisted sheaves and the question of when the Brauer group of a scheme is equal to the cohomological Brauer group. For further discussion in this direction see [**50**] and references therein.

12.4. Exercises

EXERCISE 12.A. Let C be a site.

(i) Let $f : \boldsymbol{\mu} \to \boldsymbol{\mu}'$ be a morphism of sheaves of abelian groups on C, and let $f_* : H^1(C, \boldsymbol{\mu}) \to H^1(C, \boldsymbol{\mu}')$ be the map defined in (12.1.7.1). Show that this map f_* agrees with the usual map given by the functoriality of $H^1(C, -)$.

(ii) Let

$$0 \longrightarrow \boldsymbol{\mu} \longrightarrow G \xrightarrow{\ \pi\ } H \longrightarrow 0 \tag{12.4.0.1}$$

be a short exact sequence of abelian group on C. For a global section $h \in \Gamma(C, H)$, define $\mathscr{P}_h$ to be the sheaf on C which to any $X \in C$ associates the set of elements $g \in G(X)$ mapping to $h|_X$ in $H(X)$. Show that $\mathscr{P}_h$ is a $\boldsymbol{\mu}$-torsor and the map

$$H^0(C, H) \to H^1(C, \boldsymbol{\mu})$$

sending a global section h to the class of the torsor $\mathscr{P}_h$ agrees with the usual boundary map obtained by taking cohomology of the exact sequence (12.4.0.1).

EXERCISE 12.B. Let C be a site and let $C' \subset C$ be a full subcategory satisfying the conditions in exercise 2.F, so that restriction defines an isomorphism on the associated topoi. Let $\boldsymbol{\mu}$ be a sheaf of abelian groups on C, and let $\boldsymbol{\mu}'$ be its restriction to C'.

(i) Show that if $\mathscr{G}$ is a $\boldsymbol{\mu}$-gerbe on C, then its restriction $\mathscr{G}' := \mathscr{G} \times_C C'$ to C' is a $\boldsymbol{\mu}'$-gerbe.

(ii) Show (without appealing to 12.2.8) that the association $\mathscr{G} \mapsto \mathscr{G}'$ defines a bijection between isomorphism classes of $\boldsymbol{\mu}$-gerbes on C and isomorphism classes of $\boldsymbol{\mu}'$-gerbes on C'.

EXERCISE 12.C. Fix an integer n. A *Brauer-Severi scheme of dimension n* over a scheme S is a proper smooth morphism $f : P \to S$ such that there exists an étale cover $S' \to S$ such that the base change $P \times_S S'$ is isomorphic to $\mathbb{P}^n_{S'}$.

(i) Show that a proper flat morphism $f : P \to S$ for which every geometric fiber is isomorphic to $\mathbb{P}^n$ is a Brauer-Severi scheme of dimension n over S.

(ii) Let $f : P \to S$ be a Brauer-Severi scheme of dimension n over S. Let

$$I_P : (S\text{-schemes})^{\mathrm{op}} \to \mathrm{Set}$$

be the functor sending an S-scheme $T \to S$ to the set of isomorphisms $\mathbb{P}^n_T \to P \times_S T$. Show that I_P is a PGL_n-torsor over S.

(iii) Show that (ii) defines an equivalence of categories between Brauer-Severi schemes of dimension n over S and the category of PGL_n-torsors over S.

EXERCISE 12.D. Let $\mathbb{H}$ be the quaternions over the real numbers $\mathbb{R}$, so

$$\mathbb{H} = \mathbb{R} \cdot 1 \oplus \mathbb{R} \cdot i \oplus \mathbb{R} \cdot j \oplus \mathbb{R} \cdot k,$$

with $ij = -ji = k$, and $i^2 = j^2 = -1$. Consider the functor

$$I : (\text{schemes}/\mathbb{R})^{\mathrm{op}} \to \mathrm{Set}$$

which to any scheme $T/\mathbb{R}$ associates the set of isomorphisms $M_2(\mathscr{O}_T) \simeq \mathbb{H} \otimes_{\mathbb{R}} \mathscr{O}_T$ of sheaves of $\mathscr{O}_T$-algebras on T.

(i) Show that I is representable by a smooth $\mathbb{R}$-scheme P, which has a natural PGL_2-action making P a principal PGL_2-bundle.

(ii) Show that the Brauer-Severi scheme associated to the torsor in (i) (via exercise 12.C) is isomorphic to the curve cut out by the equation $X^2 + Y^2 + Z^2 = 0$ in $\mathbb{P}^2_{\mathbb{R}}$.

(iii) Show that $H^2(\mathrm{Spec}(\mathbb{R}), \mathbb{G}_m) \simeq \mathbb{Z}/(2)$ and that the class of the gerbe associated to the PGL_2-torsor P by the construction in 12.2.5 is nontrivial.

EXERCISE 12.E. Let S be a scheme and let $\boldsymbol{\mu}$ be a smooth group scheme over S. Show that any $\boldsymbol{\mu}$-gerbe on the big étale site of S is an algebraic stack, which is Deligne-Mumford if $\boldsymbol{\mu}$ is étale over S.

EXERCISE 12.F. Let $f : \boldsymbol{\mu} \to \boldsymbol{\mu}'$ be a morphism of sheaves of abelian groups on a site C, and let $\mathscr{G}$ be a $\boldsymbol{\mu}$-gerbe. In this exercise we construct the pushout $f_*\mathscr{G}$ of the gerbe $\mathscr{G}$ along f.

(i) Let $\mathscr{G}'$ be the fibered category over C whose objects are the same as the objects of C, but where a morphism $x \to y$ over a morphism $g : X \to Y$ in C is an equivalence class of pairs (ρ, λ), where $\rho : x \to y$ is a morphism over g in $\mathscr{G}$ and $\lambda \in \boldsymbol{\mu}'(X)$. Here (ρ, λ) and (ρ', λ') are declared equivalent if there exists an element $u \in \boldsymbol{\mu}(X)$ such that $\rho' = \rho \circ \iota_x(u^{-1})$ and $\lambda' = f(u)\lambda$. Show that sending a composition

$$x \xrightarrow{(\rho,\lambda)} y \xrightarrow{(\epsilon,\eta)} z$$

over

$$X \xrightarrow{g} Y \xrightarrow{h} Z$$

to $(\epsilon \circ \rho, \lambda \cdot g^*\eta)$ defines a well-defined composition law on equivalence classes, making $\mathscr{G}'$ a fibered category over C.

(ii) Let $f_*\mathscr{G}$ be the stack associated to the fibered category $\mathscr{G}'$. Show that $f_*\mathscr{G}$ has a natural structure of a $\boldsymbol{\mu}'$-gerbe.

(iii) Show that there is a morphism of stacks $\pi : \mathscr{G} \to f_*\mathscr{G}$ which is compatible with the gerbe structures in the sense that for any object $x \in \mathscr{G}(X)$ the induced diagram

$$\begin{tikzcd} \boldsymbol{\mu} \arrow[r, "f"] \arrow[d] & \boldsymbol{\mu}' \arrow[d] \\ \underline{\mathrm{Aut}}_x \arrow[r, "\pi_*"] & \underline{\mathrm{Aut}}_{x'} \end{tikzcd}$$

commutes.

(iv) Show that if $\mathscr{H}$ is any $\boldsymbol{\mu}'$-gerbe and $t : \mathscr{G} \to \mathscr{H}$ is a morphism of stacks compatible with the gerbe structures in the sense of (iii), then t factors uniquely through a morphism of $\boldsymbol{\mu}'$-gerbes $f_*\mathscr{G} \to \mathscr{H}$.

(v) Show that the map $H^2(C, \boldsymbol{\mu}) \to H^2(C, \boldsymbol{\mu}')$ obtained by sending the class $[\mathscr{G}]$ of a $\boldsymbol{\mu}$-gerbe to the class of $f_*\mathscr{G}$ agrees with the usual map defined by the functoriality of cohomology.

EXERCISE 12.G. Let C be a site, and define a fibered category $LIEN'$ over C as follows. The objects of $LIEN'$ are pairs (U, G) where $U \in C$ is an object and G is a sheaf of groups on $C|_U$. A morphism $(U', G') \to (U, G)$ is a pair (f, f^b), where $f : U' \to U$ is a morphism in C and f^b is a conjugacy class of isomorphism of sheaves of groups $G' \to G|_{C|_{U'}}$. Let $p : LIEN' \to C$ be the functor sending (U, G) to U, and let $LIEN$ be the stack associated to $LIEN'$. The objects of $LIEN$ are called *bands* and $LIEN$ is called the *stack of bands*.

Let $\mathscr{G}$ be a stack on C satisfying the properties (G1) and (G2) in 12.2.2. Show that there is an associated band $L_{\mathscr{G}} \in LIEN$ which in the case when $\mathscr{G}$ is a $\boldsymbol{\mu}$-gerbe for some sheaf of abelian groups $\boldsymbol{\mu}$ is equal to the band associated to $\boldsymbol{\mu}$.

EXERCISE 12.H. Let $j_0 \in \mathbb{Q}$ be a rational number which is not 0 or 1728. Consider the fibered category $\mathscr{G}_{j_0}$ over $\mathbb{Q}$ whose objects are pairs $(S, E/S)$, where S is a $\mathbb{Q}$-scheme and E/S is an elliptic curve with j-invariant j_0 (see 13.1.12).

Morphisms in $\mathscr{G}_{j_0}$ are cartesian squares

$$\begin{tikzcd} E' \arrow[r, "\tilde{f}"] \arrow[d] & E \arrow[d] \\ S' \arrow[r, "f"] & S, \end{tikzcd}$$

where $\tilde{f}$ is a morphism of elliptic curves. Show that $\mathscr{G}_{j_0}$ is a $\mathbb{Z}/(2)$-gerbe over $\mathrm{Spec}(\mathbb{Q})$. Is this gerbe trivial?

EXERCISE 12.I. Let R be a ring, and let $\rho : \mathbb{G}_{m,R} \to GL_{n,R}$ be a homomorphism of group schemes.

(i) Let $\sigma : R^n \to R^n \otimes_R R[x^{\pm}]$ be the map corresponding to ρ via the correspondence in exercise 9.H (ii). Define $\sigma_i : R^n \to R^n$ by the formula

$$\sigma(e) = \sum_{i \in \mathbb{Z}} \sigma_i(e) x^i, \quad e \in R^n.$$

Show that these maps satisfy the equations

$$\sigma_i \circ \sigma_j = \begin{cases} 0 & i \neq j, \\ \sigma_i & i = j \end{cases}$$

and

$$\sum_{i \in \mathbb{Z}} \sigma_i = \mathrm{id}_{R^n}.$$

(ii) Deduce that there exists a decomposition $R^n = \bigoplus_{i \in \mathbb{Z}} V_i$, where $u \in \mathbb{G}_m$ (scheme-valued point) acts on V_i by multiplication by u^i.

EXERCISE 12.J. Let C be a site and let $\mathscr{A}$ be a sheaf of rings on C. Define the *center of* $\mathscr{A}$ to be the functor

$$Z : C^{\mathrm{op}} \to \mathrm{Set}$$

sending $U \in C$ to the set of elements $\alpha \in \mathscr{A}(U)$ such that for every morphism $f : V \to U$ in C the element $f^*\alpha \in \mathscr{A}(V)$ is in the center of the ring $\mathscr{A}(V)$. Show that Z is a sheaf of commutative subrings of $\mathscr{A}$.

EXERCISE 12.K. Let S be a scheme and let $\mathscr{E}$ be a locally free sheaf of finite rank on S.

(i) Show that the map $\mathscr{O}_S \to \mathscr{E}nd(\mathscr{E})$ induced by scalar multiplication is injective and identifies $\mathscr{O}_S$ with the center of the sheaf of noncommutative algebras $\mathscr{E}nd(\mathscr{E})$.

(ii) Let $g : \mathscr{E} \to \mathscr{E}$ be an automorphism such that conjugation by g on $\mathscr{E}nd(\mathscr{E})$ is the identity automorphism. Show that g is multiplication by a global section of $\Gamma(S, \mathscr{O}_S)$.

(iii) Let

$$Z : (\mathrm{Sch}/S)^{\mathrm{op}} \to \mathrm{Groups}$$

be the functor which to any $g : T \to S$ associates the group of algebra automorphisms of the sheaf of $\mathscr{O}_T$-algebras $\mathscr{E}nd(g^*\mathscr{E})$. Show that Z is representable by a scheme.

(iv) Let $GL(\mathscr{E})$ denote the group scheme over S representing the functor which to any S-scheme $g : T \to S$ associates the set of $\mathscr{O}_T$-module automorphisms of $g^*\mathscr{E}$. Let

$$\pi : GL(\mathscr{E}) \to Z$$

be the functor sending an automorphism $\rho : g^*\mathscr{E} \to g^*\mathscr{E}$ to the automorphism of $\mathscr{E}nd(g^*\mathscr{E})$ given by conjugation by ρ. Show that this map realizes $GL(\mathscr{E})$ a principal homogeneous space under $\mathbb{G}_m$ over Z (hint: in the case when S is the spectrum of a field, this is the classical Skolem-Noether theory. Use this case and reduce to the case when S is an artinian local ring).

(v) Assume $S = \mathrm{Spec}(R)$ with R a local ring. Deduce from (iv) that any $\mathscr{O}_S$-algebra automorphism of $\mathscr{E}nd(\mathscr{E})$ is equal to conjugation by an automorphism $g : \mathscr{E} \to \mathscr{E}$ (for details of a proof along the lines indicated here see [**37**, Exposé IV, §4 and 5] and for a more algebraic proof see [**12**]).

CHAPTER 13

Moduli of curves

This chapter is devoted to a discussion of moduli stacks of curves. The study of moduli spaces has been one of the principal driving forces behind the development of the theory of stacks, following the groundbreaking article of Deligne and Mumford [**23**], and perhaps the most widely studied of all moduli problems is the moduli of curves. In this chapter we discuss some of the basic properties of the moduli stack $\mathscr{M}_{1,1}$ of elliptic curves, and the moduli stacks $\mathscr{M}_g$ of curves of genus g (for $g \geq 2$). We do not intend to give a thorough treatment of the many aspects of these spaces (which could fill several books), but content ourselves with a discussion of their basic stack-theoretic properties. We also discuss the stacks $\overline{\mathscr{M}}_g$ of stable curves of genus g, and moduli spaces $\mathscr{K}_g(X, d)$ of stable maps. The moduli of elliptic curves is very classical. Good introductions are [**53**] and [**54**], and for an extensive treatment see [**24**]. Our treatment of the stack of stable curves follows [**23**]. The original reference for the theory of stable maps is [**47**]. Another excellent treatment is [**32**].

13.1. Moduli of elliptic curves

In this section we assume the reader is familiar with the basics of the theory of elliptic curves, as can be found, for example, in [**41**, Chapter IV, §4] and [**66**].

13.1.1. Recall that if k is an algebraically closed field, then an *elliptic curve over* k is a pair (E, e), where E/k is a smooth proper genus 1 curve, and $e \in E(k)$ is a point.

If S is a scheme, then an *elliptic curve over* S is a pair $(f : E \to S, e)$, where f is a smooth proper morphism, $e : S \to E$ is a section of f, and for every geometric point $\bar{x} : \mathrm{Spec}(k) \to S$ the pullback $(E_{\bar{x}}, e_{\bar{x}})$ is an elliptic curve over k.

Let $\mathscr{M}_{1,1}$ denote the fibered category over $\mathrm{Spec}(\mathbb{Z})$ whose objects are collections of data $(S, (E, e))$, where S is a scheme and (E, e) is an elliptic curve over S. A morphism

$$(S', (E', e')) \to (S, (E, e))$$

is a pair of morphisms (f, g) fitting into a cartesian diagram

$$\begin{tikzcd} E' \arrow[r, "g"] \arrow[d] & E \arrow[d] \\ S' \arrow[r, "f"] & S \end{tikzcd}$$

such that $g \circ e' = e \circ f$. The projection

$$\mathscr{M}_{1,1} \to (\text{schemes}), \quad (S, (E, e)) \mapsto S$$

makes $\mathscr{M}_{1,1}$ a fibered category over the category of schemes.

THEOREM 13.1.2. *The stack* $\mathscr{M}_{1,1}$ *is a smooth separated Deligne-Mumford stack of finite type over* $\mathrm{Spec}(\mathbb{Z})$.

We prove this theorem in 13.1.3–13.1.11 below.

13.1.3. The stack $\mathscr{M}_{1,1}$ is most classically understood in terms of the Weierstrass equation.

First let us consider an elliptic curve (E, e) over an algebraically closed field k. Then by Riemann-Roch the k-vector space $H^0(E, \mathscr{O}_E(3e))$ is three-dimensional and the sheaf $\mathscr{O}_E(3e)$ is very ample. Furthermore we have $H^i(E, \mathscr{O}_E(3e)) = 0$ for $i > 0$.

13.1.4. These results can be generalized to an arbitrary base scheme as follows. Let

$$f : E \to S, \quad e : S \to E$$

be an elliptic curve over a scheme S. Since f is proper, the morphism e is a closed imbedding. Let $\mathscr{I} \subset \mathscr{O}_E$ be the ideal sheaf of e. We claim that $\mathscr{I}$ defines a Cartier divisor on E. This is a local assertion so we may assume that S is the spectrum of a local ring. Let $x \in E$ be the image of the closed point under e. We then have an exact sequence

$$0 \to \mathscr{I}_x \to \mathscr{O}_{E,x} \to \mathscr{O}_S \to 0.$$

Since $\mathscr{O}_{E,x}$ is flat over $\mathscr{O}_S$ the module $\mathscr{I}_x$ is flat over $\mathscr{O}_S$, and if $\mathfrak{m} \subset \mathscr{O}_S$ is the maximal ideal then the sequence

$$0 \to \mathscr{I}_x/\mathfrak{m}\mathscr{I}_x \to \mathscr{O}_{E,x}/\mathfrak{m}\mathscr{O}_{E,x} \to \mathscr{O}_S/\mathfrak{m} \to 0$$

is exact. It follows that $\mathscr{I}_x/\mathfrak{m}\mathscr{I}_x$ is a free $\mathscr{O}_{E,x}/\mathfrak{m}\mathscr{O}_{E,x}$-module of rank 1 (being the ideal sheaf of a closed point on a smooth curve), and therefore $\mathscr{I}_x$ is by Nakayama's lemma and flatness a free module $\mathscr{O}_{E,x}$-module of rank 1. Fix a generator $f \in \mathscr{I}_x$. To prove that $\mathscr{I}_x$ defines a Cartier divisor, it suffices to show that for every $i \geq 0$ the map

$$\mathrm{Ker}(\cdot f : \mathscr{I}_x^i \to \mathscr{I}_x^i) \to \mathrm{Ker}(\cdot f : \mathscr{O}_{E,x} \to \mathscr{O}_{E,x}) \tag{13.1.4.1}$$

is an isomorphism. For then

$$\mathrm{Ker}(\cdot f : \mathscr{O}_{E,x} \to \mathscr{O}_{E,x}) \subset \bigcap_{i \geq 0} \mathscr{I}_x^i = 0.$$

To show that (13.1.4.1) is an isomorphism for every i, we proceed by induction on i. The case $i = 0$ is trivial. For the inductive step note that we have a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{I}_x^{i+1} \arrow[r] \arrow[d, "\cdot f"] & \mathscr{I}_x^i \arrow[r] \arrow[d, "\cdot f"] & \mathscr{O}_S \cdot f^i \arrow[r] \arrow[d, "\cdot 0"] & 0 \\
0 \arrow[r] & \mathscr{I}_x^{i+1} \arrow[r] & \mathscr{I}_x \arrow[r] & \mathscr{O}_S \cdot f^i \arrow[r] & 0.
\end{tikzcd}$$

The snake lemma applied to this diagram then gives that

$$\mathrm{Ker}(\cdot f : \mathscr{I}_x^{i+1} \to \mathscr{I}_x^{i+1}) \to \mathrm{Ker}(\cdot f : \mathscr{I}_x^i \to \mathscr{I}_x^i)$$

is an isomorphism, which by induction gives that (13.1.4.1) is an isomorphism.

For an integer n, let $\mathscr{O}_E(ne)$ denote $\mathscr{I}^{\otimes -n}$. By cohomology and base change [**41**, III, 12.11], the sheaf $f_*\mathscr{O}_E(3e)$ is a locally free sheaf of rank 3 on S, whose formation commutes with arbitrary base change. Moreover, the adjunction map

$f^*f_*\mathscr{O}_E(3e) \to \mathscr{O}_E(3e)$ is surjective and induces a closed imbedding $E \hookrightarrow \mathbb{P}(f_*\mathscr{O}_E(3e))$ over S, as all these statements can be verified in the geometric fibers.

From these observations it follows, just as in 4.4.13 (see also exercise 4.D) that, in fact, $\mathscr{M}_{1,1}$ is a stack with respect to the étale topology.

13.1.5. Next we show that the diagonal of $\mathscr{M}_{1,1}$ is representable. Given two elliptic curves (E, e) and (E', e') over a scheme S, let I denote the functor on the category of S-schemes which to $g : T \to S$ associates the set of isomorphisms $E_T \to E'_T$ preserving the sections. Giving such an isomorphism is equivalent to giving a closed subscheme $\Gamma \subset (E \times E')_T$ such that the two projections $\Gamma \to E$ and $\Gamma \to E'$ are isomorphisms, and such that the intersection of Γ with $\{e\} \times E'_T$ maps isomorphically to e'.

Let L denote the ample invertible sheaf $\mathrm{pr}_1^*\mathscr{O}_E(e) \otimes \mathrm{pr}_2^*\mathscr{O}_{E'}(e')$ on $E \times E'$. For the graph of an isomorphism $\Gamma \subset E \times E'$ the restriction of L to Γ has Hilbert polynomial

$$P = 2X.$$

Let H denote the Hilbert scheme of closed subschemes of $E \times E'$ with Hilbert polynomial P. We have a universal closed subscheme

$$Z \hookrightarrow (E \times E')_H$$

over H. The condition that Z maps isomorphically to E and E' is represented by an open subscheme $H^\circ \subset H$ (see exercise 8.F (i)), so over H° we have an isomorphism

$$f : E_{H^\circ} \to E'_{H^\circ}.$$

The condition that this isomorphism sends e to e' is represented by the condition that the fiber product V of the diagram

$$\begin{tikzcd} & H^\circ \arrow[d, "e'"] \\ H^\circ \arrow[r, "f\circ e"] & E' \end{tikzcd}$$

maps isomorphically to V. By exercise 8.F (i) again we get that this is represented by an open subset, therefore completing the proof that the diagonal of $\mathscr{M}_{1,1}$ is representable. Note that the proof in fact shows that the functor $\underline{\mathrm{Isom}}((E, e), (E', e'))$ is represented by a quasi-projective S-scheme.

13.1.6. To get a smooth covering of $\mathscr{M}_{1,1}$ proceed as follows. Cohomology and base change [**41**, III, 12.11] also give us that the sheaves $f_*\mathscr{O}_E(e)$ and $f_*\mathscr{O}_E(2e)$ are locally free of ranks 1 and 2, respectively, and that their formation commutes with arbitrary base change on S. Moreover, the successive quotients of the inclusions

$$f_*\mathscr{O}_E(e) \hookrightarrow f_*\mathscr{O}_E(2e) \hookrightarrow f_*\mathscr{O}_E(3e)$$

are locally free of rank 1 on S. Therefore, Zariski locally on S we can choose bases for these vector bundles (as in the case of a field [**41**, Chapter IV, 4.6])

$$1 \in f_*\mathscr{O}_E(e), \quad 1, x \in f_*\mathscr{O}_E(2), \quad 1, x, y \in f_*\mathscr{O}_E(3e).$$

Now the sheaf $f_*\mathscr{O}_E(6e)$ is locally free of rank 6, its formation commutes with arbitrary base change, and the map

$$\mathscr{O}_S^7 \to f_*\mathscr{O}_E(6e) \tag{13.1.6.1}$$

defined by the seven sections

$$1, x, x^2, x^3, y, y^2, xy,$$

is surjective, since this can be verified over the geometric fibers where it follows from the classical calculation [**41**, Chapter IV, 4.6]. After further shrinking on S, we can therefore find sections

$$\alpha_1, \alpha_2, \dots, \alpha_7 \in \mathscr{O}_S$$

such that

$$\alpha_1 + \alpha_2 x + \alpha_3 x^2 + \alpha_4 x^3 + \alpha_5 y + \alpha_6 xy + \alpha_7 y^2$$

maps to zero under the map (13.1.6.1). Furthermore, we must have $\alpha_4, \alpha_7 \in \mathscr{O}_S^*$ since this can again be verified in the geometric fibers where it follows from [**41**, Chapter IV, 4.6]. We therefore can normalize this equation by dividing through by α_4 so we may assume that $\alpha_4 = 1$.

Let $F \in \mathscr{O}_S[X, Y, Z]$ be the homogeneous polynomial

$$F := \alpha_1 Z^3 + \alpha_2 XZ^2 + \alpha_3 X^2 Z + X^3 + \alpha_5 YZ^2 + \alpha_6 XYZ + \alpha_7 Y^2 Z. \tag{13.1.6.2}$$

By construction the map $\iota_\sigma : E \hookrightarrow \mathbb{P}^2_S$ factors through $V(F) \subset \mathbb{P}^2_S$. The resulting map $\chi : E \to V(F)$ is in fact an isomorphism. Indeed, since χ induces an isomorphism in every fiber, the map χ is a bijection on points. Let $\mathscr{H} \hookrightarrow \mathscr{O}_{V(F)}$ denote the defining ideal, which is locally finitely generated since E and $V(F)$ are locally finitely presented over S. Since $\mathscr{O}_E$ is flat over $\mathscr{O}_S$, the sequence

$$0 \to \mathscr{H} \to \mathscr{O}_{V(F)} \to \mathscr{O}_E \to 0$$

remains exact after base change to any fiber over S. Therefore we must have $\mathscr{H} = 0$ by Nakayama's lemma. In summary, we have shown that any elliptic curve

$$(f : E \to S, e : S \to E)$$

is Zariski locally on S defined by a cubic equation (13.1.6.2) in $\mathbb{P}^2_S$. Furthermore, the section e is given by the section $[0 : 0 : 1]$.

Note also that by making the change of variable $Z \mapsto \alpha_7 Z$ we can arrange that $\alpha_7 = 1$. Then we see that E/S is given by an equation (where we rename the coefficients to conform with the standard form):

$$Y^2 Z + a_1 XYZ + a_3 YZ^2 = X^3 + a_2 X^2 Z + a_4 XZ^2 + a_6 Z^3. \tag{13.1.6.3}$$

For such an equation define

$$b_2 = a_1^2 + 4a_2, \quad b_4 = a_1 a_3 + 2a_4, \quad b_6 = a_3^2 + 4a_6,$$
$$b_8 = -a_1 a_3 a_4 - a_4^2 + a_1^2 + a_6 + a_2 a_3^2 + 4a_2 a_6,$$

and the *discriminant*

$$\Delta := -b_2^2 b_8 - 8b_4^3 - 27b_6^2 + 9b_2 b_4 b_6. \tag{13.1.6.4}$$

13.1.7. Conversely, given elements $\underline{a} := (a_1, a_3, a_2, a_4, a_6)$ of $\Gamma(S, \mathscr{O}_S)$, let $E_{\underline{a}} \hookrightarrow \mathbb{P}^2_S$ be the closed subscheme defined by the equation (13.1.6.3), and let $e : S \to E_{\underline{a}}$ be the section defined by $[0 : 0 : 1]$. The scheme $E_{\underline{a}}$ is flat over S. Indeed, to verify this it suffices to note that multiplication by the polynomial

$$Y^2 Z + a_1 XYZ + a_3 YZ^2 - X^3 - a_2 X^2 Z - a_4 XZ^2 - a_6 Z^3$$

on $\mathscr{O}_S[X, Y, Z] \otimes_{\mathscr{O}_S} M$ is injective for any $\mathscr{O}_S$-module M, which is immediate from looking, for example, at the $Y^2 Z$-term. Since $E_{\underline{a}} \to S$ is flat, the scheme $E_{\underline{a}}$ is smooth over S if and only if each fiber $E_{\underline{a},s}$ is smooth over $\mathrm{Spec}(k(s))$ for every point $s \in S$. From this and [**66**, III.1.4 (a)], we conclude that $E_{\underline{a}}$ is smooth over S if and only if the discriminant (13.1.6.4) is invertible on S.

13.1.8. Consider $\mathbb{A}^5_{\mathbb{Z}}$ with coordinates a_1, a_3, a_2, a_4, a_6, and let $U_{\underline{a}} \subset \mathbb{A}^5$ denote the complement of the zero locus of the discriminant (which is a polynomial in a_1, a_3, a_2, a_4, a_6). Over $U_{\underline{a}}$ we then have an elliptic curve

$$(E_{\underline{a}} \to U_{\underline{a}}, e).$$

In fact, the scheme $U_{\underline{a}}$ represents the following functor.

For a scheme S, define the *standard flag* on $\mathscr{O}_S^3$ to be the flag given by

$$\mathscr{O}_S \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \subset \mathscr{O}_S \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \oplus \mathscr{O}_S \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \subset \mathscr{O}_S^3.$$

Then it follows from the discussion above that the scheme $U_{\underline{a}}$ represents the functor F which to any scheme S associates the isomorphism classes of collections of data

$$(f : E \to S, e, \rho),$$

where (E, e) is an elliptic curve over S, and

$$\rho : \mathscr{O}_S^3 \to f_* \mathscr{O}_E(3e)$$

is an isomorphism of vector bundles on S such that ρ maps the standard flag to the flag

$$f_* \mathscr{O}_E(e) \subset f_* \mathscr{O}_E(2e) \subset f_* \mathscr{O}_E(3e).$$

Here two collections of data (E, e, ρ) and (E', e', ρ') are called equivalent if there exists an isomorphism $\lambda : (E, e) \to (E', e')$ such that the resulting diagram

$$\begin{tikzcd} \mathscr{O}_S^3 \arrow[r, "\rho"] \arrow[dr, "\rho'"'] & f_* \mathscr{O}_E(3e) \arrow[d, "\lambda"] \\ & f'_* \mathscr{O}_{E'}(3e') \end{tikzcd}$$

commutes. Note that, since $\mathscr{O}_E(3e)$ is very ample, such an isomorphism λ is unique if it exists.

13.1.9. Let

$$\pi : U_{\underline{a}} \to \mathscr{M}_{1,1}$$

denote the morphism corresponding to the elliptic curve $(E_{\underline{a}}, e)$ over $U_{\underline{a}}$. Then π is surjective, and smooth. Indeed, to verify the smoothness consider a commutative diagram of solid arrows,

$$\begin{tikzcd} T_0 \arrow[r] \arrow[d, hook] & U_{\underline{a}} \arrow[d] \\ T \arrow[r] \arrow[ur, dashed] & \mathscr{M}_{1,1}, \end{tikzcd}$$

where $T_0 \hookrightarrow T$ is a closed imbedding of affine schemes defined by a nilpotent ideal J, corresponding to an elliptic curve $(f : E \to T, e)$ over T and an isomorphism compatible with the flags

$$\rho_0 : \mathscr{O}_{T_0}^3 \to f_* \mathscr{O}_{E_0}(3e),$$

where E_0 denotes the reduction of E to T_0. Giving a dotted arrow as in the diagram then corresponds to lifting the isomorphism of flags ρ_0 to T. Clearly this is possible so a dotted arrow exists.

This completes the proof that the stack $\mathscr{M}_{1,1}$ is an algebraic stack.

13.1.10. Next let us verify that the diagonal of $\mathscr{M}_{1,1}$ is finite. Equivalently, if (E,e) and (E',e') are two elliptic curves over a scheme S, then the quasi-projective S-scheme

$$I := \underline{\mathrm{Isom}}((E,e),(E',e'))$$

is finite over S. The geometric fibers of this scheme are finite since an elliptic curve over an algebraically closed field has only finitely many automorphisms. It therefore suffices to show that I is proper over S, which we do by verifying the valuative criterion for properness. To do so assume that S is the spectrum of a discrete valuation ring with generic point $\eta \in S$ and closed point $s \in S$. Let

$$g_\eta : (E_\eta, e_\eta) \to (E'_\eta, e'_\eta)$$

be an isomorphism over the generic point. We must show that g_η extends to an isomorphism over all of S. Note that such an isomorphism is necessarily unique as its graph must be the scheme-theoretic closure of the graph of g_η in $E \times E'$. The fact that g_η does extend follows from the 'minimal model property' of elliptic curves, and can also be shown using Weierstrass equations (see [**66**, VII.1.3 (b)]).

13.1.11. Finally, to show that $\mathscr{M}_{1,1}$ is a Deligne-Mumford stack it suffices to show that elliptic curves admit no infinitesimal automorphisms by 8.3.3 . That is, if $S_0 \hookrightarrow S$ is a closed imbedding defined by a square-zero ideal J with S the spectrum of an artinian local ring, and if (E,e) is an elliptic curve over S with reduction (E_0,e_0) over S_0 and $\alpha : (E,e) \to (E,e)$ is an automorphism whose reduction to S_0 is the identity, then α must be the identity.

This can be seen as follows. The automorphisms of the scheme E that reduce to the identity but don't necessarily preserve the point e are given by

$$H^0(E_0, T_{E_0/S_0} \otimes J),$$

where T_{E_0/S_0} denotes the dual of the trivial module $\Omega^1_{E_0/S_0}$. Given such an automorphism $\beta : E \to E$ we get a new section $s_\beta : S \to E$ given by $s_\beta := \beta \circ e$ which reduces to the identity section over S_0. Such deformations of e_0 are classified by the fiber at e_0 of the tangent bundle $T_{E_0/S_0}(e_0) \otimes J$. Furthermore, the map sending β to s_β is the natural restriction map

$$H^0(E_0, T_{E_0/S_0} \otimes J) \to T_{E_0/S_0}(e_0) \otimes J.$$

To verify that there are no infinitesimal automorphisms it therefore suffices to show that the restriction map

$$H^0(E_0, T_{E_0/S_0}) \to T_{E_0/S_0}(e_0)$$

is an isomorphism, which is immediate since T_{E_0/S_0} is a trivial line bundle. This completes the proof of Theorem 13.1.2. $\square$

13.1.12. By 11.1.2, the stack $\mathscr{M}_{1,1}$ has a coarse moduli space $\pi : \mathscr{M}_{1,1} \to M_{1,1}$. We now explain how the classical theory of the j-invariant identifies this coarse space $M_{1,1}$ with the affine line $\mathbb{A}^1$.

If S is a scheme and $(E,e)/S$ is an elliptic curve over S defined by a Weiestrass equation (13.1.6.3) with $a_1, a_3, a_2, a_4, a_6 \in \Gamma(S, \mathscr{O}_S)$, define the *$j$-invariant* $j_{(E,e)} \in \Gamma(S, \mathscr{O}_S)$ to be the element

(13.1.12.1) $$j_{(E,e)} := (b_2^2 - 24b_4)^3/\Delta.$$

PROPOSITION 13.1.13. *Let S be a scheme, and $\underline{a}, \underline{a}' : S \to U_{\underline{a}}$ two morphisms defining two elliptic curves (E,e) and (E',e') over S. Let j (resp. j') be the j-invariant defined using the elements $\underline{a}$ (resp. $\underline{a}'$). If (E,e) and (E',e') are isomorphic elliptic curves then $j = j'$.*

PROOF. In the case of a field this is [**66**, III.1.4 (b)]. This implies the result for any reduced scheme S for on such a scheme the equality $j = j'$ can be verified at each generic point of S.

The general case can be reduced to the case of S reduced as follows. Consider the fiber product
$$V := U_{\underline{a}} \times_{\mathscr{M}_{1,1}} U_{\underline{a}}.$$
Let $\mathscr{E} \to U_{\underline{a}}$ be the tautological elliptic curve, and let $\varphi : \mathrm{pr}_1^*\mathscr{E} \to \mathrm{pr}_2^*\mathscr{E}$ over V. Giving an isomorphism $\lambda : (E,e) \to (E',e')$ is equivalent to giving a morphism $\gamma : S \to V$ such that
$$\mathrm{pr}_1 \circ \gamma = \underline{a}, \quad \mathrm{pr}_2 \circ \gamma = \underline{a}'.$$
It therefore suffices to verify that the j-invariants of $\mathrm{pr}_1^*\mathscr{E}$ and $\mathrm{pr}_2^*\mathscr{E}$ over V are equal. Now since $U_{\underline{a}}$ is a reduced scheme, and $\mathrm{pr}_1 : V \to U_{\underline{a}}$ is smooth, the scheme V is also reduced, and so the general case follows from the case of reduced S. $\square$

13.1.14. It follows that if S is a scheme and $(E,e)/S$ is an elliptic curve, then we have a canonically defined j-invariant $j_{(E,e)} \in \Gamma(S, \mathscr{O}_S)$. Locally on S we can choose a Weierstrass equation for E and then define the j-invariant using the formula (13.1.12.1). Proposition 13.1.13 implies that this is independent of the choice of Weierstrass equation and therefore these locally defined j-invariants agree on the overlaps to give a global section of $\mathscr{O}_S$.

Let
$$j : \mathscr{M}_{1,1} \to \mathbb{A}^1$$
be the resulting morphism. Since $\mathbb{A}^1$ is a scheme, the universal property of the coarse moduli space gives a unique factorization
$$\begin{tikzcd} \mathscr{M}_{1,1} \arrow[r, "\pi"] \arrow[rr, bend left, "j"] & M_{1,1} \arrow[r, "\bar{j}"] & \mathbb{A}^1. \end{tikzcd}$$

THEOREM 13.1.15. *The map $\bar{j}$ is an isomorphism.*

PROOF. For this, note that the morphism $j : \mathscr{M}_{1,1} \to \mathbb{A}^1_j$ is proper. Again it suffices to verify the valuative criterion for properness. This amounts to showing that if V is a discrete valuation ring, and $(E,e)/K$ is an elliptic curve over the field of fractions K of V, such that the j-invariant $j_{(E,e)}$ is an element of V, then after possibly replacing V by a finite extension, there exists an elliptic curve over V whose generic fiber is (E,e). This follows from [**66**, VII.5.5].

It follows that the map $\bar{j}$ is also proper, and since it is quasi-finite we see that the morphism $\bar{j}$ is finite. Moreover, for every geometric point $t : \mathrm{Spec}(k) \to \mathbb{A}^1$ the underlying set of $M_{1,1} \times_{\mathbb{A}^1} \mathrm{Spec}(k)$ is a single point since elliptic curves over algebraically closed fields are classified by their j-invariant. This implies that $\bar{j}$ is birational (take t to be a geometric point lying over the generic point of $\mathbb{A}^1$). Since a birational finite morphism between integral schemes with the target normal is an isomorphism, we conclude that $\bar{j}$ is an isomorphism. This completes the proof of 13.1.15. □

13.2. The stack $\overline{\mathscr{M}}_g$.

In this section we present the fundamental construction of $\overline{\mathscr{M}}_g$ of Deligne and Mumford, essentially following [**23**, §1] (see also [**27**]).

13.2.1. Let k be an algebraically closed field and let X/k be a finite type k-scheme of dimension 1. A closed point $x \in X(k)$ is called a *node* if the complete local ring $\widehat{\mathscr{O}}_{X,x}$ is isomorphic to $k[[x,y]]/(xy)$. The scheme X is called a *nodal curve* if every closed point $x \in X$ is either a smooth point or a node.

Definition 13.2.2. A *prestable curve* over a scheme S is a proper flat morphism $\pi : C \to S$ such that for every geometric point $\bar{s} \to S$ the fiber $C_{\bar{s}}$ is a connected nodal curve over $k(\bar{s})$.

Lemma 13.2.3. *Let $S = \mathrm{Spec}(k)$ be the spectrum of a field, and let C/S be a prestable curve. Then C is a projective k-scheme. If L is an invertible sheaf on C then L is ample if and only if the restriction to each irreducible component of $\widetilde{C}$ has positive degree.*

Proof. This follows from [**41**, Chapter III, Exercise 5.7 (c)] which shows that an invertible sheaf L on C is ample if and only if its pullback to the normalization $\widetilde{C}$ is ample. □

13.2.4. Let k be an algebraically closed field and let C/k be a prestable curve. Let

$$\pi : \widetilde{C} \to C$$

be the normalization of C. Then π is a finite morphism which is an isomorphism away from the nodes of C. If $p \in C(k)$ is a node, then the scheme-theoretic fiber $\pi^{-1}(p)$ is the disjoint union of two copies of $\mathrm{Spec}(k)$. Furthermore, we have an exact sequence

$$0 \to \mathscr{O}_C \to \pi_*\mathscr{O}_{\widetilde{C}} \to \oplus_{p\in C(k),\mathrm{node}} k_p \to 0, \tag{13.2.4.1}$$

where we write k_p for the skyscraper sheaf at p. Explicitly, if $p \in C(k)$ is a node, and $q_1, q_2 \in \widetilde{C}(k)$ denote the preimages of p, then

$$\pi_*\mathscr{O}_{\widetilde{C}} \otimes k(p) \simeq k(q_1) \oplus k(q_2)$$

and the image of $\mathscr{O}_{C,p}$ is a copy of k embedded diagonally.

Let L be an invertible sheaf on C such that the restriction of L to each irreducible component of $\widetilde{C}$ has positive degree. Then by 13.2.3 L is ample, and for d sufficiently big the map

$$\Gamma(\widetilde{C}, \pi^* L^{\otimes d}) \to \bigoplus_{q\in\widetilde{C}(k),\pi(q)=\mathrm{node}} \pi^* L(q)$$

is surjective. In this case we see by tensoring the exact sequence (13.2.4.1) with $L^{\otimes d}$ and taking global sections that we have an exact sequence

$$0 \to H^0(C, L) \to H^0(\widetilde{C}, \pi^*L) \to \bigoplus_{p \in C(k), \text{node}} L(p) \to 0.$$

By Riemann-Roch it follows that for d sufficiently big we have

$$h^0(C, L) = \deg(L) + 1 - \widetilde{g} - (\# \text{ of nodes}). \tag{13.2.4.2}$$

Here $\widetilde{g}$ denotes the genus of the normalization $\widetilde{C}$ and the degree $\deg(L)$ of L is defined to be the sum of the degrees of L restricted to each irreducible component of $\widetilde{C}$.

COROLLARY 13.2.5. *Let $f : C \to S$ be a prestable curve over a scheme S. Then f is étale locally on S projective.*

PROOF. Let $C^\circ \subset C$ be the maximal open subset over which f is smooth. Since f is flat, a point $x \in C$ lies in C° if and only if x is a smooth point of the fiber $C_{f(x)}$. In particular, C° is dense in every fiber. Fix a point $s \in S$, and choose points $p_1, \ldots, p_r \in C^\circ_s$ such that at least one point lies in every irreducible component of C_s. Then by 1.3.10, we can find an étale neighborhood $U \to S$ of s and sections $q_i : U \to C^\circ$ inducing the given points in the fiber. Base changing to U we may therefore assume that we have sections $q_1, \ldots, q_r : S \to C^\circ$ inducing the given sections in a fiber. Since C° is smooth over S, the image of each q_i is a Cartier divisor on C, and tensoring together the inverses of these ideal sheaves we get an invertible sheaf L on C. Since the restriction to the fiber at s is ample by 13.2.3, it follows from [**26**, III.4.7.1] that there exists a Zariski neighborhood of s in S such that L is relatively ample in that neighborhood. □

LEMMA 13.2.6. *Let k be an algebraically closed field and let C/k be a prestable curve. Then for any node $p \in C(k)$, there exists an étale morphism $U \to C$ whose image contains p and an étale morphism*

$$U \to \mathrm{Spec}(k[x, y]/(xy))$$

sending a point over p to the point $x = y = 0$.

PROOF. Let $\pi : \widetilde{C} \to C$ be the normalization, and for any étale morphism $U \to C$ let $\widetilde{U} \to U$ denote the fiber product $\widetilde{C} \times_C U$ (which is also the normalization of U). The fiber product

$$\widetilde{C} \times_C \mathrm{Spec}(\mathscr{O}_{C,\bar{p}})$$

has by 1.3.17 two connected components, where we write $\mathscr{O}_{X,\bar{p}}$ for the strict henselian local ring at p. It follows that there exists an étale neighborhood $U \to C$ of p such that $\widetilde{U} = V_1 \coprod V_2$ has two connected components with points $p_i \in V_i$ $(i = 1, 2)$ lying over p. We can further choose U to be affine, say $U = \mathrm{Spec}(R)$, in which case the V_i are also affine, say $V_i = \mathrm{Spec}(A_i)$. In this case the sequence

$$0 \to R \to A_1 \oplus A_2 \to k \to 0$$

obtained by taking the difference of the evaluation maps at p_1 and p_2 is exact, at least after localizing on R. Indeed, this can be verified after tensoring with the completion $\widehat{R}$ of R at p, in which case it is immediate. After further localization on R, we may assume that there exists an element $x \in R$ such that the image of x in A_1 maps to a uniformizer at p_1 and such that x maps to 0 in A_2, and similarly that there exists an element $y \in R$ whose image in A_1 is zero and whose image in A_2 maps to a uniformizer at p_2. Then $xy = 0$ and so we get a map

$$k[x, y]/(xy) \to R.$$

This map is étale in a neighborhood of $\bar{p}$ as it induces an isomorphism on completions at that point. $\square$

COROLLARY 13.2.7. *Let $f : C \to S$ be a prestable curve. Then f is a local complete intersection morphism.*

PROOF. In the case when $S = \mathrm{Spec}(k)$ is a field, this follows from 13.2.6 and exercise 13.A (c). The general case then follows from exercise 13.A (d). $\square$

13.2.8. In particular, it follows from this and [**40**] that if k is an algebraically closed field and C/k is a prestable curve, then there exists a dualizing sheaf $\omega_{C/k}$ on C (see also exercise 13.A). It is useful to have a more concrete description of this dualizing sheaf.

Let $\pi : \widetilde{C} \to C$ be the normalization of C, and let $D \subset \widetilde{C}$ be the preimage of the nodes in C, so $D \subset \widetilde{C}$ is a divisor on $\widetilde{C}$. Let $\tilde{j} : U \hookrightarrow \widetilde{C}$ be the complement of D, and let

$$\Omega^1_{\widetilde{C}}(\log D) \subset \tilde{j}_* \Omega^1_U$$

be the image of the natural map

$$\Omega^1_{\widetilde{C}} \otimes \mathscr{O}_{\widetilde{C}}(D) \to \tilde{j}_* \Omega^1_U.$$

If $q \in Z$ is a point and if $\pi \in \mathscr{O}_{\widetilde{C},q}$ is a uniformizer, then the stalk of $\Omega^1_{\widetilde{C}}(\log D)$ at q is the free $\mathscr{O}_{\widetilde{C},q}$-module with basis $\frac{d\pi}{\pi}$. The sheaf $\Omega^1_{\widetilde{C}}(\log D)$ is called the *sheaf of differential forms with log poles along D*.

For a point $q \in Z$ there is a *residue map*

$$\mathrm{res}_q : \Omega^1_{\widetilde{C}}(\log D) \to k(q),$$

where $k(q)$ denotes the skyscraper sheaf at q. Giving such a map is equivalent to giving a map

$$\Omega^1_{\widetilde{C}}(\log D)_q \otimes k(q) \to k(q).$$

Let $\pi \in \mathscr{O}_{\widetilde{C},q}$ be a uniformizer. Then the residue map is defined by

$$\mathrm{res}_q(\frac{d\pi}{\pi}) = 1.$$

This is independent of the choice of the uniformizer π. Indeed, if $u \in \mathscr{O}^*_{\widetilde{C},q}$ is a unit, then

$$\frac{d(u\pi)}{u\pi} = \frac{ud\pi}{u\pi} + \frac{\pi du}{u\pi} = \frac{d\pi}{\pi} + \frac{du}{u},$$

and $\frac{du}{u}$ maps to zero in $\Omega^1_{\widetilde{C}}(\log D)_q \otimes k(q)$.

The composite map

$$j := \pi\tilde{j} : U \to C$$

is also an open imbedding and identifies U with the complement of the nodes in C.

The restriction of $\omega_{C/k}$ to U is canonically isomorphic to $\Omega^1_{U/k}$ (by [**41**, III.7.11]), and so we get a morphism

$$\gamma : \omega_{C/k} \to j_* \Omega^1_{U/k} \simeq \pi_* \tilde{j}_* \Omega^1_{U/k}.$$

Proposition 13.2.9. *The map γ factors through $\pi_*\Omega^1_{\widetilde{C}}(\log D)$ and identifies $\omega_{C/k}$ with the kernel of the map*

$$\pi_*\Omega^1_{\widetilde{C}}(\log D) \to \bigoplus_{p\in C(k),\mathrm{node}} k(p) \tag{13.2.9.1}$$

obtained by choosing for each node $p \in C(k)$ an ordering q_1, q_2 of $\pi^{-1}(p)$ and taking the difference of the two residue maps res_{q_1} and res_{q_2} (note that the kernel is independent of the choice of ordering).

Proof. Let K denote the kernel of the map (13.2.9.1). The sheaf K is an invertible sheaf on C, as this can be verified étale locally on C where it is immediate. To identify K with the dualizing sheaf $\omega_{C/k}$, we construct an isomorphism between the two functors

$$\mathrm{Hom}(-, K), \quad H^1(C, -)^\wedge : (\text{line bundles on } C) \to \mathrm{Set}.$$

Since the dualizing sheaf $\omega_{C/k}$ represents the functor $H^1(C,-)^\wedge$ and is a line bundle, we then get by Yoneda's lemma an isomorphism $K \simeq \omega_{C/k}$.

So we have to identify for any line bundle L on C the group $\mathrm{Hom}(L, K) \simeq H^0(C, L^\wedge \otimes K)$ with $H^1(C, L)^\wedge$. Now observe that we have an exact sequence

$$0 \to K \to \pi_*\Omega^1_C(\log D) \to \bigoplus_{p\in C(k)} k(p) \to 0.$$

Tensoring this exact sequence with $L^\wedge$ and taking global section we an exact sequence

$$0 \to H^0(C, L^\wedge \otimes K) \to H^0(\widetilde{C}, \pi^*L^\wedge(D) \otimes \Omega^1_{\widetilde{C}}) \to \bigoplus_{p\in C,\mathrm{node}} L^\wedge(p).$$

Next observe that the map

$$\pi_*\pi^*L(-D) \to \pi_*\pi^*L$$

induced by the inclusion $\pi^*L(-D) \hookrightarrow \pi^*L$ factors through $L \subset \pi_*\pi^*L$ and we have an exact sequence

$$0 \to \pi_*\pi^*L(-D) \to L \to L(p) \to 0.$$

Indeed, all of these assertions can be verified étale locally where they are immediate. Taking cohomology we get an exact sequence

$$L(p) \to H^1(\widetilde{C}, \pi^*L(-D)) \to H^1(C, L) \to 0,$$

which upon dualizing gives an exact sequence

$$0 \to H^1(C, L)^\wedge \to H^1(\widetilde{C}, \pi^*L(-D))^\wedge \to L^\wedge(p). \tag{13.2.9.2}$$

By Serre duality for $\widetilde{C}$ we have an isomorphism

$$H^1(\widetilde{C}, \pi^*L(-D))^\wedge \simeq H^0(\widetilde{C}, \pi^*L^\wedge(D) \otimes \Omega^1_{\widetilde{C}}) \simeq H^1(\widetilde{C}, \pi^*L^\wedge \otimes \Omega^1_{\widetilde{C}}(\log D)).$$

To identify $H^1(C, L)^\wedge$ and $H^0(C, L^\wedge \otimes K)$ functorially in L, it therefore suffices to verify that the square

$$\begin{array}{ccc} H^0(\widetilde{C}, \pi^*L^\wedge \otimes \Omega^1_{\widetilde{C}}(\log D)) & \xrightarrow{\mathrm{res}} & \oplus_{p,\mathrm{node}} L^\wedge(p) \\ \downarrow{\scriptstyle S.D} & & \| \\ H^1(\widetilde{C}, \pi^*L(-D))^\wedge & \xrightarrow{(13.2.9.2)} & \oplus_{p,\mathrm{node}} L^\wedge(p) \end{array} \tag{13.2.9.3}$$

commutes. We leave this as exercise 13.B. □

COROLLARY 13.2.10. *The map* $\pi^*\omega_{C/k} \to \Omega^1_{\widetilde{C}}(\log D)$ *adjoint to the map* $\omega_{C/S} \to \pi_*\Omega^1_{\widetilde{C}}(\log D)$ *defined in* 13.2.9 *is an isomorphism.*

PROOF. This is a morphism of invertible sheaves on $\widetilde{C}$ so to check that it is an isomorphism it suffices to verify that it is surjective, which also follows from 13.2.9. □

13.2.11. For a prestable curve C over an algebraically closed field k with normalization $\pi : \widetilde{C} \to C$, we say that a point $q \in \widetilde{C}(k)$ is *special* if $\pi(q)$ is a node. By 13.2.10 we see that for any irreducible component $C_i \subset C$ with normalization $\widetilde{C}_i$, the degree of $\omega_{C/k}$ restricted to $\widetilde{C}_i$ is equal to

$$2g_i - 2 + \#\{\text{special points on } \widetilde{C}_i\}.$$

DEFINITION 13.2.12. Let k be an algebraically closed field. We say that a prestable curve C/k is *stable* if

$$2g_i - 2 + \#\{\text{special points on } \widetilde{C}_i\} > 0$$

for every irreducible component $C_i \subset C$. If $f : C \to S$ is a prestable curve over a base scheme S, we say that C is a *stable curve over* S if for every geometric point $\bar{x} \to S$ the fiber $C_{\bar{x}}$ is a stable curve over $k(\bar{x})$.

REMARK 13.2.13. If k is an algebraically closed field, and C/k is prestable curve, then the arithmetic genus g_C of C is by definition the dimension of $H^1(C, \mathscr{O}_C)$. This number can be computed using the short exact sequence

$$0 \to \mathscr{O}_C \to \pi_*\mathscr{O}_{\widetilde{C}} \to \bigoplus_{p \in C(k), \text{node}} k(p) \to 0,$$

which gives a long exact sequence

$$0 \to k \to H^0(\widetilde{C}, \mathscr{O}_{\widetilde{C}}) \to \bigoplus_{p \in C(k), \text{node}} k \to H^1(C, \mathscr{O}_C) \to H^1(\widetilde{C}, \mathscr{O}_{\widetilde{C}}) \to 0.$$

From this we see that

$$g_C = 1 + \Big(\sum_{\text{irred. comp. } C_i \subset C} (g_{\widetilde{C}_i} - 1)\Big) + \#\{\text{nodes}\}.$$

Now observe that the number of nodes is always greater than or equal to the number of irreducible components so this number is always at least 1. If C is stable then this number is at least 2.

Also note that if $f : C \to S$ is a prestable curve over a scheme S, then the arithmetic genus of the fibers is a locally constant function on S.

REMARK 13.2.14. By 13.2.3, a prestable curve C over an algebraically closed field k is stable if and only if $\omega_{C/k}$ is ample. This implies that a prestable curve $f : C \to S$ over a general base scheme S is stable if and only if the relative dualizing sheaf $\omega_{C/S}$ is relatively ample.

PROPOSITION 13.2.15. *Let* $f : C \to S$ *be a flat proper morphism locally of finite presentation, and suppose* $s \in S$ *is a point such that the fiber* C_s *is a prestable (resp. stable) curve over* S*. Then there exists a Zariski open neighborhood* $s \in U \subset S$ *of* s *such that the restriction* $f_U : C_U \to U$ *of* f *to* U *is a prestable (resp. stable) curve over* S*.*

PROOF. As remarked the condition that a prestable curve $C \to S$ is stable is equivalent to the condition that the relative dualizing sheaf $\omega_{C/S}$ is ample. This is an open condition by [**26**, III.4.7.1] so it suffices to prove the statement about prestable curves. Furthermore, we may work Zariski locally on S so may assume that S is affine, and also since f is of finite presentation we may assume that S is noetherian.

Let $\mathscr{P}$ be the property of closed subsets $Z \subset S$ that the set of points $z \in Z$ for which the fiber C_z is prestable is open. We show by noetherian induction that S has property $\mathscr{P}$. So assume property $\mathscr{P}$ holds for every proper closed subset $Z \subset S$. We then show that $\mathscr{P}$ holds for S. If the set of points $s \in S$ for which C_s is prestable is empty then there is nothing to show. So we may assume that there is such a point $s \in S$. Furthermore if S is reducible then $\mathscr{P}$ holds for each irreducible component by the noetherian induction hypothesis and therefore also for S, so it suffices to consider the case when S is integral, say $S = \mathrm{Spec}(R)$ with R an integral domain. In this case it suffices to show that there exists a nonempty open subset $U \subset S$ such that the restriction $C_U \to U$ is prestable. For this in turn it suffices to show that the generic fiber $C_\eta \to \mathrm{Spec}(k(\eta))$ is prestable. Indeed, if C_η is prestable, then we can by 13.2.6 find a finite collection of étale morphisms

$$W_i \to C$$

such that the following holds:

(1) Each W_i admits an étale morphism to $\mathrm{Spec}(R[x, y]/(xy))$ (not necessarily meeting the singular locus).
(2) The open image $V \subset C$ of $\coprod_i W_i \to C$ contains C_η.

Let $T \subset C$ be the complement of V. Since f is proper and T does not meet C_η, the image $f(T) \subset S$ is a proper closed subset, and its complement $U \subset S$ is then the desired open subset. So we are reduced to showing that the generic fiber C_η is prestable. For this we may make this base change

$$\mathrm{Spec}(\widehat{\mathscr{O}}_{S,s}) \to S,$$

and may therefore assume that R is a complete noetherian local ring with maximal ideal $\mathfrak{m} \subset R$, C/R is a flat proper scheme whose closed fiber is prestable.

Since C is proper over R, every point of C specializes to a point of the closed fiber, so the proposition therefore follows from the following lemma and exercise 13.D: □

LEMMA 13.2.16. *For every point $x \in C$ which is a node in the closed fiber, there exists an element $t \in \mathfrak{m}$ such that*

$$\widehat{\mathscr{O}}_{C,x} \simeq R[[z, w]]/(zw - t).$$

PROOF. Let A denote the ring $\widehat{\mathscr{O}}_{C,x}$, and for $n \geq 0$ let A_n (resp. R_n) denote $\widehat{\mathscr{O}}_{C,x} \otimes_R (R/\mathfrak{m}^{n+1})$ (resp. $R/\mathfrak{m}^{n+1}$) so

$$A = \varprojlim_n A_n.$$

We inductively construct elements $z_n, w_n \in A_n$ and $t_n \in R_n$ such that

$$A_n \simeq R_n[[z_n, w_n]]/(z_n w_n - t_n).$$

For $n = 0$, we can find such elements z_0, w_0 (and we take $t_0 = 0$) by the assumption that x is a node in the closed fiber.

For the inductive step we assume, given (z_n, w_n, t_n), and then lift these to $(z_{n+1}, w_{n+1}, t_{n+1})$. For this start by choosing any liftings $\tilde{z}_n, \tilde{w}_n \in A_{n+1}$, and $\tilde{t}_n \in R_{n+1}$. Then

$$\tilde{z}_n \tilde{w}_n = \tilde{t}_n + \kappa,$$

where

$$\kappa \in \mathrm{Ker}(A_{n+1} \to A_n).$$

Since A_{n+1} is flat over R_{n+1}, this kernel is isomorphic to

$$A_0 \otimes (\mathfrak{m}^{n+1}/\mathfrak{m}^{n+2}) \simeq k[[z_0, w_0]]/(z_0 w_0) \otimes (\mathfrak{m}^{n+1}/\mathfrak{m}^{n+2}).$$

It follows that we can write

$$\kappa = \kappa_0 + (\sum_{i \geq 1} \alpha_i \tilde{z}_n^i) + (\sum_{j \geq 1} \beta_j \tilde{w}_n^j),$$

where $\alpha_i, \beta_j \in \mathfrak{m}^{n+1}/\mathfrak{m}^{n+2}$. Setting

$$t_{n+1} := \tilde{t}_n + \kappa_0, \quad z_{n+1} := \tilde{z}_n + (\sum_{j \geq 1} \beta_j \tilde{w}_n^{j-1}), \quad w_{n+1} := \tilde{w}_n + (\sum_{i \geq 1} \alpha_i \tilde{z}_n^{i-1})$$

we obtain the desired liftings $(z_{n+1}, w_{n+1}, t_{n+1})$. The resulting map

$$R_{n+1}[[z_{n+1}, w_{n+1}]]/(z_{n+1} w_{n+1} - t_{n+1}) \to A_{n+1}$$

is an isomorphism since it is a morphism of flat R_{n+1}-algebras which reduces to an isomorphism modulo $\mathfrak{m}$. $\square$

The following proposition, whose statement and proof is from [**23**, 1.2], is one of the key consequences of the definition of stable curve.

PROPOSITION 13.2.17. *Let k be an algebraically closed field, and let C/k be a stable curve with arithmetic genus $g \geq 2$. Then $H^1(C, \omega_{C/k}^{\otimes n}) = 0$ if $n \geq 2$, and $\omega_{C/k}^{\otimes n}$ is very ample if $n \geq 3$.*

PROOF. Let $\pi : \widetilde{C} \to C$ be the normalization of C, and let $D \subset \widetilde{C}$ be the preimage of the nodes. Then by 13.2.10 we have $\pi^*\omega_{C/k} \simeq \Omega^1_{\widetilde{C}/k}(\log D)$, and therefore also $\pi^*\omega_{C/k}^{\otimes n} \simeq (\Omega^1_{\widetilde{C}/k}(\log D))^{\otimes n}$.

In particular, if $C_i \subset \widetilde{C}$ is the normalization of an irreducible component of C, then by definition of stable curve the restriction of $\omega_{C/k}^{\otimes n}$ to C_i has positive degree. This implies that $\omega_{C/k}$ is ample since its restriction to each irreducible component is ample, and also since $H^1(C, \omega_{C/k}^{\otimes n})$ is dual to $H^0(C, \omega_{C/k}^{\otimes 1-n})$, this implies that $H^1(C, \omega_{C/k}^{\otimes n})$ is 0 if $n \geq 2$.

It remains to see that $\omega_{C/k}^{\otimes n}$ is very ample if $n \geq 3$. For this we have to show that $\omega_{C/k}^{\otimes n}$ separates points and tangent vectors. For this in turn it suffices to show that for any two distinct closed points $x, y \in C(k)$ the map

$$H^0(C, \omega_{C/k}^{\otimes n}) \to (\omega_{C/k}^{\otimes n} \otimes k(x)) \oplus (\omega_{C/k}^{\otimes n} \otimes k(y))$$

is surjective, and for any closed point $x \in C(k)$ the map

$$H^0(C, \omega_{C/k}^{\otimes n}) \to \omega_{C/k} \otimes \mathscr{O}_{C,x}/\mathfrak{m}_x^2$$

is surjective. If I_x (resp. I_y) denotes the ideal sheaf of x (resp. y) then this will follow if we show that

$$H^1(C, I_x I_y \cdot \omega_{C/k}^{\otimes n}) = 0.$$

Note that in this formula we can take $x = y$. By Serre duality we have

$$H^1(C, I_xI_y \cdot \omega_{C/k}^{\otimes n})^\wedge \simeq \mathrm{Hom}(I_xI_y \cdot \omega_{C/k}^{\otimes n}, \omega_{C/k}) \simeq \mathrm{Hom}(I_xI_y, \omega_{C/k}^{\otimes(1-n)}).$$

It therefore suffices to show that

$$\mathrm{Hom}(I_xI_y, \omega_{C/k}^{\otimes -m}) = 0, \quad m \geq 2.$$

We verify this by treating three cases separately:

Case 1: Both x and y are smooth points. In this case, I_x and I_y are invertible sheaves and we need to show that $H^0(C, \omega_{C/k}^{\otimes -m}(x + y)) = 0$. If $\Gamma \subset \widetilde{C}$ is an irreducible component of the normalization, then the restriction of $\omega_{C/k}^{\otimes -m}(x + y)$ has negative degree, unless we have $m = 2$ and we are in one of the following two cases:

(a) The image of Γ in C has arithmetic genus 0 and meets the other components of C in exactly three points. This is clear because the restriction of $\omega_{C/k}^{\otimes -m}(x + y)$ to each irreducible component of the normalization of C has negative degree.

(b) The image of Γ in C has arithmetic genus 1 and meets the other components of C in exactly one point.

In both of these cases a global section of $\omega_{C/k}^{\otimes -m}(x+y)$ over Γ is determined by its restriction to the point where Γ meets the other components of C. Since not all the components can be of the form (a) or (b) this implies that $H^0(C, \omega_{C/k}^{\otimes -m}(x+y)) = 0$.

Case 2: The point x is a node and y is a smooth point. Let $\pi : C' \to C$ denote the blowup of C, so C' has two points $x_1, x_2 \in C'(k)$ lying over x which are both smooth. In this case it suffices by similar degree considerations as in case 1 to show that for an invertible sheaf L on C we have

$$\mathrm{Hom}(I_x, L) \simeq H^0(C', \pi^*L).$$

For this note that tensoring the exact sequence

$$0 \to \mathscr{O}_C \to \pi_*\mathscr{O}_{C'} \to k(x) \to 0$$

with L, we get an exact sequence

$$0 \to L \to \pi_*\pi^*L \to L(x) \to 0.$$

Applying $\mathrm{Hom}(I_x, -)$ we get an exact sequence

$$0 \to \mathrm{Hom}(I_x, L) \to \mathrm{Hom}(I_x, \pi_*\pi^*L) \to \mathrm{Hom}(I_x, L(x))$$

which we can also write as

$$0 \to \mathrm{Hom}(I_x, L) \to \mathrm{Hom}(\pi^*I_x, \pi^*L) \to \mathrm{Hom}(I_x/I_x^2, L(x)).$$

Let $J \subset \mathscr{O}_{C'}$ denote $\mathscr{O}_{C'}(-x_1 - x_2)$. We then have a surjection

$$\pi^*I_x \to J$$

whose kernel K is a skyscraper sheaf supported on x_1 and x_2 (in fact, a local calculation shows that $K \simeq k(x_1) \oplus k(x_2)$). Since π^*L is torsion free at x_1 and x_2 we have $\mathrm{Hom}(K, \pi^*L) = 0$, so this defines an isomorphism

$$\mathrm{Hom}(\pi^*I_x, \pi^*L) \simeq \mathrm{Hom}(J, \pi^*L) \simeq H^0(C', \pi^*L(x_1 + x_2)).$$

Notice also that we have an isomorphism $I_x/I_x^2 \simeq \pi_*(J/J^2)$ and that the resulting isomorphism

$$\mathrm{Hom}(\pi_*J/J^2, L(x)) \to \mathrm{Hom}(I_x/I_x^2, L(x))$$

identifies the kernel of the restriction map (which is $\mathrm{Hom}(I_x, L)$)

$$\mathrm{Hom}(\pi^* I_x, \pi^* L) \to \mathrm{Hom}(I_x/I_x^2, L(x))$$

with the kernel of the natural map

$$H^0(C', \pi^* L(x_1 + x_2)) \to \pi^* L(x_1) \oplus \pi^* L(x_2)$$

which from the exact sequence

$$0 \to \pi^* L \to \pi^* L(x_1 + x_2) \to \pi^* L(x_1) \oplus \pi^* L(x_2) \to 0$$

is identifed with $H^0(C', L)$. This completes the proof of case 2.

Case 3: Both x and y are nodes.

If $x = y$, then we need to show that

$$\mathrm{Hom}(I_x^2, \omega_{C/k}^{\otimes -m}) = 0$$

for $m \geq 2$. By a calculation similar to the calculation in two, if $\pi : C' \to C$ denotes the blowup of x in C and $x_1, x_2 \in C'$ are the preimages in C' of x, then

$$\mathrm{Hom}(I_x^2, \omega_{C/k}^{\otimes -m}) \simeq H^0(C', \pi^* \omega_{C/k}^{\otimes -m}(x_1 + x_2)).$$

Let $E \subset \widetilde{C}$ be an irreducible component of the normalization of C (which is also the normalization of C'). Then the restriction of $\pi^* \omega_{C/k}^{\otimes -m}(x_1 + x_2)$ to E has degree

$$-m(2g_E - 2 + \#\{\text{special points on } E\}) + \#\{x_j | x_j \in E\}.$$

This is negative unless $m = 2$, both x_1 and x_2 lie on E, and E is a rational curve meeting the other components of C in exactly one other point. Now in this case the degree of $\pi^* \omega_{C/k}^{\otimes -m}(x_1 + x_2)$ restricted to E is zero, so a global section is determined by its value at the point of E meeting the other components of C. Since not every component of $\widetilde{C}$ can be of this form, this implies that $H^0(C', \pi^* \omega_{C/k}^{\otimes -m}(x_1 + x_2)) = 0$.

If $x \neq y$, let $\pi : C' \to C$ be the result of blowing up both x and y. Then we have

$$\mathrm{Hom}(I_x I_y, \omega_{C/k}^{\otimes -m}) \simeq H^0(C', \pi^* \omega_{C/k}^{\otimes -m}),$$

and this group is zero again since the restriction of $\pi^* \omega_{C/k}^{\otimes -m}$ to any irreducible component of the normalization has negative degree. $\square$

Corollary 13.2.18. *Let S be a scheme, and let $\pi : C \to S$ be a stable curve of arithmetic genus g. Then for $n \geq 3$ the relative dualizing sheaf $\omega_{C/S}^{\otimes n}$ is relatively very ample and the sheaf $\pi_* \omega_{C/S}^{\otimes n}$ is a locally free sheaf of rank $(2n-1)(g-1)$ whose formation commutes with arbitrary base change $S' \to S$.*

Proof. Everything in the lemma follows from 13.2.17 and cohomology and base change [**41**, III, 12.11], except for the computation of the rank of $\pi_* \omega_{C/S}^{\otimes n}$. To compute this rank it suffices by cohomology and base change to consider the case when $S = \mathrm{Spec}(k)$ is the spectrum of an algebraically closed field, in which case we need to compute the dimension of $H^0(C, \omega_{C/k}^{\otimes n})$. Let $\pi : \widetilde{C} \to C$ be the normalization. Then since $H^1(C, \omega_{C/k}^{\otimes n}) = 0$ we have an exact sequence

$$0 \to H^0(C, \omega_{C/k}^{\otimes n}) \to H^0(\widetilde{C}, \pi^* \omega_{C/k}^{\otimes n}) \to \bigoplus_{x \in C, \text{node}} \omega_{C/k}^{\otimes n}(x) \to 0,$$

which gives that

$$h^0(C, \omega_{C/k}^{\otimes n}) = h^0(\widetilde{C}, \pi^* \omega_{C/k}^{\otimes n}) - \#\{\text{nodes}\}.$$

To compute $h^0(\widetilde{C}, \pi^*\omega_{C/k}^{\otimes n})$, note that by 13.2.10 we have $\pi^*\omega_{C/k}^{\otimes n} \simeq (\Omega^1_{\widetilde{C}/k}(\log D))^{\otimes n}$, where $D \subset \widetilde{C}$ is the preimage of the nodes. By Riemann-Roch we therefore get that

$$h^0(\widetilde{C}, \pi^*\omega_{C/k}^{\otimes n}) = 2n \cdot (\#\{\text{nodes}\} + \sum_{\Gamma \subset \widetilde{C}} (g_\Gamma - 1)),$$

where the sum is over the irreducible components $\Gamma \subset \widetilde{C}$. We conclude that

$$h^0(C, \omega_{C/k}^{\otimes n}) = (2n-1) \cdot \#\{\text{nodes}\} + 2n \cdot \sum_{\Gamma \subset \widetilde{C}} (g_\gamma - 1).$$

Combining this with Remark 13.2.13 we get the result. □

13.2.19. Fix $g \geq 2$. Let $\overline{\mathscr{M}}_g$ denote the fibered category over Spec($\mathbb{Z}$) whose objects are pairs $(S, f : C \to S)$, where S is a scheme and $f : C \to S$ is a stable curve of genus g. A morphism $(S' : f' : C' \to S') \to (S, f : C \to S)$ is a cartesian diagram:

$$\begin{tikzcd} C' \arrow[r] \arrow[d, "f'"] & C \arrow[d, "f"] \\ S' \arrow[r] & S. \end{tikzcd}$$

THEOREM 13.2.20. *The fibered category $\overline{\mathscr{M}}_g$ is a Deligne-Mumford stack over* Spec($\mathbb{Z}$).

The proof will be in several steps 13.2.21–13.2.24.

13.2.21. For any object $(S, C \to S) \in \overline{\mathscr{M}}_g$, we have the relative dualizing sheaf $\omega_{C/S}$ which is relatively very ample by 13.2.18. By 4.4.12 we get that $\overline{\mathscr{M}}_g$ is a stack with respect to the étale topology.

13.2.22. Next let us verify that the diagonal of $\overline{\mathscr{M}}_g$ is representable. For this let S be a scheme, and let $\pi : C \to S$ and $\pi' : C' \to S$ be two stable curves over S of genus g. The fiber product of the resulting diagram

$$\begin{tikzcd} & S \arrow[d, "(C,C')"] \\ \overline{\mathscr{M}}_g \arrow[r, "\Delta"] & \overline{\mathscr{M}}_g \times \overline{\mathscr{M}}_g \end{tikzcd}$$

is the functor I on the category of S-schemes, sending such a scheme $S' \to S$ to the set of isomorphisms of S'-schemes

$$\sigma : C_{S'} \to C'_{S'},$$

where $C_{S'}$ and $C'_{S'}$ denote the base changes to S'. To show that I is representable we proceed as in 13.1.5. Let $H \to S$ denote the Hilbert scheme of closed subschemes of $C \times_S C'$. Then an isomorphism $C_{S'} \to C'_{S'}$ corresponds to a closed subscheme $\Gamma \subset (C \times_S S')_{S'}$ such that the two projections $\Gamma \to C_{S'}$ and $\Gamma \to C'_{S'}$ are isomorphisms. As in 13.1.5, the condition that these two projections are isomorphisms is a representable condition which implies that I is represented by a subscheme of H. The same argument as in 13.1.5 also shows that I is of finite presentation over S.

13.2.23. That $\overline{\mathscr{M}}_g$ admits a smooth cover is also very similar to the proof of the algebraicity of $\mathscr{M}_g$ in 8.4.5. Namely, consider the Hilbert scheme $\mathscr{H}'$ of closed subschemes of $\mathbb{P}^{5g-6}$ all of whose fibers have Hilbert polynomial $(6g-6)T+(1-g)$. Let

$$\begin{tikzcd} \mathscr{Z} \arrow[r, hook] \arrow[dr] & \mathscr{H}' \times \mathbb{P}^{5g-6} \arrow[d] \\ & \mathscr{H}' \end{tikzcd}$$

be the universal closed subscheme. By 13.2.15, there is an open subscheme $\overline{\mathscr{W}} \subset \mathscr{H}'$ classifying closed subschemes of $\mathbb{P}^{5g-6}$ with Hilbert polynomial $(6g-6)T+(1-g)$ and such that all fibers are stable curves of genus g. Let $\overline{\mathscr{N}}_g$ denote the functor which to any scheme S associates the set of isomorphism classes of pairs $(\pi : C \to S, \iota : \mathscr{O}_S^{5g-5} \to \pi_*\omega_{C/S}^{\otimes 3})$, where $\pi : C \to S$ is a stable curve of genus g over S and ι is an isomorphism of vector bundles. There is a functor

$$\overline{\mathscr{N}}_g \to \overline{\mathscr{W}}$$

sending $(\pi : C \to S, \iota : \mathscr{O}_S^{5g-5} \to \pi_*\omega_{C/S}^{\otimes 3})$ to the closed subscheme

$$C \hookrightarrow \mathbb{P}(\pi_*\omega_{C/S}^{\otimes 3}) \xrightarrow{\iota} \mathbb{P}_S^{5g-6}.$$

By the same argument as in the proof of 8.4.5, this morphism of functors identifies $\overline{\mathscr{N}}_g$ with a quasi-projective $\overline{\mathscr{W}}$-scheme. In particular $\overline{\mathscr{N}}_g$ is a scheme.

The map

$$P : \overline{\mathscr{N}}_g \to \overline{\mathscr{M}}_g, \quad (\pi : C \to S, \iota : \mathscr{O}_S^{5g-5} \to \pi_*\omega_{C/S}^{\otimes 3}) \mapsto C/S$$

is a smooth surjection. This again is the same as in 8.4.13. This completes the proof that $\overline{\mathscr{M}}_g$ is an algebraic stack.

13.2.24. Finally, we check that $\overline{\mathscr{M}}_g$ is a Deligne-Mumford stack. By 8.3.3, it suffices to show that the diagonal is unramified. Or equivalently that if $T_0 \hookrightarrow T$ is a closed imbedding of affine schemes defined by a square-zero ideal, and if $\alpha : C \to C$ is an automorphism of a stable curve over T such that the reduction $\alpha_0 : C_0 \to C_0$ of α to T_0 is the identity, then α is also the identity. For this it suffices to consider the case when $T = \mathrm{Spec}(A)$ is the spectrum of an artinian local ring with residue field k and I annihilated by the maximal ideal of A.

Let C_k denote the reduction of C to k. Viewing $\mathscr{O}_C$ as a sheaf on $|C_k|$, such an automorphism α is given by a map

$$\mathscr{O}_C \to \mathscr{O}_C, \quad x \mapsto x + \partial(\bar{x}),$$

where $\partial : \mathscr{O}_{C_k} \to I \otimes \mathscr{O}_{C_k}$ is an k-linear derivation and for $x \in \mathscr{O}_C$ we write $\bar{x} \in \mathscr{O}_{C_k}$ for the reduction. To prove that $\overline{\mathscr{M}}_g$ is Deligne-Mumford it therefore suffices to show that

$$\mathrm{Ext}^0(\Omega^1_{C_k/k}, \mathscr{O}_{C_k}) = 0.$$

Let $\pi : \widetilde{C} \to C_k$ be the normalization, and let $D \subset \widetilde{C}$ be the preimage of the nodes. Then we have a morphism

$$\Omega^1_{C_k/k} \to \pi_*\Omega^1_{\widetilde{C}/k}. \tag{13.2.24.1}$$

In local coordinates, when we have an étale morphism to $k[x,y]/(xy)$, $\Omega^1_{C_k/k}$ is the quotient of the free module on two generators dx and dy modulo the relation

$$xdy + ydx = 0.$$

The module $\pi_*\Omega^1_{\widetilde{C}/k}$ is the module

$$k[x]\cdot dx \oplus k[y]\cdot dy.$$

From this it follows that the map (13.2.24.1) is surjective with kernel k embedded by

$$k \to \Omega^1_{C_k/k}, \quad 1 \mapsto ydx.$$

More globally this shows that there is an exact sequence

$$0 \to \bigoplus_{x\in C(k),\text{node}} k(x) \to \Omega^1_{C_k/k} \to \pi_*\Omega^1_{\widetilde{C}/k} \to 0,$$

which gives an exact sequence

$$0 \to \mathrm{Ext}^0(\pi_*\Omega^1_{\widetilde{C}/k}, \mathscr{O}_{C_k}) \to \mathrm{Ext}^0(\Omega^1_{C_k/k}, \mathscr{O}_{C_k}) \to \bigoplus_{x\in C(k),\text{node}} \mathrm{Ext}^0(k(x), \mathscr{O}_{C_k}).$$

Since $\mathrm{Ext}^0(k(x), \mathscr{O}_{C_k}) = 0$, this gives an isomorphism

$$\mathrm{Ext}^0(\Omega^1_{C_k/k}, \mathscr{O}_{C_k}) \simeq \mathrm{Ext}^0(\pi_*\Omega^1_{\widetilde{C}/k}, \mathscr{O}_{C_k}).$$

The inclusion $\mathscr{O}_{C_k} \hookrightarrow \pi_*\mathscr{O}_{\widetilde{C}}$ induces an injection

$$\begin{aligned}\mathrm{Ext}^0(\pi_*\Omega^1_{\widetilde{C}/k}, \mathscr{O}_{C_k}) &\to \mathrm{Ext}^0(\pi_*\Omega^1_{\widetilde{C}/k}, \pi_*\mathscr{O}_{\widetilde{C}})\\ &\simeq \mathrm{Ext}^0(\pi^*\pi_*\Omega^1_{\widetilde{C}/k}, \mathscr{O}_{\widetilde{C}}) \simeq \mathrm{Ext}^0(\Omega^1_{\widetilde{C}/k}, \mathscr{O}_{\widetilde{C}}),\end{aligned}$$

where the last isomorphism comes from the observation that any morphism

$$\pi^*\pi_*\Omega^1_{\widetilde{C}/k} \to \mathscr{O}_{\widetilde{C}}$$

necessarily factors through $\Omega^1_{\widetilde{C}/k}$. This inclusion identifies $\mathrm{Ext}^0(\pi_*\Omega^1_{\widetilde{C}/k}, \mathscr{O}_{C_k})$ with those sections $\partial \in H^0(\widetilde{C}, T_{\widetilde{C}})$ of the tangent bundle of $\widetilde{C}$ which vanish at each point of D. So we need to show that there are no such global sections. For components $\Gamma \in \widetilde{C}$ of genus ≥ 2 this is clear since $T_{\widetilde{C}}$ has negative degree on such a component. For a component $\Gamma \subset \widetilde{C}$ of genus 1, there are no nonzero global sections which vanish at a point, and for Γ of genus 0 there are no nonzero global sections which vanish at three points. This completes the proof that $\mathrm{Ext}^0(\Omega^1_{C_k/k}, \mathscr{O}_{C_k}) = 0$, and therefore also the proof that $\overline{\mathscr{M}}_g$ is Deligne-Mumford. □

13.2.25. Of course the real interest in $\overline{\mathscr{M}}_g$ comes from the following stronger result:

Theorem 13.2.26 ([**23**, 5.2]). *The stack $\overline{\mathscr{M}}_g$ is a proper Deligne-Mumford stack over* Spec($\mathbb{Z}$).

This result is unfortunately outside the scope of this book, because its proof requires a more thorough understanding of curve theory. However, we can state what needs to be shown as follows (see [**23**] for the proofs):

(i) (Properness of the diagonal) Let V be a discrete valuation ring with field of fractions K, and let C and C' be two stable curves of genus g over V with generic fibers C_K and C'_K. Then any isomorphism $\sigma_K : C_K \to C'_K$ extends to an isomorphism $\sigma : C \to C'$ over V.

(ii) (Properness of $\overline{\mathscr{M}}_g$) Let V be a discrete valuation ring with field of fractions K, and suppose given a stable curve C_K/K of genus g. Then after possibly replacing V by a finite extension, there exists a stable curve C/V with generic fiber C_K.

13.3. Moduli of stable maps

13.3.1. Let S be a scheme and let $(f : X \to S, L)$ be a proper S-scheme with a relatively ample invertible sheaf L. If $h : S' \to S$ is an S-scheme, then a *prestable map to X over S' of genus g* is a commutative diagram

$$\begin{tikzcd} C \arrow[r, "\tau"] \arrow[d] & X \arrow[d] \\ S' \arrow[r, "h"] & S, \end{tikzcd}$$

where C/S' is a prestable curve of arithmetic genus g. For two prestable maps to X $(i = 1, 2)$,

$$\begin{tikzcd} C_i \arrow[r, "\tau_i"] \arrow[d] & X \arrow[d] \\ S_i \arrow[r, "h_i"] & S, \end{tikzcd}$$

a morphism between them is defined to be a cartesian diagram of S-schemes,

$$\begin{tikzcd} C_1 \arrow[r, "\sigma"] \arrow[d] & C_2 \arrow[d] \\ S_1 \arrow[r, "\rho"] & S_2, \end{tikzcd}$$

such that $\sigma \circ \tau_2 = \tau_1$.

The *degree* of a prestable map $\tau : C \to X$ over S' is defined to be the degree of the invertible sheaf $\tau^* L$ (a locally constant function on S').

13.3.2. We want to obtain a Deligne-Mumford stack classifying certain prestable maps. To this end, we first analyze the infinitesimal automorphisms of prestable maps.

Let A be an artinian local ring with algebraically closed residue field k, and let $I \subset A$ be an ideal annihilated by the maximal ideal of A, so I can be viewed as a k-vector space. Suppose given a prestable map over $\mathrm{Spec}(A)$ that

$$\begin{tikzcd} C \arrow[r, "\tau"] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(A) \arrow[r] & S. \end{tikzcd}$$

We then want to understand automorphisms $\sigma : C \to C$ of this prestable map which reduce to the identity over $\mathrm{Spec}(A/I)$.

For this we may as well replace S by $\mathrm{Spec}(A)$ and X by $X \times_S \mathrm{Spec}(A)$, so that $S = \mathrm{Spec}(A)$. Viewing C as an X-scheme via τ, giving such an automorphism σ

is equivalent to giving an automorphism of C over X which reduces to the identity over $\mathrm{Spec}(A/I) \times_S X$. By the universal property of differentials 1.3.3, such automorphisms are classified by

$$\mathrm{Hom}_C(\tau^*\Omega^1_{X/S}, I \otimes \mathscr{O}_{C_0}) \simeq \mathrm{Hom}_{C_0}(\tau_0^*\Omega^1_{X_0/k}, \mathscr{O}_{C_0}) \otimes_k I,$$

where C_0, X_0, etc. denote the reductions modulo the maximal ideal of A.

Let $k[\epsilon]$ denote the ring of dual numbers over k, let $C_{k[\epsilon]}$ (resp. $X_{k[\epsilon]}$) denote the fiber product

$$C_0 \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k[\epsilon]) \quad (\text{resp. } X_0 \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k[\epsilon])),$$

and let

$$\tau_{k[\epsilon]} : C_{k[\epsilon]} \to X_{k[\epsilon]}$$

be the map obtained from g_0. Then the group

$$\mathrm{Hom}_{C_0}(\tau_0^*\Omega^1_{X_0/k}, \mathscr{O}_{C_0})$$

is isomorphic to the group of automorphisms of the prestable map $\tau_{k[\epsilon]}$ which reduce to the identity modulo ϵ.

Let $\pi : \widetilde{C} \to C_0$ be the normalization of C_0. If $\sigma : C_{k[\epsilon]} \to C_{k[\epsilon]}$ is an infinitesimal automorphism, then for every irreducible component $\Sigma \subset \widetilde{C}$ the map σ induces an automorphism

$$\sigma_\Sigma : \Sigma_{k[\epsilon]} \to \Sigma_{k[\epsilon]}$$

reducing to the identity modulo ϵ and fixing the special points on Σ. Furthermore, the maps σ_Σ determine the map σ.

Now the map σ_Σ must be the identity in all of the following cases:

(a) $g_\Sigma \geq 2$ (as noted already in 13.2.24);

(b) $g_\Sigma = 1$ and there is at least one special point on Σ. This can be seen for example by noting that the infinitesimal automorphisms of such a Σ are given by the 1-dimension k-space $H^0(\Sigma, T_\Sigma)$ and the infinitesimal automorphisms fixing a point $p \in \Sigma$ are given by the kernel of the restriction map

$$H^0(\Sigma, T_\Sigma) \to T_\Sigma(p)$$

which is zero.

(c) $g_\Sigma = 0$ and there are at least 3 special points on Σ.

(d) The map $\Sigma \to X_0$ induced by τ is not contraction to a point.

This motivates the following definition:

DEFINITION 13.3.3. A prestable map $\tau : C \to X$ over an S-scheme S' is *stable* if for every geometric point $\bar{s}' \to S'$ and every component Σ of the normalization $\widetilde{C}_{\bar{s}'}$ one of the conditions (a)-(d) above are met.

13.3.4. Now fix integers g and d and let $\mathscr{K}_g(X, d)$ be the category of pairs

$$(S', \tau : C \to X)$$

consisting of an S-scheme S' and a stable map $\tau : C \to X$ over S' such that C has genus g and the degree of the stable map is d. Morphisms in $\mathscr{K}_g(X, d)$ are morphisms of stable maps. The category $\mathscr{K}_g(X, d)$ is fibered over the category of schemes via the functor

$$\mathscr{K}_g(X, d) \to (\text{schemes}), \quad (S', \tau : C \to X) \mapsto S'.$$

THEOREM 13.3.5. *The fibered category $\mathscr{K}_g(X, d)$ is a Deligne-Mumford stack.*

The proof is in steps 13.3.6–13.3.10.

13.3.6. For an object

$$(S', \tau : C \to X) \in \mathscr{K}_g(X, d),$$

the invertible sheaf $L_\tau := \omega_{C/S'} \otimes \tau^* L^{\otimes 3}$ is relatively ample. Indeed this can be verified in the fibers, so it suffices to consider the case when $S' = \mathrm{Spec}(k)$ is the spectrum of an algebraically closed field. In this case, if $\Sigma \subset C'$ is a component, and if Σ is contracted to a point in X then the restriction of L_τ to Σ is ample since it is isomorphic to the restriction of $\omega_{C/k}$. If Σ is not contracted, then the worst-case scenario is when Σ is rational in which case $\omega_{C/k}$ restricted to Σ has degree -2 whence the degree of L_τ restricted to Σ is $-2 + 3d$ which is still positive.

From this and 4.4.12, it follows that $\mathscr{K}_g(X, d)$ is a stack with respect to the étale topology.

13.3.7. Next let us verify that the diagonal of $\mathscr{K}_g(X, d)$ is representable. So let

$$\tau : C \to X, \quad \tau' : C' \to X$$

be two stable maps of genus g over a base scheme S'. Let I denote the functor which to any S'-scheme $T \to S'$ associates the set of isomorphisms

$$C_T \to C'_T$$

over T. As in 13.3.5 this functor is representable by a scheme, which we again denote by I.

Over I, we have a tautological isomorphism:

$$\sigma : C_I \to C'_I.$$

We therefore get two maps,

$$C_I \to X,$$

from τ and $\tau' \circ \sigma$. Let $Z \subset C_I$ denote the fiber product of the resulting diagram:

$$\begin{array}{ccc} & & C_I \\ & & \big\downarrow {\scriptstyle \tau \times \tau' \circ \sigma} \\ X & \xrightarrow{\Delta} & X \times_S X. \end{array}$$

Then the fiber product P of the diagram

$$\begin{array}{ccc} & & S' \\ & & \big\downarrow {\scriptstyle (\tau, \tau')} \\ \mathscr{K}_g(X, d) & \xrightarrow{\Delta} & \mathscr{K}_g(X, d) \times \mathscr{K}_g(X, d) \end{array}$$

is the subfunctor of I, which to any $t : T \to I$ associates the unital set if the pullback $Z_T \to C_T$ is an isomorphism, and the empty set otherwise. Let $J \subset \mathscr{O}_{C_I}$ be the ideal sheaf of Z, and let r be an integer such that $J \otimes L_\tau^r$ and L_τ^r are generated by global sections and such that $f_* L_\tau^r$ is a locally free sheaf of finite rank on I whose formation commutes with arbitrary base change, where $f : C_I \to I$ is the structure morphism. Then P can also be viewed as the subfunctor of I which to any $t : T \to I$ associates the unital set if the induced map $t^*(f_*(J \otimes L_\tau^r)) \to t^* f_* L_\tau^r$ is the zero map and the empty set otherwise. From this it follows that P is represented by a closed subscheme. Indeed, locally on I we can choose a trivialization $\mathscr{O}_I^N \simeq f_* L_\tau^r$

in which case a choice of generators for $f_*(J \otimes L_\tau^r)$ gives elements $\underline{f}_i \in \mathscr{O}_I^N$, and P is represented by the zero locus of the entries of these vectors.

13.3.8. Next we show that $\mathscr{K}_g(X,d)$ admits a smooth cover by a scheme.

As above, for a stable map

$$\begin{tikzcd} C \arrow[r, "\tau"] \arrow[d, "f"] & X \arrow[d] \\ S' \arrow[r] & S \end{tikzcd} \tag{13.3.8.1}$$

over a scheme S', let L_τ denote $\omega_{C/S'} \otimes \tau^* L^{\otimes 3}$. As noted in 13.3.6 above, this line bundle on C is relatively ample. For an integer r, let

$$\mathscr{U}_r \subset \mathscr{K}_g(X,d)$$

denote the substack consisting of objects (13.3.8.1) for which the following condition holds:

($*_r$) For every geometric point $\bar{s} \to S'$ the restriction of $L_\tau^{\otimes r}$ to $C_{\bar{s}}$ is very ample and $H^1(C_{\bar{s}}, L_\tau^{\otimes r}|_{C_{\bar{s}}}) = 0$.

Notice that this condition implies that the sheaf $f_*(L_\tau^{\otimes r})$ on S' is locally free of finite rank and its formation commutes with arbitrary base change. Further it implies that $L_\tau^{\otimes r}$ is relatively very ample on C.

LEMMA 13.3.9. *The substack $\mathscr{U}_r \subset \mathscr{K}_g(X,d)$ is an open substack and*

$$\mathscr{K}_g(X,d) = \bigcup_{r \geq 1} \mathscr{U}_r.$$

PROOF. The condition that $L_\tau^{\otimes r}$ is very ample in the fibers is an open condition by [**26**, III.4.7.1], as is the vanishing of H^1 by cohomology and base change [**41**, III, 12.11]. This implies that $\mathscr{U}_r$ is an open substack of $\mathscr{K}_g(X,d)$. The fact that $\mathscr{K}_g(X,d)$ is equal to the union over all the $\mathscr{U}_r$ is clear since $\mathscr{L}_\tau$ is ample. ☐

13.3.10. To show that $\mathscr{K}_g(X,d)$ is an algebraic stack, it therefore suffices to show that for all $r \geq 1$, there exists a smooth surjection $U_r \to \mathscr{U}_r$ with U_r a scheme. In fact, we will show that $\mathscr{U}_r$ is isomorphic to a quotient stack

$$\mathscr{U}_r \simeq [U_r/GL_N]$$

for some integer N.

For this, first note that for any object (13.3.8.1) of $\mathscr{U}_r$, the rank of the sheaf $f_*(L_\tau^{\otimes r})$ is a constant N which depends only on g, d, and r. Indeed to verify this it suffices to consider the case when $S' = \mathrm{Spec}(k)$ is the spectrum of an algebraically closed field, in which case the claim follows from the Riemann-Roch theorem for singular curves [**41**, Chapter IV, exercise 1.9].

Let

$$\mathscr{F} : (S\text{-schemes})^{\mathrm{op}} \to \mathrm{Set}$$

be the functor sending S' to the set of isomorphism classes of pairs consisting of a stable map (13.3.8.1) defining a point of $\mathscr{U}_r$ and an isomorphism $\sigma : f_*(L_\tau^{\otimes r}) \simeq \mathscr{O}_S^N$. Note that given such data, we obtain an imbedding

$$C \hookrightarrow \mathbb{P}^{N-1}_{S'}$$

and therefore also an imbedding

$$C \hookrightarrow X \times_S \mathbb{P}^{N-1}_{S'}$$

over S'. Let $\mathscr{H}$ denote the Hilbert scheme of closed subschemes of $\mathbb{P}^{N-1}_X$ over S, and let

$$\mathscr{Z} \hookrightarrow \mathbb{P}^{N-1}_X \times_S \mathscr{H}$$

denote the universal closed subscheme.

The condition that $\mathscr{Z}$ is a nodal curve is an open condition by 13.2.15, and therefore represented by an open subscheme $\mathscr{H}_1 \subset \mathscr{H}$. Let $\mathscr{Z}_1 \to \mathscr{H}_1$ be the restriction of $\mathscr{Z}$ to $\mathscr{H}_1$, so we have a map

$$\tau_1 : \mathscr{Z}_1 \to X$$

and we can consider the invertible sheaf $L^{\otimes r}_{\tau_1}$. We also have the invertible sheaf $\mathscr{M}$ on $\mathscr{Z}_1$ which is the pullback of $\mathscr{O}(1)$ on $\mathbb{P}^N$. By exercise 1.L there exists a scheme $\widetilde{H}_1 \to \mathscr{H}$ classifying isomorphisms $L^{\otimes r}_{\tau_1} \to \mathscr{M}$. So the scheme $\widetilde{H}_1$ represents the functor which to any S-scheme S' associates the data of a nodal curve $f : Z \to S'$ with an imbedding

$$\tau \times i : Z \hookrightarrow X \times_S \mathbb{P}^N_{S'}$$

and an isomorphism $\sigma : L^{\otimes r}_\tau \to i^*\mathscr{O}(1)$. Next note that the condition that $L^{\otimes r}_\tau$ is very ample is represented by an open subscheme $V \subset \widetilde{H}_1$ and finally the condition that the induced map

$$\mathscr{O}^N_{S'} \longrightarrow f_* i^* \mathscr{O}_{\mathbb{P}^N(1)} \xrightarrow{\sigma} f_* L^{\otimes r}_\tau$$

is an isomorphism is also an open condition. The resulting open subscheme $U_r \subset \widetilde{H}_1$ represents $\mathscr{F}$.

Note also that there is an action of GL_N on U_r induced by the action of $\mathscr{F}$ obtained by changing the choice of isomorphism σ, and it is clear from the definition that

$$\mathscr{U}_r \simeq [U_r/GL_N].$$

This completes the proof Theorem of 13.3.5. □

As in the case of $\overline{\mathscr{M}}_g$, more is true:

THEOREM 13.3.11 ([**47**, 1.3.1]). *The stack $\mathscr{K}_g(X, d)$ is a proper Deligne-Mumford stack over S.*

Unfortunately, the proof of this theorem is again beyond the scope of this book. For more in this direction see [**32**].

13.4. Exercises

EXERCISE 13.A. Recall [**26**, IV.16.9.2] that a closed imbedding $i : X \hookrightarrow Y$ of schemes is called a *regular imbedding of codimension* d if for every point $x \in X$ there exists an affine neighborhood $\mathrm{Spec}(A) \subset Y$ containing x and such that the ideal $I \subset A$ defining $X \cap \mathrm{Spec}(A)$ is generated by a regular sequence in A of length d. Recall further that a morphism $f : X \to Y$ is called a *local complete intersection morphism of codimension* d if for any $x \in X$ there exists a neighborhood $U \subset X$ of x and a factorization

$$U \xhookrightarrow{i} P \xrightarrow{g} Y,$$

where i is a regular imbedding of codimension e and g is smooth of relative dimension $d+e$, for some e. We say that a morphism $f : X \to Y$ is a *regular imbedding* (resp. *local complete intersection*) if there exists an integer d such that f is a regular imbedding of codimension d (resp. local complete intersection morphism of codimension d).

(a) Show that if $i : X \hookrightarrow Y$ is a closed imbedding and $g : Y \to Z$ is a smooth morphism of relative dimension e such that $gi : X \to Z$ is a regular imbedding of codimension d, then i is a regular imbedding of codimension $d+e$.

(b) Suppose $f : X \to Y$ is a local complete intersection morphism. Show that for any factorization

$$X \overset{i}{\hookrightarrow} P \xrightarrow{g} Y$$

of f with i a closed imbedding and g smooth, the closed imbedding i is a regular imbedding.

(c) Let

$$\begin{tikzcd} U \arrow[r, "a"] \arrow[d, "g"] & X \arrow[d, "f"] \\ V \arrow[r, "b"] & Y \end{tikzcd}$$

be a commutative diagram of schemes with a and b étale and surjective. Show that f is a local complete intersection morphism if and only if g is a local complete intersection morphism.

(d) Let $f : X \to Y$ be a flat morphism of schemes. Suppose that for every point $y \in Y$ the fiber $f_y : X_y \to \mathrm{Spec}(k(y))$ is a local complete intersection. Show that $f : X \to Y$ is a local complete intersection.

(e) Let $f : X \to Y$ be a local complete intersection morphism of codimension d, and suppose there exists a factorization

$$X \overset{i}{\hookrightarrow} P \xrightarrow{g} Y$$

of f with i a regular imbedding of codimension e and g smooth of relative dimension $d+e$. Let $\omega_{P/Y}$ denote the $(d+e)$-th exterior power of $\Omega^1_{P/Y}$. Show that

$$i^*\mathscr{E}xt^e_{\mathscr{O}_P}(i_*\mathscr{O}_X, \omega_{P/Y})$$

is a locally free sheaf of rank 1 on X, which is independent of the choice of the factorization of f. This invertible sheaf is denoted $\omega_{X/Y}$ and is called the *relative dualizing sheaf of f*.

(f) Let $f : X \to Y$ be a proper morphism which is a local complete intersection morphism, and which étale locally on Y is projective. Show that there exists a relative dualizing sheaf $\omega_{X/Y}$ on X obtained by étale locally on Y choosing projective imbeddings of X over Y, and then using the independence of the choice of imbedding in (e) to glue the locally defined sheaves.

For further discussion see [**26**, IV, §19] and [**40**, Chapter III].

EXERCISE 13.B. Verify the commutativity of (13.2.9.3) thereby completing the proof of Proposition 13.2.9.

EXERCISE 13.C. Let k be an algebraically closed field and let C/k be a stable curve. Show that the genus g_C of C is ≥ 2.

Exercise 13.D. Let k be an algebraically closed field, and let $k \hookrightarrow \Omega$ be an inclusion into a bigger algebraically closed field. Let C/k be a proper scheme over k. Show that C is a nodal curve if and only if the base change C_Ω of C to Ω is a nodal curve over Ω.

Exercise 13.E. Fix nonnegative integers g and n with $2g-2+n>0$. Define $\overline{\mathscr{M}}_{g,n}$ to be the fibered category over the category of schemes whose objects over a scheme S are collections $(C, \sigma_1, \dots, \sigma_n)$, with C/S a prestable curve over S and $\sigma_1, \dots, \sigma_n : S \to C$ are sections of the structure morphism $C \to S$ such that for all geometric points $\bar{s} \to S$ the following conditions hold:

(1) The sections $\sigma_{1,\bar{s}}, \dots, \sigma_{n,\bar{s}} \in C_{\bar{s}}(k(\bar{s}))$ are distinct.
(2) The group of automorphisms $\rho : C_{\bar{s}} \to C_{\bar{s}}$ fixing the points $\sigma_{1,\bar{s}}, \dots, \sigma_{n,\bar{s}}$ is finite.
(3) The arithmetic genus of $C_{\bar{s}}$ is g.

We call a collection $(C, \sigma_1, \dots, \sigma_n)$ satisfying these conditions a *stable n-marked curve of genus g*. The purpose of this exercise is to show that $\overline{\mathscr{M}}_{g,n}$ is a Deligne-Mumford stack locally of finite type over $\mathrm{Spec}(\mathbb{Z})$ (in fact the stack $\overline{\mathscr{M}}_{g,n}$ is proper and smooth over $\mathrm{Spec}(\mathbb{Z})$ but this is beyond the scope of this book; see [**44**]).

(a) Let k be an algebraically closed field and let C/k be a prestable curve of arithmetic genus g. Let $\sigma_1, \dots, \sigma_n \in C(k)$ be n distinct sections in the smooth locus. Let $\widetilde{C}$ be the normalization of C, and call a point $x \in C(k)$ *special* if either x maps to a node of C or to one of the points σ_j. Show that $(C, \sigma_1, \dots, \sigma_n)$ is a stable n-marked curve if and only if every irreducible component of $\widetilde{C}$ of genus 1 contains at least one special point, and every irreducible component of $\widetilde{C}$ of genus 0 contains at least 3 special points.

(b) Let k be an algebraically closed field and let $(C, \sigma_1, \dots, \sigma_n)$ be an n-marked stable curve of genus g over k. Let ω_C be the dualizing sheaf of C. Show that the sheaf $\omega_C(\sigma_1 + \cdots + \sigma_n)$ is ample and that $\omega_C(\sigma_1 + \cdots + \sigma_n)^{\otimes m}$ is very ample if $m \geq 3$.

(c) Now using the method proving 13.2.20 show that $\overline{\mathscr{M}}_{g,n}$ is an algebraic stack locally of finite type over $\mathbb{Z}$.

(d) Show that $\overline{\mathscr{M}}_{g,n}$ is Deligne-Mumford by adapting the argument of 13.2.24 to the case of pointed curves.

APPENDIX A

Glossary of category theory

In this glossary we fix the basic terminology and notation from category theory that is needed in the text, and indicate references for the basic facts used.

A.1 Categories and functors [**48**, Chapter I, §11], [**51**, Chapter I].

A.1.1. A *category* C is a collection of data as follows:

(i) A class $\mathrm{Ob}(C)$ called the *objects* of C. We often write simply $X \in C$ instead of $X \in \mathrm{Ob}(C)$ when referring to an object of C.
(ii) For any two objects $X, Y \in C$ a set $\mathrm{Hom}_C(X, Y)$, whose elements we call *morphisms* and which we often denote by arrows $X \to Y$.
(iii) For every object $X \in C$ an element $\mathrm{id}_X \in \mathrm{Hom}_C(X, X)$, called the *identity morphism.*
(iv) For any three objects $X, Y, Z \in C$ a map of sets

$$\mathrm{Hom}_C(X, Y) \times \mathrm{Hom}_C(Y, Z) \to \mathrm{Hom}_C(X, Z), \quad (f, g) \mapsto g \circ f,$$

which we call *composition.*

This data is required to satisfy the following axioms:

(A) For three morphisms in $f : X \to Y$, $g : Y \to Z$, $h : Z \to W$ in C we have

$$h \circ (g \circ f) = (h \circ g) \circ f$$

in $\mathrm{Hom}_C(X, W)$.
(B) For every morphism $f : X \to Y$ we have $f \circ \mathrm{id}_Y = f = \mathrm{id}_X \circ f$.

A category C is called *small* if the objects $\mathrm{Ob}(C)$ form a set, and not just a class.

A.1.2. A morphism $f : X \to Y$ in a category C is called an *isomorphism* if there exists a morphism $g : Y \to X$ such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$. Such a morphism g is called an *inverse of* f, and is unique if it exists.

A.1.3. If C is a category then the *opposite category* of C, denoted C^{op}, is the category with the same objects as C but in which a morphism $X \to Y$ is an element of $\mathrm{Hom}_C(Y, X)$. Composition in C^{op} is defined using the composition law in C.

A.1.4. Let C and D be categories. A *functor* $F : C \to D$ is a rule which associates to any object $X \in C$ an object $F(X)$ of D, and to every morphism $f : X \to Y$ in C a morphism $F(f) : F(X) \to F(Y)$ in D such that the following hold:

(i) For every $X \in C$ we have $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ in $\mathrm{Hom}_D(F(X), F(X))$.
(ii) For two morphisms $f : X \to Y$ and $g : Y \to Z$ we have $F(g \circ f) = F(g) \circ F(f)$.

If $F : C \to D$ and $G : D \to E$ are functors, then their composition $G \circ F : C \to E$ is defined by sending $X \in C$ to $G(F(X))$ and a morphism $f : X \to Y$ to $G(F(f)) : G(F(X)) \to G(F(Y))$.

A.1.5. If $F, G : C \to D$ are two functors, then a *morphism of functors* $\eta : F \to G$ (sometimes called a *natural transformation*) is a rule that to every object $X \in C$ associates a morphism $\eta_X : F(X) \to G(X)$ such that for any morphism $f : X \to Y$ in C the diagram

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\eta_X"] & F(Y) \arrow[d, "\eta_Y"] \\ G(X) \arrow[r, "G(f)"] & G(Y) \end{tikzcd}$$

commutes. Note in particular that for any functor $F : C \to D$ there is a morphism of functors $\mathrm{id}_F : F \to F$, called the *identity morphism*, whose value on any object $X \in C$ is the identity map $F(X) \to F(X)$.

If $\eta : F \to G$ and $\epsilon : G \to H$ are morphisms of functors $C \to D$, then the composition $\epsilon \circ \eta$ is defined by associating to $X \in C$ the composite map

$$F(X) \xrightarrow{\eta_X} G(X) \xrightarrow{\epsilon_X} H(X).$$

A morphism of functors $\eta : F \to G$ is called an *isomorphism of functors* if there exists a morphism of functors $\epsilon : G \to F$ such that the $\epsilon \circ \eta = \mathrm{id}_F$ and $\eta \circ \epsilon = \mathrm{id}_G$. A morphism of functors $\eta : F \to G$ is an isomorphism if and only if for every object $X \in C$ the map $\eta_X : F(X) \to G(X)$ is an isomorphism in D.

A.1.6. If $F : C \to D$ is a functor between categories, then the *essential image* of F is a category whose objects are objects $d \in D$ which are isomorphic to $F(c)$ for some object $c \in C$, and whose morphisms are given by morphisms in D. Note that the functor F factors through the essential image of F.

A.2 Representable objects and Yoneda's lemma [**26**, 0_{III}, §8.1], [**51**, Chapter III, §2].

A.2.1. For a category C and object $X \in C$ we write

$$h_X : C^{\mathrm{op}} \to \mathrm{Set}$$

for the functor sending $Y \in C$ to $\mathrm{Hom}_C(Y, X)$ and a morphism $f : Y' \to Y$ in C to the map

$$\mathrm{Hom}_C(Y, X) \to \mathrm{Hom}_C(Y', X), \quad (g : Y \to X) \mapsto (g \circ f : Y' \to X).$$

A functor $F : C^{\mathrm{op}} \to \mathrm{Set}$ is called *representable* if there exists an object $X \in C$ such that $F \simeq h_X$. If $X, Y \in C$ are two objects and $g : X \to Y$ is a morphism in C, then there is an induced morphism of functors

$$h(g) : h_X \to h_X$$

sending $(f : Z \to X) \in h_X(Z)$ (for $Z \in C$) to the composite morphism $(gf : Z \to Y) \in h_Y(Z)$. Note that we recover g as the image of $\mathrm{id}_X \in h_X(X)$ under $h(g)_X$.

If $F : C^{\mathrm{op}} \to \mathrm{Set}$ is a functor and $X \in C$ is an object, then for a morphism of functors $\eta : h_X \to F$ we get an element $\tau_\eta \in F(X)$ as the image of $\mathrm{id}_X \in h_X(X)$ under η_X.

THEOREM A.2.2 (Yoneda's lemma, [**26**, 0_{III}.8.1.7]). *The collection of morphisms of functors $\eta : h_X \to F$ is a set and the map*

$$\{\textit{morphisms } h_X \to F\} \to F(X), \quad \eta \mapsto \tau_\eta$$

is a bijection. In particular, taking $F = h_Y$ for another object $Y \in C$ we get that the map

$$\mathrm{Hom}_C(X, Y) \to \{\textit{morphisms } h_X \to h_Y\}, \quad g \mapsto h(g),$$

is a bijection.

A.2.3. Yoneda's lemma implies in particular that if $F : C^{\mathrm{op}} \to \mathrm{Set}$ is a functor, then a pair (X, σ) consisting of an object $X \in C$ and an isomorphism of functors $\sigma : h_X \simeq F$ is unique up to a unique isomorphism.

A.2.4. A morphism $f : X \to Y$ in a category C is called a *monomorphism* (resp. *epimorphism*) if for every object $Z \in C$ the map

$$\mathrm{Hom}_C(Z, X) \to \mathrm{Hom}_C(Z, Y) \quad (\text{resp. } \mathrm{Hom}_C(Y, Z) \to \mathrm{Hom}_C(X, Z))$$

is an inclusion of sets.

A.3 Limits and colimits [72, A.5], [51, Chapter III, §4].

A.3.1. Let $F : I \to C$ be a functor, with I a small category. For an object X let $\kappa_X : I \to C$ denote the functor sending every object of I to X and every morphism to the identity morphism id_X. For a morphism $f : X \to Y$ in C there is an induced morphism of functors $\kappa_f : \kappa_X \to \kappa_Y$. Because I is small, for any two functors $F, G : I \to C$ the collection of morphisms of functors $F \to G$ is a set. Indeed we have an inclusion

$$\{\text{morphisms } F \to G\} \hookrightarrow \prod_{i \in I} \mathrm{Hom}_C(F(i), G(i)).$$

In particular, we can define a functor

$$\varinjlim F : C \to \mathrm{Set} \quad (\text{resp. } \varprojlim F : C^{\mathrm{op}} \to \mathrm{Set})$$

by sending $X \in C$ to the set of morphisms $F \to \kappa_X$ (resp. $\kappa_X \to F$). A pair (X, σ) consisting of an object $X \in C$ and an isomorphism $\sigma : h_X \simeq \varinjlim F$ (resp. $\sigma : h_X \simeq \varprojlim$) is called a *colimit* (resp. *limit*) of F. By Yoneda's lemma such a pair (X, σ) is unique up to a unique isomorphism, and if it exists we often write simply $\varinjlim F$ (resp. $\varprojlim F$) for the representing object.

Three important examples of limits are the following.

EXAMPLE A.3.2 (Equalizers). Let C be a category, let $X, Y \in C$ be two objects, and let $\alpha, \beta : X \to Y$ be two morphisms in C. We can think of this collection of data (X, Y, α, β) as a functor $F : I \to C$, where I is the category with two objects $\{0, 1\}$ with morphisms the identity morphisms and $\mathrm{Hom}_I(0, 1)$ consisting of two elements. In a picture:

$$I : \qquad \bullet \rightrightarrows \bullet.$$

If the limit $\varprojlim F$ is representable, then giving a representing object (Z, σ) is equivalent to giving an object Z with a morphism $\gamma : Z \to X$ such that $\alpha \circ \gamma = \beta \circ \gamma$, and such that if W is an object with a morphism $f : W \to X$ such that $\alpha \circ f = \beta \circ f$ then there exists a unique morphism $g : W \to Z$ such that $f = \gamma \circ g$:

$$\begin{tikzcd} W \arrow[r, dashed] \arrow[rr, bend left, "f"] & Z \arrow[r, "\gamma"] & X \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] & Y. \end{tikzcd}$$

Observe that if C is an abelian category, then $\mathrm{Ker}(\alpha - \beta : X \to Y)$ represents $\varprojlim F$.

EXAMPLE A.3.3 (Products). Let C be a category and let $\{X_i\}_{i \in I}$ be a collection of objects indexed by a set I. Viewing I as a category with only identity morphisms, we can think of the collection $\{X_i\}_{i \in I}$ as a functor $F : I \to C$. If the limit $\varprojlim F$ is representable, then giving a representing object (Z, σ) is equivalent to giving an object Z of C with maps $z_i : Z \to X_i$ for each $i \in I$ such that for any object $Y \in C$ the map

$$\mathrm{Hom}_C(Y, Z) \to \prod_{i \in I} \mathrm{Hom}_C(Y, X_i), \quad h \mapsto (h \circ z_i)_{i \in I}$$

is bijective. In other words, Z is the product of the X_i.

EXAMPLE A.3.4 (Fiber products). Let C be a category, and let $f : X \to Y$ and $g : Z \to Y$ be morphisms in C. We can think of these morphisms as a functor $F : I \to C$, where I is the category with three objects $\{a, b, c\}$ and morphisms the identity maps together with one morphism $a \to c$ and $b \to c$. In a picture:

$$\begin{tikzcd} & b \arrow[d] \\ a \arrow[r] & c. \end{tikzcd}$$

If the limit $\varprojlim F$ is representable, then giving a representing object (W, σ) is equivalent to giving an object $W \in C$ and morphisms $\alpha : W \to X$ and $\beta : W \to Z$ such that the square

$$\begin{tikzcd} W \arrow[r, "\alpha"] \arrow[d, "\beta"] & X \arrow[d, "f"] \\ Z \arrow[r, "g"] & Y \end{tikzcd}$$

commutes, and such that for any commutative square

$$\begin{tikzcd} V \arrow[r, "s"] \arrow[d, "t"] & X \arrow[d, "f"] \\ Z \arrow[r, "g"] & Y \end{tikzcd}$$

there exists a unique morphism $\rho : V \to W$ such that $t = \alpha \circ \rho$ and $s = \beta \circ \rho$.

EXAMPLE A.3.5 (Coequalizers). Let C be a category, let $X, Y \in C$ be two objects, and $\alpha, \beta : X \to Y$ be two morphisms in C. As in A.3.2, we can think of this collection of data (X, Y, α, β) as a functor $F : I \to C$, where I is the category with two objects $\{0, 1\}$ with morphisms the identity morphisms and $\mathrm{Hom}_I(0, 1)$ consisting of two elements. If the limit $\varinjlim F$ is representable, then giving a representing object (Z, σ) is equivalent to giving an object Z with a morphism $\gamma : X \to Z$ such that $\gamma \circ \alpha = \gamma \circ \beta$, and such that if W is an object with a morphism $f : X \to W$ such that $f \circ \alpha = f \circ \beta$, then there exists a unique morphism $g : Z \to W$ such that $f = g \circ \gamma$.

Observe that if C is an abelian category, then $\mathrm{Coker}(\alpha - \beta : X \to Y)$ represents $\varinjlim F$.

A.3.6. A category I is called *filtered* if the following conditions holds:

(i) I is nonempty.
(ii) For any two objects $x, y \in I$ there exists an object $z \in I$ and morphisms $x \to z$ and $y \to z$.
(iii) For two morphisms $f, g : x \to y$ in I there exists a morphism $h : y \to z$ such that $h \circ f = h \circ g$.

PROPOSITION A.3.7 ([**67**, Tag 04AX]). *Let I be a filtered category and let J be a category with finitely many objects and morphisms. Let*

$$F : I \times J \to Set, \quad (i, j) \mapsto F_{i,j}$$

be a functor. Then the natural map

$$\varinjlim_i \varprojlim_j F_{i,j} \to \varprojlim_j \varinjlim_i F_{i,j}$$

is an isomorphism.

A.4 Final and initial objects [**48**, p. 57].

A.4.1. If C is a category, we can consider the constant functor

$$e : C \to \mathrm{Set}$$

sending every object of C to the one point set $\{*\} \in \mathrm{Set}$. A *final object* of C is an object $X \in C$ representing this functor. Note that for such an object X there exists a unique isomorphism of functor $h_X \simeq e$. A final object $X \in C$ is characterized by the property that for any $Y \in C$ there exists a unique morphism $Y \to X$. In particular, a final object in C is unique up to a unique isomorphism.

Similarly, an *initial object* is an object $X \in C$ representing the constant functor

$$e : C^{\mathrm{op}} \to \mathrm{Set}$$

sending every object of C^{op} to the one point set. More concretely, an initial object in C is an object X such that for any $Y \in C$ there exists a unique morphism $X \to Y$.

A.5 Adjoint functors [**72**, A.6], [**51**, Chapter IV].

A.5.1. Let $F : C \to D$ and $G : D \to C$ be functors. We then obtain functors

$$\mathrm{Hom}_D(F(-), -) : C^{\mathrm{op}} \times D \to \mathrm{Set}, \quad (X, Y) \mapsto \mathrm{Hom}_D(F(X), Y),$$

and

$$\mathrm{Hom}_C(-, G(-)) : C^{\mathrm{op}} \times D \to \mathrm{Set}, \quad (X, Y) \mapsto \mathrm{Hom}_C(X, G(Y)).$$

We say that F and G are *adjoint* if there exists an isomorphism of functors

$$\tau : \mathrm{Hom}_D(F(-), -) \to \mathrm{Hom}_C(-, G(-)).$$

In this case G is called a *right adjoint of* F, and F is called a *left adjoint of* G. The collection of data (F, G, τ) is called an *adjoint pair.*

If $X \in D$ and $Y \in C$ we write

$$\tau_{Y,X} : \mathrm{Hom}_D(F(Y), X) \to \mathrm{Hom}_C(Y, G(X))$$

for the value of τ on the pair (Y, X).

A.5.2. If $F : C \to D$ is a functor, then a pair (G, τ) making (F, G, τ) an adjoint pair is unique up to unique isomorphism. Namely, for any object $X \in D$ we can consider the functor

$$\mathrm{Hom}_D(F(-), X) : C^{\mathrm{op}} \to \mathrm{Set}$$

and $G(X) \in C$ together with the isomorphism

$$\tau_{-,X} : \mathrm{Hom}_D(F(-), X) \simeq \mathrm{Hom}_C(-, G(X))$$

represents this functor. Similarly, given $G : D \to C$ the pair (F, τ) making (F, G, τ) an adjoint pair is unique up to unique isomorphism.

A.5.3. If $F : C \to D$ and $G : D \to C$ are adjoint funtors, then there are several equivalent ways of specifying the isomorphism τ. Given the isomorphism τ, we get for every $X \in D$ and isomorphism

$$\tau_{G(X),X} : \mathrm{Hom}_D(F(G(X)), X) \simeq \mathrm{Hom}_C(G(X), G(X)).$$

The identity map $G(X) \to G(X)$ therefore defines a map $\rho_X : F(G(X)) \to X$. This map is functorial in X, and therefore defines a morphism of functors $\rho : F \circ G \to \mathrm{id}_D$, called the *counit* of the adjunction. This morphism of functors determines τ. Namely, for any $X \in D$ and $Y \in C$ we get the map

$$\mathrm{Hom}_C(Y, G(X)) \xrightarrow{F} \mathrm{Hom}_D(F(Y), FG(X)) \xrightarrow{\rho_X} \mathrm{Hom}_D(F(Y), X),$$

which is the inverse of $\tau_{Y,X}$.

Similarly, from the isomorphism

$$\tau_{Y,F(Y)} : \mathrm{Hom}_D(F(Y), F(Y)) \to \mathrm{Hom}_C(Y, GF(Y))$$

we obtain a morphism of functors $\epsilon : \mathrm{id}_C \to G \circ F$, called the *unit* of the adjunction. For $X \in D$ and $Y \in C$ the map $\tau_{Y,X}$ can be recovered as the composition

$$\mathrm{Hom}_D(F(Y), X) \xrightarrow{G} \mathrm{Hom}_C(GF(Y), G(X)) \xrightarrow{\epsilon_Y} \mathrm{Hom}_C(Y, G(X)).$$

We often specify an adjoint pair by giving the functors F and G and then specifying either the unit or counit of the adjunction.

A.6 Equivalences of categories [**51**, Chapter IV, §4].

A.6.1. A functor $F : C \to D$ is called *faithful* (resp. *full*, *fully faithful*) if for every $X, Y \in C$ the map

$$\mathrm{Hom}_C(X, Y) \to \mathrm{Hom}_D(F(X), F(Y))$$

is injective (resp. surjective, bijective). A functor $F : C \to D$ is called an *equivalence of categories* if there exists a functor $G : D \to C$ and isomorphisms of functors $F \circ G \simeq \mathrm{id}_D$ and $G \circ F \simeq \mathrm{id}_C$. Such a functor G is called a *quasi-inverse for* F. A functor F is an equivalence of categories if and only if it is fully faithful and for every object $X \in D$ there exists an object $Y \in C$ such that X is isomorphic to $F(Y)$ (see [**51**, Theorem 1 on p. 93]).

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