Math 274

Lectures on

Deformation Theory

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Preface

My goal in these notes is to give an introduction to deformation theory by doing some basic constructions in careful detail in their simplest cases, by explaining why people do things the way they do, with examples, and then giving some typical interesting applications. The early sections of these notes are based on a course I gave in the Fall of 1979.

Warning: The present state of these notes is rough. The notation and numbering systems are not consistent (though I hope they are consistent within each separate section). The cross-references and references to the literature are largely missing. Assumptions may vary from one section to another. The safest way to read these notes would be as a loosely connected series of short essays on deformation theory. The order of the sections is somewhat arbitrary, because the material does not naturally fall into any linear order.

I will appreciate comments, suggestions, with particular reference to where I may have fallen into error, or where the text is confusing or misleading.

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CHAPTER 1

Getting Started

1 Introduction

Deformation theory is the local study of deformations. Or, seen from another point of view, it is the infinitesimal study of a family in the neighborhood of a given element. A typical situation would be a flat morphism of schemes $f : X \to T$. For varying $t \in T$ we regard the fibres X_t as a family of schemes. Deformation theory is the infinitesimal study of the family in the neighborhood of a special fibre X_0.

Closely connected with deformation theory is the question of existence of varieties of moduli. Suppose we try to classify some set of objects, such as curves of genus g. Not only do we want to describe the set of isomorphism classes of curves as a set, but also we wish to describe families of curves. So we seek a universal family of curves, parametrized by a *variety of moduli* M, such that each isomorphism class of curves occurs exactly once in the family. Deformation theory would then help us infer properties of the variety of moduli M in the neighborhood of a point $0 \in M$ by studying deformations of the corresponding curve X_0. Even if the variety of moduli does not exist, deformation theory can be useful for the classification problem.

The purpose of these lectures is to establish the basic techniques of deformation theory, to see how they work in various standard situations, and to give some interesting examples and applications from the literature. Here is a typical theorem which I hope to elucidate in the course of these lectures.

Theorem 1.1. *Let Y be a nonsingular closed subvariety of a nonsingular projective variety X over a field k. Then*

(a) *There exists a scheme H, called the* Hilbert scheme, *parametrizing closed subschemes of X with the same Hilbert polynomial P as Y, and there exists a universal subscheme $W \subseteq X \times H$, flat over H, such that the fibres of W over points $h \in H$ are all closed subschemes of X with the same Hilbert polynomial P and which is* universal *in the sense that if T is any other scheme, if $W' \subseteq$*

$X \times T$ is a closed subscheme, flat over T, all of whose fibres are subschemes of X with the same Hilbert polynomial P, then there exists a unique morphism $\varphi : T \to H$, such that $W' = W \times_H T$.

(b) *The Zariski tangent space to H at the point $y \in H$ corresponding to Y is given by $H^0(Y, \mathcal{N})$ where $\mathcal{N}$ is the normal bundle of Y in X.*

(c) *If $H^1(Y, \mathcal{N}) = 0$, then H is nonsingular at the point y, of dimension equal to $h^0(Y, \mathcal{N}) = \dim_k H^0(Y, \mathcal{N})$.*

(d) *In any case, the dimension of H at y is at least $h^0(Y, \mathcal{N}) - h^1(Y, \mathcal{N})$.*

Parts (a), (b), (c) of this theorem are due to Grothendieck [22]. For part (d) there are recent proofs due to Laudal [46] and Mori [54]. I do not know if there is an earlier reference.

Let me make a few remarks about this theorem. The first part (a) deals with a global existence question of a parameter variety. In these lectures I will probably not prove any global existence theorems, but I will state what is known and give references. The purpose of these lectures is rather the local theory which is relevant to parts (b), (c), (d) of the theorem. It is worthwhile noting, however, that for this particular moduli question, a parameter scheme exists, which has a universal family. In other words, the corresponding functor is *representable*.

In this case we see clearly the benefit derived from Grothendieck's insistence on the systematic use of nilpotent elements. For let $D = k[t]/t^2$ be the ring of dual numbers. Taking D as our parameter scheme, we see that the flat families $Y' \subseteq X \times D$ with closed fibre Y are in one-to-one correspondence with the morphisms of schemes $\operatorname{Spec} D \to H$ that send the unique point to y. This set $\operatorname{Hom}_y(D, H)$ in turn can be interpreted as the Zariski tangent space to H at y. Thus to prove (b) of the theorem, we have only to classify schemes $Y' \subseteq X \times D$, flat over D, whose closed fibre is Y. In §2 of these lectures we will therefore make a systematic study of structures over the dual numbers.

Part (c) of the theorem is related to *obstruction theory*. Given an infinitesimal deformation defined over an Artin ring A, to extend the deformation further there is usually some obstruction, whose vanishing is necessary and sufficient for the existence of an extended deformation.

In this case the obstructions lie in $H^1(Y, \mathcal{N})$. If that group is zero, there are no obstructions, and one can show that the corresponding moduli space is nonsingular.

Now I will describe the program of these lectures. There are several standard situations which we will keep in mind as examples of the general theory.

A. Subschemes of a fixed scheme X. The problem in this case is to deform the subscheme while keeping the ambient scheme fixed. This leads to the Hilbert scheme mentioned above.

B. Line bundles on a fixed scheme X. This leads to the Picard variety of X.

C. Deformations of nonsingular projective varieties X, in particular curves. This leads to the variety of moduli of curves.

D. Vector bundles on a fixed scheme X. Here one finds the variety of moduli of stable vector bundles. This suggests another question to investigate in these lectures. We will see that the deformations of a given vector bundle E over the dual numbers are classified by $H^1(X, \mathcal{E}nd\, E)$, where $\mathcal{E}nd\, E = \mathcal{H}om(E, E)$ is the sheaf of endomorphisms of E, and that the obstructions lie in $H^2(X, \mathcal{E}nd\, E)$. Thus we can conclude, if the functor of stable vector bundles is a representable functor, that H^1 gives the Zariski tangent space to the moduli. But what can we conclude if the functor is not representable, but only has a coarse moduli space? And what information can we obtain if E is unstable and the variety of moduli does not exist at all?

E. Deformations of singularities. In this case we consider deformations of an affine scheme to see what happens to its singularities. We will show that deformations of an affine nonsingular scheme are all trivial.

For each of these situations we will study a range of questions. The most local question is to study extensions of these structures over the dual numbers. Next we study the obstruction theory and structures over Artin rings. In the limit these give structures over complete local rings, and we will study Schlessinger's theory of prorepresentability. Then we will at least report on the question of existence of global moduli. If a fine moduli variety does not exist, we will try to understand why.

Along the way, as examples and applications of the theory, I hope to include the following.

1. Mumford's example [56] of a curve in $\mathbb{P}^3$ with obstructed deformations, i.e., whose Hilbert scheme is nonreduced.

2. Examples of rigid singularities and questions of smoothing singularities [77].

3. Mori's lower bound on the dimension of the Hilbert scheme, used in his proof that a variety with an ample tangent bundle must be projective space [54].

4. Tannenbaum's proof of the existence of irreducible plane curves of degree d and r nodes, for any $0 \leq r \leq \frac{1}{2}(d-1)(d-2)$.

5. Examples of obstructed surface deformations (Kas [36], Burns and Wahl [9]).

6. Applications to the moduli of vector bundles on projective spaces.

7. The problem of lifting schemes from characteristic p to characteristic 0. A typical question here is the following: If W is a discrete valuation ring of characteristic 0, whose residue field k is a field of characteristic $p > 0$, and if X_0 is a scheme over k, does there exist a scheme X flat over W, whose closed fibre is X_0?

2 Structures over the dual numbers

The very first deformation question to study is structures over the dual numbers $D = k[t]/t^2$. That is, one gives a structure (e.g., a scheme, or a scheme with a subscheme, or a scheme with a sheaf on it) and one seeks to classify extensions of this structure over the dual numbers. These are also called first order deformations.

To ensure that our structure is evenly spread out over the base, we will always assume that the extended structure is *flat* over D. This is the technical condition that corresponds to the intuitive idea of a deformation.

Recall that a module M is *flat* over a ring A if the functor $N \mapsto N \otimes_A M$ is exact on the category of A-modules. A morphism of schemes $f : X \to Y$ is *flat* if for every point $x \in X$, the local ring $\mathcal{O}_{x,X}$ is flat over the ring $\mathcal{O}_{f(x),Y}$. A sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ is *flat* over Y if for every $x \in X$, its stalk $\mathcal{F}_x$ is flat over $\mathcal{O}_{f(x),Y}$.

Lemma 2.1. *A module M over a noetherian ring A is flat if and only if for every prime ideal $\mathfrak{p} \subseteq A$, $\operatorname{Tor}_1^A(M, A/\mathfrak{p}) = 0$.*

Proof. The exactness of the functor $N \mapsto N \otimes_A M$ is equivalent to $\operatorname{Tor}_1(M, N) = 0$ for all A-modules N. Since Tor commutes with direct limits, it is sufficient to require $\operatorname{Tor}_1(M, N) = 0$ for all finitely generated A-modules N. Now over a noetherian ring A, a finitely generated module N has a filtration whose quotients are of the form $A/\mathfrak{p}_i$ for various prime ideals $\mathfrak{p}_i \subseteq A$. Thus, using the exact sequence of Tor, we see that $\operatorname{Tor}_1(M, A/\mathfrak{p}) = 0$ for all $\mathfrak{p}$ implies $\operatorname{Tor}_1(M, N) = 0$ for all N, hence M is flat.

Corollary 2.2.[1] *A module M over the dual numbers $D = k[t]/t^2$ is flat if and only if the natural map $M_0 \xrightarrow{t} M$ is injective, where $M_0 = M/tM$.*

Proof. If M is flat, then tensoring with the exact sequence

$$0 \to k \xrightarrow{t} D \to k \to 0$$

gives an exact sequence

$$0 \to M_0 \xrightarrow{t} M \to M_0 \to 0.$$

Conversely, if the map $M_0 \xrightarrow{t} M$ is injective, then $\operatorname{Tor}_1(M, k) = 0$, so by (2.1) M is flat over D.

Now we consider our first deformation problem, *Situation A*. Let X be a given scheme over k and let Y be a closed subscheme of X. We define a *deformation of Y over D in X* to be a closed subscheme

[1] This is a special case of the "local criterion of flatness" — cf. (6.2).

$Y' \subseteq X' = X \times D$, flat over D, such that $Y' \times_D k = Y$. We wish to classify all deformations of Y over D.

Let us consider the affine case first. Then X corresponds to a k-algebra B, and Y corresponds to an ideal $I \subseteq B$. We are seeking ideals $I' \subseteq B' = B[t]/t^2$ with B'/I' flat over D and such that the image of I' in $B = B'/tB'$ is just I. Note that $(B'/I') \otimes_D k = B/I$. Therefore by (2.2) the flatness of B'/I' over D is equivalent to the exactness of the sequence

$$0 \to B/I \xrightarrow{t} B'/I' \to B/I \to 0.$$

Suppose I' is such an ideal, and consider the diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & I & \xrightarrow{t} & I' & \to & I & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & B & \xrightarrow{t} & B' & \to & B & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & B/I & \xrightarrow{t} & B'/I' & \to & B/I & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

where the exactness of the bottom row implies the exactness of the top row.

Proposition 2.3. *In the situation above, to give $I' \subseteq B'$ such that B'/I' is flat over D and the image of I' in B is I, is equivalent to giving an element $\varphi \in \operatorname{Hom}_B(I, B/I)$. In particular, $\varphi = 0$ corresponds to the* trivial *deformation given by $I' = I \oplus tI$ inside $B' = B \oplus tB$.*

Proof. We will make use of the splitting $B' = B \oplus tB$ as B-modules, or, equivalently, of the section $\sigma : B \to B'$ given by $\sigma(b) = b + 0 \cdot t$, which makes B' into a B-module.

Take any element $x \in I$. Lift it to an element of I', which because of the splitting of B' can be written $x + ty$ for some $y \in B$. Two liftings differ by something of the form tz with $z \in I$. Thus y is not uniquely determined, but its image $\bar{y} \in B/I$ is. Now sending x to $\bar{y}$ defines a mapping $\varphi : I \to B/I$. It is clear from the construction that it is a B-module homomorphism.

Conversely, suppose $\varphi \in \mathrm{Hom}_B(I, B/I)$ is given. Define

$$I' = \{x+ty \mid x \in I,\ y \in B,\ \text{and the image of } y \text{ in } B/I \text{ is equal to } \varphi(x).\}$$

Then one checks easily that I' is an ideal of B', that the image of I' in B is I, and that there is an exact sequence

$$0 \to I \xrightarrow{t} I' \to I \to 0.$$

Therefore there is a diagram as before, where this time the exactness of the top row implies the exactness of the bottom row, and hence that B'/I' is flat over D.

These two constructions are inverse to each other, so we obtain a natural one-to-one correspondence between the set of such I' and the set $\mathrm{Hom}_B(I, B/I)$, whereby the trivial deformation $I' = I \oplus tI$ corresponds to the zero element.

Now we wish to globalize this argument to the case of a scheme X over k and a given closed subscheme Y. There are two ways to do this. One is to cover X with open affine subsets and use the above result. The construction is compatible with localization, and the correspondence is natural, so we get a one-to-one correspondence between the flat deformations $Y' \subseteq X' = X \times D$ and elements of the set $\mathrm{Hom}_X(\mathcal{I}, \mathcal{O}_Y)$, where $\mathcal{I}$ is the ideal sheaf of Y in X.

The other method is to repeat the above proof in the global case, simply dealing with sheaves of ideals and rings, on the topological space of X (which is equal to the topological space of X').

Before stating the conclusion, we will define the normal sheaf of Y in X. Note that the group $\mathrm{Hom}_X(\mathcal{I}, \mathcal{O}_Y)$ can be regarded as $H^0(X, \mathcal{H}om_X(\mathcal{I}, \mathcal{O}_Y))$ Furthermore, homomorphisms of $\mathcal{I}$ to $\mathcal{O}_Y$ factor through $\mathcal{I}/\mathcal{I}^2$, which is a sheaf on Y. So

$$\mathcal{H}om_X(\mathcal{I}, \mathcal{O}_Y) = \mathcal{H}om_Y(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_Y),$$

and this latter sheaf is called the *normal sheaf* of Y in X, and is denoted $\mathcal{N}_{Y/X}$. If X is a nonsingular and Y is locally complete intersection, then $\mathcal{I}/\mathcal{I}^2$ is locally free, and so $\mathcal{N}_{Y/X}$ is locally free also, and can be called the *normal bundle* of Y in X. This terminology derives from the fact that if Y is also nonsingular, then there is an exact sequence

$$0 \to \mathcal{J}_Y \to \mathcal{J}_X|_Y \to \mathcal{N}_{Y/X} \to 0$$

where $\mathcal{T}_Y$ and $\mathcal{T}_X$ denote the tangent sheaves to Y and X, respectively. In this case, therefore, $\mathcal{N}_{Y/X}$ is the usual normal bundle.

Summing up our results gives the following.

Theorem 2.4. *Let X be a scheme over a field k, let Y be a closed subscheme of X. Then the deformations of Y over D in X are in natural one-to-one correspondence with elements of $H^0(Y, \mathcal{N}_{Y/X})$, the zero element corresponding to the trivial deformation.*

Next we consider *Situation B*. Let X be a scheme over k and let $\mathcal{L}$ be a given invertible sheaf on X. We will study the set of isomorphism classes of invertible sheaves $\mathcal{L}'$ on $X' = X \times D$ such that $\mathcal{L}' \otimes \mathcal{O}_X \cong \mathcal{L}$. In this case flatness is automatic, because $\mathcal{L}'$ is locally free and X' is flat over A'.

Proposition 2.5. *Let X be a scheme over k, and $\mathcal{L}$ an invertible sheaf on X. The set of isomorphism classes of invertible sheaves $\mathcal{L}'$ on $X \times D$ such that $\mathcal{L}' \otimes \mathcal{O}_X \cong \mathcal{L}$ is in natural one-to-one correspondence with elements of the group $H^1(X, \mathcal{O}_X)$.*

Proof. We use the fact that on any ringed space X, the isomorphism classes of invertible sheaves are classified by $H^1(X, \mathcal{O}_X^*)$, where $\mathcal{O}_X^*$ is the sheaf of multiplicative groups of units in $\mathcal{O}_X$ [27, III, Ex. 4.5]. The exact sequence

$$0 \to \mathcal{O}_X \xrightarrow{t} \mathcal{O}_{X'} \to \mathcal{O}_X \to 0$$

gives rise to an exact sequence of sheaves of abelian groups

$$0 \to \mathcal{O}_X \xrightarrow{\alpha} \mathcal{O}_{X'}^* \to \mathcal{O}_X^* \to 0$$

where $\alpha(x) = 1 + tx$. Here $\mathcal{O}_X$ is an additive group, while $\mathcal{O}_{X'}^*$ and $\mathcal{O}_X^*$ are multiplicative groups and α is a truncated exponential map. Because the map of rings $D \to k$ has a section $k \to D$, it follows that this latter sequence is a split exact sequence of sheaves of abelian groups. So taking cohomology we obtain an exact sequence

$$0 \to H^1(X, \mathcal{O}_X) \to H^1(X', \mathcal{O}_{X'}^*) \to H^1(X, \mathcal{O}_X^*) \to 0.$$

This shows that the set of isomorphism classes of invertible sheaves on X' restricting to a given isomorphism class on X is a coset of the group $H^1(X, \mathcal{O}_X)$, as required.

Proceeding to *Situation D* we will actually consider a slightly more general situation. Let X be a given scheme over k, and let $\mathcal{F}$ be a given coherent sheaf on X. We define a *deformation* of $\mathcal{F}$ over D to be a coherent sheaf $\mathcal{F}'$ on $X' = X \times D$, flat over D, together with a homomorphism $\mathcal{F}' \to \mathcal{F}$ such that the induced map $\mathcal{F}' \otimes_D k \xrightarrow{\sim} \mathcal{F}$ is an isomorphism.

Theorem 2.6. *Let X be a scheme over k, and let $\mathcal{F}$ be a coherent sheaf on X. The deformations of $\mathcal{F}$ over D are in natural one-to-one correspondence with the elements of* $\operatorname{Ext}^1_X(\mathcal{F}, \mathcal{F})$, *with the zero-element corresponding to the trivial deformation.*

Proof. By (2.2), the flatness of $\mathcal{F}'$ over D is equivalent to the exactness of the sequence

$$0 \to \mathcal{F} \xrightarrow{t} \mathcal{F}' \to \mathcal{F} \to 0,$$

obtained by tensoring $\mathcal{F}'$ with $0 \to k \xrightarrow{t} D \to k \to 0$. Since the sequence $0 \to k \to D \to k \to 0$ splits, we have a splitting $\mathcal{O}_X \to \mathcal{O}_{X'}$, and thus we can regard this sequence as an exact sequence of $\mathcal{O}_X$-modules. This sequence gives an element $\xi \in \operatorname{Ext}^1_X(\mathcal{F}, \mathcal{F})$. Conversely, an element in that Ext group gives $\mathcal{F}'$ as an extension of $\mathcal{F}$ by $\mathcal{F}$ as $\mathcal{O}_X$-modules. To give a structure of $\mathcal{O}_{X'}$-module on $\mathcal{F}'$ we only have to specify multiplication by t. But this can be done in one and only one way compatible with the sequence above and the requirement $\mathcal{F}' \otimes_D k \cong \mathcal{F}$, namely projection from $\mathcal{F}'$ to $\mathcal{F}$ followed by the injection $t : \mathcal{F} \to \mathcal{F}'$. Note finally that $\mathcal{F}' \to \mathcal{F}$ and $\mathcal{F}'' \to \mathcal{F}$ are isomorphic as deformations of $\mathcal{F}$ if and only if the corresponding extensions ξ, ξ' are equivalent. Thus the deformations $\mathcal{F}'$ are in natural one-to-one correspondence with elements of the group $\operatorname{Ext}^1(\mathcal{F}, \mathcal{F})$.

Remark 2.6.1. Given $\mathcal{F}$ on X, we can also ask a different question like the one in (2.5), namely to classify isomorphism classes of coherent sheaves $\mathcal{F}'$ on X', flat over D, such that $\mathcal{F}' \otimes_D k$ is isomorphic to $\mathcal{F}$ (without specifying the isomorphism). This set need not be the same as the set of deformations of $\mathcal{F}$, but we can explain their relationship as follows. The group $\operatorname{Aut} \mathcal{F}$ of automorphisms of $\mathcal{F}$ acts on the set of deformations of $\mathcal{F}$ by letting $\alpha \in \operatorname{Aut} \mathcal{F}$ applied to $f : \mathcal{F}' \to \mathcal{F}$ be $\alpha f : \mathcal{F}' \to \mathcal{F}$. Now let $f : \mathcal{F}' \to \mathcal{F}$ and $g : \mathcal{F}'' \to \mathcal{F}$ be two deformations of $\mathcal{F}$. One sees easily that $\mathcal{F}'$ and $\mathcal{F}''$ are isomorphic as sheaves on X' if and only if there exists an $\alpha \in \operatorname{Aut} \mathcal{F}$ such that αf

and g are isomorphic as deformations of $\mathcal{F}$. Thus the set of such $\mathcal{F}'$ up to isomorphism as sheaves on X' is the orbit space of $\operatorname{Ext}^1_X(\mathcal{F},\mathcal{F})$ under the action of $\operatorname{Aut}\mathcal{F}$. This kind of subtle distinction will play an important role in questions of prorepresentability (Chapter 3).

Corollary 2.7. *If $\mathcal{E}$ is a vector bundle over X, then the deformations of $\mathcal{E}$ over D are in natural one-to-one correspondence with the elements of $H^1(X, \mathcal{E}nd\,\mathcal{E})$ where $\mathcal{E}nd\,\mathcal{E} = \mathcal{H}om(\mathcal{E},\mathcal{E})$ is the sheaf of endomorphisms of $\mathcal{E}$. The trivial deformation corresponds to the zero element.*

Proof. In that case, since $\mathcal{E}$ is locally free, $\operatorname{Ext}^1(\mathcal{E},\mathcal{E}) = \operatorname{Ext}^1(\mathcal{O}_X, \mathcal{E}nd\,\mathcal{E}) = H^1(X, \mathcal{E}nd\,\mathcal{E})$.

Remark 2.7.1. If $\mathcal{E}$ is a line bundle, i.e., an invertible sheaf $\mathcal{L}$ on X, then $\mathcal{E}nd\,\mathcal{E} \cong \mathcal{O}_X$, and the deformation of $\mathcal{L}$ are then classified by $H^1(\mathcal{O}_X)$. We get the same answer as in (2.5) because $\operatorname{Aut}\mathcal{L} = H^0(\mathcal{O}_X^*)$ and for any $\mathcal{L}'$ invertible on X', $\operatorname{Aut}\mathcal{L}' = H^0(\mathcal{O}_D^*)$. Now $H^0(\mathcal{O}_D^*) \to H^0(\mathcal{O}_X^*)$ is surjective, and from this it follows that two deformations $\mathcal{L}_1' \to \mathcal{L}$ and $\mathcal{L}_2' \to \mathcal{L}$ are isomorphic as deformations of $\mathcal{L}$ if and only if $\mathcal{L}_1'$ and $\mathcal{L}_2'$ are isomorphic as invertible sheaves on X'.

Remark 2.7.2. *Use of the word "natural".* In each of the main results of this section (2.4), (2.5), (2.6), (2.7), we have said that a certain set was in *natural* one-to-one correspondence with the set of elements of a certain group. We have not said exactly what we mean by this word natural (though it is possible to do so). So for the time being, you may understand it something like this: If I say there is a natural mapping from one set to another, that means I have a particular construction in mind for that mapping, and if you see my construction, you will agree that it is natural. It does not involve any unnatural choices. Use of the word natural carries with it the expectation (but not the promise) that the same construction carried out in parallel situations will give compatible results. So it should be compatible with localization, base-change, etc. However, natural does not mean unique. It is quite possible that someone else could find another mapping between these two sets, different from this one, but also natural from a different point of view.

In contrast to the natural correspondences of this section, we will see later situations where there are non-natural one-to-one correspondences. Having fixed one deformation, any other will define an element of a certain group, thus giving a one-to-one correspondence between the set of all deformations and the elements of the group, with the fixed deformation corresponding to the zero element. So there is a one-to-one correspondence, but it depends on the choice of a fixed deformation, and there may be no such choice which is natural, i.e., no one we can single out as a "trivial" deformation. In this situation we say that the set is a principal homogeneous space under the action of the group.

3 The T^i functors

In this section we will present the construction and main properties of the T^i functors introduced by Lichtenbaum and Schlessinger [47]. For any ring homomorphism $A \to B$ and any B-module M they introduce functors $T^i(B/A, M)$, for $i = 0, 1, 2$. With A and B fixed they form a cohomological functor in M, giving a 9-term exact sequence associated to a short exact sequence of modules $0 \to M' \to M \to M'' \to 0$. On the other hand, if $A \to B \to C$ are three rings and homomorphisms, and if M is a C-module, then there is a 9-term exact sequence of T^i functors associated with the three ring homomorphisms $A \to B$, $A \to C$, and $B \to C$. The principal application of these functors for us is the study of deformations of rings or affine schemes (Situation E). We will see that deformations of a ring are classified by a certain T^1 group, and that obstructions lie in a certain T^2 group.

Construction 3.1. Let $A \to B$ be a homomorphism of rings and let M be a B-module. Here we will construct the groups $T^i(B/A, M)$ for $i = 0, 1, 2$. The rings are assumed to be commutative with identity, but we do not impose any finiteness conditions yet.

First choose a polynomial ring $R = A[x]$ in a set of variables $x = \{x_i\}$ (possibly infinite) such that B can be written as a quotient of R as an A-algebra. Let I be the kernel

$$0 \to I \to R \to B \to 0,$$

which is an ideal in R.

Second choose a free R-module F and a surjection $j : F \to I \to 0$ and let Q be the kernel:

$$0 \to Q \to F \to I \to 0.$$

Having chosen R and F as above, the construction now proceeds without any further choices. Let F_0 be the submodule of F generated by all "Koszul relations" of the form $j(a)b - j(b)a$ for $a, b \in F$. Note that $j(F_0) = 0$ so $F_0 \subseteq Q$.

Now we define a complex

$$L_2 \xrightarrow{d_2} L_1 \xrightarrow{d_1} L_0$$

of B-modules as follows. Take $L_2 = Q/F_0$. Why is L_2 a B-module? A priori it is an R-module. But if $x \in I$ and $a \in Q$, then we can write $x = j(x')$ for some $x' \in F$ and then $xa = j(x')a \equiv j(a)x' \pmod{F_0}$, and $j(a) = 0$ since $a \in Q$, so we see that $xa = 0$. Therefore L_2 is a B-module.

Take $L_1 = F \otimes_R B = F/IF$, and let $d_2 : L_2 \to L_1$ be the map induced from the inclusion $Q \to F$.

Take $L_0 = \Omega_{R/A} \otimes_R B$, where Ω is the module of relative differentials. To define d_1 just map L_1 to I/I^2, then apply the natural derivation $d : R \to \Omega_{R/A}$ which induces a B-module homomorphism $I/I^2 \to L_0$.

Clearly $d_1 d_2 = 0$ so we have defined a complex of B-modules. Note also that L_1 and L_0 are *free* B-modules: L_1 is free because it is defined from the free R-module F; L_0 is free because R is a polynomial ring over A, so that $\Omega_{R/A}$ is a free R-module.

For any B-module M we now define the functors

$$T^i(B/A, M) = h^i(\operatorname{Hom}_B(L., M))$$

as the cohomology modules of the complex of homomorphisms of the complex $L.$ into M.

To show that these functors are well-defined, we must verify that they are independent of the choices made in the construction.

Lemma 3.2. *The modules $T^i(B/A, M)$ constructed above are independent of the choice of F (keeping R fixed).*

Proof. If F and F' are two choices of a free R-module mapping onto I, then $F \oplus F'$ is a third choice, so it is sufficient to compare F with $F \oplus F'$ by symmetry. Since F' is free, the map $j' : F' \to I$ factors through F, i.e., $j' = jp$ for some map $p : F' \to F$. Now changing bases in $F \oplus F'$, replacing each generator e' of F' by $e' - p(e')$, we may assume that the map $F \oplus F' \to I$ is just j on the first factor and 0 on the second factor. Thus we have the diagram

$$\begin{array}{ccccccccc} 0 & \to & Q \oplus F' & \to & F \oplus F' & \xrightarrow{(j,0)} & I & \to & 0 \\ & & \downarrow & & \downarrow pr_1 & & \downarrow id & & \\ 0 & \to & Q & \to & F & \xrightarrow{j} & I & \to & 0 \end{array}$$

showing that the kernel of $(j,0) : F \oplus F' \to I$ is just $Q \oplus F'$. Then clearly $(F \oplus F')_0 = F_0 + IF'$. Denoting $L.'$ the complex obtained from the new construction, we see that $L_2' = L_2 \oplus F'/IF'$, $L_1' = L_1 \oplus (F' \otimes_R B)$, and $L_0' = L_0$. Since $F' \otimes_R B = F'/IF'$ is a free B-module, the complex $L.'$ is obtained by taking the direct sum of $L.$ with the free acyclic complex $F' \otimes_R B \to F' \otimes_R B$. Hence when we take Hom of these complexes into M and then cohomology, the result is the same.

Lemma 3.3. *The modules $T^i(B/A, M)$ are independent of the choice of R.*

Proof. Let $R = A[x]$ and $R' = A[y]$ be two choices of polynomial rings with surjections to B. As in the previous proof, it will be sufficient to compare R with $R'' = A[x,y]$. Furthermore, the map $A[y] \to B$ can be factored through $A[x]$ by a homomorphism $p : A[y] \to A[x]$. Then, changing variables in $A[x,y]$, replacing each y_i by $y_i - p(y_i)$, we may assume in the ring homomorphism $A[x,y] \to B$ that all the y_i go to zero. So we have the diagram

$$\begin{array}{ccccccccc} 0 & \to & IR'' + yR'' & \to & R'' & \to & B & \to & 0 \\ & & \uparrow & & \uparrow\downarrow p & & \updownarrow id & & \\ 0 & \to & I & \to & R & \to & B & \to & 0 \end{array}$$

showing that the kernel of $R'' \to B$ is generated by I and all the y-variables.

Since we have already shown that the construction is independent of the choice of F, we may use any F's we like in the present proof. So take any free R-module F mapping surjectively to I. Then take F' a free R''-module on the same set of generators as F, and take G' a free R''-module on the index set of the y variables. Then we have

$$\begin{array}{ccccccccc} 0 & \to & Q' & \to & F' \oplus G' & \to & IR'' + yR'' & \to & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \to & Q & \to & F & \to & I & \to & 0. \end{array}$$

Now observe that since the y_i are independent variables, and G' has a basis e_i going to y_i, the kernel Q' in the upper row must be generated by 1) things in Q, 2) things of the form $y_i a - j(a)e_i$ with $a \in F$, and 3) things of the form $y_i e_j - y_j e_i$. Clearly the elements of types 2) and 3) are in $(F' \oplus G')_0$. Therefore $Q'/(F' \oplus G')_0$ is a B-module generated by the image of Q, so $L_2 = L_2'$.

On the other hand, $L_1' = L_1 \oplus (G' \otimes_{R'} B)$, and $L_0' = L_0 \oplus (\Omega_{A[y]/A} \otimes B)$. So L_1' has an extra free B-module generated by the e_i, and L_0' has an extra free B-module generated by the dy_i, and the map d_1 takes e_i to dy_i. So as in the previous proof we see that $L.'$ is obtained from $L.$ by taking the direct sum with a free acyclic complex, and hence the modules $T^i(B/A, M)$ are the same.

Theorem 3.4. *Let $A \to B$ be a homomorphism of rings. Then for $i = 0, 1, 2$, $T^i(B/A, \cdot)$ is a covariant, additive functor from the category of B-modules to itself. If*

$$0 \to M' \to M \to M'' \to 0$$

is a short exact sequence of B-modules, then there is a long exact sequence

$$\begin{array}{ccccccccc} 0 & \to & T^0(B/A, M') & \to & T^0(B/A, M) & \to & T^0(B/A, M'') & \to \\ & \to & T^1(B/A, M') & \to & T^1(B/A, M) & \to & T^1(B/A, M'') & \to \\ & \to & T^2(B/A, M') & \to & T^2(B/A, M) & \to & T^2(B/A, M''). \end{array}$$

Proof. We have seen that the $T^i(B/A, M)$ are well-defined. By construction they are covariant additive functors. Given a short exact sequence of modules as above, since the terms L_1 and L_0 of the complex $L.$ are free, we get a sequence of complexes

$$0 \to \mathrm{Hom}_B(L., M') \to \mathrm{Hom}_B(L., M) \to \mathrm{Hom}_B(L., M'') \to$$

which is exact except possibly for the map

$$\mathrm{Hom}_B(L_2, M) \to \mathrm{Hom}_B(L_2, M'')$$

which may not be surjective. This sequence of complexes gives the long exact sequence of cohomology above.

Theorem 3.5. *Let $A \to B \to C$ be rings and homomorphisms, and let M be a C-module. Then there is an exact sequence of C-modules*

$$\begin{array}{ccccccccc} 0 & \to & T^0(C/B,M) & \to & T^0(C/A,M) & \to & T^0(B/A,M) & \to \\ & \to & T^1(C/B,M) & \to & T^1(C/A,M) & \to & T^1(B/A,M) & \to \\ & \to & T^2(C/B,M) & \to & T^2(C/A,M) & \to & T^2(B/A,M). \end{array}$$

Proof. To prove this theorem, we will show that for suitable choices in the construction (3.1), the resulting complexes form a sequence

$$0 \to L.(B/A) \oplus_B C \to L.(C/A) \to L.(C/B) \to 0$$

which is split exact on the degree 0 and 1 terms, and which is right exact on the degree 2 terms. Given this, taking $\mathrm{Hom}(\cdot, M)$ will give a sequence of complexes which is exact on the degree 0 and 1 terms, and left exact on the degree 2 terms. Taking cohomology modules will give the 9-term exact sequence above.

First we choose a surjection $A[x] \to B \to 0$ with kernel I, and a surjection $F \to I \to 0$ with kernel Q, where F is a free $A[x]$-module, to calculate the functors $T^i(B/A, M)$.

Next choose a surjection $B[y] \to C \to 0$ with kernel J, and a surjection $G \to J \to 0$ of a free $B[y]$-module G with kernel R, to calculate $T^i(C/B, M)$.

To calculate the functors T^i for C/A, take a polynomial ring $A[x, y]$ in the x-variables and the y-variables. Then $A[x, y] \to B[y] \to C$ gives a surjection of $A[x, y] \to C$. If K is its kernel then there is an exact sequence

$$0 \to I[y] \to K \to J \to 0$$

by construction. Take F' and G' to be free $A[x, y]$-modules on the same index sets as F and G respectively. Choose a lifting of the map $G \to J$ to a map $G' \to K$. Then adding the natural map $F' \to K$ we get a surjection $F' \oplus G' \to K$. Let S be its kernel:

$$0 \to S \to F' \oplus G' \to K \to 0.$$

Now we are ready to calculate. Our of the choices thus made there are induced maps of complexes

$$L.(B/A) \otimes_B C \to L.(C/A) \to L.(C/B).$$

On the degree 0 level we have

$$\Omega_{A[x]/A} \otimes C \to \Omega_{A[x,y]/A} \otimes C \to \Omega_{B[y]/B} \otimes C.$$

These are free C-modules with bases $\{dx_i\}$ on the left, $\{dy_i\}$ on the right, and $\{dx_i, dy_i\}$ in the middle. So this sequence is clearly split exact.

On the degree 1 level we have

$$F \otimes C \to (F' \oplus G') \otimes C \to G \otimes C$$

which is split exact by construction.

On the degree 2 level we have

$$(Q/F_0) \otimes_B C \to S/(F' \oplus G')_0 \to R/G_0.$$

The right hand map is surjective because the map $S \to R$ is surjective. Clearly the composition of the two maps is 0. We make no claim of injectivity for the left hand map. So to complete our proof it remains only to show exactness in the middle.

Let $s = f' + g'$ be an element of S, and assume that its image in R is contained in G_0. We must show that s can be written as a sum of something in $(F' \oplus G')_0$ and something in the image of $Q[y]$. In the map $S \to R$, the element f' goes to 0. Let g be the image of g'. Then $g \in G_0$, so g can be written as a linear combination of expressions $j(a)b - j(b)a$ with $a, b \in G$. Lift these elements a, b to G'. Then the expressions $j(a')b' - j(b')a'$ are in S. Let g'' be g' minus a linear combination of these expressions $j(a')b' - j(b')a'$. We get a new element $s' = f' + g''$ in S, differing from s by something in $(F' \oplus G')_0$, and where now g'' is in the kernel of the map $G' \to G$, which is IG'. So we can write g'' as a sum of elements xh with $x \in I$ and $h \in G'$. Let $x' \in F$ map to x by j. Then $xh = j(x')h \equiv j(h)x' \bmod F_0$. Therefore $s' \equiv f' + \Sigma j(h)x' \bmod (F' \oplus G')_0$, and this last expression is in $F' \cap S$, and therefore is in $Q[y]$. This completes the proof.

Now we will give some special cases and remarks concerning these functors.

Proposition 3.6. *For any $A \to B$ and any M, $T^0(B/A, M) = \mathrm{Hom}_B(\Omega_{B/A}, M) = \mathrm{Der}_A(B, M)$.*

Proof. Write B as a quotient of a polynomial ring R, with kernel I. Then there is an exact sequence [27, II, 8.4A]

$$I/I^2 \xrightarrow{d} \Omega_{R/A} \otimes_R B \to \Omega_{B/A} \to 0.$$

Since $F \to I$ is surjective, there is an induced surjective map $L_1 \to I/I^2 \to 0$. Thus the sequence

$$L_1 \to L_0 \to \Omega_{B/A} \to 0$$

is exact. Taking $\operatorname{Hom}(\cdot, M)$ which is left exact, we see that $T^0(B/A, M) = \operatorname{Hom}_B(\Omega_{B/A}, M)$.

Proposition 3.7. *If B is a polynomial ring over A, then $T^i(B/A, M) = 0$ for $i = 1, 2$ and for all M.*

Proof. In this case we can take $R = B$ in the construction. Then $I = 0$, $F = 0$, so $L_2 = L_1 = 0$, and the complex $L.$ is reduced to the L_0 term. Therefore $T^i = 0$ for $i = 1, 2$ and any M.

Proposition 3.8. *If $A \to B$ is a surjective ring homomorphism with kernel I, then $T^0(B/A, M) = 0$ for all M, and $T^1(B/A, M) = \operatorname{Hom}_B(I/I^2, M)$.*

Proof. In this case we can take $R = A$, so that $L_0 = 0$. Thus $T^0 = 0$ for any M. Furthermore, the exact sequence

$$0 \to Q \to F \to I \to 0,$$

tensored with B, gives an exact sequence

$$Q \otimes_A B \to F \otimes_A B \to I/I^2 \to 0.$$

There is also a surjective map $Q \otimes_A B \to Q/F_0$, since the latter is a B-module, so we have an exact sequence

$$L_2 \to L_1 \to I/I^2 \to 0.$$

Taking $\operatorname{Hom}(\cdot, M)$ shows that $T^2(B/A, M) = \operatorname{Hom}_B(I/I^2, M)$.

A useful special case of this is the following.

Corollary 3.9. *If A is a local ring and B is a quotient A/I, where I is generated by a regular sequence $a_1, \ldots, a_r$, then also $T^2(B/A, M) = 0$ for all M.*

Proof. Indeed, in this case, since the Koszul complex of a regular sequence is exact, we find $Q = F_0$ in the construction of the T^i-functors. Thus $L_2 = 0$ and $T^2(B/A, M) = 0$ for all M.

Another useful special case is

Proposition 3.10. *Suppose $A = k[x_1, \ldots, x_n]$ and $B = A/I$. Then for any M there is an exact sequence*

$$0 \to \operatorname{Hom}(\Omega_{B/k}, M) \to \operatorname{Hom}(\Omega_{A/k}, M) \to \operatorname{Hom}(I/I^2, M) \to T^1(B/k, M) \to 0$$

and an isomorphism

$$T^2(B/A, M) \xrightarrow{\sim} T^2(B/k, M).$$

Proof. Write the long exact sequence of T^i-functors for the composition $k \to A \to B$ and use (3.6), (3.7), and (3.8).

Remark 3.11. Throughout this section so far we have not made any finiteness assumptions on the rings and modules. However, it is easy to see that if A is a noetherian ring, B a finitely generated A-algebra, and M a finitely generated B-module, then the B-modules $T^i(B/A, M)$ are also finitely generated. Indeed, we can take R to be a polynomial ring in finitely many variables over A, which is therefore noetherian. So I is finitely generated and we can take F to be a finitely generated R-module. Then the complex $L_.$ consists of finitely generated B-modules, whence the result.

Note also that the formation of the T^i-functors is compatible with localization.

4 The infinitesimal lifting property

Let us consider a scheme X of finite type over an algebraically closed ground field k. After the affine space $\mathbb{A}^n_k$ and the projective space $\mathbb{P}^n_k$, the nicest kind of scheme is a nonsingular one. The property of being nonsingular can be defined extrinsically on open affine pieces by the Jacobian criterion [27, I, §5]. Let Y be a closed subscheme of $\mathbb{A}^n$, with $\dim Y = r$. Let $f_1, \ldots, f_s \in k[x_1, \ldots, x_n]$ be a set of generators for the ideal I_Y of Y Then Y is *nonsingular* at a closed point $P \in Y$ if the rank of the Jacobian matrix $\|(\partial f_i/\partial x_j)(P)\|$ is equal to $n - r$. We say Y is nonsingular if it is nonsingular at every closed point. A scheme X is *nonsingular* if it can be covered by open affine subsets that are nonsingular.

This definition is awkward, because it is not obvious that the property of being nonsingular is independent of the affine embedding used in the definition. For this reason it is useful to have the intrinsic criterion for nonsingularity.

Proposition 4.1. *A scheme X of finite type over k is nonsingular if and only if the local ring $\mathcal{O}_{P,X}$ is a regular local ring for every point $P \in X$ [27, I,* 5.1*; II,* 8.14*A].*

Using differentials we have another characterization of nonsingular varieties.

Proposition 4.2. *Let X be a scheme over k. Then X is nonsingular if and only if the sheaf of differentials $\Omega^1_{X/k}$ is locally free of rank $n = \dim X$ at every point of X [27, II,* 8.15*].*

This result is an intrinsic characterization closely related to the original definition using the Jacobian criterion. The generalization of the Jacobian criterion describes when a closed subscheme Y of a nonsingular scheme X over k is nonsingular.

Proposition 4.3. *Let Y be an irreducible closed subscheme of a nonsingular scheme X over k, defined by a sheaf of ideals $\mathcal{I}$. Then Y is nonsingular if and only if*

1) *$\Omega_{Y/k}$ is locally free, and*

2) *the sequence of differentials [27, II,* 8.12*]*

$$0 \to \mathcal{I}/\mathcal{I}^2 \to \Omega^1_{X/k} \otimes \mathcal{O}_Y \to \Omega^1_{Y/k} \to 0$$

is exact on the left.

Furthermore, in this case $\mathcal{I}$ is locally generated by $n - r = \dim X - \dim Y$ elements, and $\mathcal{I}/\mathcal{I}^2$ is locally free on Y of rank $n - r$ [27, II, 8.17].

In this section we will see that nonsingular schemes have a special property related to deformation theory, called the infinitesimal lifting property. The general question is this. Suppose given a morphism $f : Y \to X$ of schemes, and suppose given an *infinitesimal thickening* $Y \subseteq Y'$. This means that Y is a closed subscheme of another scheme Y', and that the ideal $\mathcal{I}$ defining Y inside Y' is nilpotent. Then the question is, does there exist a *lifting* $g : Y' \to X$, i.e., a morphism such that g restricted to Y is f? Of course, there is no reason for this to hold in general, but we will see that if Y and X are affine, and X is nonsingular, then it does hold, and this property of X, for all such morphisms $f : Y \to X$, characterizes nonsingular schemes.

Proposition 4.4 (Infinitesimal Lifting Property). *Let X be a nonsingular affine scheme of finite type over k, let $f : Y \to X$ be a morphism from an affine scheme Y over k, and let $Y \subseteq Y'$ be an infinitesimal thickening of Y. Then the morphism f lifts to a morphism $g : Y' \to X$ such that $g|_Y = f$.*

Proof. (cf. [27, II, Ex. 8.6]) First we note that Y' is also affine [27, III, Ex. 3.1], so we can rephrase the problem in algebraic terms. Let $X = \operatorname{Spec} A$, let $Y = \operatorname{Spec} B$, and let $Y' = \operatorname{Spec} B'$. Then f corresponds to a ring homomorphism which (by abuse of notation) we call $f : A \to B$. On the other hand, B is a quotient of B' by an ideal I with $I^n = 0$ for some n. The problem is to find a homomorphism $g : A \to B'$ lifting f, i.e., so that g followed by the projection $B' \to B$ is f.

If we filter I by its powers and consider the sequence $B' = B'/I^n \to B'/I^{n-1} \to \cdots \to B'/I^2 \to B'/I$, it will be sufficient to lift one step at a time. Thus (changing notation) we reduce to the case $I^2 = 0$.

Since X is of finite type over k, we can write A as a quotient of a polynomial ring $P = k[x_1, \ldots, x_n]$ by an ideal J. Composing the projection $P \to A$ with f we get a homomorphism $P \to B$, which we can lift to a homomorphism $h : P \to B'$, since one can send the variables x_i to any liftings of their images in B. (This corresponds to the fact that the polynomial ring is a free object in the category of

k-algebras.)

$$\begin{array}{ccccccccc} 0 & \to & J & \to & P & \to & A & \to & 0 \\ & & & & \downarrow h & & \downarrow f & & \\ 0 & \to & I & \to & B' & \to & B & \to & 0. \end{array}$$

Now h induces a map $h : J \to I$ and since $I^2 = 0$, this gives a map $\bar{h} : J/J^2 \to I$.

Next we note that the homomorphism $P \to A$ gives an embedding of X in an affine n-space $\mathbb{A}^n_k$. Invoking Proposition 4.3 above, we obtain an exact sequence

$$0 \to J/J^2 \to \Omega^1_{P/k} \otimes_P A \to \Omega^1_{A/k} \to 0,$$

and note that these correspond to locally free sheaves on X, hence are projective A-modules. Via the maps h, f, we get a P-module structure on B', and A-module structures on B, I. Applying the functor $\mathrm{Hom}_A(\cdot, I)$ to the above sequence gives another exact sequence

$$0 \to \mathrm{Hom}_A(\Omega^1_{A/k}, I) \to \mathrm{Hom}_P(\Omega^1_{P/k}, I) \to \mathrm{Hom}_A(J/J^2, I) \to 0.$$

Let $\theta \in \mathrm{Hom}_P(\Omega^1_{P/k}, I)$ be an element whose image is $\bar{h} \in \mathrm{Hom}_A(J/J^2, I)$. We can regard θ as a k-derivation of P to the module I. Then we define a new map $h' : P \to B'$ by $h' = h - \theta$. I claim h' is a ring homomorphism lifting f and with $h'(J) = 0$. This is a consequence of Lemma 4.5 below.

Finally, to see that $h'(J) = 0$, let $y \in J$. Then $h'(y) = h(y) - \theta(y)$. We need only consider $y \bmod J^2$, and then $h(y) = \theta(y)$ by choice of θ, so $h'(y) = 0$. Now since $h'(J) = 0$, h descends to give the desired homomorphism $g : A \to B'$ lifting f.

Lemma 4.5. *Let $B' \to B$ be a surjective homomorphism of k-algebras with kernel J of square zero. Let $R \to B$ be a given homomorphism of k-algebras.*

a) *If $f, g : R \to B'$ are two liftings of the map $R \to B$ to B', then $\theta = g - f$ is a k-derivation of R to J.*

b) *Conversely, if $f : R \to B'$ is one lifting, and $\theta : R \to J$ a derivation, then $g = f + \theta$ is another homomorphism of R to B' lifting the given map $R \to B$.*

Hence, the set of liftings $R \to B$ to k-algebra homomorphisms of R to B' is a principal homogeneous space under the action of the group $\mathrm{Der}_k(R, J) = \mathrm{Hom}_R(\Omega_{R/k}, J)$. *(Note that since $J^2 = 0$, J has a natural structure of B-module and hence also of R-module.)*

Proof. a) Let $f, g : R \to B'$ and let $\theta = g - f$. As a k-linear map, θ followed by the projection $B' \to B$ is zero, so θ sends R to J. Let $x, y \in R$. Then

$$\begin{aligned} \theta(xy) &= g(xy) - f(xy) \\ &= g(x)g(y) - f(x)f(y) \\ &= g(x)(g(y) - f(y)) + f(y)(g(x) - f(x)) \\ &= g(x)\theta(y) + f(y)\theta(x) \\ &= x\theta(y) + y\theta(x), \end{aligned}$$

the last step being because $g(x)$ and $f(y)$ act in J just as x, y. Thus θ is a k-derivation of R to J.

b) Conversely, given f and θ as above, let $g = f + \theta$. Then

$$\begin{aligned} g(xy) &= f(xy) + \theta(xy) \\ &= f(x)f(y) + x\theta(y) + y\theta(x) \\ &= (f(x) + \theta(x))(f(y) + \theta(y)) \\ &= g(x)g(y), \end{aligned}$$

where we note that $\theta(x)\theta(y) = 0$ since $J^2 = 0$. Thus g is a homomorphism of $R \to B'$ lifting $R \to B$.

For the reverse implication of (4.4), we need only a special case of the infinitesimal lifting property.

Proposition 4.6. *Let X be a scheme of finite type over k. Suppose that for every morphism $f : Y \to X$ of a punctual scheme Y, finite over k, and for every infinitesimal thickening $Y \subseteq Y'$, there is a lifting $g : Y' \to X$. Then X is nonsingular.*

Proof. It is sufficient (Proposition 4.1) to show that the local ring $\mathcal{O}_{P,X}$ is a regular local ring for every closed point $P \in X$. So again we reduce to an algebraic question, namely, let A, m be a local k-algebra, essentially of finite type over k, and with residue field k. Assume that for every homomorphism $f : A \to B$, where B is a local artinian k-algebra and for every thickening $0 \to I \to B' \to B \to 0$ with $I^2 = 0$, there is a lifting $g : A \to B'$. Then A is a regular local ring.

Let $a_1, \dots, a_n$ be a minimal set of generators for the maximal ideal $\mathfrak{m}$ of A. Then there is a surjective homomorphism of the formal power series ring $P = k[[x_1, \dots, x_n]]$ to $\hat{A}$, the completion of A, sending x_i to a_i, and creating an isomorphism of $P/\mathfrak{n}^2$ to $A/\mathfrak{m}^2$, where $\mathfrak{n} = (x_1, \dots, x_n)$ is the maximal ideal of P.

Now consider the surjections $P/\mathfrak{n}^{i+1} \to P/\mathfrak{n}^i$ each defined by an ideal of square zero. Starting with the map $A \to A/\mathfrak{m}^2 \cong P/\mathfrak{n}^2$, we can lift step by step to get maps of $A \to P/\mathfrak{n}^i$ for each i, and hence a map into the inverse limit, which is P. Passing to $\hat{A}$, we have maps $P \xrightarrow{f} \hat{A} \xrightarrow{g} P$ with the property that $g \circ f$ is the identity on $P/\mathfrak{n}^2$. It follows that $g \circ f$ is an automorphism of P. Hence $g \circ f$ has no kernel, so f is injective. But f was surjective by construction, so f is an isomorphism, and $\hat{A}$ is regular. From this it follows that A is regular, as required.

Corollary 4.7. *Any infinitesimal deformation of a nonsingular affine scheme of finite type over k is trivial.*

Proof. Let X be affine nonsingular of finite type over k, and let X'/D be an infinitesimal deformation, where $D = k[\varepsilon]/(\varepsilon^2)$. Then X is a closed subscheme of X', and X' is an infinitesimal thickening of X. According to the infinitesimal lifting property (Proposition 4.4), the identity map $X \to X$ lifts to a map $X' \to X$ inducing the identity on X. Hence the deformation is trivial.

This means that nonsingular affine schemes are trivial from the point of view of deformation theory. So we will be led to study global nonsingular schemes and their deformations, which are truly global in nature. Or we may study the deformation of singular schemes as subschemes of nonsingular ones. We can also say that nonsingular affine schemes are *rigid*, in the sense that every infinitesimal deformation is trivial. Note, however, that there are other rigid schemes besides nonsingular ones (??).

The relative case

Having explained in detail the relation between the different properties of nonsingular schemes and the infinitesimal lifting property over an algebraically closed scheme, let us now review, without proofs, the corresponding results in the relative case, that is, for a morphism of schemes, without assuming an algebraically closed base field.

What we now call a smooth morphism was first introduced in Grothendieck's seminar [23] under the name of "morphisme simple". At that time the definition was that the morphism could locally be written as an étale morphism followed by an affine n-space morphism to the base. This definition is no longer used. By the time the theory appeared in [24, IV], the terminology changed to "morphisme lisse" (English smooth), and there are two equivalent definitions. We state them as a theorem-definition.

Theorem-Definition 4.8. *Let $f : X \to Y$ be a morphism. (For simplicity we will assume all schemes locally noetherian and all morphisms locally of finite type.) Then f is* smooth *if one of the following equivalent conditions is satisfied:*

(i) *f is flat, and for all $y \in Y$, the fiber X_y is* geometrically regular, *that is $X_y \times_{k(y)} \overline{k(y)}$ is a regular scheme (where $\overline{k(y)}$ is the algebraic closure of $k(y)$.*

(ii) *For any affine scheme Z together with a morphism $p : Z \to Y$, and for any closed subscheme $Z_0 \subseteq Z$ defined by a nilpotent sheaf of ideals $\mathcal{I}$, and for any morphism $g_0 : Z_0 \to X$ compatible with p, there exists a morphism $g : Z \to X$ whose restriction to Z_0 is g_0 [24, IV,* 17.5.1*].*

Condition (ii) is what we have called the infinitesimal lifting property. In [24] this property (without finiteness assumptions on f) defines the notion of X *formally smooth* over Y.

Proposition 4.9. *If $f : X \to Y$ is smooth, then $\Omega^1_{X/Y}$ is locally free on X of rank equal to the relative dimension of X over Y [**?**].*

Over a field, the local freeness of the differentials becomes an equivalent condition.

Proposition 4.10. *If X is a scheme of finite type over a field k, the following conditions are equivalent.*

(i) *X is smooth over k.*

(ii) *$\Omega^1_{X/k}$ is locally free of rank equal to* $\dim X$.

(iii) *There exists a perfect field $k' \supseteq k$ with $X_{k'}$ regular [24, IV,* 17.15.5*].*

Note that when k is not perfect, X smooth over k implies X regular, but not conversely [27, III, Ex. 10.1].

In the relative case, the Jacobian criterion of smoothness can be stated as follows:

Proposition 4.11. *Let X be smooth over a scheme S, and let Y be a closed subscheme of X, defined by a sheaf of ideals $\mathcal{I}$. Then Y is smooth over S at a point $y \in Y$ if and only if there exist $g_1, \ldots, g_r \in \mathcal{I}_y$ local generators for the ideal, such that $dg_1, \ldots, dg_r$ are linearly independent in $\Omega^1_{X/S} \otimes_S k(g)$ [24, IV,* 17.12.?*].*

Thus the whole theory of nonsingular schemes over an algebraically closed ground field extends to the relative case, and in Grothendieck's version, the infinitesimal lifting property has even become the definition of a smooth morphism [24, IV, 17.3.1]. Besides the references to [23] and [24, IV], one can find an exposition of the basic theory of smooth morphisms as presented in [23] in [4]. One can also find the purely algebraic theory of a formally smooth extension of local rings (corresponding to the extension $\mathcal{O}_{y,Y} \to \mathcal{O}_{x,X}$ where $f : X \to Y$ is a smooth morphism, $x \in X$, and $y = f(x)$) in [24, O_{IV}] and in the two books of Matsumura [50], [51].

5 Deformations of rings

In this section we will use the T^i functors to study deformations of rings over the dual numbers (Situation E). We will see that the deformations of B/k are given by $T^1(B/k, B)$.

We will also explain the special role in deformation theory played by nonsingular varieties: nonsingular affine varieties $\operatorname{Spec} B$ over k are characterized by the fact that $T^1(B/k, M) = 0$ for all M. This property is also equivalent to the "infinitesimal lifting property" discussed in Section 4.

Finally we will use this local study to analyze (5.5) the global deformations of a nonsingular variety (Situation D).

Let B be a k-algebra. We define a *deformation* of B over the dual numbers D to be a D-algebra B', flat over D, together with a ring homomorphism $B' \to B$ inducing an isomorphism $B' \otimes_D k \to B$. Because of (2.2) the flatness of B' is equivalent to the exactness of the sequence

$$0 \to B \xrightarrow{t} B' \to B \to 0. \qquad (*)$$

Here we think of B' and B on the right as rings, and B on the left as an ideal of square 0 which is a B-module. Furthermore B' is a D-algebra and B is a k-algebra. On the other hand, we can forget the D-algebra structure of B' and regard it simply as a k-algebra via the inclusion $k \subseteq D$. Then, as in (2.6), we see that the D-algebra structure of B' can be recovered in a unique way compatible with the exact sequence $(*)$. We need only specify multiplication by t, and this must be done by passing from B' to B on the right, followed by the inclusion $B \to B'$ on the left.

So we see that deformations of B over D are in one-to-one correspondence with isomorphism classes of exact sequence $(*)$ where B' and B are regarded only as k-algebras. We say in that case that B' is an extension as k-algebras of the k-algebra B by the B-module B.

This discussion suggests that we consider a more general situation. Let A be a ring, let B be an A-algebra, and let M be a B-module. We define an *extension* of B by M as A-algebras to be an exact sequence

$$0 \to M \to B' \to B \to 0$$

where $B' \to B$ is a homomorphism of A-algebras, and M is an ideal in B' with $M^2 = 0$. Two such extensions B', B'' are *equivalent* if there is an isomorphism $B' \to B''$ compatible in the exact sequences with the identity maps on B and M. The *trivial* extension is given by $B' = B \oplus M$ made into a ring by the rule $(b, m) \cdot (b_1, m_1) = (bb_1, bm_1 + b_1 m)$.

Theorem 5.1. *Let A be a ring, B an A-algebra, and M a B-module. Then the isomorphism classes of extensions of B by M as A-algebras* *are in natural one-to-one correspondence with elements of the group $T^1(B/A, M)$. The trivial extension corresponds to the zero element.*

Proof. Let $A[x] \to B$ be a surjective map of a polynomial ring over A to B, let $\{e_i\}$ be a set of generators of the B-module M, and let $y = \{y_i\}$ be a set of indeterminates with the same index set as $\{e_i\}$. Then we consider the polynomial ring $A[x, y]$, and note that if B' is any extension of B by M, then one can find a surjective ring homomorphism $f : A[x, y] \to B'$, not unique, which makes a commutative diagram

$$\begin{array}{ccccccccc}
0 & \to & (y) & \to & A[x,y] & \to & A[x] & \to & 0 \\
 & & \downarrow & & \downarrow f & & \downarrow & & \\
0 & \to & M & \to & B' & \to & B & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where the two outer vertical arrows are determined by the construction. Here (y) denotes the ideal in $A[x, y]$ generated by the y_i, and the map $(y) \to M$ sends y_i to e_i.

Now we proceed in two steps. First we classify quotients $f : A[x, y] \to B'$ which form a diagram as above. Then we ask, for a given extension B', how many different ways are there to express B' as a quotient of $A[x, y]$? Dividing out by this ambiguity will give us a description of the set of extensions B'.

For the first step, we complete the above diagram by adding a top row consisting of the kernels of the vertical arrows:

$$0 \to Q \to I' \to I \to 0.$$

Giving the quotient B' is equivalent to giving the ideal I' in $A[x, y]$. Since we have a splitting of the middle row given by the ring inclusion $A[x] \to A[x, y]$, the argument used in the proof of (2.3) shows that the set of such diagrams is in natural one-to-one correspondence with the group $\mathrm{Hom}_{A[x]}(I, M) = \mathrm{Hom}_B(I/I^2, M)$.

For the second step, we use Lemma 4.5 to see that the set of possible choices for $f : A[x, y] \to B'$ forms a principal homogeneous space under the action of $\mathrm{Der}_A(A[x], M)$.

Now write the long exact sequence of T^i functors (3.5) for the three rings $A \to A[x] \to B$ and the module M. The part that interests us is

$$T^0(A[x]/A, M) \to T^1(B/A[x], M) \to T^1(B/A, M) \to T^1(A[x]/A, M).$$

The first term here, by (3.6), is $\mathrm{Hom}_A(\Omega_{A[x]/A}, M)$ which is just the module of derivations $\mathrm{Der}_A(A[x], M)$. The second term, by (3.8), is $\mathrm{Hom}_B(I/I^2, M)$. The fourth term, by (3.7), is 0. Therefore $T^1(B/A, M)$ appears as the cokernel of a natural map

$$\mathrm{Der}_A(A[x], M) \to \mathrm{Hom}_B(I/I^2, M).$$

Under the interpretations above we see that this cokernel is the set of diagrams $A[x] \to B'$ as above, modulo the ambiguity of choice of the map $A[x] \to B'$, and so $T^1(B/A, M)$ is in one-to-one correspondence with the set of extensions B', as required.

Corollary 5.2. *Let k be a field and let B be a k-algebra. Then the set of deformations of B over the dual numbers is in natural one-to-one correspondence with the group $T^1(B/k, B)$.*

Proof. This follows from the theorem and the discussion at the beginning of this section, which showed that such deformations are in one-to-one correspondence with the k-algebra extensions of B by B.

Now we turn to nonsingular varieties. Following the terminology of [27, II, §8] we will say that a *variety*, which is an irreducible separated scheme X of finite type over an algebraically closed field k, is *nonsingular* if all of its local rings are regular local rings. Our first main result is to characterize affine nonsingular varieties in terms of the vanishing of the T^1 functor.

Theorem 5.3. *Let $X = \mathrm{Spec}\, B$ be an affine variety over k. Then X is nonsingular if and only if $T^1(B/k, M) = 0$ for all B-modules M. Furthermore, if X is nonsingular, then $T^2(B/k, M) = 0$ for all M.*

Proof. Since B is a finitely generated k-algebra, we can write B as a quotient of a polynomial ring $A = k[x]$ in finitely many variables over k. Let I be the kernel:

$$0 \to I \to A \to B \to 0.$$

Then $\mathrm{Spec}\, A$ is an affine space over k, which is nonsingular, and we can use the criterion of [27, II, 8.17] to decide when X is nonsingular. It says X is nonsingular if and only if the sequence

$$I/I^2 \xrightarrow{\alpha} \Omega_{A/k} \otimes B \to \Omega_{B/k} \to 0$$

is exact on the left also (i.e., α injective), and $\Omega_{B/k}$ is locally free on X. This latter condition is equivalent to saying $\Omega_{B/k}$ is a projective B-module. Therefore the sequence will be split exact. So we see that X is nonsingular if and only if the sequence

$$(*) \qquad 0 \to I/I^2 \xrightarrow{\alpha} \Omega_{A/k} \otimes B \to \Omega_{B/k} \to 0$$

is split exact.

Now the result about T^1 follows from (3.10), because the sequence $(*)$ is split exact if and only if for all M the map

$$\operatorname{Hom}(\Omega_{A/k}, M) \to \operatorname{Hom}(I/I^2, M)$$

is surjective.

For the vanishing of T^2, suppose that X is nonsingular. With the above notation, using (3.10) again we have $T^2(B/k, M) = T^2(B/A, M)$. Localizing at any point of X, we reduce to the case A a regular local ring, and the ideal I is generated by a regular sequence, since I/I^2 is locally free of the correct rank. Then by (3.9), $T^2(B/A, M) = 0$.

Corollary 5.4. *If $X = \operatorname{Spec} B$ is an affine nonsingular variety over k, then every deformation of B over the dual numbers is trivial.*

Proof. Combine (2) and (3). This gives another proof of (4.7).

Having seen that deformations of affine nonsingular varieties are trivial, we can classify all deformations of a nonsingular variety X/k over the dual numbers (Situation C). In general, if X is any scheme over k, we define a *deformation* of X over the dual numbers to be a scheme X', flat over D, together with a closed immersion $X \to X'$ such that the induced map $X \to X' \times_D k$ is an isomorphism.

Theorem 5.5. *Let X be a nonsingular variety over k. Then the deformations of X over the dual numbers are in natural one-to-one correspondence with the elements of the group $H^1(X, \mathcal{T}_X)$, where $\mathcal{T}_X = \mathcal{H}om_X(\Omega_{X/k}, \mathcal{O}_X)$ is the* tangent sheaf *of X.*

Proof. Let X' be a deformation of X, and let $\mathcal{U} = (U_i)$ be an open affine covering of X. Over each U_i the induced deformation U_i' is trivial by (5.4), so we can choose an isomorphism $\varphi_i : U_i \times_k D \xrightarrow{\sim} U_i'$ with the trivial deformation. Then on $U_{ij} = U_i \cap U_j$ we get an automorphism $\psi_{ij} = \varphi_j^{-1}\varphi_i$ of $U_{ij} \times_k D$. If B is the coordinate ring of U_{ij}, this means that $\psi = \psi_{ij}$ comes from an automorphism ψ of $B[t]/t^2$ inducing the identity map on B. Such an automorphism must be of the form

$$\psi(b_0 + tb_1) = b_0 + t(b_1 + \theta(b_0))$$

for some additive mapping $\theta : B \to B$.

I claim θ is a derivation. Indeed, taking $b_1 = 0$,

$$\psi(b_0 b_0') = \psi(b_0)\psi(b_0')$$

so

$$b_0 b_0' + t\theta(b_0 b_0') = (b_0 + t\theta(b_0))(b_0' + t\theta(b_0')),$$

from which it follows that

$$\theta(b_0 b_0') = b_0\theta(b_0') + b_0'\theta(b_0).$$

Thus $\theta \in \mathrm{Der}_k(B, B)$. Conversely, one sees easily that any derivation of B to B gives an automorphism of $B[t]/t^2$.

So from our deformation X' and the covering $\mathcal{U}$ we obtain elements $\theta_{ij} \in H^0(U_{ij}, \mathcal{J}_X)$, since $\mathrm{Der}_k(B, B) = \mathrm{Hom}_B(\Omega_{B/k}, B)$, so the θ's can be regarded as local sections of the tangent sheaf of X. By construction on U_{ijk} we have $\theta_{ij} + \theta_{jk} + \theta_{ki} = 0$ since composition of automorphisms of $B[t]/t^2$ corresponds to addition of derivations. Therefore (θ_{ij}) is a Čech 1-cocycle for the covering $\mathcal{U}$ and the sheaf $\mathcal{J}_X$. Note finally if we replaced the original chosen isomorphisms $\varphi_i : U_i \times_k D \xrightarrow{\sim} U_i'$ by some others φ_i', then $\varphi_i'^{-1}\varphi_i$ would be an automorphism of $U_i \times_k D$ coming from a section $\alpha_i \in H^0(U_i, J_X)$, and the new $\theta_{ij}' = \theta_{ij} + \alpha_i - \alpha_j$. So the new 1-cocycle θ_{ij}' differs from θ_{ij} by a coboundary, and we obtain a well-defined element θ in the Čech cohomology group $\check{H}^1(\mathcal{U}, J_X)$. Since $\mathcal{U}$ is an open affine covering and J_X is a coherent sheaf, this is equal to the usual cohomology group $H^1(X, \mathcal{J}_X)$. Clearly θ is independent of the covering chosen.

Reversing this process, any element $\theta \in H^1(X, \mathcal{J}_X)$ is represented on $\mathcal{U}$ by a 1-cocycle θ_{ij}, and these θ_{ij} define automorphisms of the trivial deformations $U_{ij} \times_k D$ which can be glued together to make a global deformation X' of X. So we see that the deformations of X over D are given by $H^1(X, \mathcal{J}_X)$.

Remark 5.6. Even if X is not nonsingular, the above proof shows that $H^1(X, \mathcal{J}_X)$ classifies the *locally trivial* deformations of X over D. Let $\mathrm{Def}(X/k, D)$ denote the set of all deformations of X over D. If X' is a deformation, then on each affine $U_i \subseteq X$, X' induces a deformation U_i', which by (2) corresponds to an element of $T^1(B_i/k, B_i)$, where B_i is the affine ring of U_i. The T^i functors behave well under localization, so we can define the *sheaves* $T^i(X/k, \mathcal{F})$ for $i = 0, 1, 2$, and any coherent sheaf $\mathcal{F}$ on X. Then we get an exact sequence

$$0 \to H^1(X, T^0(X/k, \mathcal{O}_X)) \to \mathrm{Def}(X/k, D) \to H^0(X, T^1(X/k, \mathcal{O}_X)) \to H^2(X, T$$

as follows: a deformation of X induces local deformations on the open sets U_i, hence the right-hand arrow from $\mathrm{Def}(X/k, D)$. The kernel of this map is the locally trivial deformations which we have seen are classified by $H^1(X, \mathcal{J}_X)$. Here we write the sheave $\mathcal{J}_X$ as $T^0(X/k, \mathcal{O}_X)$, using (3.6), for suggestive notation. We will define the arrow to $H^2(X, T^0)$ later (10.6). The sequence looks like the beginning of the exact sequence of terms of low degree of a spectral sequence...

A more thorough study of this local-global interplay will probably require some global analogue of the local T^i functors defined here.

CHAPTER 2

Higher Order Deformations

6 Higher order deformations and obstruction theory

In the previous sections we have studied deformations of a structure over the dual numbers. These are the first order infinitesimal deformations. Now we will discuss higher-order deformations, taking as our model the deformations of a closed subscheme of a given scheme. Here we encounter for the first time obstructions. Given a deformation of a certain order, it may not be possible to extend it further. So there is an obstruction, which is an element of a certain group, whose vanishing is necessary and sufficient for the existence of an extension to a higher order deformation.

In this section we will also discuss locally complete intersection subschemes of a given scheme, because these have the property that local deformations always exist.

One context in which to study higher order deformations is the following. Suppose given a structure S over a field k. S could be a scheme, or a closed subscheme of a given scheme, or a vector bundle on a given scheme, etc. We look for deformations of S over the ring $A_n = k[t]/t^{n+1}$. These would be called n-th order deformations of S. Since it is hard to classify these all at once, we consider an easier problem. Suppose S_n is a given deformation of S over A_n. Then we seek to classify all deformations S_{n+1} over A_{n+1} whose restriction to A_n is the given deformation S_n. In this case we say S_{n+1} is an extension of S_n over the ring A_{n+1}.

Typically the answer to such a problem comes in two parts: there is an obstruction to the existence of S_{n+1}, then if the obstruction is zero, the set of extensions S_{n+1} is classified by some group. However, it is not a natural correspondence as in the earlier sections. Rather it works like this: given one such extension S_{n+1}, any other S'_{n+1} determines an element of a group. We say the set of S_{n+1} is a torsor or principal homogeneous space under the action of the group, defined as follows.

Definition 6.1. Let G be a group acting on a set S, i.e., there is a map $G \times S \to S$, written $\langle g, s \rangle \mapsto g(s)$, such that for any $g, h \in G$, $(gh)(s) = g(h(s))$. We say S is a *torsor* or *principal homogeneous space* under the action of G if there exists an element $s_0 \in S$ such that the mapping $g \mapsto g(s_0)$ is a bijective mapping of G to S. Note that if there exists one such $s_0 \in S$, then the same is true for any other element $s_1 \in S$. So we see that S is a principal homogeneous space under the action of G if and only if it satisfies the conditions

1) For every $s \in S$ the induced mapping $g \mapsto g(s)$ is bijective from G to S, and

2) S is non-empty.

If condition 1) is satisfied but we do not yet know whether S is non-empty, we say S is a *pseudotorsor*.

Although it seems natural to discuss deformations over the rings $A_n = k[t]/t^{n+1}$ as described above, it will be useful for later purposes to work in a slightly more general context. Note that there is an exact sequence

$$0 \to k \xrightarrow{t^{n+1}} A_{n+1} \to A_n \to 0,$$

so that k appears as an ideal in A_{n+1} annihilated by t. More generally we will consider deformation problems over a sequence

$$0 \to J \to A' \to A \to 0$$

where A is a ring, $A' \to A$ is a ring homomorphism, and $J \subseteq A'$ is an ideal satisfying $J^2 = 0$, so that J can be considered as an A-module. We suppose some structure given over A, and we seek to classify extensions of that structure over A'. Since $J^2 = 0$ this is still a problem of infinitesimal deformations. This more general setting includes the earlier one, but also includes the non-equicharacteristic case, such as $A = \mathbb{Z}/p^n$, and $A' = \mathbb{Z}/p^{n+1}$, where A and A' are not k-algebras for any field k. This will be useful for questions of lifting from characteristic p to characteristic zero (§25). We will need a criterion for flatness over A'.

Lemma 6.2 (Local Criterion of Flatness). *Let $A' \to A$ be a surjective ring homomorphism with kernel J of square zero. Then an A'-module M' is flat over A' if and only if*

1) *$M = M' \otimes_{A'} A$ is flat over A, and*

2) *the natural map $J \otimes_A M \to M'$ is injective.*

Proof. First of all, if M' is flat over A', then M is flat over A by base extension, and the map $J \otimes_A M \to M'$ is injective, from tensoring the sequence $0 \to J \to A' \to A \to 0$ with M'.

Conversely, suppose 1) and 2) satisfied. To show M' flat over A' we must show $\mathrm{Tor}_1^{A'}(M', N') = 0$ for all A'-modules N'. For any such N' there is an exact sequence

$$0 \to K \to N' \to N \to 0$$

obtained by tensoring the sequence $0 \to J \to A' \to A \to \cdot$ with N', where N and K are A-modules. Thus using the long exact sequence of Tor, it is sufficient to show $\mathrm{Tor}_1^{A'}(M', N) = 0$ for every A-module N. This is a question of change of rings for Tor.

Now take a resolution

$$0 \to Q' \to F' \to M' \to 0$$

of M', where F' is a free A'-module. Tensoring with N gives an exact sequence

$$0 \to \mathrm{Tor}_1^{A'}(M', N) \to Q' \otimes N \to F' \otimes N.$$

On the other hand, tensoring with A gives an exact sequence

$$0 \to \mathrm{Tor}_1^{A'}(M', A) \to Q' \otimes A \to F' \otimes A \to M \to 0.$$

Now M is flat over A, and $F' \otimes A$ is free over A, so the kernel of the map $F' \otimes A \to M$ is also flat over A. Therefore tensoring this sequence by N is still exact, i.e.,

$$0 \to \mathrm{Tor}_1^{A'}(M', A) \otimes N \to Q' \otimes N \to F' \otimes N$$

is exact. We conclude that

$$\mathrm{Tor}_1^{A'}(M', N) \cong \mathrm{Tor}_1^{A'}(M', A) \otimes N.$$

Finally, the fact that $J \otimes M \to M'$ is injective implies $\mathrm{Tor}_1^{A'}(M', A) = 0$, so $\mathrm{Tor}_1^{A'}(M', N) = 0$ and M' is flat, as required.

Now we will study deformations of closed subschemes. Suppose given

$$\begin{array}{ccccccccc} & & & & & & Y & & \\ & & & & & & \downarrow & & \\ & & & & X' & \leftarrow & X & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & J & \rightarrow & A' & \rightarrow & A & \rightarrow & 0 \end{array}$$

where $A' \to A$ is a surjective ring homomorphism with kernel J of square 0; X is a scheme flat over A, X' is a scheme flat over A', $X \to X'$ is a closed immersion inducing an isomorphism $X \xrightarrow{\sim} X' \times_{A'} A$, and Y is a closed subscheme of X, flat over A. (By abuse of notation we write $X \to A$ instead of $X \to \operatorname{Spec} A$.) Then a *deformation* of Y over A' in X' is a closed subscheme Y' of X', flat over A', such that $Y \times_{A'} A = Y$.

Theorem 6.3. *In the above situation*

a) *The set of deformations of Y over A' in X' is a pseudotorsor under a natural action of the group $H^0(Y, \mathcal{N}_{Y/X} \otimes_A J)$.*

b) *If deformations of Y over A' exist locally on X, then the obstruction to global existence of Y' lies in $H^1(Y, \mathcal{N}_{Y/X} \otimes_A J)$.*

Remark. In particular, this result applies to schemes Y that are local complete intersections or Cohen–Macaulay in codimension 2 or locally Gorenstein in codimension 3—cf. §8,9 below.

Proof. We will prove this theorem in several stages. First we consider the affine case $X = \operatorname{Spec} B$, $X' = \operatorname{Spec} B'$, $Y = \operatorname{Spec} C$. Then we have a diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & J \otimes_A I & \rightarrow & I' & \rightarrow & I & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & J \otimes_A B & \rightarrow & B' & \rightarrow & B & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & J \otimes_A C & \rightarrow & C' & \rightarrow & C & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

where all parts except I' and C' are given, and we seek to classify the possible C' (resp. I') to fill in the diagram. The exactness of the bottom two rows results from flatness of B' and C' over A' and (6.2). The exactness of the first column results from flatness of C over A.

Compare this diagram with the diagram preceding (2.3). In that case we had a splitting $B \to B'$ and so were able to show that the possible diagrams were in natural one-to-one correspondence with $\mathrm{Hom}_B(I, B/I)$. In the present case we do not have a splitting. However, we can use a similar reasoning if I' and I'' are two choices of I' to fill in the diagram. Given $x \in I$, lift it to $x' \in I'$ and to $x'' \in I''$. Then $x'' - x' \in B'$ and its image in B is zero. Hence $x'' - x' \in J \otimes_A B$, and we denote its image in $J \otimes_A C$ by $\varphi(x)$. Note that the choices of x' and x'' are not unique. They are defined only up to something in $J \otimes_A I$, but this goes to 0 in $J \otimes_A C$, so φ is a well-defined additive map, in fact a B-linear homomorphism $\varphi \in \mathrm{Hom}_B(I, J \otimes_A C)$.

Conversely given I' and given $\varphi \in \mathrm{Hom}_B(I, J \otimes_A C)$, we define another ideal I'' solving our problem as follows: I'' is the set of $x'' \in B'$ whose image in B is in I, say x, and such that for any lifting x' of x to I', the image of $x'' - x'$ in $J \otimes_A C$ is equal to $\varphi(x)$.

Note finally that if I', I'', I''' are three choices of I', and if φ_1 is defined by I', I'' as above, φ_2 defined by I'', I''', and φ_3 defined by I', I'', then $\varphi_3 = \varphi_1 + \varphi_2$. Thus the operation $\langle I', \varphi \rangle \mapsto I''$ is an action of the group $\mathrm{Hom}_B(I, J \otimes_A C)$ on the set of ideals I' solving our problem, and what we have just shown is that the set of deformations of C over A' is a pseudotorsor for this group action. We have not yet discussed existence, so we cannot assert that it is a torsor.

Note also that $\mathrm{Hom}_B(I, J \otimes_A C) = \mathrm{Hom}_C(I/I^2, J \otimes_A C)$, which can be written $H^0(Y, \mathcal{N}_{Y/X} \otimes_A J)$ in the affine case.

To pass from the affine case to the global case, we note that the action of $H^0(Y, \mathcal{N}_{Y/X} \otimes_A J)$ on the set of Y', which is defined locally, is a natural action, and that it glues together on the overlaps to give a global action of $H^0(Y, \mathcal{N}_{Y/X} \otimes_A J)$ on the set of solutions. The fact of being a pseudotorsor also globalizes, so we have proved a) of the theorem.

To prove b) suppose that local deformations of Y over A' exist. In other words, we assume that there exists an open affine covering $\mathcal{U} = (U_i)$ of X, such that on each U_i there exists a deformation Y'_i of $Y \cap U_i$ in $U'_i \subseteq X'$. Choose one such Y'_i for each i. Then on $U_{ij} = U_i \cap U_j$ there are two extensions $Y'_i \cap U'_{ij}$ and $Y'_j \cap U'_{ij}$. By the previous part a) already proved, these define an element $\alpha_{ij} \in H^0(U_{ij}, \mathcal{N}_{Y/X} \otimes_A J)$. On the intersection U_{ijk} of three open sets, there are three extensions Y'_i, Y'_j, Y'_k, whose differences define elements $\alpha_{ij}, \alpha_{jk}, \alpha_{ik}$, and since by a) the set of extensions is a torsor, we have $\alpha_{ik} = \alpha_{ij} + \alpha_{jk}$ in $H^0(U_{ijk}, \mathcal{N}_{Y/X} \otimes_A J)$. So we see that (α_{ij}) is a 1-cocycle for the covering $\mathcal{U}$ and the sheaf $\mathcal{N}_{Y/X} \otimes_A J$. Finally, note that this cocycle apparently depends on the choices of deformations Y'_i over U_i. If Y''_i is another set of such choices, then Y'_i and Y''_i define an element $\beta_i \in H^0(U_i, \mathcal{N}_{Y/X} \otimes_A J)$, and the new 1-cocycle (α'_{ij}) defined using the Y''_i satisfies $\alpha''_{ij} = \alpha_{ij} + \beta_j - \beta_i$. So the cohomology class $\alpha \in H^1(Y, \mathcal{N}_{Y/X} \otimes_A J)$ is well-defined. It depends only on the given Y over A.

This α is the obstruction to the existence of a global deformation Y' of Y over A'. Indeed, if Y' exists, we can take $Y'_i = Y' \cap U'_i$ for each i. Then $\alpha_{ij} = 0$, so $\alpha = 0$. Conversely, if $\alpha = 0$ in $H^1(Y, \mathcal{N}_{Y/X} \otimes_A J)$, then it is already 0 in the Čech group $\check{H}^1(\mathcal{U}, \mathcal{N}_{Y/X} \otimes_A J)$, so the cocycle α_{ij} must be a coboundary, $\alpha_{ij} = \beta_j - \beta_i$. Then using the β_i, we modify the choices Y'_i to new choices Y''_i which then glue to form a global deformation Y'. This proves b).

Now let us study Situation B, deformations of invertible sheaves. We suppose given a scheme X flat over A and a deformation X' of X over A', where $0 \to J \to A' \to A \to 0$ and $J^2 = 0$. Let $\mathcal{L}$ be a given invertible sheaf on X. We seek to classify deformations of $\mathcal{L}$ over X', that is, invertible sheaves $\mathcal{L}'$ or X' such that $\mathcal{L}' \otimes \mathcal{O}_X \cong \mathcal{L}$.

Theorem 6.4. *In the above situation*

a) *There is an obstruction $\delta \in H^2(J \otimes_A \mathcal{O}_X)$ whose vanishing is a necessary and sufficient condition for the existence of a deformation $\mathcal{L}'$ of $\mathcal{L}$ on X'.*

b) *If a deformation exists, the group $H^1(J \otimes_A \mathcal{O}_X)$ acts transitively on the set of all isomorphism classes of deformations $\mathcal{L}'$ of $\mathcal{L}$ over X'.*

c) *The set of isomorphism classes of such deformations $\mathcal{L}'$ of $\mathcal{L}$ is a torsor under the action of $H^1(J \otimes_A \mathcal{O}_X)$ if and only if the natural map $H^0(\mathcal{O}^*_{X'}) \to H^0(\mathcal{O}^*_X)$ is surjective.*

d) *A sufficient condition for the property of c) to hold is that A is a local Artin ring, $H^0(\mathcal{O}_{X_0}) = k$, where $X_0 = X \times_A k$, and k is the residue field of A.*

Proof. As in the proof of (2.5) the exact sequence

$$0 \to J \otimes \mathcal{O}_X \to \mathcal{O}_{X'} \to \mathcal{O}_X \to 0$$

gives rise to an exact sequence of abelian groups

$$0 \to J \otimes \mathcal{O}_X \to \mathcal{O}^*_{X'} \to \mathcal{O}^*_X \to 0,$$

except that this time there is in general no splitting. Taking cohomology we obtain

$$\begin{aligned} 0 \quad &\to H^0(J \otimes \mathcal{O}_X) \to H^0(\mathcal{O}^*_{X'}) \to H^0(\mathcal{O}^*_X) \to H^1(J \otimes \mathcal{O}_X) \\ &\to H^1(\mathcal{O}^*_{X'}) \to H^1(\mathcal{O}^*_X) \to H^2(J \otimes \mathcal{O}_X) \to \dots\ . \end{aligned}$$

The given invertible sheaf $\mathcal{L}$ on X gives an element in $H^1(\mathcal{O}^*_X)$. Its image δ in $H^2(J \otimes \mathcal{O}_X)$ is the obstruction, which is 0 if and only if $\mathcal{L}$ is the restriction of an element of $H^1(\mathcal{O}^*_{X'})$, i.e., an invertible sheaf $\mathcal{L}'$ on X' with $\mathcal{L}' \otimes \mathcal{O}_X \cong \mathcal{L}$. Clearly $H^1(J \otimes \mathcal{O}_X)$ acts on the set of such $\mathcal{L}'$, but we cannot assert that it is a torsor unless the previous map $H^0(\mathcal{O}^*_{X'}) \to H^0(\mathcal{O}^*_X)$ is surjective.

Suppose now that A is a local Artin ring and $H^0(\mathcal{O}_{X_0}) = k$. Using induction on the length of A, one sees therefore that $H^0(\mathcal{O}_X) = A$ and $H^0(\mathcal{O}_{X'}) = A'$. Since $A'^* \to A^*$ is surjective, the conditions of c) follow.

Remark 6.5. One can interpret $H^0(\mathcal{O}^*_X)$ as the group of automorphisms of the invertible sheaf $\mathcal{L}$. Thus the condition of c) can be written $\operatorname{Aut} \mathcal{L}' \to \operatorname{Aut} \mathcal{L}$ is surjective. This type of condition on automorphisms appears frequently in deformation questions (cf. Chapter 3).

7 Obstruction theory for a local ring

Let A, m be a local ring with residue field k. We want to investigate properties of A in terms of homomorphism of A to local Artin rings and how these homomorphisms lift to larger Artin rings. This will be useful for studying local properties of the scheme representing a functor, for example the Hilbert scheme, because it allows us to translate properties of the local ring on the representing scheme into properties of the functor applied to local Artin rings.

We have already seen one case of this kind of analysis, where the property of A being a regular local ring is characterized by always being able to lift maps into Artin rings—the infinitesimal lifting property of smoothness (or regularity).

In this section we take this analysis one step further, by considering cases when the homomorphisms do not always lift. For this purpose we define the notion of an obstruction theory.

Definition 7.1. Let $A, \mathfrak{m}$ be a local ring with residue field k. We will consider sequences $0 \to J \to C' \to C \to 0$ where C' is a local Artin ring with residue field k, J is an ideal, $C = C'/J$ is the quotient, and J satisfies $\mathfrak{m}_{C'} J = 0$, so that J becomes a k-vector space. An *obstruction theory* for A is a vector space V over k, together with, for every sequence $0 \to J \to C' \to C \to 0$ as above, and for every homomorphism $u : A \to C$,

$$\begin{array}{ccccccccc} & & & & & & A & & \\ & & & & u' \swarrow & & \downarrow u & & \\ 0 & \to & J & \to & C' & \to & C & \to & 0 \end{array}$$

an element $\varphi(u, C') \in V \otimes J$, satisfying two properties:

a) $\varphi(u, C') = 0$ if and only if u lifts to a map $u' : A \to C'$,

b) φ is functorial in the sense that if $K \subseteq J$ is a subspace, then the element $\varphi(u, C'/K)$ associated with u and the sequence $0 \to J/K \to C'/K \to C \to 0$ is just the image of $\varphi(u, C')$ under the natural map $V \otimes J \to V \otimes J/K$.

Example 7.2. Suppose that A is a quotient of a regular local ring P by an ideal I, and assume that $I \subseteq \mathfrak{m}_P^2$. Then we can construct an obstruction theory for A as follows. Take V to be the dual vector space $(I/\mathfrak{m}_P I)^*$. Given a diagram as in the definition, we can always lift u to a homomorphism $f : P \to C'$, since P is regular (4.4)

$$\begin{array}{ccccccccc} 0 & \to & I & \to & P & \to & A & \to & 0 \\ & & \downarrow \bar{f} & & \downarrow f & & \downarrow u & & \\ 0 & \to & J & \to & C' & \to & C & \to & 0. \end{array}$$

This induces a map $\bar{f} : I \to J$, which factors through $I/\mathfrak{m}I$, since J is a k-vector space. This gives us an element $\varphi \in \operatorname{Hom}(I/\mathfrak{m}I, J) \cong V \otimes J$.

We need to show that φ is independent of the choice of lifting f. So let $f' : P \to C'$ be another lifting. We will show that $\bar{f}' - \bar{f} = 0$. Since $I \subseteq \mathfrak{m}_P^2$, it will be sufficient to show for any $x, y \in \mathfrak{m}_P$ that $f'(xy) - f(xy) = 0$. Since f' and f both lift u, we have $f'(y) - f(y) \in J$. Also since f' is a local homomorphism, $f'(x) \in \mathfrak{m}_{C'}$. But $\mathfrak{m}_{C'} J = 0$, so $f'(x)[f'(y) - f(y)] = 0$. Similarly, $[f'(x) - f(x)]f(y) = 0$. Adding these two, $f'(x)f'(y) - f(x)f(y) = 0$, which is what we want (remembering that f, f' are ring homomorphisms).

Now condition a) is clear: if $\varphi(u, C') = 0$, then f factors through A, so that u lifts. Conversely, if u lifts, this gives a lifting $f' : P \to C'$ that vanishes on I, so $\varphi = 0$.

Condition b) is obvious by construction.

Note that in this example, $\dim V = \dim(I/\mathfrak{m}I)$, which is the minimal number of generators of I.

Our main result is a converse to this example.

Theorem 7.3. *Let A be a local ring that can be written as $A \cong P/I$ with P regular local and $I \subseteq \mathfrak{m}_P^2$, and let (V, φ) be an obstruction theory for A. Then I can be generated by at most* $\dim V$ *elements.*

Proof. Note first we cannot expect to get the exact number of generators for I, because if V, φ is an obstruction theory, any bigger vector space V' containing V will also be one.

So consider any sequence $0 \to J \to C' \to C \to 0$ and map u as in the definition

$$\begin{array}{ccccccccc} 0 & \to & I & \to & P & \to & A & \to & 0 \\ & & \downarrow \bar{f} & & \downarrow f & & \downarrow u & & \\ 0 & \to & J & \to & C' & \to & C & \to & 0. \end{array}$$

As in Example 7.2, we lift u to a map $f : P \to C'$ and get a restriction $\bar{f} : I \to J$ which does not depend on the choice of f, and which is zero if and only if u lifts to a map $u' : A \to C'$. We also get an element $\varphi(u, C') \in V \otimes J$. Let $s = \dim V$ and choose a basis $v_1, \ldots, v_s$ for V. Then we can write $\varphi(u, C') = \sum v_i \otimes a_i$ for suitable $a_1, \ldots, a_s \in J$. Let K be the subspace of J generated by $a_1, \ldots, a_s$. Passing to the quotient sequence

$$0 \to J/K \to C'/K \to C \to 0,$$

according to the functoriality of φ, the induced element $\varphi(u, C'/K)$ is zero, so u lifts to give $u' : A \to C'/K$, and this implies $\bar{f}(I) \subseteq K$ and hence $\dim \bar{f}(I) \leq s$. This holds for any sequence $0 \to J \to C' \to C \to 0$ as above.

Now we apply this conclusion in a particular case. We consider the sequence, for any integer n,

$$0 \to (I + \mathfrak{m}^n)/(\mathfrak{m}I + \mathfrak{m}^n) \to P/(\mathfrak{m}I + \mathfrak{m}^n) \to A/\mathfrak{m}^n \to 0,$$

together with the natural quotient map $u : A \to A/\mathfrak{m}^n$. We conclude that the corresponding $\bar{f}(I)$ has dimension $\leq s$. But in this case $\bar{f}$ is surjective, so our conclusion is, for any n, that

$$\dim(I + \mathfrak{m}^n)/(\mathfrak{m}I + \mathfrak{m}^n) \leq s.$$

By a standard isomorphism theorem,

$$(I + \mathfrak{m}^n)/(\mathfrak{m}I + \mathfrak{m}^n) \cong I/I \cap (\mathfrak{m}I + \mathfrak{m}^n) = I/(\mathfrak{m}I + (I \cap \mathfrak{m}^n)).$$

By the Artin–Rees lemma, we have $I \cap \mathfrak{m}^n \subseteq \mathfrak{m}I$ for $n \gg 0$. Hence $\dim I/\mathfrak{m}I \leq s$, and I is generated by at most s elements.

Corollary 7.4. *Let $A, \mathfrak{m}$ be a local ring that can be written as a quotient of a regular local ring, of embedding dimension $n = \dim \mathfrak{m}/\mathfrak{m}^2$. If A has an obstruction theory in a vector space V, then $\dim A \geq n - \dim V$. Furthermore, if equality holds, then A is a local complete intersection.*

Proof. Indeed, $\dim A \geq \dim P - \#$ generators of I, and equality makes A a local complete intersection ring by definition.

Remark 7.5. As an application of this result, we will prove the classical result, stated by M. Noether [62, I,§2] for non-singular curves in $\mathbb{P}^3$, that every component of the Hilbert scheme of locally Cohen–Macaulay curves of degree d in $\mathbb{P}^3$ has dimension $\geq 4d$ (see §11 below).

References for this section. The results of this section are certainly consequences of the general deformation theories of Laudal [46], Illusie [34], Rim [73], but for a more direct approach, I have given a simplified version of the proof due to Mori [54, Prop. 3]. A slightly different proof appears in the book of Kollár [44, p. 32] hidden in a thicket of notations.

8 Cohen–Macaulay in codimension two

We have seen that smooth schemes have no non-trivial deformations locally (4.7). So we often consider deformations of a closed subscheme of a smooth scheme, so-called embedded deformations. One case we can handle well is the case of a subscheme of codimension two that is Cohen–Macaulay. In this case there is a structure theorem that allows us to track the deformations nicely. We will see that a codimension two Cohen–Macaulay scheme is locally defined by the $r \times r$ minors of an $r \times (r+1)$ matrix of functions, and that deforming the subscheme corresponds to deforming the entries of the matrix.

We start with the local case. Let A be a regular local ring of dimension n, and let $B = A/\mathfrak{a}$ be a quotient of dimension $n-2$ that is Cohen–Macaulay, i.e., depth $B = n-2$. We make use of the theorem that says if M is a finitely generated module over a regular local ring A, then depth $M + \operatorname{hd} M = n$, where $\operatorname{hd} M$ is the homological dimension of n. Thus we see $\operatorname{hd} B = 2$ as an A-module. If we take a minimal set of generators $a_1, \ldots, a_{r+1}$ for $\mathfrak{a}$, then we get a resolution

$$0 \to A^r \xrightarrow{\varphi} A^{r+1} \xrightarrow{\alpha} A \to B \to 0, \tag{1}$$

where φ is an $r \times (r+1)$ matrix of elements φ_{ij} in A, and α is the map defined by $a_1, \ldots, a_{r+1}$. Let f_i be $(-1)^i$ times the determinant of the i-th $r \times r$ minor of the matrix φ, and let $f : A^{r+1} \to A$ be defined by the f_i. Then we obtain a complex

$$A^r \xrightarrow{\varphi} A^{r+1} \xrightarrow{f} A, \tag{2}$$

because evaluating the product $f \circ \varphi$ amounts to taking determinants of $(r+1) \times (r+1)$ matrices with a repeated column, hence is zero.

We will show that the map f is the same as α, up to a unit in A. Looking at the generic point of $\operatorname{Spec} A$, i.e., tensoring with the quotient field K of A, since φ is injective, it has rank r, and at least one of its $r \times r$ minors is non-zero. Thus the map f is non-zero. Then looking at the ranks of the modules in the complex (2), we see that the homology in the middle must have rank 0. But from (1) we know that coker φ has no torsion, hence (2) is exact in the middle. Therefore the ideal $\mathfrak{a} = (a_1, \ldots, a_{r+1})$ and the ideal $(f_1, \ldots, f_m)$ are isomorphic as A-modules.

Since B has codimension 2, if we look at a point of codimension 1 in $\operatorname{Spec} A$, the resolution (1) is split exact, so one of the f_i's is a unit. Hence the support of $A/(f_1, \ldots, f_{r+1})$ is also in codimension ≥ 2. An isomorphism between ideals of A is given by multiplication by some non-zero element of A. Since both $\mathfrak{a}$ and $(f_1, \ldots, f_{r+1})$ define subsets of codimension 2, this element must be a unit in A. So, up to a change of basis, we find $\mathfrak{a} = (f_1, \ldots, f_{r+1})$. Thus we have proved the following theorem.

Theorem 8.1 (Hilbert, Burch). *Let A be a regular local ring of dimension n. Let $B = A/\mathfrak{a}$ be a Cohen–Macaulay quotient of codimension 2. Then there is an $r \times (r+1)$ matrix φ of elements of A, whose $r \times r$ minors $f_1, \ldots, f_{r+1}$ generate the ideal $\mathfrak{a}$, and there is a resolution*

$$0 \to A^r \xrightarrow{\varphi} A^{r+1} \xrightarrow{f} A \to B \to 0$$

of B over A.

Our next task is to study deformations of codimension 2 Cohen–Macaulay subschemes. We will be looking now at rings over an Artin ring as base, and we put conditions on them by specifying that they should be flat over the Artin ring and have desired properties along the closed fiber. We will see that the same structure theorem persists in this situation, so that we can always extend deformations of codimension 2 Cohen–Macaulay subschemes in this local setting, by lifting the elements of the matrix φ.

Here is the situation. We suppose given C' a local Artin ring with residue field k, an ideal $J \subseteq C'$, and its quotient $C = C'/J$. Suppose given also A' a local C'-algebra, flat over C', and with $A' \otimes_{C'} k$ a regular local ring. Let $A = A' \otimes_{C'} C$. Suppose also given $B = A/\mathfrak{a}$, flat over C, and with $B \otimes_C k$ a Cohen–Macaulay codimension 2 quotient of $A \otimes_C k$. The problem is to lift B, that is, to find a quotient $B' = A'/\mathfrak{a}'$, flat over C', with $B' \otimes_{C'} C = B$.

Theorem 8.2. *In the above situation we have*

(1) *There is an $r \times (r+1)$ matrix φ of elements of A whose $r \times r$ minors f_i generate $\mathfrak{a}$ and which gives a resolution*

$$0 \to A^r \xrightarrow{\varphi} A^{r+1} \xrightarrow{f} A \to B \to 0$$

(2) *If φ' is any lifting of the matrix φ to elements of A' and the f_i' are its minors, then the sequence*

$$0 \to A'^r \xrightarrow{\varphi'} A'^{r+1} \xrightarrow{f'} A' \to B' \to 0$$

is exact and defines a quotient B', flat over C', and with $B' \otimes_{C'} C = B$.

(3) *Any lifting of B, i.e., a quotient $B' = A'/\mathfrak{a}'$ with B' flat over C' and $B' \otimes_{C'} C = B$ arises by lifting the matrix φ, as in (2).*

Proof. Using induction on the length of C', we may assume $J^2 = 0$, or even that $J \cong k$, as needed.

We start with (2). Let φ' be any lifting of φ. Then we can consider the complex

$$L'_\bullet : A'^r \xrightarrow{\varphi'} A'^{r+1} \xrightarrow{f'} A'.$$

This is a complex for the same reason as given in the proof of Theorem 8.1 above—composition of φ' and f' amounts to evaluating determinants with a repeated column.

Since A' is flat over C', we can tensor with the exact sequence

$$0 \to J \to C' \to C \to 0$$

and obtain an exact sequence of complexes

$$0 \to L'_\bullet \otimes J \to L'_\bullet \to L'_\bullet \otimes C \to 0. \tag{3}$$

Since φ' is a lifting of φ, the complex $L'_\bullet \otimes C$ is just the complex that appears in (1), namely

$$L_\bullet : A^r \xrightarrow{\varphi} A^{r+1} \xrightarrow{f} A.$$

Since $J^2 = 0$, J is a C-module, and so $L'_\bullet \otimes_{C'} J = L_\bullet \otimes_C J$.

Because of (1), $L_\bullet$ is exact, with cokernel B. Since B is flat over C, the complex $L_\bullet \otimes_C J$ is exact with cokernel $B \otimes_C J$. Now the long exact sequence of homology of the sequence of complexes (3) shows that $L'_\bullet$ is exact. We call its cokernel B':

$$0 \to A'^r \xrightarrow{\varphi'} A'^{r+1} \xrightarrow{f'} A' \to B' \to 0. \tag{4}$$

Also the homology sequence of (3) shows that B' belongs to an exact sequence

$$0 \to B \otimes_C J \to B' \to B \to 0. \tag{5}$$

Now B is flat over C, and the sequence (5), obtained by tensoring B with $0 \to J \to C' \to C \to 0$, shows that $\mathrm{Tor}_1^{C'}(B', C) = 0$, so by the Local Criterion of Flatness (6.2) we find B' flat over C'. Hence, by (4) we find $B' \otimes_{C'} C = h_0(L'_\bullet \otimes_{C'} C) = h_0(L_\bullet) = B$, and B' is a lifting of B, as required. This completes the proof of (2).

Next we prove (3). Let $B' = A'/\mathfrak{a}'$ be a lifting of B. Lift the elements $f_i \in \mathfrak{a}$ to $g_i \in \mathfrak{a}'$. By Nakayawa's lemma, these will generate $\mathfrak{a}'$, so we can write a resolution, with kernel M

$$0 \to M \to A'^{r+1} \xrightarrow{g} A' \to B' \to 0.$$

Since B' is flat over C', so is M. And since B' lifts B and the g_i lift f_i, $M \otimes_{C'} C \cong A^r$. Hence M is free, equal to A'^r, so we get a resolution

$$0 \to A'^r \xrightarrow{\varphi'} A'^{r+1} \xrightarrow{g} A' \to B' \to 0$$

for a suitable matrix φ' lifting φ. But from (2) we also have

$$0 \to A'^r \xrightarrow{\varphi'} A'^{r+1} \xrightarrow{f'} A' \to B'' \to 0$$

where B'' is another lifting of B. We must show $B' = B''$. First we need a lemma.

Lemma 8.3. *Let A' be a C'-algebra flat over C', with $A' \otimes_C k$ normal. Let $Z \subseteq \operatorname{Spec} X = \operatorname{Spec} A'$ be a subset of codimension ≥ 2. Then $H^0(X - Z, \mathcal{O}_X) = A'$.*

Proof. By induction on length C', the case of length 1 being known, since then A' is normal. We may assume $J \cong k$. Then the result follows inductively, using the sheaf sequence associated to the exact sequence of modules

$$0 \to A' \otimes_{C'} J \to A' \to A' \otimes_{C'} C \to 0.$$

To complete the proof of (3), we note from the sequences above that the ideals $\mathfrak{a}' = (g_1, \ldots, g_{r+1})$ and $\mathfrak{a}'' = (f'_1, \ldots, f'_{r+1})$ are isomorphic as A'-modules. Let $X = \operatorname{Spec} A'$ and $Z = \operatorname{Supp} B$ (which is also the support of B' and B''). Restricting our isomorphism to $X - Z$ gives a section of $\mathcal{O}_X$, which by the lemma, is an element of A'. Since both B' and B'' have codimension 2, it must be a unit. So, up to change of basis, the $g_i = f_i'$ as required.

Finally, to prove (1), we use induction on the length of C', assuming $J \cong k$, and use (3) at each step, starting from the case length 1, which is Theorem 8.1 above.

Remark 8.3.1. Since we can always lift the deformations in this local ring case, we say deformations of codimension 2 Cohen–Macaulay subschemes are *locally unobstructed* (precise definition of unobstructed elsewhere).

Remark 8.3.2. The proof of the global analogue of Theorem 8.2 given by Ellingsrud [18], following Peskine and Szpiro [69] is slightly different. By formulating the Hilbert–Burch theorem over a more general local ring, and using the more subtle theorem of Auslander that for a module M of finite type and finite homological dimension over a local ring A, depth $M + \operatorname{hd} M =$ depth A, they avoid some of the complicated induction in the proof of Theorem 8.2. I preferred to limit the homological algebra to the regular local ring, and then proceed by induction.

The affine case

Suppose now that X is a smooth affine scheme over a field k, and that $Y \subseteq X$ is a closed subscheme of codimension 2 that is Cohen–Macaulay, i.e., all the local rings $\mathcal{O}_{y,Y}$ for $y \in Y$ are Cohen–Macaulay local rings. We cannot expect that Y should be globally defined by minors of a matrix of functions on X, but we can accomplish this on small open affines.

Theorem 8.4. *Let X be a smooth scheme over a field k, and let $Y \subseteq X$ be a closed Cohen–Macaulay subscheme of codimension 2. Then for each point $y \in Y$ there is an open affine neighborhood U of y in X and a matrix φ of regular functions on U so that the maximal minors f_i of φ generate the ideal of $Y \cap U$ and there is a resolution*

$$0 \to \mathcal{O}_U^r \xrightarrow{\varphi} \mathcal{O}_U^{r+1} \xrightarrow{f} \mathcal{O}_U \to \mathcal{O}_{Y \cap U} \to 0.$$

Proof. We apply Theorem 8.1 to the local ring $\mathcal{O}_{y,X}$ and its quotient $\mathcal{O}_{y,Y}$. This gives a matrix φ of elements of $\mathcal{O}_{y,X}$ as in Theorem 8.1.

These elements are all defined on some open affine neighborhood U of y and so determine a complex

$$\mathcal{O}_U^r \xrightarrow{\varphi} \mathcal{O}_U^{r+1} \xrightarrow{f} \mathcal{O}_U.$$

This complex is exact at y, hence also in a neighborhood of y. The elements f_i generate the ideal of Y at y, hence also in a neighborhood. So by shrinking U to a smaller affine neighborhood, we obtain the result.

For the deformation theory, once we have a resolution as in Theorem 8.3, the deformation can be accomplished over that same affine.

Theorem 8.5. (Schaps [74]) *Suppose given a smooth affine scheme X_0 over a field k and a codimension 2 Cohen–Macaulay subscheme Y_0 having a resolution*

$$0 \to \mathcal{O}_{X_0}^r \xrightarrow{\varphi_0} \mathcal{O}_{Y_0}^{r+1} \xrightarrow{f_0} \mathcal{O}_{X_0} \to \mathcal{O}_{Y_0} \to 0$$

as in Theorem 8.3. Suppose also given a local Artin ring C', an ideal $J \subseteq C'$ and its quotient $C = C'/J$. Suppose given X' flat over C' with $X' \times_{C'} k = X_0$, and let $X = X' \times_{C'} C$. Suppose given a closed subscheme $Y \subseteq X$, flat over C, such that $Y \times_C k = Y_0$. Then

(1) *There is a matrix φ of global sections of $\mathcal{O}_X$, and a resolution*

$$0 \to \mathcal{O}_X^r \xrightarrow{\varphi} \mathcal{O}_X^{r+1} \xrightarrow{f} \mathcal{O}_X \to \mathcal{O}_Y \to 0$$

where the f_i are the maximal minors of φ.

(2) *If φ' is any lifting of φ to sections of $\mathcal{O}_{X'}$, then the sequence*

$$0 \to \mathcal{O}_{X'}^r \xrightarrow{\varphi'} \mathcal{O}_{X'}^{r+1} \xrightarrow{f'} \mathcal{O}_{X'} \to \mathcal{O}_{Y'} \to 0$$

is exact and defines a closed subscheme $Y' \subseteq X'$, flat over C', with $Y' \times_{C'} C = Y$.

(3) *Any lifting of Y to a $Y' \subseteq X'$, flat over C', with $Y' \times_{C'} C = Y$ arises by lifting the matrix φ, as in (2).*

Proof. Same as the proof of Theorem 8.2.

Remark 8.5.1. In case $X = \mathbb{A}^n_k$, Schaps shows using the fact that a projective module on a polynomial ring is stably free, that one can get a global resolution, as in Theorem 8.3, over all of X. She then shows that all infinitesimal deformations of Y in X are given by lifting the matrix φ, as in Theorem 8.4.

The global projective case

This time we consider $X = \mathbb{P}^n_k$, the projective space over a field k, and a closed subscheme $Y \subseteq X$ of codimension 2. To obtain results analogous to the local and affine cases, we must put a global hypothesis on Y.

Definition 8.6. A closed subscheme $Y \subseteq \mathbb{P}^n_k$ is *arithmetically Cohen–Macaulay* (ACM) if its homogeneous coordinate ring R/I_Y is a (graded) Cohen–Macaulay ring. Here $R = k[x_0, \ldots, x_n]$, and I_Y is the (saturated) homogeneous ideal of Y.

Proposition 8.7. *Let $Y \subseteq X = \mathbb{P}^n_k$ be a closed subscheme. If $\dim Y = 0$, then Y is* ACM. *If $\dim Y \geq 1$, the following conditions are equivalent.*

(i) Y *is* ACM.

(ii) $R \to H^0_*(\mathcal{O}_Y)$ *is surjective, and* $H^i_*(\mathcal{O}_Y) = 0$ *for* $0 < i < \dim Y$.

(iii) $H^i_*(\mathcal{I}_Y) = 0$ *for* $0 < i \leq \dim Y$.

Proof. Let $\mathfrak{m} = (x_0, \ldots, x_n)$ be the irrelevant prime ideal of R. If $\dim Y = 0$, then $\dim R/I_Y = 1$, and since I_Y is the saturated ideal, it does not have $\mathfrak{m}$ as an associated prime. Hence R/I_Y has depth 1, and is a Cohen–Macaulay ring, so Y is ACM.

For $\dim Y \geq 1$, we use the exact sequence

$$0 \to H^0_{\mathfrak{m}}(R/I_Y) \to R/I_Y \to H^0_*(\mathcal{O}_Y) \to H^1_{\mathfrak{m}}(R/I_Y) \to 0$$

and the isomorphisms for $i > 0$

$$H^i_*(\mathcal{O}_Y) \cong H^{i+1}_{\mathfrak{m}}(R/I_Y),$$

together with the local cohomology criterion for depth, to obtain the equivalence of (i) and (ii).

The equivalence of (ii) and (iii) follows from the cohomology of the exact sequence

$$0 \to \mathcal{I}_Y \to \mathcal{O}_X \to \mathcal{O}_Y \to 0.$$

Theorem 8.8. *Let $Y \subseteq X = \mathbb{P}^n_k$ be an* ACM *closed subscheme of codimension* 2*. Then there is an $r \times (r+1)$ matrix φ of homogeneous elements of R whose $r \times r$ minors f_i generate I_Y, giving rise to a resolution*

$$0 \to \bigoplus_{i=1}^{r} R(-b_i) \xrightarrow{\varphi} \bigoplus_{i=1}^{r+1} R(-a_i) \xrightarrow{f} R \to R/I_Y \to 0.$$

Proof. Since R/I_Y is Cohen–Macaulay and a quotient of codimension 2 of R, it has homological dimension 2 over R. The proof then follows exactly as in the proof of the local case (Theorem 8.1), using the graded analogues of depth and homological dimension from the local case.

Although this result follows exactly as in the local case, when it comes to deformations, there is a new ingredient. We consider deformations of the subscheme Y, and some extra work is required to show that these give rise to deformations of the ring R/I_Y, so that we can apply the techniques we used in the local case. This extra work is contained in the following proposition, which only applies when $\dim Y \geq 1$.

Proposition 8.9. *Let $Y_0 \subseteq X_0 = \mathbb{P}^n_k$ be a closed subscheme, and assume that depth $R_0/I_0 \geq 2$, where $R_0 = k[x_0, \dots, x_n]$ and I_0 is the homogeneous ideal of Y_0. Let C be a local Artin ring with residue field k, let $X = \mathbb{P}^n_C$, and let $Y \subseteq X$ be a closed subscheme, flat over C, with $Y \times_C k = Y_0$. Let $R = C[x_0, \dots, x_n]$, and let $I \subseteq R$ be the ideal of Y. Then*

1) $H^1_*(\mathcal{I}_Y) = 0$

2) $R/I \cong H^0_*(\mathcal{O}_Y)$

3) R/I *is flat over* C

4) $R/I \otimes_C k = R_0/I_0$.

Proof. 1) We use induction on the length of C. If $C = k$, the result follows from depth $R_0/I_0 \geq 2$ and the exact sequence in the proof of Proposition 8.7. For the induction step, let

$$0 \to J \to C' \to C \to 0$$

with $J \cong k$, and suppose $Y' \subseteq X' = \mathbb{P}^n_{C'}$ flat over C' as above. Then $\mathcal{I}_{Y'}$ is also flat, so we get an exact sequence

$$0 \to \mathcal{I}_{Y'} \otimes J \to \mathcal{I}_{Y'} \to \mathcal{I}_Y \to 0.$$

Now the exact sequence of H^1_* and the induction hypothesis show $H^1_*(\mathcal{I}_{Y'}) = 0$.

2) This follows from the exact sequence

$$0 \to \mathcal{I}_Y \to \mathcal{O}_X \to \mathcal{O}_Y \to 0$$

which gives

$$0 \to H^0_*(\mathcal{I}_Y) \to R \to H^0_*(\mathcal{O}_Y) \to H^1_*(\mathcal{I}_Y) \to 0$$

and 1) above, since $H^0_*(\mathcal{I}_Y) = I$.

For 3) and 4), we use the isomorphism of Y. Tensoring with k we obtain a diagram

$$\begin{array}{ccc} R/I \otimes_C k & \cong & H^0_*(\mathcal{O}_Y) \otimes_C k \\ \downarrow & & \downarrow \beta \\ R/I_0 & \cong & H^0_*(\mathcal{O}_{Y_0}). \end{array}$$

Consider the functor, for any C-module M, $T(M) = H^0_*(\mathcal{O}_Y \otimes_C M)$, and consider the natural maps $\varphi(M) : T(C) \otimes_C M \to T(M)$. Note the functor T is a direct sum of functors $T_\nu(M) = H^0(\mathcal{O}_Y(\nu) \otimes M)$ for $\nu \in \mathbb{Z}$, so we can apply the theory of cohomology and base extension [27, III §12].

Note that α is surjective, since both terms are quotients of R. Hence β is surjective. But β is just $\varphi(k) : T(C) \otimes k \to T(k)$. Hence by [loc. cit. 12.10], the functor T is right exact. But it is also left exact, being an H^0, so T is exact. Hence by [loc. cit. 12.5], $\varphi(M)$ is an isomorphism for all M. This gives property 4). Also by [loc. cit. 12.6], $T(C) = R/I_n$ is flat over C, which is 3).

Now we can study deformations of codimension 2 ACM subschemes of $\mathbb{P}^n$.

Theorem 8.10 (Ellingsrud [18]). *Let $Y_0 \subseteq X_0 = \mathbb{P}^n_k$ be an* ACM *closed subscheme of codimension 2, and assume* $\dim Y_0 \geq 1$. *Suppose given a local Artin ring C', an ideal $J \subseteq C'$ and its quotient $C = C'/J$. Suppose given a closed subscheme $Y \subseteq X = \mathbb{P}^n_C$, flat over C and with $Y \times_C k = Y_0$. Then*

1) *There is an $r \times (r+1)$ matrix φ of homogeneous elements of $R = C[x_0, \ldots, x_n]$ whose $r \times r$ minors f_i generate the ideal I of Y, and give a resolution*

$$0 \to \oplus R(-b_i) \xrightarrow{\varphi} \oplus R(-a_i) \xrightarrow{f} R \to R/I \to 0.$$

2) *For any lifting φ' of φ to $R' = C'[x_0, \ldots, x_n]$, taking f' to be the $r \times r$ minors given an exact sequence*

$$0 \to \oplus R'(-b_i) \xrightarrow{\varphi'} R'(-a_i) \xrightarrow{f'} R' \to R'/I' \to 0$$

and defines a closed subscheme $Y' \subseteq X' = \mathbb{P}^n_{C'}$, flat over C', with $Y' \times_{C'} C = Y$, and

3) *Any lifting Y' of Y to X', flat over C', with $Y' \times_{C'} C = Y$ arises by lifting φ as in* 2).

Proof. We use Proposition 8.9 to conclude from the hypothesis Y flat over C that R/I is flat over C and $R/I \otimes k = R_0/I_0$. Since the reverse implication is trivial, we reduce to studying deformations of R_0/I_0, that is, ideals $I \subseteq R$ such that R/I is flat over C and $R/I \otimes k = R_0/I_0$. Now the proof of Theorem 8.2, adapted from the local to the graded case, proves the theorem.

Note that we needed the hypothesis $\dim Y_0 \geq 1$ to get depth $R_0/I_0 \geq 2$ to be able to apply Proposition 8.9.

Example 8.10.1. The conclusions of Theorem 8.10 are false in the case of a scheme Y_0 of dimension 0 in $\mathbb{P}^n$. For example, let Y_0 be a set of three collinear points in $\mathbb{P}^2$. Then there is a linear form in the ideal of Y_0. But as you deform the points in the direction of a set of three non-collinear points of $\mathbb{P}^2$, the linear form does not lift. So the deformations of Y_0 cannot all be obtained by lifting the elements of the corresponding matrix φ.

Corollary 8.11. *The Hilbert scheme at a point corresponding to a codimension* 2 ACM *closed subscheme* $Y \subseteq \mathbb{P}^n_k$ *is smooth.*

Proof. If $n = 2$, and Y is a zero scheme, then Y is contained in an affine $\mathbb{A}^2$ and Theorem 8.5 shows the deformation space is smooth. This result first appeared in Fogarty [20]. The case $\dim Y \geq 1$ follows from Theorem 8.10 and the infinitesimal lifting property of smoothness.

Remark. The representation of the ideal I of a codimension 2 closed subscheme of $X = \mathbb{P}^n_k$ given in Theorem 8.8 allows one to compute the Hilbert polynomial of Y and other numerical invariants. Ellingsrud [18] uses this information to show that the component of the Hilbert scheme with fixed numerical invariants is irreducible, and he can also compute its dimension. We refer to his paper for details.

References for this section. The statement that under certain conditions an ideal I is generated by the $r \times r$ minors of an $r \times (r+1)$ matrix, now currently known as the Hilbert–Burch theorem, appears in many forms in the literature. Hilbert, in his fundamental paper [31], where he first proves the finite generation of an ideal in a polynomial ring ("Hilbert basis theorem"), the existence of a finite free resolution for a homogeneous ideal in a polynomial ring ("Hilbert syzygy theorem"), defines the characteristic function ("Hilbert function") of an ideal, and proves that it is a polynomial ("Hilbert polynomial") for large integers, as an application of his methods, gives the structure of a homogeneous ideal I in the polynomial ring $k[x_1, x_2]$ by showing that if it is generated by $f_1, \ldots, f_n$ with no common factor, then the f_i are, up to a scalar, the $(n-1) \times (n-1)$ minors of an $(n-1) \times n$ matrix.

Burch [8] proved the same structure theorem for an ideal of homological dimension one in a local ring, referring to an earlier paper of his for the case of a local domain.

Buchsbaum [6, 3.4] proved the result for an ideal of homological dimension one in a local UFD, as a consequence of a more general result of his.

As far as I can tell from the published record, the three above discoveries are all independent of each other, except that Burch and Buchsbaum were both working within a larger context of rapidly developing results in homological algebra involving many other mathematicians.

Peskine and Szpiro [69, 3.3] extended Buchbaum's result to the case of an arbitrary local ring. Ellingsrud [18] used the result of Peskine–Szpiro in his study of Cohen–Macaulay schemes of codimension 2 in $\mathbb{P}^n$. The deformation theory in this case is new with Ellingsrud.

Meanwhile Burch's result is given as an exercise in Kaplansky [35, p. 148]. Schaps [74] gives a proof of this exercise for the case of a Cohen–Macaulay subscheme of codimension 2 in $\mathbb{A}^n$, and studies the deformation theory, new with her in this case. Artin [3] gives an account of the result, with deformation theory, saying it is due to Hilbert and Schaps.

By the time of Eisenbud's book [15, p. 502] the result for an ideal of homological dimension one in a local ring appears as the "Hilbert–Burch" theorem. Eisenbud's proof is a consequence of his more general theory of "what makes a complex exact".

For a refinement of Proposition 8.9 of this section, see Piene and Schlessinger [70].

9 Complete intersections and Gorenstein in codimension three

As in the previous section, Cohen–Macaulay in codimension two, there are some other situations in which the particular structure of the resolution of an ideal allows us to show that all deformations have the same structure, and that deformations can always be extended by lifting the corresponding resolutions. These are the case of complete intersections and Gorenstein schemes in codimension 3.

Complete intersections

Proposition 9.1. *Let A be a local Cohen–Macaulay ring, let $a_1, \dots, a_r$ be elements of A, and let $\mathfrak{a} = (a_1, \dots, a_r)$, and let $B = A/\mathfrak{a}$. The following conditions are equivalent:*

(i) *$a_1, \dots, a_r$ is a regular sequence in A*

(ii) $\dim B = \dim A - r$

(iii) *The Koszul complex $K_\bullet(a_1, \ldots, a_r)$ is exact and so gives a resolution of B over A.*

Proof. (ref. to Matsumma or Serre).

Definition 9.2. In case the equivalent conditions of the proposition are satisfied, we say that $\mathfrak{a}$ is a *complete intersection ideal* in A, or that B is a *complete intersection quotient* of A.

Thus already in our definition of complete intersection, we have the resolution

$$0 \to \wedge^r A^r \to \wedge^{r-1} A^r \to \cdots \to A^r \to A \to B \to 0.$$

The *Koszul complex* of $a_1, \ldots, a_r$ is all of this except the B at the right.

We will see that deformations of a complete intersection correspond to lifting the generators of the ideal, and that the Koszul resolution follows along.

Theorem 9.3. *Suppose given C' a local Artin ring, J an ideal, and $C = C'/J$. Suppose given A' flat over C' such that $A_0 = A' \otimes_{C'} k$ is a local Cohen–Macaulay ring. And suppose given $B = A/\mathfrak{a}$, a quotient of $A = A' \times_{C'} C$, flat over C, such that $B_0 = B \otimes_C k$ is a complete intersection quotient of A_0 of codimension r. Then*

1) *$\mathfrak{a}$ can be generated by r elements $a_1, \ldots, a_r$, and the Koszul complex $K_\bullet(A; a_1, \ldots, a_r)$ gives a resolution of B.*

2) *If $a'_1, \ldots, a'_r$ are any liftings of the a_i to A', then the Koszul complex $K_\bullet(A'; a'_1, \ldots, a'_r)$ is exact and defines a quotient $B' = A'/\mathfrak{a}'$, flat over C', with $B' \times_{C'} C = B$.*

3) *Any lifting of B to a quotient of A', flat over C', such that $B' \times_{C'} C = B$ arises by lifting the a_i, as in* 2).

Proof. The proof follows the plan of proof of (8.2) except that it is simpler. We may assume $J^2 = 0$.

For 2), suppose given the situation of 1) and let $a_1', \ldots, a_r'$ be liftings of the a_i. Then we get an exact sequence of Koszul complexes

$$0 \to K_\bullet(A'; a_1', \ldots, a_r') \otimes J \to K_\bullet(A'; a_1', \ldots, a_r') \to K_\bullet(A; a_1, \ldots, a_r) \to 0.$$

Since $J^2 = 0$, the one on the left is equal to $K_\bullet(A; a_1, \ldots, a_r) \otimes_C J$. Since B is flat over C, this complex is exact with quotient $B \otimes J$. The exact sequence of homology of this sequence of complexes shows that the middle one is exact, and its cokernel B' belongs to an exact sequence

$$0 \to B \otimes_C J \to B' \to B \to 0.$$

Now the local criterion of flatness (6.2) shows that B' is flat over C'.

For 3), we just use Nakayama's lemma to show that a' can be generated by r elements $a_1', \ldots, a_r'$. Then 1) follows from 2) and 3) by induction on length C.

We leave to the reader to formulate an affine version of this theorem, in case A is a finitely generated ring over a field k, whose localizations are all Cohen–Macaulay rings. In this case a complete intersection ideal would be $\mathfrak{a} = (a_1, \ldots, a_r)$, such that for every prime ideal $\mathfrak{p} \in \operatorname{Supp} A/\mathfrak{a}$, the a_i generate a complete intersection ideal in the local ring $A_\mathfrak{p}$. We can form a Koszul complex globally, and show that it is a resolution of $B = A/\mathfrak{a}$ by looking at its localizations. Deformations will behave exactly as in Theorem 9.3.

For the global projective case, we say a closed subscheme $Y \subseteq X = \mathbb{P}^n_k$ is a *complete intersection* if its homogeneous ideal $I_Y \subseteq R = k[x_0, \ldots, x_n]$ can be generated by $r = \operatorname{codim}(Y, X)$ homogeneous elements. These elements will then form a regular sequence in R, and the associated Koszul complex will give a resolution of R/I_Y over R.

To deal with deformations, we must again assume $\dim Y \geq 1$ so as to be able to apply (8.9).

Theorem 9.4. *Let C', J, C be as before. Let $Y \subseteq X = \mathbb{P}^n_C$ be a closed subscheme, flat over C, such that $Y \times_C k = Y_0 \subseteq X_0 = \mathbb{P}^n_k$ is a complete intersection of codimension r and assume* $\dim Y_0 \geq 1$. *Then as in Theorem* 8.3 *(we abbreviate the statement),*

1) *Y has a resolution by the Koszul complex.*

2) *Any lifting of the generators of I_Y gives a deformation $Y' \subseteq X'$.*

3) *Any deformation $Y' \subseteq X'$ of Y arises as in* 2).

We could abbreviate this further by saying that any deformation of a complete intersection of dim ≥ 1 is again a complete intersection, and that these deformations are unobstructed.

Corollary 9.5. *If $Y_0 \subseteq X_0 = \mathbb{P}^n_k$ is a complete intersection of* $\dim \geq 1$, *the Hilbert scheme at the corresponding point is smooth.*

Example 9.5.1. The conclusion of (9.4) is false for Y_0 of dimension 0. The same example mentioned above (8.10.1) of three collinear points in $\mathbb{P}^2$ is a complete intersection, but its general deformation to 3 non-collinear points is not a complete intersection.

Gorenstein in codimension 3

Here we consider a regular local ring A and a quotient $B = A/\mathfrak{a}$ that is a Gorenstein local ring and has codimension 3. The situation is analogous to the case of Cohen–Macaulay in codimension 2, but more difficult. First we have the structure theorem of Buchsbaum and Eisenbud [7].

Theorem 9.6. *Let A be a regular local ring, and let $B = A/\mathfrak{a}$ be a quotient that is Gorenstein and of codimension* 3. *Then there is a skew-symmetric matrix φ of odd order n of elements of A, whose $(n-1) \times (n-1)$ pfaffians f_i generate the ideal $\mathfrak{a}$, and gives rise to a resolution*

$$0 \to A \xrightarrow{f^\vee} A^n \xrightarrow{\varphi} A^n \xrightarrow{f} A \to A/\mathfrak{a} \to 0.$$

For the proof we refer to the paper [7, Thm. 2.1].

Using techniques analogous to those in the Cohen–Macaulay codimension 2 case, one can show that deformations of B always extend, and have resolutions of the same type (analogous to Theorem 8.2).

Theorem 9.7. (Repeat statement of (8.2) in the Gorenstein codimension 3 case.)

The same kind of resolution holds also in the graded case, and using (8.9) to pass from deformations of projective schemes to the associated graded rings, one can show in the same way

Theorem 9.8. (Repeat Theorem 8.10 in the Gorenstein codimension 3 case for *Arithmetically Gorenstein* closed subschemes, defined by requiring that R/I_Y be a graded Gorenstein ring.)

Corollary 9.9 (Miró–Roig [53]). *Let $Y \subseteq X = \mathbb{P}^n_k$ be an arithmetically Gorenstein scheme of codimension 3, and assume $\dim Y \geq 1$. Then the Hilbert scheme is smooth at the point corresponding to Y, and all nearly points also represent arithmetically Gorenstein schemes.*

References for this section. The relation between complete intersection, regular sequences, and the Koszul complex is by now classical—I first learned about it from Serre's "Algèbre Locale Multiplicités" [81].

As for the Gorenstein in codimension 3 case, the first paper was by Watanabe [88]. Then came the structure theorem of Buchsbaum and Eisenbud [7], on which all later results are based. The proof of the structure Theorem 9.6 is rather subtle. But once one has that result, the implications we have listed for deformation theory follow quite easily using the methods of the previous section.

10 Obstructions to deformations of schemes

Suppose given B_0 a finitely generated algebra over k, suppose given $0 \to J \to C' \to C \to 0$, where C' is a local Artin ring, J an ideal annihilated by $\mathfrak{m}_{C'}$. And suppose given B a deformation of B_0 over C, such that B is flat over C and $B \otimes_C k = B_0$. We ask for conditions under which there exists a deformation of B_0 over C' extending B, that is B' flat over C' with $B' \otimes_{C'} C = B$.

Theorem 10.1. *In the above situation,*

a) *there is an element $\delta \in T^2(B_0/k, B_0 \otimes J)$, called the* obstruction, *with the property that $\delta = 0$ if and only if the deformation B' exists.*

b) *If deformations exist, then the set of all pairs $B' \to B$, consisting of B' flat over C' and a morphism to B inducing an isomorphism $B' \otimes_{C'} C \cong B$, modulo isomorphisms of B' inducing the identity on B, is a torsor under the action of $T^1(B_0/k, B_0 \otimes J)$.*

Proof.

a) To define the obstruction, choose a polynomial ring $A = C[x_1, \dots, x_n]$ and a surjective mapping $A \to B$ with kernel $\mathfrak{a}$. Let $f_1, \dots, f_r \in \mathfrak{a}$ be a set of generators, let F be the free module A^r, and let Q be the kernel:

$$0 \to Q \to F \to \mathfrak{a} \to 0.$$

The idea is to lift the f_i to elements $f_i' \in A' = C'[x_1, \dots, x_n]$, define $B' = A'/\mathfrak{a}'$, where $\mathfrak{a}' = (f_i', \dots, f_n')$, and investigate whether we can make B' flat over C'. Let $F' = A'^r$, and let $Q' = \ker(F \to \mathfrak{a}')$. Tensoring with $0 \to J \to C' \to C \to 0$ we get a diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & Q' & \to & Q & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & F \otimes J & \to & F' & \to & F & \to & 0 \\
 & & \downarrow & & \downarrow f' & & \downarrow f & & \\
0 & \to & A_0 \otimes J & \to & A' & \to & A & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & B_0 \otimes J & \to & B' & \to & B & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

From the snake lemma there is a map $\delta_0 : Q \to B_0 \otimes J$, depending on the lifting f' of f. We know from the local criterion of flatness (6.2) that B' is flat over C' if and only if the map $B_0 \otimes J \to B'$ is injective, and this is equivalent to $\delta_0 = 0$. Any element in F_0 (using the notation of 3.1) is of the form $f_j e_i - f_i e_j$ in F, and this lifts to $f_j' e_i' - f_i' e_j'$ in Q', so the map δ_0 factors through Q/F_0. Thus we get a homomorphism $\delta_1 \in \mathrm{Hom}(Q/F_0, B_0 \otimes J)$. And this, using the definition of

$$T^2(B/A, B_0 \otimes J) = \mathrm{coker}(\mathrm{Hom}(F/IF, B_0 \otimes J) \to \mathrm{Hom}(Q/F_0, B_0 \otimes J))$$

gives us the desired element $\delta \in T^2(B_0/k, B_0 \otimes J)$, which is the same as $T^2(B/A, B_0 \otimes J)$.

We must show that δ is independent of all the choices made. If we make a different choice of lifting f_i'' of the f_i, then the $f_i' - f_i''$ define a map from F' to $A_0 \otimes J$ and hence from F/IF to $B_0 \otimes J$, and these go to zero in T^2. If we choose a different polynomial ring $A^* \to B$, then as in the proof of (3.3) we reduce to the case $A^* = A[y_1, \ldots, y_s]$, and the y_i go to zero in B. The contribution to δ of the y_i is then zero. Thus δ depends only on the initial situation B/C and $C' \to C$.

If the extension B'/C' does exist, we can start by picking a polynomial ring A' over C' that maps surjectively to B', and use $A = A' \otimes_{C'} C$ mapping to B in the above construction. Then the generators of $\mathfrak{a}'$ descend to generators of fa, and the diagram above shows $\delta_0 = 0$, so afortiori $\delta = 0$.

Conversely, suppose $\delta = 0$. Then we must show an extension B' exists. In fact we will show something apparently stronger, namely that having made a choice $A \to B \to 0$ as above, we can lift B to a quotient of the corresponding A'. In fact, we will show that it is possible to lift the f_i to f_i' in such a way that the map $\delta_0 : Q \to B_0 \otimes J$ is zero.

Our hypothesis is only that $\delta \in T^2$ is zero. By definition of T^2, this means that the map $\delta_1 \in \operatorname{Hom}(Q/F_0, B_0 \otimes J)$ lifts to a map $\gamma : F/IF \to B_0 \otimes J$. This defines a map $F \to B_0 \otimes J$, and since F is free it lifts to a map $F \to A_0 \otimes J$, defined by $g_1, \ldots, g_r \in A_0 \otimes J$. Now take $f_i'' = f_i' - g_i$. The g_i cancel out the images of δ_1, and so we find the new $\delta_0 = 0$, so the new B' is flat over C'.

b) Suppose one such deformation B_1' exists. Choose and fix a map $B_1' \to B$ inducing an isomorphism $B_1' \otimes_{C'} C \xrightarrow{\sim} B$. Let $R_0 = k[x_1, \ldots, x_n]$ be a polynomial ring of which B_0 is a quotient, with kernel I_0. Then the map $R_0 \to B_0$ lifts to a map of $R = R_0 \otimes_k C$ to B and to a map of $R' = R_0 \otimes_k C'$ to B_1', compatible with the map to B.

For any other deformation $B_2' \to B$, the map $R \to B$ lifts to a map $R' \to B_2'$. Thus every abstract deformation is also an embedded deformation (in many ways perhaps), and we know the embedded deformations are a torsor under the action of $\operatorname{Hom}(I/I^2, B \otimes J)$ by (6.3). So comparing with the fixed one B_1', we get an element of this group corresponding to B_2'. Now if two of these, B_2' and B_3' happen to be isomorphic as abstract deformations (inducing the identity on B), choose an isomorphism $B_2' \cong B_3'$. Using this we obtain two maps $R' \rightrightarrows B_2'$, and hence a derivation of R' to $B \otimes J$, by (4.5), which can be regarded as an element of $\operatorname{Hom}(\Omega_{R'/C'}, B \otimes J)$.

Now we use the exact sequence (3.10) determining T^1,

$$\operatorname{Hom}(\Omega_{R/C}, B \otimes J) \to \operatorname{Hom}(I/I^2, B \otimes J) \to T^1(B/A, B \otimes J) \to 0$$

to see the ambiguity of embedding is exactly resolved by the image of the derivations, and so the pairs $B' \to B$, up to isomorphism, form a torsor under $T^1(B/A, B \otimes J) = T^1(B_0/k, B_0 \otimes J)$ (???).

Remark 10.2. For future reference, we note that given a deformation B' of B over C', the group of automorphisms of B' lying over the identity of B is naturally isomorphic to the group $T^0(B_0/k, J \otimes B_0)$ of derivations of B_0 into $J \otimes B_0$. Indeed, we have only to apply (4.5) taking $R = B'$.

Corollary 10.3. *The obstruction to deforming B as an abstract k-algebra is the same as the obstruction to deforming B as a quotient of any fixed polynomial ring A.*

Proof. Follows from proof of theorem.

Proposition 10.4. *Let A be a regular local ring containing its residue field k, and let $B = A/\mathfrak{a}$. Then B is a local complete intersection in A if and only if $T^2(B/k, M) = 0$ for all B-modules M.*

Proof. Since A is regular (non-singular), we have $T^1(A/k, M) = 0$ for $i = 1, 2$ and all M. Therefore, from the long-exact sequences of (3.5) we find that $T^2(B/k, M) = T^2(B/A, M)$. To compute the latter, in the construction of the T^2 functors we can take $R = A$, and a resolution $0 \to Q \to F \to \mathfrak{a} \to 0$, and find the complex $L_\bullet$ is just

$$Q/F_0 \to F/IF \to 0.$$

Let us assume that $F \to \mathfrak{a}$ corresponds to a minimal set of generators $f_1, \ldots, f_n$ of $\mathfrak{a}$. The definition of T^2 gives an exact sequence

$$\operatorname{Hom}(F/IF, M) \overset{\alpha(M)}{\to} \operatorname{Hom}(Q/F_0, M) \to T^2(B/A, M) \to 0.$$

If B is a local complete intersection, that means that $f_1, \ldots, f_n$ is a regular sequence in A. Then the Koszul complex is a resolution, so the Koszul relations F_0 generate Q. So $Q/F_0 = 0$, and clearly $T^2(B/A, M) = 0$ for all M.

Conversely, suppose that $T^2(B/A, M) = 0$ for all M. Then the map $\alpha(M)$ is surjective. Taking $M = Q/F_0$, this means that the map $j : Q/F_0 \to F/IF$ has an inverse $\sigma : F/IF \to Q/F_0$, so that $\sigma \circ j =$ identity. But since we chose a minimal set of generators for $\mathfrak{a}$, the image of j is contained in $\mathfrak{m}(F/IF)$, where $\mathfrak{m}$ is the maximal ideal of A. Hence $Q/F_0 =$ image $\sigma \circ j \subseteq \mathfrak{m}(Q/F_0)$. By Nakayama's lemma, this implies $Q/F_0 = 0$. But Q/F_0 is exactly the first homology $h_1(K_\bullet(f_1, \ldots, f_n))$ of the Koszul complex, and the vanishing of this homology implies that the f_i form a regular sequence [81], so B is a local complete intersection in A.

Corollary 10.5. *The property of B being a local complete intersection in A is independent of the choice of the regular local ring A of which B is a quotient.*

Proof. Indeed, the criterion $T^2(B/k, M) = 0$ for all M depends only on B.

Now we pass to the global case.

Theorem 10.6. *Given a scheme X_0 over k, and given a sequence $0 \to J \to C' \to C \to 0$ with $J^2 = 0$ as before, and given a deformation X of X_0, flat over C, we consider pairs $X \hookrightarrow X'$, where X' is flat over C' and $X' \times_{C'} C \cong X$.*

a) *Given X, there are three successive obstructions to be overcome for the existence of such an X', lying in $H^0(X_0, T^2_{X_0} \otimes J)$, $H^1(X_0, T^1_{X_0} \otimes J)$ and $H^2(X_0, T^0_{X_0} \otimes J)$.*

b) *Let $\operatorname{Def}(X/C, C')$ be the set of all such $X \hookrightarrow X'$, up to isomorphism. Having fixed one such X'_1, there is an exact sequence*

$$0 \to H^1(X_0, T^0_{X_0} \otimes J) \to \operatorname{Def}(X/C, C') \to H^0(X_0, T^1_{X_0} \otimes J) \to H^2(X_0, T^0_{X_0} \otimes$$

Proof.

a) Suppose given X. For each open affine subset $U_i \subseteq X$ there is an obstruction lying in $H^0(U_i, T^2_{U_i} \otimes J)$ for the existence of a deformation U'_i over U_i, by (10.1). These patch together to give a global obstruction in $\delta_1 \in H^0(X_0, T^2_{X_0} \otimes J)$.

   If this obstruction vanishes, then for each U_i there exists a deformation U'_i over U_i. For each $U_{ij} = U_i \cap U_j$ we then have two deformations $U'_i|_{U_{ij}}$ and $U'_j|_{U_{ij}}$. By (10.1) again, their difference gives an element in $H^0(U_{ij}, T^1 \otimes J)$. The difference of three of these is zero on U_{ijk}, so we get the second obstruction $\delta_2 \in H^1(X_0, T^1 \otimes J)$.

   If this obstruction vanishes, then we can modify the deformations U'_i so that they become isomorphic on the overlap U_{ij}. Choose isomorphisms $\varphi_{ij} : U'_i|_{U_{ij}} \xrightarrow{\sim} U'_j|_{U_{ij}}$ for each ij. On the triple intersection U_{ijk}, composing three of these gives an automorphism of $U'_i|_{U_{ijk}}$, which gives an element in $H^0(U_{ijk}, T^0 \otimes J)$. On the fourfold intersection, these agree, so we get an obstruction $\delta_3 \in H^2(X_0, T^0 \otimes J)$.

   If this last obstruction also vanishes, then we can modify the isomorphisms φ_{ij} so that they agree on the U_{ijk}, and then we can glue the deformations U'_i to get a global deformation X'.

b) Suppose now given one fixed deformation X'_1 of X over C'. If X'_2 is another deformation, then on each open affine U_i we have two, and their difference gives an element of $H^0(U_i, T^1 \otimes J)$ by (10.1). These glue together to give a global element of $H^0(X, T^1 \otimes J)$. We have already seen in part a) above that conversely, given a global element of $H^0(X, T^1 \otimes J)$, it gives deformations over the open sets U_i that are isomorphic on the intersections U_{ij}, and in that case there is an obstruction in $H^2(X, T^0 \otimes J)$ to gluing this together to get a global deformation.

   Now suppose two deformations X'_2 and X'_3 give the same element in $H^0(X, T^1 \otimes J)$. This means that they are isomorphic on each open affine U_i. Choose isomorphisms $\varphi_i : X'_2|_{U_i} \xrightarrow{\sim} X'_2|_{U_i}$. On the intersection U_{ij}, we get $\psi_{ij} = \varphi_j^{-1} \circ \varphi_i$ which is an automorphism of $X'_2|_{U_{ij}}$ and so defines a section of $T^0 \otimes J$ over U_{ij}. These agree on the triple overlap, so we get an element of $H^1(X, T^0 \otimes J)$. The vanishing of this element is equivalent to the possibility of modifying the isomorphisms φ_i so that they will agree on the overlap, which is equivalent to saying X_2' and X_3' are globally isomorphic. Thus we get the exact sequence for $\mathrm{Def}(X/C, C')$ as claimed.

Remark 10.6.1. This proposition suggests the existence of a spectral sequence beginning with $H^p(X, T^q \otimes J)$ and abutting to some groups of which the H^2 would contain the obstruction to lifting X, and the $H^1 = \mathrm{Def}(X/C, C')$ would classify the extensions if they exist. But I will not attempt to say where such a spectral sequence might come from. If you really want to know, you will have to look in another book!

Remark 10.6.2. As in (10.2) we observe that given X' a deformation of X over C', the group of automorphisms of X' lying over the identity automorphism of X is naturally isomorphic to the group $H^0(X_0, T^0_{X_0} \otimes J)$. Just apply (10.2) and glue.

References for this section. The applications of the T^i functors to deformation theory are in the paper of Lichtenbaum and Schlessinger [47], including the characterization of complete intersection local rings (10.4) and (10.5).

11 Dimensions of families of space curves

A classical problem, studied by G. Halphen and M. Noether in the 1880's, and subject of considerable activity one hundred years later, is the problem of classification of algebraic space curves. Here we mean closed one-dimensional subschemes of $\mathbb{P}^n$, while the classical case was irreducible non-singular curves in $\mathbb{P}^3$. See [27, IV, §6], [28], [29], [26] for some surveys of the problem.

To begin with let us focus our attention on non-singular curves in $\mathbb{P}^3$. These form an open subset of the Hilbert scheme, so the problem is to find which pairs (d, g) can be the possible degree and genus of a curve, and then for each such (d, g) to find the irreducible components and the dimensions of the corresponding parameter space $H_{d,g}$. The problem of determining the possible (d, g) pairs has been solved by Gruson and Peskine [25] (see also [30]). Easy examples show that $H_{d,g}$ need not be irreducible [27, IV, 6.4.3]. So we consider the problem of dimension.

Max Noether's approach to the problem [62, pp. 18,19] goes like this. The choice of an abstract curve of genus g depends on $3g-3$ parameters. The choice of a divisor of degree d on the curve (up to linear equivalence) is another g parameters. If $d \geq g+3$, a general such divisor D is non-special, so by the Riemann–Roch theorem, the dimension of the complete linear system $|D|$ is $d-g$. In here we must choose a 3-dimensional subsystem, and such a choice depends on $4(d-g-3)$ parameters (the dimension of the Grassmann variety of $\mathbb{P}^3$'s in a $\mathbb{P}^{d-g}$). Now add 15 parameters for an arbitrary automorphism of $\mathbb{P}^3$. Putting these together, we find that the dimension of the family of general curves of genus g, embedded with a general linear system of degree $d \geq g+3$, is $4d$.

Refining his argument (but still speaking always of curves that are general in the variety of moduli, and general linear systems on these), Noether claimed that

a) If $d \geq \frac{3}{4}(g+4)$, the family has dimension $4d$.

b) If $d < \frac{3}{4}(g+4)$, the family has dimension $\geq 4d$.

These methods do not take into account curves whose moduli are special, and therefore may have linear systems of kinds that do not appear on general curves, and of course they do not apply to singular or reducible curves. Furthermore, Noether's method depends on knowing the dimension of the variety of moduli $(3g-3)$, the dimension of the Jacobian variety (g), the dimension of Grassmann varieties, and also depends on having confidence in the method of "counting constants", which sometimes seems more like an art than a science.

In this section we will use an entirely different method, the infinitesimal study of the Hilbert scheme, to prove the following theorem.

Theorem 11.1. *Every irreducible component of the Hilbert scheme of locally Cohen–Macaulay curves of degree d and arithmetic genus g in $\mathbb{P}^3$ has dimension $\geq 4d$.*

Proof. These curves are locally Cohen–Macaulay and of codimension 2 in $\mathbb{P}^3$, so there are no local obstructions to embedded deformations (8.2). The Zariski tangent space to the Hilbert scheme at the point corresponding to a curve C is given by $H^0(C, \mathcal{N}_C)$, where $\mathcal{N}_C = \mathcal{H}om(\mathcal{I}_C, \mathcal{O}_C)$ (2.4). Since there are no local obstructions, the obstructions to global deformations lie in $H^1(C, \mathcal{N}_C)$ (6.3). Now using the dimension theorem for a local ring with an obstruction theory (7.4) we find

$$\dim_C \mathrm{Hilb} \geq h^0(C, \mathcal{N}_C) - h^1(C, \mathcal{N}_C).$$

Thus we are reduced to the problem of evaluating the cohomology of the normal sheaf $\mathcal{N}_C$.

We do an easy case first. If C is irreducible and non-singular, then $\mathcal{N}_C$ is locally free and is just the usual normal bundle to the curve. It belongs to an exact sequence

$$0 \to \mathcal{T}_C \to \mathcal{T}_{\mathbb{P}^3}|_C \to \mathcal{N}_C \to 0$$

where $\mathcal{T}$ denotes the tangent bundle of C (resp. $\mathbb{P}^3$). On the other hand, the tangent bundle of $\mathbb{P}^3$ belongs to an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_{\mathbb{P}^3}(1)^4 \to \mathcal{T}_{\mathbb{P}^3} \to 0.$$

Restricting this sequence to C, and taking Euler characteristics of the corresponding sheaves, we find

$$\begin{aligned} h^0(\mathcal{N}_C) - h^1(\mathcal{N}_C) &= \chi(\mathcal{N}_C) \\ &= \chi(\mathcal{O}_C(1)^4) - \chi(\mathcal{O}_C) - \chi(\mathcal{T}_C). \end{aligned}$$

Now using the Riemann-Roch theorem on C, and using the fact that $\deg \mathcal{T}_C = 2 - 2g$, we find

$$\chi(\mathcal{N}_C) = 4(d + 1 - g) - (1 - g) - (2 - 2g + 1 - g) = 4d.$$

This proves the theorem for non-singular C.

The general case is a bit more technical, because we do not have the same simple relationship between the normal sheaf and the tangent sheaves. But we have assumed that C is locally Cohen–Macaulay, so there is a resolution

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{I}_C \to 0 \tag{1}$$

with $\mathcal{E}, \mathcal{F}$ being locally free sheaves on $\mathbb{P}^3$. Taking $\mathcal{H}om(\bullet, \mathcal{O}_{\mathbb{P}})$ we find

$$0 \to \mathcal{O}_{\mathbb{P}} \to \mathcal{F}^\vee \to \mathcal{E}^\vee \to \mathcal{E}xt^1(\mathcal{I}_C, \mathcal{O}_{\mathbb{P}}) \to 0. \tag{2}$$

Since C has codimension 2, this $\mathcal{E}xt^1(\mathcal{I}_C, \mathcal{O}_{\mathbb{P}})$ is isomorphic to $\mathcal{E}xt^2(\mathcal{O}_C, \mathcal{O}_{\mathbb{P}})$, which is just $w_C(4)$ where w_C is the dualizing sheaf of C.

Tensoring the sequence (2) with $\mathcal{O}_C$ we therefore get

$$\mathcal{F}^\vee \otimes \mathcal{O}_C \to \mathcal{E}^\vee \otimes \mathcal{O}_C \to w_C(4) \to 0. \tag{3}$$

On the other hand, applying $\mathcal{H}om(\bullet, \mathcal{O}_C)$ to the sequence (1) we obtain

$$0 \to \mathcal{N}_C \to \mathcal{F}^\vee \otimes \mathcal{O}_C \to \mathcal{E}^\vee \otimes \mathcal{O}_C \to \mathcal{E}xt^1(\mathcal{I}_C, \mathcal{O}_C) \to 0. \tag{4}$$

Comparing (3) and (4), we see that $\mathcal{E}xt^1(\mathcal{I}_C, \mathcal{O}_C) \cong w_C(4)$.

Now we take Euler characteristics and find

$$\chi(\mathcal{N}_C) = \chi(\mathcal{F}^\vee \otimes \mathcal{O}_C) - \chi(\mathcal{E}^\vee \otimes \mathcal{O}_C) + \chi(w_C(4)).$$

Suppose $\mathcal{E}$ has rank r and $\mathcal{F}$ has rank $r+1$. From the sequence (1) we see they both have the same first Chern class $c_1(\mathcal{E}) = c_1(\mathcal{F}) = c$. Thus the degree of the locally free sheaves $\mathcal{F}^\vee \otimes \mathcal{O}_C$ and $\mathcal{E}^\vee \otimes \mathcal{O}_C$ on C is just cd. Now applying the Riemann–Roch theorem (which works on any curve C for the restriction of locally free sheaves from $\mathbb{P}^3$), and using Serre duality to note that $\chi(w_C(4)) = -\chi(\mathcal{O}_C(-4))$, we get

$$\chi(\mathcal{N}_C) = cd + (r+1)(1-g) - [cd + r(1-g)] - [-4d + 1 - g] = 4d.$$

Remark 11.2. If we apply the same argument to (say) locally complete intersection curves of degree d and arithmetic genus g in $\mathbb{P}^4$ we find $\dim \mathrm{Hilb} \geq 5d + 1 - g$. (There is a similar formula for $\mathbb{P}^n$, $n > 4$.) This number can become negative for large values of g, making the result worthless. This led Joe Harris to ask

a) Can you find a better (sharp) lower bound for the dimension of the Hilbert scheme? and

b) Are there any “rigid” curves in $\mathbb{P}^n$, i.e., curves whose only deformations come from automorphisms of $\mathbb{P}^n$, besides the rational normal curves of degree n?

References for this section. Since the work of Noether, the fact that the dimension of the families of curves of degree d in $\mathbb{P}^3$ is $\geq 4d$ seems to have passed into folklore. The first complete proof in the case of locally Cohen–Macaulay curves, as far as I know, is the one due to Ein [14, Lemma 5], as explained to me by Rao and reproduced here. Theorem 10.1 has been generalized by Rao and Oh to the case of one-dimensional closed subschemes of $\mathbb{P}^3$ that may have isolated or embedded points [66].

12 A non-reduced component of the Hilbert scheme

In the classification of algebraic space curves, the idea that curves formed algebraic families, and that one could speak of the irreducible components and dimensions of these families, goes back well into the nineteenth century. The observation that the family of curves of given degree and genus in $\mathbb{P}^3$ need not be irreducible goes back to Weyr (1873). Tables of families with their irreducible components and dimensions were computed (independently) by Halphen and Noether in 1882. Now these notions can be made rigorous by speaking of the Chow variety or the Hilbert scheme, and for the question of irreducible components and dimensions of families of *smooth* space curves, the answer does not depend on what theory one uses.

With Grothendieck's construction of the Hilbert scheme, a new element appears: the families of curves, which up to then were described only as algebraic varieties, now have a scheme structure. In 1962, only a few years after Grothendieck introduced the Hilbert scheme, Mumford [56] surprised everyone by showing that even for such nice objects as irreducible non-singular curves in $\mathbb{P}^3$, there may be irreducible components of the Hilbert scheme that are generically non-reduced, that is to say as schemes, they have nilpotent elements in their structure sheaves all points of the scheme.

In this section we will give Mumford's example.

Theorem 12.1. *There is an irreducible component of the Hilbert scheme of smooth irreducible curves in $\mathbb{P}^3$ of degree 14 and genus 24 that is generically non-reduced.*

Proof. The argument falls into three parts:

a) We construct a certain irreducible family U of smooth curves of degree 14 and genus 24, and show that the dimension of the family is 56.

b) For any curve C in the family, we show that $H^0(C, \mathcal{N}_C)$ has dimension 57. This gives the Zariski tangent space to the family at the point C.

c) We show that the family U is not contained in any other irreducible family of curves with the same degree and genus, of dimension > 56.

Property c) shows that the family U is actually an open subset of an irreducible component of the Hilbert scheme, of dimension 56. Hence the scheme U_{red} is integral, and therefore non-singular on some open subset $V \subseteq U_{\mathrm{red}}$. If $C \in V$, then property b) shows that U is not smooth at the point C, hence $U \neq U_{\mathrm{red}}$ at the pont C. In other words, U is non-reduced along the open set V.

Step a). The construction. Let X be a non-singular cubic surface in $\mathbb{P}^3$, let H denote the hyperplane section of X, and let L be one of the 27 lines on X. We consider curves C in the linear system $|4H + 2L|$. If we take L to be the sixth exceptional curve E_6, then in the notation of [27, V, §4], the divisor class is $(12; 4, 4, 4, 4, 4, 2)$, and by [loc. cit. 4.12] this class is very ample, so the linear system contains irreducible non-singular curves C. The formulas of [loc. cit.] show that the degree is $d = 14$ and the genus is $g = 24$. The family U we wish to consider consists of all non-singular curves C in the above linear system, for all choices of X a smooth cubic surface and L a line on X.

The cubic surfaces move in an irreducible family of dimension 19, and as they move, the lines on them are permuted transitively, so that U is an irreducible family of curves. Since $\operatorname{Pic} X$ is discrete, the only algebraic families of curves on X are the linear systems. So to find the dimension of U we must add 19 to the dimension of the linear system $|C|$, which is $h^0(\mathcal{O}_X(C)) - 1$. (Note that since $d > 9$, each of our curves C is contained in a unique cubic surface.) Consider the exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(C) \to \mathcal{O}_C(C) \to 0.$$

Since $h^0(\mathcal{O}_X) = 1$ and $h^1(\mathcal{O}_X) = 0$, we find $\dim |C| = h^0(\mathcal{O}_C(C))$. The linear system $\mathcal{O}_C(C)$ on C has degree C^2, which can be computed by [loc. cit.] as 60. This is greater than $2g - 2$, so the linear system is non-special on C, and by Riemann–Roch its dimension is $60 + 1 - 24 = 37$.

Adding, we find $\dim U = 19 + 37 = 56$.

Step b). Computation of $h^0(C, \mathcal{N}_C)$. We use the exact sequence of normal bundles for the non-singular curve C on the non-singular surface X,

$$0 \to \mathcal{N}_{C/X} \to \mathcal{N}_C \to \mathcal{N}_X|_C \to 0.$$

Now $\mathcal{N}_{C/X} = \mathcal{O}_C(C)$, which as we saw above is non-special, with $h^0 = 37$. Since X is a cubic surface, $\mathcal{N}_X = \mathcal{O}_X(3)$, so we find

$$h^0(\mathcal{N}_C) = 37 + h^0(\mathcal{O}_C(3)).$$

By Riemann-Roch,

$$\begin{aligned} h^0(\mathcal{O}_C(3)) &= 3 \cdot 14 + 1 - 24 + h^1(\mathcal{O}_C(3)) \\ &= 19 + h^1(\mathcal{O}_C(3)). \end{aligned}$$

By duality on C, $h^1(\mathcal{O}_C(3)) = h^0(w_C(-3))$. By the adjunction formula on X, $w_C = \mathcal{O}_C(C + K_X) = \mathcal{O}_C(C - H) = \mathcal{O}_C(3H + 2L)$. Thus $h^1(\mathcal{O}_C(3)) = h^0(\mathcal{O}_C(2L))$. Now we use the sequence

$$0 \to \mathcal{O}_X(2L - C) \to \mathcal{O}_X(2L) \to \mathcal{O}_C(2L) \to 0.$$

Note that $2L - C = -4H$, which has $h^0 = h^1 = 0$. Hence $h^0(\mathcal{O}_C(2L)) = h^0(\mathcal{O}_X(2L)) = 1$, since the divisor $2L$ is effective, but does not move in a linear system.

Thus $h^1(\mathcal{O}_C(3)) = 1$, $h^0(\mathcal{O}_C(3)) = 20$, and $h^0(\mathcal{N}_C) = 57$.

Step c). To show that U is not contained in a larger family of dimension > 56, we proceed by contradiction. If $C' \in U'$ was a general curve in this supposed larger family U', then C' would be smooth, still of degree 14 and genus 24, but would not be contained in any cubic surface, because our family U contains all those curves that can be obtained by varying X and varying C on X. From the exact sequence

$$0 \to \mathcal{I}_{C'} \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_{C'} \to 0,$$

twisting by 4, we find

$$0 \to H^0(\mathcal{I}_{C'}(4)) \to H^0(\mathcal{O}_{\mathbb{P}^3}(4)) \to H^0(\mathcal{O}_C(4)) \to \dots$$

The dimension of the middle term is 35; that of the term on the right, by Riemann–Roch, 33. Hence $h^0(\mathcal{I}_{C'}(4)) \geq 2$. Take two independent quartic surfaces F, F' containing C'. Since C' is not contained in a cubic (or lesser degree) surface, F, F' are irreducible and distinct, so their intersection has dimension 1, and provides us with a linkage from C' to the remainder curve $D = F \cap F' \backslash C'$. Computation of degree and genus shows that $\deg D = 2$, and $p_a(D) = 0$. Note that D need not be irreducible or reduced. However, one knows [?] that locally Cohen–Macaulay curves D with $d = 2$, $p_a = 0$ are just the plane conics (possibly reducible). In particular, D is ACM. This property is preserved by linkage [?], so C' is also ACM. Now one could apply the formula of Ellingsrud [?] to show that the irreducible component of the Hilbert scheme containing C' has dimension 56, contradicting our hypotheses above.

Alternatively, one can compute the dimension of the family of curves C' as follows. The Hilbert scheme of plane conics D has dimension 8. The vector space of equations of quartics F containing D is $H^0(\mathcal{I}_D(4))$, which has dimension 26. The choice of a two-dimensional subspace (generated by F, F') is a Grassmann variety of dimension 48. Thus the dimension of the family of curves C' as above is $\leq 8 + 48 = 56$.

This completes the proof of Mumford's example.

Example 12.2. A non-singular 3-fold with obstructed deformations. In the same paper, Mumford observed that the above example, by blowing up, produces a 3-fold with obstructed deformations. We outline the argument.

Let $C \subseteq \mathbb{P}^3$ be a non-singular curve. Let $f : X \to \mathbb{P}^3$ be obtained by blowing up C. Let $E \subseteq X$ be the exceptional divisor. Then $f : E \to C$ is the projective space bundle $\mathbb{P}(\mathcal{I}/\mathcal{I}^2)$ over C, where $\mathcal{I} = \mathcal{I}_{C/\mathbb{P}^3}$.

We make use of the sequence of differentials

$$0 \to f^*\Omega^1_{\mathbb{P}^3} \to \Omega^1_X \to \Omega^1_{X/\mathbb{P}^3} \to 0,$$

the identification $\Omega^1_{X/\mathbb{P}^3} \cong \Omega^1_{E/C}$, and the Euler sequence

$$0 \to \Omega^1_{E/C} \to f^*(\mathcal{I}/\mathcal{I}^2)(-1) \to \mathcal{O}_E \to 0.$$

Dualizing we obtain sequences

$$0 \to T_X \to f^*T_{\mathbb{P}^3} \to T_{E/C}(-1) \to 0$$

and

$$0 \to \mathcal{O}_E(-1) \to f^*\mathcal{N}_C \to T_{E/C}(-1) \to 0.$$

Now the cohomology of $f^*T_{\mathbb{P}^3}$ is the same as $T_{\mathbb{P}^3}$; the cohomology of $\mathcal{O}_E(-1)$ is zero, and $f^*\mathcal{N}_C$ has the same cohomology as $\mathcal{N}_C$. Making the substitutions we find an exact sequence

$$0 \to H^0(T_X) \to H^0(T_{\mathbb{P}^3}) \to H^0(\mathcal{N}_C) \to H^1(T_X) \to 0$$

and an isomorphism
$$H^1(\mathcal{N}_C) \cong H^2(T_X).$$
The interpretation is that every infinitesimal deformation of X comes from a deformation of C, and that those deformations of C coming from automorphisms of $\mathbb{P}^3$ have no effect on X. The obstructions to deforming X are equal to the obstructions to deforming C in $\mathbb{P}^3$.

If we now take C to the curve of Mumford's example, it has obstructed deformations, and so the 3-fold X also has obstructed deformations.

References for this section. Mumford's example appeared in his second "pathologies" paper [56]. We have simplified his argument somewhat by using liaison theory instead of some delicate arguments on families of quartic surfaces. The same paper also gives (12.2).

Kleppe devoted his thesis [39] to an analysis and expansion of Mumford's example. He generalized the Hilbert scheme to the Hilbert-flag scheme, parametrizing (for example) pairs of a curve in a surface in $\mathbb{P}^3$, and made an infinitesimal study of these schemes using Laudal's deformation theory [46]. Generalizing Mumford's example, he found for every $d \geq 14$ a suitable g, and a family of non-singular curves of degree d and genus g on non-singular cubic surfaces that form a non-reduced irreducible component of the Hilbert scheme [39, 3.2.10, p. 192]. Further study of this situation appears in his papers [40] and [41].

Gruson and Peskine [25] give an example of a family of non-singular curves of degree 13 and genus 18 lying on ruled cubic surfaces that has dimension 52 and $h^0(\mathcal{N}_C) = 54$ for C in the family. They state that this is a non-reduced component of the Hilbert scheme, but say only "on peut alors montrer" to justify the hard part, which is to show that it is not contained in any larger family. I have not yet seen a complete proof of this statement.

Martin–Deschamps and Perrin [49] show that when one considers curves that need not be irreducible or reduced, then the Hilbert scheme is "almost always" non-reduced. More precisely, they show the following. Excluding the plane curves, one knows that the Hilbert scheme of locally Cohen–Macaulay curves in $\mathbb{P}^3$ of degree d and arithmetic genus g is non-empty if and only if $d \geq 2$ and $g \leq \frac{1}{2}(d-2)(d-3)$. They show that for $d \geq 6$ and $g \leq \frac{1}{2}(d-3)(d-4)+1$, the corresponding Hilbert scheme $H_{d,g}$ has at least one non-reduced irreducible component. For

$d < 6$ there are also exact statements. The non-reduced components correspond to the "extremal" curves having the largest Rao-module, which are almost always reducible, non-reduced curves. The examples of smallest degree that they find is for $d = 3$, $g = -2$. In this case the Hilbert scheme $H_{3,-2}$ has two irreducible components. One component of dimension 12 has as its general curve the disjoint union of three lines. The other, of dimension 13 has as its general curve a double structure on a line of arithmetic genus -3, plus a reduced line meeting the first with multiplicity 2. This component is non-reduced. By performing liaisons starting with non-reduced components of singular curves, Martin–Deschamps and Perrin obtain further examples of non-reduced irreducible components of the Hilbert scheme of smooth curves [49, 5.4]. Their first example is for $(d, g) = (46, 213)$.

CHAPTER 3

Formal Moduli

13 Plane curve singularities

For plane curve singularities, one can ask questions similar to the ones we have been asking for global objects. Can one describe the set of possible singularities up to isomorphism, and can one find moduli spaces parametrizing them?

First we have to decide what we mean by isomorphism. We do not mean equal as subschemes of the plane (that question is answered by the Hilbert scheme) because it is the type of the singularity, not its embedding that we are after. Nor do we mean isomorphism of a neighborhood of the point on the curve in the Zariski topology, because that involves the global moduli of the curve. We want a purely local notion, and for the moment analytic isomorphism seems to be a reasonable choice. This means we ask for isomorphism of the completion of the local rings of the points on the curves (see [27, Ex. 5.14] for some examples).

Right away we see that we cannot expect to have a moduli space of all curve singularities because of jump phenomena. The family $xy-t = 0$ is non-singular for all $t \neq 0$ but gives a node for $t = 0$. The family $y^2 - tx^2 - x^3 = 0$ gives a node for all $t \neq 0$ but a cusp for $t = 0$. Since all smooth points are analytically isomorphic, and all nodes are analytically isomorphic, there cannot be a coarse moduli space (refer to earlier discussion of jump phenomena).

One can improve the situation by considering only "equisingular" families, meaning families with roughly the same type of singularity. Then for example, the analytic isomorphism types of ordinary four-fold points can be distinguished by a cross-ratio, leading to a moduli space similar to the j-invariant of an elliptic curve. We will come back to the study of equisingular deformations later.

For the moment, instead of looking for a moduli space of singularities, we will focus our attention on a single singularity, and attempt to describe all possible local deformations of this singularity. Our goal is to find a deformation over a suitable local parameter space that is "complete" in the sense that any other local deformation can be obtained (up to isomorphism) by base extension from this one, and that is "minimal" in the sense that it is the smallest possible such. The completeness is expressed by saying it is a *versal* deformation space for the singularity, and if it is minimal, we call it *miniversal.* (The precise definition of these notions will come later when we make a systematic study of functors of Artin rings.) One could also ask for a family that is *universal* in the sense that any other family is obtained by a *unique* base extension from this one, but as we shall see, this rarely exists.

It turns out that our goal of finding such a versal or miniversal deformation of a given singularity can be accomplished only for strictly local deformations. This means over parameter spaces that are artinian or complete local rings. (There is also a complex-analytic version using convergent power series over $\mathbb{C}$, and then there are algebraization theorems in certain circumstances allowing one to weaken the condition of complete local rings to henselian local rings — but more of that later.)

So in this section, before introducing the general theory of formal deformations, we will construct explicitly some deformation spaces of curve singularities and prove directly their versal property. This will serve as an introduction to the general theory and will illustrate some of the issues we must deal with in studying local deformations.

Example 13.1. We start with a node, represented by the equation $xy = 0$ in the plane $\mathbb{A}^2 = \operatorname{Spec} k[x, y]$. We consider the family X given by $xy - t = 0$ in $\mathbb{A}^3 = \operatorname{Spec} k[x, y, t]$, together with its map to the parameter space $T = \operatorname{Spec} k[t]$. For $t \neq 0$ the fiber is a non-singular hyperbola. For $t = 0$ we recover the original nodal singularity.

We will show that this family X/T has a versal property, at least in a formal sense. Let X'/S be any flat deformation of the node over the spectrum S of a complete local ring. We will assume for simplicity that $S = \operatorname{Spec} k[[s]]$, since the case of more variables, or the quotient of a complete regular local ring can be handled similarly.

We must show that there exists a morphism $S \to T$, i.e., a homomorphism $k[t] \xrightarrow{\varphi} k[[s]]$ given by a power series $\varphi(t) = T(s)$ with $T(0) = 0$, such that the base extension of the family X becomes isomorphic to the family X'. Since deformations of local complete intersections are again local complete intersections, we may assume that X' is defined by an equation $g(x, y, s) = 0$ in $k[[s]][x, y]$ with $g(x, y, 0) = xy$. To show that X' and $X \times_T S$ are isomorphic, we need functions $X(x, y, s)$

and $Y(x, y, s)$ reducing to x and y for $s = 0$, and a unit $U(x, y, s)$ reducing to 1 for $s = 0$, such that

$$U(XY - T) = g(x, y, s). \tag{$*$}$$

We will construct T, X, Y, U as power series in s degree by degree. The constant terms (for $s = 0$) have already been prescribed: $T(0) = 0$, $X(0) = x$, $Y(0) = y$, $U(0) = 1$, and so the equation $(*)$ is satisfied for $s = 0$.

Let us write

$$\begin{aligned}
T &= \sum_{i \geq 1} a_i s^i \\
X &= x + \sum_{i \geq 1} X_i s^i \\
Y &= y + \sum_{i \geq 1} Y_i s^i \\
U &= 1 + \sum_{i \geq 1} U_i s^i \\
g &= xy + \sum_{i \geq 1} g_i s^i
\end{aligned}$$

where $a_i \in k$ and $X_i, Y_i, U_i, g_i \in k[x, y]$. Substituting and looking at the degree 1 part of the equation $(*)$ (the coefficient of s) we find

$$xY_1 + yX_1 - a_1 + xyU_1 = g_1.$$

Now g_1 is a given polynomial in x, y. Any polynomial can be expressed as a constant term $-a_1$ plus polynomial multiples of x, y, and xy, since 1 forms a vector-space basis of $k[x, y]$ divided by the ideal generated by (x, y, xy), which is just (x, y). Thus we can find a_1, X_1, Y_1, U_1 to make this equation hold, and then the equation $(*)$ is valid for the coefficients of s.

Note that in making these choices, a_1 is uniquely determined, but there is considerable flexibility in choosing polynomials X_1, Y_1, U_1.

We proceed inductively. Suppose that a_i, X_i, Y_i, U_i have been chosen for all $i < n$ so that $(*)$ is satisfied for all coefficients of s^i with $i < n$. We write out the coefficient of s^n and find

$$h(x, y) + xY_n + yX_n - a_n + xyU_n = g_n$$

where $h(x, y)$ is a polynomial consisting of all the cross products involving a_i, X_i, etc., with $i < n$, that are already determined. Then as before we can find a_n, X_n, Y_n, U_n to satisfy this equation and it follows that T, X, Y, U will satisfy $(*)$ up through the coefficient of s^n.

Proceeding in this manner, we find functions T, X, Y, U that are power series in s, with coefficients that are polynomials in x, y, i.e., elements of the ring $k[x, y][[s]]$, that make the equation $(*)$ hold.

This is not quite what we were hoping for, since the ring $k[x, y][[s]]$ is bigger than $k[[s]][x, y]$. So we have not quite found an isomorphism of X' with $X \times_T S$, but only an isomorphism of their formal completions along the closed fiber at $s = 0$. So the versality property is only true in this formal sense. In other words, we have shown that the family X/T has the following property: given a flat deformation X'/S of the node over a complete local ring S, there exists a morphism $S \to T$ so that the base extension $X \times_T S$ and X'/S have isomorphic formal completions along the fiber over the closed point of S.

Remark 13.2. While we wrote this example using polynomials in x and y, everything works just as well using power series in x and y, and in that case we obtain an analogous versal deformation property.

13.3. We saw in the calculation above that the linear coefficient a_1 of T was uniquely determined. Thus the morphism $S \to T$ induces a unique map on Zariski tangent spaces, and this implies that T is as small as possible, i.e., it is a miniversal deformation space for the node.

13.4. On the other hand, the higher coefficients of T are not uniquely determined. For example, let $u \in k[[s]]$ be a unit with constant term 1, and let u^{-1} be its inverse. Since

$$u^{-1}((xu)y - su) = xy - s$$

we get an isomorphism with $U = u^{-1}$, $X = xu$, $Y = y$, $T = su$. Taking $u = 1 - s$, $u^{-1} = 1 + s + s^2 + \dots$, this gives $T = s - s^2$, so the coefficient $a_2 = -1$, while for the trivial isomorphism $a_2 = 0$. This shows that the morphism $S \to T$ is not unique, even at the power series level, and so the deformation is not universal.

13.5. Even though the versality property is local in the formal sense, we can still draw some global conclusions from this result. For example, suppose X'/S is a family of affine curves in $\mathbb{A}^3 = \operatorname{Spec} k[x, y, s]$ over $S = \operatorname{Spec} k[s]$, and suppose that for $s = 0$ the fiber X_0 in $\mathbb{A}^2 = \operatorname{Spec} k[x, y]$ has a node at the origin. Then for some Zariski neighborhood of $s = 0$ in S and some Zariski neighborhood of the origin in $\mathbb{A}^2$, the nearby curves $s \neq 0$ are either all non-singular, or all have a node as singularity. Indeed, there is a map of $\operatorname{Spec} \hat{\mathcal{O}}_{S,0}$ to T as above, which will either be surjective or will have image just the closed point of T, and this distinguishes the two cases. (Is this obvious, or does it require more proof?)

We will now generalize the above argument to an arbitrary isolated plane curve singularity, given by an equation $f(x, y) = 0$. This may be either a polynomial or a power series. We assume that it has an isolated singularity at the origin, so that the ideal $J = \left(f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$ will be primary for the maximal ideal $\mathfrak{m} = (x, y)$.

To guess the versal deformation space of this singularity, we take a hint from the calculation of the T^1-functor, which parametrizes deformations over the dual numbers. In the case of a plane curve, letting $R = k[x, y]$ and $B = R/(f)$, we have an exact sequence (3.10)

$$\operatorname{Hom}(\Omega_R, B) \xrightarrow{\varphi} \operatorname{Hom}(I/I^2, B) \to T^1(B, B) \to 0$$

where $I = (f)$. The middle term is a free B-module of rank 1. The left term is free generated by $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$, so the image of φ is the ideal generated by $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$. Thus $T^1 = B \Big/ \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = R/J$.

Take polynomials $g_1, \ldots, g_r \in R$ whose images in R/J form a vector space basis. Then we take r new variables $t_1, \ldots, t_r$ and define a deformation X over $T = \operatorname{Spec} k[t_1, \ldots, t_r]$ by

$$F(x, y, t) = f(x, y) + \sum t_i g_i(x, y) = 0.$$

Theorem 13.6. *Given an isolated plane curve singularity $f(x, y) = 0$, the deformation X/T defined above is miniversal in the following sense:*

a) *For any other deformation X'/S, with S a complete local ring, there is a morphism $\varphi : S \to T$ such that X' and $X \times_T S$ become isomorphic after completing along the closed fiber over zero, and*

b) *although φ may not be unique, the induced map on Zariski tangent spaces of S and T is uniquely determined.*

Proof. The proof is a generalization of the one given in Example 13.1 above, once we make clear the role of the partial derivatives and of the basis g_i of R/J.

First of all, as before we may assume that X' is given by a single equation $G(x, y, s)$ for $s \in S$. Second, writing S as a quotient of a formal power series $k[[s_1, \ldots, s_m]]$, we can lift the equation G to the power series ring, obtaining a deformation over that ring. It will thus be sufficient to prove the theorem for the case of the power series ring itself.

To establish the isomorphism required in a) we must find power series T_i, $i = 1, \ldots, r$, in S and X, Y, U in $k[x, y][[s_1, \ldots, s_m]]$ such that

$$UF(X, Y, T) = G(x, y, s), \tag{$*$}$$

where T stands for $T_1, \ldots, T_r$. We will construct T, X, Y, U step by step, starting with constant terms $0, x, y, 1$ as before.

Suppose inductively that we have constructed partial power series $T^{(\nu)}, X^{(\nu)}, Y^{(\nu)}, U^{(\nu)}$ so that the equation $(*)$ holds modulo $s^{\nu+1}$. (Here we will abbreviate $s_1, \ldots, s_n$ to simply s, leaving the reader to supply missing indices as needed — so for example $s^{\nu+1}$ means the ideal $(s_1, \ldots, s_n)^{\nu+1}$.) This has just been done for $\nu = 0$.

We define a new function

$$H^{(\nu)} = U^{(\nu)} F(X^{(\nu)}, Y^{(\nu)}, T^{(\nu)}) - G(x, s).$$

By construction this function lies in the ideal $(s^{\nu+1})$. Thus $H^{(\nu)} \bmod (s^{\nu+2})$ is homogeneous in s of degree $\nu + 1$, so we can write

$$H^{(\nu)} = f(x, y)\Delta U + \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta Y + \sum_{i=1}^{r} g_i \Delta T_i$$

where the ΔT_i are polynomials in s, and $\Delta U, \Delta X, \Delta Y$ are polynomials in x, y, s, all of these being homogeneous if degree $\nu + 1$ in s. This is possible, because the coefficient of each monomial in s in H^{ν} is a polynomial in x, y, which can be expressed as a combination of linear multiples of the g_i and polynomial multiples of $f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$, since the g_i form a basis for R/J.

Now we define

$$\begin{aligned} T_i^{(\nu+1)} &= T_i^{(\nu)} - \Delta T_i \\ X^{(\nu+1)} &= X^{(\nu)} - \Delta X \\ Y^{(\nu+1)} &= Y^{(\nu)} - \Delta Y \\ U^{(\nu+1)} &= U^{(\nu)} - \Delta U, \end{aligned}$$

and I claim these new functions will satisfy the equation $(*)$ mod $s^{\nu+2}$.

This is a consequence of the following lemma.

Lemma 13.7. *Let $F(x_1, \ldots, x_n)$ be a polynomial or power series. Let $h_1, \ldots, h_n$ be new variables. Then*

$$F(x_1+h_1, \ldots, x_n+h_n) \equiv F(x_1, \ldots, x_n) + \sum h_i \frac{\partial F}{\partial x_i}(x_1, \ldots, x_n) \operatorname{mod}(h)^2.$$

The proof of the lemma is elementary, and we leave to the reader the simple verification of the claim made above, applying the lemma to the function $UF(X, Y, T_1, \ldots, T_r)$.

Thus we have constructed power series $T \in S$ and X, Y, U in $k[x,y] \otimes S$ making the required isomorphism. It is clear from the proof, as in Example 13.1 before, that the linear part of the functions $T_1, \ldots, T_r$ is uniquely determined, and so the map $S \to T$ is unique on the Zariski tangent spaces.

Remark 13.8. Exactly the same proof works for an isolated hypersurface singularity in any dimension. So if $f(x_1, \ldots, x_n) = 0$ in $\mathbb{A}^n$ has an isolated singularity at the origin, then the ideal $J = \left(f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right)$ will be primary for the maximal ideal $\mathfrak{m} = (x_1, \ldots, x_n)$. We take polynomials g_i to form a k-basis of R/J, and then the versal deformation space is defined by $F(x_1, \ldots, x_n, t_1, \ldots, t_r) = f - \Sigma t_i g_i = 0$.

Example 13.9. Let us study the cusp defined by $f(x,y) = y^2 - x^3$. The partial derivatives are $2y$ and $3x^2$, so (assuming char. $k \neq 2, 3$) we can take $1, x$ as a basis for R/J, and the versal deformation is defined by $F(x, y, t, u) = y^2 - x^3 + t + ux = 0$. Here the parameter space is 2-dimensional, given by t, u. For general values of t, u, the nearby curve will be non-singular, but for special non-zero values of t, u it may be singular. Indeed, if we set F, $\frac{\partial F}{\partial x}$, and $\frac{\partial F}{\partial y}$ equal to zero, we find a singular point at $t = -2x^3$, $u = 3x^2$. Hence there are singularities in the fiber over points on the discriminant locus $27t^2 - 4u^3 = 0$. It is easy to check this singularity is a node when $t, u \neq 0$. So the general deformation is non-singular, but some nearby deformations have nodes.

Reference for this section. Notes of lectures by Mike Schlessinger that I heard ca. 1972 (unpublished).

14 Functors of Artin rings

In this section we will formalize the idea of studying local deformations of a fixed object. We introduce the notion of pro-representable functors and of versal deformation spaces, and in the next section we will prove the theorem of Schlessinger giving a criterion for the existence of a versal deformation space. This will give us a systematic way of dealing with questions of local deformations. Although it may seem rather technical at first, the formal local study is important because it gives necessary conditions for existence of global moduli, and is often easier to deal with than the global questions. Also it gives useful local information when there is no global moduli space at all.

The typical situation is to start with a fixed object X_0, which could be a projective scheme or an affine scheme with a singular point, or any other structure, and we wish to understand all possible deformations of X_0 over local Artin rings. We can consider the functor that to each local Artin ring associates the set of deformations (up to isomorphism) of X_0 over that ring, and to each homomorphism of Artin rings associates the deformation defined by base extension. In this way we get a (covariant) functor from Artin rings to sets.

Now we describe the general situation that we will consider (that includes as a special case the deformations of a fixed object as above).

Let k be a fixed algebraically closed ground field, and let $\mathcal{C}$ be the category of local artinian k-algebras with residue field k. We consider a covariant functor F from $\mathcal{C}$ to (sets).

One example of such a functor is obtained as follows. Let R be a complete local k-algebra, and for each $A \in \mathcal{C}$, let $h_R(A)$ be the set of k-algebra homomorphisms $\mathrm{Hom}(R, A)$. For any morphism $A \to B$ in $\mathcal{C}$ we get a map of sets $h_R(A) \to h_R(B)$, so h_R is a functor from $\mathcal{C}$ to sets.

Definition 14.1. A functor $F : \mathcal{C} \to$ (sets) that is isomorphic to a functor of the form h_R for some complete local k-algebra R is called *pro-representable.*

To explain the nature of an isomorphism between h_R and F, let us consider more generally any homomorphism of functors $\varphi : h_R \to F$. In particular, for each n this will give a natural map $\varphi_n : \operatorname{Hom}(R, R/\mathfrak{m}^n) \to F(R/\mathfrak{m}^n)$, and the image of the natural map of R to $R/\mathfrak{m}^n$ gives an element $\xi_n \in F(R/\mathfrak{m}^n)$. These elements ξ_n are compatible, in the sense that the natural map $R/\mathfrak{m}^{n+1} \to R/\mathfrak{m}^n$ induces a map of sets $F(R/\mathfrak{m}^{n+1}) \to F(R/\mathfrak{m}^n)$ that send ξ_{n+1} to ξ_n. Thus the collection $\{\xi_n\}$ defines an element $\xi \in \varprojlim F(R/\mathfrak{m}^n)$.

Here it is useful to introduce the category $\hat{\mathcal{C}}$ of complete local k-algebras with residue field k. The category $\hat{\mathcal{C}}$ contains the category $\mathcal{C}$, and we can extend the functor F on $\mathcal{C}$ to a functor $\hat{F}$ from $\hat{\mathcal{C}}$ to sets by defining $\hat{F}(R) = \varprojlim F(R/\mathfrak{m}^n)$ for any $R \in \hat{\mathcal{C}}$. In this notation, the element ξ defined above is in $\hat{F}(R)$.

Conversely, any element $\xi = \{\xi_n\}$ of $\hat{F}(R)$ defines a homomorphism of functors $\varphi : h_R \to F$ as follows. For any $A \in \mathcal{C}$ and any homomorphism $f : R \to A$, since A is Artinian, if factors through $R/\mathfrak{m}^n$ for some n, say $f = g\pi$ where $\pi : R \to R/\mathfrak{m}^n$ and $g : R/\mathfrak{m}^n \to A$. Then we take $\varphi(f) =$ the image of ξ_n under the map $F(g) : F(R/\mathfrak{m}^n) \to F(A)$. It is easy to check these constructions are well-defined and inverse to each other, so we have

Lemma 14.2. *If F is a functor from $\mathcal{C}$ to (sets) and R is a complete local k-algebra with residue field k, then there is a natural bijection between the set $\hat{F}(R)$ and the set of homomorphisms of functors h_R to F.*

Thus, if F is pro-representable, there is an isomorphism $\xi : h_R \to F$ for some R, and we can think of ξ as an element of $\hat{F}(R)$. We say the pair (R, ξ) pro-represents the functor F. One can verify easily that if F is pro-representable, the pair (R, ξ) is unique up to unique isomorphism.

In many cases of interest, the functors we consider will not be pro-representable, so we define the weaker notion of having a versal family, which is a pair (R, ξ) giving a morphism $h_R \to F$ that is surjective in a strong sense.

Definition 14.3. Let $F : \mathcal{C} \to$ (sets) be a functor. A pair (R, ξ) with $R \in \hat{\mathcal{C}}$ and $\xi \in \hat{F}(R)$ is a *versal family* for F if the associated map $h_R \to F$ is surjective, i.e., for every $A \in \mathcal{C}$, $\operatorname{Hom}(R, A) \to F(A)$ is surjective; and furthermore, for every surjection $B \to A$ in $\mathcal{C}$, the map $\operatorname{Hom}(R, B) \to \operatorname{Hom}(R, A) \times_{F(A)} F(B)$ is also surjective. In other words, given a map $R \to A$ inducing an element $\eta \in F(A)$, and given $\theta \in F(B)$ mapping to η, one can lift the map $R \to A$ to a map $R \to B$ inducing θ.

If in addition the map $h_R(k[t]) \to F(k[t])$ is bijective, where $k[t]$ is the ring of dual numbers, we say that (R, ξ) is a *miniversal family*, or that the functor has a *pro-representable hull* (R, ξ). In case F is pro-representable, we also say that (R, ξ) is a *universal* family.

Lemma 14.4. *If the functor F has a miniversal family (R, ξ), then the pair (R, ξ) is uniquely determined up to isomorphism (but the isomorphism may not be unique).*

Proof. Suppose that (R, ξ) and (R', ξ') are two miniversal families. Consider the collection $\xi'_n \in F(R'/\mathfrak{m}^n)$. Because of the strong surjectivity property of the family (R, ξ), we can find compatible homomorphism $R \to R'/\mathfrak{m}^n$ for each n, and thus a map $R \to \varprojlim R'/\mathfrak{m}^n = R'$. On the other and because of the minimality of the second family (R', ξ'), we have $\operatorname{Hom}(R', k[t]) \xrightarrow{\sim} F(k[t])$. But this map factors through the induced map $\operatorname{Hom}(R', k[t]) \to \operatorname{Hom}(R, k[t])$, so this latter is injective. This implies that the dual map $\mathfrak{m}_R/\mathfrak{m}_R^2 \to \mathfrak{m}_{R'}/\mathfrak{m}_{R'}^2$ is surjective, hence the map $R \to R'$ is surjective, since both are complete local rings.

Running the same argument with R and R' interchanged, we find there exists also a surjective map $R' \to R$. Composing the two, we get a surjective map of R to itself, which must be an isomorphism. It follows that R and R' are isomorphic.

Example 14.5. Suppose that $\mathcal{F}$ is a globally defined contravariant functor from (Sch/k) to (sets). For example, think of the functor Hilb which to each scheme S/k associates the set of closed subschemes of $\mathbb{P}^n_S$, flat over S. Given a particular element $X_0 \in \mathcal{F}(k)$, we can define a local functor $F : \mathcal{C} \to$ (sets) by taking, for each $A \in \mathcal{C}$ the subset $F(A) = \mathcal{F}(\operatorname{Spec} A)$ consisting of those elements $X \in \mathcal{F}(\operatorname{Spec} A)$ that reduce to $X_0 \in \mathcal{F}(k)$.

If the global functor $\mathcal{F}$ is representable, say by some scheme M and a family $\mathcal{X} \in \mathcal{F}(M)$, it follows that the local functor F will be pro-representable. Just take $x_0 \in M$ to be the point corresponding to X_0, and let $R = \hat{\mathcal{O}}_{M,x_0}$, the completion of the local ring of x_0 on M. An element of $F(A)$, being in $\mathcal{F}(\operatorname{Spec} A)$, corresponds to a morphism of $\operatorname{Spec} A$ to M sending the closed point to x_0. This induces a homomorphism of $\mathcal{O}_{M,x_0} \to A$, and hence also of its completion R, since A is an Artin ring. Now one checks easily that F is isomorphic to the functor h_R on the category $\mathcal{C}$. Note however that the "formal family" $\xi \in \hat{F}(R)$ is not a family over R in the sense of the original global functor $\mathcal{F}$. Elements of $\hat{F}(R)$ correspond only to formal completions of elements of $\mathcal{F}(R)$.

Thus pro-representability of the local functor is a necessary condition for representability of the global functor.

Example 14.6. The converse of Example 14.5 is false: the local functor may be pro-representable when the global functor is not representable. Take for example deformations of $\mathbb{P}^1$. It is easy to see that this functor is not representable (§22). But since all local deformations over Artin rings are trivial, the local functor is pro-represented by the ring k.

Example 14.7. We have seen in the previous section that the functor of local deformations of a plane curve singularity has a miniversal deformation space. On the other hand, the functor is not pro-representable in general because of the non-uniqueness of the maps on the parameter spaces.

Example 14.8. For an example of a functor with no versal family, we note that if (R, ξ) is a versal family for the functor F, then the map $\operatorname{Hom}(R, k[t]) \to F(k[t])$ is surjective. Now if F is the functor of deformations of a k-algebra B, then $F(k[t])$ is given by $T^1(B, B)$, and this must be a finite-dimensional k-vector space. If $T^1(B, B)$ is not finite dimensional, F cannot have a versal deformation space. An example would be $B = k[x, y, z]/(xy)$. Then $T^1(B, B) = k[z]$. The trouble is that B does not have isolated singularities.

Remark 14.9. There is considerable variation in the literature concerning the exact hypotheses and terminology in setting up this theory. One need not assume k algebraically closed, for example, and then there is a choice whether to stick with local k-algebras having residue field k, or to allow finite field extensions. Also, one need not restrict to k-algebras. Sometimes it is convenient (e.g., for mixed characteristic cases) to take Artin algebras over a fixed ring such as the Witt vectors. Some people use "versal" to mean what we called "miniversal". Some call the latter "semi-universal". Some do not say universal but say only "weakly universal" for what we called universal, thinking more generally of the stack instead of the functor.

We have chosen what seems to be the most basic case, for simplicity.

15 Schlessinger's criterion

In this section we will prove Schlessinger's theorem [75], which gives a criterion for a functor of Artin rings to have a versal family.

We keep the notation of the previous section: k is a fixed algebraically closed field, $\mathcal{C}$ is the category of local artinian k-algebras, and F is a covariant functor from $\mathcal{C}$ to (sets). Note that the category $\mathcal{C}$ has fibered direct products. If $A' \to A$ and $A'' \to A$ are morphisms in $\mathcal{C}$, we take $A' \times_A A''$ to be the set-theoretical product $\{(a', a'') \mid a'$ and a'' have the same image in $A\}$. The ring operations extend naturally, giving another object of $\mathcal{C}$, and this object is also the categorical fibered direct product in $\mathcal{C}$.

It is also convenient to introduce the notation t_F for $F(k[t])$. We call this the *tangent space* of F. Similarly t_R denotes the tangent space of the functor h_R, which is just $\operatorname{Hom}_k(R, k[t])$ and is equal to the dual vector space of $\mathfrak{m}_R/\mathfrak{m}_R^2$.

A *small extension* in $\mathcal{C}$ is a surjective map $A' \to A$ whose kernel I is a one-dimensional k-vector space.

We begin with some necessary conditions.

Proposition 15.1. *If F has a versal family, then*

a) *$F(k)$ has just one element, and*

b) *for any morphisms $A' \to A$ and $A'' \to A$ in $\mathcal{C}$, the natural map*

$$F(A' \times_A A'') \to F(A') \times_{F(A)} F(A'')$$

is surjective.

If furthermore F has a miniversal family, then

c) *for any $A \in \mathcal{C}$, considering the maps $A \to k$ and $k[t] \to k$, the map of* b) *above,*

$$F(A \times_k k[t]) \to F(A) \times_{F(k)} F(k[t]),$$

is bijective.

d) *$F(k[t]) = t_F$ has a natural structure of a finite-dimensional k-vector space.*

e) *For any small extension $p : A' \to A$ and any element $\eta \in F(A)$, there is a transitive group action of the vector space t_F on the set $p^{-1}(\eta)$ (provided it is non-empty).*

Proof.

a) Since $\mathrm{Hom}(R, k) \to F(k)$ is surjective, and $\mathrm{Hom}(R, k)$ has just one element, so does $F(k)$.

b) Given elements $\eta' \in F(A')$ and $\eta'' \in F(A'')$ mapping to the same element $\eta \in F(A)$, by the strong surjective property of a versal family, there are compatible homomorphisms of R to A', A, and A'' inducing these elements. Then there is a unique map of R to the product $A' \times_A A''$ inducing the given maps of R to A' and A''. This in turn defines an element of $F(A' \times_A A'')$ that restricts to η' and η'' as required. Note that although the map of R to $A' \times_A A''$ is uniquely determined by the maps of R to A' and A'', these latter may not be uniquely determined by η' and η'', and so the resulting element in $F(A' \times_A A'')$ may not be uniquely determined.

c) Suppose given $\eta \in F(A)$ and $\xi \in F(k[t])$. We know from b) there are elements of $F(A \times_k k[t])$ lying over both η and ξ. Suppose θ_1 and θ_2 are two such. Choose a homomorphism $u : R \to A$ inducing η. Since $A \times_k k[\varepsilon] = A[\varepsilon] \to A$ is surjective, we can lift u to v_1 and $v_2 : R \to A[\varepsilon]$ inducing θ_1 and θ_2. Since θ_1 and θ_2 both lie over ξ, the projections of v_1 and v_2 to $k[t]$ both induce ξ. By the hypothesis of miniversality, $t_R \to t_F$ is bijective, so these restrictions are equal. Since v_1 and v_2 also induce the same map $u : R \to A$, we find $v_1 = v_2$ and hence $\theta_1 = \theta_2$.

d) By miniversality, $t_R \to t_F$ is bijective, so we can just carry over the vector space structure on t_R to t_F. But this structure can also be recovered intrinsically, using only the functorial properties of F and condition c) above. If $\lambda \in k$, the ring homomorphism $k[\varepsilon] \to k[\varepsilon]$ sending ε to $\lambda\varepsilon$ induces a map of sets $t_F \to t_F$. This is scalar multiplication. The ring homomorphism $k[\varepsilon_1] \times_k k[\varepsilon_2] \to k[\varepsilon]$ sending $\varepsilon_i \to \varepsilon$ for $i = 1, 2$ induces a map of sets of $k[\varepsilon_1] \times_k k[\varepsilon_2] = t_F \times t_F$ by c) to t_F. This is addition. Finite-dimensionality follows from the fact that t_R is finite-dimensional.

e) Let $A' \to A$ be a small extension with kernel $I \cong k$. Note that $A' \times_A A' \cong A' \times_k k[I]$ by sending $(x, y) \mapsto (x, x_0 + y - x)$ where $x_0 \in k$ is the residue of $x \bmod \mathfrak{m}$. Consider the surjective map

$$F(A' \times_A A') \to F(A') \times_{F(A)} F(A')$$

of b). Using the isomorphism above, and condition c), we can reinterpret the left-hand side as $F(A' \times_k k[I]) \cong F(A') \times t_F$, and we get a surjective map

$$F(A') \times t_F \to F(A') \times_{F(A)} F(A')$$

that is an isomorphism on the first factor. If we take $\eta \in F(A)$ and fix $\eta' \in p^{-1}(\eta)$ then we get a surjective map

$$\{\eta'\} \times t_F \to \{\eta'\} \times p^{-1}(\eta),$$

and this gives a transitive group action of t_F on $p^{-1}(\eta)$.

Theorem 15.2 (Schlessinger's criterion). *The functor $F : \mathcal{C} \to$ (sets) has a miniversal family if and only if*

(H_0) *$F(k)$ has just one element.*

(H_1) *$F(A' \times_A A'') \to F(A') \times_{F(A)} F(A'')$ is surjective for every small extension $A'' \to A$.*

(H_2) *The map of H_1 is bijective when $A'' = k[t]$ and $A = k$.*

(H_3) *t_F is a finite-dimensional k-vector space.*

Furthermore, F is pro-representable if and only if in addition

(H_4) *For every small extension $p : A'' \to A$ and every $\eta \in F(A)$ for which $p^{-1}(\eta)$ is non-empty, the group action of t_F on $p^{-1}(\eta)$ is bijective.*

Proof. The necessity of conditions H_0, H_1, H_2, H_3 has been seen in (15.1). If F is pro-representable, the maps in the proof of (15.1e) are all bijective, so the operation of t_F on $p^{-1}(\eta)$ is bijective whenever this set is non-empty.

So now let F be a functor satisfying conditions H_0, H_1, H_2, H_3. First we will construct a ring R and a morphism $h_R \to F$. Then we will show that it has the versal family property.

We define R as an inverse limit of rings R_q that we construct inductively for $q \geq 0$. We take $R_0 = k$. Note from the proof of (15.1d) that the vector space structure on t_F is already determined by conditions H_0, H_1, H_2, so that it makes sense to say in H_3 that it is finite-dimensional. Let $t_1, \dots, t_r$ be a basis of the dual vector space t_F^*, let S be the formal power series ring $k[[t_1, \dots, t_r]]$, with maximal ideal $\mathfrak{m}$, and take $R_1 = S/\mathfrak{m}^2$. Then $t_{R_1} \cong t_F$ by construction. Furthermore by iterating the condition H_2, we find $F(R_1) = F(k[t_1] \times \ldots \times k[t_r]) \cong t_F \otimes_k t_F^*$. The natural element here gives $\xi_1 \in F(R_1)$ inducing the isomorphism $t_{R_1} \cong t_F$.

Now suppose we have constructed a compatible sequence (R_i, ξ_i) for $i = 1, \dots, q$, with $\xi_i \in F(R_i)$, where each $R_i = S/J_i$, and for each i $\mathfrak{m}^{i+1} \leq J_i \leq J_{i-1}$, and for each i the natural map $R_i \to R_{i-1}$ sends ξ_i to ξ_{i-1}. Then, to construct R_{q+1}, we look at ideals J in S, with $\mathfrak{m}J_q \leq J \leq J_q$, and take J_{q+1} to be the minimal such ideal J with the property that $\xi_q \in F(R_q)$ lifts to an element $\xi' \in F(S/J)$. To show that there is a minimal such J, it will be sufficient to show that if J and K are two such, then their intersection $J \cap K$ is another one. By expanding J or K we may assume without loss of generality that $J + K = J_q$. In that case $S/J \cap K = (S/J) \times_{(S/J_q)} (S/K)$. Now the existence of liftings of ξ_q over S/J and S/K implies by condition H_1 the existence of a lifting over $S/J \cap K$. Note that by iteration, H_1 implies surjectivity of the given map for any surjective $A'' \to A$, since any surjective map can be factored into a finite number of small extensions. Then we take $R_{q+1} = S/J_{q+1}$, and ξ_{q+2} any lifting of ξ_q, which exists by construction.

Thus we obtain a surjective system of rings R_q and and compatible elements $\xi_q \in F(R_q)$. We define $R = \varprojlim R_q$. Since each $J_q \leq \mathfrak{m}^{q+1}$ by construction, the ideals J_q form a base for the topology of R, and so we can form $\xi = \varprojlim \xi_q \in \hat{F}(R)$.

I claim (R, ξ) is a miniversal family for F. Since $t_R \cong t_F$ by construction, we have only to show for any surjective map $A' \to A$ and any $\eta' \in A'$ restricting to $\eta \in A$, and any map $R \to A$ inducing η, that there exists a lifting to a map $R \to A'$ inducing η'. Since any surjective map factors into a sequence of small extensions, it is sufficient to treat the case of a small extension $A' \to A$.

Let $u : R \to A$ induce η. It will be sufficient to show that u lifts to some map $v : R \to A'$. For then v will induce an element $\eta'' \in F(A')$ lying over η. Because of (15.1c), whose proof used only (15.1e), which is our H_2, there is an element of t_F sending η'' to η' by the group action. This same $t_F = t_R$ acts on the set of $v : R \to A'$ restricting to u, so then we can adjust v to a homomorphism $v' : R \to A'$ inducing η'.

Thus it remains to show that for a small extension $A' \to A$, the given map $u : R \to A$ lifts to a map $v : R \to A'$. Since A is an Artin ring, u factors through R_q for some q. On the other hand, R is a quotient of the power series ring S, and the map u will lift to a map of S into A'. Thus we get a commutative diagram

$$\begin{array}{ccccc} S & \xrightarrow{w} & R_q \times_A A' & \to & A' \\ \downarrow & & \downarrow p' & & \downarrow p \\ R & \to & R_q & \to & A \end{array}$$

$$u$$

Note that $p' : R_q \times_A A' \to R_q$ is also a small extension. If this map has a section $s : R_q \to R_q \times_A A'$, then using s and the second projection we get a map $R_q \to A'$ lifting u, and we are done.

If p' does not have a section, then I claim the map w is surjective. Indeed, if w is not surjective, then $\operatorname{Im} w$ is a subring mapping surjectively to R_q. The kernel of $\operatorname{Im} w \to R_q$ must be strictly contained in $I = \ker p'$, which is a one-dimensional vector space, so this kernel is zero, the map $\operatorname{Im} w \to R_q$ is an isomorphism and this gives a section. Contradiction!

Knowing thus that w is surjective, let $J = \ker w$. Then $J \subseteq J_q$ since S maps to R_q via w. On the other hand, $J \supseteq \mathfrak{m} J_q$ since p' is a small extension. But also we have $\xi \in F(R_q)$ and there is an $\eta' \in F(A')$ lying over $\eta \in F(A)$, so by H_1, there is a $\xi' \in F(R_q \times_A A')$ lying over both of these. Since $R_q \times_A A' = S/J$, this ideal J satisfies the condition imposed in the construction. Therefore $J \supseteq J_{q+1}$ and w factors through R_{q+1}. This gives the required lifting of R to A', and completes the proof that (R, ξ) is a miniversal family for F.

Finally, suppose in addition F satisfies H_4. To show that $h_R(A) \to F(A)$ is bijective for all A, it will be sufficient to show inductively, starting with $A = k$, that for any small extension $p : A' \to A$ we have $\mathrm{Hom}(R, A') \to \mathrm{Hom}(R, A) \times_{F(A)} F(A')$ is bijective. So fix $u \in \mathrm{Hom}(R, A)$ and the corresponding $\eta \in F(A)$. If there is no map of R to A' lying over u, then there is also no $\eta' \in F(A')$ lying over η, and there is nothing to prove. On the other hand, if $p^{-1}(\eta)$ is non-empty, then the action of t_F on $p^{-1}(\eta)$ is bijective by H_4, and the action of t_R on the set of homomorphism $R \to A'$ lying over u is bijective since h_R is pro-representable, and $t_k \cong t_F$ by miniversality, so our map is bijective as required. $\square$

Remark 15.3. If (H_4) holds, then the map of (H_1) is bijective for all small extensions $A'' \to A$. In particular, (H_4) implies (H_2). Indeed, if $A'' \to A$ is a small extension, then $A' \times_A A'' \to A'$ is also a small extension. So the set of elements of $F(A' \times_A A'')$ going to a fixed element α' of $F(A')$ is in one-to-one correspondence with the set of elements of $F(A'')$ going to the image of α' in $F(A)$. Hence $F(A' \times_A A'') = F(A') \times_{F(A)} F(A'')$.

References for this section. For the proof, I have mainly followed Schlessinger's original paper [75]. The proof is also given in the Appendix of Sernesi's notes [79], and in abbreviated form in Artin's Tata lectures [3].

16 Fibred products and flatness

This section contains some technical results on fibred products and flatness that will be used in studying the pro-representability and existence of versal families for various functors. It can be skipped at a first reading and referred to as needed.

We are dealing here with fibred products of sets. If A' and A'' are sets, with maps $A' \to A$, $A'' \to A$ to a set A, then the fibred product is

$$A' \times_A A'' = \{(a', a'') \mid a' \text{ and } a'' \text{ have the same image } a \in A\}.$$

If A, A', A'' have structures of abelian groups, or rings, or rings with identity and the maps respect these structures, then $A' \times_A A''$ has a structure of the same kind. This product is categorical, namely given any set C together with maps $C \to A'$ and $C \to A''$ that compose to give the same map to A, there exists a unique map of C to $A' \times_A A''$ factoring the given maps.

Note that if we consider the schemes $\operatorname{Spec} A$, $\operatorname{Spec} A'$, $\operatorname{Spec} A''$, this is not related to the fibred product in the category of schemes. The arrows go in the opposite direction.

If M, M', M'' are modules over the rings A, A', A'' and we are given compatible maps of modules $M' \to M$, $M'' \to M$, then $M' \times_M M''$ is a module over $A' \times_A A''$.

If $\mathcal{F}, \mathcal{F}', \mathcal{F}''$ are sheaves of abelian groups on a fixed topological space X_0, and we are given maps $\mathcal{F}' \to \mathcal{F}$ and $\mathcal{F}'' \to \mathcal{F}$, the assignment of $\mathcal{F}(U) \times_{\mathcal{F}(U)} \mathcal{F}''(U)$ to each open set U is a sheaf of abelian group on X_0, which we will denote simply $\mathcal{F}' \times_{\mathcal{F}} \mathcal{F}''$.

If $\mathcal{O}_X, \mathcal{O}_{X'}, \mathcal{O}_{X''}$ are sheaves of rings on X_0, together with maps $\mathcal{O}_{X'} \to \mathcal{O}_X$ and $\mathcal{O}_{X''} \to \mathcal{O}_X$, and $\mathcal{F}, \mathcal{F}', \mathcal{F}''$ are sheaves of modules with maps over the respective sheaves of rings, then $\mathcal{F}' \times_{\mathcal{F}} \mathcal{F}''$ is a sheaf of modules over the sheaf of rings $\mathcal{O}_{X'} \times_{\mathcal{O}_X} \mathcal{O}_{X''}$.

If $\mathcal{O}_X, \mathcal{O}_{X'}, \mathcal{O}_{X''}$ define scheme structures on the topological space X_0, then so does $\mathcal{O}_{X'} \times_{\mathcal{O}_X} \mathcal{O}_{X''}$. This will be a fibred sum in the category of scheme structures on X_0. One just has to check that localization is compatible with fibred product of rings and modules.

Lemma 16.1. *Let A, A', A'' be abelian groups, with maps $A' \to A$, $A'' \to A$. In the diagram*

$$\begin{array}{ccccccc} 0 & \to & \ker u' & \to & A' \times_A A'' & \xrightarrow{u'} & A' \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \to & \ker u & \to & A'' & \xrightarrow{u} & A \end{array}$$

a) *the natural map* $\ker u' \to \ker u$ *is bijective*

b) *if u is surjective, so is u'.*

Proof. Immediate diagram chasing.

Remark 16.2. The same applies to rings, modules, and sheaves on a fixed topological space.

Proposition 16.3. *Let A, A', A'' be rings with maps as before, and let $A^* = A' \times_A A''$. Let M, M', M'' be modules over A, A', A'' respectively, with compatible maps $M' \to M$ and $M'' \to M$, and assume that $M' \otimes_{A'} A \to M$ and $M'' \otimes_{A''} A \to M$ are isomorphisms. Let $M^* = M' \times_M M''$.*

a) *Assume $A'' \to A$ is surjective. Then $M^* \otimes_{A^*} A' \to M'$ is an isomorphism.*

b) *Now assume furthermore that $J = \ker(A'' \to A)$ is an ideal of square zero, and that M', M'' are flat over A', A'' respectively. Then M^* is flat over A^*, and also $M^* \otimes_{A^*} A'' \to M''$ is an isomorphism.*

Proof.

a) Since $A'' \to A$ is surjective and $M'' \otimes_{A''} A = M$, it follows that $M'' \to M$ is surjective. Then by Lemma 16.1, $M^* \to M'$ is surjective, and hence $M^* \otimes_{A^*} A' \to M'$ is surjective. To show injectivity, we consider an element $\Sigma\langle m_i', m_i''\rangle \otimes b_i$ in the kernel of this map and show by usual properties of the tensor product that it is zero (left as an amusing exercise for the reader).

b) Since M'' is flat over A'' and $M'' \otimes_{A''} A \cong M$, and $J^2 = 0$, we have an exact sequence

$$0 \to J \otimes_A M \to M'' \to M \to 0$$

by the Local Criterion of Flatness (6.2). From Lemma 16.1 it then follows that

$$0 \to J \otimes_A M \to M^* \to M' \to 0$$

is also exact. Now M' is flat over A' by hypothesis, and $M^* \otimes_{A^*} A' \cong M'$ by part a) above, and since $M' \otimes_{A'} A \cong M$ we have also $J \otimes_A M = J \otimes_{A^*} M^* = J \otimes_{A'} M'$. Now again by (6.2) it follows that M^* is flat over A^*. (Note the kernel of $A^* \to A'$ is again J with $J^2 = 0$.)

For the last statement, we tensor the sequence $0 \to J \to A'' \to A \to 0$ with M^*, to obtain

$$0 \to J \otimes M^* \to M^* \otimes_{A^*} A'' \to M^* \otimes_{A^*} A \to 0.$$

On the right we have just M, because of part a) and the hypothesis $M' \otimes_{A'} A \cong M$, and on the left we have $J \otimes M$, so comparing with the sequence for M'' above we find $M^* \otimes_{A^*} A'' \to M''$ also an isomorphism.

Example 16.4 (Schlessinger). Without the hypothesis $A'' \to A$ surjective, the proposition may fail. For example, take $A = k[t]/(t^3)$, take $A' = A'' = k[x]/(x^2)$, and for homomorphisms send x to t^2. Then $A^* = A' = A''$. Now take $M = A$, $M' = M'' = A' = A''$ (note these are all flat) and for morphisms take $M' \to M$ the natural injection, but for $M'' \to M$ the natural injection followed by multiplication by the unit $1 + t$. Then $M' \otimes_{A'} A \cong M$ and $M^* \otimes_{A''} A \cong M$, but M^* is just $k \cdot x$ which is not flat over A^*, nor does its tensor product with A' or A'' give M' or M''.

Reference for this section. The main reference is again Schlessinger's paper [75], though he proves 16.3 only for free modules.

17 Hilb and Pic are pro-representable

There is a general theorem of Grothendieck [22, exp. 221] that the Hilbert functor, parametrizing closed subschemes of a given projective scheme over k is representable. From this it follows (14.5) that the local functor is pro-representable. However, the proof of existence of the Hilbert scheme is long and involved (and not given in this book), so it is of some interest to give an independent proof of pro-representability of the local Hilb functor.

Let X_0 be a given closed subscheme of $\mathbb{P}^n_k$. For each local artinian k-algebra A we let $F(A)$ be the set of deformations of X_0 over A, that is, the set of closed subschemes $X \subseteq \mathbb{P}^n_A$, flat over A, such that $X \times_A k \cong X_0$. (Here by abuse of notation, $X \times_A k$ means $X \times_{\operatorname{Spec} A} \operatorname{Spec} k$, the fibred product in the category of schemes, not the fibred product of sets!) Then F is a functor from the category $\mathcal{C}$ of local artinian k-algebras to sets, which we call the local Hilb functor of deformations of X_0.

Theorem 17.1. *For a given closed subscheme $X_0 \subseteq \mathbb{P}^n_k$, the local Hilb functor F is pro-representable.*

Proof. We apply Schlessinger's criterion (15.2). Condition (H_0) says $F(k)$ should have just one element, which it does, namely X_0 itself.

Condition (H_1) say for every small extension $A'' \to A$, and any map $A' \to A$, the map

$$F(A' \times_A A'') \to F(A') \times_{F(A)} F(A'')$$

should be surjective. So suppose given closed subschemes $X' \subseteq \mathbb{P}^n_{A'}$ and $X'' \subseteq \mathbb{P}^n_{A''}$, both restricting to $X \subseteq \mathbb{P}^n_A$, and both flat over A' and A'' respectively. We let X^* be the scheme structure on the topological space X_0 defined by the fibred product of sheaves of rings $\mathcal{O}_{X'} \times_{\mathcal{O}_X} \mathcal{O}_{X''}$ (§16). Letting $A^* = A' \times_A A''$, we have surjective maps of sheaves $\mathcal{O}_{\mathbb{P}^n_{A^*}}$ to $\mathcal{O}_{X'}$ and to $\mathcal{O}_{X''}$, giving the same composed map to $\mathcal{O}_X$, hence a surjective map to $\mathcal{O}_{X^*}$. Therefore X^* is a closed subscheme of $\mathbb{P}^n_{A^*}$. It is flat over A^* and restricts to $\mathcal{O}_{X'}$ and $\mathcal{O}_{X''}$ over A' and A'' by (16.3). Thus X^* is an element of $F(A^*)$ mapping to X' and X'', and (H_1) is satisfied.

Condition (H_2) is a consequence of (H_4) (15.3).

For (H_3) we note that $t_F = F(k[t])$ is the set of deformations of X_0 over $k[t]$, which by (2.4) is $H^0(X_0, \mathcal{N}_{X_0/\mathbb{P}^n})$. Since X_0 is projective, this is a finite-dimensional vector space.

For (H_4), let $\eta \in F(A)$ be given by a deformation $X \subseteq \mathbb{P}^n_A$ of X_0. Then $p^{-1}(\eta)$ consists of subschemes $X' \subseteq \mathbb{P}^n_{A'}$, flat over A', with $X' \times_{A'} A \cong X$. If such exist, they form a torsor under the action of t_F by (6.3).

Thus all the conditions are satisfied and F is pro-representable.

There is also a general theorem of Grothendieck [22, exp. 232] that the Picard functor is representable, from which it follows that the local functor is pro-representable, but here we give an independent proof.

Let X_0 be a given scheme over k, and $\mathcal{L}_0$ a given invertible sheaf on X_0. The local Picard functor F assigns to each local artinian k-algebra A the set of isomorphism classes of invertible sheaves $\mathcal{L}$ on $X = X_0 \times_k A$ for which $\mathcal{L} \otimes \mathcal{O}_{X_0} \cong \mathcal{L}_0$.

Lemma 17.2. *Assume that X_0 is projective over k and that $H^0(X_0, \mathcal{O}_{X_0}) = k$. Then the local Picard functor for a given invertible sheaf $\mathcal{L}_0$ on X_0 is pro-representable.*

Proof. We apply Schlessinger's criterion. $F(k)$ consists of the one element $\mathcal{L}_0$, so (H_0) is satisfied. For (H_1), let invertible sheaves $\mathcal{L}'$ on X' and $\mathcal{L}''$ on X'' be given such that $\mathcal{L}' \otimes \mathcal{O}_X \cong \mathcal{L}'' \otimes \mathcal{O}_X \cong \mathcal{L}$ on X.

Choose maps $\mathcal{L}' \to \mathcal{L}$ and $\mathcal{L}'' \to \mathcal{L}$ inducing these isomorphisms. Then we take $\mathcal{L}^* = \mathcal{L}' \times_{\mathcal{L}} \mathcal{L}''$ to be the fibred product of sheaves. It is an invertible sheaf on $X^* = X_0 \times_k A^*$, where $A^* = A' \times_A A''$, and by (16.3) it restricts to $\mathcal{L}'$ on X' and $\mathcal{L}''$ on X''. Thus (H_1) holds.

(H_2) is a consequence of (H_4) (15.3).

By (2.5), the tangent space t_F is $H^1(X_0, \mathcal{O}_{X_0})$, which is finite-dimensional since X_0 is projective, so (H_3) holds. (H_4) is a direct consequence of (6.4) since we have assumed $H^0(\mathcal{O}_{X_0}) = k$. Thus F is pro-representable.

References for this section. The Hilbert scheme was first constructed by Grothendieck [22]. Other proofs of its existence can be found in Mumford [57] in a special case; in the lecture notes of Sernesi [79]; and in the book of Kollár [44]. Representability of the Picard functor was also proved by Grothendieck [22].

18 Miniversal and universal deformations of schemes

In this section we will discuss the question of pro-representability or existence of a miniversal family of deformations of a scheme.

If we start with a global moduli problem, such as the moduli of curves of genus g, the global functor considers flat families X/S for a scheme S, whose geometric fibers are projective non-singular curves of genus g, up to isomorphism of families. The formal local version of this functor, around a given curve X_0/k would assign to each Artin ring A with residue field k, the set $F_1(A)$ of isomorphism classes of flat families X/A such that $X \otimes_A k$ is isomorphic to X_0.

For the purposes of this section we will consider a slightly different functor of deformations of X_0. We define a *deformation of* X_0 *over* A to be a pair (X, i) where X is a scheme flat over A, and where $i : X_0 \to X$ is a closed immersion such that the induced map $i \otimes k : X_0 \to X \otimes_A k$ is an isomorphism. We consider the functor $F(A)$ which to each A assigns the set of deformations (X, i) of X_0 over A, up to isomorphism, where an isomorphism of (X_1, i_1) and (X_2, i_2) means an isomorphism $\varphi : X_1 \to X_2$ compatible with the maps i_1, i_2 from X_0.

The effect of using the functor F instead of F_1 is to leave possible automorphisms of X_0 out of the picture and thus simplify the discussion. If X_0 has no non-trivial automorphisms, then the functors F and F_1 are equivalent. We will discuss the relation between these two functors more at the end of this section.

Theorem 18.1. *Let X_0 be a scheme over k. Then the functor F (defined above) of deformations of X_0 over local Artin rings has a miniversal family under either of the two following hypotheses:*

a) *X_0 is affine with isolated singularities.*

b) *X_0 is projective.*

Proof. We verify the conditions of Schlessinger's criterion (15.2).

(H_0) $F(k)$ consists of the single object (X_0, id). If σ is an automorphism of X_0, the object (X_0, σ) is isomorphic to (X_0, id) by the map $\sigma : X_0 \to X_0$.

(H_1) Suppose given a small extension $A'' \to A$ and any map $A' \to A$, and suppose given objects $X' \in F(A'')$, $X'' \in F(A'')$ restricting to $X \in F(A)$. Then $X' \otimes_{A'} A \cong X$, the isomorphism being compatible with the maps from X_0, so we can choose a closed immersion $X \hookrightarrow X'$ inducing this isomorphism. Similarly choose $X \hookrightarrow X''$. Then we define X^* by the fibred product of sheaves of rings $\mathcal{O}_{X^*} = \mathcal{O}_{X'} \times_{\mathcal{O}_X} \mathcal{O}_{X''}$, and X^* will be an object of $F(A^*)$ reducing to X' and X'', where $A^* = A' \times_A A''$ (16.3).

(H_2) Suppose $A = k$ in the situation of (H_1) (which effectively means $A'' \cong k[t]$) and let X^* be constructed as in (H_1). If W is any other object of $F(A^*)$ restricting to X' and X'' respectively, then we can choose immersions $X' \hookrightarrow W$ and $X'' \hookrightarrow W$ inducing these isomorphisms. Since these maps are all compatible with the immersions from X_0, they agree with the chosen maps $X \hookrightarrow X'$ and $X \hookrightarrow X''$, since in this case $X = X_0$. Now by the universal property of fibred product of rings, there is a map $X^* \to W$ compatible with the above maps. Since X^* and W are both flat over A^*, and the map becomes an isomorphism when restricted to X_0, we find X^* is isomorphic to W, and hence they are equal as elements of $F(A^*)$.

(H_3) Here is the only place in the proof that we need the hypothesis a) or b).

a) Let $X_0 = \operatorname{Spec} B$, and choose a surjective map $R = k[x_1, \dots, x_n] \to B$ with kernel I. Then there is an exact sequence (3.10)

$$0 \to T^0_B \to T^0_R \otimes B \to \operatorname{Hom}(I/I^2, B) \to T^1_B \to 0.$$

The middle map is surjective at smooth points of X_0, so T^1_B is supported at the singularities of X_0. As such it corresponds to a sheaf of finite length, so $T^1_B = t_F$ (5.2) is a finite-dimensional k-vector space.

b) For arbitrary X_0, the tangent space t_F corresponds to deformations over the dual numbers D. Because of the exact sequence (5.6)

$$0 \to H^1(X_0, T^0_{X_0}) \to \operatorname{Def}(X_0/k, D) \to H^0(X_0, T^1_{X_0}) \to \dots$$

we see that if X_0 is projective, the two outside groups are finite-dimensional vector spaces, and so $\operatorname{Def}(X_0/k, D)$ is also.

Thus conditions (H_0)–(H_3) are satisfied, and F is a miniversal family.

Examples 18.2. We have seen (13.1) that the plane curve singularity $xy = 0$ has a miniversal deformation space, but that the functor is not pro-representable. Thus the theorem cannot be improved without further hypotheses.

18.3. Example of a non-singular projective variety X_0 for which the functor is not pro-representable.

(to be added)

Next, we consider conditions under which the functor F is actually pro-representable.

Theorem 18.4. *Let X_0/k be given and assume the hypotheses of* (18.1) *satisfied. The functor F of deformations of X_0 is pro-representable if and only if for each small extension $A' \to A$, and for each deformation X' over A', restricting to a deformation X over A, the natural* *map* $\mathrm{Aut}(X'/X_0) \to \mathrm{Aut}(X/X_0)$ *of automorphisms of* X' *(resp.* X*) restricting to the identity automorphism of* X_0*, is surjective.*

Proof. Suppose that $\mathrm{Aut}(X'/X_0) \to \mathrm{Aut}(X/X_0)$ is surjective for every X' lying over X. If $X \hookrightarrow X'_1$ and $X \hookrightarrow X'_2$ are maps inducing the isomorphisms $X'_1 \otimes_{A'} A \cong X$ and $X'_2 \otimes_{A'} A \cong X$, and if X'_1 and X'_2 are isomorphic over X_0, then I claim the inclusions $X \hookrightarrow X'_1$ and $X \hookrightarrow X'_2$ are isomorphic as deformations of X over A'.

Indeed, let $u' : X'_1 \to X'_2$ be an isomorphism over X_0. Then $u = u' \otimes_{A'} A$ is an automorphism of X over X_0. By hypothesis this lifts to an automorphism σ of X'_1. Then $v = u \circ \sigma^{-1} : X'_1 \to X'_2$ is an isomorphism inducing the identity on X, so we get isomorphic elements of $\mathrm{Def}(X/A, A')$.

Now by (5.6) and (10.6) $\mathrm{Def}(X/A, A')$ is a principal homogeneous pace under the action of t_F, so condition (H_4) of Schlessinger's criterion is satisfied.

Conversely, suppose that F is pro-representable, let $X \in F(A')$ restrict to $X \in F(A)$, and choose a map $u : X \hookrightarrow X'$ inducing the isomorphism $X \xrightarrow{\sim} X' \otimes_{A'} A$. Let $\sigma \in \mathrm{Aut}(X/X_0)$. Then $u' = \sigma \circ u : X \hookrightarrow X'$ gives another element of $\mathrm{Def}(X/A, A')$, and so u and u' differ by an element of t_F, by (5.6) and (10.6). But u and u' define the same element $X' \in F(A')$, lying over X, so by condition (H_4), this element of t_F must be zero. Hence u and u' are equal as elements of $\mathrm{Def}(X/A, A')$, in other words there exists an isomorphism $\tau : X' \to X'$ over X_0 such that $u' = \tau \circ u$. Restricting to X we find $\sigma = \tau \mid X$. Thus $\tau \in \mathrm{Aut}(X'/X)$ lifts σ, and the map is surjective.

Satisfying as it may be to have a necessary and sufficient condition for pro-representability, this condition is difficult to apply in practice, so we will give some practical corollaries and examples.

Corollary 18.5. *Let* X/k *be a projective scheme with* $H^0(X, T^0_X) = 0$ *(in which case we say "*X *has no infinitesimal automorphisms"). Then the functor of deformations of* X/k *is pro-representable.*

Proof. (For the proof, we denote X by X_0.) We will show, by induction on the length of A, that for any deformation X of X_0 over A, $\mathrm{Aut}(X/X_0) = \{\mathrm{id}\}$. Then obviously the condition of (18.4) will be satisfied.

We start the induction by noting that $\mathrm{Aut}(X_0/X_0) = \{\mathrm{id}\}$. And here it is important that we are using the functor F, and not the other functor F_1 mentioned at the beginning of this section! Thus it does not matter if X_0 has automorphisms as a scheme over k.

Inductively, assume that $\mathrm{Aut}(X/X_0) = \{\mathrm{id}\}$, where X is a deformation over A. Consider a small extension $A' \to A$ and any $X' \in F(A')$ restricting to X. Choose a map $X \hookrightarrow X'$ inducing the isomorphism $X \xrightarrow{\sim} X' \otimes_{A'} A$. Any automorphism of X' restricts to the identity on X, by the induction hypothesis, so it is an automorphism of the deformation $X \hookrightarrow X'$. Since these are classified by $H^0(X_0, T^0_{X_0}) = 0$, (10.6.2) this automorphism is the identity.

Example 18.6. Let X_0 be a non-singular projective curve of genus $g \geq 2$ over k. The tangent sheaf T^0 has degree $2-2g < 0$, so $H^0(X_0, T^0)$ is 0. Thus the functor of deformations of X_0 is pro-representable. Note it does not matter if X_0 has a finite group of automorphisms. What counts here is that it has no infinitesimal automorphisms, such as might arise for example from a continuous group of automorphisms of X_0.

18.7. On the other hand, if we take $X_0 = \mathbb{P}^1_k$, the tangent bundle T^0 is isomorphic to $\mathcal{O}(2)$, and $H^0(X_0, T^0)$ has dimension 3. There are infinitesimal automorphisms. Nevertheless, the functor is pro-representable, as we have seen (14.6), reduced to a single point. Thus the condition of the corollary is not necessary for pro-representability.

For the case of elliptic curves see (18.10) below.

Surfaces of degree $d \geq 3$ in $\mathbb{P}^3$: see (21.4) below.

Now let us return to the question of comparing the functors F and F_1 mentioned at the beginning of this section. Recall that given a scheme X_0/k, F is the functor of deformations of X_0 over A, that is pairs (X, i) where X is flat over A, and $i : X_0 \hookrightarrow X$ is a morphism such that $i \otimes k : X_0 \to X \otimes k$ is an isomorphism, while F_1 is the functor of flat families X/A such that there exists an isomorphism $X \otimes k \cong X_0$, modulo isomorphism. There is a natural "forgetful" functor from F to F_1, which is clearly surjective.

The following result is proved using the same kind of arguments as in the earlier part of this section, so we leave the proof to the reader.

Theorem 18.8. *Suppose the hypotheses of* (18.1) *satisfied. Then*

a) *F_1 has a miniversal family if and only if in addition* $\operatorname{Aut} X \to \operatorname{Aut} X_0$ *is surjective for each flat family X over the dual numbers $k[\varepsilon]$. In this case $t_{F_1} = t_F$.*

b) *The following conditions are equivalent:*

   (i) *$F_1 = F$ and F is pro-representable.*

   (ii) *F_1 is pro-representable.*

   (iii) $\operatorname{Aut} X' \to \operatorname{Aut} X$ *is surjective for every small extension $A' \to A$, where X' is a flat family over A' and $X = X' \otimes_{A'} A$.*

Example 18.9. Let us take X_0 to be the affine scheme $\operatorname{Spec} k[x,y]/[xy]$. This is the node that was discussed previously (13.1).

a) It is easy to check that the automorphisms of X_0 are of two types

$$1) \quad \begin{cases} x' = ax \\ y' = by \end{cases} \qquad\qquad 2) \quad \begin{cases} x' = ay \\ y' = bx \end{cases}$$

where $a, b \in k^*$. If we attempt to lift an automorphism of type 1) to the family $xy - t$ over the dual numbers $D = \operatorname{Spec} k[t]/t^2$, we will need

$$\begin{cases} x' = ax + tf \\ y' = by + tg \end{cases}$$

for some $f, g \in k[x,y]$, satisfying $u(xy - t) = x'y' - t$ where k is a unit $u = \lambda + th$ in $D[x,y]$. To satisfy this equation, we find that $ab = 1$, $f = xf_1$, $g = yg_1$, and $h = ag_1 + bf_1$. Thus the lifted automorphism is of the form

$$\begin{aligned} x' &= (a + tf_1)x \\ y' &= (b + tg_1)y \end{aligned}$$

subject to the condition $ab = 1$, and with f_1, g_1 arbitrary.

In particular if we consider an automorphism of X_0 with $ab \neq 1$, it does not lift. Thus $\operatorname{Aut} X \to \operatorname{Aut} X_0$ is not surjective, and F_1 does not have a miniversal family.

Another way to interpret this is to let $\operatorname{Aut} X_0$ act on the set $F(D) = t_F$ of deformations of X_0 over D. Any element of $F(D)$

is defined by $xy - at = 0$ for some $a \in k$. We let $\operatorname{Aut} X_0$ act on this set by replacing $i : X_0 \hookrightarrow X$ by $i \circ \sigma : X_0 \hookrightarrow X$ for any $\sigma \in \operatorname{Aut} X_0$. The calculation above shows that this action is non-trivial. In fact, it has two orbits, corresponding to $a = 0$ and $a \neq 0$, and the set $F_1(D) = t_{F_1}$ is the quotient space consisting of two elements, the trivial deformation and the non-trivial deformation. Thus t_{F_1} is not even a vector space over k.

b) Now let us consider lifting automorphisms of the deformation X given by $xy - t$ over the dual numbers to the deformation X' given by $xy - t$ over the ring $A' = k[t]/(t^3)$.

Automorphism of X/X_0 is given by

$$\begin{aligned} x' &= (1 + tf)x \\ y' &= (1 + tg)y \end{aligned}$$

with $f, g \in k[x, y]$ arbitrary. To lift it to an automorphism of X' we need

$$\begin{aligned} x' &= (1 + tf)x + at^2 \\ y' &= (1 + tg)y + bt^2 \end{aligned}$$

for some $a, b \in k[x, y]$. A calculation similar to the one above shows that for this to be possible, there must exist a polynomial $h \in k[x, y]$ for which

$$hxy = f + g + ay + bx + fgxy.$$

In particular $f + g \in (x, y)$. So if we take $f = 1$, $g = 0$, for example, the automorphism does not lift. This confirms, by (18.4), that F is not pro-representable, as we have noted earlier (13.4).

Example 18.10 (Elliptic curves). Let X_0 be a non-singular projective curve of genus 1 over k, and let P_0 be a fixed point. Assume char. $k \neq 2, 3$. We consider two functors associated to the pair (X_0, P_0). One, $F(A)$, consists of isomorphism classes of deformations of (X_0, P_0) over A, namely, flat families X, together with a section P, and $i : X_0 \to X$ an inclusion such that $i \otimes k$ is an isomorphism of (X_0, P_0) to $(X, P) \otimes k$. The other is $F_1(A)$, which is just isomorphism classes of flat families X over A, with a section P, such that $(X, P) \otimes k \cong (X_0, P_0)$.

Repeating the analysis of [27, IV, 4.7] we find that any family of pointed curves (X, P) over the dual numbers D has an equation

$$y^2 = x(x-1)(x-\lambda)$$

with $\lambda \in D$, and that the group of automorphisms of (X, P) has order

$$\begin{cases} 6 & \text{if } \lambda = -\omega, -\omega^2 \ (j=0) \\ 4 & \text{if } \lambda = -1, \frac{1}{2}, 2 \ (j = 12^3) \\ 2 & \text{otherwise.} \end{cases}$$

If we take X_0 to be a curve with $j = 12^3$, and X to be the family over $D = k[t]/t^2$ defined by $\lambda = -1 + t$, then the group of automorphisms of (X, P) has order 2 while $\mathrm{Aut}(X_0, P_0)$ has order 4. In particular, $\mathrm{Aut}(X, P) \to \mathrm{Aut}(X_0, P_0)$ is not surjective, so F_1 does not have a miniversal family.

On the other hand, even though X_0 has infinitesimal automorphisms, since $H^0(X_0, T^0) = H^0(X_0, \mathcal{O}_{X_0}) \neq 0$, there are none leaving P_0 fixed, and so the method of (18.5) shows that F is pro-representable.

In this case $t_F = H^1(T^0)$ has dimension 1, and the deformations over D are given by the equation above with $\lambda = -1 + at$ for any $a \in k$. The action of $\mathrm{Aut}(X_0, P_0)$ on this space sends a to $-a$, so $t_{F_1} = k/\{\pm 1\}$, which is not a k-vector space.

Suppose now we take X_0 to be an elliptic curve with $j \neq 0, 12^3$. Then $\mathrm{Aut}(X_0, P_0)$ has order 2, corresponding to the automorphisms $y \mapsto \pm y$, and these lift to any deformation. So in this case $F_1 = F$ is pro-representable.

Because of the form of the equations above, we can think of the ring pro-representing F as the completion of the λ-line at the corresponding point. In the case $j \neq 0, 12^3$, the λ-line is étale over the j-line, so this is also equal to the completion of the j-line at that point.

19 Deformations of sheaves and the Quot functor

Let X_0 be a given scheme over k, and $\mathcal{F}_0$ a given coherent sheaf on X_0. If A is an Artin ring over k, and X is a deformation of X_0 over A, by a *deformation of* $\mathcal{F}_0$ *over* X we mean a coherent sheaf $\mathcal{F}$ on

X, flat over A, together with a map $\mathcal{F} \to \mathcal{F}_0$ such that the induced map $\mathcal{F} \otimes_A k \to \mathcal{F}_0$ is an isomorphism. We have seen (2.6) that if A is the dual numbers D, and $X = X_0 \times_k D$ is the trivial deformation, then such deformations $\mathcal{F}$ over X always exist, and the set of possible deformations is classified by $\operatorname{Ext}^1_{X_0}(\mathcal{F}_0, \mathcal{F}_0)$.

Now we will consider the more general situation where $0 \to k \to A' \to A \to 0$ is a small extension, where X' is a given deformation of X_0 over A', where $X = X' \otimes_{A'} A$, and $\mathcal{F}$ is a given deformation of $\mathcal{F}_0$ over A. We ask for a *deformations of $\mathcal{F}$ over X*, namely a deformation $\mathcal{F}'$ of $\mathcal{F}_0$ over X', together with a map $\mathcal{F}' \to \mathcal{F}$ such that $\mathcal{F}' \otimes_{A'} A \to \mathcal{F}$ is an isomorphism.

First we treat the case of a vector bundle, i.e., a locally free sheaf $\mathcal{F}_0$ on X_0, in which case $\mathcal{F}$ and $\mathcal{F}'$ will also be locally free because of flatness.

Theorem 19.1. *Let $A, X, \mathcal{F}$ be as above, and assume that $\mathcal{F}_0$ is locally free on X_0.*

a) *If a deformation $\mathcal{F}'$ of $\mathcal{F}$ over X' exists, then the group $\operatorname{Aut}(\mathcal{F}'/\mathcal{F})$ of automorphisms of $\mathcal{F}'$ inducing the identity automorphism of $\mathcal{F}$ is isomorphic to $H^0(X_0, \mathcal{E}nd\, \mathcal{F}_0)$.*

b) *Given $\mathcal{F}/X$, there is an obstruction in $H^2(X_0, \mathcal{E}nd\, \mathcal{F}_0)$ whose vanishing is a necessary and sufficient condition for the existence of a deformation $\mathcal{F}'$ of $\mathcal{F}$ over X'.*

c) *If a deformation $\mathcal{F}'$ of $\mathcal{F}$ over X' exists, then the set of all such is a principal homogeneous space under the action of $H^1(X_0, \mathcal{E}nd\, \mathcal{F}_0)$.*

Proof.

a) If $\mathcal{F}'$ is a deformation of $\mathcal{F}$, because of flatness there is an exact sequence
$$0 \to \mathcal{F}_0 \to \mathcal{F}' \to \mathcal{F} \to 0.$$
If $\sigma \in \operatorname{Aut}(\mathcal{F}'/\mathcal{F})$, then $\sigma - \mathrm{id}$ maps $\mathcal{F}'$ to $\mathcal{F}_0$, and this map factors through the given projection $\pi : \mathcal{F}' \to \mathcal{F}_0$, thus giving an endormophism of $\mathcal{F}_0$. Conversely, if $\tau : \mathcal{F}_0 \to \mathcal{F}_0$ is an endomorphism, then $\mathrm{id} + \tau\pi$ is an automorphism of $\mathcal{F}'$ over $\mathcal{F}$. Thus $\operatorname{Aut}(\mathcal{F}'/\mathcal{F}) = H^0(X_0, \mathcal{E}nd\, \mathcal{F}_0)$.

b) Given $\mathcal{F}$ on X choose a covering of X_0 by open sets U_i on which $\mathcal{F}$ is free. Let $\mathcal{F}'_i$ be free on X' over U_i, and $\mathcal{F}'_i \to \mathcal{F}|_{U_i}$ the natural map. Since the $\mathcal{F}'_i$ are free, we can choose isomorphisms $\gamma_{ij} : \mathcal{F}'_i|_{U_{ij}} \to \mathcal{F}'_j|_{U_{ij}}$ for each $U_{ij} = U_i \cap U_j$. On a triple intersection U_{ijk}, the composition $\delta_{ijk} = \gamma_{ik}^{-1}\gamma_{jk}\gamma_{ij}$ is an automorphism of $\mathcal{F}'_i|_{U_{ijk}}$, and so by a) above gives an element of $H^0(U_{ijk}, \mathcal{E}nd\,\mathcal{F}_0)$. These form a Čech 2-cocycle for the covering $\mathcal{U} = \{U_i\}$, and so we get an element $\delta \in H^2(X_0, \mathcal{E}nd\,\mathcal{F}_0)$. If $\delta = 0$, then we can adjust the isomorphisms γ_{ij} so that they agree on U_{ijk}, and then we can glue the deformations $\mathcal{F}'_i$ to get a global deformation $\mathcal{F}'$ of $\mathcal{F}$ over X'. Conversely, if $\mathcal{F}'$ exists, it is obvious that $\delta = 0$. So $\delta \in H^2(X_0, \mathcal{E}nd\,\mathcal{F}_0)$ is the obstruction to the existence of $\mathcal{F}'$.

c) Let $\mathcal{F}'$ and $\mathcal{F}''$ be two deformations of $\mathcal{F}$ over X'. Since they are locally free, we can choose a covering $\mathcal{U} = \{U_i\}$ of X and isomorphisms $\gamma_i : \mathcal{F}'|_{U_i} \to \mathcal{F}''|_{U_i}$ for each i. On the intersection U_{ij} we find that $\delta_{ij} = \gamma_j^{-1}\gamma_i$ is an automorphism of $\mathcal{F}'|_{U_{ij}}$ and so determines an element of $H^0(U_{ij}, \mathcal{E}nd\,\mathcal{F}_0)$. These form a Čech 1-cocycle, and so we get an element $\delta \in H^1(X_0, \mathcal{E}nd\,\mathcal{F}_0)$. This element is zero if and only if the γ_i can be adjusted to agree on the overlaps and thus glue to give an isomorphism of $\mathcal{F}'$ and $\mathcal{F}''$ over $\mathcal{F}$. By fixing one $\mathcal{F}'$ then, we see that the set of deformations $\mathcal{F}'$, if non-empty, is a principal homogeneous space under the action of $H^1(X_0, \mathcal{E}nd\,\mathcal{F}_0)$.

$\square$

Next we consider the "embedded" version of this problem, which Grothendieck calls the Quot functor. Let $X_0, \mathcal{F}_0, X, X'$ be as before, but fix a locally free sheaf $\mathcal{E}_0$ on X_0 of which $\mathcal{F}_0$ is a quotient, and fix a deformation $\mathcal{E}'$ of $\mathcal{E}_0$ over X' and let $\mathcal{E} = \mathcal{E} \otimes_{\mathcal{O}_{X'}} \mathcal{O}_X$. A deformation of the quotient $\mathcal{E}_0 \to \mathcal{F}_0 \to 0$ over X is a coherent sheaf $\mathcal{F}$ on X, flat over A, together with a surjection $\mathcal{E} \to \mathcal{F} \to 0$ and a map $\mathcal{F} \to \mathcal{F}_0$ compatible with the map $\mathcal{E} \to \mathcal{E}_0$, and inducing an isomorphism $\mathcal{F} \otimes_A k \to \mathcal{F}_0$. For simplicity we will assume the homological dimension of $\mathcal{F}_0$, $\operatorname{hd}\mathcal{F}_0 \leq 1$. This ensures that deformations exist locally. Because then $Q_0 = \ker(\mathcal{E}_0 \to \mathcal{F}_0)$ is locally free; it can be lifted locally to a locally free sheaf Q on X; and then lifting the map $Q_0 \to \mathcal{E}_0$ any way to a map $Q \to \mathcal{E}$ will give a quotient $\mathcal{F}$, locally on X, as required.

Theorem 19.2. *Given $X_0, \mathcal{E}_0 \to \mathcal{F}_0 \to 0$ in the situation as above, assuming $\mathcal{E}_0$ locally free, and $\operatorname{hd} \mathcal{F}_0 \leq 1$, we have*

a) *There is an obstruction in $H^1(X_0, \mathcal{H}om(Q_0, \mathcal{F}_0))$ whose vanishing is a necessary and sufficient condition for the existence of a deformation $\mathcal{E}' \to \mathcal{F}' \to 0$ of $\mathcal{E} \to \mathcal{F} \to 0$ on X.*

b) *If such deformations $\mathcal{E}' \to \mathcal{F}' \to 0$ exist, then the set of all such is a principal homogeneous space under the action of $H^0(X_0, \mathcal{H}om(Q_0, \mathcal{F}_0))$.*

Proof.

a) Given $\mathcal{E} \to \mathcal{F} \to 0$, because of the hypothesis $\operatorname{hd} \mathcal{F}_0 \leq 1$, the kernel Q will be locally free. Therefore on a small open set U_i it can be lifted to a locally free subsheaf Q'_i of $\mathcal{E}'$, and we let $\mathcal{F}'_i$ be the quotient. Then on the open set U_i we have locally (supressing subscripts U_i) a diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & Q_0 & \to & Q'_i & \to & Q & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{E}_0 & \to & \mathcal{E}' & \to & \mathcal{E} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{F}_0 & \to & \mathcal{F}'_i & \to & \mathcal{F} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Now on U_{ij} we have two liftings Q'_i and Q'_j (restricted to U_{ij}). Take a local section x of Q. Lift it to sections $x' \in Q'_i$ and $x'' \in Q'_j$. The difference $x'' - x'$ is then a local section of $\mathcal{E}'$ that becomes zero in $\mathcal{E}$, hence lands in $\mathcal{E}_0$. Let its image in $\mathcal{F}_0$ be y. In this way we define an element $\gamma_{ij} \in H^0(U_{ij}, \mathcal{H}om(Q_0, \mathcal{F}_0))$ since the map defined from Q factors through Q_0. The γ_{ij} defines an element $\gamma \in H^1(X_0, \mathcal{H}om(Q_0, \mathcal{F}_0))$. Now the usual argument shows that $\gamma = 0$ if and only if the Q'_i can be modified so as to patch together and thus define a global quotient $\mathcal{F}'$ of $\mathcal{E}'$.

b) A similar argument shows that if one $\mathcal{F}'$ exists, then the set of all such is a principal homogeneous space under the action of $H^0(X_0, \mathcal{H}om(Q_0, \mathcal{F}_0))$.

Remark 19.3. The hypothesis $\operatorname{hd} \mathcal{F}_0 \leq 1$ was used only to ensure the local existence of deformations. In the special case $\mathcal{E}_0 = \mathcal{O}_{X_0}$, the sheaf $\mathcal{F}_0$ is simply the structure sheaf of a closed subscheme $Y_0 \subseteq X_0$, and $\mathcal{H}om(Q_0, \mathcal{F}_0) = \mathcal{H}om(\mathcal{I}_{Y_0}, \mathcal{O}_{Y_0}) = \mathcal{N}_{Y_0/X_0}$, so we recover the result of (6.3).

Now, armed with our discussion of the embedded case, we will tackle a more difficult case of the abstract deformation problem.

Theorem 19.4. *In the same situation as Theorem* 19.1*, instead of assuming $\mathcal{F}_0$ locally free, we will assume* $\operatorname{hd} \mathcal{F}_0 \leq 1$ *and X' projective. Then*

a) *If a deformation $\mathcal{F}'$ of $\mathcal{F}$ over X' exists, then* $\operatorname{Aut}(\mathcal{F}'/\mathcal{F}) = \operatorname{Ext}^0_{X_0}(\mathcal{F}_0, \mathcal{F}_0)$.

b) *Given $\mathcal{F}$, there is an obstruction in* $\operatorname{Ext}^2_{X_0}(\mathcal{F}_0, \mathcal{F}_0)$ *to the existence of $\mathcal{F}'$.*

c) *If an $\mathcal{F}'$ exists, then the set of all such is a principal homogeneous space under the action of* $\operatorname{Ext}^1_{X_0}(\mathcal{F}_0, \mathcal{F}_0)$.

Proof.

a) The same as (19.1) since that step did not use the hypothesis $\mathcal{F}_0$ locally free, noting that $\operatorname{Ext}^0(\mathcal{F}_0, \mathcal{F}_0) = H^0(X_0, \mathcal{E}nd\, \mathcal{F}_0)$.

b) Let $\mathcal{O}_{X'}(1)$ be an ample invertible sheaf on X', with restrictions $\mathcal{O}_X(1)$, $\mathcal{O}_{X_0}(1)$ to X and X_0. Given $\mathcal{F}$ on X, for any $a \gg 0$ we can find a surjection $\mathcal{E} = \mathcal{O}_X(-a)^q \to \mathcal{F} \to 0$ for some q, and this $\mathcal{E}$ lifts to $\mathcal{E}' = \mathcal{O}_{X'}(-a)^q$ on X'.

   Note that $\operatorname{Ext}^i(\mathcal{E}_0, \mathcal{F}_0) = H^i(X_0, \mathcal{F}_0(a))^q$. So taking $a \gg 0$, we may assume these groups are zero for $i > 0$. (Here we use Serre's vanishing theorem on the projective scheme X_0.) On the other hand, since $\operatorname{hd} \mathcal{F}_0 \leq 1$, we see that Q_0 is locally free, so that $\mathcal{E}xt^i(Q_0, \mathcal{F}_0) = 0$ for $i > 0$, and hence $\operatorname{Ext}^1(Q_0, \mathcal{F}_0) = H^1(X_0, \mathcal{H}om(Q_0, \mathcal{F}_0))$. Running the exact sequence of Ext for homomorphisms of the sequence

$$0 \to Q_0 \to \mathcal{E}_0 \to \mathcal{F}_0 \to 0$$

into $\mathcal{F}_0$, we find $\operatorname{Ext}^1(Q_0, \mathcal{F}_0) \cong \operatorname{Ext}^2(\mathcal{F}_0, \mathcal{F}_0)$.

Thus the obstruction $\delta \in H^1(X_0, \mathcal{H}om(Q_0, \mathcal{F}_0))$ of (19.2) gives us an element $\delta \in \operatorname{Ext}^2(\mathcal{F}_0, \mathcal{F}_0)$. If this element is zero, then a deformation of $\mathcal{F}$ in the embedded sense exists, and we just forget the embedding to get $\mathcal{F}'$.

Conversely, if $\mathcal{F}'$ exists, we could have chosen a large enough so that $\mathcal{F}'$ is a quotient of $\mathcal{E}'$, and this shows $\delta = 0$.

c) Given any two deformations $\mathcal{F}', \mathcal{F}''$, we can again choose a large enough so that both of them appear as quotients of $\mathcal{E}'$. Then the embedded deformations of $\mathcal{F}$ are classified by $\operatorname{Hom}(Q_0, \mathcal{F}_0)$, and the ambiguity of the quotient map $\mathcal{E}' \to \mathcal{F}'$ is resolved by $\operatorname{Hom}(\mathcal{E}_0, \mathcal{F}_0)$, again the long exact sequence of Ext's shows us that the (abstract) deformations $\mathcal{F}'$ are classified by $\operatorname{Ext}^1(\mathcal{F}_0, \mathcal{F}_0)$.

Remark 19.5. I believe this theorem is still true without the hypotheses $\operatorname{hd} \mathcal{F}_0 \leq 1$ and X' projective, but it will require a different proof, and this I leave to the true devotees of the fine points of the subject.

20 Versal families of sheaves

Suppose given a scheme X_0 over k and a coherent sheaf $\mathcal{F}_0$ on X_0. For each local Artin k-algebra A, let $X = X_0 \times_k A$ be the trivial deformation of X_0. We consider the functor F which to each A assigns the set of deformations of $\mathcal{F}_0$ over A, namely $\mathcal{F}$ coherent on X, flat over A, together with a map $\mathcal{F} \to \mathcal{F}_0$ inducing an isomorphism $\mathcal{F} \otimes_A k \cong \mathcal{F}_0$, modulo isomorphisms of $\mathcal{F}$ over $\mathcal{F}_0$.

Theorem 20.1. *In the above situation, assume X_0 is projective and $\operatorname{hd} \mathcal{F}_0 \leq 1$. Then the functor F has a miniversal family.*

Proof. We apply Schlessinger's criterion (15.2), the proof being similar to the case of deformations of schemes (18.1).

(H_0) $F(k)$ has just one element $\mathcal{F}_0 \xrightarrow{\mathrm{id}} \mathcal{F}_0$.

(H_1) Given $\mathcal{F}'/X'$ and $\mathcal{F}''/X''$ restricting to the same $\mathcal{F}/X$, we can choose maps $\mathcal{F}' \to \mathcal{F}$ and $\mathcal{F}'' \to \mathcal{F}$, compatible with the given maps to $\mathcal{F}_0$, inducing isomorphisms $\mathcal{F}' \otimes A \to \mathcal{F}$ and $\mathcal{F}'' \otimes A \to \mathcal{F}$. We now take $\mathcal{F}^*$ to be the fibred product sheaf $\mathcal{F}' \times_{\mathcal{F}} \mathcal{F}''$, which will be flat over $A^* = A' \times_A A''$ by (16.3).

(H_2) In case $A = k$, the maps to $\mathcal{F} = \mathcal{F}_0$ are already specified, so, as in the proof of (18.1), $\mathcal{F}^*$ is unique.

(H_3) With the hypothesis $\operatorname{hd} \mathcal{F}_0 \leq 1$ and X_0 projective, by (19.4) we have $t_F = \operatorname{Ext}^1_{X_0}(\mathcal{F}_0, \mathcal{F}_0)$, which is finite-dimensional.

Hence F has a miniversal family.

Remark 20.2. We could also consider the functor $F_1(A)$, which is the set of isomorphism classes of $\mathcal{F}$ flat over X such that $\mathcal{F} \otimes k \cong \mathcal{F}_0$, but without specifying the map $\mathcal{F} \to \mathcal{F}_0$. As in the case of deformations of schemes (18.9), if $\operatorname{Aut} \mathcal{F} \to \operatorname{Aut} \mathcal{F}_0$ is surjective for every such $\mathcal{F}$ over the ring of dual numbers D, then F_1 will also have a miniversal family, and $t_{F_1} = t_F$.

Theorem 20.3. *Assume X_0 projective and $\operatorname{hd} \mathcal{F}_0 \leq 1$ as above, but now assume in addition that $\mathcal{F}_0$ is* simple, *i.e., $H^0(\mathcal{E}nd\, \mathcal{F}_0) = k$. Then the functors F and F_1 are equal and pro-representable.*

Proof. Since for any small extension $A' \to A$ the deformations $\mathcal{F}'$ of a given $\mathcal{F}$ over A are classified by t_F (19.4), as in the case of deformations of schemes (18.4), it is merely a matter of showing that $\operatorname{Aut} \mathcal{F}' \to \operatorname{Aut} \mathcal{F}$ is surjective for any $\mathcal{F}' \to \mathcal{F}$.

We have assumed $\mathcal{F}_0$ simple, so $H^0(\mathcal{E}nd\, \mathcal{F}_0) = k$, and $\operatorname{Aut} \mathcal{F}_0 = k^*$. Using the description of automorphisms of an extension $\mathcal{F}'/\mathcal{F}$ given in (19.4), we see, by induction on the length of A, that for any $\mathcal{F}$ over A, $\operatorname{Aut} \mathcal{F} \cong A^*$. (Here A^* means the group of units in A, not the ring mentioned in (20.1).) Now clearly $\operatorname{Aut} \mathcal{F}' \to \operatorname{Aut} \mathcal{F}$ is surjective and so F is pro-representable. In particular, $\operatorname{Aut} \mathcal{F} \to \operatorname{Aut} \mathcal{F}_0$ is surjective, and so the two functors F and F_1 are equal.

Theorem 20.4. *Let X_0 be a projective scheme over k, and let $\mathcal{E}_0 \to \mathcal{F}_0 \to 0$ be a surjective map of coherent sheaves. For any local Artin k-algebra A, let $X = X_0 \times_k A$, let $\mathcal{E} = \mathcal{E}_0 \times_k A$. Then the Quot functor F of quotients $\mathcal{E} \to \mathcal{F} \to 0$ with $\mathcal{F}$ flat over A and $\mathcal{F} \otimes_A k = \mathcal{F}_0$ is pro-representable.*

Proof. Conditions $(H_0), (H_1), (H_2)$ of Schlessinger's criterion are verified as in the previous proof. The tangent space t_F is $H^0(X_0, \mathcal{H}om(Q_0, \mathcal{F}_0))$ which is finite-dimensional since X_0 is projective. Note that statement b) of (20.3) does not make use of the hypotheses $\mathcal{E}_0$ locally free and $\operatorname{hd} \mathcal{F}_0 \leq 1$. Since there are no automorphisms of a quotient of a fixed sheaf $\mathcal{E}$, the criterion b) of (20.3) allows us to verify (H_4), and so the functor is pro-representable. Note also in this case the functors F and F_1 are the same, since there are no automorphisms.

Remark 20.5. In fact, Grothendieck [22, exp. 221] has shown that given X_0/k and $\mathcal{E}_0$ on X_0, the global Quot functor of quotients $\mathcal{E}_0 \times S \to \mathcal{F} \to 0$ on $X = X_0 \times S$, flat over S, for any base scheme S, with given Hilbert polynomial P, is representable by a scheme, projective over k.

Example 20.6. Deformations of $\mathcal{O}(-1) \oplus \mathcal{O}(1)$ on $\mathbb{P}^1_k$. Over any Artin ring A, we can define a coherent sheaf $\mathcal{F}$ on $\mathbb{P}^1_A$ by an extension

$$0 \to \mathcal{O}_{\mathbb{P}^1_A}(-1) \to \mathcal{F} \to \mathcal{O}_{\mathbb{P}^1_A}(1) \to 0.$$

These extensions are classified by $\operatorname{Ext}^1_{\mathbb{P}^1_A}(\mathcal{O}(1), \mathcal{O}(-1)) = H^1(\mathcal{O}(-2)) = A$. If we take a sheaf $\mathcal{F}$ defined by an element $f \in A$ that is contained in the maximal ideal $\mathfrak{m}_A$, then the image of f in k is 0, and so the sheaf $\mathcal{F} \otimes_A k = \mathcal{F}_0$ will be the trivial extension $\mathcal{O}(-1) \oplus \mathcal{O}(1)$ on $\mathbb{P}^1_k$.

Taking Hom of the above sequence into $\mathcal{F}$, we obtain an exact sequence

$$0 \to \operatorname{Hom}(\mathcal{O}(1), \mathcal{F}) \to \operatorname{Hom}(\mathcal{F}, \mathcal{F}) \to \operatorname{Hom}(\mathcal{O}(-1), \mathcal{F}) \to 0.$$

The group on the right is $H^0(\mathcal{F}(1))$, which is a free A-module of rank 4. The group on the left, $H^0(\mathcal{F}(-1))$ depends on the choice of extension. Taking Hom of $\mathcal{O}(1)$ into the sequence above, we get

$$0 = H^0(\mathcal{O}(-2)) \to H^0(\mathcal{F}(-1)) \to H^0(\mathcal{O}) \xrightarrow{\delta} \operatorname{Ext}^1(\mathcal{O}(1), \mathcal{O}(-1)) \to \dots$$

and the image $\delta(1)$ is the element $f \in A$ determining the extension.

a) Now let us take $A = D = k[t]/(t^2)$ the dual numbers, and take the sheaf $\mathcal{F}$ defined by $f = t$. Then $\mathcal{F} \otimes_D k \cong \mathcal{F}_0$. Furthermore, $H^0(\mathcal{F}(-1)) \cong kt$ and the map $H^0(\mathcal{F}(-1)) \to H^0(\mathcal{F}_0(-1))$ is zero. Hence $End\, \mathcal{F} \to End\, \mathcal{F}_0$ is not surjective, so $\operatorname{Aut} \mathcal{F} \to \operatorname{Aut} \mathcal{F}_0$ is not surjective, and we conclude that the functor F_1 does not have a miniversal family.

b) Next, take $A' = k[t]/t^3 \to A = D$, and let $\mathcal{F}$ over A' be defined by $f = t^2 \in A'$. Then for the same reason $\mathrm{Aut}(\mathcal{F}'/\mathcal{F}_0) \to \mathrm{Aut}(\mathcal{F}/\mathcal{F}_0)$ is not surjective, and so we see that the functor F is not pro-representable.

All of this is due to the fact that the global family over $S = \operatorname{Spec} k[t]$ defined by $f = t$ exhibits a jump phenomenon: the fibre over $t = 0$ is $\mathcal{O}(-1) \oplus \mathcal{O}(1)$, while the fiber over any point $t \neq 0$ is isomorphic to $\mathcal{O} \oplus \mathcal{O}$. So this study of the automorphisms of extensions over Artin rings is the infinitesimal reflection of a global jump phenomenon.

21 Comparison of embedded and abstract deformations

There are many situations in which it is profitable to compare one deformation problem to another. In this section we will compare deformations of a closed subscheme $X_0 \subseteq \mathbb{P}^n$ to the abstract deformations of X_0. If F_1 is the functor of Artin rings of embedded deformations, and F_2 is the functor of abstract deformations, then we have a "forgetful morphism" from F_1 to F_2, which for every Artin ring A maps $F_1(A) \to F_2(A)$ by forgetting the embedding.

We begin with a general result on morphisms of functors.

Proposition 21.1. *Let $f : F_1 \to F_2$ be a morphism of functors on Artin rings. Assume that F_1 and F_2 both have versal families corresponding to complete local rings R_1, R_2. Then there is a morphism of schemes $\bar{f} : \operatorname{Spec} R_1 \to \operatorname{Spec} R_2$ corresponding to a homomorphism of rings $\varphi : R_2 \to R_1$ such that for each Artin ring A the following diagram is commutative:*

$$\begin{array}{ccc} \mathrm{Hom}(R_1, A) & \xrightarrow{\varphi^*} & \mathrm{Hom}(R_2, A) \\ \downarrow & & \downarrow \\ F_1(A) & \xrightarrow{f} & F_2(A) \end{array}$$

where the vertical arrows are the maps expressing the versal families. Furthermore if R_1 and R_2 are miniversal families, then the map induced by $\bar{f}$ on Zariski tangent spaces $t_{R_1} \to t_{R_2}$ is just $t_{F_1} \to t_{F_2}$ given by $F_1(D) \to F_2(D)$, where D is the dual numbers.

Proof. Consider the inverse system $(R_1/\mathfrak{m}^n)$. The natural maps $R_1 \to R_1/\mathfrak{m}^n$ induce elements $\xi_n \in F_1(R_1/\mathfrak{m}^n)$ forming a compatible sequence. By f we get a compatible sequence $f(\xi_n) \in R_2(R_1/\mathfrak{m}^n)$. By the versal property of R_2 we get compatible maps $R_2 \to R_1/\mathfrak{m}^n$ and hence a homomorphism of $R_2 \to \varprojlim R_1/\mathfrak{m}^n = R_1$. The rest is straightforward.

Now we will consider the case where F_1 is the functor of embedded deformations of a projective scheme $X_0 \subseteq \mathbb{P}^n_k$, F_2 is the functor of abstract deformations of X_0, and $f : F_1 \to F_2$ is the forgetful morphism. We know that F_1 is pro-representable (17.1) by a ring R_1 and that F_2 has a miniversal family (18.1) given by a ring R_2, and so we have a morphism of families $\operatorname{Spec} R_1 \to \operatorname{Spec} R_2$.

Proposition 21.2. *Suppose that $X = X_0$ is a non-singular subscheme of $\mathbb{P}^n$. Then the exact sequence*

$$0 \to T_X \to T_{\mathbb{P}^n/X} \to \mathcal{N}_{X/\mathbb{P}^n} \to 0$$

gives rise to an exact sequence of cohomology

$$\begin{aligned} 0 &\to H^0(T_X) \to H^0(T_{\mathbb{P}^n}|_X) \to H^0(\mathcal{N}_X) \xrightarrow{\delta^0} H^1(T_X) \\ &\to H^1(T_{\mathbb{P}^n}|_X) \to H^1(\mathcal{N}_X) \xrightarrow{\delta^1} H^2(T_X) \to H^2(T_{\mathbb{P}^n}|_X) \to \dots \end{aligned}$$

in which the boundary map $\delta^0 : H^0(\mathcal{N}_X) \to H^1(T_X)$ is just the induced map on tangent spaces $t_{F_1} \to t_{F_2}$ of the deformation functors, and $\delta^1 : H^1(\mathcal{N}_X) \to H^2(T_X)$ maps the obstruction space of F_1 to the obstruction space of F_2.

Proof. The only thing to prove is the identification of δ^0 and δ^1 with the corresponding properties of the functors F_1 and F_2, and this we leave to the reader to trace through the identification of these tangent spaces and obstruction spaces discussed earlier.

Remark 21.3. Because of this exact sequence, we can interpret $H^1(T_{\mathbb{P}^n}|_X)$ as the obstructions to lifting an abstract deformation of X to an embedded deformation of X. We can also interpret the image of $H^0(T_{\mathbb{P}^n}|_X)$ in $H^0(\mathcal{N}_X)$ as those deformations of X_0 induced by automorphisms of $\mathbb{P}^n$.

Example 21.4. Let us apply this proposition to the case of a non-singular surface X of degree $d \geq 2$ in $\mathbb{P}^3$.

Restricting the Euler sequence on $\mathbb{P}^3$ to X we obtain

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(1)^4 \to T_{\mathbb{P}^3}|_X \to 0.$$

From the cohomology of this sequence we find $h^0(T_{\mathbb{P}^3}|_X) = 15$, $h^1(T_{\mathbb{P}^3}|_X) = 0$ except for the case $d = 4$, in which case it is 1; and $h^2(T_{\mathbb{P}^3}|_X) = 0$ for $d \leq 5$, but $\neq 0$ for $d \geq 6$.

Next, we observe that the map $H^0(T_{\mathbb{P}^3}|_X) \to H^0(\mathcal{N}_X)$ is surjective for $d = 2$ and injective for $d \geq 3$. Noting that $\mathcal{N}_X \cong \mathcal{O}_X(d)$ and using the sequence above, this is a consequence of the following lemma.

Lemma 21.5. *Let $f \in k[x_0, \ldots, x_n]$ be a homogeneous polynomial of degree $d \geq 3$ whose zero scheme is a non-singular hypersurface in $\mathbb{P}^n$ and assume that char. $k \nmid d$. Let f_i, $i = 0, \ldots, n$ be the partial derivatives of f. Then the forms $x_i f_j$, $i, j = 0, \ldots, n$ are linearly independent forms of degree d.*

Proof. Since the zero scheme of f is non-singular, the subset of $\mathbb{P}^n$ defined by $(f, f_0, \ldots, f_n)$ is empty. The Euler relation $d \cdot f = \sum x_i f_i$ shows that this ideal is the same as the ideal $(f_0, \ldots, f_n)$. Therefore it is primary for the maximal ideal $(x_0, \ldots, x_n)$, and the f_i form a regular sequence. Now the exactness of the Koszul complex shows that the relations among the f_i are generated by the relations $f_i f_j - f_j f_i = 0$. Since $d \geq 3$, there are no relations with linear coefficients.

Example 21.4, continued. Now, using the fact that $H^1(\mathcal{N}_X) = H^1(\mathcal{O}_X(d)) = 0$ for any surface in $\mathbb{P}^3$, we can construct the following table for the dimensions of the groups of (21.2).

d	$h^0(T_X)$	$h^0(T_{\mathbb{P}^3}\|_X)$	$h^0(\mathcal{N}_X)$	$h^1(T_X)$	$h^1(T_{\mathbb{P}^3}\|_X)$
2	6	15	9	0	0
3	0	15	19	4	0
4	0	15	34	20	1
≥ 5	0	15	large	large	0

For $d = 2$, the quadric surface X has no abstract deformations, i.e., it is rigid. On the other hand it has a 6-dimensional family of automorphisms, since $X \cong \mathbb{P}^1 \times \mathbb{P}^1$. The chart shows a 9-dimensional family of surfaces in $\mathbb{P}^3$, any two related by an automorphism of $\mathbb{P}^3$.

For $d \geq 3$ we find $h^0(T_X) = 0$. There are no infinitesimal automorphisms, and so the abstract deformation functor F_2 is also pro-representable.

Except for the case $d = 4$ (for which see below), every abstract deformation of a surface of degree d in $\mathbb{P}^3$ is realized as a deformation inside $\mathbb{P}^3$. Indeed, since both functors are pro-representable, and the map $t_{F_1} \to t_{F_2}$ is surjective, we find by induction on the length of A that $F_1(A) \to F_2(A)$ is surjective for all Artin rings A.

The same exact sequences as above show also that $h^2(T_X) = 0$ for $d \leq 5$, but $h^2(T_X) \neq 0$ for $d \geq 6$. Thus at least for $2 \leq d \leq 5$ the abstract deformations are unobstructed. For $d \geq 6$ see below.

Example 21.6. We examine more closely the case of a non-singular surface of degree 4 in $\mathbb{P}^3$, which is a $K3$ surface. The functor of embedded deformations is unobstructed, since $h^1(\mathcal{N}_X) = 0$ as noted above. Also the functor F_1 is pro-representable, so the universal family is defined by a complete regular local ring R_1 of dimension 34. The abstract deformations are also unobstructed, as noted above, so the functor F_2 is pro-represented by a complete regular local ring R_2 of dimension 20. The induced map on the Zariski tangent spaces of the morphism $\operatorname{Spec} R_1 \to \operatorname{Spec} R_2$ however is not surjective, as we see from the table above: its image has only dimension 19. Computing the image step by step, we see that the image, which corresponds to abstract deformations that lift to embedded deformations, is a smooth subspace of $\operatorname{Spec} R_2$ of dimension 19. In particular, there are abstract deformations of X_0 that cannot be realized as embedded deformations in $\mathbb{P}^3$. (Over the complex numbers, this corresponds to the fact that there are complex manifold $K3$ surfaces that are not algebraic.)

Using this fact, we can give an example of an obstructed deformation of a line bundle. Let X_0 be a quartic surface in $\mathbb{P}^3$. Let X be a deformation over the dual numbers D that does not lift to $\mathbb{P}^3$. Let $\mathcal{L}_0$ be the invertible sheaf $\mathcal{O}_{X_0}(1)$. I claim that $\mathcal{L}_0$ does not lift to X. For suppose it did lift to an invertible sheaf $\mathcal{L}$ on X. Then the exact sequence

$$0 \to \mathcal{L}_0 \to \mathcal{L} \to \mathcal{L}_0 \to 0$$

and $H^1(\mathcal{O}_{X_0}(1)) = 0$ would show that the sections $x_0, x_1, x_2, x_3 \in H^0(\mathcal{L}_0)$ that define the embedding $X_0 \subseteq \mathbb{P}^3$ lift to $\mathcal{L}$. Using these sections we would obtain a morphism of X to $\mathbb{P}^3_D$, which must be a closed immersion by flatness. Thus X lifts to $\mathbb{P}^3_D$, a contradiction. So this is an example where the obstruction in $H^2(\mathcal{O}_{X_0})$ to lifting $\mathcal{L}_0$ is non-zero (6.4).

Example 21.7. For $d \leq 5$ we have seen that the abstract deformations of X are unobstructed, since $H^2(T_X) = 0$. Here we will show also for $d \geq 6$ that the abstract deformations are unobstructed, even though $H^2(T_X) \neq 0$.

So suppose $d \geq 6$, and suppose X is a deformation of X_0 over A, which we wish to lift to a small extension $A' \to A$. Since $F_1(A) \to F_2(A)$ is surjective, we can lift X to an embedded deformation of $X_0 \subseteq \mathbb{P}^3$. Since $H^1(\mathcal{N}_{X_0}) = 0$, these are unobstructed, so X lifts to an embedded deformation $X' \subseteq \mathbb{P}^3_{A'}$. Forgetting the embedding gives the desired deformation X' over A'. Thus the abstract deformations of X_0 are unobstructed.

Example 21.8. A similar analysis of embedded versus abstract deformations of non-singular curves of degree d in $\mathbb{P}^2$ shows that for $d \leq 4$, any abstract deformation lifts to an embedded deformation, while for $d \geq 5$ this is not the case.

For example if $d = 4$, the curve X_0 is a curve of genus 3 embedded by the canonical linear system. If X is any deformation, the canonical system on X gives an embedding in $\mathbb{P}^2$.

On the other hand, for $d = 5$, we have a curve X_0 of genus 6 having a linear system g^2_5 of degree 5 and dimension 2. This argument shows that there are infinitesimal deformations of X_0 that do not have such a linear system. In fact, one knows for other reasons that a "general" curve of genus 6 does not have a g^2_5.

CHAPTER 4

Global Questions

22 Introduction to moduli questions

What is a variety of moduli or a moduli scheme? In this section we will consider the question in general and make some definitions. Then in subsequent sections we will give some elementary examples to illustrate the various issues that often arise in dealing with moduli questions.

To fix the ideas, let us work over an algebraically closed base field k (though everything that follows could be generalized to work over a fixed base scheme). Suppose we have identified a certain class of objects $\mathcal{M}$ over k that we wish to classify. You can think of closed subschemes with fixed Hilbert polynomial of $\mathbb{P}^n_k$, or curves of genus g over k, or vector bundles of given rank and given Chern classes over a fixed scheme X over k, and so on. We will deal with specific cases in the subsequent sections. But for the moment, let us just say we have focused our attention on a set of objects $\mathcal{M}$, and we have given a rule for saying when two of them are the same (usually isomorphism). We wish to classify the objects in $\mathcal{M}$.

The first step is to describe the set $\mathcal{M}$, that is, to list the possible elements of $\mathcal{M}$ up to isomorphism. This determines $\mathcal{M}$ as a set. To go further, we wish to put a structure of algebraic variety or scheme on the set $\mathcal{M}$ that should be natural in some sense. So we look for a scheme M/k, whose closed points are in one-to-one correspondence with the elements of the set $\mathcal{M}$, and such that scheme structure should reflect the possible variations of elements in $\mathcal{M}$, how they behave in families.

The next step is to define what we mean by a family of elements in $\mathcal{M}$. For a parameter scheme S, this will usually mean a scheme X/S, flat over S, with an extra structure whose fibers at closed points are elements of $\mathcal{M}$. Then for the scheme M to be a variety of moduli for the class $\mathcal{M}$, we require that for every family X/S there is a morphism $f : S \to M$ such that for each closed point $s \in S$, the image $f(s) \in M$ corresponds to the isomorphism class of the fiber X_s in $\mathcal{M}$.

But that is not enough. We want the assignment of the morphism $f : S \to M$ to the family X/S to be functorial. To explain what this means, for every scheme S/k, let $\mathcal{F}(S)$ be the set of all families X/S of elements of $\mathcal{M}$ parametrized by S. If $S' \to S$ is a morphism, then by base extension, a family X/S will give rise to a family X'/S' (note here that we should define our notion of family in each situation so that they do extend by base extension), and so the morphism $S' \to S$ gives rise to a map of sets $\mathcal{F}(S) \to \mathcal{F}(S')$. In this way $\mathcal{F}$ becomes a contravariant functor from (Sch/k) to (sets). So what we are asking for above is a morphism of functors $\varphi : \mathcal{F} \to \mathrm{Hom}(\cdot, M)$. This means that for every scheme S/k we have a map of sets $\varphi(S) : \mathcal{F}(S) \to \mathrm{Hom}(S, M)$, which to each element $X/S \in \mathcal{F}(S)$ assigns the morphism $f_{X/S} : S \to M$, and for each morphism $g : S' \to S$, if $g^*(X/S) = X'/S'$, then $f_{X'/S'} = f_{X/S} \circ g$. If we denote the functor $\mathrm{Hom}(\cdot, M)$ from (Sch/k) to (sets) by h_M, then we can say φ is a homomorphism of functors $\varphi : \mathcal{F} \to h_M$.

What we have said so far still does not determine the scheme structure on M uniquely. To make M unique, we will require that it should be the "largest possible" with the above properties. So we require that if N is any other scheme, and $\psi : \mathcal{F} \to h_N$ a morphism of functors, then there should exist a unique morphism $e : M \to N$, such that if $h_e : h_M \to h_N$ is the induced map on associated functors, then $\psi = h_e \circ \varphi$.

Summing up, we come to the following definition.

Definition 22.1. Having fixed a certain class $\mathcal{M}$ of objects over k, and having described what we mean by families of elements of $\mathcal{M}$ parametrized by a scheme S, and having said when two families are the same, so that we get a functor $\mathcal{F} : (\mathrm{Sch}/k) \to (\text{sets})$ which to each scheme S/k assigns the set $\mathcal{F}(S)$ of equivalence classes of families of elements of $\mathcal{M}$ over S, we define a *coarse moduli scheme* for the class $\mathcal{M}$ to be a scheme M/k such that

a) the closed points of M are in one-to-one correspondence with the elements of $\mathcal{M}$, i.e., the set $\mathcal{F}(k)$, and

b) there is a morphism of functors $\varphi : \mathcal{F} \to h_M$ such that for each S/k and each $X/S \in \mathcal{F}(S)$, letting $f_{X/S} : S \to M$ be the associated morphism, for each closed point $s \in S$, $f_{X/S}(s)$ is the closed point in M corresponding to the class of the fiber X_s in $\mathcal{M}$, and

c) the scheme M is universal with property b), namely if $\psi : \mathcal{F} \to h_N$ is any other morphism of $\mathcal{F}$ to a functor of the form h_N, then there exists a unique morphism $e : M \to N$ such that $\psi = h_e \circ \varphi$.

Remark 22.2. Because of property c), the scheme M is uniquely determined, if it exists.

The definition as we have stated it is somewhat redundant. We stated it that way for clarity. But in fact properties a), b) can be compressed into the single statement

ab′) There is a morphism of functors $\varphi : \mathcal{F} \to h_M$ such that $\varphi(k) : \mathcal{F}(\operatorname{Spec} k) \to h_M(\operatorname{Spec} k)$ is bijective.

The reason for this is that $\mathcal{F}(\operatorname{Spec} k)$ describes families over $\operatorname{Spec} k$, that is, just the elements of $\mathcal{M}$, while $h_M(\operatorname{Spec} k) = \operatorname{Hom}(\operatorname{Spec} k, M)$ is the set of closed points of M (assuming that M is a scheme of finite type over the algebraically closed field k). So this statement includes the original a). Now suppose $X/S \in \mathcal{F}(S)$ and $f : S \to M$ is the corresponding map. For any closed point $s \in S$, we get a morphism of $\operatorname{Spec} k \to S$, and now the functoriality tells us that the pull back of X/S is the fiber X_s/k, and this must correspond to $f(s)$ via the bijection $\mathcal{F}(k) \to h_M(k)$ above.

Continuing our discussion of a moduli problem $\mathcal{M}$, we note that a coarse moduli scheme may fail to exist. We will give examples later. If on the other hand a coarse moduli scheme does exist, then there are further properties we can ask for.

Definition 22.3. If the moduli problem $\mathcal{M}$ has a coarse moduli scheme M, we say it has a *tautological family* if there exists a family X/M such that for each closed point $m \in M$, the fiber X_m is in the class of $\mathcal{M}$ corresponding to the point m. If M is reduced, this is equivalent to saying that $X/M \in \mathcal{F}(M)$ is an element corresponding to the identity homomorphism $1_M \in h_M(M)$ via the functorial map $\varphi(M) : \mathcal{F}(M) \to h_M(M)$.

In general, when M is a coarse moduli space, the associated map of sets $\mathcal{F}(S) \to h_M(S)$ for a parameter scheme may not be either injective or surjective. But in a good case if it is bijective for all S, then $\varphi : \mathcal{F} \to h_M$ becomes an isomorphism of functors, and we say that $\mathcal{F}$ is *represented* by the scheme M and that $\mathcal{F}$ is a *representable functor*.

Definition 22.4. If $\varphi : \mathcal{F} \to h_M$ is an isomorphism of functors, i.e., $\mathcal{F}$ representable by the scheme M, then we say that M is a *fine moduli scheme* for the moduli problem $\mathcal{M}$.

Remark 22.5. In the case of a fine moduli space, the map $\mathcal{F}(M) \to h_M(M)$ is bijective, and so there exists a family X_u/M corresponding to the identity map $1_M \in h_M(M)$. We call X_u a *universal* family, because it has the property that for each family X/S, there exists a unique morphism $f : S \to M$ such that X is obtained by pulling back the universal family X_u by f. Note that the universal family X_u is also a tautological family as defined above (but the converse does not hold in general).

Proposition 22.6. *If the moduli problem $\mathcal{M}$ has a fine moduli scheme M, then M is also a coarse moduli scheme for $\mathcal{M}$.*

Proof. Since M is a fine moduli scheme, we have the morphism $\varphi : \mathcal{F} \to h_M$, which in this case happens to be an isomorphism. We need only check the universal property c) in the definition of coarse moduli. So suppose $\psi : \mathcal{F} \to h_N$ is another morphism of functors. We must show there is a morphism $e : M \to N$ such that $\psi = h_e \circ \varphi$. Since φ is an isomorphism, ψ gives us a map $h_M \to h_N$. Then the result is a consequence of the following lemma.

Lemma 22.7. *The functor h, from the category of schemes over k to the category of contravariant functors from* $(\mathrm{Sch}\,/k)$ *to* (sets) *is fully faithful, that is, for any M, N,* $\mathrm{Hom}(M, N) \to \mathrm{Hom}(h_M, h_N)$ *is bijective.*

Proof. The inverse mapping is obtained by taking the image of 1_M in the corresponding $\mathrm{Hom}(h_M(M), h_N(M))$.

Remark 22.8. In particular, since there are moduli problems having a non-reduced fine moduli scheme, this shows that a moduli problem may have a non-reduced coarse moduli scheme, even though it seems that in the definition of a coarse moduli scheme we have dealt only with closed points, and hence apparently cannot distinguish a scheme from its associated reduced scheme.

One of the great benefits of having a fine moduli space is that we can study it using infinitesimal methods.

Proposition 22.9. *Let M be a fine moduli scheme for the moduli problem $\mathcal{M}$, and let $X_0 \in \mathcal{M}$, corresponding to a point $x_0 \in M$. Then the Zariski tangent space to M at x_0 is in one-to-one correspondence with the set of families X over the dual numbers D, whose closed fibers are isomorphic to X_0.*

Proof. Indeed, the Zariski tangent space to M at x_0 can be identified with $\operatorname{Hom}_{x_0}(D, M)$, [27, II, Ex. 2.8], and this in turn corresponds to the subset of those elements of $\mathcal{F}(D)$ restricting to X_0 over k.

Furthermore, whenever one has an obstruction theory for the moduli problem $\mathcal{M}$, one can obtain further information about the fine moduli space M, such as its dimension, criteria for M to be smooth or local complete intersection, and so forth. For a coarse moduli space, such techniques are not available, unless it be for some special reason in a particular case.

In considering moduli questions, it is useful to have some criteria for the existence of a moduli scheme.

Definition 22.10. We say a contravariant functor $\mathcal{F}$ from (Sch/k) to (sets) is a *sheaf for the Zariski topology* if for every scheme S and every covering of S by open subsets $\{U_i\}$, the diagram

$$\mathcal{F}(S) \to \prod \mathcal{F}(U_i) \rightrightarrows \prod \mathcal{F}(U_i \cap U_j)$$

is exact. Spelled out, this means two things:

a) given elements $x, x' \in \mathcal{F}(S)$ whose restrictions to $\mathcal{F}(U_i)$ are equal for all i, then $x = x'$, and

b) given a collection of elements $x_i \in \mathcal{F}(U_i)$ for each i, such that for each i, j, the restrictions of x_i and x_j to $U_i \cap U_j$ are equal, then there exists an element $x \in \mathcal{F}(S)$ whose restriction to each $\mathcal{F}(U_i)$ is x_i.

Proposition 22.11. *If the moduli problem $\mathcal{M}$ has a fine moduli space, then the associated functor $\mathcal{F}$ is a sheaf for the Zariski topology.*

Proof. Indeed, if $\mathcal{F} = h_M$, then for any scheme S, $\mathcal{F}(S) = \operatorname{Hom}(S, M)$, and one knows that morphisms from one scheme to another are determined locally, and can be glued together if given locally and compatible on overlaps [27, II.3.3, Step 3].

Examples 22.12. A moduli problem that does not have a coarse moduli space: rank 2 vector bundles over $\mathbb{P}^1$ (20.6).

22.13. A moduli problem that has a coarse moduli space, but no tautological family: elliptic curves (§26).

22.14. A coarse moduli space with a tautological family that is not a fine moduli space: curves of genus 0 over an algebraically closed field (§23).

22.15. A fine moduli space: Hilbert scheme.

22.16. A problem with a coarse moduli space that is a sheaf for Zariski topology, but does not have a fine moduli space: ?

[give formal references]

23 Curves of genus zero

Any complete non-singular curve of genus 0 over an algebraically closed field k is isomorphic to $\mathbb{P}^1_k$ ([27, IV.1.3.5]), so you may think the moduli problem for curves of genus 0 is trivial. But even in this case, there are some special aspects to the problem.

So let us consider the moduli problem for non-singular projective curves of genus 0 over an algebraically closed field k. The set $\mathcal{M}$ of isomorphism classes of such curves has just one element, namely $\mathbb{P}^1_k$. A *family* of curves of genus 0 over a scheme S will be a scheme X, smooth and projective over S, whose geometric fibers are curves of genus 0. That means for each $s \in S$, if we take the fiber X_s and extend the base field to the algebraic closure $\overline{k(s)}$ of the residue field $k(s)$, then the new curve $X_{\bar{s}} = X_s \times_{k(s)} \overline{k(s)}$ is a non-singular projective curve of genus 0 over the field $\overline{k(s)}$.

Proposition 23.1. *The one point space $M = \operatorname{Spec} k$ is a coarse moduli scheme for curves of genus 0, and it has tautological family.*

Proof. The first condition a) for a coarse moduli scheme is satisfied because the one point of M corresponds to the one curve $\mathbb{P}^1_k$. We can also see right away that there is a tautological family: just take $\mathbb{P}^1_k/\operatorname{Spec} k$. For any family X/S of curves of genus 0, where S is a scheme over k, there is a unique morphism $S \to M = \operatorname{Spec} k$, and this will satisfy the second condition to be a coarse moduli scheme.

For the third condition, suppose $\psi : \mathcal{F} \to h_N$ is any morphism of functors, where $\mathcal{F}$ is our functor of families of curves of genus 0. Then in particular, the family $\mathbb{P}^1_k/M$ determines a morphism $e : M \to N$. We need to show that ψ factors through the morphism $\varphi : \mathcal{F} \to h_M$ described above that maps every scheme S/K to $\operatorname{Spec} k$.

Lemma 23.2. *If C is an Artin ring with residue field k algebraically closed, then any family $X/\operatorname{Spec} C$ of curves of genus 0 is trivial, namely isomorphic to $\mathbb{P}^1_{\operatorname{Spec} C}$.*

Proof. Since k is algebraically closed, the special fiber X_0 is just $\mathbb{P}^1_k$. Then by our infinitesimal study of deformations (???) the obstructions to deforming X_0 lie in $H^2(X_0, \mathcal{T}_{X_0}) = 0$, and the choices at each step are given by $H^1(X_0, \mathcal{T}_{X_0}) = 0$. Thus at each step there is a unique deformation, which must be equal to $\mathbb{P}^1_{\operatorname{Spec} C}$.

Proof of* (23.1)*, continued. Let X be a family of curves of genus 0 over a scheme S of finite type over k. For any closed point $s \in S$, the fiber X_s is just $\mathbb{P}^1_k$, so the point s must go to the same point $n_0 \in N$ as the image of the morphism $e : M \to N$. Thus all closed points of S go to n_0. But we need more. We need to know that the morphism $S \to N$ factors through the reduced point n_0 as a closed subscheme of N. And this follows from the lemma, because the restriction of the family on S to any artinian closed subscheme of S will be trivial, and therefore will factor through the reduced scheme $\operatorname{Spec} k$.

If S is not of finite type over k, a similar argument, making base extensions to geometric points of S and Artin rings over them shows in that case also the associated map $S \to N$ factors through the reduced point $n_0 \in N$, and so the morphism ψ factors through φ, as required.

Example 23.3. Here we show that the one-point space M is not a fine moduli space for curves of genus 0. Just think of the theory of ruled surfaces. A ruled surface is a non-singular projective surface X together with a morphism π to a non-singular projective curve C, whose fibers are copies of $\mathbb{P}^1$ and that has a section, and therefore is isomorphic to $\mathbb{P}(\mathcal{E})$ for some rank 2 vector bundle $\mathcal{E}$ on C [27, V, 2.2]. In particular, this implies that C can be covered by open subsets U_i over which X is trivial, i.e., $\pi^{-1}(U_i) \cong U_i \times \mathbb{P}^1$. On the other hand, there are many ruled surfaces X that are not trivial. Since a ruled surface is in particular a family of curves of genus 0 parametrized by C according to our definition, the functor $\mathcal{F}$ is not a sheaf for the Zariski topology: the structure of X is not determined by knowing its structure locally on C, so the moduli space cannot be a fine moduli (22.10).

Another way of putting this is if our space M were a fine moduli space, then every family of curves of genus 0 would be trivial, i.e., a product of the base with $\mathbb{P}^1$, and the ruled surfaces give examples of families that are locally trivial but not globally trivial.

Example 23.4. Here we show that families of curves of genus 0 need not even be locally trivial. Let $A = k[t, u]$, and consider the curve in $\mathbb{P}^2_A$ defined by $ux^2 + ty^2 + z^2 = 0$. We take $S = \operatorname{Spec} A - \{tu = 0\}$, and take X to be this family of curves over S. This is a family of curves of genus 0, but it is not even locally trivial. If it were, the generic fiber X_η, defined by the same equation over the field $K = k(t, u)$ would be isomorphic to $\mathbb{P}^1_k$. But X_η has no rational points over K. A rational point would be given by taking $x = f(t, u)$, $y = g(t, u)$, $z = h(t, u)$, where f, g, h are rational functions in t and u, not all zero. Clearing denominators, we may assume that f, g, h are polynomials. Then, looking at the terms of highest degree in t, u, we see that they cannot cancel in the equation, which gives a contradiction.

This is an example of an *isotrivial* family, namely a family in which all the fibers are isomorphic to each other, but the family itself is not trivial.

The phenomenon exhibited in (23.4) comes from the fact that over a non-algebraically closed field, there are other curves of genus 0 besides $\mathbb{P}^1$, that have no rational points. For example over $\mathbb{R}$ there is the conic $x^2 + y^2 + z^2 = 0$ in $\mathbb{P}^2$, which has no real points. Over the rational numbers there are many different non-isomorphic curves of genus 0. This is part of the reason for the subtleties in families. We can improve the situation by changing the moduli problem slightly to consider pointed curves.

Definition 23.5. A *pointed* curve of genus 0 over k will be a curve of genus 0 together with a choice of a point, rational over k. So the set of objects $\mathcal{M}$ we are considering still has just one element, namely $\mathbb{P}^1_k$ together with a chosen point P. (The choice of which point does not matter, since the automorphisms of $\mathbb{P}^1$ are transitive on closed points.) A *family* of pointed curves will be a flat family X/S, whose geometric fibers are all curves of genus 0, together with section $\sigma : S \to X$ (which some people call an S-point of X). The section σ induces a point on each fiber in a coherent way.

As before, we can show that the one-point space $M = \operatorname{Spec} k$ is a coarse moduli scheme for pointed curves of genus 0, and that it has a tautological family. Also as before it is not a fine moduli scheme, because of the ruled surfaces exhibited in (23.3). What is different in this case is that now all families are locally trivial.

Proposition 23.6. *Any family X/S of pointed curves of genus 0 is locally trivial, that is every point $s \in S$ has an open neighborhood U such that $\pi^{-1}(U) \cong \mathbb{P}^1_U$. In particular, a pointed curve of genus 0 over any field k (not necessarily algebraically closed) is isomorphic to $\mathbb{P}^1_k$.*

Proof. (Cf. [27, V.2.2] for a special case.) Given the family $\pi : X \to S$ and the section $\sigma : S \to X$, we let D be the scheme-theoretic image of σ. Then D is flat over S, and its restriction to any fiber is one point, so D is a Cartier divisor on X. Let $\mathcal{L}$ be the associated invertible sheaf on X. For each point $s \in S$ we then have $H^0(X_s, \mathcal{L}_s)$ is a 2-dimensional vector space, and $H^1(X_s, \mathcal{L}_s) = 0$. Now we apply cohomology and base extension [27, III, 12.11] to the maps

$$\varphi^i(s) : R^i f_*(\mathcal{L}) \otimes k(s) \to H^i(X_s, \mathcal{L}_s).$$

For $i = 1$, since $H^1(X_s, \mathcal{L}_s) = 0$, $\varphi^1(s)$ is surjective, hence an isomorphism, so $R^1 f_*(\mathcal{L}) = 0$. The zero sheaf is locally free, so we find $\varphi^0(s)$ is surjective, hence also an isomorphism. Since $\varphi^{-1}(s)$ is trivially surjective, we find $f_*\mathcal{L}$ is locally free of rank 2 on S. Call it $\mathcal{E}$. Then the natural map $\pi^*\mathcal{E} \to \mathcal{L} \to 0$ determines a morphism $X \to \mathbb{P}(\mathcal{E})$ which is an isomorphism on each fiber, hence an isomorphism. If we take $U \subseteq S$ to be an open set over which $\mathcal{E}$ is free, then $\pi^{-1}(U) \cong \mathbb{P}^1_U$ as required.

Remark 23.7. If we wish to have a fine moduli space for curves of genus 0, we must rigidify the functor further. Consider curves of genus 0 with an ordered choice of three distinct points, and families with an ordered choice of three non-intersecting sections. In this case we leave to the reader to verify that every family is trivial, and so the one-point space with universal family consisting of $\mathbb{P}^1$ with the three points $\{0, 1, \infty\}$ becomes a fine moduli space. The point is that the choice of three points rigidifies the automorphisms of $\mathbb{P}^1$, so that there is a unique way of identifying each fiber with $\mathbb{P}^1$, and this makes the families trivial.

24 Deformations of a morphism

It is often useful to consider deforming not only schemes, but also morphisms of schemes. Given a morphism $f : X \to Y$ of schemes over k, a *deformation of* f (keeping X, Y fixed) over an Artin ring A, is a morphism $f' : X \times A \to Y \times A$ such that $f' \otimes k = f$.

Lemma 24.1. *To give a deformation of a morphism $f : X \to Y$ (keeping X and Y fixed) it is equivalent to give a deformation of the graph Γ_f as a closed subscheme of $X \times Y$.*

Proof. To any deformation f' of f we associate its graph $\Gamma_{f'}$, which will be a closed subscheme of $X \times Y \times A$ that is a deformation of Γ_f. Conversely, given a deformation Z of Γ_f over A, we need only verify that it is a graph of some morphism. The projection $p_1 : Z \to X \times A$ gives an isomorphism when tensored with k, and then from flatness of Z over A it follows that p_1 is an isomorphism, and so Z is the graph of $f' = p_2 \circ p_1^{-1}$.

Proposition 24.2. *Assume that Y is non-singular. Then the tangent space to the deformation functor of $f : X \to Y$ (keeping X and Y fixed) is $H^0(X, f^*T_Y)$, and the obstructions to deforming f lie in $H^1(X, f^*T_Y)$. If X and Y are also projective, the deformation functor of f is pro-representable.*

Proof. From (24.1) we must consider the deformations of Γ_f as a closed subscheme of $X \times Y$. Note that $\Gamma_f = (f \times \mathrm{id})^* \Delta_Y$, where $\Delta_Y \subseteq Y \times Y$ is the diagonal. Since Y is non-singular, Δ_Y is a local complete intersection in $Y \times Y$, and $\mathcal{I}_\Delta / \mathcal{I}_\Delta^2 = \Omega^1_{Y/k}$. It follows that Γ_f is a local complete intersection in $X \times Y$, and that its normal bundle is f^*T_Y. Now our result follows from the corresponding discussion for the Hilbert scheme (6.3).

Theorem 24.3. *Given X, Y projective schemes over k, with Y non-singular, the global functor of families of morphisms $f : X \times S \to Y \times S$ over a scheme S is represented by a quasi-projective scheme over k.*

Proof. This follows from the existence of the Hilbert scheme of closed subschemes of $X \times Y$, and the observation that the set of subschemes Z representing graphs of morphisms is an open subset of the Hilbert scheme.

Remark 24.4. If $f : X \to Y$ is a closed immersion, there is a natural morphism of functors $\mathrm{Def}(f) \to \mathrm{Hilb}(Y)$, by assigning to f the closed subscheme image. If Y is non-singular, the exact sequence

$$\mathcal{I}/\mathcal{I}^2 \to \Omega^1_Y|_X \to \Omega^1_X \to 0$$

dualizes to give a cohomology sequence

$$0 \to H^0(T^0_X) \to H^0(f^*T_Y) \to H^0(\mathcal{N}_{X/Y}).$$

The middle group represents infinitesimal deformations of f. The image on the right is the corresponding deformation of the subscheme. If the subscheme is unchanged, then f comes from an infinitesimal deformation of X.

Remark 24.5. A special case of the Hom functor is the functor of isomorphisms. Given X, Y schemes over a base scheme S, for any base extension $T \to S$, we denote by $F(T)$ the set of isomorphisms $\varphi : X \times_S T \xrightarrow{\sim} Y \times_S T$, as schemes over T. Giving φ is equivalent to giving its graph $\Gamma_\varphi \subseteq X \times Y \times T$. Thus, if X and Y are projective over S, the representability of the Hilbert scheme shows that F is globally represented by a scheme $\mathrm{Isom}_S(X, Y)$, quasi-projective over S.

Remark 24.6. A generalization of the above discussion allows us to treat deformations of $f : X \to Y$, keeping Y fixed, but allowing both f and X to vary. We consider the functor $F = \mathrm{Def}(X, f)$ which to each Artin ring assigns a deformation X'/A, together with its closed immersion $X \hookrightarrow X'$, and a morphism $f' : X' \to Y \times A$, restriction to f on X. If X and Y are projective over k, one can apply Schlessinger's criterion as before to see that F has a miniversal family. If X and Y are both non-singular, there is an exact sequence of tangent spaces

$$0 \to H^0(f^*T_Y) \to t_F \to H^1(T_X),$$

where the right-hand arrow is the forgetful functor $\operatorname{Def}(X, f) \to \operatorname{Def}(X)$, and the kernel is those deformations of f that leave X fixed, which we studied above.

Application 24.7. Curves with no g^1_d. As an application of this section, we give a result showing how deformation theory can be used to prove the existence of objects with some desirable general property. It is easy to give schemes with a particular property by example. But when wants to prove that "a sufficiently general X does not have special property P", this may be exceedingly difficult to do by example. A famous example is the theorem of Noether saying that a general surface X of degree $d \geq 4$ in $\mathbb{P}^3$ contains only curves that are complete intersections with other surfaces in $\mathbb{P}^3$, i.e., $\operatorname{Pic} X \cong \mathbb{Z}$, generated by $\mathcal{O}_X(1)$. This can be proved by the deformation theory of Kodaira–Spencer over $\mathbb{C}$ [43], but only very recently has anyone been able to give a simgle specific example of such a surface! Here we illustrate the method with a much easier problem.

Proposition 24.8. *A general curve of genus $g > 2d-2$ does not have a g^1_d.*

Proof. Recall that a g^1_d means a linear system of dimension 1 and degree d, and we will always assume it to be without base points. A curve C has a g^1_d if and only if it admits a morphism $C \to \mathbb{P}^1$ of degree d. A curve with a g^1_2 is called *hyperelliptic*; a curve with a g^1_3 is *trigonal*, and so on. So our result says that for $g \geq 3$ a general curve of genus g is not hyperelliptic; for $g \geq 5$, a general curve of genus g is not trigonal, and so on. Of course one can give proofs by construction in special cases. For example, a non-singular plane quartic curve is a non-hyperelliptic curve of genus 3. But the examples get more difficult as the numbers get bigger, and I think a few hours spent trying to construct as many cases as you can will generate ample appreciation for the general method.

So, let us suppose for a given g, d, that every curve of genus g has a g^1_d and assume $g \geq 2$ for non-triviality. We need to presuppose the existence of a "modular" family of curves of genus g, i.e., a family X/S of curves, where S is smooth of dimension $3g-3$ and at each point $s \in S$, the corresponding family over $\operatorname{Spec} \hat{\mathcal{O}}_{S,s}$ pro-represents the local deformation functor of the fiber C of X over s (27.2).

Thinking of the graphs of g^1_d's in the fibers of X, and the Hilbert scheme of $X \times \mathbb{P}^1/S$, we may assume that there is an invertible sheaf $\mathcal{L}$ on X and sections t_0, t_1, at least over a neighborhood of a point $s \in S$, such that $\mathcal{L}, t_0, t_1$ give a g^1_d on all nearby fibers.

If C is the fiber over s and $f : C \to \mathbb{P}^1$ the morphism given by the g^1_d, we write the sequence of differentials

$$0 \to f^*\Omega^1_{\mathbb{P}^1} \to \Omega^1_C \to R \to 0$$

where R is the ramification sheaf. Dualizing we get

$$0 \to T_C \to f^*T_{\mathbb{P}^1} \to R' \to 0$$

where R' is the torsion sheaf $\mathcal{E}xt^1(R, \mathcal{O}_C)$. This gives a sequence of cohomology

$$0 \to H^0(T_C) \to H^0(f^*T_{\mathbb{P}^1}) \to H^0(R') \to H^1(T_C) \to H^1(f^*T_{\mathbb{P}^1}) \to 0.$$

Here $H^0(T_C) = 0$ since $g \geq 2$, and one can identify $H^0(R')$ with the tangent space to the deformations of the pair (C, f) described in (24.6) above.

Since our g^1_d extends over the whole modular family S by hypothesis, we conclude that the map $H^0(R') \to H^1(T_C)$ must be surjective, hence $H^1(f^*T_{\mathbb{P}^1}) = 0$. Now $T_{\mathbb{P}^1} = \mathcal{O}(2)$, so $f^*T_{\mathbb{P}^1}$ corresponds to the divisor $2D$, where D is the divisor of the g^1_d. In other words, $H^1(\mathcal{O}_C(2D)) = 0$. Furthermore, since $\dim |D| \geq 1$, it follows that $\dim |2D| \geq 2$. Then by Riemann–Roch

$$h^0(\mathcal{O}(2D)) = 2d + 1 - g \geq 3$$

and hence $g \leq 2d - 2$.

Therefore, by contradiction, we find that for $g > 2d - 2$, the general curve of genus g has no g^1_d. This argument justifies the proofs "by counting parameters" used by the ancients.

Application 24.9. Mori's theorem. A spectacular application of the deformation theory of a morphism was Mori's proof that a non-singular projective variety with ample tangent bundle is isomorphic to $\mathbb{P}^n$ ("Hartshorne's conjecture"). We will not describe how he deduced the existence of rational curves on such a variety in characteristic 0 from their existence in characteristic $p > 0$; nor will we trace the steps leading from the existence of rational curves to the final result. We will only prove the key step, which is the following criterion for the existence of a rational curve on a manifold in characteristic $p > 0$.

Theorem 24.10. (Mori) *Let X be a non-singular projective variety over an algebraically closed field k of characteristic $p > 0$. Assume that the canonical divisor K_X is not numerically effective, i.e., there exists an irreducible curve C with $C.K_X < 0$. Then X contains a rational curve, i.e., an integral curve whose normalization is isomorphic to $\mathbb{P}^1$.*

Proof. Let $C_0 \subseteq X$ be an integral curve with $C_0.K_X < 0$. Let $C_1 \to C_0$ be the normalization of C_0, and let $g =$ genus of C_1. If $g = 0$ there is nothing to prove, so we suppose $g > 0$. Since $C_0.K_X < 0$ we can find $q = p^r$ for $r \gg 0$ such that

$$-q(C_0.K_X) \geq ng + 1$$

where $n = \dim X$.

Let $f : C \to C_1$ be the q^{th} k-linear Frobenius morphism, i.e., C is the same abstract curve as C_1, but with structural morphism to k modified by q^{th} powers in k, and so that f is a purely inseparable k-morphism of degree q. Note that the genus of C is still g. We denote also by f the composed map $C \to C_1 \to C_0 \subseteq X$.

Fix a point $P \in C$. We will consider the deformation theory of the morphism $f : C \to X$, keeping C and X fixed, and also keeping fixed the image $f(P) = P_0 \in C_0$. As in (24.3) the corresponding deformation functor is represented by a scheme $\mathrm{Hom}_P(C, X)$, quasi-projective over k; its tangent space is $H^0((f^*T_X)(-P))$ and its obstructions lie in $H^1((f^*T_X)(-P))$.

Now the dimension estimate for representable functors (7.4) tells us that

$$\dim \mathrm{Hom}_P(C, X) \geq h^0((f^*T_X)(-P)) - h^1((f^*T_X)(-P)).$$

To compute this, note that T_X is locally free of rank n; the restriction $T_X|_{C_0}$ has degree $-C_0.K_X$, and so $f^*(T_X)$ has degree $-q(C_0.K_X)$. The twist $-P$ subtracts n from the degree. Then by Riemann–Roch on C we have

$$\chi((f^*T_X)(-P)) = -q(C_0.K_X) - n + n(1 - g),$$

and by our choice of q this number is at least 1, so $\dim \mathrm{Hom}_P(C, X) \geq 1$.

Thus there exists a non-singular curve D, not necessarily complete, and a morphism $F : C \times D \to X$ representing a non-constant family of morphisms of C to X, parametrized by D, all sending P to P_0.

I claim, in fact, that D is not complete. For suppose D were complete. Then $C \times D$ would be a non-singular projective surface. For any point $Q \in C$, the curve $Q \times D$ is algebraically equivalent to $P \times D$. Now let $\mathcal{L}$ be a very ample invertible sheaf on X, corresponding to a projective embedding of X in some projective space. The degree of the image curve $F(Q \times D)$ is then measured by $(Q \times D).F^*\mathcal{L}$. Since $Q \times D \sim P \times D$, and $F(P \times D) = P_0$ is a point, this degree is zero. So $F(Q \times D)$ is also a point, and this implies that F is a constant family $f : C \to X$, contrary to hypothesis. Thus D cannot be complete.

Now let $D \subseteq \bar{D}$ be a completion to a projective non-singular curve $\bar{D}$, and let $\bar{F} : C \times \bar{D} \dashrightarrow X$ be the corresponding rational map, which by the previous argument cannot be a morphism. The undefined points of $\bar{F}$ can be resolved after a finite number of blowing-ups of points $\pi : Y \to C \times \bar{D}$ into a morphism $F' : Y \to X$. Let $E \subseteq Y$ be the exceptional curve of the last blowing up that was needed to get the morphism F'. Then F' does not collapse E to a point, and the image $F'(E)$ is the required rational curve in X.

References for this section. Mori's theorem occurs in his paper [54], which is also where the proof of the dimension estimate for representable functors was proved. The study of special linear systems on curves is explained in detail in the book [1]. The Isom scheme is used by Mumford in his discussion of Picard groups of moduli problems [58]. And, of course, the general theory of the Hom scheme as a representable functor appears in the same exposé of Grothendieck's as the Hilbert scheme [22].

25 Lifting from characteristic p to characteristic 0

If we have a scheme X flat over $\operatorname{Spec} R$, where R is a ring of characteristic zero, but with residue fields of finite characteristic (for example R could be the ring of integers in an algebraic number field), then the generic fiber X_η of X over R will be a scheme over a field of characteristic zero, while a special fiber X_0 will be a scheme over a field of characteristic $p > 0$. In this case we can think of X_0 as a specialization of X_η. The lifting problem is the reverse question: Given a scheme X_0 over a field k_0 of characteristic $p > 0$, does there exist a flat family X over an integral domain R whose special fiber is X_0 and whose general fiber is a scheme over a field of characteristic zero?

To fix the ideas, let us suppose that X_0 is a non-singular projective variety over a perfect field k_0 of characteristic $p > 0$. Let us fix a complete discrete valuation ring $R, \mathfrak{m}$ of characteristic 0 with residue field k_0. Such a ring always exists, for example the ring of Witt vectors over k_0. Then we ask if there is a scheme X, flat over R, with closed fiber X_0.

For each $n \geq 1$, let $R_n = R/\mathfrak{m}^{n+1}$. Then we have exact sequences

$$0 \to \mathfrak{m}^{n-1}/\mathfrak{m}^n \to R_{n+1} \to R_n \to 0$$

and we can use our study of infinitesimal liftings to try to lift X_0 successively to a scheme X_n flat over R_n. Note that in contrast to the equicharacteristic situation of a local ring R containing its residue field k_0, there is no "trivial" deformation of X_0: already the step from X_0 to X_1 may be obstructed. But we know (???) that the obstructions to lifting at each step lie in $H^2(X_0, \mathcal{T}_{X_0})$, and when an extension exists, the set of all such is a torsor under $H^1(X_0, \mathcal{T}_{X_0})$.

Proposition 25.1. *Suppose that there is a family of liftings X_n flat over R_n for each n, with $X_n \otimes R_{n-1} = X_{n-1}$. Then the limit $\mathcal{X} = \varinjlim X_n$ is a Noetherian formal scheme.*

Proof. We define $\mathcal{X}$ to be the locally ringed space formed by taking the topological space X_0, together with the sheaf of rings $\mathcal{O}_{\mathcal{X}} = \varprojlim \mathcal{O}_{X_n}$ [27, II, 9.2]. To show that $\mathcal{X}$ is a Noetherian formal scheme, we must show that $\mathcal{X}$ has an open cover $\mathcal{U}_i$, such that on each $\mathcal{U}_i$, the induced ringed space is obtained as the formal completion of a scheme U_i along a closed subset Z_i.

Let U be an open affine subset of X_0, with $U = \operatorname{Spec} B_0$. Then for each n the restriction of X_n to U will be $\operatorname{Spec} B_n$ for a suitable ring B_n. Furthermore, the rings B_n form a surjective inverse system with $\varprojlim B_n = B_\infty$, and $H^0(U, \mathcal{O}_{\mathcal{X}}) = B_\infty$.

Take a polynomial ring $A_0 = k_0[x_1, \ldots, x_n]$ together with a surjective map $A_0 \to B_0$. For each n, let $A_n = R_n[x_1, \ldots, x_n]$. Lifting the images of x_i we get a surjective map $A_n \to B_n$, with kernel I_n. Because of the flatness of B_n over R_n, we find the inverse system $\{I_n\}$ is also surjective, and hence [27, II, 9.1] the map of inverse limits $\varprojlim A_n \to B_\infty$ is also surjective. Now $\varprojlim A_n = R\{x_1, \ldots, x_n\}$, the convergent power series in $x_1, \ldots, x_n$ over R, which is a Noetherian ring, so B_∞ is a Noetherian ring also, complete with respect to the $\mathfrak{m}B_\infty$-adic topology, and each $B_n = B_\infty/\mathfrak{m}^n B_\infty$. Thus we see that the ringed space $(U, \mathcal{O}_{\mathcal{X}}|_U)$ is just the formal completion of $\operatorname{Spec} B_\infty$ along the closed subset U. Such open sets U cover X_0, so by definition $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is a Noetherian formal scheme. $\square$

The next problem is that while $\mathcal{X}$, as a Noetherian formal scheme, is locally isomorphic to the completion of a (usual) scheme along a closed subset, it may not be globally so, in other words, it may not be *algebraizable* [27, II, 9.3.2]. In general, the algebraizability may be a difficult question, but we can deal with it in the projective case by the following theorem of Grothendieck.

Theorem 25.2. [24, III, 5.4.5] *Let $R, \mathfrak{m}$ be a complete local Noetherian ring, let $X = \mathbb{P}^n_R$, and let $\hat{X}$ be the formal completion of X along the closed fiber X_0 defined by $\mathfrak{m}$. If $\mathcal{Y}$ is a closed formal subscheme of $\hat{X}$, then $\mathcal{Y}$ is algebraizable, i.e., $\mathcal{Y} = \hat{Y}$, where Y is a closed subscheme of X, and $\hat{Y}$ is its completion along the closed fiber Y_0 defined by $\mathfrak{m}$.*

Putting these results together, we can now prove the following lifting theorem.

Theorem 25.3. *Let X_0 be a non-singular projective variety over a perfect field k_0 of characteristic $p > 0$. Assume that $H^2(X_0, \mathcal{O}_{X_0}) = 0$ and $H^2(X_0, \mathcal{T}_{X_0}) = 0$. Let $R, \mathfrak{m}$ be a complete discrete valuation ring with residue field k. Then X_0 can be lifted to a scheme X, flat over R, with closed fiber isomorphic to X_0.*

Proof. Since $H^2(X_0, \mathcal{T}_{X_0}) = 0$ the obstructions to infinitesimal lifting are zero, so we obtain a compatible sequence of liftings X_n flat over R_n. Their limit gives a Noetherian formal scheme $\mathcal{X}$ by (25.1). Since X_0 is assumed to be projective, it has an ample invertible sheaf $\mathcal{L}_0$. Replacing $\mathcal{L}_0$ by a high enough power (which corresponds to replacing a given projective embedding of X_0 by a d-uple embedding), we may assume $H^1(X_0, \mathcal{L}_0) = 0$. The obstruction to lifting an invertible sheaf lies in $H^2(X_0, \mathcal{O}_{X_0})$. Since this is zero, we may lift $\mathcal{L}_0$ to a compatible sequence of invertible sheaves $\mathcal{L}_n$ on each X_n. Comparing X_n to X_{n+1} we have an exact sequence

$$0 \to \mathcal{L}_{n+1} \oplus \mathfrak{m}^{n-1}/\mathfrak{m}^n \to \mathcal{L}_{n+1} \to \mathcal{L}_n \to 0.$$

The sheaf on the left is just $\mathcal{L}_0$, so we get map on cohomology

$$H^0(X_{n+1}, \mathcal{L}_{n+1}) \to H^0(X_n, \mathcal{L}_n) \to H^1(X_0, \mathcal{L}_0).$$

Since we assumed $H^1(X_0, \mathcal{L}_0) = 0$, we can lift the sections of $\mathcal{L}_0$ that give a projective embedding of X_0 to all the $\mathcal{L}_n$. In the limit, these give sections of the limit sheaf $\mathcal{L} = \varprojlim \mathcal{L}_n$ that determine an embedding of $\mathcal{X}$ in the completion of the corresponding projective space over R along its closed fiber.

Now we use (25.2) to conclude that $\mathcal{X}$ is algebraizable, hence comes from a scheme X flat over R with closed fiber X_0. □

Remark 25.4. We can write the proof of (25.3) slightly differently by converting it to a question of embedded deformations instead of deformations of abstract varieties. Suppose that X_0 is embedded in $\mathbb{P}^r_{k_0}$ in such a way that $H^1(X_0, \mathcal{O}_{X_0}(1)) = 0$ (i.e., after replacing by a d-uple embedding if necessary as above). Then the obstruction to lifting X_0 to a closed subscheme X_n of $\mathbb{P}^r_{R_n}$ lies in $H^1(X_0, \mathcal{N}_{X_0})$. Since X_0 is non-singular, we have an exact sequence

$$0 \to \mathcal{T}_{X_0} \to \mathcal{T}_{\mathbb{P}^r}|_{X_0} \to \mathcal{N}_{X_0} \to 0,$$

and from the standard sequence on $\mathbb{P}^r$, restricted to X_0, we get

$$0 \to \mathcal{O}_{X_0} \to \mathcal{O}_{X_0}(1)^{r+1} \to \mathcal{T}_{\mathbb{P}^r}|_{X_0} \to 0.$$

Now the hypotheses $H^1(\mathcal{O}_{X_0}(1)) = 0$ and $H^2(\mathcal{O}_{X_0}) = 0$ force $H^1(\mathcal{T}_{\mathbb{P}^r}|_{X_0}) = 0$, and this combined with $H^2(X_0, \mathcal{T}_{X_0}) = 0$ forces $H^1(X_0, \mathcal{N}_{X_0}) = 0$.

Thus the obstructions vanish, and we can lift X_0 to a succession of closed subschemes X_n of $\mathbb{P}^r_{R_n}$, flat over R_n. Taking the limit of these we obtain a closed formal subscheme $\mathcal{X}$ of $\hat{\mathbb{P}}^r_R$, and then as before this is algebraizable and comes from the desired scheme X flat over R.

Corollary 25.5. *Any non-singular projective curve over a perfect field k_0 of characteristic $p > 0$ is liftable to characteristic zero.*

We can apply the same techniques to the embedded lifting problem.

Theorem 25.6. *Let X_0 be a closed subscheme of $\mathbb{P}^r_{k_0}$, with k_0 a perfect field of characteristic p. Assume that X_0 is locally unobstructed (e.g., X_0 is locally complete intersection* (9.3)*, or X_0 is locally Cohen–Macaulay of codimension* 2 (8.5)*). Assume also that $H^1(X_0, \mathcal{N}_{X_0}) = 0$. Let R be a complete discrete valuation ring with residue field k_0. Then X_0 lifts to R, as a subscheme of $\mathbb{P}^r$, i.e., there exists a closed subscheme X of $\mathbb{P}^r_R$, flat over R with $X \times_p k_0 = X_0$.*

Proof. The method is already contained in the proof of the previous theorem. Because of $H^1(X_0, \mathcal{N}_0) = 0$, one can lift X_0 stepwise to a sequence of closed subschemes X_n of $\mathbb{P}^r$, flat over R_n. The limit of these is a projective formal subscheme of $\hat{\mathbb{P}}^r_R$, which is algebraizable by (25.2).

Remark 25.7. Without assuming $H^1(\mathcal{N}) = 0$, it seems to be an open question whether any non-singular curve in $\mathbb{P}^3_{k_0}$ (or any other $\mathbb{P}^r_{k_0}$) lifts to characteristic zero as an embedded curve (cf. [17]).

In theorems (25.3) and (25.6) above, the problem of lifting from characteristic p to characteristic 0 is solved in the *strong* sense, namely, given the object X_0 over k_0 and the valuation ring R with residue field k_0, the lifting is possible over that ring R. One can also ask the *weak* lifting problem: given X_0 over k_0, does there exist a local integral domain R of characteristic 0 and residue field k_0 over which X_0 lifts to an X flat over R? Oort has given an example [68] of a curve together with an automorphism of that curve that is not liftable over the Witt vectors, but is liftable over a ramified extension of the Witt vectors. Thus the strong and the weak lifting problems are in general not equivalent. Next we will give Serre's example that even the weak lifting problem is not always possible for non-singular projective varieties.

Theorem 25.8. (Serre) *Over an algebraically closed field k of characteristic $p \geq 5$, there is a non-singular projective 3-fold Z that cannot be lifted to characteristic 0, even in the weak sense.*

Proof. Let k be algebraically closed of characteristic $p \geq 5$. Let $r \geq 5$, and let $G = (\mathbb{Z}/p)^r$. Then G is a finite abelian group, and by choosing elements $e_1, \ldots, e_r \in k$ that are linearly independent over $\mathbb{F}_p$, we can find an additive subgroup $G \subseteq k^+$ isomorphic to the abstract group G.

Now let N be the 5×5 matrix $(a_{ij})_{i,j=0,\ldots,4}$ defined by $a_{i,i+1} = 1$ for $i = 0, 1, 2, 3$ and $a_{ij} = 0$ otherwise. Then N is a nilpotent matrix with $N^5 = 0$. For each $t \in G \subseteq k^+$, consider the matrix $e^{tN} = I + tN + \frac{1}{2}t^2N^2 + \frac{1}{6}t^3N^3 + \frac{1}{24}t^4N^4$ in $SL(5, k)$. The fractions are well-defined because we assumed characteristic $k = p \geq 5$. This gives a homomorphism of the additive group G to the multiplicative group $SL(5, k)$, and hence an action of G on $\mathbb{P}^4_k$. It is easy to check that the only fixed point of this action is $P_0 = (1, 0, 0, 0, 0)$.

Now $\mathbb{P}^4/G$ is a (singular) projective variety. Taking a suitable projective embedding we can find a smooth 3-dimensional hyperplane section Z. This is the required example.

To prove that Z is not liftable, we proceed as follows. First of all, let $Y \subseteq \mathbb{P}^4$ be the inverse image of Z under the quotient map $\mathbb{P}^4 \to \mathbb{P}^4/G$. Then Y is a hypersurface, stable under the action of G, and $Y/G \cong Z$. Since Z is smooth, it does not contain the image of the fixed point P_0, so G acts freely on Y, and the map $Y \to Z$ makes Y into an étale Galois cover of Z with group G.

Now suppose there is a local integral domain R of characteristic 0 with residue field k, and a scheme Z', flat over R, with $Z' \times_R k = Z$. First we show that the étale cover Y lifts.

Proposition 25.9. *Let Z be a scheme over a field k, let $Y \to Z$ be a finite étale cover, let R be a complete local ring with residue field k, and suppose there exists a scheme Z', flat over R, with $Z' \times_R k = Z$. Then there is a finite étale cover $Y' \to Z'$ (necessarily flat over R) with $Y' \times_R k = Y$.*

Proof. By definition of an étale cover $Y \to Z$ is a finite, affine, smooth morphism of relative dimension zero. Then, just as we showed in §3,4, that the functors $T^i(B/k, M) = 0$ for $i = 1, 2$, any M, when B is smooth over k, one can show also that for any smooth ring extension $A \to B$, the functors $T^i(B/A, M) = 0$ for all M. If $A \to B$ is étale, then we also have $\Omega_{B/A} = 0$ and so $T^0(B/A, M) = 0$.

Thus, for a finite étale morphism, over each open affine subset of the base, the obstructions in T^2 to lifting vanish. A lifting exists, and because of $T^1 = 0$, it is unique. Thus the liftings patch together, and we get a unique lifting of the entire étale cover over each $R_n = R/\mathfrak{m}^n$.

In the limit, these give an étale cover of the formal scheme $\hat{Z}'$. Since the morphism $Y \to Z$ is projective, the algebraization theorem (25.2) again gives the cover Y' of Z' desired. $\square$

Proof of* (25.8), *continued. Using (25.9), we obtain a finite étale cover Y' of Z' that reduces to Y over Z. Because of the uniqueness in each step of lifting the étale cover, the group action G extends to Y' and makes Y' a Galois covering of Z' with group G.

Next, since Y is a hypersurface in $\mathbb{P}^4$ we have $H^i(Y, \mathcal{O}_Y) = 0$ for $i = 1, 2$. Now $H^2(Y, \mathcal{O}_Y)$ contains the obstructions to lifting an invertible sheaf, and $H^1(Y, \mathcal{O}_Y)$ tells the number of ways to lift an invertible sheaf. Since both of these are zero, the invertible sheaf $\mathcal{L} = \mathcal{O}_Y(1)$ on Y lifts uniquely to an invertible sheaf $\mathcal{L}'$ on Y'. Furthermore, since $H^1(Y, \mathcal{O}_Y(1)) = 0$, the sections of $H^0(\mathcal{O}_Y(1))$ defining the projective embedding also lift, and we find that $H^0(\mathcal{L}')$ is a free R-module of rank 5.

Since the group acts on $\mathbb{P}^4_k$ sending Y to itself, G also acts on $H^0(Y, \mathcal{O}_Y(1)) = k^5$, and this action also lifts to $H^0(Y', \mathcal{L}')$. Let K be the quotient field of R. Then we get an embedding of Y'_K in $\mathbb{P}^4_K$, and the group action G extends to an action on $\mathbb{P}^4_K$. In other words, we get a homomorphism $\varphi : G \to PGL(5, K)$ compatible with the original action on $\mathbb{P}^4_k$. In particular, φ is injective. Thus $PGL(5, K)$ contains a subgroup isomorphic to G, which is impossible as long as the rank of G is $r \geq 5$.

Hence Z cannot be lifted, as was to be shown.

References and further results. Theorems (25.2) and (25.3) and Corollary (25.4) first appeared in Grothendieck's Bourbaki Seminar [21] of May, 1959. There he also raised the problem of liftability of smooth projective varieties, which was answered by Serre's example (25.8), in 1961 [80]. Mumford is said to have modified Serre's method to give an example of a non-liftable surface.

Since then, many lifting problems have been studied. The two articles of Oort [67], [68] are extremely useful. He shows that finite commutative group schemes can be lifted [60], but gives an example of a non-commutative finite group scheme that cannot be lifted (as group schemes).

Mumford [**?**] showed that one can lift any principally polarized abelian variety (as an abelian variety).

Deligne [11] showed that any $K3$ surface can be lifted. This is an interesting case, because the lifting of the abstract surface to a formal scheme may not be algebraizable: the ample invertible sheaf need not lift. It requires some extra subtlety to show that there is an algebraizable lifting. This is analogous to the complex analytic theory, where the deformation space, as complex manifolds, has dimension 20, but the algebraic $K3$ surfaces form only 19 dimensional subfamilies. This lifting result has been sharpened by Ogus [65].

Several authors have studied the problem of lifting a curve along with some of its automorphisms. One cannot expect to lift a curve with its entire group of automorphisms, because the order of that group in characteristic $p > 0$ can exceed $84(g-1)$, which is impossible in characteristic 0. However, one can lift a curve C together with a cyclic group H of automorphism, provided that p^2 does not divide the order of H [**?**].

Raynaud [72] has given examples of surfaces, the "false ruled surfaces" that cannot be lifted, and W. E. Lang [45] has generalized these.

Hirokado [32] and Schröer [78] have given examples of non-liftable Calabi–Yau threefolds, and Ekedahl [16] has shown that these examples have all their deformations limited to characteristic p. In particular, they cannot be lifted even to the Witt vectors $\bmod p^2$.

26 Moduli of elliptic curves

In this section we will apply the theory we have developed so far to elliptic curves. Our provisional definition is an *elliptic curve* over an algebraically closed field k is a non-singular projective curve of genus one. We will assume characteristic $k \neq 2, 3$ for simplicity throughout this section.

If one studies one elliptic curve at a time, there is a satisfactory theory, explained in [27, IV, §4]. To each elliptic curve C over k one can assign an element $j(C) \in k$, called the j-invariant, and two elliptic curves over k are isomorphic if and only if they have the same j-invariant. Furthermore, for any $j \in k$ there is an elliptic curve with j-invariant j. Thus the set of isomorphism classes of elliptic curves over k is in one-to-one correspondence with the set of closed points of the affine line $\mathbb{A}^1_k$ via the j-invariant.

The problem of moduli is to understand not only individual curves, but also flat families X/S whose geometric fibers are elliptic curves. In particular one can study the formal local problem of deformations over Artin rings of a given elliptic curve C/k. Our general theory tells us that this functor has a miniversal family (18.1), but since $h^0(T_C) \neq 0$, our basic result (18.5) does not guarantee that the local functor is pro-representable. On the other hand, if we consider pointed elliptic curves, namely a curve C/k with a fixed point $P \in C$, and consider deformations with a section extending the point, then we have seen (18.10) that the local deformation functor is pro-representable.

Our first result tells us that the local functor of deformations of a pointed elliptic curve over Artin rings is equivalent to the deformations of the elliptic curve without its point.

Proposition 26.1. *Let C_0 be an elliptic curve over k, and let F be the functor of deformations of C_0 over Artin rings A. Let $P_0 \in C_0$ be a closed point and let F' be the functor of deformations of the pointed curves C_0, P_0, i.e., an element of $F'(A)$ is a family C/A, flat over A, together with a section $\sigma : \operatorname{Spec} A \to C$, and a closed immersion $C_0 \subseteq C$, so that $\sigma(\mathfrak{m}) = P_0$. Then the "forgetful" morphism $F' \to F$ forgetting the section σ, is an isomorphism of functors.*

Proof. Given a deformation $C_0 \subseteq C$ and given $P_0 \in C_0$, the problem of finding a section σ of C reducing to $P_0 \in C_0$ is a question of the Hilbert scheme of P_0 in C_0. The normal sheaf $\mathcal{N}_{P_0/C_0}$ is a 1-dimensional vector space on the 1-point space P_0, so $h^1(\mathcal{N}_{P_0/C_0}) = 0$, and there are no obstructions (6.3). Hence P_0 deforms to give a section σ. Therefore the map $F'(A) \to F(A)$ is surjective for each A.

To show that $F'(A) \to F(A)$ is injective, we use induction on the length of A. For $A = k$, we note that since k is algebraically closed, every elliptic curve has a closed point, and the choice of closed point does not matter, since the group structure on the curve provides automorphisms that act transitively on the set of closed points.

Now suppose given C and a section σ over A, and suppose given C' over A', where $A' \to A$ is a small extension. Then the ambiguity in extending σ lies in $H^0(\mathcal{N}_{P_0/C_0})$. On the other hand, the automorphisms of C' leaving C fixed are given by $H^0(T_{C_0})$. One checks easily that the natural map $H^0(T_{C_0}) \to H^0(\mathcal{N}_{P_0/C_0})$ is an isomorphism. Hence there is a unique pair (C', σ') up to isomorphism for each C' given and so $F'(A') \to F(A')$ is bijective.

Remark 26.2. Since we know that the functor F' is pro-representable, it follows that the functor F of deformations of (unpointed) elliptic curves is also pro-representable, even though $h^0(T_{C_0}) \neq 0$.

Remark 26.3. Even though the formal local functors F and F' are isomorphic, the same does not hold for the global functor of isomorphism classes of families X/S of elliptic curves, because there are families having no section. Consider the family of plane curves defined by $x^3 + ty^3 + t^2z^3 = 0$ in $\mathbb{P}^2_A$ where $A = k[t, t^{-1}]$. This is a flat family of elliptic curves, but has no section, because to give a section would be to give $x = f(t)$, $y = g(t)$, $z = h(t)$, polynomials in t and t^{-1} satisfying this equation, and this is impossible (just consider the terms of highest degree in f, g, h).

For this reason when studying global moduli we must make a choice whether to consider families of unpointed or pointed elliptic curves. We choose the latter, both because it is easier to handle technically, and also because it gives a better analogy with the case of curves of genus $g \geq 2$, which have only finitely many automorphisms. So for the rest of this section we will use the following definitive definition.

Definition 26.4. An *elliptic curve* over a scheme S is a flat morphism $X \to S$ whose geometric fibers are all non-singular projective curves of genus 1, together with a section $\sigma : S \to X$. In particular, an elliptic curve over any field k is a smooth curve C of genus 1 together with a rational point $P \in C$.

Now we turn to the question of moduli. We fix k algebraically closed, and for any scheme S/k consider the functor $F(S) = \{$isomorphism classes of elliptic curves over $S\}$. We ask what kind of moduli space we can find for F.

Proposition 26.5. *The functor F does not have a fine moduli space.*

Proof. There are several reasons one can give for this. One is that the second local functor F_1 we have considered, of local families C/A such that $C \otimes_A k \cong C_0$, but without specifying the inclusion $C_0 \subseteq C_1$ is not pro-representable (18.10). We have seen that this would be a necessary condition for the global functor to be representable.

A second reason is that if F had a fine moduli space, i.e., if F were representable, then any isotrivial family (that is, a family with all isomorphic fibers) would be trivial. One way to make an isotrivial family, is to take a constant family over $\mathbb{P}^1$, identify the fibers over 0 and 1 by a non-constant automorphism τ that sends the distinguished point P to itself, and glue to get a non-constant family over a nodal curve with all isomorphic fibers.

Another way to make such a family is to write an equation such as $y^2 = x^3 + t$ over $A = k[t, t^{-1}]$. For each t we get a curve with $j = 0$, but to write an isomorphism between this one and the constant family $y^2 = x^3 + 1$, we need $t^{1/6}$, which is not in the ring A.

Proposition 26.6. *The j-line $\mathbb{A}_j$ is a coarse moduli space for the functor F.*

Proof. Recall that to be a coarse moduli space for the functor F means several things:

a) The closed points of $\mathbb{A}_j$ are in one-to-one correspondence with the isomorphism classes of elliptic curves over k. This we know from the basic theory [27, IV, 4.1].

b) For any family X/S there is a morphism $\varphi : S \to \mathbb{A}_j$ such that for each closed point $s \in S$, $\varphi(s)$ is the j-invariant of the fiber over s. This step is easy. Given X/S together with its section σ, for any open affine $U = \operatorname{Spec} A \subseteq S$, we define an embedding of $X_A \hookrightarrow \mathbb{P}^2_A$ using the divisor 3σ. Then by rational operations over the ring A as in [27, IV, §4] (and here we use the assumption that characteristic $k \neq 2, 3$) we bring the equation of the image into the form $y^2 = x^3 + ax + b$, with $a, b \in A$. Then

$$j = 12^3 \cdot \frac{4a^3}{4a^3 + 27b^2}$$

gives the desired morphism from $\operatorname{Spec} A$ to the j-line. These patch together to give $\varphi : S \to \mathbb{A}_j$.

c) Lastly, we must show that the j-line is universal with property b). So let N be some other scheme together with a morphism of the functor F to h_N, i.e., a functorial assignment, for each family X/S of a morphism $S \to N$. We consider in particular the family given by the equation $y^2 = x(x-1)(x-\lambda)$ over the λ-line $\operatorname{Spec} B$ with $B = k[\lambda, \lambda^{-1}, (\lambda-1)^{-1}]$. Then there is a morphism

$\varphi : \operatorname{Spec} B \to N$. Furthermore, by functoriality this morphism is compatible with the action of the group G of order 6 consisting of the substitutions $\{\lambda, \lambda^{-1}, 1-\lambda, (1-\lambda)^{-1}, \lambda(\lambda-1)^{-1}, (\lambda-1)\lambda^{-1}\}$, since these extend to morphisms of the family $X/\operatorname{Spec} B$. Hence the morphism φ factors through $\operatorname{Spec} B^G$, where B^G is the fixed ring of the action of G on B. All that remains is to identify B^G with $k[j]$. Clearly $j \in B^G$. Considering the function fields $k(j) \leq k(B^G) \leq k(B)$, the latter is of degree 6 over the two former, so $k(j) = k(B^G)$. Next note that B is integral over $k[j]$: the defining equation of j in terms of λ gives

$$\lambda^2(\lambda-1)^2 j = 256(\lambda^2 - \lambda + 1)^3.$$

This shows that λ is integral over $k[j]$. Rewriting this equation in terms of λ^{-1} and $(\lambda-1)^{-1}$ shows that they too are integral over $k[j]$. Therefore B^G is integral over $k[j]$. But they have the same function field, and $k[j]$ is integrally closed, so $k[j] = B^G$.

Thus we obtain a morphism $\mathbb{A}_j \to N$, so $\mathbb{A}_j$ has the desired universal property.

Remark 26.7. The coarse moduli space $\mathbb{A}_j$ does not have a tautological family. For suppose X/S is a family of elliptic curves, and $s_0 \in S$ is a point whose fiber C_0 has $j = 0$. In an affine neighborhood $\operatorname{Spec} A$ of s_0 we represent the family by $y^2 = x^3 + ax + b$ with $a, b \in A$. At the point s_0, since $j = 12^3 \cdot 4a^3/(4a^3 + 27b^2)$ we must have $a \in \mathfrak{m}$, the maximal ideal of A at the point s_0. Hence $j \in \mathfrak{m}^3$, and the morphism $S \to \mathbb{A}_j$ is ramified at the point s_0. In particular, S cannot be $\mathbb{A}_j$.

Remark 26.8. The functor F is not a sheaf for the Zariski topology. For if it were, any family X/S which becomes trivial on each open set of an open cover of S would have to be trivial. We can make a counterexample as follows. Let S be a triangle made of three copies of $\mathbb{P}^1$ meeting in pairs at points P_1, P_2, P_3. Let E be an elliptic curve over k, let τ be an automorphism of order 2 and let X be the family formed from $E \times \mathbb{P}^1$ on each line, glued by the identity at P_1 and P_2, and by τ at P_3. Then the family X is not trivial, but restricted to each of the open set $S - P_i$ it is isomorphic to the trivial family. Hence F is not a sheaf for the Zariski topology.

Note however that there is an étale morphism from a hexagon S' made of six $\mathbb{P}^1$'s to S, and that the base extension $X' = X \times_s S'$ is isomorphic to the trivial family over S'. This is a foreshadowing of the discussion to come next.

To summarize the discussion so far, we consider the functor of families of (pointed) elliptic curves. We have seen that the local deformation functor is pro-representable for each elliptic curve. The global functor does not have a fine moduli space, but it does have a coarse moduli space. The coarse moduli space does not have a tautological family. The global functor is not a sheaf for the Zariski topology.

This is about all we can say within the framework of discourse up to this point. But it is unsatisfactory since it does not give us, as in the case of a representable functor, a complete description of all possible families of elliptic curves. To go farther we must expand the range of our discourse, and this leads to the world of Grothendieck topologies, algebraic spaces, and stacks. Without explaining what any of these are, we will rather show explicitly how those theories manifest themselves in the case of elliptic curves.

The main idea is to think of replacing the Zariski topology by the étale topology. A local property will be one that holds after an étale base extension instead of on an open subset. I would like to say that the moduli functor is "representable to within étale morphisms", or that "there is a fine moduli space to within étale morphisms". To be precise, we make a definition and prove a theorem.

Definition 26.9. A *modular family* is a flat family of elliptic curves X/S, with S a scheme of finite type over k, such that

a) For each elliptic curve C/k there is at least one and at most finitely many closed points $s \in S$ whose fiber X_s is isomorphic to C, and

b) for each $s \in S$, the complete local ring $\hat{\mathcal{O}}_{s,S}$, together with the formal family induced from X, pro-represents the local functor of deformations of the fiber X_s.

Theorem 26.10.

a) *There exists a modular family $\mathcal{X}/\mathcal{S}$ of elliptic curves over k.*

b) *For any other family X/S of elliptic curves, there exists a surjective étale morphism $S' \to S$ and a morphism $S' \to \mathcal{S}$ such that $X \times_S S' \cong \mathcal{X} \times_{\mathcal{S}} S'$ as families over S'.*

c) *In particular, if $\mathcal{X}_1/\mathcal{S}_1$ and $\mathcal{X}_2/\mathcal{S}_2$ are two modular families, then there is a third modular family $\mathcal{X}_3/\mathcal{S}_3$ and surjective étale morphisms $\mathcal{S}_3 \to \mathcal{S}_1$, $\mathcal{S}_3 \to \mathcal{S}_2$ such that $\mathcal{X}_1 \times_{\mathcal{S}_1} \mathcal{S}_3 \cong \mathcal{X}_3 \cong \mathcal{X}_2 \times_{\mathcal{S}_2} \mathcal{S}_3$.*

Proof.

a) We will show that the family of plane cubic curves $y^2 = x(x-1)(x-\lambda)$ over the λ-line $\operatorname{Spec} B$, where $B = k[\lambda, \lambda^{-1}, (\lambda-1)^{-1}]$ is a modular family. First of all, we know that every elliptic curve is isomorphic to one of these for some $\lambda \neq 0, 1$, and that each isomorphism type occurs 2, 3, or 6 times.

Next we need to show that the completion of this family at any point pro-represents the local deformation functor. Since in any case by pro-representability there is a morphism from the formal family over the λ-line to the pro-representing family, and both of these are smooth and one-dimensional, it will be sufficient to show the induced map on Zariski tangent spaces is non-zero. So let $y^2 = x(x-1)(x-\lambda-t)$ be the induced family over the dual numbers $D = \operatorname{Spec} k[t]/t^2$ at the point λ. We have only to show that this family is non-trivial over D. Now two curves in $\mathbb{P}^2_D$ with equations of the form above are isomorphic if and only if their λ-values are interchanged by the six element group G. This group sends any $\lambda \in K$ to another $\lambda \in k$ and never to $\lambda + t$, hence the deformation is non-trivial.

This shows that the family over the λ-line is a modular family.

b) Now let X/S be any family of elliptic curves, and let $\mathcal{X}/\mathcal{S}$ be a modular family. Then over $S \times \mathcal{S}$ we have two families $X \times \mathcal{S}$ and $\mathcal{X} \times S$. Let $S' = \operatorname{Isom}_{S \times \mathcal{S}}(X \times \mathcal{S}, \mathcal{X} \times S)$ (24.5). Then the families become isomorphic over S', namely $X \times_S S' \cong \mathcal{X} \times_{\mathcal{S}} S'$. Furthermore, S' is universal with this property.

Now I claim $S' \to S$ is surjective and étale. For any point $s \in S$, let $C = X_s$ be the corresponding fiber. Then the isomorphism type of C occurs at least once and at most finitely many times in the family $\mathcal{X}/\mathcal{S}$, say at points $s_1, \ldots, s_n \in \mathcal{S}$. Furthermore, since the automorphism group G of C as an elliptic curve is finite, for each s_i the scheme $\mathrm{Isom}_k(X_s, \mathcal{X}_{s_i})$ is finite. Thus by the universal property of the Isom scheme, the fiber of S' over S is a finite non-empty set. Hence the map $S' \to S$ is surjective and quasi-finite.

Finally, consider a point $s' \in S'$ lying over $s \in S$. This fixes the corresponding point $s_i \in \mathcal{S}$, and also fixes the isomorphism of X_s with $\mathcal{X}_{s_i}$. For any Artin ring A, quotient of $\mathcal{O}_{s,S}$, we get an induced family over $\operatorname{Spec} A$. Since $\mathcal{S}$ is a modular family, there is a unique morphism of $\operatorname{Spec} A \to \mathcal{S}$ at the point s_i inducing an isomorphic family. Furthermore, having fixed the isomorphism on the closed fiber, there are no further automorphisms of the family over $\operatorname{Spec} A$ (recall the proof of local pro-representability (18.5)). Hence there is a unique morphism of $\operatorname{Spec} A$ to S' at the point s'. This implies that the induced homomorphism on complete local rings $\hat{\mathcal{O}}_{s,S} \to \hat{\mathcal{O}}_{s',S'}$ is an isomorphism, and hence $S' \to S$ is étale, as required.

c) Given $\mathcal{X}_1/\mathcal{S}_1$ and $\mathcal{X}_2/\mathcal{S}_2$, apply step b) to get $S' \to \mathcal{S}_1$ and $S' \to \mathcal{S}_2$ with isomorphic pull-back families and $S' \to \mathcal{S}_1$ surjective and étale (using the hypothesis $\mathcal{X}_2/\mathcal{S}_2$ modular). Since $\mathcal{X}_1/\mathcal{S}_1$ is also modular, the morphism $S' \to \mathcal{S}_2$ is also quasi-finite and surjective. Now the corresponding maps on complete local rings of all three schemes at corresponding points are isomorphisms, so $S' \to \mathcal{S}_2$ is also étale, and the induced family over S' is also modular.

Corollary 26.11. *If X/S is an isotrivial family of elliptic curves, then there exists a surjective étale map $S' \to S$ such that the base extension X'/S' is isomorphic to the trivial family.*

Proposition 26.12. *If X/S is a modular family of elliptic curves, the corresponding map of S to the coarse moduli space $\mathbb{A}_j$ is étale over points where $j \neq 0, 12^3$; ramified of order 2 over $j = 12^3$ and ramified of order 3 over $j = 0$.*

Proof. Writing

$$j = 256\frac{(\lambda+\omega)^3(\lambda+\omega^2)^3}{\lambda^2(\lambda-1)^2}$$

shows that at $\lambda = -\omega$, corresponding to $j = 0$, the map from the λ-line to the j-line is ramified of order 3. At $\lambda = -1, \frac{1}{2}, 2$, corresponding to $j = 12^3$, there are three roots, and the map is of order 6, so it is ramified of order 2. For $j \neq 0, 12^3$, there are six values of λ, so it is unramified.

Since the modular family is unique up to étale morphisms, the same holds for any modular family.

Remark 26.13. Thus we may think of a modular family as "the moduli space", uniquely determined up to étale morphisms with the universal mapping property for any family holding after an étale morphism. Or we may think of the j-line as "the moduli space", but where we need $\sqrt{j - 12^3}$ and $\sqrt[3]{j}$ as local parameters at the points $j = 12^3$ and $j = 0$. Still this does not tell us everything about the functor F, in contrast to the case of a representable functor, where the knowledge of the representing scheme and its universal family is equivalent to knowledge of the functor of families. We can ask, what further data do we need to know the functor entirely? The following remarks will reflect on this question, without however giving a complete answer.

Remark 26.14. If X/S is a modular family and $S' \to S$ is any surjective étale morphism, then $X' = X \times_S S'/S'$ is another modular family. Thus there are bigger and bigger modular families. This leads us to ask if there is a smallest modular family. The answer is no. Indeed, there is a family over the j-line minus the points $0, 12^3$, defined by the equation

$$y^2 = x^3 + ax + b, \text{ with } a = b = \frac{27}{4} \cdot \frac{j}{12^3 - j}.$$

A simple calculation shows that for any $j \neq 0, 12^3$, this defines an elliptic curve with the corresponding j-invariant. To get a modular family, we need to take a disjoint union with some patches of families containing curves with $j = 0$ and $j = 12^3$. There is no smallest such choice. On the other hand, this family does not map to the λ-line, so there is no smallest modular family.

Remark 26.15. If we confine our attention to elliptic curves with $j = 0, 12^3$, then $\mathbb{A}_j - \{0, 12^3\}$ is a coarse moduli space, and it has a tautological family, given in the previous remark. However, it is a tautological family only in the sense that the fiber at each point is a curve with the corresponding j-invariant and its is not a universal family. For any family X/S, there is a unique morphism $S \to \mathbb{A}_j - \{0, 12^3\}$ sending points $s \in S$ to the j-value of the fiber X_s, but the pull-back of our universal family may not be isomorphic to X, so the functor is still not representable, even restricting to $j \neq 0, 12^3$. To see this, note that $y^2 = x^3 + j^2ax + j^3b$, with the same a, b as above, is another tautological family over $\mathbb{A}_j - \{0, 12^3\}$, but it does not become isomorphic to the previous one until we take a double covering defined by $\sqrt{j}$. So even in this restricted case, there is no minimal modular family.

Remark 26.16. Completion of the moduli space. Having once found the coarse moduli space $\mathbb{A}_j$, a natural question is, what extra objects can we consider in order to obtain a complete moduli space. Here we will show that if in addition to elliptic curves as above, one allows irreducible nodal curves with $p_a = 1$, together with a fixed non-singular point, the whole theory extends. We consider families X/S where the fibers are elliptic curves or pointed nodal curves (the point being chosen as a smooth point of the nodal curve). The projective line $\mathbb{P}^1$ acts as a coarse moduli space, taking $j = \infty$ for the nodal curve. The family $y^2 = x(x-1)(x-\lambda)$ over the whole λ-line is a modular family in which the values $\lambda = 0, 1$ correspond to nodal curves. The proofs above all extend without difficulty, once we know the deformation theory of the nodal curve, which we explain in the next remark.

Remark 26.17. Deformation theory of the nodal elliptic curve. We consider a reduced irreducible curve C over k of arithmetic genus $p_a = 1$ having one node as its singularity (such as the curve $y^2 = x^2(x-1)$ in $\mathbb{P}^2$). The tangent space $\mathrm{Def}(C)$ to its deformation theory fits in an exact sequence

$$0 \to H^1(T_C^0) \to \mathrm{Def}(C) \to H^0(T_C^1) \to H^2(T_C^0),$$

and the three successive obstructions to deformations lie in $H^0(T_C^2)$, $H^1(T_C^1)$, $H^2(T_C^0)$ (cf. (10.6)).

Now $T_C^2 = 0$ since C is a local complete intersection scheme (10.4) so $H^0(T_C^2) = 0$. T_C^1 is a sheaf concentrated at the singular point, so

$H^1(T^1_C) = 0$. Since C is a curve, $H^2(T^0_C) = 0$. Hence there are no obstructions, and the local deformation space is smooth.

Since T^1_C is concentrated at the singular point, we know from the local discussion of deformations of a node (13.1) that $H^0(T^1_C)$ is a 1-dimensional k-vector space.

It remains to consider the sheaf T^0_C. Considering a nodal cubic curve C in $\mathbb{P}^2$ there is an exact sequence

$$0 \to T^0_C \to T_{\mathbb{P}^2}|_C \to \mathcal{N}_{C/\mathbb{P}^2} \to T^1_C \to 0.$$

One sees easily that $h^0(T_{\mathbb{P}^2}|_C) = 0$, $h^0(\mathcal{N}_{C/\mathbb{P}^2}) = 0$, $h^0(T^1_C) = 1$. Furthermore, the natural map $H^0(\mathcal{N}_{C/\mathbb{P}^2}) \to H^0(T^1_C)$ is surjective because the former measures deformations of C as a closed subscheme of $\mathbb{P}^2$, the latter measures abstract deformations of the node, and we know that there are deformations of C in $\mathbb{P}^2$ that smooth the node. From all this it follows that $h^0(T^0_C) \geq 1$.

Now let $s \in H^0(T^0_C)$ be a non-zero section. Then we get an exact sequence $0 \to \mathcal{O}_C \xrightarrow{s} T^0_C \to R \to 0$ where the cokernel R is of finite length. Furthermore R is not zero, because $T^0_C \cong \mathcal{H}om(\Omega^1_C, \mathcal{O}_C)$ is not locally free, hence not isomorphic to $\mathcal{O}_C$. Therefore $(T^0_C)^\vee$ is properly contained in $\mathcal{O}_C$, and by Serre duality on C, using the dualizing sheaf $\omega_C \cong \mathcal{O}_C$, we find $h^1(T^0_C) = h^0((T^0_C)^\vee) = 0$.

Thus $\mathrm{Def}(C)$ is one-dimensional, and the miniversal deformation space of C is smooth of dimension 1.

Finally, we compare the deformations of C to the deformations of the pointed curve (C, P), where P is a non-singular point. We find, as in the case of a smooth curve (26.1) the two functors are isomorphic, so we conclude that the deformations of (C, P) are pro-representable of dimension 1. This is all we need to complete the argument.

Remark 26.18. One might ask, why do we use the nodal curve, but not the cuspidal curve or any other connected reduced curves with $p_a = 1$? One reason is that any other singular curve besides the node has a local deformation theory of dimension ≥ 2, (13.9), and so would not fit in a modular family of elliptic curves.

Another reason is the presence of jump phenomena. Consider the family $y^2 = x^3 + t^2ax + t^3b$ over the t-line, for any fixed values of a and b such that $4a^3 + 27b^2 \neq 0$. Then for $t \neq 0$ we have non-singular elliptic curves all with the same j-invariant, while for $t = 0$ we get a cuspidal curve. Thus the cuspidal curve cannot belong to a deformation theory having a coarse moduli space. Another way of saying this is that if you try to add a point to the j-line representing the cuspidal curve, that point would have to be in the closure of every point on the j-line!

Remark 26.19. If you really want a fine moduli space for elliptic curves, you can rigidify the curves with extra structure. A *level*-2 *structure* on an elliptic curve (still assuming characteristic $k \neq 2, 3$) is an isomorphism of the group of points of order 2 in the group structure on the curve with the Klein 4-group $V \cong \mathbb{Z}/2 \times \mathbb{Z}/2$. Since $\operatorname{Aut} V = S_3$, the symmetric group on three letters, each elliptic curve has 6 possible level-2 structures. A family of elliptic curves with level-2 structures is a family X/S of elliptic curves together with the identification of their points of order 2 with a fixed V. In this case the branch points of the $2-1$ map from the curve to $\mathbb{P}^1$ can be labeled, so the invariant λ is well-defined. Thus the λ-line (minus $0, 1$) with its tautological family represents the functor of families of elliptic curves with level-2 structure.

Remark 26.20. There is one more question one could ask in trying to make sense of the functor of all possible families of elliptic curves. Though there is not a universal family, is there perhaps a small set (say finite) of modular families X_i/S_i such that for any family X/S there is a morphism $S \to S_i$ for some i such that $X \cong X_i \times_{S_i} S$? But even this last hope is dashed to the ground by the following examples of incomparable families over subsets of the j-line.

Let X_0/S_0 denote the family described in (26.14) over $S_0 = \mathbb{A}_j - \{0, 12^3\}$. For any open set $U \subseteq S_0$, let $T \to U$ be an étale cover of order 2, and let $X = X|_U \times_U T$. Now for each $u \in U$, let t_1, t_2 be the two points lying over u, and identify the fibers X_1, X_2 at t_1, t_2 via the automorphism τ of order 2. Then glue to get a new family X'_T over U. Note that we can recover T as $\operatorname{Isom}_U(X, X'_T)$. The family X'_T is isomorphic to X if and only if T is the trivial cover.

Now if $\pi : C \to \mathbb{P}^1$ is any hyperelliptic curve, and U is $\mathbb{P}^1$ minus $0, 12^3, \infty$, and the branch points of π, and $T = C|_U$, then we get a family X'_T over U. Two of these for different C, C' are isomorphic on a common open set if and only if the corresponding hyperelliptic curves are isomorphic.

Thus there is not a finite number, there is not even a collection of such families as we desired parametrized by a finite union of finite-dimensional algebraic varieties!

References for this section. I owe a special debt to Mumford's article [58] from the Purdue conference. The basic theory of elliptic curves is treated in [27, IV, §4]. Other valuable sources are the article of Deligne and Mumford [13] and [59].

27 Moduli of curves

It has been understood for a long time that there is some kind of moduli space of curves of genus $g \geq 2$. Riemann gave the dimension as $3g - 3$. Transcendental methods show that it is irreducible over the complex numbers. Fulton extended this result to characteristic $p > 0$, with some restrictions on small p, by considering the Hurwitz scheme of branched covers of $\mathbb{P}^1$. Deligne and Mumford proved irreducibility in all characteristics by introducing a compactification of the variety of moduli in which they allowed certain singular "stable" curves. They also hinted at a more sophisticated object, the moduli stack. Mumford, in his article "Picard groups of moduli problems" [58] makes the point that to investigate the more subtle properties of the moduli of curves, the coarse moduli space may not carry enough information, and so one should really work with stacks.

Our purpose here is not to prove all of these results (for which there are ample references): rather it is to disengage the issues involved, to explain why we do things the way we do, and to make some precise statements.

First we state the problem. We fix an algebraically closed field k, and we consider projective non-singular curves of genus $g \geq 2$ over k. The restriction to $g \geq 2$ is because a) we have discussed the cases of $g = 0, 1$ separately, and b) the case of $g \geq 2$ is qualitatively different in that curves of genus $g \geq 2$ can have only finitely many automorphisms.

We want to describe isomorphism classes of these curves and families of curves, so we define the *moduli functor* F, which assigns to each scheme S/k the set of isomorphism classes of flat families X/S, proper over S, whose geometric fibers are all non-singular curves of genus g. If this functor were representable, we would call the corresponding scheme a fine moduli space. But since there are curves with non-trivial automorphisms, we know (???) the functor is not representable. On the other hand, one of the main results of the theory is

Theorem 27.1. *The moduli functor F of curves of genus $g \geq 2$ over k algebraically closed has a coarse moduli space M_g that is an integral quasi-projective variety of dimension $3g-3$.*

The existence and the fact that it is quasi-projective are proved in Mumford's book [59]; the irreducibility is proved in the article of Deligne and Mumford [13].

The coarse moduli space is a variety whose closed points are in one-to-one correspondence with the set of isomorphism classes of curves in a natural way. Furthermore, for any flat family X/S, there is a morphism $f : S \to M_g$ with the property that for each k-rational point $s \in S$, the image $f(s)$ corresponds to the isomorphism class of the fiber X_s. However, there is no tautological family over M_g, and knowledge of M_g does not give us full information about all possible flat families and morphisms between them.

To give more information about families of curves, we follow the ideas of Mumford [58].

Definition 27.2. A *modular family* of curves of genus g is a flat family $\mathcal{X} \to Z$ with two properties

a) Each isomorphism type of curves of genus g occurs at least once and at most a finite number of times in the family $\mathcal{X}/Z$, and

b) for each point $z \in Z$, the completion of the family $\mathcal{X}$ over the complete local ring $R = \hat{\mathcal{O}}_{Z,z}$ pro-represents the local deformation functor of the fiber curve X_z.

Theorem 27.3.

(i) *Modular families exist (and may be taken quasi-projective).*

(ii) *If $\mathcal{X}/Z$ is a modular family, and if X/S is any other family, then there is a surjective étale morphism $S' \to S$ and a morphism $S' \to Z$ such that the two families $X \times_S S'$ and $\mathcal{X} \times_Z S'$ over S' are isomorphic.*

While this theorem does not completely describe the moduli functor F, it does suggest that we know all about flat families of curves "up to étale base extension".

Outline of Proof of (27.3)**.**

(i) On a curve of genus g, any divisor of degree $\geq 2g+1$ is non-special and very ample. In particular, if we take the tricanonical divisor $3K$, where K is the canonical divisor, then for any $g \geq 2$, its degree $d = 6g-6$ is $\geq 2g+1$, so we can use it to embed the curve in a projective space $\mathbb{P}^n$, with $n = 5g-6$, as a non-singular curve of degree d.

Now we consider the Hilbert scheme H of non-singular curves of degree d in $\mathbb{P}^n$. Since the curves are non-special, the infinitesimal study of the Hilbert scheme, using $H^1(\mathcal{N}) = 0$, shows that H is smooth of dimension $h^0(\mathcal{N}) = 25(g-1)^2 + 4(g-1)$. Of course H contains curves embedded by any divisor of degree d, not only the tricanonical divisor $3K$. One can show, however, that the subset $H' \subseteq H$ of tricanonically embedded curves is also smooth and of dimension $25(g-1)^2 + 3g - 4$, since the choice of a divisor, up to linear equivalence, is an element of the Picard scheme of C, which has dimension g.

The virtue of using the tricanonical embeddings is that if two points of H' correspond to isomorphic curves of genus g, the isomorphism preserves the tricanonical divisor, and so the two embeddings differ only by the choice of basis of $H^0(\mathcal{O}_C(3K))$, which corresponds to an automorphism of $\mathbb{P}^n$. Thus the group $G = PGL(n)$ acts on H', and the orbits of this action are closed subsets of H' in one-to-one correspondence with the isomorphism classes of curves. Note that G has dimension $(n+1)^2 - 1 = 25(g-1)^2 - 1$, so that the "orbit space", if it exists, will have dimension $3g-3$, as we expect.

The next step is to consider a particular curve C, and choose a point $P \in H'$ representing it. The orbit of G containing P, being a homogeneous space, is smooth, so we can choose a smooth, locally closed subscheme Z of H' of dimension $3g-3$ passing through P and transversal to the orbit of G at P. Replacing Z by a smaller open subset still containing P, we may assume that for every orbit of G, whenever Z intersects that orbit, if at all, it intersects in only finitely many points, and that the intersection is transversal at those points. (The idea here is just to throw away points of Z where these properties do not hold.) Since Z is contained in H' and in H, we can restrict the universal family of curves on H to Z and obtain a flat family X/Z.

By construction the given curve C occurs in the family X/Z, and also by construction, any curve appears at most finitely many times. At a point $z \in Z$, we consider the formal family induced by X over the complete local ring $R = \hat{\mathcal{O}}_{z,Z}$. We know that the local deformation functor of the corresponding curve C is pro-representable and has a smooth universal deformation space of dimension $3g-3$ with tangent space $H^1(T_C)$ (???). A standard sequence shows that $H^0(\mathcal{N}_{C/H}) \to H^1(T_C)$ is surjective, and it takes only a little work to show also that the tangent space to Z maps surjectively also, and hence isomorphically to $H^1(T_C)$. Since R and the deformation space of C are both smooth, and their tangent spaces are isomorphic, they are isomorphic.

Thus the family X/Z satisfies all the properties of a modular family except that not every curve may occur. It is easy to show that the image of $G \times Z$ in H' contains an open set. Since H' is quasi-projective, a finite number of such open sets will cover H'. Thus taking a disjoint finite union of such families X/Z, we obtain a modular family $\mathcal{X}/Z$, and we see into the bargain that Z may be taken to be quasi-projective.

(ii) For the second statement, given a modular family $\mathcal{X}/Z$ and any family X/S, we consider the two families $X \times Z$ and $S \times \mathcal{X}$ over $S \times Z$, and then we let $S' = \mathrm{Isom}_{S\times Z}(X \times Z, S \times \mathcal{X})$. As before (26.10) it then follows that $S' \to S$ is étale surjective, and that the two pulled back families by the morphisms $S' \to S$ and $S \to Z$ are isomorphic.

Remark 27.4. The existence of a coarse moduli space (at least as an algebraic space) follows directly from Theorem 27.3. All we need to do is take a modular family $\mathcal{X}/Z$, and apply condition (ii) to this family taken twice. We let $Z' = \mathrm{Isom}_{Z\times Z}(\mathcal{X} \times Z, Z \times \mathcal{X})$. Then $Z' \to Z \times Z$ and is étale over Z by both projections. This is what is known as an étale equivalence relation. The quotient of a scheme Z by an étale equivalence relation Z' need not exist in general as a scheme, but this quotient is precisely the definition of an algebraic space! That this quotient acts as a coarse moduli space follows formally. The fact that the coarse moduli space of curves is actually a scheme and quasi-projective requires more work, which we do not go into here.

Over the complex numbers, étale maps are local homeomorphisms. So the quotient always exists as a complex analytic space. Furthermore, the way the equivalence relation acts at a single point is via an action of the group of automorphisms of the curve. So if C is a curve with finite automorphism group G, the formal moduli is a regular local ring R of dimension $3g-3$ with G acting on it, and the local ring of the corresponding point on the coarse moduli space is the ring of invariants R^G. Thus the coarse moduli space has what are called "quotient singularities".

Remark 27.5. The stack approach is based on the observation that even the moduli functor does not contain all the information we want, because it ignores how two families are isomorphic. To remedy this situation, we consider instead the *moduli stack*, which is a "functor" that assigns to each base scheme S the category of flat families X/S and isomorphisms between them. Such a category, in which all morphisms are isomorphisms, is called a *groupoid*. So the stack is a "functor" in groupoids. I put "functor" in quotes because it is not really a functor in the usual sense, and to explain exactly what this means would lead us into the higher realms of abstract category theory, which I would prefer to avoid.

So a stack is just a "functor" in groupoids, with certain axioms, such as requiring that it should be a sheaf in the étale topology. An *algebraic stack* in the sense of Deligne and Mumford is one that has objects playing the role of the modular families of (27.3). For more about stacks, see the references.

The theory of stacks is a general set-up designed to formalize the situation we have just encountered in studying the moduli of curves. One can regard the category of stacks as an enlargement of the category of schemes, and extend to them many of the notions that apply to schemes such as regular, normal, noetherian, reduced, Cohen–Macaulay, or for morphisms finite type, separated, proper, etc. Then one can say without qualification that the stack $\mathcal{M}_g$ of stable curves of genus g is proper and smooth over $\operatorname{Spec}\mathbb{Z}$ [13, Thm. 5.2].

Remark 27.6. To compactify the variety of moduli, Deligne and Mumford [13] introduce stable curves. A *stable curve* of genus $g \geq 2$ is a reduced connected projective curve of arithmetic genus g with at most nodes as singularities, and having only finitely many automorphisms. This last condition is equivalent to saying that every irreducible component with $p_a = 0$ must meet the other components in at least 3 points, and every component with $p_a = 1$ must meet at least one other component.

They show then that stable curves behave like non-singular curves in the theory above. In particular, (27.3) holds also for families of stable curves, and there is a coarse moduli stack for stable curves, which they show to be projective and irreducible.

Remark 27.7. A corollary of the existence of the moduli stack is that if X/S is an isotrivial family of stable curves of genus g, i.e., a family whose fibers at closed points of S are all isomorphic to each other (assuming S of finite type over k), then there is a surjective étale base extension $S' \to S$ such that the extended family X'/S' is trivial. Indeed, by (27.3) we can find such an S', together with a compatible map $S' \to Z$ to the base of a modular family. Then each connected component of S' goes to a single point of Z, and so the pull-back family is trivial.

References for this section. Mumford [59] contains the proofs of existence of coarse moduli spaces. Deligne and Mumford [13] establish the irreducibility of the compactification of the moduli space and introduce the language of stacks. Mumford [58] explains the motivation behind the theory of stacks. Vistoli [84] has an appendix giving a good introduction to the theory of stacks. I would also like to thank Barbara Fantechi for explaining the whole theory to me in the short space of two hours.

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