An Introduction to Hochschild Cohomology

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Abstract. This is an advanced graduate level textbook, designed both as an introduction for students to the subject of Hochschild cohomology, and as a resource for mathematicians who use Hochschild cohomology in their work. The text begins with definitions, properties, and many examples. The structure of Hochschild cohomology as a Gerstenhaber algebra is explored in detail. Many other topics of current interest are presented, including smoothness and duality, algebraic deformation theory, infinity structures, support varieties, and connections to Hopf algebra cohomology. Also included is an appendix containing some needed homological background.

Contents

Introduction

Homological techniques first arose in topology, in work of Poincaré at the end of the 19th century. They appeared in algebra several decades later in the 1940's, when Hochschild [**Hoc45**] introduced (co)homology of algebras and Eilenberg and Mac Lane [**EL47**] introduced (co)homology of groups. Since that time, both Hochschild cohomology and group cohomology, as they came to be called, have become indispensable in algebra, algebraic topology, representation theory, and other fields. They remain active areas of research, with frequent discoveries of new applications. There are excellent books on group cohomology such as [**AM94, Ben91a, Ben91b, Bro82, CTVE03, Eve91**]. These are good references for those working in the field and are also important resources for those learning group cohomology in order to begin using it in their research. There are fewer such resources for Hochschild (co)homology, notwithstanding some informative chapters in the books [**Lod98, Wei94**]. This book aims to begin filling the gap.

Hochschild cohomology records meaningful information about rings and algebras. It is used to understand their structure and deformations, and to identify essential information about their representations. In this book, we take a concrete approach, giving many early examples that reappear later in various applications. We begin in Chapter 1 with Hochshild's own definitions from [**Hoc45**], only slightly rephrased in modern terminology and notation, and then expand to include arbitrary resolutions under suitable conditions. We present important contributions of Gerstenhaber [**Ger63**] beginning in the 1960's that lead us now to think of Hochschild cohomology as a Gerstenhaber algebra; that is, it has both an associative product and a nonassociative Lie bracket. Many properties of Hochschild cohomology that are used today can be seen in these classical definitions of Hochschild and

Gerstenhaber. In Chapter 3 we present several different types of examples: Koszul algebras, algebras given by quivers and relations, and algebras built from others such as skew group algebras and (twisted) tensor products.

Current applications and developments in Hochschild cohomology include the following, guiding our choice of topics to explore in detail in the rest of book.

In noncommutative geometry, notions of smoothness and some other classical geometric notions may be viewed as essentially homological properties of commutative function algebras, allowing interpretations of them in noncommutative settings via Hochschild cohomology. We present these and related ideas in Chapter 4, including noncommutative differential forms, Van den Bergh duality, Calabi-Yau algebras, Connes differential, and Batalin-Vilkovisky structure.

Understanding how some algebras may be viewed as deformations of others calls on Hochschild cohomology, as explained in Chapter 5. It is in algebraic deformation theory that the Lie structure on Hochschild cohomology arises naturally, and we spend some additional time studying this important but elusive structure in detail in Chapter 6. Further probing the associative and Lie algebra structures on Hochschild cohomology and related complexes uncovers infinity algebras. There, the traditional binary operations are layered with newer n-ary operations which in turn have important implications for the traditional algebra structure. We give a brief introduction to infinity structures and their applications to Hochschild cohomology in Chapter 7.

In representation theory, support varieties may sometimes be defined in terms of Hochschild cohomology; these are geometric spaces assigned to modules that encode representation-theoretic information. Support varieties for finite dimensional algebras are introduced and explored in Chapter 8. This theory began as an application of finite group cohomology, and there are strong connections between Hochschild cohomology and group cohomology that we analyze in the more general setting of Hopf algebras in Chapter 9. Hopf algebras are those whose categories of modules are tensor categories, and include many examples of interest such as group algebras, universal enveloping algebras of Lie algebras, and quantum groups. Relationships between Hochschild cohomology and Hopf algebra cohomology lead to better understanding of both and of all their applications.

We include an appendix with needed background material from homological algebra. The appendix is largely self-contained, however, proofs are omitted, and instead the reader is referred to standard homological algebra textbooks such as [**HS71, Wei94**] for proofs and more details.

This book is not intended to be a comprehensive treatment of the whole subject of Hochschild cohomology, as this subject has expanded well beyond the reach of a single book. Necessarily some important topics are left out. For example, we do not treat Tate-Hochschild cohomology, relative Hochschild cohomology, higher Hochschild cohomology, connections to cyclic (co)homology and K-theory, Hochschild cohomology of abelian categories, singular Hochschild cohomology, topological Hochschild cohomology, nor operads. Also, here we will almost exclusively work with algebras over a field, in order to take advantage of a great array of good properties and current applications that we wish to cover, although one can work over a ground ring that is not a field. Hochschild homology is an important subject in its own right, particularly for commutative algebras where it also has a ring structure, and we spend only a little time on it in this book. More can be found in standard references such as [**Lod98, Wei94**].

This book is written for graduate students and working mathematicians to learn about Hochschild cohomology, and for those who want a reference for many facts that are currently only found in research papers. The main prerequisite for students is a graduate course in algebra. It would also be helpful to have taken further introductory courses in homological algebra or algebraic topology and in representation theory, or else have done some reading in these subjects. However, all of the required homological algebra background is summarized in the appendix, with references, and a motivated reader might rely solely on this as homological algebra background. Beyond the first two chapters, the remaining chapters are largely independent of each other, and so there are many choices one can make to give a one semester graduate course based on this book. A one semester course could start with a treatment of Chapter 1 and selected sections from Chapter 3, possibly including material from the appendix depending on the background of the students. Then one could choose to focus on a subset of the remaining chapters: A course with a focus on noncommutative geometry inspired by important results in commutative geometry could continue with Chapter 4; a course with a focus on algebraic deformation theory and related structures could instead continue with Chapter 5 and the related Chapters 6 and 7, as time allowed; a course with a focus on Hopf algebras, group algebras, and support varieties could instead continue with Chapters 8 and 9. A full year course might include the whole book and time for a complete introduction to or review of homological algebra based on the appendix.

This book came into being as an aftereffect of some lecture series that I gave and through interactions with many people. I first thank Universidad de Buenos Aires, and especially Andrea Solotar and her students, postdocs, and colleagues, for hosting me for several weeks in 2010. During that time I gave a short course on Hopf algebra cohomology that led to an early version of Chapter 9 on which they gave me valuable feedback. I thank the Morningside Center in Beijing and the organizers and students of a workshop there in 2011 for the opportunity to give lectures on support varieties that expanded into the current Chapter 8. I thank the Mathematisches Forschungsinstitut Oberwolfach for its hospitality during several workshops where the idea for this book began in discussions with Karin Erdmann and Henning Krause.

Most of this book was written during the academic year 2016–17 that I spent at the University of Toronto visiting Ragnar-Olaf Buchweitz. I am very grateful to the University of Toronto for hosting me and to Buchweitz and his team during my time in Toronto for the many helpful conversations and stimulating seminar talks. This interaction significantly influenced some of my choices of topics for the book. I particularly thank Buchweitz and his students Benjamin Briggs and Vincent Gélinas for many pointers on smooth algebras and infinity algebras that helped me prepare Chapters 4 and 7. I also had fruitful discussions with Cris Negron and Yury Volkov in relation to the material in Chapter 6. Special thanks go to my long time collaborator Anne Shepler for many conversations over the years that led to my current point of view on Chapter 5.

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Chapter 1

Historical Definitions and Basic Properties

We begin with Hochschild's historical definition of homology and cohomology for algebras [**Hoc45**]. We then present some additional structure found by Gerstenhaber [**Ger63**] under which we say that Hochschild cohomology is a Gerstenhaber algebra. These early developments were essentially based on one choice of chain complex known as the bar complex, and we focus in this chapter on the many properties and structural results that can be derived from this: We discuss here the meaning of Hochschild (co)homology in low degrees such as connections to derivations and deformations. We present the cap product that pairs Hochschild homology and cohomology, and the shuffle product on Hochschild homology of a commutative algebra. We discuss Harrison cohomology and the Hodge decomposition arising from a symmetric group action on the bar complex. Other work invokes many other complexes, depending on the setting. We also begin to include in this chapter some examples and discussion of the structure of Hochschild cohomology that takes advantage of other such chain complexes, and then expand on this discussion in later chapters.

1.1. Definitions of Hochschild homology and cohomology

For now, let k be a commutative associative ring (with 1) and let A be a k-algebra. That is, A is an associative ring (with multiplicative identity) that is also a k-module for which multiplication is a k-bilinear map. Denote the multiplicative identity of A also by 1, identified with the multiplicative identity of k via the unit map $k \to A$ given by $c \mapsto c \cdot 1$ for all $c \in k$. We assume that this unit map is injective. Denote by A^{op} the *opposite algebra* of A; this is A as a module over k, with multiplication $a \cdot_{\mathrm{op}} b = ba$ for all $a, b \in A$. Tensor products will be taken over k, unless otherwise indicated, that is, $\otimes = \otimes_k$. Let $A^e = A \otimes A^{\mathrm{op}}$, with the tensor product multiplication:

$$(a_1 \otimes b_1) \cdot (a_2 \otimes b_2) = a_1 a_2 \otimes b_2 b_1$$

for all $a_1, a_2, b_1, b_2 \in A$. (Technically, we are really taking b_1, b_2 to be elements of A^{op}, but since the underlying k-modules are the same, we write $b_1, b_2 \in A$ where convenient.) We call A^e the *enveloping algebra* of A.

By an A-*bimodule*, we mean a k-module M that is both a left and a right A-module for which $(a_1 m)a_2 = a_1(m a_2)$ for all $a_1, a_2 \in A$ and $m \in M$, and the left and right actions of k induced by the unit map $k \hookrightarrow A$ agree. Thus an A-bimodule M is equivalent to a left A^e-module, where we define

$$(a \otimes b) \cdot m = amb$$

for all $a, b \in A$ and $m \in M$. It is also equivalent to a right A^e-module where the action is defined by $m \cdot (a \otimes b) = bma$ for all $a, b \in A$ and $m \in M$. We will use both structures in the sequel, but for simplicity, when we refer to a module we generally mean a left module unless otherwise specified.

Note that the algebra A is itself a (left) A^e-module (equivalently, an A-bimodule) under left and right multiplication: $(a \otimes b) \cdot c = acb$ for all $a, b, c \in A$. More generally, let $A^{\otimes n} = A \otimes \cdots \otimes A$ (that is, n factors of A). This tensor power of A is an A^e-module (equivalently, an A-bimodule) by letting

$$(a \otimes b) \cdot (c_1 \otimes c_2 \otimes \cdots \otimes c_{n-1} \otimes c_n) = ac_1 \otimes c_2 \otimes \cdots \otimes c_{n-1} \otimes c_n b$$

for all $a, b, c_1, \ldots, c_n \in A$.

Consider the following sequence of A-bimodules:

$$\cdots \xrightarrow{d_3} A^{\otimes 4} \xrightarrow{d_2} A^{\otimes 3} \xrightarrow{d_1} A \otimes A \xrightarrow{\pi} A \to 0, \tag{1.1.1}$$

where π is multiplication, $d_1(a \otimes b \otimes c) = ab \otimes c - a \otimes bc$ for all $a, b, c \in A$, and in general

$$d_n(a_0 \otimes a_1 \otimes \cdots \otimes a_{n+1}) = \sum_{i=0}^{n} (-1)^i a_0 \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1} \tag{1.1.2}$$

for all $a_0, \ldots, a_{n+1} \in A$. One may check directly that (1.1.1) is a complex, that is, that $d_{n-1} d_n = 0$ for all n (see Section A.1). Moreover, it is exact, as a consequence of existence of the following contracting homotopy (see Section A.1): Let s_n be the k-linear map defined by

$$s_n(a_0 \otimes \cdots \otimes a_{n+1}) = 1 \otimes a_0 \otimes \cdots \otimes a_{n+1} \tag{1.1.3}$$

for all n and all $a_0, \dots, a_{n+1} \in A$. We write $B_n(A) = A^{\otimes(n+2)}$ for $n \geq 0$ and often consider the truncated complex associated to (1.1.1):

$$(1.1.4) \qquad B(A): \qquad \cdots \xrightarrow{d_3} A^{\otimes 4} \xrightarrow{d_2} A^{\otimes 3} \xrightarrow{d_1} A \otimes A \to 0.$$

This is the *bar complex* of the A^e-module A. As a complex, its homology is concentrated in degree 0, where it is simply A, as a consequence of exactness of (1.1.1). Sometimes we use the isomorphism of left A^e-modules

$$(1.1.5) \qquad A^{\otimes(n+2)} \cong A^e \otimes A^{\otimes n},$$

given by $a_0 \otimes a_1 \otimes \cdots \otimes a_n \otimes a_{n+1} \mapsto (a_0 \otimes a_{n+1}) \otimes (a_1 \otimes \cdots \otimes a_n)$ for all $a_0, \dots, a_{n+1} \in A$. (The action of A^e on $A^e \otimes A^{\otimes n}$ is multiplication on the leftmost factor A^e.) If A is free as a k-module, one sees in this way that the terms in the bar complex (1.1.4) are free A^e-modules: $A^e \otimes A^{\otimes n} \cong \oplus_{i \in I} A^e(1 \otimes 1 \otimes \alpha_i)$ where I is some indexing set and $\{\alpha_i \mid i \in I\}$ is a basis of $A^{\otimes n}$ as a free k-module.

Remark 1.1.6. The term bar complex arose historically due to an abbreviation of a tensor product $a_1 \otimes \cdots \otimes a_n$ as $a_1|\cdots|a_n$. This convention was begun by Eilenberg and Mac Lane.

Let M be an A-bimodule. Tensor the complex (1.1.4) with M: Let

$$(1.1.7) \qquad C_*(A, M) = M \otimes_{A^e} B(A),$$

a complex of k-modules with differentials $1_M \otimes d_n$ where d_n is defined in (1.1.2) and 1_M is the identity map on M. Call $C_*(A, M)$ the space of *Hochschild chains with coefficients in* M. There is a k-module isomorphism

$$(1.1.8) \qquad M \otimes_{A^e} B_n(A) \xrightarrow{\sim} M \otimes A^{\otimes n}$$

given by

$$m \otimes_{A^e} (a_0 \otimes \cdots \otimes a_{n+1}) \mapsto a_{n+1} m a_0 \otimes a_1 \otimes \cdots \otimes a_n$$

for all $m \in M$ and $a_0, \dots, a_{n+1} \in A$. (The inverse isomorphism is given by $m \otimes a_1 \otimes \cdots \otimes a_n \mapsto m \otimes_{A^e} (1 \otimes a_1 \otimes \cdots \otimes a_n \otimes 1)$; recall the left action of A^e on $A^{\otimes(n+2)}$ involves the first and last tensor factors only, and the right action of A^e on M involves both the left and right actions of A on M.) Combined with the isomorphism (1.1.5), we find that the differential on

$$C_n(A, M) \cong M \otimes A^{\otimes n}$$

corresponding to $1_M \otimes d_n$ on $M \otimes_{A^e} A^{\otimes(n+2)}$, for $n > 0$, is given by

$$\begin{aligned} &\partial_n(m \otimes a_1 \otimes a_2 \otimes \cdots \otimes a_n) \\ &= m a_1 \otimes a_2 \otimes \cdots \otimes a_n + \sum_{i=1}^{n-1} (-1)^i m \otimes a_1 \otimes a_2 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n \\ &\qquad + (-1)^n a_n m \otimes a_1 \otimes a_2 \otimes \cdots \otimes a_{n-1} \end{aligned}$$

for all $m \in M$ and $a_1, \ldots, a_n \in A$. We define Hochschild homology to be the homology of this complex:

Definition 1.1.9. The *Hochschild homology* $\mathrm{HH}_*(A, M)$ of A with coefficients in an A-bimodule M is the homology of the complex (1.1.7), equivalently

$$\mathrm{HH}_n(A, M) = \mathrm{H}_n(M \otimes A^{\otimes \bullet}),$$

that is, $\mathrm{HH}_n(A, M) = \mathrm{Ker}(\partial_n)/\operatorname{Im}(\partial_{n+1})$ for all $n \geq 0$, taking $\partial_0 \equiv 0$, and differentials ∂_n as given above for $n > 0$. Elements of $\mathrm{Ker}(\partial_n)$ are *Hochschild n-cycles* and those in $\operatorname{Im}(\partial_{n+1})$ are *Hochschild n-boundaries.* Let

$$\mathrm{HH}_*(A, M) = \oplus_{n \geq 0} \mathrm{HH}_n(A, M),$$

an $\mathbb{N}$-graded k-module.

We have chosen the notation HH to denote Hochschild homology (and cohomology below) so as to distinguish it from other versions of (co)homology that we will use later. It is denoted with a single letter H in some other texts.

Next we will apply $\mathrm{Hom}_{A^e}(-, M)$ to the complex (1.1.4): Let

$$C^*(A, M) = \oplus_{n \geq 0} \mathrm{Hom}_{A^e}(B_n(A), M), \tag{1.1.10}$$

a complex of k-modules with differentials d_n^*, where $d_n^*(f) = f d_n$ for all functions f in $\mathrm{Hom}_{A^e}(A^{\otimes(n+1)}, M)$. Call $C^*(A, M)$ the space of *Hochschild cochains with coefficients in M*. There is a k-module isomorphism

$$\mathrm{Hom}_{A^e}(B_n(A), M) \xrightarrow{\sim} \mathrm{Hom}_k(A^{\otimes n}, M) \tag{1.1.11}$$

given by $g \mapsto (a_1 \otimes \cdots \otimes a_n \mapsto g(1 \otimes a_1 \otimes \cdots \otimes a_n \otimes 1))$ for all $g \in \mathrm{Hom}_{A^e}(B(A)_n, M)$ and $a_1, \ldots, a_n \in A$. (If $n = 0$ this is $g \mapsto (1 \mapsto g(1 \otimes 1))$. The inverse isomorphism is $g' \mapsto (a_0 \otimes \cdots \otimes a_{n+1} \mapsto a_0 g'(a_1 \otimes \cdots \otimes a_n) a_{n+1})$.) We thus have an isomorphism of complexes,

$$C^*(A, M) \cong \oplus_{n \geq 0} \mathrm{Hom}_k(A^{\otimes n}, M),$$

with differential for $n > 0$ given by (abusing notation by identifying functions that correspond under the isomorphism (1.1.11)):

$$\begin{aligned} d_n^*(h)(a_1 \otimes \cdots \otimes a_n) \; &= \; a_1 h(a_2 \otimes \cdots \otimes a_n) \\ &\quad + \sum_{i=1}^{n-1} (-1)^i h(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n) \\ &\quad + (-1)^n h(a_1 \otimes \cdots \otimes a_{n-1}) a_n \end{aligned}$$

for all $h \in \mathrm{Hom}_k(A^{\otimes(n-1)}, M)$ and $a_1, \ldots, a_n \in A$. In this expression and others, we interpret an empty tensor product to be the element 1 in k. We define Hochschild cohomology to be the cohomology of this complex:

Definition 1.1.12. The *Hochschild cohomology* $\mathrm{HH}^*(A, M)$ of A with coefficients in an A-bimodule M is the cohomology of the complex (1.1.10), equivalently

$$\mathrm{HH}^n(A, M) = \mathrm{H}^n(\mathrm{Hom}_k(A^{\otimes \bullet}, M)),$$

that is, $\mathrm{HH}^n(A, M) = \mathrm{Ker}(d^*_{n+1})/\,\mathrm{Im}(d^*_n)$ for all $n \geq 0$, where d^*_0 is taken to be the zero map, and differentials d^*_n are as given above for $n > 0$. Elements in $\mathrm{Ker}(d^*_{n+1})$ are *Hochschild n-cocycles* and those in $\mathrm{Im}(d^*_n)$ are *Hochschild n-coboundaries.* Let

$$\mathrm{HH}^*(A, M) = \oplus_{n \geq 0} \mathrm{HH}^n(A, M),$$

an $\mathbb{N}$-graded k-module.

As a special case, consider $M = A$ to be an A-bimodule under left and right multiplication. The resulting Hochschild homology and cohomology k-modules are sometimes abbreviated

$$\mathrm{HH}_*(A) = \mathrm{HH}_*(A, A) \quad \text{and} \quad \mathrm{HH}^*(A) = \mathrm{HH}^*(A, A).$$

A disadvantage of this abbreviated notation in the case of cohomology is that it appears to indicate a functor, however HH^* is not a functor: Fixing one of the two arguments, HH^* is a contravariant functor in the first and a covariant functor in the second. For simplicity, we adopt this abbreviated notation in spite of this disadvantage.

Remarks 1.1.13. (i) It follows from the definitions that Hochschild homology and cohomology may be realized as relative Tor and relative Ext [**Wei94**, Lemma 9.1.3]. See also [**Hoc56**]. We will not use these notions in this book.

(ii) In the case that k is a field (the case of focus for most of this book), the bar complex $B(A)$ given by (1.1.4) is a free left A^e-module resolution of A, called the *bar resolution.* Thus

$$\mathrm{HH}_n(A, M) \cong \mathrm{Tor}_n^{A^e}(M, A) \quad \text{and} \quad \mathrm{HH}^n(A, M) \cong \mathrm{Ext}^n_{A^e}(A, M). \tag{1.1.14}$$

(See Section A.3.) More generally, as long as A is flat over the commutative ring k, the first isomorphism holds, and as long as A is projective over k, the second holds.

We will use the equivalent definitions of Hochschild homology and cohomology given by (1.1.14) in the case that k is a field. An advantage is that we may thus choose any flat (respectively, projective) resolution of A as an A^e-module to define Hochschild homology (respectively, cohomology). Depending on the algebra A, there may be more convenient resolutions than the bar resolution, which is quite large and not conducive to explicit computation. The bar resolution may also obscure important information that stands out in other resolutions tailored more closely to specific algebras. However, the bar resolution is very useful theoretically, as we will see.

Also useful is the following variant of the bar resolution: Identify k with the k-submodule $k \cdot 1_A$ of the k-algebra A, and write $\overline{A} = A/k$, the quotient k-module. For each n, let

$$\overline{B}_n(A) = A \otimes \overline{A}^{\otimes n} \otimes A.$$

Let $p_n : B_n(A) \to \overline{B}_n(A)$ be the corresponding quotient map. A calculation shows that the kernels of the maps p_n form a subcomplex of $B(A)$, and thus the quotients $\overline{B}_n(A)$ constitute a complex $\overline{B}(A)$. The contracting homotopy (1.1.3) can be shown to factor through this quotient, implying that $\overline{B}(A)$ is a free resolution of the A^e-module A in case A is free as a k-module:

Definition 1.1.15. Assume that A is free as a k-module. The *reduced bar resolution* (or *normalized bar resolution*) is $\overline{B}(A)$, a free resolution of A as an A^e-module.

See also [**GS88**, §13.5] for more details on the reduced bar resolution.

Hochschild cohomology $\mathrm{HH}^*(A)$ contains significant information about the algebra A, some of which we will see in this chapter via the bar complex, and some in later chapters via a wider range of complexes. Hochschild cohomology is invariant under some standard equivalences on rings: In case k is a field, invariance under Morita equivalence is automatic since this is an equivalence of module categories and Hochschild cohomology is given by Ext in this case (see Remark 1.1.13(ii)). For more details, see, e.g., [**Ben91b**, Theorem 2.11.1]. For tilting and derived category equivalence, see, e.g., [**Hap89**, Theorem 4.2] and [**Ric91**].

We end this section with some examples that apply Remark 1.1.13(ii) and take advantage of resolutions smaller than the bar resolution.

Example 1.1.16. Let k be a field and $A = k[x]$. Consider the following sequence:

$$0 \longrightarrow k[x] \otimes k[x] \xrightarrow{(x\otimes 1 - 1\otimes x)\cdot} k[x] \otimes k[x] \xrightarrow{\pi} k[x] \longrightarrow 0, \tag{1.1.17}$$

where π is multiplication and the map $(x \otimes 1 - 1 \otimes x)\cdot$ is multiplication by the element $x \otimes 1 - 1 \otimes x$. As $k[x]$ is commutative, this sequence is a complex. It is in fact exact, as can be shown directly via a calculation. Alternatively, exactness can be shown by exhibiting a contracting homotopy (see Section A.1): Let $s_{-1}(x^i) = x^i \otimes 1$ and

$$s_0(x^i \otimes x^j) = -\sum_{l=1}^{j} x^{i+j-l} \otimes x^{l-1}$$

for all i, j. (We interpret an empty sum to be 0, thus $s_0(x^i \otimes 1) = 0$ for all i.) A calculation shows that $s_{\bullet}$ is a contracting homotopy for the above sequence. Note that for each i, the map s_i is left (but not right) $k[x]$-linear.

The terms in nonnegative degrees are visibly free as A^e-modules, and so the sequence (1.1.17) is a free resolution of the A^e-module A. Now apply $\mathrm{Hom}_{k[x]^e}(-, k[x])$ to the truncation of sequence (1.1.17) given by deleting the term $k[x]$. Identify $\mathrm{Hom}_{k[x]^e}(k[x] \otimes k[x], k[x]) \cong \mathrm{Hom}_k(k, k[x])$ with $k[x]$ under the isomorphism in which a function f is sent to $f(1)$. The resulting complex, with arrows reversed, becomes

$$0 \longleftarrow k[x] \longleftarrow k[x] \longleftarrow 0. \tag{1.1.18}$$

There is only one map to compute, namely composition with $(x\otimes 1 - 1\otimes x)\cdot$. Let $a \in k[x]$, identified with the function f_a in $\mathrm{Hom}_{k[x]^e}(k[x] \otimes k[x], k[x])$ taking $1\otimes 1$ to a. Composing with the differential, since f_a is a $k[x]^e$-module homomorphism,

$$f_a((x \otimes 1 - 1 \otimes x) \cdot (1 \otimes 1)) = x f_a(1 \otimes 1) - f_a(1 \otimes 1)x = xa - ax = 0,$$

as $k[x]$ is commutative. Therefore all maps in complex (1.1.18) are 0, and the homology of the complex in each degree is just the term in the complex. We thus find that $\mathrm{HH}^0(k[x]) \cong k[x]$, that $\mathrm{HH}^1(k[x]) \cong k[x]$, and $\mathrm{HH}^n(k[x]) = 0$ for $n \geq 2$. A similar argument yields Hochschild homology $\mathrm{HH}_n(k[x])$ by first applying $k[x] \otimes_{k[x]^e} -$ to the truncation of the sequence (1.1.17) and identifying $k[x] \otimes_{k[x]^e} (k[x] \otimes k[x])$ with $k[x]$.

Example 1.1.19. Let k be a field, $n \geq 2$, and $A = k[x]/(x^n)$, called a *truncated polynomial ring*. Consider the following sequence:

$$\cdots \xrightarrow{v\cdot} A^e \xrightarrow{u\cdot} A^e \xrightarrow{v\cdot} A^e \xrightarrow{u\cdot} A^e \xrightarrow{\pi} A \longrightarrow 0, \tag{1.1.20}$$

where $u = x \otimes 1 - 1 \otimes x$, $v = x^{n-1} \otimes 1 + x^{n-2} \otimes x + \cdots + 1 \otimes x^{n-1}$, and π is multiplication. This sequence is exact, as can be shown directly. Alternatively, the following is a contracting homotopy (see Section A.1): For each i, define a left A-linear map s_i by $s_{-1}(1) = 1 \otimes 1$ and for all $m \geq 0$,

$$s_{2m}(1 \otimes x^j) = -\sum_{l=1}^{j} x^{j-l} \otimes x^{l-1} \quad \text{and} \quad s_{2m+1}(1 \otimes x^j) = \delta_{j,n-1} \otimes 1$$

for all j, where $\delta_{j,n-1}$ is the Kronecker delta (that is, $\delta_{j,n-1} = 1$ if $j = n-1$ and $\delta_{j,n-1} = 0$ otherwise). The terms in nonnegative degrees are visibly free as A^e-modules, and so the sequence (1.1.20) is a free resolution of the A^e-module A.

Apply $\mathrm{Hom}_{A^e}(-, A)$ to (1.1.20), truncate by deleting $\mathrm{Hom}_{A^e}(A, A)$, and identify $\mathrm{Hom}_{A^e}(A \otimes A, A)$ with $\mathrm{Hom}_k(k, A) \cong A$. The resulting sequence may be viewed as:

$$\cdots \xleftarrow{nx^{n-1}\cdot} A \xleftarrow{0} A \xleftarrow{nx^{n-1}\cdot} A \xleftarrow{0} A \longleftarrow 0.$$

If n is divisible by the characteristic of k, then $\mathrm{HH}^n(A) \cong A$ for all n. If n is not divisible by the characteristic of k, then $\mathrm{HH}^0(A) \cong A$, $\mathrm{HH}^{2m+1}(A) \cong (x)$ (the ideal generated by x) for all $m \geq 0$, and $\mathrm{HH}^{2m}(A) \cong A/(x^{n-1})$ for all $m \geq 1$.

Exercise 1.1.21. Verify the claims that formulas given for the following differentials are induced by the differential (1.1.2) on the bar resolution:

(a) ∂_n right before Definition 1.1.9.
(b) d_n^* right before Definition 1.1.12.

Exercise 1.1.22. Show that the following maps constitute contracting homotopies as claimed:

(a) s_n defined in (1.1.3).
(b) s_{-1}, s_0 defined in Example 1.1.16.
(c) s_i defined in Example 1.1.19.

Exercise 1.1.23. Finish Example 1.1.16 by finding Hochschild homology $\mathrm{HH}_n(k[x])$ for each n.

Exercise 1.1.24. Let k be a field and let $0 \to U \to V \to W \to 0$ be an exact sequence of A-bimodules. Use Remark 1.1.13(ii) and Theorem A.4.7 (Second long exact sequence for Tor) to show that there is a long exact sequence

$$\cdots \longrightarrow \mathrm{HH}_n(A,U) \longrightarrow \mathrm{HH}_n(A,V) \longrightarrow \mathrm{HH}_n(A,W) \longrightarrow$$

$$\mathrm{HH}_{n-1}(A,U) \longrightarrow \cdots .$$

Derive a similar long exact sequence for Hochschild cohomology using Theorem A.4.4 (First long exact sequence for Ext).

Exercise 1.1.25. Check that $\overline{B}(A)$ as in Definition 1.1.15 is indeed a free resolution of A as an A^e-module in case A is free as a k-module, by verifying the claim that the kernels of the maps p_n form a subcomplex of $B(A)$ and that the contracting homotopy (1.1.3) for the bar resolution factors through $\overline{B}(A)$.

1.2. Interpretation in low degrees

The historical Definitions 1.1.9 and 1.1.12 of Hochschild homology and cohomology point directly to specific information encoded for small values of n. For example, in Hochschild cohomology we can see the center of A in degree 0, derivations on A in degree 1, and infinitesimal deformations together with obstructions to lifting these to formal deformations in degrees 2 and 3. We make some of these observations in this section, including more general statements for A-bimodules M as well as analogous statements for Hochschild homology. More detailed discussion of deformations and their obstructions is in Chapter 5.

Degree 0**.** Let M be an A-bimodule. By definition, $\mathrm{HH}^0(A, M) = \mathrm{Ker}(d_1^*)$ (notation as in Definition 1.1.12). We determine conditions on a function $f \in \mathrm{Hom}_{A^e}(A \otimes A, M)$ equivalent to being in $\mathrm{Ker}(d_1^*)$. First assume that $d_1^*(f) = 0$, that is, for all $a \in A$,

$$\begin{aligned} 0 \ &= \ d_1^*(f)(1 \otimes a \otimes 1) \ = \ f(d_1(1 \otimes a \otimes 1)) \\ &\qquad\qquad = \ f(a \otimes 1 - 1 \otimes a) \ = \ af(1 \otimes 1) - f(1 \otimes 1)a. \end{aligned}$$

Then $f(1 \otimes 1)$ is an element m of M for which $am = ma$ for all $a \in A$, and f is in fact determined by this element m: $f(b \otimes c) = bf(1 \otimes 1)c = bmc$ for all $b, c \in A$. Conversely, any such element of M defines a function in $\mathrm{Ker}(d_1^*)$, that is, given $m \in M$ for which $am = ma$ for all $a \in A$, let $f_m \in \mathrm{Hom}_{A^e}(A \otimes A, M)$ be the function given by $f_m(b \otimes c) = bmc$ for all $b, c \in A$. Then $d_1^*(f_m) = 0$. So as a k-module,

$$\mathrm{HH}^0(A, M) \cong \{m \in M \mid am = ma \text{ for all } a \in A\}.$$

In the special case $M = A$, we thus see that $\mathrm{HH}^0(A, A) \cong Z(A)$, the center of the algebra A.

Similarly, one finds that Hochschild homology in degree 0 is

$$\mathrm{HH}_0(A, M) \cong M/\operatorname{Span}_k\{am - ma \mid a \in A,\ m \in M\}. \tag{1.2.1}$$

Degree 1**.** By definition, $\mathrm{HH}^1(A, M) = \mathrm{Ker}(d_2^*)/\mathrm{Im}(d_1^*)$ (notation as in Definition 1.1.12). Let $f \in \mathrm{Ker}(d_2^*)$, that is $f \in \mathrm{Hom}_{A^e}(A^{\otimes 3}, M)$ and fd_2 is the zero map on $A^{\otimes 4}$. Equivalently, for all $a, b \in A$,

$$\begin{aligned} 0 \ &= \ d_2^*(f)(1 \otimes a \otimes b \otimes 1) \\ &= \ f(d_2(1 \otimes a \otimes b \otimes 1)) \\ &= \ f(a \otimes b \otimes 1 - 1 \otimes ab \otimes 1 + 1 \otimes a \otimes b) \\ &= \ af(1 \otimes b \otimes 1) - f(1 \otimes ab \otimes 1) + f(1 \otimes a \otimes 1)b. \end{aligned}$$

By abuse of notation, we identify f with a function in $\mathrm{Hom}_k(A, M)$ under the isomorphism $\mathrm{Hom}_{A^e}(A^{\otimes 3}, M) \cong \mathrm{Hom}_k(A, M)$ of (1.1.11), and the above equation becomes $0 = af(b) - f(ab) + f(a)b$, or

$$f(ab) = af(b) + f(a)b$$

for all $a, b \in A$. This is precisely the definition of a *k-derivation* from A to M. The space of all k-derivations from A to M is denoted

$$\mathrm{Der}(A, M).$$

Suppose in addition that $f \in \mathrm{Im}(d_1^*)$, that is $f = d_1^*(g)$ for some g in $\mathrm{Hom}_{A^e}(A^{\otimes 2}, M)$; the function g is defined by its value on $1 \otimes 1$, say m. Then

$$\begin{aligned} d_1^*(g)(1 \otimes a \otimes 1) &= g(d_1(1 \otimes a \otimes 1)) \\ &= g(a \otimes 1 - 1 \otimes a) \\ &= ag(1 \otimes 1) - g(1 \otimes 1)a \quad = \quad am - ma. \end{aligned}$$

That is, $d_1^*(g)$ is the *inner k-derivation* from A to M defined by the element m. Conversely, any inner k-derivation will be an element of $\mathrm{Im}(d_1^*)$. The space of all inner k-derivations from A to M is denoted

$$\mathrm{InnDer}(A, M).$$

We have shown that

$$\mathrm{HH}^1(A, M) \cong \mathrm{Der}(A, M)/\,\mathrm{InnDer}(A, M).$$

In particular, if $M = A$, then $\mathrm{HH}^1(A)$ is isomorphic to the space of derivations of A modulo inner derivations. If A is commutative, the zero function is the only inner derivation, so in this case, $\mathrm{HH}^1(A)$ is simply the space of derivations of A.

It can be shown that Hochschild homology in degree 1, $\mathrm{HH}_1(A, M)$, is isomorphic to the kernel of the canonical map $I \otimes_{A^e} M \to IM$ where I is the kernel of multiplication $\pi : A \otimes A \to A$. See Exercise 1.2.7 in case k is a field. For more details in the general case, see [**Wei94**, Section 9.2], where a connection to Kähler differentials of commutative algebras is also given. In Section 4.2, we will identify I with the space $\Omega^1_{nc}A$ of noncommutative Kähler differentials.

Degree 2. By definition, $\mathrm{HH}^2(A, M)$ is the quotient $\mathrm{Ker}(d_3^*)/\,\mathrm{Im}(d_2^*)$ (notation as in Definition 1.1.12). Let $f \in \mathrm{Hom}_{A^e}(A^{\otimes 4}, M)$. Then f is in $\mathrm{Ker}(d_3^*)$ if and only if for all $a, b, c \in A$,

$$\begin{aligned} 0 &= d_3^*(f)(1 \otimes a \otimes b \otimes c \otimes 1) \\ &= f(d_3(1 \otimes a \otimes b \otimes c \otimes 1)) \\ &= f(a \otimes b \otimes c \otimes 1 - 1 \otimes ab \otimes c \otimes 1 + 1 \otimes a \otimes bc \otimes 1 - 1 \otimes a \otimes b \otimes c) \\ &= af(1 \otimes b \otimes c \otimes 1) - f(1 \otimes ab \otimes c \otimes 1) + f(1 \otimes a \otimes bc \otimes 1) \\ &\quad - f(1 \otimes a \otimes b \otimes 1)c. \end{aligned}$$

Identifying f with a function in $\mathrm{Hom}_k(A^{\otimes 2}, M)$ under the isomorphism $\mathrm{Hom}_{A^e}(A^{\otimes 4}, M) \cong \mathrm{Hom}_k(A^{\otimes 2}, M)$ of (1.1.11), we find that $f \in \mathrm{Ker}(d_3^*)$ if and only if

$$af(b \otimes c) + f(a \otimes bc) = f(ab \otimes c) + f(a \otimes b)c \tag{1.2.2}$$

for all $a, b, c \in A$. A calculation shows that the image of d_2^* may be identified with the space of all functions f in $\mathrm{Hom}_k(A \otimes A, M)$ given by

$$f(a \otimes b) = ag(b) - g(ab) + g(a)b \tag{1.2.3}$$

for some $g \in \mathrm{Hom}_k(A, M)$.

We will see in Section 4.2 that Hochschild 2-cocycles, that is functions satisfying (1.2.2), give the A-bimodule $A \oplus M$ the structure of an algebra, called a square-zero extension. These extensions arise in notions of smoothness of algebras. In the case that $M = A$, equation (1.2.2) gives rise to what will be called infinitesimal deformations of A. We will discuss this connection to algebraic deformation theory in Chapter 5, and we will see there that obstructions to lifting a Hochschild 2-cocycle to a formal deformation lie in $\mathrm{HH}^3(A)$. We will also see that functions satisfying (1.2.3) give rise to deformations isomorphic to the original algebra. In this way, each formal deformation, up to isomorphism, will have associated to it an element of $\mathrm{HH}^2(A)$.

Action of the center of A on $\mathrm{HH}^n(A)$. There is an action of the center $Z(A)$ of A on $\mathrm{Hom}_{A^e}(U, V)$ for any two A^e-modules U, V, given by

$$(a \cdot f)(u) = af(u) \tag{1.2.4}$$

for all $a \in Z(A)$, $u \in U$, and $f \in \mathrm{Hom}_A(U, V)$. Taking $U = A^{\otimes(n+2)}$ and $V = A$, this action commutes with the differentials on the bar complex, inducing an action of $Z(A)$ on $\mathrm{HH}^n(A)$ under which $\mathrm{HH}^n(A)$ becomes a $Z(A)$-module. Identifying $Z(A)$ with $\mathrm{HH}^0(A)$ as described above, this is an action of $\mathrm{HH}^0(A)$ on $\mathrm{HH}^n(A)$. In the next section, this action will be extended to a graded product on $\mathrm{HH}^*(A)$. This action of $Z(A)$ has some useful consequences. For example, let $1 = e_1 + \cdots + e_i$ be an expansion of the multiplicative identity of A as the sum of a set of orthogonal central idempotents $e_1, \ldots, e_i$. (That is, each e_j is central in A and $e_j e_l = \delta_{j,l} e_j$ where $\delta_{j,l}$ is the Kronecker delta.) Then $A = \oplus_{j=1}^i Ae_j$ as a direct sum of ideals $Ae_j = e_j A = e_j A e_j$ of A. This leads to a similar decomposition $\mathrm{Hom}_{A^e}(U, A) = \oplus_{j=1}^i \mathrm{Hom}_{A^e}(U, Ae_j)$ for any A^e-module U, and further,

$$\mathrm{HH}^*(A) \cong \bigoplus_{j=1}^{i} \mathrm{HH}^*(Ae_j) \cong \bigoplus_{j=1}^{i} \mathrm{HH}^*(A)e_j.$$

Here we view the ideal Ae_j of A itself as an algebra with multiplicative identity e_j, and we may identify $\mathrm{HH}^*(A)e_j$ with $\mathrm{HH}^*(Ae_j)$ in this expansion, under the action of $Z(A)$ on $\mathrm{HH}^*(A)$.

Exercise 1.2.5. Verify the isomorphism (1.2.1).

Exercise 1.2.6. Describe all k-derivations of $k[x]$ with the help of Example 1.1.16 and the connection to Hochschild cohomology explained in this section.

Exercise 1.2.7. Let k be a field and let I be the kernel of multiplication $\pi : A \otimes A \to A$. Show that $\mathrm{HH}_1(A, M)$ is isomorphic to the kernel of the map $I \otimes_{A^e} M \to IM$ given by $\sum_i (a_i \otimes b_i) \otimes_{A^e} m \mapsto \sum_i a_i m b_i$. (Hint: Consider the short exact sequence $0 \to I \to A^e \xrightarrow{\pi} A \to 0$ and use Exercise 1.1.24, noting that $\mathrm{HH}_1(A, A^e) = 0$ since A^e is flat as an A^e-module.)

Exercise 1.2.8. Find the action of $Z(A)$ on $\mathrm{HH}^*(A)$ in each case:

(a) $A = k[x]$ as in Example 1.1.16.

(b) $A = k[x]/(x^n)$ as in Example 1.1.19.

1.3. Cup product

Hochschild cohomology $\mathrm{HH}^*(A)$ is a graded k-module by its definition. (That is, it is graded by $\mathbb{N}$, which we understand to include 0.) We will see next that it has an associative product making it into a graded commutative algebra, that is, up to a sign determined by homological degrees, homogeneous elements commute. (See Theorem 1.4.4 below.) We define this product at the chain level for functions on the bar complex (1.1.4) in this section. In fact the cup product is the unique associative product on $\mathrm{HH}^*(A)$ satisfying some basic conditions; see Sanada [**San93**]. There are many equivalent definitions of this associative product on $\mathrm{HH}^*(A)$, particularly in case k is a field, making it very versatile. We give these other definitions in Chapter 2.

We again use the isomorphism (1.1.11) to identify $\mathrm{Hom}_{A^e}(A^{\otimes(n+2)}, M)$ with $\mathrm{Hom}_k(A^{\otimes n}, M)$ as a k-module. In the notation of (1.1.10),

$$C^*(A, M) \cong \oplus_{n \geq 0} \mathrm{Hom}_k(A^{\otimes n}, M),$$

the space of Hochschild cochains on A with coefficients in M.

We start by taking $M = A$:

Definition 1.3.1. Let $f \in \mathrm{Hom}_k(A^{\otimes m}, A)$ and $g \in \mathrm{Hom}_k(A^{\otimes n}, A)$. The *cup product* $f \smile g$ is the element of $\mathrm{Hom}_k(A^{\otimes(m+n)}, A)$ defined by

$$(f \smile g)(a_1 \otimes \cdots \otimes a_{m+n}) = f(a_1 \otimes \cdots \otimes a_m) g(a_{m+1} \otimes \cdots \otimes a_{m+n}) \tag{1.3.2}$$

for all $a_1, \dots, a_{m+n} \in A$. If $m = 0$, we interpret this formula to be

$$(f \smile g)(a_1 \otimes \cdots \otimes a_n) = f(1)g(a_1 \otimes \cdots \otimes a_n),$$

and similarly if $n = 0$.

By its definition, the cup product is associative. A calculation shows that

$$d^*_{m+n+1}(f \smile g) = (d^*_{m+1}(f)) \smile g + (-1)^m f \smile (d^*_{n+1}(g)). \tag{1.3.3}$$

As a consequence, the space $C^*(A, A)$ of Hochschild cochains is a *differential graded algebra*, that is, a graded algebra (i.e., $C^i(A, A) \smile C^j(A, A) \subset C^{i+j}(A, A)$) with a graded derivation d^* of degree 1 and square 0. Another consequence of equation (1.3.3) is that this cup product $\smile$ induces a well-defined graded associative product on Hochschild cohomology, which we denote by the same symbol:

$$\smile : \ \mathrm{HH}^m(A) \times \mathrm{HH}^n(A) \to \mathrm{HH}^{m+n}(A).$$

Remark 1.3.4. More generally, if B is an A-bimodule that is also an algebra for which $a(bb') = (ab)b'$, $(bb')a = b(b'a)$, and $(ba)b' = b(ab')$ for all $a \in A$ and $b, b' \in B$, a calculation shows that a formula analogous to (1.3.2) induces a product on $\mathrm{HH}^*(A, B)$. This condition is satisfied, for example, in case A is a subalgebra of B and the A-bimodule structure of B is given by left and right multiplication.

We have seen that in degree 0, Hochschild cohomology $\mathrm{HH}^0(A)$ is isomorphic to $Z(A)$, the center of the algebra A. As a consequence of the definition of cup product, the cup product of two elements in degree 0 is precisely the product of the corresponding elements in $Z(A)$, and the cup product of an element in arbitrary degree n with a degree 0 element corresponds to multiplying the values of a corresponding function by the corresponding element in $Z(A)$. This is the $\mathrm{HH}^0(A)$-module structure on Hochschild cohomology $\mathrm{HH}^*(A)$ given by equation (1.2.4). Sometimes this is enough information to determine the full structure of $\mathrm{HH}^*(A)$ as a ring under cup product, such as in the following example.

Example 1.3.5. We return to Example 1.1.16, letting $A = k[x]$, and describe the cup product using only general properties and the graded vector space structure found there. We found that $\mathrm{HH}^0(k[x]) \cong k[x]$, $\mathrm{HH}^1(k[x]) \cong k[x]$, and $\mathrm{HH}^n(k[x]) = 0$ for $n > 1$, and so as a graded vector space,

$$\mathrm{HH}^*(k[x]) \cong k[x] \oplus k[x], \tag{1.3.6}$$

with the first copy of $k[x]$ in degree 0 and the second in degree 1. We will describe cup products on $\mathrm{HH}^*(k[x])$ in view of this expression. In degree 0, the cup product is simply multiplication on $k[x]$. Likewise, the product of an element in degree 0 with an element in degree 1 corresponds to multiplication on $k[x]$, with the result having degree 1 (see Exercise 1.2.8). Since $\mathrm{HH}^2(k[x]) = 0$, the product of two elements in degree 1 is 0. Now denote by y the copy of the multiplicative identity of $k[x]$ that is in the degree 1 component, so that we may rewrite (1.3.6) as

$$\mathrm{HH}^*(k[x]) \cong k[x] \oplus k[x]y$$

with the first summand $k[x]$ the degree 0 component and $k[x]y$ the degree 1 component (treating y as a place holder). By our above description of products, we now see that

$$\mathrm{HH}^*(k[x]) \cong k[x,y]/(y^2)$$

as a k-algebra, where $|x| = 0$ and $|y| = 1$. (The notation $|x|, |y|$ here refers to their homological degrees.)

We have noted that the cup product is associative as a direct consequence of formula (1.3.2). Associativity can also be deduced readily from each of the equivalent definitions of cup product that will be given in Chapter 2.

We next turn to our claim that the cup product is graded commutative. This may be shown in many different ways. It may be proven by induction, as in [**San93**]. Another proof uses two of the equivalent definitions of the product, namely the Yoneda product of Section 2.2 and the tensor product of Section 2.3, relying on the latter being an algebra homomorphism over the former (see [**SA04**]). Yet another proof uses tensor products of generalized extensions and an argument similar to the proof of Theorem 2.5.5 below (see [**SS04**, Theorem 1.1] and some discussion in Section 2.4). Our first proof, in the next section, uses the more concrete historical approach of Gerstenhaber [**Ger63**]. This proof is connected to the first natural appearance of the graded Lie bracket on Hochschild cohomology, also defined in the next section.

Exercise 1.3.7. Verify formula (1.3.3).

Exercise 1.3.8. Let B be an algebra and A a subalgebra of B. Consider B to be an A-bimodule under left and right multiplication. Let $f \in \mathrm{Hom}_k(A^{\otimes m}, A)$, $g \in \mathrm{Hom}_k(A^{\otimes n}, B)$. Define $f \smile g \in \mathrm{Hom}_k(A^{\otimes(m+n)}, B)$ by a formula analogous to (1.3.2). Show that this induces a well-defined product on $\mathrm{HH}^*(A, B)$. (More generally, see Remark 1.3.4.)

Exercise 1.3.9. Let M be an A-bimodule. Let $f \in \mathrm{Hom}_k(A^{\otimes m}, A)$ and $g \in \mathrm{Hom}_k(A^{\otimes n}, M)$. Define $f \cdot g \in \mathrm{Hom}_k(A^{\otimes(m+n)}, M)$ by

$$(f \cdot g)(a_1 \otimes \cdots \otimes a_{m+n}) = f(a_1 \otimes \cdots \otimes a_m)g(a_{m+1} \otimes \cdots \otimes a_{m+n})$$

for all $a_1, \ldots, a_{m+n} \in A$. Show that this induces a well-defined action of $\mathrm{HH}^*(A)$ on $\mathrm{HH}^*(A, M)$ (cf. Section 2.5 below).

1.4. Gerstenhaber bracket

In addition to the cup product, Hochschild cohomology $\mathrm{HH}^*(A)$ has another binary operation. We define this operation at the chain level on the bar complex (1.1.4) in this section, allowing k to be an arbitrary commutative ring. In Chapter 6, under the assumption that k is a field, we will examine equivalent definitions by way of other projective resolutions and exact sequences.

Definition 1.4.1. Let $f \in \mathrm{Hom}_k(A^{\otimes m}, A)$ and $g \in \mathrm{Hom}_k(A^{\otimes n}, A)$. The *Gerstenhaber bracket* $[f, g]$ is defined at the chain level as the element of $\mathrm{Hom}_k(A^{\otimes(m+n-1)}, A)$ given by

$$[f, g] = f \circ g - (-1)^{(m-1)(n-1)} g \circ f$$

where the *circle product* $f \circ g$ generalizes composition of functions and is defined by

$$\begin{aligned}&(f \circ g)(a_1 \otimes \cdots \otimes a_{m+n-1})\\ &= \sum_{i=1}^{m} (-1)^j f(a_1 \otimes \cdots \otimes a_{i-1} \otimes g(a_i \otimes \cdots \otimes a_{i+n-1}) \otimes a_{i+n} \otimes \cdots \otimes a_{m+n-1}),\end{aligned}$$

in which $j = (n-1)(i-1)$, and similarly $g \circ f$. If $m = 0$, then $f \circ g = 0$ (as indicated by the empty sum), while if $n = 0$, then the formula should be interpreted by taking the value $g(1)$ in place of $g(a_i \otimes \cdots \otimes a_{i+n-1})$ (as indicated by the empty tensor product).

The following lemmas may be proven by direct computation on the bar complex as in [**Ger63**]. For ease of notation, as is common, we leave out subscripts on the differentials.

Lemma 1.4.2. *Let* $f \in \mathrm{Hom}_k(A^{\otimes m}, A)$*,* $g \in \mathrm{Hom}_k(A^{\otimes n}, A)$*, and* $h \in \mathrm{Hom}_k(A^{\otimes p}, A)$*. Then*

(i) $[f, g] = -(-1)^{(m-1)(n-1)}[g, f]$,

(ii) $(-1)^{(m-1)(p-1)}[f, [g, h]] + (-1)^{(n-1)(m-1)}[g, [h, f]]$
$+(-1)^{(p-1)(n-1)}[h, [f, g]] = 0$,

(iii) $d^*([f, g]) = (-1)^{n-1}[d^* f, g] + [f, d^* g]$.

Property (i) is graded anti-commutativity of the bracket, where we shift the homological degrees of f, g by -1. Property (ii) is the graded Jacobi identity. These first two properties make $C^*(A, A) = \oplus_{n\geq 0} \mathrm{Hom}_k(A^{\otimes n}, A)$ into a graded Lie algebra. Property (iii) and a sign modification further makes it a *differential graded Lie algebra*, that is a graded Lie algebra with a graded derivation δ of degree 1 and square 0: Letting $\delta(f) = (-1)^{m-1} d^*(f)$ for all $f \in \mathrm{Hom}_k(A^{\otimes m}, A)$, by Lemma 1.4.2(iii) we have

$$\delta([f, g]) = [\delta(f), g] + (-1)^{m-1}[f, \delta(g)].$$

We emphasize again that the degree of an element here is shifted by one from the homological degree, so that f has degree $m-1$ when considering the Lie structure. Some authors choose notation to clarify this distinction, introducing a shift operator that shifts degree when needed.

Gerstenhaber [**Ger63**] more generally developed the notion of a pre-Lie algebra for handling the circle product and bracket operations and proving results about the Lie structure.

Let $\pi : A \otimes A \to A$ denote multiplication. The following lemma may be proven by tedious direct computation.

Lemma 1.4.3. *Let $f \in \mathrm{Hom}_k(A^{\otimes m}, A)$ and $g \in \mathrm{Hom}_k(A^{\otimes n}, A)$. Then*

(i) $f \smile g - (-1)^{mn} g \smile f$
$= (d^* g) \circ f + (-1)^m d^*(g \circ f) + (-1)^{m-1} g \circ (d^* f)$,

(ii) $[f, \pi] = -d^*(f)$.

The following theorem is a consequence of Lemma 1.4.3(i).

Theorem 1.4.4. *Let A be an associative algebra over the commutative ring k. The cup product on $\mathrm{HH}^*(A)$ is graded commutative.*

Proof. Let $f \in \mathrm{Hom}_k(A^{\otimes m}, A)$ and $g \in \mathrm{Hom}_k(A^{\otimes n}, A)$ be cocycles, that is, $d^* f = 0$ and $d^* g = 0$. By Lemma 1.4.3(i),

$$f \smile g = (-1)^{mn} g \smile f + (-1)^m d^*(g \circ f).$$

Their images $\overline{f}$ and $\overline{g}$ in $\mathrm{HH}^*(A)$ thus satisfy $\overline{f} \smile \overline{g} = (-1)^{mn} \overline{g} \smile \overline{f}$. Therefore $\mathrm{HH}^*(A)$ is graded commutative. □

As a consequence of Lemma 1.4.2(iii), the bracket [,], as defined at the cochain level, induces a well-defined operation on $\mathrm{HH}^*(A)$. Next we state a further property satisfied by this bracket on Hochschild cohomology. For a proof, see [**Ger63**, Corollary 1 of Theorem 5], where at the cochain level, the difference of the left and right sides of the stated equation in the lemma is shown to be a specific coboundary.

Lemma 1.4.5. *Let $\alpha \in \mathrm{HH}^m(A)$, $\beta \in \mathrm{HH}^n(A)$, and $\gamma \in \mathrm{HH}^p(A)$. Then*

$$[\alpha \smile \beta, \gamma] = [\alpha, \gamma] \smile \beta + (-1)^{m(p-1)} \alpha \smile [\beta, \gamma].$$

As a consequence of the lemma, for each γ, the operation $[-,\gamma]$ is a graded derivation with respect to cup product. Moreover, Hochschild cohomology is a Gerstenhaber algebra (sometimes also called a G-algebra), as we define next.

Definition 1.4.6. A *Gerstenhaber algebra* $(H, \smile, [\ ,\ ])$ is a free $\mathbb{Z}$-graded k-module H for which $(H, \smile)$ is a graded commutative associative algebra, $(H, [\ ,\ ])$ is a graded Lie algebra with bracket $[\ ,\ ]$ of degree -1 and corresponding degree shift by -1 on elements, and

$$[\alpha \smile \beta, \gamma] = [\alpha, \gamma] \smile \beta + (-1)^{|\alpha|(|\gamma|-1)} \alpha \smile [\beta, \gamma]$$

for all homogeneous α, β, γ in H.

Theorem 1.4.7. *Hochschild cohomology* $\mathrm{HH}^*(A)$ *is a Gerstenhaber algebra.*

Proof. The main properties to prove are dealt with in Theorem 1.4.4 and Lemmas 1.4.2 and 1.4.5. □

In Chapter 6 we will examine the Gerstenhaber bracket in more detail, including ways to define it on an arbitrary resolution independent of the bar resolution, and on exact sequences.

Exercise 1.4.8. Verify Lemma 1.4.2(i) by direct computation.

Exercise 1.4.9. Verify Lemma 1.4.2(ii) by direct computation.

Exercise 1.4.10. Verify all other properties of the Gerstenhaber bracket stated in this section either by direct computation or by reading the verifications in [**Ger63**].

1.5. Cap product and shuffle product

The cap product is a pairing between Hochschild cohomology and homology, that is a function

$$\mathrm{HH}^m(A) \otimes \mathrm{HH}_n(A) \xrightarrow{\frown} \mathrm{HH}_{n-m}(A),$$

defined at the chain level as follows. (We consider $\mathrm{HH}_i(A)$ to be 0 for all $i < 0$.) Let $f \in \mathrm{Hom}_k(A^{\otimes m}, A)$ be a function representing an element of $\mathrm{HH}^m(A)$. We identify $A \otimes_{A^e} \otimes A^{\otimes(n+2)}$ with $A \otimes A^{\otimes n}$ via the isomorphism (1.1.8) with $M = A$, symbolically representing an element of $\mathrm{HH}_n(A)$ at the chain level by a sum of elements of the form $a_0 \otimes \cdots \otimes a_n$ in $A \otimes A^{\otimes n}$. The *cap product* is defined by

$$f \frown (a_0 \otimes a_1 \otimes \cdots \otimes a_n) = (-1)^m a_0 f(a_1 \otimes \cdots \otimes a_m) \otimes a_{m+1} \otimes \cdots \otimes a_n.$$

This induces a well-defined function as claimed: Assuming $\sum_i a_0^i \otimes \cdots \otimes a_n^i$ is a cycle for some $a_j^i \in A$, and that $f \in \mathrm{Hom}_k(A^{\otimes m}, A)$ is a cocycle, one sees that their cap product is a cycle by rewriting the image of the differential on $\sum_i a_0^i \otimes \cdots \otimes a_n^i$ in such a way as to take advantage of the relation $d_{m+1}^*(f) = 0$. Thus the cap product of a cocycle with a cycle is a cycle. Similarly one sees that the cap product of a coboundary with a cycle, or of a cocycle with a boundary, is a boundary. By its definition, the cap product gives $\mathrm{HH}_*(A)$ the structure of an $\mathrm{HH}^*(A)$-module.

Example 1.5.1. Let $A = k[x]$, as in Example 1.1.16. By our work in that example, we see that $\mathrm{HH}_*(A)$ is a free A-module. In degree 0, $\mathrm{HH}^0(A) \cong A$ acts on the free A-module in the canonical way. Let V be the one-dimensional vector space with basis x. We identify $\mathrm{HH}^1(A)$ with $A \otimes V^*$ and $\mathrm{HH}_1(A)$ with $A \otimes V$. The resolution (1.1.17) may be embedded in the bar resolution by identifying the two resolutions in degree 0 and by mapping $a \otimes b$ in $k[x] \otimes k[x]$ in degree 1 to $a \otimes x \otimes b$ in the bar resolution. (See Section 3.4 for a more general embedding for Koszul algebras.) A calculation then shows that the action of $\mathrm{HH}^1(A) \cong A \otimes V^*$ on $\mathrm{HH}_1(A) \cong A \otimes V$ is multiplication in the first tensor factor and evaluation in the second.

For the rest of this section, assume that A is a commutative k-algebra. The shuffle product, defined next, is a product on Hochschild homology $\mathrm{HH}_*(A)$ in this commutative case. Let S_n denote the symmetric group on n symbols, and let sgn denote its sign character, that is, sgn takes even permutations to 1 and odd permutations to -1. We will need the subset $S_{p,q}$ of (p,q)-shuffles of the symmetric group S_{p+q}, defined next.

Definition 1.5.2. For nonnegative integers p and q, a (p,q)-*shuffle* is an element σ of the symmetric group S_{p+q} for which $\sigma(i) < \sigma(j)$ whenever $1 \le i < j \le p$ or $p+1 \le i < j \le p+q$. Let $S_{p,q}$ denote the subset of S_{p+q} consisting of all (p,q)-shuffles.

Identify $A \otimes_{A^e} A^{\otimes(n+2)}$ with $A^{\otimes(n+1)}$ via the isomorphism (1.1.8) with $M = A$.

Definition 1.5.3. The *shuffle product* on $\mathrm{HH}_*(A)$ is defined at the chain level by

$$\begin{aligned}(a_0 \otimes a_1 \otimes \cdots \otimes a_p) &\cdot (a_0' \otimes a_{p+1} \otimes \cdots \otimes a_{p+q}) \\ &= \sum_{\sigma \in S_{p,q}} (\operatorname{sgn}\sigma) a_0 a_0' \otimes a_{\sigma^{-1}(1)} \otimes \cdots \otimes a_{\sigma^{-1}(p+q)}\end{aligned}$$

for all $a_0, \ldots, a_{p+q} \in A$.

This product is well-defined on $\mathrm{HH}_*(A)$ since the above formula defines a chain map, as may be checked directly. Also see [**Wei94**, Section 9.4.1] for a more general context.

Theorem 1.5.4. *Let A be a commutative algebra. Then* $\mathrm{HH}_*(A)$ *is a graded commutative algebra under the shuffle product.*

Proof. Compose each σ in $S_{p,q}$ with the (p,q)-shuffle τ given by

$$\tau(1) = p+1, \ldots, \tau(q) = p+q, \tau(q+1) = 1, \ldots, \tau(p+q) = p.$$

This interchanges a shuffle product and its opposite, up to the sign $(-1)^{pq} = \operatorname{sgn} \tau$. $\square$

Exercise 1.5.5. Verify that the cap product is well-defined.

Exercise 1.5.6. Verify the details in Example 1.5.1 by explicitly calculating the action of $\mathrm{HH}^1(A)$ on $\mathrm{HH}_1(A)$.

Exercise 1.5.7. Verify that the shuffle product is well-defined.

Exercise 1.5.8. Let k be a field and let $A = k[x]$, as in Example 1.1.16. Show that $\mathrm{HH}_*(A) \cong k[x,y]/(y^2)$ as a graded algebra under shuffle product, where $|x| = 0$, $|y| = 1$.

Exercise 1.5.9. Let $A = k[x]/(x^2)$. Find the structure of $\mathrm{HH}_*(A)$ under shuffle product.

1.6. Harrison cohomology and Hodge decomposition

In this section, let k be a field of characteristic 0 and let A be a commutative k-algebra. We use the bar complex to define Harrison cohomology, a variant of Hochschild cohomology defined specifically for commutative algebras. In turn, Harrison cohomology is one summand in the Hodge decomposition of Hochschild cohomology defined in this section.

Recall Definition 1.5.2 of (p,q)-shuffles $S_{p,q}$.

Definition 1.6.1. Let k be a field of characteristic 0, let A be a commutative k-algebra, and let $f \in \mathrm{Hom}_k(A^{\otimes n}, A)$. We call f a *Harrison cochain* if

$$\sum_{\sigma \in S_{p,q}} (\operatorname{sgn} \sigma) f(a_{\sigma^{-1}(1)} \otimes \cdots \otimes a_{\sigma^{-1}(n)}) = 0$$

for each pair p, q for which $p+q = n$ and all $a_1, \ldots, a_n \in A$. It can be shown that the Harrison cochains form a subcomplex of the complex of Hochschild cochains. The cohomology of this subcomplex is the *Harrison cohomology* of A.

Details on Harrison cohomology may be found in [**GS88**, Section 4] or [**Har62**]. Barr [**Bar68**] first discovered that Harrison cohomology is a direct summand of Hochschild cohomology. The corresponding decomposition of Hochschild cohomology can be refined to obtain the Hodge decomposition (1.6.2), and this is what we examine next.

The Hodge decomposition of Hochschild cohomology is due to Gerstenhaber and Schack [**GS87**] and is inherent in work of Quillen [**Qui70**]: For each r, let

$$s_{r,n-r} = \sum_{\sigma \in S_{r,n-r}} (\operatorname{sgn} \sigma)\sigma,$$

where the sum is taken over all $(r, n-r)$-shuffles as in Definition 1.5.2. Let

$$s_n = \sum_{r=1}^{n-1} s_{r,n-r}.$$

By [**GS87**, Theorem 1.2], each s_n may be written as a sum,

$$s_n = \lambda_1 e_n(1) + \cdots + \lambda_n e_n(n),$$

where $\lambda_i = 2^i - 2$ and for each j,

$$e_n(j) = \prod_{i \neq j} (\lambda_j - \lambda_i)^{-1} \prod_{i \neq j} (s_n - \lambda_i)$$

in the group algebra kS_n. The elements $e_n(j)$ are the *Eulerian idempotents*; for each n, the set $\{e_n(1), \ldots, e_n(n)\}$ is a set of orthogonal idempotents whose sum is 1 in kS_n. These idempotents may be defined equivalently by a generating function due to Garsia [**Gar90**]:

$$\sum_{j=0}^{n} e_n(j)x^j = \frac{1}{n!} \sum_{\sigma \in S_n} (\operatorname{sgn} \sigma)(x - d_\sigma)(x - d_\sigma + 1) \cdots (x - d_\sigma + n - 1)\sigma$$

where d_σ denotes the number of descents in σ, that is the number of elements i with $\sigma(i) > \sigma(i+1)$.

Consider the action of the symmetric group S_n on $A^{\otimes(n+2)}$ by permutation of n tensor factors (excluding the outermost two), that is,

$$\sigma(a_0 \otimes a_1 \otimes \cdots \otimes a_n \otimes a_{n+1}) = a_0 \otimes a_{\sigma^{-1}(1)} \otimes \cdots \otimes a_{\sigma^{-1}(n)} \otimes a_{n+1}$$

for all $\sigma \in S_n$ and $a_0, \ldots, a_{n+1} \in A$. The resulting actions of the Eulerian idempotents on the bar complex commute with the differentials in the sense that $d_n e_n(j) = e_{n-1}(j) d_n$ for all n, j as functions on $A^{\otimes(n+2)}$ [**GS87**]. Thus the complex $C^*(A, A)$ of (1.1.10) has an induced direct sum decomposition, and so does Hochschild cohomology $\mathrm{HH}^*(A)$: For each n,

$$\mathrm{HH}^n(A) = \bigoplus_{j=1}^{n} \mathrm{HH}^n(A) e_n(j), \tag{1.6.2}$$

called the *Hodge decomposition*. The component $\mathrm{HH}^n(A)e_n(1)$ is precisely Harrison cohomology in degree n.

Other actions on $C^*(A, A)$ lead to other decompositions of Hochschild cohomology. For example, Bergeron and Bergeron [**BB92**] considered an action of signed permutations and obtained an analogous decomposition for algebras with involution.

Exercise 1.6.3. For small values of n, find the Eulerian idempotents $e_n(j)$ using either of the two equivalent definitions given in this section. Verify that $\mathrm{HH}^n(A)e_n(1)$ is indeed isomorphic to Harrison cohomology in degree n.

Chapter 2

Cup Product and Actions

In this chapter we examine in much greater detail the cup product on Hochschild cohomology given by Definition 1.3.1. Since one often works with a resolution other than the bar resolution, or with generalized extensions, definitions of the cup product involving other resolutions or extensions are essential. We give several such equivalent definitions in this chapter, and similarly define actions of Hochschild cohomology rings on Ext spaces of modules.

We make the simplifying assumption from now on that k is a field, and we use the definition (1.1.14) of Hochschild homology and cohomology of a k-algebra A in terms of Tor and Ext.

2.1. From cocycles to chain maps

The first equivalent definition of cup product, in Section 2.2, uses the following construction of chain maps from cocycles. This construction will be used elsewhere as well. It is general in that the same technique can be used to construct a chain map corresponding to a homogeneous element of any Ext space (see Exercise 2.1.2). We present here the more specific case that we will use in the next section and frequently throughout the book.

Let $P_\bullet$ be any projective resolution of A as an A^e-module. Let $g \in \operatorname{Hom}_{A^e}(P_n, A)$ be a cocycle, viewed in a diagram as:

$$\begin{tikzcd}
\cdots \arrow[r, "d_{n+1}"] & P_n \arrow[r, "d_n"] \arrow[d, "g"] & P_{n-1} \arrow[r, "d_{n-1}"] & \cdots \arrow[r, "d_1"] & P_0 \arrow[r, "\varepsilon"] & A \arrow[r] & 0 \\
& A & & & & &
\end{tikzcd}$$

Extend g to a chain map $g_\bullet : P_\bullet \to P_\bullet$ as follows. Let $K_n = \operatorname{Ker}(d_{n-1}) = \operatorname{Im}(d_n)$ for all n, and note that $K_n \cong P_n / \operatorname{Im}(d_{n+1})$. As in (A.2.4), let $\varepsilon_n : P_n \to K_n$ denote the quotient map and $i_n : K_n \to P_{n-1}$ the inclusion map. Since g is a cocycle, that is, $d_{n+1}^*(g) = 0$, it factors through K_n. Denote by $\overline{g}$ the map from K_n to A for which $\overline{g}\varepsilon_n = g$. We illustrate these maps in the following commuting diagram:
(2.1.1)

$$\begin{tikzcd}
\cdots \arrow[r, "d_{n+1}"] & P_n \arrow[rr, "d_n"] \arrow[dr, "\varepsilon_n"] \arrow[dd, "g"] & & P_{n-1} \arrow[r, "d_{n-1}"] & \cdots \arrow[r, "d_1"] & P_0 \arrow[r, "\varepsilon"] & A \arrow[r] & 0 \\
& & K_n \arrow[ur, "i_n"] \arrow[dl, "\overline{g}"] & & & & & \\
& A & & & & & &
\end{tikzcd}$$

We may consider the sequence

$$\cdots \xrightarrow{d_{n+3}} P_{n+2} \xrightarrow{d_{n+2}} P_{n+1} \xrightarrow{d_{n+1}} P_n \xrightarrow{\varepsilon_n} K_n \longrightarrow 0$$

to be a projective resolution of K_n as an A^e-module. We rewrite our earlier diagram to focus on this information, and apply the Comparison Theorem (Theorem A.2.7) which guarantees existence of maps $g_i : P_{n+i} \to P_i$ $(i \geq 0)$ that commute with the differentials. We obtain the following commuting diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & P_{n+2} \arrow[r, "d_{n+2}"] \arrow[d, dashed, "g_2"] & P_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, dashed, "g_1"] & P_n \arrow[r, "\varepsilon_n"] \arrow[d, dashed, "g_0"] \arrow[dr, "g"] & K_n \arrow[r] \arrow[d, "\overline{g}"] & 0 \\
\cdots \arrow[r] & P_2 \arrow[r, "d_2"] & P_1 \arrow[r, "d_1"] & P_0 \arrow[r, "\varepsilon"] & A \arrow[r] & 0
\end{tikzcd}$$

We give some details of the maps g_i that are guaranteed by the Comparison Theorem (Theorem A.2.7), as we will use them often: Take g_i to be the zero map if $i < 0$. Since ε maps P_0 surjectively onto A, and P_n is projective, there is a map $g_0 : P_n \to P_0$ such that $\varepsilon g_0 = g$. We claim that the image of $g_0 d_{n+1}$ is contained in K_1. To see this, note that $K_1 = \operatorname{Ker}\varepsilon$ and that $0 = g d_{n+1} = \varepsilon g_0 d_{n+1}$, so the image of $g_0 d_{n+1}$ is contained in $\operatorname{Ker}\varepsilon = K_1$. Now, since $g_0 d_{n+1}$ maps to K_1, P_1 surjects onto K_1 via d_1, and P_{n+1} is projective, there is a map $g_1 : P_{n+1} \to P_1$ such that $d_1 g_1 = g_0 d_{n+1}$. Once again, we see that the image of $g_1 d_{n+2}$ is contained in K_2, and this ensures existence of $g_2 : P_{n+2} \to P_2$, and so on. The chain map $g_\bullet$ is unique up to chain homotopy, as stated in the Comparison Theorem (Theorem A.2.7).

Exercise 2.1.2. Generalize the construction in this section: Let B be a ring. Let $P_\bullet$ and $Q_\bullet$ be projective resolutions of B-modules U and V, and let $g \in \mathrm{Hom}_B(P_n, V)$ be a cocycle, so that g represents an element of $\mathrm{Ext}^n_B(U, V)$. Let $\varepsilon : Q_0 \to V$ be the augmentation map of the resolution $Q_\bullet$. Show that there exists a chain map $g_\bullet : P_\bullet \to Q_\bullet$ for which $g = \varepsilon g_0$.

Exercise 2.1.3. Let $A = k[x]$ and let $P_\bullet$ be the resolution in Example 1.1.16. Letting g be any nonzero 1-cocycle, find g_0. Explain that g_i is the zero map for all $i \geq 1$.

2.2. Yoneda product

Our second definition of associative product on Hochschild cohomology $\mathrm{HH}^*(A)$, that is equivalent to Definition 1.3.1, is sometimes called the Yoneda product. We will define this product at the chain level on any resolution as a composition of chain maps. Then we will show that for the bar resolution, this definition is indeed equivalent to the cup product as defined by equation (1.3.2). We will use the same notation $\smile$ for this product in anticipation of our proof that it is equivalent to that in Definition 1.3.1.

Let $P_\bullet$ be any projective resolution of A as an A^e-module. Let $f \in \mathrm{Hom}_{A^e}(P_m, A)$ and $g \in \mathrm{Hom}_{A^e}(P_n, A)$ be cocycles. Extend g to a chain map $g_\bullet : P_\bullet \to P_\bullet$ as described in Section 2.1. The map $f \smile g \in \mathrm{Hom}_{A^e}(P_{m+n}, A)$ is defined to be the composition $f g_m$:

$$f \smile g = f g_m. \tag{2.2.1}$$

This composition may be viewed in a diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & P_{m+n} \arrow[r, "d_{m+n}"] \arrow[d, "g_m"] & \cdots \arrow[r, "d_{n+2}"] & P_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "g_1"] & P_n \arrow[d, "g_0"] \arrow[dr, "g"] & & \\
\cdots \arrow[r] & P_m \arrow[r, "d_m"] \arrow[d, "f"] & \cdots \arrow[r, "d_2"] & P_1 \arrow[r, "d_1"] & P_0 \arrow[r, "\varepsilon"] & A \arrow[r] & 0 \\
& A & & & & &
\end{tikzcd}$$

Note that $f \smile g$ is a cocycle because f is a cocycle and $g_\bullet$ is a chain map. Since $g_\bullet$ is unique up to chain homotopy, a different such chain map $g'_\bullet$ results in a cohomologous cocycle $f g'_m$. If f is a coboundary then $f \smile g$ is a coboundary since $g_\bullet$ is a chain map. If g is a coboundary, we claim that we may set $g_m = 0$ for all $m \geq 1$. To see this, write $g = h d_n$ for a cochain $h \in \mathrm{Hom}_{A^e}(P_{n-1}, A)$. By projectivity of P_{n-1}, there is a map $h' : P_{n-1} \to P_0$ such that $\varepsilon h' = h$. Setting $g_0 = h'd_n$ and $g_1 = 0$, we find that $d_1 g_1 = g_0 d_{n+1} = h' d_n d_{n+1} = 0$. Now we may set $g_m = 0$ for all $m > 1$ as well. Hence $f \smile g = 0$ at the chain level in this case when $m \geq 1$. We thus see that this cup product at the chain level induces a well-defined product on cohomology.

Equivalently, we may identify both cocycles f, g with choices of chain maps $f_\bullet, g_\bullet$ that they induce as described above, in which case $f \smile g$ may be identified at the chain level with a composition of chain maps, as indicated by a commuting diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & P_{m+n+1} \arrow[r, "d_{m+n+1}"] \arrow[d, "g_{m+1}"] & P_{m+n} \arrow[r, "d_{m+n}"] \arrow[d, "g_m"] & \cdots \arrow[r, "d_{n+2}"] & P_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "g_1"] & P_n \arrow[d, "g_0"] \\
\cdots \arrow[r] & P_{m+1} \arrow[r, "d_{m+1}"] \arrow[d, "f_1"] & P_m \arrow[r, "d_m"] \arrow[d, "f_0"] & \cdots \arrow[r, "d_2"] & P_1 \arrow[r, "d_1"] & P_0 \\
\cdots \arrow[r] & P_1 \arrow[r, "d_1"] & P_0 & & &
\end{tikzcd}$$

Again, at the level of cohomology, this does not depend on choices. Filling in missing components and arrows, we obtain a chain map $(f \smile g)_\bullet$, an element of $\mathrm{Hom}_{A^e}(P_\bullet, P_\bullet)$ defined componentwise by $(f \smile g)_i = f_i g_{m+i}$ as a map from P_{m+n+i} to P_i for all $i \geq 0$. We map P_j to 0 for $j < m + n$.

Returning to the definition (2.2.1) of product, we claim that it does not depend on choice of resolution $P_\bullet$: To see this, apply the Comparison Theorem (Theorem A.2.7) to $P_\bullet$ and any other projective A^e-resolution $Q_\bullet$ of A. Composition with comparison maps takes the chain map $g_\bullet$ to a corresponding chain map on $Q_\bullet$ and the composition fg_m of (2.2.1) to a corresponding composition as a function on Q_{m+n}.

Taking $P_\bullet$ to be the bar resolution $B(A)$ given by (1.1.4), we will show that the product defined by equation (2.2.1) is the same as that defined by equation (1.3.2). Then by the Comparison Theorem (Theorem A.2.7), comparing any other projective resolution to $B(A)$, definition (2.2.1) will then indeed be an equivalent definition of cup product on Hochschild cohomology $\mathrm{HH}^*(A)$. Let us apply the definition (2.2.1) when $P_n = A^{\otimes(n+2)}$ for all n. Given $g : A^{\otimes(n+2)} \to A$, a cocycle, we must choose $g_0 : A^{\otimes(n+2)} \to A \otimes A$ for which $\pi g_0 = g$, that is,

$$\pi g_0(1 \otimes a_1 \otimes \cdots \otimes a_n \otimes 1) = g(1 \otimes a_1 \otimes \cdots \otimes a_n \otimes 1)$$

for all $a_1, \ldots, a_n \in A$. One such choice is given by

$$g_0(1 \otimes a_1 \otimes \cdots a_n \otimes 1) = 1 \otimes g(1 \otimes a_1 \otimes \cdots \otimes a_n \otimes 1),$$

where we extend to an A^e-module map by defining

$$g_0(a_0 \otimes a_1 \otimes \cdots \otimes a_{n+1}) = a_0 \otimes g(1 \otimes a_1 \otimes \cdots \otimes a_n \otimes 1)a_{n+1}$$

for all $a_0, \ldots, a_{n+1} \in A$. Then a choice of $g_1 : A^{\otimes(n+3)} \to A^{\otimes 3}$ can be given by

$$g_1(1 \otimes a_1 \otimes \cdots \otimes a_{n+1} \otimes 1) = 1 \otimes a_1 \otimes g(1 \otimes a_2 \otimes \cdots \otimes a_{n+1} \otimes 1),$$

extended to an A^e-module map; a calculation shows that $g_0 d_{n+1} = d_1 g_1$ using the definition of g_0 and the assumption that g is a cocycle. In general we may choose

$$g_i(1 \otimes a_1 \otimes \cdots \otimes a_{n+i} \otimes 1) = 1 \otimes a_1 \otimes \cdots \otimes a_i \otimes g(1 \otimes a_{i+1} \otimes \cdots \otimes a_{n+i} \otimes 1).$$

We see that if $f \in \mathrm{Hom}_{A^e}(A^{\otimes(m+2)}, A)$, then

$$\begin{aligned} &(f \smile g)(1 \otimes a_1 \otimes \cdots \otimes a_{m+n} \otimes 1) \\ &= f g_m(1 \otimes a_1 \otimes \cdots \otimes a_{m+n} \otimes 1) \\ &= f(1 \otimes a_1 \otimes \cdots \otimes a_m \otimes g(1 \otimes a_{m+1} \otimes \cdots \otimes a_{m+n} \otimes 1)) \\ &= f(1 \otimes a_1 \otimes \cdots \otimes a_m \otimes 1) g(1 \otimes a_{m+1} \otimes \cdots \otimes a_{m+n} \otimes 1) \end{aligned}$$

for all $a_1, \ldots, a_{m+n} \in A$. This is equivalent to (1.3.2) under the identification of $\mathrm{Hom}_{A^e}(A^{\otimes(m+n+2)}, A)$ and $\mathrm{Hom}_k(A^{\otimes(m+n)}, A)$ via isomorphism (1.1.11). There is another choice of lifting $g_\bullet$ of g to which a comparison leads to another proof that the product on Hochschild cohomology is graded commutative; see, e.g., Solberg [**Sol06**, Theorem 2.1] (cf. Theorem 1.4.4).

We may use equation (2.2.1) to compute cup products for more examples, starting with the truncated polynomial rings $k[x]/(x^n)$ as we see next.

Example 2.2.2. We return to Example 1.1.19 in which $A = k[x]/(x^n)$, and compute cup products. Let $p = \mathrm{char}(k)$, and assume that p does not divide n. (The case where p divides n is different, and is the next example.) In degree 1, let $g \in \mathrm{Hom}_{A^e}(A^e, A)$ be the function given by $g(1 \otimes 1) = x$. By the description of the cohomology in Example 1.1.19 and of the action of $\mathrm{HH}^0(A) \cong Z(A)$ described at the end of Section 1.2, g represents an element in $\mathrm{HH}^1(A)$ and generates $\mathrm{HH}^1(A)$ as an $\mathrm{HH}^0(A)$-module. In degree 2, let $f \in \mathrm{Hom}_{A^e}(A^e, A)$ be the function given by $f(1 \otimes 1) = 1$. Then by similar reasoning, f represents an element in $\mathrm{HH}^2(A)$ and generates $\mathrm{HH}^2(A)$ as an $\mathrm{HH}^0(A)$-module. We will find cup products of these functions as compositions of chain maps, and show that other cup products are determined by these.

In relation to the resolution (1.1.20), let $K_1 = A^e/\mathrm{Im}(v\cdot)$. Since g is a cocycle, it factors through K_1 as in the following diagram. The map from A^e to K_1 in the diagram can be taken to be the quotient map, which can be identified with the map $u\cdot$ to $\operatorname{Im}(u\cdot) \cong K_1$ as a submodule of A^e in degree 0.

$$\begin{tikzcd}
\cdots \arrow[r] & A^e \arrow[r, "u\cdot"] \arrow[d, dashed, "g_2"] & A^e \arrow[r, "v\cdot"] \arrow[d, dashed, "g_1"] & A^e \arrow[r] \arrow[d, dashed, "g_0"] \arrow[dr, "g"] & K_1 \arrow[r] \arrow[d, "\overline{g}"] & \cdots \\
\cdots \arrow[r] & A^e \arrow[r, "v\cdot"] & A^e \arrow[r, "u\cdot"] & A^e \arrow[r, "\pi"] & A \arrow[r] & 0
\end{tikzcd}$$

We may find $g_0, g_1, g_2, \ldots$ via the technique described in Section 2.1, starting by finding g_0, then g_1, and so on. Set $g_{2m}(1 \otimes 1) = x \otimes 1$ and

$$g_{2m+1}(1 \otimes 1) = \sum_{i=1}^{n-1} i x^{n-1-i} \otimes x^i$$

for all $m \geq 0$. Extend to A^e-module homomorphisms. A calculation then shows that the above diagram commutes.

We use these maps g_j to find some cup products. First note $gg_1 = 0$ since $x^n = 0$ in A, and so $g \smile g = 0$. We similarly find that

$$(f \smile g)(1 \otimes 1) = fg_2(1 \otimes 1) = f(x \otimes 1) = x.$$

Comparing to the results of Example 1.1.19, we find that $f \smile g$ represents a generator for $\mathrm{HH}^3(A)$ as an $\mathrm{HH}^0(A)$-module.

Now let $K_2 = A^e / \operatorname{Im}(u\cdot)$. Since f is a cocycle, it factors through K_2 as shown in the following diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & A^e \arrow[r, "v\cdot"] \arrow[d, dashed, "f_2"] & A^e \arrow[r, "u\cdot"] \arrow[d, dashed, "f_1"] & A^e \arrow[r] \arrow[d, dashed, "f_0"] \arrow[dr, "f"] & K_2 \arrow[r] \arrow[d, "\overline{f}"] & \cdots \\
\cdots \arrow[r] & A^e \arrow[r, "v\cdot"] & A^e \arrow[r, "u\cdot"] & A^e \arrow[r, "\pi"] & A \arrow[r] & 0
\end{tikzcd}$$

Set $f_j(1 \otimes 1) = 1 \otimes 1$ for all $j \geq 0$. A calculation then shows that the above diagram commutes. We find that $(f \smile f)(1 \otimes 1) = ff_2(1 \otimes 1) = 1$, and similarly for all powers i of f, by induction:

$$f^i(1 \otimes 1) = f^{i-1} f_2(1 \otimes 1) = 1,$$

where $f^i = f \smile \cdots \smile f$ (i factors of f). Again comparing to the results of Example 1.1.19, we see that in each even degree, $\mathrm{HH}^{2i}(A)$ is generated by the image $\overline{f^i}$ of f^i in $\mathrm{HH}^{2i}(A)$ as an $\mathrm{HH}^0(A)$-module. Similarly, the image of $f^i \smile g = f^i g_2$ generates $\mathrm{HH}^{2i+1}(A)$ as an $\mathrm{HH}^0(A)$-module. Due to the structure of the resolution (1.1.20), $x^{n-1} \smile g$ and $x^{n-1} \smile f$ are coboundaries. Taking advantage of graded commutativity (Theorem 1.4.4), we claim that we have now computed all products needed to conclude the ring structure of $\mathrm{HH}^*(A)$. To see this, map the polynomial ring $k[x, y, z]$ onto $\mathrm{HH}^*(A)$ by sending x to a copy of itself in $\mathrm{HH}^0(A)$, and y and z

to the elements of cohomology represented by f and g, respectively. Our observations so far imply that this map factors through a quotient as follows:

$$k[x,y,z]/(x^n,\ x^{n-1}y,\ y^2,\ x^{n-1}z) \longrightarrow \mathrm{HH}^*(A).$$

This map is surjective since x, f, g generate $\mathrm{HH}^*(A)$ as we have seen. Comparing vector space dimensions in each degree (setting $|x| = 0$, $|y| = 1$, and $|z| = 2$), we see that this map must in fact be an isomorphism:

$$\mathrm{HH}^*(A) \cong k[x,y,z]/(x^n,\ x^{n-1}y,\ y^2,\ x^{n-1}z).$$

Example 2.2.3. Let $A = k[x]/(x^n)$ as in the previous example but now under the assumption that the characteristic p of k divides n. We again use the description of its Hochschild cohomology in Example 1.1.19. At the chain level in degree 1, let $g \in \mathrm{Hom}_{A^e}(A^e, A)$ be the function defined by $g(1 \otimes 1) = 1$. In degree 2, let $f \in \mathrm{Hom}_{A^e}(A^e, A)$ be the function defined by $f(1 \otimes 1) = 1$. Then f, g are cocycles. Let $K_1 = A^e/\operatorname{Im}(v\cdot)$. Set $g_{2m}(1 \otimes 1) = 1 \otimes 1$ and

$$g_{2m+1}(1 \otimes 1) = \sum_{i=1}^{n-1} i x^{n-1-i} \otimes x^{i-1}$$

for all $m \geq 0$. (Note that if $n = 2$, this formula yields $g_{2m+1}(1 \otimes 1) = 1 \otimes 1$.) A calculation shows that, since $n \equiv 0$ in k, $g_\bullet$ is a chain map. Thus we may use equation (2.2.1) to compute:

$$\begin{aligned} (g \smile g)(1 \otimes 1) &= g g_1(1 \otimes 1) = \frac{n(n-1)}{2} x^{n-2}, \\ (f \smile g)(1 \otimes 1) &= f g_2(1 \otimes 1) = 1. \end{aligned} \tag{2.2.4}$$

Further calculations show as in Example 2.2.2 that the images of f and g generate $\mathrm{HH}^*(A)$ as an algebra over its degree 0 component $\mathrm{HH}^0(A) \cong A$. If p is odd, the coefficient $\frac{n(n-1)}{2}$ is 0 in k. As well, if $p = 2$ and $n = 2s$ for some even integer s, the coefficient $\frac{n(n-1)}{2}$ is 0 in k. Thus we find, similarly to Example 2.2.2, that in odd characteristic p dividing n, or in case $p = 2$ and $n = 2s$ for some even integer s, there is an isomorphism of algebras,

$$\mathrm{HH}^*(A) \cong k[x,y,z]/(x^n,\ y^2),$$

where $|x| = 0$, $|y| = 1$, $|z| = 2$. If $p = 2$ and $n = 2s$ for some odd integer s, the coefficient $\frac{n(n-1)}{2}$ in (2.2.4) is 1. If $n = 2$, this implies that the image of $g \smile g$ generates $\mathrm{HH}^2(A)$ as an $\mathrm{HH}^0(A)$-module, and similarly the image of g generates $\mathrm{HH}^*(A)$ as an algebra over $A \cong \mathrm{HH}^0(A)$. If $n \neq 2$, it does not, and equation (2.2.4) yields a relation among the generators. Thus we find that if $p = 2$ and $n = 2$,

$$\mathrm{HH}^*(A) \cong k[x,y]/(x^n),$$

where $|x| = 0$, $|y| = 1$. If $p = 2$ and $n = 2s$ for some odd integer $s > 1$, then

$$\mathrm{HH}^*(A) \cong k[x, y, z]/(x^n,\ x^2y^2)$$

where $|x| = 0$, $|y| = 1$, $|z| = 2$.

Exercise 2.2.5. Verify the claims made right after equation (2.2.1). Specifically, verify that $f \smile g$ as defined in (2.2.1) is indeed a cocycle. If $g'_{\bullet} : P_{\bullet} \to P_{\bullet}$ is another chain map extending g, show that fg'_m is cohomologous to fg_m.

Exercise 2.2.6. Verify that the definition (2.2.1) of cup product on $\mathrm{HH}^*(A)$ does not depend on choice of resolution. That is, given two projective resolutions $P_{\bullet}$ and $Q_{\bullet}$ of A as an A^e-module, and a comparison map $\phi : Q_{\bullet} \to P_{\bullet}$ lifting the identity map from A to A (whose existence is guaranteed by the Comparison Theorem, Theorem A.2.7), the cup product $f \smile g$ defined on $P_{\bullet}$ and the cup product $(f\phi_m) \smile (g\phi_n)$ defined on $Q_{\bullet}$ represent the same element of $\mathrm{HH}^{m+n}(A)$.

Exercise 2.2.7. Verify that $f_{\bullet}$, $g_{\bullet}$ of Example 2.2.2 are indeed chain maps.

Exercise 2.2.8. Verify that $f_{\bullet}$, $g_{\bullet}$ of Example 2.2.3 are indeed chain maps.

2.3. Tensor product of complexes

Another definition of associative product on Hochschild cohomology $\mathrm{HH}^*(A)$, that also turns out to be equivalent to the cup product of Definition 1.3.1, is a convolution product arising from a tensor product of complexes and a diagonal map, as follows.

Let $P_{\bullet}$ be any A^e-projective resolution of A with augmentation map $\mu : P_0 \to A$. Consider the tensor product complex $P_{\bullet} \otimes_A P_{\bullet}$:

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
P_0 \otimes_A P_2 \arrow[d] & P_1 \otimes_A P_2 \arrow[l] \arrow[d] & P_2 \otimes_A P_2 \arrow[l] \arrow[d] & \cdots \arrow[l] \\
P_0 \otimes_A P_1 \arrow[d] & P_1 \otimes_A P_1 \arrow[l] \arrow[d] & P_2 \otimes_A P_1 \arrow[l] \arrow[d] & \cdots \arrow[l] \\
P_0 \otimes_A P_0 & P_1 \otimes_A P_0 \arrow[l] & P_2 \otimes_A P_0 \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

We will show that the total complex of $P_{\bullet} \otimes_A P_{\bullet}$ is also an A^e-projective resolution of A. First we must show that for each m, n, the A^e-module $P_m \otimes_A P_n$ is projective. To see this, note that since P_m, P_n are projective

A^e-modules, each is a direct summand of a direct sum of copies of A^e. Thus it suffices to show that $A^e \otimes_A A^e$ is free. This follows from the isomorphism $A^e \otimes_A A^e \xrightarrow{\sim} A^e \otimes_k A$ as an A^e-module via the map $(a_1 \otimes a_2) \otimes_A (a_3 \otimes a_4) \mapsto a_1 \otimes a_4 \otimes a_2 a_3$ for $a_1, a_2, a_3, a_4 \in A$ (note A^e acts only on the outermost two factors of $A^e \otimes_A A^e$). Since k is assumed to be a field, A is free as a k-module, and so $A^e \otimes_k A$ is free as an A^e-module.

Next, to see that the total complex $\mathrm{Tot}(P_{\bullet} \otimes_A P_{\bullet})$ of the tensor product complex $P_{\bullet} \otimes_A P_{\bullet}$ has cohomology A concentrated in degree 0, we apply the Künneth Theorem (Theorem A.5.1): The module $A \otimes A^{\mathrm{op}}$, under right multiplication by elements of A in the right factor, is a free right A-module since A is a free k-module. It follows that $P_{\bullet}$ is, by restriction, a projective resolution of the free right A-module A. Considering only the right A-module structure of the resolution $P_{\bullet}$ now, it necessarily splits as a sequence of right A-modules. That is, the augmentation map $\mu : P_0 \to A$ splits via some splitting map $\iota : A \to P_0$ such that $\iota(A) = \iota(\mu(P_0))$ is a direct summand of P_0, so that $P_0 \cong A \oplus \mathrm{Im}(d_1) = A \oplus \mathrm{Ker}(\mu)$ as a right A-module. Then $d_1 : P_1 \to P_0$ splits so that $P_1 \cong \mathrm{Im}(d_1) \oplus \mathrm{Im}(d_2)$ as a right A-module, and so on. It follows that the boundaries $d_i(P_i)$ are also all projective right A-modules, that is the hypotheses of the Künneth Theorem (Theorem A.5.1) hold. The Tor terms in the Künneth sequence (that is, the sequence in the Künneth Theorem statement) vanish: The only term in which both arguments are nonzero is $\mathrm{Tor}_1^A(\mathrm{H}_0(P), \mathrm{H}_0(P)) = \mathrm{Tor}_1^A(A, A) = 0$ (since A is free as an A-module). This implies that the total complex of $P_{\bullet} \otimes_A P_{\bullet}$ is indeed a resolution of $A \otimes_A A \cong A$ by A^e-projective modules.

By the Comparison Theorem (Theorem A.2.7) there is a chain map $\Delta : P_{\bullet} \to P_{\bullet} \otimes_A P_{\bullet}$ lifting the identity map from A to A (where we have identified $A \otimes_A A \cong A$). Such a map Δ is unique up to chain homotopy. Sometimes Δ is called a *diagonal map.*

The cup product on Hochschild cohomology may be defined via a diagonal map $\Delta : P_{\bullet} \to P_{\bullet} \otimes_A P_{\bullet}$ as a convolution product in the following way. Let $f \in \mathrm{Hom}_{A^e}(P_m, A)$, $g \in \mathrm{Hom}_{A^e}(P_n, A)$ represent elements of $\mathrm{HH}^m(A, A)$, $\mathrm{HH}^n(A, A)$. View $f \otimes g$ as a function on $P_m \otimes_A P_n$ and extend to $P_{\bullet} \otimes_A P_{\bullet}$ by setting $(f \otimes g)(x) = 0$ for all $x \in P_r \otimes_A P_s$ such that $(r, s) \neq (m, n)$. We may define

$$f \smile g = \pi(f \otimes g)\Delta, \tag{2.3.1}$$

where π is multiplication on A. Calculations show that this induces a well-defined map on Hochschild cohomology: The product of cocycles is a cocycle, and the product of a cocycle with a coboundary is a coboundary. For any two resolutions, by the Comparison Theorem (Theorem A.2.7), there are chain maps between them, and it follows that this definition of product does not depend on choices of $P_{\bullet}$ and Δ. If $P_{\bullet}$ is the bar resolution, one choice of chain map Δ induces precisely the chain level cup product (1.3.2): Define a diagonal map Δ on the bar resolution by

$$(2.3.2)\quad \Delta(1\otimes a_1\otimes\cdots\otimes a_s\otimes 1) = \sum_{i=0}^{s}(1\otimes a_1\otimes\cdots\otimes a_i\otimes 1)\otimes(1\otimes a_{i+1}\otimes\cdots\otimes a_s\otimes 1)$$

for all $a_1, \ldots, a_s \in A$. A calculation shows that formula (2.3.1), with these choices, is precisely formula (1.3.2). If $P_{\bullet}$ is not the bar resolution, one choice of chain map Δ is given by first applying a chain map from $P_{\bullet}$ to $B(A)$, then applying the diagonal map Δ on $B(A)$ given by (2.3.2), and then mapping $B(A) \otimes_A B(A)$ to $P_{\bullet} \otimes_A P_{\bullet}$ via another chain map. Thus on cohomology, our product in (2.3.1) is equivalent to the cup product (1.3.2), justifying our use of the same notation for this product as well.

Exercise 2.3.3. Verify that the convolution product (2.3.1) induces a well-defined map on Hochschild cohomology and is independent of choices, that is:

(a) The product of cocycles is a cocycle, and the product of a cocycle with a coboundary is a coboundary.

(b) The product induced on cohomology is independent of choice of projective resolution $P_{\bullet}$ and diagonal map Δ.

Exercise 2.3.4. Let $A = k[x]/(x^2)$, and let $P_{\bullet}$ be the resolution (1.1.20). Let $\xi_i = 1 \otimes 1$ in $P_i = A^e$. Show that the following defines a diagonal map on $P_{\bullet}$:

$$\Delta(\xi_i) = \sum_{j=0}^{i} \xi_j \otimes \xi_{i-j}.$$

Use this diagonal map and equation (2.3.1) to reproduce the algebra structure of $\mathrm{HH}^*(A)$ given in Examples 2.2.2 and 2.2.3.

2.4. Yoneda composition and tensor product of extensions

Two more definitions of the product on Hochschild cohomology $\mathrm{HH}^*(A)$ are described by way of generalized extensions. (See Section A.3 for a summary of n-extensions and connections with Ext.)

We begin with the *Yoneda composition* (or *Yoneda splice*). This definition uses the description of Hochschild cohomology $\mathrm{HH}^n(A) \cong \mathrm{Ext}^n_{A^e}(A, A)$ as equivalence classes of n-extensions of A by A as A^e-modules. Let $\mathbf{f}$ and

$\mathbf{g}$ be m- and n-extensions,

$$\mathbf{f}: \qquad 0 \longrightarrow A \xrightarrow{\alpha_m} M_{m-1} \xrightarrow{\alpha_{m-1}} \cdots \xrightarrow{\alpha_2} M_1 \xrightarrow{\alpha_1} M_0 \xrightarrow{\alpha_0} A \longrightarrow 0,$$

$$\mathbf{g}: \qquad 0 \longrightarrow A \xrightarrow{\beta_n} N_{n-1} \xrightarrow{\beta_{n-1}} \cdots \xrightarrow{\beta_2} N_1 \xrightarrow{\beta_1} N_0 \xrightarrow{\beta_0} A \longrightarrow 0,$$

respectively, representing elements of $\mathrm{HH}^m(A)$ and $\mathrm{HH}^n(A)$. Consider the composition $\beta_n\alpha_0$, a map from M_0 to N_{n-1}. We may use this map to combine the two sequences (by splicing them):

(2.4.1) $\qquad \mathbf{f} \smile \mathbf{g}:$

$$0 \longrightarrow A \xrightarrow{\alpha_m} M_{m-1} \xrightarrow{\alpha_{m-1}} \cdots \xrightarrow{\alpha_1} M_0 \xrightarrow{\beta_n\alpha_0} N_{n-1} \xrightarrow{\beta_{n-1}} \cdots$$

$$\xrightarrow{\beta_1} N_0 \xrightarrow{\beta_0} A \longrightarrow 0.$$

A calculation shows that this sequence is exact at M_0 and at N_{n-1}. Therefore it is an exact sequence, in other words, it is an $(m+n)$-extension of A by A. We have defined $\mathbf{f} \smile \mathbf{g}$ to be this $(m+n)$-extension, representing an element of $\mathrm{HH}^{m+n}(A)$. It can be seen to follow from the definitions and the correspondence between generalized extensions and Ext (see Section A.3) that this is equivalent to the cup product of Definition 1.3.1. This equivalence is shown in detail for example in [**Aho08**, Theorem 4.3].

We may alternatively take the tensor product, over A, of an m-extension with an n-extension to obtain an $(m+n)$-extension from the total complex. This may be seen to be equivalent to Yoneda composition, and thus to cup product, by mapping to two edges. Specifically, the tensor product complex of the m- and n-extensions $\mathbf{f}$ and $\mathbf{g}$ as above is

$$\begin{tikzcd}
A \otimes_A A \arrow[d] & M_0 \otimes_A A \arrow[l] \arrow[d] & \cdots \arrow[l] & M_{m-1} \otimes_A A \arrow[l] \arrow[d] & A \otimes_A A \arrow[l] \arrow[d] \\
A \otimes_A N_{n-1} \arrow[d] & M_0 \otimes_A N_{n-1} \arrow[l] \arrow[d] & \cdots \arrow[l] & M_{m-1} \otimes_A N_{n-1} \arrow[l] \arrow[d] & A \otimes_A N_{n-1} \arrow[l] \arrow[d] \\
\vdots \arrow[d] & \vdots \arrow[d] & & \vdots \arrow[d] & \vdots \arrow[d] \\
A \otimes_A N_1 \arrow[d] & M_0 \otimes_A N_1 \arrow[l] \arrow[d] & \cdots \arrow[l] & M_{m-1} \otimes_A N_1 \arrow[l] \arrow[d] & A \otimes_A N_1 \arrow[l] \arrow[d] \\
A \otimes_A N_0 \arrow[d] & M_0 \otimes_A N_0 \arrow[l] \arrow[d] & \cdots \arrow[l] & M_{m-1} \otimes_A N_0 \arrow[l] \arrow[d] & A \otimes_A N_0 \arrow[l] \arrow[d] \\
A \otimes_A A & M_0 \otimes_A A \arrow[l] & \cdots \arrow[l] & M_{m-1} \otimes_A A \arrow[l] & A \otimes_A A \arrow[l]
\end{tikzcd}$$

Consider this diagram to be an augmented double complex. The corresponding $(m+n)$-extension of A by A is obtained by deleting the leftmost column and bottom row, taking the total complex, and augmenting with A:

$$0 \longrightarrow A \longrightarrow M_{m-1} \oplus N_{n-1} \longrightarrow \cdots$$

$$\longrightarrow \begin{smallmatrix} (M_1 \otimes_A N_0) \oplus \\ (M_0 \otimes_A N_1) \end{smallmatrix} \longrightarrow M_0 \otimes_A N_0 \longrightarrow A \longrightarrow 0$$

The Künneth Theorem (Theorem A.5.1) implies that the above sequence is exact, since all modules in the original sequences $\mathbf{f}$ and $\mathbf{g}$ are free as right A-modules.

We next show that the total complex above is equivalent to the Yoneda splice $\mathbf{f} \smile \mathbf{g}$ of (2.4.1): Identify $A \otimes_A A$ in the tensor product diagram above with A by canonical isomorphism, $A \otimes_A N_{n-1}$ with N_{n-1}, and so on, and we delete the upper left and lower right corners, instead replacing them with compositions of the maps:

$$\begin{tikzcd}
 & M_0 \arrow[dl] \arrow[d] & \cdots \arrow[l] & M_{m-1} \arrow[l] \arrow[d] & A \arrow[l] \arrow[d] \\
N_{n-1} \arrow[d] & M_0 \otimes_A N_{n-1} \arrow[l] \arrow[d] & \cdots \arrow[l] & M_{m-1} \otimes_A N_{n-1} \arrow[l] \arrow[d] & N_{n-1} \arrow[l] \arrow[d] \\
\vdots \arrow[d] & \vdots \arrow[d] & & \vdots \arrow[d] & \vdots \arrow[d] \\
N_1 \arrow[d] & M_0 \otimes_A N_1 \arrow[l] \arrow[d] & \cdots \arrow[l] & M_{m-1} \otimes_A N_1 \arrow[l] \arrow[d] & N_1 \arrow[l] \arrow[d] \\
N_0 \arrow[d] & M_0 \otimes_A N_0 \arrow[l] \arrow[d] & \cdots \arrow[l] & M_{m-1} \otimes_A N_0 \arrow[l] \arrow[d] & N_0 \arrow[l] \arrow[dl] \\
A & M_0 \arrow[l] & \cdots \arrow[l] & M_{m-1} \arrow[l] &
\end{tikzcd}$$

Now we see that the top and left edges form the Yoneda splice $\mathbf{f} \smile \mathbf{g}$ of (2.4.1). The total complex maps to $\mathbf{f} \smile \mathbf{g}$ simply by canonical projection to the top row and canonical projection followed by maps induced by $M_0 \to A$ in the left column. In this way we see that the $(m+n)$-extension arising from the total complex is equivalent to $\mathbf{f} \smile \mathbf{g}$ as defined by (2.4.1), and thus to cup product.

This diagram also leads to another proof of graded commutativity: Up to signs, the right and lower edges form the Yoneda splice $\mathbf{g} \smile \mathbf{f}$. Recall that in the tensor product complex, vertical maps have signs attached so that for example the rightmost column maps take the sign $(-1)^m$. Additional signs arise in projecting from the total complex, and we use the convention that $-(\mathbf{g} \smile \mathbf{f})$ is the extension $\mathbf{g} \smile \mathbf{f}$ in which α_0 is replaced by $-\alpha_0$. Keeping careful track of all these signs indeed yields another proof of graded commutativity of cup product. For additional details, see Section 6.5 where these projection maps are used in connection to the Gerstenhaber bracket.

In the sequel, we will frequently exploit the fact that these associative products on $\mathrm{HH}^*(A)$ defined in Sections 2.2–2.4 and in Definition 1.3.1 all agree. In practice, in each setting we will use the version that is most convenient for that setting.

Exercise 2.4.2. Verify that (2.4.1) is indeed an exact sequence.

Exercise 2.4.3. Verify directly that the product defined by (2.4.1) is equivalent to the Yoneda product of (2.2.1) by applying the correspondence between generalized extensions and cochains described in Section A.3.

Exercise 2.4.4. Let $A = k[x]/(x^2)$ and let V be the 2-dimensional A-module with basis v_1, v_2 for which $xv_2 = v_1$ and $xv_1 = 0$. Consider a 1-extension of A-modules,

$$\mathbf{f}: \qquad 0 \longrightarrow k \longrightarrow V \longrightarrow k \longrightarrow 0.$$

(The map $k \to V$ sends 1 to v_1 and the map $V \to k$ sends v_1 to 0 and v_2 to 1.) Find $\mathbf{f} \smile \mathbf{f}$ as a Yoneda composition. Find elements of $\mathrm{Hom}_{A^e}(A^e, V)$ corresponding to $\mathbf{f}$ and to $\mathbf{f} \smile \mathbf{f}$ (see Section A.3 for the correspondence between generalized extensions and elements of Hom spaces).

Exercise 2.4.5. Verify the claim that the total complex of the tensor product of the m-extension $\mathbf{f}$ and the n-extension $\mathbf{g}$ in this section indeed results in an $(m+n)$-extension, by applying the Künneth Theorem (Theorem A.5.1).

2.5. Actions of Hochschild cohomology

For any two (left) A-modules M and N, the Hochschild cohomology ring $\mathrm{HH}^*(A)$ acts on $\mathrm{Ext}^*_A(M,N)$ in such a way that $\mathrm{Ext}^*_A(M,N)$ is an $\mathrm{HH}^*(A)$-module. (There is also an action for any two right modules.) Similarly, for any A-bimodule B, Hochschild cohomology $\mathrm{HH}^*(A)$ acts on $\mathrm{HH}^*(A,B)$. We describe these actions next.

Let M, N be A-modules. Choose an A^e-projective resolution of A as an A^e-module,

$$\cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} A \to 0, \tag{2.5.1}$$

for example, we could take the bar resolution (1.1.4). Apply $-\otimes_A M$ and the isomorphism $A \otimes_A M \cong M$ to obtain the sequence:

$$\cdots \xrightarrow{d_2 \otimes 1_M} P_1 \otimes_A M \xrightarrow{d_1 \otimes 1_M} P_0 \otimes_A M \xrightarrow{\varepsilon \otimes 1_M} M \to 0. \tag{2.5.2}$$

Since each term P_i in the sequence (2.5.1) is projective as an A^e-module, and therefore is a direct summand of a free A^e-module, each $P_i \otimes_A M$ is projective as a left A-module, where the action is on the left tensor factor only. (It suffices to prove that $(A \otimes A^{\mathrm{op}}) \otimes_A M$ is projective as a left A-module, and this is immediate from the isomorphism $(A \otimes A^{\mathrm{op}}) \otimes_A M \cong A \otimes M$.) As tensor product is right exact, the map $\varepsilon \otimes 1_M$ is surjective. Each term in the sequence (2.5.1) is projective as a right A-module, so all P_i and $\mathrm{Im}(d_i)$ are projective as right A-modules, and consequently the sequence (2.5.2) is exact. (One sees this for example in Theorem A.4.6, the first long exact sequence for Tor, applied to the short exact sequences $0 \to \mathrm{Im}(d_{i+1}) \to P_i \to \mathrm{Im}(d_i) \to 0$ of which sequence (2.5.1) may be built.) Therefore the sequence (2.5.2) is a projective resolution of M as a (left) A-module.

Let $f \in \mathrm{Hom}_{A^e}(P_i, A)$ represent an element of $\mathrm{HH}^i(A)$. Let

$$\phi_M(f) = f \otimes_A 1_M \tag{2.5.3}$$

in $\mathrm{Hom}_A(P_i \otimes_A M, M)$, representing an element of $\mathrm{Ext}^i_A(M,M)$. By the Comparison Theorem (Theorem A.2.7), in light of the isomorphism of A-modules $A \otimes_A M \xrightarrow{\sim} M$, one may lift $\phi_M(f)$ to a chain map, that is to maps $\phi_M(f)_l : P_{i+l} \otimes_A M \to P_l \otimes_A M$ for all $l \geq 0$:

$$\begin{tikzcd}
P_{i+l} \otimes_A M \arrow[r] \arrow[d, "\phi_M(f)_l"] & \cdots \arrow[r] & P_{i+1} \otimes_A M \arrow[r] \arrow[d, "\phi_M(f)_1"] & P_i \otimes_A M \arrow[d, "\phi_M(f)_0"] \arrow[dr, "\phi_M(f)"] & \\
P_l \otimes_A M \arrow[r] & \cdots \arrow[r] & P_1 \otimes_A M \arrow[r] & P_0 \otimes_A M \arrow[r] & M
\end{tikzcd}$$

Compose with any function $g \in \mathrm{Hom}_A(P_j \otimes_A M, N)$ to obtain the element $g\phi_M(f)_j$ of $\mathrm{Hom}_A(P_{i+j} \otimes_A M, N)$. One may check that this induces a well-defined map

$$\mathrm{Ext}^j_A(M,N) \otimes \mathrm{HH}^i(A) \to \mathrm{Ext}^{i+j}_A(M,N), \tag{2.5.4}$$

that is a right module action of $\mathrm{HH}^*(A)$ on $\mathrm{Ext}^*_A(M,N)$. Similarly, there is a left action of $\mathrm{HH}^*(A)$ on $\mathrm{Ext}^*_A(M,N)$: First apply $-\otimes_A N$ to the resolution $P_{\bullet}$ to obtain a map ϕ_N from $\mathrm{HH}^*(A)$ to $\mathrm{Ext}^*_A(N,N)$. Then compose chain maps corresponding to elements of $\mathrm{Ext}^*_A(N,N)$ and $\mathrm{Ext}^*_A(M,N)$. The right and left actions are the same up to a sign, as the next theorem shows. The theorem is a special case of [**SS04**, Theorem 1.1].

Theorem 2.5.5. *Let $\alpha \in \mathrm{HH}^i(A)$ and $\beta \in \mathrm{Ext}^j_A(M,N)$. Then*

$$\alpha \cdot \beta = (-1)^{ij} \beta \cdot \alpha.$$

That is, the left and right actions of $\mathrm{HH}^*(A)$ on $\mathrm{Ext}^*_A(M,N)$ agree up to a sign.

Proof. The element α may be represented by an i-extension of A by A:

$$0 \to A \xrightarrow{\iota} E \to P_{i-2} \to \cdots \to P_1 \to P_0 \to A \to 0 \tag{2.5.6}$$

for an A^e-module E and projective A^e-modules P_i. (The correspondence between i-extensions and cochains described in Section A.3 indicates that we may assume $P_1, \ldots, P_{i-2}$ are projective.) We may assume that E and all P_i are projective as right A-modules since this is an extension of A by A, both projective as right A-modules. Suppose $\beta \in \mathrm{Ext}^0_A(M,N) \cong \mathrm{Hom}_A(M,N)$. If $\alpha \in \mathrm{HH}^0(A) \cong Z(A)$, the action is by multiplication, and so the equation claimed in the theorem statement holds. If $\alpha \in \mathrm{HH}^1(A)$, then the 1-extension (2.5.6) is simply $0 \to A \xrightarrow{\iota} E \xrightarrow{\varepsilon} A \to 0$. We let

$$X = (N \oplus (E \otimes_A M))/\{(a\beta(m),\ -\iota(a) \otimes m) \mid a \in A,\ m \in M\}.$$

Consider the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & A \otimes_A M \arrow[r, "\iota\otimes 1"] \arrow[d, "1\otimes\beta"] & E \otimes_A M \arrow[r, "\varepsilon\otimes 1"] \arrow[d, "\binom{0}{1}"] & A \otimes_A M \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & A \otimes_A N \arrow[r, "\binom{1}{0}"] \arrow[d, "="] & X \arrow[r, "(0,\varepsilon\otimes 1)"] \arrow[d, "(\iota\otimes 1, 1\otimes\beta)"] & A \otimes_A M \arrow[r] \arrow[d, "1\otimes\beta"] & 0 \\
0 \arrow[r] & A \otimes_A N \arrow[r, "\iota\otimes 1"] & E \otimes_A N \arrow[r, "\varepsilon\otimes 1"] & A \otimes_A N \arrow[r] & 0
\end{tikzcd}$$

Note that X is the pushout (as defined in Section A.1) of the upper left corner, and that the diagram commutes. Therefore $\alpha \cdot \beta = \beta \cdot \alpha$. A similar argument applies for any $\alpha \in \mathrm{HH}^i(A)$.

Next suppose $\beta \in \mathrm{Ext}^1_A(M, N)$ and that $0 \to N \to X \to M \to 0$ is a corresponding 1-extension. For each l, consider the short exact sequence

$$0 \to K_l \to P_{l-1} \to K_{l-1} \to 0 \tag{2.5.7}$$

where K_l, K_{l-1} are the lth and $(l-1)$st syzygies in the sequence (2.5.6). (Take $K_0 = A$.) We may assume that each K_l is projective as a right A-module, since (2.5.6) consists entirely of modules that are projective as right A-modules. Tensor this sequence with $0 \to N \to X \to M \to 0$ over A to obtain the following commuting diagram with exact rows and columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & K_l \otimes_A N \arrow[r] \arrow[d] & P_{l-1} \otimes_A N \arrow[r] \arrow[d] & K_{l-1} \otimes_A N \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K_l \otimes_A X \arrow[r] \arrow[d] & P_{l-1} \otimes_A X \arrow[r] \arrow[d] & K_{l-1} \otimes_A X \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K_l \otimes_A M \arrow[r] \arrow[d] & P_{l-1} \otimes_A M \arrow[r] \arrow[d] & K_{l-1} \otimes_A M \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Then by [**Lan95**, Lemma VIII.3.1], the 2-extensions obtained from the edges by composing maps across the upper right and lower left corners, namely

$$0 \to K_l \otimes_A N \longrightarrow P_{l-1} \otimes_A N \longrightarrow K_{l-1} \otimes_A X \longrightarrow K_{l-1} \otimes_A M \to 0,$$

$$0 \to K_l \otimes_A N \longrightarrow K_l \otimes_A X \longrightarrow P_{l-1} \otimes_A M \longrightarrow K_{l-1} \otimes_A M \to 0,$$

are equivalent up to a sign. Considering α to correspond to the Yoneda splice of all the short exact sequences (2.5.7), we find by induction on i that $\alpha \cdot \beta = (-1)^{ij}\beta \cdot \alpha$.

Finally, if $\beta \in \mathrm{Ext}^j_A(M, N)$ for $j > 1$, view β as a Yoneda splice of j short exact sequences. By induction on j, we have $\alpha \cdot \beta = (-1)^{ij}\beta \cdot \alpha$. □

Note that $\mathrm{Ext}^*_A(M, M)$ is itself an associative algebra with product given by Yoneda product (composition of chain maps analogous to that in Section 2.2) or equivalently Yoneda composition (splice of generalized extensions analogous to that in Section 2.4). It is typically not graded commutative, however its graded center, defined as follows, is important in relation to Hochschild cohomology.

Definition 2.5.8. Let B be a graded algebra. The *graded center* of B is the subalgebra $Z_{\mathrm{gr}}(B)$ generated by all homogeneous elements $\alpha \in B$ such that $\alpha \cdot \beta = (-1)^{|\alpha||\beta|}\beta \cdot \alpha$ for all homogeneous elements $\beta \in B$.

We may now state a corollary of Theorem 2.5.5, which follows immediately (taking $N = M$):

Corollary 2.5.9. *Let M be an A-module. The map*

$$\phi_M : \mathrm{HH}^*(A) \to \mathrm{Ext}^*_A(M, M),$$

defined at the chain level by $\phi_M(f) = f \otimes 1_M$ as in (2.5.3), is a ring homomorphism whose image is contained in the graded center $Z_{\mathrm{gr}}(\mathrm{Ext}^*_A(M, M))$.

For some algebras and modules, there are general results stating precisely the image of $\mathrm{HH}^*(A)$ in $\mathrm{Ext}^*_A(M, M)$. For Koszul algebras and a canonical choice of M, the map is surjective. See [**BGMS05**, Theorem 4.1] for details; some discussion is in Section 3.4. For some more general algebras defined by quivers and relations, the image of $\mathrm{HH}^*(A)$ in $\mathrm{Ext}^*_A(M, M)$, for a canonical choice of M, is the A_∞-center, defined in Section 7.4. (Some discussion of this phenomenon is in Section 7.4, and details are in [**BG**].)

We give an example to illustrate Corollary 2.5.9 next.

Example 2.5.10. Let $A = k[x]/(x^n)$ as in Examples 1.1.19, 2.2.2, and 2.2.3. Let $M = k$, an A-module via the augmentation map $\varepsilon : A \to k$ given by $\varepsilon(x) = 0$. The following is a free resolution of k as an A-module:

$$\cdots \longrightarrow A \xrightarrow{x^{n-1}\cdot} A \xrightarrow{x\cdot} A \xrightarrow{x^{n-1}\cdot} A \xrightarrow{x\cdot} A \xrightarrow{\varepsilon} k \longrightarrow 0.$$

Applying $\mathrm{Hom}_A(-, k)$, dropping the term $\mathrm{Hom}_A(k, k)$, and identifying the space $\mathrm{Hom}_A(A, k)$ with k, we have:

$$\cdots \xleftarrow{0} k \xleftarrow{0} k \xleftarrow{0} k \xleftarrow{0} k \xleftarrow{0} k \longleftarrow 0$$

The maps are indeed all zero maps since $\varepsilon(x) = 0$ and $\varepsilon(x^{n-1}) = 0$. Therefore $\mathrm{Ext}^i_A(k,k) \cong k$ for all i. Under a Yoneda/cup product defined in an analogous way to that for Hochschild cohomology in Section 2.2, similar calculations to those in Examples 2.2.2 and 2.2.3 show that

$$\mathrm{Ext}^*_A(k,k) \cong \begin{cases} k[y], & \text{if } n = 2, \\ k[y,z]/(y^2), & \text{otherwise}, \end{cases}$$

where $|y| = 1$ and $|z| = 2$, independent of the characteristic of k. We will determine the image of $\mathrm{HH}^*(A)$ in $\mathrm{Ext}^*_A(k,k)$ under the map ϕ_k defined by (2.5.3).

First assume that n is not divisible by $p = \mathrm{char}(k)$ and note that by the definition of g in Example 2.2.2, $\phi_k(g) = 0$. If f is the function defined in Example 2.2.2 and $n = 2$, then $\phi_k(f) = y^2$, and if $n > 2$ then $\phi_k(f) = z$. Thus the image of $\mathrm{HH}^*(A)$ under ϕ_k is the subalgebra of even degree elements. Next assume that p divides n as in Example 2.2.3. If $n = 2$, then $\phi_k(g) = y$ and $\phi_k(f) = y^2$. If $n > 2$, then $\phi_k(g) = y$ and $\phi_k(f) = z$. Thus ϕ_k maps $\mathrm{HH}^*(A)$ surjectively onto $\mathrm{Ext}^*_A(k,k)$ in this case. Note that in case $n = 2$, independently of the characteristic p of k, the image of $\mathrm{HH}^*(A)$ under ϕ_k is the full graded center of $\mathrm{Ext}^*_A(k,k)$ as in Definition 2.5.8 (in case $p \neq 2$, this is the subspace of even degree elements due to signs involved in multiplying by odd degree elements). In case $n > 2$ and n is not divisible by $\mathrm{char}(k)$, the image of $\mathrm{HH}^*(A)$ under ϕ_k is not the full graded center. We will return to this example in Section 7.4. (See Theorem 7.4.4 and Example 7.4.5.)

If B is an A-bimodule, then $\mathrm{HH}^*(A)$ acts on $\mathrm{HH}^*(A,B)$ similarly to how it acts on $\mathrm{Ext}^*_A(M,N)$ as described earlier in this section: Take a Yoneda product, or first tensor with B and then take a Yoneda product. Again these two actions agree up to a sign by a proof similar to that of Theorem 2.5.5. See [**SS04**, Theorem 1.1] for a more general setting that includes as special cases both this statement and Theorem 2.5.5 (by taking one of the two rings to be the field in [**SS04**]).

Exercise 2.5.11. Verify that formula (2.5.3) indeed leads to a well-defined map (2.5.4).

Exercise 2.5.12. Verify that (2.5.4) indeed defines a right module action of $\mathrm{HH}^*(A)$ on $\mathrm{Ext}^*_A(M,N)$.

Exercise 2.5.13. Let $A = k[x]$. Find $\mathrm{Ext}^*_A(k,k)$ and the image of $\mathrm{HH}^*(A)$ under the map ϕ_k defined by (2.5.3).

Exercise 2.5.14. Let M be an A-module. Verify the claims made about a product on $\mathrm{Ext}^*_A(M,M)$:

(a) Define a Yoneda product on cochains, analogous to the definition in Section 2.2, and show that it induces a well-defined product on $\mathrm{Ext}^*_A(M, M)$.

(b) Define a Yoneda composition on generalized extensions, analogous to the definition in Section 2.4, and show that it induces a well-defined product on $\mathrm{Ext}^*_A(M, M)$.

(c) Show that the products defined in (a) and (b) are equivalent.

Chapter 3

Examples

In this chapter, we look at specific types of resolutions, designed for particular algebras, that aid both theoretical understanding and computation of Hochschild (co)homology rings. These resolutions in turn will provide a rich assortment of examples on which to draw in later chapters. Specifically, we work with tensor products and twisted tensor products of algebras, Koszul algebras, smooth commutative noetherian algebras, monomial algebras, and skew group algebras. These are some important classes both of commutative and of noncommutative algebras.

We take k to be a field and A an algebra over k.

3.1. Tensor product of algebras

Let A and B be k-algebras. Their *tensor product algebra* is $A\otimes B$ as a vector space, with multiplication determined by

$$(a \otimes b) \cdot (a' \otimes b') = aa' \otimes bb'$$

for all $a, a' \in A$ and $b, b' \in B$, extended linearly to all elements in $A \otimes B$. If A and B are $\mathbb{Z}$-graded algebras, their *graded tensor product algebra* is $A \otimes B$ as a vector space, with multiplication determined by

$$(a \otimes b) \cdot (a' \otimes b') = (-1)^{|a'||b|} aa' \otimes bb'$$

for all homogeneous $a, a' \in A$ and $b, b' \in B$, where $|a'|$ denotes the degree of a' in $\mathbb{Z}$, and similarly $|b|$. (We have used the same notation for homological degree, but this should cause no confusion, as it will be clear in each context which degree is meant. In Theorem 3.1.2 below, they are the same, since Hochschild cohomology is graded by homological degree.)

Under some finiteness conditions, the Hochschild cohomology ring of the tensor product of two algebras is the graded tensor product of their Hochschild cohomology rings, as we will see in Theorem 3.1.2 below. This will allow us to understand the Hochschild cohomology rings of algebras that are tensor products of others. First we construct a needed resolution.

Let $P_{\bullet}$ be a projective A^e-resolution of A, and $Q_{\bullet}$ a projective B^e-resolution of B. Taking their tensor product over k, consider the bicomplex

$$P_{\bullet} \otimes Q_{\bullet} \tag{3.1.1}$$

and its corresponding total complex $\mathrm{Tot}(P_{\bullet} \otimes Q_{\bullet})$ which we sometimes denote more simply by $P_{\bullet} \otimes Q_{\bullet}$ when no confusion will arise. We claim that for each i, j, the $(A \otimes B)^e$-module $P_i \otimes Q_j$ is projective. To see this, note that since P_i is a projective A^e-module, there is an A^e-module P_i' such that $P_i \oplus P_i' \cong (A^e)^{\oplus I}$ for some indexing set I. Similarly, there is a B^e-module Q_j' such that $Q_j \oplus Q_j' \cong (B^e)^{\oplus J}$ for some indexing set J. Since tensor product distributes over direct sum, $P_i \otimes Q_j$ is a direct summand of

$$(A^e)^{\oplus I} \otimes (B^e)^{\oplus J} \cong (A^e \otimes B^e)^{\oplus (I \times J)} \cong ((A \otimes B)^e)^{\oplus (I \times J)}$$

as vector spaces. The module actions of A and B commute, and so $P_i \otimes Q_j$ is in fact a direct summand of $((A \otimes B)^e)^{\oplus (I \times J)}$ as an $(A \otimes B)^e$-module. Thus $P_i \otimes Q_j$ is indeed a projective $(A \otimes B)^e$-module. By the Künneth Theorem (Theorem A.5.1), since the tensor product is over the field k, the total complex of $P_{\bullet} \otimes Q_{\bullet}$ has homology concentrated in degree 0, where it is $\mathrm{H}_0(P_{\bullet} \otimes Q_{\bullet}) \cong \mathrm{H}_0(P_{\bullet}) \otimes \mathrm{H}_0(Q_{\bullet}) \cong A \otimes B$. Finally we note that the differentials on the tensor product complex $P_{\bullet} \otimes Q_{\bullet}$ are in fact $(A \otimes B)^e$-module homomorphisms by their definitions. Thus $P_{\bullet} \otimes Q_{\bullet}$ is a projective resolution of $A \otimes B$ as an $(A \otimes B)^e$-module.

Theorem 3.1.2. *Let A and B be finite dimensional k-algebras. Then*

$$\mathrm{HH}^*(A \otimes B) \cong \mathrm{HH}^*(A) \otimes \mathrm{HH}^*(B)$$

as algebras, where the right side is a graded tensor product algebra.

If A and B are themselves $\mathbb{N}$-graded algebras, we may weaken the hypothesis in the theorem to assume instead that they are locally finite dimensional, that is, each graded component is finite dimensional. The isomorphism in the theorem is in fact an isomorphism of Gerstenhaber algebras; see [**LZ14**] for the definition of Gerstenhaber bracket on a graded tensor product of Gerstenhaber algebras, and a proof of this more general statement.

Proof of Theorem 3.1.2. Let $P_{\bullet} \otimes Q_{\bullet}$ be the bicomplex (3.1.1) defined above. The Hochschild cohomology space $\mathrm{HH}^*(A \otimes B)$ is the cohomology of the total complex of the bicomplex $\mathrm{Hom}_{(A \otimes B)^e}(P_{\bullet} \otimes Q_{\bullet}, A \otimes B)$. Without loss of generality we may assume that $P_\bullet$ and $Q_\bullet$ are free resolutions, so that $P_i \otimes Q_j \cong (A^e)^{\oplus I} \otimes (B^e)^{\oplus J} \cong ((A \otimes B)^e)^{\oplus (I \times J)}$, for some finite indexing sets I, J. Thus there are finite dimensional vector spaces P_i', Q_j' such that $P_i \cong A \otimes P_i' \otimes A$, $Q_j \cong B \otimes Q_j' \otimes B$, and so

$$\mathrm{Hom}_{(A\otimes B)^e}(P_i \otimes Q_j, A \otimes B) \cong \mathrm{Hom}_k(P_i' \otimes Q_j', A \otimes B).$$

Consider embedding $\mathrm{Hom}_k(P_i', A) \otimes \mathrm{Hom}_k(Q_j', B) \hookrightarrow \mathrm{Hom}_k(P_i' \otimes Q_j', A \otimes B)$ via tensor product of functions. Since P_i', Q_j', A, and B are finite dimensional vector spaces, this is an isomorphism, and moreover the differentials correspond under this isomorphism. By the Künneth Theorem (Theorem A.5.1), since the tensor product is taken over the field k, the cohomology is $\mathrm{HH}^*(A) \otimes \mathrm{HH}^*(B)$ as a vector space.

Next we determine the algebra structure, using definition (2.3.1) of product. Define a diagonal map $\Delta : P_\bullet \otimes Q_\bullet \to (P_\bullet \otimes Q_\bullet) \otimes_{A\otimes B} (P_\bullet \otimes Q_\bullet)$ by

$$P_\bullet \otimes Q_\bullet \xrightarrow{\Delta_A \otimes \Delta_B} (P_\bullet \otimes_A P_\bullet) \otimes (Q_\bullet \otimes_B Q_\bullet) \xrightarrow{\sim} (P_\bullet \otimes Q_\bullet) \otimes_{A\otimes B} (P_\bullet \otimes Q_\bullet),$$

where Δ_A, Δ_B are diagonal maps for $P_\bullet$, $Q_\bullet$, respectively. The second map is the isomorphism given by interchanging factors, with a sign, that is,

$$(x \otimes_A x') \otimes (y \otimes_B y') \mapsto (-1)^{|x'||y|}(x \otimes y) \otimes_{A\otimes B} (x' \otimes y')$$

for all homogeneous $x, x' \in P_\bullet$ and $y, y' \in Q_\bullet$. Then Δ is a chain map by the definition of the differential of a tensor product of complexes. Now let f, g be homogeneous elements of degrees m, n in $\mathrm{Hom}_{A^e}(P_\bullet, A)$ and f', g' be homogeneous elements of degrees m', n' in $\mathrm{Hom}_{B^e}(Q_\bullet, B)$. Then $f \otimes f'$ is an element in

$$\begin{aligned}
\mathrm{Hom}_{A^e}(P_m, A) \otimes \mathrm{Hom}_{B^e}(Q_{m'}, B) &\cong \mathrm{Hom}_k(P_m', A) \otimes \mathrm{Hom}_k(Q_{m'}', B)\\
&\cong \mathrm{Hom}_k(P_m' \otimes Q_{m'}', A \otimes B)\\
&\cong \mathrm{Hom}_{(A\otimes B)^e}(P_m \otimes Q_{m'}, A \otimes B),
\end{aligned}$$

and $g \otimes g'$ may be identified with an element in $\mathrm{Hom}_{(A\otimes B)^e}(P_n \otimes Q_{n'}, A \otimes B)$. We determine the cup product $(f \otimes f') \smile (g \otimes g')$ by using the diagonal map Δ and formula (2.3.1): Let $x \in P_r$, $y \in Q_s$ for some r, s with $r + s = m + n + m' + n'$, and suppose $\Delta_A(x)$ has component in $P_m \otimes P_n$ given by $\sum_s x_s' \otimes x_s''$ and $\Delta_B(y)$ has component in $Q_{m'} \otimes Q_{n'}$ given by $\sum_t y_t' \otimes y_t''$. (This is the only component on which the cup product potentially takes a nonzero value.) Letting π_A, π_B, and $\pi_{A\otimes B}$ denote multiplication on A, B, and $A \otimes B$, respectively,

$$\begin{aligned}
&((f \otimes f') \smile (g \otimes g'))(x \otimes y) \\
&\quad = \pi_{A\otimes B}(f \otimes f' \otimes g \otimes g')(\Delta(x \otimes y)) \\
&\quad = \pi_{A\otimes B}\left(\sum_{s,t}(-1)^{|x''_s||y'_t|} f(x'_s) \otimes f'(y'_t) \otimes g(x''_s) \otimes g'(y''_t)\right) \\
&\quad = (-1)^{m'n}(\sum_s f(x'_s)g(x''_s)) \otimes (\sum_t f'(y'_t)g'(y''_t)) \\
&\quad = (-1)^{m'n}\pi_A\big((f \otimes g)\Delta_A(x)\big) \otimes \pi_B\big((f' \otimes g')\Delta_B(y)\big) \\
&\quad = (-1)^{m'n}((f \smile g)(x)) \otimes ((f' \smile g')(y)),
\end{aligned}$$

and so $(f \otimes f') \smile (g \otimes g') = (-1)^{m'n}(f \smile g) \otimes (f' \smile g')$. □

The theorem allows us to understand Hochschild cohomology for many more algebras, as the following examples show.

Example 3.1.3. Let $A_1 = k[x_1]$, $A_2 = k[x_2]$, and $A = A_1 \otimes A_2$. Then $A \cong k[x_1, x_2]$, a polynomial ring in two indeterminates. In Example 1.3.5, we found that $\mathrm{HH}^*(A_i) \cong k[x_i, y_i]/(y_i^2)$ as a k-algebra, where $|x_i| = 0$, $|y_i| = 1$, for $i = 1, 2$. By Theorem 3.1.2, $\mathrm{HH}^*(k[x_1, x_2])$ is the graded tensor product $\mathrm{HH}^*(A_1) \otimes \mathrm{HH}^*(A_2)$. We thus see that

$$\mathrm{HH}^*(k[x_1, x_2]) \cong k[x_1, x_2] \otimes \textstyle\bigwedge(y_1, y_2)$$

as a graded algebra, where by $\bigwedge(y_1, y_2)$ we mean the exterior algebra on a vector space W with basis y_1, y_2 in degree 1 (and x_1, x_2 have degree 0).

More generally, by induction on the number m of indeterminates,

$$\mathrm{HH}^*(k[x_1, \dots, x_m]) \cong k[x_1, \dots, x_m] \otimes \textstyle\bigwedge(y_1, \dots, y_m)$$

as a graded algebra. We give another way to see this structure on Hochschild cohomology next: Let V be the vector space with basis $x_1, \dots, x_m$, and let W be the vector space with basis $y_1, \dots, y_m$. Identify W with the dual space V^* of V. We may view the Hochschild cohomology of $A = k[x_1, \dots, x_m]$ as $A \otimes \bigwedge(V^*)$. Equivalently, this view arises in a natural way in the tensor product approach of Section 2.3, as we will explain next.

The A^e-projective resolution of $A = k[x_1, \dots, x_m]$ obtained by a tensor product of m copies of resolution (1.1.17), one for each x_i, may be described as follows. As a graded vector space, let

$$P_\bullet = A \otimes \textstyle\bigwedge^\bullet(V) \otimes A. \qquad (3.1.4)$$

As a free A^e-module, we may canonically identify each P_n with the degree n component of the tensor product of m copies of resolution (1.1.17), one for each of $x_1, \dots, x_m$. We identify $1 \otimes x_{i_1} \wedge \cdots \wedge x_{i_n} \otimes 1$ in P_n with $(1 \otimes 1)^{\otimes m}$

in the tensor product complex, where the tensor factor $1 \otimes 1$ is taken to be in degree 1 when it is in any of the positions $i_1, \ldots, i_n$, and is in degree 0 otherwise. Under this identification, the differential on $P_{\bullet}$ is given by

$$d_n(1 \otimes x_{i_1} \wedge \cdots \wedge x_{i_n} \otimes 1) = \sum_{j=1}^{n} (-1)^{j-1} \big(x_{i_j} \otimes x_{i_1} \wedge \cdots \wedge \hat{x}_{i_j} \wedge \cdots \wedge x_{i_n} \otimes 1 \\ - 1 \otimes x_{i_1} \wedge \cdots \wedge \hat{x}_{i_j} \wedge \cdots \wedge x_{i_n} \otimes x_{i_j} \big),$$

where the notation $\hat{x}_{i_j}$ indicates a missing exterior factor. This extends linearly to a formula for all n-tuples of vectors:

$$d_n(1 \otimes v_1 \wedge \cdots \wedge v_n \otimes 1) = \sum_{j=1}^{n} (-1)^{j-1} \big(v_j \otimes v_1 \wedge \cdots \wedge \hat{v}_j \wedge \cdots \wedge v_n \otimes 1 \\ - 1 \otimes v_1 \wedge \cdots \wedge \hat{v}_j \wedge \cdots \wedge v_n \otimes v_j \big)$$

for all $v_1, \ldots, v_n \in V$. Applying $\mathrm{Hom}_{A^e}(-, A)$, since A is commutative, we see that the induced differentials are all 0, and so the Hochschild cohomology space in each degree n is

$$\begin{aligned} \mathrm{Hom}_{A^e}(A \otimes \textstyle\bigwedge^n(V) \otimes A, A) &\cong \mathrm{Hom}_k(\textstyle\bigwedge^n(V), A) \\ &\cong A \otimes (\textstyle\bigwedge^n(V))^* \cong A \otimes \textstyle\bigwedge(V^*), \end{aligned}$$

since V is finite dimensional. This is another way in which Hochschild cohomology $\mathrm{HH}^*(A)$ can be identified with $A \otimes \bigwedge(V^*)$, as stated above. There is a diagonal map $\Delta : P_{\bullet} \to P_{\bullet} \otimes_A P_{\bullet}$ arising from an embedding of $P_{\bullet}$ into the bar resolution $B(A)$ that will be given by (3.4.8). The diagonal map on $P_{\bullet}$ is the composition of this embedding with the diagonal map on $B(A)$ given by formula (2.3.2). This and definition (2.3.1) of cup product gives another proof that the product on Hochschild cohomology is indeed that of the tensor product $A \otimes \bigwedge(V^*)$ of the two algebras A and $\bigwedge(V^*)$.

Our next examples are truncated polynomial rings.

Example 3.1.5. Let $n_1, n_2 \geq 2$. Let $A_1 = k[x_1]/(x_1^{n_1})$, $A_2 = k[x_2]/(x_2^{n_2})$, and $A = A_1 \otimes A_2$. Then $A \cong k[x_1, x_2]/(x_1^{n_1}, x_2^{n_2})$. In Example 2.2.2, we found that if n_i is not divisible by $\mathrm{char}(k)$, then

$$\mathrm{HH}^*(A_i) \cong k[x_i, y_i, z_i]/(x_i^{n_i},\ x_i^{n_i-1} y_i,\ y_i^2,\ x_i^{n_i-1} z_i)$$

where $|x_i| = 0$, $|y_i| = 1$, $|z_i| = 2$. If neither n_1 nor n_2 is divisible by $\mathrm{char}(k)$, then by Theorem 3.1.2,

$$\mathrm{HH}^*(k[x_1, x_2]/(x_1^{n_1}, x_2^{n_2})) \\ \cong k[x_1, x_2, z_1, z_2] \otimes \textstyle\bigwedge(y_1, y_2)/(x_i^{n_i} \otimes 1,\ x_i^{n_i-1} \otimes y_i,\ x_i^{n_i-1} z_i \otimes 1).$$

The cases where one or both of n_1, n_2 is divisible by $\mathrm{char}(k)$ may be treated similarly by applying results of Example 2.2.3. More generally, we see by induction on the number m of generators that if none of $n_1, \ldots, n_m$ is divisible by $\mathrm{char}(k)$, then

$$\begin{aligned}&\mathrm{HH}^*(k[x_1, \ldots, x_m]/(x_1^{n_1}, \ldots, x_m^{n_m}))\\ &\cong k[x_1, \ldots, x_m, z_1, \ldots, z_m] \otimes \textstyle\bigwedge(y_1, \ldots, y_m)/J\end{aligned}$$

where $J = (x_i^{n_i} \otimes 1,\ x_i^{n_i-1} \otimes y_i,\ x_i^{n_i-1} z_i \otimes 1)$ and $|x_i| = 0$, $|y_i| = 1$, $|z_i| = 2$. If one or more of $n_1, \ldots, n_m$ is divisible by $\mathrm{char}(k)$, then we obtain the Hochschild cohomology ring by applying Theorem 3.1.2 repeatedly, using the results of Example 2.2.3, for a similar expression.

It is useful to describe the resolution of $A = k[x_1, \ldots, x_m]/(x_1^{n_1}, \ldots, x_m^{n_m})$ obtained as a tensor product of complexes for the factors $A_i = k[x_i]/(x_i^{n_i})$: Let V be the vector space with basis $\xi_1, \ldots, \xi_m$ and let $S(V)$ be the symmetric algebra on the symbols $\xi_1, \ldots, \xi_m$ (polynomials in these indeterminates). Let

$$P_\bullet = A \otimes S(V) \otimes A,$$

where $|\xi_i| = 1$ for each i, that is, for each j, P_j is the free A^e-module on standard monomials (that is, monomials with scalar coefficient 1) in $S(V)$ of degree j, since these form a basis for the degree j subspace of $S(V)$ as a vector space over k. Then $P_\bullet$ may be identified with the total complex of the tensor product of m copies of resolution (1.1.20), one for each x_i: The monomial $1 \otimes \xi_1^{i_1} \cdots \xi_m^{i_m} \otimes 1$ is identified with the tensor product of m copies of $1 \otimes 1$, one in each $A_i \otimes A_i$, where the exponents $i_1, \ldots, i_m$ indicate the homological degree of the corresponding factor. We see then that the differential on $P_\bullet$ corresponding to the differential on the tensor product of complexes is given by $d = \sum_{j=1}^m d_j$ where

(3.1.6)

$$\begin{aligned}&d_j(1 \otimes \xi_1^{i_1} \cdots \xi_m^{i_m} \otimes 1)\\ &= (-1)^l \begin{cases} 0, & \text{if } i_j = 0,\\ u_j \cdot (1 \otimes \xi_1^{i_1} \cdots \xi_j^{i_j-1} \cdots \xi_m^{i_m} \otimes 1), & \text{if } i_j \text{ is odd},\\ v_j \cdot (1 \otimes \xi_1^{i_1} \cdots \xi_j^{i_j-1} \cdots \xi_m^{i_m} \otimes 1), & \text{if } i_j \text{ is even and } i_j > 0\end{cases}\end{aligned}$$

with $u_j = x_j \otimes 1 - 1 \otimes x_j$, $v_j = x_j^{n_j-1} \otimes 1 + x_j^{n_j-2} \otimes x_j + \cdots + 1 \otimes x_j^{n_j-1}$, and $l = i_1 + \cdots + i_{j-1}$. Applying $\mathrm{Hom}_{A^e}(-, A)$, we find that since A is commutative, the induced differentials in odd degrees are all 0 while in even degrees they are multiplication by $n_j x_j^{n_j-1}$ (which is 0 if and only if n_j is divisible by $\mathrm{char}(k)$). This is consistent with the graded vector space structure of $\mathrm{HH}^*(A)$ described above.

Exercise 3.1.7. Let V be a vector space of dimension m, and let $n \le m$. What is the vector space dimension of $\bigwedge^n(V)$?

Exercise 3.1.8. Verify that the formula for the differential in Example 3.1.3 is indeed that of the tensor product of m copies of resolution (1.1.17), and that on applying $\mathrm{Hom}_{A^e}(-, A)$, the induced differentials are indeed all 0.

Exercise 3.1.9. Verify that the formula for the differential in Example 3.1.5 is indeed that of the tensor product of m copies of resolution (1.1.20), and that on applying $\mathrm{Hom}_{A^e}(-, A)$, the induced differentials are as claimed.

Exercise 3.1.10. Let M be an A-bimodule and N a B-bimodule. Consider $M \otimes N$ to be an $A \otimes B$-bimodule with action defined by $(a \otimes b)(m \otimes n)(a' \otimes b') = ama' \otimes bnb'$ for all $a, a' \in A$, $b, b' \in B$, $m \in M$, $n \in N$.

(a) Use the techniques of Theorem 3.1.2 to prove that

$$\mathrm{HH}^*(A \otimes B, M \otimes N) \cong \mathrm{HH}^*(A, M) \otimes \mathrm{HH}^*(B, M)$$

as graded vector spaces. What finiteness hypotheses are required?

(b) Prove a homology version of Theorem 3.1.2, using the techniques from its proof:

$$\mathrm{HH}_*(A \otimes B, M \otimes N) \cong \mathrm{HH}_*(A, M) \otimes \mathrm{HH}_*(B, N)$$

as graded vector spaces. Does this isomorphism require any finiteness hypotheses?

Exercise 3.1.11. Use Exercise 3.1.10(b) to obtain an expression for the Hochschild homology space $\mathrm{HH}_*(k[x_1, \ldots, x_m])$ similar to that for Hochschild cohomology found in Example 3.1.3.

3.2. Koszul complexes and the HKR Theorem

In this section we define Koszul complexes associated to regular sequences of central elements in an algebra. Koszul complexes are used to prove the Hochschild-Kostant-Rosenberg (HKR) Theorem that describes Hochschild homology and cohomology rings of smooth commutative algebras.

Definition 3.2.1. Let x be a central element of the algebra A. The *Koszul complex* associated to x is

$$K(x): \qquad 0 \longrightarrow A \xrightarrow{x\cdot} A \longrightarrow 0,$$

concentrated in degrees 1 and 0. More generally, let $\mathbf{x} = (x_1, \ldots, x_n)$ be a sequence of central elements $x_1, \ldots, x_n$ in A. The *Koszul complex* associated to $\mathbf{x}$ is the total complex

$$K(\mathbf{x}): \qquad \mathrm{Tot}(K(x_1) \otimes_A \cdots \otimes_A K(x_n)). \tag{3.2.2}$$

Note that if x is not a zero divisor of A, then by the definition of the Koszul complex $K(x)$ above, $\mathrm{H}_1(K(x)) = 0$ and $\mathrm{H}_0(K(x)) \cong A/(x)$. Consequently $K(x)$ is a (free) resolution of the A-module $A/(x)$. This observation generalizes to a statement about the Koszul complex $K(\mathbf{x})$ provided $\mathbf{x}$ is a regular sequence, as we define next. Given an A-module M, we say that a nonzero element $x \in A$ is a *zero divisor* of M if $xm = 0$ for some $m \neq 0$.

Definition 3.2.3. A sequence $(x_1, \ldots, x_n)$ of central elements in an algebra A is a *regular sequence* if x_1 is not a zero divisor of A and for each i, x_i is not a zero divisor of the A-module $A/(x_1, \ldots, x_{i-1})$, where here the notation $(x_1, \ldots, x_{i-1})$ refers to the ideal generated by these elements.

Theorem 3.2.4. *If $\mathbf{x} = (x_1, \ldots, x_n)$ is a regular sequence in A, then the Koszul complex $K(\mathbf{x})$ is a free resolution of the A-module $A/(x_1, \ldots, x_n)$.*

Proof. We will use induction on n. Suppose $n = 1$. Since x_1 is not a zero divisor of A, augmenting the sequence $K(x_1)$ by the term $A/(x_1)$ in degree -1 yields an exact sequence. Clearly the terms in nonnegative degrees are free as (left) A-modules. Thus the statement holds when $n = 1$.

Note that by the definition, $(x_1, \ldots, x_{n-1})$ is a regular sequence if $\mathbf{x}$ is a regular sequence. Assume that $K(x_1, \ldots, x_{n-1})$ is a free resolution of the A-module $A/(x_1, \ldots, x_{n-1})$. Consider the short exact sequence of complexes $0 \to A \to K(x_n) \to A[-1] \to 0$ in which A is considered to be a complex concentrated in degree 0, that is:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & A \arrow[r, "1_A"] \arrow[d, "x_n \cdot"] & A \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r, "1_A"] \arrow[d] & A \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Take the tensor product, over A, with $K(x_1, \ldots, x_{n-1})$. There is a canonical isomorphism $K(x_1, \ldots, x_{n-1}) \otimes_A A \cong K(x_1, \ldots, x_{n-1})$, and so this results in a short exact sequence of complexes

$$0 \longrightarrow K(x_1, \ldots, x_{n-1}) \longrightarrow K(\mathbf{x}) \longrightarrow K(x_1, \ldots, x_{n-1})[-1] \longrightarrow 0.$$

By the induction hypothesis, the corresponding homology long exact sequence (see Theorem A.4.2) ends in

$$0 \longrightarrow \mathrm{H}_1(K(\mathbf{x})) \longrightarrow \mathrm{H}_1(K(x_1, \ldots, x_{n-1})[-1]) \xrightarrow{\partial_1} \mathrm{H}_0(K(x_1, \ldots, x_{n-1})) \longrightarrow \mathrm{H}_0(K(\mathbf{x})) \longrightarrow 0.$$

The two middle terms are simply $A/(x_1, \ldots, x_{n-1})$, again as a consequence of the induction hypothesis. A diagram chase shows that the connecting homomorphism ∂_1 is multiplication by x_n. Since x_n is not a zero divisor of $A/(x_1, \ldots, x_{n-1})$ by hypothesis, the sequence

$$0 \to A/(x_1, \ldots, x_{n-1}) \xrightarrow{x_n \cdot} A/(x_1, \ldots, x_{n-1}) \to A/(x_1, \ldots, x_n) \to 0$$

is exact. Comparing the two sequences, we conclude that $\mathrm{H}_1(K(\mathbf{x})) = 0$ and $\mathrm{H}_0(K(\mathbf{x})) \cong A/(x_1, \ldots, x_n)$. A look at the rest of the long exact sequence shows that by the induction hypothesis, $\mathrm{H}_i(K(\mathbf{x})) = 0$ for all $i \geq 1$. Finally, we note that the terms in $K(\mathbf{x})$ are all direct sums of (left) A-modules of the form $A \otimes_A A \otimes_A \cdots \otimes_A A \cong A$, and so are free. $\square$

See [**Wei94**, Section 4.5] for a more general version of the above theorem.

Example 3.2.5. Let $A = k[x_1, \ldots, x_n]$. Then

$$(x_1 \otimes 1 - 1 \otimes x_1, \ldots, x_n \otimes 1 - 1 \otimes x_n)$$

is a regular sequence of A^e. The associated Koszul complex

$$K(x_1 \otimes 1 - 1 \otimes x_1, \ldots, x_n \otimes 1 - 1 \otimes x_n)$$

can be identified with resolution (3.1.4) by its definition, as a consequence of an isomorphism $A^e/(x_1 \otimes 1 - 1 \otimes x_1, \ldots, x_n \otimes 1 - 1 \otimes x_n) \cong A$. Similarly, $(x_1, \ldots, x_n)$ is a regular sequence of A and $K(x_1, \ldots, x_n)$ is a resolution of $A/(x_1, \ldots, x_n) \cong k$ as an A-module.

A classical theorem of Hochschild, Kostant, and Rosenberg [**HKR62**] gives the structure of Hochschild (co)homology of a smooth finitely generated commutative algebra. The theorem has more general versions than what we present here, see, for example, [**Wei94**, Theorem 9.4.7]. We can take as a definition of smooth algebra A one for which there is a finite projective resolution by finitely generated projective A^e-modules (see the general Definition 4.1.4 of smoothness for not necessarily commutative algebras). For the commutative rings of interest here, this is equivalent to many other definitions of smoothness, see, e.g., [**Lod98**, Proposition 3.4.2] and [**Van98**]. See also [**Mat86**]. In Chapter 4, we will explore smooth commutative and noncommutative algebras in some detail.

Recall from Section 1.2 that for a commutative algebra A, the Hochschild cohomology in degree 1 may be identified with the space of k-derivations: $\mathrm{HH}^1(A) \cong \mathrm{Der}(A)$. We will use here the standard notation $\Omega^1_{com}(A)$ for the Hochschild homology space in degree 1: $\Omega^1_{com}(A) = \mathrm{HH}_1(A)$. In Section 4.2 we will use this notation $\Omega^1_{com}(A)$ for the A-module of Kähler differentials defined there and show that it is indeed isomorphic to $\mathrm{HH}_1(A)$. Thus the following theorem implies that for a smooth finitely generated commutative algebra, Hochschild homology $\mathrm{HH}_*(A)$ and cohomology $\mathrm{HH}^*(A)$ are generated as algebras by their degree 1 components $\mathrm{HH}_1(A)$ and $\mathrm{HH}^1(A)$, respectively. The algebra structure on Hochschild homology is that given by the shuffle product (Section 1.5).

Theorem 3.2.6. *(Hochschild-Kostant-Rosenberg) Let A be a smooth finitely generated commutative algebra. There are isomorphisms of graded algebras,*

$$\mathrm{HH}^*(A) \cong \textstyle\bigwedge_A(\mathrm{Der}(A)) \quad \textit{and} \quad \mathrm{HH}_*(A) \cong \textstyle\bigwedge_A(\Omega^1_{com}(A)).$$

Proof. We will prove the cohomology isomorphism. The homology isomorphism is similar, see, e.g., [**Wei94**, Theorem 9.4.7]. There is a map

$$\psi : \textstyle\bigwedge_A(\mathrm{Der}(A)) \to \mathrm{HH}^*(A)$$

given by identifying $\bigwedge_A^0(\mathrm{Der}(A))$ with $A \cong \mathrm{HH}^0(A)$ and $\bigwedge_A^1(\mathrm{Der}(A))$ with $\mathrm{Der}(A) \cong \mathrm{HH}^1(A)$, and then extending to an algebra homomorphism. This is well-defined due to the graded commutativity of $\mathrm{HH}^*(A)$. Note that ψ is an isomorphism if and only if $\psi \otimes_A A_{\mathfrak{m}}$ is an isomorphism for every maximal ideal $\mathfrak{m}$ of A. (See Matsumura [**Mat86**] for needed properties of localization such as [**Mat86**, Theorem 4.6].)

Since Ext commutes with localization [**Wei94**, Proposition 3.3.10], there is an isomorphism $\mathrm{HH}^*(A_{\mathfrak{m}}) \cong \mathrm{HH}^*(A) \otimes_A A_{\mathfrak{m}}$ for any maximal ideal $\mathfrak{m}$ of A. We will show that $\mathrm{HH}^*(A_{\mathfrak{m}}) \cong \bigwedge_{A_{\mathfrak{m}}}(\mathrm{Der}(A_{\mathfrak{m}}))$ for every maximal ideal $\mathfrak{m}$ of A. The claimed isomorphism will follow. This will be the case once we have shown that ψ is an isomorphism whenever A is a local ring.

Now assume A is local with unique maximal ideal $\mathfrak{m}$. Let M be the inverse image of $\mathfrak{m}$ in A^e under the multiplication map π, so that M is a maximal ideal of A^e. Since A is smooth, A^e is a regular ring [**Wei94**, Proposition 9.4.6]. (See [**Mat86**] or [**Wei94**] for properties of regular rings.) Since A is local, $A \cong A_{\mathfrak{m}}$. Now $A \cong A_{\mathfrak{m}} \cong (A^e)_M/(\mathrm{Ker}\,\pi)_M$ (by [**Mat86**, Theorem 4.2]) and since $(A^e)_M$ is a regular local ring, $(\mathrm{Ker}\,\pi)_M$ is generated by a regular sequence $\mathbf{x}$ [**Wei94**, Exercise 4.4.2]. By Theorem 3.2.4, $K(\mathbf{x})$ is a free resolution of the $(A^e)_M$-module $A \cong A_{\mathfrak{m}}$. Restrict to A^e and apply $\mathrm{Hom}_{A^e}(-, A)$. The differentials are 0 since $\mathbf{x}$ is a regular sequence of elements in $(\mathrm{Ker}\,\pi)_M$, and upon taking homomorphisms to A, we multiply. Thus the cohomology is as claimed. The multiplicative structure is as stated by applying a diagonal map and definition (2.3.1) analogous to argument at the end of Example 3.1.3. □

Exercise 3.2.7. Verify the claims made in Example 3.2.5, that is, the Koszul complex associated to the regular sequence

$$(x_1 \otimes 1 - 1 \otimes x_1, \ldots, x_n \otimes 1 - 1 \otimes x_n)$$

is equivalent to resolution (3.1.4).

Exercise 3.2.8. Let $A = k[x_1, \ldots, x_m]$. Explain how Theorem 3.2.6 is consistent with our earlier calculation of $\mathrm{HH}^*(A)$ in Example 3.1.3. What is $\mathrm{HH}_*(A)$ (cf. Exercise 3.1.11)?

3.3. Twisted tensor product of algebras

Twisted tensor products generalize the tensor products and graded tensor products discussed in Section 3.1. Here we present the work of Bergh and Oppermann [**BO08**] on a twisted tensor product of resolutions when the twisting arises from a bicharacter on grading groups. More general versions of twisting algebras and resolutions are dealt with in the literature (see, e.g., [**ČSV95, GG99, LP07, SW**]).

Let A_1, A_2 be k-algebras that are graded by abelian groups Γ_1, Γ_2, respectively. That is, $A_1 = \oplus_{\gamma\in\Gamma_1}(A_1)_\gamma$, a direct sum of vector spaces, such that $(A_1)_\gamma(A_1)_{\gamma'} \subset (A_1)_{\gamma\gamma'}$ for all $\gamma, \gamma' \in \Gamma_1$, where Γ_1 is written multiplicatively, and similarly for A_2, Γ_2. Let $t : \Gamma_1 \times \Gamma_2 \to k^\times$ be a *bicharacter*, that is,

$$\begin{aligned} t(1_{\Gamma_1}, \gamma_2) = t(\gamma_1, 1_{\Gamma_2}) &= 1, \\ t(\gamma_1\gamma_1', \gamma_2) &= t(\gamma_1, \gamma_2)t(\gamma_1', \gamma_2), \\ t(\gamma_1, \gamma_2\gamma_2') &= t(\gamma_1, \gamma_2)t(\gamma_1, \gamma_2') \end{aligned}$$

for all $\gamma_i, \gamma_i' \in \Gamma_i$, in which 1_{Γ_i} denotes the identity element of Γ_i $(i = 1, 2)$. The *twisted tensor product algebra*

$$A = A_1 \otimes^t A_2$$

is $A_1 \otimes A_2$ as a vector space, and multiplication is determined by

$$(a_1 \otimes a_2) \cdot (a_1' \otimes a_2') = t(|a_1'|, |a_2|) a_1 a_1' \otimes a_2 a_2'$$

for all homogeneous elements $a_1, a_1' \in A_1$ and $a_2, a_2' \in A_2$, where $|a_1'|$ denotes the degree of a_1' in Γ_1 and similarly $|a_2|$ in Γ_2. By its definition, $A_1 \otimes^t A_2$ is graded by the group $\Gamma_1 \times \Gamma_2$: $(A_1 \otimes^t A_2)_{(\gamma_1,\gamma_2)} = (A_1)_{\gamma_1} \otimes (A_2)_{\gamma_2}$ for all $\gamma_1 \in \Gamma_1$, $\gamma_2 \in \Gamma_2$.

Our first example is the quantum plane.

Example 3.3.1. Let $A_1 = k[x_1]$ and $A_2 = k[x_2]$, each graded by $\mathbb{Z}$ in a standard way: Write $\Gamma = \langle\gamma\rangle$, a free abelian group on one generator γ, written multiplicatively, so that $\Gamma \cong \mathbb{Z}$ as a group. Let $|x_1| = |x_2| = \gamma$. Let q be any nonzero scalar. Define $t : \mathbb{Z} \times \mathbb{Z} \to k^\times$ by $t(\gamma, \gamma) = q^{-1}$. This extends uniquely to a bicharacter on Γ since γ generates the free abelian group Γ; specifically, we find $t(\gamma^m, \gamma^n) = q^{-mn}$ for all $m, n \in \mathbb{Z}$. Denote by $k\langle x_1, x_2\rangle$ the free k-algebra on x_1, x_2, so that $k\langle x_1, x_2\rangle \cong T(V) = T_k(V)$

where V is a vector space with basis x_1, x_2. The twisted tensor product $A_1 \otimes^t A_2$ of A_1 and A_2 is given by

$$A_1 \otimes^t A_2 \cong k\langle x_1, x_2\rangle/(x_1x_2 - qx_2x_1),$$

called a *skew polynomial ring* or a *quantum plane*, and is denoted $k_q[x_1, x_2]$. The latter terminology recalls the affine plane, which may be identified with the set of maximal ideals of the commutative polynomial ring $k[x_1, x_2]$ (the case $q = 1$) if k is algebraically closed. More generally we iterate this construction to obtain a *skew polynomial ring* or *quantum affine space*

$$k_{\mathbf{q}}[x_1, \ldots, x_m] = k\langle x_1, \ldots, x_m\rangle/(x_ix_j - q_{ij}x_jx_i \mid 1 \le i, j \le m)$$

determined by a set $\mathbf{q} = \{q_{ij}\}$ of nonzero scalars for which $q_{ii} = 1$ and $q_{ji} = q_{ij}^{-1}$ for all i, j $(1 \le i, j \le m)$. We call such a set $\mathbf{q}$ a *quantum system of parameters*. Sometimes $k_{\mathbf{q}}[x_1, \ldots, x_m]$ is also called a *quantum symmetric algebra*, however this term is also used both more generally and slightly differently in other contexts, so we will not use it here.

Our next example is a noncommutative version of a truncated polynomial ring.

Example 3.3.2. Let n_1, n_2 be positive integers, $n_1, n_2 \ge 2$, and let $A_1 = k[x_1]/(x_1^{n_1})$ and $A_2 = k[x_2]/(x_2^{n_2})$. Each of A_1, A_2 is $\mathbb{Z}$-graded just as in Example 3.3.1. Alternatively, A_i may be viewed as being graded by $\mathbb{Z}/n_i\mathbb{Z}$ for $i = 1, 2$. Define a bicharacter t just as in Example 3.3.1; if we view each A_i as being graded by a finite quotient of $\mathbb{Z}$ instead of by $\mathbb{Z}$ itself, we must choose q to be a suitable root of unity. The twisted tensor product algebra $A_1 \otimes^t A_2$ is isomorphic to $k\langle x_1, x_2\rangle/(x_1x_2 - qx_2x_1,\ x_1^{n_1},\ x_2^{n_2})$. More generally we iterate this construction to obtain a *truncated skew polynomial ring*,

$$k\langle x_1, \ldots, x_m\rangle/(x_ix_j - q_{ij}x_jx_i,\ x_i^{n_i} \mid 1 \le i, j \le m)$$

determined by a quantum system of parameters $\mathbf{q} = \{q_{ij}\}$ as defined in Example 3.3.1, and positive integers $n_1, \ldots, n_m$. These algebras are also called *quantum complete intersections*, recalling the special case in which all $q_{ij} = 1$ (a commutative algebra that is an example of a complete intersection).

We return to the general case of a twisted tensor product $A_1 \otimes^t A_2$ of algebras A_1, A_2 determined by abelian grading groups Γ_1, Γ_2 and a bicharacter $t : \Gamma_1 \times \Gamma_2 \to k^{\times}$. We construct a projective resolution of the $(A_1 \otimes^t A_2)^e$-module $A_1 \otimes^t A_2$ from those of A_1 and A_2. Let $P_{\bullet}$ be a graded A_1^e-projective resolution of A_1, that is, each A_1^e-module P_i is graded by Γ_1 in such a way that $(A_1)_{\gamma}(P_i)_{\gamma'}(A_1)_{\gamma''} \subset (P_i)_{\gamma\gamma'\gamma''}$ for all $\gamma, \gamma', \gamma'' \in \Gamma_1$, the differentials preserve the grading, and P_i embeds as a direct summand of a free module via a graded map (that is, that preserves Γ_1-degree). For example, the bar resolution (1.1.4) is graded (by grading a tensor product in the usual way,

$(A \otimes A)_\gamma = \oplus_{\gamma'\gamma''=\gamma}(A_{\gamma'} \otimes A_{\gamma''})$). Similarly, let $Q_\bullet$ be a graded projective resolution of the A_2^e-module A_2.

Consider the tensor product complex $P_\bullet \otimes Q_\bullet$ as a complex of vector spaces. By the Künneth Theorem (Theorem A.5.1), since the tensor product is over the field k, the total complex has homology 0 in all positive degrees, and in degree 0 its homology is $A_1 \otimes A_2$. We will put the structure of an $(A_1 \otimes^t A_2)^e$-module on each $P_i \otimes Q_j$ in such a way that it is projective and the differentials are $(A_1 \otimes^t A_2)^e$-module homomorphisms. It will follow that the total complex of $P_\bullet \otimes Q_\bullet$ is an $(A_1 \otimes^t A_2)^e$-projective resolution of $A_1 \otimes^t A_2$. For all homogeneous $x \in P_i$, $y \in Q_j$, $a_1, a_1' \in A_1$, and $a_2, a_2' \in A_2$, set

$$(a_1 \otimes a_2)(x \otimes y)(a_1' \otimes a_2') = t(|x|, |a_2|)t(|a_1'|, |a_2|)t(|a_1'|, |y|)a_1 x a_1' \otimes a_2 y a_2'.$$

A calculation shows that this gives $P_i \otimes Q_j$ the structure of an $(A_1 \otimes^t A_2)^e$-module. Moreover, it is projective: Since P_i is a graded projective A_1^e-module, it is a direct summand of a direct sum of copies of A_1^e, say $(A_1^e)^{\oplus I}$ where I is some indexing set, and the embedding is a graded map. Similarly, Q_j is a direct summand of $(A_2^e)^{\oplus J}$ where J is some indexing set. We show that $P_i \otimes Q_j$ is a direct summand of a direct sum of copies of $(A_1 \otimes^t A_2)^e$ as an $(A_1 \otimes^t A_2)^e$-module: Since tensor product distributes over direct sum, as a vector space, $P_i \otimes Q_j$ is a direct summand of $(A_1^e)^{\oplus I} \otimes (A_2^e)^{\oplus J} \cong (A_1^e \otimes A_2^e)^{\oplus (I \times J)}$. Now, $A_1^e \otimes A_2^e \cong (A_1 \otimes^t A_2)^e$ as $(A_1 \otimes^t A_2)^e$-modules by definition of the module structure. The embeddings of P_i, Q_j into free modules are graded maps, and so $P_i \otimes Q_j$ embeds into this direct sum as an $(A_1 \otimes^t A_2)^e$-module. Finally, a calculation shows that the differentials on the tensor product complex $P_\bullet \otimes Q_\bullet$ are $(A_1 \otimes^t A_2)^e$-module homomorphisms since the differentials on $P_\bullet$ and $Q_\bullet$ preserve gradings.

We have proven the following theorem of Bergh and Oppermann [**BO08**].

Theorem 3.3.3. *Let A_1 and A_2 be k-algebras graded by abelian groups Γ_1 and Γ_2, respectively. Let $P_\bullet$ (respectively, $Q_\bullet$) be a graded projective resolution of A_1 as an A_1^e-module (respectively, of A_2 as an A_2^e-module). Then* $\mathrm{Tot}(P_\bullet \otimes Q_\bullet)$ *is a projective resolution of the twisted tensor product algebra $A_1 \otimes^t A_2$ as an $(A_1 \otimes^t A_2)^e$-module.*

Remark 3.3.4. Suppose that A_1, A_2 are augmented algebras, that is, for each i, there is an algebra homomorphism $\varepsilon_i : A_i \to k$ (called an augmentation map). If these augmentation maps ε_1, ε_2 are graded maps, then the twisted tensor product $A_1 \otimes^t A_2$ is augmented by $\varepsilon = \varepsilon_1 \otimes \varepsilon_2$. Consider k itself to be a module for each of $A_1, A_2, A_1 \otimes^t A_2$ via the augmentation maps $\varepsilon_1, \varepsilon_2, \varepsilon$, respectively. A construction similar to that above leads to a projective resolution of k as $A_1 \otimes^t A_2$-module from projective resolutions of k as A_1-module and as A_2-module. See [**BO08**] for details.

The resolution given by the total complex of $P_{\bullet} \otimes Q_{\bullet}$ constructed above may be used to understand Hochschild cohomology of the twisted tensor product algebra $A_1 \otimes^t A_2$. There is however not a result such as Theorem 3.1.2 that describes the Hochschild cohomology ring of $A_1 \otimes^t A_2$ in terms of that of A_1 and A_2. (For one thing, Hochschild cohomology is graded commutative, while the twisted tensor product generally is not.) The grading on $P_{\bullet}$, $Q_{\bullet}$ by Γ_1, Γ_2 does impart some structure to the Hochschild cohomology ring of $A_1 \otimes^t A_2$, and a consequence is that there is a result analogous to Theorem 3.1.2 for the part of Hochschild cohomology graded by a subgroup of $\Gamma_1 \times \Gamma_2$ contained in the kernel of the bicharacter t. One proves this by retracing the steps of the proof of Theorem 3.1.2, noting that the sequence of isomorphisms of Hom spaces given there works for subspaces graded by a subgroup of $\Gamma_1 \times \Gamma_2$. See [**BO08**, Theorem 4.7].

We give details of the resolution of the twisted tensor product $A_1 \otimes^t A_2$ constructed above for our Examples 3.3.1 and 3.3.2.

Example 3.3.5. Let $A = k_q[x_1, x_2] \cong A_1 \otimes^t A_2$ as in Example 3.3.1. Let $P_{\bullet}$ be the resolution (1.1.17) of Example 1.1.16 for A_1. In order for $P_{\bullet}$ to be a *graded* resolution, the differentials must preserve grading, and accordingly we shift the $\mathbb{Z}$-grading of A_1^e in the homological degree 1 component so that $1 \otimes 1$ there has graded degree γ. Then the differential is indeed a graded map. Let $Q_{\bullet}$ be a similar resolution for A_2. By Theorem 3.3.3, $\mathrm{Tot}(P_{\bullet} \otimes Q_{\bullet})$ is an A^e-projective resolution of A. Apply $\mathrm{Hom}_{A^e}(-, A)$ to obtain

$$\begin{aligned} 0 \longleftarrow \mathrm{Hom}_{A^e}(P_1 \otimes Q_1, A) \longleftarrow \mathrm{Hom}_{A^e}\big((P_0 \otimes Q_1) \oplus (P_1 \otimes Q_0), A\big) \\ \longleftarrow \mathrm{Hom}_{A^e}(A^e, A) \longleftarrow 0, \end{aligned}$$

which is equivalent, under standard isomorphisms, to

$$0 \longleftarrow A \xleftarrow{\;d_2^*\;} A \oplus A \xleftarrow{\;d_1^*\;} A \longleftarrow 0.$$

Under our identifications and degree shifts, a calculation shows that the maps d_1^*, d_2^* are given by

$$\begin{aligned} d_1^*(a \otimes b) &= \big((q^{-|a|} - 1)a \otimes bx_2,\ (1 - q^{-|b|})ax_1 \otimes b\big), \\ d_2^*(a \otimes b,\ a' \otimes b') &= (1 - q^{-|b|-1})ax_1 \otimes b + (1 - q^{-|a'|-1})a' \otimes b'x_2 \end{aligned}$$

for all homogeneous $a, a' \in A_1$ and $b, b' \in A_2$. (Note that in case $q = 1$, these differentials are indeed 0, in accordance with Example 3.1.3.) We may iterate this construction to obtain a free resolution of A as an A^e-module for $A = k_{\mathbf{q}}[x_1, \dots, x_m]$.

Example 3.3.6. Another important family of algebras that may be constructed as twisted tensor products are the quantum complete intersections of Example 3.3.2. For simplicity, we focus here on the case $m = 2$ and $n_1 = n_2 = 2$: $A = k\langle x_1, x_2\rangle/(x_1x_2 - qx_2x_1,\ x_1^2,\ x_2^2)$. Similar techniques yield information about the more general case. Buchweitz, Green, Madsen, and Solberg [**BGMS05**] used precisely these examples to answer a question of Happel [**Hap89**], showing that some of these are finite dimensional algebras that have finite dimensional Hochschild cohomology, yet infinite global dimension. We give some details next.

Let $P_{\bullet}$ (respectively, $Q_{\bullet}$) be resolution (1.1.20) for $A_1 = k[x_1]/(x_1^{n_1})$ (respectively, $A_2 = k[x_2]/(x_2^{n_2})$). The total complex of $P_{\bullet} \otimes Q_{\bullet}$ has the structure of an A^e-projective resolution of A where $A = A_1 \otimes^t A_2$. This is equivalent to the projective resolution constructed in [**BGMS05**]. Consider k to be an A-module on which x_1, x_2 each act as 0, that is, A is augmented with augmentation map $\varepsilon : A \to k$ given by the algebra homomorphism sending each of x_1, x_2 to 0. Applying $- \otimes_A k$ to $\mathrm{Tot}(P_{\bullet} \otimes Q_{\bullet})$, we obtain a projective resolution of k as an A-module. This resolution may be used to show that $\mathrm{Ext}^n_A(k,k) \neq 0$ for all n, independent of the characteristic of k and of the value of q. It follows that A has infinite global dimension, since existence of a projective resolution of finite length would imply $\mathrm{Ext}^n_A(k,k) = 0$ for all n larger than the length of the resolution.

If q is not a root of unity and $\mathrm{char}(k) \neq 2$, the Hochschild cohomology ring $\mathrm{HH}^*(A)$ is 5-dimensional as a vector space: In this case, computations using our resolution $P_{\bullet} \otimes Q_{\bullet}$ show that $\mathrm{HH}^0(A)$ is a vector space spanned by 1 and x_1x_2 (the center of A), $\mathrm{HH}^1(A)$ is a vector space spanned by elements y_1 and y_2 that arise from functions at the chain level taking $(1\otimes 1)\otimes(1\otimes 1)$ in degree $(1,0)$ to $x_1 \otimes 1$ and $(1\otimes 1)\otimes(1\otimes 1)$ in degree $(0,1)$ to $1 \otimes x_2$, respectively, $\mathrm{HH}^2(A)$ is a vector space spanned by $y_1 \smile y_2$, and $\mathrm{HH}^i(A) = 0$ for all $i \geq 3$. Thus we see that A has infinite global dimension and yet finite dimensional Hochschild cohomology. The cup product of the degree 0 element x_1x_2 with y_i is 0 for $i = 1, 2$. Considering the vector space dimension in each degree, it now follows that there is an isomorphism of algebras

$$\mathrm{HH}^*(A) \cong k[x_1x_2]/((x_1x_2)^2) \times_k \textstyle\bigwedge(y_1, y_2),$$

where the latter is a fiber product. (The *fiber product* $R_1 \times_k R_2$ of two augmented k-algebras R_1, R_2 is the subring of $R_1 \oplus R_2$ consisting of pairs (r_1, r_2) such that the images of r_1 and r_2 under the respective augmentation maps are equal, i.e., the fiber product is the pullback of the two augmentation maps.) See [**BGMS05**] for details, as well as the cases where $\mathrm{char}(k) = 2$ or where $q = 0$.

If q is a root of unity, the Hochschild cohomology ring has completely different structure. In this case, it is infinite dimensional, yet there are gaps: it is 0 in infinitely many degrees. See [**BGMS05**] for details.

Exercise 3.3.7. Let $A_1 \otimes^t A_2$ be the twisted tensor product given in Example 3.3.1. Define

$$[n]_q = 1+q+q^2+\cdots+q^{n-1},\ [n]_q! = [1]_q[2]_q\cdots[n]_q,\ \left[\begin{array}{c} n \\ r \end{array}\right]_q = \frac{[n]_q!}{[r]_q![n-r]_q!},$$

the last expression defined only when $[r]_q![n-r]_q! \neq 0$.

(a) Show that when the coefficients below are defined,

$$(x_1+x_2)^n = \sum_{r=0}^{n} \left[\begin{array}{c} n \\ r \end{array}\right]_{q^{-1}} x_1^r x_2^{n-r}.$$

(b) Suppose that q is a primitive nth root of unity. Show that

$$(x_1+x_2)^n = x_1^n + x_2^n.$$

Exercise 3.3.8. Verify that the formulas for d_1^* and d_2^* given in Example 3.3.5 are indeed those arising from the differential on the tensor product of the resolutions $P_{\bullet}$ and $Q_{\bullet}$.

Exercise 3.3.9. Suppose q is not a root of unity, and let $A = k_q[x_1, x_2]$, defined as in Example 3.3.5. Use the resolution given there to find the structure of $\mathrm{HH}^*(A)$ as a graded vector space.

Exercise 3.3.10. Let $A = k_{\mathbf{q}}[x_1, \ldots, x_m]$. Use the methods of Example 3.3.5 to find a formula for the differential of a resolution of A as an A^e-module constructed as described there.

Exercise 3.3.11. Verify the claimed structure of $\mathrm{HH}^*(A)$ in Exercise 3.3.6 in case q is not a root of unity and $\mathrm{char}(k) \neq 2$.

Exercise 3.3.12. Let $A = k\langle x_1, x_2\rangle/(x_1x_2 - qx_2x_1,\ x_1^{n_1},\ x_2^{n_2})$. Use the resolution in Example 3.3.6 to find (a) $\mathrm{Ext}_A^*(k,k)$ and (b) the Hochschild homology $\mathrm{HH}_*(A)$.

3.4. Koszul algebras

Some of the algebras we have seen in this chapter are in fact Koszul algebras, introduced by Priddy [**Pri70**]. We will define Koszul algebras next, revisiting earlier examples in this light. We assume here that the algebra A is graded (by $\mathbb{N}$) and *connected*, that is $A_0 = k$. More general Koszul algebras are considered in the literature. (See, e.g., [**MV08**] for an introduction to Koszul algebras defined by quivers and relations, and further references therein.)

Let V be a finite dimensional vector space and let $T(V) = T_k(V)$ denote the *tensor algebra* of V:

$$T(V) = \oplus_{n\geq 0} T^n(V)$$

where $T^0(V) = k$, $T^1(V) = V$, and $T^n(V) = V \otimes \cdots \otimes V$ (n tensor factors). Multiplication is simply $\otimes$, that is,

$$(v_1 \otimes \cdots \otimes v_m) \cdot (v_1' \otimes \cdots \otimes v_n') = v_1 \otimes \cdots \otimes v_m \otimes v_1' \otimes \cdots \otimes v_n'$$

for all $v_1, \ldots, v_m, v_1', \ldots, v_n' \in V$. Then $T(V)$ is a graded algebra with $|v| = 1$ for all $v \in V$. Sometimes we write $v_1 \cdots v_m$ in place of $v_1 \otimes \cdots \otimes v_m$, for simplicity of notation. Let R be a subspace of $T^2(V)$, that is

$$R \subset V \otimes V,$$

and let

$$A = T(V)/(R),$$

where (R) denotes the ideal generated by R in $T(V)$. We call R the space of *relations* for A. By definition, A is a *quadratic algebra*, that is A is a graded algebra generated by elements in degree 1 with relations in degree 2.

Let $V^* = \mathrm{Hom}_k(V, k)$ be the dual vector space to V. Since V is finite dimensional, we may identify $(V \otimes V)^*$ with $V^* \otimes V^*$. Let

$$R^\perp = \{u \in V^* \otimes V^* \mid u(r) = 0 \ \text{ for all } \ r \in R\}.$$

The *quadratic dual* (or *Koszul dual*) of A is the quadratic algebra

$$A^! = T(V^*)/(R^\perp).$$

Example 3.4.1. Let V be a vector space with basis $x_1, \ldots, x_m$. Let $\mathbf{q} = \{q_{ij}\}$ be a quantum system of parameters as defined in Example 3.3.1. Let

$$R = \mathrm{Span}_k\{x_i \otimes x_j - q_{ij} x_j \otimes x_i \mid 1 \le i < j \le m\}.$$

Then $A = T(V)/(R) \cong k_{\mathbf{q}}[x_1, \ldots, x_m]$, the skew polynomial ring of Example 3.3.1. Let $x_1^*, \ldots, x_m^*$ denote the dual basis to the basis $x_1, \ldots, x_m$ of V. Calculations show that the subspace $R^\perp$ of $V^* \otimes V^*$ is

$$R^\perp = \mathrm{Span}_k\{x_i^* \otimes x_j^* + q_{ij}^{-1} x_j^* \otimes x_i^*, \ x_i^* \otimes x_i^* \mid 1 \le i < j \le m\}.$$

Setting $y_i = x_i^*$ for each i, the Koszul dual of A is thus

$$A^! \cong k\langle y_1, \ldots, y_m\rangle/(y_i y_j + q_{ij}^{-1} y_j y_i, \ y_i^2 \mid 1 \le i, j \le m),$$

sometimes denoted $\bigwedge_{\mathbf{q}^{-1}}(V^*)$, a *quantum exterior algebra.* As a special case, if $q_{ij} = 1$ for all i, j, then $A \cong k[x_1, \ldots, x_m]$ and $A^! \cong \bigwedge(V^*)$.

Example 3.4.2. Let V be a finite dimensional vector space and let $R = 0$, the zero subspace of $V \otimes V$. Then $A = T(V)/(R) \cong T(V)$ and $A^! \cong k \oplus V^*$ since $R^\perp = V^* \otimes V^*$.

Next we will define Koszul algebras to be connected quadratic algebras having particular types of resolutions. To this end, view the field k as the quotient A/A_+ where $A_+ = \oplus_{n>0} A_n$. Let $\varepsilon : A \to k$ be the quotient map. Then A is an augmented algebra via this augmentation map ε. Consider a graded free resolution $P_\bullet$ of the corresponding (left) A-module k, that is, each A-module P_i is free and thus graded via the grading on A, shifted in such a way that the differentials are graded maps. We further require each P_n to be finite dimensional in each degree; note that such a $P_\bullet$ exists as for example $B(A) \otimes_A k$, where $B(A)$ is the bar resolution (1.1.4). For each P_i, choose a free basis $\{p^i_l \mid l \in L_i\}$, for L_i some indexing set, in order to identify it with the free module $A^{\oplus L_i} \cong \oplus_{l \in L_i} A p^i_l$. The differentials may then be viewed as matrices with entries in A. Since we have chosen each P_n to be finite dimensional in each degree, the differential on each homogeneous subspace is given by a finite matrix. We say that $P_\bullet$ is *minimal* if the matrix entries are all in A_+, and *linear* if the matrix entries are all in A_1.

Definition 3.4.3. A graded connected quadratic algebra $A = T(V)/(R)$ is a *Koszul algebra* if the A-module k has a linear minimal graded free resolution.

There are many equivalent definitions of Koszul algebras, as we will see in the next theorem below. In fact, for any algebra that is graded, connected, and locally finite dimensional (i.e. having finite dimensional homogeneous components), existence of a linear minimal graded free resolution implies it is quadratic (see [**Pri70**]). Thus the assumption that the algebra is quadratic need not be part of the definition of Koszul algebra, and we include it only for convenience, as is common to do. Some of the equivalent definitions of Koszul algebras involve specific complexes, which we will construct next.

Consider the following sequence:

$$\cdots \xrightarrow{d_4} K_3(A) \xrightarrow{d_3} A \otimes R \otimes A \xrightarrow{d_2} A \otimes V \otimes A \xrightarrow{d_1} \tag{3.4.4}$$
$$A \otimes A \xrightarrow{\quad \pi \quad} A \longrightarrow 0$$

where for each $n \geq 2$, $K_n(A) = A \otimes K'_n(A) \otimes A$ with

$$K'_n(A) = \bigcap_{i+j=n-2} (V^{\otimes i} \otimes R \otimes V^{\otimes j}).$$

Write $K_0(A) = A \otimes A$ and $K_1(A) = A \otimes V \otimes A$. The differentials d_n are those of the bar resolution $B(A)$ defined in (1.1.4) under the canonical embedding of $K_\bullet(A)$ into $B(A)$; a calculation shows that the sequence (3.4.4) is indeed closed under this differential. Then (3.4.4) is a chain complex. It may or may not be a resolution of A, and it turns out that Koszul algebras are precisely those for which it is, as Theorem 3.4.6 below states.

We need some more notation to state the theorem. For each $n \geq 0$, let $\widetilde{K}_n(A) = A \otimes K'_n(A)$, a left A-module by multiplication by the leftmost factor. Note that $\widetilde{K}_n(A) \cong K_n(A) \otimes_A k$. We will be interested in the resulting sequence:

$$\cdots \longrightarrow \widetilde{K}_2(A) \longrightarrow \widetilde{K}_1(A) \longrightarrow \widetilde{K}_0(A) \longrightarrow k \longrightarrow 0. \tag{3.4.5}$$

Theorem 3.4.6. *Let $A = T(V)/(R)$ be a finitely generated graded connected quadratic algebra. The following are equivalent:*

(i) *A is a Koszul algebra.*

(ii) *$K_\bullet(A)$ is a resolution of A as an A^e-module.*

(iii) *$\widetilde{K}_\bullet(A)$ is a resolution of k as an A-module.*

(iv) *$\operatorname{Ext}^*_A(k,k)$ is generated by $\operatorname{Ext}^1_A(k,k)$ as an algebra.*

(v) *$\operatorname{Ext}^*_A(k,k) \cong A^!$ as graded algebras.*

Proof. We prove equivalence of (ii) and (iii), following [**Krä11**]. Other equivalences are also in [**Krä11**] and in earlier literature such as [**Pri70**].

Assume that (ii) holds, that is, $K_\bullet(A)$ is a resolution of A as an A^e-module. Since $\widetilde{K}_\bullet(A) = K_\bullet(A) \otimes_A k$ and sequence (3.4.4) consists of free right A-modules, it follows that $\widetilde{K}_\bullet(A)$ is exact other than in degree 0 where it has homology $A \otimes_A k \cong k$. Thus $\widetilde{K}_\bullet(A)$ is a resolution of k as an A-module, that is, (iii) holds.

Now assume (iii) holds, that is, the sequence (3.4.5) is exact. Since (3.4.4) consists of free right A-modules, it splits as a sequence of right A-modules, and so applying $- \otimes_A k$ commutes with taking homology. Consequently, as $\widetilde{K}_\bullet(A) = K_\bullet(A) \otimes_A k$, for all $n \geq 1$,

$$\mathrm{H}_n(K_\bullet(A)) \otimes_A k \cong \mathrm{H}_n(K_\bullet(A) \otimes_A k) = \mathrm{H}_n(\widetilde{K}_\bullet(A)) = 0.$$

Now for each n, we consider $\mathrm{H}_n(K_\bullet(A))$ to be a graded right A-module, with grading induced by that on A. Note that for any nonzero graded A-module M, A_+M is a proper submodule of M. As a right A-module, note that $\mathrm{H}_n(K_\bullet(A)) \otimes_A k \cong \mathrm{H}_n(K_\bullet(A))/A_+ \mathrm{H}_n(K_\bullet(A))$. It follows that $\mathrm{H}_n(K_\bullet(A)) = 0$ for all $n \geq 1$. A calculation shows that $\mathrm{H}_0(K_\bullet(A)) \cong A$. So $K_\bullet(A)$ is exact, and thus is a free resolution of A as an A^e-module. Thus (iii) implies (ii). □

We next give some standard examples.

Example 3.4.7. Let $A = k[x_1, \ldots, x_n] \cong T(V)/(R)$ where V is the vector space with basis $x_1, \ldots, x_n$ and

$$R = \operatorname{Span}_k\{v \otimes w - w \otimes v \mid v, w \in V\}.$$

We claim that $K_\bullet(A)$, for this algebra A, is equivalent to the resolution (3.1.4) given in Example 3.1.3. To see this, define $\phi : A \otimes \bigwedge^\bullet(V) \otimes A \to B_\bullet(A)$ by

$$(3.4.8) \qquad \phi(1 \otimes v_1 \wedge \cdots \wedge v_n \otimes 1) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \otimes v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)} \otimes 1$$

for all $v_1, \ldots, v_n \in V$, where S_n is the symmetric group on n letters. A calculation shows that ϕ is a chain map, ϕ is injective, and that the image of ϕ is precisely $K_\bullet(A)$. We showed in Example 3.1.3 that $A \otimes \bigwedge^\bullet(V) \otimes A$ is a free resolution of A as an A^e-module, so $K_\bullet(A)$ is a free resolution of A as an A^e-module. By Theorem 3.4.6(i) and (ii), A is a Koszul algebra.

Similarly, the skew polynomial ring $A = k_{\mathbf{q}}[x_1, \ldots, x_m]$ of Example 3.3.1 is a Koszul algebra: The resolution discussed in Example 3.3.5 is equivalent to $K_\bullet(A)$ for this algebra.

Example 3.4.9. Let $\mathbf{q}$ be a quantum system of parameters, as in Example 3.3.1. Let $A = k_{\mathbf{q}}[x_1, \ldots, x_m]/(x_1^2, \ldots, x_m^2) \cong T(V)/(R)$, where V is the vector space with basis $x_1, \ldots, x_m$ and

$$R = \operatorname{Span}_k\{x_i \otimes x_j - q_{ij} x_j \otimes x_i \mid 1 \le i, j \le m\} \oplus \operatorname{Span}_k\{x_1 \otimes x_1, \ldots, x_m \otimes x_m\}.$$

The Koszul resolution $K_\bullet(A)$ is equivalent to that constructed in Example 3.3.6, as may be shown by an argument similar to that in Example 3.4.7.

There is a close relationship between Hochschild cohomology and the Ext algebra $\operatorname{Ext}^*_A(k,k)$ when A is a Koszul algebra: Let $\phi_k : \operatorname{HH}^*(A) \to \operatorname{Ext}^*_A(k,k)$ be the map given by $- \otimes_A k$, as defined in (2.5.3). By Corollary 2.5.9, for any augmented algebra A, the image of ϕ_k is contained in the graded center $Z_{\mathrm{gr}}(\operatorname{Ext}^*_A(k,k))$ (see Definition 2.5.8). In fact, for Koszul algebras, even more is true:

Theorem 3.4.10. *Let A be a Koszul algebra. Then the image of the map $\phi_k : \operatorname{HH}^*(A) \to \operatorname{Ext}^*_A(k,k)$ defined in (2.5.3) is precisely the graded center, $Z_{\mathrm{gr}}(\operatorname{Ext}^*_A(k,k))$.*

For a proof of the theorem, see [**BGSS08**, Theorem 4.1].

Exercise 3.4.11. In Example 3.4.1, verify that $R^\perp$ is as stated there.

Exercise 3.4.12. Verify that $K_\bullet(A)$ given by (3.4.4) is indeed a subcomplex of $B_\bullet(A)$, that is, the differential on $B_\bullet(A)$ takes $K_n(A)$ to $K_{n-1}(A)$ for each n.

Exercise 3.4.13. Verify that ϕ in Example 3.4.7 is a chain map and its image is $K_\bullet(A)$.

Exercise 3.4.14. Let $A = k_{\mathbf{q}}[x_1, \ldots, x_m]$. Define a map analogous to ϕ, of Example 3.4.7, from the resolution described in Exercise 3.3.10 to the bar resolution. Conclude that $k_{\mathbf{q}}[x_1, \ldots, x_m]$ is a Koszul algebra.

Exercise 3.4.15. Verify the claim made in Example 3.4.9 via an argument similar to that in Example 3.4.7. Constructing the map ϕ may take some work. See [**BGMS05**].

3.5. Path algebras and monomial algebras

Bardzell [**Bar97**] constructed a bimodule resolution of a monomial algebra. In this section we define monomial algebras and present this construction. Chouhy and Solotar [**CS15**] generalized Bardzell's construction to algebras defined by quivers and relations. These resolutions are analogues of the Anick resolution for left modules [**Ani, GS07**].

A *quiver* Q is a directed graph, that is, Q consists of a set Q_0 of *vertices*, a set Q_1 of *arrows*, and two maps $s : Q_1 \to Q_0$, $t : Q_1 \to Q_0$, associating to each arrow α its *source* $s(\alpha)$ and *target* $t(\alpha)$. The quiver is *finite* if the sets Q_0 and Q_1 are both finite. A *path* in Q is a sequence of arrows $(\alpha_1, \dots, \alpha_l)$ for which $t(\alpha_i) = s(\alpha_{i+1})$ for all i. We denote the associated path by $\alpha_1 \cdots \alpha_l$. Its *length* is l. There is a path of length 0 associated to each vertex a of Q_0, denoted e_a. The *path algebra* of Q, denoted kQ, is the k-algebra whose underlying vector space is the set of all paths $\alpha_1 \cdots \alpha_l$ of length $l \geq 0$ with multiplication determined by

$$(\alpha_1 \cdots \alpha_l) \cdot (\beta_1 \cdots \beta_m) = \begin{cases} \alpha_1 \cdots \alpha_l \beta_1 \cdots \beta_m, & \text{if } t(\alpha_l) = s(\beta_1), \\ 0 & \text{otherwise.} \end{cases}$$

Example 3.5.1. The path algebra of the leftmost quiver below is finite dimensional as a vector space since $\alpha^2 = 0$ by definition. The path algebra of the rightmost quiver is isomorphic to $k[x]$.

$$\bullet \xrightarrow{\alpha} \bullet \qquad\qquad \begin{tikzcd} \bullet \arrow[loop left, "x"] \end{tikzcd}$$

Remark 3.5.2. Any finite dimensional algebra A is Morita equivalent to a quotient kQ/I for some quiver Q and ideal I. That is, the category of A-modules is equivalent to the category of kQ/I-modules [**ASS06**, Corollary I.6.10 and Theorem II.3.7]. Thus quiver techniques are very important in the representation theory of finite dimensional algebras.

Definition 3.5.3. A *monomial algebra* is an algebra of the form $A = kQ/I$ where Q is a finite quiver and I is the ideal generated by a finite set of paths of length at least 2.

Example 3.5.4. The algebra $k[x]/(x^n)$ is a monomial algebra with quiver Q given by the rightmost quiver in Example 3.5.1.

We need further notation and terminology for the resolution we will construct next. Let $A = kQ/I$ be a monomial algebra. Let R be a minimal set of paths, of minimal length, for which $I = (R)$. Let u, w be paths in Q. We say that y is a *subpath* of w if $w = xyz$ for some paths x, z. We write $x = l(y)$ and $z = r(y)$ (for brevity, the notation l and r suppresses the dependence on the expression xyz).

We define the *Bardzell resolution* $P_\bullet$ of A in a similar manner to the presentation of Redondo and Róman [**RR18**]. See also [**Bar97, Skö08**]. Let

$$\begin{aligned} P_0 &= A \otimes_{kQ_0} kQ_0 \otimes_{kQ_0} A \cong A \otimes_{kQ_0} A, \\ P_1 &= A \otimes_{kQ_0} kQ_1 \otimes_{kQ_0} A, \\ P_2 &= A \otimes_{kQ_0} kR \otimes_{kQ_0} A, \end{aligned}$$

where kQ_0 denotes the vector space with basis Q_0, and similarly kQ_1, kR. For $n > 2$, let $P_n = A \otimes_{kQ_0} P'_n \otimes_{kQ_0} A$ where P'_n is as follows. Let $w = \alpha_1 \cdots \alpha_l$ be any path in Q and order all vertices occurring in the path according to its layout, with multiplicities as needed:

$$s(\alpha_1) < s(\alpha_2) < \cdots < s(\alpha_l) < t(\alpha_l).$$

Let $R(w)$ be the set of paths in R that are subpaths of w. We construct sets $L_1, L_2, \ldots$ recursively: Choose $p_1 \in R(w)$. Let

$$L_1 = \{p \in R(w) \mid s(p_1) < s(p) < t(p_1)\}.$$

If $L_1 \neq \varnothing$, let $p_2 \in R(w)$ be a path for which $s(p_2)$ is minimal for paths in L_1. In general let $j \geq 1$ and assume $L_1, \ldots, L_j$ have been defined and let $p_{j+1} \in R(w)$ be a path for which $s(p_{j+1})$ is minimal for paths in L_j. Let

$$L_{j+1} = \{p \in R(w) \mid t(p_j) \leq s(p) < t(p_{j+1})\}.$$

For each n, write $w(p_1, \ldots, p_{n-1})$ for the *support* of the sequence $p_1, \ldots, p_{n-1}$, that is the path from $s(p_1)$ to $t(p_{n-1})$ that contains $p_1, \ldots, p_{n-1}$ as subpaths (note that this is a subpath of w by construction). Let P'_n be the set of all supports of all such sequences $p_1, \ldots, p_{n-1}$ for all paths w in Q, called *n-concatenations*. For each path w in P'_n, let

$$\mathrm{Sub}(w) = \{w' \in P'_{n-1} \mid w' \text{ is a subpath of } w\}.$$

Next we describe the differentials on the Bardzell resolution $P_\bullet$. The map $\pi : P_0 \to A$ is given by multiplication on $A \otimes_{kQ_0} A$. The map $d_1 : P_1 \to P_0$ is defined by

$$d_1(1 \otimes \alpha \otimes 1) = \alpha \otimes 1 - 1 \otimes \alpha$$

for all $\alpha \in Q_1$, where we have suppressed the subscript kQ_0 on the tensor symbol in expressions for elements in order to reduce clutter. The map $d_2 : P_2 \to P_1$ is given by

$$d_2(1 \otimes w \otimes 1) = \sum_{w' \in \mathrm{Sub}(w)} l(w') \otimes w' \otimes r(w').$$

More generally, in even degrees, the differential is similar:

$$d_{2m}(1 \otimes w \otimes 1) = \sum_{w' \in \mathrm{Sub}(w)} l(w') \otimes w' \otimes r(w'),$$

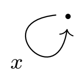
while in odd degrees we set

$$d_{2m+1}(1 \otimes w \otimes 1) = l(w_2) \otimes w_2 \otimes 1 - 1 \otimes w_1 \otimes r(w_1),$$

where $w = l(w_2)w_2 = w_1 r(w_1)$ with w_1, w_2 the supports of the corresponding $2m$-concatenations. See [**Bar97, RR18, Skö08**] for more details.

Exactness and minimality of $P_{\bullet} \to A$ are proven in [**Bar97**]. For a different proof, see Skoldberg [**Skö08**]. Some examples are given in [**Bar97**, Section 7]. Redondo and Román [**RR18**] provided chain maps between the Bardzell resolution and the bar resolution with the goal of computing some of the structure of Hochschild cohomology of monomial algebras.

Example 3.5.5. Let $A = k[x]/(x^n)$ as in Example 1.1.19. We show that the resolution (1.1.20) is equivalent to the Bardzell resolution. Let Q be the quiver with one vertex and one arrow:



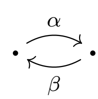
Then $A \cong kQ/I$ where $I = (x^n)$. The paths in Q are all nonnegative integer powers of x, and we may take $R = \{x^n\}$. For each m, there is a unique m-concatenation: In even degrees m, it is $x^{\frac{mn}{2}}$, and in odd degrees m it is $x^{\frac{(m-1)n}{2}+1}$. The above formula for the differentials on the Bardzell resolution agrees with the differentials in the sequence (1.1.20) once we identify $1 \otimes 1$ in each degree in (1.1.20) with these free generators in the Bardzell resolution.

Exercise 3.5.6. Verify the claims in Example 3.5.5, that is:

(a) Check that the m-concatenations are as stated.

(b) Check that the formula for the differentials on the Bardzell resolution agrees with that for (1.1.20) under suitable identifications.

Exercise 3.5.7. Find the Bardzell resolution for the path algebra kQ/I where Q is the following quiver and $I = (\alpha\beta\alpha)$.



3.6. Skew group algebras

Skew group algebras arise in considering an algebra together with a group of algebra automorphisms. The corresponding skew group algebra is a larger algebra encoding both structures. When a group acts on a geometric space such as a manifold or algebraic variety, it correspondingly acts on a suitable ring of functions on the space, and the associated skew group algebra is studied in noncommutative geometry. We will look at the special case of a finite group action here, and some techniques for studying the Hochschild homology and cohomology of resulting skew group algebras.

Let G be a finite group acting by automorphisms on an algebra A. We use left superscript to denote the action in order to distinguish it from multiplication in an algebra, that is, ${}^{g}a$ is the result of applying $g \in G$ to $a \in A$. The *skew group algebra* $A \rtimes G$ (also denoted $A\#G$ or $A\#kG$ or $A * G$) is $A \otimes kG$ as a vector space, with multiplication given by

$$(a \otimes g)(b \otimes h) = a({}^{g}b) \otimes gh$$

for all $a, b \in A$ and $g, h \in G$. Note that A is isomorphic to the subalgebra $A \otimes k$ of $A \rtimes G$ and that kG is isomorphic to the subalgebra $k \otimes kG$ of $A \rtimes G$. For simplicity of notation, we will abbreviate $a \otimes g$ by ag when it will cause no confusion. In this notation, the action of G on A is by conjugation in $A \rtimes G$: $gag^{-1} = {}^{g}a$.

There is a spectral sequence describing the Hochschild (co)homology of $A \rtimes G$ in terms of that of A and of G. This is a special case of a construction given in Section 9.6 for smash products with Hopf algebras. For now, we will work in a more specialized setting: Assume the characteristic of k does not divide the order of G, so that the group algebra kG is semisimple by Maschke's Theorem. In this case, there is a more elementary description of the Hochschild (co)homology of $A \rtimes G$ as we see next.

For any set X on which G acts, we use a superscript G to denote the set of *invariants*, that is,

$$X^{G} = \{x \in X \mid {}^{g}x = x \text{ for all } g \in G\},$$

and a subscript G to denote the set of *coinvariants*,

$$X_{G} = X/\sim \quad \text{where } x \sim y \ \text{ if } \ y = {}^{g}x \ \text{ for some } g \in G.$$

In the following, we take the algebra structure on $\mathrm{HH}^{n}(A, A \rtimes G)$ to be that described in Remark 1.3.4.

Theorem 3.6.1. *Assume the characteristic of the field k does not divide the order of the finite group G. Let A be a k-algebra on which G acts by algebra automorphisms. There are actions of G for which*

$$\mathrm{HH}^{n}(A \rtimes G) \cong \mathrm{HH}^{n}(A, A \rtimes G)^{G} \quad \text{and} \quad \mathrm{HH}_{n}(A \rtimes G) \cong \mathrm{HH}_{n}(A, A \rtimes G)_{G}$$

as graded algebras and graded vector spaces, respectively.

Proof. One proof relies on a spectral sequence for a skew group algebra, described in Section 9.6 in a more general setting. We will give a more elementary proof here. We will see in the course of the proof that the action of G is that induced by its action on the bar resolution of A (diagonal on each tensor factor) and by conjugation on $A \rtimes G$. Let

$$\mathcal{D} = \bigoplus_{g \in G} (Ag) \otimes (A^{\mathrm{op}} g^{-1}).$$

A calculation shows that $\mathcal{D}$ is a subalgebra of $(A \rtimes G)^e$, and further that $(A \rtimes G)^e$ is free as a right $\mathcal{D}$-module under multiplication, with free basis $\{1 \otimes g \mid g \in G\}$.

We claim that there is an isomorphism of $(A \rtimes G)$-bimodules,

$$A \rtimes G \xrightarrow{\sim} A \uparrow_{\mathcal{D}}^{(A \rtimes G)^e}, \tag{3.6.2}$$

where uparrow denotes tensor induction: $A \uparrow_{\mathcal{D}}^{(A \rtimes G)^e} = (A \rtimes G)^e \otimes_{\mathcal{D}} A$, on which $(A \rtimes G)^e$ acts by multiplication on the leftmost factor. This isomorphism is given by sending $a \otimes g$ to $(1 \otimes g) \otimes a$ for all $a \in A$ and $g \in G$. In light of the isomorphism (3.6.2), by the Eckmann-Shapiro Lemma (Lemma A.6.2),

$$\mathrm{Ext}^*_{(A \rtimes G)^e}(A \rtimes G, A \rtimes G) \cong \mathrm{Ext}^*_{\mathcal{D}}(A, A \rtimes G).$$

Now, any $\mathcal{D}$-projective resolution of A may be viewed, on restriction to A^e, as an A^e-projective resolution having an action of G (through the elements $g \otimes g^{-1}$ for g in G) that commutes with the differentials. For any pair of $\mathcal{D}$-modules U, V, note that $\mathrm{Hom}_{\mathcal{D}}(U, V) \cong \mathrm{Hom}_{A^e}(U, V)^G$, where the action of G on such functions is given by $({}^g f)(u) = {}^g(f({}^{g^{-1}}u))$ for $g \in G$, $f \in \mathrm{Hom}_{A^e}(U, V)$, $u \in U$. Since the characteristic of k does not divide the order of G, the space of G-invariants of any kG-module V is the image of $\frac{1}{|G|} \sum_{g \in G} g$ as an operator on V. It follows that taking G-invariants commutes with taking (co)homology, and so $\mathrm{Ext}^*_{\mathcal{D}}(A, A \rtimes G) \cong (\mathrm{Ext}^*_{A^e}(A, A \rtimes G))^G$. A calculation shows that this is an algebra isomorphism.

There is an isomorphism similar to (3.6.2): $A \rtimes G \cong A \otimes_{\mathcal{D}} (A \rtimes G)^e$ as a *right* $(A \rtimes G)^e$-module. We use this isomorphism to see similarly that

$$\mathrm{HH}_n(A \rtimes G) \cong \mathrm{Tor}_n^{\mathcal{D}}(A, A \rtimes G),$$

as follows. Let $P_{\bullet}$ be the bar resolution (1.1.4) of A as an A^e-module. It admits an action of G commuting with the differentials, and this action may be used to extend the A^e-module structure on each P_i to a $\mathcal{D}$-module structure. Tensoring with $(A \rtimes G)^e$ over $\mathcal{D}$, we obtain $P_{\bullet} \otimes_{\mathcal{D}} (A \rtimes G)^e$. Let k be the trivial kG-module, that is, each group element acts as the identity. For each i, there is an isomorphism $P_i \otimes_{\mathcal{D}} (A \rtimes G) \xrightarrow{\sim} k \otimes_{kG} (P_i \otimes_{A^e} (A \rtimes G))$, given by sending $x \otimes b$ to $1 \otimes x \otimes b$ for $x \in P_i$, $b \in A \rtimes G$. The inverse map is given by $1 \otimes x \otimes b \mapsto x \otimes b$. The action of kG on the tensor product of two modules is given as usual: g acts as $g \otimes g$ for all $g \in G$. It may be checked directly that these maps are well-defined. Finally, we see that $k \otimes_{kG} (P_i \otimes_{A^e} (A \rtimes G)) \xrightarrow{\sim} (P_i \otimes_{A^e} (A \rtimes G))_G$ by sending $1 \otimes x \otimes b$ to $G \cdot (x \otimes b)$, the orbit of $x \otimes b$. The inverse map sends $G \cdot (x \otimes b)$ to $1 \otimes x \otimes b$ (a well-defined map due to the tensor product over kG). □

We may further rewrite the expressions in the theorem: As an A^e-module, $A \rtimes G \cong \oplus_{g \in G} Ag$, which yields an isomorphism of graded vector spaces,

$$\mathrm{HH}^*(A, A \rtimes G) \cong \bigoplus_{g \in G} \mathrm{HH}^*(A, Ag). \tag{3.6.3}$$

The action of G permutes the components of the direct sum via conjugation: Letting $h \in G$, we have ${}^h(ag) = {}^h a hgh^{-1}$ for all $a \in A$ and $g \in G$, and so h takes $\mathrm{HH}^*(A, Ag)$ to $\mathrm{HH}^*(A, Ahgh^{-1})$. We may then apply h^{-1} to see that these two components are isomorphic as vector spaces, that is, the components of (3.6.3) are permuted according to the conjugation action of G on itself. The G-invariant subspace is the image of the operator $\frac{1}{|G|} \sum_{g \in G} g$ since $|G|$ is invertible in k. We choose one representative element g in each conjugacy class to rewrite the sum. By Theorem 3.6.1,

$$\mathrm{HH}^*(A \rtimes G) \cong \bigoplus_{g \in \overline{G}} (\mathrm{HH}^*(A, Ag))^{C(g)}, \tag{3.6.4}$$

where $\overline{G}$ is a set of conjugacy class representatives in G, and $C(g)$ is the centralizer in G of g. We will use this isomorphism in the example of a polynomial ring next.

Example 3.6.5. For this example, we take k to be algebraically closed, as we will need to use eigenvalues of operators. Let V be a finite dimensional kG-module, and let $A = S(V)$. The action of G on V may be extended to an action on A by algebra automorphisms. We use techniques similar to those of Farinati [**Far05**] and of Ginzburg and Kaledin [**GK04**] to find the structure of the Hochschild cohomology ring of A via Theorem 3.6.1. Let $g \in G$. We wish to find an expression for the component $\mathrm{HH}^*(S(V), S(V)g)$ of (3.6.3). We will use the Koszul resolution of $S(V)$ as an $S(V)$-bimodule, from Example 3.1.3, 3.4.7, or 3.2.5. Since the element g of G has finite order, we may choose a basis $x_1, \ldots, x_n$ of V consisting of eigenvectors of g. Let $\lambda_1, \ldots, \lambda_n \in k$ be the corresponding eigenvalues. Assume they are ordered so that $\lambda_1 = 1, \ldots, \lambda_r = 1$ and $\lambda_{r+1} \neq 1, \ldots, \lambda_n \neq 1$. The invariant subspace V^g is then the k-linear span of $x_1, \ldots, x_r$. Let

$$V_g = \mathrm{Span}_k\{x_{r+1}, \ldots, x_n\} = \mathrm{Im}(1-g) \cong V/V^g.$$

Consider the Koszul complex $K(x_1 \otimes 1 - 1 \otimes x_1, \ldots, x_n \otimes 1 - 1 \otimes x_n)$ defined in Example 3.2.5, and note that it is the tensor product over $S(V)^e$ of the two Koszul complexes $K(x_1 \otimes 1 - 1 \otimes x_1, \ldots, x_r \otimes 1 - 1 \otimes x_r)$ and

$K(x_{r+1} \otimes 1 - 1 \otimes x_{r+1}, \dots, x_n \otimes 1 - 1 \otimes x_n)$. Applying $\mathrm{Hom}_{S(V)^e}(-, S(V)g)$ and writing $S(V)g = S(V^g) \otimes S(V_g)g$, we obtain

$$\textstyle\bigwedge((V^g)^*) \otimes S(V^g) \otimes \bigwedge((V_g)^*) \otimes S(V_g)g.$$

To find the differentials, note that for each index i, for $s \in S(V)$ and $g \in G$,

$$(x_i \otimes 1 - 1 \otimes x_i) \cdot sg = x_i sg - sgx_i = x_i sg - s^g x_i g = (1 - \lambda_i)(x_i sg).$$

When $\lambda_i \neq 1$, the differential is thus just multiplication by a nonzero scalar multiple of x_i. It follows that the complex $\bigwedge((V_g)^*) \otimes S(V_g)g$ is exact other than in degree $n-r$, where it has homology $S(V_g)/(x_{r+1}, \dots, x_n)S(V_g)g \cong k$ by Theorem 3.2.4 (applying Hom reverses the arrows). We will identify this with the top exterior power $\bigwedge^{n-r}((V_g)^*)$. The complex $\bigwedge((V^g)^*) \otimes S(V^g)$, by contrast, has differentials all 0. By the Künneth Theorem (Theorem A.5.1), since the tensor product is now over the field k, the homology of the complex is $\bigwedge((V^g)^*) \otimes S(V^g) \otimes \bigwedge^{n-r}((V_g)^*)g$. Since $n - r = \operatorname{codim} V^g$, applying (3.6.4), we have

$$\mathrm{HH}^n(S(V) \rtimes G) \cong \bigoplus_{g \in \overline{G}} (S(V^g)g \otimes \textstyle\bigwedge^{n-\operatorname{codim} V^g}((V^g)^*) \otimes \bigwedge^{\operatorname{codim} V^g}((V_g)^*))^{C(g)}.$$

In this expression, the factor $\bigwedge^{\operatorname{codim} V^g}((V_g)^*)$ is isomorphic to k as a vector space, but it potentially has a nontrivial $C(g)$-action, so we retain the factor in the notation.

Exercise 3.6.6. Verify the following claims in the proof of Theorem 3.6.1:

(a) $(A \rtimes G)^e$ is free as a right $\mathcal{D}$-module.

(b) The map (3.6.2) is an $(A \rtimes G)$-bimodule isomorphism. What is the inverse map?

(c) $\mathrm{Hom}_{\mathcal{D}}(U, V) \cong \mathrm{Hom}_{A^e}(U, V)^G$ as vector spaces.

(d) $\mathrm{Ext}^*_{\mathcal{D}}(A, A \rtimes G) \cong (\mathrm{Ext}^*_{A^e}(A, A \rtimes G))^G$ as algebras.

Exercise 3.6.7. Let $k = \mathbb{C}$, let $G = S_3$, and let V be the 3-dimensional vector space that is a $\mathbb{C}G$-module by permutations of a chosen basis. Describe the graded vector space structure of $\mathrm{HH}^*(S(V) \rtimes G)$ by applying the isomorphism given at the end of Example 3.6.5.

Chapter 4

Smooth Algebras and Van den Bergh Duality

In this chapter we look at noncommutative analogs of some commutative notions, beginning with dimension, smoothness, and differential forms. These analogs are defined using Hochschild cohomology. An analog of Poincaré duality is given by Van den Bergh duality between Hochschild homology and cohomology for some types of smooth algebras. We take a closer look at Calabi-Yau algebras and skew group algebras in this light. For Calabi-Yau algebras and for symmetric algebras we further define Batalin-Vilkovisky structures on Hochschild cohomology using different types of duality.

Throughout, k will be a field and A will be a k-algebra.

4.1. Dimension and smoothness

We first use the notion of projective dimension (see Section A.2) in a definition of dimension of the algebra A.

Definition 4.1.1. The *Hochschild dimension* of A is its projective dimension as an A^e-module:

$$\dim(A) = \mathrm{pdim}_{A^e}(A).$$

Some authors simply refer to this as the *dimension* of A, and we will as well when there is no confusion possible. However there are many other types of dimension for algebras, depending on context, such as global dimension, Krull dimension, or vector space dimension. See, for example, [**MR88**] for a discussion of dimension for noncommutative rings. Note that the global dimension of A is always less than or equal to its Hochschild dimension, since given an A^e-projective resolution of A, tensoring over A with any module M yields an A-projective resolution of M, as we saw in Section 2.5.

Example 4.1.2. By our work in Example 3.1.3, $\dim(k[x_1, \ldots, x_m]) = m$. Specifically, we found there a projective resolution of A as an A^e-module of length m. There cannot exist a shorter resolution since $\mathrm{HH}^m(A) \neq 0$.

Our first result describes a relationship among the Hochschild dimensions of two algebras and that of their tensor product algebra or twisted tensor product algebra as defined in Section 3.3.

Theorem 4.1.3. *Let A and B be two algebras. Then*

(i) $\dim(A \otimes B) \leq \dim(A) + \dim(B)$, *and more generally*

(ii) *if A, B are graded by abelian groups Γ, Γ' and $t : \Gamma \times \Gamma' \to k^\times$ is a bicharacter, then* $\dim(A \otimes^t B) \leq \dim(A) + \dim(B)$ *where $A \otimes^t B$ is the twisted tensor product defined in Section 3.3.*

Proof. (i) Let $P_\bullet$ (respectively, $Q_\bullet$) be a projective resolution of A as an A^e-module (respectively, of B as a B^e-module). In Section 3.1 we showed that the total complex of the tensor product complex $P_\bullet \otimes Q_\bullet$ is a projective resolution of $A \otimes B$ as $(A \otimes B)^e$-module. The length of this total complex is the sum of the lengths of $P_\bullet$ and $Q_\bullet$. Therefore there is a projective resolution of $A \otimes B$ as $(A \otimes B)^e$-module of length $\dim(A) + \dim(B)$. It follows that $\dim(A \otimes B)$ is at most this number.

(ii) The proof of (i) applies more generally to the twisted tensor product algebra $A \otimes^t B$ by using Theorem 3.3.3. $\square$

By contrast, there is no such theorem for a more general notion of twisted tensor product that was defined by Čap, Schichl, and Vanžura [**ČSV95**]. There are conditions under which resolutions $P_\bullet$ and $Q_\bullet$ may be combined to form a complex of bimodules for this more general twisted tensor product algebra, and there are conditions under which the bimodules in such a complex are projective, but there are no general guarantees [**SW**]. Indeed, there exist algebras of Hochschild dimension 0 and a twisted tensor product of them in the general sense of [**ČSV95**] that has infinite Hochschild dimension. See, e.g., [**LP07**, Proposition 3.3.6].

The following definition is due to Van den Bergh [**Van98**].

Definition 4.1.4. The algebra A is *smooth* if its Hochschild dimension is finite and it has a finite projective resolution as an A^e-module by finitely generated projective modules.

Some authors use the term *homologically smooth* to distinguish this from other notions of smoothness. Note that if A and B are smooth, then so is

$A \otimes B$ (and more generally $A \otimes^t B$): This follows from Theorem 4.1.3 and its proof, as the (twisted) tensor product of finitely generated modules will also be finitely generated. If A is a finitely generated commutative algebra over k, this notion of smoothness is equivalent to more standard definitions as mentioned in Section 3.2 (see [**Van98**]).

Example 4.1.5. A polynomial ring $A = k[x_1, \ldots, x_m]$ is smooth by our work in Example 3.1.3 (the resolution used there consists of finitely generated free modules). A skew polynomial ring $k_{\mathbf{q}}[x_1, \ldots, x_m]$ is smooth by our work in Example 3.3.5. If G is a finite group acting on $k[x_1, \ldots, x_m]$ by degree-preserving automorphisms and $\operatorname{char}(k)$ does not divide the order of G, then the skew group algebra $k[x_1, \ldots, x_m] \rtimes G$ of Section 3.6 is smooth: The group G acts on the Koszul resolution of $k[x_1, \ldots, x_m]$, and our work in that section shows that it may be induced to a projective resolution of $k[x_1, \ldots, x_m] \rtimes G$ as a $(k[x_1, \ldots, x_m] \rtimes G)^e$-module. The quantum polynomial ring $k_{\mathbf{q}}[x_1, \ldots, x_m]$ and the skew group algebras $k[x_1, \ldots, x_m] \rtimes G$ are smooth algebras that are generally noncommutative.

Example 4.1.6. In contrast, the quantum complete intersections of Example 3.3.6, $A = k\langle x_1, x_2 \rangle (x_1x_2 - qx_2x_1,\ x_1^2,\ x_2^2)$, are not smooth: We showed there that A has infinite global dimension, forcing its Hochschild dimension to be infinite as well.

We will look closely at some algebras with Hochschild dimension 0 or 1.

Definition 4.1.7. An algebra A is *separable* if $\dim(A) = 0$. It is *quasi-free* if $\dim(A) \leq 1$.

This notion of quasi-free algebras is due to Cuntz and Quillen [**CQ95**]. Quasi-free algebras have also been called *Cuntz-Quillen smooth* or *formally smooth*.

By definition, A is separable if and only if it is projective as an A^e-module. Another equivalent condition to separability is that any derivation from A to an A-bimodule is inner: Indeed, if A is separable, then $\mathrm{HH}^1(A, M) = 0$ for all A-bimodules M. By our work in Section 1.2, the vanishing of $\mathrm{HH}^1(A, M)$ is equivalent to the statement that any derivation from A to M is inner. Conversely, suppose that A is not projective as an A^e-module. Let K_1 be the first syzygy module of A in a given projective resolution $P_{\bullet}$ of A as an A^e-module. We claim that $\mathrm{HH}^1(A, K_1) \neq 0$. To see this, note that by the definitions, $\mathrm{HH}^1(A, K_1) = \operatorname{Hom}_{A^e}(K_1, K_1)/\operatorname{Im}(i_1^*)$, where i_1 is the embedding of K_1 into P_0 as in diagram (A.2.4). This quotient is nonzero, as if not, then the identity map from K_1 to K_1 is in the image of i_1^*, which implies that the short exact sequence $0 \to K_1 \to P_0 \to A \to 0$ splits and A is projective as an A^e-module, a contradiction.

We will look at some equivalent conditions to quasi-freeness in the next section. For now, we consider some examples and implications.

Example 4.1.8. $A = k[x]$ has Hochschild dimension 1 by our work in Example 3.1.3, and so is quasi-free. However, $k[x, y]$ has Hochschild dimension 2 and so is not quasi-free. Thus the tensor product of two quasi-free algebras is not always quasi-free, and similarly for twisted tensor products. However, the free product of two quasi-free algebras is always quasi-free [**CQ95**, Proposition 5.3]. It follows that the tensor algebra $T(V)$ of a finite dimensional vector space V is quasi-free.

A quasi-free algebra is hereditary: If A is quasi-free, then there is a projective resolution $0 \to P_1 \to P_0 \to A \to 0$ of A as an A^e-module. For any A-module M, we may apply $- \otimes_A M$ to this sequence to obtain a projective resolution of M as an A-module, as explained in Section 2.5. Therefore the projective dimension $\mathrm{pdim}_A(M)$ of M is at most 1.

Example 4.1.9. Any semisimple algebra is separable, since if A is semisimple then so is A^e, and so all A^e-modules are projective. For example, if G is a finite group whose order is not divisible by the characteristic of k, then the group algebra kG is semisimple by Maschke's Theorem, and so kG is separable.

Exercise 4.1.10. Find the Hochschild dimensions of the following algebras:

(a) a skew polynomial ring $k_{\mathbf{q}}[x_1, \ldots, x_m]$.

(b) a skew group algebra $k_{\mathbf{q}}[x_1, \ldots, x_m] \rtimes G$ in case the characteristic of k does not divide the order of G.

Exercise 4.1.11. Find the Hochschild dimension of $k[x]/(x^n)$. (See Example 2.5.10.)

Exercise 4.1.12. Find the Hochschild dimension of each of the algebras $k\langle x_1, x_2\rangle/(x_1x_2 - qx_2x_1,\ x_1^{n_1},\ x_2^{n_2})$ discussed in Example 3.3.2. (See Exercise 3.3.12 and Examples 3.3.6 and 4.1.6.)

4.2. Noncommutative differential forms

We study the quasi-free algebras of Definition 4.1.7 in more detail in this section. To this end, we introduce a noncommutative version of Kähler differentials. This material and notation are from Cuntz and Quillen [**CQ95**] and Ginzburg [**Ginb**].

Let A be a k-algebra and for each $n \geq 0$, let

$$\Omega^n_{nc} A = A \otimes (\overline{A})^{\otimes n} \tag{4.2.1}$$

where $\overline{A} = A/k$ is the vector space quotient (the field k is identified with $k \cdot 1$ as a vector subspace of A). We write elements of $\overline{A}$ via notation from A, viewing $\overline{A}$ noncanonically as a vector space direct summand of A, when this will not cause confusion. We will identify the vector space $\Omega^1_{nc}A$ with the kernel of the multiplication map $\pi : A \otimes A \to A$, at the same time giving it the structure of an A-bimodule, as follows. This will indicate a connection to the Heller operator of similar notation Ω (see Section A.2) and also a comparison to Kähler differentials [**Wei94**] of similar notation Ω^1_{com} in the special case that A is commutative.

Consider $\Omega^1_{nc}A$ to be an A-bimodule under the following actions:

$$c(a \otimes b) = ca \otimes b \quad \text{and} \quad (a \otimes b)c = a \otimes bc - ab \otimes c \tag{4.2.2}$$

for all $a, b, c \in A$. Let $j : A \otimes \overline{A} \to A \otimes A$ be given by $j(a \otimes b) = ab \otimes 1 - a \otimes b$ for $a, b \in A$. Then the sequence

$$0 \to \Omega^1_{nc}A \xrightarrow{j} A \otimes A \xrightarrow{\pi} A \to 0 \tag{4.2.3}$$

is an exact sequence of A-bimodules. To see this, note that j maps $\Omega^1_{nc}A$ isomorphically onto $\mathrm{Ker}(\pi)$: Due to exactness of the bar resolution $B(A)$ of the A^e-module A given in (1.1.4), $\mathrm{Ker}(\pi) = \mathrm{Im}(d_1)$, but this is precisely the image of j. Further, j is injective: Assume that $j(\sum_i a_i \otimes b_i) = 0$ for some elements a_i, b_i. Then, since $b_i \in \overline{A}$ and

$$j(\sum_i a_i \otimes b_i) = (\sum_i a_i b_i) \otimes 1 - \sum_i a_i \otimes b_i,$$

we have $\sum_i a_i b_i = 0$ (the tensor product is over the field k). It follows that $\sum_i a_i \otimes b_i = j(\sum_i a_i \otimes b_i) = 0$. Thus the sequence (4.2.3) is exact as claimed, and $\Omega^1_{nc}A$ is a first syzygy module of A as an A-bimodule.

As one important property of the A-bimodule $\Omega^1_{nc}A$, we claim that for all A-bimodules M,

$$\mathrm{Der}(A, M) \cong \mathrm{Hom}_{A^e}(\Omega^1_{nc}A, M), \tag{4.2.4}$$

where $\mathrm{Der}(A, M)$ is the space of k-derivations from A to M, defined in Section 1.2. This isomorphism follows immediately from our work in Section 1.2 interpreting Hochschild cohomology in degree 1, as this is precisely the space of Hochschild 1-cocycles, that is the 1-cochains on the bar resolution that factor through the first syzygy module. The isomorphism (4.2.4) can be interpreted as saying that $\Omega^1_{nc}A$ represents the functor $\mathrm{Der}(A, -)$ on the category of A-bimodules.

In comparison, for commutative algebras A, we consider A-modules rather than A-bimodules. The Kähler differentials, defined next, represent the functor $\mathrm{Der}(A, -)$ on the category of A-modules [**Wei94**].

Definition 4.2.5. Let A be a commutative algebra. The A-module of *Kähler differentials* $\Omega^1_{com}A$ is the A-module with one generator da for each $a \in A$ and $dc = 0$ for all $c \in k$. Relations are

$$d(a+b) = da + db \quad \text{and} \quad d(ab) = adb + bda$$

for all $a, b \in A$.

We claim that $\Omega^1_{com}A \cong \operatorname{Ker}\pi/(\operatorname{Ker}\pi)^2 \cong (\Omega^1_{nc}A)/(\operatorname{Ker}\pi)^2$ and that $\Omega^1_{com}A \cong \mathrm{HH}_1(A)$. See Exercises 4.2.14 and 4.2.15.

Returning to the general case of a not necessarily commutative algebra A, recall the definition (4.2.1) of $\Omega^n_{nc}A$.

Definition 4.2.6. Let A be an algebra. The space of *noncommutative differential forms* on A is

$$\Omega_{nc}A = \oplus_{n\geq 0}\Omega^n_{nc}A.$$

We will see that $\Omega_{nc}A$ is a differential graded algebra with

$$\begin{aligned} d(a_0 \otimes \cdots \otimes a_n) &= 1 \otimes a_0 \otimes \cdots \otimes a_n, \\ (a_0 \otimes \cdots \otimes a_n)(a_{n+1} \otimes \cdots \otimes a_r) &= \sum_{i=0}^{n} (-1)^{n-i} a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_r \end{aligned}$$

for all $a_0, \ldots, a_r \in A$, a theorem of Cuntz and Quillen [**CQ95**, Proposition 1.1]. Moreover, $\Omega_{nc}A$ is universal with respect to differential graded algebras whose degree 0 term is the target of an algebra homomorphism from A:

Theorem 4.2.7. *The space $\Omega_{nc}A$ of noncommutative differential forms on an algebra A is a differential graded algebra with differential and multiplication given above, unique such that*

$$a_0(da_1)\cdots(da_n) = a_0 \otimes \cdots \otimes a_n$$

for all $a_0, \ldots, a_n \in A$. Moreover, for any differential graded algebra Γ and algebra homomorphism $u : A \to \Gamma^0$, there is a unique differential graded algebra homomorphism $u_* : \Omega_{nc}A \to \Gamma$ that extends u.

Proof. It may be checked directly that $\Omega_{nc}A$ is indeed a differential graded algebra. (See Exercise 4.2.16 or [**CQ95**, Proposition 1.1].)

For the second statement, let Γ be a differential graded algebra and $u : A \to \Gamma^0$ an algebra homomorphism. Define $u_* : \Omega_{nc}A \to \Gamma$ by

$$u_*(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = u(a_0)du(a_1)\cdots du(a_n).$$

It may be checked that u_* is a homomorphism of differential graded algebras, and it is uniquely determined. $\square$

Definition 4.2.8. A *square-zero extension* of A is an algebra R such that $A \cong R/I$ for some ideal I of R with $I^2 = 0$.

The ideal I in the definition is necessarily an A-bimodule since $I^2 = 0$: Given an element $a = r+I \in A$ for some $r \in R$, and given $x \in I$, define $ax = rx$ and $xa = xr$. Every A-bimodule M determines a square-zero extension: Let $R = A \oplus M$ and define $(a_1, m_1) \cdot (a_2, m_2) = (a_1a_2, a_1m_2 + m_1a_2)$ for all $a_1, a_2 \in A$ and $m_1, m_2 \in M$. This is called the *trivial extension.* More generally, let $f : A \otimes A \to M$ be a Hochschild 2-cocycle, that is,

$$af(b \otimes c) + f(a \otimes bc) = f(ab \otimes c) + f(a \otimes b)c,$$

for all $a, b, c \in A$ as in (1.2.2). Then $R = A \oplus M$ is a ring with

$$(a_1, m_1)(a_2, m_2) = (a_1a_2, a_1m_2 + m_1a_2 + f(a_1 \otimes a_2)) \tag{4.2.9}$$

for all $a_1, a_2 \in A$ and $m_1, m_2 \in M$. (Associativity is equivalent to the above Hochschild 2-cocycle condition.) In fact the following theorem holds:

Theorem 4.2.10. *Let M be an A-bimodule. Then $\mathrm{HH}^2(A, M)$ is in one-to-one correspondence with equivalence classes of square-zero extensions of A by M.*

Proof. We have already seen that a representative element of $\mathrm{HH}^2(A, M)$, at the chain level, determines a square-zero extension of A by M. A calculation shows that a coboundary corresponds to a square-zero extension that is isomorphic to the trivial extension. Conversely, given a square-zero extension R of A by $I = M$, choose an A-bimodule isomorphism $A \oplus I \to R$ that sends each element of I to itself in R and when composed with the quotient map $R \to A$, is the identity on A. This is possible due to the A-bimodule structure of I described earlier. Let $a, b \in A$ and $x, y \in I$, and identify (a, x) and (b, y) with their images in R. Then

$$\begin{aligned}
(a, x)(b, y) &= (a, 0)(b, 0) + (a, 0)(0, y) + (0, x)(b, 0) + (0, x)(0, y) \\
&= (a, 0)(b, 0) + (1, 0)a(0, y) + (0, x)b(1, 0) \\
&= (a, 0)(b, 0) + (0, ay) + (0, xb),
\end{aligned}$$

since $I^2 = 0$. Necessarily

$$(a, 0)(b, 0) = (ab, f(a \otimes b))$$

for some function $f : A \otimes A \to I$ that is a Hochschild 2-cocycle. A calculations shows that a different choice of map $A \oplus I \to R$ yields a cohomologous cocycle. $\square$

Square-zero extensions, noncommutative differential forms, and quasi-free algebras are all related as stated in the following theorem. By a *lifting* of a square-zero extension R of A, we mean an A-bimodule structure on R

extending that on I and an A-bimodule homomorphism $A \to R$ that is a section of the quotient map $R \to A$.

Theorem 4.2.11. *The following are equivalent for an algebra A:*

(i) *A is quasi-free.*

(ii) *$\Omega^1_{nc}A$ is a projective A^e-module.*

(iii) *$\mathrm{HH}^2(A, M) = 0$ for all A-bimodules M.*

(iv) *For any square-zero extension R of A, there is a lifting $A \to R$.*

Proof. We have identified $\Omega^1_{nc}A$ with the first syzygy of A as an A^e-module. If A is quasi-free, there exists a projective resolution $P_\bullet$ of A as an A^e-module of length 1, that is, $P_i = 0$ for all $i \geq 2$. Thus the first syzygy module is P_1, and so is projective. By Schanuel's Lemma (Lemma A.2.5), any other first syzygy module is projective as well, so in particular $\Omega^1_{nc}A$ is projective. Thus (i) implies (ii). Conversely, if $\Omega^1_{nc}A$ is a projective A^e-module, then the Hochschild dimension of A is at most 1, so A is quasi-free, that is, (ii) implies (i).

By dimension shifting (Theorem A.3.3),

$$\mathrm{HH}^2(A, M) \cong \mathrm{Ext}^1_{A^e}(\Omega^1_{nc}A, M).$$

So (ii) implies (iii). Conversely, assume that $\mathrm{Ext}^1_{A^e}(\Omega^1_{nc}A, M) = 0$ for all A-bimodules M, so that every A^e-extension of $\Omega^1_{nc}A$ splits. Map a projective A^e-module onto $\Omega^1_{nc}A$, and consider the extension of $\Omega^1_{nc}A$ by the kernel of this map. It splits, forcing $\Omega^1_{nc}A$ itself to be projective. So (iii) implies (ii).

Finally, if $\mathrm{HH}^2(A, M) = 0$ for all A-bimodules M, then every square-zero extension splits by Theorem 4.2.10. So if R is a square-zero extension of A, there is a lifting $A \to R$. That is, (iii) implies (iv). Conversely, let R be a square-zero extension of A by an ideal I. Assume there is a lifting $A \to R$. The lifting is a splitting of the sequence of A-bimodules, $0 \to I \to R \to A \to 0$. So if any square-zero extension lifts, then in particular the square-zero extension $A \oplus M$ lifts for any A-bimodule M, and so the sequence $0 \to M \to A \oplus M \to A \to 0$ splits as a sequence of A-bimodules. By Theorem 4.2.10, $\mathrm{HH}^2(A, M) = 0$ for all A-bimodules M. Thus (iv) implies (iii). $\square$

Schelter [**Sch86**] proposed a condition similar to (iv) in the theorem as a definition of smoothness for noncommutative algebras.

In case A is commutative, consider the related condition that for every *commutative* square-zero extension R of A, there is a lifting $A \to R$. This is equivalent to smoothness for commutative algebras [**Wei94**, Section 9.3.1] but is a weaker condition than being quasi-free. In fact, a commutative algebra is smooth in the sense of Definition 4.1.4 if and only if it is smooth in this classical sense [**Van98**].

Exercise 4.2.12. Verify that (4.2.2) does indeed give $\Omega^1_{nc}A$ the structure of an A-bimodule, that is, (a) the action is well-defined, and (b) it is an A-bimodule action.

Exercise 4.2.13. Verify that the map j in (4.2.3) is a well-defined A-bimodule map.

Exercise 4.2.14. Show that if A is a commutative algebra, then the A-module $\Omega^1_{com}A$ may equivalently be defined as

$$A \otimes A/(ab \otimes c - a \otimes bc + ac \otimes b \mid a, b, c \in A)$$

by considering the left A-module map from $A \otimes A$ to $\Omega^1_{com}A$ that sends $1 \otimes b$ to db for all $b \in A$. This can also be identified with Hochschild homology $\mathrm{HH}_1(A)$.

Exercise 4.2.15. Verify the isomorphism $\Omega^1_{com}A \cong \operatorname{Ker}\pi/(\operatorname{Ker}\pi)^2$ for a commutative algebra A by using Exercise 4.2.14 and the map $A \otimes A \to I$ that sends $a \otimes b$ to $ab \otimes 1 - a \otimes b$ for all $a, b \in A$.

Exercise 4.2.16. Verify the claims from the proof of Theorem 4.2.7:

(a) $\Omega_{nc}A$ is a differential graded algebra. In particular, check that $d^2 = 0$, the multiplication is associative, and the differential is a graded derivation.

(b) The map u_* is a homomorphism of differential graded algebras, and is unique.

Exercise 4.2.17. Verify that formula (4.2.9) defines an associative multiplication on $A \oplus M$.

4.3. Van den Bergh duality and Calabi-Yau algebras

For some smooth noncommutative algebras, there exists a duality between Hochschild homology and cohomology, analogous to Poincaré duality in geometry. We state this duality in Theorem 4.3.2 below. A special case is Corollary 4.3.8 for Calabi-Yau algebras, also defined in this section.

In general, if P is a left A^e-module then $\mathrm{Hom}_{A^e}(P, A^e)$ is a right A^e-module by setting $(f \cdot (a \otimes b))(p) = bf(p)a$ for all $a, b \in A$, $p \in P$, and $f \in \mathrm{Hom}_{A^e}(P, A^e)$. This right A^e-module structure induces the structure of a right A^e-module on the Hochschild cohomology space of A with coefficients in A^e, that is on $\mathrm{HH}^i(A, A^e)$ for each i. This action is used in the proof of Theorem 4.3.2 below.

Definition 4.3.1. An A-bimodule U is *invertible* if there is an A-bimodule V such that $U \otimes_A V \cong A$ and $V \otimes_A U \cong A$ as A-bimodules.

The invertible A-bimodules correspond one-to-one with autoequivalences of the category A-mod, that is, equivalences between the category of A-modules and itself, given by tensor product with the invertible A-bimodules. The identity autoequivalence is given by the invertible bimodule A. See Exercise 4.3.9.

An invertible A-bimodule gives a duality between Hochschild homology $\mathrm{HH}_*(A)$ and and cohomology $\mathrm{HH}^*(A)$ under some conditions, as stated in the following theorem of Van den Bergh [**Van98**, Theorem 1].

Theorem 4.3.2. *Let A be a smooth algebra. Assume that there is a positive integer d for which $\mathrm{HH}^i(A, A^e) = 0$ unless $i = d$ and that $U = \mathrm{HH}^d(A, A^e)$ is an invertible A-bimodule. Then*

$$\mathrm{HH}^n(A, M) \cong \mathrm{HH}_{d-n}(A, U \otimes_A M)$$

for all A-bimodules M and $n \in \{0, \dots, d\}$, and $\mathrm{HH}^n(A, M) = 0$ for $n > d$.

Definition 4.3.3. If the hypotheses of the theorem are satisfied, we call U the *dualizing bimodule* of A and we say that A has *Van den Bergh duality.*

Proof of Theorem 4.3.2. This proof is a special case of a proof in [**Krä07**]. Since A is smooth, there is an A^e-projective resolution $P_\bullet$ of A such that each P_i is finitely generated and $P_i = 0$ for $i > \dim(A)$. Let $Q_\bullet$ be an A^e-projective resolution of M. Since U is invertible, the functor $U \otimes_A -$ is a category equivalence, and so $U \otimes_A Q_\bullet$ is an A^e-projective resolution of $U \otimes_A M$. Let

$$C_{p,q} = \mathrm{Hom}_{A^e}(P_{-p}, Q_q)$$

for all $p \leq 0$, $q \geq 0$. We claim that since P_{-p} is finitely generated projective, for each p, q, there is an isomorphism of vector spaces,

$$\mathrm{Hom}_{A^e}(P_{-p}, A^e) \otimes_{A^e} Q_q \xrightarrow{\sim} C_{p,q} \tag{4.3.4}$$

given by $f \otimes y \mapsto (x \mapsto f(x)y)$ for all $f \in \mathrm{Hom}_{A^e}(P_{-p}, A^e)$ and $y \in Q_q$. Indeed, in case $P_{-p} = A^e$, this is clearly an isomorphism, as it is if P_{-p} is a free module of finite rank. It then follows for any finitely generated projective P_{-p}. Note that the tensor product here is taken over A^e instead of over A.

We now give two proofs of the isomorphism stated in the theorem, both using the bicomplex $C_{\bullet,\bullet}$. The proofs are essentially the same, however the first uses the Acyclic Assembly Lemma (Theorem A.5.4) in two ways, and the second uses a comparison of two spectral sequences for a bicomplex (Section A.7).

The first proof is as follows. Let $\varepsilon : Q_0 \to M$ be the augmentation of the projective resolution $Q_\bullet$ of M. There is an induced map

$$\mathrm{Tot}(C_{\bullet,\bullet}) \xrightarrow{\varepsilon_*} \mathrm{Hom}_{A^e}(P_{-\bullet}, M).$$

Similarly, using the above isomorphism (4.3.4), there is a map

$$\mathrm{Tot}(C_{\bullet,\bullet}) \longrightarrow U \otimes_{A^e} Q_\bullet.$$

We claim that this second map is a quasi-isomorphism, and with the corresponding degree shift by d, this results in the isomorphism stated in the theorem. To prove the claim, consider the columns in the following diagram:

$$\begin{tikzcd}
& \vdots \arrow[d] & & \vdots \arrow[d] & \\
0 & \mathrm{Hom}_{A^e}(P_d, Q_2) \arrow[l] \arrow[d] & \cdots \arrow[l] & \mathrm{Hom}_{A^e}(P_0, Q_2) \arrow[l] \arrow[d] & 0 \arrow[l] \\
0 & \mathrm{Hom}_{A^e}(P_d, Q_1) \arrow[l] \arrow[d] & \cdots \arrow[l] & \mathrm{Hom}_{A^e}(P_0, Q_1) \arrow[l] \arrow[d] & 0 \arrow[l] \\
0 & \mathrm{Hom}_{A^e}(P_d, Q_0) \arrow[l] \arrow[d] & \cdots \arrow[l] & \mathrm{Hom}_{A^e}(P_0, Q_0) \arrow[l] \arrow[d] & 0 \arrow[l] \\
& 0 & & 0 &
\end{tikzcd}$$

Since each P_i is projective and $Q_\bullet \xrightarrow{\varepsilon} M$ is exact, the ith column is exact when augmented with $\mathrm{Hom}_{A^e}(P_i, M)$. By the Acyclic Assembly Lemma (Theorem A.5.4), the resulting total complex is acyclic. Similarly, instead augmenting each jth row with $U \otimes_{A^e} Q_j$ on the left, by the hypothesis that $\mathrm{HH}^i(A, A^e) = 0$ for $i \neq d$ and the isomorphism (4.3.4), the rows will be exact. By the Acyclic Assembly Lemma (Theorem A.5.4), the resulting total complex is acyclic. We claim that $U \otimes_{A^e} Q_j \cong A \otimes_{A^e} (U \otimes_A Q_j)$ as A^e-modules for all j. To see this, first note that it is true for a free A^e-module since $U \xrightarrow{\sim} A \otimes_{A^e} (U \otimes_A A^e)$ via the map $u \mapsto 1 \otimes (u \otimes (1 \otimes 1))$ which has inverse $a \otimes (u \otimes (b \otimes c)) \mapsto caub$. Each Q_j is projective, so is a direct summand of a free module, and this isomorphism preserves such a direct sum. This shows that the cohomology of the total complex of $C_{\bullet,\bullet}$ is $\mathrm{HH}^\bullet(A, M)$ on the one hand, and it is $\mathrm{HH}_{d-\bullet}(A, U \otimes_A M)$ on the other, proving the theorem.

For a second proof, we may apply the two spectral sequences of the double complex $C_{\bullet,\bullet}$ described in Section A.7. Let E'' denote the first described spectral sequence in which $E''_1 \cong \mathrm{H}''(C)$, $C''_2 \cong \mathrm{H}'(\mathrm{H}''(C))$. Using the same arguments as above, we find that E''_1 consists of $\mathrm{Hom}_{A^e}(P_\bullet, M)$ in the bottom row, with 0 in all other components, and thus E_2'' consists of $\mathrm{HH}^\bullet(A,M)$ in the bottom row, with zero differentials. So the spectral sequence collapses and this is the cohomology of $C_{\bullet,\bullet}$. Let E' denote the second described spectral sequence in which $E_1' \cong \mathrm{H}'(C)$, $E_2' \cong \mathrm{H}''(\mathrm{H}'(C))$. We find that E_1' consists of $U \otimes_{A^e} Q_\bullet$ in the left column and E_2' is thus $\mathrm{Tor}^{A^e}_{d-\bullet}(U,M)$, with zero differentials. Since $U \otimes_{A^e} Q_i \cong A \otimes_{A^e} (U \otimes_A Q_i)$ for all i (see above), this is $\mathrm{HH}_{d-\bullet}(A, U \otimes_A M)$, the cohomology of $C_{\bullet,\bullet}$, completing the second proof. □

We next show that polynomial rings have Van den Bergh duality.

Example 4.3.5. Let $A = k[x]$. Consider the Koszul resolution (1.1.17) of A as an A^e-module. Apply $\mathrm{Hom}_{k[x]^e}(-, k[x]^e)$ to obtain

$$0 \longleftarrow \mathrm{Hom}_{k[x]^e}(k[x]^e, k[x]^e) \longleftarrow \mathrm{Hom}_{k[x]^e}(k[x]^e, k[x]^e) \longleftarrow 0.$$

Under the isomorphism $\mathrm{Hom}_{k[x]^e}(k[x]^e, k[x]^e) \cong \mathrm{Hom}_k(k, k[x]^e) \cong k[x]^e$, this sequence is equivalent to

$$0 \longleftarrow k[x]^e \longleftarrow k[x]^e \longleftarrow 0,$$

the nonzero map being given by multiplication by $x \otimes 1 - 1 \otimes x$. So $\mathrm{HH}^0(k[x], k[x]^e) = 0$ and $\mathrm{HH}^1(k[x], k[x]^e) \cong k[x]$, an invertible $k[x]$-bimodule. The hypotheses of Theorem 4.3.2 are satisfied and so

$$\mathrm{HH}^n(k[x], M) \cong \mathrm{HH}_{1-n}(k[x], M)$$

for all $k[x]$-bimodules M and $n = 0, 1$. A similar argument applies to a polynomial ring in more indeterminates, and we find that

$$\mathrm{HH}^m(k[x_1,\dots,x_m], k[x_1,\dots,x_m]^e) \cong k[x_1,\dots,x_m],$$

while $\mathrm{HH}^i(k[x_1,\dots,x_m], k[x_1,\dots,x_m]^e) = 0$ for $i \neq m$, and so

$$\mathrm{HH}^n([k[x_1,\dots,x_m], M) \cong \mathrm{HH}_{m-n}(k[x_1,\dots,x_m], M)$$

for all $k[x_1,\dots,x_m]$-bimodules M and $n = 0,\dots,m$.

Definition 4.3.6. A smooth algebra A is *Calabi-Yau* if it has Van den Bergh duality with dualizing bimodule $U \cong A$.

Example 4.3.7. As a consequence of our work in Example 4.3.5, polynomial rings $k[x_1,\dots,x_m]$ are Calabi-Yau.

Calabi-Yau algebras were first defined by Ginzburg [**Gina**] as an analog, in the noncommutative setting, of rings of functions on Calabi-Yau varieties. There is also a notion of a twisted Calabi-Yau algebra: In the definition of Calabi-Yau algebra, allow more generally an isomorphism $U \cong A_\sigma$, twisted by an algebra automorphism σ on the right, that is, the right action of A on A_σ is given by $a \cdot b = a\sigma(b)$ for all $a, b \in A$. See, e.g., [**GK14**].

By replacing U by A in Theorem 4.3.2 and applying the isomorphism $A \otimes_A M \cong M$, we obtain the following corollary.

Corollary 4.3.8. *If A is a Calabi-Yau algebra of Hochschild dimension d, then*

$$\mathrm{HH}^n(A, M) \cong \mathrm{HH}_{d-n}(A, M)$$

for all A-bimodules M and integers n.

Examples of noncommutative Calabi-Yau algebras include some Sklyanin algebras [**MS**] and some deformed preprojective algebras [**Ami**]. Skew group algebras can be Calabi-Yau, and we give details for some of these examples in the next section.

Exercise 4.3.9. Prove that invertible bimodules correspond with autoequivalences of the category A-mod.

Exercise 4.3.10. Let $A = k[x_1, \ldots, x_m]$. Verify the claimed structure of $\mathrm{HH}^i(A, A^e)$ stated in Example 4.3.5.

Exercise 4.3.11. Let $A = k_q[x_1, x_2]$, as defined in Example 3.3.1. Find $\mathrm{HH}^i(A, A^e)$ for each i. Is A Calabi-Yau?

Exercise 4.3.12. Let $A = k\langle x_1, x_2\rangle/(x_1x_2 - qx_2x_1,\ x_1^{n_1},\ x_2^{n_2})$, as in Example 3.3.6. Find $\mathrm{HH}^i(A, A^e)$ for each i. Is A Calabi-Yau? Cf. Exercise 3.3.12.

4.4. Skew group algebras

Let G be a finite group, let k be a field of characteristic not dividing the order of G, and let V be a kG-module of finite dimension d as a vector space. In this section we show that the skew group algebra $A = S(V) \rtimes G$ has Van den Bergh duality and determine conditions under which it is Calabi-Yau. In Section 3.6, we found expressions for the Hochschild homology and cohomology of A. Here, Van den Bergh duality leads to an alternate computation of the Hochschild cohomology ring in the case that k is algebraically closed, based on knowing Hochschild homology, due to Farinati [**Far05**].

By our work in Example 4.3.5,

$$\mathrm{HH}^d(S(V), S(V)^e) \cong S(V) \otimes \textstyle\bigwedge^d(V^*)$$

and $\mathrm{HH}^n(S(V), S(V)^e) = 0$ for $n \neq d$. We retain the tensor factor $\bigwedge^d(V^*)$ in this expression even though it is a one-dimensional vector space, as it may have a nontrivial group action. The dual vector space $V^* = \mathrm{Hom}_k(V, k)$ is a kG-module via $({}^g f)(v) = f({}^{g^{-1}}v)$ for all $g \in G$, $f \in V^*$, and $v \in V$, and G acts factorwise on the tensor product $S(V) \otimes \bigwedge^d(V^*)$.

Note that as an $S(V)^e$-module, $A^e \cong S(V)^e \otimes k(G \times G)$, where $S(V)^e$ acts only on the left tensor factor of $S(V)^e$. (Map $sg \otimes g's' \mapsto (s \otimes s') \otimes (g, (g')^{-1})$

for all $s, s' \in S(V)$ and $g, g' \in G$.) Applying first the techniques in the proof of Theorem 3.6.1, and then Van den Bergh duality for $S(V)$ as in Example 4.3.5, we obtain isomorphisms for each n:

$$\begin{aligned} \mathrm{HH}^n(A, A^e) &\cong \mathrm{HH}^n(S(V), A^e)^G \\ &\cong \mathrm{HH}^n(S(V), S(V)^e \otimes k(G \times G))^G \\ &\cong \mathrm{HH}_{d-n}(S(V), S(V)^e \otimes k(G \times G))^G. \end{aligned}$$

This is Hochschild homology, obtained from the tensor product of an $S(V)^e$-projective resolution of $S(V)$, over $S(V)^e$, with $S(V)^e \otimes k(G \times G)$. By definition, the differential is the identity on the factor $k(G \times G)$ and so tensoring with $k(G \times G)$ commutes with taking homology. That is, paying close attention to placement of parentheses,

$$\mathrm{HH}^n(A, A^e) \cong (\mathrm{HH}_{d-n}(S(V), S(V)^e) \otimes k(G \times G))^G.$$

Now apply Van den Bergh duality again to obtain

$$\mathrm{HH}^n(A, A^e) \cong (\mathrm{HH}^n(S(V), S(V)^e) \otimes k(G \times G))^G.$$

If $n \neq d$, this is 0, while if $n = d$, this is isomorphic to

$$(S(V) \otimes \textstyle\bigwedge^d(V^*) \otimes k(G \times G))^G$$

as a vector space. We claim that the A-bimodule structure on the above space is as it is on $A \otimes \bigwedge^d(V^*)$: If G acts by linear transformations of determinant 1, the isomorphism is given by

$$\begin{aligned} S(V) \rtimes G &\longrightarrow (S(V) \otimes k(G \times G))^G \\ s \otimes g &\mapsto \frac{1}{|G|} \sum_{h \in G} {}^h s \otimes (hg, h^{-1}) \end{aligned}$$

for all $s \in S(V)$ and $g \in G$. In the more general case where G does not act by linear transformations of determinant 1, a similar isomorphism also yields $\mathrm{HH}^d(A, A^e) \cong A \otimes \bigwedge^d(V^*)$. Thus in either case, A has Van den Bergh duality with dualizing bimodule $A \otimes \bigwedge^d(V^*)$, so that the following theorem holds.

Theorem 4.4.1. *Let G be a finite group, let k be a field of characteristic not dividing the order of G, and let V be a kG-module of finite dimension d as a vector space. The skew group ring $S(V) \rtimes G$ has Van den Bergh duality with dualizing bimodule $(S(V) \rtimes G) \otimes \bigwedge^d(V^*)$. If G acts on V via linear transformations of determinant 1, then $S(V) \rtimes G$ is Calabi-Yau.*

Now assume further that k is algebraically closed. We use Van den Bergh duality to compute Hochschild cohomology from Hochschild homology. We begin with a computation of Hochschild homology. By Theorem 3.6.1 and further analysis similar to that leading to expression (3.6.4) for Hochschild cohomology, the Hochschild homology of A is

$$\begin{aligned}\mathrm{HH}_n(S(V)\rtimes G) &\cong \mathrm{HH}_n(S(V),S(V)\rtimes G)_G\\ &\cong \bigoplus_{g\in\overline{G}}\mathrm{HH}_n(S(V),S(V)g)_{C(g)}\end{aligned}$$

where $\overline{G}$ is a set of representatives of conjugacy classes of G and $C(g)$ is the centralizer in G of g. Letting V^g be the subspace of V invariant under g and $V_g=\mathrm{Im}(1-g)$, we have $V=V^g\oplus V_g$ and an $S(V)$-bimodule isomorphism $S(V)g\cong S(V^g)\otimes S(V_g)g$. By Exercise 3.1.10, it now follows that

$$\begin{aligned}\mathrm{HH}_n(S(V),S(V)g)_{C(g)} &\cong \mathrm{HH}_n(S(V^g)\otimes S(V_g),S(V^g)\otimes S(V_g)g)_{C(g)}\\ &\cong \left(\bigoplus_{p+q=n}\mathrm{HH}_p(S(V^g))\otimes \mathrm{HH}_q(S(V_g),S(V_g)g)\right)_{C(g)}.\end{aligned}$$

By Exercise 3.1.11 or Theorem 3.2.6, $\mathrm{HH}_*(S(V^g))\cong S(V^g)\otimes \bigwedge(V^g)$. We compute $\mathrm{HH}_*(S(V_g),S(V_g)g)$. Diagonalize the action of g on V_g, so that V_g has a basis of eigenvectors for g: namely $x_1,\ldots,x_r$ with eigenvalues $\lambda_1,\ldots,\lambda_r$. By the Künneth Theorem (Theorem A.5.1),

$$\mathrm{HH}_*(S(V_g),S(V_g)g)\cong \otimes_{i=1}^r \mathrm{HH}_*(k[x_i],k[x_i]g).$$

We claim that $\mathrm{HH}_0(k[x_i],k[x_i]g)\cong k$ and $\mathrm{HH}_1(k[x_i],k[x_i]g)=0$ since $\lambda_i\neq 1$ (as $x_i\in V_g$): If we consider the $k[x]^e$-module $k[x]g$ where ${}^gx=\lambda x$ for some scalar λ, then applying $-\otimes_{k[x]^e}k[x]g$ to (1.1.17), we obtain

$$0\to k[x]g\to k[x]g\to 0$$

where the nonzero map is given by applying $x\otimes 1-1\otimes x$, and thus becomes multiplication by $(1-\lambda)x$. So $\mathrm{HH}_0(k[x],k[x]g)\cong k$ and $\mathrm{HH}_1(k[x],k[x]g)=0$, as claimed. Note also that $h\in C(g)$ may be simultaneously diagonalized with g. So we now have

$$\mathrm{HH}_n(S(V),S(V)g)_{C(g)}\cong (S(V^g)\otimes \textstyle\bigwedge^n(V^g))_{C(g)}.$$

Now we may compute Hochschild cohomology via Van den Bergh duality: From our earlier work, using dualizing bimodule $A\otimes\bigwedge^d(V^*)$, we have

$$\begin{aligned}\mathrm{HH}^n(S(V)\rtimes G) &\cong \mathrm{HH}_{d-n}(S(V)\rtimes G,(S(V)\rtimes G)\otimes \textstyle\bigwedge^d(V^*))\\ &\cong \bigoplus_{g\in\overline{G}}(S(V^g)\otimes \textstyle\bigwedge^{d-n}(V^g)\otimes\bigwedge^d(V^*))_{C(g)}.\end{aligned}$$

Note that

$$\begin{aligned}\textstyle\bigwedge^{d-n}(V^g)\otimes\bigwedge^d(V^*) &\cong \textstyle\bigwedge^{d-n}(V^g)\otimes\bigwedge^{\dim V^g}((V^g)^*)\wedge\bigwedge^{\operatorname{codim} V^g}((V_g)^*)\\ &\cong \textstyle\bigwedge^{n-\operatorname{codim} V^g}((V^g)^*)\otimes\bigwedge^{\operatorname{codim} V^g}((V_g)^*)\end{aligned}$$

as $kC(g)$-modules via the evaluation map pairing V and V^*, noting also that the vector space dimension of $\bigwedge^{n-\operatorname{codim} V^g}((V^g)^*)$ is the same as that of $\bigwedge^{d-n}((V^g)^*)$ since $d = \dim_k V$. In comparison to our calculation of Example 3.6.5, the duality makes the degree shift due to the factor $\bigwedge^{\operatorname{codim} V^g}((V^g)^*)$ appear very naturally. In order to compare with our result in Example 3.6.5, we identify the space of orbits under the action of $C(g)$ with its image under the map $s \mapsto \frac{1}{|C(g)|}\sum_{h\in C(g)} {}^h s$.

Exercise 4.4.2. Under the hypotheses at the start of the section, what is the inverse of the dualizing bimodule $(S(V) \rtimes G) \otimes \bigwedge^d(V^*)$?

Exercise 4.4.3. Let $k = \mathbb{C}$, let $G = S_3$, and let V be the 3-dimensional vector space that is a $\mathbb{C}G$-module by permutations of a chosen basis. Use Van den Bergh duality to determine $\mathrm{HH}^*(S(V) \rtimes G)$ and compare to the results of Exercise 3.6.7.

4.5. Connes differential and Batalin-Vilkovisky structure

We introduce the Connes differential on Hochschild homology. For Calabi-Yau algebras, we use this differential in combination with Van den Bergh duality to define a new operation on Hochschild cohomology, called a Batalin-Vilkovisky operator. For finite dimensional symmetric algebras, we use it in combination with a different duality relation to define a Batalin-Vilkovisky structure. The former method can be generalized to some twisted Calabi-Yau algebras, and the latter to some Frobenius algebras. The Connes differential arises in cyclic homology [**Lod98**].

Recall the contracting homotopy $s_\bullet$ for the bar resolution, given by maps $s : A^{\otimes(n+1)} \to A^{\otimes(n+2)}$ for each $n \geq 0$ in (1.1.3) as:

$$s(a_0 \otimes \cdots \otimes a_n) = 1 \otimes a_0 \otimes \cdots \otimes a_n$$

for all $a_0, \dots, a_n \in A$. (We suppress indices on maps here for legibility in formulas, generically writing s in place of s_{n-1}.) Define $t : A^{\otimes(n+1)} \to A^{\otimes(n+1)}$ to be the signed cyclic permutation of tensor factors given by

$$t(a_0 \otimes \cdots \otimes a_n) = (-1)^n a_n \otimes a_0 \otimes \cdots \otimes a_{n-1}$$

for all $a_0, \dots, a_n \in A$, and define $N : A^{\otimes(n+1)} \to A^{\otimes(n+1)}$ by

$$N = 1 + t + t^2 + \cdots + t^n.$$

Definition 4.5.1. The *Connes differential* $\mathcal{B} : A^{\otimes(n+1)} \to A^{\otimes(n+2)}$ is the map defined for each $n \geq 0$ by

$$\mathcal{B} = (1 - t)sN.$$

Calculations show that $\mathcal{B}$ is a chain map of degree 1, on the bar complex, and so it induces a map on Hochschild homology.

Calabi-Yau algebras. Assume A is a Calabi-Yau algebra of Hochschild dimension d so that Corollary 4.3.8 implies $\mathrm{HH}^n(A) \cong \mathrm{HH}_{d-n}(A)$ for all n. We define an operator

$$\Delta : \mathrm{HH}^n(A) \to \mathrm{HH}^{n-1}(A)$$

to be that induced by the Connes differential $\mathcal{B}$ under the Van den Bergh duality isomorphism. That is, define Δ by the following commuting diagram for all n:

$$\begin{tikzcd}
\mathrm{HH}^n(A) \arrow[r, "\Delta"] \arrow[d, "\cong"] & \mathrm{HH}^{n-1}(A) \arrow[d, "\cong"] \\
\mathrm{HH}_{d-n}(A) \arrow[r, "\mathcal{B}"] & \mathrm{HH}_{d-n+1}(A)
\end{tikzcd}$$

Definition 4.5.2. The map Δ defined by the above diagram is the *Batalin-Vilkovisky operator* on the Hochschild cohomology ring of the Calabi-Yau algebra A.

Remark 4.5.3. There is a relationship between the Batalin-Vilkovisky operator Δ and the Gerstenhaber bracket: Let A be a Calabi-Yau algebra and let α, β be homogeneous elements in $\mathrm{HH}^*(A)$. Then

$$[\alpha, \beta] = \Delta(\alpha \smile \beta) - \Delta(\alpha) \smile \beta - (-1)^{|\alpha|}\alpha \smile \Delta(\beta).$$

For a proof, see, e.g., [**Gina**, Section 9.3].

For a Calabi-Yau algebra A, we say that $(\mathrm{HH}^*(A), \smile, [\ ,\ ], \Delta)$ is a *Batalin-Vilkovisky algebra*. Such a structure on Hochschild cohomology has also been defined similarly for some twisted Calabi-Yau algebras [**KK14**].

Symmetric algebras. A different duality can be used to define a Batalin-Vilkovisky structure in the case of a finite dimensional symmetric algebra, see, e.g., [**Tra08**], and for some Frobenius algebras, see [**LZZ16, YV**]. We give a description of this structure for symmetric algebras based on ideas in [**LZZ16**].

Let A be a finite dimensional symmetric algebra, that is, there is a non-degenerate symmetric associative bilinear form $\langle\ ,\ \rangle : A \times A \to k$. Give the dual vector space $D(A) = \mathrm{Hom}_k(A, k)$ the structure of an A^e-module via $(afb)(c) = f(bca)$ for all $a, b, c \in A$ and $f \in D(A)$. (We use the notation $D(A)$ here rather than A^* to avoid confusion.) The form $\langle\ ,\ \rangle$ gives an isomorphism $A \cong D(A)$ as A^e-modules. Let $B(A)$ denote the bar resolution (1.1.4) of A as an A^e-module. For Hochschild homology, we consider the complex $A \otimes_{A^e} B(A)$ with terms $A \otimes_{A^e} B_n(A)$, which are left A-modules by multiplication on the leftmost tensor factor. Note that the corresponding tensor induced A^e-module is

$$A^e \otimes_A (A \otimes_{A^e} B_n(A)) \cong B_n(A).$$

Applying the Nakayama relations (Lemma A.6.1) twice (the first time involving coinduction from k to A and the second time involving induction from A to A^e), we find

$$\begin{aligned}\operatorname{Hom}_k(A \otimes_{A^e} B_n(A), k) &\cong \operatorname{Hom}_A(A \otimes_{A^e} B_n(A), D(A)) \\ &\cong \operatorname{Hom}_{A^e}(B_n(A), D(A)) \\ &\cong \operatorname{Hom}_{A^e}(B_n(A), A),\end{aligned}$$

the last isomorphism due to the isomorphism $D(A) \cong A$ of A^e-modules. Taking cohomology of the corresponding complexes, we obtain

$$D(\mathrm{HH}_n(A)) \cong \mathrm{HH}^n(A).$$

We use this duality to define a Batalin-Vilkovisky structure on $\mathrm{HH}^*(A)$ just as in the Calabi-Yau setting: Let $\mathcal{B}^t$ denote the transpose of the Connes differential $\mathcal{B}$, that is $\mathcal{B}^t(f) = f\mathcal{B}$ for all $f \in D(\mathrm{HH}_*(A))$. Then

$$\Delta : \mathrm{HH}^n(A) \to \mathrm{HH}^{n-1}(A)$$

is defined by the following commuting diagram:

$$\begin{tikzcd} \mathrm{HH}^n(A) \arrow[r, "\Delta"] \arrow[d, "\cong"] & \mathrm{HH}^{n-1}(A) \arrow[d, "\cong"] \\ D(\mathrm{HH}_n(A)) \arrow[r, "\mathcal{B}^t"] & D(\mathrm{HH}_{n-1}(A)) \end{tikzcd}$$

Just as in the case of a Calabi-Yau algebra, the relation between the Batalin-Vilkovisky operator and the Gerstenhaber bracket given in 4.5.3 holds. A generalization to Frobenius algebras with diagonalizable Nakayama automorphism may be found in the two papers [**LZZ16, YV**] using different methods. More details and examples may be found in these papers.

Exercise 4.5.4. Find a formula for the Connes differential $\mathcal{B}$ of Definition 4.5.1 in cases $n = 1, 2$.

Exercise 4.5.5. Verify that the Connes differential $\mathcal{B}$ is indeed a chain map.

Chapter 5

Algebraic Deformation Theory

In this chapter we will define and examine some types of deformations of associative algebras. We focus on the role played by Hochschild cohomology under the Gerstenhaber bracket. Surveys of further aspects of the theory include [**Scha**] on deformations arising in noncommutative geometry and [**Gia11**] for deformation formulas arising from bialgebra actions and for deformations of bialgebras. Here we generally discuss formal deformations, infinitesimal deformations, deformation quantization, and graded deformations. We summarize the theory of Braverman and Gaitsgory [**BG96**] for graded deformations of Koszul algebras in particular, and give their proof of the classical Poincaré-Birkhoff-Witt Theorem for Lie algebras as an application. Their theory makes heavy use of Hochschild cohomology. A different homological treatment was given by Polishchuk and Positselski [**PP05, Pos93**].

Let A be an algebra over a field k.

5.1. Formal deformations

Let t be an indeterminate. Denote by $A[[t]]$ the algebra whose elements are formal power series $\sum_{i\geq 0} a_i t^i$ with coefficients $a_i \in A$. Multiplication is given by the Cauchy product:

$$(\sum_{i\geq 0} a_i t^i)(\sum_{j\geq 0} b_j t^j) = \sum_{l\geq 0}\ \sum_{i+j=l} a_i b_j t^l. \tag{5.1.1}$$

The algebra $A[[t]]$ is a $k[[t]]$-module via the identification of k with the subalgebra $k\cdot 1$ of A. We are interested in new associative algebra structures on this $k[[t]]$-module $A[[t]]$ such that the quotient by the ideal generated by t is isomorphic to A. Precisely, we have the following definition.

Definition 5.1.2. A *formal deformation* $(A_t, *)$ of A (also called a *deformation of A over $k[[t]]$*) is an associative k-bilinear multiplication $*$ on the $k[[t]]$-module $A[[t]]$, such that modulo the ideal generated by t, the multiplication corresponds to that on A; this multiplication is required to be determined by a multiplication on elements of A and extended to $A[[t]]$ by the Cauchy product rule (5.1.1). Define similarly a *deformation of A over $k[t]$* or *over $k[t]/(t^n)$*.

To be clear, we intend in the definition a *new* multiplication on elements of A, taking values in the $k[[t]]$-module $A[[t]]$, and extended to a multiplication on $A[[t]]$ by Cauchy products. We give an explicit description via a multiplication formula (5.1.4) below.

Remark 5.1.3. There are more general types of deformations: Let R be any commutative augmented k-algebra (such as $R = k[[t_1, \dots, t_m]]$), with augmentation map $\varepsilon : R \to k$, that is complete with respect to the $(\operatorname{Ker} \varepsilon)$-adic topology. A *deformation of A over R* is an associative R-algebra A_R that is isomorphic to the completed tensor product of A with R as an R-module and for which there is a k-algebra isomorphism $A_R/(\operatorname{Ker} \varepsilon) \xrightarrow{\sim} A$. One often assumes that A_R is free as an R-module, or more generally that A_R is flat as an R-module (for a *flat deformation*). In this book we will only consider deformations over $k[[t]]$, $k[t]$, or $k[t]/(t^n)$ as in Definition 5.1.2, and we will not need this more general definition.

Any multiplication $*$ as in Definition 5.1.2 is determined by products of pairs of elements of A: For $a, b \in A$, write

$$a * b = ab + \mu_1(a \otimes b)t + \mu_2(a \otimes b)t^2 + \mu_3(a \otimes b)t^3 + \cdots , \tag{5.1.4}$$

where ab is the usual product in A and $\mu_1, \mu_2, \mu_3, \dots$ are functions from $A \otimes A$ to A giving the coefficients of $t, t^2, t^3, \dots$ as indicated. Sometimes we write $\mu_0(a \otimes b) = ab$ so that the formula becomes

$$a * b = \sum_{i \geq 0} \mu_i(a \otimes b)t^i.$$

The functions μ_i are necessarily k-linear. We call μ_i the *ith multiplication map* of the deformation. We sometimes denote the deformation $(A_t, *)$ by (A_t, μ_t), writing

$$\mu_t = \mu_0 + \mu_1 t + \mu_2 t^2 + \cdots$$

as a function from $A \otimes A$ to A_t. When needed, we extend μ_t to be a function on the tensor product of the $k[[t]]$-module $A[[t]]$ with itself, completed so that expressions with formal power series as tensor factors make sense:

$$(\sum_{i\geq 0} a_i t^i) \otimes_{k[[t]]} (\sum_{j\geq 0} b_j t^j) = \sum_{n\geq 0} (\sum_{i+j=n} a_i \otimes b_j) t^n.$$

In this case, the value of μ_t on an element $\sum_{n\geq 0} c_n t^n$, where $c_n \in A \otimes A$ for all n, is taken to be $\sum_{n\geq 0}(\sum_{i+j=n} \mu_i(c_j))t^n$. Some authors write $\widehat{\otimes}$ in place of $\otimes$ to denote a completed tensor product.

We will next derive some properties of the multiplication maps μ_i. Assume that $(A, *)$ is a formal deformation of A as in Definition 5.1.2, so that the multiplication $*$ is associative and given by maps μ_i as above. Calculating each side of the equation $(a * b) * c = a * (b * c)$ for all $a, b, c \in A$ by using formula (5.1.4), we find that

$$\begin{aligned}(a * b) * c &= abc + (\mu_1(ab \otimes c) + \mu_1(a \otimes b)c)t \\ &\quad +(\mu_2(ab \otimes c) + \mu_1(\mu_1(a \otimes b) \otimes c) + \mu_2(a \otimes b)c)t^2 + \cdots\end{aligned}$$

while

$$\begin{aligned}a * (b * c) &= abc + (\mu_1(a \otimes bc) + a\mu_1(b \otimes c))t \\ &\quad +(\mu_2(a \otimes bc) + \mu_1(a \otimes \mu_1(b \otimes c)) + a\mu_2(b \otimes c))t^2 + \cdots\end{aligned}$$

Equating coefficients of t, it follows that

$$\mu_1(ab \otimes c) + \mu_1(a \otimes b)c = \mu_1(a \otimes bc) + a\mu_1(b \otimes c) \tag{5.1.5}$$

for all $a, b, c \in A$. Comparing with equation (1.2.2), we see that μ_1 may be identified with a Hochschild 2-cocycle on the bar resolution of A. That is, $d_3^*(\mu_1) = 0$ where d_3 is a differential on the bar resolution (1.1.4), under the identification $\mathrm{Hom}_{A^e}(A^{\otimes 5}, A) \cong \mathrm{Hom}_k(A^{\otimes 3}, A)$ given by (1.1.11) with $M = A$ and $n = 3$. Equating coefficients of t^2, we have

$$\begin{aligned}&\mu_1(\mu_1(a \otimes b) \otimes c) - \mu_1(a \otimes \mu_1(b \otimes c)) \\ &= a\mu_2(b \otimes c) - \mu_2(ab \otimes c) + \mu_2(a \otimes bc) - \mu_2(a \otimes b)c\end{aligned}$$

for all $a, b, c \in A$. Comparing with Definition 1.4.1, the left side of the above equation is the circle product $\mu_1 \circ \mu_1$, which in characteristic not 2 is half of the Gerstenhaber bracket $[\mu_1, \mu_1]$ applied to $a \otimes b \otimes c$. The right side is $d_3^*(\mu_2)$ applied to $a \otimes b \otimes c$, where d_3^* again denotes a differential on the bar resolution. Thus associativity of $*$ implies that, at the chain level,

$$[\mu_1, \mu_1] = 2d_3^*(\mu_2) \tag{5.1.6}$$

in characteristic not 2, while in characteristic 2 we must express the condition as $\mu_1 \circ \mu_1 = d_3^*(\mu_2)$. A similar analysis shows that

$$[\mu_1, \mu_2] = d_3^*(\mu_3) \tag{5.1.7}$$

and more generally that

$$(5.1.8)\quad \sum_{j=1}^{i-1}(\mu_j(\mu_{i-j}(a\otimes b)\otimes c)-\mu_j(a\otimes\mu_{i-j}(b\otimes c)))=d_3^*(\mu_i)(a\otimes b\otimes c)$$

for all $a, b, c \in A$ and $i \geq 2$. Alternatively we may apply Lemma 1.4.3(ii), with $\pi = \mu_0$, to rewrite the differential on $\mathrm{Hom}_k(A^{\otimes\bullet}, A)$ as $(-1)[-, \mu_0]$ and change the indexing of the above sum to include $j = 0$ and $j = i$. Then equation (5.1.8) becomes

$$(5.1.9)\qquad \sum_{j=0}^{i}(\mu_j(\mu_{i-j}(a\otimes b)\otimes c)-\mu_j(a\otimes\mu_{i-j}(b\otimes c)))=0.$$

We have thus found that there are infinitely many conditions that must be satisfied, one for each i, given by equation (5.1.8) or (5.1.9), in order that $*$ be an associative multiplication on $A[[t]]$. We call the left side of equation (5.1.8), viewed as a function on $A^{\otimes 3}$, the *$(i-1)$st obstruction.* Considering the right side of equation (5.1.8), we find that the $(i-1)$st obstruction must represent the element 0 in degree 3 Hochschild cohomology $\mathrm{HH}^3(A)$ as a consequence of associativity of $*$. Consequently, a set of k-linear functions $\mu_i : A \otimes A \to A$ potentially defining an associative product $*$ on $A[[t]]$ via formula (5.1.4) is prevented from doing so if any of the obstructions is nonzero in $\mathrm{HH}^3(A)$.

We give several examples next. Each example is a deformation over $k[[t]]$ or $k[t]$ which is then specialized to a particular value of the parameter t where possible, a common source of new algebras arising from deformations of given algebras.

Example 5.1.10. Let $A = k[x, y]$. Define a multiplication $*$ on $A[t]$ or on $A[[t]]$ by

$$x^i * x^j = x^{i+j}, \quad y^i * y^j = y^{i+j}, \quad x^i * y^j = x^i y^j, \quad y * x = xy + t$$

for all i, j, and extend by requiring $*$ to be associative. This gives rise to a deformation of A over $k[[t]]$ or $k[t]$. Over $k[t]$, we may substitute $t = 1$ (that is, take the quotient by the ideal $(t-1)$) to obtain the *Weyl algebra*

$$A_1 = k\langle x, y\rangle/(yx - xy - 1).$$

This may be done more generally to obtain the Weyl algebra A_m on $2m$ generators, that is the algebra with generators $x_1, \ldots, x_m, y_1, \ldots, y_m$ and relations $x_i x_j = x_j x_i$, $y_i y_j = y_j y_i$, and $y_i x_j = x_j y_i + \delta_{i,j}$ for all i, j.

Example 5.1.11. Let $k = \mathbb{C}$ for this example, so that we may take advantage of convergence of the exponential function. Let $A = \mathbb{C}[x, y]$. Define a multiplication $*$ on $A[[t]]$ by

$$x^i * x^j = x^{i+j}, \quad y^i * y^j = y^{i+j}, \quad x^i * y^j = x^i y^j,$$
$$y * x = xy(1 + t + \frac{1}{2!}t^2 + \frac{1}{3!}t^3 + \cdots) = xy \cdot \exp(t)$$

for all i, j, and extend by requiring $*$ to be associative. This gives rise to a formal deformation of A. Let $t_0 \in \mathbb{C}$ and substitute $t = t_0$ in the subalgebra of $A[[t]]$ generated by x and y. Let $q = \exp(t_0)$. The resulting algebra is the quantum plane

$$\mathbb{C}_q[x, y] = \mathbb{C}\langle x, y\rangle/(yx - qxy).$$

This is done more generally to obtain a skew polynomial ring $\mathbb{C}_{\mathbf{q}}[x_1, \ldots, x_n]$ as defined in Example 3.3.1.

Example 5.1.12. Let k be a field of characteristic not 2. Let $A = k[e, f, h]$. Define a multiplication on $A[t]$ by

$$f * e = ef - ht, \quad h * e = eh + 2et, \quad h * f = fh - 2ft,$$

and all products of monomials in alphabetical order are as in A, for example, $e^i f^j * h^m = e^i f^j h^m$ for all i, j, m. Extend by requiring $*$ to be associative. Substitute $t = 1$ to obtain the universal enveloping algebra of the Lie algebra $\mathfrak{sl}_2$:

$$U(\mathfrak{sl}_2) = k\langle e, f, h\rangle/(fe - ef + h,\ he - eh - 2e,\ hf - fh + 2f).$$

The last example is essentially generalized below in Section 5.5 in a restatement of the classical Poincaré-Birkhoff-Witt (PBW) Theorem: Recall that a *Lie algebra* $\mathfrak{g}$ is a vector space with a linear map $[-,-] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ that is antisymmetric, that is, $[x, y] = -[y, x]$, and satisfies the *Jacobi identity*,

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$$

for all $x, y, z \in \mathfrak{g}$. The *universal enveloping algebra* of $\mathfrak{g}$ is

$$U(\mathfrak{g}) = T(\mathfrak{g})/(x \otimes y - y \otimes x - [x, y] \mid x, y \in \mathfrak{g}).$$

In the proof of Theorem 5.5.5 below we will see that $U(\mathfrak{g})$ is a particular type of deformation of a polynomial ring, termed a PBW deformation due to its appearance in this theorem.

Exercise 5.1.13. Derive the formula (5.1.7) by equating coefficients of t^3 in the equation $(a * b) * c = a * (b * c)$.

Exercise 5.1.14. Identify the Hochschild cocycle μ_1 inherent in Example 5.1.10. What is μ_2?

Exercise 5.1.15. Identify the Hochschild cocycle μ_1 inherent in Example 5.1.11. What is μ_2?

Exercise 5.1.16. Let B be the *Jordan plane*:

$$B = k\langle x, y\rangle/(yx - xy - x^2).$$

One may realize B as a specialization to $t = 1$ of a deformation of the polynomial ring $k[x, y]$ over $k[t]$. What is the corresponding Hochschild 2-cocycle μ_1?

Exercise 5.1.17. Let A_1 be the Weyl algebra of Example 5.1.10. Show that $\mathrm{HH}^0(A_1) \cong k$ and $\mathrm{HH}^n(A) = 0$ for all $n > 0$. (Use the Koszul resolution $K_\bullet$ for A_1: Set $K_\bullet = A_1 \otimes \bigwedge(V) \otimes A_1$ where V is a vector space with basis $\{x, y\}$. The differentials are as in the Koszul resolution for $k[x, y]$. Check that $K_\bullet$ is indeed a free resolution of A_1 as an A_1^e-module.)

Exercise 5.1.18. Let A_m be the Weyl algebra on $2m$ generators of Example 5.1.10. Use Exercise 5.1.17 and Theorem 3.1.2 to show that $\mathrm{HH}^0(A_m) \cong k$ and $\mathrm{HH}^n(A_m) = 0$ for all $n > 0$.

5.2. Infinitesimal deformations and rigidity

An algebra is rigid if it cannot be deformed, and we make this notion precise in this section. We saw in the last section that every formal deformation has associated to it a Hochschild 2-cocycle. If Hochschild cohomology vanishes in degree 2, the algebra is necessarily rigid, as stated in Proposition 5.2.8 below. Otherwise we examine more closely the cocycles that can arise, starting with the next definition.

Definition 5.2.1. A k-linear function $\mu_1 : A \otimes A \to A$ is an *infinitesimal deformation* if (5.1.5) holds, that is $d_3^*(\mu_1) = 0$ in $\mathrm{Hom}_k(A^{\otimes 3}, A)$. Its *primary* (or *first*) *obstruction vanishes* if $[\mu_1, \mu_1]$ is a coboundary in the space $\mathrm{Hom}_k(A^{\otimes 3}, A)$. It is *integrable* if there is a formal deformation (A_t, μ_t) for which μ_1 is the first multiplication map.

If μ_1 is an infinitesimal deformation, it defines an associative algebra structure on $A[t]/(t^2)$, that is, it defines a deformation of A over $k[t]/(t^2)$, the ring of dual numbers: Let

$$a * b = ab + \mu_1(a \otimes b)t$$

for all $a, b \in A$ and extend $k[t]/(t^2)$-bilinearly to $A[t]/(t^2)$. (For notational convenience, we identify t with its image under the quotient map from $k[t]$ to $k[t]/(t^2)$.) Conversely, a deformation of A over $k[t]/(t^2)$ is determined by the coefficient of t in the above equation, which necessarily satisfies equation (5.1.5). Sometimes when we refer to an infinitesimal deformation, we mean this corresponding algebra structure on $A[t]/(t^2)$.

In Section 7.1, we will more generally define infinitesimal n-deformations for any $n \geq 2$; our infinitesimal deformations here will be called infinitesimal

2-deformations there. We will show in Theorem 7.1.8 that a Hochschild n-cocycle corresponds to an infinitesimal n-deformation, generalizing the connection we have already seen among infinitesimal deformations, Hochschild 2-cocycles, and deformations over $k[t]/(t^2)$.

We will next define a notion of equivalence of formal deformations. We will see that the Hochschild 2-cocycles corresponding to equivalent deformations are cohomologous.

Definition 5.2.2. Two formal deformations (A_t, μ_t), (A'_t, μ'_t) are *equivalent* if there is a $k[[t]]$-linear function $\phi_t : A_t \to A'_t$ determined by values on A as follows:

$$\phi_t(a) = a + \phi_1(a)t + \phi_2(a)t^2 + \cdots \tag{5.2.3}$$

for k-linear functions $\phi_i : A \to A$ such that

$$\phi_t\mu_t(a \otimes b) = \mu'_t(\phi_t(a) \otimes \phi_t(b)) \tag{5.2.4}$$

for all $a, b \in A$ and extended to A_t. A formal deformation (A_t, μ_t) is *trivial* if it is equivalent to $A[[t]]$.

In the definition, we extend (5.2.3) to all elements of A_t by defining $\phi_t(\sum_{i\geq 0} a_i t^i)$ to be $\sum_{n\geq 0}(\sum_{i+j=n} \phi_i(a_j))t^n$ where we take ϕ_0 to be the identity map on A.

Note that the function ϕ_t is an isomorphism of algebras by its definition above. (Any function of the given form is necessarily invertible as a formal power series; see for example the proof of Lemma 5.2.6).) We view each function $\phi_i \in \mathrm{Hom}_k(A, A)$ as a 1-cochain on the Hochschild complex via the identification $\mathrm{Hom}_k(A, A) \cong \mathrm{Hom}_{A^e}(A^{\otimes 3}, A)$. This view gives meaning to the expression $d_2^*(\phi_1)$ in the statement of the next lemma.

Lemma 5.2.5. *If (A_t, μ_t), (A'_t, μ'_t) are equivalent via a function ϕ_t as in Definition 5.2.2, then $\mu'_1 = \mu_1 - d_2^*(\phi_1)$. In particular, if (A_t, μ_t) is trivial, then μ_1 is a coboundary.*

Proof. Expanding equation (5.2.4), we have

$$ab + (\phi_1(ab) + \mu_1(a \otimes b))t + \cdots = ab + (\mu'_1(a \otimes b) + \phi_1(a)b + a\phi_1(b))t + \cdots$$

for all $a, b \in A$. Equating coefficients of t, and using the techniques of Section 1.2 to evaluate $(d_2^*(\phi_1))(a \otimes b)$, we see that $\mu'_1 = \mu_1 - d_2^*(\phi_1)$ as claimed. If (A_t, μ_t) is trivial, then it is equivalent to (A'_t, μ'_t) with $\mu'_0 = \mu_0$ and $\mu'_i = 0$ for all $i > 0$. Thus $\mu_1 = d_2^*(\phi_1)$. □

In the next lemma, we view functions $\mu'_n \in \mathrm{Hom}_k(A\otimes A, A)$ as 2-cochains on the Hochschild complex under the standard identification $\mathrm{Hom}_k(A \otimes A, A) \cong \mathrm{Hom}_{A^e}(A^{\otimes 4}, A)$. Given a sequence $\mu'_0, \mu'_1, \ldots$ of such cochains, we will refer in the next lemma to the first nonvanishing cochain, by which we mean μ'_n such that $\mu'_n \neq 0$ and $\mu'_i = 0$ for all $i < n$.

Lemma 5.2.6. *A nontrivial formal deformation (A_t, μ_t) of A is equivalent to a formal deformation (A'_t, μ'_t) with the property that the first nonvanishing cochain μ'_n is a Hochschild 2-cocycle that is not a coboundary.*

Proof. Suppose (A_t, μ_t) is a formal deformation of A whose first nonvanishing cochain is a coboundary. Write

$$\mu_t(a \otimes b) = ab + \mu_n(a \otimes b)t^n + \mu_{n+1}(a \otimes b)t^{n+1} + \cdots$$

for all $a, b \in A$, where $\mu_n = d_2^*(\beta)$ for some $\beta \in \mathrm{Hom}_k(A, A)$. Let

$$\phi_t(a) = a + \beta(a)t^n$$

for all $a \in A$, and note that

$$\phi_t^{-1}(a) = a - \beta(a)t^n + \beta^2(a)t^{2n} - \beta^3(a)t^{3n} + \cdots ,$$

where $\beta^2(a) = \beta(\beta(a))$, $\beta^3(a) = \beta(\beta(\beta(a)))$, etc. Set $\mu'_t = \phi_t\mu_t(\phi_t^{-1} \otimes \phi_t^{-1})$. A calculation shows that μ'_t is the multiplication map for a deformation (A'_t, μ'_t) which by definition is equivalent to (A_t, μ_t). Since $\mu_n = d_2^*(\beta)$, there is some function μ'_{n+1} such that

$$\begin{aligned}
&\mu'_t(a \otimes b)\\
&= \phi_t\mu_t((a - \beta(a)t^n + \cdots) \otimes (b - \beta(b)t^n + \cdots))\\
&= \phi_t(ab + (\mu_n(a \otimes b) - a\beta(b) - \beta(a)b)t^n + \cdots)\\
&= ab + (\beta(ab) + \mu_n(a \otimes b) - a\beta(b) - \beta(a)b)t^n + \mu'_{n+1}(a \otimes b)t^{n+1} + \cdots\\
&= ab + \mu'_{n+1}(a \otimes b)t^{n+1} + \cdots .
\end{aligned}$$

If μ'_{n+1} is a coboundary, then similarly (A'_t, μ'_t) is equivalent to (A''_t, μ''_t) where

$$\mu''_t(a \otimes b) = ab + \mu''_{n+2}(a \otimes b)t^{n+2} + \cdots$$

via a function ϕ'_t with $\phi'_t(a) = a + \beta'(a)t^{n+1}$. Continuing in this fashion, we may let Φ_t be the function

$$\Phi_t(a) = a + \Phi_n(a)t^n + \Phi_{n+1}(a)t^{n+1} + \cdots$$

defined as the composition of all $\ldots, \phi''_t, \phi'_t, \phi_t$. This composition is well-defined: The coefficient function of each power of t in Φ_t is a finite polynomial in $\beta, \beta', \beta'', \ldots$. For example, if $n = 1$, then

$$\Phi_t(a) = a + \beta(a)t + \beta'(a)t^2 + (\beta''(a) + \beta'(\beta(a)))t^3 + \cdots .$$

Applying Φ_t, we see that (A_t, μ_t) is trivial as claimed: Note that for any given power of t, its coefficient function in the resulting equivalent deformation only involves a composition of finitely many such equivalences, and we find that the coefficient function is indeed 0. □

We now focus on algebras having no such deformations.

Definition 5.2.7. An algebra A is *rigid* if it has no nontrivial formal deformations.

As an immediate consequence of Lemma 5.2.6, we have the following proposition.

Proposition 5.2.8. *If* $\mathrm{HH}^2(A) = 0$*, then* A *is rigid.*

Examples of algebras A for which $\mathrm{HH}^2(A) = 0$ include separable algebras (as a consequence of their definition), universal enveloping algebras of complex semisimple Lie algebras [**Kas95**, XVIII.3], Weyl algebras (see Exercise 5.1.18), and tensor algebras over a field (see Example 4.1.8). Thus all of these algebras are rigid. We point out however that universal enveloping algebras do have bialgebra deformations [**EK96**], yielding one approach to some types of quantum groups. It is bialgebra cohomology that governs these deformations in an analogous theory [**Gia11**].

Exercise 5.2.9. For Examples 5.1.10 and 5.1.11, determine the structure of the corresponding deformations over $k[t]/(t^2)$.

Exercise 5.2.10. Justify the claim that a function ϕ_t defined as in (5.2.3) is always an isomorphism.

Exercise 5.2.11. Let $A = k\langle x, y\rangle/(yx - xy - 1)$, a Weyl algebra. Verify the claim that A is rigid by constructing or finding an A-bimodule resolution of A to determine its Hochschild cohomology in degree 2, $\mathrm{HH}^2(A)$.

5.3. Maurer-Cartan equation and Poisson bracket

For this section, we assume the characteristic of the field k is not 2. We will give another interpretation of the conditions (5.1.9) for a deformation, deriving the Maurer-Cartan equation (5.3.1) below. We will in particular examine the first obstruction condition, as defined in Section 5.1, in the case of commutative algebras.

Let (A_t, μ_t) be a formal deformation of A. Write $\mu_t = \sum_{i\geq 0} \mu_i t^i$ as before. The conditions (5.1.9) for all $i \geq 2$, together with associativity of μ_0 and the assumption that μ_1 is a Hochschild 2-cocycle, may combined and reinterpreted as stating that

$$[\mu_t, \mu_t] = 0.$$

Set

$$\mu_t = \mu_0 + \mu'.$$

Note that associativity of A is the condition that $[\mu_0, \mu_0] = 0$. Since μ_0 and μ' may be viewed as 2-cochains, the graded commutativity of the bracket gives $[\mu', \mu_0] = [\mu_0, \mu']$. So the equation $[\mu_t, \mu_t] = 0$ is equivalent to

$$2[\mu_0, \mu'] + [\mu', \mu'] = 0.$$

By Lemma 1.4.3(ii), the differential on the Hochschild complex is $(-1)[-, \mu_0]$, so this is equivalent to

$$-d^*(\mu') + \frac{1}{2}[\mu', \mu'] = 0. \tag{5.3.1}$$

With appropriate sign conventions, this is the *Maurer-Cartan equation* (also called the *Berikashvili equation*). We have shown that the deformed multiplication μ_t is associative if and only if μ' satisfies the Maurer-Cartan equation. Focusing on μ_1 and μ_2 (giving rise to coefficients of t and t^2 in expression (5.1.4)), this equation implies μ_1 is a Hochschild 2-cocycle and $d_3^*(\mu_2) = \frac{1}{2}[\mu_1, \mu_1]$, as we saw before in equation (5.1.6).

We next look more closely at commutative algebras and their potentially noncommutative deformations.

Definition 5.3.2. A *Poisson algebra* is a commutative associative algebra A that is also a Lie algebra under a binary operation $\{\ ,\ \}$ for which

$$\{a, bc\} = \{a, b\}c + b\{a, c\}$$

for all $a, b, c \in A$. We call $\{\ ,\ \}$ a *Poisson bracket.*

Note that a Poisson bracket $\{\ ,\ \}$ is by definition a Lie bracket, that is, it is anti-commutative and satisfies the Jacobi identity. One source of Poisson brackets on commutative algebras is Hochschild cohomology: Let $\mu_1 : A \otimes A \to A$ be a Hochschild 2-cocycle on A whose primary obstruction vanishes *at the chain level*, that is, $[\mu_1, \mu_1] = 0$ as a cochain on the bar resolution. Let

$$\{a, b\} = \frac{1}{2}(\mu_1(a \otimes b) - \mu_1(b \otimes a)) \tag{5.3.3}$$

for all $a, b \in A$. Calculations using commutativity of A and the Hochschild 2-cocycle condition (5.1.5) show that $\{\ ,\}$ is a Poisson bracket on A. Conversely, a Poisson bracket $\{\ ,\ \}$ on a commutative algebra A is a Hochschild 2-cocycle.

Definition 5.3.4. Let A be a Poisson algebra. A *deformation quantization* of A is a formal deformation $(A_t, *)$ for which

$$a * b - b * a = \{a, b\}t \pmod{t^2}$$

for all $a, b \in A$.

Example 5.3.5. The Weyl algebra A_1 of Example 5.1.10 is the specialization of a deformation quantization of the polynomial ring $k[x, y]$ with Poisson bracket $\{x, y\} = -1$. Similar statements hold for Weyl algebras in more indeterminates, and for the quantum plane and other skew polynomial rings described in Example 5.1.11.

We will discuss in Section 7.6 some general conditions under which it is known that a Poisson algebra has a deformation quantization. In the noncommutative setting, an infinitesimal deformation μ_1 of a not necessarily commutative algebra A is sometimes regarded as a noncommutative Poisson structure on A when its primary obstruction vanishes.

Exercise 5.3.6. Verify the claim that the condition $[\mu_t, \mu_t] = 0$ is equivalent to conditions (5.1.9) for all $i \geq 2$ together with associativity of μ_0 and the property that μ_1 is a Hochschild 2-cocycle.

Exercise 5.3.7. Verify that a Poisson bracket on a commutative algebra is a Hochschild 2-cocycle.

Exercise 5.3.8. Verify that (5.3.3) defines a Poisson bracket on a commutative algebra A.

Exercise 5.3.9. Find the Poisson bracket on the polynomial ring $\mathcal{C}[x, y]$ arising from viewing Example 5.1.11 as a deformation quantization.

5.4. Graded deformations

Let A be an $\mathbb{N}$-graded algebra. Let t be an indeterminate, and assign t the degree 1. Consider the resulting $\mathbb{N}$-graded algebra $A[t]$, in which $|at^i| = |a|+i$ for all homogeneous $a \in A$ and $i \in \mathbb{N}$. A *graded deformation* of A over $k[t]$ is a deformation A_t of A over $k[t]$ that is also a graded algebra. We will translate this condition to one on the multiplication maps μ_i of (5.1.4), viewed as elements of $\mathrm{Hom}_k(A \otimes A, A)$. First note that the grading on A induces a grading on this space of homomorphisms:

$$\mathrm{Hom}_k(A \otimes A, A)_m = \{f \in \mathrm{Hom}_k(A \otimes A, A) \mid f((A \otimes A)_i) \subset A_{i+m} \text{ for all } i\}$$

for all $m \in \mathbb{Z}$. Necessarily then, for a graded deformation, since $|t| = 1$, each function μ_j of equation (5.1.4) is homogeneous of degree $-j$, that is, $\mu_j \in \mathrm{Hom}_k(A \otimes A, A)_{-j}$. An *ith level graded deformation of* A is a graded associative algebra A_i over $k[t]/(t^{i+1})$ with underlying $k[t]/(t^{i+1})$-module $A[t]/(t^{i+1})$ and multiplication given on elements $a, b \in A$ by

$$a * b = ab + \mu_1(a \otimes b)t + \mu_2(a \otimes b)t^2 + \cdots + \mu_i(a \otimes b)t^i$$

for k-linear functions $\mu_j : A \otimes A \to A$. (For notational convenience, we identify t with its image under the quotient map from $k[t]$ to $k[t]/(t^{i+1})$.) A first level graded deformation of A then corresponds to an infinitesimal deformation of A that is graded. We say that an $(i-1)$st level graded deformation A_{i-1} of A *lifts* to an ith level graded deformation A_i of A if the jth multiplication maps of A_{i-1} and A_i agree for all $j \le i-1$. Just as in Section 5.1, a calculation shows that if A_{i-1} lifts to A_i, then equation (5.1.9) holds. Accordingly, just as in Section 5.1, we say that the obstruction to existence of such a lifting is the left side of equation (5.1.8), which must define a coboundary on the Hochschild complex as a condition for lifting.

Braverman and Gaitsgory [**BG96**] proved the following proposition regarding lifting graded deformations. We will need to use a grading on Hochschild cohomology induced by that on the Hochschild complex which generalizes the grading on $\mathrm{Hom}_k(A \otimes A, A)$ discussed above. We define this grading next.

The bar resolution of the $\mathbb{N}$-graded algebra A is itself graded, with

$$|a_0 \otimes \cdots \otimes a_{n+1}| = |a_0| + \cdots + |a_{n+1}|$$

for all homogeneous $a_0, \ldots, a_{n+1} \in A$. The A^e-module $A^{\otimes(n+2)}$ is a graded A^e-module in this way, and the differentials are graded maps, that is, they preserve the grading. A grading on $\mathrm{Hom}_{A^e}(A^{\otimes(n+2)}, A) \cong \mathrm{Hom}_k(A^{\otimes n}, A)$ for each n is defined by

$$\mathrm{Hom}_k(A^{\otimes n}, A)_m = \{f \in \mathrm{Hom}_k(A^{\otimes n}, A) \mid f((A^{\otimes n})_i) \subset A_{i+m} \text{ for all } i\},$$

and again the differentials are graded maps. Thus the cohomology inherits the grading. Hochschild cohomology $\mathrm{HH}^*(A)$ is therefore graded both by this grading from A and by homological degree. We call the grading on $\mathrm{HH}^*(A)$ coming from A its *internal grading* and that of homological degree its *homological grading*. Denote by $\mathrm{HH}^{i,j}(A)$ the subspace of $\mathrm{HH}^i(A)$ consisting of elements of internal degree j.

Proposition 5.4.1. *A first level graded deformation of an* $\mathbb{N}$*-graded algebra* A *corresponds to an element of* $\mathrm{HH}^{2,-1}(A)$*. An obstruction to lifting an* $(i-1)$*st level graded deformation of* A *to an* i*th level graded deformation of* A *is in* $\mathrm{HH}^{3,-i}(A)$*, and an* $(i-1)$*st level graded deformation lifts to an* i*th level deformation if and only if the* $(i-1)$*st obstruction given by the left side of equation (5.1.8) becomes 0 in cohomology.*

Proof. We have noted before that a first level deformation corresponds to an infinitesimal deformation, that is a Hochschild 2-cocycle. If, in addition, it is graded, then necessarily the internal degree of the Hochschild 2-cocycle is -1, as noted before. Two first level graded deformations are isomorphic if and only if their corresponding cocycles are cohomologous. Therefore the first statement of the proposition holds.

A graded deformation of A necessarily involves functions μ_j that are homogeneous of degree $-j$, as was noted at the beginning of this section, and the same is true of an ith level graded deformation. We have already noted the obstructions (5.1.8), and in this graded setting, we see that equation (5.1.8) indeed involves functions of internal degree $-i$. Thus the second statement holds. $\square$

Exercise 5.4.2. Verify the claim that the differential on the bar resolution is a graded map.

Exercise 5.4.3. Let $P_{\bullet}$ be a free resolution of an $\mathbb{N}$-graded algebra A as an A^e-module consisting of graded modules and graded differentials. Show that the internal grading on $\mathrm{HH}^*(A)$ is induced from that on $P_{\bullet}$.

Exercise 5.4.4. Let A be a Koszul algebra and let $K_{\bullet}(A)$ be the Koszul resolution given by (3.4.4). Put a grading on each $K_i(A)$ for which the differentials are graded maps, that is, preserve the grading.

Exercise 5.4.5. Let $P_{\bullet}$ be the resolution (3.1.4) of a polynomial ring $A = k[x_1, \ldots, x_m]$. Define a grading on each module P_i under which $P_{\bullet}$ becomes a graded resolution, that is the differentials are graded maps. Use this grading on $P_{\bullet}$ to describe each space $\mathrm{HH}^{i,j}(A)$.

Exercise 5.4.6. Verify, by direct calculation, the claim in the proof of Proposition 5.4.1 that two first level graded deformations are isomorphic if and only if their corresponding cocycles are cohomologous.

Exercise 5.4.7. Identify a first level graded deformation of the polynomial ring $k[e, f, h]$ corresponding to the deformation of Example 5.1.12.

5.5. Braverman-Gaitsgory theory and the PBW Theorem

We present the theory of Braverman and Gaitsgory [**BG96**] on particular types of deformations and its application to the classical Poincaré-Birkhoff-Witt (PBW) Theorem on the structure of universal enveloping algebras of Lie algebras. We only consider here the original setting of [**BG96**] where A is a connected graded Koszul algebra, although this theory has been generalized in a number of directions (see, for example, the survey [**SW15**]).

Let A be a Koszul algebra, as defined in Section 3.4. Write $A = T(V)/(R)$ where V is a finite dimensional vector space whose elements are given degree 1, and R is a subspace of $V \otimes V$. We will construct a particular type of deformation of A from a choice of k-linear maps $\alpha : R \to V$ and $\beta : R \to k$ as follows. Consider the subspace of $T_0(V) \oplus T_1(V) \oplus T_2(V)$ given by

$$P = P_{\alpha,\beta} = \{x - \alpha(x) - \beta(x) \mid x \in R\}. \tag{5.5.1}$$

Set $A' = T(V)/(P)$, where (P) denotes the ideal of $T(V)$ generated by $P = P_{\alpha,\beta}$. If α and β are both zero maps, then $P = R$ and $A' = A$ is a graded algebra. In general, A' is not graded but is a filtered algebra in the following way: Let F_nA' be the image in A' of the subspace $\oplus_{0\le i\le n}T_i(V)$ of $T(V)$. Then

$$F_0A' \subset F_1A' \subset F_2A' \subset \cdots$$

and $(F_iA')(F_jA') \subset F_{i+j}A'$. Denote the associated graded algebra to A' by $\operatorname{gr} A'$, that is, as a vector space,

$$\operatorname{gr} A' = F_0A' \oplus (F_1A'/F_0A') \oplus (F_2A'/F_1A') \oplus \cdots,$$

and the product of an element in $F_iA'/F_{i-1}A'$ with one in $F_jA'/F_{j-1}A'$ is defined by multiplying inverse images in F_iA' and F_jA', and taking the resulting image in $F_{i+j}A'/F_{i+j-1}A'$. By definition of P, since $|\alpha| = -1$ and $|\beta| = -2$ as maps, the relations R hold in $\operatorname{gr} A'$. So there is a surjective algebra homomorphism

$$A \to \operatorname{gr} A' \tag{5.5.2}$$

induced by mapping the generating space V to its image in $\operatorname{gr} A'$. In general the map is not injective, and the cases where it is are given the name PBW deformations:

Definition 5.5.3. The algebra $A' = T(V)/(P)$ is a *PBW deformation* (also called a *filtered deformation*) of A if the surjective algebra homomorphism (5.5.2) is an isomorphism.

A PBW deformation turns out to be a specialization of a graded deformation to a value of the parameter t. To see this, given P as in (5.5.1), let

$$P[t] = \{x - \alpha(x)t - \beta(x)t^2 \mid x \in R\},$$

a homogeneous subspace of $T(V)[t]$ of degree 2. The specializations of $T(V)[t]/P[t]$ to $t = 0$ and to $t = 1$ are isomorphic to A and to A', respectively. The condition that A' be a PBW deformation of A implies that $T(V)[t]/P[t]$ is a graded deformation of A over $k[t]$.

Braverman and Gaitsgory gave necessary and sufficient conditions for $A' = T(V)/(P)$ to be a PBW deformation of A. The following theorem is essentially a combination of [**BG96**, Lemma 0.4, Theorem 0.5, and Lemma 3.3].

Theorem 5.5.4. *Let $A = T(V)/(R)$ be a Koszul algebra as defined in Section 3.4, and let P be as in (5.5.1). The algebra $A' = T(V)/(P)$ is a PBW deformation of A if and only if the following conditions hold:*

(i) $P \cap F_1(T(V)) = \{0\}$,

(ii) $\operatorname{Im}(\alpha \otimes 1 - 1 \otimes \alpha) \subset R$,

(iii) $\alpha(\alpha \otimes 1 - 1 \otimes \alpha) = -(\beta \otimes 1 - 1 \otimes \beta)$,

(iv) $\beta(\alpha\otimes 1 - 1\otimes\alpha) = 0$,

where the maps $\alpha\otimes 1 - 1\otimes\alpha$ *and* $\beta\otimes 1 - 1\otimes\beta$ *are defined on the subspace* $(R\otimes V)\cap(V\otimes R)$ *of* $T(V)$.

Proof. Assume that A' is a PBW deformation of A. Then necessarily condition (i) holds, in order that the subspace $k\oplus V$ of $T(V)/(R)$ map isomorphically onto a copy of itself in $\operatorname{gr} A'$ under the map (5.5.2). We will show that the other three conditions also hold. These can be interpreted in terms of the Hochschild cohomology $\mathrm{HH}^*(A)$ as follows.

Let $K_\bullet$ be the Koszul resolution (3.4.4) of A and let $\iota_\bullet : K_\bullet \hookrightarrow B_\bullet$ be its inclusion into the bar resolution of A, as described in Section 3.4. Let $\psi_\bullet : B_\bullet \to K_\bullet$ be a chain map lifting the identity map on A. Note that $\psi_\bullet$ may be chosen so that $\psi_\bullet\iota_\bullet = 1_{K_\bullet}$ since $\iota_j(K_j)$ is a direct summand of B_j as an A^e-module for each j (a free basis of a complementary module is given by all $1\otimes x\otimes 1$ where x runs over a vector space basis of a vector space complement to K_j' in $A^{\otimes j}$). (See also [**BGSS08**]).

Now α and β are functions on R, and the degree 2 term of the Koszul resolution is $K_2 = A\otimes R\otimes A$. We identify α and β with functions on K_2 by setting $\alpha(1\otimes x\otimes 1) = \alpha(x)$ and $\beta(1\otimes x\otimes 1) = \beta(x)$ for all $x\in R$ and extending to A-bimodule homomorphisms. Set

$$\mu_1 = \alpha\psi, \quad \mu_2 = \beta\psi$$

to obtain maps in $\operatorname{Hom}_{A^e}(A^{\otimes 4}, A) \cong \operatorname{Hom}_k(A^{\otimes 2}, A)$.

If A' is a PBW deformation of A then

$$T(V)[t]/(P[t],\ t^2) = T(V)[t]/(x-\alpha(x)t,\ t^2 \mid x\in R)$$

corresponds to an infinitesimal deformation of A. Considering the elements of $(R\otimes V)\cap(V\otimes R)$, associativity implies condition (ii). Further, the quotient $T(V)[t]/(P[t],\ t^3)$ is a second level graded deformation of A, which implies that

$$\mu_1(\mu_1\otimes 1 - 1\otimes\mu_1) = d_3^*(\mu_2),$$

considered as functions on $A\otimes A\otimes A$. Applying both sides of this equation to $(R\otimes V)\cap(V\otimes R)$, since $\psi_\bullet\iota_\bullet$ is the identity map, this is equivalent to condition (iii). Similarly, there is a μ_3 such that

$$\mu_1(\mu_2\otimes 1 - 1\otimes\mu_2) + \mu_2(\mu_1\otimes 1 - 1\otimes\mu_1) = d_3^*(\mu_3).$$

Applying both sides of this equation to elements of $(R\otimes V)\cap(V\otimes R)$, for degree reasons (since $|\mu_3| = -3$, $|\mu_2| = -2$, $|\mu_1| = -1$), the terms $\mu_1(\mu_2\otimes 1 - 1\otimes\mu_2)$ have image zero, and we obtain $\beta(\alpha\otimes 1 - 1\otimes\alpha) = 0$, which is condition (iv).

Conversely, suppose conditions (i)–(iv) hold. Then $\mu_1\iota_2 = \alpha$ since $\psi_2\iota_2 = 1_{K_2}$. It follows that μ_1 is a Hochschild 2-cocycle on the bar resolution, and so defines an infinitesimal deformation of A. Similarly $\mu_2\iota_2 = \beta$. We modify μ_2 so that it satisfies condition (5.1.8) with $i = 2$ as a function on the bar resolution: Let

$$\gamma = -\mu_2 d_3 + \mu_1(\mu_1 \otimes 1 - 1 \otimes \mu_1).$$

Then $\gamma\iota_3 = -\beta d_3 + \alpha(\alpha \otimes 1 - 1 \otimes \alpha) = 0$ on K_3, which implies γ is a coboundary on the bar resolution, that is, $\gamma = \mu d_3$ for some μ. Now

$$\mu\iota_2 d_3 = \mu d_3 \iota_3 = \gamma\iota_3 = 0,$$

so $\mu\iota_2$ is a cocycle on $K_\bullet$. Consequently there is a cocycle μ', of internal degree -2, on the bar resolution with $\mu'\iota_2 = \mu\iota_2$. Then $(\mu_2 - \mu + \mu')\iota_2 = \beta$ and

$$(\mu_2 - \mu + \mu')d_3 + \mu_1(\mu_1 \otimes 1 - 1 \otimes \mu_1)$$

is zero on the bar resolution since $\mu' d_3 = 0$ and

$$-(\mu_2 - \mu)d_3 = \mu_1(\mu_1 \otimes 1 - 1 \otimes \mu_1).$$

We replace μ_2 by $\mu_2 - \mu + \mu'$.

Now, the map μ_1 and the new map μ_2 satisfy (5.1.8) with $i = 1$, $i = 2$. Thus there is a second level graded deformation of A defined by μ_1, μ_2. By condition (iv), considering internal degree, the obstruction

$$\mu_2(\mu_1 \otimes 1 - 1 \otimes \mu_1) + \mu_1(\mu_2 \otimes 1 - 1 \otimes \mu_2)$$

to lifting to a third level deformation of A becomes 0 as a cochain on $K_\bullet$, on applying ι_3. So this is a coboundary on the bar resolution, and there is a μ_3 of internal degree -3 satisfying (5.1.8) with $i = 3$.

Now by Proposition 5.4.1, the obstruction to lifting to a fourth level graded deformation lies in $\mathrm{HH}^{3,-4}(A)$. However, this is 0: $K_3 = A \otimes K_3' \otimes A$ and elements of K_3' all have degree 3, so $\mathrm{Hom}_{A^e}(K_3, A) \cong \mathrm{Hom}_k(K_3', A)$ consists of elements of internal degree at least -3. It follows that there is a μ_4 defining a fourth level deformation. The same argument shows that this may be lifted to a fifth level deformation and so on. Letting A_t denote the graded deformation obtained in this manner, we see that A' is isomorphic to $A_t|_{t=1}$: A map of vector spaces from V to $A_t|_{t=1}$ induces a map $A' \to A_t|_{t=1}$, which may be seen to be an isomorphism by counting dimensions in each degree. □

As an application, we obtain the classical Poincaré-Birkhoff-Witt Theorem for Lie algebras next. The proof shows that the universal enveloping algebra of a Lie algebra $\mathfrak{g}$ is a PBW deformation of a polynomial ring. Specifically, the polynomial ring may be identified as $S(\mathfrak{g})$, the symmetric algebra on the underlying vector space of $\mathfrak{g}$, that is,

$$S(\mathfrak{g}) = T(\mathfrak{g})/(x \otimes y - y \otimes x \mid x, y \in \mathfrak{g}).$$

Theorem 5.5.5 (Poincaré-Birkhoff-Witt Theorem)**.** *Let $\mathfrak{g}$ be a finite dimensional Lie algebra and let $U(\mathfrak{g})$ be its universal enveloping algebra. The associated graded algebra of $U(\mathfrak{g})$ is isomorphic to $S(\mathfrak{g})$.*

Proof. Let $V = \mathfrak{g}$ and $A = S(\mathfrak{g}) = T(V)/(R)$ where R is the vector subspace of $V \otimes V$ spanned by all $x \otimes y - y \otimes x$ for $x, y \in \mathfrak{g}$. Let

$$P = \{x \otimes y - y \otimes x - [x, y] \mid x, y \in \mathfrak{g}\},$$

so that $U(\mathfrak{g}) = T(\mathfrak{g})/(P)$ by definition. Set $\alpha(x \otimes y - y \otimes x) = [x, y]$ and $\beta(x \otimes y - y \otimes x) = 0$. By antisymmetry of the Lie bracket [,], condition (i) of Theorem 5.5.4 holds. A calculation shows that $(R \otimes V) \cap (V \otimes R)$ consists of linear combinations of elements of the form

$$x \otimes y \otimes z - y \otimes x \otimes z + y \otimes z \otimes x - z \otimes y \otimes x + z \otimes x \otimes y - x \otimes z \otimes y,$$

for $x, y, z \in \mathfrak{g}$, and that condition (ii) of Theorem 5.5.4 holds. Condition (iii) is equivalent to the Jacobi identity, and condition (iv) automatically holds since β is 0. By Theorem 5.5.4, $U(\mathfrak{g})$ is a PBW deformation of $S(\mathfrak{g})$, and in particular, $\operatorname{gr} U(\mathfrak{g}) \cong S(\mathfrak{g})$. □

For a survey of some of the many generalizations of this classical Poincaré-Birkhoff-Witt Theorem, as well as other methods of proof, see [**SW15**].

Exercise 5.5.6. Let $A = k\langle x, y\rangle/(yx - xy - 1)$, the Weyl algebra. What is its associated graded algebra $\operatorname{gr} A$? Show that A is a PBW deformation of $\operatorname{gr} A$.

Exercise 5.5.7. Consider $U(\mathfrak{sl}_2)$ as defined in Example 5.1.12. Show directly that $\operatorname{gr} U(\mathfrak{sl}_2)$ is isomorphic to a polynomial ring in three indeterminates.

Exercise 5.5.8. Look up Sridharan enveloping algebras [**Sri61**] and express them in the notation of this section. That is, each is a PBW deformation of a polynomial ring. What are α and β as in (5.5.1)?

Exercise 5.5.9. Verify the claims in the proof of Theorem 5.5.5, in particular that conditions (ii) and (iii) of Theorem 5.5.4 hold under the hypotheses of Theorem 5.5.5.

Chapter 6

Gerstenhaber Bracket

In Section 1.4, we introduced the Gerstenhaber bracket on Hochschild cohomology under which it becomes a graded Lie algebra. In Chapter 5 we saw applications of this Lie structure in algebraic deformation theory. In this chapter we give some equivalent definitions of this Lie structure, partially paralleling Chapter 2 which gives equivalent definitions of cup product. The definition of Gerstenhaber bracket that is most suitable will vary depending on circumstances. Some definitions we consider here lead to computational techniques, and we illustrate with some small examples.

We begin in Section 6.1 with a realization of Hochschild cohomology as the homology of a complex of coderivations on the tensor coalgebra of A. The Gerstenhaber bracket is then a graded commutator of coderivations ([**Qui89, Sta93**]). This tensor coalgebra is related to the bar resolution. We next derive a formula in Section 6.2 for brackets of elements in degree 1 with those in arbitrary degree n [**SA**]. The degree 1 elements are identified with derivations and thus with functions on the bar resolution, while degree n elements and the bracket formula are given on an arbitrary resolution. Thus our tour of Gerstenhaber bracket techniques begins to depart from the historical setting of the bar resolution, and the remainder of this chapter involves techniques for other resolutions and exact sequences. We present in Section 6.3 the notion of homotopy liftings that allow Gerstenhaber brackets to be expressed on an arbitrary resolution as essentially graded commutators of function compositions [**Vol**]. We discuss related computational techniques in Section 6.4 for resolutions with coalgebra structures, and these apply in particular to Koszul algebras [**NW16**]. For a topological approach, we outline in Section 6.5 a construction of brackets as loops in the classifying space of an extension category [**Sch98**].

We use the Koszul sign convention in this chapter: Let V, V', W, W' be graded vector spaces, and let $f : V \to V'$ and $g : W \to W'$ be k-linear graded functions, that is there is some m for which

$$f(V_n) \subset V_{n+m}$$

for all n, and similarly for g. Write $|f| = m$. The function $f \otimes g$ on $V \otimes W$ is defined by

$$(f \otimes g)(v \otimes w) = (-1)^{|g||v|} f(v) \otimes g(w) \tag{6.0.1}$$

for all homogeneous $v \in V$, $w \in W$.

6.1. Coderivations

In this section, we present Stasheff's view of the Gerstenhaber bracket on Hochschild cohomology of A as a graded commutator of coderivations on the tensor coalgebra of A. We start by defining this tensor coalgebra.

Let $T = T(A) = \oplus_{n \geq 0} A^{\otimes n}$ where we set $A^{\otimes 0} = k$, considered as a complex with differential d_T given by

$$d_T(a_1 \otimes \cdots \otimes a_n) = \sum_{i=1}^{n-1} (-1)^i a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n$$

for all $a_1, \ldots, a_n \in A$. Then $T(A)$ is a graded vector space with $T_n(A) = A^{\otimes n}$. We will use the notation $\operatorname{Hom}_k(T(A), T(A))$ to mean the graded vector space $\oplus_{n \geq 0} \operatorname{Hom}_k(A^{\otimes n}, T(A))$, with grading defined by

$$\operatorname{Hom}_k(T(A), T(A))_m = \{ f \mid f(A^{\otimes n}) \subset A^{\otimes (n+m)} \text{ for all } n \}.$$

Then $\operatorname{Hom}_k(T(A), T(A))$ is a graded algebra under composition of functions, and it is a complex with differential ∂ given by

$$\partial(f) = d_T f - (-1)^{|f|} f d_T \tag{6.1.1}$$

for all homogeneous functions f (see Section A.5). Note that $|d_T| = -1$. A calculation shows that

$$\partial(fg) = \partial(f) g + (-1)^{|f|} f \partial(g),$$

where fg denotes composition of these functions. That is, ∂ is a graded derivation on $\operatorname{Hom}_k(T(A), T(A))$. Thus $\operatorname{Hom}_k(T(A), T(A))$ is a differential graded algebra.

The complex $\operatorname{Hom}_k(T(A), T(A))$ has another binary operation given by the graded commutator:

$$[f, g] = fg - (-1)^{|f||g|} gf \tag{6.1.2}$$

for all homogeneous $f, g \in \mathrm{Hom}_k(T(A), T(A))$. By virtue of being a graded commutator, it enjoys a graded Jacobi identity just as in Lemma 1.4.2(ii). A calculation shows that

$$\partial([f,g]) = [\partial(f), g] + (-1)^{|f|}[f, \partial(g)].$$

Another calculation shows that

$$\partial(f) = [d_T, f]. \tag{6.1.3}$$

Define a k-linear map $\Delta_T : T(A) \to T(A) \otimes T(A)$ by

$$\begin{aligned} \Delta_T(a_1 \otimes \cdots \otimes a_n) &= 1 \otimes (a_1 \otimes \cdots \otimes a_n) + (a_1 \otimes \cdots \otimes a_n) \otimes 1 \\ &\quad + \sum_{i=1}^{n-1} (a_1 \otimes \cdots \otimes a_i) \otimes (a_{i+1} \otimes \cdots \otimes a_n) \end{aligned}$$

for all $a_1, \ldots, a_n \in A$. Under this map, $T(A)$ is a differential graded coalgebra, that is, Δ_T is a chain map, $(\Delta_T \otimes 1)\Delta_T = (1 \otimes \Delta_T)\Delta_T$, and $(\varepsilon \otimes 1)\Delta_T = 1 = (1 \otimes \varepsilon)\Delta_T$ where $\varepsilon : T(A) \to k$ projects onto $T(A)_0 = k$. Some authors write $T^c(A)$ for this differential graded coalgebra in order to distinguish it from the algebra on the same underlying vector space. Let $\Delta_T^{(2)} : T(A) \to T(A) \otimes T(A) \otimes T(A)$ be defined by

$$\Delta_T^{(2)} = (\Delta_T \otimes 1_T)\Delta_T = (1_T \otimes \Delta_T)\Delta_T.$$

Definition 6.1.4. A *graded coderivation* on $T(A)$ is a graded k-linear map $f : T(A) \to T(A)$, of some degree j, for which

$$\Delta_T f = (f \otimes 1_T + 1_T \otimes f)\Delta_T$$

as functions from $T(A)$ to $T(A) \otimes T(A)$. Denote by $\mathrm{Coder}(T(A))$ the vector space spanned by the graded coderivations on $T(A)$.

A calculation shows that the space $\mathrm{Coder}(T(A))$ is closed under the graded commutator bracket (6.1.2) by its definition (recalling the Koszul sign convention (6.0.1)). Also note that the differential d_T is itself a coderivation since Δ_T is a chain map. $\mathrm{Coder}(T(A))$ is thus a subcomplex of the Hom complex $(\mathrm{Hom}_k(T(A), T(A)), \partial)$ since $\partial = [d_T, -]$ as noted in (6.1.3).

The following connection with Hochschild cohomology goes back to work of Quillen [**Qui89**] and Stasheff [**Sta93**]. Let $B = B(A)$ be the bar resolution (1.1.4) of A as an A^e-module, so that $B_n = A^{\otimes(n+2)}$ for all $n \geq 0$. We take the differential d^* on $\mathrm{Hom}_k(T(A), A) \cong \mathrm{Hom}_{A^e}(B, A)$ to be that induced by the differential d on the bar resolution of A given by equation (1.1.2), which in turn is related to the differential d_T on $T(A)$:

$$\begin{aligned} &d^*(f)(a_1 \otimes \cdots \otimes a_m) \\ &= a_1 f(a_2 \otimes \cdots \otimes a_m) + f d_T(a_1 \otimes \cdots \otimes a_m) + (-1)^m f(a_1 \otimes \cdots \otimes a_{m-1}) a_m \end{aligned}$$

for all $a_1, \ldots, a_m \in A$ and $f \in \mathrm{Hom}_k(A^{\otimes m}, A)$. Note that the degree of such a function f is taken to be $m-1$ in our context here (not m as in other contexts). A calculation shows that f may be extended uniquely to a coderivation $D_f : T(A) \to T(A)[1-m]$ as follows:

$$D_f = (1_T \otimes f \otimes 1_T)\Delta_T^{(2)}, \tag{6.1.5}$$

where if $l < m$, we interpret D_f to be 0 on $A^{\otimes l}$. On elements then, applying the Koszul sign convention (6.0.1), we have

(6.1.6)

$$D_f(a_1 \otimes \cdots \otimes a_l) = \sum_{i=1}^{l-m+1} (-1)^u a_1 \otimes \cdots \otimes a_{i-1} \otimes f(a_i \otimes \cdots \otimes a_{i+m-1}) \otimes a_{i+m} \otimes \cdots \otimes a_l$$

for all $a_1, \ldots, a_l \in A$, where $u = (m-1)(i-1)$. (Existence and uniqueness of D_f is due to the corresponding truncated complex being cofree in a certain category of coalgebras; see, for example, [**MSS02**, Section II.3.7].)

Next we show that the complex of coderivations is isomorphic to the bar complex $\mathrm{Hom}_k(T(A), A) \cong \mathrm{Hom}_{A^e}(B, A)$ from which Hochschild cohomology is obtained.

Theorem 6.1.7. *The complex* $(\mathrm{Coder}(T(A)), \partial)$ *is a subcomplex of the complex* $(\mathrm{Hom}_k(T(A), T(A)), \partial)$ *that is isomorphic, as a differential graded vector space, to* $(\mathrm{Hom}_k(T(A), A), d^*)$.

Proof. We have already seen that the space $\mathrm{Coder}(T(A))$ is a subcomplex of $(\mathrm{Hom}_k(T(A), T(A)), \partial)$, since the differential d_T is a coderivation, $\partial = [d_T, -]$, and $\mathrm{Coder}(T(A))$ is closed under bracket. Given an element f of $\mathrm{Hom}_k(T(A), A)$, it extends uniquely to a coderivation D_f from $T(A)$ to $T(A)$ given by (6.1.5). On the other hand, given a coderivation from $T(A)$ to $T(A)$, its composition with projection onto $T_1(A) = A$ is an element of $\mathrm{Hom}_k(T(A), A)$. A calculation shows that the differential ∂ on $\mathrm{Coder}(T(A))$ corresponds to d^* on $\mathrm{Hom}_k(T(A), A)$. $\square$

As a consequence of the theorem, Hochschild cohomology $\mathrm{HH}^*(A)$ is the cohomology of the complex $(\mathrm{Coder}(T(A)), \partial)$. We may realize the Gerstenhaber bracket in a natural way on $\mathrm{Coder}(T(A))$ as follows. Recall the degree shift by 1 here in making comparisons to earlier sections.

Theorem 6.1.8. *The bracket (6.1.2) induces the Gerstenhaber bracket of Definition 1.4.1 on Hochschild cohomology* $\mathrm{HH}^*(A)$ *under the isomorphism of complexes given in Theorem 6.1.7.*

Proof. The isomorphism of Theorem 6.1.7 sends cochains f, g on the bar resolution $B = B(A)$ to their corresponding coderivations D_f, D_g on $T(A)$

given by formula (6.1.6). The formula (6.1.2) applied to D_f, D_g coincides with Definition 1.4.1 of Gerstenhaber bracket. To see this, note that projecting values of $[D_f, D_g] = D_f D_g - (-1)^{|D_f||D_g|} D_g D_f$ onto $T_1(A) = A$ yields the formula

$$f D_g - (-1)^{|f||g|} g D_f \tag{6.1.9}$$

for their bracket as an element of $\mathrm{Hom}_k(T(A), A)$. If $f \in \mathrm{Hom}_k(A^{\otimes m}, A)$ and $g \in \mathrm{Hom}_k(A^{\otimes n}, A)$ then $|D_f| = m - 1$ and $|D_g| = n - 1$. Applying the formula (6.1.6) for D_f and D_g in terms of f and g, and comparing to the formula for the Gerstenhaber bracket in Definition 1.4.1, we see that they are the same. $\square$

Exercise 6.1.10. Verify that Δ_T is a chain map.

Exercise 6.1.11. Verify that $\mathrm{Coder}(T(A))$ is closed under the graded commutator bracket (6.1.2).

Exercise 6.1.12. Verify that the differential d_T is a coderivation.

Exercise 6.1.13. Verify that the differential ∂ is a graded derivation with respect to the graded commutator (6.1.2).

Exercise 6.1.14. Verify that (6.1.5) (equivalently, (6.1.6)) defines a coderivation on $T(A)$.

6.2. Derivation operators

In this section, we present Suárez-Álvarez' methods from [**SA**] for computing Gerstenhaber brackets with elements of homological degree 1 via an arbitrary resolution. These methods may be used for example to find the Lie structure on degree 1 Hochschild cohomology $\mathrm{HH}^1(A)$ and the structure of its Lie module $\mathrm{HH}^*(A)$. Suárez-Álvarez worked in a broader context of derivation operators and actions on Ext. Here we consider only that part of his theory that is directly relevant to the Gerstenhaber bracket on $\mathrm{HH}^*(A)$, and refer to [**SA**] for more general results.

Let $P \xrightarrow{\mu_P} A$ be a projective resolution of the A^e-module A with differential d. Let $f : A \to A$ be a derivation, so that it represents an element of $\mathrm{HH}^1(A)$, as explained in Section 1.2. Let $f^e : A^e \to A^e$ be defined by

$$f^e = f \otimes 1 + 1 \otimes f, \tag{6.2.1}$$

and note that f^e is a derivation on A^e. Functions satisfying equation (6.2.3) below are termed *derivation operators* (or more specifically f^e*-operators*). More generally, the notion of a δ-operator, for any derivation δ on an algebra, is defined in [**SA**].

The following lemma is related to work of Gopalakrishnan and Sridharan [**GS66**].

Lemma 6.2.2. *Let $f : A \to A$ be a derivation. There is a k-linear chain map $\tilde{f}_{\bullet} : P_{\bullet} \to P_{\bullet}$ lifting f with the property that for each n,*

$$\tilde{f}_n((a \otimes b) \cdot x) = f(a)xb + a\tilde{f}_n(x)b + axf(b) \tag{6.2.3}$$

for all $a, b \in A$ and $x \in P_n$. Moreover, $\tilde{f}_{\bullet}$ is unique up to A^e-module chain homotopy.

Proof. We wish to define each $\tilde{f}_i$ so that it satisfies equation (6.2.3), and so that the following diagram commutes:

$$\begin{tikzcd}
\cdots \arrow[r, "d_2"] & P_1 \arrow[r, "d_1"] \arrow[d, "\tilde{f}_1"] & P_0 \arrow[r, "\mu_P"] \arrow[d, "\tilde{f}_0"] & A \arrow[r] \arrow[d, "f"] & 0 \\
\cdots \arrow[r, "d_2"] & P_1 \arrow[r, "d_1"] & P_0 \arrow[r, "\mu_P"] & A \arrow[r] & 0
\end{tikzcd}$$

If P_0 is free as an A^e-module, choose a free basis $\{x_j \mid j \in J\}$, where J is some indexing set. Since μ_P is surjective, for each $j \in J$, there exists a $y_j \in P_0$ such that $\mu_P(y_j) = f(\mu_P(x_j))$. Set $\tilde{f}_0(x_j) = y_j$. Extend to P_0 by requiring

$$\tilde{f}_0((a \otimes b) \cdot x_j) = f(a)x_jb + ay_jb + ax_jf(b)$$

for all $a, b \in A$ and $j \in J$. Note the rightmost square in the diagram indeed commutes since f is a derivation and the action of A^e on A is by left and right multiplication. If P_0 is not free, we may realize it as a direct summand of a free module and argue similarly.

Now $\tilde{f}_0 d_1$ has image contained in the image of d_1, since $\mu_P \tilde{f}_0 d_1 = f \mu_P d_1 = 0$. We may apply the same argument as above to define $\tilde{f}_1$, and so on. Thus we have a k-linear chain map $\tilde{f}_{\bullet}$ satisfying (6.2.3).

If $\tilde{f}_{\bullet}$ and $\tilde{f}'_{\bullet}$ are two such k-linear chain maps, then $\tilde{f}_{\bullet} - \tilde{f}'_{\bullet}$ is a chain map lifting the zero map from A to A. Since each of $\tilde{f}_{\bullet}$, $\tilde{f}'_{\bullet}$ satisfies (6.2.3), their difference $\tilde{f}_{\bullet} - \tilde{f}'_{\bullet}$ is A^e-linear, and so it is A^e-chain homotopic to 0. □

A standard example is given by functions on the bar resolution, as we explain next.

Example 6.2.4. Let B be the bar resolution on A, and let $f : A \to A$ be a derivation. For each i, let

$$\tilde{f}_i(a_0 \otimes \cdots \otimes a_{i+1}) = \sum_{j=0}^{i+1} a_0 \otimes \cdots \otimes a_{j-1} \otimes f(a_j) \otimes a_{j+1} \otimes \cdots \otimes a_{i+1}$$

for all $a_0, \ldots, a_{i+1} \in A$ and extend k-linearly. Then $\tilde{f}_{\bullet} : B \to B$ is a derivation operator, that is it satisfies equation (6.2.3).

The following theorem is due to Suárez-Álvarez [**SA**]. For Hochschild cocycles defined on a resolution P other than the bar resolution B, we define their Gerstenhaber bracket as a function on P via chain maps between P and B. See Exercise 6.2.11.

Theorem 6.2.5. *Let $f : A \to A$ be a derivation. Let P be a projective resolution of A as an A^e-module. Let $g \in \mathrm{Hom}_{A^e}(P_n, A)$ be a cocycle, and let $\tilde{f}_n : P_n \to P_n$ be a map satisfying (6.2.3). The Gerstenhaber bracket of f and g is represented by*

$$[f, g] = fg - g\tilde{f}_n \tag{6.2.6}$$

as a cocycle on P_n.

Proof. First note that if P is the bar resolution B, and $\tilde{f}_n$ is chosen as in Example 6.2.4, then formula (6.2.6) agrees with the historical formula of Definition 1.4.1 for the Gerstenhaber bracket, since f is a 1-cocycle. Since $gd = 0$ and $\tilde{f}_\bullet$ is unique up to chain homotopy as stated in Lemma 6.2.2, the element of Hochschild cohomology given by formula (6.2.6) does not depend on choice of $\tilde{f}_\bullet$.

More generally let $\theta : B \to P$ and $\iota : P \to B$ be comparison maps, that is, chain maps lifting the identity map on A. Identify the derivation f with a cocycle on B as described in Section 1.2. The Gerstenhaber bracket of f and g is by definition $[f, g\theta]\iota$, where $[f, g\theta]$ denotes the Gerstenhaber bracket defined as usual on B (see Exercise 6.2.11). Let $\tilde{f}'_\bullet : B \to B$ be a k-linear chain map satisfying (6.2.3) for B. A calculation shows that for each i, the function $\theta\tilde{f}'_i - \tilde{f}_i\theta$ is in fact an A^e-module homomorphism. Since $\theta\tilde{f}'_\bullet - \tilde{f}_\bullet\theta$ lifts the zero map from A to A, it must be A^e-chain homotopic to 0. By our arguments in the first paragraph above, $[f, g\theta] = fg\theta - g\theta\tilde{f}'_n$ represents the Gerstenhaber bracket of f and $g\theta$ at the chain level on B. Using the notation $\sim$ to indicate that cocycles are cohomologous, on P we have

$$\begin{aligned} [f, g\theta]\iota &\sim fg\theta\iota - g\theta\tilde{f}'_n\iota \\ &\sim fg\theta\iota - g\tilde{f}_n\theta\iota \\ &\sim fg - g\tilde{f}_n, \end{aligned}$$

since $\theta\iota$ is chain homotopic to the identity map and $gd = 0$. □

The proof of Theorem 6.2.5 via Lemma 6.2.2 is constructive, giving rise to a method for computing Gerstenhaber brackets with 1-cocycles. We illustrate this derivation operator method next with a small example. Other examples are in the literature, e.g. [**MNP**$^+$]. Used in combination with the relation given in Lemma 1.4.5 between cup product and Gerstenhaber bracket, the derivation operator method sometimes suffices to compute the full Gerstenhaber algebra structure on Hochschild cohomology.

Example 6.2.7. Let $A = k[x,y]$. We will find a general formula for the Gerstenhaber bracket of a 1-cocycle with a 2-cocycle on the Koszul resolution P of (3.1.4), using formula (6.2.6). Other brackets may be found similarly. Let $f = x^i y^j \frac{\partial}{\partial x}$, a derivation on A. Let $g = qx^* \wedge y^*$ for some $q \in A$. We first find derivation operators $\tilde{f}_0, \tilde{f}_1, \tilde{f}_2$ as in Lemma 6.2.2:

$$\begin{aligned}
\tilde{f}_0(a \otimes b) &= f(a) \otimes b + a \otimes f(b), \\
\tilde{f}_1(a \otimes x \otimes b) &= f(a) \otimes x \otimes b + \sum_{l=1}^{j} ax^i y^{j-l} \otimes y \otimes y^{l-1} b \\
&\quad + \sum_{l=1}^{i} ax^{i-l} \otimes x \otimes x^{l-1} y^j b + a \otimes x \otimes f(b), \\
\tilde{f}_1(a \otimes y \otimes b) &= f(a) \otimes y \otimes b + a \otimes y \otimes f(b), \\
\tilde{f}_2(a \otimes x \wedge y \otimes b) &= f(a) \otimes x \wedge y \otimes b + \sum_{l=1}^{i} ax^{i-l} \otimes x \wedge y \otimes x^{l-1} y^j b \\
&\quad + a \otimes x \wedge y \otimes f(b),
\end{aligned}$$

for all $a, b \in A$. By Theorem 6.2.5, setting $p = x^i y^j$,

$$\begin{aligned}
[f,g](x \wedge y) &= (fg - g\tilde{f}_2)(x \wedge y) \\
&= f(q) - g\left(\sum_{l=1}^{i} x^{i-l} \otimes x \wedge y \otimes x^{l-1} y^j\right) \\
&= p\frac{\partial}{\partial x}(q) - q\frac{\partial}{\partial x}(p).
\end{aligned}$$

So $[f,g] = (p\frac{\partial}{\partial x}(q) - q\frac{\partial}{\partial x}(p))x^* \wedge y^*$.

Exercise 6.2.8. Verify that f^e, defined by (6.2.1), is a derivation on A^e.

Exercise 6.2.9. Verify that $\tilde{f}_\bullet$ as defined in Example 6.2.4 is indeed a k-linear chain map on the bar resolution $B = B(A)$ and that it satisfies (6.2.3).

Exercise 6.2.10. Verify the claim in the first sentence of the proof of Theorem 6.2.5, that is, (6.2.6) agrees with the historical definition of Gerstenhaber bracket as defined on the bar resolution.

Exercise 6.2.11. Let P be a projective resolution of A as an A^e-module and let B be the bar resolution of A. Let $\theta : B \to P$ and $\iota : P \to B$ be comparison maps. Define a bilinear operation $[\ ,\ ]$ on $\mathrm{Hom}_{A^e}(P_\bullet, A)$ as follows. Let $f \in \mathrm{Hom}_{A^e}(P_m, A)$ and $g \in \mathrm{Hom}_{A^e}(P_n, A)$ be Hochschild cocycles. Let $[f,g] = [f\theta, g\theta]\iota$, where the bracket on the right side is the Gerstenhaber bracket of Definition 1.4.1. Show that this induces a well-defined operation $[\ ,]$ on Hochschild cohomology that agrees with the Gerstenhaber bracket under the isomorphism induced by the chain maps θ, ι.

Exercise 6.2.12. Let $A = k[x,y]$, notation as in Example 6.2.7. Find $[x^i y^j \frac{\partial}{\partial x}, qx^*]$ and $[x^i y^j \frac{\partial}{\partial x}, qy^*]$. What more must be computed to obtain all possible Gerstenhaber brackets among elements of $\mathrm{HH}^*(A)$?

6.3. Homotopy liftings

In this section we present Volkov's approach to brackets on Hochschild cohomology expressed directly on an arbitrary resolution, and explain how results of Sections 6.1 and 6.2 fit with this approach. More details and applications may be found in [**Vol**].

Let $P \xrightarrow{\mu_P} A$ be a projective resolution of A as an A^e-module with differential d. We work with the Hom complex $\mathrm{Hom}_{A^e}(P,P)$ in which the differential $\mathbf{d}$ is given by

$$\mathbf{d}(f) = df - (-1)^m fd$$

for all A^e-maps $f : P \to P[-m]$. (See Section A.2 for the degree shift notation $P[-m]$ and Section A.5 for the Hom complex.) The Hom complex is quasi-isomorphic to $\mathrm{Hom}_{A^e}(P,A)$ via the augmentation μ_P. We use the notation $\sim$ in this section to indicate that two functions are cohomologous in $\mathrm{Hom}_{A^e}(P,P)$. Equivalently, they are cohomologous in $\mathrm{Hom}_{A^e}(P,A)$ after application of μ_P, since μ_P induces a quasi-isomorphism between the complexes $\mathrm{Hom}_{A^e}(P,P)$ and $\mathrm{Hom}_{A^e}(P,A)$.

Let $f \in \mathrm{Hom}_{A^e}(P_m,A)$ and $g \in \mathrm{Hom}_{A^e}(P_n,A)$ be cocycles, that is, $fd = 0$ and $gd = 0$. Inspired by the expression (6.1.9) for the Gerstenhaber bracket obtained from Stasheff's coderivation approach, we aim to express the Gerstenhaber bracket $[f,g]$ analogously as a function on P similar to a graded commutator of function compositions. We will define functions $\psi_f : P \to P[1-m]$ and $\psi_g : P \to P[1-n]$ for which the Gerstenhaber bracket is represented at the chain level by

$$[f,g] = f\psi_g - (-1)^{(m-1)(n-1)} g\psi_f. \tag{6.3.1}$$

We caution that we have chosen slightly different notation from that of Volkov [**Vol**]. Our functions will differ from his by signs: Our ψ_f will be $\pm\phi_f$ in [**Vol**].

We will impose a condition on the functions ψ_f, ψ_g similar to a property of the circle product stated in Lemma 1.4.3(i). For an an m-cocycle f and an n-cocycle g on the bar resolution, this is:

$$(-1)^m (g \circ f)d = f \smile g - (-1)^{mn} g \smile f. \tag{6.3.2}$$

We wish to define an analog of the circle product, as a function on an arbitrary resolution, that has such a relationship to the cup product. With this in mind, let $\Delta_P : P \to P \otimes_A P$ be a diagonal map, that is an A^e-module chain map lifting the identity map on A. The cup product $f \smile g$ may be represented by the function $(-1)^{mn}(f \otimes g)\Delta_P$, where we have rewritten formula (2.3.1) with the Koszul sign convention (6.0.1) in mind. Accordingly, we require ψ_f to satisfy the following equation analogous to (6.3.2) for all cocycles g in $\mathrm{Hom}_{A^e}(P_n, A)$:

$$(-1)^m g\psi_f d = ((-1)^{mn} f \otimes g - g \otimes f)\Delta_P. \tag{6.3.3}$$

We impose these two conditions (6.3.1) and (6.3.3) on functions ψ_f, ψ_g, and derive from these some further conditions, leading to Definition 6.3.6 below of homotopy lifting. We will show in Theorem 6.3.11 that the conditions will be sufficient to define the Gerstenhaber bracket as (6.3.1).

We consider the second imposed condition (6.3.3) first. Fixing f, since $gd = 0$ and $|\psi_f| = m - 1$, the condition is (recalling the Koszul sign convention (6.0.1)):

$$g\mathbf{d}(\psi_f) = (-1)^m g\psi_f d = ((-1)^{mn} f \otimes g - g \otimes f)\Delta_P = g(f \otimes 1_P - 1_P \otimes f)\Delta_P$$

for all $n \geq 0$ and all n-cocycles g. This will hold if

$$\mathbf{d}(\psi_f) = (f \otimes 1_P - 1_P \otimes f)\Delta_P. \tag{6.3.4}$$

We consider the first imposed condition (6.3.1) in the case that g is the 0-cocycle μ_P, rewriting it as follows. Let B denote the bar resolution on A, and let $\theta : B \to P$ and $\iota : P \to B$ be comparison maps. Then $f\theta$ is a cocycle on B, and so is cohomologous to a cocycle taking the value 0 whenever one of the tensor factors in an argument is in the field k, by a comparison to the reduced bar resolution. Thus the Gerstenhaber bracket $[f\theta, \mu_B]$ becomes 0 in cohomology. Using the historical definition of Gerstenhaber bracket and the comparison maps ι, θ to translate to cocycles on P, the Gerstenhaber bracket of f and μ_P is

$$[f, \mu_P] = [f\theta, \mu_P\theta]\iota = [f\theta, \mu_B]\iota \sim 0.$$

So, if $[f, \mu_P]$ may be expressed as in equation (6.3.1), then setting $\psi = \psi_{\mu_P}$, we have

$$f\psi + (-1)^m \mu_P \psi_f \sim 0. \tag{6.3.5}$$

Note that by its definition, the function $f\psi + (-1)^m \mu_P\psi_f$ takes P_{m-1} to A and condition (6.3.5) is simply requiring ψ_f to take values in P_0 consistent with values of $f\psi$.

In fact these two conditions (6.3.4) and (6.3.5) are sufficient to define the bracket via formula (6.3.1), as we will see in Theorem 6.3.11. Next we will give a name to functions ψ_f having these properties, as in [**Vol**].

Definition 6.3.6. Let P be a projective resolution of A as an A^e-module, let $\Delta_P : P \to P \otimes_A P$ be a diagonal map, and let $f \in \mathrm{Hom}_{A^e}(P_m, A)$ be a cocycle. An A^e-module homomorphism $\psi_f : P \to P[1-m]$ is a *homotopy lifting of f with respect to Δ_P* if

$$\begin{aligned}\mathbf{d}(\psi_f) &= (f\otimes 1_P - 1_P\otimes f)\Delta_P \ \text{ and}\\ \mu_P\psi_f &\sim (-1)^{m-1} f\psi\end{aligned}$$

for some $\psi : P \to P[1]$ for which $\mathbf{d}(\psi) = (\mu_P\otimes 1_P - 1_P\otimes \mu_P)\Delta_P$.

We will often use the term *homotopy lifting of f* without explicit reference to Δ_P if it is clear from context which map Δ_P is intended, or in situations where the choice of Δ_P does not matter. We caution again that our homotopy lifting differs from that of Volkov [**Vol**] by signs.

It may be checked directly that if ψ_f, ψ_g are homotopy liftings for cocycles f, g with respect to Δ_P, then $[f, g]$ as defined in (6.3.1) is a cocycle. We check that if either f or g is a coboundary, then so is $[f, g]$ as defined in (6.3.1): If $f = hd$ for some cochain h, set

(6.3.7) $$\psi_f = (-1)^m (h\otimes 1_P - 1_P\otimes h)\Delta_P.$$

A calculation shows that ψ_f is a homotopy lifting for f. With this choice, $f\psi_g \sim (-1)^{(m-1)(n-1)} g\psi_f$, and so $[f, g]$ is a coboundary.

Example 6.3.8. Let $P = B$, the bar resolution of A, and let Δ_B be the standard diagonal map on B given by formula (2.3.2) (note this corresponds to the map Δ_T of Section 6.1 by writing $B = A\otimes T(A)\otimes A$ and extending Δ_T to be an A^e-module homomorphism). Then $(\mu_B\otimes 1_B - 1_B\otimes\mu_B)\Delta_B = 0$, and we may take $\psi = 0$ in Definition 6.3.6. Let $f \in \mathrm{Hom}_{A^e}(B_m, A)$ be a cocycle. We may assume without loss of generality that $f(a_0\otimes\cdots\otimes a_{m+1})$ is 0 whenever at least one of $a_1, \ldots, a_m$ is in the field k, since f is cohomologous to such a function. Let

$$\begin{aligned}&\psi_f(a_0\otimes\cdots\otimes a_{l+1})\\ &= \sum_{i=1}^{l-m+1} (-1)^u a_0\otimes\cdots\otimes a_{i-1}\otimes f(a_i\otimes\cdots\otimes a_{i+m-1})\otimes a_{i+m}\otimes\cdots\otimes a_{l+1},\end{aligned}$$

where $u = (m-1)(i-1)$, for all $l \geq m$ and $a_0, \ldots, a_{l+1} \in A$, and take ψ_f to be the zero map on B_l for $l \leq m-1$. Then ψ_f is a homotopy lifting of f with respect to Δ_B. A calculation shows that with this choice of ψ_f and a similar choice of ψ_g, the bracket $[f, g]$ as given by formula (6.3.1) is precisely the Gerstenhaber bracket as defined on the bar resolution in Definition 1.4.1.

We may view ψ_f defined by the above formula as a coderivation on B, or restrict to $T(A) \cong k\otimes T(A)\otimes k \hookrightarrow A\otimes T(A)\otimes A = B$ to obtain a coderivation $\psi_f|_{T(A)}$ on $T(A)$ as in Definition 6.1.4; see also formula (6.1.6). If f is a 1-cocycle, then $\psi_f|_{T(A)}$, viewed another way, may be extended to a derivation operator in the sense of Lemma 6.2.2 (see Example 6.2.4).

Thus homotopy liftings encompass these two views—coderivations on the tensor coalgebra and derivation operators on the bar resolution—that were introduced in Sections 6.1 and 6.2.

Lemma 6.3.9. *Let $f \in \mathrm{Hom}_{A^e}(P_m, A)$ be a cocycle and let $\Delta_P : P \to P \otimes_A P$ be a diagonal map. There is a homotopy lifting $\psi_f : P \to P[1-m]$ of f with respect to Δ_P. Moreover, it is unique up to chain homotopy.*

Proof. First we show existence of ψ, a homotopy lifting of μ_P with respect to Δ_P. Consider the function $(\mu_P \otimes 1_P - 1_P \otimes \mu_P)\Delta_P$ in the Hom complex $\mathrm{Hom}_{A^e}(P, P)$. Apply the quasi-isomorphism μ_P to $\mathrm{Hom}_{A^e}(P, A)$. Note that $\mu_P(\mu_P \otimes 1_P - 1_P \otimes \mu_P) = \mu_P \otimes \mu_P - \mu_P \otimes \mu_P = 0$, and so under the quasi-isomorphism from $\mathrm{Hom}_{A^e}(P \otimes_A P, P)$ to $\mathrm{Hom}_{A^e}(P \otimes_A P, A)$, the map $\mu_P \otimes 1_P - 1_P \otimes \mu_P$ becomes 0. Since $\mathbf{d}(\mu_P \otimes 1_P - 1_P \otimes \mu_P) = 0$, it is therefore a boundary in $\mathrm{Hom}_{A^e}(P \otimes_A P, P)$. Precomposing with the chain map Δ_P, we see that $(\mu_P \otimes 1_P - 1_P \otimes \mu_P)\Delta_P = \mathbf{d}(\psi)$ for some $\psi : P \to P[1]$, as claimed.

Next we show existence of functions ψ_f satisfying the conditions (6.3.4) and (6.3.5). Now $(f \otimes 1_P - 1_P \otimes f)\Delta_P$ is a chain map from P to $P[-m]$ since $fd = 0$. Applying μ_P, since $|\mu_P| = 0$, we have

$$\begin{aligned} \mu_P(f \otimes 1_P - 1_P \otimes f)\Delta_P &= f(1_P \otimes \mu_P - \mu_P \otimes 1_P)\Delta_P \\ &= -f\mathbf{d}(\psi) \;=\; -f\psi d, \end{aligned}$$

that is, applying the quasi-isomorphism from $\mathrm{Hom}_{A^e}(P, P)$ to $\mathrm{Hom}_{A^e}(P, A)$ given by μ_P, we find that $\mu_P(f \otimes 1_P - 1_P \otimes f)\Delta_P$ is a coboundary. Consequently, in $\mathrm{Hom}_{A^e}(P, P)$, the function $(f \otimes 1_P - 1_P \otimes f)\Delta_P$ is a coboundary, so that

$$(f \otimes 1_P - 1_P \otimes f)\Delta_P = \mathbf{d}(\psi_f)$$

for some ψ_f, that is, condition (6.3.4) holds. We will show that some of the functions ψ_f satisfying (6.3.4) also satisfy condition (6.3.5). As above we now have

$$\mu_P\mathbf{d}(\psi_f) = -f\psi d,$$

and since $\mathbf{d}(\psi_f) = d\psi_f + (-1)^m\psi_f d$ and $\mu_P d = 0$, this is equivalent to

$$((-1)^m\mu_P\psi_f + f\psi)d = 0.$$

However, we want $(-1)^m\mu_P\psi_f + f\psi \sim 0$. Set $g = (-1)^m\mu_P\psi_f + f\psi$, viewed as a map from P_{m-1} to A. We have seen that g is a cocycle, and thus it corresponds to a chain map $g_\bullet$ from P to $P[1-m]$. Define $\psi_f' = \psi_f - (-1)^m g_\bullet$. Since $g_\bullet$ is a chain map, $\mathbf{d}(\psi_f') = \mathbf{d}(\psi_f)$, and so ψ_f' also satisfies (6.3.4). Additionally we now have

$$(-1)^m\mu_p\psi_f' + f\psi = (-1)^m\mu_P\psi_f + f\psi - g = 0,$$

by definition of g, and so ψ'_f also satisfies (6.3.5). Without loss of generality then, ψ_f takes the correct values in P_0 and so (6.3.5) holds, as well as (6.3.4).

Finally, we show uniqueness up to chain homotopy. Let ψ_f and ψ'_f be two homotopy liftings of f with respect to Δ_P. Then $\mathbf{d}(\psi_f - \psi'_f) = 0$ and $\mu_P(\psi_f - \psi'_f) \sim 0$. Again, μ_P gives rise to the quasi-isomorphism from $\mathrm{Hom}_{A^e}(P, P)$ to $\mathrm{Hom}_{A^e}(P, A)$ and this implies $\psi_f - \psi'_f \sim 0$, as claimed. Note that this argument does not depend on choice of homotopy ψ for $(\mu_P \otimes 1_P - 1_P \otimes \mu_P)\Delta_P$, again as any two will be homotopic. $\square$

The following theorem and proof are [**Vol**, Theorem 4].

Theorem 6.3.10. *Let P be a projective resolution of A as an A^e-module with diagonal map $\Delta_P : P \to P \otimes_A P$. Let $f \in \mathrm{Hom}_{A^e}(P_m, A)$ and $g \in \mathrm{Hom}_{A^e}(P_n, A)$ be cocycles. The element of Hochschild cohomology $\mathrm{HH}^*(A)$ represented by $[f, g]$ as defined by formula (6.3.1) is independent of choice of resolution P, of diagonal map Δ_P, and of homotopy liftings ψ_f and ψ_g.*

Proof. We will prove independence of choices in the reverse order from what is stated. Independence of choice of ψ_f and ψ_g is immediate from the uniqueness of ψ_f and ψ_g up to chain homotopy stated in Lemma 6.3.9, since $fd = 0$ and $gd = 0$.

Let Δ_P and Δ'_P be two diagonal maps. Then $\Delta'_P - \Delta_P = \mathbf{d}(u)$ for some $u : P \to (P \otimes_A P)[1]$. Let ψ_f and ψ_g be homotopy liftings of f and g with respect to Δ_P. Let

$$\psi'_f = \psi_f + (-1)^m (f \otimes 1_P - 1_P \otimes f)u,$$

and similarly ψ'_g. A calculation shows that ψ'_f and ψ'_g are homotopy liftings of f and g with respect to Δ'_P, respectively. We find that

$$\begin{aligned}
&f\psi'_g - (-1)^{(m-1)(n-1)} g\psi'_f \\
&= f\psi_g - (-1)^{(m-1)(n-1)} g\psi_f + (-1)^n f(g \otimes 1_P - 1_P \otimes g)u \\
&\qquad - (-1)^{(m-1)(n-1)}(-1)^m g(f \otimes 1_P - 1_P \otimes f)u \\
&= f\psi_g - (-1)^{(m-1)(n-1)} g\psi_f,
\end{aligned}$$

so these two expressions give the same bracket $[f, g]$ via formula (6.3.1). Thus the formula is independent of choice of diagonal map.

Let $Q \xrightarrow{\mu_Q} A$ be another projective resolution of A as an A^e-module, and let $\Delta_Q : Q \to Q \otimes_A Q$ be a diagonal map. Let $\iota : P \to Q$ and $\theta : Q \to P$ be chain maps lifting the identity map on A. Let $f \in \mathrm{Hom}_{A^e}(P_m, A)$ and $g \in \mathrm{Hom}_{A^e}(P_n, A)$ be cocycles on P. Then $f\theta$ and $g\theta$ are cocycles on Q. Let $\psi_{f\theta}$ be a homotopy lifting for $f\theta$ with respect to Δ_Q. Set $\psi_f = \theta\psi_{f\theta}\iota$. We first check that ψ_f is a homotopy lifting for f with respect to $\Delta_P = (\theta \otimes \theta)\Delta_Q \iota$:

$$\begin{aligned}
\mathbf{d}(\psi_f) &= \theta \mathbf{d}(\psi_{f\theta})\iota \\
&= \theta(f\theta \otimes 1_Q - 1_Q \otimes f\theta)\Delta_Q \iota \\
&= (f \otimes 1_P - 1_P \otimes f)(\theta \otimes \theta)\Delta_Q \iota \\
&= (f \otimes 1_P - 1_P \otimes f)\Delta_P,
\end{aligned}$$

so ψ_f satisfies (6.3.4).

Set $\psi_P = \theta \psi_Q \iota$ where ψ_Q satisfies $\mathbf{d}(\psi_Q) = (\mu_Q \otimes 1_Q - 1_Q \otimes \mu_Q)\Delta_Q$ as well as $(-1)^m \mu_Q \psi_{f\theta} + f\theta\psi_Q \sim 0$. We may check then that

$$\begin{aligned}
\mathbf{d}(\psi_P) = \theta \mathbf{d}(\psi_Q)\iota &= \theta(\mu_Q \otimes 1_Q - 1_Q \otimes \mu_Q)\Delta_Q \iota \\
&= (\mu_P \otimes 1_P - 1_P \otimes \mu_P)(\theta \otimes \theta)\Delta_Q \iota \\
&= (\mu_P \otimes 1_P - 1_P \otimes \mu_P)\Delta_P,
\end{aligned}$$

since $\theta\mu_Q = \mu_P\theta$. Then, as $(-1)^m \mu_Q \psi_{f\theta} + f\theta\psi_Q \sim 0$, we have by the definitions of ψ_f and of ψ_P above,

$$\begin{aligned}
(-1)^m \mu_P \psi_f + f\psi_P &= (-1)^m \mu_Q \psi_{f\theta}\iota + f\theta\psi_Q \iota \\
&= ((-1)^m \mu_Q \psi_{f\theta} + f\theta\psi_Q)\iota \quad \sim \quad 0,
\end{aligned}$$

that is, ψ_f satisfies (6.3.5). Therefore ψ_f is a homotopy lifting of f with respect to Δ_P, and we may similarly define a homotopy lifting of g.

Finally, formula (6.3.1) applied to f, g on P yields

$$\begin{aligned}
[f, g] &= f\theta\psi_{g\theta}\iota - (-1)^{(m-1)(n-1)} g\theta\psi_{f\theta}\iota \\
&= [f\theta, g\theta]\iota,
\end{aligned}$$

so the chain map ι takes $[f\theta, g\theta]$ to $[f, g]$. Thus the bracket does not depend on choice of resolution. □

As a consequence of Theorem 6.3.10, the bracket given by formula (6.3.1) agrees with the Gerstenhaber bracket of Definition 1.4.1 on Hochschild cohomology:

Theorem 6.3.11. *Let P be a projective resolution of A as an A^e-module. Let $f \in \mathrm{Hom}_{A^e}(P_m, A)$ and $g \in \mathrm{Hom}_{A^e}(P_n, A)$ be cocycles on P, and let ψ_f and ψ_g be homotopy liftings of f and g, as in Definition 6.3.6. The bracket given at the chain level by*

$$[f, g] = f\psi_g - (-1)^{(m-1)(n-1)} g\psi_f$$

induces the Gerstenhaber bracket on Hochschild cohomology $\mathrm{HH}^*(A)$.

Proof. In Example 6.3.8, we saw that taking P to be the bar resolution recovers the Gerstenhaber bracket of Definition 1.4.1 from formula (6.3.1). By Theorem 6.3.10, it is independent of choices. □

Remark 6.3.12. In practice, often $(\mu_P \otimes 1_P)\Delta_P = 1_P = (1_P \otimes \mu_P)\Delta_P$, in which case Definition 6.3.6 of homotopy lifting can be simplified by taking $\psi = 0$. See Example 6.3.8, the next Section 6.4, and [**Vol**, Remark 1].

Exercise 6.3.13. Let ψ_f, ψ_g be homotopy liftings of f, g as in Definition 6.3.6.

(a) Check directly that $[f, g]$ as defined in (6.3.1) is a cocycle.

(b) Check directly that if either f or g is a coboundary then so is $[f, g]$ as defined in (6.3.1). (See (6.3.7) for a homotopy lifting of a coboundary and verify first that it is indeed a homotopy lifting.)

Exercise 6.3.14. Verify that in the context of Example 6.3.8, the formula for ψ_f indeed yields the classical Gerstenhaber bracket as defined on the bar resolution in Definition 1.4.1.

6.4. Resolutions with coalgebra structure

Some of the results of the previous sections lead to effective computational techniques for the Lie structure on Hochschild cohomology. In particular, we explain in this section some settings in which the theory of homotopy liftings can be simplified for computational purposes. Formula (6.4.2) below gives homotopy liftings for all cocycles f in terms of a diagonal map Δ_P and an additional function ϕ_P that we introduce next.

Let $\phi_P : P \otimes_A P \to P[1]$ be a homotopy for $\mu_P \otimes 1_P - 1_P \otimes \mu_P$. That is,

$$\mathbf{d}(\phi_P) = \mu_P \otimes 1_P - 1_P \otimes \mu_P. \tag{6.4.1}$$

To see that such a homotopy exists, consider the quasi-isomorphism μ_P from $\mathrm{Hom}_{A^e}(P \otimes_A P, P)$ to $\mathrm{Hom}_{A^e}(P \otimes_A P, A)$, which takes $\mu_P \otimes 1_P - 1_P \otimes \mu_P$ to 0. Since μ_P is a chain map, $\mathbf{d}(\mu_P \otimes 1_P - 1_P \otimes \mu_P) = 0$. It follows that in $\mathrm{Hom}_{A^e}(P \otimes_A P, A)$, the map $\mu_P \otimes 1_P - 1_P \otimes \mu_P$ is a cocycle that becomes 0 (which is a coboundary) after applying the quasi-isomorphism μ_P, so it must be a coboundary in $\mathrm{Hom}_{A^e}(P \otimes_A P, A)$.

Let

$$\Delta_P^{(2)} = (\Delta_P \otimes 1_P)\Delta_P.$$

Note that in general this may not be the same as $(1_P \otimes \Delta_P)\Delta_P$. Let $f \in \mathrm{Hom}_{A^e}(P_m, A)$ with $fd = 0$, and let $\psi_f : P \to P[1-m]$ be defined by

$$\psi_f = \phi_P(1_P \otimes f \otimes 1_P)\Delta_P^{(2)}. \tag{6.4.2}$$

Here the map $1_P \otimes f \otimes 1_P$ is considered to be a map from $P \otimes_A P \otimes_A P$ to $P \otimes_A P$ (upon applying the canonical isomorphism $P \otimes_A A \otimes_A P \cong P \otimes_A P$). We will see next that under some conditions, ψ_f is a homotopy lifting of f with respect to Δ_P as in Definition 6.3.6. Thus homotopy liftings ψ_f for all cocycles f may be found in terms of these two maps Δ_P and ϕ_P.

Example 6.4.3. Consider the bar resolution B of A. We may identify $B_i \otimes_A B_j$ with $A \otimes A^{\otimes i} \otimes A \otimes A^{\otimes j} \otimes A$, under the isomorphism

$$
\begin{aligned}
(A \otimes A^{\otimes i} \otimes A) \otimes_A (A \otimes A^{\otimes j} \otimes A) &\xrightarrow{\sim} A \otimes A^{\otimes i} \otimes A \otimes A^{\otimes j} \otimes A \\
(a_0 \otimes \cdots \otimes a_{i+1}) \otimes_A (a_0' \otimes \cdots \otimes a_{j+1}') &\mapsto \\
a_0 \otimes (a_1 \otimes \cdots \otimes a_i) \otimes (a_{i+1}a_0') &\otimes (a_1' \otimes \cdots \otimes a_j') \otimes a_{j+1}'
\end{aligned}
$$

for all $a_0, \ldots, a_{i+1}, a_0', \ldots, a_{j+1}' \in A$. Define $\phi_B : B \otimes_A B \to B[1]$ by

$$
\begin{aligned}
&\phi_B(a_0 \otimes (a_1 \otimes \cdots \otimes a_i) \otimes (a_{i+1}) \otimes (a_{i+2} \otimes \cdots \otimes a_{i+j+1}) \otimes a_{i+j+2}) \\
&= (-1)^i a_0 \otimes \cdots \otimes a_{i+j+2}
\end{aligned}
$$

for all $a_0, \ldots, a_{i+j+2} \in A$, that is, up to sign, we have just removed parentheses. Then ψ_f defined as in (6.4.2) agrees with that given in Example 6.3.8.

In the rest of this section we prove results under some hypotheses that will be satisfied for example by Koszul algebras as defined in Section 3.4. A result of Buchweitz, Green, Snashall and Solberg [**BGSS08**] for Koszul algebras guarantees that the standard embedding $\iota : P \to B$ of the Koszul resolution P into the bar resolution B of A preserves the diagonal map in the sense that the diagonal map Δ_B of the bar resolution given by formula (2.3.2) takes $\iota(P)$ to $\iota(P) \otimes_A \iota(P)$, and so we may define a diagonal map Δ_P on P via this embedding. It follows that Δ_P is coassociative, that is $(\Delta_P \otimes 1_P)\Delta_P = (1_P \otimes \Delta_P)\Delta_P$, and $(\mu_P \otimes 1_P)\Delta_P = 1_P = (1_P \otimes \mu_P)\Delta_P$, and we say that P is a *counital differential graded coalgebra*. In this case, in Definition 6.3.6 of homotopy lifting of an m-cocycle f on P, we may thus take $\psi = 0$, and so condition (6.3.5) becomes $\mu_P \psi_f \sim 0$. We may in fact assume that $\psi_f|_{P_{m-1}} = 0$. This simplifies the work of finding homotopy liftings (and it simplifies many of the proofs of the previous section under these hypotheses). In fact formula (6.4.2) always defines a homotopy lifting in the case that Δ_P, μ_P give P a coalgebra structure, as we see next.

Lemma 6.4.4. *Let $P \xrightarrow{\mu_P} A$ be a projective resolution of the A^e-module A that is a counital differential graded coalgebra, that is, there is a diagonal map $\Delta_P : P \to P \otimes_A P$ such that $(\Delta_P \otimes 1_P)\Delta_P = (1_P \otimes \Delta_P)\Delta_P$ and $(\mu_P \otimes 1_P)\Delta_P = 1_P = (1_P \otimes \mu_P)\Delta_P$. Let $f \in \mathrm{Hom}_{A^e}(P_m, A)$ be a cocycle, and let $\psi_f : P \to P[1-m]$ be defined by formula (6.4.2). Then ψ_f is a homotopy lifting of f.*

Proof. Let $\phi_P : P \otimes_A P \to P[1]$ be a homotopy for $\mu_P \otimes 1_P - 1_P \otimes \mu_P$, that is, equation (6.4.1) holds. Set $\Delta_P^{(2)} = (\Delta_P \otimes 1_P)\Delta_P = (1_P \otimes \Delta_P)\Delta_P$. Since

$\Delta_P^{(2)}$ and $1_P \otimes f \otimes 1_P$ are chain maps,

$$\begin{aligned}
\mathbf{d}(\psi_f) &= \mathbf{d}(\phi_P)(1_P \otimes f \otimes 1_P)\Delta_P^{(2)} \\
&= (\mu_P \otimes 1_P - 1_P \otimes \mu_P)(1_P \otimes f \otimes 1_P)\Delta_P^{(2)} \\
&= (\mu_P \otimes f \otimes 1_P - 1_P \otimes f \otimes \mu_P)\Delta_P^{(2)} \\
&= ((f \otimes 1_P)(\mu_P \otimes 1_P \otimes 1_P) - (1_P \otimes f)(1_P \otimes 1_P \otimes \mu_P))\Delta_P^{(2)} \\
&= (f \otimes 1_P - 1_P \otimes f)\Delta_P.
\end{aligned}$$

Note that $\psi_f|_{P_{m-1}} = 0$ by definition, and as explained above, we may take $\psi = 0$ in Definition 6.3.6. Therefore ψ_f is a homotopy lifting of f. $\square$

Compare the following theorem to [**NW16**, Theorem 3.2.5], which has stronger hypotheses, and to [**NW16**, Lemma 3.4.1], which has somewhat different hypotheses.

Theorem 6.4.5. *Let $P \xrightarrow{\mu_P} A$ be a projective resolution of the A^e-module A that is a counital differential graded coalgebra, that is, there is a diagonal map $\Delta_P : P \to P \otimes_A P$ satisfying the coassociative and counit properties. Let $f \in \mathrm{Hom}_{A^e}(P_m, A)$ and $g \in \mathrm{Hom}_{A^e}(P_n, A)$ be cocycles. Define ψ_f by formula (6.4.2), and similarly ψ_g. Then*

$$[f, g] = f\psi_g - (-1)^{(m-1)(n-1)} g\psi_f$$

represents the Gerstenhaber bracket of f and g on Hochschild cohomology.

Proof. This follows immediately from Lemma 6.4.4 and Theorem 6.3.11. $\square$

Remark 6.4.6. If the hypotheses of the theorem do not hold, a homotopy ϕ_P for $\mu_P \otimes 1_P - 1_P \otimes \mu_P$ may still be used to define the Gerstenhaber bracket in a similar way, with the addition of some error terms. See [**Vol**, Corollary 5 and Remark 1] in which the Gerstenhaber bracket is given generally as

(6.4.7)

$$\begin{aligned}
[f, g] = &-f\phi_P(g \otimes 1_P \otimes 1_P - 1_P \otimes g \otimes 1_P + 1_P \otimes 1_P \otimes g)\Delta_P^{(2)} \\
&+ (-1)^{(m-1)(n-1)} g\phi_P(f \otimes 1_P \otimes 1_P - 1_P \otimes f \otimes 1_P + 1_P \otimes 1_P \otimes f)\Delta_P^{(2)}.
\end{aligned}$$

We caution that the function

$$-\phi_P(f \otimes 1_P \otimes 1_P - 1_P \otimes f \otimes 1_P + 1_P \otimes 1_P \otimes f)\Delta_P^{(2)}$$

is not necessarily a homotopy lifting of f; the formula (6.4.7) instead results from a more complicated homotopy lifting as explained in the proof of [**Vol**, Corollary 5].

In the remainder of this section, we apply Theorem 6.4.5 to an example, a polynomial ring in two indeterminates. The case of n indeterminates is similar, if more notationally unwieldy, and is handled in [**NW16**, Section 4], showing that formula (6.3.1) indeed yields the familiar Gerstenhaber bracket on Hochschild cohomology of a polynomial ring. In other settings the first computation of Gerstenhaber brackets, or of a related Batalin-Vilkovisky structure, used these techniques (see, for example, [**Gri, GNW, NW, Vol**]).

Example 6.4.8. Let $A = k[x,y]$ and let P be its Koszul resolution (3.1.4), so that $P = A \otimes \bigwedge^{\bullet} V \otimes A$ where $V = \mathrm{Span}_k\{x,y\}$. Identify $P_i \otimes_A P_j$ with $A \otimes \bigwedge^i V \otimes A \otimes \bigwedge^j V \otimes A$ for each i, j, and identify $\bigwedge^0 V$ with k and $\bigwedge^1 V$ with V. Thus for example, $P_0 \otimes_A P_1 \cong A \otimes k \otimes A \otimes V \otimes A \cong A \otimes A \otimes V \otimes A$, and we use such expressions in our definitions of maps below. We first find a homotopy $\phi_P : P \otimes_A P \to P[1]$ for $\mu_P \otimes 1_P - 1_P \otimes \mu_P$. In degree 2, the map ϕ_P is necessarily 0 since $P_3 = 0$. We define ϕ_P in degrees 0 and 1 on free basis elements:

$$\begin{aligned}
\phi_P(1 \otimes x^i y^j \otimes 1) &= \sum_{l=1}^{j} x^i y^{j-l} \otimes y \otimes y^{l-1} + \sum_{l=1}^{i} x^{i-l} \otimes x \otimes x^{l-1} y^j, \\
\phi_P(1 \otimes x^i y^j \otimes x \otimes 1) &= -\sum_{l=1}^{j} x^i y^{j-l} \otimes x \wedge y \otimes y^{l-1}, \\
\phi_P(1 \otimes x^i y^j \otimes y \otimes 1) &= 0, \\
\phi_P(1 \otimes x \otimes x^i y^j \otimes 1) &= 0, \\
\phi_P(1 \otimes y \otimes x^i y^j \otimes 1) &= \sum_{l=1}^{i} x^{i-l} \otimes x \wedge y \otimes x^{l-1} y^j.
\end{aligned}$$

We use this function ϕ_P, formula (6.4.2) for ψ_f, and the formula of Theorem 6.4.5 to compute some brackets in degree 1. The diagonal map Δ_P is defined by the standard embedding (3.4.8) of P into the bar resolution, followed by the standard diagonal map (2.3.2) on the bar resolution. Consider the cocycles in degree 1 denoted by $x^i y^j x^*$ and $x^i y^j y^*$, where $\{x^*, y^*\}$ is the dual basis to $\{x, y\}$ (i.e. $x^i y^j x^*$ takes x to $x^i y^j$ and y to 0 where x, y are identified with their images in $\bigwedge^1 V$). First we find some values of $\psi_{x^i y^j x^*}$ and $\psi_{x^i y^j y^*}$ via formula (6.4.2):

$$\begin{aligned}
\psi_{x^i y^j x^*}(1 \otimes x \otimes 1) &= \phi_P(x^i y^j), & \psi_{x^i y^j x^*}(1 \otimes y \otimes 1) &= 0, \\
\psi_{x^i y^j y^*}(1 \otimes x \otimes 1) &= 0, & \psi_{x^i y^j y^*}(1 \otimes y \otimes 1) &= \phi_P(x^i y^j).
\end{aligned}$$

It follows that, for example,

$$\begin{aligned}
&[x^iy^jx^*,\ x^my^nx^*](1\otimes x\otimes 1)\\
&= x^iy^jx^*\psi_{x^my^nx^*}(1\otimes x\otimes 1) - x^my^nx^*\psi_{x^iy^jx^*}(1\otimes x\otimes 1)\\
&= x^iy^jx^*\phi_P(x^my^n) - x^my^nx^*\phi_P(x^iy^j)\\
&= \sum_{l=1}^{m} x^iy^jx^{m-l}x^{l-1}y^n - \sum_{l=1}^{i} x^my^nx^{i-l}x^{l-1}y^j\\
&= mx^iy^jx^{m-1}y^n - ix^my^nx^{i-1}y^j\\
&= x^iy^j\frac{\partial}{\partial x}(x^my^n) - x^my^n\frac{\partial}{\partial x}(x^iy^j).
\end{aligned}$$

Another calculation shows that the value of this bracket function on $1\otimes y\otimes 1$ is zero. Therefore, for all $p,q\in A$, we have

$$[px^*,\ qx^*] = (p\frac{\partial}{\partial x}(q) - q\frac{\partial}{\partial x}(p))x^*.$$

Similarly we find that

$$\begin{aligned}
[px^*,\ qy^*] &= p\frac{\partial}{\partial x}(q)y^* - q\frac{\partial}{\partial y}(p)x^*,\\
[py^*,\ qy^*] &= (p\frac{\partial}{\partial y}(q) - q\frac{\partial}{\partial y}(p))y^*.
\end{aligned}$$

We may calculate other brackets using the same techniques. (Cf. Example 6.2.7 and Exercise 6.2.12.)

Exercise 6.4.9. Verify that ψ_f as defined in Example 6.4.3 agrees with that given in Example 6.3.8.

Exercise 6.4.10. Let $P \xrightarrow{\mu_P} A$ be a projective resolution of the A^e-module A that is a counital differential graded coalgebra. Show that one choice of chain map $g_\bullet$ corresponding to a cocycle $g\in \mathrm{Hom}_{A^e}(P_n,A)$ is given by $g_m = (1\otimes g)(\Delta_P)_m$ for all m. (This shows a parallel between the cup product as given by formula (2.2.1) and the Gerstenhaber bracket as given by formula (6.3.1) using (6.4.2).)

Exercise 6.4.11. Verify the formulas for $[px^*, qy^*]$ and $[qy^*, qy^*]$ in Example 6.4.8. Find $[px^*, qx^*\wedge y^*]$ and compare with Example 6.2.7.

6.5. Extensions

In this section, we consider Schwede's exact sequence interpretation of the Lie structure on Hochschild cohomology [**Sch98**]. Hermann [**Her16c**] generalized Schwede's results to some exact monoidal categories, and gave a description of the bracket with degree 0 elements in [**Her16a**], completing

Schwede's interpretation. We refer to these papers for most of the technical details and proofs, and give just a skimming here.

Let $n \geq 1$ and let $\mathcal{E}xt^n_{A^e}(A, A)$ denote the category whose objects are n-extensions of A by A as an A^e-module, and morphisms are maps of n-extensions. (See Section A.3 for a discussion of maps of n-extensions and Section A.6 for categories.) View $\mathrm{HH}^n(A) = \mathrm{Ext}^n_{A^e}(A, A)$ as equivalence classes of objects in $\mathcal{E}xt^n_{A^e}(A, A)$. We adapt notation from [**Sch98**]: Consider an m-extension and an n-extension of A by A,

$$\mathbf{f}: \qquad 0 \longrightarrow A \xrightarrow{i_M} M_{m-1} \longrightarrow \cdots \longrightarrow M_0 \xrightarrow{\mu_M} A \longrightarrow 0,$$

$$\mathbf{g}: \qquad 0 \longrightarrow A \xrightarrow{i_N} N_{n-1} \longrightarrow \cdots \longrightarrow N_0 \xrightarrow{\mu_N} A \longrightarrow 0.$$

We will not need notation for the unlabeled maps. We assume that all M_j, N_j are projective as left A-modules, and as right A-modules, where needed. See [**Sch98**] for a discussion about such an assumption.

Let P be a projective resolution of A as an A^e-module. Let f and g be an m-cocycle and an n-cocycle on P, corresponding to generalized extensions $\mathbf{f}$ and $\mathbf{g}$, respectively. So $f \in \mathrm{Hom}_{A^e}(P_m, A)$ may be defined via the following commuting diagram, which exists by the Comparison Theorem (Theorem A.2.7); see also Section A.3. The map 1 from A to A is the identity map. We denote by $\hat{f}_{\bullet} : P_{\bullet} \to M_{\bullet}$ the chain map indicated below, so that $f = \hat{f}_m$.

$$\begin{tikzcd}
\cdots \arrow[r] & P_m \arrow[r, "d_m"] \arrow[d, "f"] & P_{m-1} \arrow[r, "d_{m-1}"] \arrow[d, "\hat{f}_{m-1}"] & \cdots \arrow[r, "d_1"] & P_0 \arrow[r, "\mu_P"] \arrow[d, "\hat{f}_0"] & A \arrow[r] \arrow[d, "1"] & 0 \\
0 \arrow[r] & A \arrow[r, "i_M"] & M_{m-1} \arrow[r] & \cdots \arrow[r] & M_0 \arrow[r, "\mu_M"] & A \arrow[r] & 0
\end{tikzcd}$$

We will later replace $\mathbf{f}$ by a specific m-extension defined via P and a pushout diagram, denoted $K(f)$.

We write $\mathbf{g}\#\mathbf{f}$ for the following Yoneda splice (see Section 2.2). Note that $\mathbf{g}\#\mathbf{f}$ here is what was denoted by $\mathbf{f} \smile \mathbf{g}$ in Section 2.2. We use this alternate notation here for ease of comparison with [**Sch98**].

$\mathbf{g}\#\mathbf{f}$:

$$0 \longrightarrow A \xrightarrow{i_M} M_{m-1} \longrightarrow \cdots \longrightarrow M_0 \xrightarrow{i_N \mu_M} N_{n-1} \longrightarrow$$

$$\cdots \longrightarrow N_0 \xrightarrow{\mu_N} A \longrightarrow 0$$

Note that an element of $\mathrm{Hom}_{A^e}(P_{m+n}, A)$ corresponding to this $(m+n)$-extension is $f \smile g$ as explained in Section 2.2.

We write $\mathbf{f} \otimes_A \mathbf{g}$ for the $(m+n)$-extension corresponding to the total complex of the tensor product of the two truncated sequences as discussed in Section 2.2. In degree $m+n-1$, for example, we have the module $(M_m \otimes_A N_{n-1}) \oplus (M_{m-1} \otimes_A N_n) \cong N_{n-1} \oplus M_{m-1}$, and the extension is

$$\mathbf{f} \otimes_A \mathbf{g} :$$

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] & M_{m-1} \oplus N_{n-1} \arrow[r] & \cdots \end{tikzcd}$$

$$\begin{tikzcd} {} \arrow[r] & \begin{smallmatrix}(M_1 \otimes_A N_0) \oplus \\ (M_0 \otimes_A N_1)\end{smallmatrix} \arrow[r] & M_0 \otimes_A N_0 \arrow[r] & A \arrow[r] & 0 \end{tikzcd}$$

The cup product of f and g corresponds to any of the $(m+n)$-extensions $\mathbf{g}\#\mathbf{f}$, $\mathbf{f} \otimes_A \mathbf{g}$, $(-1)^{mn}\mathbf{f}\#\mathbf{g}$, $(-1)^{mn}\mathbf{g} \otimes_A \mathbf{f}$. (As an additive inverse $-\mathbf{f}\#\mathbf{g}$ to the extension $\mathbf{f}\#\mathbf{g}$, we take here the extension whose modules agree with those of $\mathbf{f}\#\mathbf{g}$, the map μ_M is replaced by $-\mu_M$, and all other maps agree with those of $\mathbf{f}\#\mathbf{g}$.) These extensions are all equivalent and there are maps as indicated in the following diagram:

$$\begin{tikzcd} & \mathbf{f} \otimes_A \mathbf{g} \arrow[dl, "\lambda_{\mathbf{f},\mathbf{g}}"'] \arrow[dr, "\rho_{\mathbf{f},\mathbf{g}}"] & \\ \mathbf{g}\#\mathbf{f} & & (-1)^{mn}\mathbf{f}\#\mathbf{g} \\ & (-1)^{mn}\mathbf{g} \otimes_A \mathbf{f} \arrow[ul, "\rho_{\mathbf{g},\mathbf{f}}"] \arrow[ur, "\lambda_{\mathbf{g},\mathbf{f}}"'] & \end{tikzcd} \tag{6.5.1}$$

Such maps may be described as follows. Consider the augmented double complex:

$$\begin{tikzcd}
& M_0 \arrow[dl] \arrow[d] & \cdots \arrow[l] & M_{m-1} \arrow[l] \arrow[d] & A \arrow[l] \arrow[d] \\
N_{n-1} \arrow[d] & M_0 \otimes_A N_{n-1} \arrow[l] \arrow[d] & \cdots \arrow[l] & M_{m-1} \otimes_A N_{n-1} \arrow[l] \arrow[d] & N_{n-1} \arrow[l] \arrow[d] \\
\vdots \arrow[d] & \vdots \arrow[d] & & \vdots \arrow[d] & \vdots \arrow[d] \\
N_1 \arrow[d] & M_0 \otimes_A N_1 \arrow[l] \arrow[d] & \cdots \arrow[l] & M_{m-1} \otimes_A N_1 \arrow[l] \arrow[d] & N_1 \arrow[l] \arrow[d] \\
N_0 \arrow[d] & M_0 \otimes_A N_0 \arrow[l] \arrow[d] & \cdots \arrow[l] & M_{m-1} \otimes_A N_0 \arrow[l] \arrow[d] & N_0 \arrow[l] \arrow[dl] \\
A & M_0 \arrow[l] & \cdots \arrow[l] & M_{m-1} \arrow[l] &
\end{tikzcd}$$

All but the leftmost column and bottom row constitute the truncated double complex corresponding to $\mathbf{f} \otimes_A \mathbf{g}$, and the outermost rows and columns are $\mathbf{g}\#\mathbf{f}$ (left column and top row) and $(-1)^{mn}\mathbf{f}\#\mathbf{g}$ (right column and bottom row). The maps $\lambda_{\mathbf{f},\mathbf{g}} : \mathbf{f} \otimes_A \mathbf{g} \to \mathbf{g}\#\mathbf{f}$ and $\rho_{\mathbf{f},\mathbf{g}} : \mathbf{f} \otimes_A \mathbf{g} \to (-1)^{mn}\mathbf{f}\#\mathbf{g}$ are given as follows. See also Section 2.2 where these maps are used to show that two definitions of associative product on $\mathrm{HH}^*(A)$ agree.

For $n \leq i \leq m+n$, $\lambda_{\mathbf{f},\mathbf{g}}$ projects $(\mathbf{f} \otimes_A \mathbf{g})_i$ onto $M_{i-n} \otimes_A N_n \cong M_{i-n}$. For $0 \leq i \leq n-1$, $\lambda_{\mathbf{f},\mathbf{g}}$ first projects $(\mathbf{f} \otimes_A \mathbf{g})_i$ onto $M_0 \otimes_A N_i$, then maps to $A \otimes_A N_i \cong N_i$ via $\mu_N \otimes 1$. For $m \leq i \leq m+n$, $\rho_{\mathbf{f},\mathbf{g}}$ projects $(\mathbf{f} \otimes_A \mathbf{g})_i$ onto $(-1)^{m(m+n-i)} M_m \otimes_A N_{i-m} \cong N_{i-m}$. For $0 \leq i \leq m-1$, $\rho_{\mathbf{f},\mathbf{g}}$ projects $(\mathbf{f} \otimes_A \mathbf{g})_i$ onto $(-1)^{mn} M_i \otimes_A N_0$, then maps to $M_i \otimes_A A \cong M_i$ via $1 \otimes \mu_N$. Similarly define maps $\rho_{\mathbf{g},\mathbf{f}} : (-1)^{mn}\mathbf{g} \otimes_A \mathbf{f} \to \mathbf{g}\#\mathbf{f}$ and $\lambda_{\mathbf{g},\mathbf{f}} : (-1)^{mn}\mathbf{g} \otimes_A \mathbf{f} \to (-1)^{mn}\mathbf{f}\#\mathbf{g}$.

Diagram (6.5.1) represents a loop in the classifying space of the extension category $\mathcal{E}xt^{m+n}_{A^e}(A,A)$. This information can be realized combinatorially without reference to the topology, as explained in [**Sch98**]. Basically, paths in the space are zigzags of maps, that is, you can travel either way along a map (from the domain to the codomain or vice versa) and compose such trips (recalling the equivalence relation generated by maps of generalized extensions).

Loops in the classifying space of $\mathcal{E}xt^{m+n}_{A^e}(A,A)$ are in one-to-one correspondence with $\mathrm{Ext}^{m+n-1}_{A^e}(A,A)$ [**Ret86**]. Under this correspondence, the loop (6.5.1) corresponds to the Gerstenhaber bracket $[f, g]$ [**Sch98**]. We refer to Retakh [**Ret86**] and Schwede [**Sch98**] for most details and proofs. Here we give some of the algebraic ideas underlying Schwede's result. Specifically, for the projective resolution P of A as an A^e-module, we look closely at some maps $P \to \mathbf{g}\#\mathbf{f}$ arising from the maps comprising the loop (6.5.1).

Replace the loop (6.5.1) with another equivalent loop as follows. Starting with $f \smile g \in \operatorname{Hom}_{A^e}(P_{m+n}, A)$, define an $(m+n)$-extension of A by A by a pushout diagram as in Section A.3:

$K(f \smile g)$:

$$0 \longrightarrow A \longrightarrow K(f \smile g)_{m+n-1} \longrightarrow P_{m+n-2} \longrightarrow \cdots \longrightarrow$$

$$P_0 \longrightarrow A \longrightarrow 0$$

where

$$K(f \smile g)_{m+n-1} = (P_{m+n-1} \oplus A)/\{(-d_{m+n}(x),\ (f \smile g)(x)) \mid x \in P_{m+n}\}.$$

The proof in [**Sch98**] has $P = B$, the bar resolution, since the goal there is to show that the loop (6.5.1) corresponds to the historical definition of Gerstenhaber bracket on the bar resolution. A tedious comparison of maps shows that there is a map ε for which the rightmost quadrilateral in the following diagram commutes:

$$\begin{tikzcd}
 & \mathbf{f} \otimes_A \mathbf{g} \arrow[dl, "\lambda_{\mathbf{f},\mathbf{g}}"'] \arrow[dr, "\rho_{\mathbf{f},\mathbf{g}}"] & & \\
\mathbf{g}\#\mathbf{f} & & (-1)^{mn}\mathbf{f}\#\mathbf{g} & K(f \smile g) \arrow[ull, "f \smile g"'] \arrow[dll, "(-1)^{mn} g \smile f + \varepsilon"] \\
 & (-1)^{mn}\mathbf{g} \otimes_A \mathbf{f} \arrow[ul, "\rho_{\mathbf{g},\mathbf{f}}"] \arrow[ur, "\lambda_{\mathbf{g},\mathbf{f}}"'] & &
\end{tikzcd} \tag{6.5.2}$$

We define the map $f \smile g$ (and similarly $g \smile f$), as in [**Sch98**], to be the map induced by all $(\hat{f}_i \otimes \hat{g}_j)\Delta_B$ where Δ_B is the diagonal map (2.3.2) on the bar resolution B.

Delete the component $(-1)^{mn}\mathbf{f}\#\mathbf{g}$ from diagram (6.5.2), thus replacing loop (6.5.1) with loop (6.5.3) below. For this purpose, in Schwede's general theory [**Sch98**], commutativity of the rightmost quadrilateral in diagram (6.5.2) is needed, and so the error correction term ε in diagram (6.5.2) is crucial.

$$
\begin{tikzcd}
 & \mathbf{f} \otimes_A \mathbf{g} \arrow[dl, "\lambda_{\mathbf{f},\mathbf{g}}"'] & \\
\mathbf{g}\#\mathbf{f} & & K(f \smile g) \arrow[ul, "f \smile g"'] \arrow[dl, "(-1)^{mn} g \smile f + \varepsilon"] \\
 & \mathbf{g} \otimes_A \mathbf{f} \arrow[ul, "\rho_{\mathbf{g},\mathbf{f}}"] &
\end{tikzcd}
\tag{6.5.3}
$$

The following theorem is due to Schwede [**Sch98**].

Theorem 6.5.4. *Let* $\mathbf{f}$ *and* $\mathbf{g}$ *be an* m*- and an* n*-extension, respectively, of* A *by* A *as an* A^e*-module. Let* $f \in \mathrm{Hom}_{A^e}(B_m, A)$ *and* $g \in \mathrm{Hom}_{A^e}(B_n, A)$ *be corresponding cocycles, where* B *is the bar resolution of* A *as an* A^e*-module. There is a chain homotopy* $s : B \to \mathbf{g}\#\mathbf{f}$*, factoring through* $K(f \smile g)$*, between* $\lambda_{\mathbf{f},\mathbf{g}}(f \smile g)$ *and* $\rho_{\mathbf{g},\mathbf{f}}((-1)^{mn} g \smile f + \varepsilon)$*, for which the Gerstenhaber bracket is represented at the chain level by*

$$[f, g] = (-1)^{n-1} s_{m+n-1}.$$

Exercise 6.5.5. Verify that $\lambda_{\mathbf{f},\mathbf{g}}$, $\rho_{\mathbf{f},\mathbf{g}}$, $\lambda_{\mathbf{g},\mathbf{f}}$, $\rho_{\mathbf{g},\mathbf{f}}$ of diagram (6.5.1) are all indeed maps of $(m+n)$-extensions.

Exercise 6.5.6. Use the definitions of the maps $f \smile g$, $g \smile f$, $\rho_{\mathbf{f},\mathbf{g}}$, and $\lambda_{\mathbf{g},\mathbf{f}}$ in diagram (6.5.2) to find a map ε for which the rightmost quadrilateral commutes. (Cf. [**Sch98**, p. 71].)

Chapter 7

Infinity Algebras

There are several appearances in Hochschild cohomology of higher order operations, the original idea of which is due to Stasheff [**Sta63**]. Some of these operations extend those on underlying chain complexes, such as the cup product operation, giving rise to infinity algebras. In this chapter, we look at a few settings where such infinity algebras arise in relation to Hochschild cohomology. There are many more applications in the literature than those we present here.

Indexing and sign conventions vary somewhat in the literature; we make some of the more standard choices.

7.1. A_∞-algebras

In this section we define A_∞-algebras (also called strongly homotopy associative algebras) and their morphisms, and give some examples relevant to Hochschild cohomology.

Definition 7.1.1. An A_∞*-algebra* is a graded vector space $A = \oplus_{i\in\mathbb{Z}} A_i$ together with graded linear maps

$$m_n : A^{\otimes n} \to A$$

of degree $|m_n| = 2 - n$ for all $n \geq 1$ such that

$$\sum_{r+s+t=n} (-1)^{r+st} m_{r+1+t}(1^{\otimes r} \otimes m_s \otimes 1^{\otimes t}) = 0. \tag{7.1.2}$$

The equations (7.1.2) are called *Stasheff identities*. Sometimes we denote an A_∞-algebra as $(A, m_\bullet)$ to emphasize the notation chosen for these higher multiplication maps. If $m_1 = 0$, then A is called *minimal*.

We consider the implications of equation (7.1.2) for small values of n: If $n = 1$, we must take $s = 1$ and $r = t = 0$, and so the equation is

$$m_1^2 = 0,$$

that is, m_1 is a differential on A. We will thus sometimes write $d = m_1$. If $n = 2$, we may take $s = 2$ and $r = t = 0$, or $s = 1$ and $r + t = 1$, to obtain

$$m_1 m_2 - m_2(m_1 \otimes 1) - m_2(1 \otimes m_1) = 0.$$

To express this equation on elements of A, we may write $m_1(a) = d(a)$ and $m_2(a \otimes b) = a \cdot b$, and the equation becomes

$$d(a \cdot b) = d(a) \cdot b + (-1)^{|a|} a \cdot d(b)$$

for all homogeneous $a, b \in A$, due to the Koszul sign convention. That is, $m_1 = d$ is a graded derivation with respect to m_2. If $n = 3$, equation (7.1.2) becomes

$$\begin{aligned} &m_1 m_3 + m_2(m_2 \otimes 1) - m_2(1 \otimes m_2) \\ &+ m_3(m_1 \otimes 1^{\otimes 2}) + m_3(1 \otimes m_1 \otimes 1) + m_3(1^{\otimes 2} \otimes m_1) = 0, \end{aligned}$$

which may be rewritten

$$\delta(m_3) = m_2(1 \otimes m_2) - m_2(m_2 \otimes 1),$$

where δ is the differential induced by $m_1 = d$ on the complex $\mathrm{Hom}_k(A^{\otimes 3}, A)$. That is, m_2 is associative up to a coboundary in this Hom complex. It follows that the cohomology of (A, m_1) is a graded associative algebra with multiplication induced by m_2.

We give some examples.

Example 7.1.3. If A is any differential graded algebra, we may take m_1 to be its differential, m_2 its multiplication, and $m_n = 0$ for $n \geq 3$, to define an A_∞-algebra structure on A. In particular, an associative algebra may be viewed as a differential graded algebra with zero differential, and thus as an A_∞-algebra in this way. If an A_∞-algebra A is concentrated in degree 0, that is $A_i = 0$ for all $i \neq 0$, then the maps m_n are necessarily zero maps for all $n \neq 2$ since $|m_n| = 2 - n$, so A is simply an associative algebra.

Next are some examples with $m_n \neq 0$ for arbitrary values of n, the first due to Penkava and Schwarz [**PS95**].

Example 7.1.4. Let B be an associative algebra, let $A = B[x]/(x^2)$, and let n be a positive integer. We take A to be graded with $|b| = 0$ for all $b \in B$ and $|x| = 2 - n$. Let $g : B^{\otimes n} \to B$ be a Hochschild n-cocycle. We will put an A_∞-algebra structure on A, depending on g. Define a linear function $xg : A^{\otimes n} \to A$ by

$$(xg)(a_1 \otimes \cdots \otimes a_n) = \begin{cases} xg(a_1 \otimes \cdots \otimes a_n), & \text{if } a_1, \ldots, a_n \in B \\ 0, & \text{if } x \text{ is a factor of } a_1 \cdots a_n. \end{cases}$$

Let $\pi : A^{\otimes 2} \to A$ denote multiplication on A. If $n = 2$, let $m_2 = \pi + xg$ and $m_i = 0$ for all $i \neq 2$. If $n \neq 2$, let $m_2 = \pi$, $m_n = xg$, and $m_i = 0$ for all $i \notin \{2, n\}$. Calculations show that A is an A_∞-algebra. We will see a connection to algebraic deformation theory via Definition 7.1.7 below.

Example 7.1.5. Let n be a positive integer, $n > 2$, let $B = k[x]/(x^n)$, and let $A = \mathrm{Ext}^*_B(k, k)$. As shown in Example 2.5.10, $A \cong k[y, z]/(y^2)$ with $|y| = 1$ and $|z| = 2$. We will put an A_∞-algebra structure on A. We take m_2 to be multiplication on A, m_i to be the zero map if $i \notin \{2, n\}$, and

$$m_n(y^{i_1} z^{j_1} \otimes \cdots \otimes y^{i_n} z^{j_n}) = \begin{cases} z^{j_1 + \cdots + j_n + 1}, & \text{if } i_1 = \cdots = i_n = 1 \\ 0, & \text{otherwise} \end{cases}$$

for all nonnegative integers $j_1, \ldots, j_n$ and all $i_1, \ldots, i_n \in \{0, 1\}$. Calculations show that A is an A_∞-algebra for each $n > 2$. This example may also be constructed via the general method outlined in the proof of Theorem 7.2.2 below. In Section 7.3 we will discuss a distinction between this A_∞-structure on $\mathrm{Ext}^*_B(k, k)$ and the structure of $\mathrm{Ext}^*_B(k, k)$ when $B = k[x]/(x^2)$, which by contrast is a Koszul algebra.

We say that an A_∞-algebra A is *generated* by a subset S if A coincides with its smallest subspace containing S that is closed under all m_n.

Example 7.1.6. In Example 7.1.5, $m_n(y \otimes \cdots \otimes y) = z$. Since y, z generate A as an associative algebra, we see that as an A_∞-algebra, A is generated by y alone.

The next definition generalizes that of an infinitesimal deformation in Definition 5.2.1.

Definition 7.1.7. Let n be a positive integer. Let B be an associative algebra, and let $A = B[x]/(x^2)$ where $|b| = 0$ for all $b \in B$ and $|x| = 2 - n$. An *infinitesimal n-deformation* of B is a $k[x]/(x^2)$-multilinear A_∞-algebra structure on A that lifts the multiplication of B. That is, under composition with the vector space quotient map from A to $A/(x) \cong B$, the map $m_2|_B$ agrees with multiplication on B and $m_i|_B$ becomes 0 for all $i > 2$.

A calculation shows that an infinitesimal 2-deformation may be identified with an infinitesimal deformation as in Definition 5.2.1: Writing

$$m_2(b_1 \otimes b_2) = m_2'(b_1 \otimes b_2) + m_2''(b_1 \otimes b_2)x$$

for all $b_1, b_2 \in B$, it follows from the definitions that m_2' is the original multiplication on B and m_2'' is a Hochschild 2-cocycle. Similarly, an infinitesimal n-deformation corresponds to a Hochschild n-cocycle: Since $|x| = 2 - n$ and $|m_i| = 2 - i$, the only possible nonzero operations m_i are m_2 and m_n. Additionally, m_n takes elements of $B^{\otimes n}$ to Bx if $n > 2$. Calculations show that the resulting coefficient function of x must be an n-cocycle. Thus this observation is essentially a converse to Example 7.1.4, and is a proof of the following theorem.

Theorem 7.1.8. *Let B be an associative algebra, and let $n \geq 2$. The Hochschild n-cocycles on B are in one-to-one correspondence with the infinitesimal n-deformations of B.*

Generalizing the case $n = 2$, a calculation shows that cohomologous Hochschild n-cocycles correspond to isomorphic infinitesimal n-deformations; the appropriate notion of isomorphism is given by Definition 7.1.9 below. Specifically, let $(A_g, m_\bullet)$ be the infinitesimal n-deformation of B given in Example 7.1.4. Suppose $g' = g + hd$ for some $(n-1)$-cochain h and $(A_{g'}, m'_\bullet)$ is the infinitesimal n-deformation of B corresponding to g'. Set $f_1 : A_g \to A_{g'}$ to be the identity map on the underlying vector space, set $f_i = 0$ if $i \notin \{1, n-1\}$, and $f_{n-1} = -xh$. Recalling that $m_1 = 0$ and $m'_1 = 0$, we see that the only conditions (7.1.10) below with nonzero terms are the second and $(n-1)$st such equations. The second such equation automatically holds since f_1 is the identity map and $m_2 = m'_2$. The $(n-1)$st equation holds since $g' = g + hd$.

By way of Theorem 7.1.8, we relate Hochschild cohomology $\mathrm{HH}^*(B)$ of an associative algebra B with infinitesimal deformations of B as an A_∞-algebra, in the same way that degree 2 Hochschild cohomology $\mathrm{HH}^2(B)$ corresponds to infinitesimal deformations of B as an associative algebra.

We now return to the general setting of A_∞-algebras.

Definition 7.1.9. Let $(A, m^A_\bullet)$, $(B, m^B_\bullet)$ be A_∞-algebras. A *morphism* of A_∞-algebras $f_\bullet : (A, m^A_\bullet) \to (B, m^B_\bullet)$ consists of graded linear maps

$$f_n : A^{\otimes n} \to B$$

of degree $|f_n| = 1 - n$ for all $n \geq 1$ such that

(7.1.10)
$$\sum_{r+s+t=n} (-1)^{r+st} f_{r+1+t}(1^{\otimes r} \otimes m^A_s \otimes 1^{\otimes t}) = \sum_{i_1+\cdots+i_r=n} (-1)^u m^B_r(f_{i_1} \otimes \cdots \otimes f_{i_r})$$

where $u = (r-1)(i_1-1) + (r-2)(i_2-1) + \cdots + 2(i_{r-2}-1) + (i_{r-1}-1)$. The *identity morphism* $f_\bullet : A \to A$ is defined by $f_1 = 1_A$ and $f_n = 0$ for $n \neq 1$. The composition of two morphisms $g : A \to B$ and $f : B \to C$ is given by

$$(fg)_n = \sum_{i_1+\cdots+i_r=n} (-1)^u f_r(g_{i_1} \otimes \cdots \otimes g_{i_r})$$

for all n, with $u = u(i_1, \ldots, i_r)$ as above. An A_∞-morphism $f_\bullet$ is a *quasi-isomorphism* if f_1 is a quasi-isomorphism, that is, f_1 induces an isomorphism on cohomology, $\mathrm{H}^*(A) \xrightarrow{\sim} \mathrm{H}^*(B)$.

We interpret the definition of A_∞-morphism for small values of n: If $n = 1$, equation (7.1.10) is

$$f_1 m_1^A = m_1^B f_1,$$

in other words, f_1 is a cochain map. If $n = 2$, it is

$$f_1 m_2^A = m_2^B (f_1 \otimes f_1) + m_1^B f_2 + f_2 (m_1^A \otimes 1 + 1 \otimes m_1^A),$$

which may be rewritten

$$f_1 m_2^A = m_2^B (f_1 \otimes f_1) + \delta(f_2).$$

That is, up to the coboundary $\delta(f_2)$, the map f_1 is an algebra homomorphism with respect to multiplication m_2.

Exercise 7.1.11. Find an expression for the Stasheff identity (7.1.2) when $n = 4$.

Exercise 7.1.12. Verify that $A = B[x]/(x^2)$ of Example 7.1.4 is indeed an A_∞-algebra.

Exercise 7.1.13. Verify that $A = \mathrm{Ext}_B^*(k, k)$ of Example 7.1.5 is indeed an A_∞-algebra.

Exercise 7.1.14. Prove Theorem 7.1.8 following the outline indicated in the text preceding its statement.

7.2. Minimal models

Recall that an A_∞-algebra is called minimal if $m_1 = 0$. For some applications, an A_∞-algebra may be replaced by a minimal A_∞-algebra to which it is quasi-isomorphic (see Definition 7.1.9). We outline this technique here.

Definition 7.2.1. Let A be an A_∞-algebra. A *minimal model* for A is a minimal A_∞-algebra B together with a quasi-isomorphism of A_∞-algebras $f_\bullet : B \to A$.

Let A be an A_∞-algebra, and $\mathrm{H}^*(A)$ its cohomology. The following theorem of Kadeishvili [**Kad82**] states that the cohomology $\mathrm{H}^*(A)$ has the structure of a minimal A_∞-algebra. The theorem further implies existence of a minimal model.

Theorem 7.2.2. *The cohomology* $\mathrm{H}^*(A)$ *of an* A_∞*-algebra* A *may be given the structure of an* A_∞*-algebra under which it is a minimal model for* A*. This structure is unique up to isomorphism of* A_∞*-algebras.*

Proof. We give a proof only in the special case that $(A, m_\bullet)$ is a differential graded algebra, that is $m_n = 0$ for $n > 2$. For the general case, see [**Kad82**].

We will define maps $m'_n : \mathrm{H}^*(A)^{\otimes n} \to \mathrm{H}^*(A)$, for each n, under which $(\mathrm{H}^*(A), m'_\bullet)$ becomes an A_∞-algebra. At the same time we will define maps $f_n : \mathrm{H}^*(A)^{\otimes n} \to A$ that will constitute a quasi-isomorphism $f_\bullet : (\mathrm{H}^*(A), m'_\bullet) \to (A, m_\bullet)$. Set $m'_1 = 0$ and let $f_1 : \mathrm{H}^*(A) \to A$ be any k-linear section of the surjection $p : \mathrm{Z}^*(A) \to \mathrm{H}^*(A)$ from the space of cocycles $\mathrm{Z}^*(A)$ to the cohomology $\mathrm{H}^*(A)$ of A. That is, f_1 takes values in $Z^*(A)$ and $pf_1 = 1_{\mathrm{H}^*(A)}$. Let m'_2 be multiplication on $\mathrm{H}^*(A)$ as induced by multiplication m_2 on A. Then by definition, for each $\alpha, \beta \in \mathrm{H}^*(A)$, the elements $m_2(f_1(\alpha) \otimes f_1(\beta))$ and $f_1(m'_2(\alpha \otimes \beta))$ are cohomologous in A. Put another way, letting $\Phi_2 = m_2(f_1 \otimes f_1)$, we see that $f_1 m'_2 - \Phi_2$ is a coboundary, that is there is some k-linear map $f_2 : \mathrm{H}^*(A)^{\otimes 2} \to A$ for which

$$f_1 m'_2 - \Phi_2 = m_1 f_2.$$

Since $m'_1 = 0$, we may rewrite this as

$$\Phi_2 = f_1 m'_2 - \delta(f_2),$$

the required condition (7.1.10), with $n = 2$, for an A_∞-morphism. Since $m'_1 = 0$ and m'_2 is associative, condition (7.1.2) holds with $n = 3$.

The remainder of the proof proceeds by induction on n. We explain the case $n = 3$ first for clarity. Let

$$\Phi_3 = m_2(f_1 \otimes f_2 - f_2 \otimes f_1) + f_2(1 \otimes m'_2 - m'_2 \otimes 1).$$

A calculation shows that Φ_3 takes values in the space $Z^*(A)$ of cocycles of A. Let $m'_3 : \mathrm{H}^*(A)^{\otimes 3} \to \mathrm{H}^*(A)$ be a k-linear function such that $m'_3(\alpha \otimes \beta \otimes \gamma)$ represents $\Phi_3(\alpha \otimes \beta \otimes \gamma)$ for all $\alpha, \beta, \gamma \in \mathrm{H}^*(A)$. Then by definition of m'_3, the elements $f_1 m'_3(\alpha \otimes \beta \otimes \gamma)$ and $\Phi_3(\alpha \otimes \beta \otimes \gamma)$ are cohomologous. It follows that

$$f_1 m'_3 - \Phi_3 = m_1 f_3$$

for some $f_3 : \mathrm{H}^*(A)^{\otimes 3} \to A$. Thus equation (7.1.10) holds when $n = 3$. Now consider the left side of equation (7.1.2) with $n = 4$ for $m'_1 = 0$, m'_2, m'_3:

$$m'_2(-m'_3 \otimes 1 - 1 \otimes m'_3) + m'_3(m'_2 \otimes 1 \otimes 1 - 1 \otimes m'_2 \otimes 1 + 1 \otimes 1 \otimes m'_2). \tag{7.2.3}$$

Compose with f_1 and apply (7.1.10) repeatedly to obtain a coboundary in A. Since f_1 is a section of the quotient map from $Z^*(A)$ to $\mathrm{H}^*(A)$, this implies that the expression (7.2.3) is equal to 0.

More generally, let $n > 3$ and suppose we have defined m'_i, f_i for all $i < n$. Let

$$\Phi_n = \sum_{i_1 + i_2 = n} (-1)^{i_1 - 1} m_2(f_{i_1} \otimes f_{i_2}) - \sum_{\substack{r+s+t=n \\ s>1,\ r+t>0}} (-1)^{r+st} f_{r+1+t}(1^{\otimes r} \otimes m'_s \otimes 1^{\otimes t}),$$

that is the difference of the right and left sides of equation (7.1.10), excluding the terms $f_1 m'_n$ and $m_1 f_n$ (since $m_i = 0$ for $i > 2$). A calculation shows that Φ_n takes values in the space $Z^*(A)$ of cocycles. Let $m'_n : \mathrm{H}^*(A)^{\otimes n} \to \mathrm{H}^*(A)$

be a k-linear function such that $m'_n(\alpha_1 \otimes \cdots \alpha_n)$ represents $\Phi_n(\alpha_1 \otimes \cdots \otimes \alpha_n)$ for all $\alpha_1, \ldots, \alpha_n \in \mathrm{H}^*(A)$. Then $f_1 m'_n(\alpha_1 \otimes \cdots \otimes \alpha_n)$ and $\Phi_n(\alpha_1 \otimes \cdots \alpha_n)$ are cohomologous, so

$$f_1 m'_n - \Phi_n = m_1 f_n$$

for some $f_n : \mathrm{H}^*(A)^{\otimes n} \to A$. Thus equation (7.1.10) holds for n.

By construction, $f_\bullet$ is an A_∞-morphism. Condition (7.1.2) for $(\mathrm{H}^*(A), m'_\bullet)$ automatically holds when $n = 1$ or $n = 2$ since $m'_1 = 0$, and as we saw before, it holds when $n = 3$ since m'_2 is an associative multiplication on $\mathrm{H}^*(A)$. More generally, apply f_1 to the left side of equation (7.1.2) for $m'_\bullet$. The result is seen to be a coboundary in A by repeated use of equation (7.1.10) to eliminate terms involving $m'_\bullet$ from the expression, as explained for $n = 4$ above. Since f_1 is a quasi-isomorphism and $m'_1 = 0$, the left side of equation (7.1.2) for $m'_\bullet$ is indeed 0 for each n.

The uniqueness statement can be proven by invoking a k-linear projection from A to $\mathrm{H}^*(A)$ whose composition with f_1 is the identity on $\mathrm{H}^*(A)$. This may be extended to an A_∞-morphism from A to $\mathrm{H}^*(A)$; in fact any A_∞-quasi-isomorphism has a homotopy inverse [**Hue10**]. Given two copies of $\mathrm{H}^*(A)$, with possibly different higher multiplications $m'_\bullet$ and A_∞-morphism $f_\bullet$, by mapping each to A and then to the other, we obtain maps between the two copies whose compositions in both directions must be identity maps. □

Remark 7.2.4. For a more conceptual proof that $(\mathrm{H}^*(A), m'_\bullet)$ is an A_∞-algebra, equivalent to the above proof but avoiding explicit calculations, observe that $f_\bullet$ constitutes an injective coalgebra morphism from the reduced bar construction of $\mathrm{H}^*(A)$ to that of A. Since $m_\bullet$ satisfies the conditions (7.1.2), it follows that $m'_\bullet$ does as well.

As a consequence of Theorem 7.2.2, Hochschild cohomology of an associative algebra B has higher multiplication maps under which it is an A_∞-algebra, and thus a minimal model of the differential graded algebra $C^*(B, B)$ viewed as an A_∞-algebra.

We illustrate Theorem 7.2.2 with a specific example.

Example 7.2.5. Let $B = k[x]/(x^n)$ and let $P_\bullet$ be the standard periodic free resolution of k as a B-module as in Example 2.5.10. Let $A = \mathrm{Hom}_B(P_\bullet, P_\bullet)$, so that $\mathrm{H}^*(A) \cong \mathrm{Ext}^*_B(k, k) \cong k[y, z]/(y^2)$. Applying the algorithm suggested by the proof of Theorem 7.2.2 leads to the A_∞-algebra structure on $\mathrm{H}^*(A)$ that was given in Example 7.1.5.

Remark 7.2.6. In Theorem 7.2.2, letting B be an associative algebra, we may take A to be the differential graded algebra $\oplus_{i \ge 0} \mathrm{Hom}_{B^e}(B^{\otimes(i+2)}, B)$, that is, $C^*(B, B)$. Then $\mathrm{H}^*(A)$ is the Hochschild cohomology $\mathrm{HH}^*(B)$, and Theorem 7.2.2 implies that we may realize this Hochschild cohomology as a minimal model. The proof of the theorem indicates how to define the needed higher operations. See also [**Hue10**] for a proof using homological perturbation and further results.

Exercise 7.2.7. In the proof of Theorem 7.2.2, verify that Φ_3 takes values in $Z^*(A)$. More generally, verify that Φ_n takes values in $Z^*(A)$.

Exercise 7.2.8. In the proof of Theorem 7.2.2, verify that applying f_1 to the left side of equation (7.1.2) for $m'_\bullet$ results in a coboundary on A. First check the case $n = 4$, then general n.

7.3. Formality and Koszul algebras

In this section, we present a special case of results of Keller [**Kel02**] on Koszul algebras and formality as defined next.

Definition 7.3.1. An A_∞-algebra A is *formal* if its minimal model has the property that $m_n = 0$ for all $n \geq 3$.

We will see next that a large class of formal A_∞-algebras is given by cohomology of Koszul algebras.

Let $B = T(V)/(R)$ for a finite dimensional vector space V with homogeneous relations $R \subset \oplus_{n\geq 2} T_n(V)$. Consider k to be a B-module on which each element of V acts as 0, and let $P_\bullet$ be a projective resolution of k as a B-module. Then $\mathrm{Hom}_B(P_\bullet, P_\bullet)$ is a differential graded algebra with homology $\mathrm{Ext}^*_B(k,k)$. View $\mathrm{Hom}_B(P_\bullet, P_\bullet)$ as an A_∞-algebra with higher multiplication maps 0. Theorem 7.2.2 implies that $\mathrm{Ext}^*_B(k,k)$ has the structure of an A_∞-algebra for which it is the minimal model for $\mathrm{Hom}_B(P_\bullet, P_\bullet)$. In general, $\mathrm{Ext}^*_B(k,k)$ will have higher multiplication maps. When we mention the A_∞-algebra $\mathrm{Ext}^*_B(k,k)$, it is this A_∞-structure that is intended. A use of comparison maps between resolutions shows that, up to isomorphism, this A_∞-structure will not depend on choice of resolution $P_\bullet$.

We will need the following lemma about a generating set for this A_∞-algebra.

Lemma 7.3.2. *The A_∞-algebra $\mathrm{Ext}^*_B(k,k)$ is generated in degree 1.*

A proof uses homological properties of such algebras; see, e.g., [**Kel02**, §2.2] for some techniques.

Example 7.3.3. Let $B = k[x]/(x^n)$ as in Example 7.1.6. As we saw there, $\mathrm{Ext}^*_B(k,k)$ is generated by a single element y as an A_∞-algebra. The degree of y is 1.

Recall from Theorem 3.4.6 that the Ext algebra $\mathrm{Ext}^*_B(k,k)$ of a Koszul algebra B is generated in degree 1 as an associative algebra, and this is essentially a defining characteristic of Koszul algebras. We thus obtain the following formality result.

Theorem 7.3.4. *Let B be a finitely generated graded connected algebra. The A_∞-algebra $\mathrm{Ext}^*_B(k,k)$ is formal if and only if B is a Koszul algebra.*

Proof. First assume that B is Koszul. The Koszul resolution $\widetilde{K}_\bullet = \widetilde{K}_\bullet(B)$ defined by (3.4.5) gives rise to the complex $\mathrm{Hom}_B(\widetilde{K}_\bullet, \widetilde{K}_\bullet)$, which has grading both by homological degree and that induced by the grading on the algebra B. There is a subcomplex of $\mathrm{Hom}_B(\widetilde{K}_\bullet, \widetilde{K}_\bullet)$ consisting of all elements whose homological degree is the opposite of grading degree. By Theorem 3.4.6, since B is Koszul, $\mathrm{Ext}^i_B(k,k) = \mathrm{Ext}^{i,-i}_B(k,k)$, and we see that it embeds into $\mathrm{Hom}_B(\widetilde{K}_\bullet, \widetilde{K}_\bullet)$ as an associative algebra. Therefore $\mathrm{Ext}^*_B(k,k)$ is formal.

Now assume that $\mathrm{Ext}^*_B(k,k)$ is formal. By Lemma 7.3.2, it is generated in degree 1 as an A_∞-algebra. Since $\mathrm{Ext}^*_B(k,k)$ is formal, we conclude that it is generated in degree 1 as an associative algebra. By Theorem 3.4.6, B is Koszul. $\square$

We illustrate Theorem 7.3.4 with an example.

Example 7.3.5. See Example 7.1.5 for a distinction made by the theorem: Let $B = k[x]/(x^n)$. If $n = 2$, then B is Koszul, and $\mathrm{Ext}^*_B(k,k)$ is formal. If $n > 2$, then B is not Koszul and $\mathrm{Ext}^*_B(k,k)$ is not formal. Accordingly we had found a nonzero higher multiplication m_n in this latter case.

Exercise 7.3.6. Let $B = k[x]/(x^n)$. Let $P_\bullet$ be the free resolution of k as a B-module given in Example 2.5.10. Verify that the A_∞-algebra structure on $\mathrm{Ext}^*_B(k,k) = \mathrm{H}^*(\mathrm{Hom}_B(P_\bullet, P_\bullet))$ given in the proof of Theorem 7.2.2 agrees with Example 7.1.5.

7.4. A_∞-center

There is a notion of center of an A_∞-algebra that plays an important role in Hochschild cohomology, as we will see in Theorem 7.4.4. There is more than one reasonable way to define the center of an A_∞-algebra. We follow Briggs and Gélinas [**BG**] for a definition, in the special case of a minimal A_∞-algebra, that is invariant under quasi-isomorphism.

We introduce the graded symmetric and exterior algebras that will be used for the rest of this chapter. The notation $S(V)$ and $\bigwedge(V)$ below agree with earlier uses of the notation in this book in the case that V is concentrated in degree 0. We assume in this section that the characteristic of the field k is not 2.

Let V be a graded vector space over k. The *graded symmetric algebra* is

$$S(V) = T(V)/(u \otimes v - (-1)^{|u||v|} v \otimes u \mid u, v \text{ homogeneous elements of } V).$$

The *graded exterior algebra* is

$$\bigwedge(V) = T(V)/(u \otimes v + (-1)^{|u||v|} v \otimes u \mid u, v \text{ homogeneous elements of } V).$$

It follows from the definitions that $S(V)$ is universal with respect to graded symmetric maps and $\bigwedge(V)$ is universal with respect to graded anti-symmetric maps (see Exercise 7.4.6).

Remark 7.4.1. If instead the characteristic of k is 2, the above definitions of $S(V)$ and $\bigwedge(V)$ may be modified to satisfy these universal properties. In the former case we must additionally mod out by all $v \otimes v$ for which $|v|$ is odd, and in the latter by all $v \otimes v$ for which $|v|$ is even. There is some resulting redundancy in these larger sets of relations.

In the rest of this chapter, all our symmetric and exterior algebras will be graded in the above sense.

Let S_n denote the symmetric group on n symbols. For each $\sigma \in S_n$ and homogeneous $v_1, \ldots, v_n \in V$, define the scalar $\chi(\sigma; v_1, \ldots, v_n)$ by the following equation involving elements of the graded exterior algebra $\bigwedge(V)$:

$$v_{\sigma(1)} \cdots v_{\sigma(n)} = \chi(\sigma; v_1, \ldots, v_n) v_1 \cdots v_n. \tag{7.4.2}$$

We sometimes write $\chi(\sigma)$ when it is clear which vectors $v_1, \ldots, v_n$ are involved. If V is concentrated in degree 0, then $\chi(\sigma)$ is simply $\mathrm{sgn}(\sigma)$.

For an A_∞-algebra A, define *higher commutators* $[-;-]_{p,q} : A^{\otimes p} \otimes A^{\otimes q} \to A$ by

$$\begin{aligned} &[a_1, \ldots, a_p; a_{p+1}, \ldots, a_{p+q}]_{p,q} \\ &= \sum_{\sigma \in S_{p,q}} \chi(\sigma; a_1, \ldots, a_{p+q}) m_{p+q}(a_{\sigma^{-1}(1)} \otimes \cdots \otimes a_{\sigma^{-1}(p+q)}) \end{aligned}$$

for all homogeneous $a_1, \ldots, a_n \in A$, where $S_{p,q}$ is the set of all (p, q)-shuffles in the symmetric group S_{p+q} as in Definition 1.5.2. Note that $[-;-]_{1,1}$ is the usual commutator for m_2, that is,

$$[a_1; a_2]_{1,1} = m_2(a_1 \otimes a_2) - (-1)^{|a_1||a_2|} m_2(a_2 \otimes a_1).$$

Definition 7.4.3. Let A be a minimal A_∞-algebra and let a be a homogeneous element of A. Then a is *central* in A if for all $n \geq 1$, there are k-linear maps $p_i : A^{\otimes i} \to A$ of degrees $|p_i| = |a| - i$ $(i \geq 1)$ for which

$$\begin{aligned} [a; -]_{1,n} = &\sum_{r+s+t=n} (-1)^{r(|a|+s)+t(|a|+1)} m_{r+1+t}(1^{\otimes r} \otimes p_s \otimes 1^{\otimes t}) \\ &- (-1)^{|a|}(-1)^{rs+t} p_{r+1+t}(1^{\otimes r} \otimes m_s \otimes 1^{\otimes t}). \end{aligned}$$

Put more concisely, $\mathrm{ad}(a) = \partial(p)$, for suitable notions of adjoint map ad and differential ∂ (see [**BG**, Definition 3.7] for this and more details). The A_∞-*center* of A, denoted $Z_\infty(A)$, is the vector space spanned by all homogeneous central elements of A.

The maps $p_\bullet$ are called *homotopy derivations*. The related notion of *strong homotopy derivation* [**KS06**] is such a map $p_\bullet$ for which the right side of the equation in the definition above is 0, that is, $\partial(p) = 0$. Note that the higher commutator on the left side of the equation has the effect of inserting the element a between factors in all possible ways. For example,

$$[a; a_1, a_2]_{1,2} = m_3(a \otimes a_1 \otimes a_2) \\ \quad - (-1)^{|a||a_1|} m_3(a_1 \otimes a \otimes a_2) + (-1)^{|a|(|a_1|+|a_2|)} m_3(a_1 \otimes a_2 \otimes a).$$

Recalling that A is assumed to be minimal, it follows from Definition 7.4.3 of a central element a in an A_∞-algebra A that $[a; -]_{1,1} = 0$, and thus a is in fact in the graded center of A (see Definition 2.5.8). In other words, $Z_\infty(A) \subseteq Z_{\mathrm{gr}}(A)$.

The following theorem is due to Briggs and Gélinas [**BG**], as a consequence of more general results. For a proof, see [**BG**]. Recall that an augmented algebra B is one for which there is an algebra homomorphism to k, and k is considered to be a module via this map. We will use the map from Hochschild cohomology $\mathrm{HH}^*(B)$ to $\mathrm{Ext}^*_B(k,k)$ given at the chain level by (2.5.3).

Theorem 7.4.4. *Let B be an augmented algebra. The image of Hochschild cohomology $\mathrm{HH}^*(B)$ in the Ext algebra $\mathrm{Ext}^*_B(k,k)$ is precisely the A_∞-center, $Z_\infty(\mathrm{Ext}^*_B(k,k))$.*

The theorem generalizes a result of Buchweitz, Green, Snashall, and Solberg [**BGSS08**] from Koszul algebras to augmented algebras, as follows: If B is a Koszul algebra, then the image of Hochschild cohomology $\mathrm{HH}^*(B)$ in the Ext algebra $\mathrm{Ext}^*_B(k,k)$ is the graded center $Z_{\mathrm{gr}}(\mathrm{Ext}^*_B(k,k))$ [**BGSS08**]. As we saw in the last section, the Ext algebra of a Koszul algebra B is formal, and thus the graded center of $\mathrm{Ext}^*_B(k,k)$ coincides with its A_∞-center.

Example 7.4.5. Let $B = k[x]/(x^n)$, $n > 2$. Then $\mathrm{Ext}^*_B(k,k) \cong k[y,z]/(y^2)$ and

$$Z_\infty(\mathrm{Ext}^*_B(k,k)) = \begin{cases} k[z], & \text{if } \mathrm{char}(k) \nmid n \\ k[y,z]/(y^2), & \text{if } \mathrm{char}(k) \mid n. \end{cases}$$

(Recall that $\mathrm{char}(k) \neq 2$. See [**BG**, Example 4.8] for details.)

There are many more applications of A_∞-algebras, as well as the dual notion of A_∞-coalgebras. See, for example, [**Her**].

Exercise 7.4.6. Verify the universality of the graded symmetric algebra $S(V)$ and the graded exterior algebra $\bigwedge(V)$ as defined in this section in case $\mathrm{char}(k) \neq 2$:

(a) Given a graded vector space V and a k-algebra B, a *graded symmetric* map $f : V \to B$ is a k-linear map such that $f(v)f(w) =$

$(-1)^{|v||w|}f(w)f(v)$ for all homogeneous $v, w \in V$. Show that for all k-algebras B and graded symmetric maps $f : V \to B$, there is a unique k-algebra homomorphism $F : S(V) \to B$ such that $F|_V = f$.

(b) Given a graded vector space V and a k-algebra B, a *graded anti-symmetric* map $f : V \to B$ is a k-linear map such that $f(v)f(w) = -(-1)^{|v||w|}f(w)f(v)$ for all homogeneous $v, w \in V$. Show that for all k-algebras B and graded anti-symmetric maps $f : V \to B$, there is a unique k-algebra homomorphism $F : \bigwedge(V) \to B$ such that $F|_V = f$.

Exercise 7.4.7. Let k be a field of characteristic 2. Define $S(V)$ and $\bigwedge(V)$ by universal properties such as in Exercise 7.4.6(a) and (b). Deduce that $S(V)$ and $\bigwedge(V)$ are isomorphic to particular quotients of $T(V)$ as outlined in Remark 7.4.1.

Exercise 7.4.8. Verify that $Z_\infty(A) \subseteq Z_{\mathrm{gr}}(A)$ by comparing the definitions of these two notions of center of an A_∞-algebra.

Exercise 7.4.9. Find an expression for $[a_1, a_2; a]_{2,1}$ as a sum over $(2,1)$-shuffles.

7.5. L_∞-algebras

Analogous to A_∞-algebras are L_∞-algebras in which the operations are higher order Lie brackets. This notion first appeared in the paper [**SS85**] by Schlessinger and Stasheff.

Definition 7.5.1. An *L_∞-algebra* is a graded vector space L together with graded linear maps

$$\ell_n : \bigwedge^n L \to L$$

of degree $|\ell_n| = 2 - n$ for all n such that

$$\sum_{i=1}^{n} \sum_{\sigma \in S_{i,n-i}} (-1)^{i(n-i)} \chi(\sigma) \ell_{n-i+1}(\ell_i(v_{\sigma(1)}, \ldots, v_{\sigma(i)}), v_{\sigma(i+1)}, \ldots, v_{\sigma(n)}) = 0$$

for all homogeneous $v_1, \ldots, v_n \in L$, where $\chi(\sigma) = \chi(\sigma; v_1, \ldots, v_n)$ is defined by (7.4.2) and $S_{i,n-i}$ is the set of all $(i, n-i)$-shuffles in the symmetric group S_n as in Definition 1.5.2. For simplicity of notation, we have separated elements of L by commas when their product in $\bigwedge(L)$ is the argument of a function. Sometimes we denote the L_∞-algebra by $(L, \ell_\bullet)$ to emphasize the notation chosen for the higher operations. An L_∞-algebra is *minimal* if $\ell_1 = 0$.

Remark 7.5.2. In the literature, the elements of $S_{i,n-i}$ are sometimes called *i-unshuffles* in this context. Also, a graded symmetric algebra sometimes appears in definitions of L_∞-structures, in place of the graded exterior algebra we have here, due to different choices of grading and indexing.

We interpret these equations in low degrees. Write $d(v) = \ell_1(v)$ and $[u, v] = \ell_2(u, v)$, $[u, v, w] = \ell_3(u, v, w)$, etc., for elements u, v, w of L. If $n = 1$, the equation in Definition 7.5.1 is simply $d^2(v) = 0$ for all $v \in L$, and so d is a differential on L. If $n = 2$, the condition is

$$d([u, v]) = [d(u), v] + (-1)^{|u|}[u, d(v)]$$

for all homogeneous $u, v \in L$, that is, d is a graded derivation with respect to $\ell_2 = [\ ,\ ]$. If $n = 3$, the condition may be written

$$\begin{aligned}&(-1)^{|u||w|}[[u, v], w] + (-1)^{|u||v|}[[v, w], u] + (-1)^{|v||w|}[[w, u], v]\\ &= (-1)^{|u||w|}(d\ell_3 + \ell_3 d)(u, v, w)\end{aligned}$$

for all homogeneous $u, v, w \in L$, that is, up to homotopy, the graded Jacobi identity holds.

Example 7.5.3. If L is a differential graded Lie algebra, we may take ℓ_1 to be its differential, ℓ_2 to be its Lie bracket, and $\ell_n = 0$ for all $n \geq 3$, for an L_∞-structure on L. In particular, a graded Lie algebra may be viewed as a differential graded Lie algebra with zero differential, and thus as an L_∞-algebra in this way. If an L_∞-algebra is concentrated in degree 0, that is $L_i = 0$ for all $i \neq 0$, the maps ℓ_n are necessarily zero maps for all $n \neq 2$ since $|\ell_n| = 2 - n$, so L is simply a Lie algebra.

A large class of examples is provided by A_∞-algebras together with graded commutators, as the following theorem of Lada and Markl [**LM95**] shows. This generalizes a relationship between associative algebras and Lie algebras. The theorem may be proven by direct computation, or by invoking a connection between L_∞-structures and coderivations as in [**LM95**].

Theorem 7.5.4. *Let $(A, m_\bullet)$ be an A_∞-algebra. Let*

$$\ell_n(a_1, \ldots, a_n) = \sum_{\sigma \in S_n} \chi(\sigma) m_n(a_{\sigma(1)}, \ldots, a_{\sigma(n)})$$

for all homogeneous $a_1, \ldots, a_n \in A$. Then $(A, \ell_\bullet)$ is an L_∞-algebra.

Morphisms of L_∞-algebras involve the following generalization of shuffles. There are more conceptual alternative descriptions, as well as equivalent formulas in characteristic 0 that instead involve sums over all permutations and division by factorials. See, for example, [**All10**] or [**KS06**].

Let $i_1, \ldots, i_t$ be positive integers. A permutation σ of $S_{i_1+\cdots+i_j}$ is an $(i_1, \ldots, i_t)$-*shuffle* if

$$\begin{aligned} &\sigma(1) < \cdots < \sigma(i_1), \\ &\sigma(i_1+1) < \cdots < \sigma(i_1+i_2), \ldots, \\ \text{and } &\sigma(i_1+\cdots+i_{t-1}+1) < \cdots < \sigma(i_1+\cdots+i_t). \end{aligned}$$

Definition 7.5.5. Let $(L, \ell_\bullet)$, $(L', \ell'_\bullet)$ be L_∞-algebras. A *morphism* of L_∞-algebras $f_\bullet : L \to L'$ consists of graded linear maps

$$f_n : \bigwedge^n L \to L'$$

of degree $|f_n| = 1-n$ for all $n \geq 1$ such that for all homogeneous $v_1, \ldots, v_n \in L$,

$$\begin{aligned} &\sum_{i=1}^{n} \sum_{\sigma \in S_{i,n-i}} (-1)^{i(n-i)} \chi(\sigma) f_{n-i+1}(\ell_i \otimes 1^{\otimes(n-i)})(v_{\sigma(1)}, \ldots, v_{\sigma(n)}) \\ &= \sum_{\substack{1 \leq r \leq n \\ i_1+\cdots+i_r=n}} \sum_{\tau} (-1)^u \chi(\tau) \ell'_t(f_{i_1} \otimes \cdots \otimes f_{i_r})(v_{\tau(1)}, \ldots, v_{\tau(n)}) \end{aligned}$$

where τ runs over all $(i_1, \ldots, i_t)$-shuffles for which

$$\tau(i_1+\cdots+i_{l-1}+1) < \tau(i_1+\cdots+i_l+1)$$

if $i_l = i_{l+1}$, and $u = (r-1)(i_1-1)+(r-2)(i_2-1)+\cdots+2(i_{r-2}-1)+(i_{r-1}-1)$. An L_∞-morphism $f_\bullet$ is a *quasi-isomorphism* if f_1 is a quasi-isomorphism.

We interpret the definition for some small values of n: If $n = 1$, the equation is

$$f_1 \ell_1 = \ell'_1 f_1,$$

that is, f_1 is a cochain map. If $n = 2$, we obtain

$$\begin{aligned} &- f_2(d(v_1), v_2) - (-1)^{|v_1||v_2|} f_2(d(v_2), v_1) + f_1([v_1, v_2]) \\ &= d'(f_2(v_1, v_2)) + [f_1(v_1), f_1(v_2)] \end{aligned}$$

for all homogeneous $v_1, v_2 \in L$. Rewriting, the equation is

$$\begin{aligned} &f_1([v_1, v_2]) \\ &= [f_1(v_1), f_1(v_2)] + d'(f_2(v_1, v_2)) + f_2(d(v_1), v_2) + (-1)^{|v_1||v_2|} f_2(d(v_2), v_1), \end{aligned}$$

that is, f_1 preserves the bracket up to the coboundary $\partial(f_2)$.

Just as for A_∞-algebras, there is a notion of minimal model (see Definition 7.2.1) and a version of the existence Theorem 7.2.2 (see [**HS02, Schb**]). For more general results, see for example [**Hin03**, Lemma 4.2.1].

Exercise 7.5.6. Verify that the conditions on ℓ_n given in Definition 7.5.1 for $n = 1, 2, 3$ are as claimed in this section (that is, ℓ_1 is a differential that is a graded derivation with respect to ℓ_2 and ℓ_2 satisfies the graded Jacobi identity up to homotopy).

Exercise 7.5.7. Understand the proof of Theorem 7.5.4 either by direct computation or by looking up the proof technique used in [**LM95**].

Exercise 7.5.8. Verify that the conditions on f_n given in Definition 7.5.5 for $n = 1, 2$ are as claimed in this section (that is, f_1 is a cochain map and preserves the bracket up to addition of a coboundary).

7.6. Formality and algebraic deformations

Just as for A_∞-algebras, there is a notion of formality, and in this section we discuss it specifically in the case of Hochschild cohomology. We assume that the characteristic of the field k is 0 so that exponential maps are defined. The main ideas we present here are due to Kontsevich [**Kon03**]. Let A be an associative algebra and let $C^*(A, A) = \oplus_{i\geq 0} \operatorname{Hom}_k(A^{\otimes i}, A)$, a differential graded Lie algebra as described in Section 1.4.

Definition 7.6.1. An *HKR map* is a graded k-linear injective map

$$\phi : \mathrm{HH}^*(A) \to C^*(A, A),$$

of degree $|\phi| = 0$, with image contained in the space of cocycles, that is a section of the quotient map from the space of cocycles to $\mathrm{HH}^*(A)$.

By Lemma 1.4.2, we may view both $\mathrm{HH}^*(A)$ and $C^*(A, A)$ as differential graded Lie algebras, the former having differential 0. An HKR map is in general not a morphism of differential graded Lie algebras. However, viewing $\mathrm{HH}^*(A)$ and $C^*(A, A)$ as L_∞-algebras (with higher brackets 0), an HKR map can sometimes be extended to a quasi-isomorphism of L_∞-algebras, as we will see below. In this case we say that A is formal:

Definition 7.6.2. The associative algebra A is *formal* if there is a quasi-isomorphism of L_∞-algebras $\Phi_\bullet : (\mathrm{HH}^*(A), \ell_\bullet) \to (C^*(A, A), \ell'_\bullet)$ for which Φ_1 is an HKR map. Such a map Φ is called a *formality map*.

Note that this definition is analogous to Definition 7.3.1 when we replace the A_∞-structure with the L_∞-structure, taking $\mathrm{HH}^*(A)$ to be the minimal model of $C^*(A, A)$. Which version of formality is intended should be clear from context.

Note that the grading on $\mathrm{HH}^*(A)$ and on $C^*(A, A)$ in Definition 7.6.2 is shifted by 1 since we are dealing with the Lie structure. So for example, the degree of a Hochschild 2-cocycle is now 1, an important distinction to make in the proof of the next theorem.

Theorem 7.6.3. *Let A be a formal associative algebra over a field of characteristic 0, and let α be an infinitesimal deformation of A for which the first obstruction (5.1.8) vanishes as a cochain. Then α lifts to a formal deformation of A.*

Proof. Let α be a Hochschild 2-cocycle on A for which the first obtruction vanishes as a cochain, that is, $[\alpha, \alpha] = 0$. Consider the following element in $S(\mathrm{HH}^*(A))[[t]]$:

$$\exp(t\alpha) = 1 + t\alpha + \frac{1}{2!}t^2\alpha^2 + \frac{1}{3!}t^3\alpha^3 + \cdots$$

where we consider the ith term to be in $S^i(\mathrm{HH}^*(A))[[t]]$, starting with $i = 0$. Let Φ be a formality map for A. Consider the image of $\exp(t\alpha)$ under Φ extended to formal power series in t. Explicitly, we write this as

$$\Phi(\exp(t\alpha)) = 1 + t\Phi_1(\alpha) + \frac{1}{2!}t^2\Phi_2(\alpha^2) + \frac{1}{3!}t^3\Phi_3(\alpha^3) + \cdots ,$$

where $\Phi_2(\alpha^2)$ may also be written $\Phi_2(\alpha, \alpha)$, $\Phi_3(\alpha^3)$ as $\Phi_3(\alpha, \alpha, \alpha)$, and so on. Due to the degree requirement on L_∞-morphisms, the elements $\Phi_i(\alpha^i)$ each have degree 1, that is, they are Hochschild 2-cochains. Since Φ is an L_∞-morphism and $[\alpha, \alpha] = 0$, it follows from the definitions that $\Phi(\exp(t\alpha))$ satisfies the Maurer-Cartan equation (5.3.1). Write $\mu_i = \frac{1}{i!}\Phi_i(\alpha^i)$ and $\mu = \mu_0 + \mu_*$ where $\mu_* = t\mu_1 + t^2\mu_2 + \cdots$. Then $(A[[t]], \mu)$ is a formal deformation of (A, μ_0). □

The following is a special case of general results of Kontsevich [**Kon03**] about Poisson manifolds. See also [**DTT07**]. We take k to be $\mathbb{C}$ or $\mathbb{R}$ here.

Theorem 7.6.4. *Let V be a finite dimensional vector space. Its (ungraded) symmetric algebra $S(V)$ is formal.*

For an outline of the proof, see [**BM08**, §5.2], where the theorem is then generalized to universal enveloping algebras of some Lie algebras. The proof is combinatorial and geometric, involving sums over some planar graphs. For more details and geometric context, see the excellent survey [**Scha**].

There are further infinity structures arising on Hochschild cohomology $\mathrm{HH}^*(A)$ and the Hochschild complex $C^*(A, A)$. In particular, combining the A_∞- and L_∞-structures we obtain a G_∞-algebra, that is an infinity analog of a Gerstenhaber algebra (see Definition 1.4.6). Deligne conjectured that the Hochschild complex $C^*(A, A)$ is an algebra over an operad of little disks. An algebraic version of Deligne's conjecture states that the Hochschild complex $C^*(A, A)$ is a G_∞-algebra. This is now a theorem and there are various proofs in the literature, for example [**KS00, MS02, Tam, Vor00**].

Exercise 7.6.5. Let $A = k[x]$. Find an HKR map $\phi : \mathrm{HH}^*(A) \to C^*(A, A)$. (Recall that $\mathrm{HH}^*(A)$ was found in Example 1.1.16 and that Hochschild 1-cocycles correspond to derivations as discussed in Section 1.2.)

Exercise 7.6.6. Look up a proof of a version or generalization of Theorem 7.6.4, for example in [**BM08, DTT07, Kon03, Scha**].

Chapter 8

Support Varieties for Finite Dimensional Algebras

In this chapter we present an application of Hochschild cohomology in representation theory of finite dimensional algebras, that is algebras that are finite dimensional as vector spaces over the field k. For many finite dimensional algebras A, though not all, Hochschild cohomology $\mathrm{HH}^*(A)$ is finitely generated as an algebra. Consequently its maximal ideal spectrum is an affine algebraic variety. This variety has subvarieties associated to A-modules through actions of Hochschild cohomology $\mathrm{HH}^*(A)$, and the geometry of these subvarieties has implications in representation theory. These "varieties for modules" based on Hochschild cohomology were introduced by Snashall and Solberg [**SS04**] to mimic support varieties for finite groups (based on group cohomology). The theory is particularly well-behaved for self-injective algebras, as further developed by Erdmann, Holloway, Snashall, Solberg, and Taillefer [**EHS$^+$04**]. See also Solberg's excellent survey [**Sol06**].

We begin this chapter by briefly introducing the needed geometric notions and finite generation conditions. We take A to be a finite dimensional algebra and k an algebraically closed field, although support variety theory has been developed for other algebras and fields as well. In some of the references, A is assumed to be indecomposable. We do not make this assumption since we aim at more general applications, and there are some minor differences. We refer the reader to [**GJ89, MR88**] for the general theory of noncommutative noetherian rings, although we will only need this theory in the graded commutative setting where it is essentially the same as for commutative rings.

8.1. Affine varieties

In this section we give a very brief introduction to the geometry that we will use in this chapter. For more details, see any text on algebraic geometry or commutative algebra, or [**Ben91b**, Section 5.4]. Let k be an algebraically closed field.

Let H be a finitely generated commutative algebra over k. Equivalently, $H \cong k[x_1, \dots, x_n]/I$ for some ideal I of a polynomial ring $k[x_1, \dots, x_n]$. Let $\mathrm{Max}(H)$ denote the set of maximal ideals of H, so $\mathrm{Max}(H)$ is in one-to-one correspondence with the set of maximal ideals of $k[x_1, \dots, x_n]$ containing I. In particular, since k is algebraically closed,

$$\mathrm{Max}(k[x_1, \dots, x_n]) = \{(x_1 - a_1, \dots, x_n - a_n) \mid a_1, \dots, a_n \in k\},$$

so that the set of maximal ideals of the polynomial ring $k[x_1, \dots, x_n]$ is in one-to-one correspondence with k^n.

The set $\mathrm{Max}(H)$ becomes a topological space under the *Zariski topology*: Closed sets are the sets

$$V(J) = \{J' \in \mathrm{Max}(H) \mid J' \supset J\},$$

determined by ideals J of H. Sometimes we write $V_H(J)$ in place of $V(J)$ to emphasize dependence on H. These sets satisfy the relations

$$V(J_1 J_2) = V(J_1) \cup V(J_2) \quad \text{and} \quad V\left(\sum_\alpha J_\alpha\right) = \bigcap_\alpha V(J_\alpha), \tag{8.1.1}$$

where α ranges over an indexing set and J_1, J_2, J_α are ideals of H. We call $\mathrm{Max}(H)$ with this topology the *maximal ideal spectrum* of H, also called an *affine variety*. In particular, $\mathrm{Max}(k[x_1, \dots, x_n])$ is the *affine space* k^n. Projective varieties and prime ideals are also of interest in representation theory, but here we will focus on affine varieties and maximal ideals.

The following lemma will be used in a definition of dimension of an affine variety.

Lemma 8.1.2 (Noether Normalization Lemma)**.** *Let H be a finitely generated commutative algebra over k. There are elements $y_1, \dots, y_n \in H$ generating a subalgebra of H that is isomorphic to the polynomial ring $k[y_1, \dots, y_n]$ and over which H is finitely generated as a module.*

For a proof, see [**Mat86**, §33]. By its definition, the integer n in the lemma is unique.

Definition 8.1.3. Let H be a finitely generated commutative algebra over k, and $\operatorname{Max}(H)$ its maximal ideal spectrum. The *dimension* of $\operatorname{Max}(H)$ is the integer n of the Noether Normalization Lemma (Lemma 8.1.2).

It can be shown that if H is finitely generated, then the dimension of $\operatorname{Max}(H)$ defined above is the same as the Krull dimension of H, defined next. (See [**Mat86**].)

Definition 8.1.4. For any commutative ring H, its *Krull dimension* is the largest nonnegative integer n for which there exist prime ideals

$$I_0 \supset I_1 \supset \cdots \supset I_n$$

of H such that $I_j \neq I_{j+1}$ for $0 \leq j \leq n-1$.

There is another notion of size that we will need:

Definition 8.1.5. Let $V = \oplus_{i\geq 0} V_i$ be a graded vector space. The *rate of growth* $\gamma(V)$ is the smallest nonnegative integer c such that there is a real number b and a positive integer m for which $\dim_k V_n \leq bn^{c-1}$ for all $n \geq m$.

The Krull dimension of a finitely generated graded commutative algebra is precisely its rate of growth. To see this, note that a polynomial ring in n indeterminates has rate of growth n. Now apply Lemma 8.1.2 and the above comments.

Exercise 8.1.6. Let $A = k[x, y]$, $J_1 = (x)$, and $J_2 = (y)$. Find $V(J_1J_2)$ and $V(J_1 + J_2)$ and verify that (8.1.1) holds.

Exercise 8.1.7. Let $A = k[x, y]/(xy)$. What is the Krull dimension of A? Find elements of A satisfying the conclusion of the Noether Normalization Lemma.

Exercise 8.1.8. Show directly that the Krull dimension of $k[x]$ is 1 by applying the definition.

Exercise 8.1.9. Justify the statement that $k[x_1, \ldots, x_n]$ has rate of growth n by examining an expression for $\dim_k(k[x_1, \ldots, x_n]_m)$ for each m.

8.2. Finiteness properties

Let $\mathfrak{r} = \operatorname{rad}(A)$, the Jacobson radical of the finite dimensional algebra A, that is the intersection of all maximal left ideals. For details on the Jacobson radical and related representation theory of finite dimensional algebras over algebraically closed fields, see e.g. [**Alp86**, Sections 1 and 2] or [**ARS95**].

We will use the action of $\mathrm{HH}^*(A)$ on $\operatorname{Ext}^*_A(M, M)$ for an A-module M, as described in Section 2.5, beginning with the special case $M = A/\mathfrak{r}$. We will assume in Sections 8.3 through 8.5 that A satisfies the following finite generation condition:

(fg) $\mathrm{HH}^*(A)$ is a noetherian ring and $\mathrm{Ext}^*_A(A/\mathfrak{r}, A/\mathfrak{r})$ is a finitely generated $\mathrm{HH}^*(A)$-module.

Since $\mathrm{HH}^*(A)$ is graded commutative, if $\mathrm{HH}^*(A)$ is noetherian, then it is finitely generated as an algebra by homogeneous elements. Conversely, if $\mathrm{HH}^*(A)$ is finitely generated as an algebra, then it is noetherian, since free graded commutative rings are noetherian and quotients of noetherian rings are noetherian. We will refine these statements in Theorem 8.2.3 below.

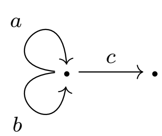
There are many finite dimensional algebras A that satisfy condition (fg), as well as some that do not, as we will see.

Example 8.2.1. Let $A = k[x_1, \ldots, x_m]/(x_1^{n_1}, \ldots, x_m^{n_m})$. We saw in Example 3.1.5 that $\mathrm{HH}^*(A)$ is finitely generated. (See also Example 3.3.6 for some related noncommutative examples.) The Jacobson radical $\mathfrak{r}$ of A is generated by $x_1, \ldots, x_m$, and so $A/\mathfrak{r} \cong k$, the trivial module. If $m = 1$, as a consequence of our work in Example 2.5.10, the Ext algebra $\mathrm{Ext}^*_A(k, k)$ is finitely generated as an $\mathrm{HH}^*(A)$-module. The same is seen to be true if $m > 1$ by combining the techniques of Examples 2.5.10 and 3.1.5.

Many other algebras satisfy condition (fg), such as finite group algebras [**Eve61, Gol59, Ven59**] and Hecke algebras [**Lin11**]. Others satisfy related finiteness conditions, such as monomial algebras [**GS06**], self-injective algebras of finite representation type [**GSS03**], and algebras of finite global dimension [**Hap89**]. Some types of Hopf algebras satisfy condition (fg); Hopf algebras generally are discussed in Chapter 9.

We next give examples of algebras that do not satisfy condition (fg). In characteristic 2, the example below is due to Xu [**Xu08**], who presented it as the first counterexample to a related conjecture of Snashall and Solberg. Here we describe a generalization of Xu's example to arbitrary characteristic, due to Snashall [**Sna09**]. It was generalized further by Gawell and Xantcha [**GX16**] to many more algebras defined by quivers and relations.

Example 8.2.2. Let $A = kQ/I$ where Q is the quiver



with two vertices as indicated, arrows a, b, c, and $I = (a^2,\ b^2,\ ab - ba,\ ac)$. Then $\mathrm{Ext}^*_A(A/\mathfrak{r}, A/\mathfrak{r}) \cong kQ^{\mathrm{op}}/I'$ where Q^{op} is the quiver Q with arrows reversed, labeled α, β, γ, and $I' = (\alpha\beta + \beta\alpha,\ \beta\gamma)$ [**Sna09**]. A calculation shows that the graded center of $\mathrm{Ext}^*_A(A/\mathfrak{r}, A/\mathfrak{r})$ is $k \oplus k[\alpha, \beta]\beta$ if $\mathrm{char}(k) = 2$, and $k \oplus k[\alpha^2, \beta^2]\beta^2$ if $\mathrm{char}(k) \neq 2$, where β has degree 1, $\alpha\beta$ has degree 2, β^2 has degree 2, and $\alpha^2\beta^2$ has degree 4. By [**BGSS08**], the image of $\mathrm{HH}^*(A)$ under the map $\phi_{A/\mathfrak{r}}$ defined by (2.5.3) is precisely this graded center. This is not finitely generated as an algebra, since for each i, the element $\alpha^i\beta$ in characteristic 2 (respectively, $\alpha^{2i}\beta^2$ in characteristic not 2) is not in any subalgebra generated by elements of lower homological degree. Therefore $\mathrm{HH}^*(A)$ is not finitely generated as an algebra, and consequently does not satisfy condition (fg).

For the purpose of defining affine varieties, one can essentially ignore nilpotent elements, and indeed Snashall and Solberg [**SS04**] had originally conjectured that the quotient of $\mathrm{HH}^*(A)$ by its ideal generated by all homogeneous nilpotent elements is noetherian. Example 8.2.2 is a counterexample. Hermann [**Her16b**] asked if a weaker condition might be satisfied by more finite dimensional algebras: Replace the condition that $\mathrm{HH}^*(A)$ be noetherian with the condition that the quotient by its Gerstenhaber ideal generated by all homogeneous nilpotent elements be noetherian. (By this ideal, we mean the ideal generated via the binary operations of cup product and Gerstenhaber bracket.) We will not consider these weaker conditions here.

We will prove the following theorem from [**EHS$^+$04**, Proposition 1.4] and [**Sol06**, Proposition 5.7]. We will then use it to define support varieties in the next section. The flexibility in choosing an algebra H satisfying the conditions of the theorem below will be helpful. We will use the action of $\mathrm{HH}^*(A)$ on $\mathrm{Ext}^*_A(M, N)$ for any two A-modules M, N that is described in Section 2.5.

Theorem 8.2.3. *The finite dimensional algebra A satisfies condition (fg) if and only if there exists a graded subalgebra H of $\mathrm{HH}^*(A)$ such that*

(fg1) *H is finitely generated commutative and $H^0 = \mathrm{HH}^0(A, A)$, and*

(fg2) *$\mathrm{Ext}^*_A(A/\mathfrak{r}, A/\mathfrak{r})$ is a finitely generated H-module.*

There is a trade-off between these two conditions (fg1) and (fg2): Condition (fg1) says that H is small enough for some geometric applications, and condition (fg2) says that H is large enough for others. The two taken together say that H is just right for the theory of support varieties that we will define in Section 8.3.

In order to prove the theorem, we need the following lemmas. If M, N are A-modules, then $\mathrm{Hom}_k(M, N)$ is an A-bimodule with action given by

$$(afb)(m) = a(f(bm))$$

for all $f \in \mathrm{Hom}_k(M, N)$, $a, b \in A$, and $m \in M$. The action of $\mathrm{HH}^*(A)$ on $\mathrm{HH}^*(A, B)$ for any A-bimodule B is given by Yoneda product as described in Section 2.5.

Lemma 8.2.4. *For all finite dimensional A-modules M, N, there is a graded vector space isomorphism*

$$\mathrm{Ext}_A^*(M, N) \cong \mathrm{HH}^*(A, \mathrm{Hom}_k(M, N)).$$

Moreover, the actions of Hochschild cohomology $\mathrm{HH}^*(A)$ correspond under this isomorphism.

Proof. Let $P_\bullet \to A$ be a projective resolution of A as an A^e-module. For each i, define a function $\phi_i : \mathrm{Hom}_A(P_i \otimes_A M, N) \to \mathrm{Hom}_{A^e}(P_i, \mathrm{Hom}_k(M, N))$ by $\phi_i(f)(x)(m) = f(x \otimes m)$ for all $f \in \mathrm{Hom}_A(P_i \otimes_A M, N)$, $x \in P_i$, and $m \in M$. A calculation shows that $\phi_i(f)$ is an A^e-module homomorphism and $\phi_\bullet$ is a cochain map. The inverse maps are $\psi_i : \mathrm{Hom}_{A^e}(P_i, \mathrm{Hom}_k(M, N)) \to \mathrm{Hom}_A(P_i \otimes_A M, N)$ given by $\psi_i(f)(x \otimes m) = f(x)(m)$. A calculation now shows that the actions of $\mathrm{HH}^*(A)$ correspond. $\square$

The next lemma is [**EHS$^+$04**, Proposition 2.4].

Lemma 8.2.5. *Let H be a finitely generated commutative subalgebra of* $\mathrm{HH}^*(A)$. *The following are equivalent:*

(i) $\mathrm{Ext}_A^*(A/\mathfrak{r}, A/\mathfrak{r})$ *is a finitely generated H-module.*

(ii) $\mathrm{Ext}_A^*(M, N)$ *is a finitely generated H-module for all finite dimensional A-modules M, N.*

(iii) $\mathrm{HH}^*(A, B)$ *is a finitely generated H-module for all finite dimensional A-bimodules B.*

Proof. By Lemma 8.2.4, $\mathrm{Ext}_A^*(M, N) \cong \mathrm{HH}^*(A, \mathrm{Hom}_k(M, N))$ and the actions of $\mathrm{HH}^*(A)$ on these two spaces correspond, so (iii) implies (ii). By setting $M = N = A/\mathfrak{r}$, we see that (ii) implies (i). It remains to show that (i) implies (iii). By Lemma 8.2.4 with $M = N = A/\mathfrak{r}$, there is an isomorphism

$$\mathrm{Ext}_A^*(A/\mathfrak{r}, A/\mathfrak{r}) \cong \mathrm{HH}^*(A, \mathrm{Hom}_k(A/\mathfrak{r}, A/\mathfrak{r})).$$

Each simple A-module S is a direct summand of $A/\mathfrak{r}$, and simple A^e-modules are all of the form $\mathrm{Hom}_k(S, T)$ for simple A-modules S, T. Letting B be any finite dimensional A^e-module, it has a composition series with simple factors B_i. By (i) and the above observations, $\mathrm{HH}^*(A, B_i)$ is a finitely generated H-module for each i. By induction on the length of the composition series of B and the first long exact sequence for Ext (Theorem A.4.4), the H-module $\mathrm{HH}^*(A, B)$ is finitely generated. Specifically, for the induction step, suppose $0 \to U \to V \to W \to 0$ is a short exact sequence of A^e-modules and

$\mathrm{HH}^*(A, U)$, $\mathrm{HH}^*(A, W)$ are both finitely generated H-modules. Choose a finite set of generators for $\mathrm{HH}^*(A, U)$, and consider their images in $\mathrm{HH}^*(A, V)$ under the map induced by $U \to V$. Since H is noetherian and the image of $\mathrm{HH}^*(A, V)$ in $\mathrm{HH}^*(A, W)$, under the map induced by $V \to W$, is an H-submodule, the image of $\mathrm{HH}^*(A, V)$ in $\mathrm{HH}^*(A, W)$ is finitely generated. Choose a finite set of generators of this image, and choose an inverse image of each one in $\mathrm{HH}^*(A, V)$. Then the finite set of all these generators taken together generates $\mathrm{HH}^*(A, V)$. □

Proof of Theorem 8.2.3. Assume A satisfies (fg). If $\mathrm{char}(k) = 2$, let $H = \mathrm{HH}^*(A)$, and if $\mathrm{char}(k) \neq 2$, let $H = \mathrm{HH}^{\mathrm{ev}}(A)$, the subalgebra of $\mathrm{HH}^*(A)$ generated by all homogeneous elements of even degree. Then $H^0 = \mathrm{HH}^0(A)$ and H is commutative since $\mathrm{HH}^*(A)$ is graded commutative. In addition, H is finitely generated: Take a finite set of homogeneous generators of $\mathrm{HH}^*(A)$, and replace those of odd degree by all products of pairs of odd degree generators. The resulting finite set generates H. So (fg1) holds. Condition (fg2) also holds since H is a subalgebra of $\mathrm{HH}^*(A)$ and we have assumed that $\mathrm{Ext}^*_A(A/\mathfrak{r}, A/\mathfrak{r})$ is finitely generated over the noetherian ring $\mathrm{HH}^*(A)$.

Conversely, assume there is a graded subalgebra H of $\mathrm{HH}^*(A)$ that satisfies (fg1) and (fg2). Note that (fg2) is precisely condition (i) of Lemma 8.2.5. By Lemma 8.2.5(iii) with $B = A$, $\mathrm{HH}^*(A)$ is a finitely generated H-module, and so it is finitely generated as an algebra (take the algebra generators of H together with the H-module generators of $\mathrm{HH}^*(A)$). Since $\mathrm{Ext}^*_A(A/\mathfrak{r}, A/\mathfrak{r})$ is finitely generated as an H-module, it will be finitely generated as an $\mathrm{HH}^*(A)$-module, and so (fg) holds. □

We will be interested in the maximal ideal spectrum of H and of $\mathrm{HH}^*(A)$. Due to (graded) commutativity of these algebras, in either case the nilpotent elements constitute an ideal that is contained in all maximal ideals. Thus for the purpose of considering the maximal ideal spectrum, we may as well work modulo this ideal. The following theorem, due to Snashall and Solberg [**SS04**, Proposition 4.6], gives some information about nilpotent elements of $\mathrm{HH}^*(A)$. Recall that the *radical* of an ideal I of a (graded) commutative ring R is

$$\sqrt{I} = \{x \in R \mid x^n \in I \text{ for some positive integer } n\}.$$

By Corollary 2.5.9 with $M = A/\mathfrak{r}$, the map $\phi_{A/\mathfrak{r}}$ defined in (2.5.3) is a ring homomorphism from $\mathrm{HH}^*(A)$ to $\mathrm{Ext}^*_A(A/\mathfrak{r}, A/\mathfrak{r})$ with image in the graded center. We now examine this ring homomorphism further.

Theorem 8.2.6. *Let $\mathcal{N}$ be the ideal of* $\mathrm{HH}^*(A)$ *generated by all homogeneous nilpotent elements. Then*

$$\mathcal{N} = \sqrt{\mathrm{Ker}(\phi_{A/\mathfrak{r}})}.$$

Proof. A nilpotent element is in the radical of every ideal by definition, so the containment $\mathcal{N} \subset \sqrt{\mathrm{Ker}(\phi_{A/\mathfrak{r}})}$ is automatic. It remains to prove that $\mathrm{Ker}(\phi_{A/\mathfrak{r}}) \subset \mathcal{N}$ (the containment $\sqrt{\mathrm{Ker}(\phi_{A/\mathfrak{r}})} \subset \mathcal{N}$ will follow since $\mathcal{N}$ contains all nilpotent elements). Let η be a homogeneous element in $\mathrm{Ker}(\phi_{A/\mathfrak{r}})$, and suppose η is represented by a function $f : P_m \to A$ where $P_{\bullet}$ is a projective resolution of A as an A^e-module. Since $\phi_{A/\mathfrak{r}}(\eta) = 0$, at the chain level, $\phi_{A/\mathfrak{r}}(f)$ has image in $\mathfrak{r}$. For each m, the power η^m is represented by f^m, whose image is thus in $\mathfrak{r}^m$. Since $\mathfrak{r}$ is a nilpotent ideal, it follows that η is nilpotent. □

Exercise 8.2.7. Let $A = k[x]/(x^n)$. Use the results of Example 2.5.10 to give the structure of $\mathrm{HH}^*(A)$, of $\mathrm{Ext}^*_A(A/\mathfrak{r}, A/\mathfrak{r})$, and the action of the former on the latter, thus verifying directly that A satisfies condition (fg).

Exercise 8.2.8. Let $A = k[x]/(x^n)$. Find a graded subalgebra H of $\mathrm{HH}^*(A)$ that satisfies conditions (fg1) and (fg2) of Theorem 8.2.3.

Exercise 8.2.9. Verify the last statement in the proof of Lemma 8.2.4, that the actions correspond.

Exercise 8.2.10. Let $A = k[x]/(x^n)$. What is the ideal $\mathcal{N}$ of $\mathrm{HH}^*(A)$ generated by all homogeneous nilpotent elements?

8.3. Support varieties

We are now ready to define support varieties. Assume the finite dimensional algebra A satisfies condition (fg) of the previous section. Applying Theorem 8.2.3, we fix a subalgebra H of $\mathrm{HH}^*(A)$ for which (fg1) and (fg2) hold. We do not assume that A is indecomposable, as is done in some of the references. The main difference is that in applications, it may be necessary to keep track of extra points in support varieties of nonindecomposable modules, corresponding to idempotents in A that are not in the annihilators of the modules. Specifically, as in Section 1.2, the Hochschild cohomology ring of A decomposes as a direct sum of Hochschild cohomology rings of the algebras e_jA where $\{e_1, \ldots, e_i\}$ is a set of orthogonal central idempotents of A for which $1 = e_1 + \cdots + e_i$. We will assume that each e_j is primitive, that is, it is not the sum of two nonzero orthogonal idempotents. The algebras e_jA are then indecomposable.

For finite dimensional A-modules M, N, let $I_H(M,N)$ be the annihilator in H of $\mathrm{Ext}^*_A(M,N)$, that is

$$I_H(M,N) = \{\alpha \in H \mid \alpha \cdot \beta = 0 \ \text{ for all } \ \beta \in \mathrm{Ext}^*_A(M,N)\},$$

where the (left) action of the subalgebra H of $\mathrm{HH}^*(A)$ on $\mathrm{Ext}^*_A(M,N)$ is defined in Section 2.5. By its definition, $I_H(M,N)$ is an ideal of H. Let $I_H(M) = I_H(M,M)$. There is an analogous definition of support variety of a right module, and we will sometimes use it as well.

Definition 8.3.1. Let M, N be finite dimensional A-modules. The *support variety* of the pair M, N is

$$V_H(M,N) = V_H(I_H(M,N)) \cong \mathrm{Max}(H/I_H(M,N)),$$

the maximal ideal spectrum of the quotient ring $H/I_H(M,N)$. The *support variety* of M is $V_H(M) = V_H(M,M)$.

Note that by its definition, a support variety is a subvariety of $\mathrm{Max}(H)$, the maximal ideal spectrum of H.

Example 8.3.2. Let $A = k[x]/(x^n)$. Let $M = k$, the trivial module (on which x acts as 0). As we saw in Example 2.5.10, $\mathrm{Ext}^*_A(k,k)$ is isomorphic to $k[y]$ in case $n = 2$ and is isomorphic to $k[y,z]/(y^2)$ otherwise. We also found the image of $\mathrm{HH}^*(A)$ in $\mathrm{Ext}^*_A(k,k)$ under the map ϕ_k defined in (2.5.3). Let $H = \mathrm{HH}^*(A)$ in characteristic 2 and $H = \mathrm{HH}^{\mathrm{ev}}(A)$ otherwise. In both cases, $H/I_H(k)$ has Krull dimension 1 and so $V_H(k)$ is a line.

The following lemma gives a relationship between the support variety of a pair of modules and the support varieties of the modules.

Lemma 8.3.3. *For all finite dimensional A-modules M and N,*

$$V_H(M,N) \subset V_H(M) \cap V_H(N).$$

Proof. This is true by the definitions, since the (left) action of H on the space $\mathrm{Ext}^*_A(M,N)$ factors through that on $\mathrm{Ext}^*_A(M,M)$, and by Theorem 2.5.5 is the same, up to a sign, as a right action factoring through that on $\mathrm{Ext}^*_A(N,N)$. □

Let $\mathrm{Irr}\, A$ denote a set of representatives of isomorphism classes of simple A-modules. The following lemma gives a relationship between the support variety of a module and those of the simple A-modules, and a relationship among support varieties for modules in a short exact sequence. Recall that $\mathfrak{r}$ denotes the Jacobson radical of A.

Proposition 8.3.4. *Let M, M_1, M_2, M_3 be finite dimensional A-modules. Then*

(i) $V_H(M) = \cup_{S\in \mathrm{Irr}(A)} V_H(M,S) = \cup_{S\in \mathrm{Irr}(A)} V_H(S,M)$.

(ii) *If* $0 \to M_1 \to M_2 \to M_3 \to 0$ *is a short exact sequence, then*

$$V_H(M_i) \subset V_H(M_j) \cup V_H(M_l)$$

whenever $\{i,j,l\} = \{1,2,3\}$.

(iii) $V_H(M) = V_H(M, A/\mathfrak{r}) = V_H(A/\mathfrak{r}, M)$.

Proof. If $0 \to U' \to U \to U'' \to 0$ is a short exact sequence of A-modules, then $V_H(M,U) \subset V_H(M,U') \cup V_H(M,U'')$ since the annihilator in H of $\mathrm{Ext}^*_A(M,U)$ contains the product of the annihilators of $\mathrm{Ext}^*_A(M,U')$ and $\mathrm{Ext}^*_A(M,U'')$ in light of the first long exact sequence for Ext (Theorem A.4.4). Therefore $V_H(M,N) \subset \cup_S V_H(M,S)$, the union over all simple modules in a composition series for M, and similarly $V_H(M,N) \subset \cup_S V_H(S,N)$. Thus (i) holds. For (ii), the above argument using the first long exact sequence for Ext (Theorem A.4.4) shows that $V_H(M_i) \subset V_H(M_i, M_j) \cup V_H(M_i, M_l)$, and this is contained in $V_H(M_j) \cup V_H(M_l)$ by Lemma 8.3.3. By the above arguments, $V_H(M) \subset V_H(M, A/\mathfrak{r})$ and $V_H(M) \subset V_H(A/\mathfrak{r}, M)$. On the other hand, by Lemma 8.3.3, the reverse inclusions also are true. Thus (iii) holds. □

We will need the notions of projective cover and minimal projective resolution. For a finite dimensional algebra A with Jacobson radical $\mathfrak{r} = \mathrm{rad}(A)$, we may take the following definitions, equivalent to those in Section A.2. For an A-module M, write $\mathrm{rad}(M) = \mathrm{rad}(A)M$, the radical of M. A projective cover of M is a projective A-module P for which $P/\mathrm{rad}(P) \cong M/\mathrm{rad}(M)$. A minimal projective resolution $P_{\bullet}$ of M is one for which P_0 is a projective cover of M and P_n is a projective cover of K_n for all $n \geq 1$, where K_n is the nth syzygy module (notation as in Section A.2). Note this implies that $K_n \hookrightarrow \mathrm{rad}(P_{n-1})$ for all $n \geq 1$. We will also need the following property of projective modules for finite dimensional algebras: A finite dimensional projective A-module P is a direct sum of copies of the projective covers of simple modules. (See, e.g., [**ARS95**, Corollary 4.5].)

Next we define the complexity of an A-module M and derive a connection to the dimension of its support variety.

Definition 8.3.5. Let $\cdots \to P_1 \to P_0 \to M \to 0$ be a minimal projective resolution of a finite dimensional A-module M, and view $\oplus_i P_i$ as a graded vector space. The *complexity* of M is $\mathrm{cx}_A(M) = \gamma(P_{\bullet}) = \gamma(\oplus_i P_i)$, the rate of growth of the resolution (see Definition 8.1.5).

Recall that the dimension of $V_H(M)$ is given by Definition 8.1.3, equivalently by Definition 8.1.4 as the Krull dimension of $H/I_H(M)$, equivalently by Definition 8.1.5 as the rate of growth of $H/I_H(M)$. The following theorem makes a connection with the rate of growth of a minimal projective resolution.

Theorem 8.3.6. *Let M be a finite dimensional A-module. Then*

$$\dim V_H(M) = \mathrm{cx}_A(M).$$

Proof. The proof is essentially the same as that of [**Ben91b**, Proposition 5.7.2], which is the case that A is a group algebra of a finite group. By Lemma 8.2.5(ii), $\mathrm{Ext}_A^*(M,M)$ is finitely generated as a module over $H/I_H(M)$. It follows that

$$\dim V_H(M) = \gamma(H/I_H(M)) = \gamma(\mathrm{Ext}_A^*(M,M)).$$

We will show that $\gamma(\mathrm{Ext}_A^*(M,M)) = \gamma(P_\bullet)$, where $P_\bullet$ is a minimal projective resolution of the A-module M. This is by definition $\mathrm{cx}_A(M)$.

Let K_n denote the nth syzygy module of $P_\bullet$. For any simple A-module S, the multiplicity of its projective cover $P(S)$ as a direct summand of P_n is

$$\dim_k(\mathrm{Hom}_A(P_n,S)) = \dim_k(\mathrm{Hom}_A(K_n,S)) = \dim_k(\mathrm{Ext}_A^n(M,S))$$

since $S \cong P(S)/\operatorname{rad}(P(S))$, the radical $\mathfrak{r}$ acts trivially on S, and $K_n \hookrightarrow \operatorname{rad}(P_{n-1})$ for all $n \geq 1$. So

$$\dim_k P_n = \sum_{S\in \mathrm{Irr}(A)} \dim_k P(S)\cdot \dim_k(\mathrm{Ext}_A^n(M,S)).$$

It follows that

$$\gamma(P_\bullet) \leq \max\{\gamma(\mathrm{Ext}_A^*(M,S)) \mid S \in \mathrm{Irr}(A)\}.$$

Now for each simple A-module S, the action of H on $\mathrm{Ext}_A^*(M,S)$ factors through $\mathrm{Ext}_A^*(M,M)$, and both are finitely generated as H-modules. Therefore

$$\gamma(\mathrm{Ext}_A^*(M,S)) \leq \gamma(\mathrm{Ext}_A^*(M,M))$$

for each S. It follows that $\gamma(P_\bullet) \leq \gamma(\mathrm{Ext}_A^*(M,M))$. On the other hand, by definition of Ext,

$$\dim_k \mathrm{Ext}_A^n(M,M) \leq \dim_k \mathrm{Hom}_k(P_n,M) = \dim_k(P_n)\dim_k(M)$$

for each n, and so $\gamma(\mathrm{Ext}_A^*(M,M)) \leq \gamma(P_\bullet)$. Therefore $\gamma(\mathrm{Ext}_A^*(M,M)) = \gamma(P_\bullet)$, and this is by definition the complexity of M. □

Remark 8.3.7. For completeness, we mention a duality result, although we will not use it: For each A-module M, let $D(M) = \mathrm{Hom}_k(M,k)$, an A^{op}-module with action given by $(a\cdot f)(m) = f(a\cdot m)$ for all $a \in A$, $f \in D(M)$, and $m \in M$. It can be shown that $V_H(M) = V_H(D(M))$ for all finite dimensional A-modules M (see [**SS04**, Proposition 3.5]).

Exercise 8.3.8. Verify some details of Example 8.3.2: Find $I_H(k)$ explicitly and show that $H/I_H(k)$ has Krull dimension 1.

Exercise 8.3.9. Let $A = k[x]/(x^n)$, and let k be the A-module on which x acts as 0. Let H be as in Example 8.3.2. Verify directly that $\dim V_H(k) = \mathrm{cx}_A(k)$ by finding the rate of growth of a minimal projective resolution of k and comparing with Exercise 8.3.8.

8.4. Self-injective algebras and realization

For the rest of this chapter, assume A is a finite dimensional self-injective algebra satisfying condition (fg) from Section 8.2. Recall that a self-injective algebra A is one for which the left A-module A under multiplication is an injective A-module. Examples of self-injective algebras are finite group algebras and finite dimensional Hopf algebras.

Fix a subalgebra H of $\mathrm{HH}^*(A)$ satisfying (fg1) and (fg2), in accordance with Theorem 8.2.3. Under these assumptions, in Theorem 8.4.4 below we state a tensor product property that has some important consequences. First we characterize the support varieties of projective modules.

Lemma 8.4.1. *Let M be a finite dimensional A-module. Then M is projective if and only if* $\dim V_H(M) = 0$.

Proof. If M is projective, then $0 \to M \to M \to 0$ is a projective resolution of M, and so it must be that $\mathrm{cx}_A(M) = 0$. By Theorem 8.3.6, $\dim V_H(M) = \mathrm{cx}_A(M) = 0$. Conversely, assume $\dim V_H(M) = 0$, that is, $\mathrm{cx}_A(M) = 0$, so that if $P_\bullet$ is a minimal projective resolution of M, then $P_n = 0$ for some n. That is,

$$0 \to P_{n-1} \to \cdots \to P_0 \to M \to 0$$

is a projective resolution of M. Since A is self-injective, each P_i is also injective, so the sequence splits, and thus M is projective. □

The *Heller operator* $\Omega = \Omega_A$ of the following lemma is defined in Section A.2. We assume that it refers specifically to the syzygy module of a projective cover in this setting of self-injective algebras.

Lemma 8.4.2. *Let M be a finite dimensional indecomposable A-module. Then $V_H(M) = V_H(\Omega_A M)$.*

Proof. Let P be a projective cover of the A-module M, so that $P/\operatorname{rad}(P) \cong M/\operatorname{rad}(M)$. Since M is indecomposable, there is a unique primitive central idempotent e_j of A for which $e_jM = M$ and $e_jP = P$. By the Wedderburn Theorem, $V_H(P)$ consists of precisely one point corresponding to e_j that is also in $V_H(M)$ and in $V_H(\Omega_A M)$. Now apply Proposition 8.3.4(ii) to the sequence $0 \to \Omega_A M \to P \to M \to 0$. □

Next we give a realization result, namely that any closed homogeneous subvariety of $\mathrm{Max}(H)$ is the support variety of some A-module. This makes use of some bimodules attached to elements of H, analogous to Carlson's modules L_ζ in the group algebra case, and defined in [**EHS$^+$04**] for self-injective algebras.

Let $P_\bullet$ be a minimal resolution of A as an A^e-module. Let $\eta \in \mathrm{HH}^n(A)$, $n \geq 1$, so that η is represented by an element $\hat{\eta}$ of the space $\mathrm{Hom}_{A^e}(\Omega^n_{A^e}A, A)$. Define the A^e-module M_η to be the pushout (see Section A.1) of the inclusion $\Omega^n_{A^e}A \hookrightarrow P_{n-1}$ and $\hat{\eta} : \Omega^n_{A^e}A \to A$. By definition of pushout, there is a commuting diagram with both rows exact:

$$\begin{tikzcd}
0 \arrow[r] & \Omega^n_{A^e}A \arrow[r] \arrow[d, "\hat{\eta}"] & P_{n-1} \arrow[r] \arrow[d] & \Omega^{n-1}_{A^e}A \arrow[r] \arrow[d, "1"] & 0 \\
0 \arrow[r] & A \arrow[r] & M_\eta \arrow[r] & \Omega^{n-1}_{A^e}A \arrow[r] & 0
\end{tikzcd}$$

Note that M_η is projective as a right A-module and as a left A-module by its definition, since both A and $\Omega^{n-1}_{A^e}A$ have this property. We will frequently use the bottom row of the diagram, the short exact sequence

$$0 \to A \to M_\eta \to \Omega^{n-1}_{A^e}A \to 0. \tag{8.4.3}$$

In the following theorem, by $V_H(\eta)$, we mean the variety of the ideal (η) generated by η, that is, all maximal ideals containing η.

Theorem 8.4.4. *Let M be an A-module and $\eta \in \mathrm{HH}^n(A)$, $n \geq 1$. Then*

$$V_H(M_\eta \otimes_A M) = V_H(\eta) \cap V_H(M).$$

Proof. For the proof, we may assume that M is indecomposable so that it is a module for Ae_j for some primitive central idempotent e_j of A, and that $\eta \in \mathrm{HH}^*(e_jA)$. (See Section 1.2 for the decomposition of $\mathrm{HH}^*(A)$ into a direct sum of such components.)

By the definitions, a maximal ideal $\mathfrak{m}$ of H is in $V_H(M, N)$ if and only if $I_H(M, N) \subset \mathfrak{m}$ if and only if $\mathrm{Ext}^*_A(M, N)_\mathfrak{m} \neq 0$, where $\mathrm{Ext}^*_A(M, N)_\mathfrak{m}$ is the localization at $\mathfrak{m}$ of $\mathrm{Ext}^*_A(M, N)$.

By Proposition 8.3.4(i), $V_H(M_\eta \otimes_A M) = \cup_{S \in \mathrm{Irr}(A)} V_H(M_\eta \otimes_A M, S)$. We will first show that for each simple A-module S,

$$V_H(\eta) \cap V_H(M, S) \subset V_H(M_\eta \otimes_A M, S),$$

from which it will follow that $V_H(\eta) \cap V_H(M) \subset V_H(M_\eta \otimes_A M)$.

Let $\mathfrak{m}$ be a maximal ideal in $V_H(\eta) \cap V_H(M, S)$, that is $\mathfrak{m}$ contains (η) and $I_H(M, S)$. We want to show that $I_H(M_\eta \otimes_A M, S) \subset \mathfrak{m}$. Suppose this is not true. Then after localization, $\mathrm{Ext}^*_A(M_\eta \otimes_A M, S)_\mathfrak{m} = 0$. Apply $- \otimes_A M$

to the short exact sequence (8.4.3). Since each A^e-module in the sequence is free as a right A-module, we obtain a short exact sequence of A-modules,

$$0 \to M \to M_\eta \otimes_A M \to \Omega_{A^e}^{n-1} A \otimes_A M \to 0. \tag{8.4.5}$$

For each simple A-module S, apply $\mathrm{Ext}_A^*(-, S)$ and consider the corresponding second long exact sequence for Ext (Theorem A.4.5). By dimension shifting (Theorem A.3.3) and the observation that $\Omega_{A^e}^{n-1} A \otimes_A M$ and $\Omega_A^{n-1} M$ agree up to projective direct summands, it is:

$$\cdots \to \mathrm{Ext}_A^{i+n-1}(M, S) \xrightarrow{\phi} \mathrm{Ext}_A^i(M_\eta \otimes_A M, S) \longrightarrow \mathrm{Ext}_A^i(M, S) \xrightarrow{\tilde{\eta}} \mathrm{Ext}_A^{i+n}(M, S) \to \cdots \tag{8.4.6}$$

where $\tilde{\eta}$ is the action of η on $\mathrm{Ext}_A^*(M, S)$. Let $z \in \mathrm{Ext}_A^{i+n-1}(M, S)$, and consider $\phi(z) \in \mathrm{Ext}_A^i(M_\eta \otimes_A M, S)$. Since $\mathrm{Ext}_A^*(M_\eta \otimes_A M, S)_{\mathfrak{m}} = 0$ by assumption, there is a homogeneous element $a \notin \mathfrak{m}$ such that $\phi(az) = a\phi(z) = 0$. In light of the above long exact sequence, $az = \tilde{\eta}(y)$ for some $y \in \mathrm{Ext}_A^{|a|+i}(M, S)$. Upon localizing, we have $z = a^{-1}\tilde{\eta}(y)$, so

$$\mathrm{Ext}_A^*(M, S)_{\mathfrak{m}} = \tilde{\eta}(\mathrm{Ext}_A^*(M, S)_{\mathfrak{m}}).$$

Since $\eta \in \mathfrak{m}$ and $\mathrm{Ext}_A^*(M, S)$ is finitely generated over H, it now follows by Nakayama's Lemma that $\mathrm{Ext}_A^*(M, S)_{\mathfrak{m}} = 0$. This contradicts the assumption that $I_H(M, S) \subset \mathfrak{m}$. So $I_H(M_\eta \otimes_A M, S) \subset \mathfrak{m}$, and therefore $V_H(\eta) \cap V_H(M, S) \subset V_H(M_\eta \otimes_A M, S)$, as claimed. Thus $V_H(\eta) \cap V_H(M) \subset V_H(M_\eta \otimes_A M)$.

To prove the reverse inclusion $V_H(M_\eta \otimes_A M) \subset V_H(\eta) \cap V_H(M)$, we will first show that $V_H(M_\eta \otimes_A M) \subset V_H(\eta)$. By Proposition 8.3.4(i), it suffices to show that $V_H(S, M_\eta \otimes_A M) \subset V_H(\eta)$ for each simple A-module S. Equivalently, that $\mathrm{Ext}_A^*(S, M_\eta \otimes_A M)_{\mathfrak{m}} \neq 0$ for a maximal ideal $\mathfrak{m}$ of H implies $\eta \in \mathfrak{m}$. Suppose $\eta \notin \mathfrak{m}$. Then multiplication by η induces an isomorphism on $\mathrm{Ext}_A^*(S, M_\eta \otimes_A M)_{\mathfrak{m}}$ since it is invertible in $H_{\mathfrak{m}}$. As localization is exact, existence of the short exact sequence (8.4.3) implies that $\mathrm{Ext}_A^*(S, M_\eta \otimes_A M)_{\mathfrak{m}}$ is the kernel of the isomorphism $\eta : \mathrm{Ext}_A^*(S, M)_{\mathfrak{m}} \to \mathrm{Ext}_A^{*+n}(S, M)_{\mathfrak{m}}$. So $\mathrm{Ext}_A^*(S, M_\eta \otimes_A M)_{\mathfrak{m}} = 0$.

Finally we will show that $V_H(M_\eta \otimes_A M) \subset V_H(M)$. This is true by Proposition 8.3.4(ii) and Lemma 8.4.2 applied to the sequence (8.4.5) since $\Omega_{A^e}^n A \otimes_A M$ is the same as $\Omega_A^n M$ up to projective direct summands. □

We obtain the aforementioned realization result as a corollary.

Corollary 8.4.7. *For any homogeneous ideal I of H, there exists an A-module M such that $V_H(M) = V_H(I)$.*

Proof. Let I be a homogeneous ideal of H. Since H is noetherian, I is finitely generated by homogeneous elements, say $I = (\eta_1, \dots, \eta_r)$. Let

$$M = M_{\eta_1} \otimes_A \cdots \otimes_A M_{\eta_r} \otimes_A A/\mathfrak{r}.$$

Then $V_H(M) = V_H(I)$ by Theorem 8.4.4. $\square$

Exercise 8.4.8. Let $A = k[x]/(x^n)$. Find $\Omega_A k$, where k is an A-module on which x acts as 0. Letting H be as in Example 8.3.2, verify directly that $V_H(k) = V_H(\Omega_A k)$.

Exercise 8.4.9. In the long exact sequence (8.4.6), verify that $\tilde{\eta}$ is indeed the action of η on $\mathrm{Ext}^*_A(M, S)$ as claimed.

8.5. Self-injective algebras and indecomposable modules

We continue under the assumption that A is a finite dimensional self-injective algebra. A consequence of Theorem 8.5.6 below is that the support variety of an indecomposable module is connected. (More precisely, the corresponding projective variety is connected.) We will introduce periodic modules and see that the indecomposable periodic modules are those whose support varieties have dimension 1. These results appeared in [**EHS$^+$04**], based on techniques from support variety theory for finite groups in [**Car77, Car83, Car84, Eis80**].

We will need some lemmas. For each A-module M, let

$$M^{\#} = \mathrm{Hom}_A(M, A),$$

an A^{op}-module under the action $(af)(m) = f(am)$ for all $a \in A^{\mathrm{op}}$, $m \in M$, and $f \in \mathrm{Hom}_A(M, A)$. We may view $M^{\#}$ alternatively as a right A-module via $(fa)(m) = f(am)$, and consider the action of H on $\mathrm{Ext}^*_A(M^{\#}, M^{\#})$. Note that $(M^{\#})^{\#} \cong M$.

Lemma 8.5.1. *Let M be a finite dimensional A-module. Then*

$$V_H(M) = V_H(M^{\#}).$$

Proof. The proof uses properties of the duality $(\ )^{\#}$ and adjoint functors, similarly to the proof of Lemma 8.2.4; see [**SS04**, Proposition 3.6] for details. We give only an outline here. Let $P_{\bullet}$ be a minimal projective resolution of A as an A^e-module. Let $\eta \in I_H(M)$, so that $\eta \otimes 1_M : P_n \otimes_A M \to M$ factors through $d_n \otimes 1_M : P_n \otimes_A M \to P_{n-1} \otimes_A M$. Consider $1_{M^{\#}} \otimes_A \eta : M^{\#} \otimes_A P_n \to M^{\#}$. We wish to show that $1_{M^{\#}} \otimes_A \eta$ factors through $1_{M^{\#}} \otimes d_n$. This is a consequence of a sequence of isomorphisms for all i:

$$\begin{aligned}
\mathrm{Hom}_{A^{\mathrm{op}}}(M^{\#} \otimes_A P_i, M^{\#}) &\cong \mathrm{Hom}_A(M, \mathrm{Hom}_{A^{\mathrm{op}}}(M^{\#} \otimes_A P_i, A)) \\
&\cong \mathrm{Hom}_A(M, \mathrm{Hom}_A(P_i, M)) \\
&\cong \mathrm{Hom}_A(P_i \otimes_A M, M).
\end{aligned}$$

□

For the next lemma, recall the definition of the modules M_η in (8.4.3).

Lemma 8.5.2. *Let η be a homogeneous element of H of positive degree n and let M be a finite dimensional A-module. Then $\eta \in I_H(M)$ if and only if*

$$M_\eta \otimes_A M \cong M \oplus \Omega_A^{n-1}(M) \oplus Q$$

for some projective A-module Q.

Proof. We may assume M is nonprojective, as the lemma is automatically true in case M is projective. Let $\eta \in H^n$, represented by an element of $\mathrm{Hom}_{A^e}(\Omega_{A^e}^n A, A) \cong \mathrm{Ext}_{A^e}^1(\Omega_{A^e}^{n-1} A, A)$, as in Theorem A.3.3. Note that $\eta \in I_H(M)$ if and only if the sequence (8.4.5) splits, that is, if and only if

$$M_\eta \otimes_A M \cong M \oplus (\Omega_{A^e}^{n-1} A \otimes_A M).$$

Since $\Omega_{A^e}^{n-1} A \otimes_A M$ and $\Omega_A^{n-1} M$ agree up to projective direct summands, the statement follows. □

As a consequence of the lemma, we next show a connection between the intersection of support varieties of two modules and their generalized extensions.

Lemma 8.5.3. *Let M, N be finite dimensional A-modules. If*

$$\dim(V_H(M) \cap V_H(N)) = 0,$$

then $\mathrm{Ext}_A^i(M, N) = 0$ for all $i > 0$.

Proof. Since H is noetherian, $I_H(M)$ is finitely generated by homogeneous elements. Suppose $I_H(M) = (\eta_1, \dots, \eta_r)$. By Lemma 8.5.2 and induction on r, up to projective direct summands, the A-module M is a direct summand of

$$M_{\eta_1} \otimes_A \cdots \otimes_A M_{\eta_r} \otimes_A M.$$

So $\mathrm{Ext}_A^i(M, N)$ is a direct summand of

$$\mathrm{Ext}_A^i(M_{\eta_1} \otimes_A \cdots \otimes_A M_{\eta_r} \otimes_A M, N),$$

and this Ext space is isomorphic to

$$\mathrm{Ext}_A^i(M, (M_{\eta_1} \otimes_A \cdots \otimes_A M_{\eta_r})^\# \otimes_A N)$$

by an argument similar to the proof of Lemma 8.2.4. Also,

$$\begin{aligned} &\mathrm{Ext}_A^i(A/\mathfrak{r}, (M_{\eta_1} \otimes_A \cdots \otimes_A M_{\eta_r})^\# \otimes_A N) \\ &\cong \mathrm{Ext}_A^i(M_{\eta_1} \otimes_A \cdots \otimes_A M_{\eta_r} \otimes_A A/\mathfrak{r}, N). \end{aligned}$$

So $V_H((M_{\eta_1} \otimes_A \cdots \otimes_A M_{\eta_r})^\# \otimes_A N)$ is contained in the intersection of the varieties $V_H(N)$ and $V_H((M_{\eta_1} \otimes_A \cdots \otimes_A M_{\eta_r}) \otimes_A A/\mathfrak{r})$, by Lemma 8.3.3 and

Proposition 8.3.4(iii). The latter is contained in $V_H(M)$ by Theorem 8.4.4, since $I_H(M) = (\eta_1, \ldots, \eta_r)$, so by hypothesis, the variety of the A-module $(M_{\eta_1} \otimes_A \cdots \otimes_A M_{\eta_r})^{\#} \otimes_A N$ has dimension 0. It follows that the module is projective by Lemma 8.4.1, and therefore it is injective, and so

$$\mathrm{Ext}^i_A(M, (M_{\eta_1} \otimes_A \cdots \otimes_A M_{\eta_r})^{\#} \otimes_A N) = 0$$

for all $i > 0$. The same is then true of the direct summand $\mathrm{Ext}^i_A(M, N)$. $\square$

We need one more lemma before stating the main theorem of this section. The lemma is on existence of a short exact sequence corresponding to two homogeneous elements of H.

Lemma 8.5.4. *Let $\eta_1 \in H^m$ and $\eta_2 \in H^n$. There is a projective A^e-module P for which there is a short exact sequence*

$$0 \to \Omega^n_{A^e}(M_{\eta_1}) \to M_{\eta_2\eta_1} \oplus P \to M_{\eta_2} \to 0.$$

Proof. Starting with the short exact sequence (8.4.3) with n replaced by m and η by η_1, there are projective A^e-modules Q, Q' for which there is an exact sequence

$$0 \to \Omega^1_{A^e}(M_{\eta_1}) \oplus Q' \to \Omega^m_{A^e} A \oplus Q \to A \to 0, \tag{8.5.5}$$

where the map $\Omega^m_{A^e} A \to A$ is $\hat{\eta}_1$. Now $\widehat{\eta_2\eta_1}$ is the composition $\hat{\eta}_2(\hat{\eta}_1)_n$, where $(\hat{\eta}_1)_n$ may be viewed as a map from $\Omega^{m+n}_{A^e} A$ to $\Omega^m_{A^e} A$. Therefore, there is a commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "1"] & M_{\eta_2\eta_1} \arrow[r] \arrow[d] & \Omega^{m+n-1}_{A^e} A \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] & M_{\eta_2} \arrow[r] & \Omega^{n-1}_{A^e} A \arrow[r] & 0
\end{tikzcd}$$

Now apply the functor $\Omega^{n-1}_{A^e}$ to the sequence (8.5.5) to obtain an exact sequence

$$0 \to \Omega^n_{A^e} M_{\eta_1} \oplus Q' \to \Omega^{m+n-1}_{A^e} A \oplus Q \to \Omega^{n-1}_{A^e} A \to 0.$$

Apply the Horseshoe Lemma (Lemma A.4.3) to this sequence to see that there is a projective module P such that the map

$$\Omega^{m+n-1}_{A^e} A \oplus P \to \Omega^{n-1}_{A^e} A$$

is surjective, with kernel $\Omega^n_{A^e} A$. Now take $M_{\eta_2\eta_1} \oplus P$ in the middle column to obtain the desired sequence. $\square$

We are now ready to state the main theorem of this section.

Theorem 8.5.6. *Let M be a finite dimensional A-module for which $V_H(M) = V_1 \cup V_2$ for some homogeneous varieties V_1 and V_2 with $\dim(V_1 \cap V_2) = 0$. Then there are A-modules M_1 and M_2 with $V_H(M_1) = V_1$ and $V_H(M_2) = V_2$ and*

$$M \cong M_1 \oplus M_2.$$

A consequence of the theorem is that the projective variety of an indecomposable module is connected, where the projective variety of H is the space of lines through the origin in $\mathrm{Max}(H)$ and that of a module M is lines through the origin in $\mathrm{Max}(H/I_H(M))$. This makes sense as $I_H(M)$ is a homogeneous ideal by its definition.

Proof. Let $m_1 = \dim V_1$ and $m_2 = \dim V_2$. We will prove the statement by induction on $m_1 + m_2$. If $m_1 = 0$ or $m_2 = 0$, the result is clear, so assume $m_1 > 0$ and $m_2 > 0$. Then there are homogeneous elements η_1 and η_2 of H such that $V_1 \subset V_H(\eta_1)$, $V_2 \subset V_H(\eta_2)$, and

$$\dim(V_2 \cap V_H(\eta_1)) = m_2 - 1, \quad \dim(V_1 \cap V_H(\eta_2)) = m_1 - 1.$$

Now $V_H(\eta_2\eta_1) = V_H(\eta_1) \cup V_H(\eta_2) \supset V_1 \cup V_2 = V_H(M)$, so $(\eta_2\eta_1)^s \in I_H(M)$ for some s. We may assume $\eta_2\eta_1 \in I_H(M)$ by replacing each η_i with η_i^s if $s > 1$. By Lemma 8.5.2,

$$M_{\eta_2\eta_1} \otimes_A M \cong M \oplus \Omega_A^{n-1} M \oplus Q$$

for some projective A-module Q. By Lemma 8.5.4, there is a short exact sequence

$$0 \to \Omega^n_{A^e} M_{\eta_1} \to M_{\eta_2\eta_1} \oplus P \to M_{\eta_2} \to 0$$

for some projective A^e-module P. Apply $- \otimes_A M$ to this sequence to obtain

$$0 \to \Omega^n_{A^e} M_{\eta_1} \otimes_A M \to (M_{\eta_2\eta_1} \otimes_A M) \oplus (P \otimes_A M) \to M_{\eta_2} \otimes_A M \to 0.$$

Replacing $M_{\eta_2\eta_1} \otimes_A M$ via the above isomorphism, we have a short exact sequence

$$\begin{aligned} 0 \to \Omega^n_{A^e} M_{\eta_1} \otimes_A M \to & M \oplus \Omega_A^{n-1} M \oplus Q \oplus (P \otimes_A M) \\ & \to M_{\eta_2} \otimes_A M \to 0. \end{aligned} \tag{8.5.7}$$

By Theorem 8.4.4,

$$\begin{aligned} V_H(M_{\eta_2} \otimes_A M) &= V_H(\eta_2) \cap V_H(M) \\ &= V_H(\eta_2) \cap (V_1 \cup V_2) \\ &= (V_H(\eta_2) \cap V_1) \cup V_2, \end{aligned}$$

as $V_2 \subset V_H(\eta_2)$. The intersection of the varieties $V_H(\eta_2) \cap V_1$ and V_2 is contained in $V_1 \cap V_2$, and thus has dimension 0. Further, the sum of the dimensions of these two varieties is $m_1 - 1 + m_2$. By the induction hypothesis,

$\Omega^n_{A^e} M_{\eta_1} \otimes_A M \cong N_1 \oplus N_2$ for A-modules N_1, N_2 with $V_H(N_1) = V_H(\eta_2) \cap V_1$ and $V_H(N_2) = V_2$. We also find that by Theorem 8.4.4 similarly,

$$V_H(\Omega^n_{A^e} M_{\eta_1} \otimes_A M) = V_H(\eta_1) \cap V_H(M) = V_1 \cup (V_H(\eta_1) \cap V_2),$$

and so $\Omega^n_{A^e} M_{\eta_1} \otimes_A M \cong N_1' \oplus N_2'$ for A-modules N_1', N_2' with $V_H(N_1') = V_1$ and $V_H(N_2') = V_H(\eta_1) \cap V_2$. Thus the sequence (8.5.7) may be rewritten

$$0 \to N_1 \oplus N_2 \to X \to N_1' \oplus N_2' \to 0 \tag{8.5.8}$$

for $X = M \oplus \Omega^{n-1}_A M \oplus Q \oplus (P \otimes_A M)$.

Now $V_H(N_1') \cap V_H(N_2) = V_1 \cap V_2$, which has dimension 0, and so by Lemma 8.5.3, $\mathrm{Ext}^i_A(N_1', N_2) = 0$ for all $i > 0$. Similarly, $V_H(N_1) \cap V_H(N_2') \subset V_1 \cap V_2$, and so has dimension 0, so $\mathrm{Ext}^i_A(N_2', N_1) = 0$ for all $i > 0$. Then

$$\mathrm{Ext}^1_A(N_1' \oplus N_2', N_1 \oplus N_2) \cong \mathrm{Ext}^1_A(N_1', N_1) \oplus \mathrm{Ext}^1_A(N_2', N_2)$$

and so (8.5.8) is a direct sum of two sequences $0 \to N_1 \to N_1'' \to N_1' \to 0$ and $0 \to N_2 \to N_2'' \to N_2' \to 0$. Moreover, $V_H(N_1'') \subset V_1$ and $V_H(N_2'') \subset V_2$. Rewriting X as a direct sum:

$$M \oplus \Omega^{n-1}_A M \oplus Q \oplus (P \otimes_A M) \cong N_1'' \oplus N_2''.$$

Now $P \otimes_A M$ is a projective A-module, so

$$M \oplus \Omega^{n-1}_A M \oplus Q'' \cong N_1'' \oplus N_2''$$

for some projective A-module Q''. By the Krull-Schmidt Theorem, $M \cong M_1 \oplus M_2$ for some M_1 and M_2 with $V_H(M_1) \subset V_H(N_1'') \subset V_1$ and $V_H(M_2) \subset V_H(N_2'') \subset V_2$. By hypothesis, $V_H(M) = V_1 \cup V_2$, and this forces $V_H(M_1) = V_1$ and $V_H(M_2) = V_2$. □

As a final topic, we consider periodic indecomposable modules.

Definition 8.5.9. An A-module M is *periodic* if $\Omega^i_A M \cong M$ as A-modules for some i.

Example 8.5.10. The trivial module k for $A = k[x]/(x^n)$ is periodic, as can be seen from our work in Example 2.5.10. If $n = 2$, then the period is $i = 1$, and if $n > 2$, the period is $i = 2$.

The following result characterizes indecomposable periodic modules.

Theorem 8.5.11. *Let A be a finite dimensional self-injective algebra satisfying condition (fg). Let M be a finite dimensional indecomposable A-module. Then M is periodic if and only if* $\dim V_H(M) = 1$.

Proof. Assume M is periodic, so that $\Omega^n_A M \cong M$ for some n. Let $\zeta \in \mathrm{Ext}^n_A(M, M)$ corresponding to an isomorphism $\hat{\zeta} : \Omega^n_A(M) \xrightarrow{\sim} M$. Let $\eta \in \mathrm{Ext}^r_A(M, M)$ be any homogeneous element and $\hat{\eta} : \Omega^r_A(M) \to M$ be the corresponding map. Since $\Omega^{nr}_A M \cong M$, the function $\hat{\eta}^n : \Omega^{nr}_A M \to M$

can be identified with an element of $\mathrm{Hom}_A(M,M)$, a local ring since M is indecomposable. Accordingly, $\hat{\eta}^n$ is either an isomorphism or is nilpotent, and moreover if it is an isomorphism, it must be a sum of a scalar multiple of $\hat{\zeta}^r$ and a nilpotent element. It follows that $\mathrm{Ext}^*_A(M,M)$ is a direct sum of $k[\zeta]$ and a nilpotent ideal. Thus $\dim(V_H(M)) = 1$.

Conversely, assume $\dim(V_H(M)) = 1$. Then $H/I_H(M)$ has Krull dimension 1. By hypothesis, for all simple A-modules S, $\mathrm{Ext}^*_A(M,S)$ is finitely generated as a module over $H/I_H(M)$, and so is finitely generated over a polynomial subring $k[\zeta]$ of H, for ζ a homogeneous element of some positive degree n. By the classification of finitely generated modules over a principal ideal domain, action by ζ is an isomorphism

$$\mathrm{Ext}^i_A(M,S) \xrightarrow{\sim} \mathrm{Ext}^{i+n}_A(M,S)$$

for sufficiently large i and all simple A-modules S. Let $P_{\bullet}$ be a minimal projective resolution of the A-module M. Then $\mathrm{Ext}^i_A(M,S) \cong \mathrm{Hom}_A(P_i,S)$ for all i, and so there is an isomorphism

$$\mathrm{Hom}_A(P_i,S) \xrightarrow{\sim} \mathrm{Hom}_A(P_{i+n},S)$$

for all i sufficiently large and all simple A-modules S. These homomorphism spaces uniquely determine the projective A-modules P_i, and so $P_i \cong P_{i+n}$ for all i sufficiently large, which further implies that $\Omega^i_A M \cong \Omega^{i+n}_A M$. Choosing just one such value of i, and applying Ω^{-i}_A to this isomorphism, we have $M \cong \Omega^n_A M$. Thus M is periodic. $\square$

There are many further applications of support varieties. For example, all modules in a connected stable component of the Auslander-Reiten quiver have the same variety [**SS04**, Theorem 3.7]. There is a connection to representation type through complexity of modules [**EHS$^+$04**, Section 5]. For algebras A whose category of modules is a tensor category, such as when A is a Hopf algebra, in the best known settings the structure of the tensor category given by its tensor ideals can be understood using support variety theory.

Exercise 8.5.12. Let $A = k[x]/(x^n)$. Let $M = k[x]/(x^m)$ for some integer m ($2 \le m < n$), considered to be an A-module via a quotient map and multiplication. (The case $m = 1$ corresponds to the trivial module k.) Show that M is periodic by examining its minimal projective resolution, in accordance with Theorem 8.5.11.

Exercise 8.5.13. Suppose $\mathrm{char}(k) = 2$ and $A = k[x,y]/(x^2,y^2)$. Let k be an A-module on which x and y both act as 0. Let $H = \mathrm{HH}^*(A)$.

(a) Show that k is not periodic by showing that $\dim V_H(k) = 2$ and appealing to Theorem 8.5.11.

(b) Show that k is not periodic by examining its minimal projective resolution.

Chapter 9

Hopf Algebras

Hopf algebras are an important class of algebras that include group algebras, universal enveloping algebras of Lie algebras, and quantum groups. They act on rings, giving rise to extension rings such as smash and crossed product algebras and Hopf Galois extensions. In this chapter, we collect some techniques for understanding homological information about Hopf algebras and related ring extensions. We introduce the Hopf algebra cohomology ring and prove that it embeds into the Hochschild cohomology ring. Such embeddings have many applications, for example: They aid understanding of support varieties of modules, as defined in Chapter 8. They lead to better understanding of cup products in Hochschild cohomology by way of cup products in Hopf algebra cohomology, for example the finite group algebras that we examine in this chapter. They provide tools for constructing spectral sequences, for example a spectral sequence relating the Hochschild cohomology of a smash product to cohomology of its components as we present here.

9.1. Hopf algebras and actions on rings

In this section we give a brief introduction to Hopf algebras and smash product algebras. For details and more general notions, see [**Mon93, Rad12, Sch06**].

Let A be an algebra over the field k. Let $\pi : A \otimes A \to A$ denote its multiplication map and let $\eta : k \to A$ denote its unit map (that is, the embedding of the field k into A as scalar multiples of the multiplicative identity). We consider $A \otimes A$ to be an algebra with factorwise multiplication:

$(a \otimes b)(c \otimes d) = ac \otimes bd$ for all $a, b, c, d \in A$. In the following definition, we canonically identify the spaces $k \otimes A$ and $A \otimes k$ with A.

Definition 9.1.1. A *Hopf algebra* is an algebra A over k together with algebra homomorphisms $\Delta : A \to A \otimes A$ (called *coproduct* or *comultiplication*) and $\varepsilon : A \to k$ (called *counit* or *augmentation*) and an algebra anti-homomorphism (that is, reversing order of multiplication) $S : A \to A$ (called *coinverse* or *antipode*) such that

$$\begin{aligned} (\Delta \otimes 1)\Delta &= (1 \otimes \Delta)\Delta, \\ (\varepsilon \otimes 1)\Delta &= 1 = (1 \otimes \varepsilon)\Delta, \\ \pi(S \otimes 1)\Delta &= \eta\varepsilon = \pi(1 \otimes S)\Delta, \end{aligned}$$

as maps on $A \otimes A$, where 1 denotes the identity map on A. The first equation above is called *coassociativity* and we may view the last as stating that S is the convolution inverse of the identity map. We say that A is *cocommutative* if $\tau\Delta = \Delta$, where $\tau : A \otimes A \to A \otimes A$ is the map that interchanges tensor factors, that is $\tau(a \otimes b) = b \otimes a$ for all $a, b \in A$.

It can be shown that the antipode S is a coalgebra anti-homomorphism, that is, $\Delta S = (S \otimes S)\tau\Delta$ and $\varepsilon S = \varepsilon$. (See, for example, [**Rad12**, Theorem 7.1.14].)

Some examples of Hopf algebras are the following.

Example 9.1.2. (Group algebras.) Let G be a group and $A = kG$, its group algebra. Let $\Delta(g) = g \otimes g$, $\varepsilon(g) = 1$, and $S(g) = g^{-1}$ for all $g \in G$. Then A is a cocommutative Hopf algebra.

Example 9.1.3. (Universal enveloping algebras of Lie algebras.) Let $\mathfrak{g}$ be a Lie algebra and let $A = U(\mathfrak{g})$, its universal enveloping algebra. That is,

$$U(\mathfrak{g}) = T(\mathfrak{g})/(x \otimes y - y \otimes x - [x, y] \mid x, y \in \mathfrak{g}),$$

where $T(\mathfrak{g})$ denotes the tensor algebra of the underlying vector space of $\mathfrak{g}$. Let $\Delta(x) = x \otimes 1 + 1 \otimes x$, $\varepsilon(x) = 0$, and $S(x) = -x$ for all $x \in \mathfrak{g}$. The maps Δ and ε are extended to be algebra homomorphisms, and S to be an algebra anti-homomorphism. Then A is a cocommutative Hopf algebra.

Example 9.1.4. (Quantum enveloping algebras.) Let $A = U_q(\mathfrak{g})$, a quantum enveloping algebra. See, for example, [**GK93**] for the definition in the general case. Here we give just one small example explicitly: Let q be a nonzero scalar, $q^2 \neq 1$. Let $U_q(\mathfrak{sl}_2)$ be the k-algebra generated by E, F, K with $KE = q^2 EK$, $KF = q^{-2} FK$, and

$$EF = FE + \frac{K - K^{-1}}{q - q^{-1}}.$$

Let $\Delta(E) = E \otimes 1 + K \otimes E$, $\Delta(F) = F \otimes K^{-1} + 1 \otimes F$, $\Delta(K) = K \otimes K$, $\varepsilon(E) = 0$, $\varepsilon(F) = 0$, $\varepsilon(K) = 1$, $S(E) = -K^{-1}E$, $S(F) = -FK$, and $S(K) = K^{-1}$. Then A is a noncocommutative Hopf algebra. If q is a primitive nth root of unity, $n > 2$, we may consider the quotient of $U_q(\mathfrak{sl}_2)$ by the ideal generated by E^n, F^n, $K^n - 1$, denoted by $u_q(\mathfrak{sl}_2)$. This quotient is a finite dimensional (that is, finite dimensional as a vector space) noncocommutative Hopf algebra, called a small quantum group.

Example 9.1.5. (Quantum elementary abelian groups.) Let m and n be positive integers, $n \geq 2$. Let q be a primitive nth root of unity, and let A be the k-algebra generated by $x_1, \ldots, x_m, g_1, \ldots, g_m$ with relations $x_i^n = 0$, $g_i^n = 1$, $x_i x_j = x_j x_i$, $g_i g_j = g_j g_i$, and $g_i x_j = q^{\delta_{i,j}} x_j g_i$ for all i, j. Let $\Delta(x_i) = x_i \otimes 1 + g_i \otimes x_i$, $\Delta(g_i) = g_i \otimes g_i$, $\varepsilon(x_i) = 0$, $\varepsilon(g_i) = 1$, and $S(x_i) = -g_i^{-1} x_i$, $S(g_i) = g_i^{-1}$ for all i. Then A is a finite dimensional noncocommutative Hopf algebra.

We will use some standard notation for the coproduct, called *Sweedler notation*: Write

$$\Delta(a) = \sum_{(a)} a_1 \otimes a_2,$$

or simply $\sum a_1 \otimes a_2$, where the notation a_1, a_2 for tensor factors is symbolic. Some authors dispense with the summation symbol, writing $a_1 \otimes a_2$ to denote this sum. Similarly write $\Delta(1 \otimes \Delta)(a) = \sum a_1 \otimes a_2 \otimes a_3$, and note this is the same as $\Delta(\Delta \otimes 1)(a)$ by coassociativity.

If A is a finite dimensional Hopf algebra, then $A^* = \mathrm{Hom}_k(A, k)$ is also a Hopf algebra under the duals of the defining maps of A. That is, identifying $(A \otimes A)^*$ canonically with $A^* \otimes A^*$, multiplication on A^* is Δ^*, comultiplication is π^*, the unit map is ε^*, the counit map is η^*, and the antipode is S^*. If A is an infinite dimensional Hopf algebra, there are meaningful dual Hopf algebras such as the finite dual (see [**Mon93**]). If A and B are Hopf algebras, then $A \otimes B$ is a Hopf algebra with factorwise product, coproduct, and other maps, and A^{op} is a Hopf algebra with the same coproduct, counit, and antipode as A. In this way, $A^e = A \otimes A^{\mathrm{op}}$ is also a Hopf algebra.

An element h of a Hopf algebra A is a *left integral* of A if $a \cdot h = \varepsilon(a)h$ for all $a \in A$. A right integral is defined similarly. It can be shown that the spaces of left and right integrals of a finite dimensional Hopf algebra are one dimensional and are interchanged by the antipode. Maschke's Theorem for Hopf algebras states that a finite dimensional Hopf algebra A is semisimple if, and only if, $\varepsilon(h) \neq 0$ for a nonzero left (respectively, right) integral h.

A finite dimensional Hopf algebra is a Frobenius algebra, and therefore self-injective (see [**Mon93**]). Specifically, a nonzero left integral λ in the dual Hopf algebra A^* is a Frobenius homomorphism of A: A nondegenerate associative bilinear form on A is given by $(a,b) \mapsto \lambda(ab)$ for all $a, b \in A$. The left A-module A is isomorphic to the left A-module given by the dual of the right A-module A, via the map that takes a to ϕ_a where $\phi_a(b) = \lambda(ba)$ for all $a, b \in A$.

The quantum elementary abelian groups in Example 9.1.5 are examples of skew group rings as defined in Section 3.6. We generalize skew group rings by defining smash products next.

Definition 9.1.6. Let A be a Hopf algebra. An A-*module algebra* is an algebra R over k that is also an A-module for which

$$\begin{aligned} a \cdot 1 &= \varepsilon(a)1, \\ a \cdot (rr') &= \sum (a_1 \cdot r)(a_2 \cdot r') \end{aligned}$$

for all $a \in A$ and $r, r' \in R$. The *smash product* $R\#A$ of A with R is an algebra that is $R \otimes A$ as a vector space, with multiplication defined by

$$(r \otimes a)(r' \otimes a') = \sum r(a_1 \cdot r') \otimes a_2 a'$$

for all $a, a' \in A$ and $r, r' \in R$.

By definition of multiplication on the smash product algebra, R and A are both subalgebras of $R\#A$. So it should cause no confusion if we sometimes abuse notation and write r for $r \otimes 1$ and a for $1 \otimes a$ as elements of $R\#A$, for $r \in R$ and $a \in A$.

A more general construction than the smash product is a crossed product involving cocycles, and both are examples of yet more general Hopf Galois extensions. We will not use these more general notions in this book.

We rewrite the action of A on R as an internal action within the smash product $R\#A$: For each $a \in A$ and $r \in R$, by the properties of Hopf algebras and definition of multiplication in the smash product,

$$\begin{aligned} (a \cdot r) \otimes 1 &= \sum ((a_1 \varepsilon(a_2)) \cdot r) \otimes 1 &&= \sum a_1 \cdot r \otimes \varepsilon(a_2) \\ & &&= \sum a_1 \cdot r \otimes a_2 S(a_3) \\ & &&= \sum (1 \otimes a_1)(r \otimes 1)(1 \otimes S(a_2)). \end{aligned}$$

We call this an *adjoint action* of A, by analogy to the adjoint action of a Lie algebra.

The quantum elementary abelian groups of Example 9.1.5 are smash products of $R = k[x_1, \dots, x_m]/(x_1^n, \dots, x_m^n)$ with $A = kG$ where $G = \langle g_1, \dots, g_m \rangle$ is a direct product of cyclic groups $\langle g_i \rangle$, each of order n, acting by $g_i \cdot x_j = q^{\delta_{ij}} x_j$. Skew group algebras (Section 3.6) are all smash product algebras. For an example in which the Hopf algebra is not a group algebra, take $A = U_q(\mathfrak{sl}_2)$ from Example 9.1.4, $R = k\langle x, y \rangle/(xy - qyx)$, and $E \cdot x = 0$,

$E \cdot y = x$, $F \cdot x = y$, $F \cdot y = 0$, $K^{\pm 1} \cdot u = q^{\pm 1} u$, $K^{\pm 1} \cdot v = q^{\mp 1} v$. These actions extend to an action of A on R under which R is an A-module algebra, and we may form the smash product algebra $A \# R$.

Exercise 9.1.7. Let G be a finite group and $A = kG$. What is the structure of the dual Hopf algebra A^*? Describe multiplication, unit, comultiplication, counit, and antipode in terms of the basis dual to G.

Exercise 9.1.8. Let G be a finite group and $A = kG$. Show that $\sum_{g \in G} g$ is a left integral, and that any left integral is a scalar multiple of this one.

Exercise 9.1.9. Verify that R and A are indeed both subalgebras of the smash product algebra $R \# A$.

Exercise 9.1.10. Let G be a group. Show that a kG-module algebra R is simply an algebra R with action of G by algebra automorphisms, and that the smash product $R \# kG$ given by Definition 9.1.6 is precisely the skew group algebra defined in Section 3.6.

Exercise 9.1.11. Verify that the quantum plane $R = k\langle x, y \rangle / (xy - qyx)$ is indeed a $U_q(\mathfrak{sl}_2)$-module algebra under the action given above.

9.2. Modules for Hopf algebras

Modules for Hopf algebras have some important properties: Tensor products (over k) of modules are modules, and Homs (over k) of modules are modules. These structures make categories of A-modules into tensor categories [**EGNO15**]. In this section, we define these module structures and relationships among them.

We work primarily with left modules as before, and the term module will mean left module if not otherwise specified. However, we will also need right modules in Section 9.6, and we include some discussion in this section of the properties of right modules that we will need there.

Given a Hopf algebra A, the field k is an A-module via the counit ε, sometimes called the *trivial module*, that is, $a \cdot c = \varepsilon(a)c$ for all $a \in A$, $c \in k$. For any two A-modules V and W, their vector space tensor product $V \otimes W$ is an A-module via the coproduct Δ:

$$a \cdot (v \otimes w) = \sum (a_1 \cdot v) \otimes (a_2 \cdot w) \tag{9.2.1}$$

for all $a \in A$, $v \in V$, and $w \in W$. Similarly, $\mathrm{Hom}_k(V, W)$ is an A-module via

$$(a \cdot f)(v) = \sum a_1 f(S(a_2)v)$$

for all $a \in A$, $f \in \mathrm{Hom}_k(V, W)$, and $v \in V$. In particular, if V is an A-module, its dual vector space $V^* = \mathrm{Hom}_k(V, k)$ has an A-module structure given by

$$(a \cdot f)(v) = f(S(a)v)$$

for all $a \in A$, $v \in V$, and $f \in V^*$. Note that the tensor product of the trivial module k with V is isomorphic to V: $k \otimes V \cong V$ and $V \otimes k \cong V$ due to the properties of the maps Δ, ε.

For any A-module V, let V^A denote the A-submodule of V on which A acts via ε, that is,

$$V^A = \{v \in V \mid a \cdot v = \varepsilon(a)v \text{ for all } a \in A\},$$

called the submodule of *A-invariants* of V. We will use the same notation for A-invariants of a right A-module.

Lemma 9.2.2. *Let V, W be A-modules. There is an isomorphism of vector spaces*

$$\mathrm{Hom}_A(V, W) \cong (\mathrm{Hom}_k(V, W))^A.$$

Proof. This is [**Sch06**, Lemma 4.1]. Let $f \in \mathrm{Hom}_A(V, W)$. Then

$$(a \cdot f)(v) \;=\; \sum a_1 f(S(a_2)v) \;=\; \sum a_1 S(a_2) f(v) \;=\; \varepsilon(a) f(v)$$

for all $a \in A$, $v \in V$. So $f \in (\mathrm{Hom}_k(V, W))^A$. Conversely, let $f \in (\mathrm{Hom}_k(V, W))^A$. Then

$$\begin{aligned} f(a \cdot v) \;=\; \sum f(\varepsilon(a_1)a_2 \cdot v) &= \sum \varepsilon(a_1) f(a_2 \cdot v) \\ &= \sum a_1 f(S(a_2) \cdot (a_3 \cdot v)) \\ &= \sum a_1 f(\varepsilon(a_2)v) \\ &= \sum a_1 \varepsilon(a_2) f(v) \;=\; a \cdot (f(v)) \end{aligned}$$

for all $a \in A$, $v \in V$. So $f \in \mathrm{Hom}_A(V, W)$. □

For a Hopf algebra having a bijective antipode, there is an alternative action on Hom, as we describe in the next remark. All finite dimensional Hopf algebras have bijective antipodes, as well as many infinite dimensional Hopf algebras.

Remark 9.2.3. Under the assumption that the antipode S is bijective with inverse map S^{-1}, we may alternatively define a dual module to V as ${}^*V = \mathrm{Hom}_k(V, k)$ with action $(a \cdot f)(v) = f(S^{-1}(a)v)$ for all $a \in A$, $v \in V$, and $f \in {}^*V$. Similarly give $\mathrm{Hom}_k(V, W)$ the alternative A-module structure $(a \cdot f)(v) = \sum a_2 f(S^{-1}(a_1)v)$ for all $a \in A$, $f \in \mathrm{Hom}_k(V, W)$, $v \in V$. Lemma 9.2.2 holds under this alternative action, that is, the subspace of A-homomorphisms in $\mathrm{Hom}_k(V, W)$ is also equal to the A-invariant subspace of $\mathrm{Hom}_k(V, W)$ under this alternative action. To see this, apply S

to $\sum a_2 S^{-1}(a_1)$ to obtain $\sum a_1 S(a_2) = \varepsilon(a)$, implying that $\sum a_2 S^{-1}(a_1) = \varepsilon(a)$ for all $a \in A$.

There are right module versions of all of these actions as well:

Remark 9.2.4. If V, W are right A-modules, then $\mathrm{Hom}_k(V,W)$ is a right A-module via $(f \cdot a)(v) = \sum f(vS(a_1))a_2$ for all $f \in \mathrm{Hom}_A(V,W)$, $a \in A$, $v \in V$. There is a right module version of Lemma 9.2.2. If S is bijective, then $\mathrm{Hom}_k(V,W)$ is also a right A-module via $(f \cdot a)(v) = \sum f(vS^{-1}(a_2))a_1$ for all $f \in \mathrm{Hom}_k(V,W)$, $a \in A$, $v \in V$, and again there is a corresponding version of Lemma 9.2.2.

We next establish relations among the A-modules obtained by taking Hom, $\otimes$, and duals.

Lemma 9.2.5. *Let U, V, and W be A-modules. There is a natural isomorphism of A-modules*

$$\mathrm{Hom}_k(U \otimes V, W) \cong \mathrm{Hom}_k(U, \mathrm{Hom}_k(V,W)),$$

and a natural isomorphism of vector spaces

$$\mathrm{Hom}_A(U \otimes V, W) \cong \mathrm{Hom}_A(U, \mathrm{Hom}_k(V,W)).$$

Proof. Define functions $\phi : \mathrm{Hom}_k(U \otimes V, W) \to \mathrm{Hom}_k(U, \mathrm{Hom}_k(V,W))$ by

$$(\phi(f)(u))(v) = f(u \otimes v),$$

and $\psi : \mathrm{Hom}_k(U, \mathrm{Hom}_k(V,W)) \to \mathrm{Hom}_k(U \otimes V, W)$ by

$$(\psi(g))(u \otimes v) = (g(u))(v).$$

By its definition, ψ is inverse to ϕ. We check that ϕ is an A-module homomorphism. Let $a \in A$ and $f \in \mathrm{Hom}_k(U \otimes V, W)$. Then, as S reverses the order of comultiplication, for all $u \in U$ and $v \in V$,

$$\begin{aligned} (\phi(a \cdot f)(u))(v) &= (a \cdot f)(u \otimes v) \\ &= \sum a_1(f(S(a_2) \cdot (u \otimes v))) \\ &= \sum a_1(f(S(a_3)u \otimes S(a_2)v)). \end{aligned}$$

On the other hand,

$$\begin{aligned} (a \cdot \phi(f))(u)(v) &= \sum (a_1(\phi(f)(S(a_2)u)))(v) \\ &= \sum a_1((\phi(f))(S(a_3)u)(S(a_2)v) \\ &= \sum a_1(f(S(a_3)u \otimes S(a_2)v)). \end{aligned}$$

Therefore $\phi(a \cdot f) = a \cdot \phi(f)$. These isomorphisms are natural by their definitions.

The second statement now follows by Lemma 9.2.2. $\square$

Remark 9.2.6. A similar proof shows that if U, V, W are right A-modules, then there is a (slightly different) right A-module isomorphism

$$\mathrm{Hom}_k(U \otimes V, W) \cong \mathrm{Hom}_k(V, \mathrm{Hom}_k(U, W))$$

under the first action defined in Remark 9.2.4. Taking A-invariants, we obtain an isomorphism

$$\mathrm{Hom}_A(U \otimes V, W) \cong \mathrm{Hom}_A(V, \mathrm{Hom}_k(U, W)).$$

The next lemma gives a relationship between Hom and dual.

Lemma 9.2.7. *Let V, W be A-modules. If V is finite dimensional as a vector space over k, there is an A-module isomorphism*

$$\mathrm{Hom}_k(V, W) \cong W \otimes V^*.$$

Proof. Let $\phi : W \otimes V^* \to \mathrm{Hom}_k(V, W)$ and $\psi : \mathrm{Hom}_k(V, W) \to W \otimes V^*$ be defined by $(\phi(w \otimes f))(v) = f(v)w$ and $\psi(f) = \sum_i f(v_i) \otimes v_i^*$, where $\{v_i\}, \{v_i^*\}$ are dual bases for V, V^*, for all $v \in V$, $w \in W$, and $f \in V^*$. Let $a \in A$. Then

$$\begin{aligned} \phi(a \cdot (w \otimes f))(v) &= \sum \phi(a_1 w \otimes (a_2 \cdot f))(v) \\ &= \sum ((a_2 \cdot f)(v))(a_1 w) \;\; = \;\; \sum f(S(a_2)v) a_1 w. \end{aligned}$$

On the other hand,

$$\begin{aligned} (a \cdot (\phi(w \otimes f)))(v) &= \sum a_1(\phi(w \otimes f)(S(a_2)v)) \\ &= \sum a_1(f(S(a_2)v)w) \;\; = \;\; \sum f(S(a_2)v) a_1 w. \end{aligned}$$

Therefore $\phi(a \cdot (w \otimes f)) = a \cdot (\phi(w \otimes f))$. A calculation shows that ϕ is inverse to ψ. $\square$

Remark 9.2.8. Under the alternative A-module structure described in Remark 9.2.3, there are (slightly different) A-module isomorphisms

$$\mathrm{Hom}_k(V, W) \cong {}^*V \otimes W$$

and

$$\mathrm{Hom}_k(U \otimes V, W) \cong \mathrm{Hom}_k(V, \mathrm{Hom}_k(U, W)).$$

Now consider only finite dimensional A-modules. This observation combined with Lemma 9.2.7 implies that V^* is a *left dual* of V in the category of finite dimensional A-modules and ${}^*V$ is a *right dual* of V in this category (notation and terminology as in [**EGNO15**]). It follows that ${}^*(V^*) \cong V$ and $({}^*V)^* \cong V$ for all finite dimensional A-modules V; these isomorphisms can also be deduced directly from the definitions. We caution that the terms left and right dual are interchanged in comparison to [**BK01**].

Lemma 9.2.9. *Let P be a projective A-module, and V any A-module. Then $P \otimes V$ is a projective A-module. If the antipode S is bijective, then $V \otimes P$ is a projective A-module. Similar statements hold for right modules, as well as when "projective" is replaced by "flat".*

Proof. We give two proofs of the first two statements. The first proof is essentially that of [**Ben91a**, Proposition 3.1.5]: The projective module P is a direct summand of a free module, so it suffices to prove that $A \otimes V$ and $V \otimes A$ are both free as A-modules. There is an isomorphism $A \otimes V \xrightarrow{\sim} A \otimes V_{tr}$, where V_{tr} is the underlying vector space of V, but with the trivial A-module structure (via ε). This isomorphism is similar to one in [**Mon93**, Theorem 1.9.4], and is given by $a \otimes v \mapsto \sum a_1 \otimes S(a_2)v$, the inverse function by $a \otimes v \mapsto \sum a_1 \otimes a_2 v$, for all $v \in V$, $a \in A$. Now V_{tr} is a direct sum of copies of the trivial module k, and so $A \otimes V_{tr}$ is a free A-module. Similarly, there is an isomorphism of left A-modules, $V \otimes A \xrightarrow{\sim} V_{tr} \otimes A$, via the A-module homomorphism $v \otimes a \mapsto \sum \overline{S}(a_1)v \otimes a_2$ whose inverse is $v \otimes a \mapsto \sum a_1 v \otimes a_2$, and so $V \otimes A$ is a free A-module.

The second proof uses properties of functors: As V is projective over the field k and P is projective over A, $\mathrm{Hom}_A(P, \mathrm{Hom}_k(V, -))$ is an exact functor. By Lemma 9.2.5, this is the same as $\mathrm{Hom}_A(P \otimes V, -)$. Therefore $P \otimes V$ is projective. A similar argument applies to $V \otimes P$, using Remark 9.2.8 and the alternative A-action on Hom given in Remark 9.2.3.

The last statement, for right modules, may be proven similarly, and the statement about flat modules follows since flat modules are direct limits of finitely generated free modules. □

Exercise 9.2.10. Let A be a Hopf algebra. Verify that $k \otimes V \cong V$ and $V \otimes k \cong V$ for all A-modules V, where k is the trivial A-module.

Exercise 9.2.11. Verify the claim in Remark 9.2.3 that the subspace of A-homomorphisms in $\mathrm{Hom}_k(V, W)$ is its A-invariant subspace under the alternative action defined there.

Exercise 9.2.12. Prove the two isomorphisms stated in Remark 9.2.6.

Exercise 9.2.13. Prove the two isomorphisms stated in Remark 9.2.8.

Exercise 9.2.14. Verify the isomorphism $A \otimes V \cong A \otimes V_{tr}$ in the proof of Lemma 9.2.9.

Exercise 9.2.15. Formulate the last statement of Lemma 9.2.9 more precisely and prove it.

9.3. Hopf algebra cohomology and actions on Ext

In this section we define Hopf algebra cohomology and derive many properties of Ext spaces for modules of a Hopf algebra A. We will connect these ideas with Hochschild cohomology in the next section.

Lemma 9.3.1. *Let U, V, W be A-modules. There is an isomorphism of graded vector spaces,*

$$\mathrm{Ext}^*_A(U \otimes V, W) \cong \mathrm{Ext}^*_A(U, W \otimes V^*).$$

Proof. Let $P_\bullet$ be a projective resolution of the A-module U. By Lemma 9.2.9, $P_i \otimes V$ is a projective A-module for all i. Since the tensor product is over the field k, the functor $- \otimes V$ is exact, and so $P_\bullet \otimes V$ is a projective resolution of $U \otimes V$. The natural isomorphisms of Lemmas 9.2.5 and 9.2.7 yield a chain homotopy equivalence between $\mathrm{Hom}_A(P_\bullet, W \otimes V^*)$ and $\mathrm{Hom}_A(P_\bullet \otimes V, W)$, and thus the claimed isomorphism on Ext spaces. $\square$

Remark 9.3.2. Using the alternative A-action on Hom described in Remark 9.2.3, we similarly find that

$$\mathrm{Ext}^*_A(U \otimes V, W) \cong \mathrm{Ext}^*_A(V, {}^*U \otimes W).$$

Now let M, M', N, N' be A-modules. We will next define a *cup product* for each $i, j \geq 0$,

$$\smile : \mathrm{Ext}^i_A(M, M') \times \mathrm{Ext}^j_A(N, N') \longrightarrow \mathrm{Ext}^{i+j}_A(M \otimes N, M' \otimes N'). \tag{9.3.3}$$

Let $P_\bullet$ be a projective resolution of M, and let $Q_\bullet$ be a projective resolution of N. Consider the total complex of the tensor product complex $P_\bullet \otimes Q_\bullet$, with action of A on each $P_i \otimes Q_j$ given by the coproduct Δ on A as in (9.2.1). Note that the differentials on the total complex of $P_\bullet \otimes Q_\bullet$ are A-module homomorphisms since the differentials on $P_\bullet$ and on $Q_\bullet$ are A-module homomorphisms. By Lemma 9.2.9, each module in this tensor product complex is projective. By the Künneth Theorem (Theorem A.5.1), since the tensor product is over the field k and Tor^k_1 is 0, the total complex of $P_\bullet \otimes Q_\bullet$ is a projective resolution of the A-module $M \otimes N$.

Let $f \in \mathrm{Hom}_A(P_i, M')$, $g \in \mathrm{Hom}_A(Q_j, N')$ represent elements in the spaces $\mathrm{Ext}^i_A(M, M')$, $\mathrm{Ext}^j_A(N, N')$, respectively. Then

$$f \otimes g \in \mathrm{Hom}_A(P_i \otimes Q_j, M' \otimes N')$$

and this function may be extended to an element of

$$\mathrm{Hom}_A\left(\bigoplus_{r+s=i+j} (P_r \otimes Q_s), M' \otimes N'\right)$$

by defining it to be the zero map on all components other than $P_i \otimes Q_j$. By definition, $d(f \otimes g) = d(f) \otimes g + (-1)^{|f|} f \otimes d(g)$. It follows that if f and g are cocycles, then their tensor product $f \otimes g$ is again a cocycle, and the tensor product of a cocycle with a coboundary is a coboundary. Therefore this tensor product of functions induces a well-defined product on cohomology, namely the cup product denoted $\smile$.

We will need the following result giving an equivalent definition of this cup product via a Yoneda composition, paralleling equivalent definitions of cup product on Hochschild cohomology in Section 2.4. See also [**Ben91a**, Proposition 3.2.1].

Lemma 9.3.4. *If M, M', N, N' are left A-modules and $\alpha \in \mathrm{Ext}_A^m(M, M')$, $\beta \in \mathrm{Ext}_A^n(N, N')$, then the cup product*

$$\alpha \smile \beta \in \mathrm{Ext}_A^{m+n}(M \otimes N, M' \otimes N')$$

of (9.3.3) is equal to the Yoneda composite of

$$\alpha \otimes 1_{N'} \quad \text{and} \quad 1_M \otimes \beta$$

in $\mathrm{Ext}_A^m(M \otimes N', M' \otimes N')$ and $\mathrm{Ext}_A^n(M \otimes N, M \otimes N')$, respectively.

Proof. Let

$$\mathbf{f}: \qquad 0 \longrightarrow M' \longrightarrow M_{m-1} \longrightarrow \cdots M_0 \longrightarrow M \longrightarrow 0,$$

$$\mathbf{g}: \qquad 0 \longrightarrow N' \longrightarrow N_{n-1} \longrightarrow \cdots N_0 \longrightarrow N \longrightarrow 0$$

be an m- and an n-extension representing α and β, respectively (see Section A.3). By the Künneth Theorem (Theorem A.5.1), taking their tensor product over the field k results in an $(m+n)$-extension of $M \otimes N$ by $M' \otimes N'$:

$$\begin{array}{ccccccccc}
M_0 \otimes N' & \longleftarrow & M_1 \otimes N' & \longleftarrow & \cdots \longleftarrow & M_{m-1} \otimes N' & \longleftarrow & M' \otimes N' \\
\downarrow & & \downarrow & & & \downarrow & & \downarrow \\
M_0 \otimes N_{n-1} & \longleftarrow & M_1 \otimes N_{n-1} & \longleftarrow & \cdots \longleftarrow & M_{m-1} \otimes N_{n-1} & \longleftarrow & M' \otimes N_{n-1} \\
\downarrow & & \downarrow & & & \downarrow & & \downarrow \\
\vdots & & \vdots & & & \vdots & & \vdots \\
\downarrow & & \downarrow & & & \downarrow & & \downarrow \\
M_0 \otimes N_1 & \longleftarrow & M_1 \otimes N_1 & \longleftarrow & \cdots \longleftarrow & M_{m-1} \otimes N_1 & \longleftarrow & M' \otimes N_1 \\
\downarrow & & \downarrow & & & \downarrow & & \downarrow \\
M_0 \otimes N_0 & \longleftarrow & M_1 \otimes N_0 & \longleftarrow & \cdots \longleftarrow & M_{m-1} \otimes N_0 & \longleftarrow & M' \otimes N_0
\end{array}$$

That is, the cup product $\alpha \smile \beta$ is represented by the total complex of this tensor product complex, augmented by $M \otimes N$, denoted $\mathbf{f} \otimes \mathbf{g}$. This can be deduced from examination of the relationship between generalized extensions and elements of Ext as in Section A.3. Just as in Sections 2.4 and 6.5, there is a map from the total complex of this bicomplex to the Yoneda composite of $\mathbf{f} \otimes N'$ with $M \otimes \mathbf{g}$: Project the bicomplex onto the leftmost column followed by the map $M_0 \to M$ and onto the top row. Specifically, in degrees $i+j$ with $0 \le i+j \le n-1$, send $M_i \otimes N_j$ to 0 if $i > 0$ and to $M \otimes N_j$ as a projection from $M_0 \otimes N_j$ if $i = 0$. In degrees $i+j$ with $n \le i+j \le m+n-1$, send $M_i \otimes N_j$ to 0 if $j < n$ and to $M_{i+j-n} \otimes N'$ if $j = n$. It can be checked that this is a chain map. □

Recall the trivial module for A is k with action given by the counit ε. Set $M = M' = N = N' = k$ in Lemma 9.3.4 and identify $k \otimes k$ with k in the following definition.

Definition 9.3.5. The *Hopf algebra cohomology ring* of the Hopf algebra A over the field k is

$$\mathrm{H}^*(A, k) = \mathrm{Ext}^*_A(k, k)$$

under cup product. More generally, we write $\mathrm{H}^*(A, M) = \mathrm{Ext}^*_A(k, M)$ for any A-module M, a graded vector space that is an $\mathrm{H}^*(A, k)$-module under cup product.

We could equally well have defined Hopf algebra cohomology via right A-modules, and we will use this right module version in Section 9.6. Left and right modules are interchanged by applying the antipode, since it is an algebra anti-homomorphism.

By Lemma 9.2.2 and the definitions, the degree 0 Hopf algebra cohomology is

$$\mathrm{H}^0(A, M) \cong \mathrm{Hom}_A(k, M) \cong (\mathrm{Hom}_k(k, M))^A \cong M^A.$$

Remark 9.3.6. Another proof of Lemma 9.3.4 uses a product given by Yoneda composition the Eckmann-Hilton argument, which simultaneously shows that the product is graded commutative, that is

$$\alpha \smile \beta = (-1)^{|\alpha||\beta|} \beta \smile \alpha$$

for all $\alpha, \beta \in \mathrm{H}^*(A, k)$. See Suarez-Alvarez [**SA04**] for a general context for this type of argument. More generally, when $M = N = k$ and $M' = N' = B$ is an A-module algebra, we may compose the cup product with the map induced by multiplication $B \otimes B \to B$ to obtain a ring structure on $\mathrm{H}^*(A, B)$. In the next section, we will let B be the algebra A itself, under the adjoint action of A, as defined there.

Example 9.3.7. Let G be a group and let $A = kG$, the group algebra. Then $\mathrm{H}^*(kG, k) = \mathrm{Ext}^*_{kG}(k, k)$ is the group cohomology of G with coefficients in k, also written $\mathrm{H}^*(G, k)$, and the cup product gives it the structure of a graded commutative ring. As a small example, let p be a prime, let k be a field of characteristic p, and $G = \mathbb{Z}/p\mathbb{Z}$. Then the group cohomology of G with coefficients in k is

$$\mathrm{H}^*(G, k) \cong \begin{cases} k[y], & \text{if } p = 2 \\ k[y, z]/(y^2), & \text{if } p > 2 \end{cases}$$

where $|y| = 1$, $|z| = 2$. (Note that $kG \cong k[x]/(x^p)$ since $\mathrm{char}(k) = p$, which can be seen by taking $x = g - 1$. Then see Example 2.5.10.)

Let $M = M' = k$ and $N' = N$ in Lemma 9.3.4. By composing with the isomorphism $k \otimes N \cong N$, we obtain an action of $\mathrm{H}^*(A, k)$ on $\mathrm{Ext}^*_A(N, N)$, via $- \otimes N$ followed by Yoneda composition. On the other hand, we have an action of $\mathrm{H}^*(A, k)$ on $\mathrm{Ext}^*_A(k, N \otimes N^*)$ by Yoneda composition.

In the following statement, we apply Lemma 9.3.1 with $U = k$, $V = N$:

Theorem 9.3.8. *Let N be a left A-module. The action of $\mathrm{H}^*(A, k)$ on $\mathrm{Ext}^*_A(N, N)$, given by $- \otimes N$ followed by Yoneda composition, corresponds to that on $\mathrm{Ext}^*_A(k, N \otimes N^*)$, given by Yoneda composition, under the isomorphism*

$$\mathrm{Ext}^*_A(N, N) \cong \mathrm{Ext}^*_A(k, N \otimes N^*).$$

Proof. Let $P_{\bullet}$ be a projective resolution of k, so that $P_{\bullet} \otimes N$ is a projective resolution of $k \otimes N \cong N$. We must check that the following diagram commutes for each m, n, where ϕ_m, ϕ_{m+n} are the isomorphisms given by Lemma 9.2.5 with $V = N$ and $U = P_m, P_{m+n}$, respectively, and the horizontal maps are the chain level maps corresponding to the cup product of Lemma 9.3.4.

$$\begin{tikzcd}
\mathrm{Hom}_A(P_m, k) \otimes \mathrm{Hom}_A(P_n \otimes N, N) \arrow[r] \arrow[d, "1 \otimes \phi_n"] & \mathrm{Hom}_A(P_{m+n} \otimes N, N) \arrow[d, "\phi_{m+n}"] \\
\mathrm{Hom}_A(P_m, k) \otimes \mathrm{Hom}_A(P_n, \mathrm{Hom}_k(N, N)) \arrow[r] & \mathrm{Hom}_A(P_{m+n}, \mathrm{Hom}_k(N, N))
\end{tikzcd}$$

Let $\zeta \in \mathrm{Ext}^m_A(k, k)$ and $\eta \in \mathrm{Ext}^n_A(N, N)$, represented by $f \in \mathrm{Hom}_A(P_m, k)$ and $g \in \mathrm{Hom}_A(P_n \otimes N, N)$, respectively. Identify f with the corresponding function from an mth syzygy module $\Omega^m(k)$ to k, and extend to a chain map $f_{\bullet}$ with $f_i \in \mathrm{Hom}_A(P_{m+i}, P_i)$. The top horizontal map takes $f \otimes g$ to $g(f_n \otimes 1)$, and applying ϕ_{m+n} we have

$$\phi_{m+n}(g(f_n \otimes 1))(x)(v) = g(f_n(x) \otimes v)$$

for all $x \in P_{m+n}$, $v \in N$. On the other hand, $(1 \otimes \phi_n)(f \otimes g) = f \otimes \phi_n(g)$, and applying the bottom horizontal map we find

$$(\phi_n(g) f_n)(x)(v) = g(f_n(x) \otimes v).$$

Therefore the diagram commutes. □

There is another action of $\mathrm{H}^*(A,k)$ on $\mathrm{Ext}^*_A(M,M)$, given by $M \otimes -$ followed by Yoneda composition. In case A is cocommutative, this action is the same as that given by $- \otimes M$. (More generally, this is true if A is *quasitriangular*, that is there are functorial isomorphisms $M \otimes N \cong N \otimes M$ for all A-modules M, N.) In general these actions will not be the same; see, for example, [**BW14**]. We state next a counterpart of Theorem 9.3.8 for this action under the assumption that the antipode S is bijective, and include a proof to highlight this subtle distinction. Let $\mathrm{Hom}'_k(V,W)$ denote the A-module that is $\mathrm{Hom}_k(V,W)$ as a vector space, but with action as described in Remark 9.2.3. Let ${}^*V = \mathrm{Hom}_k(V,k)$, with A-module structure as described in Remark 9.2.3. Then, by Remark 9.2.8, there are isomorphisms of A-modules:

$$\mathrm{Hom}'_k(U \otimes V, W) \cong \mathrm{Hom}'_k(V, \mathrm{Hom}'_k(U,W)) \cong \mathrm{Hom}'_k(V, {}^*U \otimes W).$$

It follows that

$$\mathrm{Hom}'_A(U \otimes V, W) \cong \mathrm{Hom}'_A(V, {}^*U \otimes W), \tag{9.3.9}$$

and consequently

$$\mathrm{Ext}^*_A(U \otimes V, W) \cong \mathrm{Ext}^*_A(V, {}^*U \otimes W).$$

Theorem 9.3.10. *Assume the antipode S of A is bijective and let M be an A-module. The action of $\mathrm{H}^*(A,k)$ on $\mathrm{Ext}^*_A(M,M)$, given by $M \otimes -$ followed by Yoneda composition, corresponds to that on $\mathrm{Ext}^*_A(k, {}^*M \otimes M)$, given by Yoneda composition, under the isomorphism*

$$\mathrm{Ext}^*_A(M,M) \cong \mathrm{Ext}^*_A(k, {}^*M \otimes M).$$

Proof. Let $P_\bullet$ be a projective resolution of k, so that $M \otimes P_\bullet$ is a projective resolution of $M \otimes k \cong M$. We must check that the following diagram commutes for each m, n, where ϕ_m, ϕ_{m+n} are the isomorphisms given in (9.3.9), with $U = M$ and $V = P_n, P_{m+n}$, respectively, and the horizontal maps are the chain level maps corresponding to the cup product of Lemma 9.3.4.

$$\begin{tikzcd}
\mathrm{Hom}_A(P_m,k) \otimes \mathrm{Hom}_A(M \otimes P_n, M) \arrow[r] \arrow[d, "1 \otimes \phi_n"] & \mathrm{Hom}_A(M \otimes P_{m+n}, M) \arrow[d, "\phi_{m+n}"] \\
\mathrm{Hom}_A(P_m,k) \otimes \mathrm{Hom}_A(P_n, \mathrm{Hom}'_k(M,M)) \arrow[r] & \mathrm{Hom}_A(P_{m+n}, \mathrm{Hom}'_k(M,M))
\end{tikzcd}$$

Let $\zeta \in \mathrm{Ext}^m_A(k,k)$ and $\eta \in \mathrm{Ext}^n_A(M,M)$, represented by $f \in \mathrm{Hom}_A(P_m,k)$ and $g \in \mathrm{Hom}_A(M \otimes P_n, M)$, respectively. Identify f with the corresponding function from an mth syzygy module $\Omega^m(k)$ to k, and extend to a chain map $f_\bullet$ with $f_i \in \mathrm{Hom}_A(P_{m+i},P_i)$. The top horizontal map takes $f \otimes g$ to $g(1_M \otimes f_n)$, and applying ϕ_{m+n} we have

$$\phi_{m+n}(g(1_M \otimes f_n))(x)(v) = g(v \otimes f_n(x))$$

for all $x \in P_{m+n}$, $v \in M$. On the other hand, $(1 \otimes \phi_n)(f \otimes g) = f \otimes \phi_n(g)$, and applying the bottom horizontal map we find

$$(\phi_n(g)f_n)(x)(v) = g(v \otimes f_n(x)).$$

Therefore the diagram commutes. □

As noted in Remark 9.2.8, for any A-modules M, N, there are A-module isomorphisms ${}^*(N^*) \cong N$ and $({}^*M)^* \cong M$. This provides further relationships among various actions: Set $M = N^*$, so that ${}^*M \cong N$. Applying the isomorphisms in the statements of Theorems 9.3.8 and 9.3.10, we see that

$$\mathrm{Ext}^*_A(N,N) \cong \mathrm{Ext}^*_A(k, N \otimes M) \cong \mathrm{Ext}^*_A(M,M).$$

In this way, the second described action in each part of Theorem 9.3.11 below makes sense.

Theorem 9.3.11. *Assume the antipode S of A is bijective.*

(i) *Let N be a finite dimensional A-module. The action of* $\mathrm{H}^*(A,k)$ *on* $\mathrm{Ext}^*_A(N,N)$*, given by $- \otimes N$ followed by Yoneda composition, corresponds to the action given by $N^* \otimes -$ followed by Yoneda composition.*

(ii) *Let M be a finite dimensional A-module. The action of* $\mathrm{H}^*(A,k)$ *on* $\mathrm{Ext}^*_A(M,M)$*, given by $M \otimes -$ followed by Yoneda composition, corresponds to the action given by $- \otimes {}^*M$ followed by Yoneda composition.*

Proof. For (i), let $M = N^*$, and so ${}^*M \cong N$, as noted above. Apply Theorems 9.3.8 and 9.3.10. The proof of (ii) is similar. □

Exercise 9.3.12. Verify details in the proof of Lemma 9.3.1, in particular that there is a chain homotopy equivalence between $\mathrm{Hom}_A(P_\bullet, W \otimes V^*)$ and $\mathrm{Hom}_A(P_\bullet \otimes V, W)$.

Exercise 9.3.13. Verify details in the proof of Lemma 9.3.4, in particular that the cup product $\alpha \smile \beta$ is represented by the $(m+n)$-extension $\mathbf{f} \otimes \mathbf{g}$.

Exercise 9.3.14. Let A be a Hopf algebra and let M, N be A-modules. Show that there is a well-defined action of $\mathrm{H}^*(A,k)$ on $\mathrm{Ext}^*_A(M,N)$ given by $M \otimes -$ followed by Yoneda composition. Similarly show that there is an action given by $N \otimes -$ followed by Yoneda composition. Are these two actions the same? Compare with actions that begin with $- \otimes M$ or $- \otimes N$.

9.4. Bimodules and Hochschild cohomology

In this section, we describe connections among Hochshild cohomology, Hopf algebra cohomology, and their actions on Ext spaces. We begin by finding some relationships among the Hopf algebras A and A^e and their modules. Some of these arise from the following embedding of A as a subalgebra of A^e.

Lemma 9.4.1. *Let $\delta : A \to A^e$ be the function defined by*

$$\delta(a) = \sum a_1 \otimes S(a_2)$$

for all $a \in A$. Then δ is an injective algebra homomorphism.

Proof. First note that $\delta(1) = 1 \otimes 1$, the identity in A^e. Let $a, b \in A$. Since S is an algebra anti-homomorphism,

$$\begin{aligned}
\delta(ab) &= \sum a_1 b_1 \otimes S(a_2 b_2) \\
&= \sum a_1 b_1 \otimes S(b_2) S(a_2) \\
&= (\sum a_1 \otimes S(a_2))(\sum b_1 \otimes S(b_2)) = \delta(a)\delta(b),
\end{aligned}$$

since multiplication in the second factor is opposite that in A.

To see that δ is injective, compose with the k-linear function $\psi : A^e \to A$ defined by $\psi(a \otimes b) = a\varepsilon(b)$. We have, for all $a \in A$,

$$\psi\delta(a) = \psi(\sum a_1 \otimes S(a_2)) = \sum a_1 \varepsilon(S(a_2)) = \sum a_1 \varepsilon(a_2) = a,$$

that is $\psi\delta$ is the identity map on A. This implies that δ is injective. □

We will identify A with the subalgebra $\delta(A)$ of A^e. This will allow us to induce modules from A to A^e, using tensor products: Let M be an A-module, and consider A^e to be a right A-module via right multiplication by elements of $\delta(A)$. Then the vector space $A^e \otimes_A M$ is an A^e-module, the action given by left multiplication on the factor A^e. We next determine the A^e-module induced from the trivial A-module k in this way.

Lemma 9.4.2. *There is an isomorphism of A^e-modules*

$$A \cong A^e \otimes_A k,$$

where $A^e \otimes_A k$ is the A^e-module induced from the trivial A-module k via the embedding of A into A^e given by the map δ of Lemma 9.4.1.

Proof. Let $f : A \to A^e \otimes_A k$ be the function defined by $f(a) = a \otimes 1 \otimes 1$, and let $g : A^e \otimes_A k \to A$ be the function defined by $g(a \otimes b \otimes 1) = ab$ for all $a, b \in A$. We will check that f is an A^e-module homomorphism, with inverse function g.

Let $a, b, c \in A$. Then, since $c = \sum c_1 \varepsilon(c_2)$, we have

$$\begin{aligned} f((b \otimes c)(a)) &= f(bac) = bac \otimes 1 \otimes 1 \\ &= \sum bac_1 \otimes \varepsilon(c_2) \otimes 1 \\ &= \sum bac_1 \otimes S(c_2)c_3 \otimes 1. \end{aligned}$$

Now identifying A with $\delta(A) \subset A^e$, since the rightmost factor is in k with action of A given by ε, and the tensor product is over A, we may rewrite this as

$$\begin{aligned} \sum ba \otimes c_2 \otimes \varepsilon(c_1) &= \sum ba \otimes \varepsilon(c_1)c_2 \otimes 1 \\ &= ba \otimes c \otimes 1 \\ &= (b \otimes c)(a \otimes 1 \otimes 1) = (b \otimes c)f(a). \end{aligned}$$

Therefore f is an A^e-module homomorphism.

Now let $a, b \in A$. We have

$$\begin{aligned} gf(a) &= g(a \otimes 1 \otimes 1) = a, \\ \text{and } fg(a \otimes b \otimes 1) &= f(ab) = ab \otimes 1 \otimes 1 \\ &= \sum ab_1\varepsilon(b_2) \otimes 1 \otimes 1 \\ &= \sum ab_1 \otimes \varepsilon(b_2) \otimes 1 \\ &= \sum ab_1 \otimes S(b_2)b_3 \otimes 1 \\ &= \sum a \otimes b_2 \otimes \varepsilon(b_1) \\ &= \sum a \otimes \varepsilon(b_1)b_2 \otimes 1 = a \otimes b \otimes 1. \end{aligned}$$

Therefore f and g are inverse functions. □

Remark 9.4.3. There is also a right module version: Let $\delta' : A \to A^{\mathrm{op}} \otimes A$ be the function defined by

$$\delta'(a) = \sum S(a_1) \otimes a_2 \tag{9.4.4}$$

for all $a \in A$. Then δ' is an injective algebra homomorphism, by a similar proof to that of Lemma 9.4.1. There is an isomorphism of right $A^{\mathrm{op}} \otimes A$-modules (equivalently, left A^e-modules)

$$A \cong k \otimes_{\delta'(A)} (A^{\mathrm{op}} \otimes A),$$

by a proof similar to that of Lemma 9.4.2.

We will consider A to be an A-module under the left adjoint action, that is for all $a, b \in A$,

$$a \cdot b = \sum a_1 b S(a_2).$$

Denote this A-module by A^{ad}. More generally, if M is any A-bimodule, denote by M^{ad} the A-module with action given by $a \cdot m = \sum a_1 m S(a_2)$ for all $a \in A$, $m \in M$.

The following theorem is due to Ginzburg and Kumar [**GK93**]; our proof is from [**PW09, SW99**].

Theorem 9.4.5. *Let A be a Hopf algebra over k with bijective antipode S. There is an isomorphism of algebras*

$$\mathrm{HH}^*(A) \cong \mathrm{H}^*(A, A^{ad}).$$

Proof. By Lemma 9.2.9, A^e is a projective right A-module. Thus we may apply the Eckmann-Shapiro Lemma (Lemma A.6.2) with $B = A^e$, $M = k$, and $N = A$ and Lemma 9.4.2 to obtain an isomorphism of vector spaces,

$$\mathrm{Ext}^*_{A^e}(A, A) \cong \mathrm{Ext}^*_A(k, A^{ad}).$$

It remains to prove that cup products are preserved by this isomorphism. This follows from the proof of [**SW99**, Proposition 3.1], valid more generally in this context, as we explain next.

Let $P_{\bullet}$ denote a projective resolution of k as an A-module. Let $X_{\bullet} = A^e \otimes_A P_{\bullet}$, a projective resolution of $A^e \otimes_A k \cong A$ as an A^e-module, since A^e is flat as a right module over $A \cong \delta(A)$.

There is a chain map $\iota : P_{\bullet} \to X_{\bullet}$ of A-modules defined by $\iota(p) = (1 \otimes 1) \otimes p$ for all $p \in P_i$. Let $f \in \mathrm{Hom}_{A^e}(X_i, A)$ be a cocycle representing a cohomology class in $\mathrm{Ext}^*_{A^e}(A, A)$. The corresponding cohomology class in $\mathrm{Ext}^*_A(k, A^{ad})$ is represented by $f \circ \iota$.

Since $k \otimes k \cong k$, the Künneth Theorem (Theorem A.5.1) implies that $P_{\bullet} \otimes P_{\bullet}$ is also a projective resolution of k as an A-module. By the Comparison Theorem (Theorem A.2.7), there is a chain map $D : P_{\bullet} \to P_{\bullet} \otimes P_{\bullet}$ of A-modules lifting the identity map on k. Note that D induces an isomorphism on cohomology. It also induces a chain map $D' : X_{\bullet} \to X_{\bullet} \otimes_A X_{\bullet}$ as follows. There is a map of chain complexes $\theta : A^e \otimes_A (P_{\bullet} \otimes P_{\bullet}) \to X_{\bullet} \otimes_A X_{\bullet}$ of A^e-modules, given by

$$\theta((a \otimes b) \otimes (p \otimes q)) = ((a \otimes 1) \otimes p) \otimes ((1 \otimes b) \otimes q).$$

Take the map from $A^e \otimes_A P_{\bullet}$ to $A^e \otimes_A (P_{\bullet} \otimes P_{\bullet})$ induced by D. Let D' be the composition of this map with θ. Again D' is unique up to homotopy.

Now let $f \in \mathrm{Hom}_{A^e}(X_i, A)$, $g \in \mathrm{Hom}_{A^e}(X_j, A)$ be cocycles. The above observations imply that the following diagram commutes:

$$\begin{array}{ccccccc} X_{\bullet} & \xrightarrow{D'} & X_{\bullet} \otimes_A X_{\bullet} & \xrightarrow{f\otimes g} & A \otimes_A A & \xrightarrow{\sim} & A \\ \iota \uparrow & & & & & & \| \\ P_{\bullet} & \xrightarrow{D} & P_{\bullet} \otimes P_{\bullet} & \xrightarrow{f\iota\otimes g\iota} & A \otimes A & \xrightarrow{\pi} & A \end{array}$$

where π is multiplication. The top row yields the product in $\mathrm{Ext}^*_{A^e}(A, A)$ and the bottom row yields the product in $\mathrm{Ext}^*_A(k, A^{ad})$. $\square$

The following consequence is due to Linckelmann [**Lin00**].

Corollary 9.4.6. *Let A be a commutative Hopf algebra. There is an isomorphism of algebras*

$$\mathrm{HH}^*(A) \cong A \otimes \mathrm{H}^*(A, k).$$

Proof. Since A is commutative, it acts trivially on the A-module A^{ad}, that is, $(A^{ad})^A = A^{ad}$. By the Universal Coefficients Theorem (Theorem A.5.3), Theorem 9.4.5, and analysis of the cup products, the statement holds. $\square$

Another consequence of Theorem 9.4.5 is the following.

Corollary 9.4.7. *Let $I = \mathrm{Ker}(\varepsilon)$, the augmentation ideal of the Hopf algebra A. Then as an algebra,*

$$\mathrm{HH}^*(A) \cong \mathrm{H}^*(A, k) \oplus \mathrm{H}^*(A, I^{ad}),$$

a direct sum of the subalgebra $\mathrm{H}^*(A, k)$ and the ideal $\mathrm{H}^*(A, I^{ad})$.

We may thus view $\mathrm{H}^*(A, k)$ as both a quotient and a subalgebra of $\mathrm{HH}^*(A)$.

Proof. Under the left adjoint action of A on itself, the trivial module k is isomorphic to the submodule of A^{ad} given by all scalar multiples of the identity 1. In fact k is a direct summand of A^{ad}, its complement being $I = \mathrm{Ker}(\varepsilon)$. As $\mathrm{Ext}^*_A(k, -)$ is additive, the result follows. $\square$

Remark 9.4.8. There is a Tor version that is easier: $\mathrm{HH}_i(A) \cong \mathrm{H}_i(A, A^{ad})$ as abelian groups. It follows that $\mathrm{H}_i(A, k)$ is a direct summand of $\mathrm{HH}_i(A)$.

There is a connection between the actions of Hopf algebra cohomology and of Hochschild cohomology on Ext spaces, as noted in [**PW09**]:

Proposition 9.4.9. *The following diagram commutes:*

$$\begin{tikzcd} \mathrm{H}^*(A,k) \arrow[d, "A^e\otimes_A -"'] \arrow[dr, "-\otimes M"] & \\ \mathrm{HH}^*(A) \arrow[r, "-\otimes_A M"'] & \mathrm{Ext}^*_A(M,M) \end{tikzcd}$$

Thus the action of $\mathrm{H}^*(A,k)$ *on* $\mathrm{Ext}^*_A(M,M)$*, given by* $-\otimes M$ *followed by Yoneda composition, factors through the action of* $\mathrm{HH}^*(A)$ *on* $\mathrm{Ext}^*_A(M,M)$*.*

Proof. Let $0 \to k \to P_{n-1} \to \cdots \to P_0 \to k \to 0$ be an n-extension of A-modules. For any A-module X, consider the function

$$f_X : (A^e \otimes_A X) \otimes_A M \to X \otimes M$$

given by

$$f_X(a \otimes b \otimes x \otimes m) = \sum a_1 x \otimes a_2 bm$$

for all $a, b \in A$, $x \in X$, $m \in M$. By construction, the action of A on $(A^e \otimes_A X) \otimes_A M$ is on the leftmost factor of A only. It can be checked that f_X is a functorial A-module isomorphism, and thus the actions are equivalent as stated. $\square$

Remark 9.4.10. One consequence of the proposition, under some finiteness assumptions, is a connection between the support variety theory of Chapter 8 and that defined via Hopf algebra cohomology: The support variety of an A-module M can be defined alternatively as the maximal ideal spectrum of the quotient of $\mathrm{H}^*(A,k)$ by the annihilator of $\mathrm{Ext}^*_A(M,M)$. (In odd characteristic, $\mathrm{H}^*(A,k)$ may be replaced by its subalgebra generated by even degree elements, a commutative ring.) To compare to Definition 8.3.1, choose H to be the subalgebra of $\mathrm{HH}^*(A)$ generated by the even degree elements of $\mathrm{H}^*(A,k)$ (or all elements in characteristic 2) and $\mathrm{HH}^0(A) \cong Z(A)$. This choice yields a direct relationship between the varieties for modules given by Hochschild cohomology and by Hopf algebra cohomology. For more details, see the survey [**Wit**] and references therein.

Exercise 9.4.11. Let G be a finite group and $A = kG$. Identify the subalgebra $\delta(kG)$ of $(kG)^e$ where δ is defined in Lemma 9.4.1.

Exercise 9.4.12. Let k be a field of characteristic $p > 0$ and let G be a group of order p. Use Example 9.3.7 and Corollary 9.4.6 to describe the Hochschild cohomology $\mathrm{HH}^*(kG)$ as an algebra under cup product.

9.5. Finite group algebras

In this section, we apply Theorem 9.4.5 to the special case $A = kG$, a group algebra of a finite group G. This leads to a decomposition of Hochschild cohomology $\mathrm{HH}^*(kG)$ into a vector space direct sum of group cohomology rings of centralizer subgroups in (9.5.3) below. Burghelea [**Bur85**] first gave a related result on Hochschild homology. Techniques from group cohomology yield a description of the product on Hochschild cohomology $\mathrm{HH}^*(kG)$ in terms of products on group cohomology. We briefly introduce the needed techniques from group cohomology here. For details, see [**Ben91a, Ben91b, Eve61**].

We will use the notation

$$\mathrm{H}^*(G,k) = \mathrm{H}^*(kG,k) = \mathrm{Ext}^*_{kG}(k,k)$$

for the group cohomology of G with coefficients in k, and write

$$\mathrm{H}^*(G,M) = \mathrm{H}^*(kG,M) = \mathrm{Ext}^*_{kG}(k,M)$$

for any kG-module.

By Theorem 9.4.5,

$$\mathrm{HH}^*(kG) \cong \mathrm{H}^*(G, kG^{ad}). \tag{9.5.1}$$

We interpret the kG-module kG^{ad}: Let $g, h \in G$. The action of g on the element h viewed as being in kG^{ad} is given by

$$g \cdot h = ghS(g) = ghg^{-1},$$

since $\Delta(g) = g \otimes g$ and $S(g) = g^{-1}$. Thus the kG-module kG^{ad} has vector space basis G on which elements of G act by conjugation. It then decomposes into a direct sum of kG-submodules corresponding to conjugacy classes. Let $g_1, \ldots, g_r$ be a set of conjugacy class representatives. For each i, the G-set that is the conjugacy class of g_i may be written $G \cdot g_i \cong G/C(g_i)$, where $C(g_i)$ is the centralizer in G of g_i. The kG-submodule of kG^{ad} having basis the conjugacy class of g_i may thus be written

$$kG \cdot g_i \cong kG \otimes_{kC(g_i)} k, \tag{9.5.2}$$

that is, this module is the trivial module k for $kC(g_i)$ induced to kG. For finite group algebras, induction and coinduction yield isomorphic modules since kG is isomorphic to $(kG)^*$ as a kG-module. So by the Eckmann-Shapiro Lemma (Lemma A.6.2),

$$\begin{aligned} \mathrm{H}^*(G, kG \otimes_{kC(g_i)} k) &= \mathrm{Ext}^*_{kG}(k, kG \otimes_{kC(g_i)} k) \\ &\cong \mathrm{Ext}^*_{kC(g_i)}(k,k) = \mathrm{H}^*(C(g_i), k). \end{aligned}$$

We may thus rewrite Hochschild cohomology $\mathrm{HH}^*(kG)$, first applying the isomorphisms (9.5.1) and (9.5.2):

$$\mathrm{HH}^*(kG) \cong \oplus_{i=1}^r \mathrm{H}^*(G, kG\cdot g_i) \cong \oplus_{i=1}^r \mathrm{H}^*(C(g_i), k). \tag{9.5.3}$$

That is, the Hochschild cohomology of kG is isomorphic, as a graded vector space, to the direct sum of group cohomology rings for centralizer subgroups, one summand for each element in a set of conjugacy class representatives. Analyzing the two applications of the Eckmann-Shapiro Lemma used in this description, we see that we may write this isomorphism explicitly as follows. We identify elements in $\mathrm{HH}^*(kG)$ with elements in $\mathrm{H}^*(G, kG^{ad})$, at the chain level, by restriction from $(kG)^e$ to kG under the map δ of Lemma 9.4.1. Our description of the product will be for such elements in $\mathrm{H}^*(G, kG^{ad})$. For any subgroup H of G, and any kG-module M, there is the structure of a kH-module on M by restriction. Since kG is free as a kH-module, restriction takes a kG-projective resolution of k to a kH-projective resolution of k, thus inducing a well-defined map

$$\mathrm{res}^G_H : \mathrm{H}^*(G, M) \to \mathrm{H}^*(H, M).$$

Let $\pi_i : \mathrm{H}^*(C(g_i), kG \otimes_{kC(g_i)} k) \to \mathrm{H}^*(C(g_i), k)$ denote the map induced by the projection of $kG \otimes_{kC(g_i)} k$ onto the $kC(g_i)$-direct summand $k \otimes_{kC(g_i)} k \cong k$. Then the isomorphism above is given in one direction by

$$\begin{array}{ccc} \mathrm{H}^*(G, kG^{ad}) & \xrightarrow{\sim} & \oplus_{i=1}^r \mathrm{H}^*(C(g_i), k) \\ \zeta & \mapsto & (\pi_i \, \mathrm{res}^G_{C(g_i)} \, \zeta)_i. \end{array}$$

In order to describe products, we will also need an expression for the inverse isomorphism. This will be expressed in terms of corestriction maps on group cohomology that we define next: The map

$$\mathrm{cores}^G_H : \mathrm{H}^*(H, M) \to \mathrm{H}^*(G, M)$$

is defined at the chain level as follows. Let P be a projective resolution of k as a kG-module, and let $f \in \mathrm{Hom}_{kH}(P_n, M)$. Then

$$\mathrm{cores}^G_H(f)(x) = \sum_{g \in [G/H]} g \cdot f(g^{-1} \cdot x)$$

for all $x \in P_n$, where $[G/H]$ denotes a set of coset representatives of H in G. Since f is a kH-module homomorphism, the values of this function do not depend on choice of coset representatives. This map also commutes with the differential and so induces a well-defined map on cohomology, for which we use the same notation.

Now let $\iota : \mathrm{H}^*(C(g_i), k) \to \mathrm{H}^*(C(g_i), kG \otimes_{kC(g_i)} k)$ denote the map induced by embedding k into $kG \otimes_{kC(g_i)} k$ as $k \otimes_{kC(g_i)} k$. Then the desired inverse isomorphism may be described as follows:

$$\begin{array}{ccc} \mathrm{H}^*(G, kG^{ad}) & \xleftarrow{\sim} & \oplus_{i=1}^r \mathrm{H}^*(C(g_i), k) \\ \displaystyle\sum_{i=1}^r \mathrm{cores}^G_{C(g_i)} \, \iota(\alpha_i) & \leftarrow\!\shortmid & (\alpha_i)_i. \end{array}$$

For notational convenience, set

$$\gamma_i(\alpha_i) = \mathrm{cores}^G_{C(g_i)} \, \iota(\alpha_i).$$

We need one more map on group cohomology: If H is a subgroup of G and $g \in G$, write ${}^gH = \{ghg^{-1} \mid h \in H\}$ for the conjugate subgroup. There is a map $g^* : \mathrm{H}^*(H, k) \to \mathrm{H}^*({}^gH, k)$ given at the chain level by

$$g^*(f)(x) = g \cdot (f(g^{-1} \cdot x))$$

for all $x \in P_n$ and $f \in \mathrm{Hom}_{kH}(P_n, k)$.

The following is a special case of [**SW99**, Theorem 5.1], and gives the product on Hochschild cohomology $\mathrm{HH}^*(kG)$ in terms of the vector space direct sum (9.5.3).

Theorem 9.5.4. *Let* $\alpha \in \mathrm{H}^*(C(g_i)), k)$ *and* $\beta \in \mathrm{H}^*(C(g_j), k)$. *Then*

$$\gamma_i(\alpha) \smile \gamma_j(\beta) = \sum_{x \in D} \gamma_l(\mathrm{cores}_{W(x)}^{C(g_k)}(\mathrm{res}_{W(x)}^{{}^yC(g_i)}\, y^*\alpha \smile \mathrm{res}_{W(x)}^{{}^{yx}C(g_j)}(yx)^*\beta))$$

where D *is a set of double coset representatives for* $C(g_i)\backslash G/C(g_j)$, *and the integer* $l = l(x)$ *and the group element* $y = y(x)$ *are chosen so that* $g_l = ({}^yg_i)({}^{yx}g_j)$, *and* $W(x) = {}^yC(g_i) \cap {}^{yx}C(g_j)$.

For a proof and some examples, see [**SW99**]. The main idea of the proof is that group cohomology, together with the maps restriction, corestriction, and conjugation, is a Green functor [**Eve61**]. The formula in the theorem is precisely the product formula for a Green functor arising from a Mackey decomposition, interpreted in this notation and setting of Hochschild cohomology.

The following corollary, due to Cibils and Solotar [**CS97**], is a special case of Corollary 9.4.6, and can also be proven directly.

Corollary 9.5.5. *Let* G *be a finite abelian group. The Hochschild cohomology of* kG *is isomorphic, as an algebra, to the tensor product of* kG *and group cohomology* $\mathrm{H}^*(G,k)$:

$$\mathrm{H}^*(kG) \cong kG \otimes \mathrm{H}^*(G,k).$$

Exercise 9.5.6. Let k be a field of characteristic $p > 0$. Let G be an elementary abelian p-group, that is, $G \cong \mathbb{Z}/p\mathbb{Z} \times \cdots \times \mathbb{Z}/p\mathbb{Z}$. Describe $\mathrm{HH}^*(kG)$ and $\mathrm{H}^*(G,k)$.

Exercise 9.5.7. Let k be a field of characteristic 3. Let $G = S_3$, the symmetric group on 3 symbols. Apply (9.5.3) to describe the graded vector space structure of Hochschild cohomology $\mathrm{HH}^*(kG)$ as a direct sum of group cohomology spaces.

9.6. A spectral sequence for a smash product

In this section, we focus on smash product algebras $R\#A$, as in Definition 9.1.6, constructing a spectral sequence relating its Hochschild cohomology with that of R and Hopf algebra cohomology of A. This spectral sequence and a more general version for Hopf Galois extensions is due to Ştefan [**Şte95**]. In Section 9.1 we introduced smash products arising from a *left* action of A on R, and it turns out that as a result we will need to work with *right* A-module cohomology here. First we need a lemma about some special types of right A-modules.

For any $(R\#A)^e$-module N, the space $\mathrm{Hom}_{R^e}(R,N)$ is a right A-module under the action

$$(f\cdot a)(r) = \sum rS(a_1)f(1)a_2 \tag{9.6.1}$$

for all $a\in A$, $f\in \mathrm{Hom}_{R^e}(R,N)$, and $r\in R$. To see this, note that an element of $\mathrm{Hom}_{R^e}(R,N)$ is determined by its value on 1, which must be an element x of N such that $rx = xr$ for all $r\in R$. A calculation shows that for any $a\in A$, the element $\sum S(a_1)xa_2$ also then has this property. This action generalizes the Miyashita-Ulbrich action from groups to Hopf algebras.

Lemma 9.6.2. *Let A be a Hopf algebra, let R be an A-module algebra, and let M be an $(R\#A)^e$-module. There is an isomorphism of right A-modules,*

$$\mathrm{Hom}_{R^e}(R,\mathrm{Hom}_k((R\#A)^e,M)) \cong \mathrm{Hom}_k(A^{\mathrm{op}}\otimes(R\#A),M).$$

The $(R\#A)^e$-module $\mathrm{Hom}_k((R\#A)^e,M)$ in the lemma is the coinduced module (see Section A.6) of M from k to $(R\#A)^e$. Similarly we may view the A-module $\mathrm{Hom}_k(A^{\mathrm{op}}\otimes(R\#A),M)$ as a coinduced module from k to $A^{\mathrm{op}}\otimes(R\#A)$, and A acts via the embedding δ' of A into $A^{\mathrm{op}}\otimes A$ given by (9.4.4).

Proof. Let $f\in \mathrm{Hom}_{R^e}(R,\mathrm{Hom}_k((R\#A)^e,M))$ and define an element $\phi(f)$ of $\mathrm{Hom}_k(A^{\mathrm{op}}\otimes(R\#A),M)$ by

$$\phi(f)(a\otimes ra') = f(1)(a\otimes ra')$$

for all $a,a'\in A$ and $r\in R$. Then ϕ is an A-module homomorphism: The function f is determined by $f(1)$, which has the property that $rf(1) = f(1)r$. Now $\mathrm{Hom}_k((R\#A)^e,M)$ is the module M coinduced from k to $(R\#A)^e$, so this property is equivalent to the property that $f(1)(xr\otimes y) = f(1)(x\otimes ry)$ for all $r\in R$ and $x,y\in R\#A$. Paying close attention to which factors have opposite multiplication, it follows that for all $a,a',a''\in A$ and $r\in R$,

$$\begin{aligned}(\phi(f\cdot a))(a'\otimes ra'') &= (f\cdot a)(1)(a'\otimes ra'')\\ &= \sum S(a_1)f(1)a_2(a'\otimes ra'')\\ &= \sum f(1)(a'S(a_1)\otimes a_2ra'')\end{aligned}$$

and using the embedding δ' of A into $A^{\mathrm{op}}\otimes A$,

$$\begin{aligned}(\phi(f)\cdot a)(a'\otimes ra'') &= \sum \phi(f)(a'S(a_1)\otimes a_2ra'')\\ &= \sum f(1)(a'S(a_1)\otimes a_2ra''),\end{aligned}$$

and so ϕ is an A-module homomorphism.

We next show that ϕ is bijective: Let $g\in \mathrm{Hom}_k(A^{\mathrm{op}}\otimes(R\#A),M)$ and define an element $\psi(g)$ of $\mathrm{Hom}_{R^e}(R,\mathrm{Hom}_k((A\#R)^e,M))$ by

$$\psi(g)(1)(ar\otimes r'a') = g(a\otimes rr'a').$$

Since $\mathrm{Hom}_k((A\#R)^e, M)$ is the coinduced module, $\psi(g)$ is indeed an R^e-homomorphism:

$$\psi(g)(1)(arr'' \otimes r'a') = \psi(g)(1)(ar \otimes r''r'a')$$

for all $a, a' \in A$ and $r, r', r'' \in R$. A calculation shows that ψ is an inverse map to ϕ. $\square$

Now we are ready to construct the spectral sequence for Hochschild cohomology of bimodules over the smash product $R\#A$. The Hopf algebra cohomology in the next theorem is that of right A-modules. If M is an $(R\#A)^e$-module, the Hochschild cohomology $\mathrm{HH}^*(R, M)$ is a right A-module under the action induced by that of equation (9.6.1): Take an injective resolution $Q_{\bullet}$ of M as an $(R\#A)^e$-module, take the right A-action on each $\mathrm{Hom}_{R^e}(R, Q_q)$ given by (9.6.1), and check that this action of A commutes with the differentials and thus induces an action on Hochschild cohomology as claimed.

Theorem 9.6.3. *Let A be a Hopf algebra, let R be an A-module algebra, and let M be an $(R\#A)^e$-module. There is a right action of A on* $\mathrm{HH}^q(R, M)$ *and a spectral sequence*

$$E_2^{p,q} = \mathrm{H}^p(A, \mathrm{HH}^q(R, M)) \implies \mathrm{HH}^{p+q}(R\#A, M).$$

We give a direct proof of the theorem in our setting. A proof in the more general setting of Hopf Galois extensions is in [**Şte95**]. We emphasize that the Hopf algebra cohomology in the theorem statement is that of the *right* A-module $\mathrm{HH}^q(R, M)$.

Proof. Let $P_{\bullet}$ be a projective resolution of k as a right A-module.

Let $Q_{\bullet}$ be an injective resolution of M as an $(R\#A)^e$-module. By restricting to the subalgebra R^e of $(R\#A)^e$, since $(R\#A)^e$ is free over R^e, $Q_{\bullet}$ becomes an R^e-injective resolution of M. To see this, note that by the Nakayama relations in Lemma A.6.1,

$$\mathrm{Hom}_{R^e}(U, I) \cong \mathrm{Hom}_{(R\#A)^e}((R\#A)^e \otimes_{R^e} U, I)$$

for all R^e-modules U and $(R\#A)^e$-modules I, so $\mathrm{Hom}_{R^e}(-, I)$ is an exact functor when I is an injective $(R\#A)^e$-module, implying that the restriction of I to R^e is also injective.

Let

$$C^{p,q} = \mathrm{Hom}_A(P_p, \mathrm{Hom}_{R^e}(R, Q_q)),$$

a double complex that we may view as:

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \\
\mathrm{Hom}_A(P_0, \mathrm{Hom}_{R^e}(R, Q_2)) \arrow[r] \arrow[d] & \mathrm{Hom}_A(P_1, \mathrm{Hom}_{R^e}(R, Q_2)) \arrow[r] \arrow[d] & \cdots \\
\mathrm{Hom}_A(P_0, \mathrm{Hom}_{R^e}(R, Q_1) \arrow[r] \arrow[d] & \mathrm{Hom}_A(P_1, \mathrm{Hom}_{R^e}(R, Q_1)) \arrow[r] \arrow[d] & \cdots \\
\mathrm{Hom}_A(P_0, \mathrm{Hom}_{R^e}(R, Q_0)) \arrow[r] & \mathrm{Hom}_A(P_1, \mathrm{Hom}_{R^e}(R, Q_0)) \arrow[r] & \cdots
\end{tikzcd}$$

We will analyze the two first quadrant spectral sequences associated to this double complex in the notation of Section A.7. They each converge to $\mathrm{H}^{\bullet}(C)$ by Theorem A.7.6. Since each P_i is projective as a right A-module, $\mathrm{Hom}_A(P_i, -)$ is exact, and we have

$$\mathrm{H}''(C) = \mathrm{Hom}_A(P_{\bullet}, \mathrm{HH}^{\bullet}(R, M))$$

and so

$$\mathrm{H}'(\mathrm{H}''(C)) = \mathrm{H}^{\bullet}(A, \mathrm{HH}^{\bullet}(R, M)).$$

On the other hand,

$$\mathrm{H}'(C) \cong \mathrm{H}^{\bullet}(A, \mathrm{Hom}_{R^e}(R, Q_{\bullet})). \tag{9.6.4}$$

We claim that for each q, $\mathrm{H}^p(A, \mathrm{Hom}_{R^e}(R, Q_q)) = 0$ whenever $p > 0$. This is a special case of [**Şte95**, Proposition 3.2]. To see this in our setting, first note that there is an injective $(R\#A)^e$-module homomorphism

$$i : Q_q \to \mathrm{Hom}_k((R\#A)^e, Q_q)$$

given by $i(m)(x \otimes y) = xmy$ for all $m \in Q_q$ and $x, y \in R\#A$. Since Q_q is an injective $(R\#A)^e$-module, this map splits, and so Q_q is a direct summand of $\mathrm{Hom}_k((R\#A)^e, Q_q)$ as an $(R\#A)^e$-module. This implies that $\mathrm{Hom}_{R^e}(R, Q_q)$ is a direct summand of $\mathrm{Hom}_{R^e}(R, \mathrm{Hom}_k((R\#A)^e, Q_q))$ as a right A-module. So to prove our claim, it suffices to show that

$$\mathrm{H}^p(A, \mathrm{Hom}_{R^e}(R, \mathrm{Hom}_k((R\#A)^e, Q_q)) = 0$$

whenever $p > 0$. Now this is the degree p cohomology of the complex

$$\mathrm{Hom}_A(P_{\bullet}, \mathrm{Hom}_{R^e}(R, \mathrm{Hom}_k((R\#A)^e, Q_q))),$$

and by Lemma 9.6.2, this is isomorphic to

$$\mathrm{Hom}_A(P_{\bullet}, \mathrm{Hom}_k(A^{\mathrm{op}} \otimes (R\#A), Q_q)).$$

In turn, this is isomorphic to $\mathrm{Hom}_k(P_{\bullet} \otimes_A (A^{\mathrm{op}} \otimes (R\#A)), Q_q)$ as a graded vector space: An isomorphism is given in each degree p by sending an element f of $\mathrm{Hom}_A(P_p, \mathrm{Hom}_k(A^{\mathrm{op}} \otimes (R\#A), Q_q))$ to the element $\phi(f)$ of $\mathrm{Hom}_k(P_p \otimes (R\#A), Q_q)$ such that $\phi(f)(x \otimes y) = f(x)(y)$ for all $x \in P_p$, $y \in R\#A$. An inverse map is given by taking $g \in \mathrm{Hom}_k(P_p \otimes_A (A^{\mathrm{op}} \otimes (R\#A)), Q_q)$ to $\psi(g)$ where $\psi(g)(x)(y) = g(x \otimes y)$. A calculation shows that for each g, the function $\psi(g)$ is a homomorphism of right A-modules, and that ϕ, ψ are indeed inverse maps. Now $\mathrm{Hom}_k(P_\bullet \otimes_A (A^{\mathrm{op}} \otimes (R\#A)), Q_q)$ has homology 0 in positive degrees, since $- \otimes_A (A^{\mathrm{op}} \otimes (R\#A))$ and $\mathrm{Hom}_k(-, Q_q)$ are exact functors.

As a result, the cohomology $\mathrm{H}'(C)$ of (9.6.4) is concentrated in the leftmost column. In the qth position, by Remark 9.2.4 (see Lemma 9.2.2), it is

$$\begin{aligned} \mathrm{H}^0(A, \mathrm{Hom}_{R^e}(R, Q_q)) &\cong \mathrm{Hom}_A(k, \mathrm{Hom}_{R^e}(R, Q_q)) \\ &\cong \mathrm{Hom}_{R^e}(R, Q_q)^A. \end{aligned}$$

Now we claim there is an isomorphism of vector spaces

$$\mathrm{Hom}_{R^e}(R, Q_q)^A \cong \mathrm{Hom}_{(R\#A)^e}(R\#A, Q_q) \tag{9.6.5}$$

for each q. To see this, consider an element f of $\mathrm{Hom}_{R^e}(R, Q_q)^A$, which is determined by $f(1)$. By hypothesis, $rf(1) = f(1)r$ and $af(1) = f(1)a$ for all $a \in A$. (The latter equation holds since $f(1)a = \sum a_1 S(a_2) f(1) a_3 = \sum a_1 \varepsilon(a_2) f(1) = af(1)$.) Define $\gamma(f)(ra) = (ra)f(1)$ for all $r \in R$ and $a \in A$. Then $\gamma(f)$ is an $(R\#A)^e$-module homomorphism. The map γ has inverse given by sending a function g in $\mathrm{Hom}_{(R\#A)^e}(R\#A, Q_q)$ to the function taking 1 to $g(1)$. It follows from the isomorphism (9.6.5) that $\mathrm{H}''(\mathrm{H}'(C)) = \mathrm{HH}^\bullet(R\#A, M)$, as claimed. $\square$

In the special case of a semisimple Hopf algebra, we have the following consequence.

Corollary 9.6.6. *Let A be a semisimple Hopf algebra, let R be an A-module algebra, and let M be an $(R\#A)^e$-module. Then*

$$\mathrm{HH}^*(R\#A, M) \cong (\mathrm{HH}^*(R, M))^A.$$

Proof. Since A is semisimple, the spectral sequence of Theorem 9.6.3 collapses: The E_2 page is concentrated in the leftmost column, which is

$$\mathrm{H}^0(A, \mathrm{HH}^*(R, M)) \cong (\mathrm{HH}^*(R, M))^A.$$

$\square$

Exercise 9.6.7. Verify that (9.6.1) gives a well-defined right A-module structure to $\mathrm{Hom}_{R^e}(R, N)$.

Exercise 9.6.8. Verify the last statement in the proof of Lemma 9.6.2, that ψ is an inverse map to ϕ.

Appendix A

Homological Algebra Background

We collect here some homological algebra terminology, notation, and results that we will use throughout the book. Proofs and more details may be found in standard homological algebra texts such as [**Ben91a, HS71, Osb00, Rot09, Wei94**].

A.1. Complexes

Let R be a ring. We always assume that R has multiplicative identity 1 and all modules are unital modules, that is, 1 acts as the identity map. Modules will be left modules unless otherwise specified. We will often take $R = A \otimes A^{\mathrm{op}}$ where A is an algebra over a field k and $\otimes = \otimes_k$.

A *complex* $C_\bullet$ of R-modules, also written $(C_\bullet, d_\bullet)$, is a sequence of R-modules and R-module homomorphisms, called *differentials*,

$$C_\bullet : \qquad \cdots \longrightarrow C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \xrightarrow{d_0} C_{-1} \xrightarrow{d_{-1}} C_{-2} \longrightarrow \cdots$$

where $d_{n-1}d_n = 0$ for all $n \in \mathbb{Z}$. The *degree* (or *dimension*) of an element x of C_n is n, and we write $|x| = n$. The differential then is considered to have degree -1 as a map. For each n, the kernel $\mathrm{Ker}(d_n)$ is the R-submodule of C_n consisting of *n-cycles*, the image $\mathrm{Im}(d_{n+1})$ is the R-submodule of C_n consisting of *n-boundaries*, and $\mathrm{H}_n(C_\bullet) = \mathrm{Ker}(d_n)/\,\mathrm{Im}(d_{n+1})$ is the *nth homology* of $C_\bullet$. Two n-cycles x and y are *homologous* if $x - y$ is an n-boundary.

We will sometimes omit the subscript on $C_\bullet$, writing C instead, to denote the complex. We will also sometimes identify $C_\bullet$ with the R-module $\oplus_{n\in\mathbb{Z}} C_n$

and $d_\bullet$ with the homomorphism from this R-module to itself that agrees with d_n on each C_n.

We take a *chain complex* to be a complex for which $C_n = 0$ for $n < 0$, and a *cochain complex* to be a complex for which $C_n = 0$ for $n > 0$. Some authors use these terms more generally to refer to complexes. Some complexes may be indexed differently, replacing n by $-n$ in $C_\bullet$ above, with the maps still oriented as shown so that the indexing agrees with the left to right ordering of integers on a standard number line. A cochain complex then has differential of degree $+1$, and we may choose to write the index as a superscript:

$$C^\bullet : \qquad 0 \longrightarrow C^0 \xrightarrow{d_0} C^1 \xrightarrow{d_1} C^2 \xrightarrow{d_2} \cdots$$

In this context, elements in the kernel of d_n are *n-cocycles*, and elements in the image of d_{n-1} are *n-coboundaries*. Two n-cocycles are *cohomologous* if their difference is an n-coboundary. We write $\mathrm{H}^n(C^\bullet) = \mathrm{Ker}(d_n)/\,\mathrm{Im}(d_{n-1})$ and refer to this as the *cohomology* of the cochain complex $C^\bullet$. The following definitions and statements may be rephrased in terms of this other indexing choice.

We say that $C_\bullet$ is *acyclic*, or *exact*, if $\mathrm{H}_n(C_\bullet) = 0$ for all n. A *short exact sequence* is an exact complex of the form $0 \to U \to V \to W \to 0$.

Let (C, d) be a complex and $n \in \mathbb{Z}$. The *shifted* (or *translated*) *complex* $C[n]$ has

$$C[n]_i = C_{i+n}$$

and differentials $(-1)^n d$, so for example, $C[n]_0 = C_n$. (For a cochain complex, we take instead $C[n]^i = C^{i-n}$.)

Let (C, d) and (C', d') be complexes. A *chain map* $f_\bullet : C_\bullet \to C'_\bullet$ consists of an R-module homomorphism $f_n : C_n \to C'_n$ for which $f_{n-1}d_n = d'_n f_n$ for each n, that is, the following diagram commutes:

$$\begin{tikzcd}
\cdots \arrow[r] & C_1 \arrow[r, "d_1"] \arrow[d, "f_1"] & C_0 \arrow[r, "d_0"] \arrow[d, "f_0"] & C_{-1} \arrow[r] \arrow[d, "f_{-1}"] & \cdots \\
\cdots \arrow[r] & C'_1 \arrow[r, "d'_1"] & C'_0 \arrow[r, "d'_0"] & C'_{-1} \arrow[r] & \cdots
\end{tikzcd}$$

A chain map induces a map on homology, and is a *quasi-isomorphism* if this induced map is an isomorphism.

Two chain maps $f_\bullet, g_\bullet : C_\bullet \to C'_\bullet$ are *chain homotopic* if there are R-module homomorphisms $s_n : C_n \to C'_{n+1}$ such that

$$f_n - g_n = s_{n-1}d_n + d'_{n+1}s_n \tag{A.1.1}$$

for all n. We call $s_\bullet$ a *homotopy* for $f_\bullet - g_\bullet$. Chain homotopy is an equivalence relation, and two chain homotopic maps induce the same maps on homology. If $g_\bullet$ is the zero map, we call $s_\bullet$ a *chain contraction* of $f_\bullet$. If there is a chain contraction of the identity map on $C_\bullet$, it is also sometimes called a *contracting homotopy*, and it follows that $C_\bullet$ is acyclic. This conclusion holds under the weaker hypothesis that there are set maps $s_n : C_n \to C'_{n+1}$ (not necessarily R-module homomorphisms) satisfying equation (A.1.1). Sometimes when R is a k-algebra, there are k-linear maps s_n satisfying this hypothesis that are not R-module homomorphisms.

Differential graded algebras. Let k be a field. A *differential graded algebra* is a complex (C, d) for which C is a graded k-algebra and the Leibniz rule holds: For all $x \in C_i$ and $y \in C_j$,

$$d(xy) = d(x)y + (-1)^i x d(y).$$

A *differential graded Lie algebra* is a complex (C, d) for which C is a graded Lie algebra over k and for all $x \in C_i$ and $y \in C_j$,

$$d([x, y]) = [d(x), y] + (-1)^i [x, d(y)].$$

A *differential graded coalgebra* is a complex (C, d) for which C is a graded coalgebra over k, i.e. there is a graded k-linear map $\Delta : C \to C \otimes C$ that is coassociative (i.e. $(\Delta \otimes 1)\Delta = (1 \otimes \Delta)\Delta$) and a k-linear map $\varepsilon : C \to k$ such that $(\varepsilon \otimes 1)\Delta = 1 = (1 \otimes \varepsilon)\Delta$, and

$$(d \otimes 1 + 1 \otimes d)\Delta = \Delta d.$$

Pushout and pullback. Let A, B, Y be R-modules and let $\alpha : Y \to A$, $\beta : Y \to B$ be R-module homomorphisms. A *pushout* of α, β is an R-module X together with R-module homomorphisms $\phi : A \to X$, $\psi : B \to X$ such that $\phi\alpha = \psi\beta$ and for any R-module Z and R-homomorphisms $\tilde{\phi} : A \to Z$, $\tilde{\psi} : B \to Z$ for which $\tilde{\phi}\alpha = \tilde{\psi}\beta$, there is a unique R-module homomorphism $\eta : X \to Z$ such that $\tilde{\phi} = \eta\phi$, $\tilde{\psi} = \eta\psi$:

$$\begin{tikzcd} Y \arrow[r, "\alpha"] \arrow[d, "\beta"] & A \arrow[d, "\phi"] \\ B \arrow[r, "\psi"] & X \end{tikzcd}$$

Note that we may take

$$X = A \oplus B / \{(-\alpha(y), \beta(y)) \mid y \in Y\}$$

and ϕ, ψ to be the maps induced by inclusion into $A \oplus B$.

Let A, B, X be R-modules and let $\phi : A \to X$, $\psi : B \to X$ be R-module homomorphisms. A *pullback* of ϕ, ψ is an R-module Y together with R-module homomorphisms $\alpha : Y \to A$, $\beta : Y \to B$ such that $\phi\alpha = \psi\beta$ and for any R-module Z and R-module homomorphisms $\tilde{\alpha} : Z \to A$, $\tilde{\beta} : Z \to B$

for which $\phi\tilde{\alpha} = \psi\tilde{\beta}$, there is a unique R-module homomorphism $\eta : Z \to Y$ such that $\tilde{\alpha} = \alpha\eta$, $\tilde{\beta} = \beta\eta$. Note that we may take

$$Y = \{(a, b) \in A \oplus B \mid \phi(a) = \psi(b)\}$$

and α, β to be the maps induced by projection from $A \oplus B$.

A.2. Resolutions and dimensions

An R-module P is *projective* if for every surjective R-module homomorphism $f : U \to V$ and R-module homomorphism $g : P \to V$, there exists an R-module homomorphism $h : P \to U$ such that $fh = g$:

(A.2.1)
$$\begin{tikzcd} & P \arrow[dl, dashed, "h"'] \arrow[d, "g"] & \\ U \arrow[r, "f"] & V \arrow[r] & 0 \end{tikzcd}$$

There are many equivalent definitions of projective module, for example, an R-module is projective if and only if it is a direct summand of a free module (i.e. $R^{\oplus I}$ for some indexing set I). A *projective cover* of an R-module M is a projective R-module P together with a surjective R-module homomorphism $\varepsilon : P \to M$ such that for any projective R-module P' and surjective R-module homomorphism $\nu : P' \to M$, there is a surjective R-module homomorphism $\psi : P' \to P$ such that $\nu = \varepsilon\psi$.

An R-module I is *injective* if for every injective R-module homomorphism $f : V \to U$ and R-module homomorphism $g : V \to I$, there exists an R-module homomorphism $h : U \to I$ such that $hf = g$:

(A.2.2)
$$\begin{tikzcd} & I & \\ U \arrow[ur, dashed, "h"] & V \arrow[l, "f"] \arrow[u, "g"'] & 0 \arrow[l] \end{tikzcd}$$

An R-module F is *flat* if for every short exact sequence of right R-modules $0 \to U \to V \to W \to 0$, the induced sequence of abelian groups $0 \to U \otimes_R F \to V \otimes_R F \to W \otimes_R F \to 0$ is exact. Every projective module is flat.

Let M be an R-module. A *projective resolution* of M is a chain complex $P_\bullet$ consisting of projective R-modules P_n $(n \geq 0)$ for which $\mathrm{H}_0(P_\bullet) \cong M$ and $\mathrm{H}_n(P_\bullet) = 0$ for all $n \neq 0$. Thus $P_\bullet$ is quasi-isomorphic to the complex that is M concentrated in degree 0 and 0 elsewhere, with all maps 0:

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r] \arrow[d] & P_1 \arrow[r] \arrow[d] & P_0 \arrow[r] \arrow[d, "\varepsilon"] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & M \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

As a consequence of the definition, the following sequence is exact:

$$\cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} M \longrightarrow 0. \tag{A.2.3}$$

We refer to the complex (A.2.3) as the *augmented complex* of $P_\bullet$. Sometimes this augmented complex is abbreviated $P_\bullet \xrightarrow{\varepsilon} M$ and referred to as the projective resolution of M, when it is clear from context what is meant. We also sometimes refer to $P_\bullet$ as the *truncated complex* of (A.2.3).

Note that projective resolutions always exist: Every R-module M is a homomorphic image of a projective R-module, for example, the free module on a set of generators of M. One may use this fact to build projective resolutions as follows. Let P_0 be a projective R-module having M as a homomorphic image under a map ε. Let $K_1 = \operatorname{Ker}(\varepsilon)$. Then K_1 is a homomorphic image of a projective R-module P_1 via some map $\varepsilon_1 : P_1 \to K_1$. Denote by i_1 the inclusion map $i_1 : K_1 \to P_0$ and set $d_1 = i_1\varepsilon_1$. Let $K_2 = \operatorname{Ker}(d_1)$ and continue.

(A.2.4)

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[rr, "d_2"] \arrow[dr, "\varepsilon_2"'] & & P_1 \arrow[rr, "d_1"] \arrow[dr, "\varepsilon_1"'] & & P_0 \arrow[r, "\varepsilon"] & M \arrow[r] & 0 \\
& & K_2 \arrow[ur, "i_2"'] & & K_1 \arrow[ur, "i_1"'] & & &
\end{tikzcd}$$

We call K_i an *ith syzygy module* of M. It depends on some choices, and Lemma A.2.6 below is a precise statement about these choices. It follows from Schanuel's Lemma:

Lemma A.2.5. *[Schanuel's Lemma] Let* $0 \to K \to P \xrightarrow{\varepsilon} M \to 0$ *and* $0 \to K' \to P' \xrightarrow{\varepsilon'} M \to 0$ *be two short exact sequences of R-modules with P, P' projective. Then* $K \oplus P' \cong K' \oplus P$.

Schanuel's Lemma immediately implies the following.

Lemma A.2.6. *If K_i and K_i' are ith syzygy modules of the R-module M, then there are projective R-modules P, P' such that* $K_i \oplus P \cong K_i' \oplus P'$.

Another way to state Lemma A.2.6 is to say that K_i and K_i' are equivalent under the following equivalence relation: Two R-modules U and V are equivalent if $U \oplus P \cong V \oplus P'$ for some projective R-modules P, P'.

The *Heller operator* Ω is defined by $\Omega(M) = K_1$ (notation from diagram (A.2.4)), understood to take values in an equivalence class of R-modules. Sometimes we write $\Omega_R = \Omega$ to emphasize the choice of ring R. This operator is often used in settings where projective direct summands do not matter, such as Theorem A.3.3 below. In some contexts, $\Omega(M)$ may instead be defined more specifically and uniquely up to isomorphism, for example by taking $P_{\bullet}$ to be a minimal resolution. It should be clear from context which is meant. Sometimes this operator is iterated, and we write $\Omega^1 = \Omega$ and $\Omega^n(M) = \Omega(\Omega^{n-1}(M))$ for all $n \geq 2$.

The next theorem in particular implies a relation among projective resolutions.

Theorem A.2.7 (Comparison Theorem). *Let $(P_{\bullet}, d_{\bullet})$ and $(Q_{\bullet}, d'_{\bullet})$ be chain complexes with $M = \mathrm{H}_0(P_{\bullet})$, $N = \mathrm{H}_0(Q_{\bullet})$, and let $\varepsilon : P_0 \to M$ and $\varepsilon' : Q_0 \to N$ be corresponding augmentation maps. Assume that the augmented complex $\cdots \to Q_1 \to Q_0 \to N \to 0$ is exact and that P_i is projective for each i. If $f : M \to N$ is an R-module homomorphism, then there is a chain map $f_{\bullet} : P_{\bullet} \to Q_{\bullet}$ for which $f\varepsilon = \varepsilon' f_0$. This chain map is unique up to chain homotopy.*

In particular, if $P_{\bullet}, Q_{\bullet}$ are projective resolutions of M, N, respectively, the Comparison Theorem states that there is a chain map $f_{\bullet} : P_{\bullet} \to Q_{\bullet}$ *lifting* $f : M \to N$.

A projective resolution $P_{\bullet}$ of an R-module M is *minimal* if for every projective resolution $P'_{\bullet} \to M$, there is a chain map $f_{\bullet} : P'_{\bullet} \to P_{\bullet}$ lifting the identity map on M such that f_n is surjective for each n. Equivalently, P_0 is a projective cover of M and P_i is a projective cover of K_i for each i.

An *injective resolution* of an R-module M is a cochain complex $I_{\bullet}$ consisting of injective R-modules I_n for which $\mathrm{H}_0(I_{\bullet}) \cong M$ and $\mathrm{H}_n(I_{\bullet}) = 0$ for all $n \neq 0$. In other words, $M \cong \mathrm{Ker}(d_0)$ and the following sequence is exact:

$$0 \longrightarrow M \xrightarrow{\iota} I_0 \xrightarrow{d_0} I_1 \xrightarrow{d_1} \cdots , \tag{A.2.8}$$

where ι is an isomorphism from M to $\mathrm{Ker}(d_0)$ followed by inclusion into I_0. We refer to the complex (A.2.8) as the *augmented complex* of $I_{\bullet}$, and sometimes as the injective resolution of M when it is clear from context what is intended. We also sometimes refer to $I_{\bullet}$ as the *truncated complex* of (A.2.8).

Note that injective resolutions always exist: Baer's Theorem states that every R-module can be embedded in an injective R-module, and one builds an injective resolution in a similar fashion to that described for a projective resolution above: Let $L_1 = \mathrm{Coker}(\iota) = I_0/\mathrm{Im}(\iota)$, then embed L_1 into an injective module I_1 via $\iota_1 : L_1 \to I_1$, set $\delta_0 = \iota_1\pi_0$ where $\pi_0 : I_0 \to L_1$ is the quotient map, and so on.

$$
\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "\iota"] & I_0 \arrow[rr, "\delta_0"] \arrow[dr, "\pi_0"'] & & I_1 \arrow[rr, "\delta_1"] \arrow[dr, "\pi_1"'] & & I_2 \arrow[r] & \cdots \\
& & & L_1 \arrow[ur, "\iota_1"'] & & L_2 \arrow[ur, "\iota_2"'] & &
\end{tikzcd}
\tag{A.2.9}
$$

Again, the module L_1 is unique up to injective direct summands, due to a dual version of Schanuel's Lemma, which states that if $0 \to N \to I \to L \to 0$ and $0 \to N \to I' \to L' \to 0$ are exact sequences with I, I' injective, then there is an isomorphism $L \oplus I' \cong L' \oplus I$.

We define the operator Ω^{-1} to be $\Omega^{-1}(M) = L_1$, understood to take values in an equivalence class of modules, and call it a *first cosyzygy module* of M. Similarly, $L_i = \Omega^{-i}M$ is an *ith cosyzygy module* of M. This notation is chosen with the following setting in mind: Assume R is a *self-injective* algebra over a field k, that is, R is injective as an R-module (under left multiplication). Then projective A-modules are also injective, and vice versa. Combining diagrams (A.2.4) and (A.2.9) we obtain

$$
\begin{tikzcd}
\cdots \arrow[r] & P_1 \arrow[rr, "d_1"] \arrow[dr, "\varepsilon_1"'] & & P_0 \arrow[rr, "\iota\varepsilon"] \arrow[dr, "\varepsilon"'] & & I_0 \arrow[rr, "\delta_0"] \arrow[dr, "\pi_0"'] & & I_1 \arrow[r] & \cdots \\
& & K_1 \arrow[ur, "i_1"'] & & M \arrow[ur, "\iota"'] & & L_1 \arrow[ur, "\iota_1"'] & &
\end{tikzcd}
$$

The modules in the top row are all projective and injective and may be viewed alternately as terms in projective and injective resolutions of the modules in the bottom row. It follows that $\Omega^{-1}(\Omega(M))$ is equivalent to M: There are projective modules P, P' such that

$$\Omega^{-1}(\Omega(M)) \oplus P \cong M \oplus P'.$$

Other types of resolutions may be defined similarly, for example, flat resolutions.

The *projective dimension* $\operatorname{pdim}_R(M)$ of an R-module M is the smallest integer n such that there is a projective resolution of M:

$$0 \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0.$$

If no such n exists we write $\operatorname{pdim}_R(M) = \infty$. We similarly define injective dimension and flat dimension. Note that $\operatorname{pdim}_R(M)$ is the smallest integer n such that an nth syzygy module of M is projective. Similarly we see that $\operatorname{pdim}_R(\Omega M) = \operatorname{pdim}_R(M) - 1$ if M is not projective.

The *left global dimension* of R is

$$\operatorname{gldim}_l R = \sup\{\operatorname{pdim}_R(M) \mid M \text{ is a left } R\text{-module}\}.$$

We may similarly define right global dimension, gldim_r. Authors often use the notation gldim in contexts where it is understood which is meant.

An important class of examples is provided by the following theorem.

Theorem A.2.10 (Hilbert's Syzygy Theorem)**.** *Let k be a field. Then*

$$\operatorname{gldim} k[x_1, \ldots, x_n] = n.$$

A ring R is (left) *hereditary* if every left ideal is projective, equivalently if $\operatorname{gldim}_l R \leq 1$. Thus for example, $k[x]$ is hereditary.

A.3. Ext and Tor

Let M and N be R-modules. Let $P \xrightarrow{\varepsilon} M$ be a projective resolution of M. Applying $\operatorname{Hom}_R(-, N)$ to the sequence $\cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \to 0$, we obtain

(A.3.1) $$0 \longrightarrow \operatorname{Hom}_R(P_0, N) \xrightarrow{d_1^*} \operatorname{Hom}_R(P_1, N) \xrightarrow{d_2^*} \cdots$$

where $d_i^*(f) = f d_i$ for all i and $f \in \operatorname{Hom}_R(P_{i-1}, N)$. We set $d_0^* = 0$. Note that $d_{i+1}^* d_i^* = 0$ since $d_i d_{i+1} = 0$, so the sequence (A.3.1) is a (cochain) complex of abelian groups (that is, $\mathbb{Z}$-modules). If R is commutative, it is a complex of R-modules. If R is an algebra over a field k, it is a complex of k-vector spaces. We define $\operatorname{Ext}_R^*(M, N)$ to be the cohomology of this complex:

$$\operatorname{Ext}_R^n(M, N) = H^n(\operatorname{Hom}_R(P_\bullet, N)) = \operatorname{Ker}(d_{n+1}^*)/\operatorname{Im}(d_n^*)$$

for $n \geq 0$, and $\operatorname{Ext}_R^*(M, N) = \oplus_{n \geq 0} \operatorname{Ext}_R^n(M, N)$. An application of the Comparison Theorem (Theorem A.2.7) shows that $\operatorname{Ext}_R^*(M, N)$ does not depend on choice of projective resolution of M. Note that

$$\operatorname{Ext}_R^0(M, N) \cong \operatorname{Hom}_R(M, N).$$

By construction, if M is itself a projective R-module, then $\operatorname{Ext}_R^n(M, N) = 0$ for all $n > 0$.

Equivalently, we may define $\operatorname{Ext}_R^*(M, N)$ via an injective resolution: Let $N \xrightarrow{\iota} I$ be an injective resolution of N. Apply $\operatorname{Hom}_R(M, -)$ to the sequence $0 \to I_0 \xrightarrow{d_0} I_1 \xrightarrow{d_1} \cdots$ to obtain a sequence

(A.3.2) $$0 \longrightarrow \operatorname{Hom}_R(M, I_0) \xrightarrow{(d_0)_*} \operatorname{Hom}_R(M, I_1) \xrightarrow{(d_1)_*} \cdots$$

where $(d_i)_*(f) = d_i f$ for all i and $f \in \operatorname{Hom}_R(M, I_i)$. Set $(d_{-1})_* = 0$. Then $(d_{i+1})_*(d_i)_* = 0$ for all $i \geq -1$, so the sequence (A.3.2) is a (cochain) complex of abelian groups. It can be shown that

$$\operatorname{Ext}_R^n(M, N) \cong \mathrm{H}^n(\operatorname{Hom}_R(M, I_\bullet)) = \operatorname{Ker}((d_n)_*)/\operatorname{Im}((d_{n-1})_*).$$

If N is an injective R-module, we now see that $\operatorname{Ext}_R^n(M,N) = 0$ for all $n > 0$.

For each n, the group $\operatorname{Ext}_R^n(M,N)$ has an interpretation in terms of exact sequences: An *n-extension* of M by N is an exact sequence of R-modules

$$\mathbf{f}: \qquad 0 \longrightarrow N \longrightarrow U_{n-1} \longrightarrow \cdots \longrightarrow U_1 \longrightarrow U_0 \longrightarrow M \longrightarrow 0.$$

If $\mathbf{g}$ is another n-extension of M by N, a map from $\mathbf{f}$ to $\mathbf{g}$ is a chain map which on M and on N is an identity map (denoted 1):

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] \arrow[d, "1"] & U_{n-1} \arrow[r] \arrow[d, "\phi_{n-1}"] & \cdots \arrow[r] & U_1 \arrow[r] \arrow[d, "\phi_1"] & U_0 \arrow[r] \arrow[d, "\phi_0"] & M \arrow[r] \arrow[d, "1"] & 0 \\
0 \arrow[r] & N \arrow[r] & V_{n-1} \arrow[r] & \cdots \arrow[r] & V_1 \arrow[r] & V_0 \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

Maps of n-extensions generate an equivalence relation. There is a well-defined binary operation on equivalence classes, called the Baer sum, under which the set of equivalence classes of n-extensions is an abelian group. A representative of the additive inverse of the n-extension $\mathbf{f}$ above can be taken to be the sequence with the same modules in which the map $U_0 \to M$ is replaced by its additive inverse, and all other maps are the same. The group $\operatorname{Ext}_R^n(M,N)$ is isomorphic to the group of equivalence classes of n-extensions of M by N. We outline this one-to-one correspondence next.

Let $\mathbf{f}$ be the above n-extension of M by N. We will define an element of $\operatorname{Ext}_R^n(M,N)$ corresponding to it. Let $P \to M$ be a projective resolution of M. By the Comparison Theorem (Theorem A.2.7), there is a chain map $\hat{f}_\bullet$:

$$\begin{tikzcd}
P_{n+1} \arrow[r, "d_{n+1}"] \arrow[d] & P_n \arrow[r, "d_n"] \arrow[d, "\hat{f}_n"] & P_{n-1} \arrow[r] \arrow[d, "\hat{f}_{n-1}"] & \cdots \arrow[r] & P_1 \arrow[r, "d_1"] \arrow[d, "\hat{f}_1"] & P_0 \arrow[r, "\varepsilon"] \arrow[d, "\hat{f}_0"] & M \arrow[r] \arrow[d, "1"] & 0 \\
0 \arrow[r] & N \arrow[r] & U_{n-1} \arrow[r] & \cdots \arrow[r] & U_1 \arrow[r] & U_0 \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

Then $\hat{f}_n \in \operatorname{Hom}_R(P_n, N)$ and $\hat{f}_n d_{n+1} = 0$, so $\hat{f}_n$ is a cocycle. Set $f = \hat{f}_n$. Since $\hat{f}_\bullet$ is unique up to chain homotopy, any two such maps represent the same element of $\operatorname{Ext}_R^n(M,N)$.

Conversely, let $f \in \operatorname{Hom}_R(P_n, N)$ for which $f d_{n+1} = 0$. We will define an n-extension $\mathbf{f}$ of M by N corresponding to f. Let X be a pushout of $P_n \xrightarrow{d_n} P_{n-1}$ and $P_n \xrightarrow{f} N$. We may take

$$X = (P_{n-1} \oplus N)/\{(-d_n(x), f(x)) \mid x \in P_n\}.$$

Then the following diagram commutes:

$$\begin{tikzcd}
P_{n+1} \arrow[r, "d_{n+1}"] \arrow[d] & P_n \arrow[r, "d_n"] \arrow[d, "f"] & P_{n-1} \arrow[r, "d_{n-1}"] \arrow[d, "\binom{1}{0}"] & P_{n-2} \arrow[r] \arrow[d, "1"] & \cdots \arrow[r] & P_0 \arrow[r, "\varepsilon"] \arrow[d, "1"] & M \arrow[r] \arrow[d, "1"] & 0 \\
0 \arrow[r] & N \arrow[r, "\binom{0}{1}"] & X \arrow[r, "(1,0)"] & P_{n-2} \arrow[r, "d_{n-2}"] & \cdots \arrow[r] & P_0 \arrow[r, "\varepsilon"] & M \arrow[r] & 0
\end{tikzcd}$$

(Equivalently, we may replace P_n by $K_n = \mathrm{Ker}(d_{n-1})$ in the pushout diagram.) The lower sequence is an n-extension of M by N.

The following theorem is called "dimension shifting" since it allows any Ext^n group to be expressed as an Ext^1 group (shifting degree, or dimension, from n to 1). It follows from close inspection of diagrams (A.2.4) and (A.2.9).

Theorem A.3.3 (Dimension shifting)**.** *Let $\Omega^i M$ denote an ith syzygy module, and $\Omega^{-i}M$ an ith cosyzygy module of M. Then*

$$\begin{aligned}
\mathrm{Ext}^n_R(M,N) &\cong \mathrm{Ext}^1_R(\Omega^{n-1}M,N),\\
\mathrm{Ext}^n_R(M,N) &\cong \mathrm{Ext}^1_R(M,\Omega^{1-n}N)
\end{aligned}$$

for all $n \geq 2$.

As observed in the last section, if A is self-injective, then $\Omega^{-1}\Omega N$ is equivalent to N (that is, isomorphic up to projective direct summands). Thus we have the following corollary.

Corollary A.3.4. *If A is a self-injective algebra, then*

$$\mathrm{Ext}^n_A(M,N) \cong \mathrm{Ext}^n_A(\Omega M, \Omega N)$$

for all A-modules M, N and $n \geq 1$.

Now suppose M is a right R-module and N is a left R-module. Let $P \to M$ be a (right R-module) projective resolution of M. Apply $-\otimes_R N$ to obtain a sequence of $\mathbb{Z}$-modules:

$$\cdots \longrightarrow P_2 \otimes_R N \xrightarrow{d_2 \otimes 1_N} P_1 \otimes_R N \xrightarrow{d_1 \otimes 1_N} P_0 \otimes_R N \longrightarrow 0.$$

Here, as elsewhere, in order to minimize notational clutter, we suppress the subscript R on the tensor symbol $\otimes$, for maps and elements, when it is clear from context that they involve tensor products over R. We set $d_0 = 0$. This is a chain complex and we define $\mathrm{Tor}^R_n(M,N)$ to be its homology:

$$\mathrm{Tor}^R_n(M,N) = \mathrm{H}_n(P_{\bullet} \otimes_R N) = \mathrm{Ker}(d_n \otimes 1_N)/\,\mathrm{Im}(d_{n+1} \otimes 1_N).$$

By the Comparison Theorem (Theorem A.2.7), $\mathrm{Tor}^R_n(M,N)$ does not depend on choice of projective resolution of M. Note that $\mathrm{Tor}^R_0(M,N) \cong M \otimes_R N$.

Equivalently, we may define $\mathrm{Tor}_n^R(M,N)$ via a (left R-module) projective resolution of N: Let $Q \to N$ be a projective resolution of N and apply $M \otimes_R -$ to obtain a sequence

$$\cdots \longrightarrow M \otimes_R Q_2 \xrightarrow{1_M \otimes d_2} M \otimes_R Q_1 \xrightarrow{1_M \otimes d_1} M \otimes_R Q_0 \longrightarrow 0.$$

It can be shown that $\mathrm{Tor}_n^R(M,N) \cong \mathrm{H}_n(M \otimes_R Q_\bullet)$. By construction then, if either M or N is flat as an R-module, then $\mathrm{Tor}_n^R(M,N) = 0$ for all $n > 0$.

A.4. Long exact sequences

The following lemma is called the Snake Lemma since the diagram in the statement can be extended to include the indicated homomorphism ∂ and drawn with a snake curving from the top right to the bottom left.

Lemma A.4.1 (Snake Lemma). *Let U, U', V, V', W, W' be R-modules for which there is a commuting diagram with exact rows:*

$$\begin{tikzcd} & U' \arrow[r] \arrow[d, "f"] & V' \arrow[r, "p"] \arrow[d, "g"] & W' \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & U \arrow[r, "i"] & V \arrow[r] & W & \end{tikzcd}$$

There is an exact sequence

$$\mathrm{Ker}(f) \to \mathrm{Ker}(g) \to \mathrm{Ker}(h) \xrightarrow{\partial} \mathrm{Coker}(f) \to \mathrm{Coker}(g) \to \mathrm{Coker}(h)$$

where $\partial = i^{-1}gp^{-1}(w')$ for all $w' \in \mathrm{Ker}(h)$. If the map $U' \to V'$ is injective, then $\mathrm{Ker}(f) \to \mathrm{Ker}(g)$ is injective, and if $V \to W$ is surjective, then $\mathrm{Coker}(g) \to \mathrm{Coker}(h)$ is surjective.

By the notation $p^{-1}(w')$ in the lemma, we mean any element in the inverse image of w'. Its value under $i^{-1}g$ followed by projection to $\mathrm{Coker}(f)$ will not depend on the choice.

A consequence of the Snake Lemma (Lemma A.4.1) is the following theorem. A short exact sequence of complexes is a sequence

$$0 \to U_\bullet \xrightarrow{f_\bullet} V_\bullet \xrightarrow{g_\bullet} W_\bullet \to 0$$

where $f_\bullet$, $g_\bullet$ are chain maps and for each i, f_i is injective, g_i is surjective, and $\mathrm{Im}(f_i) = \mathrm{Ker}(g_i)$.

Theorem A.4.2. *Let $0 \to U_\bullet \xrightarrow{f_\bullet} V_\bullet \xrightarrow{g_\bullet} W_\bullet \to 0$ be a short exact sequence of complexes. For each n, there is an abelian group homomorphism ∂_n : $\mathrm{H}_n(W_\bullet) \to \mathrm{H}_{n-1}(U_\bullet)$ such that*

$$\cdots \longrightarrow \mathrm{H}_{n+1}(W) \xrightarrow{\partial_{n+1}} \mathrm{H}_n(U) \xrightarrow{\overline{f}_n} \mathrm{H}_n(V) \xrightarrow{\overline{g}_n} \mathrm{H}_n(W) \xrightarrow{\partial_n} \cdots$$

is an exact sequence, where $\overline{f}_n, \overline{g}_n$ *denote the maps induced by* f_n, g_n.

The homomorphisms ∂_n in the theorem are called *connecting homomorphisms.*

The following lemma is called the Horseshoe Lemma due to the shape of the diagram in the statement.

Lemma A.4.3 (Horseshoe Lemma)**.** *Let* U', U, U'' *be* R*-modules for which there is an exact sequence* $0 \to U' \to U \to U'' \to 0$*, and let* $P'_\bullet \to U'$*,* $P''_\bullet \to U''$ *be projective resolutions of* U' *and* U''*:*

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
\cdots \arrow[r] & P'_1 \arrow[r] & P'_0 \arrow[r] & U' \arrow[r] \arrow[d] & 0 \\
 & & & U \arrow[d] & \\
\cdots \arrow[r] & P''_1 \arrow[r] & P''_0 \arrow[r] & U'' \arrow[r] \arrow[d] & 0 \\
 & & & 0 &
\end{tikzcd}$$

For each n*, let* $P_n = P'_n \oplus P''_n$*. Then there are differentials* d_i *for which* $\cdots \to P_1 \xrightarrow{d_1} P_0 \to U \to 0$ *is a projective resolution of* U*, and the right column lifts to an exact sequence of complexes* $0 \to P'_\bullet \xrightarrow{\iota_\bullet} P_\bullet \xrightarrow{\pi_\bullet} P''_\bullet \to 0$ *with* $\iota_\bullet, \pi_\bullet$ *the standard inclusion and projection maps, respectively.*

The Horseshoe Lemma (Lemma A.4.3) is used in conjunction with Theorem A.4.2 to obtain the following four long exact sequences.

Theorem A.4.4 (First long exact sequence for Ext)**.** *Let* U *be an* R*-module and let* $0 \to V' \to V \to V'' \to 0$ *be an exact sequence of* R*-modules. There is an exact sequence*

$$0 \to \mathrm{Hom}_R(U, V') \longrightarrow \mathrm{Hom}_R(U, V) \longrightarrow \mathrm{Hom}_R(U, V'') \longrightarrow$$

$$\mathrm{Ext}^1_R(U, V') \longrightarrow \mathrm{Ext}^1_R(U, V) \longrightarrow \mathrm{Ext}^1_R(U, V'') \longrightarrow \mathrm{Ext}^2_R(U, V') \cdots$$

Theorem A.4.5 (Second long exact sequence for Ext)**.** *Let* V *be an* R*-module and let* $0 \to U' \to U \to U'' \to 0$ *be an exact sequence. There is an* *exact sequence*

$$0 \to \mathrm{Hom}_R(U'', V) \longrightarrow \mathrm{Hom}_R(U, V) \longrightarrow \mathrm{Hom}_R(U', V) \longrightarrow$$

$$\mathrm{Ext}^1_R(U'', V) \longrightarrow \mathrm{Ext}^1_R(U, V) \longrightarrow \mathrm{Ext}^1_R(U', V) \longrightarrow \mathrm{Ext}^2_R(U'', V) \cdots$$

Theorem A.4.6 (First long exact sequence for Tor)**.** *Let V be a left R-module and let $0 \to U' \to U \to U'' \to 0$ be an exact sequence of right R-modules. There is an exact sequence*

$$\cdots \longrightarrow \mathrm{Tor}^R_2(U'', V) \longrightarrow \mathrm{Tor}^R_1(U', V) \longrightarrow \mathrm{Tor}^R_1(U, V) \longrightarrow$$

$$\mathrm{Tor}^R_1(U'', V) \longrightarrow U' \otimes_R V \longrightarrow U \otimes_R V \longrightarrow U'' \otimes_R V \longrightarrow 0.$$

Theorem A.4.7 (Second long exact sequence for Tor)**.** *Let U be a right R-module and let $0 \to V' \to V \to V'' \to 0$ be an exact sequence of left R-modules. There is an exact sequence*

$$\cdots \longrightarrow \mathrm{Tor}^R_2(U, V'') \longrightarrow \mathrm{Tor}^R_1(U, V') \longrightarrow \mathrm{Tor}^R_1(U, V) \longrightarrow$$

$$\mathrm{Tor}^R_1(U, V'') \longrightarrow U \otimes_R V' \longrightarrow U \otimes_R V \longrightarrow U \otimes_R V'' \longrightarrow 0.$$

A.5. Bicomplexes

Let R be a ring. Sometimes we will take $R = \mathbb{Z}$ or $R = k$ (a field).

A *bicomplex* (or *double complex*) of R-modules is a set $B = \{B_{i,j}\}_{i,j \in \mathbb{Z}}$ of R-modules B_{ij} with maps

$$d^h_{i,j} : B_{i,j} \to B_{i-1,j} \quad \text{and} \quad d^v_{i,j} : B_{i,j} \to B_{i,j-1}$$

(called *horizontal* and *vertical* differentials, respectively) such that $d^h d^h = 0$, $d^v d^v = 0$, and $d^v d^h + d^h d^v = 0$. We say that B is *bounded* if for each n, there are finitely many $B_{i,j}$ with $i + j = n$ that are nonzero. The *total complex* of the bicomplex B is

$$\mathrm{Tot}(B)_n = \oplus_{i+j=n} B_{i,j}$$

with differential $d = d^h + d^v$. (To be more precise, we may write $\mathrm{Tot}^{\oplus}(B)_n = \oplus_{i+j=n} B_{i,j}$ and $\mathrm{Tot}^{\Pi}(B)_n = \prod_{i+j=n} B_{ij}$. We will generally work with bounded complexes, in which case there is no distinction.) When we refer to the homology of the bicomplex B, we mean the homology of its total complex.

Important examples of bicomplexes are tensor product complexes and Hom complexes, as we define next.

Tensor product complexes. Let $(C_\bullet, d_\bullet^C)$, $(D_\bullet, d_\bullet^D)$ be complexes of right and left R-modules, respectively. Let $B_{i,j} = C_i \otimes_R D_j$ with

$$d_{i,j}^h(x \otimes y) = d_i^C(x) \otimes y \quad \text{and} \quad d_{i,j}^v(x \otimes y) = (-1)^i x \otimes d_j^D(y)$$

for all $x \in C_i$, $y \in D_j$. Written a different way, we have

$$d_{i,j}^h = d_i^C \otimes 1_D \quad \text{and} \quad d_{i,j}^v = 1_C \otimes d_j^D.$$

(We use the Koszul sign convention that

$$(f \otimes g)(v \otimes w) = (-1)^{|g||v|} f(v) \otimes g(w)$$

whenever V, W are graded R-modules, f, g are homogeneous graded R-module homomorphisms, and $v \in V$, $w \in W$ are homogeneous elements.) Then $B_{\bullet,\bullet}$ is a bicomplex of $\mathbb{Z}$-modules. (If R is commutative, then B is a bicomplex of R-modules.) We sometimes denote $B_{\bullet,\bullet}$ by $C_\bullet \otimes_R D_\bullet$ or $C \otimes_R D$.

Theorem A.5.1 (Künneth Theorem)**.** *Let $C_\bullet$ and $D_\bullet$ be complexes of right and left R-modules, respectively, for which C_n and $d(C_n)$ are flat R-modules for all $n \in \mathbb{Z}$. Then for all $n \in \mathbb{Z}$, there is a short exact sequence:*

$$0 \longrightarrow \oplus_{i+j=n} \mathrm{H}_i(C) \otimes_R \mathrm{H}_j(D) \longrightarrow \mathrm{H}_n(C \otimes_R D) \longrightarrow \oplus_{i+j=n} \mathrm{Tor}_1^R(\mathrm{H}_i(C), \mathrm{H}_j(D)) \longrightarrow 0.$$

Remark A.5.2. The hypothesis that C_n and $d(C_n)$ are flat can be replaced by the hypothesis that D_n and $d(D_n)$ are flat as left R-modules.

Viewing a module as a complex concentrated in degree 0, with differentials all 0, we obtain the following corollary.

Theorem A.5.3 (Universal Coefficients Theorem)**.** *Let C be a complex of right R-modules in which all C_n, $d(C_n)$ are flat, and let M be a left R-module. There is a short exact sequence*

$$0 \longrightarrow \mathrm{H}_n(C) \otimes_R M \longrightarrow \mathrm{H}_n(C \otimes_R M) \longrightarrow \mathrm{Tor}_1^R(\mathrm{H}_{n-1}(C), M) \longrightarrow 0.$$

If C is quasi-isomorphic to C' and D is quasi-isomorphic to D', then $C \otimes_R D$ is quasi-isomorphic to $C' \otimes_R D'$, via tensor product maps.

Hom complexes. Let $(C_\bullet, D_\bullet^C)$, $(D_\bullet, d_\bullet^D)$ be complexes of left R-modules. Let $B_{i,j} = \mathrm{Hom}_R(C_i, D_j)$ with

$$d_{i,j}^h(f) = (-1)^{i-j+1} f d_{i+1}^C \quad \text{and} \quad d_{i,j}^v(f) = d_j^D f$$

for all $f \in \mathrm{Hom}_R(C_i, D_j)$. Then $B_{\bullet,\bullet}$ is a bicomplex of $\mathbb{Z}$-modules. We caution that there are other choices of sign conventions on Hom complexes in the literature. It is common instead first to reindex so that either C or D becomes a cocomplex, but this choice will suffice for our purposes. We sometimes denote $B_{\bullet,\bullet}$ by $\mathrm{Hom}_R(C_\bullet, D_\bullet)$ or $\mathrm{Hom}_R(C, D)$.

If C is quasi-isomorphic to C' and D is quasi-isomorphic to D', then $\mathrm{Hom}_R(C, D)$ is quasi-isomorphic to $\mathrm{Hom}_R(C', D')$.

A standard Hom complex arises as follows. Let $P_\bullet \to M$ be a projective resolution of an R-module M. Then $\mathrm{Hom}_R(P_\bullet, P_\bullet)$ is a bicomplex as described above. Since $P_\bullet$ is quasi-isomorphic to M as a complex concentrated in degree 0, $\mathrm{Hom}_R(P_\bullet, P_\bullet)$ is quasi-isomorphic to $\mathrm{Hom}_R(P_\bullet, M)$, whose cohomology is $\mathrm{Ext}^*_R(M, M)$. Moreover, a calculation shows that $\mathrm{Hom}_R(P_\bullet, P_\bullet)$ is a differential graded algebra under composition of functions.

Theorem A.5.4 (Acyclic Assembly Lemma)**.** *Let B be a bounded bicomplex of R-modules. Then* $\mathrm{Tot}(B)$ *is acyclic if one of the following four conditions holds: $B_{i,j} = 0$ for all $j < 0$ and B has exact columns or exact rows, or $B_{i,j} = 0$ for all $i < 0$ and B has exact columns or exact rows.*

See [**Wei94**, Lemma 2.7.3] for a precise statement for unbounded complexes.

A.6. Categories, functors, derived functors

A *category* $\mathcal{C}$ is a collection of *objects* $\mathrm{Obj}(\mathcal{C})$ together with a set of *morphisms* $\mathrm{Hom}_{\mathcal{C}}(A, B)$ for each pair of objects A, B of $\mathcal{C}$, including an *identity morphism* $1_A \in \mathrm{Hom}_{\mathcal{C}}(A, A)$ for each object A and a binary operation called *composition* $\circ : \mathrm{Hom}_{\mathcal{C}}(A, B) \times \mathrm{Hom}_{\mathcal{C}}(B, C) \to \mathrm{Hom}_{\mathcal{C}}(A, C)$ for every triple A, B, C of objects of $\mathcal{C}$, such that

$$(hg)f = h(gf) \quad \text{ and } \quad 1_B f = f 1_A$$

for all $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, $g \in \mathrm{Hom}_{\mathcal{C}}(B, C)$, $h \in \mathrm{Hom}_{\mathcal{C}}(C, D)$ and objects A, B, C, D of $\mathcal{C}$. (Here, as elsewhere, we have written gf in place of $g \circ f$ to denote composition.)

A morphism $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ is an *isomorphism* if there is a morphism $g \in \mathrm{Hom}_{\mathcal{C}}(B, A)$ such that $gf = 1_A$ and $fg = 1_B$.

We work primarily with categories of left or right modules or bimodules for a ring. The morphisms are module homomorphisms, the identity morphism is the identity homomorphism, and composition is function composition. If R is a ring, we use the notation R-Mod (respectively, R-mod) to denote the categories of all left R-modules (respectively, all finitely generated left R-modules). The notation Mod-R (respectively, mod-R) denotes similar categories of right R-modules. We abbreviate $\mathrm{Hom}_{R\text{-Mod}}$ (respectively, $\mathrm{Hom}_{R\text{-mod}}$, $\mathrm{Hom}_{\text{Mod-}R}$, $\mathrm{Hom}_{\text{mod-}R}$) by Hom_R in all these cases, and it will always be clear from context which is meant. Note that for any pair of R-modules A, B, the set $\mathrm{Hom}_R(A, B)$ is in fact an abelian group under addition of functions.

Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *functor* $F : \mathcal{C} \to \mathcal{D}$ assigns an object $F(A)$ of $\mathcal{D}$ to each object A of $\mathcal{C}$, and a morphism $F(f) \in \mathrm{Hom}_{\mathcal{D}}(F(A), F(B))$ to each morphism $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ for each pair of objects A, B of $\mathcal{C}$ in such a way that $F(1_A) = 1_{F(A)}$ for all A and $F(g \circ f) = F(g) \circ F(f)$ for all morphisms f, g that can be composed. The identity functor $1_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ is given by $1_{\mathcal{C}}(A) = A$ and $1_{\mathcal{C}}(f) = f$ for all objects A and morphisms f of $\mathcal{C}$. We have in fact defined a *covariant functor*, to be more precise. A *contravariant functor* similarly assigns to each object A of $\mathcal{C}$ an object $F(A)$ of $\mathcal{D}$ and to each morphism $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ a morphism $F(f) \in \mathrm{Hom}_{\mathcal{D}}(F(B), F(A))$ such that $F(1_A) = 1_{F(A)}$ and $F(g \circ f) = F(f) \circ F(g)$.

Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{C} \to \mathcal{D}$ be functors. A *natural transformation* $\eta : F \to G$ assigns a morphism $\eta_A : F(A) \to G(A)$ to each object A of $\mathcal{C}$ in such a way that $G(f) \circ \eta_A = \eta_B \circ F(f)$ for all objects A, B of $\mathcal{C}$ and morphisms $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, that is the following diagram commutes:

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "\eta_A"'] & F(B) \arrow[d, "\eta_B"] \\ G(A) \arrow[r, "G(f)"] & G(B) \end{tikzcd}$$

If η_A is an isomorphism for each object A, we say that η is a *natural isomorphism*, and write $F \cong G$.

Two categories $\mathcal{C}$ and $\mathcal{D}$ are *equivalent* if there are functors $F : \mathcal{C} \to D$, $G : \mathcal{D} \to \mathcal{C}$ and natural isomorphisms $FG \cong 1_{\mathcal{D}}$, $GF \cong 1_{\mathcal{C}}$.

Let R and S be rings. A functor $F : R\text{-Mod} \to S\text{-Mod}$ is *additive* if F induces homomorphisms of abelian groups $\mathrm{Hom}_R(A, B) \cong \mathrm{Hom}_S(F(A), F(B))$ for all R-modules A, B. The rings R and S are *Morita equivalent* if R-Mod and S-Mod are equivalent categories via additive functors $F : R\text{-Mod} \to S\text{-Mod}$ and $G : S\text{-Mod} \to R\text{-Mod}$. Note that the original definition of Morita equivalence requires F and G to have a particular form; this definition is equivalent.

Let $\mathcal{C}$, $\mathcal{D}$ be categories and let $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$ be functors. Suppose that there are natural isomorphisms

$$\mathrm{Hom}_{\mathcal{D}}(FX, Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(X, GY)$$

for each object X of $\mathcal{C}$ and each object Y of $\mathcal{D}$. Then we say that F is a *left adjoint* to G, and that G is a *right adjoint* to F.

Examples of adjoint functors are provided by induction and coinduction of modules as follows. Let A be a k-algebra and let B be a k-subalgebra of A. Let N be a B-module. The *induced* (also called *tensor induced*) A-module is $A \otimes_B N$ with action of A given by multiplication on the left factor

A. The *coinduced* A-module is $\mathrm{Hom}_B(A, N)$ with action of A given by

$$(a \cdot f)(a') = f(a'a)$$

for all $a, a' \in A$ and $f \in \mathrm{Hom}_B(A, N)$.

The following lemma is a statement about adjoint functors.

Lemma A.6.1 (Nakayama relations)**.** *Let A be a k-algebra and let B be a k-subalgebra of A. Let M be an A-module and let N be a B-module. Then*

$$\begin{aligned} \mathrm{Hom}_B(N, M) &\cong \mathrm{Hom}_A(A \otimes_B N, M), \\ \mathrm{Hom}_B(M, N) &\cong \mathrm{Hom}_A(M, \mathrm{Hom}_B(A, N)). \end{aligned}$$

That is, restriction from A to B has a left adjoint given by induction and a right adjoint given by coinduction.

The following lemma is a consequence of Lemma A.6.1.

Lemma A.6.2 (Eckmann-Shapiro Lemma)**.** *Let A be a k-algebra and let B be a k-subalgebra of A such that A is projective as a right B-module. Let M be an A-module and let N be a B-module. Then*

$$\begin{aligned} \mathrm{Ext}^n_B(N, M) &\cong \mathrm{Ext}^n_A(A \otimes_B N, M), \\ \mathrm{Ext}^n_B(M, N) &\cong \mathrm{Ext}^n_A(M, \mathrm{Hom}_B(A, N)). \end{aligned}$$

More generally we will be interested in additive functors on abelian categories, which are a generalization of categories of modules that retain enough structure for homological algebra. We define these next after some other needed definitions.

A *zero object* of a category $\mathcal{C}$ is an object A such that $|\mathrm{Hom}_{\mathcal{C}}(A, B)| = 1$ and $|\mathrm{Hom}_{\mathcal{C}}(B, A)| = 1$ for all objects B of $\mathcal{C}$ (or in other words, A is both an *initial* and a *terminal* object). We often write 0 instead of A.

Let $\{A_i\}_{i \in I}$ be a set of objects A_i of $\mathcal{C}$ indexed by some set I. A *product* $\prod_{i \in I} A_i$ is an object A, together with morphisms $\pi_i \in \mathrm{Hom}_{\mathcal{C}}(A, A_i)$ for all $i \in I$ satisfying the following universal property: If B is an object of $\mathcal{C}$ and $\psi_i \in \mathrm{Hom}_{\mathcal{C}}(B, A_i)$ for all $i \in I$ then there is a unique $\theta \in \mathrm{Hom}_{\mathcal{C}}(B, A)$ such that the following diagram commutes for all $i \in I$:

$$\begin{tikzcd} & A \arrow[d, "\pi_i"] \\ B \arrow[ur, "\theta"] \arrow[r, "\psi_i"'] & A_i \end{tikzcd}$$

A *coproduct* $\coprod_{i \in I} A_i$ is an object A together with morphisms $\iota_i \in \mathrm{Hom}_{\mathcal{C}}(A_i, A)$ satisfying: If B is an object of $\mathcal{C}$ and $\phi_i \in \mathrm{Hom}_{\mathcal{C}}(A_i, B)$ for all $i \in I$, then there is a unique $\tau \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ such that the following diagram commutes for all $i \in I$:

$$\begin{tikzcd} A \arrow[dr, "\tau"] & \\ A_i \arrow[u, "\iota_i"] \arrow[r, "\phi_i"'] & B \end{tikzcd}$$

For categories of modules, product is direct product and coproduct is direct sum.

A category $\mathcal{C}$ is *additive* if $\mathrm{Hom}_{\mathcal{C}}(A, B)$ is an abelian group for every object A, B in $\mathcal{C}$, composition of morphisms is $\mathbb{Z}$-bilinear, and $\mathcal{C}$ has a zero object, finite products and coproducts.

Let $\mathcal{C}$ be an additive category and $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ for objects A, B of $\mathcal{C}$. A *kernel* of f is an object K in $\mathcal{C}$ and a morphism $j \in \mathrm{Hom}_{\mathcal{C}}(K, A)$ such that $fj = 0$, and whenever C is an object and $g \in \mathrm{Hom}_{\mathcal{C}}(C, A)$ satisfies $fg = 0$, there is a unique $\overline{g} \in \mathrm{Hom}_{\mathcal{C}}(C, K)$ such that $j\overline{g} = g$. That is, the following diagram commutes:

$$\begin{tikzcd} K \arrow[r, "j"] & A \arrow[r, "f"] & B \\ & C \arrow[ul, "\overline{g}"] \arrow[u, "g"'] & \end{tikzcd}$$

A *cokernel* of f is an object D in $\mathcal{C}$ and a morphism $p \in \mathrm{Hom}_{\mathcal{C}}(B, D)$ such that $pf = 0$, and whenever C is an object and $g \in \mathrm{Hom}_{\mathcal{C}}(B, C)$ satisfies $gf = 0$, there is a unique $\overline{g} \in \mathrm{Hom}_{\mathcal{C}}(D, C)$ such that $\overline{g}p = g$. That is, the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, "p"] \arrow[d, "g"'] & D \arrow[dl, "\overline{g}"] \\ & C & \end{tikzcd}$$

Let $\mathcal{C}$ be a category. Let A, B be objects of $\mathcal{C}$ and $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$. Then f is a *monomorphism* if whenever C is an object of $\mathcal{C}$ and $g, h \in \mathrm{Hom}_{\mathcal{C}}(C, A)$, if $fg = fh$ then $g = h$. The morphism f is an *epimorphism* if whenever C is an object of $\mathcal{C}$ and $g, h \in \mathrm{Hom}_{\mathcal{C}}(B, C)$, if $gf = hg$ then $g = h$.

A category $\mathcal{C}$ is *abelian* if it is additive, every morphism has both a kernel and a cokernel, every monomorphism is a kernel, and every epimorphism is a cokernel. Categories of R-modules are abelian.

Let $\mathcal{C}$ be an abelian category. A sequence $A \xrightarrow{f} B \xrightarrow{g} C$ of morphisms f, g in $\mathcal{C}$ is *exact* if f is a kernel of g, and g is a cokernel of f. Projective and injective objects of $\mathcal{C}$ can be defined via diagrams (A.2.1) and (A.2.2), as well as projective and injective resolutions (which do not always exist in general). Many of the standard homological constructions and properties of the previous sections make sense in any abelian category.

Let $\mathcal{C}, \mathcal{D}$ be abelian categories. A covariant functor $F : \mathcal{C} \to \mathcal{D}$ is *left exact* (respectively, *right exact*) if for every exact sequence $0 \to A \to B \to C$ (respectively, $A \to B \to C \to 0$), the sequence

$$0 \to F(A) \to F(B) \to F(C)$$

is exact (respectively, $F(A) \to F(B) \to F(C) \to 0$ is exact). For example, $\mathrm{Hom}_R(D, -)$ is left exact. A contravariant functor $F : \mathcal{C} \to \mathcal{D}$ is *left exact* (respectively, *right exact*) if for every exact sequence $A \to B \to C \to 0$ (respectively, $0 \to A \to B \to C$), the sequence

$$0 \to F(C) \to F(B) \to F(A)$$

is exact (respectively, $F(C) \to F(B) \to F(A) \to 0$ is exact). For example, $\mathrm{Hom}_R(-, D)$ is left exact. In either case, F is *exact* if it is both left and right exact.

Let $\mathcal{C}, \mathcal{D}$ be abelian categories and $F : \mathcal{C} \to \mathcal{D}$ an additive (covariant) functor. Assume $\mathcal{C}$ *has enough projectives*, that is, assume that for each object A of $\mathcal{C}$, there is an epimorphism from a projective object in $\mathcal{C}$ to A. For each object A in $\mathcal{C}$ choose a projective resolution $P_\bullet$ of A, which exists since $\mathcal{C}$ has enough projectives. Apply the functor F:

$$F(P_\bullet): \qquad \cdots \longrightarrow F(P_2) \xrightarrow{F(d_2)} F(P_1) \xrightarrow{F(d_1)} F(P_0) \longrightarrow 0.$$

Then $F(P_\bullet)$ is a complex and we define the *left derived functor* of F to be $L_\bullet F$ where $L_n F(A) = \mathrm{H}_n(F(P_\bullet))$. Note that if F is right exact, then $L_0 F(A) \cong F(A)$. (The adjective "left" here indicates the objects are on the left with 0 at the end.) A typical example is: Let R be a ring, $\mathcal{C} = $ Mod-R, $\mathcal{D} = \mathbb{Z}$-Mod, B an object of R-Mod, and F the functor $- \otimes_R B$. Then

$$L_n F(A) = \mathrm{Tor}_n^R(A, B).$$

Assume $\mathcal{D}$ *has enough injectives*, that is, assume that for each object A of $\mathcal{C}$, there is a monomorphism from A to an injective object. For each object B in $\mathcal{C}$ choose an injective resolution $I_\bullet$ of B. Apply the functor F:

$$F(I_\bullet): \qquad 0 \longrightarrow F(I_0) \longrightarrow F(I_1) \longrightarrow F(I_2) \longrightarrow \cdots$$

Then $F(I_\bullet)$ is a complex and we define the *right derived functor* of F to be $R^\bullet F$ where $R^n F(B) = \mathrm{H}^n(F(I_\bullet))$. Note that if F is left exact, then $R^0 F(B) \cong F(B)$. (The adjective "right" here indicates the objects are on the right with 0 at the beginning.) A typical example is: Let R be a ring, $\mathcal{C} = R$-Mod, $\mathcal{D} - \mathbb{Z}$-Mod, A an object of $\mathcal{C}$, and $F(B) = \mathrm{Hom}_R(A, B)$. Then

$$R^n F(B) = \mathrm{Ext}_R^n(A, B).$$

Similarly one defines derived functors of contravariant functors: For a right derived functor, use a projective resolution, and for a left derived functor, use an injective resolution.

A.7. Spectral sequences

We will define cohomology spectral sequences here; homology spectral sequences are similar but with arrows reversed.

Definition A.7.1. A *cohomology spectral sequence* in an abelian category $\mathcal{C}$ is a set $\{E_r^{pq} \mid p,q,r \in \mathbb{Z},\ r \geq 0\}$ of objects in $\mathcal{C}$, together with morphisms

$$d_r^{pq} : E_r^{pq} \to E_r^{p+r,q-r+1}$$

for which $d_r^2 = 0$ and $E_{r+1} \cong \mathrm{H}^\bullet(E_r)$, that is, for all p, q, r,

$$E_{r+1}^{p,q} \cong \mathrm{Ker}(d_r^{pq}) / \mathrm{Im}(d_r^{p-r,q+r-1}).$$

For each r, the set E_r of objects E_r^{pq} together with the morphisms d_r^{pq} is the *rth page* of the spectral sequence. A page can be visualized in a plane, such as pages E_0, E_1, E_2 below.

$\underline{E_0}$

$$\begin{tikzcd}
 & & \\
E_0^{02} \arrow[u] & E_0^{12} \arrow[u] & E_0^{22} \arrow[u] \\
E_0^{01} \arrow[u] & E_0^{11} \arrow[u] & E_0^{21} \arrow[u] \\
E_0^{00} \arrow[u] & E_0^{10} \arrow[u] & E_0^{20} \arrow[u]
\end{tikzcd}$$

$\underline{E_1}$

$$\begin{tikzcd}
 \arrow[r] & E_1^{02} \arrow[r] & E_1^{12} \arrow[r] & E_1^{22} \arrow[r] & \\
 \arrow[r] & E_1^{01} \arrow[r] & E_1^{11} \arrow[r] & E_1^{21} \arrow[r] & \\
 \arrow[r] & E_1^{00} \arrow[r] & E_1^{10} \arrow[r] & E_1^{20} \arrow[r] &
\end{tikzcd}$$

$$\underline{E_2}$$

$$\begin{tikzcd}
E_2^{02} \arrow[drr] & E_2^{12} \arrow[drr] & E_2^{22} \arrow[drr] & E_2^{32} & E_2^{42} \\
E_2^{01} \arrow[drr] & E_2^{11} \arrow[drr] & E_2^{21} \arrow[drr] & E_2^{31} & E_2^{41} \\
E_2^{00} & E_2^{10} & E_2^{20} & E_2^{30} & E_2^{40}
\end{tikzcd}$$

Definition A.7.2. A spectral sequence (E, d) is *bounded* if for each $n \in \mathbb{Z}$ there are finitely many nonzero E_0^{pq} with $p + q = n$.

By its definition, in case (E, d) is bounded, for each pair p, q, there is an r_0 (depending on p, q) such that $E_r^{pq} \cong E_{r_0}^{pq}$ for all $r \geq r_0$. In this case, we write

$$E_\infty^{pq} = E_{r_0}^{pq}.$$

Definition A.7.3. A bounded spectral sequence (E, d) *converges* if there is a family $\{H^n \mid n \in \mathbb{Z}\}$ of objects of $\mathcal{C}$, each having a finite filtration

$$0 = F^t H^n \subset \cdots \subset F^{p+1} H^n \subset F^p H^n \subset F^{p-1} H^n \subset \cdots \subset F^s H^n = H^n,$$

and isomorphisms $E_\infty^{pq} \cong F^p H^{p+q} / F^{p+1} H^{p+q}$ for all p, q.

In this book we will use spectral sequences associated to double complexes, defined next. In this context, common notation for a double complex is $B = (B^{pq}, d', d'')$ where d' and d'' are the horizontal and vertical differentials. For each n, write $B^n = \mathrm{Tot}(B)_n = \oplus_{p+q=n} B^{p,q}$, a complex with $d = d' + d''$. The notation B will then sometimes refer to this complex, when no confusion will arise.

For each p, n, let

$$F^p B^n = \oplus_{p' \geq p} B^{p', n-p'}. \tag{A.7.4}$$

That is, we truncate the double complex at the pth column, replacing all objects to the left of this column by 0, and then sum over diagonal lines whose indices sum to a fixed value n.

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \\
B^{p,2} \arrow[u] \arrow[r] & B^{p+1,2} \arrow[u] \arrow[r] & B^{p+2,2} \arrow[u] \arrow[r] & \cdots \\
B^{p,1} \arrow[u] \arrow[r] & B^{p+1,1} \arrow[u] \arrow[r] & B^{p+2,1} \arrow[u] \arrow[r] & \cdots \\
B^{p,0} \arrow[u] \arrow[r] & B^{p+1,0} \arrow[u] \arrow[r] & B^{p+2,0} \arrow[u] \arrow[r] & \cdots \\
\vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] &
\end{tikzcd}$$

If B is bounded, then for each n, this yields a finite filtration of B^n. If B is a first quadrant double complex (that is, $B^{pq} = 0$ whenever $p < 0$ or $q < 0$), then $F^0 B^n = B^n$ and $F^p B^n = 0$ for all $p > n$.

For each p, q, r, let

$$C_r^{pq} = \{x \in F^p B^{p+q} \mid d(x) \in F^{p+r} B^{p+q+1}\}.$$

In particular, $C_0^{pq} = F^p B^{p+q}$ by definition. Also by definition, if $x \in C_r^{pq}$, then $d(x)$ has component 0 within the band $p \leq p' \leq p + r$, in order that $d(x)$ be in $F^{p+r} B^{p+q+1}$ as specified. Let $E_0^{pq} = C_0^{pq}$ and

(A.7.5)
$$E_r^{pq} = \frac{C_r^{pq} + F^{p+1} B^{p+q}}{d(C_{r-1}^{p-r+1,q+r-2}) + F^{p+1} B^{p+q}}$$

for each $r > 0$ and $p, q \in \mathbb{Z}$. By the definitions there are induced morphisms

$$d_r^{pq} : E_r^{pq} \to E_r^{p+r,q-r+1}$$

for which $d_r^2 = 0$. By the definitions,

$$\mathrm{H}^*(E_r) \cong E_{r+1}.$$

Note that $E_1^{p,q}$ is often written $\mathrm{H}''(B)^{p,q}$, that is the cohomology of B with vertical differentials only, and E_2^{pq} is often written $\mathrm{H}' \mathrm{H}''(B)^{p,q}$, the cohomology of $\mathrm{H}''(B)^{pq}$ with respect to the differential induced by horizontal differentials on B only.

We have assumed B is bounded, and so for each p, q, there is an r_0 for which the differentials $d_{r_0}^{pq}$ as well as $d_{r_0}^{p-r_0,q+r_0-1}$ (that is, those starting and ending at position p, q) are zero maps. So $E_\infty^{pq} = E_{r_0}^{pq}$.

Let $p, q \in \mathbb{Z}$, $n = p + q$, and let $x \in B^{p,q}$ be a cocycle such that $x \notin F^{p+1} B^n$. Then x determines an element of E_r^{pq} for all $r \geq 1$ and d_r is 0 on the corresponding element of E_∞^{pq}. This describes a morphism from $F^p\,\mathrm{H}^{p+q}(B)$, which is the image of $\mathrm{H}^{p+q}(F^pB)$ in $\mathrm{H}^{p+q}(B)$, to E_∞^{pq}. Moreover, this is an epimorphism since $E_\infty^{pq} = E_{r_0}^{pq}$ for some r_0. The kernel of the epimorphism is $F^{p+1}\,\mathrm{H}^{p+q}(B)$ by the definitions. As a consequence, $\mathrm{H}^\bullet(B)$ is filtered with filtration given by $F^p\,\mathrm{H}^\bullet(B)$ and

$$F^p\,\mathrm{H}^n(B)/F^{p+1}\,\mathrm{H}^n(B) \cong E_\infty^{pq}(B)$$

for fixed $p+q=n$. That is, E_r converges to $\mathrm{H}^\bullet(B)$:

Theorem A.7.6. *Let B be a bounded bicomplex. Give B the filtration (A.7.4) and let E be the corresponding spectral sequence given by (A.7.5). Then E_r converges to* $\mathrm{H}^*(B)$*.*

The spectral sequence E is *multiplicative* if E_0 has a bigraded product

$$E_0^{p,q} \times E_0^{p',q'} \to E_0^{p+p',q+q'}$$

satisfying the Leibniz rule:

$$d(xy) = d(x)y + (-1)^{p+q}xd(y)$$

for $x \in E_0^{p,q}, y \in E_0^{p',q'}$. It follows that for all r, E_r has a bigraded product such that the Leibniz rule holds. In the context of Theorem A.7.6, if E is multiplicative, then it converges to the associated graded algebra of $\mathrm{H}^*(B)$.

Note that we could have chosen to filter the complex instead by truncating rows, resulting in another spectral sequence. Comparison of these two spectral sequences can be useful.

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