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Nonlinear backstepping tracking control of DC motor driven inverted pendulum

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Abstract. The paper presents nonlinear backstepping approach for nonlinear control design applied to a DC motor driven inverted pendulum system. The nonlinear closed-loop system designed simultaneously tracks exponentially desired reference trajectory with overdamped response specifications. A simulation example is given for illustration.

1. Introduction

Backstepping approach is a recursive procedure for systematic construction of feedback control laws with respective Lyapunov functions. The designs are related to feedback linearization [10] but backstepping allows flexibilities in the cancelation of the nonlinearities. The designer has the freedom to eliminate only the destabilizing nonlinearities, which leads to robustness of the nonlinear control law. This advantage over the feedback linearization method makes the backstepping approach preferable in industrial applications [8].

The idea of the approach has appeared in the early 1987 [1]. Since then the method has been adopted for development by various authors [5][7][9] in nonlinear [2] and in adaptive control [3][4] techniques. The first book describing the backstepping design [6] discusses SISO systems adaptive and nonlinear control with extensions to MIMO systems.

In this paper the backstepping approach is presented for nonlinear trajectory tracking control design. This allows performance specifications of the nonlinear closed loop system using reference model. The method is applied on a DC motor driven inverted pendulum nonlinear system (figure 1).
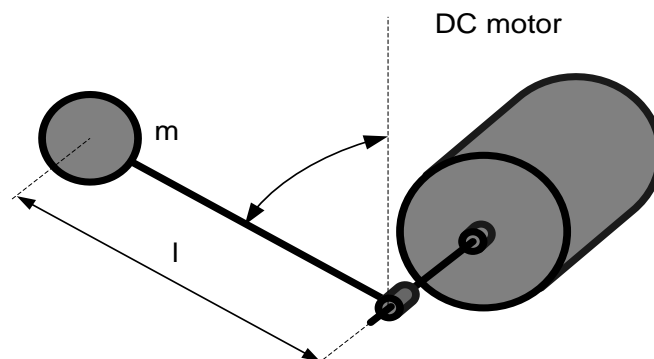

Figure 1. Inverted pendulum system.

The performance of the designed nonlinear control system is illustrated via dynamic simulation and the results are shown graphically.

The paper is organized as follows: the next section gives introductory example to the backstepping design procedure, section 3 is devoted to the inverted pendulum system modelling, section 4 - application of the approach on the objective system, section 5 is for dynamic simulation of the closed loop system and the last section is dedicated to conclusions.

2. Introduction to integrator backstepping

Demonstration of the approach as in [7] is given by considering a second order scalar system of the form

$$\dot{x}_1 = f(x_1) + x_2$$
$$\dot{x}_2 = u \tag{1}$$

The design objective is stabilization of $x_1 \to 0$ as $t \to \infty$. The only equilibrium point with $x_1 = 0$ is $(x_1, x_2) = (0, -f(0))$ corresponding to $\dot{x}_1 = 0$. The second order system requires two steps recursive design procedure. The backstepping design introduces recursively change of coordinates during the design.

Step 1

The objective is to regulate x_1 to zero, thus the first backstepping variable is chosen as

$$z_1 = x_1 \tag{2}$$

In the first equation of system (1) x_2 will be selected as *virtual control* input. The virtual control is defined as

$$x_2 = \alpha_1 + z_2 \tag{3}$$

where α_1 is called *stabilizing function* and z_2 is a new state variable. Hence the dynamics of (2) reads

$$\dot{z}_1 = f(x_1) + \alpha_1 + z_2 \tag{4}$$

The stabilizing function α_1 must provide the necessary feedback for the z_1-system. Choosing it as

$$\alpha_1 = -f(x_1) - k_1 z_1 \tag{5}$$

yields:

$$\dot{z}_1 = -k_1 z_1 + z_2$$

A candidate Lyapunov function (CLF) for the z_1-system is:

$$V_1 = \frac{1}{2} z_1^2$$

$$\dot{V}_1 = z_1 \dot{z}_1$$
$$= -k_1 z_1^2 + z_1 z_2$$

with $k_1 > 0$ the feedback gain. Hence system (4) is stabilized.

Step 2

The dynamics of z_2 is computed from (3) by time differentiation:

$$\dot{z}_2 = \dot{x}_2 - \dot{\alpha}_1$$
$$= u - \dot{\alpha}_1$$

A Lyapunov function candidate for the z_2-system is:

$$V_2 = V_1 + \frac{1}{2} z_2^2$$

$$\dot{V}_2 = \dot{V}_1 + z_2 \dot{z}_2$$
$$= (k_1 z_1^2 + z_1 z_2) + \dot{z}_2 z_2$$
$$= -k_1 z_1^2 + z_2 (u - \dot{\alpha}_1 + z_1)$$

Since the control u appears in this step no further steps are required. By choosing the control law as:

$$u = \dot{\alpha}_1 - z_1 - k_2 z_2 \tag{6}$$

with $k_2 > 0$, reads

$$\dot{V}_2 = -k_1 z_1^2 - k_2 z_2^2 < 0, \forall (z_1 \neq 0,\ z_2 \neq 0)$$

Instead of using the derivative $\dot{\alpha}_1$ in (6) we can take its analytical solution from (5) as

$$\dot{\alpha}_1 = -\frac{\partial f(x_1)}{\partial x_1} \dot{x}_1 - k_1 \dot{x}_1$$

$$= -\left(\frac{\partial f(x_1)}{\partial x_1} + k_1 \right)(f(x_1) + x_2)$$

Then the final expression for the control law is:

$$u = -\left(\frac{\partial f(x_1)}{\partial x_1} + k_1 \right)(f(x_1) + x_2) - x_1 - k_2(x_2 + f(x_1) + k_1 x_1)$$

where the z variables are transformed in original coordinates using

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 - f(x_1) - k_1 x_1 \end{bmatrix}$$

which is called *backstepping coordinate transformation* with inverse

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 + f(z_1) + k_1 z_1 \end{bmatrix}.$$

3. DC motor driven inverted pendulum modelling

The objective system model is derived using the pendulum kinematics and electromechanical energy conversion of DC motor.

The equation of motion is:

$$J_d \frac{d\omega}{dt} = T_m + T_p, \tag{7}$$

where J_d is motor shaft inertia moment, ω is the angular speed, T_m is the electromagnetic torque of the motor and T_p is the pendulum load torque. The load torque for the objective system has the form

$$T_p = -l^2 m \ddot{\theta} - m g l \sin(\theta) - b\dot{\theta}$$

where b is viscose friction coefficient, m is the attached ball mass, l is the link length, g is the gravitational constant and θ is the pendulum rotation angle.

The electromagnetic torque of the DC motor is

$$T_m = k_m i$$

where k_m is the motor torque constant.

The pendulum kinematics based on the last three equations reads

$$J\ddot{\theta} = k_m i - m g l \sin\theta - b\dot{\theta} \tag{8}$$

where

$$J = J_d + m l^2$$

is the total moment of inertia.

The electrical part of the system is modelled as current differential equation, which for DC motor is

$$\frac{di}{dt} = -\frac{k_e}{L}\dot{\theta} - \frac{R}{L}i + \frac{1}{L}u \tag{9}$$

where i is the armature current, k_e is the motor back EMF constant, R and L are the motor resistance and inductance and u is the control voltage.

The state space model state variables are selected as

$$x_1 = \theta$$

$$x_2 = \dot{\theta} = \omega$$

$$x_3 = i$$

Replacing the above in equations (8) and (9), yields the state space model of the objective system as

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -c_1 \sin x_1 + c_2 x_3 - c_6 x_2 \qquad (10a)$$

$$\dot{x}_3 = -c_3 x_2 - c_4 x_3 + c_5 u$$

$$y = x_1 \qquad (10b)$$

where y is the system output and the model parameters are

$$c_1 = \frac{mgl}{J}, c_2 = \frac{k_t}{J}, c_6 = \frac{f_m}{J}, c_3 = \frac{k_e}{L}, c_4 = \frac{R}{L}, c_5 = \frac{1}{L}. \qquad (11)$$

4. Nonlinear backstepping tracking control design

The control goal is the objective system output (the pendulum angle) (10b) to track exponentially a reference trajectory. This allows via the trajectory settings to control the performance specifications of the objective nonlinear system. The reference trajectory model is defined as

$$\dot{x}_{1r} = x_{2r}$$

$$\dot{x}_{2r} = x_{3r} \qquad (12)$$

$$\dot{x}_{3r} = v$$

where

$$v = -k_{3r} x_{3r} - k_{2r} x_{2r} - k_{1r} (x_{1r} - y_d)$$

with y_d the desired pendulum position and $y_r = x_{1r}$. The objective nonlinear system (10 is third order system, hence three recursive steps has to be performed according to the backstepping procedure. Before proceeding with these steps the tracking task is transformed into stabilization task to make the design easier. For this purpose, an error tracking system has to be designed as

$$e_1 = x_{1r} - x_1$$

$$e_2 = x_{2r} - x_2 \qquad (13)$$

$$e_3 = x_{3r} - x_3$$

with dynamics

$$\dot{e}_1 = e_2$$

$$\dot{e}_2 = x_{3r} + c_6 x_2 - c_2 x_3 + c_1 \sin(x_1) \qquad (14)$$

$$\dot{e}_3 = -c_5 u + v + c_3 x_2 + c_4 x_3$$

Now the task is to stabilize the output tracking error e_1 to zero.

Step 1:

Since the objective is to regulate the error e_1 to zero the first backstepping variable is chosen as

$$z_1 = e_1 \qquad (15)$$

In the first equation of system (14) e_2 is the selected *virtual control* input as

$$e_2 = z_2 + \alpha_1 \qquad (16)$$

Hence the dynamics of (15) reads

$$\dot{z}_1 = z_2 + \alpha_1$$

The stabilizing function

$$\alpha_1 = -k_1 z_1 \qquad (17)$$

where $k_1 > 0$ is chosen to stabilize the z_1 dynamics as

$\dot{z}_1 = -k_1 z_1 + z_2$

A candidate Lyapunov function for the z_1-system is:

$V_1 = \dfrac{1}{2} z_1^2$

$\dot{V}_1 = z_1 \dot{z}_1$

$\quad = -k_1 z_1^2 + z_1 z_2$

The new state variable z_2 will not be used in the first step, but its presence is important since z_2 has to couple the z_1-system to the system considered in the next step.

Step 2:

The dynamics of z_2 is estimated by time differentiation of (16) as

$\dot{z}_2 = x_{3r} + c_6 x_2 - c_2 x_3 - \dot{\alpha}_1 + c_1 \sin(x_1)$

in this case the virtual control is chosen to be x_3 as

$x_3 = z_3 + \alpha_2$ $\hspace{6cm}$ (18)

thus the z_2-dynamics yields

$\dot{z}_2 = c_6 x_2 + x_{3r} - c_2 (z_3 + \alpha_2) - \dot{\alpha}_1 + c_1 \sin(x_1)$.

From this step further the selection of stabilizing functions is done by recursively designed CLF. In this case the CLF is selected as

$V_2 = V_1 + \dfrac{1}{2} z_2^2$

$\dot{V}_2 = \dot{V}_1 + z_2 \dot{z}_2$

$\quad = -k_1 z_1^2 + z_1 z_2 + z_2 \left(c_6 x_2 + x_{3r} - c_2 (z_3 + \alpha_2) - \dot{\alpha}_1 + c_1 \sin(x_1) \right)$

$\quad = -k_1 z_1^2 + z_2 \left(z_1 + c_6 x_2 + x_{3r} - c_2 (z_3 + \alpha_2) - \dot{\alpha}_1 + c_1 \sin(x_1) \right)$

Now the stabilizing function α_2 is selected to make $\dot{V}_2$ negative definite as

$\alpha_2 = \dfrac{k_2 z_2}{c_2} + \dfrac{c_6 x_2 + x_{3r} + z_1 - \dot{\alpha}_1 + c_1 \sin(x_1)}{c_2}$ $\hspace{3cm}$ (19)

with $k_2 > 0$. Then the derivative $\dot{V}_2$ reads

$\dot{V}_2 = -k_1 z_1^2 - k_2 z_2^2 - c_2 z_2 z_3$

The $-c_2 z_2 z_3$ term will be eliminated in the next step.

Step 3:

The dynamics of z_3 is estimated from (18) with time differentiation as

$\dot{z}_3 = c_5 u - c_3 x_2 - c_4 x_3 - \dot{\alpha}_2$

It is evident that the control u appears in the z_3 dynamics. This shows that the design is in the final step. The last CLF is chosen as

$V_3 = V_2 + \dfrac{1}{2} z_3^2$

$\dot{V}_3 = \dot{V}_2 + z_3 \dot{z}_3$

$\quad = -k_1 z_1^2 - k_2 z_2^2 - c_2 z_2 z_3 + z_3 \left(c_5 u - c_3 x_2 - c_4 x_3 - \dot{\alpha}_2 \right)$

The final goal is to make $\dot{V}_3$ negative definite by selecting the appropriate control u. The choice of

$u = \dfrac{-k_3 z_3}{c_5} + \dfrac{c_3 x_2 + c_4 x_3 + c_2 z_2 + \dot{\alpha}_2}{c_5}$ $\hspace{4cm}$ (20)

turns the CLF to

$$\dot{V}_3 = -k_1 z_1^2 - k_2 z_2^2 - k_3 z_3^2$$

This implies that tracking control system error **z** is asymptotically stable. Moreover $\dot{V}_3$ yields

$$\dot{V}_3 = -\mathbf{z}^T \mathbf{K} \mathbf{z} \tag{21}$$

where

$$K = \begin{bmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{bmatrix} \text{ and } \mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

Since the matrix **K** is positive definite the following relation holds

$$\lambda_{min}[\mathbf{K}]\mathbf{I}_{3x3} \le \mathbf{K} \le \lambda_{max}[\mathbf{K}]\mathbf{I}_{3x3}$$

where $\lambda_{min}[\cdot]$ and $\lambda_{max}[\cdot]$ denote the minimal and the maximal eigenvalues. The slowest component of **z** is associated with the smallest eigenvalue, respectively. Considering the last relation, the slowest component of (21) will be

$$\dot{V}_3 = -\lambda_{min}[\mathbf{K}]\mathbf{z}^T\mathbf{z}$$

Thus, the considered slowest dynamics of $\dot{V}_3$ can be represented as a homogeneous equation with respect to V_3 with initial time t_0 as

$$\dot{V}_3 \le -2\lambda_{min}[\mathbf{K}] V_3, \ V(t_0) = V_0 .$$

The solution of this homogeneous equation is

$$V(t) \le e^{-2\lambda_{min}[\mathbf{K}](t-t_0)} V_0 .$$

The response $V(t)$ implies that the steady state error will be less than $1\% V_0$ when

$$\lambda_{min}[\mathbf{K}]T_E = \frac{5}{2}$$

The time T_E is the desired response time

$$T_E = \frac{5}{\lambda_{min}[\mathbf{K}]2}$$

Thus, the **z** error is exponentially stable with maximum transient response time T_E, which can be controlled by the smallest eigenvalue of the matrix **K**. The larger $\lambda_{min}[\mathbf{K}]$ leads to faster response.

In order to obtain the final form of the control input the analytical solutions of the stabilizing functions derivatives are required. Differentiating equations (17) and (19) the derivatives read

$$\dot{\alpha}_1 = k_1^2 z_1 - k_1 z_2$$

$$\dot{\alpha}_2 = c_6 x_3 + \frac{(-2k_1 + k_1^3 - k_2)z_1}{c_2} + \frac{(1 - k_1^2 - k_1 k_2 - k_2^2)z_2}{c_2} + \frac{(-c_2 k_1 - c_2 k_2)z_3}{c_2} + \frac{x_2(-c_6^2 + c_1 \cos(x_1))}{c_2}$$

$$+ \frac{v - c_1 c_6 \sin(x_1)}{c_2}$$

Then by replacing $\dot{\alpha}_2$ in equation (20) the final control input yields

$$u = \frac{(c_4 + c_6)x_3}{c_5} + \frac{(-2k_1 + k_1^3 - k_2)z_1}{c_2 c_5} + \frac{(1 + c_2^2 - k_1^2 - k_1 k_2 - k_2^2)z_2}{c_2 c_5} - \frac{(k_1 + k_2 + k_3)z_3}{c_5}$$

$$+ \frac{x_2(c_2 c_3 - c_6^2 + c_1 \cos(x_1))}{c_2 c_5} + \frac{v - c_1 c_6 \sin(x_1)}{c_2 c_5}$$

It is evident that the control depends on the backstepping variables z_1, z_2 and z_3. These variables have to be transformed in original coordinates. This is achieved by applying the backstepping diffeomorphism

$$\mathbf{z} = \mathbf{z}(\mathbf{e}, \mathbf{x}, \mathbf{x_r}) = \begin{bmatrix} e_1 \\ e_2 + k_1 e_1 \\ -\dfrac{k_1 k_2 + 1}{c_2} e1 - \dfrac{k_1 + k_2}{c_2} e_2 - \dfrac{c_6}{c_2} x_2 + x_3 - \dfrac{x_{3r}}{c_2} - \dfrac{c_1 \sin(x_1)}{c_2} \end{bmatrix} \quad (22)$$

By applying the above transformation, the closed loop system control in original coordinates yields

$$u = \frac{1}{c_2 c_5} \Big[e_1 (c_2^2 k_1 + k_3 + k_1 k_2 k_3) + e_2 (1 + c_2^2 + k_2 k_3 + k_1 (k_2 + k_3)) + v + c_2 c_3 x_2 - c_6^2 x_2$$
$$+ c_6 k_1 x_2 + c_6 k_2 x_2 + c_6 k_3 x_2 + c_2 (c_4 + c_6 - k_1 - k_2 - k_3) x_3 + (k_1 + k_2 + k_3) x_{3r} \quad (23)$$
$$+ c_1 (x_2 \cos(x1) + (-c_6 + k_1 + k_2 + k_3) \sin(x_1)) \Big]$$

With this the task for nonlinear tracking control backstepping design is fulfilled with exponential stability. The transition response time is controllable via the control law coefficients k_1, k_2 and k_3. The overall performance specifications of the closed loop system are set by the reference trajectory.

The analysis of (22) when the error vector $\mathbf{z}$ converges to zero is

$$0 = e_1$$
$$0 = e_2 + k_1 e_1 \Rightarrow e_2 = 0$$
$$0 = -\frac{k_1 k_2 + 1}{c_2} e1 - \frac{k_1 + k_2}{c_2} e_2 - \frac{c_6}{c_2} x_2 + x_3 - \frac{x_{3r}}{c_2} - \frac{c_1 \sin(x_1)}{c_2}$$
$$= -\frac{c_6}{c_2} x_2 + x_3 - \frac{x_{3r}}{c_2} - \frac{c_1 \sin(x_1)}{c_2}$$

From the last equation the expected value x_{3d} of the virtual control x_3 can be estimated as

$$x_{3d} = \frac{x_{3r} + c_1 \sin(x_1) + c_6}{c_2} . \quad (24)$$

This analysis implies that the tracking errors e_1 and e_2 will converge to zero, while the state variable x_3 selected as virtual control will converge to x_{3d} with the convergence of $\mathbf{z} \to 0$. This will be proven graphically in the next section.

5. Dynamic simulation
Equations (10), (12), (13) and (23) are simulated in the MATLAB environment. The objective system parameters are

Table 1. Inverted pendulum system parameters.

Parameter	Value	Unit
m	0.5	kg
l	0.5	m
g	9.81	$m.s^{-2}$
k_t	0.4775	V.s
k_e	0.02	Nm.s
f_m	0.3	Ω
R	2.7	mH
L	3.7×10^{-3}	$kg.\,m^2$
J_d	0.4775	V.s

The pendulum system initial conditions are selected as $x_0 = [-\pi/2, 0, 0]^T$. The reference model coefficients are chosen by pole placement to be $k_{1r} = 343$, $k_{2r} = 147$, $k_{3r} = 21$ providing triple pole $p_d = -7$. Its initial conditions are $x_{0r} = [0, 0, 0]^T$; .The backstepping control parameters are set as $k_1 = k_2 = k_3 = 10$. The closed loop system is simulated for T=4 seconds with desired angle $y_d = \pi/2$ which is changed when $t \geq 2$ to $y_d = -\pi/2$.

The desired output tracking is shown on figure **2** together with its tracking error. It is evident that the
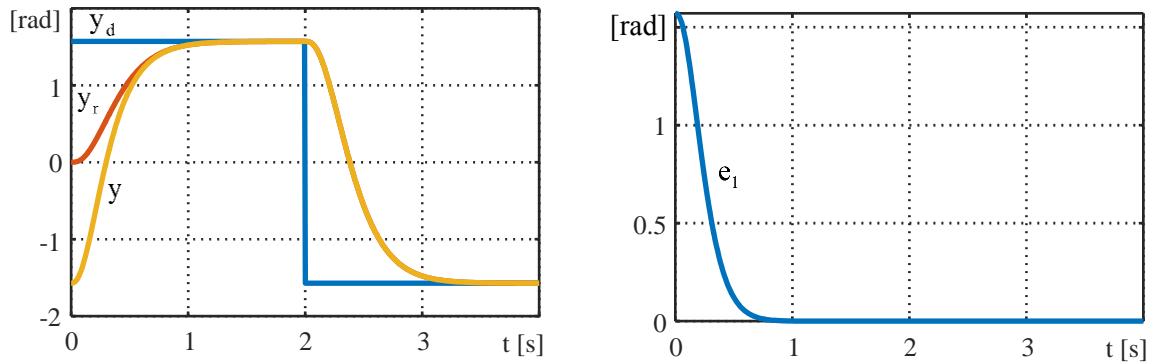

Figure 2. Output angle tracking.

system output y tracks the desired trajectory y_r without error after its convergence even if the desired angle y_d changes. Figure **3** depicts the speed tracking performance which has the same performance.
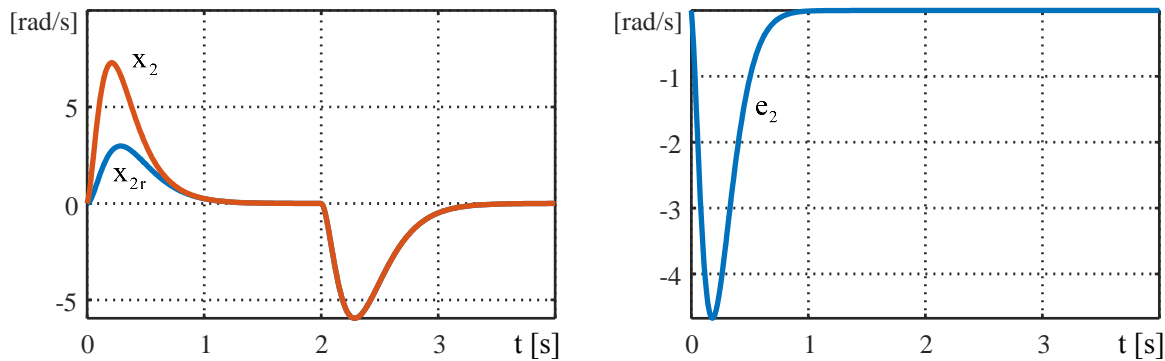

Figure 3. Speed tracking.

On figure **4** we see that the virtual control x_3 tracks it desired value according to (24). Its tracking is the reason of the zero tracking errors of the above variables. On the last graphics **5** the motor control
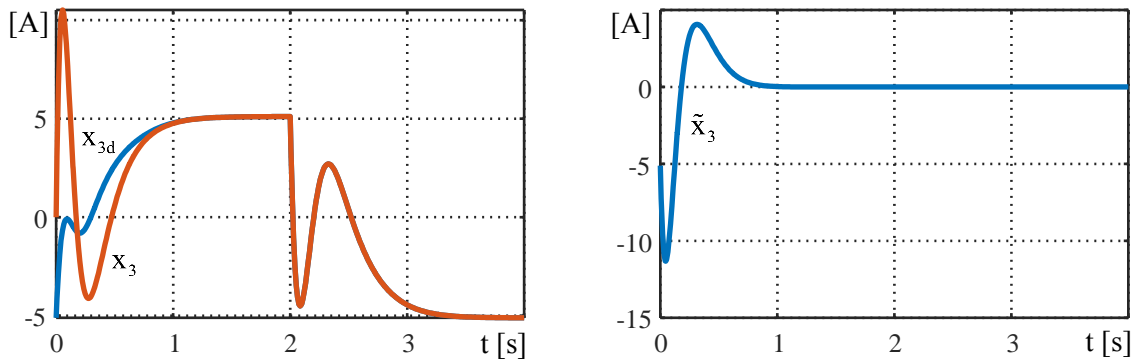

Figure 4. Current tracking.

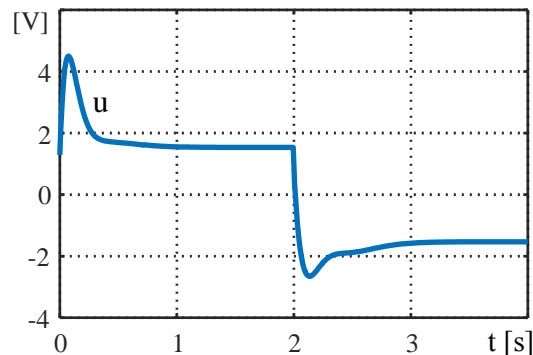voltage is shown. The last two figure are useful to check if the current and the voltage to be applied

Figure 5. Control input voltage.

are in the motor specification range. For this specific motor the maximum voltage is $V_{max} = 110[V]$ and the maximum current is $I_{max} = 100[A]$ with continuous current $I_n = 12.5[A]$ thus the control is in the motor specification range.

6. Conclusions

The paper has presented trajectory tracking control design procedure for DC motor driven inverted pendulum system using nonlinear backstepping approach. The tracking control allows specification of the controlled system output performance via reference trajectory design.

The designed nonlinear closed loop system is dynamically simulated for two transition responses of the pendulum angle. The simulation results show the overdamped performance of the output specified by the reference trajectory. The control design achieves fast pendulum system output response without exceeding the motor maximum allowed armature current.

Acknowledgement

References

[1]   Koditschek D 1987 Adaptive techniques for mechanical systems *Proc. 5th Yale Workshop on Adaptive Systems* New Haven pp 259–265
[2]   Byrnes C and Isidori I 1989 New Results and examples in nonlinear feedback stabilization *Systems & Control Letters* vol 12 pp 437–442
[3]   Narendra K and Annaswamy A 1989 *Stable adaptive systems* (New Jersy: Prentice Hall)
[4]   Landau Y 1977 *Adaptive control* (New York: Marcel Dekker)
[5]   Kanellakopoulos I, Kokotovic P V and Morse A S 1992 A Toolkit for nonlinear feedback design *Systems & Control Letters* vol 18 pp 83–92
[6]   Krstic M Kanellakopoulos I and Kokotovic P V 1995 *Nonlinear and adaptive control design* (New York: John Wiley & Sons Ltd)
[7]   Fossen T I and Strand J P 1999 Tutorial on nonlinear backstepping: Application to ship control *Modeling, identification and control* vol 20 iss 2 pp 83–134
[8]   Merzoug M S and Benalla H 2010 Nonlinear backstepping control of permanent magnet synchronous motor (PMSM), *Int. J. of Systems Control* vol 1 iss 1 pp 30–34
[9]   Sepulchre R Jankovic M and Kokotovic P 1997 *Constructive nonlinear control* (Berlin: Springer Verlag)
[10]   Astolfi A, Karagianiss D and Ortega R 2008 *Nonlinear and adaptive control with applications* (London: Springer-Verlag)