Review

Midterm Exam

- **4/26 Tuesday: 16:00**~18:45
- **E3-1(CS Bldg) Room# 1501**

- Readings: Chapter 1 ~ 12
- Topics:
  - Rendering Pipeline
  - Geometric Transformation
    - What can be done in Vertex Shader
  - 3D Rotation representation
    - Interpolation

Homework #3
Due postponed to **May 2 (Monday)** 11:59 PM

For your midterm

- Try Exercise problems in each chapter.
- Try to understand the concept behind it.
- We'll use notation given by the textbook.

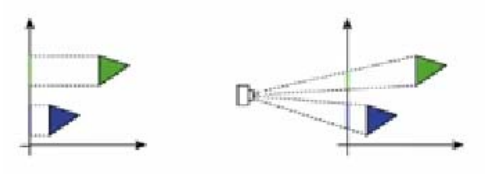
- You can bring **1 page A4-size hand-written** memo sheet for your own use in the exam.
  - No extra post-in type attachment please
  - You will have to hand-in your sheet at the end of the exam. (It will be returned!)

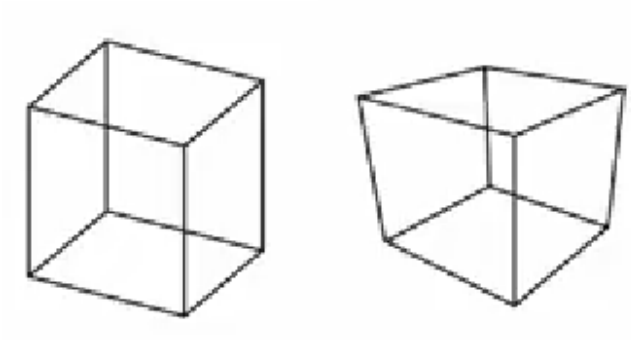
Review Topics

- Projection
- Quaternion

Projection

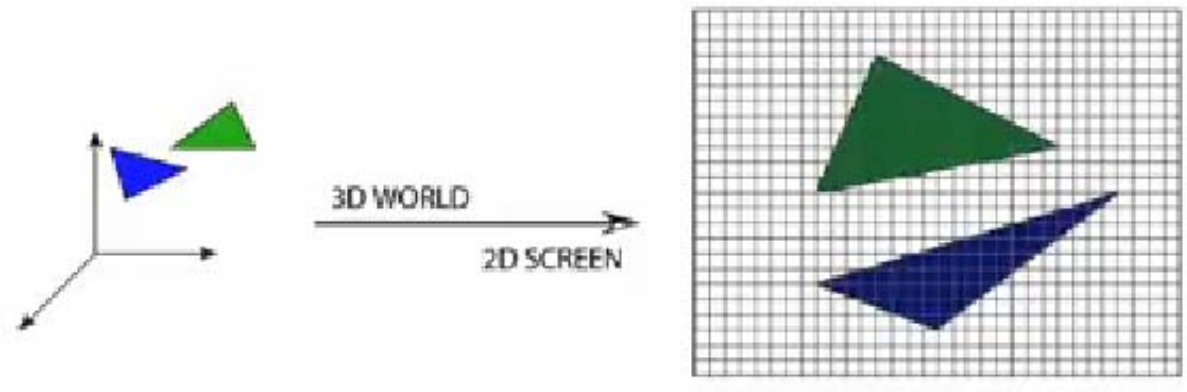
- Linear perspective
  - Linear line remains linear line (c.f. fisheye)
- Parallel projection
- Perspective projection





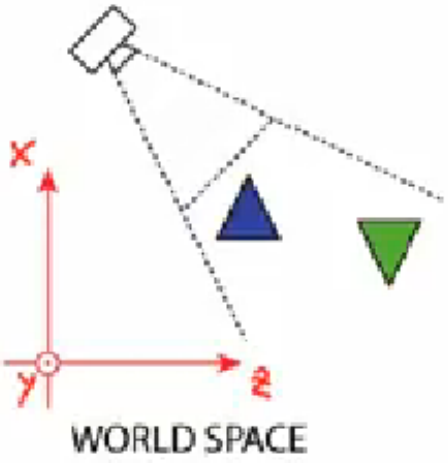
Slides captured from the lecture by Prof. Wolfgang Huerst Utrecht University, The Netherlands

- **Perspective projection**



World space

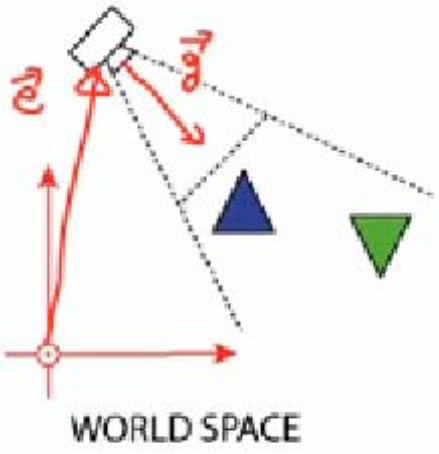
- Our 3D scene is given in world space, i.e. linear combinations of the base vectors $\vec{x}$, $\vec{y}$, and $\vec{z}$
- Given an arbitrary camera position, we want to display our 3D world in a 2D image using perspective projection



Camera position

The camera position is specified by

- the eye vector $\vec{e}$ (it's location)
- the gaze vector $\vec{g}$ (it's direction)
- the image plane (it's field of view (FOV) and distance)

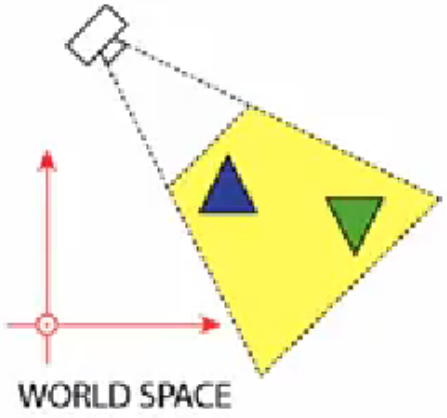


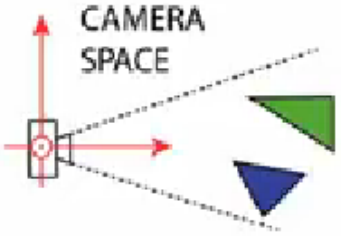
View frustum

The view frustum (aka view volume) specifies everything that the camera can see. It's defined by

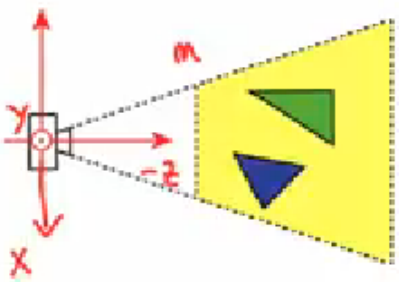
- the left plane l
- the right plane r
- the near plane n
- the far plane f
- the top plane t
- the bottom plane b

Note: for now, let's assume all our objects are completely within the view frustum





Per convention, we look into the direction of the negative Z-axis

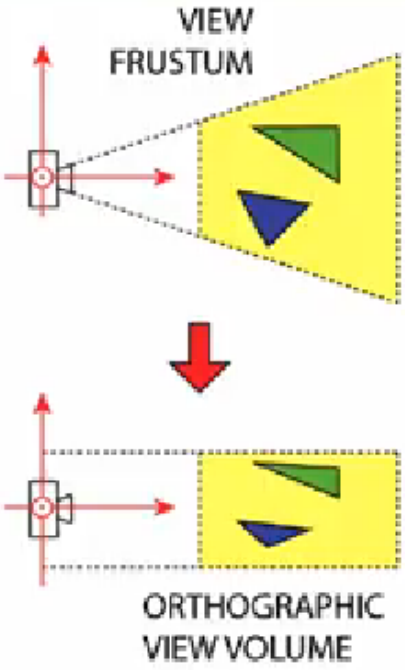


Orthographic projection

Hmm, it would be much easier if we could do parallel projection . . .

We can do that by transforming the view frustum to the orthographic view volume.

Again, this is just a matrix multiplication (but this time, it's not that simple, cf. later).

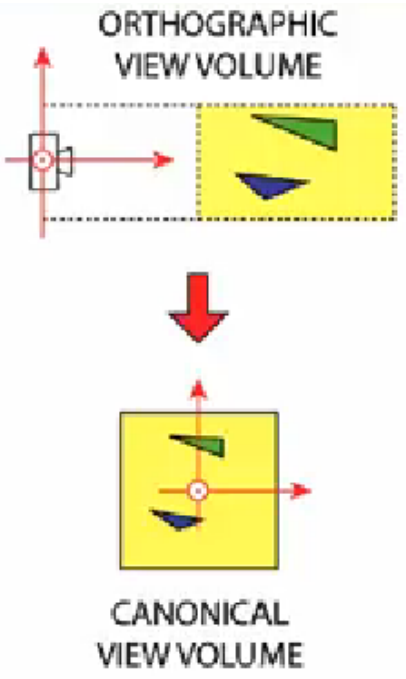


Canonical view volume

Hmm, it would be much easier if our values were between -1 and 1 . . .

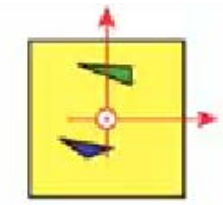
We can do that by transforming the orthographic view volume to the canonical view volume.

Again, this is just a (simple) matrix multiplication (cf. later).

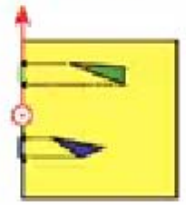


Windowing transform

Now all that's left is a parallel projection along the Z-axis (every easy) and ...



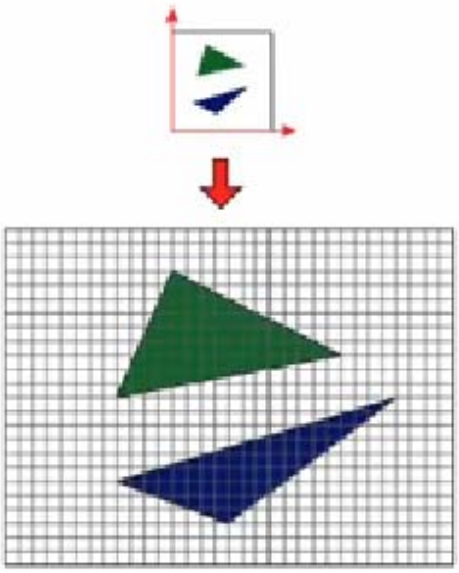
CANONICAL VIEW VOLUME



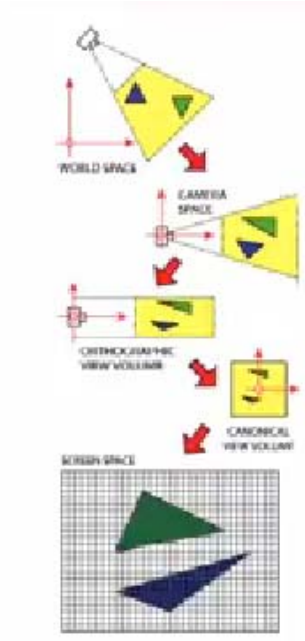
SCREEN SPACE

... a windowing transformation in order to display the square $[-1, 1]^2$ onto an $n_x \times n_y$ image.

Again, these are just some (simple) matrix multiplications (cf. later).

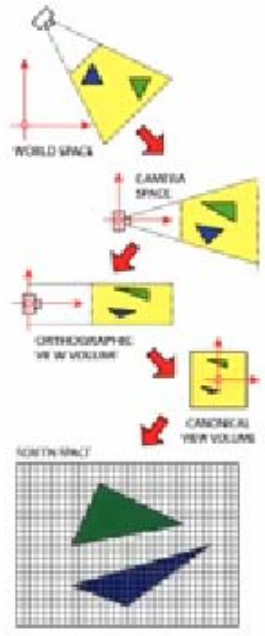


Graphics pipeline



Notice that every step in this sequence can be represented by a matrix operation, so the whole process can be applied by performing a single matrix operation! (well, almost . . .)

We call this sequence a graphics pipeline = a special software or hardware subsystem that efficiently draws 3D primitives in perspective.



Let's start with the easier stuff, e.g.

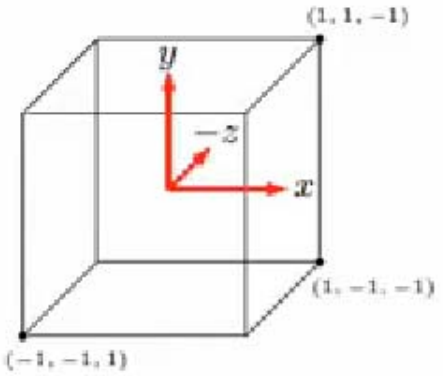
Windowing transformation
(aka viewport transformation)

How do we get the data from the canonical view volume to the screen?

The canonical view volume is a $2 \times 2 \times 2$ box, centered at the origin.

The view frustum is transformed to this box (and the objects within the view frustum undergo the same transformation).

Vertices in the canonical view volume are orthographically projected onto an $n_x \times n_y$ image.

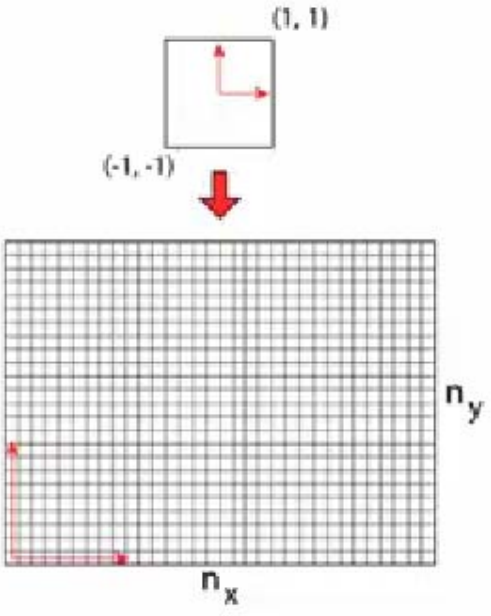


Mapping the canonical view volume

We need to map the square $[-1, 1]^2$ onto a rectangle $[0, n_x] \times [0, n_y]$.

The following matrix takes care of that:

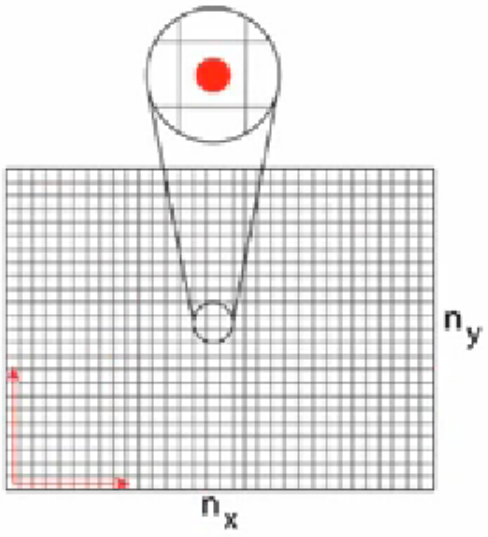
$$\begin{pmatrix} \frac{n_x}{2} & 0 & \frac{n_x}{2} \\ 0 & \frac{n_y}{2} & \frac{n_y}{2} \\ 0 & 0 & 1 \end{pmatrix}$$



In practice, pixels represent unit squares centered at integer coordinates, so we actually have to map to the rectangle $[-\frac{1}{2}, n_x - \frac{1}{2}] \times [-\frac{1}{2}, n_y - \frac{1}{2}]$.

Hence, our matrix becomes:

$$\begin{pmatrix} \frac{n_x}{2} & 0 & \frac{n_x}{2} - \frac{1}{2} \\ 0 & \frac{n_y}{2} & \frac{n_y}{2} - \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix}$$

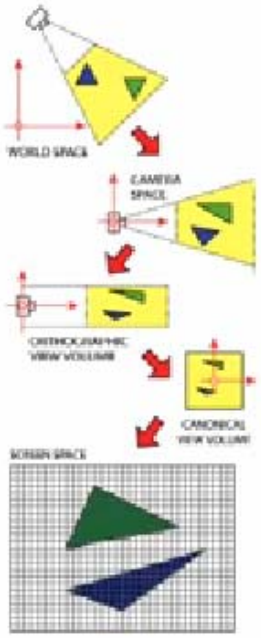


Notice that we did orthographic projection by "throwing away" the z-coordinate.

But since we want to combine all matrices in the end, we need a 4×4 matrix, so we add a row and colum that "doesn't change z".

Our final matrix for the windowing or viewport transformation is

$$M_{vp} = \begin{pmatrix} \frac{n_x}{2} & 0 & 0 & \frac{n_x}{2} - \frac{1}{2} \\ 0 & \frac{n_y}{2} & 0 & \frac{n_y}{2} - \frac{1}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



Hence, our last step will be

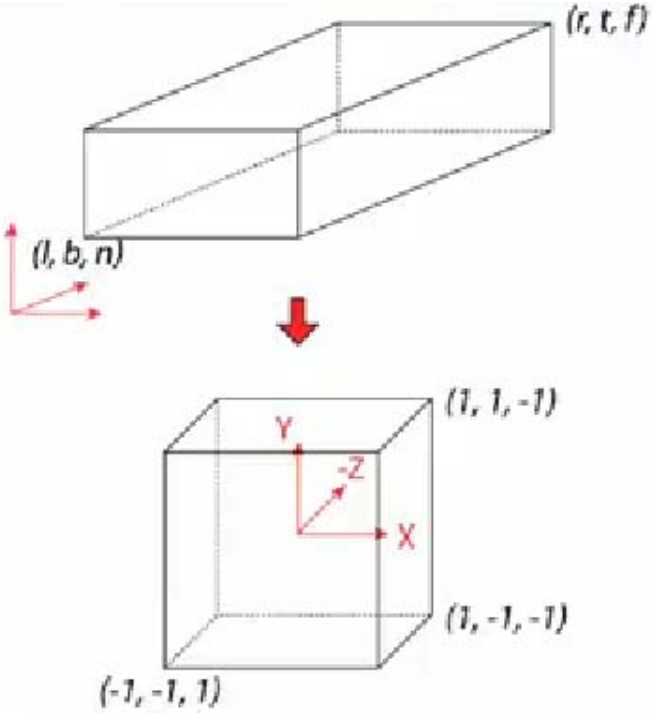
$$\begin{pmatrix} x_{screen} \\ y_{screen} \\ z_{canonical} \\ 1 \end{pmatrix} = M_{vp} \begin{pmatrix} x_{canonical} \\ y_{canonical} \\ z_{canonical} \\ 1 \end{pmatrix}$$

Ok, now let's work our way up:

How do we get the data from the orthographic view volume to the canonical view volume, i.e. ...

The orthographic view volume

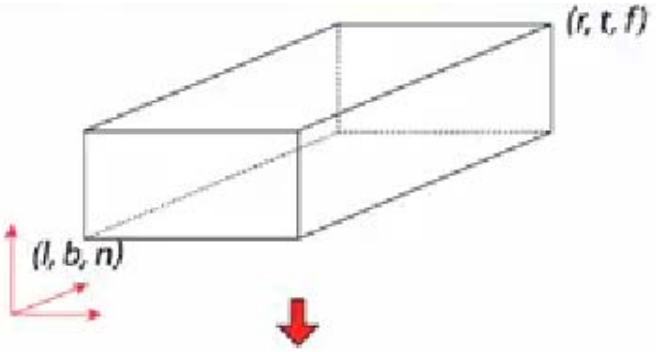
...how do we get the data from the axis-aligned box $[l, r] \times [b, t] \times [n, f]$ to a $2 \times 2 \times 2$ box around the origin?



The orthographic view volume

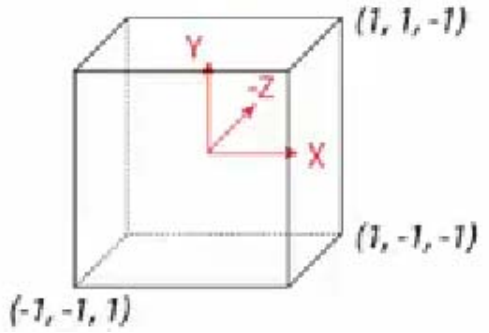
First we need to move the center to the origin:

$$\begin{pmatrix} 1 & 0 & 0 & -\frac{l+r}{2} \\ 0 & 1 & 0 & -\frac{b+t}{2} \\ 0 & 0 & 1 & -\frac{n+f}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



Then we have to scale everything to $[-1, 1]$:

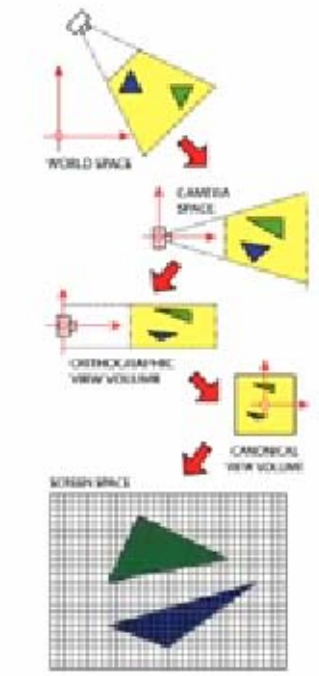
$$\begin{pmatrix} \frac{2}{r-l} & 0 & 0 & 0 \\ 0 & \frac{2}{t-b} & 0 & 0 \\ 0 & 0 & \frac{2}{n-f} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



Since these are just matrix multiplications (associative!), we can combine them into one matrix:

$$M_{orth} = \begin{pmatrix} \frac{2}{r-l} & 0 & 0 & 0 \\ 0 & \frac{2}{t-b} & 0 & 0 \\ 0 & 0 & \frac{2}{n-f} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & -\frac{l+r}{2} \\ 0 & 1 & 0 & -\frac{b+t}{2} \\ 0 & 0 & 1 & -\frac{n+f}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{2}{r-l} & 0 & 0 & -\frac{r+l}{r-l} \\ 0 & \frac{2}{t-b} & 0 & -\frac{t+b}{t-b} \\ 0 & 0 & \frac{2}{n-f} & -\frac{n+f}{n-f} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



Hence, our last step becomes

$$\begin{pmatrix} x_{pixel} \\ y_{pixel} \\ z_{canonical} \\ 1 \end{pmatrix} = M_{vp} M_{orth} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

Now, how do we get the data in the orthographic view volume?

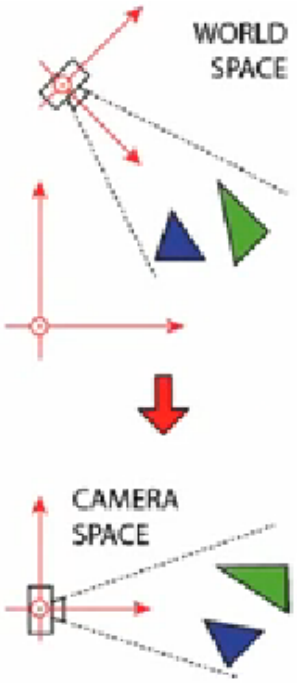
That's more difficult, so let's look at camera transformation first.

Aligning coordinate systems

How do we get the camera to the origin, i.e. how do we move from world space to camera space?

Remember:

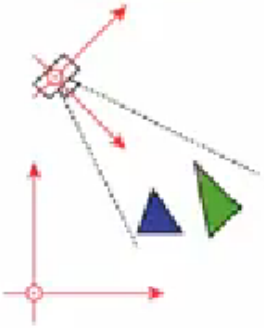
- **world space** is expressed by the base vectors $\vec{x}$, $\vec{y}$, and $\vec{z}$
- the **camera** is specified by eye vector $\vec{e}$ and gaze vector $\vec{g}$

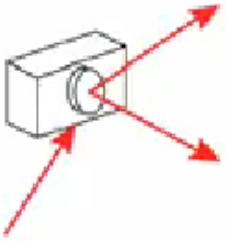


To map one space to another, we need a coordinate system for both spaces.

We can easily get that using a view up vector $\vec{t}$ i.e. a vector in the plane bisecting the viewer's head into left and right halves and "pointing to the sky"

This gives us an orthonormal base $(\vec{u}, \vec{v}, \vec{w})$ of our camera coordinate system (how?)

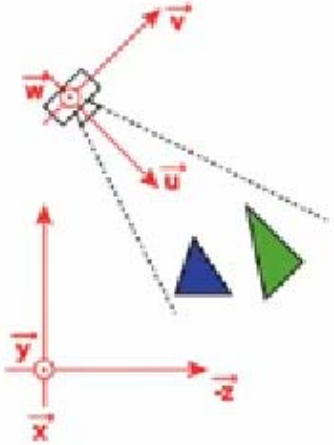


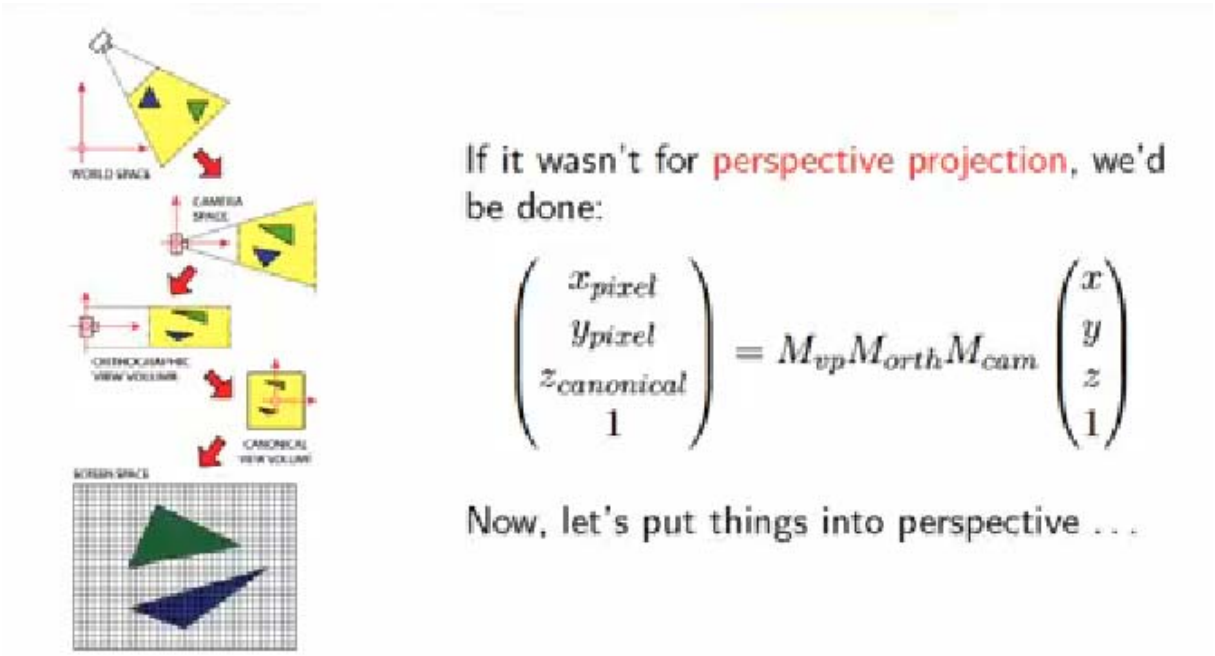


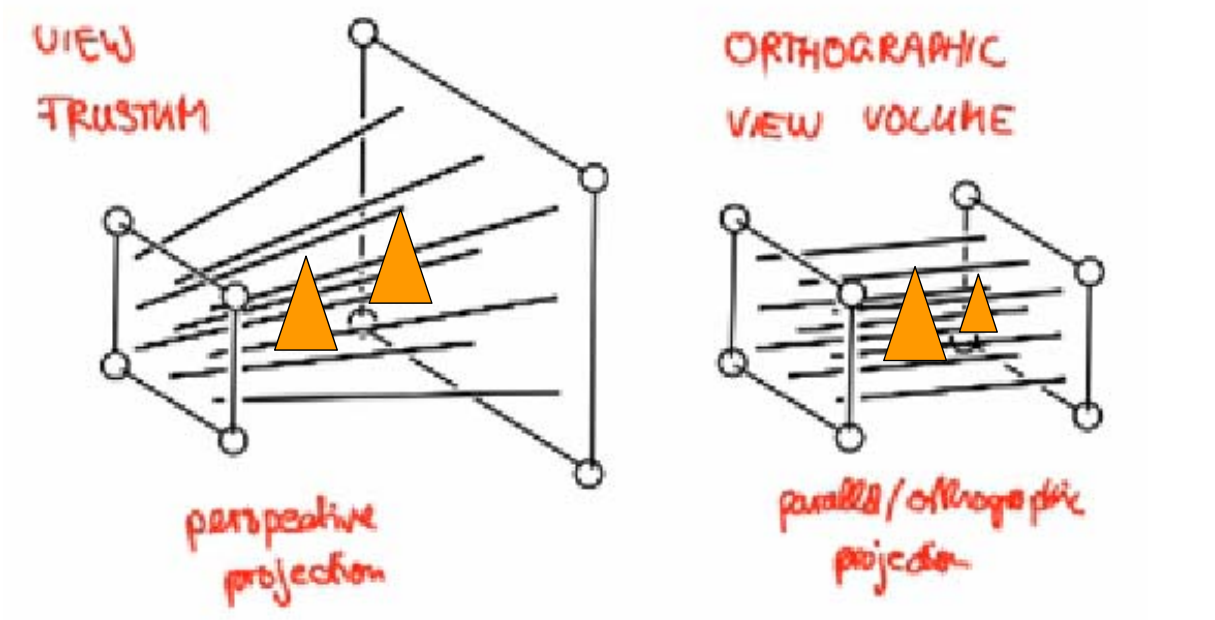
The required transformation is done by

$$M_{cam} = \begin{pmatrix} \vec{u} & \vec{v} & \vec{w} & \vec{e} \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1}$$

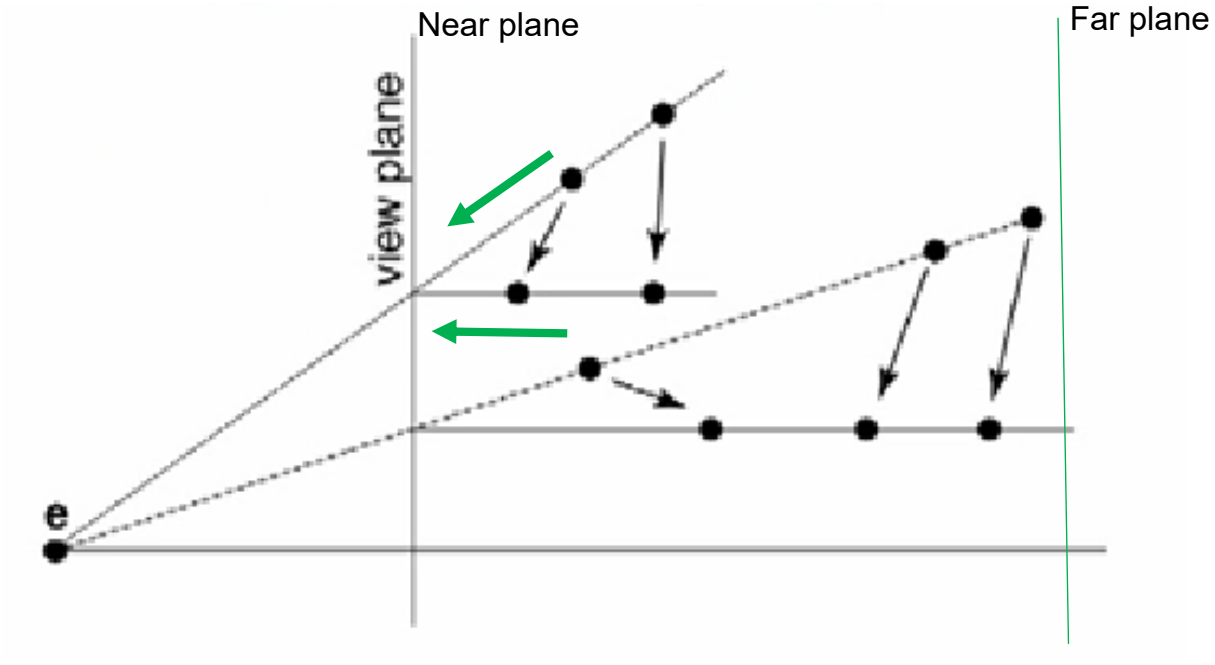
$$= \begin{pmatrix} x_u & y_u & z_u & 0 \\ x_v & y_v & z_v & 0 \\ x_w & y_w & z_w & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & -x_e \\ 0 & 1 & 0 & -y_e \\ 0 & 0 & 1 & -z_e \\ 0 & 0 & 0 & 1 \end{pmatrix}$$




If it wasn't for perspective projection, we'd be done:
$$\begin{pmatrix} x_{pixel} \\ y_{pixel} \\ z_{canonical} \\ 1 \end{pmatrix} = M_{vp} M_{orth} M_{cam} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$
Now, let's put things into perspective . . .

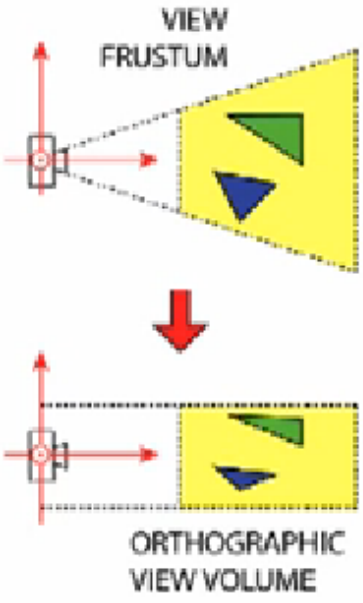

VIEW
FRUSTUM
perspective
projection
ORTHOGRAPHIC
VIEW VOLUME
parallel / orthographic
projection

Transforming the view frustum

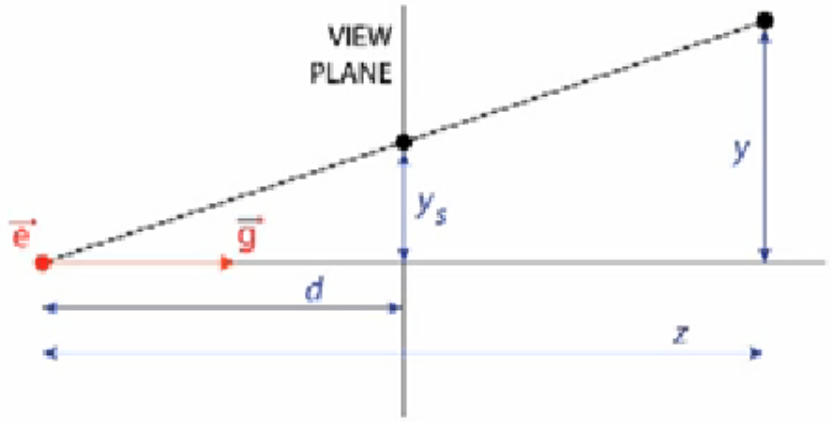


We have to transform the view frustum into the orthographic view volume. The transformation needs to

- Map lines through the origin to lines parallel to the z axis
- Map points on the viewing plane to themselves.
- Map points on the far plane to (other) points on the far plane.
- Preserve the near-to-far order of points on a line.



How do we calculate this?



From basic geometry we know:

$$\frac{y_s}{y} = \frac{d}{z} \quad \text{and thus} \quad y_s = \frac{d}{z}y$$

So we need a matrix that gives us

- $x_s = \frac{dx}{z}$
- $y_s = \frac{dy}{z}$

and a z-value that

- stays the same for all points on the near and fare planes
- does not change the order along the z-axis for all other points

Problem: we can't do division with matrix multiplication

Extending homogeneous coordinates

Remember: matrix multiplication is a linear transformation, i.e. it can only produce values such as:

$$x' = ax + by + cz$$

Introducing homogeneous coordinates and representing points as $(x, y, z, 1)$, enables us to do affine transformations, i.e. create values such as:

$$x' = ax + by + cz + d$$

Now we introduce projective transformation (aka homography) that allows us to create values such as:

$$x' = \frac{ax+by+cz+d}{ex+fy+gz+h}$$

We do this by replacing "the one" in the forth coordinate with a value w that serves as denominator, i.e. the homogeneous vector

$$(x, y, z, w) \quad \text{represents the point} \quad (x/w, y/w, z/w).$$

Matrix transformation becomes:

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \\ \tilde{w} \end{pmatrix} = \begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ e & f & g & h \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

and

$$(x', y', z') = (\tilde{x}/\tilde{w}, \tilde{y}/\tilde{w}, \tilde{z}/\tilde{w})$$

Matrix multiplication:

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \\ \tilde{w} \end{pmatrix} = \begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ e & f & g & h \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \begin{pmatrix} a_1x + b_1y + c_1z + d_1 \\ a_2x + b_2y + c_2z + d_2 \\ a_3x + b_3y + c_3z + d_3 \\ ex + fy + gz + h \end{pmatrix}$$

Perspective transform matrix

Using this extension, the following matrix solves our problem:

$$\begin{pmatrix} n & 0 & 0 & 0 \\ 0 & n & 0 & 0 \\ 0 & 0 & n+f & -fn \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Notice that

- we are looking in negative Z-direction
- n, f denote the near and far plane of the view frustum
- n serves as projection plane

Let's verify that . . .

$$\begin{pmatrix} n & 0 & 0 & 0 \\ 0 & n & 0 & 0 \\ 0 & 0 & n+f & -fn \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z\frac{n+f}{n} - f \\ \frac{z}{n} \end{pmatrix} \xrightarrow{\text{homogenize}} \begin{pmatrix} \frac{nx}{z} \\ \frac{ny}{z} \\ n+f-\frac{fn}{z} \\ 1 \end{pmatrix}$$

Indeed, that gives the correct values for x_s and y_s.

But what about z? Remember our requirements for z:

- stays the same for all points on the near and fare planes
- does not change the order along the z-axis for all other points

We have $z_s = n + f - \frac{fn}{z}$ and need to prove that ...

- values on the near plane stay on the near plane, i.e. if $z = n$, then $z_s = n$:

$$z_s = n + f - f = n = z$$

- values on the far plane stay on the far plane, i.e. if $z = f$, then $z_s = n$:

$$z_s = n + f - n - f = z$$

and ...

We have $z_s = n + f - \frac{fn}{z}$ and need to prove that ...

- values within the view frustum stay within the view frustum, i.e. if $z > n$ then $z_s > n$:

$$z_s = n + f - \frac{fn}{z} > n + f - \frac{fn}{n} = n$$

and if $z < f$ then $z_s < f$:

$$z_s = n + f - \frac{fn}{z} < n + f - \frac{fn}{f} = f$$

and ...

We have $z_s = n + f - \frac{fn}{z}$ and need to prove that ...

- the order along the Z-axis is preserved, i.e. if $z_1 > z_2$ then $z_{1s} > z_{2s}$:

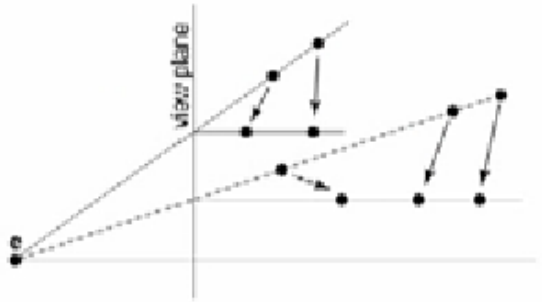
$$z_{1s} = n + f - \frac{fn}{z_1} > n + f - \frac{fn}{z_2} = z_{2s}$$

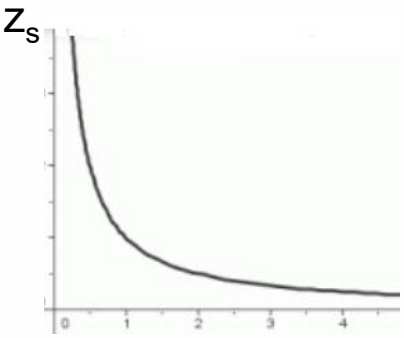
$$-\frac{1}{z_1} < -\frac{1}{z_2}$$

$$z_1 > z_2$$

and we are done.

Hence, the order is preserved. But how?

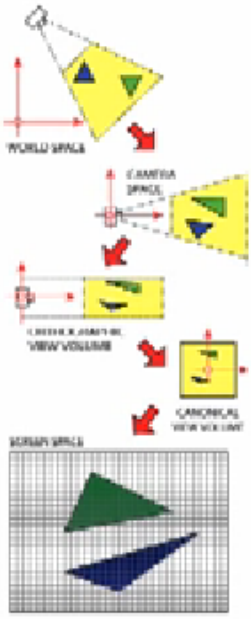




$z_s = n + f - \frac{fn}{z}$,
so z_s is proportional to $-\frac{1}{z}$

With this, we got our final matrix P. To map the perspective view frustum to the orthographic view volume, we need to combine it with the orthographic projection matrix M_{orth}, i.e. M_{per} =

$$M_{orth}P = M_{orth}\begin{pmatrix} n & 0 & 0 & 0 \\ 0 & n & 0 & 0 \\ 0 & 0 & n+f & -fn \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{2n}{r-l} & 0 & \frac{l+r}{l-r} & 0 \\ 0 & \frac{2n}{t-b} & \frac{b+t}{b-t} & 0 \\ 0 & 0 & \frac{f+n}{n-f} & \frac{2fn}{f-n} \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

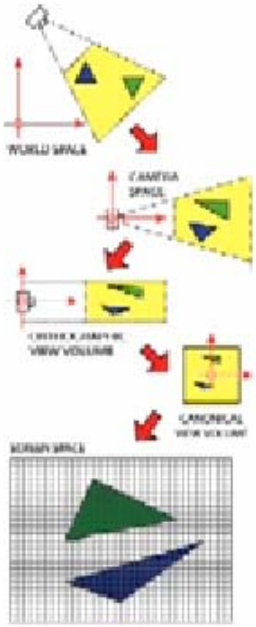


The following achieved parallel projection:

$$\begin{pmatrix} x_{pixel} \\ y_{pixel} \\ z_{canonical} \\ 1 \end{pmatrix} = M_{vp} M_{orth} M_{cam} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

We just have to replace M_{orth} with M_{per} now to get perspective projection:

$$\begin{pmatrix} x_{pixel} \\ y_{pixel} \\ z_{canonical} \\ 1 \end{pmatrix} = M_{vp} M_{per} M_{cam} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$



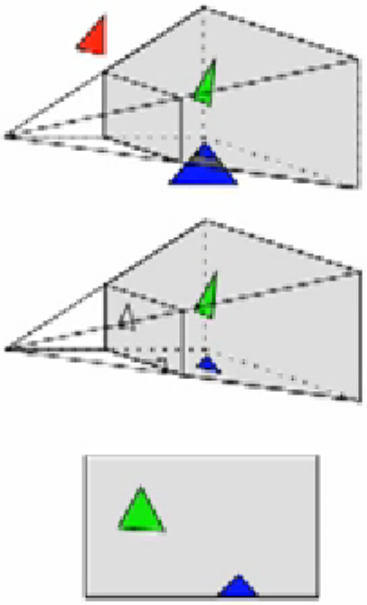
To draw lines on the screen, we can use the following pseudo code:

```
compute M_vp  ///view port
compute M_per  ///persp.  proj.
compute M_cam  ///camera space
M = M_vp M_per M_cam
for each line segment (a_i, b_i) do
  p = Ma_i
  q = Mb_i
  drawline(x_p/w_p, y_p/w_p, x_q/w_q, y_q/w_q)
```

End of
Slides captured from the lecture
by Prof. Wolfgang Huerst
Utrecht University, The Netherlands

Now we can draw points and lines.
But there's more ...

- Triangles that lie (partly) outside the view frustum need not be projected, and are clipped.
- The remaining triangles are projected if they are front facing.
- Projected triangles have to be shaded and/or textured.



Quaternion