Project 12

Perona–Malik Model

Tsz Nga Au, Emily

Image Denoising

??

Content

- Heat Equation
  - Formulation
  - Model
  - Weakness
- Perona–Malik Model
  - Formulation
  - Model
  - Weakness
- Proposed methods improve Perona–Malik Model

Heat Equation

Widely used in Image Processing for smoothing and denoising

$$\frac{\partial u(x,t)}{\partial t} = \underbrace{\Delta u}_{\text{Laplacian}} = \underbrace{div(\nabla u(x,t))}_{\text{Divergence of gradient}}$$

Heat Equation

To apply the heat equation on image,

Let u be the function of intensity,

$x = (x_1, x_2)$ be pixel in the image,

Ω be the image domain

Heat Equation

Initial Condition

Boundary Condition

$$
\begin{cases}
\frac{\partial u(x,t)}{\partial t} = div(\nabla u(x,t)) = \frac{\partial^2 u}{\partial {x_1}^2} + \frac{\partial^2 u}{\partial {x_2}^2} & \text{for } t > 0 \text{ and } x \in \Omega \\
u(x,0) = u_0(x) & \text{for } x \in \Omega \\
\frac{\partial u(x,t)}{\partial t} = 0 & \text{for } x \in \partial\Omega
\end{cases}
$$

Implement Heat Equation Model

–> Discretize the model by Finite Difference Scheme
(We use Explicit difference Scheme)

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u}{\partial {x_1}^2} + \frac{\partial^2 u}{\partial {x_2}^2}$$

Fix t, denote $u(x_1, x_2, t)$ by let $f(x_1, x_2)$

$$\frac{\partial u(x_1,x_2)}{\partial t} \approx \lim_{h\to 0} \frac{f(x_1-h,x_2)+f(x_1+h,x_2)+f(x_1,x_2-h)+f(x_1,x_2-h)-4f(x_1,x_2)}{h^2}$$

–> Solve it by Euler's method

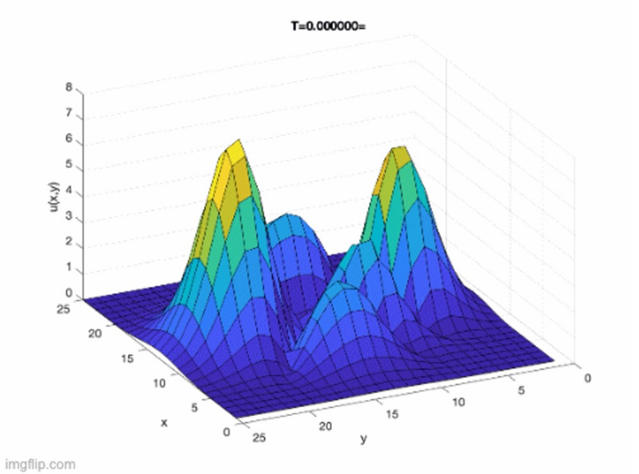
$$u(x_1,x_2,t+1) = u(x_1,x_2,t) + \alpha * \frac{\partial u(x_1,x_2,t)}{\partial t}$$

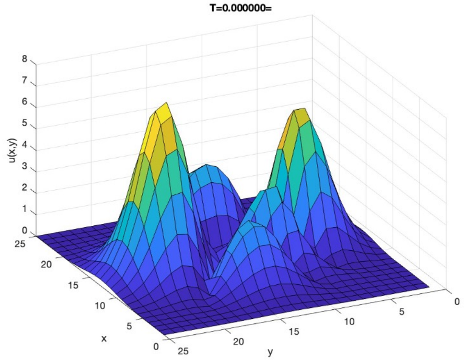
diffusion constant

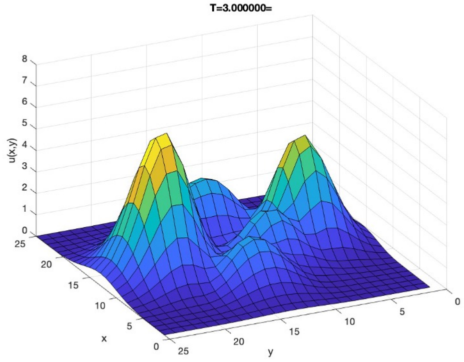
Heat Equation

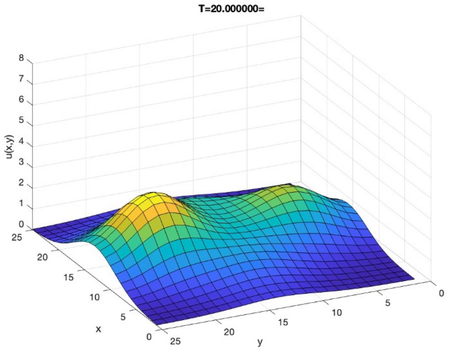
$$u(x_1, x_2, t+1) = u(x_1, x_2, t) + \alpha * dt * \frac{\partial u(x_1, x_2, t)}{\partial t}$$

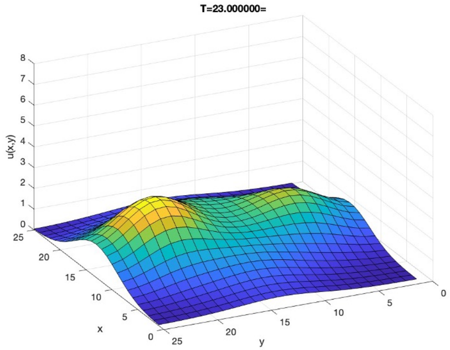
Set α=0.2, dt=0.1







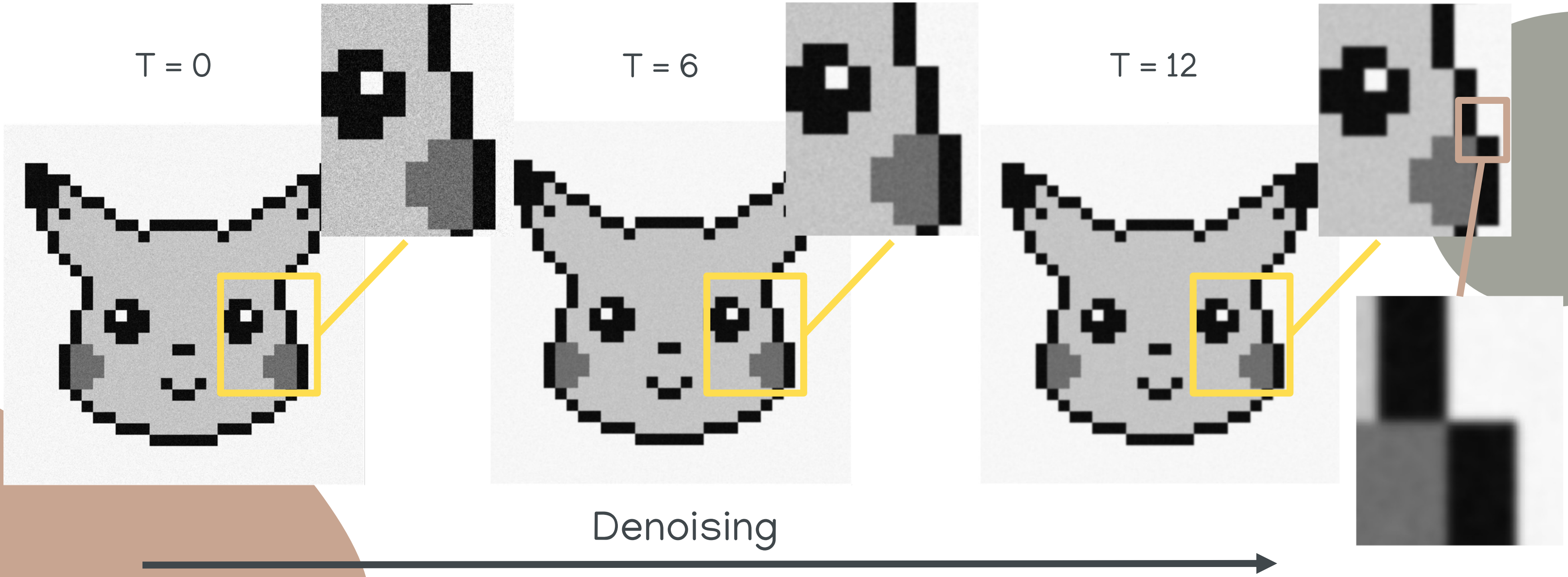




Apply Heat Equation on Image

Image with Gaussian Noise

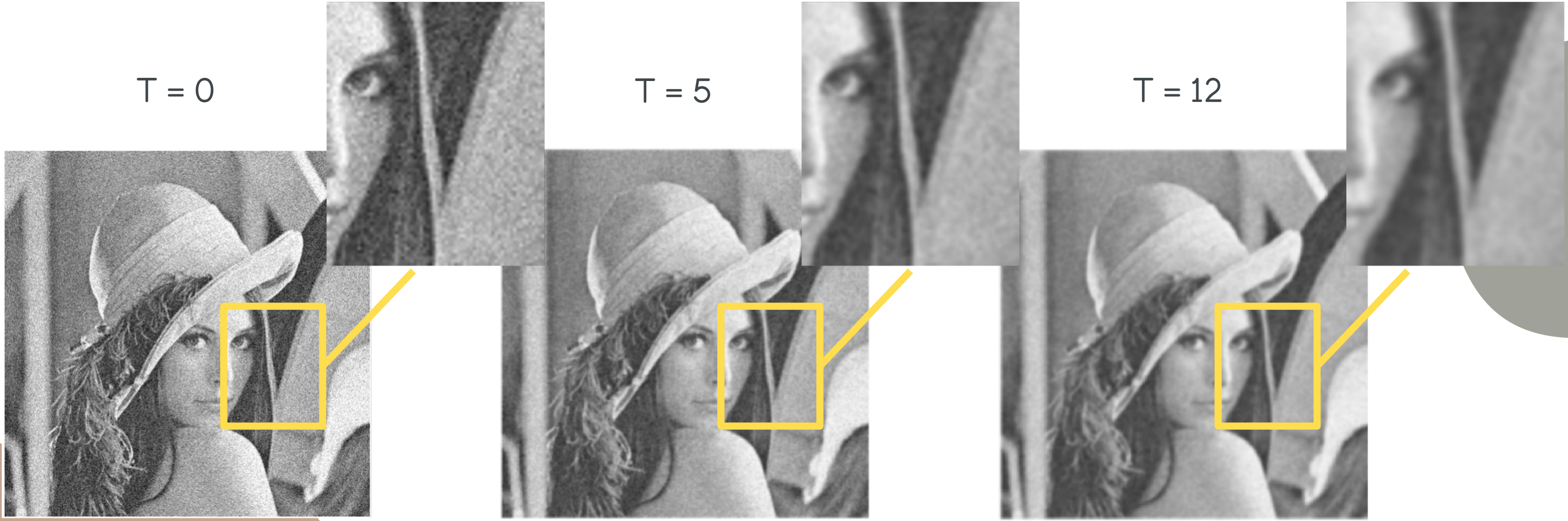
Set $dt=0.1$ and $\alpha=0.6$



Apply Heat Equation on Image

Image with Gaussian Noise

Set dt=0.1 and α=0.2

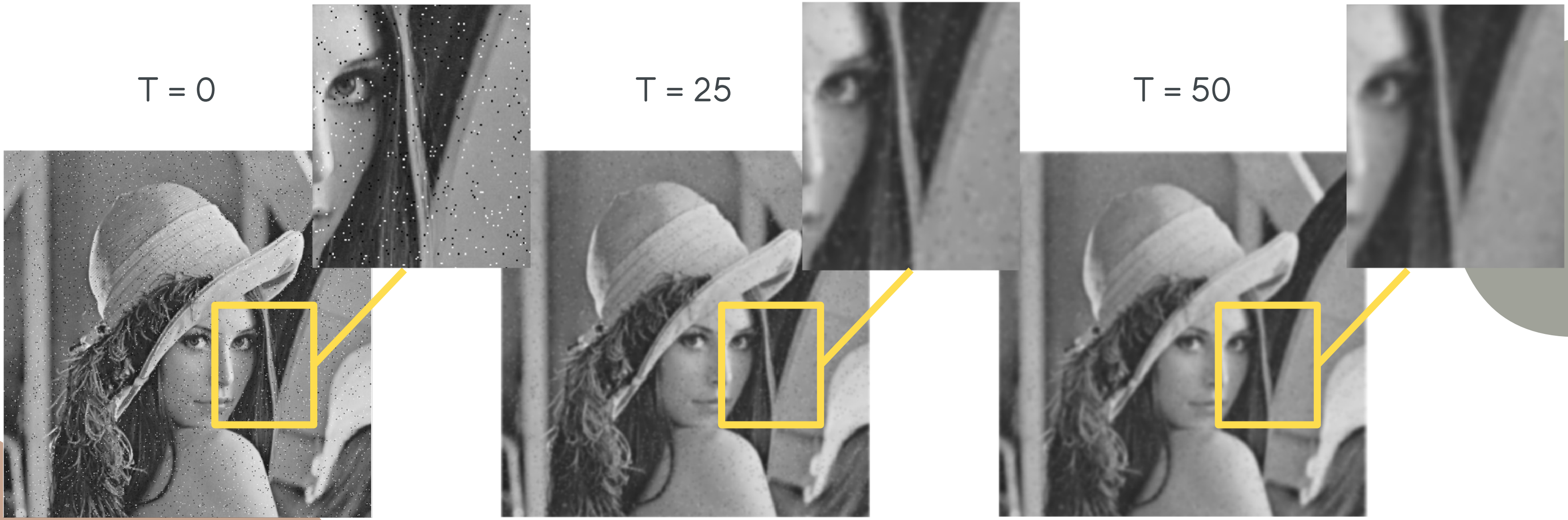


Denoising

Apply Heat Equation on Image

Image with Salt and Pepper Noise

Set dt=0.1 and α=0.2



Denoising

Summary of Heat Equation

Isotropic diffusion – apply smoothing to the image equally in all directions

Pros:

- Intuitive and easy to implement

Cons:

- Uniform smoothing
    - Edge are smoothed out

To preserve the edge –> use the Perona–Malik Model

Perona-Malik Model

Heat equation:

$$\frac{\partial u(x,t)}{\partial t} = \Delta u = div(\nabla u(x,t))$$

The Perona–Malik equation:

$$\frac{\partial u(x,t)}{\partial t} = div(g(|\nabla u(x,t)|)\nabla u(x,t)),$$

Diffusion coefficient / Edge Stopping Function

where

$$g(s) = \frac{1}{1+\frac{s^2}{\lambda^2}}.$$

alternative $g(s) = e^{-\frac{s^2}{\lambda^2}}$

Diffusion Coefficient in Perona-Malik equation

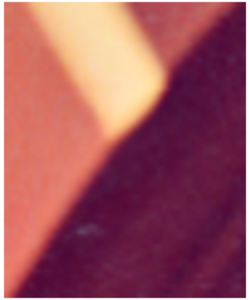
$$g(|\nabla u(x,t)|) = \frac{1}{1 + \frac{|\nabla u(x,t)|^2}{\lambda^2}}$$

$$\frac{\partial u(x,t)}{\partial t} = div(g(|\nabla u(x,t)|)\nabla u(x,t))$$

Edge -> $|\nabla u(x,t)|$ ⇧

-> $g(|\nabla u(x,t)|) = \frac{1}{1 + \frac{|\nabla u(x,t)|^2}{\lambda^2}}$ ⇩

-> $\frac{\partial u(x,t)}{\partial t} = div(g(|\nabla u(x,t)|)\nabla u(x,t))$ ⇩



Anisotropic Diffusion

$$\frac{\partial u(x,t)}{\partial t} = div(g(|\nabla u(x,t)|)\nabla u(x,t))$$

$$g(|\nabla u(x,t)|) = \frac{1}{1 + \frac{|\nabla u(x,t)|^2}{\lambda^2}}$$

When $|\nabla u(x,t)|$ is large, $\frac{\partial u(x,t)}{\partial t}$ would be small and edge can be preserved

When $|\nabla u(x,t)|$ is small, $\frac{\partial u(x,t)}{\partial t} \approx div(\nabla u(x,t))$ and behave like heat equation

=> Reducing noise while maintaining the significant parts of image

Implement Perona-Malik Model

Similar to the heat equation, we can discretize and solve it by finite difference scheme and Euler Method.

$$\frac{\partial u(x,t)}{\partial t} = div(g(|\nabla u(x,t)|)\nabla u(x,t)),$$

Iteration:

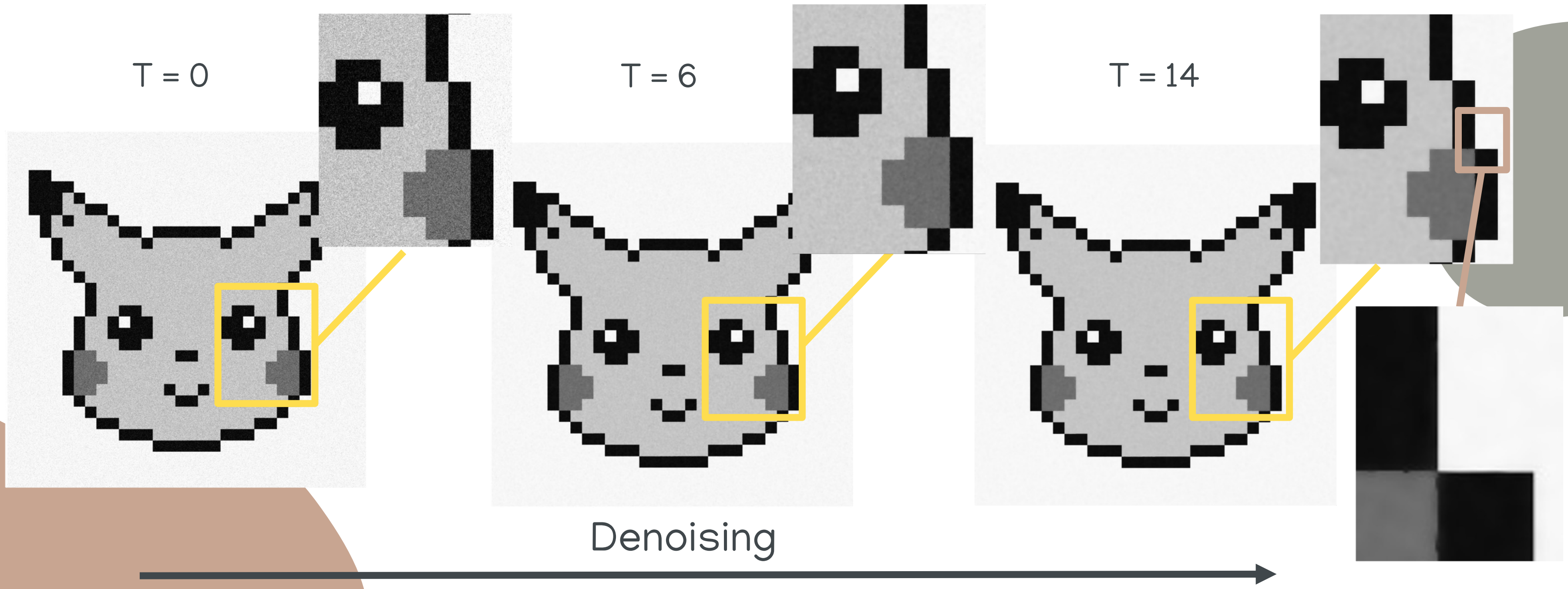
$$u(x_1, x_2, t+1) = u(x_1, x_2, t) + \alpha * \frac{\partial u(x_1, x_2, t)}{\partial t}$$

diffusion constant

Apply Perona-Malik Equation on Image

Image with Gaussian Noise

Set dt=0.1 and α=15



Apply Perona-Malik Equation on Image

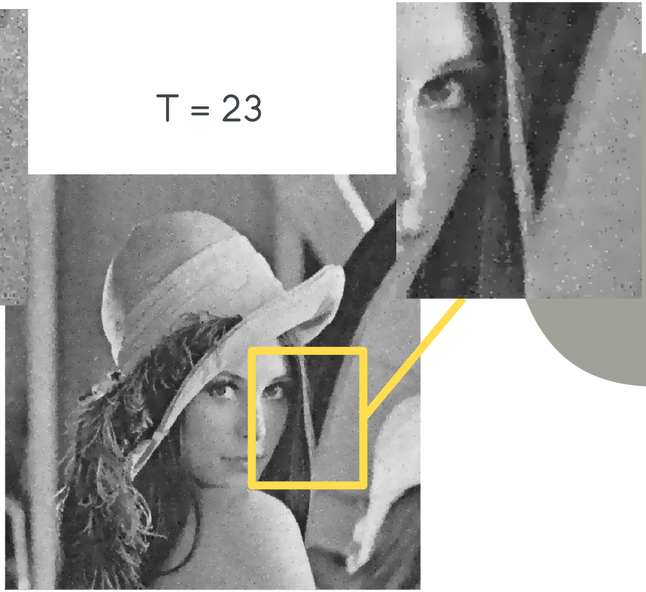
Image with Gaussian Noise

Set dt=0.1 and α=4.5

T = 0

T = 12

T = 23



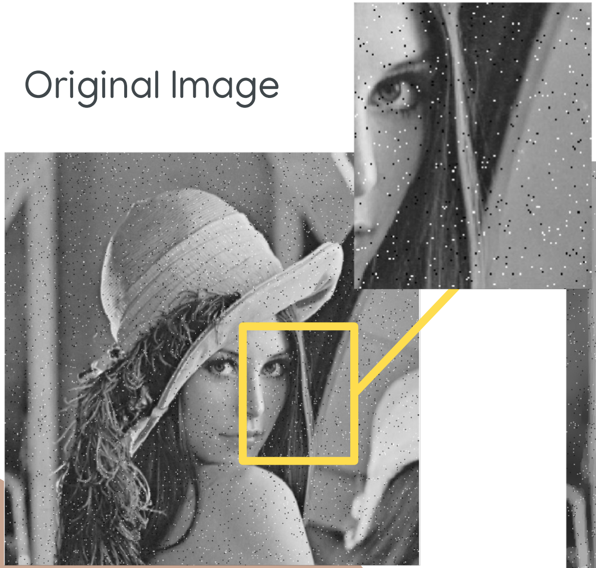
Denoising

Apply Perona-Malik Equation on Image

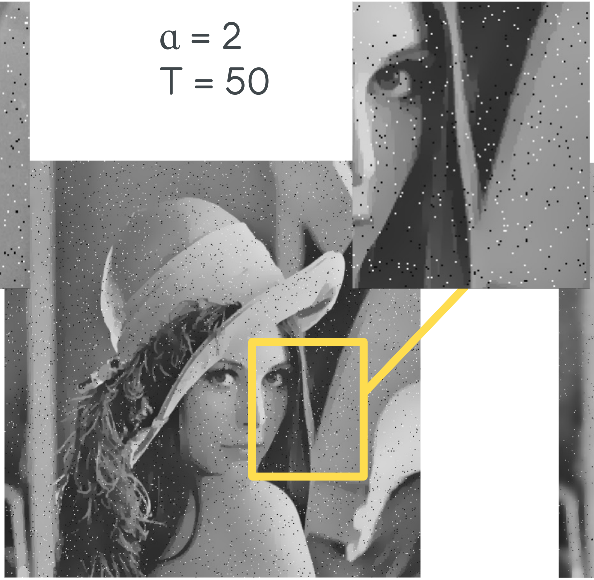
Image with Salt and Pepper Noise

Set dt=0.1

Original Image



$\alpha = 2$
$T = 50$



$\alpha = 15$
$T = 10$

Comparison between 2 Models

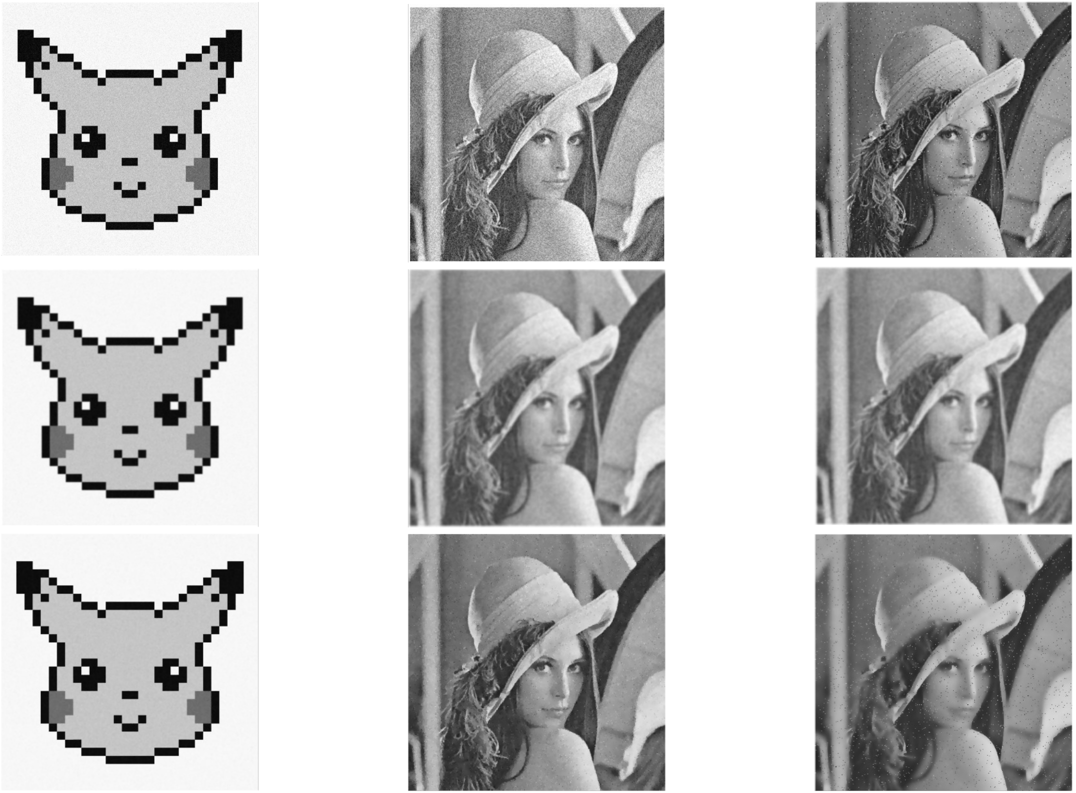
Gaussian Noise

Salt and Pepper Noise

Original Image

Heat Equation

Perona-Malik Equation



Summary of Perona-Malik Model

Anisotropic Diffusion – Apply smoothing to reduce the noises in image while maintaining the significant details

Pros:

- Reduce the noise
- Maintain the edges with great intensity difference

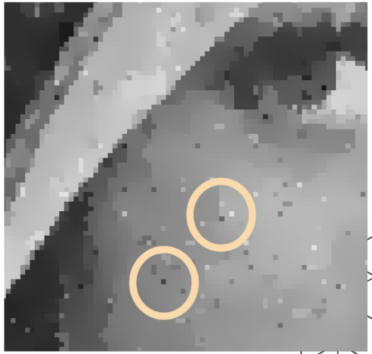
Cons:

- Cannot maintain the edge with low intensity difference
- Cannot reduce noise with great intensity difference, i.e. salt and pepper noise

Further Improvement

Model:

- Add filter to reduce noise with great intensity difference, i.e. salt and pepper noise



Implementation:

- Control the number of time step
- Control the diffusion constant

Take Home Message

Heat Equation – uniform smoothing

Perona–Malik Equation – smoothing while preserving the edge with great intensity difference

Every models have their own strengths and we can choose a better model if we have more prior knowledge

Reference

- Aubert, Gilles, Pierre Kornprobst, and Giles Aubert. Mathematical problems in image processing: partial differential equations and the calculus of variations. Vol. 147. New York: Springer, 2006.
- https://srome.github.io/Why-Blurring-an-Image-is-Similar-to-Warming-Your-Coffee/
- https://web.stanford.edu/class/math220b/handouts/heateqn.pdf
- https://www.mathworks.com/matlabcentral/fileexchange/65889-image-blurring-using-2d-heat-equation
- https://www.mathworks.com/matlabcentral/fileexchange/62274-modified-perona-malik-model