Multiple Conclusion Intuitionistic Linear Logic and Cut Elimination

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Abstract

Full Intuitionistic Linear Logic (FILL) was first introduced by Hyland and de Paiva as one of the results of their investigation into a categorical understanding of Gödel's Dialectica interpretation. FILL went against current beliefs that it was not possible to incorporate all of the linear connectives, e.g. tensor, par, and implication, into an intuitionistic linear logic. They showed that it is natural to support all of the connectives given sequents that have multiple hypotheses and multiple conclusions. To enforce intuitionism de Paiva's original formalization of FILL used the well-known Dragalin restriction, forcing the implication right rule to have only a single conclusion in its premise, but Schellinx showed that this results in a failure of cut-elimination. To overcome this failure Hyland and de Paiva introduced a term assignment for FILL that eliminated the need for the strong restriction. The main idea was to first relax the restriction by assigning variables to each hypothesis and terms to each conclusion. Then when introducing an implication on the right enforcing that the variable annotating the hypothesis being discharged is only free in the term annotating the conclusion of the implication. Bierman showed in a short note that this formalization of FILL still did not enjoy cut-elimination, because of a flaw in the left rule for par. However, Bellin proposed an alternate left rule for par and conjectured that by adopting his rule cut-elimination is restored. In this note we show that adopting Bellin's proposed rule one does obtain cut-elimination for FILL, as suggested. Additionally, we show that FILL can be modeled by a new form of dialectica category called order-enriched dialectica category, and discuss future work giving FILL a semantics in terms of Lorenzen games.

1 Introduction

A commonly held belief during the early history of linear logic was that the linear-connective par could not be incorporated into an intuitionistic linear logic. This belief was challenged when de Paiva gave a categorical understanding of Gödel's Dialectica interpretation in terms of dialectica categories [4, 3]. Upon setting out on her investigation she initially believed that dialectica categories would end up being a model of intuitionistic logic, but to her surprise they are actually models of intuitionistic linear logic, containing the linear connectives:

tensor, par, implication, and their units. Furthermore, unlike other models at that time the units did not collapse into a single object.

Armed with this semantic insight de Paiva gave the first formalization of Full Intuitionistic Linear Logic (FILL) [3]. FILL is a sequent calculus with multiple conclusions in addition to multiple hypotheses. Logics of this type go back to Gentzen's work on the sequent calculi LK and LJ, and Maehara's work on LJ' [10, 15]. The sequents in these types of logics usually have the form $\Gamma \vdash \Delta$ where Γ and Δ are multisets of formulas. Sequents such as these are read as "the conjunction of the formulas in Γ imply the disjunction of the formulas in Δ". For a brief, but more complete history of logics with multiple conclusions see the introduction to [6].

Gentzen showed that to obtain intuitionistic logic one could start with the logic LK and then place a cardinality restriction on the righthand side of sequents, however, this is not the only means of enforcing intuitionism. Maehara showed that in the propositional case one could simply place the cardinality restriction on the premise of the implication right rule, and leave all of the other rules of LK unrestricted. This restriction is sometimes called the Dragalin restriction, as it appeared in his AMS textbook [7]. The classical implication right rule has the form:

$$\frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \multimap B, \Delta}\ \textsc{impR}$$

By placing the Dragalin restriction on the previous rule we obtain:

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B}\ \textsc{impR}$$

de Paiva's first formalization of FILL used the Dragalin restriction, see [3] p. 58, but Schellinx showed that this restriction has the unfortunate consequence of breaking cut-elimination [14].

Later, Hyland and de Paiva gave an alternate formalization of FILL in the hopes to regain cut-elimination [8]. This new formalization lifted the Dragalin restriction by decorating sequents with a term assignment. Hypotheses were assigned variables, and the conclusions were assigned terms. Then using these terms one can track the use of hypotheses throughout a derivation. They proposed a new implication right rule:

$$\frac{\Gamma, x : A \vdash t : B, \Delta \qquad x \notin \mathsf{FV}(\Delta)}{\Gamma \vdash \lambda x.t : A \multimap B, \Delta}\ \textsc{impR}$$

Intuitionism is enforced in this rule by requiring that the variable being discharged, x, is potentially free in only one term annotating a conclusion. Unfortunately, this formalization did not enjoy cut-elimination either.

Bierman was able to give a counterexample to cut-elimination [1]. As Bierman explains the problem was with the left rule for par. The original rule was as follows:

$$\frac{\Gamma, x : A \vdash \Delta \qquad \Gamma', y : B \vdash \Delta'}{\Gamma, \Gamma', z : A \parr B \vdash \mathsf{let}\, z\, \mathsf{be}\, (x \parr -)\, \mathsf{in}\, \Delta \mid \mathsf{let}\, z\, \mathsf{be}\, (- \parr y)\, \mathsf{in}\, \Delta'}\ \textsc{parL}$$

In this rule the pattern variables x and y are bound in each term of L and L' respectively. Notice that the variable z becomes free in every term in L and L'. Bierman showed that this rule mixed with the restriction on implication right prevents the usual cut-elimination step that commutes cut with the left rule for par. The main idea behind the counterexample is that in the derivation before commuting the cut it is possible to discharge z using implication right, but after the cut is commuted past the left rule for par, the variable z becomes free in more than one conclusion, and thus, can no longer be discharged.

In the conclusion of Bierman's note he gives an alternate left rule for par that he attributes to Bellin. This new left-rule is as follows:

$$\frac{\Gamma, x : A \vdash \Delta \quad \Gamma', y : B \vdash \Delta'}{\Gamma, \Gamma', z : A \parr B \vdash \text{let-pat}\, z\, (x \parr -)\, \Delta \mid \text{let-pat}\, z\, (- \parr y)\, \Delta'}\ \textsc{Parl}$$

In this rule let-pat $z\,(x \parr -)\, t$ and let-pat $z\,(- \parr y)\, t'$ only let-bind z in t or t' if $x \in FV(t)$ or $y \in FV(t')$. Otherwise the terms are left unaltered. Bellin conjectured that adopting this rule results in FILL regaining cut-elimination. However, no proof has been given.

Contributions. In this paper our main contribution is to confirm Bellin's conjecture by adopting his proposed rule (Section 2) and proving cut-elimination (Section 3). In addition, we show that FILL can be modeled by a new form of dialectica category called order-enriched dialectica category (Section 4).

Related Work. The first formalization of FILL with cut-elimination was due to Brauner and de Paiva [2]. Their formalization can be seen as a linear version of LK with a sophisticated meta-level dependency tracking system. A proof of a FILL sequent in their formalization amounts to a classical derivation, π, invariant in a what they call the FILL property:

- The hypothesis discharged by an application of the implication right rule in π is a dependency of only the conclusion of the implication being introduced.

They were able to show that their formalization is sound, complete, and enjoys cut-elimination. The one point in favor of the term assignment formalization given here over Brauner and de Paiva's formalization is that the dependency tracking system complicates both the definition of the logic and its use. However, we do not wish to detract from the importance of their work.

de Paiva and Pereira used annotations on the sequents of LK to arrive at full intuitionistic logic (FIL) with multiple conclusion that enjoys cut-elimination [6]. They annotate hypothesis with natural number indices, and conclusions with finite sets of indices. The sets on conclusions correspond to the collection of the hypotheses that the conclusion depends on. Then they have a similar property to that of the FILL property from Brauner and de Paiva's formalization. In fact, the dependency tracking system is very similar to this formalization, but the dependency tracking has been collapsed into the object language instead of being at the meta-level.

2 Full Intuitionistic Linear Logic (FILL)

In this section we give a brief description of FILL. We first give the syntax of formulas, patterns, terms, and contexts. Following the syntax we define several meta-functions that will be used when defining the inference rules of the logic.

Definition 1. *The syntax for FILL is as follows:*

(Formulas)	$A, B, C, D, E ::= \top \mid \perp \mid A \multimap B \mid A \otimes B \mid A \parr B$
(Patterns)	$p ::= * \mid - \mid x \mid p_1 \otimes p_2 \mid p_1 \parr p_2$
(Terms)	$t, e ::= x \mid * \mid \circ \mid t_1 \otimes t_2 \mid t_1 \parr t_2 \mid \lambda x.t \mid \mathsf{let}\, t\, \mathsf{be}\, p\, \mathsf{in}\, e \mid t_1\, t_2$
(Left Contexts)	$\Gamma ::= \cdot \mid x : A \mid \Gamma_1, \Gamma_2$
(Right Contexts)	$\Delta ::= \cdot \mid t : A \mid \Delta_1, \Delta_2$

The formulas of FILL are standard, but we denote the unit of tensor as $\top$ and the unit of par as $\perp$. Patterns are used to distinguish between the various let-expressions for tensor, par, and their units. There are three different let-expressions:

$$\begin{array}{rl} \text{Tensor:} & \mathsf{let}\, t\, \mathsf{be}\, p_1 \otimes p_2\, \mathsf{in}\, e \\ \text{Par:} & \mathsf{let}\, t\, \mathsf{be}\, p_1 \parr p_2\, \mathsf{in}\, e \\ \text{Tensor Unit:} & \mathsf{let}\, t\, \mathsf{be}\, *\, \mathsf{in}\, e \end{array}$$

In addition, each of these will have their own equational rules, see Figure 2. The role each term plays in the overall logic will become clear after we introduce the inference rules.

At this point we introduce some syntax and mete-level functions that will be used in the definition of the inference rules for FILL. Left contexts are multisets of formulas labeled with a variable, and right contexts are multisets of formulas labeled with a term. We will often write $\Delta_1 \mid \Delta_2$ as syntactic sugar for Δ_1, Δ_2. The former should be read as "Δ_1 or Δ_2." We denote the usual capture-avoiding substitution by $[t/x]t'$, and its straightforward extension to right contexts as $[t/x]\Delta$. Similarly, we find it convenient to be able to do this style of extension for the let-binding as well.

Definition 2. *We extend let-binding terms to right contexts as follows:*

$$\begin{array}{l} \mathsf{let}\, t\, \mathsf{be}\, p\, \mathsf{in} \cdot = \cdot \\ \mathsf{let}\, t\, \mathsf{be}\, p\, \mathsf{in}\, (t' : A) = (\mathsf{let}\, t\, \mathsf{be}\, p\, \mathsf{in}\, t') : A \\ \mathsf{let}\, t\, \mathsf{be}\, p\, \mathsf{in}\, (\Delta_1 \mid \Delta_2) = (\mathsf{let}\, t\, \mathsf{be}\, p\, \mathsf{in}\, \Delta_1) \mid (\mathsf{let}\, t\, \mathsf{be}\, p\, \mathsf{in}\, \Delta_2) \end{array}$$

Lastly, we denote the usual function that computes the set of free variables in a term by $\mathsf{FV}(t)$, and its straightforward extension to right contexts as $\mathsf{FV}(\Delta)$.

The inference rules for FILL are defined in Figure 1. The Parl rule depends on the function let-pat $z\, p\, \Delta$ which we define next.

Definition 3. *The function* let-pat $z\, p\, t$ *is defined as follows:*

$$\begin{array}{lll} \text{let-pat}\, z\, (x \parr -)\, t = t & \text{let-pat}\, z\, (- \parr y)\, t = t & \text{let-pat}\, z\, p\, t = \mathsf{let}\, z\, \mathsf{be}\, p\, \mathsf{in}\, t \\ \quad \textit{where } x \notin \mathsf{FV}(t) & \quad \textit{where } y \notin \mathsf{FV}(t) & \end{array}$$

$$\frac{}{x : A \vdash x : A}\ \textsc{Ax} \qquad \frac{\Gamma \vdash t : A \mid \Delta \quad \Gamma', y : A \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta \mid [t/y]\Delta'}\ \textsc{Cut} \qquad \frac{\Gamma \vdash \Delta}{\Gamma, x : \top \vdash \mathsf{let}\ x\ \mathsf{be}\ *\ \mathsf{in}\ \Delta}\ \textsc{Il}$$

$$\frac{}{\cdot \vdash * : \top}\ \textsc{Ir} \qquad \frac{\Gamma, x : A, y : B \vdash \Delta}{\Gamma, z : A \otimes B \vdash \mathsf{let}\ z\ \mathsf{be}\ x \otimes y\ \mathsf{in}\ \Delta}\ \textsc{Tl}$$

$$\frac{\Gamma \vdash e : A \mid \Delta \quad \Gamma' \vdash f : B \mid \Delta'}{\Gamma, \Gamma' \vdash e \otimes f : A \otimes B \mid \Delta \mid \Delta'}\ \textsc{Tr} \qquad \frac{}{x : \perp \vdash \cdot}\ \textsc{Pl} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \circ : \perp \mid \Delta}\ \textsc{Pr}$$

$$\frac{\Gamma, x : A \vdash \Delta \quad \Gamma', y : B \vdash \Delta'}{\Gamma, \Gamma', z : A \parr B \vdash \text{let-pat}\ z\ (x \parr -)\ \Delta \mid \text{let-pat}\ z\ (- \parr y)\ \Delta'}\ \textsc{Parl}$$

$$\frac{\Gamma \vdash \Delta \mid e : A \mid f : B \mid \Delta'}{\Gamma \vdash \Delta \mid e \parr f : A \parr B \mid \Delta'}\ \textsc{Parr} \qquad \frac{\Gamma \vdash e : A \mid \Delta \quad \Gamma', x : B \vdash \Delta'}{\Gamma, y : A \multimap B, \Gamma' \vdash \Delta \mid [y\ e/x]\Delta'}\ \textsc{Impl}$$

$$\frac{\Gamma, x : A \vdash e : B \mid \Delta \quad x \notin \mathsf{FV}(\Delta)}{\Gamma \vdash \lambda x.e : A \multimap B \mid \Delta}\ \textsc{Impr} \qquad \frac{\Gamma, x : A, y : B \vdash \Delta}{\Gamma, y : B, x : A \vdash \Delta}\ \textsc{Exl}$$

$$\frac{\Gamma \vdash \Delta_1 \mid t_1 : A \mid t_2 : B \mid \Delta_2}{\Gamma \vdash \Delta_1 \mid t_2 : B \mid t_1 : A \mid \Delta_2}\ \textsc{Exr}$$

Figure 1: Inference rules for FILL

It is straightforward to extend the previous definition to right-contexts, and we denote this extension by let-pat $z\ p\ \Delta$*.*

The motivation behind this function is that it only binds the pattern variables in $x \parr -$ and $- \parr y$ if and only if those pattern variables are free in the body of the let. This over comes the counterexample given by Bierman in [1]. Throughout the sequel we will denote derivations of the previous rules by π.

Similarly to λ-calculi the terms of FILL are equipped with an equivalence relation. This equivalence on terms is defined in Figure 2. However, these rules should not be considered as computational rules, but rather are only necessary for the cut-elimination procedure. The rules are highly motivated by the semantic interpretation of FILL into symmetric monoidal categories. There are a number of α, β, and η like rules as well as several rules we call naturality rules. These rules are similar to the rules presented in [8].

3 Cut-elimination

FILL can be viewed from two different angles: i. as an intuitionistic linear logic with par, or ii. as a restricted form of classical linear logic. Thus, to prove cut-elimination of FILL one only need to start with the cut-elimination procedure for intuitionistic linear logic, and then dualize all of the steps in the procedure for tensor and its unit to obtain the steps for par and its unit. Similarly, one could just as easily start with the cut-elimination procedure for classical linear logic, and then apply the restriction on the implication right rule producing a cut-elimination procedure for FILL.

The major difference between proving cut-elimination of FILL from classical or intuitionistic linear logic is that we must prove an invariant across each step in the procedure. The invariant is that if a derivation π is transformed into a derivation π', then the terms in the conclusion of the final rule applied in π must be equivalent to the terms in the conclusion of the final rule applied in π' using the rules from Figure 2. For example, consider the following case in the cut-elimination procedure for FILL:

The proof

$$
\dfrac{\dfrac{\pi_1}{\Gamma \vdash t : A \mid \Delta} \quad \dfrac{\dfrac{\pi_2}{\Gamma_1, x : A, \Gamma_2 \vdash \Delta_1 \mid t_1 : B \mid t_2 : C \mid \Delta_2}}{\Gamma_1, x : A, \Gamma_2 \vdash \Delta_1 \mid t_1 \parr t_2 : B \parr C \mid \Delta_2}\ \textsc{Parr}}{\Gamma_1, \Gamma, \Gamma_2 \vdash \Delta \mid [t/x]\Delta_1 \mid [t/x](t_1 \parr t_2) : B \parr C \mid [t/x]\Delta_2}\ \textsc{Cut}
$$

transforms into the proof

$$
\dfrac{\dfrac{\dfrac{\pi_1}{\Gamma \vdash t : A \mid \Delta} \quad \dfrac{\pi_2}{\Gamma_1, x : A, \Gamma_2 \vdash \Delta_1 \mid t_1 : B \mid t_2 : C \mid \Delta_2}}{\Gamma_1, \Gamma, \Gamma_2 \vdash \Delta \mid [t/x]\Delta_1 \mid [t/x]t_1 : B \mid [t/x]t_2 : C \mid [t/x]\Delta_2}\ \textsc{Cut}}{\Gamma_1, \Gamma, \Gamma_2 \vdash \Delta \mid [t/x]\Delta_1 \mid [t/x]t_1 \parr [t/x]t_2 : B \parr C \mid [t/x]\Delta_2}\ \textsc{Parr}
$$

Now we must show that $\Delta = \Delta$, $[t/x]\Delta_1 = [t/x]\Delta_1$, $[t/x](t_1 \parr t_2) = ([t/x]t_1) \parr [t/x]t_2$, and $[t/x]\Delta_2 = [t/x]\Delta_2$, but the only non-trivial case is whether $[t/x](t_1 \parr t_2) = ([t/x]t_1) \parr [t/x]t_2$ holds, but this clearly holds by a simple property of capture avoiding substitution. Every case of the cut-elimination procedure proceeds just as this example does.

The cut elimination procedure requires the following two basic results:

Lemma 4 (Substitution Distribution)**.** *For any terms t, t_1, and t_2, $[t_1/x][t_2/y]t = [[t_1/x]t_2/y][t_2/x]t$.*

Proof. This proof holds by straightforward induction on the form of t. □

Lemma 5 (Let-pat Distribution)**.** *For any terms t, t_1, and t_2, and pattern p,* $\text{let-pat}\ t\ p\ [t_1/y]t_2 = [\text{let-pat}\ t\ p\ t_1/y]t_2$.

Proof. This proof holds by case splitting over p, and then using the naturality equations for the respective pattern. □

We finally arrive at cut-elimination.

Theorem 6. *If $\Gamma \vdash t_1 : A_1, \ldots, t_i : A_i$ steps to $\Gamma \vdash t'_1 : A_1, \ldots, t'_i : A_i$ using the cut-elimination procedure, then $t_j = t'_j$ for $1 \leq j \leq i$.*

Proof. The cut-elimination procedure given here is the standard cut-elimination procedure for classical linear logic except the cases involving the implication right rule have the FILL restriction. The structure of our procedure follows the structure of the procedure found in [11]. Due to space limitations we only show two of the most interesting cases, but for the entire proof see the companion report [**?**].

Case: commuting conversion cut vs cut (first case). The following proof

$$
\dfrac{
  \dfrac{\begin{array}{c}\pi_1\\ \vdots\end{array}}{\Gamma \vdash t : A \mid \Delta}
  \qquad
  \dfrac{
    \dfrac{\begin{array}{c}\pi_2\\ \vdots\end{array}}{\Gamma_2, x : A, \Gamma_3 \vdash t_1 : B \mid \Delta_1}
    \qquad
    \dfrac{\begin{array}{c}\pi_3\\ \vdots\end{array}}{\Gamma_1, y : B, \Gamma_4 \vdash \Delta_2}
  }{\Gamma_1, \Gamma_2, x : A, \Gamma_3, \Gamma_4 \vdash \Delta_1 \mid [t_1/y]\Delta_2}\ \textsc{Cut}
}{\Gamma_1, \Gamma_2, \Gamma, \Gamma_3, \Gamma_4 \vdash \Delta \mid [t/x]\Delta_1 \mid [t/x][t_1/y]\Delta_2}\ \textsc{Cut}
$$

is transformed into the proof

$$
\dfrac{
  \dfrac{
    \dfrac{\begin{array}{c}\pi_1\\ \vdots\end{array}}{\Gamma \vdash t : A \mid \Delta}
    \qquad
    \dfrac{\begin{array}{c}\pi_2\\ \vdots\end{array}}{\Gamma_2, x : A, \Gamma_3 \vdash t_1 : B \mid \Delta_1}
  }{\Gamma_2, \Gamma, \Gamma_3 \vdash [t/x]t_1 : B \mid [t/x]\Delta_1}
  \qquad
  \dfrac{\begin{array}{c}\pi_3\\ \vdots\end{array}}{\Gamma_1, y : B, \Gamma_4 \vdash \Delta_2}
}{\Gamma_1, \Gamma_2, \Gamma, \Gamma_3, \Gamma_4 \vdash \Delta \mid [t/x]\Delta_1 \mid [[t/x]t_1/y]\Delta_2}\ \textsc{Cut}
$$

First, if Δ_2 is empty, then all the terms in the conclusion of the previous two derivations are equivalent. So suppose $\Delta_2 = t_2 : C \mid \Delta'_2$. Then we know that the term $[t/x][t_1/y]t_2$ in the first derivation above is equivalent to $[[t/x]t_1/y][t/x]t_2$ by Lemma 4. Furthermore, by inspecting the first derivation we can see that $x \notin \mathsf{FV}(t_2)$, and thus, $[[t/x]t_1/y][t/x]t_2 = [[t/x]t_1/y]t_2$. This argument may be repeated for any term in Δ'_2, and thus, we know $[t/x][t_1/y]\Delta_2 = [[t/x]t_1/y]\Delta_2$.

Case: commuting conversion cut vs. cut (second case). The second commuting conversion on cut begins with the proof

$$
\dfrac{
  \dfrac{\begin{array}{c}\pi_1\\ \vdots\end{array}}{\Gamma \vdash t : A \mid \Delta}
  \qquad
  \dfrac{
    \dfrac{\begin{array}{c}\pi_2\\ \vdots\end{array}}{\Gamma' \vdash t' : B \mid \Delta'}
    \qquad
    \dfrac{\begin{array}{c}\pi_3\\ \vdots\end{array}}{\Gamma_1, x : A, \Gamma_2, y : B, \Gamma_3 \vdash \Delta_1}
  }{\Gamma_1, x : A, \Gamma_2, \Gamma', \Gamma_3 \vdash \Delta' \mid [t'/y]\Delta_1}\ \textsc{Cut}
}{\Gamma_1, \Gamma, \Gamma_2, \Gamma', \Gamma_3 \vdash \Delta \mid [t/x]\Delta' \mid [t/x][t'/y]\Delta_1}\ \textsc{Cut}
$$

is transformed into the following proof:

$$\begin{array}{c}\pi_2\\ \vdots\\ \dfrac{}{\Gamma' \vdash t' : B \mid \Delta'}\end{array}$$

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\end{array}}{\Gamma \vdash t : A \mid \Delta} \qquad \dfrac{\begin{array}{c}\pi_3\\ \vdots\end{array}}{\Gamma_1, x : A, \Gamma_2, y : B, \Gamma_3 \vdash \Delta_1}}{\Gamma_1, \Gamma, \Gamma_2, y : B, \Gamma_3 \vdash \Delta \mid [t/x]\Delta_1}\,\textsc{Cut}}{\Gamma_1, \Gamma, \Gamma_2, \Gamma', \Gamma_3 \vdash \Delta' \mid [t'/y]\Delta \mid [t'/y][t/x]\Delta_1}\,\textsc{Cut}$$

$$\dfrac{\Gamma_1, \Gamma, \Gamma_2, \Gamma', \Gamma_3 \vdash \Delta' \mid [t'/y]\Delta \mid [t'/y][t/x]\Delta_1}{\Gamma_1, \Gamma, \Gamma_2, \Gamma', \Gamma_3 \vdash [t'/y]\Delta \mid \Delta' \mid [t'/y][t/x]\Delta_1}\,\textsc{Series of Exchanges}$$

We know $x, y \notin \mathsf{FV}(\Delta)$ by inspection of the first derivation, and so we know that $\Delta = [t'/y]\Delta$ and $\Delta' = [t/x]\Delta'$. Without loss of generality suppose $\Delta_1 = t_1 : C \mid \Delta_1'$. Then we know that $x, y \notin \mathsf{FV}(t)$ and $x, y \notin \mathsf{FV}(t')$. Thus, by this fact and Lemma 4, we know that $[t/x][t'/y]t_1 = [[t/x]t'/y][t/x]t_1 = [t'/y][t/x]t_1$. This argument can be repeated for any term in Δ_1', hence, $[t/x][t'/y]\Delta_1 = [t'/y][t/x]\Delta_1$.

Case: the η-expansion cases: tensor. The proof

$$\dfrac{}{x : A \otimes B \vdash x : A \otimes B}\,\textsc{Ax}$$

is transformed into the proof

$$\dfrac{\dfrac{\dfrac{}{y : A \vdash y : A}\,\textsc{Ax} \qquad \dfrac{}{z : B \vdash z : B}\,\textsc{Ax}}{y : A, z : B \vdash y \otimes z : A \otimes B}\,\textsc{Tr}}{x : A \otimes B \vdash \mathsf{let}\, x\, \mathsf{be}\, y \otimes z\, \mathsf{in}\, (y \otimes z) : A \otimes B}\,\textsc{Tl}$$

By the rule $\textsc{eq\_EtaTensor}$ we know $\mathsf{let}\, x\, \mathsf{be}\, y \otimes z\, \mathsf{in}\, (y \otimes z) = x$.

Case: the η-expansion cases: par. The proof

$$\dfrac{}{x : A \parr B \vdash x : A \parr B}\,\textsc{Ax}$$

is transformed into the proof

$$\dfrac{\dfrac{\dfrac{}{y : A \vdash y : A}\,\textsc{Ax} \qquad \dfrac{}{z : B \vdash z : B}\,\textsc{Ax}}{x : A \parr B \vdash \mathsf{let}\, x\, \mathsf{be}\, (y \parr -)\, \mathsf{in}\, y : A \mid \mathsf{let}\, x\, \mathsf{be}\, (- \parr z)\, \mathsf{in}\, z : B}\,\textsc{ParL}}{x : A \parr B \vdash (\mathsf{let}\, x\, \mathsf{be}\, (y \parr -)\, \mathsf{in}\, y) \parr (\mathsf{let}\, x\, \mathsf{be}\, (- \parr z)\, \mathsf{in}\, z) : A \parr B}\,\textsc{ParR}$$

By rule Eq_EtaPar we know $((\mathsf{let}\ x\ \mathsf{be}\ (y \parr -)\ \mathsf{in}\ y) \parr (\mathsf{let}\ x\ \mathsf{be}\ (- \parr z)\ \mathsf{in}\ z)) = x$.

Case: the η-expansion cases: implication. The proof

$$\dfrac{}{x : A \multimap B \vdash x : A \multimap B}\ \textsc{Ax}$$

transforms into the proof

$$\dfrac{\dfrac{\dfrac{}{y : A \vdash y : A}\ \textsc{Ax} \qquad \dfrac{}{z : B \vdash z : B}\ \textsc{Ax}}{y : A, x : A \multimap B \vdash x\ y : B}\ \textsc{ImpL}}{x : A \multimap B \vdash \lambda y.x\ y : A \multimap B}\ \textsc{ImpR}$$

All terms in the two derivations are equivalent, because $(\lambda y.x\ y) = x$ by the Eq_EtaFun rule.

Case: the η-expansion cases: tensor unit. The proof

$$\dfrac{}{x : \top \vdash x : \top}\ \textsc{Ax}$$

transforms into the proof

$$\dfrac{\dfrac{}{\cdot \vdash * : \top}\ \textsc{Ir}}{x : \top \vdash \mathsf{let}\ x\ \mathsf{be}\ *\ \mathsf{in} * : \top}\ \textsc{Il}$$

We know $x = \mathsf{let}\ x\ \mathsf{be}\ *\ \mathsf{in} *$ by Eq_EtaI.

Case: the η-expansion cases: par unit. The proof

$$\dfrac{}{x : \perp \vdash x : \perp}\ \textsc{Ax}$$

transforms into the proof

$$\dfrac{\dfrac{}{x : \perp \vdash \cdot}\ \textsc{Pl}}{x : \perp \vdash \circ : \perp}\ \textsc{Pr}$$

We know $x = \circ$ by Eq_EtaParU.

Case: the axiom steps: the axiom step. The proof

$$\dfrac{\dfrac{}{x : A \vdash x : A}\;\textsc{Ax} \qquad \dfrac{\begin{matrix}\pi \\ \vdots\end{matrix}}{\Gamma_1, y : A, \Gamma_2 \vdash \Delta}}{\Gamma_1, x : A, \Gamma_2 \vdash [x/y]\Delta}\;\textsc{Cut}$$

transforms into the proof

$$\dfrac{\begin{matrix}\pi \\ \vdots\end{matrix}}{\Gamma_1, y : A, \Gamma_2 \vdash \Delta}$$

By Eq_Alpha, we know, for any t in Δ, $t = [x/y]t$, and hence $\Delta = [x/y]\Delta$.

Case: the axiom steps: conclusion vs. axiom. The proof

$$\dfrac{\dfrac{\begin{matrix}\pi \\ \vdots\end{matrix}}{\Gamma \vdash t : A \mid \Delta} \qquad \dfrac{}{x : A \vdash x : A}\;\textsc{Ax}}{\Gamma \vdash \Delta \mid [t/x]x : A}\;\textsc{Cut}$$

transforms into

$$\dfrac{\dfrac{\begin{matrix}\pi \\ \vdots\end{matrix}}{\Gamma \vdash t : A \mid \Delta}}{\Gamma \vdash \Delta \mid t : A}\;\textsc{Series of Exchanges}$$

By the definition of the substitution function we know $t = [t/x]x$.

Case: the exchange steps: conclusion vs. left-exchange (the first case). The proof

$$\dfrac{\dfrac{\begin{matrix}\pi_1 \\ \vdots\end{matrix}}{\Gamma \vdash t : A \mid \Delta} \qquad \dfrac{\dfrac{\begin{matrix}\pi_2 \\ \vdots\end{matrix}}{\Gamma_1, x : A, y : B, \Gamma_2 \vdash \Delta'}}{\Gamma_1, y : B, x : A, \Gamma_2 \vdash \Delta'}\;\textsc{ExL}}{\Gamma_1, y : B, \Gamma, \Gamma_2 \vdash \Delta \mid [t/x]\Delta'}\;\textsc{Cut}$$

transforms into the proof

$$\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\\ \hline \Gamma \vdash t : A \mid \Delta\end{matrix} \qquad \begin{matrix}\pi_2\\ \vdots\\ \hline \Gamma_1, x : A, y : B, \Gamma_2 \vdash \Delta'\end{matrix}}{\Gamma_1, \Gamma, y : B, \Gamma_2 \vdash \Delta \mid [t/x]\Delta'}\ \textsc{Cut}}{\Gamma_1, y : B, \Gamma, \Gamma_2 \vdash \Delta \mid [t/x]\Delta'}\ \textsc{Series of Exchanges}$$

Clearly, all terms are equivalent.

Case: the exchange steps: conclusion vs. left-exchange (the second case). The proof

$$\dfrac{\begin{matrix}\pi_1\\ \vdots\\ \hline \Gamma \vdash t : B \mid \Delta\end{matrix} \qquad \dfrac{\begin{matrix}\pi_2\\ \vdots\\ \hline \Gamma_1, x : A, y : B, \Gamma_2 \vdash \Delta'\end{matrix}}{\Gamma_1, y : B, x : A, \Gamma_2 \vdash \Delta'}\ \textsc{Exl}}{\Gamma_1, \Gamma, x : A, \Gamma_2 \vdash \Delta \mid [t/y]\Delta'}\ \textsc{Cut}$$

transforms into the proof

$$\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\\ \hline \Gamma \vdash t : B \mid \Delta\end{matrix} \qquad \begin{matrix}\pi_2\\ \vdots\\ \hline \Gamma_1, x : A, y : B, \Gamma_2 \vdash \Delta'\end{matrix}}{\Gamma_1, x : A, \Gamma, \Gamma_2 \vdash \Delta \mid [t/y]\Delta'}\ \textsc{Cut}}{\Gamma_1, \Gamma, x : A, \Gamma_2 \vdash \Delta \mid [t/y]\Delta'}\ \textsc{Series of Exchanges}$$

Clearly, all terms are equivalent.

Case: the exchange steps: conclusion vs. right-exchange The proof

$$\dfrac{\begin{matrix}\pi_1\\ \vdots\\ \hline \Gamma \vdash t : A \mid \Delta\end{matrix} \qquad \dfrac{\begin{matrix}\pi_2\\ \vdots\\ \hline \Gamma_1, x : A, \Gamma_2 \vdash \Delta_1 \mid t_1 : B \mid t_2 : C \mid \Delta'\end{matrix}}{\Gamma_1, x : A, \Gamma_2 \vdash \Delta_1 \mid t_2 : C \mid t_1 : B \mid \Delta'}\ \textsc{Exr}}{\Gamma_1, \Gamma, \Gamma_2 \vdash \Delta \mid [t/x]\Delta_1 \mid [t/x]t_2 : C \mid [t/x]t_1 : B \mid [t/x]\Delta'}\ \textsc{Cut}$$

transforms into this proof

$$\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\\ \hline \Gamma \vdash t : A \mid \Delta\end{matrix} \qquad \begin{matrix}\pi_2\\ \vdots\\ \hline \Gamma_1, x : A, \Gamma_2 \vdash \Delta_1 \mid t_1 : B \mid t_2 : C \mid \Delta'\end{matrix}}{\Gamma_1, \Gamma, \Gamma_2 \vdash \Delta \mid [t/x]\Delta_1 \mid [t/x]t_1 : B \mid [t/x]t_2 : C \mid [t/x]\Delta'}\ \textsc{Cut}}{\Gamma_1, \Gamma, \Gamma_2 \vdash [t/x]\Delta_1 \mid [t/x]t_2 : C \mid [t/x]t_1 : B \mid [t/x]\Delta'}\ \textsc{Exr}$$

Clearly, all terms are equivalent.

Case: principal formula vs. principal formula: tensor. The proof

$$
\dfrac{\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\end{array}}{\Gamma_1 \vdash t_1 : A \mid \Delta_1} \qquad \dfrac{\begin{array}{c}\pi_2\\ \vdots\end{array}}{\Gamma_2 \vdash t_2 : B \mid \Delta_2}}{\Gamma_1, \Gamma_2 \vdash t_1 \otimes t_2 : A \otimes B \mid \Delta_1 \mid \Delta_2}\,\textsc{Tr} \qquad \dfrac{\dfrac{\begin{array}{c}\pi_3\\ \vdots\end{array}}{\Gamma_3, x : A, y : B, \Gamma_4 \vdash \Delta_3}}{\Gamma_3, z : A \otimes B, \Gamma_4 \vdash \mathsf{let}\, z\, \mathsf{be}\, x \otimes y\, \mathsf{in}\, \Delta_3}\,\textsc{Tl}}{\Gamma_3, \Gamma_1, \Gamma_2, \Gamma_4 \vdash \Delta_1 \mid \Delta_2 \mid [t_1 \otimes t_2/z](\mathsf{let}\, z\, \mathsf{be}\, x \otimes y\, \mathsf{in}\, \Delta_3)}\,\textsc{Cut}
$$

is transformed into the proof

$$
\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\end{array}}{\Gamma_1 \vdash t_1 : A \mid \Delta_1} \qquad \dfrac{\dfrac{\begin{array}{c}\pi_2\\ \vdots\end{array}}{\Gamma_2 \vdash t_2 : B \mid \Delta_2} \qquad \dfrac{\begin{array}{c}\pi_3\\ \vdots\end{array}}{\Gamma_3, x : A, y : B, \Gamma_4 \vdash \Delta_3}}{\Gamma_3, x : A, \Gamma_2, \Gamma_4 \vdash \Delta_2 \mid [t_2/y]\Delta_3}\,\textsc{Cut}}{\Gamma_3, \Gamma_1, \Gamma_2, \Gamma_4 \vdash \Delta_1 \mid \Delta_2 \mid [t_1/x][t_2/y]\Delta_3}\,\textsc{Cut}
$$

Without loss of generality suppose $\Delta_3 = t_3 : C, \Delta'_3$. We can see that $[t_1 \otimes t_2/z](\mathsf{let}\, z\, \mathsf{be}\, x \otimes y\, \mathsf{in}\, t_3) = \mathsf{let}\, t_1 \otimes t_2\, \mathsf{be}\, x \otimes y\, \mathsf{in}\, t_3$ by the definition of substitution, and by using the Eq_Beta1Tensor rule we obtain $\mathsf{let}\, t_1 \otimes t_2\, \mathsf{be}\, x \otimes y\, \mathsf{in}\, t_3 = [t_1/x][t_2/y]t_3$. This argument can be repeated for any term in $[t_1 \otimes t_2/z](\mathsf{let}\, z\, \mathsf{be}\, x \otimes y\, \mathsf{in}\, \Delta'_3)$, and thus, $[t_1 \otimes t_2/z](\mathsf{let}\, z\, \mathsf{be}\, x \otimes y\, \mathsf{in}\, \Delta_3) = [t_1/x][t_2/y]\Delta_3$.

Note that in the second derivation of the above transformation we first cut on B, and then A, but we could have cut on A first, and then B, but this would yeild equivalent derivations as above by using Lemma 4.

Case: principal formula vs. principal formula: par. The proof

$$
\dfrac{\textsc{Parr}\,\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\end{array}}{\Gamma_1 \vdash \Delta_1 \mid t_1 : A \mid t_2 : B \mid \Delta_2}}{\Gamma_1 \vdash \Delta_1 \mid t_1 \parr t_2 : A \parr B \mid \Delta_2} \quad \dfrac{\dfrac{\begin{array}{c}\pi_2\\ \vdots\end{array}}{\Gamma_2, x : A \vdash \Delta_3} \qquad \dfrac{\begin{array}{c}\pi_3\\ \vdots\end{array}}{\Gamma_3, y : B \vdash \Delta_4}}{\Gamma_2, \Gamma_3, z : A \parr B \vdash \text{let-pat}\, z\, (x \parr -)\, \Delta_3 \mid \text{let-pat}\, z\, (- \parr y)\, \Delta_4}\,\textsc{Parl}}{\Gamma_2, \Gamma_3, \Gamma_1 \vdash \Delta_1 \mid \Delta_2 \mid [t_1 \parr t_2/z](\text{let-pat}\, z\, (x \parr -)\, \Delta_3) \mid [t_1 \parr t_2/z](\text{let-pat}\, z\, (- \parr y)\, \Delta_4)}\,\textsc{Cut}
$$

is transformed into the proof

$$
\dfrac{\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{\Gamma_1 \vdash \Delta_1 \mid t_1 : A \mid t_2 : B \mid \Delta_2} \quad \dfrac{\begin{matrix}\pi_3\\ \vdots\end{matrix}}{\Gamma_3, y : B \vdash \Delta_4}}{\Gamma_3, \Gamma_1 \vdash \Delta_1 \mid t_1 : A \mid \Delta_2 \mid [t_2/y]\Delta_4}\,\textsc{Cut} \quad \dfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{\Gamma_2, x : A \vdash \Delta_3}}{\dfrac{\Gamma_2, \Gamma_3, \Gamma_1 \vdash \Delta_1 \mid \Delta_2 \mid [t_2/y]\Delta_4 \mid [t_1/x]\Delta_3}{\Gamma_2, \Gamma_3, \Gamma_1 \vdash \Delta_1 \mid \Delta_2 \mid [t_1/x]\Delta_3 \mid [t_2/y]\Delta_4}\,\textsc{Series of Exchanges}}\,\textsc{Cut}
$$

Without loss of generality consider the case when $\Delta_3 = t_3 : C_1 \mid \Delta_3'$ and $\Delta_4 = t_4 : C_2 \mid \Delta_4'$. First, $[t_1 \parr t_2/z](\text{let-pat}\, z\, (x \parr -)\, t_3) = \text{let-pat}\, (t_1 \parr t_2)\, (x \parr -)\, t_3$, and by $\textsc{Eq\_Beta1Par}$ we know $\text{let-pat}\, (t_1 \parr t_2)\, (x \parr -)\, t_3 = [t_1/x]t_3$ if $x \in \mathsf{FV}(t_3)$ or $\text{let-pat}\, (t_1 \parr t_2)\, (x \parr -)\, t_3 = t_3$ otherwise. In the latter case we can see that $t_3 = [t_1/x]t_3$, thus, in both cases $\text{let-pat}\, (t_1 \parr t_2)\, (x \parr -)\, t_3 = [t_1/x]t_3$. This argument can be repeated for any terms in Δ_3', and hence $[t_1 \parr t_2/z](\text{let-pat}\, z\, (x \parr -)\, \Delta_3) = \text{let-pat}\, (t_1 \parr t_2)\, (x \parr -)\, \Delta_3 = [t_1/x]\Delta_3$. We can apply a similar argument for $[t_1 \parr t_2/z](\text{let-pat}\, z\, (- \parr y)\, t_4)$ and $[t_1 \parr t_2/z](\text{let-pat}\, z\, (- \parr y)\, \Delta_4)$.

Note that just as we mentioned about tensor we could have first cut on A, and then on B in the second derivation, but we would have arrived at the same result just with potentially more exchanges on the right.

Case: principal formula vs. principal formula: implication. The proof

$$
\dfrac{\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{\Gamma, x : A \vdash t : B \mid \Delta} \quad x \notin \mathsf{FV}(\Delta)}{\Gamma \vdash \lambda x.t : A \multimap B \mid \Delta}\,\textsc{ImpR} \quad \dfrac{\dfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{\Gamma_1 \vdash t_1 : A \mid \Delta_1} \quad \dfrac{\begin{matrix}\pi_3\\ \vdots\end{matrix}}{\Gamma_2, y : B \vdash \Delta_2}}{\Gamma_1, z : A \multimap B, \Gamma_2 \vdash \Delta_1 \mid [z\, t_1/y]\Delta_2}\,\textsc{ImpL}}{\Gamma_1, \Gamma, \Gamma_2 \vdash \Delta \mid [\lambda x.t/z]\Delta_1 \mid [\lambda x.t/z][z\, t_1/y]\Delta_2}\,\textsc{Cut}
$$

transforms into the proof

$$
\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{\Gamma_1 \vdash t_1 : A \mid \Delta_1} \quad \dfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{\Gamma, x : A \vdash t : B \mid \Delta} \quad x \notin \mathsf{FV}(\Delta)}{\Gamma, \Gamma_1 \vdash \Delta_1 \mid [t_1/x]t : B \mid [t_1/x]\Delta}\,\textsc{Cut} \quad \dfrac{\begin{matrix}\pi_3\\ \vdots\end{matrix}}{\Gamma_2, y : B \vdash \Delta_2}}{\Gamma_2, \Gamma, \Gamma_1 \vdash \Delta_1 \mid [t_1/x]\Delta \mid [[t_1/x]t/y]\Delta_2}\,\textsc{Cut}}{\Gamma_1, \Gamma, \Gamma_2 \vdash [t_1/x]\Delta \mid \Delta_1 \mid [[t_1/x]t/y]\Delta_2}\,\textsc{Series of Exchanges}
$$

Without loss of generality consider the case when $\Delta_2 = t_2 : C \mid \Delta_2'$. First, by hypothesis we know $x \notin \mathsf{FV}(\Delta)$, and so we know $\Delta = [t_1/x]\Delta$. We can see that $[\lambda x.t/z][z\, t_1/y]t_2 = [(\lambda x.t)\, t_1/y]t_2 = [[t_1/x]t/y]t_2$ by using the congruence rules of equality and the rule $\textsc{Eq\_BetaFun}$. This argument can be repeated for any term in $[\lambda x.t/z][z\, t_1/y]\Delta_2'$, and so $[\lambda x.t/z][z\, t_1/y]\Delta_2 = [[t_1/x]t/y]\Delta_2$. Finally, by inspecting the previous derivations we can see that $z \notin \mathsf{FV}(\Delta_1)$, and thus, $\Delta_1 = [\lambda x.t/z]\Delta_1$.

Case: principal formula vs. principal formula: tensor unit. The proof

$$\dfrac{\dfrac{}{\cdot \vdash * : \top}\ \textsc{Ir} \qquad \dfrac{\dfrac{\pi \atop \vdots}{\Gamma \vdash \Delta}}{\Gamma, x : \top \vdash \mathsf{let}\, x\, \mathsf{be}\, *\, \mathsf{in}\, \Delta}\ \textsc{Il}}{\Gamma \vdash [*/x](\mathsf{let}\, x\, \mathsf{be}\, *\, \mathsf{in}\, \Delta)}\ \textsc{Cut}$$

is transformed into the proof

$$\dfrac{\pi \atop \vdots}{\Gamma \vdash \Delta}$$

Without loss of generality suppose $\Delta = t : A \mid \Delta'$. We can see that $[*/x](\mathsf{let}\, x\, \mathsf{be}\, *\, \mathsf{in}\, t) = \mathsf{let}\, *\, \mathsf{be}\, *\, \mathsf{in}\, t = t$ by the definition of substitution and the $\textsc{Eq\_EtaI}$ rule. This argument can be repeated for any term in $[*/x](\mathsf{let}\, x\, \mathsf{be}\, *\, \mathsf{in}\, \Delta')$, and hence, $[*/x](\mathsf{let}\, x\, \mathsf{be}\, *\, \mathsf{in}\, \Delta) = \Delta$.

Case: principal formula vs. principal formula: par unit. The proof

$$\dfrac{\dfrac{\dfrac{\pi \atop \vdots}{\Gamma \vdash \Delta}}{\Gamma \vdash \circ : \perp \mid \Delta}\ \textsc{Pr} \qquad \dfrac{}{x : \perp \vdash \cdot}\ \textsc{Pl}}{\Gamma \vdash \Delta \mid [\circ/x]\cdot}\ \textsc{Cut}$$

transforms into the proof

$$\dfrac{\pi \atop \vdots}{\Gamma \vdash \Delta}$$

Clearly, $[\circ/x]\cdot = \cdot$.

Case: secondary conclusion: left introduction of implication. The proof

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ \hline \Gamma \vdash t_1 : A \mid \Delta\end{array} \qquad \begin{array}{c}\pi_2\\ \vdots\\ \hline \Gamma_1, x : B, \Gamma_2 \vdash t_2 : C \mid \Delta_2\end{array}}{\Gamma, y : A \multimap B, \Gamma_1, \Gamma_2 \vdash \Delta \mid [y\, t_1/x]t_2 : C \mid [y\, t_1/x]\Delta_2}\ \textsc{Impl} \qquad \begin{array}{c}\pi_3\\ \vdots\\ \hline \Gamma_3, z : C, \Gamma_4 \vdash \Delta_3\end{array}}{\Gamma_3, \Gamma, y : A \multimap B, \Gamma_1, \Gamma_2, \Gamma_4 \vdash \Delta \mid [y\, t_1/x]\Delta_2 \mid [[y\, t_1/x]t_2/z]\Delta_3}\ \textsc{Cut}$$

transforms into the proof

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ \hline \Gamma \vdash t_1 : A \mid \Delta\end{array} \qquad \dfrac{\begin{array}{c}\pi_2\\ \vdots\\ \hline \Gamma_1, x : B, \Gamma_2 \vdash t_2 : C \mid \Delta_2\end{array} \qquad \begin{array}{c}\pi_3\\ \vdots\\ \hline \Gamma_3, z : C, \Gamma_4 \vdash \Delta_3\end{array}}{\Gamma_3, \Gamma_1, x : B, \Gamma_2, \Gamma_4 \vdash \Delta_2 \mid [t_2/z]\Delta_3}\ \textsc{Cut}}{\Gamma, y : A \multimap B, \Gamma_3, \Gamma_1, \Gamma_2, \Gamma_4 \vdash \Delta \mid [y\, t_1/x]\Delta_2 \mid [y\, t_1/x][t_2/z]\Delta_3}\ \textsc{Impl}}{\Gamma_3, \Gamma, y : A \multimap B, \Gamma_1, \Gamma_2, \Gamma_4 \vdash \Delta \mid [y\, t_1/x]\Delta_2 \mid [y\, t_1/x][t_2/z]\Delta_3}\ \textsc{Series of Exchanges}$$

This case is similar to Section **??**. Thus, we can prove that $[y\, t_1/x][t_2/z]\Delta_3 = [[y\, t_1/x]t_2/z]\Delta_3$ by Lemma 4 and the fact that $x \notin \mathsf{FV}(\Delta_3)$.

Case: secondary conclusion: left introduction of exchange. The proof

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ \hline \Gamma, y : B, x : A, \Gamma' \vdash t : C \mid \Delta\end{array}}{\Gamma, x : A, y : B, \Gamma' \vdash t : C \mid \Delta}\ \textsc{ExL} \qquad \begin{array}{c}\pi_2\\ \vdots\\ \hline \Gamma_1, z : C, \Gamma_2 \vdash \Delta_2\end{array}}{\Gamma_1, \Gamma, x : A, y : B, \Gamma', \Gamma_2 \vdash \Delta \mid [t/z]\Delta_2}\ \textsc{Cut}$$

transforms into the proof

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ \hline \Gamma, y : B, x : A, \Gamma' \vdash t : C \mid \Delta\end{array} \qquad \begin{array}{c}\pi_2\\ \vdots\\ \hline \Gamma_1, z : C, \Gamma_2 \vdash \Delta_2\end{array}}{\Gamma_1, \Gamma, y : B, x : A, \Gamma', \Gamma_2 \vdash \Delta \mid [t/z]\Delta_2}\ \textsc{Cut}}{\Gamma_1, \Gamma, x : A, y : B, \Gamma', \Gamma_2 \vdash \Delta \mid [t/z]\Delta_2}\ \textsc{ExL}$$

Clearly, all terms are equivalent.

Case: secondary conclusion: left introduction of tensor. The proof

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ \hline \Gamma, x : A, y : B \vdash t : C \mid \Delta\end{array}}{\Gamma, z : A \otimes B \vdash \mathsf{let}\, z\, \mathsf{be}\, x \otimes y\, \mathsf{in}\, t : C \mid \mathsf{let}\, z\, \mathsf{be}\, x \otimes y\, \mathsf{in}\, \Delta}\ \textsc{Tl} \qquad \begin{array}{c}\pi_2\\ \vdots\\ \hline \Gamma_1, w : C, \Gamma_2 \vdash \Delta_2\end{array}}{\Gamma_1, \Gamma, z : A \otimes B, \Gamma_2 \vdash \mathsf{let}\, z\, \mathsf{be}\, x \otimes y\, \mathsf{in}\, \Delta \mid [\mathsf{let}\, z\, \mathsf{be}\, x \otimes y\, \mathsf{in}\, t/w]\Delta_2}\ \textsc{Cut}$$

transforms into the proof

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ \hline \Gamma, x : A, y : B \vdash t : C \mid \Delta\end{array} \qquad \begin{array}{c}\pi_2\\ \vdots\\ \hline \Gamma_1, w : C, \Gamma_2 \vdash \Delta_2\end{array}}{\Gamma_1, \Gamma, x : A, y : B, \Gamma_2 \vdash \Delta \mid [t/w]\Delta_2}\ \textsc{Cut}}{\Gamma_1, \Gamma, z : A \otimes B, \Gamma_2 \vdash \mathsf{let}\, z\, \mathsf{be}\, x \otimes y\, \mathsf{in}\, \Delta \mid \mathsf{let}\, z\, \mathsf{be}\, x \otimes y\, \mathsf{in}\, ([t/w]\Delta_2)}\ \textsc{Tl}$$

It suffices to show that $\mathsf{let}\, z\, \mathsf{be}\, x \otimes y\, \mathsf{in}\, ([t/w]\Delta_2) = [\mathsf{let}\, z\, \mathsf{be}\, x \otimes y\, \mathsf{in}\, t/w]\Delta_2$. This is a simple consequence of the rule Eq_NatTensor.

Case: secondary conclusion: left introduction of Par The proof

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ \hline \Gamma, x : A \vdash \Delta\end{array} \qquad \begin{array}{c}\pi_2\\ \vdots\\ \hline \Gamma', y : B \vdash t' : C \mid \Delta'\end{array}}{\Gamma, \Gamma', z : A \parr B \vdash \text{let-pat}\, z\, (x \parr -)\, \Delta \mid \text{let-pat}\, z\, (- \parr y)\, t' : C \mid \text{let-pat}\, z\, (- \parr y)\, \Delta'}\ \textsc{Parl} \qquad \begin{array}{c}\pi_3\\ \vdots\\ \hline \Gamma_1, w : C, \Gamma_2 \vdash \Delta_2\end{array}}{\Gamma_1, \Gamma, \Gamma', z : A \parr B, \Gamma_2 \vdash \text{let-pat}\, z\, (x \parr -)\, \Delta \mid \text{let-pat}\, z\, (- \parr y)\, \Delta' \mid [\text{let-pat}\, z\, (- \parr y)\, t'/w]\Delta_2}\ \textsc{Cut}$$

is transformed into the proof

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ \hline \Gamma, x : A \vdash \Delta\end{array} \qquad \dfrac{\begin{array}{c}\pi_2\\ \vdots\\ \hline \Gamma', y : B \vdash t' : C \mid \Delta'\end{array} \qquad \begin{array}{c}\pi_3\\ \vdots\\ \hline \Gamma_1, w : C, \Gamma_2 \vdash \Delta_2\end{array}}{\Gamma_1, \Gamma', y : B, \Gamma_2 \vdash \Delta' \mid [t'/w]\Delta_2}\ \textsc{Cut}}{\Gamma, \Gamma_1, \Gamma', \Gamma_2, z : A \parr B \vdash \text{let-pat}\, z\, (x \parr -)\, \Delta \mid \text{let-pat}\, z\, (- \parr y)\, \Delta' \mid \text{let-pat}\, z\, (- \parr y)\, [t'/w]\Delta_2}\ \textsc{Parl}}{\Gamma_1, \Gamma, \Gamma', z : A \parr B, \Gamma_2 \vdash \text{let-pat}\, z\, (x \parr -)\, \Delta \mid \text{let-pat}\, z\, (- \parr y)\, \Delta' \mid \text{let-pat}\, z\, (- \parr y)\, [t'/w]\Delta_2}\ \textsc{Series of Exchanges}$$

It suffices to show that let-pat $z\, (- \parr y)\, [t'/w]\Delta_2 = [\text{let-pat}\, z\, (- \parr y)\, t'/w]\Delta_2$. This follows from the rule Eq_Nat2Par.

Case: secondary conclusion: left introduction of tensor unit. The proof

$$\dfrac{\dfrac{\dfrac{\overset{\pi_1}{\vdots}}{\Gamma \vdash t : C \mid \Delta}}{\Gamma, x : \top \vdash t : C \mid \Delta}\,\textsc{Il} \qquad \dfrac{\overset{\pi_2}{\vdots}}{\Gamma_1, w : C, \Gamma_2 \vdash \Delta_1}}{\Gamma_1, \Gamma, x : \top, \Gamma_2 \vdash \Delta \mid [t/w]\Delta_1}\,\textsc{Cut}$$

is transformed into the following:

$$\dfrac{\dfrac{\dfrac{\dfrac{\overset{\pi_1}{\vdots}}{\Gamma \vdash t : C \mid \Delta} \qquad \dfrac{\overset{\pi_2}{\vdots}}{\Gamma_1, w : C, \Gamma_2 \vdash \Delta_1}}{\Gamma_1, \Gamma, \Gamma_2 \vdash \Delta \mid [t/w]\Delta_1}\,\textsc{Cut}}{\Gamma_1, \Gamma, \Gamma_2, x : \top \vdash \Delta \mid [t/w]\Delta_1}\,\textsc{Il}}{\Gamma_1, \Gamma, x : \top, \Gamma_2 \vdash \Delta \mid [t/w]\Delta_1}\,\textsc{Series of Exchanges}$$

Clearly, all terms are equivalent. Note that we do not give a case for secondary conclusion of the left introduction of par's unit, because it can only be introduced given an empty right context, and thus there is no cut formula.

Case: secondary hypothesis: left introduction of tensor. The proof

$$\dfrac{\dfrac{\overset{\pi_1}{\vdots}}{\Gamma \vdash t : A \mid \Delta} \qquad \dfrac{\dfrac{\overset{\pi_2}{\vdots}}{\Gamma_1, x : A, \Gamma_2, y : B, z : C, \Gamma_3 \vdash t_1 : D \mid \Delta_1}}{\Gamma_1, x : A, \Gamma_2, w : B \otimes C, \Gamma_3 \vdash \mathsf{let}\, w\, \mathsf{be}\, y \otimes z\, \mathsf{in}\, t_1 : D \mid \mathsf{let}\, w\, \mathsf{be}\, y \otimes z\, \mathsf{in}\, \Delta_1}\,\textsc{Tl}}{\Gamma_1, \Gamma, \Gamma_2, w : B \otimes C, \Gamma_3 \vdash \Delta \mid [t/x](\mathsf{let}\, w\, \mathsf{be}\, y \otimes z\, \mathsf{in}\, t_1) : D \mid [t/x](\mathsf{let}\, w\, \mathsf{be}\, y \otimes z\, \mathsf{in}\, \Delta_1)}\,\textsc{Cut}$$

transforms into the proof

$$\dfrac{\dfrac{\dfrac{\overset{\pi_1}{\vdots}}{\Gamma \vdash t : A \mid \Delta} \qquad \dfrac{\overset{\pi_2}{\vdots}}{\Gamma_1, x : A, \Gamma_2, y : B, z : C, \Gamma_3 \vdash t_1 : D \mid \Delta_1}}{\Gamma_1, \Gamma, \Gamma_2, y : B, z : C, \Gamma_3 \vdash \Delta \mid [t/x]t_1 : D \mid [t/x]\Delta_1}\,\textsc{Cut}}{\Gamma_1, \Gamma, \Gamma_2, w : B \otimes C, \Gamma_3 \vdash \mathsf{let}\, w\, \mathsf{be}\, x \otimes y\, \mathsf{in}\, \Delta \mid \mathsf{let}\, w\, \mathsf{be}\, x \otimes y\, \mathsf{in}\, [t/x]t_1 : D \mid \mathsf{let}\, w\, \mathsf{be}\, x \otimes y\, \mathsf{in}\, [t/x]\Delta_1}\,\textsc{Tl}$$

First, we can see by inspection of the previous derivations that $x, y \notin \mathsf{FV}(\Delta)$, thus, by using similar reasoning as above we can use the EtaTensor rule to obtain $\mathsf{let}\, w\, \mathsf{be}\, x \otimes y\, \mathsf{in}\, \Delta = \Delta$. It is a well-known property of substitution that $[t/x](\mathsf{let}\, w\, \mathsf{be}\, x \otimes y\, \mathsf{in}\, t_1) = \mathsf{let}\, [t/x]w\, \mathsf{be}\, x \otimes y\, \mathsf{in}\, [t/x]t_1 = \mathsf{let}\, w\, \mathsf{be}\, x \otimes y\, \mathsf{in}\, [t/x]t_1$.

Case: secondary hypothesis: right introduction of tensor (first case). The proof

$$\dfrac{\dfrac{\overset{\pi_1}{\vdots}}{\Gamma \vdash t : A \mid \Delta} \qquad \dfrac{\dfrac{\overset{\pi_2}{\vdots}}{\Gamma_1, x : A, \Gamma_2 \vdash t_1 : B \mid \Delta_1} \qquad \dfrac{\overset{\pi_3}{\vdots}}{\Gamma_3 \vdash t_2 : C \mid \Delta_2}}{\Gamma_1, x : A, \Gamma_2, \Gamma_3 \vdash t_1 \otimes t_2 : B \otimes C \mid \Delta_1 \mid \Delta_2}\,\textsc{Tr}}{\Gamma_1, \Gamma, \Gamma_2, \Gamma_3 \vdash \Delta \mid [t/x](t_1 \otimes t_2) : B \otimes C \mid [t/x]\Delta_1 \mid [t/x]\Delta_2}\,\textsc{Cut}$$

transforms into the proof

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{\Gamma \vdash t : A \mid \Delta} \qquad \dfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{\Gamma_1, x : A, \Gamma_2 \vdash t_1 : B \mid \Delta_1}}{\Gamma_1, \Gamma, \Gamma_2 \vdash \Delta \mid [t/x]t_1 : B \mid [t/x]\Delta_1}\,\textsc{Cut} \qquad \dfrac{\begin{matrix}\pi_3\\ \vdots\end{matrix}}{\Gamma_3 \vdash t_2 : C \mid \Delta_2}}{\Gamma_1, \Gamma, \Gamma_2, \Gamma_3 \vdash [t/x]t_1 \otimes t_2 : B \otimes C \mid \Delta \mid [t/x]\Delta_1 \mid \Delta_2}\,\textsc{Tr}}{\Gamma_1, \Gamma, \Gamma_2, \Gamma_3 \vdash \Delta \mid ([t/x]t_1) \otimes t_2 : B \otimes C \mid [t/x]\Delta_1 \mid \Delta_2}\,\text{Series of Exchanges}$$

By inspection of the previous derivations we can see that $x \notin \mathsf{FV}(t_2)$ and $x \notin \mathsf{FV}(\Delta_2)$. Thus, $[t/x]\Delta_2 = \Delta_2$ and $[t/x](t_1 \otimes t_2) = ([t/x]t_1) \otimes ([t/x]t_2) = ([t/x]t_1) \otimes t_2$.

Case: secondary hypothesis: right introduction of tensor (second case). The proof

$$\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{\Gamma \vdash t : A \mid \Delta} \qquad \dfrac{\dfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{\Gamma_1 \vdash t_1 : B \mid \Delta_1} \qquad \dfrac{\begin{matrix}\pi_3\\ \vdots\end{matrix}}{\Gamma_2, x : A, \Gamma_3 \vdash t_2 : C \mid \Delta_2}}{\Gamma_1, \Gamma_2, x : A, \Gamma_3 \vdash t_1 \otimes t_2 : B \otimes C \mid \Delta_1 \mid \Delta_2}\,\textsc{Tr}}{\Gamma_1, \Gamma, \Gamma_2, \Gamma_3 \vdash \Delta \mid [t/x](t_1 \otimes t_2) : B \otimes C \mid [t/x]\Delta_1 \mid [t/x]\Delta_2}\,\textsc{Cut}$$

transforms into the proof

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{\Gamma_1 \vdash t_1 : B \mid \Delta_1} \qquad \dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{\Gamma \vdash t : A \mid \Delta} \qquad \dfrac{\begin{matrix}\pi_3\\ \vdots\end{matrix}}{\Gamma_2, x : A, \Gamma_3 \vdash t_2 : C \mid \Delta_2}}{\Gamma_2, \Gamma, \Gamma_3 \vdash \Delta \mid [t/x]t_2 : C \mid [t/x]\Delta_2}\,\textsc{Cut}}{\Gamma_1, \Gamma_2, \Gamma, \Gamma_3 \vdash t_1 \otimes ([t/x]t_2) : B \otimes C \mid \Delta_1 \mid \Delta \mid [t/x]\Delta_2}\,\textsc{Tr}}{\Gamma_1, \Gamma, \Gamma_2, \Gamma_3 \vdash \Delta \mid t_1 \otimes ([t/x]t_2) : B \otimes C \mid \Delta_1 \mid [t/x]\Delta_2}\,\text{Series of Exchanges}$$

This case is similar to the previous case.

Case: secondary hypothesis: right introduction of par. The proof

$$\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{\Gamma \vdash t : A \mid \Delta} \qquad \dfrac{\dfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{\Gamma_1, x : A, \Gamma_2 \vdash \Delta_1 \mid t_1 : B \mid t_2 : C \mid \Delta_2}}{\Gamma_1, x : A, \Gamma_2 \vdash \Delta_1 \mid t_1 \parr t_2 : B \parr C \mid \Delta_2}\,\textsc{Parr}}{\Gamma_1, \Gamma, \Gamma_2 \vdash \Delta \mid [t/x]\Delta_1 \mid [t/x](t_1 \parr t_2) : B \parr C \mid [t/x]\Delta_2}\,\textsc{Cut}$$

transforms into the proof

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{\Gamma \vdash t : A \mid \Delta} \qquad \dfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{\Gamma_1, x : A, \Gamma_2 \vdash \Delta_1 \mid t_1 : B \mid t_2 : C \mid \Delta_2}}{\Gamma_1, \Gamma, \Gamma_2 \vdash \Delta \mid [t/x]\Delta_1 \mid [t/x]t_1 : B \mid [t/x]t_2 : C \mid [t/x]\Delta_2}\,\textsc{Cut}}{\Gamma_1, \Gamma, \Gamma_2 \vdash \Delta \mid [t/x]\Delta_1 \mid [t/x]t_1 \parr [t/x]t_2 : B \parr C \mid [t/x]\Delta_2}\,\textsc{Parr}$$

Clearly, $[t/x](t_1 \parr t_2) = ([t/x]t_1) \parr [t/x]t_2$.

Case: secondary hypothesis: left introduction of par (first case). The proof

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\end{array}}{\Gamma \vdash t : A \mid \Delta} \qquad \dfrac{\dfrac{\begin{array}{c}\pi_2\\ \vdots\end{array}}{\Gamma_1, x : A, \Gamma_2, y : B \vdash \Delta_1} \qquad \dfrac{\begin{array}{c}\pi_3\\ \vdots\end{array}}{\Gamma_3, z : C \vdash \Delta_2}}{\Gamma_1, x : A, \Gamma_2, \Gamma_3, w : B \parr C \vdash \text{let-pat}\, w\, (y \parr -)\, \Delta_1 \mid \text{let-pat}\, w\, (- \parr z)\, \Delta_2}\ \textsc{ParL}}{\Gamma_1, \Gamma, \Gamma_2, \Gamma_3, w : B \parr C \vdash \Delta \mid [t/x](\text{let-pat}\, w\, (y \parr -)\, \Delta_1) \mid [t/x](\text{let-pat}\, w\, (- \parr z)\, \Delta_2)}\ \textsc{Cut}$$

transforms into the proof

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\end{array}}{\Gamma \vdash t : A \mid \Delta} \qquad \dfrac{\begin{array}{c}\pi_2\\ \vdots\end{array}}{\Gamma_1, x : A, \Gamma_2, y : B \vdash \Delta_1}}{\Gamma_1, \Gamma, \Gamma_2, y : B \vdash \Delta \mid [t/x]\Delta_1}\ \textsc{Cut} \qquad \dfrac{\begin{array}{c}\pi_3\\ \vdots\end{array}}{\Gamma_3, z : C \vdash \Delta_2}}{\Gamma_1, \Gamma, \Gamma_2, \Gamma_3, w : B \parr C \vdash \text{let-pat}\, w\, (y \parr -)\, \Delta \mid \text{let-pat}\, w\, (y \parr -)\, [t/x]\Delta_1 \mid \text{let-pat}\, w\, (- \parr z)\, \Delta_2}\ \textsc{ParL}$$

First, by inspection of the previous proofs we can see that $x \notin \mathsf{FV}(\Delta)$ and $x \notin \mathsf{FV}(\Delta_2)$. Thus, $\text{let-pat}\, w\, (y \parr -)\, \Delta = \Delta$, and $[t/x](\text{let-pat}\, w\, (- \parr z)\, \Delta_2) = \text{let-pat}\, w\, (- \parr z)\, \Delta_2$. It suffices to show that $[t/x](\text{let-pat}\, w\, (y \parr -)\, \Delta_1) = \text{let-pat}\, w\, (y \parr -)\, [t/x]\Delta_1$ but this easily follows from a simple distributing the substitution into the let-pat, and then simplifying using the fact that $w \neq x$.

Case: secondary hypothesis: left introduction of par (second case). The proof

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\end{array}}{\Gamma \vdash t : A \mid \Delta} \qquad \dfrac{\dfrac{\begin{array}{c}\pi_2\\ \vdots\end{array}}{\Gamma_1, y : B \vdash \Delta_1} \qquad \dfrac{\begin{array}{c}\pi_3\\ \vdots\end{array}}{\Gamma_2, x : A, \Gamma_3, z : C \vdash \Delta_2}}{\Gamma_1, \Gamma_2, x : A, \Gamma_3, w : B \parr C \vdash \text{let-pat}\, w\, (y \parr -)\, \Delta_1 \mid \text{let-pat}\, w\, (- \parr z)\, \Delta_2}\ \textsc{ParL}}{\Gamma_1, \Gamma_2, \Gamma, \Gamma_3, w : B \parr C \vdash \Delta \mid [t/x](\text{let-pat}\, w\, (y \parr -)\, \Delta_1) \mid [t/x](\text{let-pat}\, w\, (- \parr z)\, \Delta_2)}\ \textsc{Cut}$$

transforms into the proof

$$\dfrac{\dfrac{\begin{array}{c}\pi_2\\ \vdots\end{array}}{\Gamma_1, y : B \vdash \Delta_1} \qquad \dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\end{array}}{\Gamma \vdash t : A \mid \Delta} \qquad \dfrac{\begin{array}{c}\pi_3\\ \vdots\end{array}}{\Gamma_2, x : A, \Gamma_3, z : C \vdash \Delta_2}}{\Gamma_2, \Gamma, \Gamma_3, z : C \vdash \Delta \mid [t/x]\Delta_2}\ \textsc{Cut}}{\Gamma_1, \Gamma_2, \Gamma, \Gamma_3, w : B \parr C \vdash \text{let-pat}\, w\, (y \parr -)\, \Delta_1 \mid \text{let-pat}\, w\, (- \parr z)\, \Delta \mid \text{let-pat}\, w\, (- \parr z)\, [t/x]\Delta_2}\ \textsc{ParL}$$

Similar to the previous case.

Case: secondary hypothesis: left introduction of implication (first case). The proof

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\end{array}}{\Gamma \vdash t : A \mid \Delta} \qquad \dfrac{\dfrac{\begin{array}{c}\pi_2\\ \vdots\end{array}}{\Gamma_1, x : A, \Gamma_2 \vdash t_1 : B \mid \Delta_1} \qquad \dfrac{\begin{array}{c}\pi_3\\ \vdots\end{array}}{\Gamma_3, y : C \vdash \Delta_2}}{\Gamma_1, x : A, \Gamma_2, \Gamma_3, z : B \multimap C \vdash \Delta_1 \mid [z\, t_1/y]\Delta_2}\ \textsc{ImpL}}{\Gamma_1, \Gamma, \Gamma_2, \Gamma_3, z : B \multimap C \vdash \Delta \mid [t/x]\Delta_1 \mid [t/x][z\, t_1/y]\Delta_2}\ \textsc{Cut}$$

transforms into the proof

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{\Gamma \vdash t : A \mid \Delta} \qquad \dfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{\Gamma_1, x : A, \Gamma_2 \vdash t_1 : B \mid \Delta_1}}{\Gamma_1, \Gamma, \Gamma_2 \vdash \Delta \mid [t/x]t_1 : B \mid [t/x]\Delta_1}\ \text{Cut} \qquad \dfrac{\begin{matrix}\pi_3\\ \vdots\end{matrix}}{\Gamma_3, y : C \vdash \Delta_2}}{\Gamma_1, \Gamma, \Gamma_2, \Gamma_3, z : B \multimap C \vdash \Delta \mid [t/x]\Delta_1 \mid [z\ ([t/x]t_1)/y]\Delta_2}\ \text{Impl}$$

By inspection of the above derivations we can see that $x \notin \mathsf{FV}(\Delta_2)$, and hence, by this fact and substitution distribution (Lemma 4) we know $[t/x][z\ t_1/y]\Delta_2 = [([t/x]z)\ ([t/x]t_1)/y][t/x]\Delta_2 = [z\ ([t/x]t_1)/y]\Delta_2$.

Case: secondary hypothesis: left introduction of implication (second case). The proof

$$\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{\Gamma \vdash t : A \mid \Delta} \qquad \dfrac{\dfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{\Gamma_1 \vdash t_1 : B \mid \Delta_1} \qquad \dfrac{\begin{matrix}\pi_3\\ \vdots\end{matrix}}{\Gamma_2, x : A, \Gamma_3, y : C \vdash \Delta_2}}{\Gamma_1, \Gamma_2, x : A, \Gamma_3, z : B \multimap C \vdash \Delta_1 \mid [z\ t_1/y]\Delta_2}\ \text{Impl}}{\Gamma_1, \Gamma_2, \Gamma, \Gamma_3, z : B \multimap C \vdash \Delta \mid [t/x]\Delta_1 \mid [t/x][z\ t_1/y]\Delta_2}\ \text{Cut}$$

transforms into the proof

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{\Gamma_1 \vdash t_1 : B \mid \Delta_1} \qquad \dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{\Gamma \vdash t : A \mid \Delta} \qquad \dfrac{\begin{matrix}\pi_3\\ \vdots\end{matrix}}{\Gamma_2, x : A, \Gamma_3, y : C \vdash \Delta_2}}{\Gamma_2, \Gamma, \Gamma_3, y : C \vdash \Delta \mid [t/x]\Delta_2}\ \text{Cut}}{\Gamma_1, \Gamma_2, \Gamma, \Gamma_3, z : B \multimap C \vdash \Delta_1 \mid [z\ t_1/y]\Delta \mid [z\ t_1/y][t/x]\Delta_2}\ \text{Impl}}{\Gamma_1, \Gamma_2, \Gamma, \Gamma_3, z : B \multimap C \vdash [z\ t_1/y]\Delta \mid \Delta_1 \mid [z\ t_1/y][t/x]\Delta_2}\ \text{Series of Exchanges}$$

By inspection of the above proofs we can see that $y \notin \mathsf{FV}(\Delta)$. Thus, $[z\ t_1/y]\Delta = \Delta$. The same can be said for the variable x and context Δ_1, and hence, $[t/x]\Delta_1 = \Delta_1$. Finally, by inspection of the above proofs $x \notin \mathsf{FV}(t_1)$ and so by substitution distribution (Lemma 4) we know $[t/x][z\ t_1/y]\Delta_2 = [z\ t_1/y][t/x]\Delta_2$.

Case: secondary hypothesis: left introduction of implication (third case). The proof

$$\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{\Gamma \vdash t : A \mid \Delta} \qquad \dfrac{\dfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{\Gamma_1 \vdash t_1 : B \mid \Delta_1} \qquad \dfrac{\begin{matrix}\pi_3\\ \vdots\end{matrix}}{\Gamma_2, y : C, \Gamma_3, x : A \vdash \Delta_2}}{\Gamma_1, \Gamma_2, z : B \multimap C, \Gamma_3, x : A \vdash \Delta_1 \mid [z\ t_1/y]\Delta_2}\ \text{Impl}}{\Gamma_1, \Gamma_2, z : B \multimap C, \Gamma_3, \Gamma \vdash \Delta \mid [t/x]\Delta_1 \mid [t/x][z\ t_1/y]\Delta_2}\ \text{Cut}$$

transforms into the proof

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{\Gamma_1 \vdash t_1 : B \mid \Delta_1} \qquad \dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{\Gamma \vdash t : A \mid \Delta} \qquad \dfrac{\begin{matrix}\pi_3\\ \vdots\end{matrix}}{\Gamma_2, y : C, \Gamma_3, x : A \vdash \Delta_2}}{\Gamma_2, y : C, \Gamma_3, \Gamma \vdash \Delta \mid [t/x]\Delta_2}\ \text{Cut}}{\Gamma_1, \Gamma_2, z : B \multimap C, \Gamma_3, \Gamma \vdash \Delta_1 \mid [z\ t_1/y]\Delta \mid [z\ t_1/y][t/x]\Delta_2}\ \text{Impl}}{\Gamma_1, \Gamma_2, z : B \multimap C, \Gamma_3, \Gamma \vdash [z\ t_1/y]\Delta \mid \Delta_1 \mid [z\ t_1/y][t/x]\Delta_2}\ \text{Series of Exchanges}$$

Similar to the previous case.

Case: secondary hypothesis: right introduction of implication. The proof

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\end{array}}{\Gamma \vdash t : A \mid \Delta} \qquad \dfrac{\dfrac{\begin{array}{c}\pi_2\\ \vdots\end{array}}{\Gamma_1, x : A, \Gamma_2, y : B \vdash t_1 : C \mid \Delta_1} \quad y \notin \mathsf{FV}(\Delta_1)}{\Gamma_1, x : A, \Gamma_2 \vdash \lambda y.t_1 : B \multimap C \mid \Delta_1}\,\textsc{ImpR}}{\Gamma_1, \Gamma, \Gamma_2 \vdash \Delta \mid [t/x](\lambda y.t_1) : B \multimap C \mid [t/x]\Delta_1}\,\textsc{Cut}$$

transforms into the proof

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\end{array}}{\Gamma \vdash t : A \mid \Delta} \qquad \dfrac{\begin{array}{c}\pi_2\\ \vdots\end{array}}{\Gamma_1, x : A, \Gamma_2, y : B \vdash t_1 : C \mid \Delta_1}}{\Gamma_1, \Gamma, \Gamma_2, y : B \vdash \Delta \mid [t/x]t_1 : C \mid [t/x]\Delta_1}\,\textsc{Cut}}{\Gamma_1, \Gamma, \Gamma_2 \vdash \Delta \mid \lambda y.[t/x]t_1 : B \multimap C \mid [t/x]\Delta_1}\,\textsc{ImpR}$$

Clearly, $[t/x](\lambda y.t_1) = \lambda y.[t/x]t_1$.

Case: secondary hypothesis: left introduction of tensor unit. The proof

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\end{array}}{\Gamma \vdash t : A \mid \Delta} \qquad \dfrac{\dfrac{\begin{array}{c}\pi_2\\ \vdots\end{array}}{\Gamma_1, x : A, \Gamma_2 \vdash \Delta_1}}{\Gamma_1, x : A, \Gamma_2, y : \top \vdash \mathsf{let}\, y\, \mathsf{be} * \mathsf{in}\, \Delta_1}\,\textsc{Il}}{\Gamma_1, \Gamma, \Gamma_2, y : \top \vdash \Delta \mid [t/x](\mathsf{let}\, y\, \mathsf{be} * \mathsf{in}\, \Delta_1)}\,\textsc{Cut}$$

transforms into the proof

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\end{array}}{\Gamma \vdash t : A \mid \Delta} \qquad \dfrac{\begin{array}{c}\pi_2\\ \vdots\end{array}}{\Gamma_1, x : A, \Gamma_2 \vdash \Delta_1}}{\Gamma_1, \Gamma, \Gamma_2 \vdash \Delta \mid [t/x]\Delta_1}\,\textsc{Cut}}{\Gamma_1, \Gamma, \Gamma_2, y : \top \vdash \mathsf{let}\, y\, \mathsf{be} * \mathsf{in}\, \Delta \mid \mathsf{let}\, y\, \mathsf{be} * \mathsf{in}\, [t/x]\Delta_1}\,\textsc{Il}$$

It suffices to show that $\Delta = \mathsf{let}\, y\, \mathsf{be} * \mathsf{in}\, \Delta$ and $[t/x](\mathsf{let}\, y\, \mathsf{be} * \mathsf{in}\, \Delta_1) = \mathsf{let}\, y\, \mathsf{be} * \mathsf{in}\, [t/x]\Delta_1$. Without loss of generality suppose $\Delta = t : B, \Delta'$. We know that it must be the case that $y \notin \mathsf{FV}(t)$, and we know that $[y/z]t = t$ when $z \notin \mathsf{FV}(t)$. Then by Eq_Eta2I we have $t = \mathsf{let}\, y\, \mathsf{be} * \mathsf{in}\, t$. This argument can be repeated for any other term in Δ'. Thus, $\Delta = \mathsf{let}\, y\, \mathsf{be} * \mathsf{in}\, \Delta$. It is easy to see that $[t/x](\mathsf{let}\, y\, \mathsf{be} * \mathsf{in}\, \Delta_1) = \mathsf{let}\, y\, \mathsf{be} * \mathsf{in}\, [t/x]\Delta_1$ using the rule Eq_NatI.

Case: secondary hypothesis: right introduction of par unit. The proof

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\end{array}}{\Gamma \vdash t : A \mid \Delta} \qquad \dfrac{\dfrac{\begin{array}{c}\pi_2\\ \vdots\end{array}}{\Gamma_1, x : A, \Gamma_2 \vdash \Delta_1}}{\Gamma_1, x : A, \Gamma_2 \vdash \circ : \perp \mid \Delta_1}\,\textsc{Pr}}{\Gamma_1, \Gamma, \Gamma_2 \vdash \Delta \mid [t/x]\circ : \perp \mid [t/x]\Delta_1}\,\textsc{Cut}$$

transforms into the proof

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\end{array}}{\Gamma \vdash t : A \mid \Delta} \qquad \dfrac{\begin{array}{c}\pi_2\\ \vdots\end{array}}{\Gamma_1, x : A, \Gamma_2 \vdash \Delta_1}}{\Gamma_1, \Gamma, \Gamma_2 \vdash \Delta \mid [t/x]\Delta_1}\,\textsc{Cut}}{\Gamma_1, \Gamma, \Gamma_2 \vdash \circ : \perp \mid \Delta \mid [t/x]\Delta_1}\,\textsc{Pr}}{\Gamma_1, \Gamma, \Gamma_2 \vdash \Delta \mid \circ : \perp \mid [t/x]\Delta_1}\,\textsc{Series of Exchanges}$$

Clearly, $[t/x]\circ = \circ$.

Case: secondary hypothesis: left introduction of exchange. The proof

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\end{array}}{\Gamma \vdash t : A \mid \Delta} \qquad \dfrac{\dfrac{\begin{array}{c}\pi_2\\ \vdots\end{array}}{\Gamma_1, x : A, \Gamma_2, w : B, y : C, \Gamma_3 \vdash \Delta_1}}{\Gamma_1, x : A, \Gamma_2, y : C, w : B, \Gamma_3 \vdash \Delta_1}\,\textsc{ExL}}{\Gamma_1, \Gamma, \Gamma_2, y : C, w : B, \Gamma_3 \vdash \Delta \mid [t/x]\Delta_1}\,\textsc{Cut}$$

tranforms into the proof

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\end{array}}{\Gamma \vdash t : A \mid \Delta} \qquad \dfrac{\begin{array}{c}\pi_2\\ \vdots\end{array}}{\Gamma_1, x : A, \Gamma_2, w : B, y : C, \Gamma_3 \vdash \Delta_1}}{\Gamma_1, \Gamma, \Gamma_2, w : B, y : C, \Gamma_3 \vdash \Delta \mid [t/x]\Delta_1}\,\textsc{Cut}}{\Gamma_1, \Gamma, \Gamma_2, y : C, w : B, \Gamma_3 \vdash \Delta \mid [t/x]\Delta_1}\,\textsc{ExL}$$

Clearly, all terms are equivalent.

Case: secondary hypothesis: right introduction of exchange. The proof

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\end{array}}{\Gamma \vdash t : A \mid \Delta} \qquad \dfrac{\dfrac{\begin{array}{c}\pi_2\\ \vdots\end{array}}{\Gamma_1, x : A, \Gamma_2 \vdash \Delta_1 \mid t_1 : B \mid t_2 : C \mid \Delta_2}}{\Gamma_1, x : A, \Gamma_2 \vdash \Delta_1 \mid t_2 : C \mid t_1 : B \mid \Delta_2}\,\textsc{ExR}}{\Gamma_1, \Gamma, \Gamma_2 \vdash \Delta \mid [t/x]\Delta_1 \mid [t/x]t_2 : C \mid [t/x]t_1 : B \mid [t/x]\Delta_2}\,\textsc{Cut}$$

is transformed into

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\end{array}}{\Gamma \vdash t : A \mid \Delta} \qquad \dfrac{\begin{array}{c}\pi_2\\ \vdots\end{array}}{\Gamma_1, x : A, \Gamma_2 \vdash \Delta_1 \mid t_1 : B \mid t_2 : C \mid \Delta_2}}{\Gamma_1, \Gamma, \Gamma_2 \vdash \Delta \mid [t/x]\Delta_1 \mid [t/x]t_1 : B \mid [t/x]t_2 : C \mid [t/x]\Delta_2}\,\textsc{Cut}}{\Gamma_1, \Gamma, \Gamma_2 \vdash \Delta \mid [t/x]\Delta_1 \mid [t/x]t_2 : C \mid [t/x]t_1 : B \mid [t/x]\Delta_2}\,\textsc{ExR}$$

Clearly, all terms are equivalent.

$\square$

Corollary 7 (Cut-Elimination). *Cut-elimination holds for FILL.*

4 Order-Enriched Dialectica Categories

[5]

5 Conclusion and Future Work

We first gave the definition of full intuitionistic linear logic using the left rule for par proposed by Bellin in Section 2. We then proved cut-elimination of FILL in Section 3 by adapting the well-known cut-elimination procedure for classical linear logic to FILL. Finally, we gave a semantics of FILL interms of order-enriched dialectica categories in Section 4.

Future work. Lorenzen games are a particular type of game semantics for various logics developed by Lorenz, Felscher, and Rahman et al. See [9, 13] for an introduction. Lorenzen games consist of a first-order language consisting of the logical connectives of the logic one wishes to study. Then the structure of the games are defined by two types of rules: particle rules and structural rules. The particle rules describe how formulas can be attacked or defended based on the formulas main connective. Then the structural rules orchestrate the particle rules as the game progresses. They describe the overall organization of the game.

Rahman showed that Lorenzen games could be defined for classical linear logic [12]. The main difficulty is enforcing linearity, but due to the flexibility inherit in the definition of Lorenzen games it can be enforced in the particle rules. He was able to define a sound and complete semantics in Lorenzen games for classical linear logic, but he does mention that one could adopt a particular structural rule that enforces intuitionism. We plan to show that by adopting this rule we actually obtain a sound and complete semantics in Lorenzen games for FILL. Furthermore, we hope to show that these strategies of the games are also compositional, and thus, can be used as the morphisms of a category of Lorezen games.

References

[1] Gavin Bierman. A note on full intuitionistic linear logic. *Annals of Pure and Applied Logic*, 79(3):281 – 287, 1996.

[2] Torben Brauner and Valeria Paiva. A formulation of linear logic based on dependency-relations. In Mogens Nielsen and Wolfgang Thomas, editors, *Computer Science Logic*, volume 1414 of *Lecture Notes in Computer Science*, pages 129–148. Springer Berlin Heidelberg, 1998.

[3] Valeria de Paiva. *The Dialectica Categories.* PhD thesis, University of Cambridge, 1988.

[4] Valeria de Paiva. Dialectica categories. In J. Gray and A. Scedrov, editors, *Categories in Computer Science and Logic*, volume 92, pages 47–62. Amerian Mathemaitcal Society, 1989.

[5] Valeria de Paiva. Bounded dialectica interpretation: categorically. Under review, 2015.

[6] Valeria de Paiva and Luiz Carlos Pereira. A short note on intuitionistic propositional logic with multiple conclusions. *MANUSCRITO - Rev. Int. Fil.*, 28(2):317 – 329, jul-dez 2005.

[7] A. G. Dragalin. *Mathematical Intuitionism. Introduction to Proof Theory*, volume 67 of *Translations of Mathematical Monographs*. Amerian Mathemaitcal Society, 1988.

[8] Martin Hyland and Valeria de Paiva. Full intuitionistic linear logic (extended abstract). *Annals of Pure and Applied Logic*, 64(3):273 – 291, 1993.

[9] Laurent Keiff. Dialogical logic. In Edward N. Zalta, editor, *The Stanford Encyclopedia of Philosophy*. Summer 2011 edition, 2011.

[10] S. Maehara. Eine darstellung der intuitionistischen logik in der klassischen. *Nagoya Mathematical Journal*, 7:45–64, 1954.

[11] Paul-Andre Mellies. *Categorical Semantics of Linear Logic*. 2009.

[12] S. Rahman. Un desafio para las teorias cognitivas de la competencia log- ica: los fundamentos pragmaticos de la semantica de la logica linear. *Manuscrito*, XXV(2):381–432, October 2002.

[13] Shahid Rahman and Laurent Keiff. On how to be a dialogician. In Daniel Vanderveken, editor, *Logic, Thought and Action*, volume 2 of *Logic, Epistemology, and the Unity of Science*, pages 359–408. Springer Netherlands, 2005.

[14] Harold Schellinx. Some syntactical observations on linear logic. *Journal of Logic and Computation*, 1(4):537–559, 1991.

[15] G. Takeuti. *Proof Theory*. Amsterdam: North-Holland, 1975.

A The full specification of FILL

term_var, w, x, y, z, v			
index_var, i, j, k			
form, A, B, C, D, E	::=		
	|	$\top$	
	|	$\bot$	
	|	$A \multimap B$	
	|	$A \otimes B$	
	|	$A \parr B$	
	|	(A)	S
patterns, p	::=		
	|	$*$	

	$\vert$	x	
	$\vert$	$p_1 \otimes p_2$	
	$\vert$	$p_1 \parr p_2$	
	$\vert$	$-$	
	$\vert$	(p)	S
$term,\ t,\ e,\ d,\ f,\ g,\ u$	$::=$		
	$\vert$	x	
	$\vert$	$*$	
	$\vert$	$\circ$	
	$\vert$	$e_1 \otimes e_2$	
	$\vert$	$e_1 \parr e_2$	
	$\vert$	$\lambda x.t$	
	$\vert$	let t be p in e	
	$\vert$	$f\ e$	
	$\vert$	let-pat $t\ p\ e$	M
	$\vert$	$[t/x]t'$	M
	$\vert$	$[t/x, e/y]t'$	M
	$\vert$	(t)	S
	$\vert$	t	M
	$\vert$	t	M
Γ	$::=$		
	$\vert$	$x : A$	
	$\vert$	$\cdot$	
	$\vert$	Γ, Γ'	
	$\vert$	$x : A$	
Δ	$::=$		
	$\vert$	$t : A$	
	$\vert$	$\cdot$	
	$\vert$	$\Delta \mid \Delta'$	
	$\vert$	Δ	
	$\vert$	Δ, Δ'	
	$\vert$	$[t/x]\Delta$	
	$\vert$	let t be p in Δ	
	$\vert$	(Δ)	
	$\vert$	let-pat $t\ p\ \Delta$	M
	$\vert$	$t_1 : A_1, \ldots, t_i : A_i$	M

formula	::=	
	|	*judgement*
	|	$formula_1 \quad formula_2$
	|	$(formula)$
	|	$x \notin \mathsf{FV}(\Delta)$
	|	$x \in \mathsf{FV}(t)$
	|	$x, y \notin \mathsf{FV}(\Delta)$
	|	$x \notin \mathsf{FV}(t)$
	|	$x, y \notin \mathsf{FV}(t)$
	|	$\Delta_1 = \Delta_2$
	|	$\mathsf{FV}(t)$
	|	$\mathsf{FV}(\Delta)$
InferRules	::=	
	|	$\Gamma \vdash \Delta$
	|	$f = e$
judgement	::=	
	|	*InferRules*
user_syntax	::=	
	|	*term_var*
	|	*index_var*
	|	*form*
	|	*patterns*
	|	*term*
	|	Γ
	|	Δ
	|	*formula*

$\boxed{\Gamma \vdash \Delta}$

$$\frac{}{x : A \vdash x : A}\ \textsc{Ax}$$

$$\frac{\Gamma \vdash t : A \mid \Delta \quad \Gamma', y : A \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta \mid [t/y]\Delta'}\ \textsc{Cut}$$

$$\frac{\Gamma \vdash \Delta}{\Gamma, x : \top \vdash \mathsf{let}\, x\, \mathsf{be}\, * \,\mathsf{in}\, \Delta}\ \textsc{Il}$$

$$\frac{}{\cdot \vdash * : \top}\ \textsc{Ir}$$

$$\frac{\Gamma, x : A, y : B \vdash \Delta}{\Gamma, z : A \otimes B \vdash \mathsf{let}\, z\, \mathsf{be}\, x \otimes y\, \mathsf{in}\, \Delta}\ \textsc{Tl}$$

$$\frac{\Gamma \vdash e : A \mid \Delta \quad \Gamma' \vdash f : B \mid \Delta'}{\Gamma, \Gamma' \vdash e \otimes f : A \otimes B \mid \Delta \mid \Delta'}\ \textsc{Tr}$$

$$\frac{}{x : \perp \vdash \cdot}\ \textsc{Pl}$$

$$\frac{\Gamma \vdash \Delta}{\Gamma \vdash \circ : \perp \mid \Delta}\ \textsc{Pr}$$

$$\frac{\Gamma, x : A \vdash \Delta \quad \Gamma', y : B \vdash \Delta'}{\Gamma, \Gamma', z : A \parr B \vdash \text{let-pat}\, z\, (x \parr -)\, \Delta \mid \text{let-pat}\, z\, (- \parr y)\, \Delta'}\ \textsc{ParL}$$

$$\frac{\Gamma \vdash \Delta \mid e : A \mid f : B \mid \Delta'}{\Gamma \vdash \Delta \mid e \parr f : A \parr B \mid \Delta'}\ \textsc{ParR}$$

$$\frac{\Gamma \vdash e : A \mid \Delta \quad \Gamma', x : B \vdash \Delta'}{\Gamma, y : A \multimap B, \Gamma' \vdash \Delta \mid [y\ e/x]\Delta'}\ \textsc{ImpL}$$

$$\frac{\Gamma, x : A \vdash e : B \mid \Delta \quad x \notin \mathsf{FV}(\Delta)}{\Gamma \vdash \lambda x.e : A \multimap B \mid \Delta}\ \textsc{ImpR}$$

$$\frac{\Gamma, x : A, y : B \vdash \Delta}{\Gamma, y : B, x : A \vdash \Delta}\ \textsc{ExL}$$

$$\frac{\Gamma \vdash \Delta_1 \mid t_1 : A \mid t_2 : B \mid \Delta_2}{\Gamma \vdash \Delta_1 \mid t_2 : B \mid t_1 : A \mid \Delta_2}\ \textsc{ExR}$$

$\boxed{f = e}$

$$\frac{y \notin \mathsf{FV}(t)}{t = [y/x]t}\ \textsc{Alpha}$$

$$\frac{x \notin \mathsf{FV}(f)}{(\lambda x.f\ x) = f}\ \textsc{EtaFun}$$

$$\frac{}{(\lambda x.e)\ e' = [e'/x]e}\ \textsc{BetaFun}$$

$$\frac{}{\mathsf{let}\ *\ \mathsf{be}\ *\ \mathsf{in}\ e = e}\ \textsc{Eta1I}$$

$$\frac{y \notin \mathsf{FV}(f)}{f = \mathsf{let}\ y\ \mathsf{be}\ *\ \mathsf{in}\ f}\ \textsc{Eta2I}$$

$$\frac{}{\mathsf{let}\ u\ \mathsf{be}\ *\ \mathsf{in}\ [*/z]f = [u/z]f}\ \textsc{BetaI}$$

$$\frac{}{[\mathsf{let}\ u\ \mathsf{be}\ *\ \mathsf{in}\ e/y]f = \mathsf{let}\ u\ \mathsf{be}\ *\ \mathsf{in}\ [e/y]f}\ \textsc{NatI}$$

$$\frac{x, y \notin \mathsf{FV}(t)}{\mathsf{let}\ t'\ \mathsf{be}\ x \otimes y\ \mathsf{in}\ t = t}\ \textsc{EtaTen}$$

$$\frac{}{\mathsf{let}\ e \otimes t\ \mathsf{be}\ x \otimes y\ \mathsf{in}\ u = [e/x, t/y]u}\ \textsc{Beta1Ten}$$

$$\frac{}{\mathsf{let}\ u\ \mathsf{be}\ x \otimes y\ \mathsf{in}\ [x \otimes y/z]f = [u/z]f}\ \textsc{Beta2Ten}$$

$$\frac{}{[\mathsf{let}\ u\ \mathsf{be}\ x \otimes y\ \mathsf{in}\ g/w]f = \mathsf{let}\ u\ \mathsf{be}\ x \otimes y\ \mathsf{in}\ [g/w]f}\ \textsc{NatTen}$$

$$\frac{}{u = \circ}\ \textsc{EtaParU}$$

$$\frac{}{(\mathsf{let}\ u\ \mathsf{be}\ x \parr -\ \mathsf{in}\ x) \parr (\mathsf{let}\ u\ \mathsf{be}\ - \parr y\ \mathsf{in}\ y) = u}\ \textsc{EtaPar}$$

$$\frac{}{\mathsf{let}\ u \parr t\ \mathsf{be}\ x \parr -\ \mathsf{in}\ e = [u/x]e}\ \textsc{Beta1Par}$$

$$\frac{}{\mathsf{let}\ u \parr t\ \mathsf{be}\ - \parr y\ \mathsf{in}\ e = [t/y]e}\ \textsc{Beta2Par}$$

$$\frac{}{\mathsf{let}\ t\ \mathsf{be}\ x \parr -\ \mathsf{in}\ [u/x]f = [\mathsf{let}\ t\ \mathsf{be}\ x \parr -\ \mathsf{in}\ u/x]f}\ \textsc{Nat1Par}$$

$$\frac{}{\mathsf{let}\ t\ \mathsf{be}\ - \parr y\ \mathsf{in}\ [v/y]f = [\mathsf{let}\ t\ \mathsf{be}\ - \parr y\ \mathsf{in}\ v/y]f}\ \textsc{Nat2Par}$$

$$\frac{t = t'}{\lambda x.t = \lambda x.t''}\ \textsc{Lam}$$

$$\frac{t_1 = t_1'}{t_1\ t_2 = t_1'\ t_2}\ \textsc{App1}$$

$$\frac{t_2 = t_2'}{t_1\ t_2 = t_1\ t_2'}\ \textsc{App2}$$

$$\frac{t_1 = t_1'}{t_1 \otimes t_2 = t_1' \otimes t_2}\ \textsc{Ten1}$$

$$\frac{t_2 = t_2'}{t_1 \otimes t_2 = t_1 \otimes t_2'}\ \textsc{Ten2}$$

$$\frac{t_1 = t_1'}{t_1 \parr t_2 = t_1' \parr t_2}\ \textsc{Par1}$$

$$\frac{t_2 = t_2'}{t_1 \parr t_2 = t_1 \parr t_2'}\ \textsc{Par2}$$

$$\frac{t = t'}{\mathsf{let}\ t\ \mathsf{be}\ p\ \mathsf{in}\ e = \mathsf{let}\ t'\ \mathsf{be}\ p\ \mathsf{in}\ e}\ \textsc{Let1}$$

$$\frac{e = e'}{\mathsf{let}\ t\ \mathsf{be}\ p\ \mathsf{in}\ e = \mathsf{let}\ t\ \mathsf{be}\ p\ \mathsf{in}\ e'}\ \textsc{Let2}$$

$$\frac{}{t = t}\ \textsc{Refl}$$

$$\frac{t = t'}{t' = t}\ \textsc{Sym}$$

$$\frac{t_1 = t_2 \quad t_2 = t_3}{t_1 = t_3}\ \textsc{Trans}$$

$$\frac{y \notin \mathsf{FV}(t)}{t = [y/x]t}\ \textsc{Alpha} \qquad \frac{x \notin \mathsf{FV}(f)}{(\lambda x.f\ x) = f}\ \textsc{EtaFun} \qquad \frac{}{(\lambda x.e)\ e' = [e'/x]e}\ \textsc{BetaFun}$$

$$\frac{}{\mathsf{let}\ *\ \mathsf{be}\ *\ \mathsf{in}\ e = e}\ \textsc{Eta1I} \qquad \frac{y \notin \mathsf{FV}(f)}{f = \mathsf{let}\ y\ \mathsf{be}\ *\ \mathsf{in}\ f}\ \textsc{Eta2I}$$

$$\frac{}{\mathsf{let}\ u\ \mathsf{be}\ *\ \mathsf{in}\ [*/z]f = [u/z]f}\ \textsc{BetaI} \qquad \frac{}{[\mathsf{let}\ u\ \mathsf{be}\ *\ \mathsf{in}\ e/y]f = \mathsf{let}\ u\ \mathsf{be}\ *\ \mathsf{in}\ [e/y]f}\ \textsc{NatI}$$

$$\frac{x, y \notin \mathsf{FV}(t)}{\mathsf{let}\ t'\ \mathsf{be}\ x \otimes y\ \mathsf{in}\ t = t}\ \textsc{EtaTen} \qquad \frac{}{\mathsf{let}\ e \otimes t\ \mathsf{be}\ x \otimes y\ \mathsf{in}\ u = [e/x, t/y]u}\ \textsc{Beta1Ten}$$

$$\frac{}{\mathsf{let}\ u\ \mathsf{be}\ x \otimes y\ \mathsf{in}\ [x \otimes y/z]f = [u/z]f}\ \textsc{Beta2Ten}$$

$$\frac{}{[\mathsf{let}\ u\ \mathsf{be}\ x \otimes y\ \mathsf{in}\ g/w]f = \mathsf{let}\ u\ \mathsf{be}\ x \otimes y\ \mathsf{in}\ [g/w]f}\ \textsc{NatTen} \qquad \frac{}{u = \circ}\ \textsc{EtaParU}$$

$$\frac{}{(\mathsf{let}\ u\ \mathsf{be}\ x \parr -\ \mathsf{in}\ x) \parr (\mathsf{let}\ u\ \mathsf{be}\ - \parr y\ \mathsf{in}\ y) = u}\ \textsc{EtaPar}$$

$$\frac{}{\mathsf{let}\ u \parr t\ \mathsf{be}\ x \parr -\ \mathsf{in}\ e = [u/x]e}\ \textsc{Beta1Par} \qquad \frac{}{\mathsf{let}\ u \parr t\ \mathsf{be}\ - \parr y\ \mathsf{in}\ e = [t/y]e}\ \textsc{Beta2Par}$$

$$\frac{}{\mathsf{let}\ t\ \mathsf{be}\ x \parr -\ \mathsf{in}\ [u/x]f = [\mathsf{let}\ t\ \mathsf{be}\ x \parr -\ \mathsf{in}\ u/x]f}\ \textsc{Nat1Par}$$

$$\frac{}{\mathsf{let}\ t\ \mathsf{be}\ - \parr y\ \mathsf{in}\ [v/y]f = [\mathsf{let}\ t\ \mathsf{be}\ - \parr y\ \mathsf{in}\ v/y]f}\ \textsc{Nat2Par} \qquad \frac{t = t'}{\lambda x.t = \lambda x.t''}\ \textsc{Lam}$$

$$\frac{t_1 = t_1'}{t_1\ t_2 = t_1'\ t_2}\ \textsc{App1} \qquad \frac{t_2 = t_2'}{t_1\ t_2 = t_1\ t_2'}\ \textsc{App2} \qquad \frac{t_1 = t_1'}{t_1 \otimes t_2 = t_1' \otimes t_2}\ \textsc{Ten1}$$

$$\frac{t_2 = t_2'}{t_1 \otimes t_2 = t_1 \otimes t_2'}\ \textsc{Ten2} \qquad \frac{t_1 = t_1'}{t_1 \parr t_2 = t_1' \parr t_2}\ \textsc{Par1} \qquad \frac{t_2 = t_2'}{t_1 \parr t_2 = t_1 \parr t_2'}\ \textsc{Par2}$$

$$\frac{t = t'}{\mathsf{let}\ t\ \mathsf{be}\ p\ \mathsf{in}\ e = \mathsf{let}\ t'\ \mathsf{be}\ p\ \mathsf{in}\ e}\ \textsc{Let1} \qquad \frac{e = e'}{\mathsf{let}\ t\ \mathsf{be}\ p\ \mathsf{in}\ e = \mathsf{let}\ t\ \mathsf{be}\ p\ \mathsf{in}\ e'}\ \textsc{Let2} \qquad \frac{}{t = t}\ \textsc{Refl}$$

$$\frac{t = t'}{t' = t}\ \textsc{Sym} \qquad \frac{t_1 = t_2 \quad t_2 = t_3}{t_1 = t_3}\ \textsc{Trans}$$

Figure 2: Equivalence on terms