# Yet Another Short Note on Full Intuitionistic Linear Logic

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**Abstract**

Full Intuitionistic Linear Logic (FILL) was first introduced by Hyland and de Paiva as one of the results of their investigation into a categorical understanding of Gödel's Dialectica interpretation. FILL went against current beliefs that it was not possible to incorporate all of the linear connectives, e.g. tensor, par, and implication, into an intuitionistic linear logic. However, they showed that it is natural to support all of the connectives given that sequents have multiple hypothesis and multiple conclusions. To enforce intuitionism de Paiva's original formalization of FILL used the well-known Dragalin restriction forcing the implication right rule to have only a single conclusion in its premise, but Schellinx later showed that this results in a failure of cut-elimination. To overcome this failure Hyland and de Paiva introduced a term assignment to FILL that eliminated the need for the restriction. The main idea was to lift the restriction, assign variables to each hypothesis and terms to each conclusion, and then add the property that the variable annotating the hypothesis being discharged when applying the implication right rule can only be free in the term annotating the conclusion of the implication being introduced. Unfortunately, Bierman was able to show in his short note that this formalization of FILL still did not enjoy cut-elimination, because of a flaw in the left rule for par. However, Bellin proposed an alternate left rule for par and conjectured that by adopting his rule cut-elimination is restored. In this note we show that by adopting Bellin's proposed rule one obtains cut-elimination for FILL. Additionally, we show that this new formalization can be modeled by a new form of dialectica category called order-enriched dialectica categories of de Paiva, and discuss future work giving FILL a semantics in Lorenzen games.

## 1 Introduction

[2] In [3] Martin Hyland and Veleria de Paiva give a term formalization of Full Intuitionistic Linear Logic (FILL), but later Bierman was able to give a counterexample to cut-elimination [1]. As Bierman explains the problem was that the left rule for par introduced a fresh variable into to many terms on the right-side of the conclusion. This resulted in a counterexample where this fresh variable became bound in one term, but is left free in another. This resulted from first doing a commuting conversion on cut, and then $\lambda$-binding the fresh variable. Thus, cut-elmination failed. In the conclusion of Bierman's paper he gives an alternate left-par rule which he attributes to Bellin, and states that this alternate rule should fix the problem with cut-elimination [1]. In this note we adopt Bellin's rule, and then show cut-elimination in Section 3.

## 2 Full Intuitionistic Linear Logic (FILL)

In this section we give a brief description of Full Intuitionistic Linear Logic (FILL) in the style found in [3]. However, we use a slightly different presentation that we feel provides a more elegant description of the logic. We first give the syntax of formulas, patterns, terms, and contexts. Following the syntax we define several meta-functions that will be used when defining the inference rules of the logic.

**Definition 1.** *The syntax for FILL is as follows:*

$$
\begin{array}{rl}
\textit{(Formulas)} & A, B, C, D, E ::= I \mid \perp \mid A \multimap B \mid A \otimes B \mid A \parr B \\
\textit{(Patterns)} & p ::= * \mid - \mid x \mid p_1 \otimes p_2 \mid p_1 \parr p_2 \\
\textit{(Terms)} & t, e ::= x \mid * \mid \circ \mid t_1 \otimes t_2 \mid t_1 \parr t_2 \mid \lambda x.t \mid \mathsf{let}\ t\ \mathsf{be}\ p\ \mathsf{in}\ e \mid t_1\ t_2 \\
\textit{(Left Contexts)} & \Gamma ::= \cdot \mid x : A \mid \Gamma_1, \Gamma_2 \\
\textit{(Right Contexts)} & \Delta ::= \cdot \mid t : A \mid \Delta_1, \Delta_2
\end{array}
$$

At this point we introduce some basic syntax and definitions to facilitate readability, and presentation of the inference rules. First, we will often write $\Delta_1 \mid \Delta_2$ as syntactic sugar for $\Delta_1, \Delta_2$. The former syntax should be read as "$\Delta_1$ or $\Delta_2$." This will help readability of the sequent we will introduce below. We denote the usual capture-avoiding substitution by $[t/x]t'$, and its extension to right contexts as $[t/x]\Delta$.

The previous extension will make conducting substitutions across a sequence of terms in an inference rule easier. Similarly, we find it convenient to be able to do this style of extension for the let-binding as well.

**Definition 2.** *We extend let-binding terms to right contexts as follows:*

$$
\begin{array}{l}
\mathsf{let}\ t\ \mathsf{be}\ p\ \mathsf{in} \cdot = \cdot \\
\mathsf{let}\ t\ \mathsf{be}\ p\ \mathsf{in}\ (t' : A) = (\mathsf{let}\ t\ \mathsf{be}\ p\ \mathsf{in}\ t') : A \\
\mathsf{let}\ t\ \mathsf{be}\ p\ \mathsf{in}\ (\Delta_1 \mid \Delta_2) = (\mathsf{let}\ t\ \mathsf{be}\ p\ \mathsf{in}\ \Delta_1) \mid (\mathsf{let}\ t\ \mathsf{be}\ p\ \mathsf{in}\ \Delta_2)
\end{array}
$$

We denote the usual function that computes the set of free variables in a term by $\mathsf{FV}(t)$, and its straightforward extension to right contexts as $\mathsf{FV}(\Delta)$. Finally, we arrive at the inference rules of FILL.

**Definition 3.** *The inference rules for derivability in FILL are as follows:*

$$
\frac{}{x : A \vdash x : A}\ \textsc{Ax}
\qquad
\frac{\Gamma \vdash t : A \mid \Delta \quad y : A, \Gamma' \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta \mid [t/y]\Delta'}\ \textsc{Cut}
\qquad
\frac{\Gamma \vdash \Delta}{\Gamma, x : I \vdash \mathsf{let}\ x\ \mathsf{be}\ *\ \mathsf{in}\ \Delta}\ \textsc{Il}
\qquad
\frac{}{\cdot \vdash * : I}\ \textsc{Ir}
$$

$$
\frac{\Gamma, x : A, y : B \vdash \Delta}{\Gamma, z : A \otimes B \vdash \mathsf{let}\ z\ \mathsf{be}\ x \otimes y\ \mathsf{in}\ \Delta}\ \textsc{Tl}
\qquad
\frac{\Gamma \vdash e : A \mid \Delta \quad \Gamma' \vdash f : B \mid \Delta'}{\Gamma, \Gamma' \vdash e \otimes f : A \otimes B \mid \Delta \mid \Delta'}\ \textsc{Tr}
\qquad
\frac{}{x : \perp \vdash \cdot}\ \textsc{Pl}
\qquad
\frac{\Gamma \vdash \Delta}{\Gamma \vdash \circ : \perp \mid \Delta}\ \textsc{Pr}
$$

$$
\frac{\Gamma, x : A \vdash \Delta \quad \Gamma', y : B \vdash \Delta'}{\Gamma, \Gamma', z : A \parr B \vdash \text{let-pat}\ z\ (x \parr -)\ \Delta \mid \text{let-pat}\ z\ (- \parr y)\ \Delta'}\ \textsc{Parl}
\qquad
\frac{\Gamma \vdash \Delta \mid e : A \mid f : B \mid \Delta'}{\Gamma \vdash \Delta \mid e \parr f : A \parr B \mid \Delta'}\ \textsc{Parr}
$$

$$
\frac{\Gamma \vdash e : A \mid \Delta \quad \Gamma', x : B \vdash \Delta'}{\Gamma, y : A \multimap B, \Gamma' \vdash \Delta \mid [y\ e/x]\Delta'}\ \textsc{Impl}
\qquad
\frac{\Gamma, x : A \vdash e : B \mid \Delta \quad x \notin \mathsf{FV}(\Delta)}{\Gamma \vdash \lambda x.e : A \multimap B \mid \Delta}\ \textsc{Impr}
\qquad
\frac{\Gamma, x : A, y : B \vdash \Delta}{\Gamma, y : B, x : A \vdash \Delta}\ \textsc{Exl}
$$

$$
\frac{\Gamma \vdash \Delta_1 \mid t_1 : A \mid t_2 : B \mid \Delta_2}{\Gamma \vdash \Delta_1 \mid t_2 : B \mid t_1 : A \mid \Delta_2}\ \textsc{Exr}
$$

The Parl rule depends on a function let-pat $z\ p\ \Delta$. We define this function next.

**Definition 4.** *The function* let-pat $z\ p\ t$ *is defined as follows:*

$$
\begin{array}{lll}
\text{let-pat}\ z\ (x \parr -)\ t = t & \quad \text{let-pat}\ z\ (- \parr y)\ t = t & \quad \text{let-pat}\ z\ p\ t = \mathsf{let}\ z\ \mathsf{be}\ p\ \mathsf{in}\ t \\
\quad \textit{where } x \notin \mathsf{FV}(t) & \quad\quad \textit{where } y \notin \mathsf{FV}(t) &
\end{array}
$$

*It is straightforward to extend the previous definition to right-contexts, and we denote this extension by* let-pat $z\ p\ \Delta$*.*

The motivation behind this function is that it only binds the pattern variables in $p$ in a term if and only if those pattern variables are free in the term. This over comes the counterexample given by Bierman in [1]. Throughout the sequel we will denote derivations of the previous rules by $\pi$.

# 3 Cut-elimination

The usual proof of cut-elimination for intuitionistic and classical linear logic should suffice for FILL. Thus, in this section we simply give the cut-elimination procedure for FILL following the development in [4]. However, there is one invariant that must be verified across each derivation transformation. The invariant is that if a derivation $\pi$ is transformed into a derivation $\pi'$, then the terms in the conclusion of the final rule applied in $\pi$ must be equivalent to the terms in the conclusion of the final rule applied in $\pi'$, but using what notion of equivalence?

**Definition 5.** *Equivalence on terms is defined as follows:*

$$\frac{y \notin \mathsf{FV}(t)}{t = [y/x]t}\ \textsc{Alpha} \qquad \frac{x \notin \mathsf{FV}(f)}{(\lambda x.f\ x) = f}\ \textsc{EtaFun} \qquad \frac{}{(\lambda x.e)\ e' = [e'/x]e}\ \textsc{BetaFun} \qquad \frac{}{\mathsf{let}\ *\ \mathsf{be}\ *\ \mathsf{in}\ e = e}\ \textsc{Eta1I}$$

$$\frac{}{\mathsf{let}\ u\ \mathsf{be}\ *\ \mathsf{in}\ [*/z]f = [u/z]f}\ \textsc{BetaI} \qquad \frac{}{[\mathsf{let}\ u\ \mathsf{be}\ *\ \mathsf{in}\ e/y]f = \mathsf{let}\ u\ \mathsf{be}\ *\ \mathsf{in}\ [e/y]f}\ \textsc{NatI}$$

$$\frac{}{\mathsf{let}\ e \otimes t\ \mathsf{be}\ x \otimes y\ \mathsf{in}\ u = [e/x, t/y]u}\ \textsc{Beta1Ten} \qquad \frac{}{\mathsf{let}\ u\ \mathsf{be}\ x \otimes y\ \mathsf{in}\ [x \otimes y/z]f = [u/z]f}\ \textsc{Beta2Ten}$$

$$\frac{}{[\mathsf{let}\ u\ \mathsf{be}\ x \otimes y\ \mathsf{in}\ g/w]f = \mathsf{let}\ u\ \mathsf{be}\ x \otimes y\ \mathsf{in}\ [g/w]f}\ \textsc{NatTen} \qquad \frac{}{u = \circ}\ \textsc{EtaParU}$$

$$\frac{}{(\mathsf{let}\ u\ \mathsf{be}\ x \parr -\ \mathsf{in}\ x) \parr (\mathsf{let}\ u\ \mathsf{be}\ - \parr y\ \mathsf{in}\ y) = u}\ \textsc{EtaPar} \qquad \frac{}{\mathsf{let}\ u \parr t\ \mathsf{be}\ x \parr -\ \mathsf{in}\ e = [u/x]e}\ \textsc{Beta1Par}$$

$$\frac{}{\mathsf{let}\ u \parr t\ \mathsf{be}\ - \parr y\ \mathsf{in}\ e = [t/y]e}\ \textsc{Beta2Par} \qquad \frac{}{\mathsf{let}\ t\ \mathsf{be}\ x \parr -\ \mathsf{in}\ [u/x]f = [\mathsf{let}\ t\ \mathsf{be}\ x \parr -\ \mathsf{in}\ u/x]f}\ \textsc{Nat1Par}$$

$$\frac{}{\mathsf{let}\ t\ \mathsf{be}\ - \parr y\ \mathsf{in}\ [v/y]f = [\mathsf{let}\ t\ \mathsf{be}\ - \parr y\ \mathsf{in}\ v/y]f}\ \textsc{Nat2Par}$$

The cut elimination procedure requires the following two basic results:

**Lemma 6** (Substitution Distribution)**.** *For any terms $t$, $t_1$, and $t_2$, $[t_1/x][t_2/y]t = [[t_1/x]t_2/y][t_2/x]t$.*

*Proof.* This proof holds by straightforward induction on the form of $t$. □

**Lemma 7** (Let-pat Distribution)**.** *For any terms $t$, $t_1$, and $t_2$, and pattern $p$,*
let-pat $t\ p\ [t_1/y]t_2 = [\text{let-pat}\ t\ p\ t_1/y]t_2$.

*Proof.* This proof holds by case splitting over $p$, and then using the naturality equations for the respective pattern. □

Throughout the remainder of this section we present a particular step in the cut-elimination procedure, and then give a short proof that equality of terms are preserved across the particular transformation on derivations. Many of the transformations are trivial, and follow directly from the traditional proof. Thus, we only present here the most interesting cases. The full proof can be found in the companion report [?].

## 3.1 Principle formula vs. principle formula

### 3.1.1 Par

The proof

$$\dfrac{\dfrac{\dfrac{\pi_1}{\vdots}}{\Gamma_1 \vdash \Delta_1 \mid t_1 : A \mid t_2 : B \mid \Delta_2}}{\Gamma_1 \vdash \Delta_1 \mid t_1 \parr t_2 : A \parr B \mid \Delta_2}\ \textsc{Parr} \qquad \dfrac{\dfrac{\dfrac{\pi_2}{\vdots}}{\Gamma_2, x : A \vdash \Delta_3} \qquad \dfrac{\dfrac{\pi_3}{\vdots}}{\Gamma_3, y : B \vdash \Delta_4}}{\Gamma_2, \Gamma_3, z : A \parr B \vdash \text{let-pat}\ z\ (x \parr -)\ \Delta_3 \mid \text{let-pat}\ z\ (- \parr y)\ \Delta_4}\ \textsc{Parl}$$
$$\overline{\Gamma_2, \Gamma_3, \Gamma_1 \vdash \Delta_1 \mid \Delta_2 \mid [t_1 \parr t_2/z](\text{let-pat}\ z\ (x \parr -)\ \Delta_3) \mid [t_1 \parr t_2/z](\text{let-pat}\ z\ (- \parr y)\ \Delta_4)}\ \textsc{Cut}$$

is transformed into the proof

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\pi_1}{\vdots}}{\Gamma_1 \vdash \Delta_1 \mid t_1 : A \mid t_2 : B \mid \Delta_2} \qquad \dfrac{\dfrac{\pi_3}{\vdots}}{\Gamma_3, y : B \vdash \Delta_4}}{\Gamma_3, \Gamma_1 \vdash \Delta_1 \mid t_1 : A \mid \Delta_2 \mid [t_2/y]\Delta_4}\ \textsc{Cut} \qquad \dfrac{\dfrac{\pi_2}{\vdots}}{\Gamma_2, x : A \vdash \Delta_3}}{\Gamma_2, \Gamma_3, \Gamma_1 \vdash \Delta_1 \mid \Delta_2 \mid [t_2/y]\Delta_4 \mid [t_1/x]\Delta_3}\ \textsc{Cut}}{\Gamma_2, \Gamma_3, \Gamma_1 \vdash \Delta_1 \mid \Delta_2 \mid [t_1/x]\Delta_3 \mid [t_2/y]\Delta_4}\ \textsc{Series of Exchanges}$$

Without loss of generality consider the case when $\Delta_3 = t_3 : C_1 \mid \Delta'_3$ and $\Delta_4 = t_4 : C_2 \mid \Delta'_4$. First, $[t_1 \parr t_2/z](\text{let-pat}\, z\, (x \parr -)\, t_3) = \text{let-pat}\, (t_1 \parr t_2)\, (x \parr -)\, t_3$, and by $\textsc{Eq\_Beta1Par}$ we know $\text{let-pat}\, (t_1 \parr t_2)\, (x \parr -)\, t_3 = [t_1/x]t_3$ if $x \in \mathsf{FV}(t_3)$ or $\text{let-pat}\, (t_1 \parr t_2)\, (x \parr -)\, t_3 = t_3$ otherwise. In the latter case we can see that $t_3 = [t_1/x]t_3$, thus, in both cases $\text{let-pat}\, (t_1 \parr t_2)\, (x \parr -)\, t_3 = [t_1/x]t_3$. This argument can be repeated for any terms in $\Delta'_3$, and hence $[t_1 \parr t_2/z](\text{let-pat}\, z\, (x \parr -)\, \Delta_3) = \text{let-pat}\, (t_1 \parr t_2)\, (x \parr -)\, \Delta_3 = [t_1/x]\Delta_3$. We can apply a similar argument for $[t_1 \parr t_2/z](\text{let-pat}\, z\, (- \parr y)\, t_4)$ and $[t_1 \parr t_2/z](\text{let-pat}\, z\, (- \parr y)\, \Delta_4)$.

Note that we could have first cut on $A$, and then on $B$ in the second derivation, but we would have arrived at the same result just with potentially more exchanges on the right.

### 3.1.2 Implication

The proof

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\end{array}}{\Gamma, x : A \vdash t : B \mid \Delta} \quad x \notin \mathsf{FV}(\Delta)}{\Gamma \vdash \lambda x.t : A \multimap B \mid \Delta}\textsc{ImpR} \quad \dfrac{\dfrac{\begin{array}{c}\pi_2\\ \vdots\end{array}}{\Gamma_1 \vdash t_1 : A \mid \Delta_1} \quad \dfrac{\begin{array}{c}\pi_3\\ \vdots\end{array}}{\Gamma_2, y : B \vdash \Delta_2}}{\Gamma_1, z : A \multimap B, \Gamma_2 \vdash \Delta_1 \mid [z\, t_1/y]\Delta_2}\textsc{ImpL}}{\Gamma_1, \Gamma, \Gamma_2 \vdash \Delta \mid [\lambda x.t/z]\Delta_1 \mid [\lambda x.t/z][z\, t_1/y]\Delta_2}\textsc{Cut}$$

transforms into the proof

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\pi_2\\ \vdots\end{array}}{\Gamma_1 \vdash t_1 : A \mid \Delta_1} \quad \dfrac{\begin{array}{c}\pi_1\\ \vdots\end{array}}{\Gamma, x : A \vdash t : B \mid \Delta} \quad x \notin \mathsf{FV}(\Delta)}{\Gamma, \Gamma_1 \vdash \Delta_1 \mid [t_1/x]t : B \mid [t_1/x]\Delta}\textsc{Cut} \quad \dfrac{\begin{array}{c}\pi_3\\ \vdots\end{array}}{\Gamma_2, y : B \vdash \Delta_2}}{\Gamma_2, \Gamma, \Gamma_1 \vdash \Delta_1 \mid [t_1/x]\Delta \mid [[t_1/x]t/y]\Delta_2}\textsc{Cut}}{\Gamma_1, \Gamma, \Gamma_2 \vdash [t_1/x]\Delta \mid \Delta_1 \mid [[t_1/x]t/y]\Delta_2}\textsc{Series of Exchanges}$$

Without loss of generality consider the case when $\Delta_2 = t_2 : C \mid \Delta'_2$. First, by hypothesis we know $x \notin \mathsf{FV}(\Delta)$, and so we know $\Delta = [t_1/x]\Delta$. We can see that $[\lambda x.t/z][z\, t_1/y]t_2 = [(\lambda x.t)\, t_1/y]t_2 = [[t_1/x]t/y]t_2$ by using the congruence rules of equality and the rule $\textsc{Eq\_BetaFun}$. This argument can be repeated for any term in $[\lambda x.t/z][z\, t_1/y]\Delta'_2$, and so $[\lambda x.t/z][z\, t_1/y]\Delta_2 = [[t_1/x]t/y]\Delta_2$. Finally, by inspecting the previous derivations we can see that $z \notin \mathsf{FV}(\Delta_1)$, and thus, $\Delta_1 = [\lambda x.t/z]\Delta_1$.

# References


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