Cut-elimination of the APAL term assignment formulation of FILL

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In this short note I give the proof of the term assignment formulation of FILL first given in [2].

1 The Fix

Consider the Dparl rule in the dependency-relation formalization:

$$\dfrac{\begin{array}{c}\Gamma_1, A \vdash \Delta_1 \\ \Gamma_3, B \vdash \Delta_2\end{array}}{\Gamma_1, \Gamma_3, A \parr B \vdash \Delta_1, \Delta_2}\ \textsc{Dparl} \qquad \mathit{Dep}(\tau) = \{(A \parr B, A), (A \parr B, B)\} \star (\mathit{Dep}(\tau_1) \cup \mathit{Dep}(\tau_2))$$

If anything in Δ_1 and Δ_2 depend on A or B then this will be accounted for in $\mathit{Dep}(\tau_1)$ and $\mathit{Dep}(\tau_2)$. Thus, in the term formalization when binding pattern variables across the righthand side of the sequent we should do so if and only if there is a dependency. In fact, if a formula on the righthand side depends on a formula in the lefthand side, then the variable associated with that hypnosis must be free in the term associated with the formula on the right. This evidence suggests that to fix the term formalization we must modify the Parl rule.

The new Parl rule as follows:

$$\dfrac{\Gamma, x : A \vdash d_i : C_i \quad \Gamma', y : B \vdash f_j : D_j}{\text{<<multiple parses>>}}\ \textsc{NParl}$$

The previous rule depends on a function which we define as follows:

$$\begin{array}{l}\text{let-pat}\, z\, (x \parr -)\, e = e \\ \quad \text{where } x \notin \mathsf{FV}(e) \\ \\ \text{let-pat}\, z\, (- \parr y)\, e = e \\ \quad \text{where } y \notin \mathsf{FV}(e) \\ \\ \text{let-pat}\, z\, p\, e = \mathsf{let}\, z\, \mathsf{be}\, p\, \mathsf{in}\, e\end{array}$$

Note that in the definition of let-pat $z\,p\,e$ the final case is a catchall case. Now the new Parl rule only pattern matches on z in the righthand side if there is a dependency between the variables in the pattern and the term in the pattern match. A similar rule to the above was proposed by Bellin in the conclusion of [1].

This rule recovers from the counterexample. The first derivation given in the counter example above is unchanged, so we only give the second:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{v : A \vdash v : A}\,\textsc{Ax}}{v : A \vdash v : A \mid \circ :\perp}\,\textsc{Pr} \qquad \dfrac{\dfrac{}{x : A \vdash x : A}\,\textsc{Ax} \quad \dfrac{}{y : B \vdash y : B}\,\textsc{Ax}}{x : A, y : B \vdash x \otimes y : A \otimes B}\,\textsc{Tr}}{y : B, v : A \vdash v \otimes y : A \otimes B \mid \circ :\perp}\,\textsc{Cut} \qquad \dfrac{}{w : C \vdash w : C}\,\textsc{Ax}}{v : A, z : B \parr C \vdash \mathsf{let}\, z\, \mathsf{be}\, y \parr - \,\mathsf{in}\, v \otimes y : A \otimes B \mid \mathsf{let}\, z\, \mathsf{be}\, - \parr w\, \mathsf{in}\, w : C \mid \circ :\perp}\,\textsc{Parl}}{v : A, z : B \parr C \vdash ((\mathsf{let}\, z\, \mathsf{be}\, y \parr - \,\mathsf{in}\, v \otimes y) \parr (\mathsf{let}\, z\, \mathsf{be}\, - \parr w\, \mathsf{in}\, w)) : (A \otimes B) \parr C \mid \circ :\perp}\,\textsc{Parr}}{v : A \vdash \lambda z.((\mathsf{let}\, z\, \mathsf{be}\, y \parr - \,\mathsf{in}\, v \otimes y) \parr (\mathsf{let}\, z\, \mathsf{be}\, - \parr w\, \mathsf{in}\, w)) : (B \parr C) \multimap ((A \otimes B) \parr C) \mid \circ :\perp}\,\textsc{Impr}$$

This new derivation is now correct, and mirrors that of the dependency-relation formalization.

2 Basic Results

Lemma 1 (Substitution Distribution)**.** *For any terms t, t_1, and t_2, $[t_1/x][t_2/y]t = [[t_1/x]t_2/y][t_2/x]t$.*

Proof. This proof holds by straightforward induction on the form of t. ☐

3 Cut-elimination

3.1 Commuting conversion cut vs cut (first case)

The following proof

$$\dfrac{\dfrac{\overset{\pi_1}{\vdots}}{\Gamma \vdash t : A \mid \Delta} \qquad \dfrac{\dfrac{\overset{\pi_2}{\vdots}}{\Gamma_2, x : A, \Gamma_3 \vdash t_1 : B \mid \Delta_1} \qquad \dfrac{\overset{\pi_3}{\vdots}}{\Gamma_1, y : B, \Gamma_4 \vdash t_2 : C \mid \Delta_2}}{\Gamma_1, \Gamma_2, x : A, \Gamma_3, \Gamma_4 \vdash \Delta_1 \mid [t_1/y]t_2 : C \mid [t_1/y]\Delta_2}\ \text{\textsc{Cut}}}{\Gamma_1, \Gamma_2, \Gamma, \Gamma_3, \Gamma_4 \vdash \Delta \mid [t/x]\Delta_1 \mid [t/x][t_1/y]t_2 : C \mid [t/x][t_1/y]\Delta_2}\ \text{\textsc{Cut}}$$

is transformed into the proof

$$\dfrac{\dfrac{\dfrac{\overset{\pi_1}{\vdots}}{\Gamma \vdash t : A \mid \Delta} \qquad \dfrac{\overset{\pi_2}{\vdots}}{\Gamma_2, x : A, \Gamma_3 \vdash t_1 : B \mid \Delta_1}}{\Gamma_2, \Gamma, \Gamma_3 \vdash [t/x]t_1 : B \mid [t/x]\Delta_1} \qquad \dfrac{\overset{\pi_3}{\vdots}}{\Gamma_1, y : B, \Gamma_4 \vdash t_2 : C \mid \Delta_2}}{\Gamma_1, \Gamma_2, \Gamma, \Gamma_3, \Gamma_4 \vdash \Delta \mid [t/x]\Delta_1 \mid ([[t/x]t_1/y]t_2) : C \mid [[t/x]t_1/y]\Delta_2}\ \text{\textsc{Cut}}$$

In order for the previous two proofs to be considered equal, we have to show that the final terms in the conclusion of the above derivations are equivalent. First, we know that the term $[t/x][t_1/y]t_2$ in the first derivation above is equivalent to $[[t/x]t_1/y][t/x]t_2$ by Lemma 1. Furthermore, by inspecting the first derivation we can see that $x \notin \mathsf{FV}(t_2)$, and thus, $[[t/x]t_1/y][t/x]t_2 = [[t/x]t_1/y]t_2$. This argument may be repeated for any term in Δ_2, and thus, we know $[t/x][t_1/y]\Delta_2 = [[t/x]t_1/y]\Delta_2$.

3.2 Commuting conversion cut vs. cut (second case)

The second commuting conversion on cut begins with the proof

$$\dfrac{\dfrac{\overset{\pi_1}{\vdots}}{\Gamma \vdash t : A \mid \Delta} \qquad \dfrac{\dfrac{\overset{\pi_2}{\vdots}}{\Gamma' \vdash t' : B \mid \Delta'} \qquad \dfrac{\overset{\pi_3}{\vdots}}{\Gamma_1, x : A, \Gamma_2, y : B, \Gamma_3 \vdash t_1 : C \mid \Delta_1}}{\Gamma_1, x : A, \Gamma_2, \Gamma', \Gamma_3 \vdash \Delta' \mid [t'/y]t_1 : C \mid [t'/y]\Delta_1}\ \text{\textsc{Cut}}}{\Gamma_1, \Gamma, \Gamma_2, \Gamma', \Gamma_3 \vdash \Delta \mid [t/x]\Delta' \mid [t/x][t'/y]t_1 : C \mid [t/x][t'/y]\Delta_1}\ \text{\textsc{Cut}}$$

is transformed into the following proof:

$$\dfrac{\dfrac{\dfrac{\overset{\pi_2}{\vdots}}{\Gamma' \vdash t' : B \mid \Delta'} \qquad \dfrac{\dfrac{\overset{\pi_1}{\vdots}}{\Gamma \vdash t : A \mid \Delta} \qquad \dfrac{\overset{\pi_3}{\vdots}}{\Gamma_1, x : A, \Gamma_2, y : B, \Gamma_3 \vdash t_1 : C \mid \Delta_1}}{\Gamma_1, \Gamma, \Gamma_2, y : B, \Gamma_3 \vdash \Delta \mid [t/x]t_1 : C \mid [t/x]\Delta_1}\ \text{\textsc{Cut}}}{\Gamma_1, \Gamma, \Gamma_2, \Gamma', \Gamma_3 \vdash \Delta' \mid [t'/y]\Delta \mid [t'/y][t/x]t_1 : C \mid [t'/y][t/x]\Delta_1}\ \text{\textsc{Cut}}}{\Gamma_1, \Gamma, \Gamma_2, \Gamma', \Gamma_3 \vdash [t'/y]\Delta \mid \Delta' \mid [t'/y][t/x]t_1 : C \mid [t'/y][t/x]\Delta_1}\ \text{\textsc{Series of Exchanges}}$$

Now, because we know $x, y \notin \mathsf{FV}(\Delta)$ by inspection of the first derivation, we know that $\Delta = [t'/y]\Delta$ and $\Delta' = [t/x]\Delta'$. Similarly, we know that $x, y \notin \mathsf{FV}(t)$ and $x, y \notin \mathsf{FV}(t')$. Thus, by this fact and Lemma 1, we know that $[t/x][t'/y]t_1 = [[t/x]t'/y][t/x]t_1 = [t'/y][t/x]t_1$. This argument can be repeated for any term in Δ_1, hence, $[t/x][t'/y]\Delta_1 = [t'/y][t/x]\Delta_1$. Therefore, both of the previous derivations are equivalent.

3.3 The η-expansion cases

3.3.1 Tensor

The proof

$$\dfrac{}{x : A \otimes B \vdash x : A \otimes B}\ \textsc{Ax}$$

is transformed into the proof

$$\dfrac{\dfrac{\dfrac{}{y : A \vdash y : A}\ \textsc{Ax} \qquad \dfrac{}{z : B \vdash z : B}\ \textsc{Ax}}{y : A, z : B \vdash y \otimes z : A \otimes B}\ \textsc{Tr}}{x : A \otimes B \vdash \mathsf{let}\, x\, \mathsf{be}\, y \otimes z\, \mathsf{in}\, (y \otimes z) : A \otimes B}\ \textsc{Tl}$$

Now by the rule EQ_T2 we know $\mathsf{let}\, x\, \mathsf{be}\, y \otimes z\, \mathsf{in}\, (y \otimes z) = x$.

3.3.2 Par

The proof

$$\dfrac{}{x : A \parr B \vdash x : A \parr B}\ \textsc{Ax}$$

is transformed into the proof

$$\dfrac{\dfrac{\dfrac{}{y : A \vdash y : A}\ \textsc{Ax} \qquad \dfrac{}{z : B \vdash z : B}\ \textsc{Ax}}{x : A \parr B \vdash \mathsf{let}\, x\, \mathsf{be}\, (y \parr -)\, \mathsf{in}\, y : A \mid \mathsf{let}\, x\, \mathsf{be}\, (- \parr z)\, \mathsf{in}\, z : B}\ \textsc{ParL}}{x : A \parr B \vdash (\mathsf{let}\, x\, \mathsf{be}\, (y \parr -)\, \mathsf{in}\, y) \parr (\mathsf{let}\, x\, \mathsf{be}\, (- \parr z)\, \mathsf{in}\, z) : A \parr B}\ \textsc{ParR}$$

Just as we saw in the previous case by rule Eq_P3 we know $((\mathsf{let}\, x\, \mathsf{be}\, (y \parr -)\, \mathsf{in}\, y) \parr (\mathsf{let}\, x\, \mathsf{be}\, (- \parr z)\, \mathsf{in}\, z)) = x$.

3.3.3 Implication

The proof

$$\dfrac{}{x : A \multimap B \vdash A \multimap B}\ \textsc{Ax}$$

transforms into the proof

$$\dfrac{\dfrac{\dfrac{}{y : A \vdash y : A}\ \textsc{Ax} \qquad \dfrac{}{z : B| - z : B}\ \textsc{Ax}}{y : A, x : A \multimap B \vdash x\, y : B}\ \textsc{ImpL}}{x : A \multimap B \vdash \lambda y.x\, y : A \multimap B}\ \textsc{ImpR}$$

Finally, the two derivations are equivalent, because $(\lambda y.x\, y) = x$ by the Eq_Eta rule.

3.3.4 Tensor unit

The proof

$$\dfrac{}{x : I \vdash x : I}\ \textsc{Ax}$$

transforms into the proof

$$\dfrac{\dfrac{}{\cdot \vdash * : I}\ \textsc{Ir}}{x : I \vdash \mathsf{let}\, x\, \mathsf{be}\ *\ \mathsf{in} * : I}\ \textsc{Il}$$

Lastly, we know $x = \mathsf{let}\, x\, \mathsf{be}\ *\ \mathsf{in} *$ by Eq_I, therefore, the previous two proofs are equivalent.

4 The axiom steps

4.1 The axiom step

The proof

$$\dfrac{\dfrac{}{x : A \vdash x : A}\ \textsc{Ax} \qquad \dfrac{\begin{matrix}\pi \\ \vdots\end{matrix}}{\Gamma_1, y : A, \Gamma_2 \vdash t : B \mid \Delta}}{\Gamma_1, x : A, \Gamma_2 \vdash [x/y]t : B \mid [x/y]\Delta}\ \textsc{Cut}$$

transforms into the proof

$$\dfrac{\begin{matrix}\pi \\ \vdots\end{matrix}}{\Gamma_1, y : A, \Gamma_2 \vdash t : B \mid \Delta}$$

By Eq_Alpha we know $t = [x/y]t$ and $\Delta = [x/y]\Delta$, therefore the previous two proofs are equivalent.

4.2 Conclusion vs. axom

The proof

$$\dfrac{\dfrac{\begin{matrix}\pi \\ \vdots\end{matrix}}{\Gamma \vdash t : A \mid \Delta} \quad \dfrac{}{x : A \vdash x : A}\ \textsc{Ax}}{\Gamma \vdash \Delta \mid [t/x]x : A}\ \textsc{Cut}$$

transforms into

$$\dfrac{\dfrac{\begin{matrix}\pi \\ \vdots\end{matrix}}{\Gamma \vdash t : A \mid \Delta}}{\Gamma \vdash \Delta \mid t : A}\ \textsc{Series of Exchanges}$$

By the definition of the substitution function we know $t = [t/x]x$. Therefore, the previous two proofs are equivalent.

4.3 The exchange steps

4.3.1 Conclusion vs. left-exchange (the first case)

The proof

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\end{array}}{\Gamma \vdash t : A \mid \Delta} \qquad \dfrac{\dfrac{\begin{array}{c}\pi_2\\ \vdots\end{array}}{\Gamma_1, x : A, y : B, \Gamma_2 \vdash t' : C \mid \Delta'}}{\Gamma_1, y : B, x : A, \Gamma_2 \vdash t' : C \mid \Delta'}\ \textsc{Exl}}{\Gamma_1, y : B, \Gamma, \Gamma_2 \vdash \Delta \mid [t/x]t' : C \mid [t/x]\Delta'}\ \textsc{Cut}$$

transforms into the proof

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\end{array}}{\Gamma \vdash t : A \mid \Delta} \qquad \dfrac{\begin{array}{c}\pi_2\\ \vdots\end{array}}{\Gamma_1, x : A, y : B, \Gamma_2 \vdash t' : C \mid \Delta'}}{\Gamma_1, \Gamma, y : B, \Gamma_2 \vdash \Delta \mid [t/x]t' : C \mid [t/x]\Delta'}\ \textsc{Cut}}{\Gamma_1, y : B, \Gamma, \Gamma_2 \vdash \Delta \mid [t/x]t' : C \mid [t/x]\Delta'}\ \textsc{Series of Exchanges}$$

Clearly, all terms are equivalent, and so the previous two proofs are equivalent.

4.3.2 Conclusion vs. left-exchange (the second case)

The proof

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\end{array}}{\Gamma \vdash t : B \mid \Delta} \qquad \dfrac{\dfrac{\begin{array}{c}\pi_2\\ \vdots\end{array}}{\Gamma_1, x : A, y : B, \Gamma_2 \vdash t' : C \mid \Delta'}}{\Gamma_1, y : B, x : A, \Gamma_2 \vdash t' : C \mid \Delta'}\ \textsc{Exl}}{\Gamma_1, \Gamma, x : A, \Gamma_2 \vdash \Delta \mid [t/y]t' : C \mid [t/y]\Delta'}\ \textsc{Cut}$$

transforms into the proof

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\end{array}}{\Gamma \vdash t : B \mid \Delta} \qquad \dfrac{\begin{array}{c}\pi_2\\ \vdots\end{array}}{\Gamma_1, x : A, y : B, \Gamma_2 \vdash t' : C \mid \Delta'}}{\Gamma_1, x : A, \Gamma, \Gamma_2 \vdash \Delta \mid [t/y]t' : C \mid [t/y]\Delta'}\ \textsc{Cut}}{\Gamma_1, \Gamma, x : A, \Gamma_2 \vdash \Delta \mid [t/y]t' : C \mid [t/y]\Delta'}\ \textsc{Series of Exchanges}$$

Clearly, all terms are equivalent, and so the previous two proofs are equivalent.

4.3.3 Conclusion vs. right-exchange

The proof

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\end{array}}{\Gamma \vdash t : A \mid \Delta} \qquad \dfrac{\dfrac{\begin{array}{c}\pi_2\\ \vdots\end{array}}{\Gamma_1, x : A, \Gamma_2 \vdash \Delta_1 \mid t_1 : B \mid t_2 : C \mid \Delta'}}{\Gamma_1, x : A, \Gamma_2 \vdash \Delta_1 \mid t_2 : C \mid t_1 : B \mid \Delta'}\ \textsc{Exr}}{\Gamma_1, \Gamma, \Gamma_2 \vdash \Delta \mid [t/x]\Delta_1 \mid [t/x]t_2 : C \mid [t/x]t_1 : B \mid [t/x]\Delta'}\ \textsc{Cut}$$

transforms into this proof

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{\Gamma \vdash t : A \mid \Delta} \qquad \dfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{\Gamma_1, x : A, \Gamma_2 \vdash \Delta_1 \mid t_1 : B \mid t_2 : C \mid \Delta'}}{\Gamma_1, \Gamma, \Gamma_2 \vdash \Delta \mid [t/x]\Delta_1 \mid [t/x]t_1 : B \mid [t/x]t_2 : C \mid [t/x]\Delta'}\,\textsc{Cut}}{\Gamma_1, \Gamma, \Gamma_2 \vdash [t/x]\Delta_1 \mid [t/x]t_2 : C \mid [t/x]t_1 : B \mid [t/x]\Delta'}\,\textsc{Exr}$$

4.4 Principle formula vs. principle formula

4.4.1 Tensor

The proof

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{\Gamma_1 \vdash t_1 : A \mid \Delta_1} \qquad \dfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{\Gamma_2 \vdash t_2 : B \mid \Delta_2}}{\Gamma_1, \Gamma_2 \vdash t_1 \otimes t_2 : A \otimes B \mid \Delta_1 \mid \Delta_2}\,\textsc{Tr} \qquad \dfrac{\dfrac{\begin{matrix}\pi_3\\ \vdots\end{matrix}}{\Gamma_3, x : A, y : B, \Gamma_4 \vdash t_3 : C \mid \Delta_3}}{\Gamma_3, z : A \otimes B, \Gamma_4 \vdash \mathsf{let}\ z\ \mathsf{be}\ x \otimes y\ \mathsf{in}\ t_3 : C \mid \mathsf{let}\ z\ \mathsf{be}\ x \otimes y\ \mathsf{in}\ \Delta_3}\,\textsc{Tl}}{\Gamma_3, \Gamma_1, \Gamma_2, \Gamma_4 \vdash \Delta_1 \mid \Delta_2 \mid [t_1 \otimes t_2/z](\mathsf{let}\ z\ \mathsf{be}\ x \otimes y\ \mathsf{in}\ t_3) : C \mid [t_1 \otimes t_2/z](\mathsf{let}\ z\ \mathsf{be}\ x \otimes y\ \mathsf{in}\ \Delta_3)}\,\textsc{Cut}$$

is transformed into the proof

$$\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{\Gamma_1 \vdash t_1 : A \mid \Delta_1} \qquad \dfrac{\dfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{\Gamma_2 \vdash t_2 : B \mid \Delta_2} \qquad \dfrac{\begin{matrix}\pi_3\\ \vdots\end{matrix}}{\Gamma_3, x : A, y : B, \Gamma_4 \vdash t_3 : C \mid \Delta_3}}{\Gamma_3, x : A, \Gamma_2, \Gamma_4 \vdash \Delta_2 \mid [t_2/y]t_3 : C \mid [t_2/y]\Delta_3}\,\textsc{Cut}}{\Gamma_3, \Gamma_1, \Gamma_2, \Gamma_4 \vdash \Delta_1 \mid \Delta_2 \mid [t_1/x][t_2/y]t_3 : C \mid [t_1/x][t_2/y]\Delta_3}\,\textsc{Cut}$$

Now we can see that $[t_1 \otimes t_2/z](\mathsf{let}\ z\ \mathsf{be}\ x \otimes y\ \mathsf{in}\ t_3) = \mathsf{let}\ t_1 \otimes t_2\ \mathsf{be}\ x \otimes y\ \mathsf{in}\ t_3$ by the definition of substitution, and by using the $\textsc{Eq\_T1}$ rule we obtain $\mathsf{let}\ t_1 \otimes t_2\ \mathsf{be}\ x \otimes y\ \mathsf{in}\ t_3 = [t_1/x][t_2/y]t_3$. This argument can be repeated for any term in $[t_1 \otimes t_2/z](\mathsf{let}\ z\ \mathsf{be}\ x \otimes y\ \mathsf{in}\ \Delta_3)$, and thus, $[t_1 \otimes t_2/z](\mathsf{let}\ z\ \mathsf{be}\ x \otimes y\ \mathsf{in}\ \Delta_3) = [t_1/x][t_2/y]\Delta_3$. Therefore, the previous two proofs are equivalent.

Note that in the second derivation of the above transformation we first cut on B, and then A, but we could have cut on A first, and then B, but this would yeild equivalent derivations as above by using Lemma 1.

4.4.2 Par

The proof

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{\Gamma_1 \vdash \Delta_1 \mid t_1 : A \mid t_2 : B \mid \Delta_2}}{\Gamma_1 \vdash \Delta_1 \mid t_1 \parr t_2 : A \parr B \mid \Delta_2}\,\textsc{Parr} \qquad \dfrac{\dfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{\Gamma_2, x : A \vdash t_3 : C_1 \mid \Delta_3} \qquad \dfrac{\begin{matrix}\pi_3\\ \vdots\end{matrix}}{\Gamma_3, y : B \vdash t_4 : C_2 \mid \Delta_4}}{\Gamma_2, \Gamma_3, z : A \parr B \vdash \text{let-pat}\ z\ (x \parr -)\ t_3 : C_1 \mid \text{let-pat}\ z\ (- \parr y)\ t_4 : C_2 \mid \Delta_5}\,\textsc{Parl}}{\Gamma_2, \Gamma_3, \Gamma_1 \vdash \Delta_1 \mid \Delta_2 \mid [t_1 \parr t_2/z](\text{let-pat}\ z\ (x \parr -)\ t_3) : C_1 \mid [t_1 \parr t_2/z](\text{let-pat}\ z\ (- \parr y)\ t_4) : C_2 \mid [t_1 \parr t_2/z]\Delta_5}\,\textsc{Cut}$$

where $\Delta_4 = \text{let-pat}\ z\ (x \parr -)\ \Delta_3, \text{let-pat}\ z\ (- \parr y)\ \Delta_4$ is transformed into the proof

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{\Gamma_1 \vdash \Delta_1 \mid t_1 : A \mid t_2 : B \mid \Delta_2} \qquad \dfrac{\begin{matrix}\pi_3\\ \vdots\end{matrix}}{\Gamma_3, y : B \vdash t_4 : C_2 \mid \Delta_4}}{\Gamma_3, \Gamma_1 \vdash \Delta_1 \mid t_1 : A \mid \Delta_2 \mid [t_2/y]t_4 : C_2 \mid [t_2/y]\Delta_4}\,\textsc{Cut} \qquad \dfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{\Gamma_2, x : A \vdash t_3 : C_1 \mid \Delta_3}}{\Gamma_2, \Gamma_3, \Gamma_1 \vdash \Delta_1 \mid \Delta_2 \mid [t_2/y]t_4 : C_2 \mid [t_2/y]\Delta_4 \mid [t_1/x]t_3 : C_1 \mid [t_1/x]\Delta_3}\,\textsc{Cut}}{\Gamma_2, \Gamma_3, \Gamma_1 \vdash \Delta_1 \mid \Delta_2 \mid [t_1/x]t_3 : C_1 \mid [t_2/y]t_4 : C_2 \mid [t_1/x]\Delta_3 \mid [t_2/y]\Delta_4}\,\textsc{Series of Exr}$$

First, $[t_1 \parr t_2/z](\text{let-pat}\, z\, (x \parr -)\, t_3) = \text{let-pat}\, (t_1 \parr t_2)\, (x \parr -)\, t_3$, and $\text{let-pat}\, (t_1 \parr t_2)\, (x \parr -)\, t_3 = [t_1/x]t_3$ if $x \in \mathsf{FV}(t_3)$ or $\text{let-pat}\, (t_1 \parr t_2)\, (x \parr -)\, t_3 = t_3$ otherwise. In the latter case we can see that $t_3 = [t_1/x]t_3$, thus, in both cases $\text{let-pat}\, (t_1 \parr t_2)\, (x \parr -)\, t_3 = [t_1/x]t_3$. This argument can be repeated for any terms in Δ_3, and hence, $[t_1 \parr t_2/z](\text{let-pat}\, z\, (x \parr -)\, \Delta_3) = \text{let-pat}\, (t_1 \parr t_2)\, (x \parr -)\, \Delta_3 = [t_1/x]\Delta_3$. We can apply a similar argument for $[t_1 \parr t_2/z](\text{let-pat}\, z\, (- \parr y)\, t_4)$ and $[t_1 \parr t_2/x](\text{let-pat}\, z\, (- \parr y)\, \Delta_4)$. Thus, all terms in the previous derivations are equivalent, and therefore, the previous proofs are equivalent.

Note that just as we mentioned about tensor we could have first cut on A, and then on B in the second derivation, but we would have arrived as the same result just with potentially more exchanges on the right.

References

[1] G.M. Bierman. A note on full intuitionistic linear logic. *Annals of Pure and Applied Logic*, 79(3):281 – 287, 1996.

[2] Martin Hyland and Valeria de Paiva. Full intuitionistic linear logic (extended abstract). *Annals of Pure and Applied Logic*, 64(3):273 – 291, 1993.

A The full specification of FILL

term_var, w, x, y, z, v			
index_var, i, j, k			
form, A, B, C, D, E	::=		
	|	I	
	|	$\perp$	
	|	$A \multimap B$	
	|	$A \otimes B$	
	|	$A \parr B$	
	|	(A)	S
patterns, p	::=		
	|	$*$	
	|	x	
	|	$p_1 \otimes p_2$	
	|	$p_1 \parr p_2$	
	|	$-$	
	|	(p)	S
term, t, e, d, f, g, u	::=		
	|	x	
	|	$*$	
	|	$\circ$	
	|	$e_1 \otimes e_2$	
	|	$e_1 \parr e_2$	
	|	$\lambda x.t$	
	|	let t be p in e	
	|	$f\ e$	
	|	let-pat $t\ p\ e$	M
	|	$[t/x]t'$	M

	$\|$	$[t/x, e/y]t'$	M
	$\|$	(t)	S
	$\|$	t	M
	$\|$	t	M
Γ	$::=$		
	$\|$	$x : A$	
	$\|$	$\cdot$	
	$\|$	Γ, Γ'	
	$\|$	$x : A$	
	$\|$	A	
Δ	$::=$		
	$\|$	$t : A$	
	$\|$	$\cdot$	
	$\|$	$\Delta \mid \Delta'$	
	$\|$	Δ	
	$\|$	A	
	$\|$	Δ, Δ'	
	$\|$	$[t/x]\Delta$	
	$\|$	$\mathsf{let}\ t\ \mathsf{be}\ p\ \mathsf{in}\ \Delta$	
	$\|$	(Δ)	
	$\|$	$\text{let-pat}\ t\ p\ \Delta$	M
$formula$	$::=$		
	$\|$	$judgement$	
	$\|$	$formula_1 \quad formula_2$	
	$\|$	$(formula)$	
	$\|$	$x \notin \mathsf{FV}(\Delta)$	
	$\|$	$x \in \mathsf{FV}(t)$	
	$\|$	$x, y \notin \mathsf{FV}(\Delta)$	
	$\|$	$x \notin \mathsf{FV}(t)$	
	$\|$	$x, y \notin \mathsf{FV}(t)$	
	$\|$	$\Delta_1 = \Delta_2$	
$InferRules$	$::=$		
	$\|$	$\Gamma \vdash \Delta$	
	$\|$	$f = e$	
$judgement$	$::=$		
	$\|$	$InferRules$	
$user\_syntax$	$::=$		
	$\|$	$term\_var$	
	$\|$	$index\_var$	

| *form*
| *patterns*
| *term*
| Γ
| Δ
| *formula*

$\boxed{\Gamma \vdash \Delta}$

$$\frac{}{x : A \vdash x : A} \quad \textsc{Ax}$$

$$\frac{\Gamma \vdash t : A \mid \Delta \quad y : A, \Gamma' \vdash f_i : B_i}{\Gamma, \Gamma' \vdash \Delta \mid [t/y]f_i : B_i} \quad \textsc{Cut}$$

$$\frac{\Gamma \vdash e_i : A_i}{\Gamma, x : I \vdash \mathsf{let}\ x\ \mathsf{be}\ *\ \mathsf{in}\ e_i : A_i} \quad \textsc{Il}$$

$$\frac{}{\cdot \vdash * : I} \quad \textsc{Ir}$$

$$\frac{\Gamma, x : A, y : B \vdash f_i : C_i}{\Gamma, z : A \otimes B \vdash \mathsf{let}\ z\ \mathsf{be}\ x \otimes y\ \mathsf{in}\ f_i : C_i} \quad \textsc{Tl}$$

$$\frac{\Gamma \vdash e : A \mid \Delta \quad \Gamma' \vdash f : B \mid \Delta'}{\Gamma, \Gamma' \vdash e \otimes f : A \otimes B \mid \Delta \mid \Delta'} \quad \textsc{Tr}$$

$$\frac{}{x : \perp \vdash \cdot} \quad \textsc{Pl}$$

$$\frac{\Gamma \vdash \Delta}{\Gamma \vdash \circ : \perp \mid \Delta} \quad \textsc{Pr}$$

$$\frac{\Gamma, x : A \vdash d_i : C_i \quad \Gamma', y : B \vdash f_j : D_j}{\Gamma, \Gamma', z : A \parr B \vdash \mathsf{let}\ z\ \mathsf{be}\ x \parr -\ \mathsf{in}\ d_i : C_i \mid \mathsf{let}\ z\ \mathsf{be}\ - \parr y\ \mathsf{in}\ f_j : D_j} \quad \textsc{ParL}$$

$$\frac{\Gamma, x : A \vdash d_i : C_i \quad \Gamma', y : B \vdash f_j : D_j}{\texttt{<<multiple parses>>}} \quad \textsc{NParL}$$

$$\frac{\Gamma \vdash \Delta \mid e : A \mid f : B \mid \Delta'}{\Gamma \vdash \Delta \mid e \parr f : A \parr B \mid \Delta'} \quad \textsc{ParR}$$

$$\frac{\Gamma \vdash e : A \mid \Delta \quad \Gamma', x : B \vdash f_i : C_i}{\Gamma, y : A \multimap B, \Gamma' \vdash [y\ e/x]f_i : C_i \mid \Delta} \quad \textsc{ImpL}$$

$$\frac{\Gamma, x : A \vdash e : B \mid \Delta \quad x \notin \mathsf{FV}(\Delta)}{\Gamma \vdash \lambda x.e : A \multimap B \mid \Delta} \quad \textsc{ImpR}$$

$$\frac{\Gamma, x : A, y : B \vdash \Delta}{\Gamma, y : B, x : A \vdash \Delta} \quad \textsc{ExL}$$

$$\frac{\Gamma \vdash \Delta_1 \mid t_1 : A \mid t_2 : B \mid \Delta_2}{\Gamma \vdash \Delta_1 \mid t_2 : B \mid t_1 : A \mid \Delta_2} \quad \textsc{ExR}$$

$$\frac{}{A' \vdash A''} \quad \textsc{dax}$$

$$\frac{\begin{array}{c}\Gamma_1 \vdash B', \Delta_1 \\ \Gamma_2, B'' \vdash \Delta_2\end{array}}{\Gamma_1, \Gamma_2 \vdash \Delta_2, \Delta_1} \quad \textsc{dcut}$$

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \otimes B \vdash \Delta}\ \textsc{Dtl}$$

$$\frac{\begin{array}{c}\Gamma_1 \vdash A, \Delta_1 \\ \Gamma_2 \vdash B, \Delta_2\end{array}}{\Gamma_1, \Gamma_2 \vdash A \otimes B, \Delta_1, \Delta_2}\ \textsc{Dtr}$$

$$\frac{\Gamma \vdash \Delta}{\Gamma, I \vdash \Delta}\ \textsc{Dil}$$

$$\frac{}{\cdot \vdash I}\ \textsc{Dir}$$

$$\frac{\begin{array}{c}\Gamma_1, A \vdash \Delta_1 \\ \Gamma_3, B \vdash \Delta_2\end{array}}{\Gamma_1, \Gamma_3, A \parr B \vdash \Delta_1, \Delta_2}\ \textsc{Dparl}$$

$$\frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \parr B, \Delta}\ \textsc{Dparr}$$

$$\frac{}{\perp \vdash \cdot}\ \textsc{Dpl}$$

$$\frac{\Gamma \vdash \Delta}{\Gamma \vdash \perp, \Delta}\ \textsc{Dpr}$$

$$\frac{\begin{array}{c}\Gamma_1 \vdash A, \Delta_1 \\ \Gamma_2, B \vdash \Delta_2\end{array}}{\Gamma_1, \Gamma_2, A \multimap B \vdash \Delta_1, \Delta_2}\ \textsc{Dimpl}$$

$$\frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \multimap B, \Delta}\ \textsc{Dimpr}$$

$\boxed{f = e}$

$$\frac{y \notin \mathsf{FV}(t)}{t = [y/x]t}\ \textsc{Eq\_Alpha}$$

$$\frac{}{(\lambda x.e)\ e' = [e'/x]e}\ \textsc{Eq\_Beta}$$

$$\frac{}{(\lambda x.f\ x) = f}\ \textsc{Eq\_Eta}$$

$$\frac{}{\mathsf{let}\ *\ \mathsf{be}\ *\ \mathsf{in}\ e = e}\ \textsc{Eq\_I}$$

$$\frac{}{\mathsf{let}\ u\ \mathsf{be}\ *\ \mathsf{in}\ [*/z]f = [u/z]f}\ \textsc{Eq\_Stp}$$

$$\frac{}{\mathsf{let}\ e \otimes t\ \mathsf{be}\ x \otimes y\ \mathsf{in}\ u = [e/x, t/y]u}\ \textsc{Eq\_T1}$$

$$\frac{}{\mathsf{let}\ u\ \mathsf{be}\ x \otimes y\ \mathsf{in}\ [x \otimes y/z]f = [u/z]f}\ \textsc{Eq\_T2}$$

$$\frac{}{\mathsf{let}\ u \parr t\ \mathsf{be}\ x \parr -\ \mathsf{in}\ e = [u/x]e}\ \textsc{Eq\_P1}$$

$$\frac{}{\mathsf{let}\ u \parr t\ \mathsf{be}\ - \parr y\ \mathsf{in}\ e = [t/y]e}\ \textsc{Eq\_P2}$$

$$\frac{}{(\mathsf{let}\ x\ \mathsf{be}\ x \parr -\ \mathsf{in}\ x) \parr (\mathsf{let}\ u\ \mathsf{be}\ - \parr y\ \mathsf{in}\ y) = u}\ \textsc{Eq\_P3}$$