

# Cut-elimination of the APAL term assignment formulation of FILL

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In this short note I give the proof of the term assignment formulation of FILL first given in [2].

## 1 The Fix

Consider the DPARL rule in the dependency-relation formalization:

$$\frac{\frac{\Gamma_1, A \vdash \Delta_1 \quad \Gamma_3, B \vdash \Delta_2}{\Gamma_1, \Gamma_3, A \wp B \vdash \Delta_1, \Delta_2}}{\text{DPARL}} \quad \text{Dep}(\tau) = \{(A \wp B, A), (A \wp B, B)\} \star (\text{Dep}(\tau_1) \cup \text{Dep}(\tau_2))$$

If anything in  $\Delta_1$  and  $\Delta_2$  depend on  $A$  or  $B$  then this will be accounted for in  $\text{Dep}(\tau_1)$  and  $\text{Dep}(\tau_2)$ . Thus, in the term formalization when binding pattern variables across the righthand side of the sequent we should do so if and only if there is a dependency. In fact, if a formula on the righthand side depends on a formula in the lefthand side, then the variable associated with that hypothesis must be free in the term associated with the formula on the right. This evidence suggests that to fix the term formalization we must modify the PARL rule.

The new PARL rule as follows:

$$\frac{\Gamma, x : A \vdash d_i : C_i \quad \Gamma', y : B \vdash f_j : D_j}{\Gamma, \Gamma', z : A \wp B \vdash \text{let-pat } z (x \wp -) d_i \mid \text{let-pat } z (- \wp y) f_j : D_j} \text{NPARL}$$

The previous rule depends on a function which we define as follows:

$$\begin{aligned} \text{let-pat } z (x \wp -) e &= e \\ \text{where } x &\notin \text{FV}(e) \end{aligned}$$

$$\begin{aligned} \text{let-pat } z (- \wp y) e &= e \\ \text{where } y &\notin \text{FV}(e) \end{aligned}$$

$$\text{let-pat } z p e = \text{let } z \text{ be } p \text{ in } e$$

Note that in the definition of  $\text{let-pat } z p e$  the final case is a catchall case. Now the new PARL rule only pattern matches on  $z$  in the righthand side if there is a dependency between the variables in the pattern and the term in the pattern match. A similar rule to the above was proposed by Bellin in the conclusion of [1].

This rule recovers from the counterexample. The first derivation given in the counter example above is unchanged, so we only give the second:

$$\frac{\frac{\frac{v : A \vdash v : A}{\text{Ax}} \quad \frac{v : A \vdash v : A \mid \circ : \perp}{\text{Pr}} \quad \frac{\frac{x : A \vdash x : A}{\text{Ax}} \quad \frac{y : B \vdash y : B}{\text{Ax}}}{x : A, y : B \vdash x \otimes y : A \otimes B} \text{Tr}}{\frac{y : B, v : A \vdash v \otimes y : A \otimes B \mid \circ : \perp}{\text{Cut}} \quad \frac{w : C \vdash w : C}{\text{Ax}}}{\frac{v : A, z : B \wp C \vdash \text{let } z \text{ be } y \wp - \text{in } v \otimes y : A \otimes B \mid \text{let } z \text{ be } - \wp w \text{ in } w : C \mid \circ : \perp}{\text{PARL}} \quad \frac{v : A, z : B \wp C \vdash ((\text{let } z \text{ be } y \wp - \text{in } v \otimes y) \wp (\text{let } z \text{ be } - \wp w \text{ in } w)) : (A \otimes B) \wp C \mid \circ : \perp}{\text{PARR}}}{v : A \vdash \lambda z. ((\text{let } z \text{ be } y \wp - \text{in } v \otimes y) \wp (\text{let } z \text{ be } - \wp w \text{ in } w)) : (B \wp C) \multimap ((A \otimes B) \wp C) \mid \circ : \perp} \text{IMPR}$$

This new derivation is now correct, and mirrors that of the dependency-relation formalization.

## 2 Basic Results

**Lemma 1** (Substitution Distribution). *For any terms  $t$ ,  $t_1$ , and  $t_2$ ,  $[t_1/x][t_2/y]t = [[t_1/x]t_2/y][t_2/x]t$ .*

*Proof.* This proof holds by straightforward induction on the form of  $t$ . □

**Lemma 2** (Left and Right Exchange). *If  $\Gamma, \Gamma' \vdash D, D'$ , then  $\Gamma', \Gamma \vdash D', D$ .*

*Proof.* This proof holds by straightforward induction on the assumed derivation. □

## 3 Cut-elimination

### 3.1 Commuting conversion cut vs cut (first case)

The following proof

$$\frac{\frac{\pi_1}{\vdots} \quad \frac{\frac{\pi_2}{\vdots} \quad \frac{\pi_3}{\vdots}}{\frac{\Gamma_2, x : A, \Gamma_3 \vdash t_1 : B \mid \Delta_1 \quad \Gamma_1, y : B, \Gamma_4 \vdash t_2 : C \mid \Delta_2}{\Gamma_1, \Gamma_2, x : A, \Gamma_3, \Gamma_4 \vdash \Delta_1 \mid [t_1/y]t_2 : C \mid [t_1/y]\Delta_2} \text{CUT}} \quad \frac{\Gamma \vdash t : A \mid \Delta}{\Gamma_1, \Gamma_2, \Gamma, \Gamma_3, \Gamma_4 \vdash \Delta \mid [t/x]\Delta_1 \mid [t/x][t_1/y]t_2 : C \mid [t/x][t_1/y]\Delta_2} \text{CUT}$$

is transformed into the proof

$$\frac{\frac{\pi_1}{\vdots} \quad \frac{\pi_2}{\vdots}}{\frac{\Gamma \vdash t : A \mid \Delta \quad \Gamma_2, x : A, \Gamma_3 \vdash t_1 : B \mid \Delta_1}{\Gamma_2, \Gamma, \Gamma_3 \vdash [t/x]t_1 : B \mid [t/x]\Delta_1} \quad \frac{\pi_3}{\vdots}} \quad \frac{\Gamma_2, \Gamma, \Gamma_3 \vdash [t/x]t_1 : B \mid [t/x]\Delta_1 \quad \Gamma_1, y : B, \Gamma_4 \vdash t_2 : C \mid \Delta_2}{\Gamma_1, \Gamma_2, \Gamma, \Gamma_3, \Gamma_4 \vdash \Delta \mid [t/x]\Delta_1 \mid ([t/x]t_1/y)t_2 : C \mid [[t/x]t_1/y]\Delta_2} \text{CUT}$$

In order for the previous two proofs to be considered equal, we have to show that the final terms in the conclusion of the above derivations are equivalent. First, we know that the term  $[t/x][t_1/y]t_2$  in the first derivation above is equivalent to  $[[t/x]t_1/y][t/x]t_2$  by Lemma 1. Furthermore, by inspecting the first derivation we can see that  $x \notin \text{FV}(t_2)$ , and thus,  $[[t/x]t_1/y][t/x]t_2 = [[t/x]t_1/y]t_2$ . This argument may be repeated for any term in  $\Delta_2$ , and thus, we know  $[t/x][t_1/y]\Delta_2 = [[t/x]t_1/y]\Delta_2$ .

### 3.2 Commuting conversion cut vs. cut (second case)

The second commuting conversion on cut begins with the proof

$$\frac{\frac{\pi_1}{\vdots} \quad \frac{\pi_2}{\vdots} \quad \frac{\pi_3}{\vdots}}{\frac{\Gamma \vdash t : A \mid \Delta \quad \frac{\Gamma' \vdash t' : B \mid \Delta'}{\Gamma_1, x : A, \Gamma_2, \Gamma', \Gamma_3 \vdash \Delta' \mid [t'/y]t_1 : C \mid [t'/y]\Delta_1} \text{CUT}} \quad \frac{\Gamma_1, x : A, \Gamma_2, y : B, \Gamma_3 \vdash t_1 : C \mid \Delta_1}{\Gamma_1, \Gamma, \Gamma_2, \Gamma', \Gamma_3 \vdash \Delta \mid [t/x]\Delta' \mid [t/x][t'/y]t_1 : C \mid [t/x][t'/y]\Delta_1} \text{CUT}$$

is transformed into the following proof:

$$\frac{\frac{\pi_2}{\vdots} \quad \frac{\pi_1}{\vdots} \quad \frac{\pi_3}{\vdots}}{\frac{\Gamma' \vdash t' : B \mid \Delta' \quad \frac{\Gamma \vdash t : A \mid \Delta}{\Gamma_1, \Gamma, \Gamma_2, y : B, \Gamma_3 \vdash \Delta \mid [t/x]t_1 : C \mid [t/x]\Delta_1} \text{CUT}} \quad \frac{\Gamma_1, \Gamma, \Gamma_2, \Gamma', \Gamma_3 \vdash \Delta' \mid [t'/y]\Delta \mid [t'/y][t/x]t_1 : C \mid [t'/y][t/x]\Delta_1}{\Gamma_1, \Gamma, \Gamma_2, \Gamma', \Gamma_3 \vdash \Delta' \mid [t'/y]\Delta \mid [t'/y][t/x]t_1 : C \mid [t'/y][t/x]\Delta_1} \text{CUT}$$

Now, because we know  $x, y \notin \text{FV}(\Delta)$  by inspection of the first derivation, we know that  $\Delta = [t'/y]\Delta$  and  $\Delta' = [t/x]\Delta'$ . Similarly, we know that  $x, y \notin \text{FV}(t)$  and  $x, y \notin \text{FV}(t')$ . Thus, by this fact and Lemma 1, we know that  $[t/x][t'/y]t_1 = [[t/x]t'/y][t/x]t_1 = [t'/y][t/x]t_1$ . This argument can be repeated for any term in  $\Delta_1$ , hence,  $[t/x][t'/y]\Delta_1 = [t'/y][t/x]\Delta_1$ . Finally, by Lemma 2 we can exchange  $\Delta'$  and  $[t'/y]\Delta$  in the last proof to obtain the same order of terms on the right in both derivations. Therefore, both of the previous derivations are equivalent.

### 3.3 The $\eta$ -expansion cases

#### 3.3.1 Tensor

The proof

$$\frac{}{x : A \otimes B \vdash x : A \otimes B} \text{Ax}$$

is transformed into the proof

$$\frac{\frac{\frac{}{y : A \vdash y : A} \text{Ax} \quad \frac{}{z : B \vdash z : B} \text{Ax}}{y : A, z : B \vdash y \otimes z : A \otimes B} \text{Tr}}{x : A \otimes B \vdash \text{let } x \text{ be } y \otimes z \text{ in } (y \otimes z) : A \otimes B} \text{TL}$$

Now by the rule EQ\_T2 we know  $\text{let } x \text{ be } y \otimes z \text{ in } (y \otimes z) = x$ .

#### 3.3.2 Par

The proof

$$\frac{}{x : A \wp B \vdash x : A \wp B} \text{Ax}$$

is transformed into the proof

$$\frac{\frac{\frac{}{y : A \vdash y : A} \text{Ax} \quad \frac{}{z : B \vdash z : B} \text{Ax}}{x : A \wp B \vdash \text{let } x \text{ be } (y \wp -) \text{ in } y : A \mid \text{let } x \text{ be } (- \wp z) \text{ in } z : B} \text{PARL}}{x : A \wp B \vdash (\text{let } x \text{ be } (y \wp -) \text{ in } y) \wp (\text{let } x \text{ be } (- \wp z) \text{ in } z) : A \wp B} \text{PARR}$$

Just as we saw in the previous case by rule EQ\_P3 we know  $((\text{let } x \text{ be } (y \wp -) \text{ in } y) \wp (\text{let } x \text{ be } (- \wp z) \text{ in } z)) = x$ .

#### 3.3.3 Implication

The proof

$$\frac{}{x : A \multimap B \vdash A \multimap B} \text{Ax}$$

transforms into the proof

$$\frac{\frac{\frac{}{y : A \vdash y : A} \text{Ax} \quad \frac{}{z : B \mid - z : B} \text{Ax}}{y : A, x : A \multimap B \vdash x y : B} \text{IMPL}}{x : A \multimap B \vdash \lambda y. x y : A \multimap B} \text{IMPR}$$

Finally, the two derivations are equivalent, because  $(\lambda y. x y) = x$  by the EQ\_ETA rule.

### 3.3.4 Tensor unit

The proof

$$\frac{}{x : I \vdash x : I} \text{Ax}$$

transforms into the proof

$$\frac{\frac{}{\cdot \vdash * : I} \text{IR}}{x : I \vdash \text{let } x \text{ be } * \text{ in } * : I} \text{IL}$$

Lastly, we know  $x = \text{let } x \text{ be } * \text{ in } *$  by EQ-I.

## References

- [1] G.M. Bierman. A note on full intuitionistic linear logic. *Annals of Pure and Applied Logic*, 79(3):281 – 287, 1996.
- [2] Martin Hyland and Valeria de Paiva. Full intuitionistic linear logic (extended abstract). *Annals of Pure and Applied Logic*, 64(3):273 – 291, 1993.

## A The full specification of FILL

*term\_var*,  $w, x, y, z, v$

*index\_var*,  $i, j, k$

*form*,  $A, B, C, D, E$  ::=

$I$	
$\perp$	
$A \multimap B$	
$A \otimes B$	
$A \wp B$	
$(A)$	S

*patterns*,  $p$

::=

$*$	
$x$	
$p_1 \otimes p_2$	
$p_1 \wp p_2$	
$\text{—}$	
$(p)$	S

*term*,  $t, e, d, f, g, u$

::=

$x$	
$*$	
$\circ$	
$e_1 \otimes e_2$	
$e_1 \wp e_2$	
$\lambda x. t$	

		$\text{let } t \text{ be } p \text{ in } e$ $f \ e$ $\text{let-pat } z \ p \ e$ $[t/x]t'$ $[t/x, e/y]t'$ $(t)$ $t$ $t$	   M M M S M M
$\Gamma$	$::=$	$x : A$ $\cdot$ $\Gamma, \Gamma'$ $x : A$ $A$	
$\Delta$	$::=$	$t : A$ $\cdot$ $\Delta \mid \Delta'$ $\Delta$ $A$ $\Delta, \Delta'$ $[t/x]\Delta$	
<i>formula</i>	$::=$	$\text{judgement}$ $\text{formula}_1 \ \text{formula}_2$ $(\text{formula})$ $x \notin \text{FV}(\Delta)$ $x, y \notin \text{FV}(\Delta)$ $x \notin \text{FV}(t)$ $x, y \notin \text{FV}(t)$ $\Delta_1 = \Delta_2$	
<i>InferRules</i>	$::=$	$\Gamma \vdash \Delta$ $f = e$	
<i>judgement</i>	$::=$	$\text{InferRules}$	
<i>user_syntax</i>	$::=$	$\text{term\_var}$ $\text{index\_var}$	

	<i>form</i>
	<i>patterns</i>
	<i>term</i>
	$\Gamma$
	$\Delta$
	<i>formula</i>

$\boxed{\Gamma \vdash \Delta}$

$$\begin{array}{c}
\frac{}{x : A \vdash x : A} \text{Ax} \\
\frac{\Gamma \vdash t : A \mid \Delta \quad y : A, \Gamma' \vdash f_i : B_i}{\Gamma, \Gamma' \vdash \Delta \mid [t/y]f_i : B_i} \text{CUT} \\
\frac{\Gamma \vdash e_i : A_i}{\Gamma, x : I \vdash \text{let } x \text{ be } * \text{ in } e_i : A_i} \text{IL} \\
\frac{}{\cdot \vdash * : I} \text{IR} \\
\frac{\Gamma, x : A, y : B \vdash f_i : C_i}{\Gamma, z : A \otimes B \vdash \text{let } z \text{ be } x \otimes y \text{ in } f_i : C_i} \text{TL} \\
\frac{\Gamma \vdash e : A \mid \Delta \quad \Gamma' \vdash f : B \mid \Delta'}{\Gamma, \Gamma' \vdash e \otimes f : A \otimes B \mid \Delta \mid \Delta'} \text{TR} \\
\frac{}{x : \perp \vdash \cdot} \text{PL} \\
\frac{\Gamma \vdash \Delta}{\Gamma \vdash \circ : \perp \mid \Delta} \text{PR} \\
\frac{\Gamma, x : A \vdash d_i : C_i \quad \Gamma', y : B \vdash f_j : D_j}{\Gamma, \Gamma', z : A \wp B \vdash \text{let } z \text{ be } x \wp - \text{ in } d_i : C_i \mid \text{let } z \text{ be } - \wp y \text{ in } f_j : D_j} \text{PARL} \\
\frac{\Gamma, x : A \vdash d_i : C_i \quad \Gamma', y : B \vdash f_j : D_j}{\Gamma, \Gamma', z : A \wp B \vdash \text{let-pat } z (x \wp -) d_i : C_i \mid \text{let-pat } z (- \wp y) f_j : D_j} \text{NPARL} \\
\frac{\Gamma \vdash \Delta \mid e : A \mid f : B \mid \Delta'}{\Gamma \vdash \Delta \mid e \wp f : A \wp B \mid \Delta'} \text{PARR} \\
\frac{\Gamma \vdash e : A \mid \Delta \quad \Gamma', x : B \vdash f_i : C_i}{\Gamma, y : A \multimap B, \Gamma' \vdash [y e/x]f_i : C_i \mid \Delta} \text{IMPL} \\
\frac{\Gamma, x : A \vdash e : B \mid \Delta \quad x \notin \text{FV}(\Delta)}{\Gamma \vdash \lambda x. e : A \multimap B \mid \Delta} \text{IMPR} \\
\frac{}{A' \vdash A''} \text{DAX} \\
\frac{\Gamma_1 \vdash B', \Delta_1 \quad \Gamma_2, B'' \vdash \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_2, \Delta_1} \text{DCUT} \\
\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \otimes B \vdash \Delta} \text{DTL} \\
\frac{\Gamma_1 \vdash A, \Delta_1 \quad \Gamma_2 \vdash B, \Delta_2}{\Gamma_1, \Gamma_2 \vdash A \otimes B, \Delta_1, \Delta_2} \text{DTR}
\end{array}$$

$$\begin{array}{c}
\frac{\Gamma \vdash \Delta}{\Gamma, I \vdash \Delta} \quad \text{DIL} \\
\\
\frac{}{\cdot \vdash I} \quad \text{DIR} \\
\\
\frac{\Gamma_1, A \vdash \Delta_1 \quad \Gamma_3, B \vdash \Delta_2}{\Gamma_1, \Gamma_3, A \wp B \vdash \Delta_1, \Delta_2} \quad \text{DPARL} \\
\\
\frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \wp B, \Delta} \quad \text{DPARR} \\
\\
\frac{}{\perp \vdash \cdot} \quad \text{DPL} \\
\\
\frac{\Gamma \vdash \Delta}{\Gamma \vdash \perp, \Delta} \quad \text{DPR} \\
\\
\frac{\Gamma_1 \vdash A, \Delta_1 \quad \Gamma_2, B \vdash \Delta_2}{\Gamma_1, \Gamma_2, A \multimap B \vdash \Delta_1, \Delta_2} \quad \text{DIMPL} \\
\\
\frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \multimap B, \Delta} \quad \text{DIMPR}
\end{array}$$

$$\boxed{f = e}$$

$$\begin{array}{c}
\frac{}{(\lambda x. e) e' = [e'/x]e} \quad \text{EQ\_BETA} \\
\\
\frac{}{(\lambda x. f x) = f} \quad \text{EQ\_ETA} \\
\\
\frac{}{\text{let } * \text{ be } * \text{ in } e = e} \quad \text{EQ\_I} \\
\\
\frac{}{\text{let } u \text{ be } * \text{ in } [* / z]f = [u / z]f} \quad \text{EQ\_STP} \\
\\
\frac{}{\text{let } e \otimes t \text{ be } x \otimes y \text{ in } u = [e / x, t / y]u} \quad \text{EQ\_T1} \\
\\
\frac{}{\text{let } u \text{ be } x \otimes y \text{ in } [x \otimes y / z]f = [u / z]f} \quad \text{EQ\_T2} \\
\\
\frac{}{\text{let } u \wp t \text{ be } x \wp - \text{ in } e = [u / x]e} \quad \text{EQ\_P1} \\
\\
\frac{}{\text{let } u \wp t \text{ be } - \wp y \text{ in } e = [t / y]e} \quad \text{EQ\_P2} \\
\\
\frac{}{(\text{let } x \text{ be } x \wp - \text{ in } x) \wp (\text{let } u \text{ be } - \wp y \text{ in } y) = u} \quad \text{EQ\_P3}
\end{array}$$