Multiple Conclusion Intuitionistic Linear Logic and Cut Elimination

Harley Eades III and Valeria de Paiva

February 16, 2015

Abstract

Full Intuitionistic Linear Logic (FILL) was first introduced by Hyland and de Paiva as one of the results of their investigation into a categorical understanding of Gödel's Dialectica interpretation. FILL went against current beliefs that it was not possible to incorporate all of the linear connectives, e.g. tensor, par, and implication, into an intuitionistic linear logic. They showed that it is natural to support all of the connectives given sequents that have multiple hypotheses and multiple conclusions. To enforce intuitionism de Paiva's original formalization of FILL used the well-known Dragalin restriction, forcing the implication right rule to have only a single conclusion in its premise, but Schellinx showed that this results in a failure of cut-elimination. To overcome this failure Hyland and de Paiva introduced a term assignment for FILL that eliminated the need for the strong restriction. The main idea was to first relax the restriction by assigning variables to each hypothesis and terms to each conclusion. Then when introducing an implication on the right enforcing that the variable annotating the hypothesis being discharged is only free in the term annotating the conclusion of the implication. Bierman showed in a short note that this formalization of FILL still did not enjoy cut-elimination, because of a flaw in the left rule for par. However, Bellin proposed an alternate left rule for par and conjectured that by adopting his rule cut-elimination is restored. In this note we show that adopting Bellin's proposed rule one does obtain cut-elimination for FILL, as suggested. Additionally, we show that this new formalization can be modeled by a new form of dialectica category called order-enriched dialectica category, and discuss future work giving FILL a semantics in terms of Lorenzen games.

1 Introduction

A commonly held belief during the early history of linear logic was that the linear-connective par could not be incorporated into an intuitionistic linear logic. This belief was challenged when de Paiva gave a categorical understanding of Gödel's Dialectica interpretation in her thesis [**?**].

[?] In [?] Martin Hyland and Veleria de Paiva give a term formalization of Full Intuitionistic Linear Logic (FILL), but later Bierman was able to give a counterexample to cut-elimination [?]. As Bierman explains the problem was that the left rule for par introduced a fresh variable into to many terms on the right-side of the conclusion. This resulted in a counterexample where this fresh variable became bound in one term, but is left free in another. This resulted from first doing a commuting conversion on cut, and then λ-binding the fresh variable. Thus, cut-elmination failed. In the conclusion of Bierman's paper he gives an alternate left-par rule which he attributes to Bellin, and states that this alternate rule should fix the problem with cut-elimination [?]. In this note we adopt Bellin's rule, and then show cut-elimination in Section **??**.

2 Full Intuitionistic Linear Logic (FILL)

In this section we give a brief description of Full Intuitionistic Linear Logic (FILL) in the style found in [?]. However, we use a slightly different presentation that we feel provides a more elegant description of the logic. We first give the syntax of formulas, patterns, terms, and contexts. Following the syntax we define several meta-functions that will be used when defining the inference rules of the logic.

Definition 1. *The syntax for FILL is as follows:*

$$
\begin{array}{ll}
\textit{(Formulas)} & A, B, C, D, E ::= I \mid \perp \mid A \multimap B \mid A \otimes B \mid A \parr B \\
\textit{(Patterns)} & p ::= * \mid - \mid x \mid p_1 \otimes p_2 \mid p_1 \parr p_2 \\
\textit{(Terms)} & t, e ::= x \mid * \mid \circ \mid t_1 \otimes t_2 \mid t_1 \parr t_2 \mid \lambda x.t \mid \mathsf{let}\, t\, \mathsf{be}\, p\, \mathsf{in}\, e \mid t_1\; t_2 \\
\textit{(Left Contexts)} & \Gamma ::= \cdot \mid x : A \mid \Gamma_1, \Gamma_2 \\
\textit{(Right Contexts)} & \Delta ::= \cdot \mid t : A \mid \Delta_1, \Delta_2
\end{array}
$$

At this point we introduce some basic syntax and definitions to facilitate readability, and presentation of the inference rules. First, we will often write $\Delta_1 \mid \Delta_2$ as syntactic sugar for Δ_1, Δ_2. The former syntax should be read as "Δ_1 or Δ_2." This will help readability of the sequent we will introduce below. We denote the usual capture-avoiding substitution by $[t/x]t'$, and its extension to right contexts as $[t/x]\Delta$.

The previous extension will make conducting substitutions across a sequence of terms in an inference rule easier. Similarly, we find it convenient to be able to do this style of extension for the let-binding as well.

Definition 2. *We extend let-binding terms to right contexts as follows:*

$$
\begin{array}{l}
\mathsf{let}\, t\, \mathsf{be}\, p\, \mathsf{in} \cdot = \cdot \\
\mathsf{let}\, t\, \mathsf{be}\, p\, \mathsf{in}\, (t' : A) = (\mathsf{let}\, t\, \mathsf{be}\, p\, \mathsf{in}\, t') : A \\
\mathsf{let}\, t\, \mathsf{be}\, p\, \mathsf{in}\, (\Delta_1 \mid \Delta_2) = (\mathsf{let}\, t\, \mathsf{be}\, p\, \mathsf{in}\, \Delta_1) \mid (\mathsf{let}\, t\, \mathsf{be}\, p\, \mathsf{in}\, \Delta_2)
\end{array}
$$

We denote the usual function that computes the set of free variables in a term by $\mathsf{FV}(t)$, and its straightforward extension to right contexts as $\mathsf{FV}(\Delta)$. Finally, we arrive at the inference rules of FILL.

Definition 3. *The inference rules for derivability in FILL are as follows:*

$$\frac{}{x : A \vdash x : A}\ \textsc{Ax} \qquad \frac{\Gamma \vdash t : A \mid \Delta \quad y : A, \Gamma' \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta \mid [t/y]\Delta'}\ \textsc{Cut} \qquad \frac{\Gamma \vdash \Delta}{\Gamma, x : I \vdash \mathsf{let}\, x\, \mathsf{be} * \mathsf{in}\, \Delta}\ \textsc{Il}$$

$$\frac{}{\cdot \vdash * : I}\ \textsc{Ir} \qquad \frac{\Gamma, x : A, y : B \vdash \Delta}{\Gamma, z : A \otimes B \vdash \mathsf{let}\, z\, \mathsf{be}\, x \otimes y\, \mathsf{in}\, \Delta}\ \textsc{Tl}$$

$$\frac{\Gamma \vdash e : A \mid \Delta \quad \Gamma' \vdash f : B \mid \Delta'}{\Gamma, \Gamma' \vdash e \otimes f : A \otimes B \mid \Delta \mid \Delta'}\ \textsc{Tr} \qquad \frac{}{x : \perp \vdash \cdot}\ \textsc{Pl} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \circ : \perp \mid \Delta}\ \textsc{Pr}$$

$$\frac{\Gamma, x : A \vdash \Delta \quad \Gamma', y : B \vdash \Delta'}{\Gamma, \Gamma', z : A \parr B \vdash \text{let-pat}\, z\, (x \parr -)\, \Delta \mid \text{let-pat}\, z\, (- \parr y)\, \Delta'}\ \textsc{Parl}$$

$$\frac{\Gamma \vdash \Delta \mid e : A \mid f : B \mid \Delta'}{\Gamma \vdash \Delta \mid e \parr f : A \parr B \mid \Delta'}\ \textsc{Parr} \qquad \frac{\Gamma \vdash e : A \mid \Delta \quad \Gamma', x : B \vdash \Delta'}{\Gamma, y : A \multimap B, \Gamma' \vdash \Delta \mid [y\, e/x]\Delta'}\ \textsc{Impl}$$

$$\frac{\Gamma, x : A \vdash e : B \mid \Delta \quad x \notin \mathsf{FV}(\Delta)}{\Gamma \vdash \lambda x.e : A \multimap B \mid \Delta}\ \textsc{Impr} \qquad \frac{\Gamma, x : A, y : B \vdash \Delta}{\Gamma, y : B, x : A \vdash \Delta}\ \textsc{Exl}$$

$$\frac{\Gamma \vdash \Delta_1 \mid t_1 : A \mid t_2 : B \mid \Delta_2}{\Gamma \vdash \Delta_1 \mid t_2 : B \mid t_1 : A \mid \Delta_2}\ \textsc{Exr}$$

The Parl rule depends on a function let-pat $z\, p\, \Delta$. We define this function next.

Definition 4. *The function* let-pat $z\, p\, t$ *is defined as follows:*

$$\begin{array}{lll} \text{let-pat}\, z\, (x \parr -)\, t = t & \text{let-pat}\, z\, (- \parr y)\, t = t & \text{let-pat}\, z\, p\, t = \mathsf{let}\, z\, \mathsf{be}\, p\, \mathsf{in}\, t \\ \quad \textit{where } x \notin \mathsf{FV}(t) & \quad \textit{where } y \notin \mathsf{FV}(t) & \end{array}$$

It is straightforward to extend the previous definition to right-contexts, and we denote this extension by let-pat $z\, p\, \Delta$*.*

The motivation behind this function is that it only binds the pattern variables in p in a term if and only if those pattern variables are free in the term. This over comes the counterexample given by Bierman in [**?**]. Throughout the sequel we will denote derivations of the previous rules by π.

3 Cut-elimination

The usual proof of cut-elimination for intuitionistic and classical linear logic should suffice for FILL. Thus, in this section we simply give the cut-elimination procedure for FILL following the development in [**?**]. However, there is one invariant that must be verified across each derivation transformation. The invariant is that if a derivation π is transformed into a derivation π', then the terms in the conclusion of the final rule applied in π must be equivalent to the terms in the conclusion of the final rule applied in π', but using what notion of equivalence?

Definition 5. *Equivalence on terms is defined as follows:*

$$\frac{y \notin \mathsf{FV}(t)}{t = [y/x]t}\ \textsc{Alpha} \qquad \frac{x \notin \mathsf{FV}(f)}{(\lambda x.f\ x) = f}\ \textsc{EtaFun} \qquad \frac{}{(\lambda x.e)\ e' = [e'/x]e}\ \textsc{BetaFun}$$

$$\frac{}{\mathsf{let}\ *\ \mathsf{be}\ *\ \mathsf{in}\ e = e}\ \textsc{Eta1I} \qquad \frac{y \notin \mathsf{FV}(f)}{f = \mathsf{let}\ y\ \mathsf{be}\ *\ \mathsf{in}\ f}\ \textsc{Eta2I}$$

$$\frac{}{\mathsf{let}\ u\ \mathsf{be}\ *\ \mathsf{in}\ [*/z]f = [u/z]f}\ \textsc{BetaI} \qquad \frac{}{[\mathsf{let}\ u\ \mathsf{be}\ *\ \mathsf{in}\ e/y]f = \mathsf{let}\ u\ \mathsf{be}\ *\ \mathsf{in}\ [e/y]f}\ \textsc{NatI}$$

$$\frac{x, y \notin \mathsf{FV}(t)}{\mathsf{let}\ t'\ \mathsf{be}\ x \otimes y\ \mathsf{in}\ t = t}\ \textsc{EtaTen} \qquad \frac{}{\mathsf{let}\ e \otimes t\ \mathsf{be}\ x \otimes y\ \mathsf{in}\ u = [e/x, t/y]u}\ \textsc{Beta1Ten}$$

$$\frac{}{\mathsf{let}\ u\ \mathsf{be}\ x \otimes y\ \mathsf{in}\ [x \otimes y/z]f = [u/z]f}\ \textsc{Beta2Ten}$$

$$\frac{}{[\mathsf{let}\ u\ \mathsf{be}\ x \otimes y\ \mathsf{in}\ g/w]f = \mathsf{let}\ u\ \mathsf{be}\ x \otimes y\ \mathsf{in}\ [g/w]f}\ \textsc{NatTen} \qquad \frac{}{u = \circ}\ \textsc{EtaParU}$$

$$\frac{}{(\mathsf{let}\ u\ \mathsf{be}\ x \parr -\ \mathsf{in}\ x) \parr (\mathsf{let}\ u\ \mathsf{be}\ - \parr y\ \mathsf{in}\ y) = u}\ \textsc{EtaPar}$$

$$\frac{}{\mathsf{let}\ u \parr t\ \mathsf{be}\ x \parr -\ \mathsf{in}\ e = [u/x]e}\ \textsc{Beta1Par} \qquad \frac{}{\mathsf{let}\ u \parr t\ \mathsf{be}\ - \parr y\ \mathsf{in}\ e = [t/y]e}\ \textsc{Beta2Par}$$

$$\frac{}{\mathsf{let}\ t\ \mathsf{be}\ x \parr -\ \mathsf{in}\ [u/x]f = [\mathsf{let}\ t\ \mathsf{be}\ x \parr -\ \mathsf{in}\ u/x]f}\ \textsc{Nat1Par}$$

$$\frac{}{\mathsf{let}\ t\ \mathsf{be}\ - \parr y\ \mathsf{in}\ [v/y]f = [\mathsf{let}\ t\ \mathsf{be}\ - \parr y\ \mathsf{in}\ v/y]f}\ \textsc{Nat2Par} \qquad \frac{t = t'}{\lambda x.t = \lambda x.t''}\ \textsc{Lam}$$

$$\frac{t_1 = t_1'}{t_1\ t_2 = t_1'\ t_2}\ \textsc{App1} \qquad \frac{t_2 = t_2'}{t_1\ t_2 = t_1\ t_2'}\ \textsc{App2} \qquad \frac{t_1 = t_1'}{t_1 \otimes t_2 = t_1' \otimes t_2}\ \textsc{Ten1}$$

$$\frac{t_2 = t_2'}{t_1 \otimes t_2 = t_1 \otimes t_2'}\ \textsc{Ten2} \qquad \frac{t_1 = t_1'}{t_1 \parr t_2 = t_1' \parr t_2}\ \textsc{Par1} \qquad \frac{t_2 = t_2'}{t_1 \parr t_2 = t_1 \parr t_2'}\ \textsc{Par2}$$

$$\frac{t = t'}{\mathsf{let}\ t\ \mathsf{be}\ p\ \mathsf{in}\ e = \mathsf{let}\ t'\ \mathsf{be}\ p\ \mathsf{in}\ e}\ \textsc{Let1} \qquad \frac{e = e'}{\mathsf{let}\ t\ \mathsf{be}\ p\ \mathsf{in}\ e = \mathsf{let}\ t\ \mathsf{be}\ p\ \mathsf{in}\ e'}\ \textsc{Let2} \qquad \frac{}{t = t}\ \textsc{Refl}$$

$$\frac{t = t'}{t' = t}\ \textsc{Sym} \qquad \frac{t_1 = t_2 \quad t_2 = t_3}{t_1 = t_3}\ \textsc{Trans}$$

The cut elimination procedure requires the following two basic results:

Lemma 6 (Substitution Distribution)**.** *For any terms t, t_1, and t_2, $[t_1/x][t_2/y]t = [[t_1/x]t_2/y][t_2/x]t$.*

Proof. This proof holds by straightforward induction on the form of t. $\square$

Lemma 7 (Let-pat Distribution)**.** *For any terms t, t_1, and t_2, and pattern p,* let-pat $t\ p\ [t_1/y]t_2 = [\text{let-pat}\ t\ p\ t_1/y]t_2$.

Proof. This proof holds by case splitting over p, and then using the naturality equations for the respective pattern. $\square$

Throughout the remainder of this section we present a particular step in the cut-elimination procedure, and then give a short proof that equality of terms are preserved across the particular transformation on derivations.

3.1 Commuting conversion cut vs cut (first case)

The following proof

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\end{array}}{\Gamma \vdash t : A \mid \Delta} \qquad \dfrac{\dfrac{\begin{array}{c}\pi_2\\ \vdots\end{array}}{\Gamma_2, x : A, \Gamma_3 \vdash t_1 : B \mid \Delta_1} \qquad \dfrac{\begin{array}{c}\pi_3\\ \vdots\end{array}}{\Gamma_1, y : B, \Gamma_4 \vdash \Delta_2}}{\Gamma_1, \Gamma_2, x : A, \Gamma_3, \Gamma_4 \vdash \Delta_1 \mid [t_1/y]\Delta_2}\ \textsc{Cut}}{\Gamma_1, \Gamma_2, \Gamma, \Gamma_3, \Gamma_4 \vdash \Delta \mid [t/x]\Delta_1 \mid [t/x][t_1/y]\Delta_2}\ \textsc{Cut}$$

is transformed into the proof

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\end{array}}{\Gamma \vdash t : A \mid \Delta} \qquad \dfrac{\begin{array}{c}\pi_2\\ \vdots\end{array}}{\Gamma_2, x : A, \Gamma_3 \vdash t_1 : B \mid \Delta_1}}{\Gamma_2, \Gamma, \Gamma_3 \vdash [t/x]t_1 : B \mid [t/x]\Delta_1} \qquad \dfrac{\begin{array}{c}\pi_3\\ \vdots\end{array}}{\Gamma_1, y : B, \Gamma_4 \vdash \Delta_2}}{\Gamma_1, \Gamma_2, \Gamma, \Gamma_3, \Gamma_4 \vdash \Delta \mid [t/x]\Delta_1 \mid [[t/x]t_1/y]\Delta_2}\ \textsc{Cut}$$

First, if Δ_2 is empty, then all the terms in the conclusion of the previous two derivations are equivalent. So suppose $\Delta_2 = t_2 : C \mid \Delta_2'$. Then we know that the term $[t/x][t_1/y]t_2$ in the first derivation above is equivalent to $[[t/x]t_1/y][t/x]t_2$ by Lemma **??**. Furthermore, by inspecting the first derivation we can see that $x \notin \mathsf{FV}(t_2)$, and thus, $[[t/x]t_1/y][t/x]t_2 = [[t/x]t_1/y]t_2$. This argument may be repeated for any term in Δ_2', and thus, we know $[t/x][t_1/y]\Delta_2 = [[t/x]t_1/y]\Delta_2$.

3.2 Commuting conversion cut vs. cut (second case)

The second commuting conversion on cut begins with the proof

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\end{array}}{\Gamma \vdash t : A \mid \Delta} \qquad \dfrac{\dfrac{\begin{array}{c}\pi_2\\ \vdots\end{array}}{\Gamma' \vdash t' : B \mid \Delta'} \qquad \dfrac{\begin{array}{c}\pi_3\\ \vdots\end{array}}{\Gamma_1, x : A, \Gamma_2, y : B, \Gamma_3 \vdash \Delta_1}}{\Gamma_1, x : A, \Gamma_2, \Gamma', \Gamma_3 \vdash \Delta' \mid [t'/y]\Delta_1}\ \textsc{Cut}}{\Gamma_1, \Gamma, \Gamma_2, \Gamma', \Gamma_3 \vdash \Delta \mid [t/x]\Delta' \mid [t/x][t'/y]\Delta_1}\ \textsc{Cut}$$

is transformed into the following proof:

$$
\dfrac{\dfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{\Gamma' \vdash t' : B \mid \Delta'} \qquad \dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{\Gamma \vdash t : A \mid \Delta} \qquad \dfrac{\begin{matrix}\pi_3\\ \vdots\end{matrix}}{\Gamma_1, x : A, \Gamma_2, y : B, \Gamma_3 \vdash \Delta_1}}{\Gamma_1, \Gamma, \Gamma_2, y : B, \Gamma_3 \vdash \Delta \mid [t/x]\Delta_1}\ \textsc{Cut}}{\dfrac{\Gamma_1, \Gamma, \Gamma_2, \Gamma', \Gamma_3 \vdash \Delta' \mid [t'/y]\Delta \mid [t'/y][t/x]\Delta_1}{\Gamma_1, \Gamma, \Gamma_2, \Gamma', \Gamma_3 \vdash [t'/y]\Delta \mid \Delta' \mid [t'/y][t/x]\Delta_1}\ \textsc{Series of Exchanges}}\ \textsc{Cut}
$$

We know $x, y \notin \mathsf{FV}(\Delta)$ by inspection of the first derivation, and so we know that $\Delta = [t'/y]\Delta$ and $\Delta' = [t/x]\Delta'$. Without loss of generality suppose $\Delta_1 = t_1 : C \mid \Delta'_1$. Then we know that $x, y \notin \mathsf{FV}(t)$ and $x, y \notin \mathsf{FV}(t')$. Thus, by this fact and Lemma **??**, we know that $[t/x][t'/y]t_1 = [[t/x]t'/y][t/x]t_1 = [t'/y][t/x]t_1$. This argument can be repeated for any term in Δ'_1, hence, $[t/x][t'/y]\Delta_1 = [t'/y][t/x]\Delta_1$.

3.3 The η-expansion cases

3.3.1 Tensor

The proof

$$
\dfrac{}{x : A \otimes B \vdash x : A \otimes B}\ \textsc{Ax}
$$

is transformed into the proof

$$
\dfrac{\dfrac{\dfrac{}{y : A \vdash y : A}\ \textsc{Ax} \qquad \dfrac{}{z : B \vdash z : B}\ \textsc{Ax}}{y : A, z : B \vdash y \otimes z : A \otimes B}\ \textsc{Tr}}{x : A \otimes B \vdash \mathsf{let}\, x\, \mathsf{be}\, y \otimes z\, \mathsf{in}\, (y \otimes z) : A \otimes B}\ \textsc{Tl}
$$

By the rule $\textsc{eq\_EtaTensor}$ we know $\mathsf{let}\, x\, \mathsf{be}\, y \otimes z\, \mathsf{in}\, (y \otimes z) = x$.

3.3.2 Par

The proof

$$
\dfrac{}{x : A \parr B \vdash x : A \parr B}\ \textsc{Ax}
$$

is transformed into the proof

$$
\dfrac{\dfrac{\dfrac{}{y : A \vdash y : A}\ \textsc{Ax} \qquad \dfrac{}{z : B \vdash z : B}\ \textsc{Ax}}{x : A \parr B \vdash \mathsf{let}\, x\, \mathsf{be}\, (y \parr -)\, \mathsf{in}\, y : A \mid \mathsf{let}\, x\, \mathsf{be}\, (- \parr z)\, \mathsf{in}\, z : B}\ \textsc{ParL}}{x : A \parr B \vdash (\mathsf{let}\, x\, \mathsf{be}\, (y \parr -)\, \mathsf{in}\, y) \parr (\mathsf{let}\, x\, \mathsf{be}\, (- \parr z)\, \mathsf{in}\, z) : A \parr B}\ \textsc{ParR}
$$

By rule $\textsc{Eq\_EtaPar}$ we know $((\mathsf{let}\, x \,\mathsf{be}\, (y \parr -) \,\mathsf{in}\, y) \parr (\mathsf{let}\, x \,\mathsf{be}\, (- \parr z) \,\mathsf{in}\, z)) = x$.

3.3.3 Implication

The proof

$$\dfrac{}{x : A \multimap B \vdash x : A \multimap B}\ \textsc{Ax}$$

transforms into the proof

$$\dfrac{\dfrac{\dfrac{}{y : A \vdash y : A}\ \textsc{Ax} \qquad \dfrac{}{z : B \vdash z : B}\ \textsc{Ax}}{y : A, x : A \multimap B \vdash x\, y : B}\ \textsc{ImpL}}{x : A \multimap B \vdash \lambda y.x\, y : A \multimap B}\ \textsc{ImpR}$$

All terms in the two derivations are equivalent, because $(\lambda y.x\, y) = x$ by the $\textsc{Eq\_EtaFun}$ rule.

3.3.4 Tensor unit

The proof

$$\dfrac{}{x : I \vdash x : I}\ \textsc{Ax}$$

transforms into the proof

$$\dfrac{\dfrac{}{\cdot \vdash * : I}\ \textsc{Ir}}{x : I \vdash \mathsf{let}\, x \,\mathsf{be}\, * \,\mathsf{in}\, * : I}\ \textsc{Il}$$

We know $x = \mathsf{let}\, x \,\mathsf{be}\, * \,\mathsf{in}\, *$ by $\textsc{Eq\_EtaI}$.

3.3.5 Par unit

The proof

$$\dfrac{}{x : \bot \vdash x : \bot}\ \textsc{Ax}$$

transforms into the proof

$$\dfrac{\dfrac{}{x : \bot \vdash \cdot}\ \textsc{Pl}}{x : \bot \vdash \circ : \bot}\ \textsc{Pr}$$

We know $x = \circ$ by $\textsc{Eq\_EtaParU}$.

3.4 The axiom steps

3.4.1 The axiom step

The proof

$$\dfrac{\dfrac{}{x : A \vdash x : A}\ \textsc{Ax} \qquad \dfrac{\begin{matrix}\pi \\ \vdots\end{matrix}}{\Gamma_1, y : A, \Gamma_2 \vdash \Delta}}{\Gamma_1, x : A, \Gamma_2 \vdash [x/y]\Delta}\ \textsc{Cut}$$

transforms into the proof

$$\dfrac{\begin{matrix}\pi \\ \vdots\end{matrix}}{\Gamma_1, y : A, \Gamma_2 \vdash \Delta}$$

By $\textsc{Eq\_Alpha}$, we know, for any t in Δ, $t = [x/y]t$, and hence $\Delta = [x/y]\Delta$.

3.4.2 Conclusion vs. axom

The proof

$$\dfrac{\dfrac{\begin{matrix}\pi \\ \vdots\end{matrix}}{\Gamma \vdash t : A \mid \Delta} \quad \dfrac{}{x : A \vdash x : A}\ \textsc{Ax}}{\Gamma \vdash \Delta \mid [t/x]x : A}\ \textsc{Cut}$$

transforms into

$$\dfrac{\dfrac{\begin{matrix}\pi \\ \vdots\end{matrix}}{\Gamma \vdash t : A \mid \Delta}}{\Gamma \vdash \Delta \mid t : A}\ \textsc{Series of Exchanges}$$

By the definition of the substitution function we know $t = [t/x]x$.

3.5 The exchange steps

3.5.1 Conclusion vs. left-exchange (the first case)

The proof

$$\dfrac{\dfrac{\begin{matrix}\pi_1 \\ \vdots\end{matrix}}{\Gamma \vdash t : A \mid \Delta} \qquad \dfrac{\dfrac{\begin{matrix}\pi_2 \\ \vdots\end{matrix}}{\Gamma_1, x : A, y : B, \Gamma_2 \vdash \Delta'}}{\Gamma_1, y : B, x : A, \Gamma_2 \vdash \Delta'}\ \textsc{Exl}}{\Gamma_1, y : B, \Gamma, \Gamma_2 \vdash \Delta \mid [t/x]\Delta'}\ \textsc{Cut}$$

transforms into the proof

$$
\dfrac{\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\end{array}}{\Gamma \vdash t : A \mid \Delta} \qquad \dfrac{\begin{array}{c}\pi_2\\ \vdots\end{array}}{\Gamma_1, x : A, y : B, \Gamma_2 \vdash \Delta'}}{\Gamma_1, \Gamma, y : B, \Gamma_2 \vdash \Delta \mid [t/x]\Delta'}\ \textsc{Cut}}{\Gamma_1, y : B, \Gamma, \Gamma_2 \vdash \Delta \mid [t/x]\Delta'}\ \textsc{Series of Exchanges}
$$

Clearly, all terms are equivalent.

3.5.2 Conclusion vs. left-exchange (the second case)

The proof

$$
\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\end{array}}{\Gamma \vdash t : B \mid \Delta} \qquad \dfrac{\dfrac{\begin{array}{c}\pi_2\\ \vdots\end{array}}{\Gamma_1, x : A, y : B, \Gamma_2 \vdash \Delta'}}{\Gamma_1, y : B, x : A, \Gamma_2 \vdash \Delta'}\ \textsc{Exl}}{\Gamma_1, \Gamma, x : A, \Gamma_2 \vdash \Delta \mid [t/y]\Delta'}\ \textsc{Cut}
$$

transforms into the proof

$$
\dfrac{\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\end{array}}{\Gamma \vdash t : B \mid \Delta} \qquad \dfrac{\begin{array}{c}\pi_2\\ \vdots\end{array}}{\Gamma_1, x : A, y : B, \Gamma_2 \vdash \Delta'}}{\Gamma_1, x : A, \Gamma, \Gamma_2 \vdash \Delta \mid [t/y]\Delta'}\ \textsc{Cut}}{\Gamma_1, \Gamma, x : A, \Gamma_2 \vdash \Delta \mid [t/y]\Delta'}\ \textsc{Series of Exchanges}
$$

Clearly, all terms are equivalent.

3.5.3 Conclusion vs. right-exchange

The proof

$$
\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\end{array}}{\Gamma \vdash t : A \mid \Delta} \qquad \dfrac{\dfrac{\begin{array}{c}\pi_2\\ \vdots\end{array}}{\Gamma_1, x : A, \Gamma_2 \vdash \Delta_1 \mid t_1 : B \mid t_2 : C \mid \Delta'}}{\Gamma_1, x : A, \Gamma_2 \vdash \Delta_1 \mid t_2 : C \mid t_1 : B \mid \Delta'}\ \textsc{Exr}}{\Gamma_1, \Gamma, \Gamma_2 \vdash \Delta \mid [t/x]\Delta_1 \mid [t/x]t_2 : C \mid [t/x]t_1 : B \mid [t/x]\Delta'}\ \textsc{Cut}
$$

transforms into this proof

$$
\cfrac{\cfrac{\cfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{\Gamma \vdash t : A \mid \Delta} \qquad \cfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{\Gamma_1, x : A, \Gamma_2 \vdash \Delta_1 \mid t_1 : B \mid t_2 : C \mid \Delta'}}{\Gamma_1, \Gamma, \Gamma_2 \vdash \Delta \mid [t/x]\Delta_1 \mid [t/x]t_1 : B \mid [t/x]t_2 : C \mid [t/x]\Delta'}\ \textsc{Cut}}{\Gamma_1, \Gamma, \Gamma_2 \vdash [t/x]\Delta_1 \mid [t/x]t_2 : C \mid [t/x]t_1 : B \mid [t/x]\Delta'}\ \textsc{ExR}
$$

Clearly, all terms are equivalent.

3.6 Principle formula vs. principle formula

3.6.1 Tensor

The proof

$$
\cfrac{\cfrac{\cfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{\Gamma_1 \vdash t_1 : A \mid \Delta_1} \qquad \cfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{\Gamma_2 \vdash t_2 : B \mid \Delta_2}}{\Gamma_1, \Gamma_2 \vdash t_1 \otimes t_2 : A \otimes B \mid \Delta_1 \mid \Delta_2}\ \textsc{Tr} \qquad \cfrac{\cfrac{\begin{matrix}\pi_3\\ \vdots\end{matrix}}{\Gamma_3, x : A, y : B, \Gamma_4 \vdash \Delta_3}}{\Gamma_3, z : A \otimes B, \Gamma_4 \vdash \mathsf{let}\, z\, \mathsf{be}\, x \otimes y\, \mathsf{in}\, \Delta_3}\ \textsc{Tl}}{\Gamma_3, \Gamma_1, \Gamma_2, \Gamma_4 \vdash \Delta_1 \mid \Delta_2 \mid [t_1 \otimes t_2/z](\mathsf{let}\, z\, \mathsf{be}\, x \otimes y\, \mathsf{in}\, \Delta_3)}\ \textsc{Cut}
$$

is transformed into the proof

$$
\cfrac{\cfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{\Gamma_1 \vdash t_1 : A \mid \Delta_1} \qquad \cfrac{\cfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{\Gamma_2 \vdash t_2 : B \mid \Delta_2} \qquad \cfrac{\begin{matrix}\pi_3\\ \vdots\end{matrix}}{\Gamma_3, x : A, y : B, \Gamma_4 \vdash \Delta_3}}{\Gamma_3, x : A, \Gamma_2, \Gamma_4 \vdash \Delta_2 \mid [t_2/y]\Delta_3}\ \textsc{Cut}}{\Gamma_3, \Gamma_1, \Gamma_2, \Gamma_4 \vdash \Delta_1 \mid \Delta_2 \mid [t_1/x][t_2/y]\Delta_3}\ \textsc{Cut}
$$

Without loss of generality suppose $\Delta_3 = t_3 : C, \Delta_3'$. We can see that $[t_1 \otimes t_2/z](\mathsf{let}\, z\, \mathsf{be}\, x \otimes y\, \mathsf{in}\, t_3) = \mathsf{let}\, t_1 \otimes t_2\, \mathsf{be}\, x \otimes y\, \mathsf{in}\, t_3$ by the definition of substitution, and by using the $\textsc{Eq\_Beta1Tensor}$ rule we obtain $\mathsf{let}\, t_1 \otimes t_2\, \mathsf{be}\, x \otimes y\, \mathsf{in}\, t_3 = [t_1/x][t_2/y]t_3$. This argument can be repeated for any term in $[t_1 \otimes t_2/z](\mathsf{let}\, z\, \mathsf{be}\, x \otimes y\, \mathsf{in}\, \Delta_3')$, and thus, $[t_1 \otimes t_2/z](\mathsf{let}\, z\, \mathsf{be}\, x \otimes y\, \mathsf{in}\, \Delta_3) = [t_1/x][t_2/y]\Delta_3$.

Note that in the second derivation of the above transformation we first cut on B, and then A, but we could have cut on A first, and then B, but this would yeild equivalent derivations as above by using Lemma **??**.

3.6.2 Par

The proof

$$
\dfrac{
\textsc{ParR}\ \dfrac{\begin{array}{c}\pi_1\\ \vdots\\ \hline \Gamma_1 \vdash \Delta_1 \mid t_1 : A \mid t_2 : B \mid \Delta_2\end{array}}{\Gamma_1 \vdash \Delta_1 \mid t_1 \parr t_2 : A \parr B \mid \Delta_2}
\quad
\dfrac{\begin{array}{c}\pi_2\\ \vdots\\ \hline \Gamma_2, x : A \vdash \Delta_3\end{array} \qquad \begin{array}{c}\pi_3\\ \vdots\\ \hline \Gamma_3, y : B \vdash \Delta_4\end{array}}{\Gamma_2, \Gamma_3, z : A \parr B \vdash \text{let-pat}\, z\, (x \parr -)\, \Delta_3 \mid \text{let-pat}\, z\, (- \parr y)\, \Delta_4}\ \textsc{ParL}
}{
\Gamma_2, \Gamma_3, \Gamma_1 \vdash \Delta_1 \mid \Delta_2 \mid [t_1 \parr t_2/z](\text{let-pat}\, z\, (x \parr -)\, \Delta_3) \mid [t_1 \parr t_2/z](\text{let-pat}\, z\, (- \parr y)\, \Delta_4)
}\ \textsc{Cut}
$$

is transformed into the proof

$$
\dfrac{
\dfrac{
\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ \hline \Gamma_1 \vdash \Delta_1 \mid t_1 : A \mid t_2 : B \mid \Delta_2\end{array} \qquad \begin{array}{c}\pi_3\\ \vdots\\ \hline \Gamma_3, y : B \vdash \Delta_4\end{array}}{\Gamma_3, \Gamma_1 \vdash \Delta_1 \mid t_1 : A \mid \Delta_2 \mid [t_2/y]\Delta_4}\ \textsc{Cut}
\qquad
\begin{array}{c}\pi_2\\ \vdots\\ \hline \Gamma_2, x : A \vdash \Delta_3\end{array}
}{
\Gamma_2, \Gamma_3, \Gamma_1 \vdash \Delta_1 \mid \Delta_2 \mid [t_2/y]\Delta_4 \mid [t_1/x]\Delta_3
}\ \textsc{Cut}
}{
\Gamma_2, \Gamma_3, \Gamma_1 \vdash \Delta_1 \mid \Delta_2 \mid [t_1/x]\Delta_3 \mid [t_2/y]\Delta_4
}\ \textsc{Series of Exchanges}
$$

Without loss of generality consider the case when $\Delta_3 = t_3 : C_1 \mid \Delta'_3$ and $\Delta_4 = t_4 : C_2 \mid \Delta'_4$. First, $[t_1 \parr t_2/z](\text{let-pat}\, z\, (x \parr -)\, t_3) = \text{let-pat}\, (t_1 \parr t_2)\, (x \parr -)\, t_3$, and by $\textsc{Eq\_Beta1Par}$ we know $\text{let-pat}\, (t_1 \parr t_2)\, (x \parr -)\, t_3 = [t_1/x]t_3$ if $x \in \mathsf{FV}(t_3)$ or $\text{let-pat}\, (t_1 \parr t_2)\, (x \parr -)\, t_3 = t_3$ otherwise. In the latter case we can see that $t_3 = [t_1/x]t_3$, thus, in both cases $\text{let-pat}\, (t_1 \parr t_2)\, (x \parr -)\, t_3 = [t_1/x]t_3$. This argument can be repeated for any terms in Δ'_3, and hence $[t_1 \parr t_2/z](\text{let-pat}\, z\, (x \parr -)\, \Delta_3) = \text{let-pat}\, (t_1 \parr t_2)\, (x \parr -)\, \Delta_3 = [t_1/x]\Delta_3$. We can apply a similar argument for $[t_1 \parr t_2/z](\text{let-pat}\, z\, (- \parr y)\, t_4)$ and $[t_1 \parr t_2/z](\text{let-pat}\, z\, (- \parr y)\, \Delta_4)$.

Note that just as we mentioned about tensor we could have first cut on A, and then on B in the second derivation, but we would have arrived at the same result just with potentially more exchanges on the right.

3.6.3 Implication

The proof

$$
\dfrac{
\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ \hline \Gamma, x : A \vdash t : B \mid \Delta\end{array} \qquad x \notin \mathsf{FV}(\Delta)}{\Gamma \vdash \lambda x.t : A \multimap B \mid \Delta}\ \textsc{ImpR}
\quad
\dfrac{\begin{array}{c}\pi_2\\ \vdots\\ \hline \Gamma_1 \vdash t_1 : A \mid \Delta_1\end{array} \qquad \begin{array}{c}\pi_3\\ \vdots\\ \hline \Gamma_2, y : B \vdash \Delta_2\end{array}}{\Gamma_1, z : A \multimap B, \Gamma_2 \vdash \Delta_1 \mid [z\, t_1/y]\Delta_2}\ \textsc{ImpL}
}{
\Gamma_1, \Gamma, \Gamma_2 \vdash \Delta \mid [\lambda x.t/z]\Delta_1 \mid [\lambda x.t/z][z\, t_1/y]\Delta_2
}\ \textsc{Cut}
$$

transforms into the proof

$$
\dfrac{
\dfrac{
\dfrac{\begin{array}{c}\pi_2\\ \vdots\\ \hline \Gamma_1 \vdash t_1 : A \mid \Delta_1\end{array} \quad \begin{array}{c}\pi_1\\ \vdots\\ \hline \Gamma, x : A \vdash t : B \mid \Delta\end{array} \quad x \notin \mathsf{FV}(\Delta)}{\Gamma, \Gamma_1 \vdash \Delta_1 \mid [t_1/x]t : B \mid [t_1/x]\Delta}\ \textsc{Cut}
\qquad
\begin{array}{c}\pi_3\\ \vdots\\ \hline \Gamma_2, y : B \vdash \Delta_2\end{array}
}{
\Gamma_2, \Gamma, \Gamma_1 \vdash \Delta_1 \mid [t_1/x]\Delta \mid [[t_1/x]t/y]\Delta_2
}\ \textsc{Cut}
}{
\Gamma_1, \Gamma, \Gamma_2 \vdash [t_1/x]\Delta \mid \Delta_1 \mid [[t_1/x]t/y]\Delta_2
}\ \textsc{Series of Exchanges}
$$

Without loss of generality consider the case when $\Delta_2 = t_2 : C \mid \Delta'_2$. First, by hypothesis we know $x \notin \mathsf{FV}(\Delta)$, and so we know $\Delta = [t_1/x]\Delta$. We can see that $[\lambda x.t/z][z\, t_1/y]t_2 = [(\lambda x.t)\, t_1/y]t_2 = [[t_1/x]t/y]t_2$ by using the congruence rules of equality and the rule $\textsc{Eq\_BetaFun}$. This argument can be repeated for any term in $[\lambda x.t/z][z\, t_1/y]\Delta'_2$, and so $[\lambda x.t/z][z\, t_1/y]\Delta_2 = [[t_1/x]t/y]\Delta_2$. Finally, by inspecting the previous derivations we can see that $z \notin \mathsf{FV}(\Delta_1)$, and thus, $\Delta_1 = [\lambda x.t/z]\Delta_1$.

3.6.4 Tensors Unit

The proof

$$
\dfrac{\dfrac{}{\cdot \vdash * : I}\ \textsc{Ir} \qquad \dfrac{\dfrac{\begin{array}{c}\pi \\ \vdots\end{array}}{\Gamma \vdash \Delta}}{\Gamma, x : I \vdash \mathsf{let}\, x\, \mathsf{be}\, *\, \mathsf{in}\, \Delta}\ \textsc{Il}}{\Gamma \vdash [*/x](\mathsf{let}\, x\, \mathsf{be}\, *\, \mathsf{in}\, \Delta)}\ \textsc{Cut}
$$

is transformed into the proof

$$
\dfrac{\begin{array}{c}\pi \\ \vdots\end{array}}{\Gamma \vdash \Delta}
$$

Without loss of generality suppose $\Delta = t : A \mid \Delta'$. We can see that $[*/x](\mathsf{let}\, x\, \mathsf{be}\, *\, \mathsf{in}\, t) = \mathsf{let}\, *\, \mathsf{be}\, *\, \mathsf{in}\, t = t$ by the definition of substitution and the $\textsc{Eq\_EtaI}$ rule. This argument can be repeated for any term in $[*/x](\mathsf{let}\, x\, \mathsf{be} * \mathsf{in}\, \Delta')$, and hence, $[*/x](\mathsf{let}\, x\, \mathsf{be}\, *\, \mathsf{in}\, \Delta) = \Delta$.

3.6.5 Pars Unit

The proof

$$
\dfrac{\dfrac{\dfrac{\begin{array}{c}\pi \\ \vdots\end{array}}{\Gamma \vdash \Delta}}{\Gamma \vdash \circ :\perp \mid \Delta}\ \textsc{Pr} \qquad \dfrac{}{x :\perp \vdash \cdot}\ \textsc{Pl}}{\Gamma \vdash \Delta \mid [\circ/x]\cdot}\ \textsc{Cut}
$$

transforms into the proof

$$
\dfrac{\begin{array}{c}\pi \\ \vdots\end{array}}{\Gamma \vdash \Delta}
$$

Clearly, $[\circ/x]\cdot = \cdot$.

3.7 Secondary conclusion

3.7.1 Left introduction of implication

The proof

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\end{array}}{\Gamma \vdash t_1 : A \mid \Delta} \qquad \dfrac{\begin{array}{c}\pi_2\\ \vdots\end{array}}{\Gamma_1, x : B, \Gamma_2 \vdash t_2 : C \mid \Delta_2}}{\Gamma, y : A \multimap B, \Gamma_1, \Gamma_2 \vdash \Delta \mid [y\ t_1/x]t_2 : C \mid [y\ t_1/x]\Delta_2}\ \textsc{Impl} \qquad \dfrac{\begin{array}{c}\pi_3\\ \vdots\end{array}}{\Gamma_3, z : C, \Gamma_4 \vdash \Delta_3}}{\Gamma_3, \Gamma, y : A \multimap B, \Gamma_1, \Gamma_2, \Gamma_4 \vdash \Delta \mid [y\ t_1/x]\Delta_2 \mid [[y\ t_1/x]t_2/z]\Delta_3}\ \textsc{Cut}$$

transforms into the proof

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\end{array}}{\Gamma \vdash t_1 : A \mid \Delta} \qquad \dfrac{\dfrac{\begin{array}{c}\pi_2\\ \vdots\end{array}}{\Gamma_1, x : B, \Gamma_2 \vdash t_2 : C \mid \Delta_2} \qquad \dfrac{\begin{array}{c}\pi_3\\ \vdots\end{array}}{\Gamma_3, z : C, \Gamma_4 \vdash \Delta_3}}{\Gamma_3, \Gamma_1, x : B, \Gamma_2, \Gamma_4 \vdash \Delta_2 \mid [t_2/z]\Delta_3}\ \textsc{Cut}}{\Gamma, y : A \multimap B, \Gamma_3, \Gamma_1, \Gamma_2, \Gamma_4 \vdash \Delta \mid [y\ t_1/x]\Delta_2 \mid [y\ t_1/x][t_2/z]\Delta_3}\ \textsc{Impl}}{\Gamma_3, \Gamma, y : A \multimap B, \Gamma_1, \Gamma_2, \Gamma_4 \vdash \Delta \mid [y\ t_1/x]\Delta_2 \mid [y\ t_1/x][t_2/z]\Delta_3}\ \textsc{Series of Exchanges}$$

This case is similar to Section **??**. Thus, we can prove that $[y\ t_1/x][t_2/z]\Delta_3 = [[y\ t_1/x]t_2/z]\Delta_3$ by Lemma **??** and the fact that $x \notin \mathsf{FV}(\Delta_3)$.

3.7.2 Left introduction of exchange

The proof

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\end{array}}{\Gamma, y : B, x : A, \Gamma' \vdash t : C \mid \Delta}}{\Gamma, x : A, y : B, \Gamma' \vdash t : C \mid \Delta}\ \textsc{Exl} \qquad \dfrac{\begin{array}{c}\pi_2\\ \vdots\end{array}}{\Gamma_1, z : C, \Gamma_2 \vdash \Delta_2}}{\Gamma_1, \Gamma, x : A, y : B, \Gamma', \Gamma_2 \vdash \Delta \mid [t/z]\Delta_2}\ \textsc{Cut}$$

transforms into the proof

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\end{array}}{\Gamma, y : B, x : A, \Gamma' \vdash t : C \mid \Delta} \qquad \dfrac{\begin{array}{c}\pi_2\\ \vdots\end{array}}{\Gamma_1, z : C, \Gamma_2 \vdash \Delta_2}}{\Gamma_1, \Gamma, y : B, x : A, \Gamma', \Gamma_2 \vdash \Delta \mid [t/z]\Delta_2}\ \textsc{Cut}}{\Gamma_1, \Gamma, x : A, y : B, \Gamma', \Gamma_2 \vdash \Delta \mid [t/z]\Delta_2}\ \textsc{Exl}$$

Clearly, all terms are equivalent.

3.7.3 Left introduction of tensor

The proof

$$\dfrac{\dfrac{\dfrac{\overset{\pi_1}{\vdots}}{\Gamma, x : A, y : B \vdash t : C \mid \Delta}}{\Gamma, z : A \otimes B \vdash \mathsf{let}\, z\, \mathsf{be}\, x \otimes y\, \mathsf{in}\, t : C \mid \mathsf{let}\, z\, \mathsf{be}\, x \otimes y\, \mathsf{in}\, \Delta}\,\textsc{Tl} \qquad \dfrac{\overset{\pi_2}{\vdots}}{\Gamma_1, w : C, \Gamma_2 \vdash \Delta_2}}{\Gamma_1, \Gamma, z : A \otimes B, \Gamma_2 \vdash \mathsf{let}\, z\, \mathsf{be}\, x \otimes y\, \mathsf{in}\, \Delta \mid [\mathsf{let}\, z\, \mathsf{be}\, x \otimes y\, \mathsf{in}\, t/w]\Delta_2}\,\textsc{Cut}$$

transforms into the proof

$$\dfrac{\dfrac{\dfrac{\overset{\pi_1}{\vdots}}{\Gamma, x : A, y : B \vdash t : C \mid \Delta} \qquad \dfrac{\overset{\pi_2}{\vdots}}{\Gamma_1, w : C, \Gamma_2 \vdash \Delta_2}}{\Gamma_1, \Gamma, x : A, y : B, \Gamma_2 \vdash \Delta \mid [t/w]\Delta_2}\,\textsc{Cut}}{\Gamma_1, \Gamma, z : A \otimes B, \Gamma_2 \vdash \mathsf{let}\, z\, \mathsf{be}\, x \otimes y\, \mathsf{in}\, \Delta \mid \mathsf{let}\, z\, \mathsf{be}\, x \otimes y\, \mathsf{in}\, ([t/w]\Delta_2)}\,\textsc{Tl}$$

It suffices to show that $\mathsf{let}\, z\, \mathsf{be}\, x \otimes y\, \mathsf{in}\, ([t/w]\Delta_2) = [\mathsf{let}\, z\, \mathsf{be}\, x \otimes y\, \mathsf{in}\, t/w]\Delta_2$. This is a simple consequence of the rule Eq_NatTensor.

3.7.4 Left introduction of Par

The proof

$$\dfrac{\dfrac{\dfrac{\overset{\pi_1}{\vdots}}{\Gamma, x : A \vdash \Delta} \qquad \dfrac{\overset{\pi_2}{\vdots}}{\Gamma', y : B \vdash t' : C \mid \Delta'}}{\Gamma, \Gamma', z : A \parr B \vdash \text{let-pat}\, z\, (x \parr -)\, \Delta \mid \text{let-pat}\, z\, (- \parr y)\, t' : C \mid \text{let-pat}\, z\, (- \parr y)\, \Delta'}\,\textsc{ParL} \qquad \dfrac{\overset{\pi_3}{\vdots}}{\Gamma_1, w : C, \Gamma_2 \vdash \Delta_2}}{\Gamma_1, \Gamma, \Gamma', z : A \parr B, \Gamma_2 \vdash \text{let-pat}\, z\, (x \parr -)\, \Delta \mid \text{let-pat}\, z\, (- \parr y)\, \Delta' \mid [\text{let-pat}\, z\, (- \parr y)\, t'/w]\Delta_2}\,\textsc{Cut}$$

is transformed into the proof

$$\dfrac{\dfrac{\dfrac{\overset{\pi_1}{\vdots}}{\Gamma, x : A \vdash \Delta} \qquad \dfrac{\dfrac{\overset{\pi_2}{\vdots}}{\Gamma', y : B \vdash t' : C \mid \Delta'} \qquad \dfrac{\overset{\pi_3}{\vdots}}{\Gamma_1, w : C, \Gamma_2 \vdash \Delta_2}}{\Gamma_1, \Gamma', y : B, \Gamma_2 \vdash \Delta' \mid [t'/w]\Delta_2}\,\textsc{Cut}}{\Gamma, \Gamma_1, \Gamma', \Gamma_2, z : A \parr B \vdash \text{let-pat}\, z\, (x \parr -)\, \Delta \mid \text{let-pat}\, z\, (- \parr y)\, \Delta' \mid \text{let-pat}\, z\, (- \parr y)\, [t'/w]\Delta_2}\,\textsc{ParL}}{\Gamma_1, \Gamma, \Gamma', z : A \parr B, \Gamma_2 \vdash \text{let-pat}\, z\, (x \parr -)\, \Delta \mid \text{let-pat}\, z\, (- \parr y)\, \Delta' \mid \text{let-pat}\, z\, (- \parr y)\, [t'/w]\Delta_2}\,\textsc{Series of Exchanges}$$

It suffices to show that $\text{let-pat}\, z\, (- \parr y)\, [t'/w]\Delta_2 = [\text{let-pat}\, z\, (- \parr y)\, t'/w]\Delta_2$. This follows from the rule Eq_Nat2Par.

3.7.5 Left introduction of tensor unit

The proof

$$\dfrac{\dfrac{\dfrac{\pi_1}{\Gamma \vdash t : C \mid \Delta}}{\Gamma, x : I \vdash t : C \mid \Delta}\ \textsc{Il} \qquad \dfrac{\pi_2}{\Gamma_1, w : C, \Gamma_2 \vdash \Delta_1}}{\Gamma_1, \Gamma, x : I, \Gamma_2 \vdash \Delta \mid [t/w]\Delta_1}\ \textsc{Cut}$$

is transformed into the following:

$$\dfrac{\dfrac{\dfrac{\dfrac{\pi_1}{\Gamma \vdash t : C \mid \Delta} \qquad \dfrac{\pi_2}{\Gamma_1, w : C, \Gamma_2 \vdash \Delta_1}}{\Gamma_1, \Gamma, \Gamma_2 \vdash \Delta \mid [t/w]\Delta_1}\ \textsc{Cut}}{\Gamma_1, \Gamma, \Gamma_2, x : I \vdash \Delta \mid [t/w]\Delta_1}\ \textsc{Il}}{\Gamma_1, \Gamma, x : I, \Gamma_2 \vdash \Delta \mid [t/w]\Delta_1}\ \textsc{Series of Exchanges}$$

Clearly, all terms are equivalent. Note that we do not give a case for secondary conclusion of the left introduction of par's unit, because it can only be introduced given an empty right context, and thus there is no cut formula.

3.8 Secondary hypothesis

3.8.1 Left introduction of tensor

The proof

$$\dfrac{\dfrac{\pi_1}{\Gamma \vdash t : A \mid \Delta} \qquad \dfrac{\dfrac{\pi_2}{\Gamma_1, x : A, \Gamma_2, y : B, z : C, \Gamma_3 \vdash t_1 : D \mid \Delta_1}}{\Gamma_1, x : A, \Gamma_2, w : B \otimes C, \Gamma_3 \vdash \mathsf{let}\ w\ \mathsf{be}\ y \otimes z\ \mathsf{in}\ t_1 : D \mid \mathsf{let}\ w\ \mathsf{be}\ y \otimes z\ \mathsf{in}\ \Delta_1}\ \textsc{Tl}}{\Gamma_1, \Gamma, \Gamma_2, w : B \otimes C, \Gamma_3 \vdash \Delta \mid [t/x](\mathsf{let}\ w\ \mathsf{be}\ y \otimes z\ \mathsf{in}\ t_1) : D \mid [t/x](\mathsf{let}\ w\ \mathsf{be}\ y \otimes z\ \mathsf{in}\ \Delta_1)}\ \textsc{Cut}$$

transforms into the proof

$$\dfrac{\dfrac{\dfrac{\pi_1}{\Gamma \vdash t : A \mid \Delta} \qquad \dfrac{\pi_2}{\Gamma_1, x : A, \Gamma_2, y : B, z : C, \Gamma_3 \vdash t_1 : D \mid \Delta_1}}{\Gamma_1, \Gamma, \Gamma_2, y : B, z : C, \Gamma_3 \vdash \Delta \mid [t/x]t_1 : D \mid [t/x]\Delta_1}\ \textsc{Cut}}{\Gamma_1, \Gamma, \Gamma_2, w : B \otimes C, \Gamma_3 \vdash \mathsf{let}\ w\ \mathsf{be}\ x \otimes y\ \mathsf{in}\ \Delta \mid \mathsf{let}\ w\ \mathsf{be}\ x \otimes y\ \mathsf{in}\ [t/x]t_1 : D \mid \mathsf{let}\ w\ \mathsf{be}\ x \otimes y\ \mathsf{in}\ [t/x]\Delta_1}\ \textsc{Tl}$$

First, we can see by inspection of the previous derivations that $x, y \notin \mathsf{FV}(\Delta)$, thus, by using similar reasoning as above we can use the EtaTensor rule to obtain $\mathsf{let}\, w \,\mathsf{be}\, x \otimes y \,\mathsf{in}\, \Delta = \Delta$. It is a well-known property of substitution that $[t/x](\mathsf{let}\, w \,\mathsf{be}\, x \otimes y \,\mathsf{in}\, t_1) = \mathsf{let}\, [t/x]w \,\mathsf{be}\, x \otimes y \,\mathsf{in}\, [t/x]t_1 = \mathsf{let}\, w \,\mathsf{be}\, x \otimes y \,\mathsf{in}\, [t/x]t_1$.

3.8.2 Right introduction of tensor (first case)

The proof

$$\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{\Gamma \vdash t : A \mid \Delta} \qquad \dfrac{\dfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{\Gamma_1, x : A, \Gamma_2 \vdash t_1 : B \mid \Delta_1} \qquad \dfrac{\begin{matrix}\pi_3\\ \vdots\end{matrix}}{\Gamma_3 \vdash t_2 : C \mid \Delta_2}}{\Gamma_1, x : A, \Gamma_2, \Gamma_3 \vdash t_1 \otimes t_2 : B \otimes C \mid \Delta_1 \mid \Delta_2}\,\textsc{Tr}}{\Gamma_1, \Gamma, \Gamma_2, \Gamma_3 \vdash \Delta \mid [t/x](t_1 \otimes t_2) : B \otimes C \mid [t/x]\Delta_1 \mid [t/x]\Delta_2}\,\textsc{Cut}$$

transforms into the proof

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{\Gamma \vdash t : A \mid \Delta} \qquad \dfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{\Gamma_1, x : A, \Gamma_2 \vdash t_1 : B \mid \Delta_1}}{\Gamma_1, \Gamma, \Gamma_2 \vdash \Delta \mid [t/x]t_1 : B \mid [t/x]\Delta_1}\,\textsc{Cut} \qquad \dfrac{\begin{matrix}\pi_3\\ \vdots\end{matrix}}{\Gamma_3 \vdash t_2 : C \mid \Delta_2}}{\Gamma_1, \Gamma, \Gamma_2, \Gamma_3 \vdash [t/x]t_1 \otimes t_2 : B \otimes C \mid \Delta \mid [t/x]\Delta_1 \mid \Delta_2}\,\textsc{Tr}}{\Gamma_1, \Gamma, \Gamma_2, \Gamma_3 \vdash \Delta \mid ([t/x]t_1) \otimes t_2 : B \otimes C \mid [t/x]\Delta_1 \mid \Delta_2}\,\textsc{Series of Exchanges}$$

By inspection of the previous derivations we can see that $x \notin \mathsf{FV}(t_2)$ and $x \notin \mathsf{FV}(\Delta_2)$. Thus, $[t/x]\Delta_2 = \Delta_2$ and $[t/x](t_1 \otimes t_2) = ([t/x]t_1) \otimes ([t/x]t_2) = ([t/x]t_1) \otimes t_2$.

3.8.3 Right introduction of tensor (second case)

The proof

$$\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{\Gamma \vdash t : A \mid \Delta} \qquad \dfrac{\dfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{\Gamma_1 \vdash t_1 : B \mid \Delta_1} \qquad \dfrac{\begin{matrix}\pi_3\\ \vdots\end{matrix}}{\Gamma_2, x : A, \Gamma_3 \vdash t_2 : C \mid \Delta_2}}{\Gamma_1, \Gamma_2, x : A, \Gamma_3 \vdash t_1 \otimes t_2 : B \otimes C \mid \Delta_1 \mid \Delta_2}\,\textsc{Tr}}{\Gamma_1, \Gamma, \Gamma_2, \Gamma_3 \vdash \Delta \mid [t/x](t_1 \otimes t_2) : B \otimes C \mid [t/x]\Delta_1 \mid [t/x]\Delta_2}\,\textsc{Cut}$$

transforms into the proof

$$
\dfrac{\dfrac{\dfrac{\pi_2}{\Gamma_1 \vdash t_1 : B \mid \Delta_1} \qquad \dfrac{\dfrac{\pi_1}{\Gamma \vdash t : A \mid \Delta} \qquad \dfrac{\pi_3}{\Gamma_2, x : A, \Gamma_3 \vdash t_2 : C \mid \Delta_2}}{\Gamma_2, \Gamma, \Gamma_3 \vdash \Delta \mid [t/x]t_2 : C \mid [t/x]\Delta_2}\ \textsc{Cut}}{\Gamma_1, \Gamma_2, \Gamma, \Gamma_3 \vdash t_1 \otimes ([t/x]t_2) : B \otimes C \mid \Delta_1 \mid \Delta \mid [t/x]\Delta_2}\ \textsc{Tr}}{\Gamma_1, \Gamma, \Gamma_2, \Gamma_3 \vdash \Delta \mid t_1 \otimes ([t/x]t_2) : B \otimes C \mid \Delta_1 \mid [t/x]\Delta_2}\ \textsc{Series of Exchanges}
$$

This case is similar to the previous case.

3.8.4 Right introduction of par

The proof

$$
\dfrac{\dfrac{\pi_1}{\Gamma \vdash t : A \mid \Delta} \qquad \dfrac{\dfrac{\pi_2}{\Gamma_1, x : A, \Gamma_2 \vdash \Delta_1 \mid t_1 : B \mid t_2 : C \mid \Delta_2}}{\Gamma_1, x : A, \Gamma_2 \vdash \Delta_1 \mid t_1 \parr t_2 : B \parr C \mid \Delta_2}\ \textsc{ParR}}{\Gamma_1, \Gamma, \Gamma_2 \vdash \Delta \mid [t/x]\Delta_1 \mid [t/x](t_1 \parr t_2) : B \parr C \mid [t/x]\Delta_2}\ \textsc{Cut}
$$

transforms into the proof

$$
\dfrac{\dfrac{\dfrac{\pi_1}{\Gamma \vdash t : A \mid \Delta} \qquad \dfrac{\pi_2}{\Gamma_1, x : A, \Gamma_2 \vdash \Delta_1 \mid t_1 : B \mid t_2 : C \mid \Delta_2}}{\Gamma_1, \Gamma, \Gamma_2 \vdash \Delta \mid [t/x]\Delta_1 \mid [t/x]t_1 : B \mid [t/x]t_2 : C \mid [t/x]\Delta_2}\ \textsc{Cut}}{\Gamma_1, \Gamma, \Gamma_2 \vdash \Delta \mid [t/x]\Delta_1 \mid [t/x]t_1 \parr [t/x]t_2 : B \parr C \mid [t/x]\Delta_2}\ \textsc{ParL}
$$

Clearly, $[t/x](t_1 \parr t_2) = ([t/x]t_1) \parr [t/x]t_2$.

3.8.5 Left introduction of par (first case)

The proof

$$\dfrac{\dfrac{\pi_1}{\Gamma \vdash t : A \mid \Delta} \quad \dfrac{\dfrac{\pi_2}{\Gamma_1, x : A, \Gamma_2, y : B \vdash \Delta_1} \quad \dfrac{\pi_3}{\Gamma_3, z : C \vdash \Delta_2}}{\Gamma_1, x : A, \Gamma_2, \Gamma_3, w : B \parr C \vdash \text{let-pat}\, w\, (y \parr -)\, \Delta_1 \mid \text{let-pat}\, w\, (- \parr z)\, \Delta_2} \text{ParL}}{\Gamma_1, \Gamma, \Gamma_2, \Gamma_3, w : B \parr C \vdash \Delta \mid [t/x](\text{let-pat}\, w\, (y \parr -)\, \Delta_1) \mid [t/x](\text{let-pat}\, w\, (- \parr z)\, \Delta_2)} \text{Cut}$$

transforms into the proof

$$\dfrac{\dfrac{\dfrac{\pi_1}{\Gamma \vdash t : A \mid \Delta} \quad \dfrac{\pi_2}{\Gamma_1, x : A, \Gamma_2, y : B \vdash \Delta_1}}{\Gamma_1, \Gamma, \Gamma_2, y : B \vdash \Delta \mid [t/x]\Delta_1} \text{Cut} \quad \dfrac{\pi_3}{\Gamma_3, z : C \vdash \Delta_2}}{\Gamma_1, \Gamma, \Gamma_2, \Gamma_3, w : B \parr C \vdash \text{let-pat}\, w\, (y \parr -)\, \Delta \mid \text{let-pat}\, w\, (y \parr -)\, [t/x]\Delta_1 \mid \text{let-pat}\, w\, (- \parr z)\, \Delta_2} \text{ParL}$$

First, by inspection of the previous proofs we can see that $x \notin \mathsf{FV}(\Delta)$ and $x \notin \mathsf{FV}(\Delta_2)$. Thus, $\text{let-pat}\, w\, (y \parr -)\, \Delta = \Delta$, and $[t/x](\text{let-pat}\, w\, (- \parr z)\, \Delta_2) = \text{let-pat}\, w\, (- \parr z)\, \Delta_2$. It suffices to show that $[t/x](\text{let-pat}\, w\, (y \parr -)\, \Delta_1) = \text{let-pat}\, w\, (y \parr -)\, [t/x]\Delta_1$ but this easily follows from a simple distributing the substitution into the let-pat, and then simplifying using the fact that $w \neq x$.

3.8.6 Left introduction of par (second case)

The proof

$$\dfrac{\dfrac{\pi_1}{\Gamma \vdash t : A \mid \Delta} \quad \dfrac{\dfrac{\pi_2}{\Gamma_1, y : B \vdash \Delta_1} \quad \dfrac{\pi_3}{\Gamma_2, x : A, \Gamma_3, z : C \vdash \Delta_2}}{\Gamma_1, \Gamma_2, x : A, \Gamma_3, w : B \parr C \vdash \text{let-pat}\, w\, (y \parr -)\, \Delta_1 \mid \text{let-pat}\, w\, (- \parr z)\, \Delta_2} \text{ParL}}{\Gamma_1, \Gamma_2, \Gamma, \Gamma_3, w : B \parr C \vdash \Delta \mid [t/x](\text{let-pat}\, w\, (y \parr -)\, \Delta_1) \mid [t/x](\text{let-pat}\, w\, (- \parr z)\, \Delta_2)} \text{Cut}$$

transforms into the proof

$$
\dfrac{\dfrac{\begin{array}{c}\pi_2\\ \vdots\end{array}}{\Gamma_1, y : B \vdash \Delta_1} \qquad \dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\end{array}}{\Gamma \vdash t : A \mid \Delta} \quad \dfrac{\begin{array}{c}\pi_3\\ \vdots\end{array}}{\Gamma_2, x : A, \Gamma_3, z : C \vdash \Delta_2}}{\Gamma_2, \Gamma, \Gamma_3, z : C \vdash \Delta \mid [t/x]\Delta_2}\,\textsc{Cut}}{\Gamma_1, \Gamma_2, \Gamma, \Gamma_3, w : B \parr C \vdash \text{let-pat}\, w\, (y \parr -)\, \Delta_1 \mid \text{let-pat}\, w\, (- \parr z)\, \Delta \mid \text{let-pat}\, w\, (- \parr z)\, [t/x]\Delta_2}\,\textsc{Parl}
$$

Similar to the previous case.

3.8.7 Left introduction of implication (first case)

The proof

$$
\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\end{array}}{\Gamma \vdash t : A \mid \Delta} \qquad \dfrac{\dfrac{\begin{array}{c}\pi_2\\ \vdots\end{array}}{\Gamma_1, x : A, \Gamma_2 \vdash t_1 : B \mid \Delta_1} \quad \dfrac{\begin{array}{c}\pi_3\\ \vdots\end{array}}{\Gamma_3, y : C \vdash \Delta_2}}{\Gamma_1, x : A, \Gamma_2, \Gamma_3, z : B \multimap C \vdash \Delta_1 \mid [z\, t_1/y]\Delta_2}\,\textsc{Impl}}{\Gamma_1, \Gamma, \Gamma_2, \Gamma_3, z : B \multimap C \vdash \Delta \mid [t/x]\Delta_1 \mid [t/x][z\, t_1/y]\Delta_2}\,\textsc{Cut}
$$

transforms into the proof

$$
\dfrac{\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\end{array}}{\Gamma \vdash t : A \mid \Delta} \quad \dfrac{\begin{array}{c}\pi_2\\ \vdots\end{array}}{\Gamma_1, x : A, \Gamma_2 \vdash t_1 : B \mid \Delta_1}}{\Gamma_1, \Gamma, \Gamma_2 \vdash \Delta \mid [t/x]t_1 : B \mid [t/x]\Delta_1}\,\textsc{Cut} \qquad \dfrac{\begin{array}{c}\pi_3\\ \vdots\end{array}}{\Gamma_3, y : C \vdash \Delta_2}}{\Gamma_1, \Gamma, \Gamma_2, \Gamma_3, z : B \multimap C \vdash \Delta \mid [t/x]\Delta_1 \mid [z\, ([t/x]t_1)/y]\Delta_2}\,\textsc{Impl}
$$

By inspection of the above derivations we can see that $x \notin \mathsf{FV}(\Delta_2)$, and hence, by this fact and substitution distribution (Lemma **??**) we know $[t/x][z\, t_1/y]\Delta_2 = [([t/x]z)\, ([t/x]t_1)/y][t/x]\Delta_2 = [z\, ([t/x]t_1)/y]\Delta_2$.

3.8.8 Left introduction of implication (second case)

The proof

$$
\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\end{array}}{\Gamma \vdash t : A \mid \Delta} \qquad \dfrac{\dfrac{\begin{array}{c}\pi_2\\ \vdots\end{array}}{\Gamma_1 \vdash t_1 : B \mid \Delta_1} \quad \dfrac{\begin{array}{c}\pi_3\\ \vdots\end{array}}{\Gamma_2, x : A, \Gamma_3, y : C \vdash \Delta_2}}{\Gamma_1, \Gamma_2, x : A, \Gamma_3, z : B \multimap C \vdash \Delta_1 \mid [z\, t_1/y]\Delta_2}\,\textsc{Impl}}{\Gamma_1, \Gamma_2, \Gamma, \Gamma_3, z : B \multimap C \vdash \Delta \mid [t/x]\Delta_1 \mid [t/x][z\, t_1/y]\Delta_2}\,\textsc{Cut}
$$

transforms into the proof

$$
\dfrac{\dfrac{\begin{array}{c}\pi_2\\ \vdots\\ \hline \Gamma_1 \vdash t_1 : B \mid \Delta_1\end{array} \qquad \dfrac{\begin{array}{c}\pi_1\\ \vdots\\ \hline \Gamma \vdash t : A \mid \Delta\end{array} \qquad \begin{array}{c}\pi_3\\ \vdots\\ \hline \Gamma_2, x : A, \Gamma_3, y : C \vdash \Delta_2\end{array}}{\Gamma_2, \Gamma, \Gamma_3, y : C \vdash \Delta \mid [t/x]\Delta_2}\ \textsc{Cut}}{\Gamma_1, \Gamma_2, \Gamma, \Gamma_3, z : B \multimap C \vdash \Delta_1 \mid [z\ t_1/y]\Delta \mid [z\ t_1/y][t/x]\Delta_2}\ \textsc{Impl}}{\Gamma_1, \Gamma_2, \Gamma, \Gamma_3, z : B \multimap C \vdash [z\ t_1/y]\Delta \mid \Delta_1 \mid [z\ t_1/y][t/x]\Delta_2}\ \textsc{Series of Exchanges}
$$

By inspection of the above proofs we can see that $y \notin \mathsf{FV}(\Delta)$. Thus, $[z\ t_1/y]\Delta = \Delta$. The same can be said for the variable x and context Δ_1, and hence, $[t/x]\Delta_1 = \Delta_1$. Finally, by inspection of the above proofs $x \notin \mathsf{FV}(t_1)$ and so by substitution distribution (Lemma **??**) we know $[t/x][z\ t_1/y]\Delta_2 = [z\ t_1/y][t/x]\Delta_2$.

3.8.9 Left introduction of implication (second case)

The proof

$$
\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ \hline \Gamma \vdash t : A \mid \Delta\end{array} \qquad \dfrac{\begin{array}{c}\pi_2\\ \vdots\\ \hline \Gamma_1 \vdash t_1 : B \mid \Delta_1\end{array} \qquad \begin{array}{c}\pi_3\\ \vdots\\ \hline \Gamma_2, y : C, \Gamma_3, x : A \vdash \Delta_2\end{array}}{\Gamma_1, \Gamma_2, z : B \multimap C, \Gamma_3, x : A \vdash \Delta_1 \mid [z\ t_1/y]\Delta_2}\ \textsc{Impl}}{\Gamma_1, \Gamma_2, z : B \multimap C, \Gamma_3, \Gamma \vdash \Delta \mid [t/x]\Delta_1 \mid [t/x][z\ t_1/y]\Delta_2}\ \textsc{Cut}
$$

transforms into the proof

$$
\dfrac{\dfrac{\begin{array}{c}\pi_2\\ \vdots\\ \hline \Gamma_1 \vdash t_1 : B \mid \Delta_1\end{array} \qquad \dfrac{\begin{array}{c}\pi_1\\ \vdots\\ \hline \Gamma \vdash t : A \mid \Delta\end{array} \qquad \begin{array}{c}\pi_3\\ \vdots\\ \hline \Gamma_2, y : C, \Gamma_3, x : A \vdash \Delta_2\end{array}}{\Gamma_2, y : C, \Gamma_3, \Gamma \vdash \Delta \mid [t/x]\Delta_2}\ \textsc{Cut}}{\Gamma_1, \Gamma_2, z : B \multimap C, \Gamma_3, \Gamma \vdash \Delta_1 \mid [z\ t_1/y]\Delta \mid [z\ t_1/y][t/x]\Delta_2}\ \textsc{Impl}}{\Gamma_1, \Gamma_2, z : B \multimap C, \Gamma_3, \Gamma \vdash [z\ t_1/y]\Delta \mid \Delta_1 \mid [z\ t_1/y][t/x]\Delta_2}\ \textsc{Series of Exchanges}
$$

Similar to the previous case.

3.8.10 Right introduction of implication

The proof

$$\dfrac{\dfrac{\pi_1}{\Gamma \vdash t : A \mid \Delta} \qquad \dfrac{\dfrac{\pi_2}{\Gamma_1, x : A, \Gamma_2, y : B \vdash t_1 : C \mid \Delta_1} \qquad y \notin \mathsf{FV}(\Delta_1)}{\Gamma_1, x : A, \Gamma_2 \vdash \lambda y.t_1 : B \multimap C \mid \Delta_1}\ \textsc{ImpR}}{\Gamma_1, \Gamma, \Gamma_2 \vdash \Delta \mid [t/x](\lambda y.t_1) : B \multimap C \mid [t/x]\Delta_1}\ \textsc{Cut}$$

transforms into the proof

$$\dfrac{\dfrac{\dfrac{\pi_1}{\Gamma \vdash t : A \mid \Delta} \qquad \dfrac{\pi_2}{\Gamma_1, x : A, \Gamma_2, y : B \vdash t_1 : C \mid \Delta_1}}{\Gamma_1, \Gamma, \Gamma_2, y : B \vdash \Delta \mid [t/x]t_1 : C \mid [t/x]\Delta_1}\ \textsc{Cut}}{\Gamma_1, \Gamma, \Gamma_2 \vdash \Delta \mid \lambda y.[t/x]t_1 : B \multimap C \mid [t/x]\Delta_1}\ \textsc{ImpR}$$

Clearly, $[t/x](\lambda y.t_1) = \lambda y.[t/x]t_1$.

3.8.11 Left introduction of tensor unit

The proof

$$\dfrac{\dfrac{\pi_1}{\Gamma \vdash t : A \mid \Delta} \qquad \dfrac{\dfrac{\pi_2}{\Gamma_1, x : A, \Gamma_2 \vdash \Delta_1}}{\Gamma_1, x : A, \Gamma_2, y : I \vdash \mathsf{let}\, y\, \mathsf{be}\, *\, \mathsf{in}\, \Delta_1}\ \textsc{Il}}{\Gamma_1, \Gamma, \Gamma_2, y : I \vdash \Delta \mid [t/x](\mathsf{let}\, y\, \mathsf{be}\, *\, \mathsf{in}\, \Delta_1)}\ \textsc{Cut}$$

transforms into the proof

$$\dfrac{\dfrac{\dfrac{\pi_1}{\Gamma \vdash t : A \mid \Delta} \qquad \dfrac{\pi_2}{\Gamma_1, x : A, \Gamma_2 \vdash \Delta_1}}{\Gamma_1, \Gamma, \Gamma_2 \vdash \Delta \mid [t/x]\Delta_1}\ \textsc{Cut}}{\Gamma_1, \Gamma, \Gamma_2, y : I \vdash \mathsf{let}\, y\, \mathsf{be}\, *\, \mathsf{in}\, \Delta \mid \mathsf{let}\, y\, \mathsf{be}\, *\, \mathsf{in}\, [t/x]\Delta_1}\ \textsc{Il}$$

It suffices to show that $\Delta = \mathsf{let}\, y\, \mathsf{be} * \mathsf{in}\, \Delta$ and $[t/x](\mathsf{let}\, y\, \mathsf{be} * \mathsf{in}\, \Delta_1) = \mathsf{let}\, y\, \mathsf{be} * \mathsf{in}\, [t/x]\Delta_1$. Without loss of generality suppose $\Delta = t : B, \Delta'$. We know that it must be the case that $y \notin \mathsf{FV}(t)$, and we know that $[y/z]t = t$ when $z \notin \mathsf{FV}(t)$. Then by $\textsc{Eq\_Eta2I}$ we have $t = \mathsf{let}\, y\, \mathsf{be}\, *\, \mathsf{in}\, t$. This argument can be repeated for any other term in Δ'. Thus, $\Delta = \mathsf{let}\, y\, \mathsf{be}\, *\, \mathsf{in}\, \Delta$. It is easy to see that $[t/x](\mathsf{let}\, y\, \mathsf{be}\, *\, \mathsf{in}\, \Delta_1) = \mathsf{let}\, y\, \mathsf{be}\, *\, \mathsf{in}\, [t/x]\Delta_1$ using the rule $\textsc{Eq\_NatI}$.

3.8.12 Right introduction of par unit

The proof

$$
\dfrac{\dfrac{\pi_1}{\vdots}\atop \Gamma \vdash t : A \mid \Delta \qquad \dfrac{\dfrac{\pi_2}{\vdots}\atop \Gamma_1, x : A, \Gamma_2 \vdash \Delta_1}{\Gamma_1, x : A, \Gamma_2 \vdash \circ :\perp\mid \Delta_1}\ \textsc{Pr}}{\Gamma_1, \Gamma, \Gamma_2 \vdash \Delta \mid [t/x]\circ :\perp\mid [t/x]\Delta_1}\ \textsc{Cut}
$$

transforms into the proof

$$
\dfrac{\dfrac{\dfrac{\dfrac{\pi_1}{\vdots}\atop \Gamma \vdash t : A \mid \Delta \qquad \dfrac{\pi_2}{\vdots}\atop \Gamma_1, x : A, \Gamma_2 \vdash \Delta_1}{\Gamma_1, \Gamma, \Gamma_2 \vdash \Delta \mid [t/x]\Delta_1}\ \textsc{Cut}}{\Gamma_1, \Gamma, \Gamma_2 \vdash \circ :\perp\mid \Delta \mid [t/x]\Delta_1}\ \textsc{Pr}}{\Gamma_1, \Gamma, \Gamma_2 \vdash \Delta \mid \circ :\perp\mid [t/x]\Delta_1}\ \textsc{Series of Exchanges}
$$

Clearly, $[t/x]\circ = \circ$.

3.8.13 Left introduction of exchange

The proof

$$
\dfrac{\dfrac{\pi_1}{\vdots}\atop \Gamma \vdash t : A \mid \Delta \qquad \dfrac{\dfrac{\pi_2}{\vdots}\atop \Gamma_1, x : A, \Gamma_2, w : B, y : C, \Gamma_3 \vdash \Delta_1}{\Gamma_1, x : A, \Gamma_2, y : C, w : B, \Gamma_3 \vdash \Delta_1}\ \textsc{Exl}}{\Gamma_1, \Gamma, \Gamma_2, y : C, w : B, \Gamma_3 \vdash \Delta \mid [t/x]\Delta_1}\ \textsc{Cut}
$$

tranforms into the proof

$$
\dfrac{\dfrac{\dfrac{\pi_1}{\vdots}\atop \Gamma \vdash t : A \mid \Delta \qquad \dfrac{\pi_2}{\vdots}\atop \Gamma_1, x : A, \Gamma_2, w : B, y : C, \Gamma_3 \vdash \Delta_1}{\Gamma_1, \Gamma, \Gamma_2, w : B, y : C, \Gamma_3 \vdash \Delta \mid [t/x]\Delta_1}\ \textsc{Cut}}{\Gamma_1, \Gamma, \Gamma_2, y : C, w : B, \Gamma_3 \vdash \Delta \mid [t/x]\Delta_1}\ \textsc{Exl}
$$

Clearly, all terms are equivalent.

3.8.14 Right introduction of exchange

The proof

$$
\dfrac{\dfrac{\pi_1}{\Gamma \vdash t : A \mid \Delta} \qquad \dfrac{\dfrac{\pi_2}{\Gamma_1, x : A, \Gamma_2 \vdash \Delta_1 \mid t_1 : B \mid t_2 : C \mid \Delta_2}}{\Gamma_1, x : A, \Gamma_2 \vdash \Delta_1 \mid t_2 : C \mid t_1 : B \mid \Delta_2}\,\textsc{Exr}}{\Gamma_1, \Gamma, \Gamma_2 \vdash \Delta \mid [t/x]\Delta_1 \mid [t/x]t_2 : C \mid [t/x]t_1 : B \mid [t/x]\Delta_2}\,\textsc{Cut}
$$

is transformed into

$$
\dfrac{\dfrac{\dfrac{\pi_1}{\Gamma \vdash t : A \mid \Delta} \qquad \dfrac{\pi_2}{\Gamma_1, x : A, \Gamma_2 \vdash \Delta_1 \mid t_1 : B \mid t_2 : C \mid \Delta_2}}{\Gamma_1, \Gamma, \Gamma_2 \vdash \Delta \mid [t/x]\Delta_1 \mid [t/x]t_1 : B \mid [t/x]t_2 : C \mid [t/x]\Delta_2}\,\textsc{Cut}}{\Gamma_1, \Gamma, \Gamma_2 \vdash \Delta \mid [t/x]\Delta_1 \mid [t/x]t_2 : C \mid [t/x]t_1 : B \mid [t/x]\Delta_2}\,\textsc{Exr}
$$

Clearly, all terms are equivalent.

A The full specification of FILL

term_var, w, x, y, z, v			
index_var, i, j, k			
form, A, B, C, D, E	::=		
	|	I	
	|	$\perp$	
	|	$A \multimap B$	
	|	$A \otimes B$	
	|	$A \parr B$	
	|	(A)	S
patterns, p	::=		
	|	$*$	
	|	x	
	|	$p_1 \otimes p_2$	
	|	$p_1 \parr p_2$	
	|	$-$	
	|	(p)	S
term, t, e, d, f, g, u	::=		
	|	x	
	|	$*$	
	|	$\circ$	
	|	$e_1 \otimes e_2$	
	|	$e_1 \parr e_2$	

	|	$\lambda x.t$	
	|	let t be p in e	
	|	$f\ e$	
	|	let-pat $t\ p\ e$	M
	|	$[t/x]t'$	M
	|	$[t/x, e/y]t'$	M
	|	(t)	S
	|	t	M
	|	t	M
Γ	$::=$		
	|	$x : A$	
	|	$\cdot$	
	|	Γ, Γ'	
	|	$x : A$	
Δ	$::=$		
	|	$t : A$	
	|	$\cdot$	
	|	$\Delta \mid \Delta'$	
	|	Δ	
	|	Δ, Δ'	
	|	$[t/x]\Delta$	
	|	let t be p in Δ	
	|	(Δ)	
	|	let-pat $t\ p\ \Delta$	M
formula	$::=$		
	|	*judgement*	
	|	$formula_1 \quad formula_2$	
	|	$(formula)$	
	|	$x \notin \mathsf{FV}(\Delta)$	
	|	$x \in \mathsf{FV}(t)$	
	|	$x, y \notin \mathsf{FV}(\Delta)$	
	|	$x \notin \mathsf{FV}(t)$	
	|	$x, y \notin \mathsf{FV}(t)$	
	|	$\Delta_1 = \Delta_2$	
	|	$\mathsf{FV}(t)$	
	|	$\mathsf{FV}(\Delta)$	

InferRules ::=
| $\Gamma \vdash \Delta$
| $f = e$

judgement ::=
| *InferRules*

user_syntax ::=
| *term_var*
| *index_var*
| *form*
| *patterns*
| *term*
| Γ
| Δ
| *formula*

$\boxed{\Gamma \vdash \Delta}$

$$\frac{}{x : A \vdash x : A}\ \textsc{Ax}$$

$$\frac{\Gamma \vdash t : A \mid \Delta \quad y : A, \Gamma' \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta \mid [t/y]\Delta'}\ \textsc{Cut}$$

$$\frac{\Gamma \vdash \Delta}{\Gamma, x : I \vdash \mathsf{let}\, x\, \mathsf{be}\, *\ \mathsf{in}\, \Delta}\ \textsc{Il}$$

$$\frac{}{\cdot \vdash * : I}\ \textsc{Ir}$$

$$\frac{\Gamma, x : A, y : B \vdash \Delta}{\Gamma, z : A \otimes B \vdash \mathsf{let}\, z\, \mathsf{be}\, x \otimes y\, \mathsf{in}\, \Delta}\ \textsc{Tl}$$

$$\frac{\Gamma \vdash e : A \mid \Delta \quad \Gamma' \vdash f : B \mid \Delta'}{\Gamma, \Gamma' \vdash e \otimes f : A \otimes B \mid \Delta \mid \Delta'}\ \textsc{Tr}$$

$$\frac{}{x : \perp \vdash \cdot}\ \textsc{Pl}$$

$$\frac{\Gamma \vdash \Delta}{\Gamma \vdash \circ : \perp \mid \Delta}\ \textsc{Pr}$$

$$\frac{\Gamma, x : A \vdash \Delta \quad \Gamma', y : B \vdash \Delta'}{\Gamma, \Gamma', z : A \parr B \vdash \text{let-pat}\, z\, (x \parr -)\, \Delta \mid \text{let-pat}\, z\, (- \parr y)\, \Delta'}\ \textsc{Parl}$$

$$\frac{\Gamma \vdash \Delta \mid e : A \mid f : B \mid \Delta'}{\Gamma \vdash \Delta \mid e \parr f : A \parr B \mid \Delta'}\ \textsc{Parr}$$

$$\frac{\Gamma \vdash e : A \mid \Delta \quad \Gamma', x : B \vdash \Delta'}{\Gamma, y : A \multimap B, \Gamma' \vdash \Delta \mid [y\ e/x]\Delta'}\ \textsc{Impl}$$

$$\frac{\Gamma, x : A \vdash e : B \mid \Delta \quad x \notin \mathsf{FV}(\Delta)}{\Gamma \vdash \lambda x.e : A \multimap B \mid \Delta} \textsc{ImpR}$$

$$\frac{\Gamma, x : A, y : B \vdash \Delta}{\Gamma, y : B, x : A \vdash \Delta} \textsc{ExL}$$

$$\frac{\Gamma \vdash \Delta_1 \mid t_1 : A \mid t_2 : B \mid \Delta_2}{\Gamma \vdash \Delta_1 \mid t_2 : B \mid t_1 : A \mid \Delta_2} \textsc{ExR}$$

$\boxed{f = e}$

$$\frac{y \notin \mathsf{FV}(t)}{t = [y/x]t} \textsc{Alpha}$$

$$\frac{x \notin \mathsf{FV}(f)}{(\lambda x.f\ x) = f} \textsc{EtaFun}$$

$$\frac{}{(\lambda x.e)\ e' = [e'/x]e} \textsc{BetaFun}$$

$$\frac{}{\mathsf{let}\ *\ \mathsf{be}\ *\ \mathsf{in}\ e = e} \textsc{Eta1I}$$

$$\frac{y \notin \mathsf{FV}(f)}{f = \mathsf{let}\ y\ \mathsf{be}\ *\ \mathsf{in}\ f} \textsc{Eta2I}$$

$$\frac{}{\mathsf{let}\ u\ \mathsf{be}\ *\ \mathsf{in}\ [*/z]f = [u/z]f} \textsc{BetaI}$$

$$\frac{}{[\mathsf{let}\ u\ \mathsf{be}\ *\ \mathsf{in}\ e/y]f = \mathsf{let}\ u\ \mathsf{be}\ *\ \mathsf{in}\ [e/y]f} \textsc{NatI}$$

$$\frac{x, y \notin \mathsf{FV}(t)}{\mathsf{let}\ t'\ \mathsf{be}\ x \otimes y\ \mathsf{in}\ t = t} \textsc{EtaTen}$$

$$\frac{}{\mathsf{let}\ e \otimes t\ \mathsf{be}\ x \otimes y\ \mathsf{in}\ u = [e/x, t/y]u} \textsc{Beta1Ten}$$

$$\frac{}{\mathsf{let}\ u\ \mathsf{be}\ x \otimes y\ \mathsf{in}\ [x \otimes y/z]f = [u/z]f} \textsc{Beta2Ten}$$

$$\frac{}{[\mathsf{let}\ u\ \mathsf{be}\ x \otimes y\ \mathsf{in}\ g/w]f = \mathsf{let}\ u\ \mathsf{be}\ x \otimes y\ \mathsf{in}\ [g/w]f} \textsc{NatTen}$$

$$\frac{}{u = \circ} \textsc{EtaParU}$$

$$\frac{}{(\mathsf{let}\ u\ \mathsf{be}\ x \parr -\ \mathsf{in}\ x) \parr (\mathsf{let}\ u\ \mathsf{be}\ - \parr y\ \mathsf{in}\ y) = u} \textsc{EtaPar}$$

$$\frac{}{\mathsf{let}\ u \parr t\ \mathsf{be}\ x \parr -\ \mathsf{in}\ e = [u/x]e} \textsc{Beta1Par}$$

$$\frac{}{\mathsf{let}\ u \parr t\ \mathsf{be}\ - \parr y\ \mathsf{in}\ e = [t/y]e} \textsc{Beta2Par}$$

$$\frac{}{\mathsf{let}\ t\ \mathsf{be}\ x \parr -\ \mathsf{in}\ [u/x]f = [\mathsf{let}\ t\ \mathsf{be}\ x \parr -\ \mathsf{in}\ u/x]f} \textsc{Nat1Par}$$

$$\frac{}{\mathsf{let}\ t\ \mathsf{be}\ -\parr y\ \mathsf{in}\ [v/y]f = [\mathsf{let}\ t\ \mathsf{be}\ -\parr y\ \mathsf{in}\ v/y]f}\ \textsc{Nat2Par}$$

$$\frac{t = t'}{\lambda x.t = \lambda x.t''}\ \textsc{Lam}$$

$$\frac{t_1 = t_1'}{t_1\ t_2 = t_1'\ t_2}\ \textsc{App1}$$

$$\frac{t_2 = t_2'}{t_1\ t_2 = t_1\ t_2'}\ \textsc{App2}$$

$$\frac{t_1 = t_1'}{t_1 \otimes t_2 = t_1' \otimes t_2}\ \textsc{Ten1}$$

$$\frac{t_2 = t_2'}{t_1 \otimes t_2 = t_1 \otimes t_2'}\ \textsc{Ten2}$$

$$\frac{t_1 = t_1'}{t_1 \parr t_2 = t_1' \parr t_2}\ \textsc{Par1}$$

$$\frac{t_2 = t_2'}{t_1 \parr t_2 = t_1 \parr t_2'}\ \textsc{Par2}$$

$$\frac{t = t'}{\mathsf{let}\ t\ \mathsf{be}\ p\ \mathsf{in}\ e = \mathsf{let}\ t'\ \mathsf{be}\ p\ \mathsf{in}\ e}\ \textsc{Let1}$$

$$\frac{e = e'}{\mathsf{let}\ t\ \mathsf{be}\ p\ \mathsf{in}\ e = \mathsf{let}\ t\ \mathsf{be}\ p\ \mathsf{in}\ e'}\ \textsc{Let2}$$

$$\frac{}{t = t}\ \textsc{Refl}$$

$$\frac{t = t'}{t' = t}\ \textsc{Sym}$$

$$\frac{t_1 = t_2 \quad t_2 = t_3}{t_1 = t_3}\ \textsc{Trans}$$