Cut-elimination of the APAL term assignment formulation of FILL

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In this short note I give the proof of the term assignment formulation of FILL first given in [2].

1 The Fix

Consider the Dparl rule in the dependency-relation formalization:

$$\dfrac{\begin{array}{c}\Gamma_1, A \vdash \Delta_1 \\ \Gamma_3, B \vdash \Delta_2\end{array}}{\Gamma_1, \Gamma_3, A \parr B \vdash \Delta_1, \Delta_2}\ \textsc{Dparl} \qquad \mathit{Dep}(\tau) = \{(A \parr B, A), (A \parr B, B)\} \star (\mathit{Dep}(\tau_1) \cup \mathit{Dep}(\tau_2))$$

If anything in Δ_1 and Δ_2 depend on A or B then this will be accounted for in $\mathit{Dep}(\tau_1)$ and $\mathit{Dep}(\tau_2)$. Thus, in the term formalization when binding pattern variables across the righthand side of the sequent we should do so if and only if there is a dependency. In fact, if a formula on the righthand side depends on a formula in the lefthand side, then the variable associated with that hypnosis must be free in the term associated with the formula on the right. This evidence suggests that to fix the term formalization we must modify the Parl rule.

The new Parl rule as follows:

$$\dfrac{\Gamma, x : A \vdash d_i : C_i \quad \Gamma', y : B \vdash f_j : D_j}{\Gamma, \Gamma', z : A \parr B \vdash \text{let-pat}\, z\, (x \parr -)\, d_i : C_i \mid \text{let-pat}\, z\, (- \parr y)\, f_j : D_j}\ \textsc{NParl}$$

The previous rule depends on a function which we define as follows:

$$\begin{array}{l}\text{let-pat}\, z\, (x \parr -)\, e = e \\ \quad \text{where } x \notin \mathsf{FV}(e) \\ \\ \text{let-pat}\, z\, (- \parr y)\, e = e \\ \quad \text{where } y \notin \mathsf{FV}(e) \\ \\ \text{let-pat}\, z\, p\, e = \mathsf{let}\, z\, \mathsf{be}\, p\, \mathsf{in}\, e\end{array}$$

Note that in the definition of let-pat $z\, p\, e$ the final case is a catchall case. Now the new Parl rule only pattern matches on z in the righthand side if there is a dependency between the variables in the pattern and the term in the pattern match. A similar rule to the above was proposed by Bellin in the conclusion of [1].

This rule recovers from the counterexample. The first derivation given in the counter example above is unchanged, so we only give the second:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{v : A \vdash v : A}\ \textsc{Ax}}{v : A \vdash v : A \mid \circ : \perp}\ \textsc{Pr} \quad \dfrac{\dfrac{}{x : A \vdash x : A}\ \textsc{Ax} \quad \dfrac{}{y : B \vdash y : B}\ \textsc{Ax}}{x : A, y : B \vdash x \otimes y : A \otimes B}\ \textsc{Tr}}{y : B, v : A \vdash v \otimes y : A \otimes B \mid \circ : \perp}\ \textsc{Cut} \quad \dfrac{}{w : C \vdash w : C}\ \textsc{Ax}}{v : A, z : B \parr C \vdash \mathsf{let}\, z\, \mathsf{be}\, y \parr - \,\mathsf{in}\, v \otimes y : A \otimes B \mid \mathsf{let}\, z\, \mathsf{be}\, - \parr w\, \mathsf{in}\, w : C \mid \circ : \perp}\ \textsc{Parl}}{v : A, z : B \parr C \vdash ((\mathsf{let}\, z\, \mathsf{be}\, y \parr - \,\mathsf{in}\, v \otimes y) \parr (\mathsf{let}\, z\, \mathsf{be}\, - \parr w\, \mathsf{in}\, w)) : (A \otimes B) \parr C \mid \circ : \perp}\ \textsc{Parr}}{v : A \vdash \lambda z.((\mathsf{let}\, z\, \mathsf{be}\, y \parr - \,\mathsf{in}\, v \otimes y) \parr (\mathsf{let}\, z\, \mathsf{be}\, - \parr w\, \mathsf{in}\, w)) : (B \parr C) \multimap ((A \otimes B) \parr C) \mid \circ : \perp}\ \textsc{Impr}$$

This new derivation is now correct, and mirrors that of the dependency-relation formalization.

2 Basic Results

Lemma 1 (Substitution Distribution)**.** *For any terms t, t_1, and t_2,* $[t_1/x][t_2/y]t = [[t_1/x]t_2/y][t_2/x]t$.

Proof. This proof holds by straightforward induction on the form of t. ☐

Lemma 2 (Left and Right Exchange)**.** *If* $\Gamma, \Gamma' \vdash D, D'$*, then* $\Gamma', \Gamma \vdash D', D$.

Proof. This proof holds by straightforward induction on the assumed derivation. ☐

3 Cut-elimination

3.1 Commuting conversion cut vs cut (first case)

The following proof

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\end{array}}{\Gamma \vdash t : A \mid \Delta} \qquad \dfrac{\dfrac{\begin{array}{c}\pi_2\\ \vdots\end{array}}{\Gamma_2, x : A, \Gamma_3 \vdash t_1 : B \mid \Delta_1} \qquad \dfrac{\begin{array}{c}\pi_3\\ \vdots\end{array}}{\Gamma_1, y : B, \Gamma_4 \vdash t_2 : C \mid \Delta_2}}{\Gamma_1, \Gamma_2, x : A, \Gamma_3, \Gamma_4 \vdash \Delta_1 \mid [t_1/y]t_2 : C \mid [t_1/y]\Delta_2}\,\text{Cut}}{\Gamma_1, \Gamma_2, \Gamma, \Gamma_3, \Gamma_4 \vdash \Delta \mid [t/x]\Delta_1 \mid [t/x][t_1/y]t_2 : C \mid [t/x][t_1/y]\Delta_2}\,\text{Cut}$$

is transformed into the proof

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\end{array}}{\Gamma \vdash t : A \mid \Delta} \qquad \dfrac{\begin{array}{c}\pi_2\\ \vdots\end{array}}{\Gamma_2, x : A, \Gamma_3 \vdash t_1 : B \mid \Delta_1}}{\Gamma_2, \Gamma, \Gamma_3 \vdash [t/x]t_1 : B \mid [t/x]\Delta_1} \qquad \dfrac{\begin{array}{c}\pi_3\\ \vdots\end{array}}{\Gamma_1, y : B, \Gamma_4 \vdash t_2 : C \mid \Delta_2}}{\Gamma_1, \Gamma_2, \Gamma, \Gamma_3, \Gamma_4 \vdash \Delta \mid [t/x]\Delta_1 \mid ([[t/x]t_1/y]t_2) : C \mid [[t/x]t_1/y]\Delta_2}\,\text{Cut}$$

In order for the previous two proofs to be considered equal, we have to show that the final terms in the conclusion of the above derivations are equivalent. First, we know that the term $[t/x][t_1/y]t_2$ in the first derivation above is equivalent to $[[t/x]t_1/y][t/x]t_2$ by Lemma 1. Furthermore, by inspecting the first derivation we can see that $x \notin \mathsf{FV}(t_2)$, and thus, $[[t/x]t_1/y][t/x]t_2 = [[t/x]t_1/y]t_2$. This argument may be repeated for any term in Δ_2, and thus, we know $[t/x][t_1/y]\Delta_2 = [[t/x]t_1/y]\Delta_2$.

3.2 Commuting conversion cut vs. cut (second case)

The second commuting conversion on cut begins with the proof

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\end{array}}{\Gamma \vdash t : A \mid \Delta} \qquad \dfrac{\dfrac{\begin{array}{c}\pi_2\\ \vdots\end{array}}{\Gamma' \vdash t' : B \mid \Delta'} \qquad \dfrac{\begin{array}{c}\pi_3\\ \vdots\end{array}}{\Gamma_1, x : A, \Gamma_2, y : B, \Gamma_3 \vdash t_1 : C \mid \Delta_1}}{\Gamma_1, x : A, \Gamma_2, \Gamma', \Gamma_3 \vdash \Delta' \mid [t'/y]t_1 : C \mid [t'/y]\Delta_1}\,\text{Cut}}{\Gamma_1, \Gamma, \Gamma_2, \Gamma', \Gamma_3 \vdash \Delta \mid [t/x]\Delta' \mid [t/x][t'/y]t_1 : C \mid [t/x][t'/y]\Delta_1}\,\text{Cut}$$

is transformed into the following proof:

$$\dfrac{\dfrac{\begin{array}{c}\pi_2\\ \vdots\end{array}}{\Gamma' \vdash t' : B \mid \Delta'} \qquad \dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\end{array}}{\Gamma \vdash t : A \mid \Delta} \qquad \dfrac{\begin{array}{c}\pi_3\\ \vdots\end{array}}{\Gamma_1, x : A, \Gamma_2, y : B, \Gamma_3 \vdash t_1 : C \mid \Delta_1}}{\Gamma_1, \Gamma, \Gamma_2, y : B, \Gamma_3 \vdash \Delta \mid [t/x]t_1 : C \mid [t/x]\Delta_1}\,\text{Cut}}{\Gamma_1, \Gamma, \Gamma_2, \Gamma', \Gamma_3 \vdash \Delta' \mid [t'/y]\Delta \mid [t'/y][t/x]t_1 : C \mid [t'/y][t/x]\Delta_1}\,\text{Cut}$$

Now, because we know $x, y \notin \mathsf{FV}(\Delta)$ by inspection of the first derivation, we know that $\Delta = [t'/y]\Delta$ and $\Delta' = [t/x]\Delta'$. Similarly, we know that $x, y \notin \mathsf{FV}(t)$ and $x, y \notin \mathsf{FV}(t')$. Thus, by this fact and Lemma 1, we know that $[t/x][t'/y]t_1 = [[t/x]t'/y][t/x]t_1 = [t'/y][t/x]t_1$. This argument can be repeated for any term in Δ_1, hence, $[t/x][t'/y]\Delta_1 = [t'/y][t/x]\Delta_1$. Finally, by Lemma 2 we can exchange Δ' and $[t'/y]\Delta$ in the last proof to obtain the same order of terms on the right in both derivations. Therefore, both of the previous derivations are equivalent.

References

[1] G.M. Bierman. A note on full intuitionistic linear logic. *Annals of Pure and Applied Logic*, 79(3):281 – 287, 1996.

[2] Martin Hyland and Valeria de Paiva. Full intuitionistic linear logic (extended abstract). *Annals of Pure and Applied Logic*, 64(3):273 – 291, 1993.

A The full specification of FILL

term_var, w, x, y, z, v			
index_var, i, j, k			
form, A, B, C, D, E	::=		
	|	I	
	|	$\perp$	
	|	$A \multimap B$	
	|	$A \otimes B$	
	|	$A \parr B$	
	|	(A)	S
patterns, p	::=		
	|	$*$	
	|	x	
	|	$p_1 \otimes p_2$	
	|	$p_1 \parr p_2$	
	|	$-$	
	|	(p)	S
term, t, e, d, f, g, u	::=		
	|	x	
	|	$*$	
	|	$\circ$	
	|	$e_1 \otimes e_2$	
	|	$e_1 \parr e_2$	
	|	$\lambda x.t$	
	|	$\mathsf{let}\ t\ \mathsf{be}\ p\ \mathsf{in}\ e$	
	|	$f\ e$	
	|	let-pat $z\ p\ e$	M
	|	$[t/x]t'$	M
	|	$[t/x, e/y]t'$	M
	|	(t)	S

	$\|$	t	M
	$\|$	t	M
Γ	$::=$		
	$\|$	$x : A$	
	$\|$	$\cdot$	
	$\|$	Γ, Γ'	
	$\|$	$x : A$	
	$\|$	A	
Δ	$::=$		
	$\|$	$t : A$	
	$\|$	$\cdot$	
	$\|$	$\Delta \mid \Delta'$	
	$\|$	Δ	
	$\|$	A	
	$\|$	Δ, Δ'	
	$\|$	$[t/x]\Delta$	
$formula$	$::=$		
	$\|$	$judgement$	
	$\|$	$formula_1 \quad formula_2$	
	$\|$	$(formula)$	
	$\|$	$x \notin \mathsf{FV}(\Delta)$	
	$\|$	$x, y \notin \mathsf{FV}(\Delta)$	
	$\|$	$x \notin \mathsf{FV}(t)$	
	$\|$	$x, y \notin \mathsf{FV}(t)$	
	$\|$	$\Delta_1 = \Delta_2$	
$InferRules$	$::=$		
	$\|$	$\Gamma \vdash \Delta$	
	$\|$	$f = e$	
$judgement$	$::=$		
	$\|$	$InferRules$	
$user\_syntax$	$::=$		
	$\|$	$term\_var$	
	$\|$	$index\_var$	
	$\|$	$form$	
	$\|$	$patterns$	
	$\|$	$term$	
	$\|$	Γ	
	$\|$	Δ	
	$\|$	$formula$	

$\boxed{\Gamma \vdash \Delta}$

$$\frac{}{x : A \vdash x : A}\ \textsc{Ax}$$

$$\frac{\Gamma \vdash t : A \mid \Delta \quad y : A, \Gamma' \vdash f_i : B_i}{\Gamma, \Gamma' \vdash \Delta \mid [t/y]f_i : B_i}\ \textsc{Cut}$$

$$\frac{\Gamma \vdash e_i : A_i}{\Gamma, x : I \vdash \mathsf{let}\, x\, \mathsf{be}\, * \,\mathsf{in}\, e_i : A_i}\ \textsc{Il}$$

$$\frac{}{\cdot \vdash * : I}\ \textsc{Ir}$$

$$\frac{\Gamma, x : A, y : B \vdash f_i : C_i}{\Gamma, z : A \otimes B \vdash \mathsf{let}\, x\, \mathsf{be}\, x \otimes y\, \mathsf{in}\, f_i : C_i}\ \textsc{Tl}$$

$$\frac{\Gamma \vdash e : A \mid \Delta \quad \Gamma' \vdash f : B \mid \Delta'}{\Gamma, \Gamma' \vdash e \otimes f : A \otimes B \mid \Delta \mid \Delta'}\ \textsc{Tr}$$

$$\frac{}{x : \perp \vdash \cdot}\ \textsc{Pl}$$

$$\frac{\Gamma \vdash \Delta}{\Gamma \vdash \circ : \perp \mid \Delta}\ \textsc{Pr}$$

$$\frac{\Gamma, x : A \vdash d_i : C_i \quad \Gamma', y : B \vdash f_j : D_j}{\Gamma, \Gamma', z : A \parr B \vdash \mathsf{let}\, z\, \mathsf{be}\, x \parr - \,\mathsf{in}\, d_i : C_i \mid \mathsf{let}\, z\, \mathsf{be}\, - \parr y\, \mathsf{in}\, f_j : D_j}\ \textsc{ParL}$$

$$\frac{\Gamma, x : A \vdash d_i : C_i \quad \Gamma', y : B \vdash f_j : D_j}{\Gamma, \Gamma', z : A \parr B \vdash \text{let-pat}\, z\, (x \parr -)\, d_i : C_i \mid \text{let-pat}\, z\, (- \parr y)\, f_j : D_j}\ \textsc{NParL}$$

$$\frac{\Gamma \vdash \Delta \mid e : A \mid f : B \mid \Delta'}{\Gamma \vdash \Delta \mid e \parr f : A \parr B \mid \Delta'}\ \textsc{ParR}$$

$$\frac{\Gamma \vdash e : A \mid \Delta \quad \Gamma', x : B \vdash f_i : C_i}{\Gamma, y : A \multimap B, \Gamma' \vdash [y\ e/x]f_i : C_i \mid \Delta}\ \textsc{ImpL}$$

$$\frac{\Gamma, x : A \vdash e : B \mid \Delta \quad x \notin \mathsf{FV}(\Delta)}{\Gamma \vdash \lambda x.e : A \multimap B \mid \Delta}\ \textsc{ImpR}$$

$$\frac{}{A' \vdash A''}\ \textsc{dax}$$

$$\frac{\begin{array}{c}\Gamma_1 \vdash B', \Delta_1 \\ \Gamma_2, B'' \vdash \Delta_2\end{array}}{\Gamma_1, \Gamma_2 \vdash \Delta_2, \Delta_1}\ \textsc{dcut}$$

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \otimes B \vdash \Delta}\ \textsc{Dtl}$$

$$\frac{\begin{array}{c}\Gamma_1 \vdash A, \Delta_1 \\ \Gamma_2 \vdash B, \Delta_2\end{array}}{\Gamma_1, \Gamma_2 \vdash A \otimes B, \Delta_1, \Delta_2}\ \textsc{Dtr}$$

$$\frac{\Gamma \vdash \Delta}{\Gamma, I \vdash \Delta}\ \textsc{Dil}$$

$$\frac{}{\cdot \vdash I}\ \textsc{Dir}$$

$$\frac{\Gamma_1, A \vdash \Delta_1 \qquad \Gamma_3, B \vdash \Delta_2}{\Gamma_1, \Gamma_3, A \parr B \vdash \Delta_1, \Delta_2}\ \textsc{DparL}$$

$$\frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \parr B, \Delta}\ \textsc{DparR}$$

$$\frac{}{\perp \vdash \cdot}\ \textsc{DpL}$$

$$\frac{\Gamma \vdash \Delta}{\Gamma \vdash \perp, \Delta}\ \textsc{DpR}$$

$$\frac{\Gamma_1 \vdash A, \Delta_1 \qquad \Gamma_2, B \vdash \Delta_2}{\Gamma_1, \Gamma_2, A \multimap B \vdash \Delta_1, \Delta_2}\ \textsc{DimpL}$$

$$\frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \multimap B, \Delta}\ \textsc{DimpR}$$

$\boxed{f = e}$

$$\frac{}{(\lambda x.e)\ e' = [e'/x]e}\ \textsc{Eq\_Beta}$$

$$\frac{}{\lambda x.f\ x = f}\ \textsc{Eq\_Eta}$$

$$\frac{}{\mathsf{let}\ *\ \mathsf{be}\ *\ \mathsf{in}\ e = e}\ \textsc{Eq\_I}$$

$$\frac{}{\mathsf{let}\ u\ \mathsf{be}\ *\ \mathsf{in}\ [*/z]f = [u/z]f}\ \textsc{Eq\_Stp}$$

$$\frac{}{\mathsf{let}\ e \otimes t\ \mathsf{be}\ x \otimes y\ \mathsf{in}\ u = [e/x, t/y]u}\ \textsc{Eq\_T1}$$

$$\frac{}{\mathsf{let}\ u\ \mathsf{be}\ x \otimes y\ \mathsf{in}\ [x \otimes y/z]f = [u/z]f}\ \textsc{Eq\_T2}$$

$$\frac{}{\mathsf{let}\ u \parr t\ \mathsf{be}\ x \parr -\ \mathsf{in}\ e = [u/x]e}\ \textsc{Eq\_P1}$$

$$\frac{}{\mathsf{let}\ u \parr t\ \mathsf{be}\ - \parr y\ \mathsf{in}\ e = [t/y]e}\ \textsc{Eq\_P2}$$

$$\frac{}{(\mathsf{let}\ x\ \mathsf{be}\ x \parr -\ \mathsf{in}\ x) \parr (\mathsf{let}\ u\ \mathsf{be}\ - \parr y\ \mathsf{in}\ y) = u}\ \textsc{Eq\_P3}$$