TNT: A Dynamite Prespective of Termination/Non-Termination as Effects

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Abstract

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1 Introduction

The Trellys Project was a large scale project to develop a general purpose dependently type functional programming language [**?**]. The most important language design decision was to have the language fragmented into two parts: a programmatic fragment and a logical fragment. The former is used to write general purpose programs, and the latter is used to verify the correctness of the programs written in the programmatic fragment. This implies that there must be a means for the logical fragment to take programs of the programmatic fragment – which we will call simply programs – in as input to be able to write predicates and proofs about those programs, but this must be done so as to not compromise consistency of the logical fragment. On the other side of the coin the programmatic fragment should have the ability to construct programs using the logical fragment. This property is called **freedom of speech**, because both fragments have the ability to talk about each other without sacrificing their integrity.

This paper approaches this problem from a categorical perspective. Suppose we have a cartesian closed category, $\mathcal{L}$, and a cartesian closed category with fixpoints, $\mathcal{P}$. It is well-known that $\mathcal{L}$ is a model of the simply-typed λ-calculus [**?**], and $\mathcal{P}$ is a model of the simply-typed λ-calculus with the Y combinator [**?**]. The category $\mathcal{L}$ will model our logical fragment, and $\mathcal{P}$ our programmatic fragment.

Now suppose we have a symmetric monoidal adjunction $\mathcal{L} : T \dashv D : \mathcal{P}$. We can think of this adjunction as a translation between the two fragments. It is a well-known fact about adjunctions that the functor $\uparrow = DT : \mathcal{L} \to \mathcal{L}$ is a monad, and the functor $\downarrow = TD : \mathcal{P} \to \mathcal{P}$ is a comonad. The former treats non-termination as a monadic effect allowing the logical fragment to reason about non-terminating programs, and the latter treats termination as a comonadic effect allowing the programmatic fragment to restrict itself to terminating programs. Thus, we can see freedom of speech as an adjoint situation. It can also be shown that we have a strong monad:

$$A \times \uparrow B \xrightarrow{\mathsf{m\text{-}st}_{A,B}} \uparrow(A \times B)$$

The previous strength can be defined using the fact that we have a symmetric monoidal adjunction.

One application where this is useful is in programming languages such as Haskell where programmers by default program in potentially diverging setting. However, to gain assurances it is often preferred to be able to pick and choose which programs one would want to be terminating, and have it enforced by the type checker. A termination comonad would allow the programmer to develop their program under the comonad to insure termination. Furthermore, a termination comonad supports a lighter weight verification where some properties of potentially diverging programs hold with the restriction that their inputs terminate.

Create example of this

We will construct typed λ-calculi corresponding to both $\mathcal{L}$ and $\mathcal{P}$ respectively. Then we will bring these two calculi together through a syntactic adjunction, and consider various programmatic questions. For example, every program of type $\downarrow X$ should terminate, but proofs of type $\uparrow A$ should be allowed to diverge. In addition to these syntactic questions we discuss novel contributions of these adjoint categorical models.

2 The Fragments and Their Semantics

We first must define the two fragments and give their semantics. The logic fragment will be kept simple and amounts to the simply-typed λ-calculus, but the programmatic fragment will be slightly more interesting. Since we are interested in programming it will contain the natural numbers, with a natural number eliminator in the form of a case analysis, and the Y-combinator. Each of the theories will be presented using the style of Crole [**?**].

2.1 The Logical Fragment

The logical fragment is defined in Figure 1 and Figure 2. It is the simply typed λ-calculus with pairs. The following is a well-known fact [**?**].

Lemma 1. *The logical fragment has a sound and complete interpretation into cartesian closed categories.*

2.2 The Programmatic Fragment

The programmatic fragment is a bit more interesting, and corresponds to the simply-typed λ-calculus with natural numbers and the Y-combinator. The typing relation is defined in Figure 3, and the terms-in-context inference rules are

$$\frac{}{\Gamma_1, x : A, \Gamma_2 \vdash_{\mathcal{L}} x : A}\ \textsc{L\_Ax} \qquad \frac{}{\Gamma \vdash_{\mathcal{L}} * : \top}\ \textsc{L\_True}$$

$$\frac{\Gamma \vdash_{\mathcal{L}} t_1 : A \quad \Gamma \vdash_{\mathcal{L}} t_2 : B}{\Gamma \vdash_{\mathcal{L}} (t_1, t_2) : A \times B}\ \textsc{L\_Prod} \qquad \frac{\Gamma \vdash_{\mathcal{L}} t : A \times B}{\Gamma \vdash_{\mathcal{L}} \mathsf{proj}_1\, t : A}\ \textsc{L\_Proj1}$$

$$\frac{\Gamma \vdash_{\mathcal{L}} t : A \times B}{\Gamma \vdash_{\mathcal{L}} \mathsf{proj}_2\, t : B}\ \textsc{L\_Proj2} \qquad \frac{\Gamma, x : A \vdash_{\mathcal{L}} t : B}{\Gamma \vdash_{\mathcal{L}} \lambda x : A.t : A \to B}\ \textsc{L\_Fun}$$

$$\frac{\begin{array}{l}\Gamma \vdash_{\mathcal{L}} t_2 : A \\ \Gamma \vdash_{\mathcal{L}} t_1 : A \to B\end{array}}{\Gamma \vdash_{\mathcal{L}} t_1\, t_2 : B}\ \textsc{L\_App}$$

Figure 1: The Logical Fragment: Typing Relation

defined in Figure 4. The natural number eliminator, case, can be mixed with the Y-combinator to write generally recursive programs on the natural numbers. For example, the following defines natural number addition:

$$\mathsf{fix}\,(\lambda r : \mathsf{Nat} \to \mathsf{Nat} \to \mathsf{Nat}.\lambda n_1 : \mathsf{Nat}.\lambda n_2 : \mathsf{Nat}.\mathsf{case}\, n_2\, \{0 \to n_1, \mathbf{suc}\, x \to \mathbf{suc}\,(r\, n_1\, x)\})$$

The question now becomes how do we model fix and case categorically? To model fix we endow a cartesian closed category with fixpoints [**?**].

Definition 2. *Suppose $\mathcal{C}$ is a cartesian closed category. Then we say $\mathcal{C}$ **has fixpoints** if for any object X of $\mathcal{C}$, there is a morphism $\mathsf{fix}_X : (X \to X) \longrightarrow X$ such that the following diagram commutes:*

$$\begin{tikzcd}
X \to X \arrow[r, "\langle \mathsf{id}_{X \to X}, \mathsf{fix}_X \rangle"] \arrow[dr, "\mathsf{fix}_X"'] & (X \to X) \times X \arrow[d, "\mathsf{app}_{X,X}"] \\
& X
\end{tikzcd}$$

The commuting diagram in the above definition models the reduction rule for fix, and so it is pretty easy to see that this definition will allow us to model fix categorically.

Perhaps more interestingly we use a novel approach to modeling natural numbers with their eliminator.

Definition 3. *Suppose $\mathcal{C}$ is a cartesian closed category. A **Scott natural number object (SNNO)** is an object Nat of $\mathcal{C}$ and morphisms $\mathsf{z} : 1 \longrightarrow \mathsf{Nat}$ and $\mathbf{suc} : \mathsf{Nat} \longrightarrow \mathsf{Nat}$ of $\mathcal{C}$, such that, for any morphisms $f : Y \longrightarrow X$ and $g : Y \times \mathsf{Nat} \longrightarrow X$ of $\mathcal{C}$ there is a unique morphism $\mathsf{case}_X : Y \times \mathsf{Nat} \longrightarrow X$*

$$\frac{}{\Gamma_1, x : A, \Gamma_2 \vdash_{\mathcal{L}} x = x : A}\ \textsc{LEq\_Ax} \qquad \frac{\Gamma \vdash_{\mathcal{L}} t : \top}{\Gamma \vdash_{\mathcal{L}} t = * : \top}\ \textsc{LEq\_Unit}$$

$$\frac{\Gamma \vdash_{\mathcal{L}} t_1 : A \quad \Gamma \vdash_{\mathcal{L}} t_2 : B}{\Gamma \vdash_{\mathcal{L}} \mathsf{proj}_1\,(t_1, t_2) = t_1 : A}\ \textsc{LEq\_Proj1} \qquad \frac{\Gamma \vdash_{\mathcal{L}} t_1 : A \quad \Gamma \vdash_{\mathcal{L}} t_2 : B}{\Gamma \vdash_{\mathcal{L}} \mathsf{proj}_2\,(t_1, t_2) = t_2 : B}\ \textsc{LEq\_Proj2}$$

$$\frac{\Gamma \vdash_{\mathcal{L}} t : A \times B}{\Gamma \vdash_{\mathcal{L}} (\mathsf{proj}_1\, t, \mathsf{proj}_2\, t) = t : A \times B}\ \textsc{LEq\_EtaP}$$

$$\frac{\Gamma \vdash_{\mathcal{L}} t = t' : A \times B}{\Gamma \vdash_{\mathcal{L}} \mathsf{proj}_1\, t = \mathsf{proj}_1\, t' : A}\ \textsc{LEq\_Proj1C} \qquad \frac{\Gamma \vdash_{\mathcal{L}} t = t' : A \times B}{\Gamma \vdash_{\mathcal{L}} \mathsf{proj}_2\, t = \mathsf{proj}_2\, t' : B}\ \textsc{LEq\_Proj2C}$$

$$\frac{\Gamma \vdash_{\mathcal{L}} t_1 = t_1' : A \quad \Gamma \vdash_{\mathcal{L}} t_2 = t_2' : B}{\Gamma \vdash_{\mathcal{L}} (t_1, t_2) = (t_1, t_2') : A \times B}\ \textsc{LEq\_PairC}$$

$$\frac{\begin{array}{c}\Gamma \vdash_{\mathcal{L}} t' : B \\ \Gamma, x : A \vdash_{\mathcal{L}} t : B\end{array}}{\Gamma \vdash_{\mathcal{L}} (\lambda x : A.t)\, t' = [t'/x]t : B}\ \textsc{LEq\_Beta}$$

$$\frac{\Gamma \vdash_{\mathcal{L}} t : A \to B \quad x \notin \mathsf{FV}(t)}{\Gamma \vdash_{\mathcal{L}} (\lambda x : A.t\, x) = t : A \to B}\ \textsc{LEq\_Eta}$$

$$\frac{\Gamma, x : A \vdash_{\mathcal{L}} t = t' : B}{\Gamma \vdash_{\mathcal{L}} \lambda x : A.t = \lambda x : A.t' : B}\ \textsc{LEq\_FunC}$$

$$\frac{\Gamma \vdash_{\mathcal{L}} t : A \quad \Gamma, x : A \vdash_{\mathcal{L}} t_1 = t_2 : B}{\Gamma \vdash_{\mathcal{L}} [t/x]t_1 = [t/x]t_2 : B}\ \textsc{LEq\_Subst} \qquad \frac{\Gamma \vdash_{\mathcal{L}} t_1 = t_2 : A}{\Gamma, \Gamma' \vdash_{\mathcal{L}} t_1 = t_2 : A}\ \textsc{LEq\_Weak}$$

$$\frac{\Gamma_1, x : A, y : B, \Gamma_2 \vdash_{\mathcal{L}} t_1 = t_2 : C}{\Gamma_1, y : B, x : A, \Gamma_2 \vdash_{\mathcal{L}} t_1 = t_2 : C}\ \textsc{LEq\_Ex} \qquad \frac{\Gamma \vdash_{\mathcal{L}} t : A}{\Gamma \vdash_{\mathcal{L}} t = t : A}\ \textsc{LEq\_Refl}$$

$$\frac{\Gamma \vdash_{\mathcal{L}} t_1 = t_2 : A}{\Gamma \vdash_{\mathcal{L}} t_2 = t_1 : A}\ \textsc{LEq\_Sym} \qquad \frac{\Gamma \vdash_{\mathcal{L}} t_1 = t_2 : A \quad \Gamma \vdash_{\mathcal{L}} t_2 = t_3 : A}{\Gamma \vdash_{\mathcal{L}} t_1 = t_3 : A}\ \textsc{LEq\_Trans}$$

Figure 2: The Logical Fragment: Equality

$$\frac{}{\Delta_1, x : X, \Delta_2 \vdash_{\mathcal{P}} x : X}\ \text{P\_Ax} \qquad \frac{}{\Delta \vdash_{\mathcal{P}} * : \top}\ \text{P\_Unit} \qquad \frac{}{\Delta \vdash_{\mathcal{P}} 0 : \mathsf{Nat}}\ \text{P\_Zero}$$

$$\frac{}{\Delta \vdash_{\mathcal{P}} \mathbf{suc} : \mathsf{Nat} \to \mathsf{Nat}}\ \text{P\_Suc} \qquad \frac{}{\Delta \vdash_{\mathcal{P}} \mathsf{fix} : (X \to X) \to X}\ \text{P\_Fix}$$

$$\frac{\Delta \vdash_{\mathcal{P}} t_1 : X \quad \Delta \vdash_{\mathcal{P}} t_2 : Y}{\Delta \vdash_{\mathcal{P}} (t_1, t_2) : X \times Y}\ \text{P\_Prod} \qquad \frac{\Delta \vdash_{\mathcal{P}} t : X \times Y}{\Delta \vdash_{\mathcal{P}} \mathsf{proj}_1\, t : X}\ \text{P\_Proj1}$$

$$\frac{\Delta \vdash_{\mathcal{P}} t : X \times Y}{\Delta \vdash_{\mathcal{P}} \mathsf{proj}_2\, t : X}\ \text{P\_Proj2} \qquad \frac{\Delta, x : X \vdash_{\mathcal{P}} t : Y}{\Delta \vdash_{\mathcal{P}} \lambda x : X.t : X \to Y}\ \text{P\_Fun}$$

$$\frac{\begin{array}{l}\Delta \vdash_{\mathcal{P}} t_2 : X \\ \Delta \vdash_{\mathcal{P}} t_1 : X \to Y\end{array}}{\Delta \vdash_{\mathcal{P}} t_1\, t_2 : Y}\ \text{P\_App} \qquad \frac{\begin{array}{l}\Delta \vdash_{\mathcal{P}} t : \mathsf{Nat} \\ \Delta \vdash_{\mathcal{P}} t_1 : X \quad \Delta, x : \mathsf{Nat} \vdash_{\mathcal{P}} t_2 : X\end{array}}{\Delta \vdash_{\mathcal{P}} \mathsf{case}\, t\, \{0 \to t_1, \mathbf{suc}\, x \to t_2\} : X}\ \text{P\_Case}$$

Figure 3: Programmatic Fragment: Typing Relation

making the following diagrams commute:

$$\begin{tikzcd}
Y \times \mathsf{Nat} \arrow[r, "\mathsf{id}_Y \times \mathsf{z}"] \arrow[dr, "\pi_1; f"'] & Y \times \mathsf{Nat} \arrow[d, "\mathsf{case}_X"] & Y \times \mathsf{Nat} \arrow[l, "\mathsf{id}_Y \times \mathbf{suc}"'] \arrow[dl, "g"] \\
& X &
\end{tikzcd}$$

Informally, the two diagrams essentially assert that we can define u as follows:

$$\begin{array}{rcl} u\, y\, 0 & = & f\, y \\ u\, y\, (\mathbf{suc}\, x) & = & g\, y\, x \end{array}$$

This formalization of natural numbers is inspired by the definition of Scott Numerals [**?**] where the notion of a case distinction is built into the encoding. We can think of Y in the source object of case as the type of additional inputs that will be passed to both f and g, but we can think of Nat in the source object of case as the type of the scrutiny. Thus, since in the base case there is no predecessor, f, will not require the scrutiny, and so it is ignored.

One major difference between SNNOs and the more traditional natural number objects is that in the definition of the latter g is defined by well-founded recursion. However, SNNOs do not allow this, but in the presence of fixpoints we are able to regain this feature without having to bake it into the definition of natural number objects. However, to allow this we have found that when combining fixpoints and case analysis to define terminating functions on the natural numbers it is necessary to uniformly construct the input to both f and g due to the reduction rule of fix. Thus, we extend the type of f to $Y \times \mathsf{Nat}$, but then ignore the second projection when reaching the base case.

$$\frac{}{\Delta_1, x : X, \Delta_2 \vdash_{\mathcal{P}} x = x : X}\ \textsc{PEq\_Ax} \qquad \frac{\Delta \vdash_{\mathcal{P}} t : \top}{\Delta \vdash_{\mathcal{P}} t = * : \top}\ \textsc{PEq\_Unit}$$

$$\frac{\Delta \vdash_{\mathcal{P}} t_1 : X \quad \Delta \vdash_{\mathcal{P}} t_2 : Y}{\Delta \vdash_{\mathcal{P}} \mathsf{proj}_1\,(t_1, t_2) = t_1 : X}\ \textsc{PEq\_Proj1} \qquad \frac{\Delta \vdash_{\mathcal{P}} t_1 : X \quad \Delta \vdash_{\mathcal{P}} t_2 : Y}{\Delta \vdash_{\mathcal{P}} \mathsf{proj}_2\,(t_1, t_2) = t_2 : Y}\ \textsc{PEq\_Proj2}$$

$$\frac{\Delta \vdash_{\mathcal{P}} t : X \times Y}{\Delta \vdash_{\mathcal{P}} (\mathsf{proj}_1\,t, \mathsf{proj}_2\,t) = t : X \times Y}\ \textsc{PEq\_EtaP}$$

$$\frac{\Delta \vdash_{\mathcal{P}} t = t' : X \times Y}{\Delta \vdash_{\mathcal{P}} \mathsf{proj}_1\,t = \mathsf{proj}_1\,t' : X \times Y}\ \textsc{PEq\_Proj1C}$$

$$\frac{\Delta \vdash_{\mathcal{P}} t = t' : X \times Y}{\Delta \vdash_{\mathcal{P}} \mathsf{proj}_2\,t = \mathsf{proj}_2\,t' : X \times Y}\ \textsc{PEq\_Proj2C}$$

$$\frac{\Delta \vdash_{\mathcal{P}} t_1 = t_1' : X \quad \Delta \vdash_{\mathcal{P}} t_2 = t_2' : Y}{\Delta \vdash_{\mathcal{P}} (t_1, t_2) = (t_1, t_2') : X \times Y}\ \textsc{PEq\_PairC}$$

$$\frac{\begin{array}{l}\Delta \vdash_{\mathcal{P}} t' : Y \\ \Delta, x : X \vdash_{\mathcal{P}} t : Y\end{array}}{\Delta \vdash_{\mathcal{P}} (\lambda x : X.t)\,t' = [t'/x]t : Y}\ \textsc{PEq\_Beta}$$

$$\frac{\Delta \vdash_{\mathcal{P}} t : X \to Y \quad x \notin \mathsf{FV}(t)}{\Delta \vdash_{\mathcal{P}} (\lambda x : X.t\,x) = t : X \to Y}\ \textsc{PEq\_Eta}$$

$$\frac{\Delta, x : X \vdash_{\mathcal{P}} t = t' : Y}{\Delta \vdash_{\mathcal{P}} \lambda x : X.t = \lambda x : X.t' : Y}\ \textsc{PEq\_FunC} \qquad \frac{\Delta \vdash_{\mathcal{P}} t : X \to X}{\Delta \vdash_{\mathcal{P}} \mathsf{fix}\,t = t\,(\mathsf{fix}\,t) : X}\ \textsc{PEq\_Fix}$$

$$\frac{\Delta \vdash_{\mathcal{P}} t = t' : X \to X}{\Delta \vdash_{\mathcal{P}} \mathsf{fix}\,t = \mathsf{fix}\,t' : X}\ \textsc{PEq\_FixC} \qquad \frac{\Delta \vdash_{\mathcal{P}} t = t' : \mathsf{Nat}}{\Delta \vdash_{\mathcal{P}} \mathbf{suc}\,t = \mathbf{suc}\,t' : \mathsf{Nat}}\ \textsc{PEq\_Suc}$$

$$\frac{\Delta \vdash_{\mathcal{P}} t_1 : X \quad \Delta, x : \mathsf{Nat} \vdash_{\mathcal{P}} t_2 : X}{\Delta \vdash_{\mathcal{P}} \mathsf{case}\,0\,\{0 \to t_1, \mathbf{suc}\,x \to t_2\} = t_1 : X}\ \textsc{PEq\_CaseB}$$

$$\frac{\begin{array}{l}\Delta \vdash_{\mathcal{P}} n : \mathsf{Nat} \\ \Delta \vdash_{\mathcal{P}} t_1 : X \quad \Delta, x : \mathsf{Nat} \vdash_{\mathcal{P}} t_2 : X\end{array}}{\Delta \vdash_{\mathcal{P}} \mathsf{case}\,(\mathbf{suc}\,n)\,\{0 \to t_1, \mathbf{suc}\,x \to t_2\} = [n/x]t_2 : X}\ \textsc{PEq\_CaseS}$$

$$\frac{\begin{array}{l}\Delta \vdash_{\mathcal{P}} t_1 = t_1' : X \\ \Delta, x : \mathsf{Nat} \vdash_{\mathcal{P}} t_2 = t_2' : X \quad \Delta \vdash_{\mathcal{P}} t = t' : \mathsf{Nat}\end{array}}{\Delta \vdash_{\mathcal{P}} \mathsf{case}\,t\,\{0 \to t_1, \mathbf{suc}\,x \to t_2\} = \mathsf{case}\,t'\,\{0 \to t_1', \mathbf{suc}\,x \to t_2'\} : X}\ \textsc{PEq\_CaseC}$$

$$\frac{\Delta \vdash_{\mathcal{P}} t : X \quad \Delta, x : X \vdash_{\mathcal{P}} t_1 = t_2 : Y}{\Delta \vdash_{\mathcal{P}} [t/x]t_1 = [t/x]t_2 : Y}\ \textsc{PEq\_Subst} \qquad \frac{\Delta \vdash_{\mathcal{P}} t_1 = t_2 : X}{\Delta, \Delta' \vdash_{\mathcal{P}} t_1 = t_2 : X}\ \textsc{PEq\_Weak}$$

$$\frac{\Delta_1, x : X, y : Y, \Delta_2 \vdash_{\mathcal{P}} t_1 = t_2 : Z}{\Delta_1, y : Y, x : X, \Delta_2 \vdash_{\mathcal{P}} t_1 = t_2 : Z}\ \textsc{PEq\_Ex} \qquad \frac{\Delta \vdash_{\mathcal{P}} t : X}{\Delta \vdash_{\mathcal{P}} t = t : X}\ \textsc{PEq\_Refl}$$

$$\frac{\Delta \vdash_{\mathcal{P}} t_1 = t_2 : X}{\Delta \vdash_{\mathcal{P}} t_2 = t_1 : X}\ \textsc{PEq\_Sym} \qquad \frac{\Delta \vdash_{\mathcal{P}} t_1 = t_2 : X \quad \Delta \vdash_{\mathcal{P}} t_2 = t_3 : X}{\Delta \vdash_{\mathcal{P}} t_1 = t_3 : X}\ \textsc{PEq\_Trans}$$

Figure 4: Programmatic Fragment: Equality

We now have all the necessary structure to describe the model of the programmatic fragment.

Definition 4. *A model of the programmatic fragment, called a $\mathcal{P}$-**model**, consists of a cartesian closed category, $\mathcal{C}$, with fixpoints and a SNNO.*

The following definition gives an interpretation of types and typing contexts into a $\mathcal{P}$-model.

Definition 5. *Suppose $\mathcal{P}$ is a $\mathcal{P}$-model.*

- *The interpretation of types as objects is as follows:*

$$\begin{array}{lcl} [\![\top]\!] & = & 1 \\ [\![\mathsf{Nat}]\!] & = & \mathsf{Nat} \\ [\![X \times Y]\!] & = & [\![X]\!] \times [\![Y]\!] \\ [\![X \to Y]\!] & = & [\![X]\!] \to [\![Y]\!] \end{array}$$

- *The interpretation of typing contexts is as follows:*

$$\begin{array}{lcl} [\![\cdot]\!] & = & 1 \\ [\![\Delta, x_1 : X_1]\!] & = & [\![\Delta]\!] \times [\![X_1]\!] \end{array}$$

We always assume that the interpretation of typing contexts is right-associated.

The following result is mostly known, but we give the cases of the proof that are most interesting. The cases for SNNOs are novel, but straightforward given their definition.

Theorem 6 (Soundness)**.** *Suppose $\mathcal{P}$ is a $\mathcal{P}$-model.*

i. If $\Delta \vdash_{\mathcal{P}} t : X$, then there is a morphism, $[\![t]\!] : [\![\Delta]\!] \longrightarrow [\![X]\!]$, in $\mathcal{P}$, and

ii. If $\Delta \vdash_{\mathcal{P}} t = t' : X$, then $[\![t]\!] = [\![t']\!]$ in $\mathcal{P}$.

Proof. Please see the full proof in Appendix B.1. □

3 A Termination/Non-Termination Adjoint Model

We have so far laid out both the syntactic and categorical formalizations of the logical fragment and the programmatic fragment. At this point we turn to joining these two fragments together using an adjunction, but before this we must acquaint ourselves with the basic notions involved in the construction of such a model.

3.1 The Basic Tools

We have been working in cartesian closed categories, but much of the basic categorical tools we will employ arose from the study of symmetric monoidal closed categories which are at the heart of categorical models of linear logic. This section consists of well-known categorical tools and is based on the Benton's presentation; see [**?**] for references on the history of these notions.

A symmetric monoidal closed category, $(\mathcal{C}, \otimes, I, \multimap, \lambda, \rho, \beta, \alpha)$, consists of a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, called the tensor product of $\mathcal{C}$, which is associative, there is a natural isomorphism, $\alpha_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$, has a unit I, that is, there are natural isomorphisms $\lambda_A : A \otimes I \to I$ and $\rho_A : I \otimes A \to A$, the tensor product is symmetric and hence there is a natural isomorphism $\beta_{A,B} : A \otimes B \to B \otimes A$, and finally, the category is closed. That is, the following is a natural bijection:

$$\mathsf{Hom}_{\mathcal{C}}(A \otimes B, C) \cong \mathsf{Hom}_{\mathcal{C}}(A, B \multimap C)$$

Furthermore, the natural transformations in the definition of a symmetric monoidal category are subject to several coherence conditions. For the complete definition see Appendix A.

A functor, (F, m), between the two symmetric monoidal categories $(\mathcal{M}_1, \otimes_1, I_1)$ and $(\mathcal{M}_2, \otimes_2, I_2)$ is an ordinary functor $F : \mathcal{M}_1 \longrightarrow \mathcal{M}_2$ such that there is a natural transformation $\mathsf{m}_{A,B} : FA \otimes_2 FB \longrightarrow F(A \otimes_1 B)$ and a morphism $\mathsf{m}_I : I_2 \longrightarrow FI_1$. These are also subject to some coherence conditions; see Appendix A. Natural transformations between symmetric monoidal functors, (F, m) and (G, n) are ordinary natural transformations $f : F \to G$ subject to the following coherence conditions:

$$\begin{tikzcd}
FA \otimes_2 FB \arrow[r, "m_{A,B}"] \arrow[d, "f_A \otimes_2 f_B"'] & F(A \otimes_1 B) \arrow[d, "f_{A \otimes_1 B}"] \\
GA \otimes_2 GB \arrow[r, "n_{A,B}"'] & G(A \otimes_1 B)
\end{tikzcd}
\qquad
\begin{tikzcd}
FI_1 \arrow[rr, "f_{I_1}"] & & GI_1 \\
& I_2 \arrow[ul, "m_{I_1}"] \arrow[ur, "n_{I_1}"'] &
\end{tikzcd}$$

We will be making heavy use of symmetric monoidal adjunctions, and so we give their full definition.

Definition 7. *Suppose $(\mathcal{M}_1, I_1, \otimes_1)$ and $(\mathcal{M}_2, I_2, \otimes_2)$ are SMCs, and (F, m) is a symmetric monoidal functor between $\mathcal{M}_1$ and $\mathcal{M}_2$ and (G, n) is a symmetric monoidal functor between $\mathcal{M}_2$ and $\mathcal{M}_1$. Then a* ***symmetric monoidal adjunction*** *is an ordinary adjunction $\mathcal{M}_1 : F \dashv G : \mathcal{M}_2$ such that the unit, $\varepsilon : A \to GFA$, and the counit, $\eta_A : FGA \to A$, are symmetric monoidal natural* *transformations. Thus, the following diagrams must commute:*

$$\begin{tikzcd}
FGA \otimes_1 FGB \arrow[r, "\mathsf{q}_{A,B}"] \arrow[dr, "\eta_A \otimes_1 \eta_B"'] & FG(A \otimes_1 B) \arrow[d, "\eta_{A \otimes_1 B}"] \\
& A \otimes_1 B
\end{tikzcd}
\qquad
\begin{tikzcd}
FGI_1 \arrow[rr, "\eta_{I_1}"] & & I_1 \\
& I_1 \arrow[ul, "\mathsf{q}_{I_1}"] \arrow[ur, equal] &
\end{tikzcd}$$

$$\begin{tikzcd}
& A \otimes_2 B \arrow[dl, "\varepsilon_A \otimes_2 \varepsilon_B"'] \arrow[d, "\varepsilon_{A \otimes_2 B}"] \\
GFA \otimes_2 GFB \arrow[r, "\mathsf{p}_{A,B}"'] & GF(A \otimes_2 B)
\end{tikzcd}
\qquad
\begin{tikzcd}
I_2 \arrow[rr, "\varepsilon_{I_2}"] \arrow[dr, equal] & & GFI_2 \\
& I_2 \arrow[ur, "\mathsf{p}_{I_2}"'] &
\end{tikzcd}$$

Note that p *and* q *exist because* (FG, q) *and* (GF, p) *are symmetric monoidal functors.*

In addition, we make use of symmetric monoidal monads and comonads. We give their definition next.

Definition 8. *A* ***symmetric monoidal monad*** *on a symmetric monoidal category* $\mathcal{C}$ *is a triple* (T, η, μ)*, where* (T, n) *is a symmetric monoidal endofunctor on* $\mathcal{C}$*,* $\eta_A : A \longrightarrow TA$ *and* $\mu_A : T^2A \rightarrow TA$ *are symmetric monoidal natural transformations, which make the following diagrams commute:*

$$\begin{tikzcd}
T^3A \arrow[r, "\mu_{TA}"] \arrow[d, "T\mu_A"'] & T^2A \arrow[d, "\mu_A"] \\
T^2A \arrow[r, "\mu_A"'] & TA
\end{tikzcd}
\qquad
\begin{tikzcd}
& TA & \\
TA \arrow[ur, equal] \arrow[r, "\eta_{TA}"'] & T^2A \arrow[u, "\mu_A"] & TA \arrow[ul, equal] \arrow[l, "T\eta_A"]
\end{tikzcd}$$

Definition 9. *A* ***symmetric monoidal comonad*** *on a symmetric monoidal category* $\mathcal{C}$ *is a triple* (T, ε, δ)*, where* (T, m) *is a symmetric monoidal endofunctor on* $\mathcal{C}$*,* $\varepsilon_A : TA \longrightarrow A$ *and* $\delta_A : TA \rightarrow T^2A$ *are symmetric monoidal natural transformations, which make the following diagrams commute:*

$$\begin{tikzcd}
TA \arrow[r, "\delta_A"] \arrow[d, "\delta_A"'] & T^2A \arrow[d, "T\delta_A"] \\
T^2A \arrow[r, "\delta_{TA}"'] & T^3A
\end{tikzcd}
\qquad
\begin{tikzcd}
& TA \arrow[dl, equal] \arrow[d, "\delta_A"] \arrow[dr, equal] & \\
TA & T^2A \arrow[l, "\varepsilon_{TA}"] \arrow[r, "T\varepsilon_A"'] & TA
\end{tikzcd}$$

The use of symmetric monoidal natural transformations in the previous two definitions packs a lot of punch. See what these conditions look like in Appendix A.

Specializing the basic tools just presented to cartesian categories has surprising implications especially when considering symmetric monoidal adjunctions.

3.2 Symmetric Monoidal Adjunctions on Cartesian Categories

First, it is well-known that any cartesian category is a symmetric monoidal category. The following lemma summarizes this structure.

Lemma 10 (CCCs are SMCs)**.** *Suppose $(\mathcal{C}, 1, \times, \Rightarrow)$ is a cartesian closed category. Then the following defines the structure of a symmetric monoidal category:*

$$
\begin{array}{rcl}
\lambda_A & = & \pi_2 : 1 \times A \longrightarrow A \\
\lambda_A^{-1} & = & \langle \mathsf{t}_A, \mathsf{id}_A \rangle : A \longrightarrow 1 \times A \\
\\
\rho_A & = & \pi_1 : A \times 1 \longrightarrow A \\
\rho_A^{-1} & = & \langle \mathsf{id}_A, \mathsf{t}_A \rangle : A \longrightarrow A \times 1 \\
\\
\alpha_{A,B,C} & = & \langle \pi_1; \pi_1, \pi_2 \times \mathsf{id}_C \rangle : (A \times B) \times C \longrightarrow A \times (B \times C) \\
\alpha_{A,B,C}^{-1} & = & \langle \mathsf{id}_A \times \pi_1, \pi_2; \pi_2 \rangle : A \times (B \times C) \longrightarrow (A \times B) \times C \\
\\
\beta_{A,B} & = & \langle \pi_2, \pi_1 \rangle : A \times B \longrightarrow B \times A
\end{array}
$$

Each of the above morphisms satisfy the appropriate diagrams.

Symmetric monoidal functors can be seen as functors that essentially preserve the monoidal structure, and this kind of preservation can be given to other types of functors. The following definition describes how symmetric monoidal functors can be modified to preserve the cartesian structure of a cartesian category. To the knowledge of the author product functors are novel.

Definition 11. *A* ***product functor****, $(F, m) : \mathcal{C}_1 \to \mathcal{C}_2$, between two cartesian categories consists of an ordinary functor $F : \mathcal{C}_1 \to \mathcal{C}_2$, a natural transformation $m_{A,B} : FA \times FB \longrightarrow F(A \times B)$, and a map $m_1 : 1 \to F1$ subject to the following coherence diagrams:*

$$
\begin{tikzcd}
FA \times FB \arrow[d, "m_{A.B}"'] \arrow[dr, "\pi_1"] & \\
F(A \times B) \arrow[r, "F\pi_1"] & FA
\end{tikzcd}
\qquad
\begin{tikzcd}
FA \times FB \arrow[d, "m_{A.B}"'] \arrow[dr, "\pi_2"] & \\
F(A \times B) \arrow[r, "F\pi_2"] & FB
\end{tikzcd}
$$

$$
\begin{tikzcd}
FC \arrow[d, "\langle Ff, Fg \rangle"'] \arrow[dr, "F(\langle f, g \rangle)"] & \\
FA \times FB \arrow[r, "m_{A,B}"] & F(A \times B)
\end{tikzcd}
\qquad
\begin{tikzcd}
FA \arrow[d, "\mathsf{t}_{FA}"'] \arrow[dr, "F\mathsf{t}_A"] & \\
1 \arrow[r, "m_1"] & F1
\end{tikzcd}
$$

It is not surprising that product functors are symmetric monoidal functors.

Lemma 12 (Product Functors are Symmetric Monoidal)**.** *If $(F, m) : \mathcal{C}_1 \to \mathcal{C}_2$ is a product functor betwen cartesian categories, then (F, m) is a symmetric monoidal functor.*

A symmetric monoidal functor, (F, m), is called **strong** if $\mathsf{m}_{A,B} : FA \otimes_2 FB \longrightarrow F(A \otimes_1 B)$ and $\mathsf{m}_I : I_2 \longrightarrow FI_1$ are isomorphisms. It turns out that product functors are in fact strong.

Lemma 13 (Product Functors Isomorphisms)**.** *If $(F, m) : \mathcal{C}_1 \longrightarrow \mathcal{C}_2$ is a product functor, then $m_{A,B} : FA \times FB \longrightarrow F(A \times B)$ and $m_1 : 1 \longrightarrow F1$ are isomorphisms.*

Surprisingly, strong symmetric monoidal functors between cartesian categories preserve the cartesian structure, and hence, are in fact product functors.

Lemma 14 (Strong Symmetric Monoidal Functors are Product Functors)**.** *If $(F, m) : \mathcal{C}_1 \longrightarrow \mathcal{C}_2$ is a strong symmetric monoidal functor between cartesian closed categories, then (F, m) is a product functor.*

The following lemma shows that the left adjoint in any symmetric monoidal adjunction is strong, and hence, when both categories in the adjunction are cartesian we know the left adjoint preserves the cartesian structure. This was first shown by Benton but for LNL models where one of the categories is cartesian closed, and thus, the following result is a generalization [**?**].

Lemma 15 (Symmetric Monoidal Adjunctions have Strong Left-adjoints)**.** *If $(F, m) \dashv (G, n) : \mathcal{C}_1 \longrightarrow \mathcal{C}_2$ is any symmetric monoidal adjunction, then F is strong.*

Proof. In [**?**] Benton shows that this result holds for an LNL model, but his exact proof works for the case when both categories in the adjunction are symmetric monoidal. □

If we have a symmetric monoidal adjunction between cartesian categories we know from the previous result that the left adjoint is a product functor, but it turns out that we also know the right adjoint must be a product functor as well, because right adjoints preserve limits.

Lemma 16 (Symmetric Monoidal Adjunctions and Product Functors)**.** *Suppose $(F, m) : \mathcal{C}_1 \to \mathcal{C}_2$ and $(G, n) : \mathcal{C}_2 \to \mathcal{C}_1$ are symmetric monoidal functors between cartesian-closed categories, and $F \vdash G$ is a symmetric monoidal adjunction. Then (F, m) and (G, n) are product functors.*

Proof. We know from Lemma 15 (F, m) is strong, and hence, by Lemma 14 it is a product functor. Now we must show that (G, n) is strong, but it is a right adjoint, and hence, preserves all limits. Thus, it is strong. □

The remainder of the paper will will make heavy use of product functors due to the last result. To make the exposition more clear we will call symmetric monoidal natural transformations between product functors **product natural transformations** and symmetric monoidal adjunctions **product adjunctions**.

Following Moggi's [**?**] lead we must be sure that the symmetric monoidal monad arising from the symmetric monoidal adjunction is strong for it to be programmaticly useful. However, monadic strength is part of the monoidal structure.

Lemma 17 (Monadic Strength)**.** *Suppose (T, η, μ) is a monoidal monad on a monoidal category $\mathcal{C}$. Then the strength maps exists:*

$$\mathsf{st}_{A,B} : A \otimes TB \longrightarrow T(A \otimes B)$$

Furthermore, the following diagrams commute:

$$\begin{tikzcd}
I \otimes TA \arrow[r, "\mathsf{st}_{I,A}"] \arrow[dr, "\rho_A"'] & T(I \otimes A) \arrow[d, "T\rho_A"] \\
& TA
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes B \arrow[r, "\mathsf{id}_A \otimes \eta_B"] \arrow[dr, "\eta_{A \times B}"'] & A \otimes TB \arrow[d, "\mathsf{st}_{A,B}"] \\
& T(A \otimes B)
\end{tikzcd}$$

$$\begin{tikzcd}
(A \otimes B) \otimes TC \arrow[rr, "\mathsf{st}_{A \otimes B,C}"] \arrow[d, "\alpha_{A,B,TC}"'] & & T((A \otimes B) \otimes C) \arrow[d, "T\alpha_{A,B,C}"] \\
A \otimes (B \otimes TC) \arrow[r, "\mathsf{id}_A \otimes \mathsf{st}_{B,C}"] & A \otimes T(B \otimes C) \arrow[r, "\mathsf{st}_{A,T(B \otimes C)}"] & T(A \otimes (B \otimes C))
\end{tikzcd}$$

$$\begin{tikzcd}
A \otimes T^2B \arrow[rr, "\mathsf{id}_A \otimes \mu_B"] \arrow[d, "\mathsf{st}_{A,TB}"'] & & A \otimes TB \arrow[d, "\mathsf{st}_{A,B}"] \\
T(A \otimes TB) \arrow[r, "T\mathsf{st}_{A,B}"] & T^2(A \otimes B) \arrow[r, "\mu_{A \otimes B}"] & T(A \otimes B)
\end{tikzcd}$$

Proof. The strength map is defined as follows:

$$\mathsf{st}_{A,B} = (\eta_A \otimes \mathsf{id}_{TB}); \mathsf{m}_{A,B} : A \otimes TB \longrightarrow T(A \otimes B)$$

where $\mathsf{m}_{A,B} : TA \otimes TB \longrightarrow T(A \otimes B)$ is the natural transformation arising from the fact that T is a monoidal endofunctor on $\mathcal{C}$. See Appendix B.2 for the rest of the proof. $\square$

Brookes and Van Stone [**?**] define comonadic strength similarly to the monadic strength above, $\mathsf{cm\text{-}st}_{A,B} : A \otimes SB \longrightarrow S(A \otimes B)$ given a monoidal comonad (S, ε, δ), but this is not constructible from the monoidal structure like monadic strength, because we have no means of injecting A into the comonad. Just as there is no way out of a monad, there is no way into a comonad. So if comonadic strength is necessary it will need to be taken as additional structure.

3.3 TNT Model

We have finally laid out everything we need to define the model of the theory we will be programming in.

Definition 18. *A* ***termination/non-termination adjoint model (TNT model)*** *is a symmetric monoidal adjunction $\mathcal{L} : T \vdash D : \mathcal{P}$ between a $\mathcal{P}$-model, $\mathcal{P}$, and a cartesian closed category $\mathcal{L}$. Thus, we know the following:*

i. *$\mathcal{P}$ is a cartesian closed category with fixpoints and a SNNO,*

ii. *(D, m) and (T, n) are product functors,*

iii. *$\uparrow = DT : \mathcal{L} \to \mathcal{L}$ is a symmetric monoidal monad, and*

iv. *$\downarrow = TD : \mathcal{P} \to \mathcal{P}$ is a symmetric monoidal comonad.*

In this model, the monad and comonad have some nice properties. First, by Lemma 17, we know $\uparrow : \mathcal{L} \to \mathcal{L}$ is strong, and hence, there is a natural transformation:

$$\mathsf{st}_{A,B} : A \times \uparrow B \longrightarrow \uparrow(A \times B)$$

In addition, due to $\uparrow : \mathcal{L} \to \mathcal{L}$ and $\downarrow : \mathcal{P} \to \mathcal{P}$ being defined as the composition of two product functors we know they themselves are product functors, and hence, we have the following natural isomorphisms:

$$\uparrow A \times \uparrow B \cong \uparrow(A \times B)$$

$$\downarrow A \times \downarrow B \cong \downarrow(A \times B)$$

4 TNT Calculus

5 Related Work

Write

6 Conclusion

Write

References

A Symmetric Monoidal Closed Categories

Definition 19. *A* ***symmetric monoidal category (SMC)*** *is a category, $\mathcal{M}$, with the following data:*

- *An object I of $\mathcal{M}$,*
- *A bi-functor $\otimes : \mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{M}$,*
- *The following natural isomorphisms:*

$$\begin{aligned}&\lambda_A : I \otimes A \longrightarrow A\\&\rho_A : A \otimes I \longrightarrow A\\&\alpha_{A,B,C} : (A \otimes B) \otimes C \longrightarrow A \otimes (B \otimes C)\end{aligned}$$

- *A natural transformation:*

$$\beta_{A,B} : A \otimes B \longrightarrow B \otimes A$$

- *Subject to the following coherence diagrams:*

$$\begin{tikzcd}
((A \otimes B) \otimes C) \otimes D \arrow[r, "\alpha_{A,B,C} \otimes \mathsf{id}_D"] \arrow[d, "\alpha_{A \otimes B, C, D}"] & (A \otimes (B \otimes C)) \otimes D \arrow[dd, "\alpha_{A, B \otimes C, D}"] \\
(A \otimes B) \otimes (C \otimes D) \arrow[d, "\alpha_{A,B,C \otimes D}"] & \\
A \otimes (B \otimes (C \otimes D)) & A \otimes ((B \otimes C) \otimes D) \arrow[l, "\mathsf{id}_A \otimes \alpha_{B,C,D}"']
\end{tikzcd}$$

$$\begin{tikzcd}
(A \otimes B) \otimes C \arrow[r, "\alpha_{A,B,C}"] \arrow[d, "\beta_{A,B} \otimes \mathsf{id}_C"] & A \otimes (B \otimes C) \arrow[r, "\beta_{A, B \otimes C}"] & (B \otimes C) \otimes A \arrow[d, "\alpha_{B,C,A}"] \\
(B \otimes A) \otimes C \arrow[r, "\alpha_{B,A,C}"] & B \otimes (A \otimes C) \arrow[r, "\mathsf{id}_B \otimes \beta_{A,C}"] & B \otimes (C \otimes A)
\end{tikzcd}$$

$$\begin{tikzcd}
(A \otimes I) \otimes B \arrow[rr, "\alpha_{A,I,B}"] \arrow[dr, "\rho_A"'] & & A \otimes (I \otimes B) \arrow[dl, "\lambda_B"] \\
& A \otimes B &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes B \arrow[d, "\beta_{A,B}"'] \arrow[dr, "\mathsf{id}_{A \otimes B}"] & \\
B \otimes A \arrow[r, "\beta_{B,A}"'] & A \otimes B
\end{tikzcd}$$

$$\begin{tikzcd}
I \otimes A \arrow[rr, "\beta_{I,A}"] \arrow[dr, "\lambda_A"'] & & A \otimes I \arrow[dl, "\rho_A"] \\
& A &
\end{tikzcd}$$

Definition 20. *A **symmetric monoidal closed category (SMCC)** is a symmetric monoidal category, $(\mathcal{M}, I, \otimes)$, such that, for any object B of $\mathcal{M}$, the functor $- \otimes B : \mathcal{M} \longrightarrow \mathcal{M}$ has a specified right adjoint. Hence, for any objects A and C of $\mathcal{M}$ there is an object $A \multimap B$ of $\mathcal{M}$ and a natural bijection:*

$$\mathsf{Hom}_{\mathcal{M}}(A \otimes B)C \cong \mathsf{Hom}_{\mathcal{M}}(A)B \multimap C$$

Definition 21. *Suppose $(\mathcal{M}_1, I_1, \otimes_1, \alpha_1, \lambda_1, \rho_1, \beta_1)$ and $(\mathcal{M}_2, I_2, \otimes_2, \alpha_2, \lambda_2, \rho_2, \beta_2)$ are SMCs. Then a **symmetric monoidal functor** is a functor $F : \mathcal{M}_1 \longrightarrow \mathcal{M}_2$, a map $m_I : I_2 \longrightarrow FI_1$ and a natural transformation*

$m_{A,B} : FA \otimes_2 FB \longrightarrow F(A \otimes_1 B)$ *subject to the following coherence conditions:*

$$\begin{tikzcd}
(FA \otimes_2 FB) \otimes_2 FC \arrow[r, "\alpha_{2\,FA,FB,FC}"] \arrow[d, "m_{A,B} \otimes \mathsf{id}_{FC}"] & FA \otimes_2 (FB \otimes_2 FC) \arrow[d, "\mathsf{id}_{FA} \otimes m_{B,C}"] \\
F(A \otimes_1 B) \otimes_2 FC \arrow[d, "m_{A \otimes_1 B, C}"] & FA \otimes_2 F(B \otimes_1 C) \arrow[d, "m_{A, B \otimes_1 C}"] \\
F((A \otimes_1 B) \otimes_1 C) \arrow[r, "F\alpha_{1\,A,B,C}"] & F(A \otimes_1 (B \otimes_1 C))
\end{tikzcd}$$

$$\begin{tikzcd}
I_2 \otimes_2 FA \arrow[r, "\lambda_{2\,FA}"] \arrow[d, "m_I \otimes \mathsf{id}_{FA}"] & FA \\
FI_1 \otimes_2 FA \arrow[r, "m_{I_1,A}"] & F(I_1 \otimes_1 A) \arrow[u, "F\lambda_{1\,A}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
FA \otimes_2 I_2 \arrow[r, "\rho_{2\,FA}"] \arrow[d, "\mathsf{id}_{FA} \otimes m_I"] & FA \\
FA \otimes_2 FI_1 \arrow[r, "m_{A,I_1}"] & F(A \otimes_1 I_1) \arrow[u, "F\rho_{1\,A}"]
\end{tikzcd}$$

$$\begin{tikzcd}
FA \otimes_2 FB \arrow[r, "\beta_{2\,FA,FB}"] \arrow[d, "m_{A,B}"] & FB \otimes_2 FA \arrow[d, "m_{B,A}"] \\
F(A \otimes_1 B) \arrow[r, "F\beta_{1\,A,B}"] & F(B \otimes_1 A)
\end{tikzcd}$$

Definition 22. *Suppose $(\mathcal{M}_1, I_1, \otimes_1)$ and $(\mathcal{M}_2, I_2, \otimes_2)$ are SMCs, and (F, m) and (G, n) are a symmetric monoidal functors between $\mathcal{M}_1$ and $\mathcal{M}_2$. Then a **symmetric monoidal natural transformation** is a natural transformation, $f : F \longrightarrow G$, subject to the following coherence diagrams:*

$$\begin{tikzcd}
FA \otimes_2 FB \arrow[r, "m_{A,B}"] \arrow[d, "f_A \otimes_2 f_B"'] & F(A \otimes_1 B) \arrow[d, "f_{A \otimes_1 B}"] \\
GA \otimes_2 GB \arrow[r, "n_{A,B}"'] & G(A \otimes_1 B)
\end{tikzcd}
\qquad
\begin{tikzcd}
FI_1 \arrow[rr, "f_{I_1}"] & & GI_1 \\
& I_2 \arrow[ul, "m_{I_1}"] \arrow[ur, "n_{I_1}"'] &
\end{tikzcd}$$

Definition 23. *Suppose $(\mathcal{M}_1, I_1, \otimes_1)$ and $(\mathcal{M}_2, I_2, \otimes_2)$ are SMCs, and (F, m) is a symmetric monoidal functor between $\mathcal{M}_1$ and $\mathcal{M}_2$ and (G, n) is a symmetric monoidal functor between $\mathcal{M}_2$ and $\mathcal{M}_1$. Then a **symmetric monoidal adjunction** is an ordinary adjunction $\mathcal{M}_1 : F \dashv G : \mathcal{M}_2$ such that the unit, $\eta : A \to GFA$, and the counit, $\varepsilon_A : FGA \to A$, are symmetric monoidal natural* *transformations. Thus, the following diagrams must commute:*

$$
\begin{tikzcd}
FGA \otimes_1 FGB \arrow[r, "\mathsf{q}_{A,B}"] \arrow[dr, "\varepsilon_A \otimes_1 \varepsilon_B"'] & FG(A \otimes_1 B) \arrow[d, "\varepsilon_{A \otimes_1 B}"] \\
& A \otimes_1 B
\end{tikzcd}
\qquad
\begin{tikzcd}
FGI_1 \arrow[rr, "\varepsilon_{I_1}"] & & I_1 \\
& I_1 \arrow[ul, "\mathsf{q}_{I_1}"] \arrow[ur, equal] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
& A \otimes_2 B \arrow[dl, "\eta_A \otimes_2 \eta_B"'] \arrow[d, "\eta_{A \otimes_2 B}"] \\
GFA \otimes_2 GFB \arrow[r, "\mathsf{p}_{A,B}"'] & GF(A \otimes_2 B)
\end{tikzcd}
\qquad
\begin{tikzcd}
I_2 \arrow[rr, "\eta_{I_2}"] \arrow[dr, equal] & & GFI_2 \\
& I_2 \arrow[ur, "p_{I_2}"'] &
\end{tikzcd}
$$

Note that p *and* q *exist because* (FG, q) *and* (GF, p) *are symmetric monoidal functors.*

Definition 24. *A* ***symmetric monoidal monad*** *on a symmetric monoidal category* $\mathcal{C}$ *is a triple* (T, η, μ)*, where* (T, n) *is a symmetric monoidal endofunctor on* $\mathcal{C}$*,* $\eta_A : A \longrightarrow TA$ *and* $\mu_A : T^2A \rightarrow TA$ *are symmetric monoidal natural transformations, which make the following diagrams commute:*

$$
\begin{tikzcd}
T^3A \arrow[r, "\mu_{TA}"] \arrow[d, "T\mu_A"'] & T^2A \arrow[d, "\mu_A"] \\
T^2A \arrow[r, "\mu_A"'] & TA
\end{tikzcd}
\qquad
\begin{tikzcd}
& TA & \\
TA \arrow[ur, equal] \arrow[r, "\eta_{TA}"'] & T^2A \arrow[u, "\mu_A"] & TA \arrow[ul, equal] \arrow[l, "T\eta_A"]
\end{tikzcd}
$$

The assumption that η *and* μ *are symmetric monoidal natural transformations amount to the following diagrams commuting:*

$$
\begin{tikzcd}
& A \otimes B \arrow[dl, "\eta_A \otimes \eta_B"'] \arrow[d, "\eta_A"] \\
TA \otimes TB \arrow[r, "\mathsf{n}_{A,B}"'] & T(A \otimes B)
\end{tikzcd}
\qquad
\begin{tikzcd}
I \arrow[rr, "\eta_I"] \arrow[dr, equal] & & TI \\
& I \arrow[ur, "\mathsf{n}_I"'] &
\end{tikzcd}
$$

$$\begin{tikzcd}
T^2A \otimes T^2B \arrow[r, "\mathsf{n}_{TA,TB}"] \arrow[d, "\mu_A \otimes \mu_B"'] & T(TA \otimes TB) \arrow[r, "T\mathsf{n}_{A,B}"] & T^2(A \otimes B) \arrow[d, "\mu_{A \otimes B}"] \\
TA \otimes TB \arrow[rr, "\mathsf{n}_{A,B}"] & & T(A \otimes B)
\end{tikzcd}$$

$$\begin{tikzcd}
I \arrow[r, "\mathsf{n}_I"] \arrow[d, "\mathsf{n}_T"'] & TI \arrow[d, "T\mathsf{n}_I"] \\
TI & TI \arrow[l, "\mu_I"]
\end{tikzcd}$$

Definition 25. *A **symmetric monoidal comonad** on a symmetric monoidal category $\mathcal{C}$ is a triple (T, ε, δ), where (T, m) is a symmetric monoidal endofunctor on $\mathcal{C}$, $\varepsilon_A : TA \longrightarrow A$ and $\delta_A : TA \to T^2A$ are symmetric monoidal natural transformations, which make the following diagrams commute:*

$$\begin{tikzcd}
TA \arrow[r, "\delta_A"] \arrow[d, "\delta_A"'] & T^2A \arrow[d, "T\delta_A"] \\
T^2A \arrow[r, "\delta_{TA}"] & T^3A
\end{tikzcd}
\qquad
\begin{tikzcd}
& TA \arrow[dl, equal] \arrow[d, "\delta_A"] \arrow[dr, equal] & \\
TA & T^2A \arrow[l, "\varepsilon_{TA}"] \arrow[r, "T\varepsilon_A"'] & TA
\end{tikzcd}$$

The assumption that ε and δ are symmetric monoidal natural transformations amount to the following diagrams commuting:

$$\begin{tikzcd}
TA \otimes TB \arrow[r, "\mathsf{m}_{A,B}"] \arrow[dr, "\varepsilon_A \otimes \varepsilon_B"'] & T(A \otimes B) \arrow[d, "\varepsilon_{A \otimes B}"] \\
& A \otimes B
\end{tikzcd}
\qquad
\begin{tikzcd}
TI \arrow[rr, "\varepsilon_I"] & & I \\
& I \arrow[ul, "\mathsf{m}_I"] \arrow[ur, equal] &
\end{tikzcd}$$

$$
\begin{tikzcd}
TA \otimes TB \arrow[rr, "\mathsf{m}_{A,B}"] \arrow[d, "\delta_A \otimes \delta_B"] & & T(A \otimes B) \arrow[d, "\delta_{A \otimes B}"] \\
T^2A \otimes T^2B \arrow[r, "\mathsf{m}_{TA,TB}"'] & T(TA \otimes TB) \arrow[r, "T\mathsf{m}_{A,B}"'] & T^2(A \otimes B)
\end{tikzcd}
$$

$$
\begin{tikzcd}
I \arrow[r, "\mathsf{m}_I"] \arrow[d, "\mathsf{m}_I"'] & TI \arrow[d, "\delta_I"] \\
TI \arrow[r, "T\mathsf{m}_I"'] & T^2I
\end{tikzcd}
$$

i

B Proofs

B.1 Proof of Soundness (Theorem 6)

We will prove part two explicitly which depends on part one in such away that proving part one and then part two would require a lot of repeated constructions. We proceed by induction on the form of the assumed term-in-context derivation, but since the majority of the programmatic fragment is well-known to have a sound and complete interpretation into a cartesian closed category we only show the cases for natural numbers.

First, recall that β-equality can be defined as follows in the model – for any morphisms $f : B \times C \longrightarrow D$, $g : A \longrightarrow C$, and $h : A \to B$:

$$\begin{array}{rcl} \langle h; \mathsf{cur}(f), g\rangle; \mathsf{app}_{B,C} & = & \langle h, g\rangle; (\mathsf{cur}(f) \times \mathsf{id}_C); \mathsf{app}_{B,C} \\ & = & \langle h, g\rangle; f \end{array}$$

Using β-equality we can show what we call fix-equality – for any morphisms $f : A \to B$ and $g : B \to C$:

$$\begin{array}{rcl} \langle \mathsf{cur}(\pi_2; g), f\rangle; \mathsf{app}_{B,C} & = & \langle \mathsf{id}_A, f\rangle; \pi_2; g \\ & = & f; g \end{array}$$

Case.

$$\frac{\Delta \vdash_{\mathcal{P}} t : X \to X}{\Delta \vdash_{\mathcal{P}} \mathsf{fix}\, t = t\,(\mathsf{fix}\, t) : X}\ \textsc{Fix}$$

By part one we know there is a morphism:

$$[\![\Delta]\!] \xrightarrow{[\![t]\!]} [\![X]\!] \to [\![X]\!]$$

of $\mathcal{P}$. In addition, since any $\mathcal{P}$-model contains fixpoints we know there exists a morphism:

$$([\![X]\!] \to [\![X]\!]) \xrightarrow{\mathsf{fix}_{[\![X]\!]}} [\![X]\!]$$

The interpretation of $\Delta \vdash_{\mathcal{P}} \mathsf{fix} : (X \to X) \to X$ is as follows:

$$[\![\Delta]\!] \xrightarrow{[\![\Delta \vdash_{\mathcal{P}} \mathsf{fix} : (X \to X) \to X]\!] = \mathsf{cur}(\pi_2; \mathsf{fix}_{[\![X]\!]})} ([\![X]\!] \to [\![X]\!]) \to [\![X]\!]$$

We obtain our result by the following equational reasoning:

$$\begin{array}{ll} [\![\Delta \vdash_{\mathcal{P}} \mathsf{fix}\, t : X]\!] & \\ \quad = \langle \mathsf{cur}(\pi_2; \mathsf{fix}_{[\![X]\!]}), [\![t]\!]\rangle; \mathsf{app}_{[\![X]\!] \to [\![X]\!], [\![X]\!]} & \text{(Definition)} \\ \quad = [\![t]\!]; \mathsf{fix}_{[\![X]\!]} & \text{(fix-equality)} \\ \quad = [\![t]\!]; \langle \mathsf{id}_{[\![X]\!] \to [\![X]\!]}, \mathsf{fix}_{[\![X]\!]}\rangle; \mathsf{app}_{[\![X]\!], [\![X]\!]} & \text{(Fixpoint)} \\ \quad = \langle [\![t]\!], [\![t]\!]; \mathsf{fix}_{[\![X]\!]}\rangle; \mathsf{app}_{[\![X]\!], [\![X]\!]} & \text{(Cartesian)} \\ \quad = \langle [\![t]\!], \langle \mathsf{cur}(\pi_2; \mathsf{fix}_{[\![X]\!]}), [\![t]\!]\rangle; \mathsf{app}_{[\![X]\!] \to [\![X]\!], [\![X]\!]}\rangle; \mathsf{app}_{[\![X]\!], [\![X]\!]} & \text{(fix-equality)} \\ \quad = \langle [\![t]\!], \langle [\![\Delta \vdash_{\mathcal{P}} \mathsf{fix} : (X \to X) \to X]\!], [\![t]\!]\rangle; \mathsf{app}_{[\![X]\!] \to [\![X]\!], [\![X]\!]}\rangle; \mathsf{app}_{[\![X]\!], [\![X]\!]} & \text{(Definition)} \\ \quad = \langle [\![t]\!], [\![\Delta \vdash_{\mathcal{P}} \mathsf{fix}\, t : X]\!]\rangle; \mathsf{app}_{[\![X]\!], [\![X]\!]} & \text{(Definition)} \\ \quad = [\![\Delta \vdash_{\mathcal{P}} t\,(\mathsf{fix}\, t) : X]\!] & \text{(Definition)} \end{array}$$

Case.

$$\frac{\Delta \vdash_{\mathcal{P}} t_1 : X \quad \Delta, x : \mathsf{Nat} \vdash_{\mathcal{P}} t_2 : X}{\Delta \vdash_{\mathcal{P}} \mathsf{case}\, 0\, \{0 \to t_1, \mathbf{suc}\, x \to t_2\} = t_1 : X} \text{ CASEB}$$

We have the following morphisms by part one:

$$[\![\Delta]\!] \xrightarrow{[\![t_1]\!]} [\![X]\!]$$

$$[\![\Delta]\!] \times \mathsf{Nat} \xrightarrow{[\![t_2]\!]} [\![X]\!]$$

Then by the definition of SNNO we know there exists an unique morphism

$$Y \times \mathsf{Nat} \xrightarrow{\mathsf{case}_{[\![X]\!]}} [\![X]\!]$$

such that the following is one equation that holds:

$$(\mathsf{id}_{[\![\Delta]\!]} \times \mathsf{z}); \mathsf{case}_{[\![X]\!]} = \pi_1; [\![t_1]\!]$$

We also have the following interpretation for any $[\![\Delta \vdash_{\mathcal{P}} t : \mathsf{Nat}]\!]$:

$$[\![\mathsf{case}\, t\, \{0 \to t_1, \mathbf{suc}\, x \to t_2\}]\!] = \langle \mathsf{id}_{[\![\Delta]\!]}, [\![t]\!]\rangle; \mathsf{case}_{[\![X]\!]} : [\![\Delta]\!] \to [\![X]\!]$$

Thus, we obtain our desired equality because $[\![\Delta \vdash_{\mathcal{P}} 0 : \mathsf{Nat}]\!] = \mathsf{t}_{[\![\Delta]\!]}; \mathsf{z}$:

$$\begin{array}{lcl} \langle \mathsf{id}_{[\![\Delta]\!]}, \mathsf{t}_{[\![\Delta]\!]}; \mathsf{z}\rangle; \mathsf{case}_{[\![X]\!]} & = & \langle \mathsf{id}_{[\![\Delta]\!]}, \mathsf{t}_{[\![\Delta]\!]}\rangle; (\mathsf{id}_{[\![\Delta]\!]} \times \mathsf{z}); \mathsf{case}_{[\![X]\!]} \\ & = & \langle \mathsf{id}_{[\![\Delta]\!]}, \mathsf{t}_{[\![\Delta]\!]}\rangle; \pi_1; [\![t_1]\!] \\ & = & [\![t_1]\!]. \end{array}$$

Case.

$$\frac{\begin{array}{l} \Delta \vdash_{\mathcal{P}} n : \mathsf{Nat} \\ \Delta \vdash_{\mathcal{P}} t_1 : X \quad \Delta, x : \mathsf{Nat} \vdash_{\mathcal{P}} t_2 : X \end{array}}{\Delta \vdash_{\mathcal{P}} \mathsf{case}\, (\mathbf{suc}\, n)\, \{0 \to t_1, \mathbf{suc}\, x \to t_2\} = [n/x]t_2 : X} \text{ CASES}$$

We have the following morphisms by part one:

$$[\![\Delta]\!] \xrightarrow{[\![n]\!]} \mathsf{Nat}$$

$$[\![\Delta]\!] \xrightarrow{[\![t_1]\!]} [\![X]\!]$$

$$[\![\Delta]\!] \times \mathsf{Nat} \xrightarrow{[\![t_2]\!]} [\![X]\!]$$

Then by the definition of SNNO we know there exists an unique morphism

$$Y \times \mathsf{Nat} \xrightarrow{\mathsf{case}_{[\![X]\!]}} [\![X]\!]$$

such that the following is one equation that holds:

$$(\mathsf{id}_{[\![\Delta]\!]} \times \mathsf{suc}); \mathsf{case}_{[\![X]\!]} = [\![t_2]\!]$$

We also have the following interpretation for any $[\![\Delta \vdash_{\mathcal{P}} t : \mathsf{Nat}]\!]$:

$$[\![\mathsf{case}\, t\, \{0 \to t_1, \mathbf{suc}\, x \to t_2\}]\!] = \langle \mathsf{id}_{[\![\Delta]\!]}, [\![t]\!] \rangle; \mathsf{case}_{[\![X]\!]} : [\![\Delta]\!] \to [\![X]\!]$$

We obtain our result because $[\![\Delta \vdash_{\mathcal{P}} \mathbf{suc}\, n : \mathsf{Nat}]\!] = [\![n]\!]; \mathsf{suc}$:

$$\begin{array}{rcl}
\langle \mathsf{id}_{[\![\Delta]\!]}, [\![n]\!]; \mathsf{suc} \rangle; \mathsf{case}_{[\![X]\!]} & = & \langle \mathsf{id}_{[\![\Delta]\!]}, [\![n]\!] \rangle; (\mathsf{id}_{[\![\Delta]\!]} \times \mathsf{suc}); \mathsf{case}_{[\![X]\!]} \\
& = & \langle \mathsf{id}_{[\![\Delta]\!]}, [\![n]\!] \rangle; [\![t_2]\!].
\end{array}$$

B.2 Proof of Monadic Strength (Lemma 17)

The strength map is defined as follows:

$$\mathsf{st}_{A,B} = (\eta_A \otimes \mathsf{id}_{TB}); \mathsf{m}_{A,B} : A \otimes TB \longrightarrow T(A \otimes B)$$

where $\mathsf{m}_{A,B} : TA \otimes TB \longrightarrow T(A \otimes B)$ is the natural transformation arising from the fact that T is a monoidal endofunctor on $\mathcal{C}$.

The first two diagrams are straightforward to prove using the monoidal structure. We give explicit proofs of the final two diagrams.

Case 1. Monadic Join and Strength:

$$\begin{tikzcd}
A \otimes T^2 B \arrow[rr, "\mathsf{id}_A \otimes \mu_B"] \arrow[d, "\mathsf{st}_{A,TB}"] & & A \otimes TB \arrow[d, "\mathsf{st}_{A,B}"] \\
T(A \otimes TB) \arrow[r, "T\mathsf{st}_{A,B}"] & T^2(A \otimes B) \arrow[r, "\mu_{A \otimes B}"] & T(A \otimes B)
\end{tikzcd}$$

The previous diagram commutes because the following diagram commutes:

$$
\begin{tikzcd}
A \otimes T^2 B \arrow[rrr, "\mathsf{id}_A \otimes \mu_B"] \arrow[d, "\eta_A \otimes \mathsf{id}_{T^2 B}"'] & & & A \otimes TB \arrow[d, "\eta_A \otimes \mathsf{id}_{TB}"] \\
TA \otimes T^2 B \arrow[r, "T\eta_A \otimes \mathsf{id}_{T^2 B}"] \arrow[d, "\mathsf{m}_{A,TB}"'] & T^2 A \otimes T^2 B \arrow[rr, "\mu_A \otimes \mu_B"] \arrow[d, "\mathsf{m}_{TA,TB}"] \arrow[r, phantom, "3"' near start, shift left=6ex] \arrow[dl, phantom, "1"] & & TA \otimes TB \arrow[d, "\mathsf{m}_{A,B}"] \\
T(A \otimes TB) \arrow[r, "T(\eta_A \otimes \mathsf{id}_{TB})"'] & T(TA \otimes TB) \arrow[r, "T\mathsf{m}_{A,B}"'] & T^2(A \otimes B) \arrow[r, "\mu_{A \otimes B}"'] \arrow[u, phantom, "2"] & T(A \otimes B)
\end{tikzcd}
$$

Diagram 1 commutes by naturality of m, diagram 2 commutes because T is a monoidal endofunctor, and diagram 3 commutes by using both the monoidal structure of the monad, and the fact that tensor is a functor.

Case 2. Monadic Unit and Strength:

$$
\begin{tikzcd}
A \otimes B \arrow[r, "\mathsf{id}_A \otimes \eta_B"] \arrow[dr, "\eta_{A \times B}"'] & A \otimes TB \arrow[d, "\mathsf{st}_{A,B}"] \\
 & T(A \otimes B)
\end{tikzcd}
$$

The previous diagram commutes because the following diagram commutes:

$$\begin{tikzcd}
A \otimes B \arrow[rr, "\mathsf{id}_A \otimes \eta_B"] \arrow[d, "\eta_{A\otimes B}"] \arrow[drr, "\eta_A \otimes \eta_B"] & & A \otimes TB \arrow[d, "\eta_A \otimes \mathsf{id}_{TB}"] \\
T(A \otimes B) & & TA \otimes TB \arrow[ll, "\mathsf{m}_{A,B}"]
\end{tikzcd}$$

The lower triangle commutes because η is a monoidal natural transformation, and the upper triangle commutes, because tensor is a functor.