$$
\begin{array}{llcll}
\text{(Terms)} & t, A, B, C, D & ::= & \mathsf{Type}_l & \text{(Universe of Types)} \\
& & | & (x : A) \multimap B & \text{(Dependent Linear Arrow)} \\
& & | & (x : A) \otimes B & \text{(Dependent Tensor Product)} \\
& & | & x & \text{(Variable)} \\
& & | & \lambda x.t & \text{(Lambda Expression)} \\
& & | & t_1\, t_2 & \text{(Function Application)} \\
& & | & (t_1, t_2) & \text{(Linear Pair)} \\
& & | & \mathsf{let}\,(x, y) = t_1\,\mathsf{in}\,t_2 & \text{(Linear Pair Eliminator)} \\
\\
\text{(Levels)} & l & ::= & 0 & \text{(Level 0)} \\
& & | & 1 & \text{(Level 1)} \\
& & | & l_1 \sqcup l_2 & \text{(Maximum)} \\
\\
\text{(Typing Contexts)} & \Gamma & ::= & \varnothing & \text{(Empty Context)} \\
& & | & x : A & \text{(Context Element)} \\
& & | & \Gamma_1, \Gamma_2 & \text{(Context Extension)}
\end{array}
$$

Figure 1: LDTT Expression Syntax

1 Linear Dependent Types

1.1 Metatheorems

This section houses all of our metatheorems. First, we consider any two distinctly named contexts to be disjoint. Throughout the following proofs we work up to dependent reordering of contexts. That is, we do not consider a typing derivation, $\Gamma_1, x : A, y : B, \Gamma_2 \vdash t : C$, different from the typing derivation, $\Gamma_1, y : B, x : A, \Gamma_2 \vdash t : C$, when $x \notin \mathsf{FV}(B)$.

1.1.1 Basic Metatheory

Lemma 1.1 (Inversion Principles)**.**

i. If $\Gamma \vdash (x : A) \multimap B : \mathsf{Type}_l$*, then* $\Gamma_1 \vdash A : \mathsf{Type}_{l_1}$, $\Gamma_1, \Gamma_2, x : A \vdash B : \mathsf{Type}_{l_2}$*, and* $l = l_1 \sqcup l_2$ *for some contexts* Γ_1 *and* Γ_2*, and levels* l_1 *and* l_2*.*

Lemma 1.2 (Well-Formed Context Dependency Append)**.** *If* $\vdash \Gamma_1$ *and* $\vdash \Gamma_2$*, then* $\vdash (\Gamma_1, \Gamma_2)$*.*

Lemma 1.3 (Well-formed Context Dependency)**.** *If* $\Gamma \vdash t : A$*, then* $\vdash \Gamma$*.*

Lemma 1.4 (Well-Formed Linear Contexts Append)**.** $\dashv \Gamma_1$ *and* $\dashv \Gamma_2$ *iff* $\dashv (\Gamma_1, \Gamma_2)$*.*

Lemma 1.5 (Well-Formed Linear Contexts Extension)**.** *If* $\dashv (\Gamma, x : A)$*, then* $\dashv \Gamma$*.*

Lemma 1.6 (Well-formed Linear Context)**.** *If* $\Gamma \vdash t : A$*, then* $\dashv \Gamma$*.*

Corollary 1.6.1 (Well-Formed Contexts)**.** *If* $\Gamma \vdash t : A$*, then* $\Gamma\,\mathsf{Ok}$*.*

Lemma 1.7 (Substitution for Typing)**.** *Suppose* $\Gamma_2 \vdash A : \mathsf{Type}_l$, $\Gamma_1, \Gamma_2, x : A, \Gamma_4 \vdash t_2 : B$*, and* $\Gamma_2, \Gamma_3 \vdash t_1 : A$*. Then* $\Gamma_1, \Gamma_2, \Gamma_3, [t_1/x]\Gamma_3 \vdash [t_1/x]t_2 : [t_1/x]B$*.*

Lemma 1.8 (Kinding for Typing)**.** *If* $\Gamma \vdash t : A$*, then* $\Gamma' \vdash A : \mathsf{Type}_l$ *for some* $\Gamma' \subseteq \Gamma$ *and level* l*.*

Lemma 1.9 (Arrow Kinding)**.** *If* $\Gamma \vdash t : (x : A) \multimap B$*, then* $\Gamma', x : A \vdash B : \mathsf{Type}_l$ *for some subcontext* $\Gamma' \subseteq \Gamma$ *and level* l*.*

$$\frac{l_1 < l_2}{\varnothing \vdash \mathsf{Type}_{l_1} : \mathsf{Type}_{l_2}}\ \mathsf{Type} \qquad \frac{\Gamma_1 \vdash A : \mathsf{Type}_{l_1} \qquad \Gamma_1, \Gamma_2, x : A \vdash B : \mathsf{Type}_{l_2}}{\Gamma_1, \Gamma_2 \vdash (x : A) \multimap B : \mathsf{Type}_{l_1 \sqcup l_2}}\ \multimap$$

$$\frac{\Gamma_1 \vdash A : \mathsf{Type}_{l_1} \qquad \Gamma_1, \Gamma_2, x : A \vdash B : \mathsf{Type}_{l_2}}{\Gamma_1, \Gamma_2 \vdash (x : A) \otimes B : \mathsf{Type}_{l_1 \sqcup l_2}}\ \otimes \qquad \frac{\Gamma \vdash A : \mathsf{Type}_{l} \qquad \dashv (\Gamma, x : A)}{\Gamma, x : A \vdash x : A}\ \mathsf{Var}$$

$$\frac{\Gamma_1, x : A \vdash B : \mathsf{Type}_{l} \qquad \Gamma_1, \Gamma_2, x : A \vdash t : B}{\Gamma_1, \Gamma_2 \vdash \lambda x.t : (x : A) \multimap B}\ \multimap_i \qquad \frac{\Gamma_2 \vdash t_1 : (y : A) \multimap B \qquad \Gamma_1 \vdash t_2 : A}{\Gamma_1, \Gamma_2 \vdash t_1\, t_2 : [t_2/y]B}\ \multimap_e$$

$$\frac{\begin{array}{ll} x \notin \mathsf{FV}\,(t_2) & \\ \Gamma_1 \vdash B_1 : \mathsf{Type}_{l_1} & \Gamma_1, \Gamma_2 \vdash t_1 : B_1 \\ \Gamma_1, \Gamma_3, x : B_1 \vdash B_2 : \mathsf{Type}_{l_2} & \Gamma_1, \Gamma_3, \Gamma_4, x : B_1 \vdash t_2 : B_2 \end{array}}{\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4 \vdash (t_1, t_2) : (x : B_1) \otimes B_2}\ \otimes_i$$

$$\frac{\begin{array}{ll} x, y \notin \mathsf{FV}\,(B_3) & \\ \Gamma_1 \vdash B_1 : \mathsf{Type}_{l_1} & \Gamma_1, \Gamma_2, \Gamma_3 \vdash t_1 : (x : B_1) \otimes B_2 \\ \Gamma_1, \Gamma_2, x : B_1 \vdash B_2 : \mathsf{Type}_{l_2} & \Gamma_1, \Gamma_2, \Gamma_4, x : B_1, y : B_2 \vdash t_2 : B_3 \end{array}}{\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4 \vdash \mathsf{let}\,(x, y) = t_1\ \mathsf{in}\ t_2 : B_3}\ \otimes_e$$

Figure 2: LDTT Typing Rules

1.1.2 Linearity

Definition 1.10. *Suppose $\Gamma_1 \vdash t : A$ and Γ_2 is a second context. Then we say t and Γ_2 are* disjoint*, denoted $t \rtimes \Gamma_2$, if and only if $\mathsf{FV}(t) \cap |\Gamma_2| = \varnothing$.*

Lemma 1.11 (Distributivity of Disjointness)**.** *Suppose t is a term, and that $t \rtimes \Gamma$ holds for some context Γ. Furthermore, assume that the names in the domain of Γ differ from the names of any bound variables in t. Then by case analysis over the structure of t, the following distributivity properties hold:*

i. $((x : A) \multimap B) \rtimes \Gamma$ *if and only if* $A \rtimes \Gamma$ *and* $B \rtimes \Gamma$

ii. $((x : A) \otimes B) \rtimes \Gamma$ *if and only if* $A \rtimes \Gamma$ *and* $B \rtimes \Gamma$

iii. $(\lambda x.t) \rtimes \Gamma$ *if and only if* $t \rtimes \Gamma$

iv. $(t_1\, t_2) \rtimes \Gamma$ *if and only if* $t_1 \rtimes \Gamma$ *and* $t_2 \rtimes \Gamma$

v. $(t_1, t_2) \rtimes \Gamma$ *if and only if* $t_1 \rtimes \Gamma$ *and* $t_2 \rtimes \Gamma$

vi. $(\mathsf{let}\,(x, y) = t_1\ \mathsf{in}\ t_2) \rtimes \Gamma$ *if and only if* $t_1 \rtimes \Gamma$ *and* $t_2 \rtimes \Gamma$

Lemma 1.12 (Every Variable Must be Used)**.** *If $\Gamma \vdash t : A$, then for every $x : A \in \Gamma$, $x \in \mathsf{FV}(\Gamma)$ or $x \in \mathsf{FV}\,(t)$ or $x \in \mathsf{FV}\,(A)$.*

Lemma 1.13. *If $\Gamma \vdash t : A$ and $\{t, A\} \rtimes \Gamma$, then $\Gamma = \varnothing$.*

Lemma 1.14 (Disjointness)**.** *If $\Gamma \vdash t : B$, $x : A \in \Gamma$, $\Gamma_1 \subseteq \Gamma$, and $\Gamma_1 \vdash A : \mathsf{Type}_l$, then $t \rtimes \Gamma_1$.*

Theorem 1.15 (Linearity)**.** *If $\Gamma \vdash t : B$, then for every $x : A \in \Gamma$, x appears only once in Γ, or only once in t, or only once in B.*

Corollary 1.15.1. *If $\Gamma, x : A, \Gamma' \vdash t : B$, then x appears only once in Γ', or only once in t, or only once in B.*

1.1.3 Trivialization

Lemma 1.16 (λ-Bound Variables Must be Used). *If* $\Gamma \vdash \lambda x.t : (x : A) \multimap B$, *then* $x \in \mathsf{FV}(t)$ *and* $x \in \mathsf{FV}(B)$.

Lemma 1.17 (Closed Types). *If* $\Gamma \vdash t : B$, $x : A \in \Gamma$, $\Gamma_1 \subseteq \Gamma$, $\Gamma_1 \vdash A : \mathsf{Type}_l$, *and* $B \times \Gamma_1$, *then* $\Gamma_1 = \varnothing$.

Lemma 1.18 (Arrow Bound Variables Must be Used). *If* $\Gamma \vdash t : (x : A) \multimap B$, *then* $x \in \mathsf{FV}(B)$.

Lemma 1.19 (No Type Dependency). *For any type* A *and level* l, *there is no type* B *such that* $x : A \vdash B : \mathsf{Type}_l$.

Lemma 1.20 (All Types are Type). *If* $\varnothing \vdash A : \mathsf{Type}_{l'}$, *then* A *is* Type_l *for some level* l.

Theorem 1.21 (Trivialization). *If* $\varnothing \vdash t : A$, *then* t *is* Type_{l_1} *and* A *is* Type_{l_2} *for some* l_1 *and* l_2.

2 LDTT Complete Specification

l	::=		Level
	$\mid$ 0		Lowest Level
	$\mid$ 1		First Level
	$\mid$ $l_1 \sqcup l_2$		Least Upper Bound
	$\mid$ $l_1 \sqcap l_2$		Greatest Lower Bound
	$\mid$ $l_1 + l_2$		Addition
$t,\ A,\ B,\ C,\ D,\ E,\ K,\ T,\ R,\ S$	::=		Term
	$\mid$ Type_l		Universe of Types
	$\mid$ $(x : A) \multimap B$		Linear Dependent Products
	$\mid$ $(x : A) \otimes B$		Dependent Tensor
	$\mid$ x		Variable
	$\mid$ $\lambda x.t$		λ-abstraction
	$\mid$ $t_1\ t_2$		Application
	$\mid$ (t_1, t_2)		Type-level term tensor introduction
	$\mid$ $\mathsf{let}\,(x, y) = t_1\ \mathsf{in}\ t_2$		Term tensor elimination
	$\mid$ (t)	S	
Γ	::=		Context
	$\mid$ $\varnothing$		Empty Context
	$\mid$ $x : A$		Term Variable
	$\mid$ Γ_1, Γ_2		Extension
	$\mid$ (Γ)	S	

$\boxed{\vdash \Gamma}$ Context Dependency is Well Formed

$$\frac{}{\vdash \varnothing}\ \text{P\_E}$$

$$\frac{\begin{array}{l}\vdash \Gamma \\ \Gamma_1 \subseteq \Gamma \\ \Gamma_1 \vdash A : \mathsf{Type}_l\end{array}}{\vdash \Gamma, x : A}\ \text{P\_X}$$

$\boxed{\dashv \Gamma}$

$$\frac{}{\dashv \varnothing}\ \text{J\_E}$$

$$\frac{\dashv \Gamma \qquad |\mathsf{FV}[\mathsf{G}]|_x \leq 1}{\dashv x : A, \Gamma}\ \text{J\_X}$$

$\boxed{\Gamma\,\mathsf{Ok}}$ Well-formed Contexts

$$\frac{\vdash \Gamma \quad \dashv \Gamma}{\Gamma\,\mathsf{Ok}}\ \text{OK\_K}$$

$\boxed{\Gamma \vdash t : A}$ Typing

$$\frac{l_1 < l_2}{\varnothing \vdash \mathsf{Type}_{l_1} : \mathsf{Type}_{l_2}}\ \text{T\_TYPE}$$

$$\frac{\Gamma_1 \vdash A : \mathsf{Type}_{l_1} \qquad \Gamma_1, \Gamma_2, x : A \vdash B : \mathsf{Type}_{l_2}}{\Gamma_1, \Gamma_2 \vdash (x : A) \multimap B : \mathsf{Type}_{l_1 \sqcup l_2}}\ \text{T\_ARROW}$$

$$\frac{\Gamma_1 \vdash A : \mathsf{Type}_{l_1} \qquad \Gamma_1, \Gamma_2, x : A \vdash B : \mathsf{Type}_{l_2}}{\Gamma_1, \Gamma_2 \vdash (x : A) \otimes B : \mathsf{Type}_{l_1 \sqcup l_2}}\ \text{T\_TEN}$$

$$\frac{\dashv (\Gamma, x : A) \qquad \Gamma \vdash A : \mathsf{Type}_l}{\Gamma, x : A \vdash x : A}\ \text{T\_VAR}$$

$$\frac{\Gamma_1, x : A \vdash B : \mathsf{Type}_l \qquad \Gamma_1, \Gamma_2, x : A \vdash t : B}{\Gamma_1, \Gamma_2 \vdash \lambda x.t : (x : A) \multimap B}\ \text{T\_FUN}$$

$$\frac{\Gamma_1 \vdash t_2 : A \qquad \Gamma_2 \vdash t_1 : (y : A) \multimap B}{\Gamma_1, \Gamma_2 \vdash t_1\ t_2 : [t_2/y]B}\ \text{T\_APP}$$

$$\frac{x \notin \mathsf{FV}(t_2) \qquad \Gamma_1 \vdash B_1 : \mathsf{Type}_{l_1} \qquad \Gamma_1, \Gamma_3, x : B_1 \vdash B_2 : \mathsf{Type}_{l_2} \qquad \Gamma_1, \Gamma_2 \vdash t_1 : B_1 \qquad \Gamma_1, \Gamma_3, \Gamma_4, x : B_1 \vdash t_2 : B_2}{\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4 \vdash (t_1, t_2) : (x : B_1) \otimes B_2}\ \text{T\_PAIR}$$

$$\frac{x, y \notin \mathsf{FV}(B_3) \qquad \Gamma_1 \vdash B_1 : \mathsf{Type}_{l_1} \qquad \Gamma_1, \Gamma_2, x : B_1 \vdash B_2 : \mathsf{Type}_{l_2} \qquad \Gamma_1, \Gamma_2, \Gamma_3 \vdash t_1 : (x : B_1) \otimes B_2 \qquad \Gamma_1, \Gamma_2, \Gamma_4, x : B_1, y : B_2 \vdash t_2 : B_3}{\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4 \vdash \mathsf{let}\,(x, y) = t_1\ \mathsf{in}\ t_2 : B_3}\ \text{T\_LET}$$