DUALIZED SIMPLE TYPE THEORY

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Abstract. We propose a new bi-intuitionistic type theory called Dualized Type Theory (DTT). It is a simple type theory with perfect intuitionistic duality, and corresponds to a single-sided polarized sequent calculus. We prove DTT strongly normalizing, and prove type preservation. DTT is based on a new propositional bi-intuitionistic logic called Dualized Intuitionistic Logic (DIL) that builds on Pinto and Uustalu's logic L. DIL is a simplification of L which removes several admissible inference rules from L while maintaining consistency and completeness. Furthermore, DIL is defined using a dualized syntax by labeling formulas and logical connectives with polarities thus reducing the number of inference rules needed to define the logic. We give a direct proof of consistency, but prove completeness by reduction to L.

1. Introduction

Classical logic is rich with duality. Using the De Morgan dualities it is straightforward to prove that conjunction is dual to disjunction and negation is self dual. In addition, it is also possible to prove that $\neg A \wedge B$ is dual to implication. In intuitionistic logic these dualities are no longer provable, but in [20] Rauszer gives a conservative extension of the Kripke semantics for intuitionistic logic that not only models conjunction, disjunction, negation, and implication, but also the dual to implication, by introducing a new logical connective. The usual interpretation of implication in a Kripke model is as follows:

$$[\![A \to B]\!]_w = \forall w'.w \leq w' \to [\![A]\!]_{w'} \to [\![B]\!]_{w'}$$

Now Rauszer took the dual of the previous interpretation to obtain the following:

$$[\![A - B]\!]_w = \exists w'.w' \leq w \wedge \neg[\![A]\!]_{w'} \wedge [\![B]\!]_{w'}$$

This is called subtraction or exclusion. Propositional bi-intuitionistic logic is a conservative extension of propositional intuitionistic logic with perfect duality. That is, it contains the logical connectives for disjunction, conjunction, implication, and subtraction, and it is sound and complete with respect to the Rauszer's extended Kripke semantics.

Propositional bi-intuitionistic (BINT) logic is fairly unknown in computer science. Filinski studied a fragment of BINT logic in his investigation into first class continuations in

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[11]. Crolard introduced a logic and corresponding type theory called subtractive logic, and showed it can be used to study constructive coroutines in [7, 8]. He initially defined subtractive logic in sequent style with the Dragalin restriction, and then defined the corresponding type theory in natural deduction style by imposing a restriction on Parigot's $\lambda\mu$-calculus in the form of complex dependency tracking. Just as linear logicians have found – for example in [21] – Pinto and Uustalu were able to show that imposing the Dragalin restriction in subtractive logic results in a failure of cut elimination [19]. They recover cut elimination by proposing a new BINT logic called L that lifts the Dragalin restriction by labeling formulas and sequents with nodes and graphs respectively; this labeling corresponds to placing constraints on the sequents where the graphs can be seen as abstract Kripke models. Goré et. al. also proposed a new BINT logic that enjoys cut elimination using nested sequents; however it is currently unclear how to define a type theory with nested sequents [13]. Bilinear logic in its intuitionistic form is a linear version of BINT and has been studied by Lambek in [16, 17]. Biasi and Aschieri propose a term assignment to polarized bi-intuitionistic logic in [6]. One can view the polarities of their logic as an internalization of the polarities of the logic we propose in this article. Bellin has studied BINT similar to that of Biasi and Aschieri from a philosophical perspective in [2, 3, 4], and he defined a linear version of Crolard's subtractive logic for which he was able to construct a categorical model using linear categories in [5].

Contributions. The contributions of this paper are a new formulation of Pinto and Uustalu's BINT labeled sequent calculus L called Dualized Intuitionistic Logic (DIL) and a corresponding type theory called Dualized Type Theory (DTT). DIL is a single-sided polarized formulation of Pinto and Uustalu's L, and builds on L by removing the following rules (see Section 2 for a complete definition of L):

$$\frac{\Gamma \vdash_{G \cup \{(n,n)\}} \Delta}{\Gamma \vdash_G \Delta}\ \textsc{refl} \qquad \frac{\begin{array}{l} n_1 G n_2 \\ n_2 G n_3 \\ \Gamma \vdash_{G \cup \{(n_1,n_3)\}} \Delta \end{array}}{\Gamma \vdash_G \Delta}\ \textsc{trans}$$

$$\frac{\begin{array}{l} nGn' \\ \Gamma, n : T, n' : T \vdash_G \Delta \end{array}}{\Gamma, n : T \vdash_G \Delta}\ \textsc{monL} \qquad \frac{\begin{array}{l} n'Gn \\ \Gamma \vdash_G n' : T, n : T, \Delta \end{array}}{\Gamma \vdash_G n : T, \Delta}\ \textsc{monR}$$

We show that in the absence of the previous rules DIL still maintains consistency and completeness. Furthermore, DIL is defined using a dualized syntax which reduces the number of inference rules needed to define the logic. Again, DIL is a single-sided sequent calculus with multiple conclusions and thus must provide a means of moving conclusions from left to right. This is done in DIL using cuts on hypotheses. We call these types of cuts "axiom cuts."

Now we consider BINT logic to be the closest extension of intuitionistic logic to classical logic while maintaining constructivity. BINT has two forms of negation, one defined as usual, $\neg A \stackrel{\mathsf{def}}{=} A \to \perp$, and a second defined interms of subtraction, $\sim A \stackrel{\mathsf{def}}{=} \top - A$. The latter we call "non-A". Now in BINT it is possible to prove $A \vee \sim A$ for any A [7]. Furthermore, when the latter is treated as a type in DTT, the inhabitant is a continuation without a canonical form, because the inhabitant contains as a subexpression an axiom cut. Thus, the presence of these continuations prevents the canonicity result for a type theory – like DTT – from holding. Thus, if general cut elimination was a theorem of DIL, then

$$\begin{array}{rrcl} \text{(formulas)} & A, B, C & ::= & \top \mid \perp \mid A \supset B \mid A \prec B \mid A \wedge B \mid A \vee B \\ \text{(graphs)} & G & ::= & \cdot \mid (n, n') \mid G, G' \\ \text{(contexts)} & \Gamma & ::= & \cdot \mid n : A \mid \Gamma, \Gamma' \end{array}$$

Figure 1: Syntax of L.

$A\vee \sim A$ would not be provable. So DIL must contain cuts that cannot be eliminated. This implies that DIL does not enjoy general cut elimination, but all cuts other than axiom cuts can be eliminated. Throughout the sequel we define "cut elimination" as the elimination of all cuts other than axiom cuts, and we call DIL "cut free" with respect to this definition of cut elimination. The latter point is similar to Wadler's dual calculus [24].

Finally, we present a computer-checked proof – in Agda – of consistency for DIL with respect to Rauszer's Kripke semantics for BINT logic, prove completeness of DIL by reduction to Pinto and Uustalu's L, and show type preservation and strong normalization for DTT. We show the latter using a version of Krivine's classical realizability by translating DIL into a classical logic.

The contributions of this article are subgoals of a larger one. Due to the rich duality in BINT logic we believe it shows promise of being a logical foundation for induction and coinduction, because induction is dual to coinduction. Our working hypothesis is that a logical foundation based on intuitionistic duality will allow the semantic duality between induction and coinduction to be expressed in type theory, yielding a solution to the problems with these important features in existing systems. For example, Agda restricts how inductive and coinductive types can be nested (see the discussion in [1]), while Coq supports general mixed inductive and coinductive data, but in doing so, sacrifices type preservation.

The rest of this paper is organized as follows. We first introduce Pinto and Uustalu's L calculus in Section 2, and then DIL in Section 3. Then we prove DIL consistent and complete (with only axiom cuts) in Section 3.1 and Section 3.2 respectively. Following DIL we introduce DTT in Section 4, and its metatheory in Section 5. All of the mathematical content of this paper was typeset with the help of Ott [22].

2. Pinto and Uustalu's L

In this section we briefly introduce Pinto and Uustalu's L from [19]. The syntax for formulas, graphs, and contexts of L are defined in Figure 1, while the inference rules are defined in Figure 2. The formulas include true and false denoted $\top$ and $\perp$ respectively, implication and subtraction denoted $A \supset B$ and $A \prec B$ respectively, and finally, conjunction and disjunction denoted $A \wedge B$ and $A \vee B$ respectively. So we can see that for every logical connective its dual is a logical connective of the logic. This is what we meant by BINT containing perfect intuitionistic duality in the introduction. Sequents have the form $\Gamma \vdash_G n : A, \Delta$, where Γ and Δ are multisets of formulas labeled by a node, G is the abstract Kripke model or sometimes referred to as simply the graph of the sequent, and n is a node in G.

Graphs are treated as sets of edges and we denote $(n_1, n_2) \in G$ by $n_1 G n_2$. Furthermore, we denote the union of two graphs G and G' as $G \cup G'$. Now each formula present in a sequent is labelled with a node in the graph. This labeling is denoted $n : A$ and should be read as the formula A is true at the node n. We denote the operation of constructing the list of nodes in a graph or context by $|G|$ and $|\Gamma|$ respectively. The reader should note that

$$\frac{\Gamma \vdash_{G\cup\{(n,n)\}} \Delta}{\Gamma \vdash_G \Delta}\ \textsc{refl} \qquad \frac{\begin{array}{l} n_1 G n_2 \\ n_2 G n_3 \\ \Gamma \vdash_{G\cup\{(n_1,n_3)\}} \Delta \end{array}}{\Gamma \vdash_G \Delta}\ \textsc{trans} \qquad \frac{}{\Gamma, n : T \vdash_G n : T, \Delta}\ \textsc{hyp}$$

$$\frac{\begin{array}{l} nGn' \\ \Gamma, n : T, n' : T \vdash_G \Delta \end{array}}{\Gamma, n : T \vdash_G \Delta}\ \textsc{mon}\mathrm{L} \qquad \frac{\begin{array}{l} n'Gn \\ \Gamma \vdash_G n' : T, n : T, \Delta \end{array}}{\Gamma \vdash_G n : T, \Delta}\ \textsc{mon}\mathrm{R}$$

$$\frac{\Gamma \vdash_G \Delta}{\Gamma, n : \top \vdash_G \Delta}\ \textsc{true}\mathrm{L} \qquad \frac{}{\Gamma \vdash_G n : \top, \Delta}\ \textsc{true}\mathrm{R} \qquad \frac{}{\Gamma, n : \perp \vdash_G \Delta}\ \textsc{false}\mathrm{L}$$

$$\frac{\Gamma \vdash_G \Delta}{\Gamma \vdash_G n : \perp, \Delta}\ \textsc{false}\mathrm{R} \qquad \frac{\Gamma, n : T_1, n : T_2 \vdash_G \Delta}{\Gamma, n : T_1 \wedge T_2 \vdash_G \Delta}\ \textsc{and}\mathrm{L}$$

$$\frac{\begin{array}{c} \Gamma \vdash_G n : T_1, \Delta \\ \Gamma \vdash_G n : T_2, \Delta \end{array}}{\Gamma \vdash_G n : T_1 \wedge T_2, \Delta}\ \textsc{and}\mathrm{R} \qquad \frac{\begin{array}{c} \Gamma, n : T_1 \vdash_G \Delta \\ \Gamma, n : T_2 \vdash_G \Delta \end{array}}{\Gamma, n : T_1 \vee T_2 \vdash_G \Delta}\ \textsc{disj}\mathrm{L}$$

$$\frac{\Gamma \vdash_G n : T_1, n : T_2, \Delta}{\Gamma \vdash_G n : T_1 \vee T_2, \Delta}\ \textsc{disj}\mathrm{R} \qquad \frac{\begin{array}{l} nGn' \\ \Gamma \vdash_G n' : T_1, \Delta \\ \Gamma, n' : T_2 \vdash_G \Delta \end{array}}{\Gamma, n : T_1 \supset T_2 \vdash_G \Delta}\ \textsc{imp}\mathrm{L}$$

$$\frac{\begin{array}{l} n' \notin |G|, |\Gamma|, |\Delta| \\ \Gamma, n' : T_1 \vdash_{G\cup\{(n,n')\}} n' : T_2, \Delta \end{array}}{\Gamma \vdash_G n : T_1 \supset T_2, \Delta}\ \textsc{imp}\mathrm{R} \qquad \frac{\begin{array}{l} n' \notin |G|, |\Gamma|, |\Delta| \\ \Gamma, n' : T_1 \vdash_{G\cup\{(n,n')\}} n' : T_2, \Delta \end{array}}{\Gamma, n' : T_1 \prec T_2 \vdash_G \Delta}\ \textsc{sub}\mathrm{L}$$

$$\frac{\begin{array}{l} n'Gn \\ \Gamma \vdash_G n' : T_1, \Delta \\ \Gamma, n' : T_2 \vdash_G \Delta \end{array}}{\Gamma \vdash_G n : T_1 \prec T_2, \Delta}\ \textsc{sub}\mathrm{R}$$

Figure 2: Inference Rules for L.

it is possible for some nodes in the sequent to not appear in the graph. For example, the sequent $n : A \vdash_{\cdot} n : A, \cdot$ is a derivable sequent. The complete graph can always be recovered if needed by using the graph structural rules REFL, TRANS, MONL, and MONR.

The labeling on formulas essentially adds constraints to the set of Kripke models. This is evident in the proof of consistency for DIL in Section 3.1; see the definition of validity. Consistency of L is stated in [19] without a detailed proof, but is proven complete with respect to Rauszer's Kripke semantics using a counter model construction. In Section 3 we give a translation of the formulas of L into the formulas of DIL, and in Section 3.2 we will give a translation of the rest of L into DIL which will be used to conclude completeness of DIL.

$$\begin{array}{rrcl}
\text{(polarities)} & p & ::= & + \mid - \\
\text{(formulas)} & A, B, C & ::= & \mathbf{a} \mid \langle p \rangle \mid A \to_p B \mid A \wedge_p B \\
\text{(graphs)} & G & ::= & \cdot \mid n \preccurlyeq_p n' \mid G, G' \\
\text{(contexts)} & \Gamma & ::= & \cdot \mid p\ A \mathbin{@} n \mid \Gamma, \Gamma'
\end{array}$$

Figure 3: Syntax for DIL.

$$\frac{G \vdash n \preccurlyeq_p^* n'}{G; \Gamma, p\ A \mathbin{@} n, \Gamma' \vdash p\ A \mathbin{@} n'}\ \textsc{ax} \qquad \frac{}{G; \Gamma \vdash p\ \langle p \rangle \mathbin{@} n}\ \textsc{unit}$$

$$\frac{G; \Gamma \vdash p\ A \mathbin{@} n \qquad G; \Gamma \vdash p\ B \mathbin{@} n}{G; \Gamma \vdash p\ (A \wedge_p B) \mathbin{@} n}\ \textsc{and} \qquad \frac{G; \Gamma \vdash p\ A_d \mathbin{@} n}{G; \Gamma \vdash p\ (A_1 \wedge_{\bar{p}} A_2) \mathbin{@} n}\ \textsc{andBar}$$

$$\frac{\begin{array}{l} n' \notin |G|, |\Gamma| \\ (G, n \preccurlyeq_p n'); \Gamma, p\ A \mathbin{@} n' \vdash p\ B \mathbin{@} n' \end{array}}{G; \Gamma \vdash p\ (A \to_p B) \mathbin{@} n}\ \textsc{imp}$$

$$\frac{\begin{array}{l} G \vdash n \preccurlyeq_{\bar{p}}^* n' \\ G; \Gamma \vdash \bar{p}\ A \mathbin{@} n' \qquad G; \Gamma \vdash p\ B \mathbin{@} n' \end{array}}{G; \Gamma \vdash p\ (A \to_{\bar{p}} B) \mathbin{@} n}\ \textsc{impBar}$$

$$\frac{G; \Gamma, \bar{p}\ A \mathbin{@} n \vdash +\ B \mathbin{@} n' \qquad G; \Gamma, \bar{p}\ A \mathbin{@} n \vdash -\ B \mathbin{@} n'}{G; \Gamma \vdash p\ A \mathbin{@} n}\ \textsc{cut}$$

Figure 4: Inference Rules for DIL.

3. Dualized Intuitionistic Logic (DIL)

The syntax for polarities, formulas, and graphs of DIL is defined in Figure 3, where $\mathbf{a}$ ranges over atomic formulas. The following definition shows that DIL's formulas are simply polarized versions of L's formulas.

Definition 1. The following defines a translation of formulas of L to formulas of DIL:

$$\begin{array}{rclcrclcrcl}
\ulcorner \top \urcorner & = & \langle + \rangle & \quad & \ulcorner A \wedge B \urcorner & = & \ulcorner A \urcorner \wedge_+ \ulcorner B \urcorner & \quad & \ulcorner A \supset B \urcorner & = & \ulcorner A \urcorner \to_+ \ulcorner B \urcorner \\
\ulcorner \perp \urcorner & = & \langle - \rangle & & \ulcorner A \vee B \urcorner & = & \ulcorner A \urcorner \wedge_- \ulcorner B \urcorner & & \ulcorner B \prec A \urcorner & = & \ulcorner A \urcorner \to_- \ulcorner B \urcorner
\end{array}$$

We represent graphs as lists of edges denoted $n_1 \preccurlyeq_p n_2$, where we consider the edge $n_1 \preccurlyeq_+ n_2$ to mean that there is a path from n_1 to n_2, and the edge $n_1 \preccurlyeq_- n_2$ to mean that there is a path from n_2 to n_1. Lastly, contexts denoted Γ are represented as lists of formulas. Throughout the sequel we denote the opposite of a polarity p by $\bar{p}$. This is defined by $\bar{+} = -$ and $\bar{-} = +$. The inference rules for DIL are in Figure 4.

The sequent has the form $G; \Gamma \vdash p\ A \mathbin{@} n$ which when p is positive (resp. negative) can be read as the formula A is true (resp. false) at node n in the context Γ with respect to the graph G. Note that the metavariable d in the premise of the AndBar rule ranges over the set $\{1, 2\}$ and prevents the need for two rules. The inference rules depend on a reachability judgment that provides a means of proving when a node is reachable from another within some graph G. This judgment is defined in Figure 5. In addition, the imp rule depends

$$\frac{}{G, n \preccurlyeq_p n', G' \vdash n \preccurlyeq_p^* n'}\ \text{REL\_AX} \qquad \frac{}{G \vdash n \preccurlyeq_p^* n}\ \text{REL\_REFL}$$

$$\frac{G \vdash n \preccurlyeq_p^* n' \qquad G \vdash n' \preccurlyeq_p^* n''}{G \vdash n \preccurlyeq_p^* n''}\ \text{REL\_TRANS} \qquad \frac{G \vdash n' \preccurlyeq_{\bar{p}}^* n}{G \vdash n \preccurlyeq_p^* n'}\ \text{REL\_FLIP}$$

Figure 5: Reachability Judgment for DIL.

on the operations $|G|$ and $|\Gamma|$ which simply compute the list of all the nodes in G and Γ respectively. The condition $n' \notin |G|, |\Gamma|$ in the IMP rule is required for consistency.

The most interesting inference rules of DIL are the rules for implication and coimplication from Figure 4. Let us consider these two rules in detail. These rules mimic the definitions of the interpretation of implication and coimplication in a Kripke model. The IMP rule states that the formula $p\,(A \rightarrow_p B)$ holds at node n if assuming $p\,A \mathbin{@} n'$ for an arbitrary node n' reachable from n, then $p\,B \mathbin{@} n'$ holds. Notice that when p is positive n' will be a future node, but when p is negative n' will be a past node. Thus, universally quantifying over past and future worlds is modeled here by adding edges to the graph. Now the IMPBAR rule states the formula $p\,(A \rightarrow_{\bar{p}} B)$ is derivable if there exists a node n' that is provably reachable from n, $\bar{p}\,A$ is derivable at node n', and $p\,B \mathbin{@} n'$ is derivable at node n'. When p is positive n' will be a past node, but when p is negative n' will be a future node. This is exactly dual to implication. Thus, existence of past and future worlds is modeled by the reachability judgment.

Before moving on to proving consistency and completeness of DIL we first show that the formula $A \wedge_{-} \sim A$ has a proof in DIL that contains a cut that cannot be eliminated. This also serves as an example of a derivation in DIL. Consider the following where we leave off the reachability derivations for clarity and $\Gamma' \equiv -\,(A \wedge_{-} \sim A) \mathbin{@} n, -\,A \mathbin{@} n$:

$$\cfrac{\cfrac{\cfrac{\cfrac{}{G;\Gamma,\Gamma' \vdash -\,A \mathbin{@} n}\ \text{AX} \qquad \cfrac{}{G;\Gamma,\Gamma' \vdash +\,\langle + \rangle \mathbin{@} n}\ \text{UNIT}}{G;\Gamma,\Gamma' \vdash +\,\sim A \mathbin{@} n}\ \text{IMPBAR}}{G;\Gamma,\Gamma' \vdash +\,(A \wedge_{-} \sim A) \mathbin{@} n}\ \text{ANDBAR} \qquad \cfrac{}{G;\Gamma,\Gamma' \vdash -\,(A \wedge_{-} \sim A) \mathbin{@} n}\ \text{AX}}{\cfrac{G;\Gamma, -\,(A \wedge_{-} \sim A) \mathbin{@} n \vdash +\,A \mathbin{@} n}{G;\Gamma, -\,(A \wedge_{-} \sim A) \mathbin{@} n \vdash +\,(A \wedge_{-} \sim A) \mathbin{@} n}\ \text{ANDBAR}}\ \text{CUT}$$

Now using only an axiom cut we may conclude the following derivation:

$$\frac{G;\Gamma, -\,(A \wedge_{-} \sim A) \mathbin{@} n \vdash +\,(A \wedge_{-} \sim A) \mathbin{@} n \qquad \cfrac{}{G;\Gamma, -\,(A \wedge_{-} \sim A) \mathbin{@} n \vdash -\,(A \wedge_{-} \sim A) \mathbin{@} n}\ \text{AX}}{G;\Gamma \vdash +\,(A \wedge_{-} \sim A) \mathbin{@} n}\ \text{CUT}$$

The reader should take notice to the fact that all cuts within the previous two derivations are axiom cuts, where the inner most cut uses the hypothesis of the outer cut. Therefore, neither can be eliminated.

3.1. **Consistency of DIL.**

In this section we prove consistency of DIL with respect to Rauszer's Kripke semantics for BINT logic. All of the results in this section have been formalized in the Agda proof assistant[1]. We begin by first defining a Kripke frame.

[1] Agda source code is available at `https://github.com/heades/DIL-consistency`

Definition 2. A **Kripke frame** is a pair (W, R) of a set of worlds W, and a preorder R on W.

Then we extend the notion of a Kripke frame to include an evaluation for atomic formulas resulting in a Kripke model.

Definition 3. A **Kripke model** is a tuple (W, R, V), such that, (W, R) is a Kripke frame, and V is a binary monotone relation on W and the set of atomic formulas of DIL.

Now we can interpret formulas in a Kripke model as follows:

Definition 4. The interpretation of the formulas of DIL in a Kripke model (W, R, V) is defined by recursion on the structure of the formula as follows:

$$\begin{array}{rcl} [\![\langle+\rangle]\!]_w & = & \top \\ [\![\langle-\rangle]\!]_w & = & \bot \\ [\![\mathbf{a}]\!]_w & = & V\,w\,\mathbf{a} \end{array} \qquad \begin{array}{rcl} [\![A \wedge_+ B]\!]_w & = & [\![A]\!]_w \wedge [\![B]\!]_w \\ [\![A \wedge_- B]\!]_w & = & [\![A]\!]_w \vee [\![B]\!]_w \\ [\![A \to_+ B]\!]_w & = & \forall w' \in W. R\,w\,w' \to [\![A]\!]_{w'} \to [\![B]\!]_{w'} \\ [\![A \to_- B]\!]_w & = & \exists w' \in W. R\,w'\,w \wedge \neg[\![A]\!]_{w'} \wedge [\![B]\!]_{w'} \end{array}$$

The interpretation of formulas really highlights the fact that implication is dual to coimplication. Monotonicity holds for this interpretation.

Lemma 5 (Monotonicity)**.** Suppose (W, R, V) is a Kripke model, A is some DIL formula, and $w, w' \in W$. Then $R\,w\,w'$ and $[\![A]\!]_w$ imply $[\![A]\!]_{w'}$.

At this point we must set up the mathematical machinery which allows for the interpretation of sequents in a Kripke model. This will require the interpretation of graphs, and hence, nodes. We interpret nodes as worlds in the model using a function we call a node interpreter.

Definition 6. Suppose (W, R, V) is a Kripke model and S is a set of nodes of an abstract Kripke model G. Then a **node interpreter** on S is a function from S to W.

Now using the node interpreter we can interpret edges as statements about the reachability relation in the model. Thus, the interpretation of a graph is just the conjunction of the interpretation of its edges.

Definition 7. Suppose (W, R, V) is a Kripke model, G is an abstract Kripke model, and N is a node interpreter on the set of nodes of G. Then the interpretation of G in the Kripke model is defined by recursion on the structure of the graph as follows:

$$\begin{array}{lcl} [\![\cdot]\!]_N & = & \top \\ [\![n_1 \preccurlyeq_+ n_2, G]\!]_N & = & R\,(N\,n_1)\,(N\,n_2) \wedge [\![G]\!]_N \\ [\![n_1 \preccurlyeq_- n_2, G]\!]_N & = & R\,(N\,n_2)\,(N\,n_1) \wedge [\![G]\!]_N \end{array}$$

The reachability judgment of DIL provides a means to prove that two particular nodes are reachable in the abstract Kripke graph, but this proof is really just a syntactic proof of transitivity. The following lemma makes this precise.

Lemma 8 (Reachability Interpretation)**.** Suppose (W, R, V) is a Kripke model, and $[\![G]\!]_N$ for some abstract Kripke graph G. Then

i. if $G \vdash n_1 \preccurlyeq_+ n_2$, then $R\,(N\,n_1)\,(N\,n_2)$, and
ii. if $G \vdash n_1 \preccurlyeq_- n_2$, then $R\,(N\,n_2)\,(N\,n_1)$.

We have everything we need to interpret abstract Kripke models. The final ingredient to the interpretation of sequents is the interpretation of contexts.

Definition 9. If F is some meta-logical formula, we define $p\,F$ as follows:

$$+\,F = F \quad \text{and} \quad -\,F = \neg F.$$

Definition 10. Suppose (W, R, V) is a Kripke model, Γ is a context, and N is a node interpreter on the set of nodes in Γ. The interpretation of Γ in the Kripke model is defined by recursion on the structure of the context as follows:

$$\begin{array}{lcl} [\![\cdot]\!]_N & = & \top \\ [\![p\ A \mathbin{@} n, \Gamma]\!]_N & = & p[\![A]\!]_{(N\,n)} \wedge [\![\Gamma]\!]_N \end{array}$$

Combining these interpretations results in the following definition of validity.

Definition 11. Suppose (W, R, V) is a Kripke model, Γ is a context, and N is a node interpreter on the set of nodes in Γ. The interpretation of sequents is defined as follows:

$$[\![G; \Gamma \vdash p\ A \mathbin{@} n]\!]_N \quad = \quad \text{if } [\![G]\!]_N \text{ and } [\![\Gamma]\!]_N, \text{ then } p[\![A]\!]_{(N\,n)}.$$

Notice that in the definition of validity the graph G is interpreted as a set of constraints imposed on the set of Kripke models, thus reinforcing the fact that the graphs on sequents really are abstract Kripke models. Finally, using the previous definition of validity we can prove consistency.

Theorem 12 (Consistency)**.** Suppose $G; \Gamma \vdash p\ A \mathbin{@} n$. Then for any Kripke model (W, R, V) and node interpreter N on $|G|$, $[\![G; \Gamma \vdash p\ A \mathbin{@} n]\!]_N$.

3.2. **Completeness of DIL.** Completeness can be shown by showing that every derivable sequent in L can be translated to a derivable sequent of DIL. We will call a sequent in L a L-sequent and a sequent in DIL a DIL-sequent. Throughout this section we assume without loss of generality that all L-sequents have non-empty right-hand sides. That is, for every L-sequent, $\Gamma \vdash_G \Delta$, we assume that $\Delta \neq \cdot$. We do not loose generality because it is possible to prove that $\Gamma \vdash_G \cdot$ holds if and only if $\Gamma \vdash_G n :\perp$ for any node n (proof omitted).

Along the way, we will see admissibility of the analogues of the rules we mentioned in Section 3. The proof of consistency was with respect to DIL including the cut rule, but we prove completeness with respect to DIL where the general cut rule has been replaced with the following two inference rules, which can be seen as restricted instances of the cut rule:

$$\frac{p\ B \mathbin{@} n' \in (\Gamma, \bar{p}\ A \mathbin{@} n) \qquad G; \Gamma, \bar{p}\ A \mathbin{@} n \vdash \bar{p}\ B \mathbin{@} n'}{G; \Gamma \vdash p\ A \mathbin{@} n}\ \textsc{axCut}$$

$$\frac{\bar{p}\ B \mathbin{@} n' \in (\Gamma, \bar{p}\ A \mathbin{@} n) \qquad G; \Gamma, \bar{p}\ A \mathbin{@} n \vdash p\ B \mathbin{@} n'}{G; \Gamma \vdash p\ A \mathbin{@} n}\ \textsc{axCutBar}$$

These two rules are required for the crucial left-to-right lemma. This lemma depends on the following admissible rule:

Lemma 13 (Weakening)**.** If $G; \Gamma \vdash p\ B \mathbin{@} n$ is derivable, then $G; \Gamma, p_1\ A \mathbin{@} n_1 \vdash p_2\ B \mathbin{@} n_1$ is derivable.

Proof. This holds by straightforward induction on the assumed typing derivation. $\square$

Note that we will use admissible rules as if they are inference rules of the logic throughout the sequel.

Lemma 14 (Left-to-Right)**.** If $G; \Gamma_1, \bar{p}\ A \mathbin{@} n, \Gamma_2 \vdash \bar{p'}\ B \mathbin{@} n'$ is derivable, then so is $G; \Gamma_1, \Gamma_2, p'\ B \mathbin{@} n' \vdash p\ A \mathbin{@} n$.

Proof. Suppose $G; \Gamma_1, \bar{p}\ A \mathbin{@} n, \Gamma_2 \vdash \bar{p'}\ B \mathbin{@} n'$ is derivable and $\Gamma_3 =^{\text{def}} \Gamma_1, \bar{p}\ A \mathbin{@} n, \Gamma_2$. Then we derive $G; \Gamma_1, \Gamma_2, p'\ B \mathbin{@} n' \vdash p\ A \mathbin{@} n$ as follows:

$$\dfrac{p'\ B \mathbin{@} n' \in (\Gamma_3, p'\ B \mathbin{@} n') \qquad \dfrac{G; \Gamma_3 \vdash \bar{p'}\ B \mathbin{@} n'}{G; \Gamma_3, p'\ B \mathbin{@} n' \vdash \bar{p'}\ B \mathbin{@} n'}\ \textsc{Weakening}}{G; \Gamma_1, \Gamma_2, p'\ B \mathbin{@} n' \vdash p\ A \mathbin{@} n}\ \textsc{axCut}$$

Thus, we obtain our result. □

We mentioned above that DIL avoids analogs of a number of rules from L. To be able to translate every derivable sequent of L to DIL, we must show admissibility of those rules in DIL. The first of these admissible rules are the rules for reflexivity and transitivity.

Lemma 15 (Reflexivity)**.** If $G, m \preccurlyeq_{p'} m; \Gamma \vdash p\ A \mathbin{@} n$ is derivable, then so is $G; \Gamma \vdash p\ A \mathbin{@} n$.

Proof. This holds by a straightforward induction on the form of the assumed derivation. □

Lemma 16 (Transitivity)**.** If $G, n_1 \preccurlyeq_{p'} n_3; \Gamma \vdash p\ A \mathbin{@} n$ is derivable, $n_1 \preccurlyeq_{p'} n_2 \in G$ and $n_2 \preccurlyeq_{p'} n_3 \in G$, then $G; \Gamma \vdash p\ A \mathbin{@} n$ is derivable.

Proof. This holds by a straightforward induction on the form of the assumed derivation. □

There is not a trivial correspondence between conjunction in DIL and conjunction in L, because of the use of polarities in DIL. Hence, we must show that L's left rule for conjunction is indeed admissible in DIL.

Lemma 17 (AndL)**.** If $G; \Gamma, \bar{p}\ A \mathbin{@} n \vdash p\ B \mathbin{@} n$ is derivable, then $G; \Gamma \vdash p\ (A \wedge_{\bar{p}} B) \mathbin{@} n$ is derivable.

Proof. Suppose $G; \Gamma, \bar{p}\ A \mathbin{@} n \vdash p\ B \mathbin{@} n$ is derivable. By weakening we know $G; \Gamma, \bar{p}\ (A \wedge_{\bar{p}} B) \mathbin{@} n, \bar{p}\ B \mathbin{@} n, \bar{p}\ A \mathbin{@} n \vdash p\ B \mathbin{@} n$. Then $G; \Gamma \vdash p\ (A \wedge_{\bar{p}} B) \mathbin{@} n$ is derivable as follows:

$$\dfrac{\dfrac{\dfrac{D_1 \qquad D_2}{G; \Gamma, \bar{p}\ (A \wedge_{\bar{p}} B) \mathbin{@} n \vdash p\ B \mathbin{@} n}\ \textsc{Cut}}{G; \Gamma, \bar{p}\ (A \wedge_{\bar{p}} B) \mathbin{@} n \vdash p\ (A \wedge_{\bar{p}} B) \mathbin{@} n}\ \textsc{AndBar} \qquad \dfrac{}{G; \Gamma, \bar{p}\ (A \wedge_{\bar{p}} B) \mathbin{@} n \vdash \bar{p}\ (A \wedge_{\bar{p}} B) \mathbin{@} n}\ \textsc{ax}}{G; \Gamma \vdash p\ (A \wedge_{\bar{p}} B) \mathbin{@} n}\ \textsc{Cut}$$

where we have the following subderivations:

D_0 :

$$\dfrac{\dfrac{}{G; \Gamma, \bar{p}\ (A \wedge_{\bar{p}} B) \mathbin{@} n, \bar{p}\ B \mathbin{@} n, p\ A \mathbin{@} n \vdash p\ A \mathbin{@} n}\ \textsc{ax}}{G; \Gamma, \bar{p}\ (A \wedge_{\bar{p}} B) \mathbin{@} n, \bar{p}\ B \mathbin{@} n, p\ A \mathbin{@} n \vdash p\ (A \wedge_{\bar{p}} B) \mathbin{@} n}\ \textsc{AndBar}$$

D_1 :

$$G; \Gamma, \bar{p}\ (A \wedge_{\bar{p}} B) \mathbin{@} n, \bar{p}\ B \mathbin{@} n, \bar{p}\ A \mathbin{@} n \vdash p\ B \mathbin{@} n$$

$$\dfrac{\dfrac{}{G; \Gamma, \bar{p}\ (A \wedge_{\bar{p}} B) \mathbin{@} n, \bar{p}\ B \mathbin{@} n, \bar{p}\ A \mathbin{@} n \vdash \bar{p}\ B \mathbin{@} n}\ \textsc{ax}}{G; \Gamma, \bar{p}\ (A \wedge_{\bar{p}} B) \mathbin{@} n, \bar{p}\ B \mathbin{@} n \vdash p\ A \mathbin{@} n}\ \textsc{Cut}$$

D_2 :

$$\dfrac{\dfrac{}{G;\Gamma,\bar{p}\,(A\wedge_{\bar{p}}B)\mathbin{@}n,\bar{p}\,B\mathbin{@}n,p\,A\mathbin{@}n\vdash\bar{p}\,(A\wedge_{\bar{p}}B)\mathbin{@}n}\ \text{AX}\qquad D_0}{G;\Gamma,\bar{p}\,(A\wedge_{\bar{p}}B)\mathbin{@}n,\bar{p}\,B\mathbin{@}n\vdash\bar{p}\,A\mathbin{@}n}\ \textsc{Cut}$$

□

L has several structural rules. The following lemmata show that all of these are admissible in DIL.

Lemma 18 (Exchange)**.** If $G;\Gamma \vdash p\ A \mathbin{@} n$ is derivable and π is a permutation of Γ, then $G;\pi\,\Gamma \vdash p\ A \mathbin{@} n$ is derivable.

Proof. This holds by a straightforward induction on the form of the assumed derivation. □

Note that we often leave the application of exchange implicit for readablity.

Lemma 19 (Contraction)**.** If $G;\Gamma, p\ A \mathbin{@} n, p\ A \mathbin{@} n, \Gamma' \vdash p'\ B \mathbin{@} n'$, then $G;\Gamma, p\ A \mathbin{@} n, \Gamma' \vdash p'\ B \mathbin{@} n'$.

Proof. This holds by a straightforward induction on the form of the assumed derivation. □

Monotonicity is taken as a primitive in L, but we have decided to leave monotonicity as an admissible rule in DIL. To show that it is admissible in DIL we need to be able to move nodes forward in the abstract Kripke graph. This is necessary to be able to satisfy the graph constraints in the rules IMP and IMPBAR when proving general monotonicity (Lemma 26). The next result is just weakening for the reachability judgment.

Lemma 20 (Graph Weakening)**.** If $G \vdash n_1 \preccurlyeq_p^* n_2$, then $G, n_3 \preccurlyeq_{p'} n_4 \vdash n_1 \preccurlyeq_p^* n_2$.

Proof. This holds by a straightforward induction on the form of the assumed derivation. □

The function raise is an operation on abstract Kripke graphs that takes in two nodes n_1 and n_2, where n_2 is reachable from n_1, and then moves all the edges in an abstract Kripke graph forward to n_2. This essentially performs monotonicity on the given edges. It will be used to show that nodes in the context of a DIL-sequent can be moved forward using monotonicity resulting in a lemma called raising the lower bound logically (Lemma 25).

Definition 21. We define the function raise on abstract graphs as follows:

$$\begin{array}{lcl}
\mathsf{raise}\,(n_1,n_2,\cdot) & = & \cdot\\
\mathsf{raise}\,(n_1,n_2,(n_1 \preccurlyeq_p m,G)) & = & n_2 \preccurlyeq_p m, \mathsf{raise}\,(n_1,n_2,G)\\
\mathsf{raise}\,(n_1,n_2,(m \preccurlyeq_{\bar{p}} n_1,G)) & = & m \preccurlyeq_{\bar{p}} n_2, \mathsf{raise}\,(n_1,n_2,G)\\
\mathsf{raise}\,(n_1,n_2,(m \preccurlyeq_p m',G)) & = & m \preccurlyeq_p m', \mathsf{raise}\,(n_1,n_2,G), \text{where } m \not\equiv n_1 \text{ and } m' \not\equiv n_1.\\
\mathsf{raise}\,(n_1,n_2,(m \preccurlyeq_{\bar{p}} m',G)) & = & m \preccurlyeq_{\bar{p}} m', \mathsf{raise}\,(n_1,n_2,G), \text{where } m \not\equiv n_1 \text{ and } m' \not\equiv n_1.
\end{array}$$

The proof of raising the lower bound depends on the following lemmas:

Lemma 22 (RelAssumFlip)**.** If $G_1, n_1 \preccurlyeq_p n_2, G_2 \vdash m \preccurlyeq_{p'} m'$, then $G_1, n_2 \preccurlyeq_{\bar{p}} n_1, G_2 \vdash m \preccurlyeq_{p'} m'$.

Proof. This is a proof by induction on the form of the assumed derivation. We only consider the case of the AX rule, because the remainder of the cases hold either trivially or by simple applicaitons of the induction hypothesis followed by the rule in the corresponding case.

Case.

$$\overline{G, n \preccurlyeq_{p''} n', G' \vdash n \preccurlyeq^*_{p''} n'} \text{ AX}$$

We only consider the non-trivial case when $G, n \preccurlyeq_{p''} n', G' \equiv G_1, n_1 \preccurlyeq_p n_2, G_2$. This implies that $n \equiv m \equiv n_1$, $n' \equiv m' \equiv n_2$, and $p'' \equiv p' \equiv p$. It suffices to show $G_1, n_2 \preccurlyeq_{\bar{p}} n_1, G_2 \vdash n_1 \preccurlyeq^*_p n_2$. Clearly, we know by the REL_AX rule, $G_1, n_2 \preccurlyeq_{\bar{p}} n_1, G_2 \vdash n_2 \preccurlyeq^*_{\bar{p}} n_1$, and then by the REL_FLIP rule we know $G_1, n_2 \preccurlyeq_{\bar{p}} n_1, G_2 \vdash n_1 \preccurlyeq^*_p n_2$.

□

Lemma 23 (Raising the Lower Bound)**.** If $G \vdash n_1 \preccurlyeq^*_p n_2$ and $G, G_1 \vdash m \preccurlyeq^*_{p'} m'$, then $G, \mathsf{raise}\,(n_1, n_2, G_1) \vdash m \preccurlyeq^*_{p'} m'$.

Proof. This is a proof by induction on the form of $G, G_1 \vdash m \preccurlyeq^*_{p'} m'$.

Case.

$$\overline{G', m \preccurlyeq_{p'} m', G'' \vdash m \preccurlyeq^*_{p'} m'} \text{ AX}$$

Note that it is the case that $G', m \preccurlyeq_{p'} m', G'' \equiv G, G_1$. If $m \preccurlyeq_{p'} m' \in G$, then we obtain our result, so suppose $m \preccurlyeq_{p'} m' \in G_1$. Suppose $p \equiv p'$. Now if $m \not\equiv n_1$, then clearly, we obtain our result. Consider the case where $m \equiv n_1$. Then it suffices to show $G, \mathsf{raise}\,(n_1, n_2, G'_1), n_2 \preccurlyeq_p m', \mathsf{raise}\,(n_1, n_2, G''_1) \vdash n_1 \preccurlyeq^*_p m'$ where $G_1 \equiv G'_1, n_1 \preccurlyeq_p m', G''_1$. This holds by the following derivation:

$$\dfrac{G \vdash n_1 \preccurlyeq^*_p n_2 \qquad \dfrac{}{G, \mathsf{raise}\,(n_1, n_2, G'_1), n_2 \preccurlyeq_p m', \mathsf{raise}\,(n_1, n_2, G''_1) \vdash n_2 \preccurlyeq^*_p m'} \text{ REL\_AX}}{G, \mathsf{raise}\,(n_1, n_2, G'_1), n_2 \preccurlyeq_p m', \mathsf{raise}\,(n_1, n_2, G''_1) \vdash n_1 \preccurlyeq^*_p m'} \text{ REL\_TRANS}$$

Now suppose $p' \equiv \bar{p}$. if $m' \not\equiv n_1$, then clearly, we obtain our result. Consider the case where $m' \equiv n_1$. Then it suffices to show $G, \mathsf{raise}\,(n_1, n_2, G'_1), m \preccurlyeq_{\bar{p}} n_2, \mathsf{raise}\,(n_1, n_2, G''_1) \vdash m \preccurlyeq^*_{\bar{p}} n_1$ where $G_1 \equiv G'_1, m \preccurlyeq_{\bar{p}} n_1, G''_1$. This holds by the following derivation:

$$\dfrac{\dfrac{}{G, \mathsf{raise}\,(n_1, n_2, G'_1), m \preccurlyeq_{\bar{p}} n_2, \mathsf{raise}\,(n_1, n_2, G''_1) \vdash m \preccurlyeq^*_{\bar{p}} n_2} \text{ REL\_AX} \qquad \dfrac{G \vdash n_1 \preccurlyeq^*_p n_2}{G \vdash n_2 \preccurlyeq^*_{\bar{p}} n_1} \text{ REL\_FLIP}}{G, \mathsf{raise}\,(n_1, n_2, G'_1), m \preccurlyeq_{\bar{p}} n_2, \mathsf{raise}\,(n_1, n_2, G''_1) \vdash m \preccurlyeq^*_{\bar{p}} n_1} \text{ REL\_TRANS}$$

Case.

$$\overline{G, G_1 \vdash m \preccurlyeq^*_{p'} m} \text{ REFL}$$

Note that in this case $m' \equiv m$. Our result follows from simply an application of the REL_REFL rule.

Case.

$$\dfrac{G, G_1 \vdash m \preccurlyeq^*_{p'} m'' \qquad G, G_1 \vdash m'' \preccurlyeq^*_{p'} m'}{G, G_1 \vdash m \preccurlyeq^*_{p'} m'} \text{ REL\_TRANS}$$

This case holds by two applications of the induction hypothesis followed by applying the REL_TRANS rule.

Case.

$$\frac{G, G_1 \vdash m' \preccurlyeq^*_{\bar{p}'} m}{G, G_1 \vdash m \preccurlyeq^*_{p'} m'} \text{ FLIP}$$

It suffices to show $G, \mathsf{raise}\,(n_1, n_2, G) \vdash m \preccurlyeq^*_{p'} m'$. We know $G \vdash n_1 \preccurlyeq^*_p n_2$, so by the induction hypothesis we know $G, \mathsf{raise}\,(n_2, n_1, G) \vdash m' \preccurlyeq^*_{\bar{p}} m$. So it suffices to show that $G, \mathsf{raise}\,(n_2, n_1, G) \vdash m' \preccurlyeq^*_{\bar{p}'} m$ implies $G, \mathsf{raise}\,(n_1, n_2, G) \vdash m \preccurlyeq^*_{p'} m'$, but this easily follows by repeated applications of Lemma 22.

□

Lemma 24 (Graph Node Containment)**.** If $G \vdash n_1 \preccurlyeq^*_p n_2$ and n_1 and n_2 are unique, then $n_1, n_2 \in |G|$.

Proof. This holds by straightforward induction on the form of $G \vdash n_1 \preccurlyeq^*_p n_2$. □

Finally, we arrive to raising the lower bound logically and general monotonicity. The latter depending on the former. These are the last of the admissiblilty results before showing that all derivable L-sequents are derivable in DIL.

Lemma 25 (Raising the Lower Bound Logically)**.** If $G, G_1, G'; \Gamma \vdash p\, A \,@\, n$ and $G, G' \vdash n_1 \preccurlyeq^*_p n_2$, then $G, \mathsf{raise}\,(n_1, n_2, G_1), G'; \Gamma \vdash p\, A \,@\, n$.

Proof. This is a proof by induction on the form of $G, G_1, G'; \Gamma \vdash p\, A \,@\, n$. We assume with out loss of generality that $n_1 \in |G_1|$, and that $n_1 \not\equiv n_2$. If this is not the case then $\mathsf{raise}\,(n_1, n_2, G_1) = G_1$, and the result holds trivially.

Case.

$$\frac{G, G_1, G' \vdash n' \preccurlyeq^*_p n}{G, G_1, G'; \Gamma, p\, A \,@\, n' \vdash p\, A \,@\, n} \text{ AX}$$

Clearly, if $G, G_1, G' \vdash n' \preccurlyeq^*_p n$, then $G, G', G_1 \vdash n' \preccurlyeq^*_p n$. Thus, this case follows by raising the lower bound (Lemma 23), and applying the AX rule.

Case.

$$\frac{}{G, G_1, G'; \Gamma \vdash p\, \langle p \rangle \,@\, n} \text{ UNIT}$$

Trivial.

Case.

$$\frac{G, G_1, G'; \Gamma \vdash p\, A_1 \,@\, n \qquad G, G_1, G'; \Gamma \vdash p\, A_2 \,@\, n}{G, G_1, G'; \Gamma \vdash p\, (A_1 \wedge_p A_2) \,@\, n} \text{ AND}$$

This case holds by two applications of the induction hypothesis, and then applying the AND rule.

Case.

$$\frac{G, G_1, G'; \Gamma \vdash p\ A_d \ @\ n}{G, G_1, G'; \Gamma \vdash p\ (A_1 \wedge_{\bar{p}} A_2)\ @\ n}\ \textsc{andBar}$$

Similar to the previous case.

Case.

$$\frac{\begin{array}{l} n' \notin |G, G_1, G'|, |\Gamma| \\ (G, G_1, G', n \preccurlyeq_p n'); \Gamma, p\ A_1 \ @\ n' \vdash p\ A_2 \ @\ n' \end{array}}{G, G_1, G'; \Gamma \vdash p\ (A_1 \rightarrow_p A_2)\ @\ n}\ \textsc{imp}$$

Since we know $n_1 \not\equiv n_2$, then by Lemma 24 we know $n_1, n_2 \in |G, G'|$. Thus, $n' \not\equiv n_1 \not\equiv n_2$. Now by the induction hypothesis we know $(G, \mathsf{raise}\,(n_1, n_2, G_1), G', n \preccurlyeq_p n'); \Gamma, p\ A_1 \ @\ n' \vdash p\ A_2 \ @\ n'$. This case then follows by the application of the IMP rule to the former.

Case.

$$\frac{\begin{array}{l} G, G_1, G' \vdash n \preccurlyeq^*_{\bar{p}} n' \\ G, G_1, G'; \Gamma \vdash \bar{p}\ A_1 \ @\ n' \qquad G, G_1, G'; \Gamma \vdash p\ A_2 \ @\ n' \end{array}}{G, G_1, G'; \Gamma \vdash p\ (A_1 \rightarrow_{\bar{p}} A_2)\ @\ n}\ \textsc{impBar}$$

Clearly, $G, G_1, G' \vdash n \preccurlyeq^*_{\bar{p}} n'$ implies $G, G', G_1 \vdash n \preccurlyeq^*_{\bar{p}} n'$, and by raising the lower bound (Lemma 23) we know $G, G', \mathsf{raise}\,(n_1, n_2, G_1) \vdash n \preccurlyeq^*_{\bar{p}} n'$ which implies $G, \mathsf{raise}\,(n_1, n_2, G_1), G' \vdash n \preccurlyeq^*_{\bar{p}} n'$.

Case.

$$\frac{p\ T' \ @\ n' \in \Gamma \qquad G, G_1, G'; \Gamma, \bar{p}\ T \ @\ n \vdash \bar{p}\ T' \ @\ n'}{G, G_1, G'; \Gamma \vdash p\ T \ @\ n}\ \textsc{axCut}$$

This case follows by a simple application of the induction hypothesis, and then reapplying the rule.

Case.

$$\frac{\bar{p}\ T' \ @\ n' \in \Gamma \qquad G, G_1, G'; \Gamma, \bar{p}\ T \ @\ n \vdash p\ T' \ @\ n'}{G, G_1, G'; \Gamma \vdash p\ T \ @\ n}\ \textsc{axCutBar}$$

Similar to the previous case.

□

Lemma 26 (General Monotonicity)**.** If $G \vdash n_1 \preccurlyeq^*_{p_1} n'_1, \ldots, G \vdash n_i \preccurlyeq^*_{p_i} n'_i$, $G \vdash m \preccurlyeq^*_p m'$, and $G; \bar{p}_1\ A_1 \ @\ n_1, \ldots, \bar{p}_i\ A_i \ @\ n_i \vdash p\ B \ @\ m$, then $G; \bar{p}_1 A_1 @ n'_1, \ldots, \bar{p}_i A_i @ n'_i \vdash p\ B \ @\ m'$.

Proof. This is a proof by induction on the form of $G; \bar{p}_1\ A_1 \ @\ n_1, \ldots, \bar{p}_i\ A_i \ @\ n_i \vdash p\ B \ @\ m$. We assume without loss of generality that all of $n_1, n'_1 \ldots, n_i, n'_i$ are unique. Thus, they are all member of $|G|$.

Case.

$$\frac{G \vdash n_i \preccurlyeq^*_{\bar{p}_i} n'}{G; \bar{p}_1\ A_1 \ @\ n_1, \ldots, \bar{p}_i\ A_i \ @\ n_i \vdash \bar{p}_i\ A_i \ @\ n'}\ \textsc{ax}$$

It must be the case that $p\ B \ @\ m \equiv \bar{p}_i\ A \ @\ n'$. In addition we know $G \vdash n_i \preccurlyeq^*_{p_i} n'_i$ and $G \vdash n' \preccurlyeq^*_{\bar{p}_i} m'$. It suffices to show $G; \bar{p}_1 A_1 @ n'_1, \ldots, \bar{p}_i A_i @ n'_i \vdash \bar{p}_i\ A_i \ @\ m'$. This is derivable as follows:

$$\cfrac{\cfrac{\cfrac{\cfrac{G \vdash n_i \preccurlyeq^*_{p_i} n'_i}{G \vdash n'_i \preccurlyeq^*_{\bar{p}_i} n_i}\ \textsc{rel\_flip} \quad \cfrac{G \vdash n_i \preccurlyeq^*_{\bar{p}_i} n' \qquad G \vdash n' \preccurlyeq^*_{\bar{p}_i} m'}{G \vdash n_i \preccurlyeq^*_{\bar{p}_i} m'}\ \textsc{rel\_trans}}{G \vdash n'_i \preccurlyeq^*_{\bar{p}_i} m'}\ \textsc{rel\_trans}}{G; \bar{p_1} A_1 @ n'_1, \ldots, \bar{p_i} A_i @ n'_i \vdash \bar{p_i}\, A_i \mathbin{@} m'}\ \textsc{ax}}{}$$

Case.

$$\frac{}{G; \bar{p_1}\ A_1 \mathbin{@} n_1,\ \ldots, \bar{p_i}\ A_i \mathbin{@} n_i \vdash p\ \langle p \rangle \mathbin{@} m_1}\ \textsc{unit}$$

Trivial.

Case.

$$\frac{\begin{array}{c} G; \bar{p_1}\ A_1 \mathbin{@} n_1,\ \ldots, \bar{p_i}\ A_i \mathbin{@} n_i \vdash p\ B_1 \mathbin{@} m \\ G; \bar{p_1}\ A_1 \mathbin{@} n_1,\ \ldots, \bar{p_i}\ A_i \mathbin{@} n_i \vdash p\ B_2 \mathbin{@} m \end{array}}{G; \bar{p_1}\ A_1 \mathbin{@} n_1,\ \ldots, \bar{p_i}\ A_i \mathbin{@} n_i \vdash p\ (B_1 \wedge_p B_2) \mathbin{@} m}\ \textsc{and}$$

This case follows easily by applying the induction hypothesis to each premise and then applying the AND rule.

Case.

$$\frac{G; \bar{p_1}\ A_1 \mathbin{@} n_1,\ \ldots, \bar{p_i}\ A_i \mathbin{@} n_i \vdash p\ B_d \mathbin{@} m}{G; \bar{p_1}\ A_1 \mathbin{@} n_1,\ \ldots, \bar{p_i}\ A_i \mathbin{@} n_i \vdash p\ (B_1 \wedge_{\bar{p}} B_2) \mathbin{@} m}\ \textsc{andBar}$$

This case follows easily by the induction hypothesis and then applying ANDBAR.

Case.

$$\frac{\begin{array}{l} n' \notin |G|, |\bar{p_1}\ A_1 \mathbin{@} n_1,\ \ldots, \bar{p_i}\ A_i \mathbin{@} n_i| \\ (G, m_1 \preccurlyeq_p n'); \bar{p_1}\ A_1 \mathbin{@} n_1,\ \ldots, \bar{p_i}\ A_i \mathbin{@} n_i, p\ B_1 \mathbin{@} n' \vdash p\ B_2 \mathbin{@} n' \end{array}}{G; \bar{p_1}\ A_1 \mathbin{@} n_1,\ \ldots, \bar{p_i}\ A_i \mathbin{@} n_i \vdash p\ (B_1 \to_p B_2) \mathbin{@} m}\ \textsc{imp}$$

We know by assumption $G \vdash n_1 \preccurlyeq^*_{p_1} n'_1, \ldots, G \vdash n_i \preccurlyeq^*_{p_i} n'_i$, and by graph weakening (Lemma 20) $G, m \preccurlyeq_p n' \vdash n_1 \preccurlyeq^*_{p_1} n'_1, \ldots, G, m \preccurlyeq_p n' \vdash n_i \preccurlyeq^*_{p_i} n'_i$. We also know by applying the REL_REFL rule that $G, m \preccurlyeq_p n' \vdash n' \preccurlyeq^*_{\bar{p}} n'$ and $G, m \preccurlyeq_p n' \vdash n' \preccurlyeq^*_p n'$. Thus, by the induction hypothesis we know $(G, m \preccurlyeq_p n'); \bar{p_1} A_1 @ n'_1, \ldots, \bar{p_i} A_i @ n'_i, p\, B_1 \mathbin{@} n' \vdash p\, B_2 \mathbin{@} n'$. Now we can raise the lower bound logically (Lemma 25) with $G_1 \equiv m \preccurlyeq_p n'$ and the assumption $G \vdash m \preccurlyeq^*_p m'$ to obtain
$(G, \mathsf{raise}\,(m, m', m \preccurlyeq_p n')); \bar{p_1} A_1 @ n'_1, \ldots, \bar{p_i} A_i @ n'_i, p\, B_1 \mathbin{@} n' \vdash p\, B_2 \mathbin{@} n'$, but this is equivalent to $(G, m \preccurlyeq_p n'); \bar{p_1} A_1 @ n'_1, \ldots, \bar{p_i} A_i @ n'_i, p\, B_1 \mathbin{@} n' \vdash p\, B_2 \mathbin{@} n'$. Finally, using the former, we obtain our result by applying the IMP rule.

Case.

$$\frac{\begin{array}{l} G \vdash m \preccurlyeq^*_{\bar{p}} n' \\ G; \bar{p_1}\ A_1 \mathbin{@} n_1,\ \ldots, \bar{p_i}\ A_i \mathbin{@} n_i \vdash \bar{p}\ B_1 \mathbin{@} n' \\ G; \bar{p_1}\ A_1 \mathbin{@} n_1,\ \ldots, \bar{p_i}\ A_i \mathbin{@} n_i \vdash p\ B_2 \mathbin{@} n' \end{array}}{G; \bar{p_1}\ A_1 \mathbin{@} n_1,\ \ldots, \bar{p_i}\ A_i \mathbin{@} n_i \vdash p\ (B_1 \to_{\bar{p}} B_2) \mathbin{@} m}\ \textsc{impBar}$$

We can easily derive $G \vdash m' \preccurlyeq^*_{\bar{p}} n'$ as follows:

$$\dfrac{\dfrac{\dfrac{G \vdash m \preccurlyeq^*_{\bar{p}} n'}{G \vdash n' \preccurlyeq^*_{p} m} \text{REL\_FLIP} \qquad G \vdash m \preccurlyeq^*_{p} m'}{G \vdash n' \preccurlyeq^*_{p} m'} \text{REL\_TRANS}}{G \vdash m' \preccurlyeq^*_{\bar{p}} n'} \text{REL\_FLIP}$$

This case then follows by applying the induction hypothesis twice to both $G; \bar{p_1}\ A_1 @ n_1, \dots, \bar{p_i}\ A_i @ n_i, \bar{p}\ B_{\bar{d}} @ m \vdash \bar{p}\ B_1 @ n'$ and $G; \bar{p_1}\ A_1 @ n_1, \dots, \bar{p_i}\ A_i @ n_i, \bar{p}\ B_{\bar{d}} @ m \vdash p\ B_2 @ n'$ using the assumptions $G \vdash n_1 \preccurlyeq^*_{p_1} n'_1, \dots, G \vdash n_i \preccurlyeq^*_{p_i} n'_i$, $G \vdash m \preccurlyeq^*_{p} m'$, and the fact that we know $G \vdash n' \preccurlyeq^*_{p} n'$ and $G \vdash n' \preccurlyeq^*_{\bar{p}} n'$.

Case.

$$\dfrac{\begin{array}{c}\bar{p_j}\ A_j @ n_j \in (\bar{p_1}\ A_1 @ n_1, \dots, \bar{p_i}\ A_i @ n_i)\\ G; \bar{p_1}\ A_1 @ n_1, \dots, \bar{p_i}\ A_i @ n_i, \bar{p}\ B @ m \vdash p_j\ A_j @ n_j\end{array}}{G; \bar{p_1}\ A_1 @ n_1, \dots, \bar{p_i}\ A_i @ n_i \vdash p\ B @ m} \text{AXCUT}$$

We know by assumption that $G \vdash n_1 \preccurlyeq^*_{p_1} n'_1, \dots, G \vdash n_i \preccurlyeq^*_{p_i} n'_i$, and $G \vdash m \preccurlyeq^*_{p} m'$. In particular, we know $G \vdash n_j \preccurlyeq^*_{p_j} n'_j$. It is also the case that if $\bar{p_j}\ A_j @ n_j \in (\bar{p_1}\ A_1 @ n_1, \dots, \bar{p_i}\ A_i @ n_i)$, then $\bar{p_j}\ A_j @ n'_j \in (\bar{p_1} A_1 @ n', \dots, \bar{p_i} A_i @ n'_i)$. This case then follows by applying the induction hypothesis to $G; \bar{p_1}\ A_1 @ n_1, \dots, \bar{p_i}\ A_i @ n_i, \bar{p}\ B @ m \vdash p_j\ A_j @ n_j$, to obtain, $G; \bar{p_1} A_1 @ n'_1, \dots, \bar{p_i} A_i @ n'_i, \bar{p}\ B @ m'_1 \vdash p_j\ A_j @ n'_j$, followed by applying the AXCUT rule.

Case.

$$\dfrac{\begin{array}{c}\bar{p_j}\ A_j @ n_j \in (\bar{p_1}\ A_1 @ n_1, \dots, \bar{p_i}\ A_i @ n_i)\\ G; \bar{p_1}\ A_1 @ n_1, \dots, \bar{p_i}\ A_i @ n_i, \bar{p}\ B @ m \vdash p_j\ A_j @ n_j\end{array}}{G; \bar{p_1}\ A_1 @ n_1, \dots, \bar{p_i}\ A_i @ n_i \vdash p\ B @ m} \text{AXCUTBAR}$$

Similar to the previous case.

□

The following are corollaries of general monotonicity. The latter two corollaries show that the monotonicity rules of L are admissible in DIL.

Corollary 27 (Monotonicity)**.** Suppose $G \vdash n_1 \preccurlyeq^*_{p} n_2$. Then

i. if $G; \Gamma, \bar{p}\ A @ n_1, \Gamma' \vdash p'\ B @ n'$, then $G; \Gamma, \bar{p}\ A @ n_2, \Gamma' \vdash p'\ B @ n'$, and
ii. if $G; \Gamma \vdash p\ A @ n_1$, then $G; \Gamma \vdash p\ A @ n_2$.

Proof. This result follows easily from Lemma 26. □

Corollary 28 (MonoL)**.** If $G; \Gamma, p\ A @ n_1, p\ A @ n_2, \Gamma' \vdash p'\ B @ n'$ is derivable and $n_1 \preccurlyeq_p n_2 \in G$, then $G; \Gamma, p\ A @ n_1, \Gamma' \vdash p'\ B @ n'$ is derivable.

Proof. This result easily follows by part one of Corollary 27, and contraction (Lemma 19). □

Corollary 29 (MonoR)**.** If $G; \Gamma, \bar{p}\ A @ n_1, \Gamma' \vdash p\ A @ n_2$ and $n_1 \preccurlyeq_p n_2 \in G$, then $G; \Gamma, \Gamma' \vdash p\ A @ n_2$ is derivable.

Proof. Suppose $G; \Gamma, \bar{p}\ A @ n_1, \Gamma' \vdash p\ A @ n_2$ and $n_1 \preccurlyeq_p n_2 \in G$. Then by part one of monotonicity (Corollary 27) we know $G; \Gamma, \bar{p}\ A @ n_2, \Gamma' \vdash p\ A @ n_2$. Finally, we know by the axiom cut rule that $G; \Gamma, \Gamma' \vdash p\ A @ n_2$. □

We now have everything we need to prove that every derivable sequent of L can be translated to a derivable sequent in DIL. Using the translation of formulas given in Definition 1 we can easily translate contexts. Right contexts, Γ, in L are translated to positive hypotheses, while left contexts – not including the formula chosen as the active formula – are translated into negative hypotheses. The following definition defines the translation of both types of contexts.

Definition 30. We extend the translation of formulas to contexts Γ and Δ with respect to a polarity p as follows:

$$\begin{array}{lll} \ulcorner \cdot \urcorner^{p} & = & \cdot \\ \\ \ulcorner n : A, \Gamma \urcorner^{p} & = & p \ulcorner A \urcorner \ @ \ n, \ulcorner \Gamma \urcorner^{p} \end{array}$$

Abstract Kripke models are straightforward to translate.

Definition 31. We define the translation of graphs G in L to graphs in DIL as follows:

$$\begin{array}{lll} \ulcorner \cdot \urcorner & = & \cdot \\ \\ \ulcorner (n_1, n_2), G \urcorner & = & n_1 \preccurlyeq_{+} n_2, \ulcorner G \urcorner \end{array}$$

The previous definition implies the following result:

Lemma 32 (Reachability)**.** If $n_1 G n_2$, then $\ulcorner G \urcorner \vdash n_1 \preccurlyeq_{+}^{*} n_2$.

The translation of a derivable L-sequent is a DIL-sequent which requires a particular formula as the active formula. We define such a translation in the following definition.

Definition 33. An activation of a derivable L-sequent $\Gamma \vdash_G \Delta$ is a DIL-sequent $\ulcorner G \urcorner; \ulcorner \Gamma \urcorner^{+}, \ulcorner \Delta_1, \Delta_2 \urcorner^{-} \vdash + \ulcorner A \urcorner \ @ \ n$, where $\Delta = \Delta_1, n : A, \Delta_2$.

The following theorem is the main result of this section. It shows that every derivable L-sequent can be translated into a derivable DIL-sequent. We do this by considering an arbitrary activation of the L-sequent, and then show that this arbitrary activation is derivable in DIL, but if it so happens that this is not the correct activation, the we can always get the correct one by using the left-to-right lemma (Lemma 14) to switch out the active formula.

Theorem 34 (Containment of L in DIL)**.** If $\ulcorner G \urcorner; \Gamma' \vdash + A \ @ \ n$ is an activation of the derivable L-sequent $\Gamma \vdash_G \Delta$, then $\ulcorner G \urcorner; \Gamma' \vdash + A \ @ \ n$ is derivable.

Proof. This is a proof by induction on the form of the sequent $\Gamma \vdash_G \Delta$.

Case.

$$\frac{\Gamma \vdash_{G,(n,n)} \Delta}{\Gamma \vdash_G \Delta} \text{ REFL}$$

We know by the induction hypothesis that every activation of $\Gamma \vdash_{G,(n,n)} \Delta$ is derivable. Suppose that $\ulcorner G, (n, n) \urcorner; \Gamma' \vdash + A \ @ \ n$ is an arbitrary activation, where $\ulcorner \Delta \urcorner^{-} \equiv \ulcorner \Delta_1 \urcorner^{-}, - A \ @ \ n, \ulcorner \Delta_2 \urcorner^{-}$ and $\Gamma' \equiv \ulcorner \Delta_1 \urcorner^{-}, \ulcorner \Delta_2 \urcorner^{-}$. This is equivalent to $\ulcorner G \urcorner, n \preccurlyeq_{+} n; \Gamma' \vdash + A \ @ \ n$, and by the admissible rule for reflexivity (Lemma 15) we have $\ulcorner G \urcorner; \Gamma' \vdash + A \ @ \ n$.

Case.

$$\dfrac{\begin{array}{l} n_1 G n_2 \\ n_2 G n_3 \\ \Gamma \vdash_{G,(n_1,n_3)} \Delta \end{array}}{\Gamma \vdash_G \Delta}\ \textsc{Trans}$$

We know by the induction hypothesis that every activation of $\Gamma \vdash_{G,(n_1,n_3)} \Delta$ is derivable. Suppose that $\ulcorner G,(n_1,n_3)\urcorner; \Gamma' \vdash +\, A @ n$ is an arbitrary activation, where $\ulcorner \Delta \urcorner^- \equiv \ulcorner \Delta_1 \urcorner^-, -\, A @ n, \ulcorner \Delta_2 \urcorner^-$ and $\Gamma' \equiv \ulcorner \Delta_1 \urcorner^-, \ulcorner \Delta_2 \urcorner^-$. This sequent is equivalent to $\ulcorner G \urcorner, n_1 \preccurlyeq_+ n_3; \Gamma' \vdash +\, A @ n$. Furthermore, it is clear by definition that if $n_1 G n_2$ and $n_2 G n_3$, then $n_1 \preccurlyeq_+ n_2 \in \ulcorner G \urcorner$ and $n_2 \preccurlyeq_+ n_3 \in \ulcorner G \urcorner$. Thus, by the admissible rule for transitivity (Lemma 16) we have $\ulcorner G \urcorner; \Gamma' \vdash +\, A @ n$, and we obtain our result.

Case.

$$\dfrac{}{\Gamma, n : A \vdash_G n : A, \Delta}\ \textsc{hyp}$$

It suffices to show that every activation of $\Gamma, n : A \vdash_G n : A, \Delta$ is derivable. Clearly, $\ulcorner G \urcorner; \ulcorner \Gamma \urcorner^+, +\ulcorner A \urcorner @ n, \ulcorner \Delta \urcorner^- \vdash +\ulcorner A \urcorner @ n$ is a activation of $\Gamma, n : A \vdash_G n : A, \Delta$. In addition, it is derivable:

$$\dfrac{\dfrac{\dfrac{}{\ulcorner G \urcorner \vdash n \preccurlyeq_+^* n}\ \textsc{Refl}}{\ulcorner G \urcorner; \ulcorner \Gamma \urcorner^+, \ulcorner \Delta \urcorner^-, +\ulcorner A \urcorner @ n \vdash +\ulcorner A \urcorner @ n}\ \textsc{Ax}}{\ulcorner G \urcorner; \ulcorner \Gamma \urcorner^+, +\ulcorner A \urcorner @ n, \ulcorner \Delta \urcorner^- \vdash +\ulcorner A \urcorner @ n}\ \textsc{Exchange}$$

In the previous derivation we make use of the exchange rule which is admissible by Lemma 18.

Now consider any other activation $\ulcorner G \urcorner; \Gamma' \vdash +\, B @ n'$. It must be the case that $\Gamma' = \ulcorner \Gamma \urcorner^+, +\, A @ n, \ulcorner \Delta_1 \urcorner^-, -\ulcorner A \urcorner @ n, \ulcorner \Delta_2 \urcorner^-$ for some Δ_1 and Δ_2. This sequent is then derivable as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\ulcorner G \urcorner \vdash n \preccurlyeq_+^* n}\ \textsc{Refl}}{\ulcorner G \urcorner; \ulcorner \Gamma \urcorner^+, \ulcorner \Delta_1 \urcorner^-, \ulcorner \Delta_2 \urcorner^-, -\, B @ n', +\, A @ n \vdash +\ulcorner A \urcorner @ n}\ \textsc{Ax}}{\ulcorner G \urcorner; \ulcorner \Gamma \urcorner^+, +\, A @ n, \ulcorner \Delta_1 \urcorner^-, \ulcorner \Delta_2 \urcorner^-, -\, B @ n' \vdash +\ulcorner A \urcorner @ n}\ \textsc{Exchange}}{\ulcorner G \urcorner; \ulcorner \Gamma \urcorner^+, +\, A @ n, \ulcorner \Delta_1 \urcorner^-, -\ulcorner A \urcorner @ n, \ulcorner \Delta_2 \urcorner^- \vdash +\, B @ n'}\ \textsc{Left-to-Right}$$

Thus, we obtain our result.

Case.

$$\dfrac{\begin{array}{l} n_1 G n_2 \\ \Gamma, n_1 : A, n_2 : A \vdash_G \Delta \end{array}}{\Gamma, n_1 : A \vdash_G \Delta}\ \textsc{monL}$$

Certainly, if $n_1 G n_2$, then $n_1 \preccurlyeq_+ n_2 \in \ulcorner G \urcorner$. We know by the induction hypothesis that all activations of $\Gamma, n_1 : A, n_2 : A \vdash_G \Delta$ are derivable. Suppose $\ulcorner G \urcorner; \Gamma' \vdash +\, B@n$ is an arbitrary activation. Then it must be the case that $\Gamma' \equiv \ulcorner \Gamma \urcorner^+, +\, A @ n_1, +\, A @ n_2, \ulcorner \Delta_1 \urcorner^-, \ulcorner \Delta_2 \urcorner^-$, where $\ulcorner \Delta \urcorner^- \equiv \ulcorner \Delta_1 \urcorner^-, -\, B@n, \ulcorner \Delta_2 \urcorner^-$. Now we apply the monoL admissible rule (Lemma 28) to obtain $\ulcorner G\urcorner;\ulcorner\Gamma\urcorner^{+}, +\,A@n_1, \ulcorner\Delta_1\urcorner^{-}, \ulcorner\Delta_2\urcorner^{-} \vdash +\,B@n$, which is an arbitrary activation of $\Gamma, n_1 : A \vdash_G \Delta$.

Case.

$$\frac{\begin{array}{l} n_1 G n_2 \\ \Gamma \vdash_G n_1 : A, n_2 : A, \Delta \end{array}}{\Gamma \vdash_G n_2 : A, \Delta}\ \textsc{monR}$$

If n_1Gn_2, then $n_1 \preccurlyeq_{+} n_2 \in \ulcorner G\urcorner$. We know by the induction hypothesis that all activations of $\Gamma \vdash_G n_1 : A, n_2 : A, \Delta$ are derivable. In particular, the activation (modulo exchange (Lemma 18)) $\ulcorner G\urcorner;\ulcorner\Gamma\urcorner^{+}, \ulcorner\Delta\urcorner^{-}, -\,A @ n_1 \vdash +\,A @ n_2$ is derivable. It suffices to show that $\ulcorner G\urcorner;\ulcorner\Gamma\urcorner^{+}, \ulcorner\Delta\urcorner^{-} \vdash +\,A @ n_2$. This follows from the monoR admissible rule (Lemma 29). Finally, any other activation of $\Gamma \vdash_G n_2 : A, \Delta$ can be activated into $\ulcorner G\urcorner;\ulcorner\Gamma\urcorner^{+}, \ulcorner\Delta\urcorner^{-} \vdash +\,A @ n_2$ (Lemma 14). Thus, we obtain our result.

Case.

$$\frac{\Gamma \vdash_G \Delta}{\Gamma, n' : \top \vdash_G \Delta}\ \textsc{trueL}$$

We know by the induction hypothesis that all activations of $\Gamma \vdash_G \Delta$ are derivable. Suppose $\ulcorner G\urcorner;\Gamma' \vdash +\,A@n$ is an arbitrary activation of $\Gamma \vdash_G \Delta$. Then it must be the case that $\Gamma' = \ulcorner\Gamma\urcorner^{+}, \ulcorner\Delta_1\urcorner^{-}, \ulcorner\Delta_2\urcorner^{-}$, where $\ulcorner\Delta\urcorner^{-} \equiv \ulcorner\Delta_1\urcorner^{-}, -\,A @ n, \ulcorner\Delta_2\urcorner^{-}$. Now by weakening (Lemma 13) we know $\ulcorner G\urcorner;\Gamma', +\,\langle+\rangle @ n' \vdash +\,A @ n$, and by exchange (Lemma 18) $\ulcorner G\urcorner; +\,\langle+\rangle @ n', \Gamma' \vdash +\,A @ n$, which is exactly an arbitrary activation of $\Gamma, n' : \top \vdash_G \Delta$.

Case.

$$\frac{}{\Gamma \vdash_G n : \top, \Delta}\ \textsc{trueR}$$

It suffices to show that every activation of $\Gamma \vdash_G n : \top, \Delta$ is derivable. Consider the activation $\ulcorner G\urcorner;\ulcorner\Gamma\urcorner^{+}, \ulcorner\Delta\urcorner^{-} \vdash +\ulcorner\top\urcorner @ n$. This is easily derivable by applying the UNIT rule. Now any other activation of $\Gamma \vdash_G n : \top, \Delta$ implies $\ulcorner G\urcorner;\ulcorner\Gamma\urcorner^{+}, \ulcorner\Delta\urcorner^{-} \vdash +\ulcorner\top\urcorner @ n$ is derivable by Lemma 14, and hence, are derivable.

Case.

$$\frac{}{\Gamma, n : \perp \vdash_G \Delta}\ \textsc{falseL}$$

Suppose $\ulcorner G\urcorner;\ulcorner\Gamma\urcorner^{+}, +\ulcorner\perp\urcorner @ n, \ulcorner\Delta_1\urcorner^{-}, \ulcorner\Delta_2\urcorner^{-} \vdash +\,A @ n'$ is an arbitrary activation of $\Gamma, n : \perp \vdash_G \Delta$, where $\ulcorner\Delta\urcorner^{-} \equiv \ulcorner\Delta_1\urcorner^{-}, -\,A @ n', \ulcorner\Delta_2\urcorner^{-}$. We can easily see that by definition $\ulcorner G\urcorner;\ulcorner\Gamma\urcorner^{+}, +\ulcorner\perp\urcorner @ n, \ulcorner\Delta_1\urcorner^{-}, \ulcorner\Delta_2\urcorner^{-} \vdash +\,A @ n'$ is equivalent to $\ulcorner G\urcorner;\ulcorner\Gamma\urcorner^{+}, +\,\langle-\rangle @ n, \ulcorner\Delta_1\urcorner^{-}, \ulcorner\Delta_2\urcorner^{-} \vdash +\,A @ n'$. We can derive the latter as follows:

$$\frac{+\,\langle-\rangle @ n \in \Gamma', -\,A @ n' \qquad \dfrac{}{\ulcorner G\urcorner;\Gamma', -\,A @ n' \vdash -\,\langle-\rangle @ n}\ \textsc{unit}}{\ulcorner G\urcorner;\ulcorner\Gamma\urcorner^{+}, +\,\langle-\rangle @ n, \ulcorner\Delta_1\urcorner^{-}, \ulcorner\Delta_2\urcorner^{-} \vdash +\,A @ n'}\ \textsc{axCutBar}$$

In the previous derivation $\Gamma' \equiv \ulcorner\Gamma\urcorner^{+}, +\,\langle-\rangle @ n, \ulcorner\Delta_1\urcorner^{-}, \ulcorner\Delta_2\urcorner^{-}$. Thus, any activation of $\Gamma, n : \perp \vdash_G \Delta$ is derivable.

Case.

$$\frac{\Gamma \vdash_G \Delta}{\Gamma \vdash_G n' : \perp, \Delta}\ \text{falseR}$$

We know by the induction hypothesis that all activations of $\Gamma \vdash_G \Delta$ are derivable. Suppose $\ulcorner G \urcorner; \Gamma' \vdash +A \mathbin{@} n$ is an arbitrary activation of $\Gamma \vdash_G \Delta$. Then it must be the case that $\Gamma' = \ulcorner\Gamma\urcorner^+, \ulcorner\Delta\urcorner^-$. Now by weakening (Lemma 13) we know $\ulcorner G \urcorner; \Gamma', -\langle - \rangle \mathbin{@} n' \vdash +A \mathbin{@} n$, and by the left-to-right lemma (Lemma 14) $\ulcorner G \urcorner; \Gamma', -A \mathbin{@} n \vdash +\langle - \rangle \mathbin{@} n'$, which – modulo exchange – is equivalent to $\ulcorner G \urcorner; \ulcorner\Gamma\urcorner^+, \ulcorner\Delta\urcorner^- \vdash +\ulcorner\perp\urcorner \mathbin{@} n'$. Thus, we obtain our result.

Case.

$$\frac{\Gamma, n : T_1, n : T_2 \vdash_G \Delta}{\Gamma, n : T_1 \wedge T_2 \vdash_G \Delta}\ \text{andL}$$

We know by the induction hypothesis that all activations of $\Gamma, n : T_1, n : T_2 \vdash_G \Delta$ are derivable. In particular, we know $\ulcorner G \urcorner; \ulcorner\Gamma\urcorner^+, +T_1 @ n, +T_2 @ n, \ulcorner\Delta_1\urcorner^-, \ulcorner\Delta_2\urcorner^- \vdash +A \mathbin{@} n'$ where $\ulcorner\Delta\urcorner^- = \ulcorner\Delta_1\urcorner^-, -A@n', \ulcorner\Delta_2\urcorner^-$. Using exchange we know $\ulcorner G \urcorner; \ulcorner\Gamma\urcorner^+, \ulcorner\Delta_1\urcorner^-, \ulcorner\Delta_2\urcorner^-, +T_1 \mathbin{@} n, +T_2 \mathbin{@} n \vdash +A \mathbin{@} n'$, and by the left-to-right lemma $\ulcorner G \urcorner; \ulcorner\Gamma\urcorner^+, \ulcorner\Delta_1\urcorner^-, \ulcorner\Delta_2\urcorner^-, +T_1 \mathbin{@} n, -A \mathbin{@} n' \vdash -T_2 \mathbin{@} n$, and finally by one more application of exchange $\ulcorner G \urcorner; \ulcorner\Gamma\urcorner^+, \ulcorner\Delta_1\urcorner^-, \ulcorner\Delta_2\urcorner^-, -A \mathbin{@} n', +T_1 \mathbin{@} n \vdash -T_2 \mathbin{@} n$. At this point we know $\ulcorner G \urcorner; \ulcorner\Gamma\urcorner^+, \ulcorner\Delta_1\urcorner^-, \ulcorner\Delta_2\urcorner^-, -A \mathbin{@} n' \vdash -T_1 \wedge_+ T_2 \mathbin{@} n$ by the using the admissible andL rule (Lemma 17). Now using left-to-right $\ulcorner G \urcorner; \ulcorner\Gamma\urcorner^+, \ulcorner\Delta_1\urcorner^-, \ulcorner\Delta_2\urcorner^-, +T_1 \wedge_+ T_2 \mathbin{@} n \vdash +A \mathbin{@} n'$ is derivable. Lastly, by exchange $\ulcorner G \urcorner; \ulcorner\Gamma\urcorner^+, +T_1 \wedge_+ T_2 \mathbin{@} n, \ulcorner\Delta_1\urcorner^-, \ulcorner\Delta_2\urcorner^- \vdash +A \mathbin{@} n'$ is derivable, which is clearly an arbitrary activation of $\Gamma, n : T_1 \wedge T_2 \vdash_G \Delta$.

Case.

$$\frac{\begin{array}{c}\Gamma \vdash_G n : A, \Delta \\ \Gamma \vdash_G n : B, \Delta\end{array}}{\Gamma \vdash_G n : A \wedge B, \Delta}\ \text{andR}$$

We know by the induction hypothesis that all activations of $\Gamma \vdash_G n : A, \Delta$ and $\Gamma \vdash_G n : B, \Delta$ are derivable. In particular, $\ulcorner G \urcorner; \ulcorner\Gamma\urcorner^+, \ulcorner\Delta\urcorner^- \vdash +A \mathbin{@} n$ and $\ulcorner G \urcorner; \ulcorner\Gamma\urcorner^+, \ulcorner\Delta\urcorner^- \vdash +B \mathbin{@} n$ are derivable. Now by applying the and rule we obtain $\ulcorner G \urcorner; \ulcorner\Gamma\urcorner^+, \ulcorner\Delta\urcorner^- \vdash +A \wedge_+ B \mathbin{@} n$, which is a particular activation of $\Gamma \vdash_G n : A \wedge B, \Delta$. Finally, consider any other activation, then that sequent implies $\ulcorner G \urcorner; \ulcorner\Gamma\urcorner^+, \ulcorner\Delta\urcorner^- \vdash +A \wedge_+ B \mathbin{@} n$ is derivable using Lemma 14. Thus, we obtain our result.

Case.

$$\frac{\begin{array}{c}\Gamma, n : A \vdash_G \Delta \\ \Gamma, n : A \vdash_G \Delta\end{array}}{\Gamma, n : A \vee B \vdash_G \Delta}\ \text{disjL}$$

We know by the induction hypothesis that all activations of $\Gamma, n : A \vdash_G \Delta$ and $\Gamma, n : B \vdash_G \Delta$ are derivable. So suppose $\ulcorner G \urcorner; \ulcorner\Gamma\urcorner^+, +\ulcorner A\urcorner @ n, \ulcorner\Delta'\urcorner^- \vdash +C @ n'$ and $\ulcorner G \urcorner; \ulcorner\Gamma\urcorner^+, +\ulcorner B\urcorner \mathbin{@} n, \ulcorner\Delta'\urcorner^- \vdash +E \mathbin{@} n''$ are particular activations, where $\ulcorner\Delta\urcorner^- \equiv \ulcorner\Delta_1\urcorner^-, -C \mathbin{@} n', \ulcorner\Delta_2\urcorner^-, -E \mathbin{@} n'', \ulcorner\Delta_3\urcorner^-$, and $\ulcorner\Delta'\urcorner^- \equiv \ulcorner\Delta_1\urcorner^-, \ulcorner\Delta_2\urcorner^-, \ulcorner\Delta_3\urcorner^-$. By

exchange (Lemma 18) we know
$\ulcorner G\urcorner; \ulcorner\Gamma\urcorner^{+}, \ulcorner\Delta'\urcorner^{-}, + \ulcorner A\urcorner @ n \vdash + C @ n'$ and $\ulcorner G\urcorner; \ulcorner\Gamma\urcorner^{+}, \ulcorner\Delta'\urcorner^{-}, + \ulcorner B\urcorner @ n \vdash + E @ n''$. Now by the left-to-right lemma (Lemma 14) we know $\ulcorner G\urcorner; \ulcorner\Gamma\urcorner^{+}, \ulcorner\Delta'\urcorner^{-}, - C @ n' \vdash - \ulcorner A\urcorner @ n$ and $\ulcorner G\urcorner; \ulcorner\Gamma\urcorner^{+}, \ulcorner\Delta'\urcorner^{-}, - E @ n'' \vdash - \ulcorner B\urcorner @ n$, and by applying weakening (and exchange) we know $\ulcorner G\urcorner; \ulcorner\Gamma\urcorner^{+}, \ulcorner\Delta'\urcorner^{-}, - C @ n', - E @ n'' \vdash - \ulcorner A\urcorner @ n$ and $\ulcorner G\urcorner; \ulcorner\Gamma\urcorner^{+}, \ulcorner\Delta'\urcorner^{-}, - C @ n', - E @ n'' \vdash - \ulcorner B\urcorner @ n$. At this point we can apply the AND rule to obtain $\ulcorner G\urcorner; \ulcorner\Gamma\urcorner^{+}, \ulcorner\Delta'\urcorner^{-}, - C @ n', - E @ n'' \vdash - \ulcorner A\urcorner \wedge_{-} \ulcorner B\urcorner @ n$ to which we can apply the left-to-right lemma to and obtain
$\ulcorner G\urcorner; \ulcorner\Gamma\urcorner^{+}, \ulcorner\Delta'\urcorner^{-}, - E @ n'', + \ulcorner A\urcorner \wedge_{-} \ulcorner B\urcorner @ n \vdash + C @ n'$. Finally, we can apply exchange again to obtain $\ulcorner G\urcorner; \ulcorner\Gamma\urcorner^{+}, + \ulcorner A\urcorner \wedge_{-} \ulcorner B\urcorner @ n, \ulcorner\Delta'\urcorner^{-}, - E @ n'' \vdash + C @ n'$, which – modulo exchange – is an arbitrary activation of $\Gamma, n : A \vee B \vdash_G \Delta$. Thus, we obtain our result.

Case.

$$\frac{\Gamma \vdash_G x : T_1, x : T_2, \Delta}{\Gamma \vdash_G x : T_1 \vee T_2, \Delta}\ \textsc{disjR}$$

This case is similar to the case of ANDR case, except, it makes use of the ANDBAR rule.

Case.

$$\frac{\begin{array}{l} n_1 G n_2 \\ \Gamma \vdash_G n_2 : T_1, \Delta \\ \Gamma, n_2 : T_2 \vdash_G \Delta \end{array}}{\Gamma, n_1 : T_1 \supset T_2 \vdash_G \Delta}\ \textsc{impL}$$

We know by the induction hypothesis that all activations of $\Gamma \vdash_G y : T_1, \Delta$ and $\Gamma, y : T_2 \vdash_G \Delta$ are derivable. In particular, we know $\ulcorner G\urcorner; \ulcorner\Gamma\urcorner^{+}, \ulcorner\Delta\urcorner^{-} \vdash + \ulcorner T_1\urcorner @ n_2$ is derivable, and so is $\ulcorner G\urcorner; \ulcorner\Gamma\urcorner^{+}, \ulcorner\Delta\urcorner^{-} \vdash - \ulcorner T_2\urcorner @ n_2$. The latter being derivable by applying the induction hypothesis followed by exchange (Lemma 18) and the left-to-right lemma (Lemma 14). We know $n_1 G n_2$ by assumption and so by Lemma 32 $\ulcorner G\urcorner \vdash n_1 \preccurlyeq^*_+ n_2$. Thus, by applying the IMPBAR rule we obtain $\ulcorner G\urcorner; \ulcorner\Gamma\urcorner^{+}, \ulcorner\Delta\urcorner^{-} \vdash - \ulcorner T_1\urcorner \rightarrow_+ \ulcorner T_2\urcorner @ n_1$. At this point we can apply left-to-right to the previous sequent and obtain and activation of $\Gamma, n_1 : T_1 \supset T_2 \vdash_G \Delta$, thus we obtain our result.

Case.

$$\frac{\begin{array}{l} n_2 \notin |G|, |\Gamma|, |\Delta| \\ \Gamma, n_2 : T_1 \vdash_{G \cup \{(n_1, n_2)\}} n_2 : T_2, \Delta \end{array}}{\Gamma \vdash_G n_1 : T_1 \supset T_2, \Delta}\ \textsc{impR}$$

This case follows the same pattern as the previous cases. We know by the induction hypothesis that all activations of $\Gamma, y : T_1 \vdash_{G \cup \{(x,y)\}} y : T_2, \Delta$ are derivable. In particular, $\ulcorner G\urcorner, n_1 \preccurlyeq_+ n_2; \ulcorner\Gamma\urcorner^{+}, + \ulcorner T_1\urcorner @ n_2, \ulcorner\Delta\urcorner^{-} \vdash + \ulcorner T_2\urcorner @ n_2$ is derivable. By exchange (Lemma 18)
$\ulcorner G\urcorner, n_1 \preccurlyeq_+ n_2; \ulcorner\Gamma\urcorner^{+}, \ulcorner\Delta\urcorner^{-}, + \ulcorner T_1\urcorner @ n_2 \vdash + \ulcorner T_2\urcorner @ n_2$ is derivable, and by applying the IMP rule we obtain $\ulcorner G\urcorner; \ulcorner\Gamma\urcorner^{+}, \ulcorner\Delta\urcorner^{-} \vdash + \ulcorner T_1\urcorner \rightarrow_+ \ulcorner T_2\urcorner @ n_1$, which is a particular activation of $\Gamma \vdash_G n_1 : T_1 \supset T_2, \Delta$. Note that in the previous application of IMP we use the fact that if $n_2 \notin |G|, |\Gamma|, |\Delta|$, then $n_2 \notin |\ulcorner G\urcorner|, |\ulcorner\Gamma\urcorner^{+}, \ulcorner\Delta\urcorner^{-}|$.

Lastly, any other activation of $\Gamma \vdash_G n_1 : T_1 \supset T_2, \Delta$ implies $\ulcorner G \urcorner; \ulcorner \Gamma \urcorner^+, \ulcorner \Delta \urcorner^- \vdash + \ulcorner T_1 \urcorner \to_+ \ulcorner T_2 \urcorner @\, n_1$ is derivable by the left-to-right lemma, and hence is derivable.

Case.

$$\frac{\begin{array}{l} n_1 \notin |G|, |\Gamma|, |\Delta| \\ \Gamma, n_1 : T_1 \vdash_{G \cup \{(n_1, n_2)\}} n_1 : T_2, \Delta \end{array}}{\Gamma, n_2 : T_1 \prec T_2 \vdash_G \Delta}\ \textsc{sub}\mathrm{L}$$

We know by the induction hypothesis that all activation of $\Gamma, n_1 : T_1 \vdash_{G \cup \{(n_1,n_2)\}} n_1 : T_2, \Delta$ are derivable. In particular, $\ulcorner G \urcorner, n_1 \preccurlyeq_+ n_2; \ulcorner \Gamma \urcorner^+, + \ulcorner T_1 \urcorner @\, n_1, \ulcorner \Delta \urcorner^- \vdash + \ulcorner T_2 \urcorner @\, n_1$ is derivable. By exchange (Lemma 18) $\ulcorner G \urcorner, n_1 \preccurlyeq_+ n_2; \ulcorner \Gamma \urcorner^+, \ulcorner \Delta \urcorner^-, + \ulcorner T_1 \urcorner @\, n_1 \vdash + \ulcorner T_2 \urcorner @\, n_1$ is derivable. Now by the left-to-right lemma we know $\ulcorner G \urcorner, n_1 \preccurlyeq_+ n_2; \ulcorner \Gamma \urcorner^+, \ulcorner \Delta \urcorner^-, - \ulcorner T_2 \urcorner @\, n_1 \vdash - \ulcorner T_1 \urcorner @\, n_1$, and by assumption we know $y \notin |G|, |\Gamma|, |\Delta|$ which implies $n_1 \notin |\ulcorner G \urcorner|, |\ulcorner \Gamma \urcorner^+, \ulcorner \Delta \urcorner^-|$ is derivable. Thus, by applying the IMP rule we know $\ulcorner G \urcorner, n_1 \preccurlyeq_+ n_2; \ulcorner \Gamma \urcorner^+, \ulcorner \Delta \urcorner^- \vdash - \ulcorner T_2 \urcorner \to_- \ulcorner T_1 \urcorner @\, n_2$ is derivable. Clearly, this is a particular activation of $\Gamma, n_2 : T_1 \prec T_2 \vdash_G \Delta$, and any other activation implies $\ulcorner G \urcorner, n_1 \preccurlyeq_+ n_2; \ulcorner \Gamma \urcorner^+, \ulcorner \Delta \urcorner^- \vdash - \ulcorner T_2 \urcorner \to_- \ulcorner T_1 \urcorner @\, n_2$ is derivable by the left-to-right lemma, and hence are derivable.

Case.

$$\frac{\begin{array}{l} yGx \\ \Gamma \vdash_G y : T_1, \Delta \\ \Gamma, y : T_2 \vdash_G \Delta \end{array}}{\Gamma \vdash_G x : T_1 \prec T_2, \Delta}\ \textsc{sub}\mathrm{R}$$

This case follows in the same way as the case for IMPL, except the particular activation of $\Gamma, y : T_2 \vdash_G \Delta$ has to have the active formulas such that the rule IMPBAR can be applied.

$\square$

Corollary 35 (Completeness)**.** DIL is complete.

Proof. Completeness of L is proved in [19], and by Theorem 34 we know that every derivable sequent of L is derivable in DIL. $\square$

4. Dualized Type Theory (DTT)

In this section we give DIL a term assignment yielding Dualized Type Theory (DTT). First, we introduce DTT, and give several examples illustrating how to program in DTT. Then we present the metatheory of DTT.

The syntax for DTT is defined in Figure 6. Polarities, types, and graphs are all the same as they were in DIL. Contexts differ only by the addition of labeling each hypothesis with a variable. Terms, denoted t, consist of introduction forms, together with cut terms $\nu\, x.t \bullet t'$[2]. We denote variables as x, y, z, ... The term **triv** is the introduction form for units, (t, t') is the introduction form for pairs, similarly the terms $\mathbf{in}_1\, t$ and $\mathbf{in}_2\, t$ introduce

[2]In classical type theories the symbol μ usually denotes cut, but we have reserved that symbol – indexed by a polarity – to be used with inductive (positive polarity) and coinductive (negative polarity) types in future work.

$$
\begin{array}{rrcl}
\text{(indices)} & d & ::= & 1 \mid 2 \\
\text{(polarities)} & p & ::= & + \mid - \\
\text{(types)} & A, B, C & ::= & \langle p \rangle \mid A \rightarrow_p B \mid A \wedge_p B \\
\text{(terms)} & t & ::= & x \mid \mathbf{triv} \mid (t, t') \mid \mathbf{in}_d\, t \mid \lambda x.t \mid \langle t, t' \rangle \mid \nu\, x.t \bullet t' \\
\text{(canonical terms)} & c & ::= & x \mid \mathbf{triv} \mid (t, t') \mid \mathbf{in}_d\, t \mid \lambda x.t \mid \langle t, t' \rangle \\
\text{(graphs)} & G & ::= & \cdot \mid n \preccurlyeq_p n' \mid G, G' \\
\text{(contexts)} & \Gamma & ::= & \cdot \mid x : p\, A @ n \mid \Gamma, \Gamma'
\end{array}
$$

Figure 6: Syntax for DTT.

$$
\frac{G \vdash n \preccurlyeq_p^* n'}{G; \Gamma, x : p\, A @ n, \Gamma' \vdash x : p\, A @ n'}\ \textsc{Ax} \qquad \frac{}{G; \Gamma \vdash \mathbf{triv} : p\, \langle p \rangle @ n}\ \textsc{Unit}
$$

$$
\frac{G; \Gamma \vdash t_1 : p\, A @ n \qquad G; \Gamma \vdash t_2 : p\, B @ n}{G; \Gamma \vdash (t_1, t_2) : p\, (A \wedge_p B) @ n}\ \textsc{And}
$$

$$
\frac{G; \Gamma \vdash t : p\, A_d @ n}{G; \Gamma \vdash \mathbf{in}_d\, t : p\, (A_1 \wedge_{\bar{p}} A_2) @ n}\ \textsc{AndBar}
$$

$$
\frac{\begin{array}{l} n' \notin |G|, |\Gamma| \\ (G, n \preccurlyeq_p n'); \Gamma, x : p\, A @ n' \vdash t : p\, B @ n' \end{array}}{G; \Gamma \vdash \lambda x.t : p\, (A \rightarrow_p B) @ n}\ \textsc{Imp}
$$

$$
\frac{\begin{array}{l} G \vdash n \preccurlyeq_{\bar{p}}^* n' \\ G; \Gamma \vdash t_1 : \bar{p}\, A @ n' \qquad G; \Gamma \vdash t_2 : p\, B @ n' \end{array}}{G; \Gamma \vdash \langle t_1, t_2 \rangle : p\, (A \rightarrow_{\bar{p}} B) @ n}\ \textsc{ImpBar}
$$

$$
\frac{\begin{array}{l} G; \Gamma, x : \bar{p}\, A @ n \vdash t_1 : +\, B @ n' \\ G; \Gamma, x : \bar{p}\, A @ n \vdash t_2 : -\, B @ n' \end{array}}{G; \Gamma \vdash \nu\, x.t_1 \bullet t_2 : p\, A @ n}\ \textsc{Cut}
$$

Figure 7: Type-Assignment Rules for DTT.

disjunctions, $\lambda x.t$ introduces implication, and $\langle t, t' \rangle$ introduces coimplication. The type-assignment rules are defined in Figure 7, and result from a simple term assignment to the rules for DIL. Finally, the reduction rules for DTT are defined in Figure 8. The reduction rules should be considered rewrite rules that can be applied anywhere within a term. (The congruence rules are omitted.)

Programming in DTT is not functional programming as usual, so we now give several illustrative examples. The reader familiar with type theories based on sequent calculi will find the following very familiar. The encodings are similar to that of Curien and Herbelin's $\bar{\lambda}\mu\tilde{\mu}$-calculus [9]. The locus of computation is the cut term, so naturally, function application

$$\frac{}{\nu\, z.\lambda x.t \bullet \langle t_1, t_2\rangle \rightsquigarrow \nu\, z.[t_1/x]t \bullet t_2}\ \textsc{RImp}$$

$$\frac{}{\nu\, z.\langle t_1, t_2\rangle \bullet \lambda x.t \rightsquigarrow \nu\, z.t_2 \bullet [t_1/x]t}\ \textsc{RImpBar} \qquad \frac{}{\nu\, z.(t_1, t_2) \bullet \mathbf{in}_1\, t \rightsquigarrow \nu\, z.t_1 \bullet t}\ \textsc{RAnd1}$$

$$\frac{}{\nu\, z.(t_1, t_2) \bullet \mathbf{in}_2\, t \rightsquigarrow \nu\, z.t_2 \bullet t}\ \textsc{RAnd2} \qquad \frac{}{\nu\, z.\mathbf{in}_1\, t \bullet (t_1, t_2) \rightsquigarrow \nu\, z.t \bullet t_1}\ \textsc{RAndBar1}$$

$$\frac{}{\nu\, z.\mathbf{in}_2\, t \bullet (t_1, t_2) \rightsquigarrow \nu\, z.t \bullet t_2}\ \textsc{RAndBar2} \qquad \frac{x \notin \mathsf{FV}(t)}{\nu\, x.t \bullet x \rightsquigarrow t}\ \textsc{RRet}$$

$$\frac{}{\nu\, z.(\nu\, x.t_1 \bullet t_2) \bullet t \rightsquigarrow \nu\, z.[t/x]t_1 \bullet [t/x]t_2}\ \textsc{RBetaL}$$

$$\frac{}{\nu\, z.c \bullet (\nu\, x.t_1 \bullet t_2) \rightsquigarrow \nu\, z.[c/x]t_1 \bullet [c/x]t_2}\ \textsc{RBetaR}$$

Figure 8: Reduction Rules for DTT.

is modeled using cuts. Suppose

$$\begin{array}{lll} D_1 & =^{\text{def}} & G; \Gamma \vdash \lambda x.t : +\,(A \to_+ B) \mathbin{@} n \\ D_2 & =^{\text{def}} & G; \Gamma \vdash t' : +\,A \mathbin{@} n \\ \Gamma' & =^{\text{def}} & \Gamma, y : -\,B \mathbin{@} n \end{array}$$

Then we can construct the following typing derivation:

$$\dfrac{D_1 \qquad \dfrac{D_2 \qquad \dfrac{}{G; \Gamma' \vdash y : -\,B \mathbin{@} n}\ \textsc{ax}}{G; \Gamma' \vdash \langle t', y\rangle : -\,(A \to_+ B) \mathbin{@} n}\ \textsc{impBar}}{G; \Gamma \vdash \nu\, y.\lambda x.t \bullet \langle t', y\rangle : +\,B \mathbin{@} n}\ \textsc{cut}$$

Implication was indeed eliminated, yielding the conclusion.

There is some intution one can use while thinking about this style of programming. In [14] Kimura and Tatsuta explain how we can think of positive variables as input ports, and negative variables as output ports. Clearly, these notions are dual. Then a cut of the form $\nu\, z.t \bullet t'$ can be intuitively understood as a device capable of routing information. We think of this term as first running the term t, and then plugging its value into the continuation t'. Thus, negative terms are continuations. Now consider the instance of the previous term $\nu\, z.t \bullet y$ where t is a positive term and y is a negative variable (an output port). This can be intuitively understood as after running t, route its value through the output port y. Now consider the instance $\nu\, z.t \bullet z$. This term can be understood as after running the term t, route its value through the output part z, but then capture this value as the return value. Thus, the cut term reroutes output ports into the actual return value of the cut.

There is one additional bit of intuition we can use when thinking about programming in DTT. We can think of cuts of the form $\nu z.(\lambda x_1 \cdots \lambda x_i.t) \bullet \langle t_1, \langle t_2, \cdots \langle t_i, z\rangle \cdots \rangle$ as an abstract machine, where $\lambda x_1 \cdots \lambda x_i.t$ is the functional part of the machine, and $\langle t_1, \langle t_2, \cdots \langle t_i, z\rangle \cdots \rangle$ is the stack of inputs the abstract machine will apply the function to ultimately routing the final result of the application through z, but rerouting this into the return value. This intuition is not new, but was first observed by Curien and Herbelin in [9]; see also [10].

Similarly to the eliminator for implication we can define the eliminator for disjunction in the form of the usual case analysis. Suppose $G;\Gamma \vdash t : +\,(A \wedge_{-} B) @ n$, $G;\Gamma, x : +\,A @ n \vdash t_1 : +\,C @ n$, and $G;\Gamma, x : +\,B @ n \vdash t_2 : +\,C @ n$ are all admissible. Then we can derive the usual eliminator for disjunction. Define $\mathbf{case}\,t\,\mathbf{of}\,x.t_1, x.t_2 =^{\mathrm{def}} \nu\, z_0.(\nu\, z_1.(\nu\, z_2.t \bullet (z_1, z_2)) \bullet (\nu\, x.t_2 \bullet z_0)) \bullet (\nu\, x.t_1 \bullet z_0)$. Then we have the following result.

Lemma 36. The following rule is admissible:

$$\dfrac{\begin{array}{l} G;\Gamma, x : p\,A @ n \vdash t_1 : p\,C @ n \\ G;\Gamma, x : p\,B @ n \vdash t_2 : p\,C @ n \qquad G;\Gamma \vdash t : p\,(A \wedge_{\bar{p}} B) @ n \end{array}}{G;\Gamma \vdash \mathbf{case}\,t\,\mathbf{of}\,x.t_1, x.t_2 : p\,C @ n}\ \textsc{case}$$

Proof. Due to the size of the derivation in question we give several derivations which compose together to form the typing derivation of $G;\Gamma \vdash \mathbf{case}\,t\,\mathbf{of}\,x.t_1, x.t_2 : p\,C @ n$.

The typing derivation begins using cut as follows:

$$\dfrac{D_0 \qquad D_1}{G;\Gamma \vdash \nu\, z_0.(\nu\, z_1.(\nu\, z_2.t \bullet (z_1, z_2)) \bullet (\nu\, x.t_2 \bullet z_0)) \bullet (\nu\, x.t_1 \bullet z_0) : +\,C @ n}\ \textsc{cut}$$

Then the remainder of the derivation depends on the following sub-derivations:

D_0 :

$$\dfrac{D_3 \qquad D_4}{G;\Gamma, z_0 : -\,C @ n \vdash \nu\, z_1.(\nu\, z_2.t \bullet (z_1, z_2)) \bullet (\nu\, x.t_2 \bullet z_0) : +\,A @ n}\ \textsc{cut}$$

D_1 :

$$\dfrac{D_2 \qquad \dfrac{}{G;\Gamma, z_0 : -\,C @ n, x : +\,A @ n \vdash z_0 : -\,C @ n}\ \textsc{ax}}{G;\Gamma, z_0 : -\,C @ n \vdash \nu\, x.t_1 \bullet z_0 : -\,A @ n}\ \textsc{cut}$$

D_2 :

$$\dfrac{G;\Gamma, x : +\,A @ n \vdash t_1 : +\,C @ n}{G;\Gamma, z_0 : -\,C @ n, x : +\,A @ n \vdash t_1 : +\,C @ n}\ \textsc{weakening}$$

D_4 :

$$\dfrac{D_5 \qquad G;\Gamma, z_0 : -\,C @ n, z_1 : -\,A @ n, x : +\,B @ n \vdash z_0 : -\,C @ n}{G;\Gamma, z_0 : -\,C @ n, z_1 : -\,A @ n \vdash \nu\, x.t_2 \bullet z_0 : -\,B @ n}\ \textsc{cut}$$

D_3 :

$$\dfrac{D_6 \qquad D_7}{G;\Gamma, z_0 : -\,C @ n, z_1 : -\,A @ n \vdash \nu\, z_2.t \bullet (z_1, z_2) : +\,B @ n}\ \textsc{cut}$$

D_5 :

$$\dfrac{G;\Gamma, x : +\,B @ n \vdash t_2 : +\,C @ n}{G;\Gamma, z_0 : -\,C @ n, z_1 : -\,A @ n, x : +\,B @ n \vdash t_2 : +\,C @ n}\ \textsc{weakening}$$

D_6 :

$$\dfrac{G;\Gamma \vdash t : +\,(A \wedge_{-} B) @ n}{G;\Gamma, z_0 : -\,C @ n, z_1 : -\,A @ n, z_2 : -\,B @ n \vdash t : +\,(A \wedge_{-} B) @ n}\ \textsc{weakening}$$

D_7 :

$$\dfrac{D_8 \qquad D_9}{G;\Gamma, z_0 : -\,C @ n, z_1 : -\,A @ n, z_2 : -\,B @ n \vdash (z_1, z_2) : -\,(A \wedge_{-} B) @ n}\ \textsc{and}$$

D_8 :

$$\dfrac{}{G; \Gamma, z_0 : -\, C @ n, z_1 : -\, A @ n, z_2 : -\, B @ n \vdash z_1 : -\, A @ n}\ \text{AX}$$

D_9 :

$$\dfrac{}{G; \Gamma, z_0 : -\, C @ n, z_1 : -\, A @ n, z_2 : -\, B @ n \vdash z_2 : -\, B @ n}\ \text{AX}$$

□

Now consider the term $\nu\, x.\mathbf{in}_1\, (\nu\, y.\mathbf{in}_2\, \langle y, \mathbf{triv}\rangle \bullet x) \bullet x$. This term is the inhabitant of the type $A \wedge_{-} \sim A$, and its typing derivation follows from the derivation given in Section 3. We can see by looking at the syntax that the cuts involved are indeed on the axiom x, thus this term has no canonical form. In [8] Crolard shows that inhabitants such as these amount to a constructive coroutine. That is, it is a restricted form of a continuation.

We now consider several example reductions in DTT. In the following examples we underline non-top-level redexes. The first example simply α-converts the function $\lambda x.x$ into $\lambda z.z$ as follows:

$$\begin{array}{lcl} \lambda z.\underline{\nu\, y.\lambda x.x \bullet \langle z, y\rangle} & \overset{(\text{RImp})}{\rightsquigarrow} & \lambda z.\underline{\nu\, y.z \bullet y} \\ & \overset{(\text{RRet})}{\rightsquigarrow} & \lambda z.z \end{array}$$

A more involved example is the application of the function $\lambda x.(\lambda y.y)$ to the arguments **triv** and **triv**.

$$\begin{array}{lcl} \nu\, z.\lambda x.(\lambda y.y) \bullet \langle \mathbf{triv}, \langle \mathbf{triv}, z\rangle\rangle & \overset{(\text{RImp})}{\rightsquigarrow} & \nu\, z.\lambda y.y \bullet \langle \mathbf{triv}, z\rangle \\ & \overset{(\text{RImp})}{\rightsquigarrow} & \nu\, z.\mathbf{triv} \bullet z \\ & \overset{(\text{RRet})}{\rightsquigarrow} & \mathbf{triv} \end{array}$$

5. Metatheory of DTT

We now present the basic metatheory of DTT, starting with type preservation. We begin with the inversion lemma which is necessary for proving type preservation.

Lemma 37 (Inverstion)**.**

i. If $G; \Gamma, x : p\, A @ n, \Gamma' \vdash x : p\, A @ n'$, then $G \vdash n \preccurlyeq_p^* n'$.
ii. If $G; \Gamma \vdash (t_1, t_2) : p\, (A \wedge_p B) @ n$, then $G; \Gamma \vdash t_1 : p\, A @ n$ and $G; \Gamma \vdash t_2 : p\, B @ n$.
iii. If $G; \Gamma \vdash \mathbf{in}_d\, t : p\, (A_1 \wedge_{\bar{p}} A_2) @ n$, then $G; \Gamma \vdash t : p\, A_d @ n$.
iv. If $G; \Gamma \vdash \lambda x.t : p\, (A \to_p B) @ n$, then $(G, n \preccurlyeq_p n'); \Gamma, x : p\, A @ n' \vdash t : p\, B @ n'$ for any $n' \notin |G|, |\Gamma|$.
v. If $G; \Gamma \vdash \langle t_1, t_2\rangle : p\, (A \to_{\bar{p}} B) @ n$, then $G \vdash n \preccurlyeq_{\bar{p}}^* n'$, $G; \Gamma \vdash t_1 : \bar{p}\, A @ n'$, and $G; \Gamma \vdash t_2 : p\, B @ n'$ for some node n'.

Proof. Each case of the above lemma holds by a trivial proof by induction on the assumed typing derivation. □

The lemmas node substitution for typing and substitution for typing are essential for the cases of type preservation that reduce a top-level redex. Node substitution, denoted $[n_1/n_2]n$, is defined as follows:

$$\begin{array}{lcl} [n_1/n_2]n_2 & = & n_1 \\ [n_1/n_2]n & = & n \text{ where } n \text{ is distinct from } n_2 \end{array}$$

The following lemmas are necessary in the proof of node substitution for typing.

Lemma 38 (Node Renaming)**.** If $G_1, G_2 \vdash n_1 \preccurlyeq^*_p n_3$, then for any nodes n_4 and n_5, where n_5 is distinct from n_1 and n_3, we have $[n_4/n_5]G_1, [n_4/n_5]G_2 \vdash n_1 \preccurlyeq^*_p n_3$.

Proof. This is a proof by induction on the assumed reachability derivation. Throughout each case suppose we have nodes n_4 and n_5.

Case.

$$\frac{}{G, n_1 \preccurlyeq_p n_3, G' \vdash n_1 \preccurlyeq^*_p n_3}\ \text{AX}$$

Trivial.

Case.

$$\frac{}{G_1, G_2 \vdash n \preccurlyeq^*_p n}\ \text{REFL}$$

Trivial.

Case.

$$\frac{G_1, G_2 \vdash n_1 \preccurlyeq^*_p n' \qquad G_1, G_2 \vdash n' \preccurlyeq^*_p n_3}{G_1, G_2 \vdash n_1 \preccurlyeq^*_p n_3}\ \text{TRANS}$$

By the induction hypothesis we know that for any nodes n'_4 and n'_5 that $[n'_4/n'_5]G_1, [n'_4/n'_5]G_2 \vdash n_1 \preccurlyeq^*_p n'$, and for any nodes n''_4 and n''_5 that $[n''_4/n''_5]G_1, [n''_4/n''_5]G_2 \vdash n' \preccurlyeq^*_p n_3$. Choose n_4 for n'_4 and n''_4 and n_5 for n'_5 and n''_5 to obtain $[n_4/n_5]G_1, [n_4/n_5]G_2 \vdash n_1 \preccurlyeq^*_p n'$ and $[n_4/n_5]G_1, [n_4/n_5]G_2 \vdash n' \preccurlyeq^*_p n_3$. Finally, this case follows by reapplying the rule to the previous two facts.

Case.

$$\frac{G \vdash n' \preccurlyeq^*_{\bar{p}} n}{G \vdash n \preccurlyeq^*_p n'}\ \text{FLIP}$$

Similar to the previous case.

□

Lemma 39 (Node Substitution for Reachability)**.** If $G, n_1 \preccurlyeq_{p_1} n_2, G' \vdash n_4 \preccurlyeq^*_p n_5$ and $G, G' \vdash n_1 \preccurlyeq^*_{p_1} n_3$, then $[n_3/n_2]G, [n_3/n_2]G' \vdash [n_3/n_2]n_4 \preccurlyeq^*_p [n_3/n_2]n_5$.

Proof. This is a proof by induction on the form of the assumed reachability derivation. Throughout the following cases we assume $G, G' \vdash n_1 \preccurlyeq^*_{p_1} n_3$ holds.

Case.

$$\frac{}{G_1, n_4 \preccurlyeq_p n_5, G_2 \vdash n_4 \preccurlyeq^*_p n_5}\ \text{AX}$$

Suppose $G_1, n_4 \preccurlyeq_p n_5, G_2 = G, n_1 \preccurlyeq_{p_1} n_2, G'$. Then either $n_1 \preccurlyeq_{p_1} n_2 \in G_1$, $n_1 \preccurlyeq_{p_1} n_2 \in G_2$, or $n_1 \preccurlyeq_{p_1} n_2 \equiv n_4 \preccurlyeq_p n_5$. Suppose $n_1 \preccurlyeq_{p_1} n_2 \in G_1$, then $G_1 = G'_1, n_1 \preccurlyeq_p n_2, G''_1$. Then it is easy to see that $[n_3/n_2]G'_1, [n_3/n_2]G''_1, [n_3/n_2]n_4 \preccurlyeq_p [n_3/n_2]n_5, [n_3/n_2]G_2 \vdash [n_3/n_2]n_4 \preccurlyeq^*_p [n_3/n_2]n_5$ is derivable by applying Ax. The case where $n_1 \preccurlyeq_{p_1} n_2 \in G_2$ is similar.

Now suppose $n_1 \preccurlyeq_{p_1} n_2 \equiv n_4 \preccurlyeq_p n_5$. Then we know by assumption that

$$\frac{}{G_1, n_1 \preccurlyeq_p n_2, G_2 \vdash n_1 \preccurlyeq^*_p n_2}\ \text{AX}$$

Then it suffices to show $[n_3/n_2]G_1, [n_3/n_2]G_2 \vdash [n_3/n_2]n_1 \preccurlyeq^*_p [n_3/n_2]n_2$, which is equivalent to $[n_3/n_2]G_1, [n_3/n_2]G_2 \vdash [n_3/n_2]n_1 \preccurlyeq^*_p n_3$. Now if n_1 is equivalent to n_2, then $[n_3/n_2]G_1, [n_3/n_2]G_2 \vdash [n_3/n_2]n_1 \preccurlyeq^*_p n_3$ holds by reflexivity, and if n_1 is distinct from n_2, then $[n_3/n_2]G_1, [n_3/n_2]G_2 \vdash [n_3/n_2]n_1 \preccurlyeq^*_p n_3$ is equivalent to $[n_3/n_2]G_1, [n_3/n_2]G_2 \vdash n_1 \preccurlyeq^*_p n_3$. We know by assumption that $G, G' \vdash n_1 \preccurlyeq^*_{p_1} n_3$ holds, which is equivalent to $G_1, G_2 \vdash n_1 \preccurlyeq^*_p n_3$. Now if n_3 is equal to n_2, then $[n_3/n_2]G_1, [n_3/n_2]G_2 \vdash n_1 \preccurlyeq^*_p n_3$ is equivalent to $G_1, G_2 \vdash n_1 \preccurlyeq^*_p n_3$. So suppose n_3 is distinct from n_2, then by Lemma 38 we know $[n_3/n_2]G_1, [n_3/n_2]G_2 \vdash n_1 \preccurlyeq^*_p n_3$.

Case.

$$\frac{}{G, n_1 \preccurlyeq_{p_1} n_2, G' \vdash n \preccurlyeq^*_p n}\ \text{REFL}$$

Trivial.

Case.

$$\frac{G, n_1 \preccurlyeq_{p_1} n_2, G' \vdash n_4 \preccurlyeq^*_p n_6 \qquad G \vdash n_6 \preccurlyeq^*_p n_5}{G, n_1 \preccurlyeq_{p_1} n_2, G' \vdash n_4 \preccurlyeq^*_p n_5}\ \text{TRANS}$$

This case by applying the induction to each premise, and then reapplying the rule.

Case.

$$\frac{G, n_1 \preccurlyeq_{p_1} n_2, G' \vdash n_5 \preccurlyeq^*_{\bar{p}} n_4}{G, n_1 \preccurlyeq_{p_1} n_2, G' \vdash n_4 \preccurlyeq^*_p n_5}\ \text{FLIP}$$

This case holds by applying the induction hypothesis to the premise, and then reapplying the rule.

□

Lemma 40 (Node Substitution for Typing)**.** If $G, n_1 \preccurlyeq_{p_1} n_2, G'; \Gamma \vdash t : p_2\, A \,@\, n_3$ and $G, G' \vdash n_1 \preccurlyeq^*_{p_1} n_4$, then $[n_4/n_2]G, [n_4/n_2]G'; [n_4/n_2]\Gamma \vdash t : p_2\, A \,@\, [n_4/n_2]n_3$.

Proof. This is a proof by induction on the form of the assumed typing derivation. Throughout each of the following cases we assume $G, G' \vdash n_1 \preccurlyeq^*_{p_1} n_4$ holds.

Case.

$$\frac{G, n_1 \preccurlyeq_{p_1} n_2, G' \vdash n \preccurlyeq^*_p n_3}{G, n_1 \preccurlyeq_{p_1} n_2, G'; \Gamma_1, y : p_2\, A \,@\, n, \Gamma_2 \vdash y : p_2\, A \,@\, n_3}\ \text{Ax}$$

First, by node substitution for reachability (Lemma 39) we know $[n_4/n_2]G, [n_4/n_2]G' \vdash [n_4/n_2]n \preccurlyeq^*_p [n_4/n_2]n_3$. Thus, by applying the Ax rule we may derive $[n_4/n_2]G, [n_4/n_2]G'; [n_4/n_2]\Gamma_1, y : p_2\, A \,@\, [n_4/n_2]n, [n_4/n_2]\Gamma_2 \vdash y : p_2\, A \,@\, [n_4/n_2]n_3$.

Case.

$$\frac{}{G, n_1 \preccurlyeq_{p_1} n_2, G'; \Gamma \vdash \mathbf{triv} : p_2\, \langle p_2 \rangle \mathbin{@} n_3}\ \textsc{Unit}$$

Trivial.

Case.

$$\frac{G, n_1 \preccurlyeq_{p_1} n_2; \Gamma \vdash t_1 : p_2\, A_1 \mathbin{@} n_3 \qquad G, n_1 \preccurlyeq_{p_1} n_2; \Gamma \vdash t_2 : p_2\, A_2 \mathbin{@} n_3}{G, n_1 \preccurlyeq_{p_1} n_2; \Gamma \vdash (t_1, t_2) : p_2\, (A_1 \wedge_{p_2} A_2) \mathbin{@} n_3}\ \textsc{And}$$

This case holds by applying the induction hypothesis to each premise, and then reapplying the rule.

Case.

$$\frac{G, n_1 \preccurlyeq_{p_1} n_2; \Gamma \vdash t' : p_2\, A_d \mathbin{@} n_3}{G, n_1 \preccurlyeq_{p_1} n_2; \Gamma \vdash \mathbf{in}_d\, t' : p_2\, (A_1 \wedge_{\bar{p_2}} A_2) \mathbin{@} n_3}\ \textsc{AndBar}$$

This case holds by applying the induction hypothesis to the premise, and then reapplying the rule.

Case.

$$\frac{\begin{array}{l} n' \notin |G, n_1 \preccurlyeq_{p_1} n_2, G'|, |\Gamma| \\ (G, n_1 \preccurlyeq_{p_1} n_2, G', n_3 \preccurlyeq_{p} n'); \Gamma, x : p_2\, A_1 \mathbin{@} n' \vdash t' : p_2\, A_2 \mathbin{@} n' \end{array}}{G, n_1 \preccurlyeq_{p_1} n_2, G'; \Gamma \vdash \lambda x.t' : p_2\, (A_1 \to_{p_2} A_2) \mathbin{@} n_3}\ \textsc{Imp}$$

First, if $n' \notin |G, n_1 \preccurlyeq_{p_1} n_2, G'|, |\Gamma|$, then $n' \notin |G, G'|, |\Gamma|$. Furthermore, we know that $[n_4/n_2]n' \notin |[n_4/n_2]G, [n_4/n_2]G'|, |[n_4/n_2]\Gamma|$, because we know n' is distinct from n_2 by assumption, and if n' is equal to n_4, then $n' \notin |G, n_1 \preccurlyeq_{p_1} n_2, G'|, |\Gamma|$ implies that n_1 must also be n_4, because we know by assumption that $G, G' \vdash n_1 \preccurlyeq^*_{p_1} n_4$ which could only be derivide by reflexivity since $n' \notin |G, G'|, |\Gamma|$, but we know by assumption that $n' \notin |G, n_1 \preccurlyeq_{p_1} n_2, G'|, |\Gamma|$ which implies that n' must be distinct from n_1, and hence a contradiction, thus n' cannot be n_4. Therefore, we know $n' \notin |[n_4/n_2]G, [n_4/n_2]G'|, |[n_4/n_2]\Gamma|$.

By the induction hypothesis we know

$$([n_4/n_2]G, [n_4/n_2]G', [n_4/n_2]n_3 \preccurlyeq_p [n_4/n_2]n'); [n_4/n_2]\Gamma, x : p_2\, A_1 \mathbin{@} [n_4/n_2]n' \vdash t' : p_2\, A_2 \mathbin{@} [n_4/n_2]n'$$

which is equivalent to

$$([n_4/n_2]G, [n_4/n_2]G', [n_4/n_2]n_3 \preccurlyeq_p n'); [n_4/n_2]\Gamma, x : p_2\, A_1 \mathbin{@} n' \vdash t' : p_2\, A_2 \mathbin{@} n'.$$

Finally, this case follows by applying the Imp rule using $n' \notin |[n_4/n_2]G, [n_4/n_2]G'|, |[n_4/n_2]\Gamma|$ and the previous fact.

Case.

$$\frac{\begin{array}{l} G, n_1 \preccurlyeq_{p_1} n_2, G' \vdash n_3 \preccurlyeq^*_{\bar{p_2}} n' \\ G, n_1 \preccurlyeq_{p_1} n_2, G'; \Gamma \vdash t_1 : \bar{p_2}\, A_1 \mathbin{@} n' \qquad G, n_1 \preccurlyeq_{p_1} n_2, G'; \Gamma \vdash t_2 : p_2\, A_2 \mathbin{@} n' \end{array}}{G, n_1 \preccurlyeq_{p_1} n_2, G'; \Gamma \vdash \langle t_1, t_2 \rangle : p_2\, (A_1 \to_{\bar{p_2}} A_2) \mathbin{@} n_3}\ \textsc{ImpBar}$$

We now by assumption that $G, G' \vdash n_1 \preccurlyeq^*_{p_1} n_4$ holds. So by node substitution for reachability (Lemma 39) we know $[n_4/n_2]G, [n_4/n_2]G' \vdash [n_4/n_2]n_3 \preccurlyeq^*_{\bar{p_2}} [n_4/n_2]n'$. Now by the induction hypothesis we know $[n_4/n_2]G, [n_4/n_2]G'; [n_4/n_2]\Gamma \vdash t_1 :$

$\bar{p_2}\, A_1 \mathbin{@} [n_4/n_2]n'$ and
$[n_4/n_2]G, [n_4/n_2]G'; [n_4/n_2]\Gamma \vdash t_2 : p_2\, A_2 \mathbin{@} [n_4/n_2]n'$. This case then follows by applying the rule ImBar to the previous three facts.

Case.

$$\frac{\begin{array}{c}G, n_1 \preccurlyeq_{p_1} n_2, G'; \Gamma, y : \bar{p_2}\, A \mathbin{@} n_3 \vdash t_1 : +\, C \mathbin{@} n\\ G, n_1 \preccurlyeq_{p_1} n_2, G'; \Gamma, y : \bar{p_2}\, A \mathbin{@} n_3 \vdash t_2 : -\, C \mathbin{@} n\end{array}}{G, n_1 \preccurlyeq_{p_1} n_2, G'; \Gamma \vdash \nu\, x.t_1 \bullet t_2 : p_2\, A \mathbin{@} n_3}\ \textsc{Cut}$$

This case follows by applying the induction hypothesis to each premise, and then reapplying the rule.

□

The next lemma is crucial for type preservation.

Lemma 41 (Substitution for Typing)**.** If $G; \Gamma \vdash t_1 : p_1\, A \mathbin{@} n_1$ and $G; \Gamma, x : p_1\, A \mathbin{@} n_1, \Gamma' \vdash t_2 : p_2\, B \mathbin{@} n_2$, then $G; \Gamma, \Gamma' \vdash [t_1/x]t_2 : p_2\, B \mathbin{@} n_2$.

Proof. This proof holds by a straightforward induction on the second assumed typing relation.

Case.

$$\frac{G \vdash n \preccurlyeq^*_p n'}{G; \Gamma_1, y : p\, C \mathbin{@} n, \Gamma_2 \vdash y : p\, C \mathbin{@} n'}\ \textsc{Ax}$$

Trivial.

Case.

$$\frac{}{G; \Gamma_1 \vdash \mathbf{triv} : p\, \langle p \rangle \mathbin{@} n}\ \textsc{Unit}$$

Trivial.

Case.

$$\frac{G; \Gamma_1 \vdash t_1' : p\, A \mathbin{@} n \qquad G; \Gamma_1 \vdash t_2' : p\, B \mathbin{@} n}{G; \Gamma_1 \vdash (t_1', t_2') : p\, (C_1 \wedge_p C_2) \mathbin{@} n}\ \textsc{And}$$

Suppose $\Gamma_1 \equiv \Gamma, x : p_1\, A \mathbin{@} n_1, \Gamma'$. Then this case follows from applying the induction hypothesis to each premise and then reapplying the rule.

Case.

$$\frac{G; \Gamma_1 \vdash t : p\, C_d \mathbin{@} n}{G; \Gamma_1 \vdash \mathbf{in}_d\, t : p\, (C_1 \wedge_{\bar{p}} C_2) \mathbin{@} n}\ \textsc{AndBar}$$

Suppose $\Gamma_1 \equiv \Gamma, x : p_1\, A \mathbin{@} n_1, \Gamma'$. Then this case follows from applying the induction hypothesis to the premise and then reapplying the rule.

Case.

$$\frac{\begin{array}{l}n' \notin |G|, |\Gamma_1|\\ (G, n \preccurlyeq_p n'); \Gamma_1, x : p\, C_1 \mathbin{@} n' \vdash t : p\, C_2 \mathbin{@} n'\end{array}}{G; \Gamma_1 \vdash \lambda x.t : p\, (C_1 \to_p C_2) \mathbin{@} n}\ \textsc{Imp}$$

Similarly to the previous case.

Case.

$$\frac{\begin{array}{l} G \vdash n \preccurlyeq^*_{\bar{p}} n' \\ G; \Gamma_1 \vdash t'_1 : \bar{p}\ C_1 \mathbin{@} n' \qquad G; \Gamma_1 \vdash t'_2 : p\ C_2 \mathbin{@} n' \end{array}}{G; \Gamma_1 \vdash \langle t'_1, t'_2 \rangle : p\,(C_1 \rightarrow_{\bar{p}} C_2) \mathbin{@} n}\ \textsc{ImpBar}$$

Suppose $\Gamma_1 \equiv \Gamma, x : p_1\ A \mathbin{@} n_1, \Gamma'$. Then this case follows from applying the induction hypothesis to each premise and then reapplying the rule.

Case.

$$\frac{\begin{array}{l} G; \Gamma_1, y : \bar{p}\ C \mathbin{@} n \vdash t'_1 : +\ C' \mathbin{@} n' \\ G; \Gamma_1, y : \bar{p}\ C \mathbin{@} n \vdash t'_2 : -\ C' \mathbin{@} n' \end{array}}{G; \Gamma_1 \vdash \nu\, x.t'_1 \bullet t'_2 : p\ C \mathbin{@} n}\ \textsc{Cut}$$

Similarly to the previous case.

□

Lemma 42 (Type Preservation)**.** If $G; \Gamma \vdash t : p\ A \mathbin{@} n$, and $t \rightsquigarrow t'$, then $G; \Gamma \vdash t' : p\ A \mathbin{@} n$.

Proof. This is a proof by induction on the form of the assumed typing derivation. We only consider non-trivial cases. All the other cases either follow directly from assumptions or are similar to the cases we provide below.

Case.

$$\frac{\begin{array}{l} G; \Gamma, x : \bar{p}\ A \mathbin{@} n \vdash t_1 : +\ B \mathbin{@} n' \\ G; \Gamma, x : \bar{p}\ A \mathbin{@} n \vdash t_2 : -\ B \mathbin{@} n' \end{array}}{G; \Gamma \vdash \nu\, x.t_1 \bullet t_2 : p\ A \mathbin{@} n}\ \textsc{Cut}$$

The interesting cases are the ones where the assumed cut is a redex itself, otherwise this case holds by the induction hypothesis. Thus, we case split on the form of this redex.

Case. Suppose $\nu\, x.t_1 \bullet t_2 \equiv \nu\, x.\lambda y.t'_1 \bullet \langle t'_2, t''_2 \rangle$, thus, $t_1 \equiv \lambda y.t'_1$ and $t_2 \equiv \langle t'_2, t''_2 \rangle$. This then implies that $B \equiv B_1 \rightarrow_+ B_2$ for some B_1 and B_2. Then

$$t \equiv \nu\, x.t_1 \bullet t_2 \equiv \nu\, x.\lambda y.t'_1 \bullet \langle t'_2, t''_2 \rangle \rightsquigarrow \nu\, x.[t'_2/y]t'_1 \bullet t''_2 \equiv t'.$$

Now by inversion we know the following:

(1) $G, (n' \preccurlyeq_+ n''); \Gamma, x : \bar{p}\ A \mathbin{@} n, y : +\ B_1 \mathbin{@} n'' \vdash t'_1 : +\ B_2 \mathbin{@} n''$
for some $n'' \notin |G|, |\Gamma, x : \bar{p}\ A \mathbin{@} n|$

(2) $G; \Gamma, x : \bar{p}\ A \mathbin{@} n \vdash t'_2 : +\ B_1 \mathbin{@} n'''$

(3) $G; \Gamma, x : \bar{p}\ A \mathbin{@} n \vdash t''_2 : -\ B_2 \mathbin{@} n'''$

(4) $G \vdash n' \preccurlyeq^*_+ n'''$

Using (1) and (4) we may apply node substitution for typing (Lemma 40) to obtain

$$(5)\ [n'''/n'']G; [n'''/n'']\Gamma, x : \bar{p}\ A \mathbin{@} n, y : +\ B_1 \mathbin{@} n''' \vdash t'_1 : +\ B_2 \mathbin{@} n'''.$$

Finally, by applying substitution for typing using (2) and (5) we obtain

$$(6)\ [n'''/n'']G; [n'''/n'']\Gamma, x : \bar{p}\ A \mathbin{@} n \vdash [t'_2/y]t'_1 : +\ B_2 \mathbin{@} n''',$$

and since n'' is a fresh in G and Γ we know (6) is equivalent to

$$(7)\ G; \Gamma, x : \bar{p}\ A \mathbin{@} n \vdash [t'_2/y]t'_1 : +\ B_2 \mathbin{@} n'''.$$

$$\frac{}{\Gamma, x : p\,A, \Gamma' \vdash_c x : p\,A}\ \textsc{ClassAx} \qquad \frac{}{\Gamma \vdash_c \mathbf{triv} : p\,\langle p \rangle}\ \textsc{ClassUnit}$$

$$\frac{\Gamma \vdash_c t_1 : p\,A \qquad \Gamma \vdash_c t_2 : p\,B}{\Gamma \vdash_c (t_1, t_2) : p\,(A \wedge_p B)}\ \textsc{ClassAnd}$$

$$\frac{\Gamma \vdash_c t : p\,A_d}{\Gamma \vdash_c \mathbf{in}_d\, t : p\,(A_1 \wedge_{\bar{p}} A_2)}\ \textsc{ClassAndBar} \qquad \frac{\Gamma, x : p\,A \vdash_c t : p\,B}{\Gamma \vdash_c \lambda x.t : p\,(A \to_p B)}\ \textsc{ClassImp}$$

$$\frac{\Gamma \vdash_c t_1 : \bar{p}\,A \qquad \Gamma \vdash_c t_2 : p\,B}{\Gamma \vdash_c \langle t_1, t_2 \rangle : p\,(A \to_{\bar{p}} B)}\ \textsc{ClassImpBar} \qquad \frac{\begin{array}{c}\Gamma, x : \bar{p}\,A \vdash_c t_1 : +\,B \\ \Gamma, x : \bar{p}\,A \vdash_c t_2 : -\,B\end{array}}{\Gamma \vdash_c \nu\, x.t_1 \bullet t_2 : p\,A}\ \textsc{ClassCut}$$

Figure 9: Classical typing of DTT terms

$$\begin{array}{lcl}
t \in [\![A]\!]^{+} & \Leftrightarrow & \forall x \in \mathit{Var}.\ \forall t' \in [\![A]\!]^{-}.\ \nu\, x.t \bullet t' \in \mathbf{SN} \\
t \in [\![A]\!]^{-} & \Leftrightarrow & \forall x \in \mathit{Var}.\ \forall t' \in [\![A]\!]^{+c}.\ \nu\, x.t' \bullet t \in \mathbf{SN} \\
t \in [\![\langle + \rangle]\!]^{+c} & \Leftrightarrow & t \in \mathit{Var}\ \vee\ t \equiv \mathbf{triv} \\
t \in [\![\langle - \rangle]\!]^{+c} & \Leftrightarrow & t \in \mathit{Var} \\
t \in [\![A \to_{+} B]\!]^{+c} & \Leftrightarrow & t \in \mathit{Var}\ \vee\ \exists x, t'.t \equiv \lambda x.\, t'\ \wedge\ \forall t'' \in [\![A]\!]^{+}.\ [t''/x]t' \in [\![B]\!]^{+} \\
t \in [\![A \to_{-} B]\!]^{+c} & \Leftrightarrow & t \in \mathit{Var}\ \vee\ \exists t_1 \in [\![A]\!]^{-}, t_2 \in [\![B]\!]^{+}.\ t \equiv \langle t_1, t_2 \rangle \\
t \in [\![A \wedge_{+} B]\!]^{+c} & \Leftrightarrow & t \in \mathit{Var}\ \vee\ \exists t_1 \in [\![A]\!]^{+}, t_2 \in [\![B]\!]^{+}.\ t \equiv (t_1, t_2) \\
t \in [\![A_1 \wedge_{-} A_2]\!]^{+c} & \Leftrightarrow & t \in \mathit{Var}\ \vee\ \exists d.\exists t' \in [\![A_d]\!]^{+}.\ t \equiv \mathbf{in}_d\, t'
\end{array}$$

Figure 10: Interpretations of types

Finally, by applying the Cut rule using (7) and (3) we obtain

$$G; \Gamma \vdash \nu\, x.[t_2'/y]t_1' \bullet t_2'' : p\,A @ n.$$

□

A more substantial property is strong normalization of reduction for typed terms. To prove this result, we will prove a stronger property, namely strong normalization for reduction of terms which are typable using the system of classical typing rules in Figure 9 [7]. This is justified by the following easy result (proof omitted), where $\ulcorner\Gamma\urcorner$ just drops the world annotations from assumptions in Γ:

Theorem 43. If $G; \Gamma \vdash t : p\,A @ n$, then $\ulcorner\Gamma\urcorner \vdash_c t : p\,A$

Let **SN** be the set of terms which are strongly normalizing with respect to the reduction relation. Let *Var* be the set of term variables, and let us use x and y as metavariables for variables. We will prove strong normalization for classically typed terms using a version of Krivine's classical realizability [15]. We define three interpretations of types in Figure 10. The definition is by mutual induction, and can easily be seen to well-founded, as the definition of $[\![A]\!]^{+}$ invokes the definition of $[\![A]\!]^{-}$ with the same type, which in turn invokes the definition of $[\![A]\!]^{+c}$ with the same type; and the definition of $[\![A]\!]^{+c}$ may invoke either of the other definitions at a strictly smaller type. The reader familiar with such proofs will also recognize the debt owed to Girard [12].

Lemma 44 (Step interpretations)**.** If $t \in \llbracket A \rrbracket^+$ and $t \rightsquigarrow t'$, then $t' \in \llbracket A \rrbracket^+$; and similiarly if $t \in \llbracket A \rrbracket^-$ or $t \in \llbracket A \rrbracket^{+c}$.

Proof. The proof is by a mutual well-founded induction. Assume $t \in \llbracket A \rrbracket^+$ and $t \rightsquigarrow t'$. We must show $t' \in \llbracket A \rrbracket^+$. For this, it suffices to assume $y \in \mathit{Var}$ and $t'' \in \llbracket A \rrbracket^-$, and show $\nu\, y.t' \bullet t'' \in \mathbf{SN}$. From the assumption that $t \in \llbracket A \rrbracket^+$, we have

$$\nu\, y.t \bullet t'' \in \mathbf{SN}$$

which indeed implies that

$$\nu\, y.t' \bullet t'' \in \mathbf{SN}$$

A similar argument applies if $t \in \llbracket A \rrbracket^-$.

For the last part of the lemma, assume $t \in \llbracket A \rrbracket^{+c}$ with $t \rightsquigarrow t'$, and show $t' \in \llbracket A \rrbracket^{+c}$. The only possible cases are the following, where $t \notin \mathit{Vars}$.

If $A \equiv A_1 \to_+ A_2$, then t is of the form $\lambda x.t_a$ for some x and t_a, where for all $t_b \in \llbracket A_1 \rrbracket^+$, we have $[t_b/x]t_a \in \llbracket A_2 \rrbracket^+$. Since $t \rightsquigarrow t'$, t' must be $\lambda x.t'_a$ for some t'_a with $t_a \rightsquigarrow t'_a$. It suffices now to assume an arbitrary $t_b \in \llbracket A_1 \rrbracket^+$, and show $[t_b/x]t'_a \in \llbracket A_2 \rrbracket^+$. But $[t_b/x]t_a \rightsquigarrow [t_b/x]t'_a$ follows from $t_a \rightsquigarrow t'_a$, so by our IH, we have $[t_b/x]t'_a \in \llbracket A_2 \rrbracket^+$, as required.

If $A \equiv A_1 \to_- A_2$, then t is of the form $\langle t_1, t_2 \rangle$ for some $t_1 \in \llbracket A_1 \rrbracket^-$ and $t_2 \in \llbracket A_2 \rrbracket^+$; and $t' \equiv \langle t'_1, t'_2 \rangle$ where either $t'_1 \equiv t_1$ and $t_2 \rightsquigarrow t'_2$ or else $t_1 \rightsquigarrow t'_1$ and $t'_2 \equiv t_2$. Either way, we have $t'_1 \in \llbracket A_1 \rrbracket^-$ and $t'_2 \in \llbracket A_2 \rrbracket^+$ by our IH, so we have $\langle t'_1, t'_2 \rangle \in \llbracket A_1 \to_- A_2 \rrbracket^{+c}$ as required.

The other cases for $A \equiv A_1 \wedge_p A_2$ are similar to the previous one. □

Lemma 45 (SN interpretations)**.**
1. $\llbracket A \rrbracket^+ \subseteq \mathbf{SN}$
2. $\mathit{Vars} \subseteq \llbracket A \rrbracket^-$
3. $\llbracket A \rrbracket^- \subseteq \mathbf{SN}$
4. $\llbracket A \rrbracket^{+c} \subseteq \mathbf{SN}$

Proof. For purposes of this proof and subsequent ones, define $\delta(t)$ to be the length of the longest reduction sequence from t to a normal form, for $t \in \mathbf{SN}$.

The proof of the lemma is by mutual well-founded induction on the pair (A, n), where n is the number of the proposition in the statement of the lemma; the well-founded ordering in question is the lexicographic combination of the structural ordering on types (for A) and the ordering $1 > 2 > 4 > 3$ (for n).

For proposition (1): assume $t \in \llbracket A \rrbracket^+$, and show $t \in \mathbf{SN}$. Let x be a variable. By IH(2), $x \in \llbracket A \rrbracket^-$, so by the definition of $\llbracket A \rrbracket^+$, we have

$$\nu\, x.t \bullet x \in \mathbf{SN}$$

This implies $t \in \mathbf{SN}$.

For proposition (2): assume $x \in \mathit{Vars}$, and show $x \in \llbracket A \rrbracket^-$. For the latter, it suffices to assume arbitrary $y \in \mathit{Vars}$ and $t' \in \llbracket A \rrbracket^{+c}$, and show $\nu\, y.t' \bullet x \in \mathbf{SN}$. We will prove this by inner induction on $\delta(t')$, which is defined by IH(4). By the definition of $\llbracket A \rrbracket^{+c}$ for the various cases of A, we see that $\nu\, y.t' \bullet x$ cannot be a redex itself, as t' cannot be a cut. If t' is a normal form we are done. If $t \rightsquigarrow t''$, then we have $t'' \in \llbracket A \rrbracket^{+c}$ by Lemma 44, and we may apply the inner induction hypothesis.

For proposition (3): assume $t \in \llbracket A \rrbracket^-$, and show $t \in \mathbf{SN}$. By the definition of $\llbracket A \rrbracket^-$ and the fact that $\mathit{Vars} \subseteq \llbracket A \rrbracket^{+c}$ by definition of $\llbracket A \rrbracket^{+c}$, we have

$$\nu\, y.y \bullet t \in \mathbf{SN}$$

This implies $t \in \mathbf{SN}$ as required.

For proposition (4): assume $t \in \llbracket A \rrbracket^{+c}$, and consider the following cases. If $t \in \mathit{Vars}$ or $A \equiv \langle + \rangle$, then t is normal and the result is immediate. So suppose $A \equiv A_1 \to_+ A_2$.

Then $t \equiv \lambda x.t'$ for some x and t' where for all $t'' \in [\![A_1]\!]^+$, $[t''/x]t' \in [\![A_2]\!]^+$. By IH(2), the variable x itself is in $[\![A_1]\!]^+$, so we know that $t' \equiv [x/x]t' \in [\![A_2]\!]^+$. Then by IH(1) we have $t' \in \mathbf{SN}$, which implies $\lambda x.t' \in \mathbf{SN}$. If $A \equiv A_1 \rightarrow_- A_2$, then $t \equiv \langle t_1, t_2 \rangle$ for some $t_1 \in [\![A_1]\!]^-$ and $t_2 \in [\![A_2]\!]^+$. By IH(3) and IH(1), $t_1 \in \mathbf{SN}$ and $t_2 \in \mathbf{SN}$, which implies $\langle t_1, t_2 \rangle \in \mathbf{SN}$. The cases for $A \equiv A_1 \wedge_p A_2$ are similar to this one. □

Definition 46 (Interpretation of contexts)**.** $[\![\Gamma]\!]$ is the set of substitutions σ such that for all $x : p\ A \in \Gamma$, $\sigma(x) \in [\![A]\!]^p$.

Lemma 47 (Canonical positive is positive)**.** $[\![A]\!]^{+c} \subseteq [\![A]\!]^+$

Proof. Assume $t \in [\![A]\!]^{+c}$ and show $t \in [\![A]\!]^+$. For the latter, assume arbitrary $x \in \mathit{Vars}$ and $t' \in [\![A]\!]^-$, and show $\nu\, x.t \bullet t' \in \mathbf{SN}$. This follows immediately from the assumption that $t' \in [\![A]\!]^-$. □

Theorem 48 (Soundness)**.** If $\Gamma \vdash_c t : p\ A$ then for all $\sigma \in [\![\Gamma]\!]$, $\sigma t \in [\![A]\!]^p$.

Proof. The proof is by induction on the derivation of $\Gamma \vdash_c t : p\ A$. We consider the two possible polarities for the conclusion of the typing judgment separately.

Case.

$$\frac{}{\Gamma, x : p\ A, \Gamma' \vdash_c x : p\ A}\ \textsc{ClassAx}$$

Since $\sigma \in [\![\Gamma, x : p\ A, \Gamma']\!]$, $\sigma(x) \in [\![A]\!]^p$ as required.

Case.

$$\frac{}{\Gamma \vdash_c \mathbf{triv} : + \langle + \rangle}\ \textsc{ClassUnit}$$

We have $\mathbf{triv} \in [\![\langle + \rangle]\!]^{+c}$ by definition.

Case.

$$\frac{}{\Gamma \vdash_c \mathbf{triv} : - \langle - \rangle}\ \textsc{ClassUnit}$$

To prove $\mathbf{triv} \in [\![\langle - \rangle]\!]^-$, it suffices to assume arbitrary $y \in \mathit{Vars}$ and $t \in [\![\langle - \rangle]\!]^{+c}$, and show $\nu\, y.t \bullet \mathbf{triv} \in \mathbf{SN}$. By definition of $[\![\langle - \rangle]\!]^{+c}$, $t \in \mathit{Vars}$, and then $\nu\, y.t \bullet \mathbf{triv}$ is in normal form.

Case.

$$\frac{\Gamma \vdash_c t_1 : +\ A \qquad \Gamma \vdash_c t_2 : +\ B}{\Gamma \vdash_c (t_1, t_2) : +\ A \wedge_+ B}\ \textsc{ClassAnd}$$

By Lemma 47, it suffices to show $(\sigma t_1, \sigma t_2) \in [\![A \wedge_+ B]\!]^{+c}$. This follows directly from the definition of $[\![A \wedge_+ B]\!]^{+c}$, since the IH gives us $\sigma t_1 \in [\![A]\!]^+$ and $\sigma t_2 \in [\![B]\!]^+$.

Case.

$$\frac{\Gamma \vdash_c t_1 : -\ A_1 \qquad \Gamma \vdash_c t_2 : -\ A_2}{\Gamma \vdash_c (t_1, t_2) : -\ A_1 \wedge_- A_2}\ \textsc{ClassAnd}$$

It suffices to assume arbitrary $y \in \mathit{Vars}$ and $t' \in [\![A_1 \wedge_- A_2]\!]^{+c}$, and show $\nu\, y.t' \bullet (\sigma\, t_1, \sigma\, t_2) \in \mathbf{SN}$. If $t' \in \mathit{Vars}$, then this follows by Lemma 45 from the facts that $\sigma t_1 \in [\![A_1]\!]^+$ and $\sigma t_2 \in [\![A2]\!]^+$, which we have by the IH. So suppose t' is of the form $\mathbf{in}_d\ t''$ for some d and some $t'' \in [\![A_d]\!]^+$. By the definition of $\mathbf{SN}$, it suffices to show that all one-step successors t_a of the term in question are $\mathbf{SN}$. The proof of this is by inner induction on $\delta(t'') + \delta(\sigma t_1) + \delta(\sigma t_2)$, which exists by Lemma 45, using also Lemma 44. Suppose that we step to t_a by stepping t'', σt_1, or σt_2. Then the result holds by the inner IH. So consider the step

$$\nu\, y.\mathbf{in}_d\ t'' \bullet (\sigma\, t_1, \sigma\, t_2) \rightsquigarrow \nu\, y.t'' \bullet \sigma\, t_d$$

We then have $\nu\, y.t'' \bullet \sigma\, t_d \in \mathbf{SN}$ from the facts that $t'' \in [\![A_d]\!]^+$ and $\sigma t_d \in [\![A_d]\!]^-$, by the definition of $[\![A_d]\!]^+$.

Case.

$$\frac{\Gamma \vdash_c t : +\, A_d}{\Gamma \vdash_c \mathbf{in}_d\, t : +\, A_1 \wedge_- A_2}\ \textsc{ClassAndBar}$$

By Lemma 47, it suffices to prove $\mathbf{in}_d\, \sigma\, t \in [\![A_1 \wedge_- A_2]\!]^+$, but by the definition of $[\![A_1 \wedge_- A_2]\!]^+$, this follows directly from $\sigma t \in [\![A_d]\!]^+$, which we have by the IH.

Case.

$$\frac{\Gamma \vdash_c t : -\, A_d}{\Gamma \vdash_c \mathbf{in}_d\, t : -\, A_1 \wedge_+ A_2}\ \textsc{ClassAndBar}$$

To prove $\mathbf{in}_d\, \sigma\, t \in [\![A_1 \wedge_+ A_2]\!]^-$, it suffices to assume arbitrary $y \in \mathit{Vars}$ and $t' \in [\![A_1 \wedge_+ A_2]\!]^{+c}$, and show $\nu\, y.t' \bullet \mathbf{in}_d\, \sigma\, t \in \mathbf{SN}$. If $t' \in \mathit{Vars}$, then this follows from the fact that $\sigma t \in \mathbf{SN}$, which we have by Lemma 45 from $\sigma t \in [\![A_d]\!]^-$ (which the IH gives us). So suppose t' is of the form (s_1, s_2) for some $s_1 \in [\![A_1]\!]^+$ and $s_2 \in [\![A_2]\!]^+$. It suffices to prove that all one-step successors of the term in question are in $\mathbf{SN}$, as we did in a previous case above. Lemma 45 lets us proceed by inner induction on $\delta(\sigma t) + \delta(s_1) + \delta(s_2)$, using also Lemma 44. If we step σt, s_1 or s_2, then the result holds by inner IH. Otherwise, we have the step

$$\nu\, y.(s_1, s_2) \bullet \mathbf{in}_d\, \sigma\, t \rightsquigarrow \nu\, y.s_d \bullet \sigma\, t$$

And this successor is in $\mathbf{SN}$ by the facts that $s_d \in [\![A_d]\!]^+$ and $\sigma t \in [\![A_d]\!]^-$, from the definition of $[\![A_d]\!]^+$.

Case.

$$\frac{\Gamma, x : +\, A \vdash_c t : +\, B}{\Gamma \vdash_c \lambda x.t : +\, A \to_+ B}\ \textsc{ClassImp}$$

By Lemma 47, it suffices to assume arbitrary $y \in \mathit{Vars}$ and $t' \in [\![A]\!]^+$, and prove $[t'/x](\sigma t) \in [\![B]\!]^+$. But this follows immediately from the IH, since $[t'/x](\sigma t) \equiv (\sigma[x \mapsto t'])t$ and $\sigma[x \mapsto t] \in [\![\Gamma, x : +\, A]\!]$.

Case.

$$\frac{\Gamma, x : -\, A \vdash_c t : -\, B}{\Gamma \vdash_c \lambda x.t : -\, A \to_- B}\ \textsc{ClassImp}$$

It suffices to assume arbitrary $y \in \mathit{Vars}$ and $t' \in [\![A \to_- B]\!]^{+c}$, and show $\nu\, y.t' \bullet \lambda x.\sigma\, t \in \mathbf{SN}$. Let us first observe that $\sigma\, t \in \mathbf{SN}$, because by the IH, for all $\sigma' \in [\![\Gamma, x : -\, A]\!]$, we have $\sigma' t \in [\![B]\!]^-$, and $[\![B]\!]^- \subseteq \mathbf{SN}$ by Lemma 45. We may instantiate this with $\sigma[x \mapsto x]$, since by Lemma 45, $x \in [\![A]\!]^-$. Since $\sigma\, t \in \mathbf{SN}$, we also have $\lambda x.\sigma\, t \in \mathbf{SN}$. Now let us consider cases for the assumption $t' \in [\![A \to_- B]\!]^{+c}$. If $t' \in \mathit{Vars}$ then we directly have $\nu\, y.t' \bullet \lambda x.\sigma\, t \in \mathbf{SN}$ from $\lambda x.\sigma\, t \in \mathbf{SN}$. So assume $t' \equiv \langle t_1, t_2 \rangle$ for some $t_1 \in [\![A]\!]^-$ and $t_2 \in [\![B]\!]^+$. By Lemma 45 again, we may reason by inner induction on $\delta(t_1) + \delta(t_2) + \delta(\sigma t)$ to show that all one-step successors of $\nu\, y.\langle t_1, t_2 \rangle \bullet \lambda x.\sigma\, t$ are in $\mathbf{SN}$, using also Lemma 44. If t_1, t_2, or σt steps, then the result follows by the inner IH. So suppose we have the step

$$\nu\, y.\langle t_1, t_2 \rangle \bullet \lambda x.\sigma\, t \rightsquigarrow \nu\, y.t_2 \bullet [t_1/x](\sigma\, t)$$

Since $t_1 \in [\![A]\!]^-$, the substitution $\sigma[x \mapsto t_1]$ is in $[\![\Gamma, x : -\, A]\!]$. So we may apply the IH to obtain $[t_1/x](\sigma t) \equiv \sigma[x \mapsto t_1] \in [\![B]\!]^-$. Then since $t_2 \in [\![B]\!]^+$, we have $\nu\, y.t_2 \bullet [t_1/x](\sigma\, t)$ by definition of $[\![B]\!]^+$.

Case.

$$\frac{\Gamma \vdash_c t_1 : -\,A \qquad \Gamma \vdash_c t_2 : +\,B}{\Gamma \vdash_c \langle t_1, t_2 \rangle : +\,(A \rightarrow_- B)}\ \textsc{ClassImpBar}$$

By Lemma 47, as in previous cases of positive typing, it suffices to prove $\langle \sigma\, t_1, \sigma\, t_2 \rangle \in [\![A \rightarrow_- B]\!]^{+c}$. By the definition of $[\![A \rightarrow_- B]\!]^{+c}$, this follows directly from $\sigma t_1 \in [\![A]\!]^-$ and $\sigma t_2 \in [\![B]\!]^+$, which we have by the IH.

Case.

$$\frac{\Gamma \vdash_c t_1 : +\,A \qquad \Gamma \vdash_c t_2 : -\,B}{\Gamma \vdash_c \langle t_1, t_2 \rangle : -\,(A \rightarrow_+ B)}\ \textsc{ClassImpBar}$$

It suffices to assume arbitrary $y \in \mathit{Vars}$ and $t' \in [\![A \rightarrow_+ B]\!]^{+c}$, and show $\nu\, y.t' \bullet \langle \sigma\, t_1, \sigma\, t_2 \rangle \in \mathbf{SN}$. By the IH, we have $\sigma t_1 \in [\![A]\!]^+$ and $\sigma t_2 \in [\![B]\!]^-$, and hence $\sigma t_1 \in \mathbf{SN}$ and $\sigma t_2 \in \mathbf{SN}$ by Lemma 45. If $t' \in \mathit{Vars}$, then these facts are sufficient to show the term in question is in $\mathbf{SN}$. So suppose $t' \equiv \lambda x.t_3$, for some $x \in \mathit{Vars}$ and t'' such that for all $t_4 \in [\![A]\!]^+$, $[t_4/x]t_3 \in [\![B]\!]^+$. By similar reasoning as in a previous case, we have $t_3 \in \mathbf{SN}$. So we may proceed by inner induction on $\delta(t_1)+\delta(t_2)+\delta(t_3)$ to show that all one-step successors of $\nu\, y.\lambda x.t_3 \bullet \langle \sigma\, t_1, \sigma\, t_2 \rangle$ are in $\mathbf{SN}$, using also Lemma 44. If it is t_3, σt_1, or σt_2 which steps, then the result follows by the inner IH. So consider this step:

$$\nu\, y.\lambda x.t_3 \bullet \langle \sigma\, t_1, \sigma\, t_2 \rangle \rightsquigarrow \nu\, y.[\sigma\, t_1/x]t_3 \bullet \sigma\, t_2$$

Since we have that $\sigma t_1 \in [\![A]\!]^+$, the assumption about substitution instances of t_3 gives us that $[\sigma t_1/x]t_3 \in [\![B]\!]^+$, which is then sufficient to conclude $\nu\, y.[\sigma\, t_1/x]t_3 \bullet \sigma\, t_2 \in \mathbf{SN}$ by the definition of $[\![B]\!]^+$.

Case.

$$\frac{\Gamma, x : -\,A \vdash_c t_1 : +\,B \qquad \Gamma, x : -\,A \vdash_c t_2 : -\,B}{\Gamma \vdash_c \nu\, x.t_1 \bullet t_2 : +\,A}\ \textsc{ClassCut}$$

It suffices to assume arbitrary $y \in \mathit{Vars}$ and $t' \in [\![A]\!]^-$, and show $\nu\, y.(\nu\, x.\sigma\, t_1 \bullet \sigma\, t_2) \bullet t' \in \mathbf{SN}$. By the IH and part 2 of Lemma 45, we know that $\sigma t_1 \in [\![B]\!]^+$ and $\sigma t_2 \in [\![B]\!]^-$. By Lemma 45 again, we have $t' \in \mathbf{SN}$, $\sigma t_1 \in \mathbf{SN}$, and $\sigma t_2 \in \mathbf{SN}$. So we may reason by induction on $\delta(t') + \delta(\sigma t_1) + \delta(\sigma t_2)$ to show that all one-step successors of $\nu\, y.(\nu\, x.\sigma\, t_1 \bullet \sigma\, t_2) \bullet t'$ are in $\mathbf{SN}$, using also Lemma 44. If it is t', σt_1, or σt_2 which steps, then the result follows by the inner IH. The only possible other reduction is by the RBetaL reduction rule (Figure 8). And then, since $t' \in [\![A]\!]^-$, we may apply the IH to conclude that $[t'/x](\sigma t_1) \in [\![B]\!]^+$ and $[t'/x](\sigma t_2) \in [\![B]\!]^-$. By the definition of $\in [\![B]\!]^+$, this suffices to prove $\nu\, y.[t'/x]\sigma\, t_1 \bullet [t'/x]\sigma\, t_2 \in \mathbf{SN}$, as required.

Case.

$$\frac{\Gamma, x : -\,A \vdash_c t_1 : +\,B \qquad \Gamma, x : -\,A \vdash_c t_2 : -\,B}{\Gamma \vdash_c \nu\, x.t_1 \bullet t_2 : -\,A}\ \textsc{ClassCut}$$

It suffices to consider arbitrary $y \in \mathit{Vars}$ and $t' \in [\![A]\!]^{+c}$, and show $\nu\, y.t' \bullet (\nu\, x.\sigma\, t_1 \bullet \sigma\, t_2) \in \mathbf{SN}$. By the IH and part 2 of Lemma 45, we have $\sigma t_1 \in [\![B]\!]^+$ and $\sigma t_2 \in [\![B]\!]^-$, which implies $\sigma t_1 \in \mathbf{SN}$ and $\sigma t_2 \in \mathbf{SN}$ by Lemma 45 again. We proceed by inner induction on $\delta(t') + \delta(\sigma t_1) + \delta(\sigma t_2)$, using Lemma 44, to show that all one-step successors of $\nu\, y.t' \bullet (\nu\, x.\sigma\, t_1 \bullet \sigma\, t_2)$ are in $\mathbf{SN}$. If it is t', σt_1, or σt_2 which steps, then the result holds by inner IH. The only other reduction possible is by RBetaR, since t' cannot be a cut term by the definition of $[\![A]\!]^{+c}$. In this case, the IH gives us $[t'/x]\sigma t_1 \in [\![B]\!]^+$ and $[t'/x]\sigma t_2 \in [\![B]\!]^-$, and we then have $\nu\, y.[t'/x]\sigma\, t_1 \bullet [t'/x]\sigma\, t_2 \in \mathbf{SN}$ by the definition of $[\![B]\!]^+$.

□

Corollary 49 (Strong Normalization)**.** If $G; \Gamma \vdash t : p\ A \mathbin{@} n$, then $t \in \mathbf{SN}$.

Proof. This follows easily by putting together Theorems 43 and 48, with Lemma 45. □

Corollary 50 (Cut Elimination)**.** If $G; \Gamma \vdash t : p\ A \mathbin{@} n$, then there is normal t' with $t \rightsquigarrow^* t'$ and t' containing only cut terms of the form $\nu\, x.y \bullet t$ or $\nu\, x.t \bullet y$, for y a variable.

Lemma 51 (Local Confluence)**.** The reduction relation of Figure 8 is locally confluent.

Proof. We may view the reduction rules as higher-order pattern rewrite rules. It is easy to confirm that all critical pairs (e.g., between RBetaR and the rules RImp, RImpBar, RAnd1, RAndBar1, RAnd2, and RAndBar2) are joinable. Local confluence then follows by the higher-order critical pair lemma [18]. □

Theorem 52 (Confluence for Typable Terms)**.** The reduction relation restricted to terms typable in DTT is confluent.

Proof. Suppose $G; \Gamma \vdash t : p\ A \mathbin{@} n$ for some G, Γ, p, and A. By Lemma 42, any reductions in the unrestricted reduction relation from t are also in the reduction relation restricted to typable terms. The result now follows from Newman's Lemma, using Lemma 51 and Theorem 49. □

6. Conclusion

We have presented a new type theory for bi-intuitionistic logic. We began with a compact dualized formulation of the logic, Dualized Intuitionistic Logic (DIL), and showed soundness with respect to a standard Kripke semantics (in Agda), and completeness with respect to Pinto and Uustalu's system L. We then presented Dualized Type Theory (DTT), and showed type preservation, strong normalization, and confluence for typable terms. Future work includes further additions to DTT, for example with polymorphism and inductive types. It would also be interesting to obtain a Canonicity Theorem as in [23], identifying some set of types where closed normal forms are guaranteed to be canonical values (as canonicity fails in general in DIL/DTT, as in other bi-intuitionistic systems).

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