Difference in Complete Atomic Boolean Algebras

Prof. Harley Eades (heades@gru.edu).

1 Complete Atomic Boolean Algebras

Definition 1. *Suppose $(A, \leqslant, \perp)$ is a poset with a least element $\perp$. An element $a \in A$ is called* ***atomic*** *if for any $b \leqslant a$ we have $b = \perp$ or $b = a$.*

Definition 2. *A* ***complete atomic boolean algebra (CABA)*** *is defined by the following axioms on a poset $(B, \leqslant)$:*

- *(Top) There is an element $\top \in B$ such that for any $x \in B$, $x \leqslant \top$.*
- *(Bottom) There is an element $\perp \in B$ such that for any $x \in B$, $\perp \leqslant x$.*
- *(Meet) Given elements $a, b \in B$, there is an element $a \times b \in B$, such that, $x \leqslant a$ and $x \leqslant b$ iff $x \leqslant a \times b$.*
- *(Join) Given elements $a, b \in B$, there is an element $a + b \in B$, such that, $a \leqslant x$ and $b \leqslant x$ iff $a + b \leqslant x$.*
- *(Complement) Given an element $a \in B$ there is an element $\neg a \in B$, such that, $a \times \neg a \leqslant \perp$ and $\top \leqslant a + \neg a$.*
- *(Distributivity) Given elements $a, b, c \in B$ the following must hold $a \times (b + c) \leqslant (a \times b) + (a \times c)$.*
- *(Atomic) Given any element $b \in B$ we have that $b = +_{i \in I} a_i$ where each element of $a_{i i \in I} \subseteq B$ is atomic.*
- *(Complete) Every $B' \subseteq B$ has a supremum (least-upper bound).*

2 CABAs are Powerset Algebras

Given a CABA $(B, +, \times, \neg, \perp, \top)$ we can define a powerset algebra $(\mathcal{P}(X), \cup, \cap, \overline{}, \varnothing, X)$. Suppose $A \subseteq B$ is the set of atomic elements. Then set $X = A$. So we will define the powerset algebra $P = (\mathcal{P}(A), \cup, \cap, \overline{}, \varnothing, A)$.

(**Atomicity**) Notice that we have an isomorphism between A and the set of single element subsets of A. Thus, we define any subset of $X \subseteq A$ as the union of single element subsets of A. This, implies that P is atomic.

(**Completeness**) We can also show that P is complete. Given any set $L \subseteq \mathcal{P}(A)$ the sup of L is defined to be $L \subseteq \bigcup L$. This is clearly least among all sets containing L.

(**Greatest Element**) The greatest element is defined to be A because all subsets of A are contained in A.

(**Least Element**) The least element is defined to be the empty set, $\varnothing$, which is trivially contained in every set.

(**Meet**) Suppose $b_1 \times b_2 \in B$ then by atomicity we know $b_1 = +_{i \in I}\{a_i\}$ and $b_1 = +_{j \in J}\{a_j\}$ for atomic variables a_i and a_j. We map $\times$ to $\cap$, and b_1 to the set $\{a_i\}$ and b_2 to the set $\{a_j\}$. Thus, we have $\{a_i\} \cap \{a_j\} \in P$, and the meet condition is clearly satisfied.

(**Join**) Mapping an element $b_1 + b_2 \in B$ is similar to the previous case.

(**Complement**) Given an element $L \in \mathcal{P}(A)$ we define $\overline{L} \in \mathcal{P}(A)$ as the complement relative to A. Thus, $\overline{\varnothing} = A$ and $\overline{A} = \varnothing$. Furthermore, for $L \in P$, $L \cap \overline{L} = \varnothing$ and $L \cup \overline{L} = A$. Finally, mapping any element $\neg b \in B$ can be mapped to the complement of the set of atomic elements defining $\neg b$.

(**Distributivity**) It is an elementary fact of set theory that intersection distributes over union.

These facts show that any CABA can be mapped to a powerset algebra. It is also easy to see that every powerset algebra is a CABA. Thus, an isomorphism.

3 The Category of CABAs

We now define the category of CABAs by taking CABAs as objects and CABA homomorphisms as morphisms. We call this category $\mathsf{CABACat}$.

Definition 3. *A **CABA homomorphism** between CABAs $(B_1, +, \times, \neg, \bot, \top)$ and $(B_2, +, \times, \neg, \bot, \top)$ is a function $H : B_1 \longrightarrow B_2$ such that the following holds:*

1. $H(\top) = \top$
2. $H(\bot) = \bot$
3. $H(b_1 \times b_2) = H(b_1) \times H(b_2)$
4. $H(b_1 + b_2) = H(b_1) + H(b_2)$
5. $H(\neg b) = \neg H(b)$
6. *H maps sups of B_1 to sups of B_2*

It is easy to see that we can define identity CABA homomorphisms, and composition of CABA homomorphisms is simply set theoretic composition, which, is associative and respects identities.

3.1 Coproducts in $\mathsf{CABACat}$

The coproduct construction on CABAs is not obvious at first, but by working with powerset algebras we can more easily see how the construction should be done.

Given two powerset algebras $(\mathcal{P}(X), \cup, \cap, \overline{}, \varnothing, X)$ and $(\mathcal{P}(Y), \cup, \cap, \overline{}, \varnothing, Y)$ we can define their coproduct as the powerset algebra $(\mathcal{P}(X \times Y), \cup, \cap, \overline{}, \varnothing, X \times Y)$. Both injections can be defined as follows:

$$\begin{array}{l} i_1 : \mathcal{P}(X) \longrightarrow \mathcal{P}(X \times Y) \\ i_1(S \subseteq X) = \{(x, y) \in X \times Y \mid x \in S\} \\ \\ i_2 : \mathcal{P}(Y) \longrightarrow \mathcal{P}(X \times Y) \\ i_2(S \subseteq Y) = \{(x, y) \in X \times Y \mid y \in S\} \end{array}$$

It is easy to see that the previous injections are homomorphisms that preserve the structure of the powerset algebra.

At this point we must make the following diagram commute:

$$\begin{tikzcd}
& \mathcal{P}(Z) & \\
\mathcal{P}(X) \arrow[ur, "f"] \arrow[r, "i_1"'] & \mathcal{P}(X \times Y) \arrow[u, dashed, "h"] & \mathcal{P}(Y) \arrow[ul, "g"'] \arrow[l, "i_2"]
\end{tikzcd}$$

First, note that any subset $S \in \mathcal{P}(X \times Y)$ can be written as $S = \mathsf{in}_1(S_X) \cup \mathsf{in}_2(S_Y)$ where $S_X \subseteq X$ and $S_Y \subseteq Y$. We define h as follows:

$$\begin{array}{l} h : \mathcal{P}(X \times Y) \longrightarrow \mathcal{P}(Z) \\ h(S \subseteq X \times Y) = f(\pi_1(S)) \cup g(\pi_2(S)) \end{array}$$

Given the previous definitions we can now see that the previous diagram commutes by construction.

The definition of coproduct for powersets gives us a hint of how to do it for general CABAs, certainly, we know they must exist in $\mathsf{CABACat}$ because it is isomorphic to the category of powerset algebras. The coproduct construction just given shows that we must define the coproduct in terms of the disjoint union of the sets of atomic elements. In addition, notice that the least elements of both powerset algebras in the coproduct are the same, and the greatest element of the coproduct is defined in terms of the greatest elements of the two component powerset algebras. This implies that we must collapse the greatest and lowest elements of two arbitrary CABAs in order to construct the coproduct.

Suppose $(B_1, +_1, \times_1, \neg_1, \perp_1, \top_1)$ and $(B_2, +_2, \times_2, \neg_2, \perp_2, \top_2)$ are two CABAs. Furthermore, suppose A_1 and A_2 are the sets of atomic elements of B_1 and B_2 respectively. Now take the set $A_1 \oplus A_2$ to be the coequalizer:

$$\begin{tikzcd}
1 \arrow[r, shift left, "{[\mathsf{in}_1(\perp_1), \mathsf{in}_2(\top_1)]}"] \arrow[r, shift right, "{[\mathsf{in}_1(\perp_2), \mathsf{in}_2(\top_2)]}"'] & A_1 + A_2 \arrow[r] & A_1 \oplus A_2
\end{tikzcd}$$

Thus, the set $A_1 \oplus A_2$ is the set of equivalence classes where the only identified elements are the least and greatest elements. That is, $\mathsf{in}_1(\perp_1) = \mathsf{in}_2(\perp_2)$ and $\mathsf{in}_1(\top_1) = \mathsf{in}_2(\top_2)$. Every other equivalence class consists of a single element of $A_1 + A_2$. We denote each element, $[\mathsf{in}_1(a_i)]$ and $[\mathsf{in}_2(a_j)]$, by simply a_i and a_j, and then denote the collapsed least and greatest elements by $\perp_\oplus$ and $\top_\oplus$. We can now define the preorder on $A_1 \oplus A_2$, denoted by $\leqslant_\oplus$, as follows:

$$\begin{array}{ll} a_1 \leqslant_\oplus a_2 & \text{iff } a_1 \leqslant_1 a_2 \text{ or } a_1 \leqslant_2 a_2 \\ \perp_\oplus \leqslant_\oplus a & \text{iff } a \in A_1 \text{ or } a \in A_2 \\ a \leqslant_\oplus \top_\oplus & \text{iff } a \in A_1 \text{ or } a \in A_2 \end{array}$$

3.2 Cointernal Hom in $\mathsf{CABACat}$

The internal hom in Set is defined by $A \Rightarrow B = \{f \mid f : A \longrightarrow B \text{ is a set function}\}$. Curry can be defined as follows:

$$\begin{array}{l} \mathsf{cur} : \mathsf{Hom}(A \times B, C) \longrightarrow \mathsf{Hom}(A, B \Rightarrow C) \\ \mathsf{cur}(f : A \times B \longrightarrow C) = \lambda a.\lambda b.f(a, b). \end{array}$$

It is easy to see that cur is indeed an isomorphism. The evaluator can be defined as follows:

$$\begin{array}{l} \mathsf{eval} : (A \Rightarrow B) \times A \longrightarrow B \\ \mathsf{eval} = \mathsf{cur}^{-1}(\mathsf{id}_{A \Rightarrow B}) \end{array}$$

The following contravariant functor is called the **powerset functor**:

$$\begin{array}{l} P : \mathsf{Set} \longrightarrow \mathsf{Set}^{op} \\ P(X) = \mathcal{P}(X) \\ P(f : A \longrightarrow B)(B' \subset B) = f^{-1}(B') = \{a \in A \mid \exists b \in B'.f(a) = b\} \end{array}$$

Now we can compute the opposite of the internal hom:

$$P(A \Rightarrow B) = \mathcal{P}(A \Rightarrow B)$$

$$\mathsf{cocur} : \mathsf{Hom}(\mathcal{P}(B \Rightarrow C), \mathcal{P}(A)) \longrightarrow \mathsf{Hom}(\mathcal{P}(C), \mathcal{P}(A \times B))$$