

*vars, a, x, y, z, w, m, o*

*ivar, n, i, k, j, l*

$R, S, T ::=$   
 $| 0$   
 $| S + T$   
 $| S - T$   
 $| HA$

$A, B, C ::=$   
 $| \perp$   
 $| A \oplus B$   
 $| A \bullet B$   
 $| JS$

$s, t, r ::=$  non-linear terms  
 $| x$   
 $| \text{connect}_w \text{ to } t$   
 $| t_1 \cdot t_2$   
 $| \text{false } t$   
 $| x(t)$   
 $| \text{mkc}(t, x)$   
 $| \text{postp}(x \mapsto t_1, t_2)$   
 $| \text{inl } t$   
 $| \text{inr } t$   
 $| \text{case } t_1 \text{ of } x.t_2, y.t_3$   
 $| H e$   
 $| \text{let } J x = e \text{ in } t$   
 $| \text{let } H x = t_1 \text{ in } t_2$   
 $| (t) \quad S$

$e, u, p ::=$  linear terms  
 $| x$   
 $| \text{connect}_\perp \text{ to } e$   
 $| \text{postp}_\perp e$   
 $| \text{postp}(x \mapsto e_1, e_2)$   
 $| \text{mkc}(e, x)$   
 $| x(e)$   
 $| e_1 \oplus e_2$   
 $| \text{casel } e$   
 $| \text{caser } e$   
 $| J t$   
 $| (e) \quad S$

$\Psi, \Theta ::=$   
 $| \cdot$

		$T$	
		$t : T$	
		$\Psi, \Theta$	
		$(\Psi)$	S
		$\text{let } Jx = e \text{ in } \Psi$	
$\Gamma, \Delta$	::=		
		$\cdot$	
		$A$	
		$e : A$	
		$\Gamma, \Gamma'$	
		$(\Gamma)$	S
		$\text{postp}(x \mapsto e_1, e_2)$	

$$\boxed{S \vdash_{\mathbf{C}} \Psi}$$

$$\begin{array}{c}
\frac{}{S \vdash_{\mathbf{C}} S} \text{C\_ID} \\
\frac{S \vdash_{\mathbf{C}} \Psi}{S \vdash_{\mathbf{C}} T, \Psi} \text{C\_WK} \\
\frac{S \vdash_{\mathbf{C}} T, T, \Psi}{S \vdash_{\mathbf{C}} T, \Psi} \text{C\_CR} \\
\frac{R \vdash_{\mathbf{C}} \Psi_1, S, T, \Psi_2}{R \vdash_{\mathbf{C}} \Psi_1, T, S, \Psi_2} \text{C\_EX} \\
\frac{}{0 \vdash_{\mathbf{C}} \Psi} \text{C\_FL} \\
\frac{T_1 \vdash_{\mathbf{C}} \Psi_1 \quad T_2 \vdash_{\mathbf{C}} \Psi_2}{T_1 + T_2 \vdash_{\mathbf{C}} \Psi_1, \Psi_2} \text{C\_DL} \\
\frac{R \vdash_{\mathbf{C}} \Psi, T_1}{R \vdash_{\mathbf{C}} \Psi, T_1 + T_2} \text{C\_DR1} \\
\frac{R \vdash_{\mathbf{C}} \Psi, T_2}{R \vdash_{\mathbf{C}} \Psi, T_1 + T_2} \text{C\_DR2} \\
\frac{T_1 \vdash_{\mathbf{C}} T_2, \Psi}{T_1 - T_2 \vdash_{\mathbf{C}} \Psi} \text{C\_SL} \\
\frac{S \vdash_{\mathbf{C}} \Psi_1, T_1 \quad T_2 \vdash_{\mathbf{C}} \Psi_2}{S \vdash_{\mathbf{C}} \Psi_1, \Psi_2, T_1 - T_2} \text{C\_SR} \\
\frac{S \vdash_{\mathbf{C}} \Psi_1, T \quad T \vdash_{\mathbf{C}} \Psi_2}{S \vdash_{\mathbf{C}} \Psi_1, \Psi_2} \text{C\_CUT} \\
\frac{S \vdash_{\mathbf{C}} \Psi, S'' \quad S \vdash_{\mathbf{C}} \Psi'}{S \vdash_{\mathbf{C}} \Psi, \Psi'} \text{C\_MCUT} \\
\frac{A \vdash_{\mathbf{L}} \cdot; \Psi}{HA \vdash_{\mathbf{C}} \Psi} \text{C\_HL} \\
\frac{T_1 \vdash_{\mathbf{C}} \Psi \quad T_2 \vdash_{\mathbf{C}} \Psi}{T_1 + T_2 \vdash_{\mathbf{C}} \Psi} \text{C\_ADL}
\end{array}$$

$$\boxed{A \vdash_L \Delta; \Psi}$$

$$\begin{array}{c}
\frac{}{A \vdash_L A; \cdot} \text{L\_ID} \\
\frac{A \vdash_L \Delta; \Psi}{A \vdash_L \Delta; T, \Psi} \text{L\_WK} \\
\frac{A \vdash_L \Delta; T, T, \Psi}{A \vdash_L \Delta; T, \Psi} \text{L\_CTR} \\
\frac{A \vdash_L \Delta_1, A, B, \Delta_2; \Psi}{A \vdash_L \Delta_1, B, A, \Delta_2; \Psi} \text{L\_EX} \\
\frac{A \vdash_L \Delta; \Psi_1, S, T, \Psi_2}{A \vdash_L \Delta; \Psi_1, T, S, \Psi_2} \text{L\_CEX} \\
\frac{A \vdash_L \Delta_1, B; \Psi_1 \quad B \vdash_L \Delta_2; \Psi_2}{A \vdash_L \Delta_1, \Delta_2; \Psi_1, \Psi_2} \text{L\_CUT} \\
\frac{A \vdash_L \Delta; \Psi_1, T \quad T \vdash_C \Psi_2}{A \vdash_L \Delta; \Psi_1, \Psi_2} \text{L\_CCUT} \\
\frac{}{\perp \vdash_L \cdot; \cdot} \text{L\_FLL} \\
\frac{A \vdash_L \Delta; \Psi}{A \vdash_L \perp, \Delta; \Psi} \text{L\_FLR} \\
\frac{A \vdash_L \Delta; \Psi, T_1}{A \vdash_L \Delta; \Psi, T_1 + T_2} \text{L\_DR1} \\
\frac{A \vdash_L \Delta; \Psi, T_2}{A \vdash_L \Delta; \Psi, T_1 + T_2} \text{L\_DR2} \\
\frac{B_1 \vdash_L \Delta_1; \Psi_1 \quad B_2 \vdash_L \Delta_2; \Psi_2}{B_1 \oplus B_2 \vdash_L \Delta_1, \Delta_2; \Psi_1, \Psi_2} \text{L\_PL} \\
\frac{A \vdash_L \Delta, B, C; \Psi}{A \vdash_L \Delta, B \oplus C; \Psi} \text{L\_PR} \\
\frac{B_1 \vdash_L B_2, \Delta; \Psi}{B_1 \bullet B_2 \vdash_L \Delta; \Psi} \text{L\_SL} \\
\frac{A \vdash_L B_1, \Delta_1; \Psi_1 \quad B_2 \vdash_L \Delta_2; \Psi_2}{A \vdash_L B \bullet C, \Delta_1, \Delta_2; \Psi_1, \Psi_2} \text{L\_SR} \\
\frac{A \vdash_L \Delta; \Psi_1, T_1 \quad T_2 \vdash_C \Psi_2}{A \vdash_L \Delta; \Psi_1, \Psi_2, T_1 - T_2} \text{L\_CSR} \\
\frac{T \vdash_C \Psi}{JT \vdash_L \cdot; \Psi} \text{L\_JL} \\
\frac{A \vdash_L \Delta; T, \Psi}{A \vdash_L \Delta, JT; \Psi} \text{L\_JR} \\
\frac{A \vdash_L \Delta, B; \Psi}{A \vdash_L \Delta; HB, \Psi} \text{L\_HR} \\
\frac{A \vdash_L \Delta; \Psi, S^n \quad S \vdash_C \Psi'}{A \vdash_L \Delta; \Psi, \Psi'} \text{L\_CMCUT}
\end{array}$$

$S \vdash_C \Psi$  Non-linear Natural Deduction

$$\begin{array}{c}
\frac{}{S \vdash_C S} \text{NC\_ID} \\
\frac{S \vdash_C 0, \Psi \quad S_1 \vdash_C \Psi_1, \dots, S_n \vdash_C \Psi_n}{S \vdash_C \Psi, \Psi_1, \dots, \Psi_n} \text{NC\_zE} \\
\frac{S \vdash_C \Psi, T_1}{S \vdash_C \Psi, T_1 + T_2} \text{NC\_dI1} \\
\frac{S \vdash_C \Psi, T_2}{S \vdash_C \Psi, T_1 + T_2} \text{NC\_dI2} \\
\frac{S \vdash_C \Psi_1, T_1 + T_2 \quad T_1 \vdash_C \Psi_2 \quad T_2 \vdash_C \Psi_2}{S \vdash_C \Psi_1, \Psi_2} \text{NC\_dE} \\
\frac{S \vdash_C \Psi_1, T_1 \quad T_2 \vdash_C \Psi_2}{S \vdash_C \Psi_1, \Psi_2, T_1 - T_2} \text{NC\_subI} \\
\frac{S \vdash_C \Psi_1, T_1 - T_2 \quad T_1 \vdash_C T_2, \Psi_2}{S \vdash_C \Psi_1, \Psi_2} \text{NC\_subE} \\
\frac{S \vdash_C \Psi_1, \text{HA} \quad A \vdash_L \cdot; \Psi_2}{S \vdash_C \Psi_1, \Psi_2} \text{NC\_HE} \\
\frac{S \vdash_C \Psi}{S \vdash_C T, \Psi} \text{NC\_WEAK} \\
\frac{S \vdash_C T, T, \Psi}{S \vdash_C T, \Psi} \text{NC\_CONTR} \\
\frac{S \vdash_C \Psi_1, T \quad T \vdash_C \Psi_2}{S \vdash_C \Psi_1, \Psi_2} \text{NC\_CUT}
\end{array}$$

$A \vdash_L \Delta; \Psi$  Linear Natural Deduction

$$\begin{array}{c}
\frac{}{A \vdash_L A; \cdot} \text{NL\_ID} \\
\frac{A \vdash_L \Delta; \Psi}{A \vdash_L \Delta, \perp; \Psi} \text{NL\_PI} \\
\frac{A \vdash_L \perp, \Delta; \cdot}{A \vdash_L \Delta; \cdot} \text{NL\_PE} \\
\frac{A \vdash_L \Delta, B_1, B_2; \Psi}{A \vdash_L \Delta, B_1 \oplus B_2; \Psi} \text{NL\_parI} \\
\frac{A \vdash_L \Delta, B_1 \oplus B_2; \Psi \quad B_1 \vdash_L \Delta_1; \Psi_1 \quad B_2 \vdash_L \Delta_2; \Psi_2}{A \vdash_L \Delta, \Delta_1, \Delta_2; \Psi, \Psi_1, \Psi_2} \text{NL\_parE} \\
\frac{A \vdash_L \Delta_1, B_1; \Psi_1 \quad B_2 \vdash_L \Delta_2; \Psi_2}{A \vdash_L B_1 \multimap B_2, \Delta_1, \Delta_2; \Psi_1, \Psi_2} \text{NL\_subI} \\
\frac{A \vdash_L \Delta_1, B_1 \multimap B_2; \Psi_1 \quad B_1 \vdash_L B_1, \Delta_2; \Psi_2}{A \vdash_L \Delta_1, \Delta_2; \Psi_1, \Psi_2} \text{NL\_subE} \\
\frac{A \vdash_L \Delta; T, \Psi}{A \vdash_L \Delta, \text{J } T; \Psi} \text{NL\_JI}
\end{array}$$

$$\begin{array}{c}
\frac{A \vdash_L \Delta, \mathbb{J}T; \Psi_1 \quad T \vdash_C \Psi_2}{A \vdash_L \Delta; \Psi_1, \Psi_2} \text{NL\_JE} \\
\frac{A \vdash_L \Delta, B; \Psi}{A \vdash_L \Delta; \mathbb{H}B, \Psi} \text{NL\_HI} \\
\frac{A \vdash_L \Delta; \Psi_1, \mathbb{H}A \quad A \vdash_L \cdot; \Psi_2}{A \vdash_L \Delta; \Psi_1, \Psi_2} \text{NL\_HE} \\
\frac{A \vdash_L \Delta; \Psi}{A \vdash_L \Delta; T, \Psi} \text{NL\_WEAK} \\
\frac{A \vdash_L \Delta; T, T, \Psi}{A \vdash_L \Delta; T, \Psi} \text{NL\_CONTR} \\
\frac{A \vdash_L \Delta; \Psi_1, T \quad T \vdash_C \Psi_2}{A \vdash_L \Delta; \Psi_1, \Psi_2} \text{NL\_CCUT} \\
\frac{A \vdash_L \Delta_1, B; \Psi_1 \quad B \vdash_L \Delta_2; \Psi_2}{A \vdash_L \Delta_1, \Delta_2; \Psi_1, \Psi_2} \text{NL\_CUT}
\end{array}$$

$$\boxed{x : S \vdash_C \Psi}$$

$$\begin{array}{c}
\frac{}{x : S \vdash_C x : S} \text{TC\_ID} \\
\frac{x : S \vdash_C t : 0, \Psi \quad x_1 : S_1 \vdash_C \Psi_1, \dots, x_n : S_n \vdash_C \Psi_n}{x : S \vdash_C \Psi, [\text{false } t/x_1]\Psi_1, \dots, [\text{false } t/x_n]\Psi_n} \text{TC\_zI} \\
\frac{x : S \vdash_C \Psi, t : T_1}{x : S \vdash_C \Psi, \text{inl } t : T_1 + T_2} \text{TC\_dI1} \\
\frac{x : S \vdash_C \Psi, t : T_2}{x : S \vdash_C \Psi, \text{inr } t : T_1 + T_2} \text{TC\_dI2} \\
\frac{x : S \vdash_C \Psi_1, t : T_1 + T_2 \quad y : T_1 \vdash_C \Psi_2 \quad z : T_2 \vdash_C \Psi_2 \quad |\Psi_1| = |\Psi_2|}{x : S \vdash_C \Psi_1, \Psi_2} \text{TC\_dE} \\
\frac{x : S \vdash_C \Psi_1, t : T_1 \quad y : T_2 \vdash_C \Psi_2}{x : S \vdash_C \Psi_1, \text{mkc}(t, y) : T_1 - T_2, [y(t)/y]\Psi_2} \text{TC\_subI} \\
\frac{x : S \vdash_C \Psi_1, s : T_1 - T_2 \quad y : T_1 \vdash_C t : T_2, \Psi_2}{x : S \vdash_C \Psi_1, \text{postp}(y \rightarrow t, s), [y(s)/y]\Psi_2} \text{TC\_subE} \\
\frac{x : S \vdash_C \Psi_1, t : \mathbb{H}A \quad y : A \vdash_L \cdot; \Psi_2 \quad |\Psi_1| = |\Psi_2|}{x : S \vdash_C \Psi_1 \cdot (\text{let } \mathbb{H}y = t \text{ in } \Psi_2)} \text{TC\_HE} \\
\frac{x : S \vdash_C \Psi \quad s : T' \in \Psi}{x : S \vdash_C \text{connect}_w \text{ to } s : T, \Psi} \text{TC\_WEAK} \\
\frac{x : S \vdash_C t_1 : T, t_2 : T, \Psi}{x : S \vdash_C (t_1 \cdot t_2) : T, \Psi} \text{TC\_CONTR} \\
\frac{x : S \vdash_C \Psi_1, t : T \quad y : T \vdash_C \Psi_2}{x : S \vdash_C \Psi_1, [t/y]\Psi_2} \text{TC\_CUT}
\end{array}$$

$$\boxed{x : A \vdash_L \Delta; \Psi}$$

$$\frac{}{x : A \vdash_L x : A; \cdot} \text{TL\_ID}$$

$$\begin{array}{c}
\frac{x : A \vdash_{\mathbb{L}} \Delta; \Psi \quad e : B \in \Delta}{x : A \vdash_{\mathbb{L}} \Delta, \text{connect}_{\perp} \text{ to } e : \perp; \Psi} \quad \text{TL\_PI} \\
\frac{x : A \vdash_{\mathbb{L}} e : \perp, \Delta; \cdot}{x : A \vdash_{\mathbb{L}} \text{postp}_{\perp} e, \Delta; \cdot} \quad \text{TL\_PE} \\
\frac{x : A \vdash_{\mathbb{L}} \Delta, e_1 : B_1, e_2 : B_2; \Psi}{x : A \vdash_{\mathbb{L}} \Delta, e_1 \oplus e_2 : B_1 \oplus B_2; \Psi} \quad \text{TL\_PARI} \\
\frac{x : A \vdash_{\mathbb{L}} \Delta, e : B_1 \oplus B_2; \Psi \quad y : B_1 \vdash_{\mathbb{L}} \Delta_1; \Psi_1 \quad z : B_2 \vdash_{\mathbb{L}} \Delta_2; \Psi_2}{x : A \vdash_{\mathbb{L}} \Delta, [\text{caseI}(e)/y]\Delta_1, [\text{caseI}(e)/z]\Delta_2; \Psi, [\text{caseI}(e)/y]\Psi_1, [\text{caseI}(e)/z]\Psi_2} \quad \text{TL\_PARE} \\
\frac{x : A \vdash_{\mathbb{L}} \Delta_1, e : B_1; \Psi_1 \quad y : B_2 \vdash_{\mathbb{L}} \Delta_2; \Psi_2}{x : A \vdash_{\mathbb{L}} \text{mkc}(e, y) : B_1 \bullet B_2, \Delta_1, [y(e)/y]\Delta_2; \Psi_1, [y(e)/y]\Psi_2} \quad \text{TL\_SUBI} \\
\frac{x : A \vdash_{\mathbb{L}} \Delta_1, e_1 : B_1 \bullet B_2; \Psi_1 \quad y : B_1 \vdash_{\mathbb{L}} e_2 : B_1, \Delta_2; \Psi_2}{x : A \vdash_{\mathbb{L}} \Delta_1, \text{postp}(y \mapsto e_2, e_1), \Delta_2; \Psi_1, \Psi_2} \quad \text{TL\_SUBE} \\
\frac{x : A \vdash_{\mathbb{L}} \Delta; t : T, \Psi}{x : A \vdash_{\mathbb{L}} \Delta, \text{J } t : \text{J } T; \Psi} \quad \text{TL\_JI} \\
\frac{x : A \vdash_{\mathbb{L}} \Delta, e : \text{J } T; \Psi_1 \quad y : T \vdash_{\mathbb{C}} \Psi_2}{x : A \vdash_{\mathbb{L}} \Delta; \Psi_1, \text{let J } y = e \text{ in } \Psi_2} \quad \text{TL\_JE} \\
\frac{x : A \vdash_{\mathbb{L}} \Delta, e : B; \Psi}{x : A \vdash_{\mathbb{L}} \Delta; \text{H } e : \text{H } B, \Psi} \quad \text{TL\_HI} \\
\frac{x : A \vdash_{\mathbb{L}} \Delta; \Psi_1, t : \text{H } A \quad y : A \vdash_{\mathbb{L}} \cdot; \Psi_2}{x : A \vdash_{\mathbb{L}} \Delta; \Psi_1, \text{let H } y = t \text{ in } \Psi_2} \quad \text{TL\_HE} \\
\frac{x : A \vdash_{\mathbb{L}} \Delta; \Psi \quad r : T' \in \Psi \text{ or } r : B \in \Delta}{x : A \vdash_{\mathbb{L}} \Delta; \text{connect}_{\Psi} \text{ to } r : T, \Psi} \quad \text{TL\_WEAK} \\
\frac{x : A \vdash_{\mathbb{L}} \Delta; t_1 : T, t_2 : T, \Psi}{x : A \vdash_{\mathbb{L}} \Delta; t_1 \cdot t_2 : T, \Psi} \quad \text{TL\_CONTR} \\
\frac{x : A \vdash_{\mathbb{L}} \Delta; \Psi_1, t : T \quad y : T \vdash_{\mathbb{C}} \Psi_2}{x : A \vdash_{\mathbb{L}} \Delta; \Psi_1, [t/y]\Psi_2} \quad \text{TL\_CCUT} \\
\frac{x : A \vdash_{\mathbb{L}} \Delta_1, e : B; \Psi_1 \quad y : B \vdash_{\mathbb{L}} \Delta_2; \Psi_2}{x : A \vdash_{\mathbb{L}} \Delta_1, [e/y]\Delta_2; \Psi_1, [e/y]\Psi_2} \quad \text{TL\_CUT}
\end{array}$$