A COINTUITIONISTIC ADJOINT LOGIC

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Abstract.

1. Introduction

Bi-intuitionistic logic (BINT) is a conservative extension of intuitionistic logic with perfect duality. That is, BINT contains the usual intuitionistic logical connectives such as true, conjunction, and implication, but also their duals false, disjunction, and coimplication. One leading question with respect to BINT is, what does BINT look like across the three arcs – logic, typed λ-calculi, and category theory – of the Curry-Howard-Lambek correspondence? A non-trivial (does not degenerate to a poset) categorical model of BINT is currently an open problem. This paper directly contributes to the solution of this open problem by giving a new categorical model based on adjunctions for cointuitionistic logic, and then proposing a new categorical model for BINT.

BINT can be seen as a mixing of two worlds: the first being intuitionistic logic (IL), which is modeled categorically by a cartesian closed category (CCC), and the second being the dual to intuitionistic logic called cointuitionistic logic (coIL), which is modeled by a cocartesian coclosed category (coCCC). Crolard [6] showed that combining these two categories into the same category results in it degenerating to a poset, i.e. there is at most one morphism between any two objects; we review this result in Section 2.2. However, this degeneration does not occur when both logics are linear.

We propose that IL and coIL need to be separated, and then mixed in a controlled way using the modalities from linear logic. This separation can be ultimately achieved by an adjoint formalization of bi-intuitionistic logic. This formalization consists of three worlds instead of two: the first is intuitionistic logic, the second is linear bi-intuitionistic (Bi-ILL), and the third is cointuitionistic logic. They are then related via two adjunctions as depicted by the following diagram:

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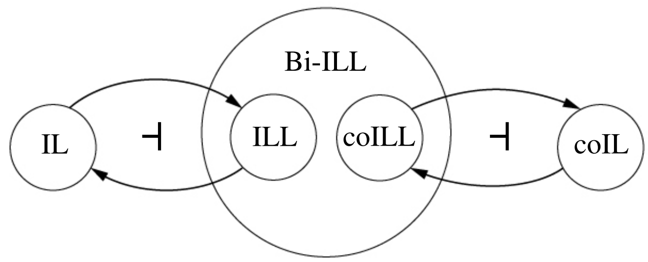

The adjoint between IL and ILL is known as a Linear/Non-linear model (LNL model) of ILL, and is due to Benton [2]. However, the dual to LNL models which would amount to the adjoint between coILL and coIL has yet to appear in the literature.

Suppose $(\mathcal{I}, 1, \times, \rightarrow)$ is a cartesian closed category, and $(\mathcal{L}, \top, \otimes, \multimap)$ is a symmetric monoidal closed category. Then relate these two categories with a symmetric monoidal adjunction $\mathcal{I} : \mathsf{F} \dashv \mathsf{G} : \mathcal{L}$ (Definition 11), where F and G are symmetric monoidal functors. The later point implies that there are natural transformations $\mathsf{m}_{X,Y} : \mathsf{F}X \otimes \mathsf{F}Y \longrightarrow \mathsf{F}(X \times Y)$ and $\mathsf{n}_{A,B} : \mathsf{G}A \times \mathsf{G}B \longrightarrow \mathsf{G}(A \otimes B)$, and maps $\mathsf{m}_\top : \top \longrightarrow \mathsf{F}1$ and $\mathsf{n}_1 : 1 \longrightarrow \mathsf{G}\top$ subject to several coherence conditions; see Definition 7. Furthermore, the functor F is strong which means that $\mathsf{m}_{X,Y}$ and $\mathsf{m}_\top$ are isomorphisms. This setup turns out to be one of the most beautiful models of intuitionistic linear logic called a LNL model due to Benton [2]. In fact, the linear modality of-course can be defined by $!A = \mathsf{F}(\mathsf{G}(A))$ which defines a symmetric monoidal comonad using the adjunction; see Section 2.2 of [2]. This model is much simpler than other known models, and resulted in a logic called LNL logic which supports mixing intuitionistic logic with linear logic. The main contribution of this paper is the definition and study of the dual to Benton's LNL models as models of cointuitionistic logic.

Taking the dual of the previous model results in what we call dual LNL models. They consist of a cocartesian coclosed category, $(C, 0, +, -)$ where $- : C \times C \longrightarrow C$ is left adjoint to the coproduct, a symmetric monoidal coclosed category (Definition 4), $(\mathcal{L}', \perp, \oplus, \bullet\!\!-)$, where $\bullet\!\!- : \mathcal{L}' \times \mathcal{L}' \longrightarrow \mathcal{L}'$ is left adjoint to cotensor (sometimes called parr), and a symmetric comonoidal adjunction (Definition 12) $\mathcal{L}' : \mathsf{H} \dashv \mathsf{J} : C$, where H and J are symmetric comonoidal functors. Dual to the above, this implies that there are natural transformations $\mathsf{m}_{X,Y} : \mathsf{J}(X + Y) \longrightarrow \mathsf{J}X \oplus \mathsf{J}Y$ and $\mathsf{n}_{A,B} : \mathsf{H}(A \oplus B) \longrightarrow \mathsf{H}A + \mathsf{H}B$, and maps $\mathsf{m}_0 : \mathsf{J}0 \longrightarrow \perp$ and $\mathsf{n}_\perp : \mathsf{H} \perp \longrightarrow 0$ subject to several coherence conditions; see Definition 8. In fact, one can define Girard's exponential why-not by $?A = \mathsf{JH}A$, and hence, is the monad induced by the adjunction.

Bellin [1] was the first to propose the dual to Bierman's [3] linear categories which he names dual linear categories as a model of cointuitionistic linear logic. We conduct a similar analysis to that of Benton for dual LNL models by showing that dual LNL models are dual linear categories (Section 2.3.2), and that from a dual linear category we may obtain a dual LNL model (Section 2.3.3). Following this we give the definition of bi-LNL models by combining our dual LNL models with Benton's LNL models to obtain a categorical model of bi-intuitionistic logic (Section 2.4), but we leave its analysis and corresponding logic to a future paper. Following the categorical model we define and analyze a new sequent calculus (Section 3.1), natural deduction formalization (Section 3.2), and a term assignment (Section 3.3) for dual LNL logic.

Introduce dual LNL logic.

2. The Adjoint Model

****short section intro*****

2.1. **Symmetric (co)Monoidal Categories.** We now introduce the necessary definitions related to symmetric monoidal categories that our model will depend on. Most of these definitions are equivalent to the ones given by Benton [2], but we give a lesser known definition of symmetric comonoidal functors due to Bellin [1]. In this section we also introduce distributive categories, the notion of cocloser, and finally, the definition of bilinear categories. The reader may wish to simply skim this section, but refer back to it when they encounter a definition or result they do not know.

Definition 1. A **symmetric monoidal category (SMC)** is a category, $\mathcal{M}$, with the following data:
- An object $\top$ of $\mathcal{M}$,
- A bi-functor $\otimes : \mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{M}$,
- The following natural isomorphisms:

$$\begin{aligned} \lambda_A &: \top \otimes A \longrightarrow A \\ \rho_A &: A \otimes \top \longrightarrow A \\ \alpha_{A,B,C} &: (A \otimes B) \otimes C \longrightarrow A \otimes (B \otimes C) \end{aligned}$$

- A symmetry natural transformation:

$$\beta_{A,B} : A \otimes B \longrightarrow B \otimes A$$

- Subject to the following coherence diagrams:

$$\begin{tikzcd}
((A \otimes B) \otimes C) \otimes D \arrow[r, "\alpha_{A,B,C} \otimes \mathsf{id}_D"] \arrow[d, "\alpha_{A \otimes B,C,D}"] & (A \otimes (B \otimes C)) \otimes D \arrow[dd, "\alpha_{A,B \otimes C,D}"] \\
(A \otimes B) \otimes (C \otimes D) \arrow[d, "\alpha_{A,B,C \otimes D}"] & \\
A \otimes (B \otimes (C \otimes D)) & A \otimes ((B \otimes C) \otimes D) \arrow[l, "\mathsf{id}_A \otimes \alpha_{B,C,D}"']
\end{tikzcd}$$

$$\begin{tikzcd}
(A \otimes B) \otimes C \arrow[r, "\alpha_{A,B,C}"] \arrow[d, "\beta_{A,B} \otimes \mathsf{id}_C"] & A \otimes (B \otimes C) \arrow[r, "\beta_{A,B \otimes C}"] & (B \otimes C) \otimes A \arrow[d, "\alpha_{B,C,A}"] \\
(B \otimes A) \otimes C \arrow[r, "\alpha_{B,A,C}"] & B \otimes (A \otimes C) \arrow[r, "\mathsf{id}_B \otimes \beta_{A,C}"] & B \otimes (C \otimes A)
\end{tikzcd}$$

$$
\begin{tikzcd}
(A \otimes \top) \otimes B \arrow[rr, "\alpha_{A,\top,B}"] \arrow[dr, "\rho_A"'] & & A \otimes (\top \otimes B) \arrow[dl, "\lambda_B"] \\
& A \otimes B &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes B \arrow[d, "\beta_{A,B}"'] \arrow[dr, "\mathsf{id}_{A\otimes B}"] & \\
B \otimes A \arrow[r, "\beta_{B,A}"'] & A \otimes B
\end{tikzcd}
$$

$$
\begin{tikzcd}
\top \otimes A \arrow[rr, "\beta_{\top,A}"] \arrow[dr, "\lambda_A"'] & & A \otimes \top \arrow[dl, "\rho_A"] \\
& A &
\end{tikzcd}
$$

Categorical modeling implication requires that the model be closed; which can be seen as an internalization of the notion of a morphism.

Definition 2. A **symmetric monoidal closed category (SMCC)** is a symmetric monoidal category, $(\mathcal{M}, \top, \otimes)$, such that, for any object B of $\mathcal{M}$, the functor $- \otimes B : \mathcal{M} \longrightarrow \mathcal{M}$ has a specified right adjoint. Hence, for any objects A and C of $\mathcal{M}$ there is an object $B \multimap C$ of $\mathcal{M}$ and a natural bijection:

$$\mathsf{Hom}_{\mathcal{M}}(A \otimes B, C) \cong \mathsf{Hom}_{\mathcal{M}}(A, B \multimap C)$$

We call the functor $\multimap : \mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{M}$ the internal hom of $\mathcal{M}$.

Symmetric monoidal closed categories can be seen as a model of intuitionistic linear logic with a tensor product and implication. What happens when we take the dual? First, we have the following result:

Lemma 3 (Dual of Symmetric Monoidal Categories)**.** If $(\mathcal{M}, \top, \otimes)$ is a symmetric monoidal category, then $\mathcal{M}^{\mathsf{op}}$ is also a symmetric monoidal category.

The previous result follows from the fact that the structures making up symmetric monoidal categories are isomorphisms, and so naturally taking their opposite will yield another symmetric monoidal category. To emphasize when we are thinking about a symmetric monoidal category in the opposite we use the notation $(\mathcal{M}, \perp, \oplus)$ which gives the suggestion of $\oplus$ corresponding to a disjunctive tensor product which we call the *cotensor* of $\mathcal{M}$. The next definition describes when a symmetric monoidal category is coclosed.

Definition 4. A **symmetric monoidal coclosed category (SMCCC)** is a symmetric monoidal category, $(\mathcal{M}, \perp, \oplus)$, such that, for any object B of $\mathcal{M}$, the functor $- \oplus B : \mathcal{M} \longrightarrow \mathcal{M}$ has a specified left adjoint. Hence, for any objects A and C of $\mathcal{M}$ there is an object $C \mathrel{\bullet\!\!-} B$ of $\mathcal{M}$ and a natural bijection:

$$\mathsf{Hom}_{\mathcal{M}}(C, A \oplus B) \cong \mathsf{Hom}_{\mathcal{M}}(C \mathrel{\bullet\!\!-} B, A)$$

We call the functor $\mathrel{\bullet\!\!-} : \mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{M}$ the internal cohom of $\mathcal{M}$.

We combine a symmetric monoidal closed category with a symmetric monoidal coclosed category in a single category. First, we define the notion of a distributive category due to Cockett and Seely [5].

Definition 5. We call a symmetric monoidal category, $(\mathcal{M}, \top, \otimes, \perp, \oplus)$ equipped with the structure of a cotensor $(\mathcal{M}, \perp, \oplus)$, a **distributive category** if there are natural transformations:

$$
\begin{array}{l}
\delta^L_{A,B,C} : A \otimes (B \oplus C) \longrightarrow (A \otimes B) \oplus C \\
\delta^R_{A,B,C} : (B \oplus C) \otimes A \longrightarrow B \oplus (C \otimes A)
\end{array}
$$

subject to several coherence diagrams. Due to the large number of coherence diagrams we do not list them here, but they all can be found in Cockett and Seely's paper [5].

Requiring that the tensor and cotensor products have the corresponding right and left adjoints results in the following definition.

Definition 6. A **bilinear category** is a distributive category $(\mathcal{M}, \top, \otimes, \perp, \oplus)$ such that $(\mathcal{M}, \top, \otimes)$ is closed, and $(\mathcal{M}, \perp, \oplus)$ is coclosed. We will denote bi-linear categories by $(\mathcal{M}, \top, \otimes, \multimap, \perp, \oplus, \bullet\!\!-)$.

Originally, Lambek defined bilinear categories to be similar to the previous definition, but the tensor and cotensor were non-commutative [4], however, the bilinear categories given here are. We retain the name in homage to his original work. As we will see below bilinear categories form the core of a categorical model for bi-intuitionism.

A symmetric monoidal category is a category with additional structure subject to several coherence diagrams. Thus, an ordinary functor is not enough to capture this structure, and hence, the introduction of symmetric monoidal functors.

Definition 7. Suppose we are given two symmetric monoidal categories $(\mathcal{M}_1, \top_1, \otimes_1, \alpha_1, \lambda_1, \rho_1, \beta_1)$ and $(\mathcal{M}_2, \top_2, \otimes_2, \alpha_2, \lambda_2, \rho_2, \beta_2)$. Then a **symmetric monoidal functor** is a functor $F : \mathcal{M}_1 \longrightarrow \mathcal{M}_2$, a map $m_{\top_1} : \top_2 \longrightarrow F\top_1$ and a natural transformation $m_{A,B} : FA \otimes_2 FB \longrightarrow F(A \otimes_1 B)$ subject to the following coherence conditions:

$$\begin{tikzcd}
(FA \otimes_2 FB) \otimes_2 FC \arrow[r, "\alpha_{2FA,FB,FC}"] \arrow[d, "m_{A,B} \otimes \mathsf{id}_{FC}"] & FA \otimes_2 (FB \otimes_2 FC) \arrow[d, "\mathsf{id}_{FA} \otimes m_{B,C}"] \\
F(A \otimes_1 B) \otimes_2 FC \arrow[d, "m_{A \otimes_1 B,C}"] & FA \otimes_2 F(B \otimes_1 C) \arrow[d, "m_{A,B \otimes_1 C}"] \\
F((A \otimes_1 B) \otimes_1 C) \arrow[r, "F\alpha_{1A,B,C}"] & F(A \otimes_1 (B \otimes_1 C))
\end{tikzcd}$$

$$\begin{tikzcd}
\top_2 \otimes_2 FA \arrow[r, "\lambda_{2FA}"] \arrow[d, "m_{\top_1} \otimes \mathsf{id}_{FA}"] & FA \\
F\top_1 \otimes_2 FA \arrow[r, "m_{\top_1,A}"] & F(\top_1 \otimes_1 A) \arrow[u, "F\lambda_{1A}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
FA \otimes_2 \top_2 \arrow[r, "\rho_{2FA}"] \arrow[d, "\mathsf{id}_{FA} \otimes m_{\top_1}"] & FA \\
FA \otimes_2 F\top_1 \arrow[r, "m_{A,\top_1}"] & F(A \otimes_1 \top_1) \arrow[u, "F\rho_{1A}"]
\end{tikzcd}$$

$$\begin{tikzcd}
FA \otimes_2 FB \arrow[r, "\beta_{2FA,FB}"] \arrow[d, "m_{A,B}"] & FB \otimes_2 FA \arrow[d, "m_{B,A}"] \\
F(A \otimes_1 B) \arrow[r, "F\beta_{1A,B}"] & F(B \otimes_1 A)
\end{tikzcd}$$

The following is dual to the previous definition.

Definition 8. Suppose we are given two symmetric monoidal categories $(\mathcal{M}_1, \perp_1, \oplus_1, \alpha_1, \lambda_1, \rho_1, \beta_1)$ and $(\mathcal{M}_2, \perp_2, \oplus_2, \alpha_2, \lambda_2, \rho_2, \beta_2)$. Then a **symmetric comonoidal functor**

is a functor $F : \mathcal{M}_1 \longrightarrow \mathcal{M}_2$, a map $m_{\perp_1} : F\ \perp_1 \longrightarrow \perp_2$ and a natural transformation $m_{A,B} : F(A \oplus_1 B) \longrightarrow FA \oplus_2 FB$ subject to the following coherence conditions:

$$\begin{tikzcd}
F((A \oplus_1 B) \oplus_1 C) \arrow[r, "m_{A\oplus_1 B,C}"] \arrow[d, "F\alpha_{A,B,C}"'] & F(A \oplus_1 B) \oplus_2 FC \arrow[d, "m_{A,B}\oplus_2 \mathsf{id}_{FC}"] \\
F(A \oplus_1 (B \oplus_1 C)) \arrow[d, "m_{A,B\oplus_1 C}"'] & (FA \oplus_2 FB) \oplus_2 FC \arrow[d, "\alpha_{FA,FB,FC}"] \\
FA \oplus_2 F(B \oplus_1 C)) \arrow[r, "\mathsf{id}_{FA}\oplus_2 m_{B,C}"] & FA \oplus_2 (FB \oplus_2 FC)
\end{tikzcd}$$

$$\begin{tikzcd}
F(\perp_1 \oplus_1 A) \arrow[r, "m_{\perp_1,A}"] \arrow[d, "F\lambda_{1A}"'] & F\ \perp_1 \oplus_2 FA \arrow[d, "m_{\perp_1}\oplus \mathsf{id}_{FA}"] \\
FA \arrow[r, "\lambda_2^{-1}{}_{FA}"] & \perp_2 \oplus_2 FA
\end{tikzcd}
\qquad
\begin{tikzcd}
F(A \oplus_1 \perp_1) \arrow[r, "m_{A,\perp_1}"] \arrow[d, "F\rho_{1A}"'] & FA \oplus_2 F\ \perp_1 \arrow[d, "\mathsf{id}_{FA}\oplus m_{\perp_1}"] \\
FA \arrow[r, "\rho_2^{-1}{}_{FA}"] & FA \oplus_2 \perp_2
\end{tikzcd}$$

$$\begin{tikzcd}
F(A \oplus_1 B) \arrow[r, "m_{A,B}"] \arrow[d, "F\beta_{1A,B}"'] & FA \oplus_2 FB \arrow[d, "\beta_{2FA,FB}"] \\
F(B \oplus_1 A) \arrow[r, "m_{B,A}"] & FB \oplus_2 FA
\end{tikzcd}$$

Naturally, since functors are enhanced to handle the additional structure found in a symmetric monoidal category we must also extend natural transformations, and adjunctions.

Definition 9. Suppose $(\mathcal{M}_1, \top_1, \otimes_1)$ and $(\mathcal{M}_2, \top_2, \otimes_2)$ are SMCs, and (F, m) and (G, n) are a symmetric monoidal functors between $\mathcal{M}_1$ and $\mathcal{M}_2$. Then a **symmetric monoidal natural transformation** is a natural transformation, $f : F \longrightarrow G$, subject to the following coherence diagrams:

$$\begin{tikzcd}
FA \otimes_2 FB \arrow[r, "m_{A,B}"] \arrow[d, "f_A \otimes_2 f_B"'] & F(A \otimes_1 B) \arrow[d, "f_{A\otimes_1 B}"] \\
GA \otimes_2 GB \arrow[r, "n_{A,B}"'] & G(A \otimes_1 B)
\end{tikzcd}
\qquad
\begin{tikzcd}
F\top_1 \arrow[rr, "f_{\top_1}"] & & G\top_1 \\
& \top_2 \arrow[ul, "m_{\top_1}"] \arrow[ur, "n_{\top_1}"'] &
\end{tikzcd}$$

Definition 10. Suppose $(\mathcal{M}_1, \perp_1, \oplus_1)$ and $(\mathcal{M}_2, \perp_2, \oplus_2)$ are SMCs, and (F, m) and (G, n) are a symmetric comonoidal functors between $\mathcal{M}_1$ and $\mathcal{M}_2$. Then a **symmetric comonoidal natural transformation** is a natural transformation, $f : F \longrightarrow G$, subject to the following coherence diagrams:

$$\begin{tikzcd}
F(A \oplus_1 B) \arrow[r, "m_{A,B}"] \arrow[d, "f_{A\oplus_1 B}"'] & FA \oplus_2 FB \arrow[d, "f_A \oplus_2 f_B"] \\
G(A \oplus_1 B) \arrow[r, "n_{A,B}"'] & GA \oplus_2 GB
\end{tikzcd}
\qquad
\begin{tikzcd}
\perp_2 & & G\ \perp_1 \arrow[ll, "n_{\perp_1}"'] \\
& F\ \perp_1 \arrow[ul, "m_{\perp_1}"] \arrow[ur, "f_{\perp_1}"'] &
\end{tikzcd}$$

Definition 11. Suppose $(\mathcal{M}_1, \top_1, \otimes_1)$ and $(\mathcal{M}_2, \top_2, \otimes_2)$ are SMCs, and (F, m) is a symmetric monoidal functor between $\mathcal{M}_1$ and $\mathcal{M}_2$ and (G, n) is a symmetric monoidal functor between $\mathcal{M}_2$ and $\mathcal{M}_1$. Then a **symmetric monoidal adjunction** is an ordinary adjunction $\mathcal{M}_1 : F \dashv G : \mathcal{M}_2$ such that the unit, $\eta_A : A \to GFA$, and the counit, $\varepsilon_A : FGA \to A$, are symmetric monoidal natural transformations. Thus, the following diagrams must commute:

$$\begin{tikzcd}
FGA \otimes_2 FGB \arrow[r, "m_{GA,GB}"] \arrow[d, "\varepsilon_A \otimes_1 \varepsilon_B"'] & F(GA \otimes_1 GB) \arrow[d, "Fn_{A,B}"] \\
A \otimes_2 B & FGA \otimes_2 FGB \arrow[l, "\varepsilon_{A \otimes_1 B}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
F\top_1 \arrow[r, "Fn_{\top_2}"] & FG\top_2 \arrow[d, "\varepsilon_{\top_1}"] \\
\top_2 \arrow[u, "m_{\top_1}"] \arrow[r, equal] & \top_2
\end{tikzcd}$$

$$\begin{tikzcd}
GFA \otimes_1 GFB \arrow[d, "n_{FA,FB}"'] & A \otimes_1 B \arrow[l, "\eta_A \otimes_1 \eta_B"'] \arrow[d, "\eta_{A \otimes_1 B}"] \\
G(FA \otimes_2 FB) \arrow[r, "m_{A,B}"] & GF(A \otimes_1 B)
\end{tikzcd}
\qquad
\begin{tikzcd}
G\top_2 \arrow[r, "Gm_{\top_1}"] & GF\top_1 \\
\top_1 \arrow[u, "n_{\top_2}"] \arrow[r, equal] & \top_1 \arrow[u, "\eta_{\top_1}"']
\end{tikzcd}$$

Definition 12. Suppose $(\mathcal{M}_1, \perp_1, \oplus_1)$ and $(\mathcal{M}_2, \perp_2, \oplus_2)$ are SMCs, and (F, m) is a symmetric comonoidal functor between $\mathcal{M}_1$ and $\mathcal{M}_2$ and (G, n) is a symmetric comonoidal functor between $\mathcal{M}_2$ and $\mathcal{M}_1$. Then a **symmetric comonoidal adjunction** is an ordinary adjunction $\mathcal{M}_1 : F \dashv G : \mathcal{M}_2$ such that the unit, $\eta_A : A \to GFA$, and the counit, $\varepsilon_A : FGA \to A$, are symmetric comonoidal natural transformations. Thus, the following diagrams must commute:

$$\begin{tikzcd}
A \oplus_1 B \arrow[r, "\eta_{A \oplus_1 B}"] \arrow[d, "\eta_A \oplus_1 \eta_B"'] & GF(A \oplus_1 B) \arrow[d, "Gm_{A,B}"] \\
GFA \oplus_1 GFB & G(FA \oplus_2 FB) \arrow[l, "m_{FA,FB}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
GF \perp_1 \arrow[r, "Gm_{\perp_1}"] & G \perp_2 \arrow[d, "n_{\perp_2}"] \\
\perp_1 \arrow[u, "\eta_{\perp_1}"] \arrow[r, equal] & \perp_1
\end{tikzcd}$$

$$\begin{tikzcd}
FG(A \oplus_2 B) \arrow[r, "Fn_{A,B}"] \arrow[d, "\varepsilon_{A \oplus_2 B}"'] & F(GA \oplus_1 GB) \arrow[d, "m_{GA,GB}"] \\
A \oplus_2 B & FGA \oplus_2 FGB \arrow[l, "\varepsilon_A \oplus_2 \varepsilon_B"']
\end{tikzcd}
\qquad
\begin{tikzcd}
FG \perp_2 \arrow[r, "\varepsilon_{\perp_2}"] \arrow[d, equal] & \perp_2 \\
FG \perp_2 \arrow[r, "Fn_{\perp_2}"] & F \perp_1 \arrow[u, "m_{\perp_1}"']
\end{tikzcd}$$

We will be defining, and making use of the why-not exponentials from linear logic, but these correspond to a symmetric comonoidal monad. In addition, whenever we have a symmetric comonoidal adjunction, we immediately obtain a symmetric comonoidal comonad on the left, and a symmetric comonoidal monad on the right.

Definition 13. A **symmetric comonoidal monad** on a symmetric monoidal category C is a triple (T, η, μ), where (T, n) is a symmetric comonoidal endofunctor on C, $\eta_A : A \longrightarrow TA$ and $\mu_A : T^2A \to$

TA are symmetric comonoidal natural transformations, which make the following diagrams commute:

$$\begin{tikzcd}
T^3A \arrow[r, "\mu_{TA}"] \arrow[d, "T\mu_A"] & T^2A \arrow[d, "\mu_A"] \\
T^2A \arrow[r, "\mu_A"'] & TA
\end{tikzcd}
\qquad
\begin{tikzcd}
 & TA & \\
TA \arrow[ur, equal] \arrow[r, "\eta_{TA}"'] & T^2A \arrow[u, "\mu_A"] & TA \arrow[ul, equal] \arrow[l, "T\eta_A"]
\end{tikzcd}$$

The assumption that η and μ are symmetric comonoidal natural transformations amount to the following diagrams commuting:

$$\begin{tikzcd}
A \oplus B \arrow[r, "\eta_A \oplus \eta_B"] \arrow[d, "\eta_A"] & TA \oplus TB \\
T(A \oplus B) \arrow[ur, "\mathsf{n}_{A,B}"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\perp \arrow[rr, "\eta_\perp"] \arrow[dr, equal] & & T\perp \arrow[dl, "\mathsf{n}_\perp"] \\
 & \perp &
\end{tikzcd}$$

$$\begin{tikzcd}
T^2(A \oplus B) \arrow[r, "T\mathsf{n}_{A,B}"] \arrow[d, "\mu_{A\oplus B}"] & T(TA \oplus TB) \arrow[r, "\mathsf{n}_{TA,TB}"] & T^2A \oplus T^2B \arrow[d, "\mu_A \oplus \mu_B"] \\
T(A \oplus B) \arrow[rr, "\mathsf{n}_{A,B}"] & & TA \oplus TB
\end{tikzcd}
\quad
\begin{tikzcd}
T^2\perp \arrow[r, "T\mathsf{n}_\perp"] \arrow[d, "\mu_\perp"] & T\perp \arrow[d, "\mathsf{n}_\perp"] \\
T\perp \arrow[r, "\mathsf{n}_\perp"'] & \perp
\end{tikzcd}$$

Finally, the dual concept of a symmetric comonoidal comonad.

Definition 14. A **symmetric comonoidal comonad** on a symmetric monoidal category C is a triple (T, ε, δ), where (T, m) is a symmetric comonoidal endofunctor on C, $\varepsilon_A : TA \longrightarrow A$ and $\delta_A : TA \to T^2A$ are symmetric comonoidal natural transformations, which make the following diagrams commute:

$$\begin{tikzcd}
TA \arrow[r, "\delta_A"] \arrow[d, "\delta_A"] & T^2A \arrow[d, "T\delta_A"] \\
T^2A \arrow[r, "\delta_{TA}"] & T^3A
\end{tikzcd}
\qquad
\begin{tikzcd}
 & TA \arrow[dl, equal] \arrow[d, "\delta_A"] \arrow[dr, equal] & \\
TA & T^2A \arrow[l, "\varepsilon_{TA}"] \arrow[r, "T\varepsilon_A"'] & TA
\end{tikzcd}$$

The assumption that ε and δ are symmetric monoidal natural transformations amount to the following diagrams commuting:

$$\begin{tikzcd}
T(A \oplus B) \arrow[r, "\mathsf{m}_{A,B}"] \arrow[dr, "\varepsilon_{A\oplus B}"'] & TA \oplus TB \arrow[d, "\varepsilon_A \oplus \varepsilon_B"] \\
 & A \oplus B
\end{tikzcd}
\qquad
\begin{tikzcd}
T\perp \arrow[rr, "\varepsilon_\perp"] \arrow[dr, equal] & & \perp \\
 & T\perp \arrow[ur, "\mathsf{m}_\perp"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
T(A \oplus B) \arrow[rr, "\mathsf{m}_{A,B}"] \arrow[d, "\delta_{A\oplus B}"'] & & TA \oplus TB \arrow[d, "\delta_A \oplus \delta_B"] \\
T^2(A \oplus B) \arrow[r, "T\mathsf{m}_{A,B}"'] & T(TA \oplus TB) \arrow[r, "\mathsf{m}_{TA,TB}"'] & T^2A \oplus T^2B
\end{tikzcd}
\qquad
\begin{tikzcd}
T\perp \arrow[r, "\mathsf{m}_\perp"] \arrow[d, "\delta_\perp"'] & \perp \\
T^2\perp \arrow[r, "T\mathsf{m}_\perp"] & T\perp \arrow[u, "\mathsf{m}_\perp"']
\end{tikzcd}$$

2.2. **Cartesian Closed and Cocartesian Coclosed Categories.** The notion of a cartesian closed category is well-known, but for completeness we define them here. However, their dual is lesser known, especially in computer science, and so we given their full definition. We also review some know results concerning cocartesian coclosed categories and categories that are both cartesian closed and cocartesian coclosed.

Definition 15. A **cartesian category** is a category, $(C, 1, \times)$, with an object, 1, and a bi-functor, $\times : C \times C \longrightarrow C$, such that for any object A there is exactly one morphism $\diamond : A \to 1$, and for any morphisms $f : C \longrightarrow A$ and $g : C \longrightarrow B$ there is a morphism $\langle f, g\rangle : C \to A \times B$ subject to the following diagram:

$$\begin{tikzcd}
 & C \arrow[dl, "f"'] \arrow[d, "\langle f,g\rangle"] \arrow[dr, "g"] & \\
A & A \times B \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & B
\end{tikzcd}$$

A cartesian category models conjunction by the product functor, $\times : C \times C \longrightarrow C$, and the unit of conjunction by the terminal object. As we mention above modeling implication requires closer, and since it is well-known that any cartesian category is also a symmetric monoidal category the definition of closer for a cartesian category is the same as the definition of closer for a symmetric monoidal category (Definition 2). We denote the internal hom for cartesian closed categories by $A \to B$.

The dual of a cartesian category is a cocartesian category. They are a model of intuitionistic logic with disjunction and its unit.

Definition 16. A **cocartesian category** is a category, $(C, 0, +)$, with an object, 0, and a bi-functor, $+ : C \times C \longrightarrow C$, such that for any object A there is exactly one morphism $\square : 0 \to A$, and for any morphisms $f : A \longrightarrow C$ and $g : B \longrightarrow C$ there is a morphism $[f, g] : A + B \longrightarrow C$ subject to the following diagram:

$$\begin{tikzcd}
 & C & \\
A \arrow[ur, "f"] \arrow[r, "\iota_1"'] & A + B \arrow[u, "{[f,g]}"] & B \arrow[ul, "g"'] \arrow[l, "\iota_2"]
\end{tikzcd}$$

Cocloser, just like closer for cartesian categories, is defined in the same way that cocloser is defined for symmetric monoidal categories, because cocartesian categories are also symmetric monoidal categories. Thus, a cocartesian category is coclosed if there is a specified left-adjoint, which we denote $S - T$, to the coproduct.

There are many examples of cocartesian coclosed categories. Basically, any interesting cartesian category has an interesting dual, and hence, induces an interesting cocartesian coclosed category. The opposite of the category of sets and functions between them is isomorphic to the category of complete atomic boolean algebras, and both of which, are examples of cocartesian coclosed categories. As we mentioned above bi-linear categories [4] are models of bi-linear logic where the left adjoint to the cotensor models coimplication. Similarly, cocartesian coclosed categories model cointuitionistic logic with disjunction and intuitionistic coimplication [6, 1].

Put more examples in here.

We might now ask if a category can be both cartesian closed and cocartesian coclosed just as bi-linear categories, but this turns out to be where the matter meets antimatter in such away that the category degenerates to a preorder. That is, every homspace contains at most one morphism. We recall this proof here, which is due to Crolard [6]. We need a couple basic facts about cartesian closed categories with initial objects.

Lemma 17. In any cartesian category $\mathcal{C}$, if 0 is an initial object in $\mathcal{C}$ and $\mathsf{Hom}_{\mathcal{C}}(A, 0)$ is non-empty, then $A \cong A \times 0$.

Proof. This follows easily from the universal mapping property for products. $\square$

Lemma 18. In any cartesian closed category $\mathcal{C}$, if 0 is an initial object in $\mathcal{C}$, then so is $0 \times A$ for any object A of $\mathcal{C}$.

Proof. We know that the universal morphism for the initial object is unique, and hence, the homspace $\mathsf{Hom}_{\mathcal{C}}(0, A \Rightarrow B)$ for any object B of $\mathcal{C}$ contains exactly one morphism. Then using the right adjoint to the product functor we know that $\mathsf{Hom}_{\mathcal{C}}(0, A \Rightarrow B) \cong \mathsf{Hom}_{\mathcal{C}}(0 \times A, B)$, and hence, there is only one arrow between $0 \times A$ and B. $\square$

The following lemma is due to Joyal [**?**], and is key to the next theorem.

Lemma 19 (Joyal's)**.** In any cartesian closed category $\mathcal{C}$, if 0 is an initial object in $\mathcal{C}$ and $\mathsf{Hom}_{\mathcal{C}}(A, 0)$ is non-empty, then A is an initial object in $\mathcal{C}$.

Proof. Suppose $\mathcal{C}$ is a cartesian closed category, such that, 0 is an initial object in $\mathcal{C}$, and A is an arbitrary object in $\mathcal{C}$. Furthermore, suppose $\mathsf{Hom}_{\mathcal{C}}(A, 0)$ is non-empty. By the first basic lemma above we know that $A \cong A \times 0$, and by the second $A \times 0$ is initial, thus A is initial. $\square$

Finally, the following theorem shows that any category that is both cartesian closed and cocartesian coclosed is a preorder.

Theorem 20 ((co)Cartesian (co)Closed Categories are Preorders (Crolard[6]))**.** If $\mathcal{C}$ is both cartesian closed and cocartesian coclosed, then for any two objects A and B of $\mathcal{C}$, $\mathsf{Hom}_{\mathcal{C}}(A, B)$ has at most one element.

Proof. Suppose $\mathcal{C}$ is both cartesian closed and cocartesian coclosed, and A and B are objects of $\mathcal{C}$. Then by using the basic fact that the initial object is the unit to the coproduct, and the coproducts left adjoint we know the following:

$$\mathsf{Hom}_{\mathcal{C}}(A, B) \cong \mathsf{Hom}_{\mathcal{C}}(A, 0 + B) \cong \mathsf{Hom}_{\mathcal{C}}(B - A, 0)$$

Therefore, by Joyal's theorem above $\mathsf{Hom}_{\mathcal{C}}(A, B)$ has at most one element. $\square$

Notice that the previous result hinges on the fact that there are initial and terminal objects, and thus, this result does not hold for bi-linear categories, because the units to the tensor and cotensor are not initial nor terminal.

The repercussions of this result are that if we do not want to work with preorders, but do want to work with all of the structure, then we must separate the two worlds. Thus, this result can be seen as the motivation for the current work. We enforce the separation using linear logic, but through the power of linear logic this separation is not far.

2.3. **A Mixed Linear/Non-Linear Model for Co-Intuitionistic Logic.** Benton [2] showed that from a LNL model it is possible to construct a linear category, and vice versa. Bellin [1] showed that the dual to linear categories are sufficient to model co-intuitionistic linear logic. We show that from the dual to a LNL model we can construct the dual to a linear category, and vice versa, thus, carrying out the same program for co-intuitionistic linear logic as Benton did for intuitionistic linear logic.

Combining a symmetric monoidal coclosed category with a cocartesian coclosed category via a symmetric comonoidal adjunction defines a dual LNL model.

Definition 21. A **mixed linear/non-linear model for co-intuitionistic logic (dual LNL model)**, $\mathcal{L} : \mathsf{H} \dashv \mathsf{J} : \mathcal{C}$, consists of the following:

i. a symmetric monoidal coclosed category $(\mathcal{L}, \perp, \oplus, \bullet\!\!-)$,
ii. a cocartesian coclosed category $(\mathcal{C}, 0, +, -)$, and
iv. a symmetric comonoidal adjunction $\mathcal{L} : \mathsf{H} \dashv \mathsf{J} : \mathcal{C}$, where $\eta_A : A \longrightarrow \mathsf{JH}A$ and $\varepsilon_R : \mathsf{HJ}R \longrightarrow R$ are the unit and counit of the adjunction respectively.

It is well-known that an adjunction $\mathcal{L} : \mathsf{H} \dashv \mathsf{J} : \mathcal{C}$ induces a monad $\mathsf{H};\mathsf{J} : \mathcal{L} \longrightarrow \mathcal{L}$, but when the adjunction is symmetric comonoidal we obtain a symmetric comonoidal monad, in fact, $\mathsf{H};\mathsf{J}$ defines the linear exponential why-not denoted $?A = \mathsf{JH}A$. By the definition of dual LNL models we know that both H and J are symmetric comonoidal functors, and hence, are equipped with natural transformations $\mathsf{h}_{A,B} : \mathsf{H}(A \oplus B) \longrightarrow \mathsf{H}A + \mathsf{H}B$ and $\mathsf{j}_{R,S} : \mathsf{J}(R+S) \longrightarrow \mathsf{J}R \oplus \mathsf{J}S$, and maps $\mathsf{h}_\perp : \mathsf{H}\perp \longrightarrow 0$ and $\mathsf{j}_0 : \mathsf{J}0 \longrightarrow \perp$. We will make heavy use of these maps throughout the sequel.

Compare this definition with that of Bellin's dual linear category from [1], and we can easily see that the definition of dual LNL models – much like LNL models – is more succinct.

Definition 22. A **dual linear category**, $\mathcal{L}$, consists of the following data:

i. A symmetric monoidal coclosed category $(\mathcal{L}, \oplus, \perp, \bullet\!\!-)$ with
ii. a symmetric co-monoidal monad $(?, \eta, \mu)$ on $\mathcal{L}$ such that
   a. each free ?-algebra carries naturally the structure of a commutative $\oplus$-monoid. This implies that there are distinguished symmetric monoidal natural transformations $\mathsf{w}_A : \perp \longrightarrow ?A$ and $\mathsf{c}_A : ?A \oplus ?A \longrightarrow ?A$ which form a commutative monoid and are ?-algebra morphisms.
   b. whenever $f : (?A, \mu_A) \longrightarrow (?B, \mu_B)$ is a morphism of free ?-algebras, then it is also a monoid morphism.

2.3.1. *A Useful Isomorphism.* One useful property of Benton's LNL model is that the maps associated with the symmetric monoidal left adjoint in the model are isomorphisms. Since dual LNL models are dual we obtain similar isomorphisms with respect to the right adjoint.

Lemma 23 (Symmetric Comonoidal Isomorphisms)**.** Given any dual LNL model $\mathcal{L} : \mathsf{H} \dashv \mathsf{J} : \mathcal{C}$, then there are the following isomorphisms:

$$\mathsf{J}(R+S) \cong \mathsf{J}R \oplus \mathsf{J}S \quad \text{and} \quad \mathsf{J}0 \cong \perp$$

Furthermore, the former is natural in R and S.

Proof. Suppose $\mathcal{L} : \mathsf{H} \dashv \mathsf{J} : C$ is a dual LNL model. Then we can define the following family of maps:

$$\mathsf{j}^{-1}_{R,S} := \mathsf{J}R \oplus \mathsf{J}S \xrightarrow{\eta} \mathsf{JH}(\mathsf{J}R \oplus \mathsf{J}S) \xrightarrow{\mathsf{Jh}_{A,B}} \mathsf{J}(\mathsf{HJ}R + \mathsf{HJ}S) \xrightarrow{\mathsf{J}(\varepsilon_R + \varepsilon_S)} \mathsf{J}(R + S)$$

$$\mathsf{j}^{-1}_0 := \perp \xrightarrow{\eta} \mathsf{JH}\perp \xrightarrow{\mathsf{Jh}_\perp} \mathsf{J}0$$

It is easy to see that $\mathsf{j}^{-1}_{R,S}$ is natural, because it is defined in terms of a composition of natural transformations. All that is left to be shown is that $\mathsf{j}^{-1}_{R,s}$ and j^{-1}_0 are mutual inverses with $\mathsf{j}_{R,S}$ and j_0; for the details see Appendix A.1. □

Just as Benton we also do not have similar isomorphisms with respect to the functor H. One fact that we can point out, that Benton did not make explicit – because he did not use the notion of symmetric comonoidal functor – is that j^{-1} makes J also a symmetric monoidal functor.

Corollary 24. Given any dual LNL model $\mathcal{L} : \mathsf{H} \dashv \mathsf{J} : C$, the functor $(\mathsf{J}, \mathsf{j}^{-1})$ is symmetric monoidal.

Proof. This holds by straightforwardly reducing the diagrams defining a symmetric monoidal functor, Definition 7, to the diagrams defining a symmetric comonoidal functor, Definition 8, using the fact that j^{-1} is an isomorphism. □

2.3.2. *Dual LNL Model Implies Dual Linear Category.* The next result shows that any dual LNL model induces a symmetric comonoidal monad.

Lemma 25 (Symmetric Comonoidal Monad)**.** Given a dual LNL model $\mathcal{L} : \mathsf{H} \dashv \mathsf{J} : C$, the functor, $? = H; J$, defines a symmetric comonoidal monad.

Proof. Suppose (H, h) and (J, j) are two symmetric comonoidal functors, such that, $\mathcal{L} : \mathsf{H} \dashv \mathsf{J} : C$ is a dual LNL model. We can easily show that $?A = \mathsf{JH}A$ is symmetric monoidal by defining the following maps:

$$\mathsf{r}_\perp := ?\perp = \mathsf{JH}\perp \xrightarrow{\mathsf{Jh}_\perp} \mathsf{J}0 \xrightarrow{\mathsf{j}_\perp} \perp$$

$$\mathsf{r}_{A,B} := ?(A \oplus B) = \mathsf{JH}(A \oplus B) \xrightarrow{\mathsf{Jh}_{A,B}} \mathsf{J}(\mathsf{H}A + \mathsf{H}B) \xrightarrow{\mathsf{j}_{\mathsf{H}A,\mathsf{H}B}} \mathsf{JH}A \oplus \mathsf{JH}B = ?A \oplus ?B$$

The fact that these maps satisfy the appropriate symmetric comonoidal functor diagrams from Definition 8 is obvious, because symmetric comonoidal functors are closed under composition.

We have a dual LNL model, and hence, we have the symmetric comonoidal natural transformations $\eta_A : A \longrightarrow \mathsf{JH}A$ and $\varepsilon_R : \mathsf{HJ}R \longrightarrow R$ which correspond to the unit and counit of the adjunction respectfully. Define $\mu_A := \mathsf{J}\varepsilon_{\mathsf{H}A} : \mathsf{JHJH}A \longrightarrow \mathsf{JH}A$. This implies that we have maps $\eta_A : A \longrightarrow ?A$ and $\mu_A : ??A \longrightarrow ?A$, and thus, we can show that $(?, \eta, \mu)$ is a symmetric comonoidal monad. All the diagrams defining a symmetric comonoidal monad hold by the structure given by the adjunction. For the complete proof see Appendix A.2. □

The monad from the previous result must be equipped with the additional structure to model the right weakening and contraction structural rules.

Lemma 26 (Right Weakening and Contraction)**.** Given a dual LNL model $\mathcal{L} : \mathsf{H} \dashv \mathsf{J} : C$, then for any $?A$ there are distinguished symmetric comonoidal natural transformations $\mathsf{w}_A : \perp \longrightarrow ?A$ and $\mathsf{c}_A : ?A \oplus ?A \longrightarrow ?A$ that form a commutative monoid, and are $?$-algebra morphisms with respect to the canonical definitions of the algebras $?A$, $\perp$, $?A \oplus ?A$.

Proof. Suppose (H, h) and (J, j) are two symmetric comonoidal functors, such that, $\mathcal{L} : \mathsf{H} \dashv \mathsf{J} : C$ is a dual LNL model. Again, we know $?A = H; J : \mathcal{L} \longrightarrow \mathcal{L}$ is a symmetric comonoidal monad by Lemma 25.

We define the following morphisms:

$$\mathsf{w}_A := \perp \xrightarrow{\mathsf{j}_\perp^{-1}} \mathsf{J}0 \xrightarrow{\mathsf{J}\diamond_{\mathsf{H}A}} \mathsf{JH}A = ?A$$

$$\mathsf{c}_A := ?A \oplus ?A = \mathsf{JH}A \oplus \mathsf{JH}A \xrightarrow{\mathsf{j}^{-1}_{\mathsf{H}A,\mathsf{H}A}} \mathsf{J}(\mathsf{H}A + \mathsf{H}A) \xrightarrow{\mathsf{J}\nabla_{\mathsf{H}A}} \mathsf{JHA} = ?A$$

The remainder of the proof is by carefully checking all of the required diagrams. Please see Appendix A.3 for the complete proof. □

Lemma 27 (?-Monoid Morphisms)**.** Suppose $\mathcal{L} : \mathsf{H} \dashv \mathsf{J} : C$ is a dual LNL model. Then if $f : (?A, \mu_A) \longrightarrow (?B, \mu_B)$ is a morphism of free ?-algebras, then it is a monoid morphism.

Proof. Suppose $\mathcal{L} : \mathsf{H} \dashv \mathsf{J} : C$ is a dual LNL model. Then we know $?A = \mathsf{JH}A$ is a symmetric comonoidal monad by Lemma 25. Bellin [1] remarks that by Maietti, Maneggia de Paiva and Ritter's Proposition 25 [7], it suffices to show that $\mu_A : ??A \longrightarrow ?A$ is a monoid morphism. For the details see the complete proof in Appendix B. □

Finally, we may now conclude the following corollary.

Corollary 28. Every dual LNL model is a dual linear category.

2.3.3. *Dual Linear Category implies Dual LNL Model.* This section shows essentially the inverse to the result from the previous section. That is, that from any dual linear category we may construct a dual LNL model. By exploiting the duality between LNL models and dual LNL models this result follows straightforwardly from Benton's result. The proof of this result must first find a symmetric monoid coclosed category, a cocartesian coclosed category, and finally, a symmetric comonoidal adjunction between them. Take the symmetric monoid coclosed category to be an arbitrary dual linear category $\mathcal{L}$. Then we may define the following categories.

- The Eilenberg-Moore category, $\mathcal{L}^?$, has as objects all ?-algebras, $(A, h_A : ?A \longrightarrow A)$, and as morphisms all ?-algebra morphisms.
- The Kleisli category, $\mathcal{L}_?$, is the full subcategory of $\mathcal{L}^?$ of all free ?-algebras $(?A, \mu_A : ??A \longrightarrow ?A)$.

The previous three categories are related by a pair of adjunctions:

$$\begin{tikzcd}
\mathcal{L} \arrow[r, bend left, "F"] \arrow[d, equal] & \mathcal{L}^? \arrow[l, bend left, "U"] \\
\mathcal{L} \arrow[r, bend left, "F"] & \mathcal{L}_? \arrow[l, bend left, "U"] \arrow[u, "i"]
\end{tikzcd}$$

The functor $F(A) = (?\,A, \mu_A)$ is the free functor, and the functor $U(A, h_A) = A$ is the forgetful functor. Note that we, just as Benton did, are overloading the symbols F and U. Lastly, the functor $i : \mathcal{L}_? \longrightarrow \mathcal{L}^?$ is the injection of the subcategory of free ?-algebras into its parent category.

We are now going to show that both $\mathcal{L}^?$ and $\mathcal{L}_?$ are induce two cocartesian coclosed categories. Then we could take either of those when constructing a dual LNL model from a dual linear category. First, we show $C^?$ is cocartesian.

Lemma 29. If $\mathcal{L}$ is a dual linear category, then $\mathcal{L}^?$ has finite coproducts.

Proof. We give a proof sketch of this result, because the proof is essentially by duality of Benton's corresponding proof for LNL models (see Lemma 9, [2]). Suppose $\mathcal{L}$ is a dual linear category. Then we first need to identify the initial object which is defined by the ?-algebra $(\perp, \mathsf{r}_\perp : ?\perp \longrightarrow \perp)$. The unique map between the initial map and any other ?-algebra $(A, h_A : ?\,A \longrightarrow A)$ is defined by $\perp \xrightarrow{\mathsf{w}_A} ?\,A \xrightarrow{h_A} A$. The coproduct of the ?-algebras $(A, h_A : ?\,A \longrightarrow A)$ and $(B, h_B : ?\,B \longrightarrow B)$ is $(A \oplus B, \mathsf{r}_{A,B}; (h_A \oplus h_B))$. Injections and the codiagonal map are defined as follows:

- Injections:

$$\iota_1 := A \xrightarrow{\rho_A} A \oplus \perp \xrightarrow{\mathsf{id}_A \oplus \mathsf{w}_B} A \oplus ?\,B \xrightarrow{\mathsf{id} \oplus h_b} A \oplus B$$
$$\iota_2 := B \xrightarrow{\lambda_A} \perp \oplus B \xrightarrow{\mathsf{w}_A \oplus \mathsf{id}_B} ?\,A \oplus B \xrightarrow{h_A \oplus \mathsf{id}_B} A \oplus B$$

- Codiagonal map:

$$\nabla := A \oplus A \xrightarrow{\eta_A \oplus \eta_A} ?\,A \oplus ?\,A \xrightarrow{\mathsf{c}_A} ?\,A \xrightarrow{h_A} A$$

Showing that these respect the appropriate diagrams is straightforward. □

Notice as a direct consequence of the previous result we know the following.

Corollary 30. The Kleisli category, $\mathcal{L}_?$, has finite coproducts.

Thus, both $\mathcal{L}^?$ and $\mathcal{L}_?$ are cocartesian, but we need a cocartesian coclosed category, and in general these are not coclosed, and so we follow Benton's lead and show that there are actually two subcategories of $\mathcal{L}^?$ that are coclosed.

Definition 31. We call an object, A, of a category, $\mathcal{L}$, **subtractable** if for any object B of $\mathcal{L}$, the internal cohom $A \bullet\!- B$ exists.

We now have the following results:

Lemma 32. In $\mathcal{L}^?$, all the free ?-algebras are subtractable, and the internal cohom is a free ?-algebra.

Proof. The internal cohom is defined as follows:

$$(?\,A, \delta_A) \bullet\!- (B, h_B) := (?(A \bullet\!- B), \delta_{A \bullet\!- B})$$

We can capitalize on the adjunctions involving F and U from above to lift the internal cohom of $\mathcal{L}$ into $\mathcal{L}^?$:

$$\begin{array}{rcl}
\mathsf{Hom}_{\mathcal{L}^?}((?(A \bullet\!- B), \delta_{A \bullet\!- B}), (C, h_C)) & = & \mathsf{Hom}_{\mathcal{L}^?}(\mathsf{F}(A \bullet\!- B), (C, h_C)) \\
& \cong & \mathsf{Hom}_{\mathcal{L}}(A \bullet\!- B, \mathsf{U}(C, h_C)) \\
& = & \mathsf{Hom}_{\mathcal{L}}(A \bullet\!- B, C) \\
& \cong & \mathsf{Hom}_{\mathcal{L}}(A, C \oplus B) \\
& = & \mathsf{Hom}_{\mathcal{L}}(A, \mathsf{U}(C \oplus B, h_{C \oplus B})) \\
& \cong & \mathsf{Hom}_{\mathcal{L}^?}(\mathsf{F}A, (C \oplus B, h_{C \oplus B})) \\
& = & \mathsf{Hom}_{\mathcal{L}^?}((?\,A, \delta_A), (C \oplus B, h_{C \oplus B}))
\end{array}$$

The previous equation holds for any $h_{C\oplus B}$ making $C \oplus B$ a ?-algebra, in particular, the co-product in $\mathcal{L}^?$ (Lemma 29), and hence, we may instantiate the final line of the previous equation with the following:

$$\mathsf{Hom}_{\mathcal{L}^?}((?A, \delta_A), (C, h_c) + (B, \delta_A))$$

Thus, we obtain our result. □

Lemma 33. We have the following cocartesian coclosed categories:

i. The full subcategory, $\mathsf{Sub}(\mathcal{L}^?)$, of $\mathcal{L}^?$ consisting of objects the subtractable ?-algebras is cocartesian coclosed, and contains the Kleisli category.
ii. The full subcategory, $\mathcal{L}^*_?$, of $\mathsf{Sub}(\mathcal{L}^?)$ consisting of finite coproducts of free ?-algebras is cocartesian coclosed.

Let C be either of the previous two categories. Then we must exhibit a adjunction between C and $\mathcal{L}$, but this is easily done.

Lemma 34. The adjunction $\mathcal{L} : \mathsf{F} \vdash \mathsf{U} : C$, with the free functor, F, and the forgetful functor, U, is symmetric comonoidal.

Proof. Showing that F and U are symmetric comonoidal follows similar reasoning to Benton's result, but in the opposite; see Lemma 13 and Lemma 14 of [2]. Lastly, showing that the unit and the counit of the adjunction are comonoidal natural transformations is straightforward, and we leave it to the reader. The reasoning is similar to Benton's, but in the opposite; see Lemma 15 and Lemma 16 of [2]. □

Corollary 35. Any dual linear category gives rise to a dual LNL model.

2.4. **A Mixed Bilinear/Non-Linear Model.** The main goal of our research program is to give a non-trivial categorical model of bi-intuitionistic logic. In this section we give a introduction of the model we have in mind, but leave the details and the study of the logical and programmatic sides to future work.

The naive approach would be to try and define a LNL-style model of bi-intuitionistic logic as an adjunction between a bilinear category and a bi-cartesian bi-closed category, but this results in a few problems. First, should the adjunction be monoidal or comonoidal? Furthermore, we know bi-cartesian bi-closed categories are trivial (Theorem 20), and hence, this model is not very interesting nor incorrect. We must separate the two worlds using two dual adjunctions, and hence, we arrive at the following definition.

Definition 36. A **mixed bilinear/non-linear model** consists of the following:

i. a bilinear category $(\mathcal{L}, \top, \otimes, \multimap, \perp, \oplus, \bullet\!\!-)$,
ii. a cartesian closed category $(\mathcal{I}, 1, \times, \to)$,
iii. a cocartesian coclosed category $(C, 0, +, -)$,
iv. a LNL model $\mathcal{I} : F \dashv G : \mathcal{L}$, and
v. a dual LNL model $\mathcal{L} : H \dashv J : C$.

Since $\mathcal{L}$ is a bilinear category then it is also a linear category, and a dual linear category. Thus, the LNL model intuitively corresponds to an adjunction between $\mathcal{I}$ and the linear subcategory of $\mathcal{L}$, and the dual LNL model corresponds to an adjunction between the dual linear subcategory of $\mathcal{L}$ and C. In addition, both intuitionistic logic and cointuitionistic logic can be embedded into $\mathcal{L}$ via the linear modalities of-course, $!A$, and why-not, $?A$, using the well-known Girard embeddings. This

C identity

$$S \vdash_C S$$

CC Cut

$$\frac{S \vdash_C \Psi_1, T \quad T \vdash_C \Psi_2}{S \vdash_C \Psi_1, \Psi_2}$$

C exchange *C weakening* *C contraction*

$$\frac{S \vdash_C \Psi_1, S, T, \Psi_2}{S \vdash_C \Psi_1, T, S, \Psi_2} \qquad \frac{S \vdash_C \Psi}{S \vdash_C T, \Psi} \qquad \frac{S \vdash_C T, T, \Psi}{S \vdash_C T, \Psi}$$

C zero

$$0 \vdash_C \Psi$$

C subtraction R *C subtraction L*

$$\frac{S \vdash_C \Psi_1, T_1 \qquad T_2 \vdash_C \Psi_2}{S \vdash_C \Psi_1, \Psi_2, T_1 - T_2} \qquad \frac{T_1 \vdash_C T_2, \Psi}{T_1 - T_2 \vdash_C \Psi}$$

C disjunction R_1 *C disjunction R_2*

$$\frac{S \vdash_C \Psi, T_1}{S \vdash_C \Psi, T_1 + T_2} \qquad \frac{S \vdash_C \Psi, T_2}{S \vdash_C \Psi, T_1 + T_2}$$

C disjunction L

$$\frac{T_1 \vdash_C \Psi_1 \qquad T_2 \vdash_C \Psi_2}{T_1 + T_2 \vdash_C \Psi_1, \Psi_2}$$

H rules

H R *H L*

$$\frac{A \vdash_L, \Delta, B; \Psi}{A \vdash_L, \Delta; HB, \Psi} \qquad \frac{A \vdash_L; \Psi}{HA \vdash_C \Psi_2}$$

Table 1: Non-linear **co-LNL**

implies that we have a very controlled way of mixing $\mathcal{I}$ and $\mathcal{C}$ within $\mathcal{L}$, and hence, linear logic is the key.

3. Dual LNL Logic

?

3.1. Sequent Calculus. We follow Benton almost verbatim in our sequent calculus for **co-LNL**. We use n-ary cuts:

$$\frac{T \vdash_C \Psi, S^n \qquad S \vdash_C \Psi'}{T \vdash_C \Psi, \Psi'}\ \text{CC-cut}_n \qquad \frac{A \vdash_L \Delta; \Psi, S^n \qquad S \vdash_C \Psi'}{A \vdash_C \Delta; \Psi, \Psi'}\ \text{LC-cut}_n$$

L identity

$$A \vdash_L A$$

LL Cut / *LC Cut*

$$\frac{A \vdash_L \Delta_1, B; \Psi_1 \quad B \vdash_L \Delta_2; \Psi_2}{A \vdash_L \Delta_1, \Delta_2; \Psi_1, \Psi_2} \qquad \frac{A \vdash_L \Delta; \Psi_1, T \quad T \vdash_C \Psi_2}{A \vdash_L \Delta; \Psi_1, \Psi_2}$$

LL exchange / *LC exchange*

$$\frac{A \vdash_L \Delta_1, B_1, B_2, \Delta_2; \Psi}{A \vdash_L \Delta_1, B_2, B_1, \Delta_2; \Psi} \qquad \frac{A \vdash_L \Delta; \Psi_1, S, T, \Psi_2}{A \vdash_L \Delta; \Psi_1, T, S, \Psi_2}$$

L weakening / *L contraction*

$$\frac{A \vdash_L \Delta; \Psi}{A \vdash_L \Delta; T, \Psi} \qquad \frac{A \vdash_L \Delta; T, T, \Psi}{A \vdash_L \Delta; T, \Psi}$$

⊥ R / ⊥ L

$$\frac{A \vdash_L \Delta; \Psi}{A \vdash_L \bot, \Delta; \Psi} \qquad \bot \vdash_L$$

L subtraction R / *L subtraction L*

$$\frac{A \vdash_L \Delta_1; \Psi_1, B_1 \qquad B_2 \vdash_L \Delta_2; \Psi_2}{A \vdash_L B_1 \bullet\!\!- B_2 \Delta_1, \Delta_2; \Psi_1, \Psi_2} \qquad \frac{B_1 \vdash_L B_2, \Delta; \Psi}{B_1 \bullet\!\!- B_2 \vdash_L \Delta; \Psi}$$

L disjunction R / *L disjunction L_2*

$$\frac{A \vdash_L \Delta, B_1, B_2; \Psi}{A \vdash_L \Delta, B_1 \oplus B_2; \Psi} \qquad \frac{B_1 \vdash_L \Delta_1; \Psi_1 \qquad B_2 \vdash_L \Delta_2; \Psi_2}{B_1 \oplus B_2 \vdash_L \Delta_1, \Delta_2; \Psi_1, \Psi_2}$$

J rules

J R / *J L*

$$\frac{A \vdash_L, \Delta; T, \Psi}{A \vdash_L, \Delta, JT; \Psi} \qquad \frac{T \vdash_C \Psi}{JT \vdash_L; \Psi}$$

Table 2: Linear **co-LNL**

where $S^n = S, \ldots, S$ n-times. We call $\mathbf{co-LNL^+}$ the system **co-LNL** with n-cuts replacing ordinary 1-cuts. Such cuts are admissible in **co-LNL** and cut-elimination for $\mathbf{co-LNL^+}$ implies cut-elimination for **co-LNL**.

We have the familiar definitions:

- the *rank* $|A|$ or $|S|$ of a formula A or S is the number of the logical symbols in it.
- the *cut-rank* $c(\Pi)$ of a proof Π is the maximum of the ranks of the cut formulas in Π plus one; if Π is cut-free its cut rank is 0.
- the *depth* $d(\Pi)$ of a proof Π is the length of the longest path in the proof tree.

Lemma 37.

(**Cut Reduction**)

(1) If Π_1 is a derivation of $T \vdash_C \Psi, S^n$ and Π_2 is a derivation of $S \vdash_C \Psi'$ with $c(\Pi_1), c(\Pi_2) \leq |S|$, then there exists a derivation Π of $T \vdash_C \Psi, \Psi'$ with $c(\Pi) \leq |S|$;

(2) If Π_1 is a derivation of $T \vdash_L \Delta; \Psi, S^n$ and Π_2 is a derivation of $S \vdash_C \Psi'$ with $c(\Pi_1), c(\Pi_2) \leq |S|$, then there exists a derivation Π of $T \vdash_L \Delta; \Psi, \Psi'$ with $c(\Pi) \leq |S|$;
(3) If Π_1 is a derivation of $B \vdash_L \Delta; \Psi, A^n$ and Π_2 is a derivation of $A \vdash_L \Delta', \Psi'$ with $c(\Pi_1), c(\Pi_2) \leq |S|$, then there exists a derivation Π of $B \vdash_C \Delta, \Delta', \Psi, \Psi'$ with $c(\Pi) \leq |A|$.

Proof. By induction on $d(\Pi_1) + d(\Pi_2)$.
We consider only the case where the last inferences of Π_1 and Π_2 are logical inferences. The other cases are handled mainly by permutation of inferences and use of the inductive hypothesis; we refer to Benton's text for them.

J right / J left. We have

$$\Pi_1 = \cfrac{\begin{matrix}\pi_1 \\ A \vdash_L \Delta, JT^n; T, \Psi\end{matrix}}{A \vdash_L \Delta, JT^{n+1}; \Psi}\, JR \qquad \Pi_2 = \cfrac{\begin{matrix}\pi_2 \\ T \vdash_C \Psi'\end{matrix}}{JT \vdash_L; \Delta, JT^{n+1}, \Psi'}\, JL$$

By the inductive hypothesis appled to Π_2 and π_1 there exists a proof Π' of $A \vdash_L \Delta; T, \Psi, \Psi'$ with $c(\Pi') \leq |JT| = |T| + 1$. Then the following derivation

$$\Pi_1 = \cfrac{\cfrac{\begin{matrix}\Pi' \\ A \vdash_L \Delta; T, \Psi, \Psi'\end{matrix} \qquad \begin{matrix}\pi_2 \\ T \vdash_C \Psi'\end{matrix}}{A \vdash_L \Delta, \Psi, \Psi', \Psi'}\, cut}{A \vdash_L \Delta, \Psi, \Psi'}\, contractions$$

has cut rank $max(|T| + 1, c(\Pi'), c(\pi_2)) = |T| + 1 = |JT|$.

H right / H left. We have

$$\Pi_1 = \cfrac{\begin{matrix}\pi_1 \\ B \vdash_L \Delta, A; HA^n; \Psi\end{matrix}}{B \vdash_L \Delta; HA^{n+1}, \Psi}\, HR \qquad \Pi_2 = \cfrac{\begin{matrix}\pi_2 \\ A \vdash_L; \Psi'\end{matrix}}{HA \vdash_C; \Psi'}\, HL$$

By the inductive hypothesis applied to Π_2 and π_1 there exists a proof Π' of $B \vdash_L \Delta; A, \Psi, \Psi'$ with $c(\Pi') \leq |HA| = |A| + 1$. Then the following derivation

$$\Pi_1 = \cfrac{\cfrac{\begin{matrix}\Pi' \\ B \vdash_L \Delta; A, \Psi, \Psi'\end{matrix} \qquad \begin{matrix}\pi_2 \\ A \vdash_L; \Psi'\end{matrix}}{B \vdash_L \Delta, \Psi, \Psi', \Psi'}\, cut}{B \vdash_L \Delta, \Psi, \Psi'}\, contractions$$

has cut rank $max(|A| + 1, c(\Pi'), c(\pi_2)) = |A| + 1 = |HA|$.

+ right$_1$ / + left. We have

$$\Pi_1 = \cfrac{\begin{matrix}\pi_1 \\ S \vdash_C T_1, (T_1 + T_2)^n, \Psi\end{matrix}}{S \vdash_C (T_1 + T_2)^{n+1}, \Psi}\, +R \qquad \Pi_2 = \cfrac{\begin{matrix}\pi_2 \\ T_1 \vdash_C \Psi_1\end{matrix} \qquad \begin{matrix}\pi_3 \\ T_2 \vdash_C \Psi_2\end{matrix}}{T_1 + T_2 \vdash_C; \Psi_1, \Psi_2}\, +L$$

If $n = 0$, then the reduction is as follows:

$$\cfrac{\Pi_1 = \cfrac{\begin{matrix}\pi_1 \\ S \vdash_C T_1, \Psi\end{matrix}}{S \vdash_C T_1 + T_2, \Psi}\, +R \quad \Pi_2 = \cfrac{\begin{matrix}\pi_2 \\ T_1 \vdash_C \Psi_1\end{matrix} \qquad \begin{matrix}\pi_3 \\ T_2 \vdash_C \Psi_2\end{matrix}}{T_1 + T_2 \vdash_C \Psi_1, \Psi_2}\, +L}{S \vdash_C \Psi, \Psi_1, \Psi_2}\, cut$$

reduces to

$$\Pi\ \cfrac{\cfrac{\begin{matrix}\pi_1 \\ S \vdash_C T_1, \Psi\end{matrix} \qquad \begin{matrix}\pi_2 \\ T_1 \vdash_C \Psi_1\end{matrix}}{S \vdash_C \Psi, \Psi_1}\, cut}{S \vdash_C \Psi, \Psi_1, \Psi_2}\, weakening$$

Here $c(\Pi) = max(|T_1 + 1|, c(\pi_1), c(\pi_2)) \leq |T_1 + T_2|$.

If $n > 0$, then by the inductive hypothesis applied to Π_2 and π_1 there exists a proof Π' of $S \vdash_C T_1, \Psi, \Psi_1, \Psi_2$ with $c(\Pi') \leq |T_1 + T_2| = |T_1| + |T_2| + 1$. Then the following derivation

$$\Pi = \cfrac{\cfrac{\cfrac{\overset{\Pi'}{S \vdash_C T_1, \Psi, \Psi_1, \Psi_2} \qquad \overset{\pi_2}{T_1 \vdash_C \Psi_1}}{S \vdash_C \Psi, \Psi_1, \Psi_1, \Psi_2}\, cut}{}}{S \vdash_C \Psi, \Psi_1, \Psi_2}\, contractions$$

has cut rank $max(|T_1| + 1, c(\Pi'), c(\pi_2)) \leq |T_1 + T_2|$.

$\bullet\!\!-$ right / $\bullet\!\!-$ left. We have

$$\cfrac{\Pi_1 = \cfrac{\overset{\pi_1}{A \vdash_L \Delta_1; \Psi_1, B_1} \qquad \overset{\pi_2}{B_2 \vdash_L \Delta_2; \Psi_2}}{A \vdash_L B_1 \bullet\!\!- B_2, \Delta_1, \Delta_2; \Psi_1, \Psi_2}\, \bullet\!\!- R \qquad \Pi_2 = \cfrac{\overset{\pi_3}{B_1 \vdash_L B_2, \Delta; \Psi}}{B_1 \bullet\!\!- B_2 \vdash_L \Delta; \Psi}\, \bullet\!\!- L}{A \vdash_L \Delta_1, \Delta_2, \Delta; \Psi_1, \Psi_2, \Psi}\, cut$$

reduces to Π

$$\cfrac{\cfrac{\overset{\pi_1}{A \vdash_L \Delta_1, B_1; \Psi_1} \qquad \overset{\pi_3}{B_1 \vdash_L B_2, \Delta; \Psi}}{A \vdash_L \Delta_1, \Delta, B_2; \Psi_1, \Psi}\, cut_1 \qquad \overset{\pi_2}{B_2 \vdash_L \Delta_2; \Psi_2}}{A \vdash_L \Delta_1, \Delta_2, \Delta; \Psi_1, \Psi_2, \Psi}\, cut_2$$

The resulting derivation Π has cut rank $c(\Pi) = max(|B_1| + 1, c(\pi_1), c(\pi_2), |B_2| + 1, c(\pi_3)) \leq |B_1 \bullet\!\!- B_2|$.

Lemma 38. Let Π be a **co-LNL**$^+$ proof of a sequent $S \vdash_C \Psi$ or $A \vdash_L \Delta; \Psi$ with $c(\Pi) > 0$. Then there exists a proof Π' of the same sequent with $c(\Pi') < c(\Pi)$.

Proof. By induction on $d(\Pi)$. If the last inference is not a cut, then we apply the inductive hypothesis. If it the last inference is a cut on a formula A, but A is not of maximal rank among the cut formulas, so that $c(\Pi) > |A| + 1$, then we apply the induction hypothesis. Finally, if the last inference is a cut on A and $c(\Pi) = |A| + 1$ we have

$$\Pi = \cfrac{\overset{\Pi_1}{B \vdash_L \Delta, A; \Psi} \qquad \overset{\Pi_2}{A \vdash_L \Delta'; \Psi'}}{B \vdash_L \Delta, \Delta'; \Psi, \Psi'}\, LL\ cut$$

Since $c(\Pi_1), c(\Pi_2) \leq |A| + 1$ by induction we can construct derivations Π_1', Π_2' of the sequent premises of the cut with $c(\Pi_1'), c(\Pi_2') \leq |A|$. Then by the Cut-reduction Lemma we can construct a Π' proving $B \vdash_L \Delta, \Delta'; \Psi, \Psi'$ with $c(\Pi') \leq |A|$ as required.

Theorem 39. Let Π be a proof of a sequent $S \vdash_C \Psi$ or $A \vdash_L \Delta; \Psi$ such that $c(\Pi) > 0$. There is an algorithm which yields a cut free proof of the same sequent.

Proof By induction on $c(\Pi)$ using the previous lemma.

Remark 40. Contexts are treated multiplicatively. Some cases would be best treated with additive contexts, for instance non-linear disjunction elimination to match the categorical interpretation of disjunction as coproduct. Of course additive contexts can be simulated using weakening and contraction.

additive disjunction elimination

$$\cfrac{T_1 \vdash_C \Psi \qquad T_2 \vdash_C \Psi}{T_1 + T_2 \vdash \Psi}$$

C assumption
$$S \vdash_C S$$

C zero
$$\frac{S \vdash_C 0, \Psi \qquad S_1 \vdash_C \Psi_1 \quad \ldots \quad S_n \vdash \Psi_n}{S \vdash_C \Psi, \Psi_1 \quad \ldots \quad \Psi_n}$$

C subtraction intro
$$\frac{S \vdash_C \Psi_1, T_1 \qquad T_2 \vdash_C \Psi_2}{S \vdash_C \Psi_1, \Psi_2, T_1 - T_2}$$

C subtraction elim
$$\frac{S \vdash_C \Psi_1, T_1 - T_2 \qquad T_1 \vdash_C T_2, \Psi_2}{S \vdash_C \Psi_1, \Psi_2}$$

C disjunction intro$_1$
$$\frac{S \vdash_C \Psi, T_1}{S \vdash_C \Psi, T_1 + T_2}$$

C disjunction intro$_2$
$$\frac{S \vdash_C \Psi, T_2}{S \vdash_C \Psi, T_1 + T_2}$$

C disjunction elim
$$\frac{S \vdash_C \Psi, T_1 + T_2 \qquad T_1 \vdash_C \Psi' \quad T_2 \vdash_C \Psi'}{S \vdash_C \Psi, \Psi'}$$

H elim
$$\frac{S \vdash_C \Psi, HA \quad A \vdash_L ; \Psi}{S \vdash_C \Psi}$$

Table 3: Natural deduction for non-linear *co-ND*

3.2. **Sequent-style Natural Deduction.** In Tables 3,4 we give the rules of a sequent-style natural deduction system *co-ND* for linear - non linear co-intuitionistic logic, corresponding to the sequent calculus $\mathbf{co-LNL}$. Some rules are formulated here *additively* (see remark 40).

Lemma 41. The following rules are admissible in the natural deduction system *co-ND*:

C weakening
$$\frac{S \vdash_C \Psi}{S \vdash_C T, \Psi}$$

C contraction
$$\frac{S \vdash_C T, T, \Psi}{S \vdash_C T, \Psi}$$

CC substitution
$$\frac{S \vdash_C \Psi_1, T \qquad T \vdash_C \Psi_2}{S \vdash_C \Psi_1, \Psi_2}$$

L weakening
$$\frac{A \vdash_L \Delta; \Psi}{A \vdash_L \Delta; T, \Psi}$$

L contraction
$$\frac{A \vdash_L \Delta; T, T, \Psi}{A \vdash_L \Delta; T, \Psi}$$

LC substitution
$$\frac{A \vdash_L \Delta; \Psi_1, T \qquad T \vdash_C \Psi_2}{A \vdash_L \Delta; \Psi_1, \Psi_2}$$

LL substitution
$$\frac{B \vdash_L \Delta_1, A; \Psi_1 \qquad A \vdash_L \Delta_2; \Psi_2}{B \vdash_L \Delta_1, \Delta_2, \Psi_1, \Psi_2}$$

Proposition 42. There are functions $\mathcal{S} : co-ND \to \mathbf{co-LNL}$ and $\mathcal{N} : \mathbf{co-LNL} \to co-ND$ from natural deduction to sequent calculus derivations.

L identity

$$A \vdash_L A$$

⊥ intro

$$\frac{A \vdash_L \Delta;\Psi}{A \vdash_L \bot,\Delta;\Psi}$$

⊥ elim

$$\frac{A \vdash_L \bot,\Delta}{A \vdash_L \Delta}$$

L subtraction intro

$$\frac{A \vdash_L \Delta_1;\Psi_1,B_1 \qquad B_2 \vdash_L \Delta_2;\Psi_2}{A \vdash_L B_1 \bullet\!\!- B_2,\Delta_1,\Delta_2;\Psi_1,\Psi_2}$$

L subtraction elim

$$\frac{A \vdash_L \Delta_1,B_1 \bullet\!\!- B_2,\Psi_1 \qquad B_1 \vdash_L B_2,\Delta_2;\Psi_2}{B_1 \bullet\!\!- B_2 \vdash_L \Delta_1,\Delta_2;\Psi_1,\Psi_2}$$

L disjunction intro

$$\frac{A \vdash_L \Delta,B_1,B_2;\Psi}{A \vdash_L \Delta,B_1 \oplus B_2;\Psi}$$

L disjunction elim$_2$

$$\frac{A \vdash_L \Delta,B_1 \oplus B_2;\Psi \qquad B_1 \vdash_L \Delta_1;\Psi_1 \qquad B_2 \vdash_L \Delta_2;\Psi_2}{A \vdash_L \Delta,\Delta_1,\Delta_2;\Psi,\Psi_1,\Psi_2}$$

J intro

$$\frac{A \vdash_L,\Delta;T,\Psi}{A \vdash_L,\Delta,JT;\Psi}$$

J elim

$$\frac{S \vdash_L \Delta,JT;\Psi_1 \qquad T \vdash_C \Psi_2}{S \vdash_L \Delta;\Psi_1,\Psi_2}$$

H intro

$$\frac{A \vdash_L,\Delta,B;\Psi}{A \vdash_L,\Delta;HB,\Psi}$$

H elim$_1$

$$\frac{B \vdash_L \Delta;\Psi_1,HA_2 \qquad A \vdash_L;\Psi_2}{B \vdash_L \Delta;\Psi_1,\Psi_2}$$

Table 4: Natural Deduction for linear *co-ND*

Notice that the right rules of the sequent calculus and the introductions of natural deduction have the same form. Elimination rules are derivable from left rules with *cut* and left rules are derivable using substitutions. For instance, the *C zero* rule

C zero

$$\frac{S \vdash_C 0,\Psi \qquad S_1 \vdash_C \Psi_1 \quad \dots \quad S_n \vdash \Psi_n}{S \vdash_C \Psi,\Psi_1 \quad \dots \quad \Psi_n}$$

is derivable in the sequent calculus as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{S \vdash_C 0,\Psi \qquad 0 \vdash S_1,\dots S_n}{S \vdash_C \Psi,S_1,\dots S_n}\,cut \qquad S_1 \vdash_C \Psi_1}{S \vdash_C \Psi,\Psi_1,S_2,\dots,S_n}\,cut \qquad \vdots}{S \vdash_C \Psi,\Psi_1,\dots\Psi_{n-1},S_n} \qquad S_n \vdash \Psi_n}{S \vdash_C \Psi,\Psi_1,\dots\Psi_{n-1},\Psi_n}\,cut$$

3.3. **Term Assignment.**

3.3.1. *Terms.* The term calculus is given by the following grammar:

non linear terms:

$$\begin{array}{lll} s, t & ::= & x \mid \mathtt{connect}_w\ \mathtt{to}\ t \mid t_1 \cdot t_2 \mid \mathtt{false}\ t \\ & & \mid x(t) \mid \mathtt{mkc}(t, x) \mid \mathtt{inl}\ t \mid \mathtt{inr}\ t \mid \mathtt{case}\ t_1\ \mathtt{of}\ x.t_2, y.t_3 \\ & & \mid He \mid \mathtt{let} Jx\ =\ e\ \mathtt{in}\ t_2 \mid \mathtt{let} Hx\ =\ t_1\ \mathtt{in}\ t_2 \\ & & \mid \mathtt{postp}(x \mapsto t_1, t_2) \mid (t) \mid (e) \end{array}$$

linear terms:

$$\begin{array}{lll} e, u & ::= & x \mid \mathtt{connect}_\perp\ \mathtt{to}\ e \mid \\ & & \mid \mathtt{mkc}(e, x) \mid x(e) \mid e_1 \oplus e_2 \mid \mathtt{casel}\ e \mid \mathtt{caser}\ e \mid Jt \\ & & \mid \mathtt{postp}_\perp\ e \mid \mathtt{postp}(x \mapsto e_1, e_2) \end{array}$$

Remark 43. Let us call the terms of the form $\mathtt{postp}(x \mapsto t_1, t_2)$, $\mathtt{postp}_\perp\ e$ and $\mathtt{postp}(x \mapsto e_1, e_2)$ *p terms*. Let us say that a term t is *p-normal* if t does not contain any p term as a proper subterm. Thus the class of *p-normal* terms r is defined by the following grammar:

$$\begin{array}{lll} s, t & ::= & x \mid \mathtt{connect}_w\ \mathtt{to}\ t \mid t_1 \cdot t_2 \mid \mathtt{false}\ t \\ & & \mid x(t) \mid \mathtt{mkc}(t, x) \mid \mathtt{inl}\ t \mid \mathtt{inr}\ t \mid \mathtt{case}\ t_1\ \mathtt{of}\ x.t_2, y.t_3 \\ & & \mid He \mid \mathtt{let} Jx\ =\ e\ \mathtt{in}\ t_2 \mid \mathtt{let} Hx\ =\ t_1\ \mathtt{in}\ t_2 \\ p & ::= & \mathtt{postp}(x \mapsto t_1, t_2) \\ e, u & ::= & x \mid \mathtt{connect}_\perp\ \mathtt{to}\ e \mid \\ & & \mid \mathtt{mkc}(e, x) \mid x(e) \mid e_1 \oplus e_2 \mid \mathtt{casel}\ e \mid \mathtt{caser}\ e \mid Jt \\ p & ::= & \mathtt{postp}_\perp\ e \mid \mathtt{postp}(x \mapsto e_1, e_2) \\ r & ::= & t \mid e \mid p \end{array}$$

When the calculus is typed, linear p-terms can be typed with $\perp$, non-linear p terms can be typed with 0. In presence of the $\perp$ rule and of the 0-rule it is possible to replace a non-p-normal p-term removed by a β-reduction with a $\perp$ or 0-rule. However this creates an anomaly in the proof theory since the newly introduced unit rule is unnecessary and could be eliminated in the case of p-normal terms. Hence we choose to leave the typing of p terms implicit in the syntax and enforce the requirement of p-normality.

Definition 44. The free variables $FV(e)$ of a linear term e are defined as follows:

$$\begin{array}{rcl} FV(x) & = & \{x\} \\ FV(\mathtt{connect}_\perp \mathtt{to}\ e) & = & FV(e) \\ FV(x(e)) & = & FV(e) \\ FV(\mathtt{mkc}(e, y) & = & FV(e) \\ FV(e_1 \oplus e_2) & = & FV(e_1) \cup FV(e_2) \\ FV(\mathtt{casel}\ e) = FV(\mathtt{caser}\ e) & = & FV(e) \end{array}$$

The free variables of a p-term are defined s follows:

$$FV(\mathtt{postp}_\perp\ e) = FV(e);\ FV(\mathtt{postp}(x \mapsto e_1, e_2)) = FV(e_1) \setminus \{x\} \cup FV(e_2)$$

and similarly for terms $\mathtt{postp}(x \mapsto t_1, t_2)$. The free variables $FV(t)$ of a non-linear term t are defined as follows:

$$
\begin{array}{rcl}
FV(x) & = & \{x\} \\
FV(\mathtt{connect}_w \mathtt{to}\ t) & = & FV(t) \\
FV(t_1 \cdot t_2) & = & FV(t_1) \cup FV(t_2) \\
FV(\mathtt{false}\ t) & = & FV(t) \\
FV(x(t)) & = & FV(t) \\
FV(\mathtt{mkc}(t, y) & = & FV(t) \\
FV(\mathtt{inl}\ t) = FV(\mathtt{inr}\ t) & = & FV(t) \\
FV(\mathtt{case}\ t_1 \mathtt{of}\ x.t_2, y.t_3) & = & FV(t_1) \cup FV(t_2) \setminus \{x\} \cup FV(t_3) \setminus \{y\} \\
FV(\mathtt{let}\ Jy = e\ \mathtt{in}\ t) & = & FV(e) \cup FV(t) \setminus \{y\} \\
FV(\mathtt{let}\ Hy = t\ \mathtt{in}\ e) & = & FV(t) \cup FV(e) \setminus \{y\} \\
FV(t) & = & FV(t)
\end{array}
$$

3.3.2. *Contexts.* We type terms by assigning terms to natural deduction derivations in *co-ND*. To denote a linear context we write $\Delta = e_1 : A_1, \dots, e_n : A_n$ and $|\Delta| = A_1, \dots, A_n$ for the list of its types. Similarly we use $\Psi = t_1 : S_1, \dots, t_n : S_n$ and $|\Psi| = S_1, \dots, S_n$. The grammar of contexts is as follows:

$$
\begin{array}{rcl}
\Psi, \Pi & ::= & | \cdot \mid t : T \mid \Psi, \Pi \\
\Gamma, \Delta & ::= & | \cdot \mid e : A | \, \Gamma, \Gamma'
\end{array}
$$

We also use the notation

$$\Delta = e_1 : A_1 \| \dots \| e_n : A_n \qquad \text{and} \qquad \Psi = t_1 : S_1, \dots, t_n : S_n$$

for contexts. Here the operation $\|$ (“*parallel composition*)” is regarded as associative, commutative and having the empty context as identity.

Definition 45. A context Ψ or Γ is *correct* if

(1) every expression $e : A$ or $t : S$ in it contains exactly one free variable;
(2) in every term of the form $\mathtt{mkc}(e, y)$ the variable y does not occur free in e; similarly for terms $\mathtt{mkc}(t, y)$;
(3) in every term of the form $\mathtt{postp}(y \mapsto e_1, e_2)$ the term e_1 contains exactly one free variable y and $y \notin e_2$; similarly, for terms $\mathtt{postp}(y \mapsto t_1, t_2)$.

We may leave the type notation implicit in contexts and write $\mathcal{S}_x$ for a correct untyped context whose only free variable is x.

We extend the notation of `let` terms and `case of` terms to contexts as follows:

$$
\begin{array}{rcl}
\mathtt{let}\ p\ = t\ \mathtt{in} \cdot & = & \cdot \\
\mathtt{let}\ p\ = t\ \mathtt{in}\,(t' : A) & = & (\mathtt{let}\ p\ = t) : A \\
\mathtt{let}\ p\ = t\ \mathtt{in}\,(\Delta_1 \| \Delta_2) & = & (\mathtt{let}\ p\ = t \mathtt{in}\ \Delta_1) \| (\mathtt{let}\ p\ = t\ \mathtt{in}\ \Delta_2)
\end{array}
$$

(where $p = Hy$ or Jy). Terms `case of` are handled similarly.

Finally, the admissible rule of contraction is also generalized to contexts. Let t_1 and t_2 be multisets of terms, let $t_1 \cdot t_2$ is the sum of multisets. (If multisets are represented as lists, then the sum is representable as the appending of the lists). Then we can define $\Psi_1 \cdot \Psi_2$ recursively as follows:

$$
\begin{array}{rcll}
(\cdot) \cdot (\cdot) & = & (\cdot) & \\
(t_1 : S) \cdot (t_2 : S) & = & t_1 \cdot t_2 : S & \\
(\Psi_1 \| \Pi_1) \cdot (\Psi_2 \| Pi_2) & = & (\Psi_1 \cdot \Psi_2) \| (\Pi_1 \cdot \Pi_2), & \qquad \text{where } |\Psi_1| = |\Psi_2|, |\Pi_1| = |\Pi_2|.
\end{array}
$$

3.3.3. *Bound variables and substitution.*

Definition 46. We define the substitution of an e term for a free variable in an e term and in a p term of a correct context as follows.

- $[e/x]y = e$ if $x = y$, $[e/x]y = y$ otherwise;
- $[e'/x]\texttt{connect}_\perp\texttt{to}\ e = \texttt{connect}_\perp\texttt{to}\ [e'/x]e$;
- $[e'/x]\texttt{mkc}(e, y) = \texttt{mkc}([e'/x]e, y)$;
- $[e'/x]y(e) = y([e'/x]e)$;
- $[e'/x](e_1 \oplus e_2) = ([e'/x]e_1) \oplus ([e'/x]e_2)$;
- $[e'/x]\texttt{casel}\ e = \texttt{casel}\ [e'/x]e$; $\quad [e'/x]\texttt{caser}\ e = \texttt{caser}\ [e'/x]e$;
- $[t'/x]J(t) = J[t'/x]t$;
- $[e'/x]\texttt{postp}_\perp\ e = \texttt{postp}_\perp\ e[e'/x]$;
- $[e'/x]\texttt{postp}(x \mapsto e_1, e_2) = \texttt{postp}(x \mapsto e_1, [e'/x]e_2$

Substitution in a t term is defined similarly, but here we have to define substitution of a multiset of terms in a term:

(1) $[t_1 \cdot \ldots \cdot t_n/z]s = [t_1]s \cdot \ldots \cdot [t_n/z]s$;
(2) $[t_1 \cdot \ldots \cdot t_n/z]p = [t_1]p \| \ldots \| [t_n/z]p$, where p is a p-term.

The operation of substitution is generalized to contexts in the obvious way:

- $[e'/x](e_1 \| \ldots \| e_n) = ([e'/x]e_1 \| \ldots \| [e'/x]e_n)$;
- $[t/x](s_1 \| \ldots \| s_n) = ([t/x]s_1 \| \ldots \| [t/x]s_n)$;
- $[t_1 \cdot \ldots \cdot t_n/x]s = [t_1/x]s \| \ldots \| [t_n]s$.

Variable binding is of two kinds, local and global.

Definition 47. (1) The variable y is locally bound in $\texttt{mkc}(e, y)$, $\texttt{mkc}(t, y)$; y is locally bound in $\texttt{postp}(y \mapsto e_1, e_2)$ and in $\texttt{postp}(y \mapsto t_1, t_2)$.
(2) the variable x is locally bound in $\texttt{let}\ Jx = e_1\ \texttt{in}\ e_2$, $\texttt{let}\ Hx = t_1\ \texttt{in}\ t_2$.
(3) x is bound in t_2 and y in t_3 in the term $\texttt{case}\ t_1\ \texttt{of}\ x.t_2, y.t_3$.

Global binding occurs when two contexts are merged at a subtraction introduction with term $\texttt{mkc}(e, y)$, or at a subtraction elimination, with term $\texttt{postp}(y \mapsto e_1, e)$ for the occurrences of y in the context outside the term $\texttt{mkc}(e, y)$ or $\texttt{postp}(y \mapsto e_1, e)$ where y is locally bounded. All occurrences of y in the other terms of the context are globally bound, and such binding is represented by substituting $y(e)$ for y.

In tables 5, 6 we give the term assignment to *co-ND*.

Definition 48. The term assignment for the admissible rules of the calculus is as follows:

$$\textit{C weakening}\quad \frac{x : S \vdash_C \Psi \qquad s : T' \in \Psi}{x : S \vdash_C \texttt{connect}_w\texttt{to}\ s : T, \Psi} \qquad \textit{C contraction}\quad \frac{x : S \vdash_C t_1 : T, t_2 : T, \Psi}{x : S \vdash_C (t_1 \cdot t_2) : T, \Psi}$$

$$\textit{CC substitution}\quad \frac{x : S \vdash_C \Psi_1, t : T \qquad y : T \vdash_C \Psi_2}{x : S \vdash_C \Psi_1, [t/y]\Psi_2}$$

$$\textit{L weakening}\quad \frac{x : A \vdash_L \Delta; \Psi \qquad r : T' \in \Psi \text{ or } r : B \in \Delta}{x : A \vdash_L \Delta; \texttt{connect}_w\texttt{to}\ r : T, \Psi} \qquad \textit{L contraction}\quad \frac{x : A \vdash_L \Delta; t_1 : T, t_2 : T, \Psi}{x : A \vdash_L \Delta; (t_1 \cdot t_2) : T, \Psi}$$

$$\textit{LC substitution}\quad \frac{x : A \vdash_L \Delta; \Psi_1, s : T \qquad y : T \vdash_C \Psi_2}{x : A \vdash_L \Delta; \Psi_1, [s/y]\Psi_2} \qquad \textit{LL substitution}\quad \frac{x : B \vdash_L \Delta_1, e : A; \Psi_1 \qquad y : A \vdash_L \Delta_2; \Psi_2}{x : B \vdash_L \Delta_1, [e/y]\Delta_2; \Psi_1, [e/y]\Psi_2}$$

C assumption

$$x : S \vdash_C x : S$$

C zero

$$\frac{x : S \vdash_C t : 0, \Psi \qquad x_1 : S_1 \vdash_C \Psi_1 \quad \ldots \quad x_n : S_n \vdash \Psi_n}{x : S \vdash_C \Psi, [\texttt{false}\ t/x_1]\Psi_1 \quad \ldots \quad [\texttt{false}\ t/x_n]\Psi_n}$$

C subtraction intro

$$\frac{x : S \vdash_C \Psi_1, t : T_1 \qquad y : T_2 \vdash_C \Psi_2}{x : S \vdash_C \Psi_1, \texttt{mkc}(t, y) : T_1 - T_2, [y(t)/y]\Psi_2}$$

C subtraction elim

$$\frac{x : S \vdash_C \Psi_1, s : T_1 - T_2 \qquad y : T_1 \vdash_C t : T_2, \Psi_2}{x : S \vdash_C \Psi_1, \texttt{postp}(y \mapsto t, s), [y(s)/y]\Psi_2}$$

C disjunction intro$_1$

$$\frac{x : S \vdash_C \Psi, t_1 : T_1}{x : S \vdash_C \Psi, \texttt{inl}\ t_1 : T_1 + T_2}$$

C disjunction intro$_2$

$$\frac{x : S \vdash_C \Psi, t_2 : T_2}{x : S \vdash_C \Psi, \texttt{inr}\ t_2 : T_1 + T_2}$$

C disjunction elim

$$\frac{x : S \vdash_C \Psi, t : T_1 + T_2 \qquad y : T_1 \vdash_C \Psi_1 \quad z : T_2 \vdash_C \Psi_2 \quad |\Psi_1| = |\Psi_2|}{x : S \vdash_C \Psi, \texttt{case}\ t\ \texttt{of}\ y.\Psi_1, z.\Psi_2}$$

H elim$_2$

$$\frac{x : S \vdash_C \Psi_1, t : HA \qquad y : A \vdash_L ; \Psi_2 \qquad |\Psi_1| = |\Psi_2|}{S \vdash_C \Psi_1 \cdot (\texttt{let}\ Hy = t\ \texttt{in}\ \Psi_2)}$$

Table 5: Term Assignment to non-linear *co-ND*

3.3.4. *β reductions.* **Non-linear terms**

$-$ **Redex:**

$$\frac{\dfrac{x : S \vdash_C \Psi_1, t : T_1 \qquad y : T_2 \vdash_C \Psi_2}{x : S \vdash_C \Psi_1, \texttt{mkc}(t, y) : T_1 - T_2, \Psi_1, [y(t)/y]\Psi_2} \qquad z : T_1 \vdash_L t_2 : T_2, \Psi_3}{x : S \vdash_C \Psi_1, \texttt{postp}(z \mapsto t_2, \texttt{mkc}(t_1, y)), [y(e_1)/y]\Psi_2, z(\texttt{mkc}(e_1, y))] : \Psi_3}$$

$-$ **Reductum:**

$$\frac{\dfrac{x : S \vdash_C \Psi_1, t : T_1 \qquad z : T_1 \vdash_C t_2 : T_2, \Psi_3}{x : S \vdash_C \Psi_1, [t/z]t_2 : T_2; \Psi_1, [t/z]\Psi_3} \qquad T_2 \vdash_C \Psi_2}{x : S \vdash_C \Psi_1, [[t/z]t_2/y]\Psi_2,\ [t/z]\Psi_3}$$

$+_1$ **Redex:**

$$\frac{\dfrac{x : S \vdash_C \Psi, t_1 : T_1}{x : S \vdash_C \Psi, \texttt{inl}\ t_1 : T_1 + T_2} \qquad y : T_1 \vdash_C \Psi_1 \quad z : T_2 \vdash_C \Psi_2 \qquad |\Psi_1| = |\Psi_2|}{x : S \vdash_C \Psi, \texttt{case inl}\ t_1\ \texttt{of}\ y.\Psi_1, z.\Psi_2}$$

$+_1$ **Reductum:**

$$\frac{x : S \vdash_C \Psi, t : T_1 \qquad y : T_1 \vdash_C \Psi_1}{x : S \vdash_C \Delta, [t_1/y]\Psi_1}$$

L identity

$$x : A \vdash_L x : A$$

$\perp$ *intro*

$$\frac{x : A \vdash_L \Delta; \Psi \qquad e : B \in \Delta}{x : A \vdash_L \texttt{connect}_\perp\ \texttt{to}\ e : \perp, \Delta; \Psi}$$

$\perp$ elim

$$\frac{x : A \vdash_L e : \perp, \Delta; \Psi}{x : A \vdash_L \texttt{postp}_\perp\ e, \Delta; \Psi}$$

L subtraction intro

$$\frac{x : A \vdash_L \Delta_1, e : B_1; \Psi_1 \qquad y : B_2 \vdash_L \Delta_2; \Psi_2}{x : A \vdash_L \texttt{mkc}(e, y) : B_1 \bullet\!\!- B_2, \Delta_1, [y(e)]/y]\Delta_2; \Psi_1, [y(e)]/y]\Psi_2}$$

L subtraction elim

$$\frac{x : A \vdash_L \Delta_1, e : B_1 \bullet\!\!- B_2; \Psi_1 \qquad y : B_1 \vdash_L e' : B_2, \Delta_2; \Psi_2}{x : A \vdash_L \Delta_1, \texttt{postp}(y \mapsto e', e), \Delta_2; \Psi_1, \Psi_2}$$

L disjunction intro

$$\frac{x : A \vdash_L \Delta, e_1 : B_1, e_2 : B_2; \Psi}{x : A \vdash_L \Delta, e_1 \oplus e_2 : B_1 \oplus B_2; \Psi}$$

L disjunction elim$_2$

$$\frac{x : A \vdash_L \Delta, e : B_1 \oplus B_2; \Psi \qquad y : B_1 \vdash_L \Delta_1; \Psi_1 \quad z : B_2 \vdash_L \Delta_2; \Psi_2}{x : A \vdash_L \Delta, [\texttt{casel}(e)/y]\Delta_1, [\texttt{caser}(e)/z]\Delta_2; \Psi, [\texttt{casel}(e)/y]\Psi_1, [\texttt{caser}(e)/z]\Psi_2}$$

J intro

$$\frac{A \vdash_L, \Delta; T, \Psi}{A \vdash_L, \Delta, JT; \Psi}$$

J elim

$$\frac{x : A \vdash_L \Delta, e : JT; \Psi_1 \qquad y : T \vdash_C \Psi_2}{x : A \vdash_L \Delta; \Psi_1, \texttt{let}\ Jy = e\ \texttt{in}\ \Psi_2}$$

H intro

$$\frac{x : A \vdash_L, \Delta, e : B; \Psi}{x : A \vdash_L, \Delta; He : HB, \Psi}$$

H elim$_1$

$$\frac{x : B \vdash_L \Delta; \Psi_1, t : HA \quad y : A \vdash_L; \Psi_2}{x : B \vdash_L \Delta; \Psi_1, \texttt{let}Hy = t\ \texttt{in}\ \Psi_2}$$

Table 6: Term Assignment to linear *co-ND*

$+_2$ **Redex:**

$$\frac{\dfrac{x : S \vdash_C \Psi, t_2 : T_2}{x : S \vdash_C \Psi, \texttt{inr}\ t_2 : T_1 + T_2} \qquad y : T_1 \vdash_C \Psi_1 \quad z : T_2 \vdash_C \Psi_2 \qquad |\Psi_1| = |\Psi_2|}{x : S \vdash_C \Psi, \texttt{case}\ \texttt{inr}\ t_2\ \texttt{of}\ y.\Psi_1, z.\Psi_2}$$

$+_2$ **Reductum:**

$$\frac{x : S \vdash_C \Psi, t_2 : T_2 \qquad z : T_2 \vdash_C \Psi_2}{x : S \vdash_C [t_2/z]\Psi_2}$$

H_2 **Redex**

$$\frac{\dfrac{y : HB \vdash_C y : HB \qquad \dfrac{x : B \vdash_L e : A; \Psi_1}{x : B \vdash_L; He : HA, \Psi_1}\ H\ intro}{y : HB \vdash_C (\texttt{let}Hx = y\ \texttt{in}\ \Psi_1), (\texttt{let}Hx = y\ \texttt{in}\ He) : HA}\ H\ elim_2 \qquad z : A \vdash_L; \Psi_2}{x : B \vdash_C; (\texttt{let}Hx = y\ \texttt{in}\ \Psi_1) \cdot (\texttt{let}\ Hz = (\texttt{let}Hx = y\ \texttt{in}\ He)\ \texttt{in}\ \Psi_2)}\ H\ elim_2$$

H_2 Reductum

$$\dfrac{y : HB \vdash_C y : HB \qquad \dfrac{x : B \vdash_L e : A; \Psi_1 \qquad z : A \vdash_L ; \Psi_2}{x : B \vdash_L \Psi_1 \cdot [e/z]\Psi_2}}{y : HB \vdash_C (\mathtt{let} Hx = y\ \mathtt{in}\ \Psi_1) \cdot (\mathtt{let} Hx = y\ \mathtt{in}\ [e/z]\Psi_2}\ H\ elim_2$$

Linear terms

$\perp$ Redex

$$\dfrac{\dfrac{x : A \vdash_L \Delta; \Psi \qquad e : B \in \Delta}{x : A \vdash_L \mathtt{connect}_\perp\ \mathtt{to}\ e : \perp, \Delta; \Psi}}{x : A \vdash_L \mathtt{postp}_\perp \mathtt{connect}_\perp\ \mathtt{to}\ e, \Delta; \Psi}$$

$\perp$ Reductum

$$x : A \vdash_L \Delta; \Psi$$

$\bullet\!\!-$ Redex:

$$\dfrac{\dfrac{x : B \vdash_L e_1 : A_1, \Delta_1; \Psi_1 \qquad y : A_2 \vdash_L \Delta_2; \Psi_2}{x : B \vdash_L \mathtt{mkc}(e_1, y) : A_1 \bullet\!\!- A_2, \Delta_1, [y(e_1)/y]\Delta_2; \Psi_1, [y(e_1)/y]\Psi_2} \qquad z : A_1 \vdash_L e_2 : A_2, \Delta_3; \Psi_3}{\begin{array}{r} x : B \vdash_L \Delta_1, \mathtt{postp}(z \mapsto e_2, \mathtt{mkc}(e_1, y))[y(e_1)/y]\Delta_2, [z(\mathtt{mkc}(e_1, y))/z]\Delta_3; \\ \Psi_1, [y(e_1)/y]\Psi_2, z(\mathtt{mkc}(e_1, y))] : \Psi_3 \end{array}}$$

$\bullet\!\!-$ Reductum:

$$\dfrac{\dfrac{x : B \vdash_L e_1 : A_1, \Delta_1; \Psi_1 \qquad z : A_1 \vdash_L e_2 : A_2, \Delta_3; \Psi_3}{x : B \vdash_L \Delta_1, [e_1/z]\Delta_3, [e_1/z]e_2 : A_2; \Psi_1, [e_1/z]\Psi_3} \qquad y : A_2 \vdash_L \Delta_2; \Psi_2}{x : B \vdash_L \Delta_1, [[e_1/z]e_2/y]\Delta_2,\ [e_1/z]\Delta_3; \Psi_1, [[e_1/z]e_2]\Psi_2,\ [e_1/z]\Psi_3}$$

$\oplus$ Redex:

$$\dfrac{\dfrac{x : A \vdash_L \Delta, e_1 : B_1, e_2 : B_2; \Psi}{x : A \vdash_L \Delta, e_1 \oplus e_2 : B_1 \oplus B_2; \Psi} \qquad y : B_1 \vdash_L \Delta_1; \Psi_1 \quad z : B_2 \vdash_L \Delta_2; \Psi_2}{\begin{array}{r} x : A \vdash_L \Delta, [\mathtt{casel}(e_1 \oplus e_2)/y]\Delta_1, [\mathtt{caser}(e_1 \oplus e_2)/z]\Delta_2; \\ \Psi, [\mathtt{casel}(e_1 \oplus e_2)/y]\Psi_1, [\mathtt{caser}(e_1 \oplus e_2)/z]\Psi_2 \end{array}}$$

$\oplus$ Reductum:

$$\dfrac{\dfrac{x : A \vdash_L \Delta, e_1 : B_1, e_2 : B_2; \Psi \qquad y : B_1 \vdash_L \Delta_1; \Psi_1}{x : A \vdash_L \Delta, [e_1/y]\Delta_1, e_2 : B_2; \Psi, [e_1/y]\Psi_1} \qquad z : B_2 \vdash_L \Delta_2; \Psi_2}{x : A \vdash_L \Delta, [e_1/y]\Delta_1, [e_2/z]\Delta_2; \Psi, [e_1/y]\Psi_1, [e_2/z]\Psi_2}$$

***J* Redex:**

$$\dfrac{\dfrac{x : A \vdash_L, \Delta; t : T, \Psi}{x : A \vdash_L \Delta, Jt : JT; \Psi}\ J\text{-}intro \qquad y : T \vdash_C \Psi_2 \text{ where } |\Psi_1| = |\Psi_2|}{x : A \vdash_L \Delta; \Psi_1 \cdot \mathtt{let}\ Jy = Jt\ \mathtt{in}\ \Psi_2}\ J\text{-}elim$$

***J* Reductum:**

$$\dfrac{x : A \vdash_L \Delta; t : T, \Psi \qquad y : T \vdash_C \Psi_2 \text{ where } |\Psi_1| = |\Psi_2|}{x : A \vdash_L \Delta; \Psi_1, [t/y]\Psi_2}$$

H_1 Redex

$$\dfrac{\dfrac{x : B \vdash_L \Delta, e : A; \Psi_1}{x : B \vdash_L \Delta; He : HA, \Psi_1} \qquad y : A \vdash_L ; \Psi_2}{x : B \vdash_L \Delta; \Psi_1 \cdot (\mathtt{let} Hy = He\ \mathtt{in}\ \Psi_2)}$$

H_1 Reductum

$$\frac{x : B \vdash_L \Delta, e : A; \Psi_1 \qquad y : A \vdash_L ; \Psi_2}{x : B \vdash_L \Delta; \Psi_1 \cdot [e/y]\Psi_2}$$

weakening commutations

$$\frac{x : S \vdash_C \mathtt{connect}_w\ \mathtt{to}\ t : T, \Psi \qquad t \in \Psi}{y : S \vdash C\mathtt{connect}_w\ \mathtt{to}\ t' : T', \Psi' \qquad t' \in \Psi'}$$

where the double line stands for any n-ary inference, $n = 1, 2, 3$ or no inference (it is possible to "rewire" the attachment within the same context).

$$\frac{x : B \vdash_L \Delta; \mathtt{connect}_w\ \mathtt{to}\ e : T, \Psi \qquad e \in \Delta}{y : S \vdash C\Delta'; \mathtt{connect}_w\ \mathtt{to}\ e' : T', \Psi' \qquad e' \in \Delta'}$$

where the double line stands for any n-ary inference, $n = 1, 2, 3$ or no inference. The weakening formula could be attached also to a term in the non-linear part.

$\perp$ commutations

$$\frac{x : A \vdash_L \mathtt{connect}_\perp\ \mathtt{to}\ e : \perp, \Delta; \Psi \qquad e \in \Delta}{y : B \vdash L\mathtt{connect}_\perp\ \mathtt{to}\ e' : \perp, \Delta'; \Psi' \qquad e \in \Delta'}$$

where the double line stands for any n-ary inference, $n = 1, 2, 3$.

There is also a zero commutation rule, which we omit here.

3.4. **Categorical interpretation of rules.** Given a *signature* Sg, consisting of a collection of types σ_i, where $\sigma_i = A$ or S, and a collection of *sorted function symbols* $f_j : \sigma_1, \ldots, \sigma_n \to \tau$ and given a Symmetric Monoidal Category (SMC) $(\mathbb{C}, \bullet, 1, \alpha, \lambda, \rho, \gamma)$, a *structure* $\mathcal{M}$ for Sg is an assignment of an object $[\![\sigma]\!]$ of $\mathcal{L}$ for each type σ and of a morphism $[\![f]\!] : [\![\sigma_1]\!] \bullet \ldots \bullet [\![\sigma_n]\!] \to [\![\tau]\!]$ for each function $f : \sigma_1, \ldots, \sigma_n \to \tau$ of Sg.[1]

The types of terms in context $\Delta = [e_1 : A_1, \ldots, e_n : A_n]$ or $\Delta = [t_1 : T_1, \ldots, t_n : T_n]$ are interpreted as $[\![\sigma_1, \sigma_2, \ldots, \sigma_n]\!] = (\ldots([\![\sigma_1]\!] \bullet [\![\sigma_2]\!]) \ldots) \bullet [\![\sigma_n]\!]$; left associativity is also intended for concatenations of type sequences Γ, Δ. Thus we need the "book-keeping" functions $\mathtt{Split}(\Gamma, \Delta) : [\![\Gamma, \Delta]\!] \to [\![\Gamma]\!] \bullet [\![\Delta]\!]$ and $\mathtt{Join}(\Gamma, \Delta) : [\![\Gamma]\!] \bullet [\![\Delta]\!] \to [\![\Gamma, \Delta]\!]$ inductively defined using the associativity laws α and their inverse α^{-1} (cfr Bierman 1994, Bellin 2015).

The semantics of terms in context is then specified by induction on terms:

$$[\![x : A \vdash_L x : A]\!] =_{df} id_{[\![\sigma]\!]}$$

$$[\![x : A \vdash_L f(e_1, \ldots, e_n) : B]\!] =_{df} [\![x : A \vdash e_1 : A_1]\!] \bullet \ldots \bullet [\![x : \sigma \triangleright e_n : A_n]\!]; [\![f]\!]$$

and similarly with non-linear types. Next one proves by induction on the type derivation that substitution in the term calculus corresponds to composition in the category ([**?**], Lemma 13).

In a Linear Non-Linear Co-intuitionistic logic sequent, mixed sequents $x : A \vdash_L \Delta; \Psi, t : T$ non-linear terms are interpreted through the functor $J : \mathcal{C} \to \mathcal{L}$. Here we use the same symbol for the functor J in the structure and the name of the functor in the syntax of the langage. Thus we have

$$[\![x : A \vdash_L \Delta; \Psi, t : T]\!] \quad = \quad [\![x : A \vdash_L \Delta, J\Psi, Jt : J(T)]\!]$$

[1] In this subsection only we use the symbol $\bullet$ and 1 for the monoidal binary operation and its unit in the categorical structure, distinguished from the $\oplus$ and $\perp$ symbols in the formal language. We shall show that the interpretation of $\oplus$ is isomorphic to the operation $\bullet$, so we shall be able to identify them (and similarly for $\perp$ and 1).

Let $\mathcal{M}$ be a structure for a signature Sg in a SMC $\mathcal{L}$. Given an equation in context for Sg

$$x : A \vdash_L \overline{e} : \Gamma, e_1 = e_2 : B; \Psi$$

we say that the structure *satisfies* the equation if it assigns the same morphisms to $x : A \vdash_L \overline{e} : \Gamma, e_1 : B; \Psi$. and to $A \vdash \overline{e} : \Gamma, e_2 : B; \Psi$. Similarly, $\mathcal{M}$ satisfies $x : S \vdash_L \overline{t} : \Psi, t_1 = t_2$ if it assigns the same morphism to $x : S \vdash_L \overline{t} : \Psi, t_1 : T$ and to $x : S \vdash_L \overline{t} : \Psi, t_2 : Y$. Then given an algebraic theory $Th = (Sg, Ax)$, a structure $\mathcal{M}$ for Sg is a *model* for Th if it satisfies all the axioms in Ax.

We now go through some cases of the rules in the sequent-style Natural Deduction calculus to specify their categorical interpretation so as to satify the equations in context and to prove consistency of the LNL co-intuitionistic logic in the model thus defined.

3.4.1. *Linear disjunction Par.* 3.4.1.1. *Par introduction.* The introduction rule for Par is of the form

$$\frac{x : A \vdash_L \Delta_1, e_1 : B, e_2 : C; \Psi}{x : A \vdash_L \Delta_1, e_1 \oplus e_2 : B \oplus C, \Psi} \oplus \mathrm{I}$$

This suggests an operation on Hom-sets of the form

$$\Phi_{A,\Delta J\Psi} : \mathcal{L}(A, \Delta \bullet A \bullet B \bullet J\Psi) \to \mathcal{L}(A, \Delta \bullet B \oplus C \bullet J\Psi)$$

natural in Δ, A and $J\Psi$. Given

$$e : A \to \Delta \bullet B \bullet C \bullet J\Psi,$$

and $d : A' \to A$ $h : \Delta \to \Delta'$ and $p : J\Psi \to J\Psi'$, naturality yields

$$\Phi_{A',\Delta'}(d; e; h \bullet id_B \bullet id_C \bullet p) = d; \Phi_{A,\Delta}(e); h \bullet id_{B \oplus C} \bullet p$$

In particular, letting

$$e = id_\Delta \bullet id_B \bullet id_C, \quad d : A \to \Delta \bullet B \bullet C,$$

$h = id_\Delta$ and $p = id_{J\Psi}$ we have

$$\Phi_{A,\Delta}(d) = d; \Phi_\Delta(id_\Delta \bullet id_B \bullet id_C)$$

By functoriality of $\bullet$ we have $id_B \bullet id_C = id_{B\bullet C}$. Hence, writing $\bigoplus$ for $\Phi_\Delta(id_\Delta \bullet id_{B\bullet C})$ we have $\Phi_{A,\Delta}(d) = d; \bigoplus$. We define

$$[\![x : A \vdash_L : \Delta, e_1 \oplus e_2 : B \oplus C]\!] =_{df} [\![x : A \vdash_L \Delta, e_1 : B, e_2 : C,]\!]; \bigoplus.$$

3.4.1.2. *Par elimination*. The Par elimination rule has the form

$$\frac{z : A \vdash_L e : B \oplus C, \Delta_1; \Psi_1 \qquad x : B \vdash_L \Delta_2; \Psi_2 \qquad y : C \vdash_L \Delta_3; \Psi_3}{z : A \vdash_L \Delta_1, [\mathtt{casel}z/x]\Delta_2, [\mathtt{caser}z/y]\Delta_3; \Psi_1 \cdot \Psi_2 \cdot \Psi_3} \oplus \mathrm{E}$$

This suggests an operation on Hom-sets of the form

$$\Psi_{A,\Delta,J\Psi} : \mathcal{L}(A, B \oplus C \bullet \Delta_1 \bullet J\Psi_1) \times \mathcal{L}(B, \Delta_2 \bullet J\Psi_2) \times \mathcal{L}(C, \Delta_3 \bullet J\Psi_3) \to \mathcal{L}(A, \Delta \bullet J\Psi)$$

natural in $A, \Delta, J\Psi$ where we write $\Delta = \Delta_1, \Delta_2, \Delta_3$, $J\Psi = J\Psi_1, J\Psi_2, J\Psi_3$. Given morphisms

$$g : A \to \Delta_1 \bullet J\Psi_1 \bullet B \wp C, \quad e : B \to \Delta_2 \bullet J\Psi_2 \quad \text{and } f : C \to \Delta_3 \bullet J\Psi_3$$

and also $a : A' \to A$, $d_1 : \Delta_1 \to \Delta'_1$, $d_2 : \Delta_2 \to \Delta'_2$ and $d_3 : \Delta_3 \to \Delta'_3$, $p_1 : J\Psi_1 \to J\Psi'_1$, $p_2 : J\Psi_2 \to J\Psi'_2$ and $p_3 : J\Psi_3 \to J\Psi'_3$ naturality yields

$$\begin{aligned} \Psi_{A',\Delta',\Gamma',J\Psi'}((a; g; d_1 \bullet id_{B\oplus C}), (e; d_2 \bullet p_2), \quad (f; d_3 \bullet p_3)) = \\ a; \Psi_{A,\Delta,J\Psi}(g, e, f); \quad d_1 \bullet d_2 \bullet d_3 \bullet p_1 \bullet p_2 \bullet p_3; \mathtt{Join}(\Delta', J\Psi'). \end{aligned}$$

In particular, set

$$e = id_B, \quad f = id_C$$

and also $a = id_A$, $d_i = id_{\Delta_i}$, $p_i = id_{J\Psi_i}$ we get

$$\Psi_{A,\Delta J\Psi}(g, e, f) = \Psi_{A,\Delta,J\Psi}(g, id_B, id_C); id_\Delta \bullet c \bullet d; \mathtt{Join}(\Delta, J\Psi)$$

where the operation $\mathtt{Join}$ establishes left associativity. Writing $(g)^*$ for $\Psi_{D,\Delta}(g, id_B, id_C)$ we define

$$[\![z : A \vdash_L \Delta_1, [\mathtt{casel}\ e/x]\Delta_2, [\mathtt{caser}\ e/y]\Delta_3; \Psi_2, \Psi_3]\!] =_{df}$$
$$[\![z : A \vdash_L \Delta_1, e, B \oplus C; \Psi_1]\!]^*; id_{\Delta_1} \bullet [\![x : B \vdash_L \Delta_2; \Psi_2]\!] \bullet [\![y : C \vdash_L \Delta_3; \Psi_3]\!]; \mathtt{Join}(\Delta, J\Psi).$$

3.4.1.3. *Equations in context.* We have equations in context of the form

$$\oplus - \beta \text{ rules:}$$

$$\frac{\begin{array}{l} e \equiv \mathtt{casel}(e_1 \oplus e_2) \quad e' \equiv \mathtt{caser}(e_1 \oplus e_2) \\ \quad |\Delta_1| = |\Delta_1'|\ |\Psi_1| = |\Psi_1'| \qquad\qquad x : A_1 \vdash_L \Delta_2; J\Psi_2 = \Delta_2'; J\Psi_2' \\ \quad |\Delta_2| = |\Delta_2'|\ |\Psi_2| = |\Psi_2'| \qquad\qquad y : A_2 \vdash_L \Delta_3; J\Psi_3 = \Delta_3'; J\Psi_3' \\ \quad |\Delta_3| = |\Delta_3'|\ |\Psi_3| = |\Psi_3'| \quad z : B \vdash_L e_1 : A_1, e_2 : A_2, \Delta_1; J\Psi_1 = e_1' : A_1, e_2' : A_2, \Delta_1'; J\Psi_1' \end{array}}{\begin{array}{l} z : B \vdash_L \Delta_1, [e/x]\Delta_2, [e'/x]\Delta_3; \Psi_1, [e/x]\Psi_2, [e'/x]\Psi_3 = \\ \quad = \Delta_1', [e_1'/x]\Delta_2, [e_2'/x]\Delta_3; \Psi_1, [e_1'/x]\Psi_2', [e_2'/x]\Psi_3' \end{array}} \tag{3.1}$$

Let

$$q : B \to A_1 \bullet A_2 \bullet \Delta_1 \bullet J\Psi, \qquad m : A_1 \to \Delta_2 \bullet J\Psi_2 \quad \text{and} \quad n : A_2 \to \Delta_3 \bullet J\Psi_3.$$

Then to satisfy the above equations in context we need that the following diagram commutes:

$$\begin{tikzcd} B \arrow[r, "q"] & \Delta_1 \bullet J\Psi_1 \bullet (A_1 \bullet A_2) \arrow[r, "id_{\Delta_1} \bullet m \bullet n"] \arrow[d, "\oplus"'] & \Delta_1 \bullet J\Psi_1 \bullet \Delta_2 \bullet J\Psi_2 \bullet \Delta_3 \bullet J\Psi_3 \\ & \Delta_1 \bullet J\Psi_1 \bullet A \oplus B \arrow[r, "*"'] & \Delta_1 \bullet J\Psi_1 \bullet A \bullet B \arrow[u, "id_{\Delta_1} \bullet m \bullet n"'] \end{tikzcd}$$

We make the assumption that the above decomposition is unique. Moreover, supposing Δ_1 to be empty and $m = id_A$, $n = id_B$, $q = id_A \bullet id_B = id_{A \bullet B}$ we obtain $(id_A \bullet id_B; \bigoplus)^* = id_A \bullet id_B$ and similarly $(id_{A \oplus B})^*; \bigoplus = id_{A \oplus B}$; hence we may conclude that there is a natural isomorphism

$$\frac{D \to \Gamma \bullet A \bullet B}{\overline{D \to \Gamma \bullet A \oplus B}}$$

so we can identify $\bullet$ and $\oplus$. Finally we see that the following η equation in context is also satisfied:

$$\boxed{\begin{array}{c} \oplus - \eta \text{ rule} \\ \dfrac{|\Delta| = |\Delta'| \quad |\Psi| = |\Psi'| \quad z : B \vdash_L \Delta; \Psi = \Delta'; \Psi'}{z : B \vdash_L (\mathtt{casel}\ \mathrm{e} \oplus \mathtt{caser}\ e) : A_1 \oplus A_2, \Delta; \Psi = e : A_1 \oplus A_2, \Delta'; \Psi'} \end{array}} \tag{3.2}$$

3.4.2. *Linear subtraction.* 3.4.2.1. *Subtraction introduction.* The introduction rule for subtraction has the form

$$\frac{x : A \vdash_L \Delta_1, e : B; \Psi_1 \qquad y : C \vdash_L \Delta_2; \Psi_2 \quad |\Psi_1| = |\Psi_2|}{x : A \vdash_L \Delta_1, \mathtt{mkc}(e, y) : B \bullet\!\!- C, [y(e)/y]\Delta_2; \Psi_1 \cdot [y(e)/y]\Psi_2}\ LsubI$$

This suggests a natural transformation with components

$$\Phi_{A,\Delta,J\Psi} : \mathcal{L}(A, \Delta_1 \bullet B \bullet J\Psi_1) \times \mathcal{L}(C, \Delta_2 \bullet J\Psi_2) \to \mathcal{L}(A, \Delta_1 \bullet (B \bullet\!\!- C) \bullet \Delta_2 \bullet J\Psi_1 \bullet J\Psi_2)$$

natural in $A, \Delta_1, \Delta_2, J\Psi_1, J\Psi_2$. Taking morphisms

$$e : A \to \Delta_1 \bullet B \bullet J\Psi_1, \quad f : C \to \Delta_2 \bullet J\Psi_2$$

and also $a : A' \to A$, $d_1 : \Delta_1 \to \Delta'_1$, $d_2 : \Delta_2 \to \Delta'_2$, $p_1 : J\Psi_1 \to J\Psi'_1$, $p_2 : J\Psi_2 \to J\Psi'_2$, by naturality we have

$$\begin{aligned}&\Phi_{A',\Delta'_1,\Delta'_2,J\Psi'_1 J\Psi'_2}((a; e; d_1 \bullet id_B \bullet p_1), (f; d_2; p_2)) = \\ &= a; \Phi_{A,\Delta,J\Psi}(e, f); d_1 \bullet d_2 \bullet id_{B \bullet\!- C}; \mathtt{Join}(\Delta'_1, \Delta'_2, B \bullet\!\!- C, J\Psi'_1, J\Psi'_2)\end{aligned}$$

In particular, taking $a = id_A$, $d_1 = id_{\Delta_1}$, $p_1 = id_{J\Psi_1}$, $p_2 = id_{J\Psi_2}$ but $d_2 : C \to \Delta_2 \cdot J\Psi_2$ and $f = id_C$ we have:

$$\begin{aligned}\Phi_{A,\Delta_1,\Delta_2,J\Psi_1,J\Psi_2}(e, d_2) = \Phi_{A,\Delta_1}(e, id_C); id_{\Delta_1} \bullet d_2 \bullet id_{A \bullet\!- B} \bullet id_{J\Psi_1} \bullet idJ\Psi_2; \\ \mathtt{Join}(\Delta_1, \Delta_2, A \bullet\!\!- B, J\Psi_1, J\Psi_2)\end{aligned}$$

Writing $\mathbf{MKC}^C_{A,\Delta_1,J\Psi_1}(e)$ for $\Phi_{A,\Delta_1 J\Psi_1}(e, id_C)$, $\Phi_{A,\Delta,J\Psi}(e, d_2)$ can be expressed as the composition

$$\mathbf{MKC}^C_{A,\Delta_1,J\Psi_1}(e); id_{\Delta_1} \bullet d_2 \bullet id_{B \bullet\!- C}$$

where $\mathbf{MKC}^C_{A,\Delta_1,J\Psi_1}$ is a natural transformation with components

$$\mathcal{L}(A, \Delta_1 \bullet B \bullet J\Psi) \times \mathcal{L}(C, C) \to \mathcal{L}(A, \Delta_1 \bullet C \bullet C \bullet\!\!- C)$$

so we make the definition

$$\begin{aligned}&[\![x : A \vdash_L \Delta_1, \mathtt{mkc}(e, y) : B \bullet\!\!- C, [y(e)/y + \Delta_2; \Psi_1 \cdot [y(e)/y]\Psi_2]\!] =_{df} \\ &\mathbf{MKC}^C_{A,\Delta_1,J\Psi_1}[\![x : A \vdash_L \Delta_1, e_1 : B]\!]; id_\Delta \bullet [\![y : C \vdash_L \Delta_2; \Psi_2]\!] \bullet id_{B \bullet\!- C}; \\ &\qquad \mathtt{Join}(\Delta_1, \Delta_2, B \bullet\!\!- C, J\Psi_1, J\Psi_2)\end{aligned}$$

Notice that $\mathbf{MKC}^C_{A,\Delta_1,J\Psi_1}$ corresponds to the one-premise form of the subtraction introduction rule

$$\frac{x : A \vdash_L \Delta_1, e : B; \Psi_1}{x : A \vdash_L \Delta_1, \mathtt{mkc}(e, y) : B \bullet\!\!- C, y(e) : C; \Psi_1}\ LsubI$$

which is equivalent in terms of provability to the more general form considered here (cfr Crolard 2004).

3.4.2.2. *Subtraction elimination.* The subtraction elimination rule has the form

$$\frac{x : A \vdash_L \Delta_1, e_1 : B \bullet\!\!- C; \Psi_1 \qquad y : B \vdash_L e_2 : C, \Delta_2; \Psi_2 \quad |\Psi_1| = |\Psi_2|}{x : A \vdash_L \mathtt{postp}(y \mapsto e_2, e_1), \Delta_1, [y(e_1)/y]\Delta_2; \Psi_1, [y(e_1)/y]\Psi_2} \setminus \mathrm{E}$$

This suggests a natural transformation with components

$$\Psi_{A,\Delta_1,\Delta_2,J\Psi_1,J\Psi_2} : \mathcal{L}(A, \Delta_1 \bullet (B \bullet\!\!- C) \bullet J\Psi_1) \times \mathcal{L}(B, C \bullet \Delta_2 \bullet J\Psi_2) \to \mathcal{L}(A, \Delta_1 \bullet \Delta_2 \bullet J\Psi_1 \bullet J\Psi_2)$$

natural in $A, \Delta_1, \Delta_2, J\Psi_1, J\Psi_2$. Here $\mathtt{postp}(y \mapsto e_2, e_1)$ is given type 1 and an application of left identity λ_{1,Δ_2} is assumed implicitly.

Given

$$e : A \to \Delta_1 \bullet (B \bullet\!\!- C) \bullet J\Psi_1, \qquad f : B \to C \bullet \Delta_2 \bullet J\Psi_2$$

and also $a : A' \to A$, $d_1 : \Delta_1 \to \Delta'_1$, $d_2 : \Delta_2 \to \Delta'_2$, $p_1 : J\Psi_1 \to J\Psi'_1$ $p_2 : J\Psi_2 \to J\Psi'_2$ naturality yields

$$\Psi_{A',\Delta_1',\Delta_2',J\Psi_1',J\Psi_2'}((a;e;d_1 \bullet id_{B\bullet\!\!-C} \bullet p_1),(f;id_C \bullet d_2 \bullet p_2)) =$$
$$a;\Psi_{A,\Delta_1,\Delta_2,J\Psi_1}(e,f);\lambda_{1,\Delta_1} \bullet d_1 \bullet d_2 \bullet p_1 \bullet p_2;\mathtt{Join}(\Delta_1',\Delta_2',J\Psi_1,J\Psi_2)$$

In particular, taking $a : A \to \Delta_1 \bullet (B \bullet\!\!- C)$, $e = id_{\Delta_1\bullet(B\bullet\!\!-C)}$, $d_1 = id_{\Delta_1}$, $d_2 : id_{\Delta_2}$, $p_1 = id_{J\Psi_1}$, $p_2 = idJ\Psi_2$ we obtain

$$\Psi_{A,\Delta_1,\Delta_2}(a,f) = a;\Psi_{A,\Delta_1,\Delta_2}(id_{\Delta_1\bullet(C\bullet\!\!-D)\bullet id_{J\Psi_1}},f);\mathtt{Join}(\Delta_1,\Delta_2,J\Psi_1,J\Psi_2)$$

Writing $\mathbf{POSTP}(f)$ for $\Psi_{A,\Delta_1,\Delta_2,J\Psi_1,J\Psi_2}(id_{\Delta_1\bullet(B\bullet\!\!-C)\bullet J\Psi_1},f)$ we define

$$[\![x : A \vdash_L \Delta_1, \mathtt{postp}(y \mapsto e_2, e_1), [y(e_1)/y]\Delta_2; \Psi_1, [y(e_1)/y], \Psi_2]\!] =_{df}$$
$$[\![x : A \vdash_L \Delta_1, e_1 : B \bullet\!\!- C]\!]; id_{\Delta_1} \bullet \mathbf{POSTP}[\![y : B \vdash_L e_2 : C, \Delta_2; \Psi_2]\!]; \mathtt{Join}(\Delta_1,\Delta_2,J\Psi_1,J\Psi_2)$$

3.4.2.3. *Equations in context.* We have equations in context of the form

$$\boxed{\dfrac{\begin{array}{c} \bullet\!\!- - \beta \text{ rules:} \\ e_p \equiv \mathtt{postp}(z \mapsto e_2, \mathtt{mkc}(e_1,y)) \quad e_z \equiv z(\mathtt{mkc}(e_1,y)) \\ |\Delta_1| = |\Delta_1'|\ |\Psi_1| = |\Psi_1'| \qquad x : B \vdash_L e_1 : A_1, \Delta_1; \Psi_1 = e_1' : A_1, \Delta_1'; \Psi_1' \\ |\Delta_2| = |\Delta_2'|\ |\Psi_2| = |\Psi_2'| \qquad y : A_2 \vdash_L \Delta_2; \Psi_2 = \Delta_2'; \Psi_2' \\ z : A_1 \vdash_L e_2 : A_2, \Delta_3; \Psi_3 = e_2' : A_2, \Delta_3'; \Psi_3' \end{array}}{\begin{array}{c} x : B \vdash_L \Delta_1.e_p, [y(e_1)/y]\Delta_2, [e_z/z]\Delta_3; \Psi_1, [y(e_1))y]\Psi_2, [e_z/z]\Psi_3 = \\ = \Delta_1', [[e_1'/z]e_2'/y]\Delta_2, [e_1'/z]\Delta_3'; \Psi_1', [[e_1'/z]e_2'/y]\Psi_2', [e_1'/z]\Psi_3' \end{array}}} \tag{3.3}$$

We repeat the derivations of the redex and of the reductum.

Redex:

$$\dfrac{\dfrac{x : B \vdash_L e_1 : A_1, \Delta_1; \Psi_1 \qquad y : A_2 \vdash_L \Delta_2; \Psi_2}{x : B \vdash_L \mathtt{mkc}(e_1,y) : A_1 \bullet\!\!- A_2, \Delta_1, [y(e_1)/y]\Delta_2; \Psi_1, [y(e_1)/y]\Psi_2} \qquad z : A_1 \vdash_L e_2 : A_2, \Delta_3; \Psi_3}{\begin{array}{r} x : B \vdash_L \Delta_1, \overbrace{\mathtt{postp}(z \mapsto e_2, \mathtt{mkc}(e_1,y))}^{e_p}, [y(e_1)/y]\Delta_2, [\overbrace{z(\mathtt{mkc}(e_1,y))}^{e_z}/z]\Delta_3; \\ \Psi_1, [y(e_1)/y]\Psi_2, [z(\mathtt{mkc}(e_1,y)/z)] : \Psi_3 \end{array}}$$

Reductum:

$$\dfrac{\dfrac{x : B \vdash_L e_1' : A_1, \Delta_1'; \Psi_1' \qquad z : A_1 \vdash_L e_2' : A_2, \Delta_3'; \Psi_3'}{x : B \vdash_L \Delta_1', [e_1'/z]\Delta_3', [e_1'/z]e_2' : A_2; \Psi_1', [e_1'/z]\Psi_3'} \qquad y : A_2 \vdash_L \Delta_2; \Psi_2}{x : B \vdash_L \Delta_1', [[e_1'/z]e_2'/y]\Delta_2',\ [e_1'/z]\Delta_3'; \Psi_1', [[e_1'/z]e_2']\Psi_2',\ [e_1'/z]\Psi_3'}$$

Given morphisms $n : B \to \Delta_1 \bullet A_1$ and $m : A_1 \to \Delta_3 \bullet A_2$, for these equations to be satisfied we need the following diagram to commute (omitting non-linear terms):

$$\begin{tikzcd} B \arrow[r, "n"] \arrow[d, "\mathbf{MKC}^{A_2}(n)"'] & \Delta_1 \bullet A_1 \arrow[d, "id_{\Delta_1} \bullet m"] \\ \Delta_1 \bullet (A_1 \bullet\!\!- A_2) \bullet A_2 \arrow[r, "\mathbf{POSTP}(m) \bullet id_{A_2}"'] & \Delta_1 \bullet \Delta_3 \bullet A_2 \end{tikzcd}$$

in particular, taking $n = id_{A_1}$ we have

$$\begin{tikzcd} A_1 \arrow[r, "m"] \arrow[d, "\mathbf{MKC}_2^A(id_{A_1})"'] & \Delta_3 \bullet A_2 \\ (A_1 \bullet\!\!- A_2) \bullet A_2 \arrow[ur, "\mathbf{POSTP}(m) \bullet id_{A_2}"'] & \end{tikzcd}$$

Assuming the above decomposition to be unique, we can show that the η equation in context is also satisfied:

$$\boxed{\frac{|\Delta| = |\Delta'| \quad |\Psi| = |\Psi'| \quad z; B \vdash_L \Delta, \Psi = \Delta', \Psi'}{z : B \vdash_L \texttt{postp}(x \mapsto y, e), \texttt{mkc}(x(e), y) : A_1 \bullet\!\!- A_2, \Delta, \Psi = e : A_1 \bullet\!\!- A_2, \Delta'; \Psi'}} \tag{3.4}$$

and conclude that there is a natural isomorphism between the maps

$$\frac{A \to \Delta \bullet B}{\overline{A \setminus B \to \Delta}}$$

i.e., that $\setminus$ is the left adjoint to the bifunctor $\bullet$.

3.4.3. *Unit rules.* The introduction and elimination rules for the unit $\perp$ are

$$\begin{array}{cc} \perp \textit{ introduction} & \perp \textit{ elimination} \\ \dfrac{x : A \vdash_L \Delta; \Psi \quad r : B \in Delta, \text{ or } r : S \in \Psi}{x : A \vdash_L \Delta, \texttt{connect}_\perp \texttt{ to}(r) : \perp; \Psi} & \dfrac{x : A \vdash_L e : \perp, \Delta; \Psi}{x : A \vdash_L \texttt{postp}_\perp(e), \Delta; \Psi} \end{array}$$

Both the introduction and the elimination rules can be interpreted by a map $a : A \to 1$, if we give the term $\texttt{postp}_\perp(e)$ the type $\perp$. However, it is a syntactic requirement that in the case of an elimination the type $\perp$ cannot be a subtype of another type.

The introduction rule requires a natural transformation with components

$$\Phi_{A,\Delta,J\Psi} : \mathcal{L}(A, \Delta, J\Psi) \to \mathcal{L}(A, \Delta \bullet \perp \bullet J\Psi)$$

natural in A, Δ, $J\Psi$. Given morphisms $e : A \to \Delta \bullet J\Psi$, $d : A' \to A$, $c : \Delta \to \Delta'$ and $p : J\Psi \to J\Psi'$, naturality yields

$$\Phi_{A',\Delta',J\Psi}(d; e; c \bullet id_{J\Psi}) = d; \Phi_{A,\Delta}(e); c \bullet id_{J\Psi}.$$

Letting $d : A \to \Delta, J\Psi$ and $e = id_{\Delta \bullet J\Psi}$, $c = id_{\Delta \bullet \perp}$ we have

$$\Phi_{A,\Delta,J\Psi}(d) = d; \mathbf{Bot}_{\Delta,J\Psi}$$

where we write $\mathbf{Bot}_{\Delta,J\Psi}$ for $\Phi_{\Delta,J\Psi}(id_\Delta, id_{J\Psi})$. We define

$$[\![x : A \vdash_L \Delta\texttt{connect to}(x) : \perp; J\Psi]\!] =_{df} [\![x : A \vdash_L \Delta, J\Psi]\!]; \mathbf{Bot}_{\Delta,J\Psi}.$$

3.4.3.2. *Equations in context.*

The equation in context

$$\boxed{\begin{array}{c} \perp - \beta \textbf{ rule} \\ \dfrac{x : A \vdash_L \Delta; \Psi = \Delta'; \Psi'}{x : A \vdash_L \Delta, \texttt{postp}_\perp(\texttt{connect}_\perp \texttt{ to}(x)); \Psi = \Delta'; \Psi'} \end{array}} \tag{3.5}$$

requires that for any $m : D \to \Gamma$ the following diagram commutes:

$$\begin{tikzcd} D \arrow[r, "m;\mathbf{Bot}"] \arrow[dr, "m"'] & \Gamma \bullet \perp \arrow[d, "id_\Gamma \bullet \langle\rangle;\lambda_A"] \\ & \Gamma. \end{tikzcd}$$

Assuming that this decomposition is unique and taking $m = id_A$ we have that $\mathbf{Bot}_A; id_A \bullet \langle\rangle; \lambda_A = id_A$. Arguing as before, we see that there is a natural isomorphism

$$\frac{D \to \Gamma \bullet 1}{\overline{D \to \Gamma \bullet \perp}}$$

(so we identify $\perp$ and 1) and that the following equation in context is satisfied:

$$\boxed{\begin{array}{c}\perp - \eta \textbf{ rule:}\\ \dfrac{z : B \vdash_L \Delta; \Psi = \Delta'; \Psi'}{z : B \vdash_L \mathtt{connect}_\perp\ \mathtt{to}(x) : \perp, \mathtt{postp}_\perp(e)\Delta; \Psi = \perp e : \perp, \Delta'; \Psi'}\end{array}} \tag{3.6}$$

3.4.4. *Functors.* Recall that a model of Linear-Non Linear co-intuitionistic logic consists of a symmetric comonoidal adjunction $\mathcal{L} : H \dashv J : C$ where $\mathcal{L} = (\mathcal{L}, \perp, \oplus, \multimap)$ is a symmetric monoidal coclosed category and $C = (C, 0, +, -)$ is a cocartesian coclosed category.

We use the same symbols for the functors $J : C \to \mathcal{L}$ and $\mathcal{H} : \mathcal{L} \to C$ in the models and for the operators that represent them in the language.

3.4.4.1 *rules for $J : C \to \mathcal{L}$.*

$$\boxed{\begin{array}{c}\textit{J introduction}\\ \dfrac{x : A \vdash_L, \Delta; t : T, \Psi}{x : A \vdash_L \Delta, Jt : JT; \Psi}\end{array}} \tag{3.7}$$

If $\Delta = \overline{R} : \Delta$ and $\Psi = \overline{S} : \Psi$, then the categorical interpetation of the rule is an application of α^{-1}:

$$\dfrac{A \xrightarrow{\overline{R} \bullet Jt \bullet J\overline{S}} \Delta \bullet JT \bullet J\Psi}{A \xrightarrow{(\overline{R} \bullet Jt) \bullet J\overline{S}} (\Delta \bullet JT) \bullet J\Psi}$$

$$\boxed{\begin{array}{c}\textit{J elimination}\\ \dfrac{x : A \vdash_L \Delta, e : JT; \Psi_1 \qquad y : T \vdash_C \Psi_2 \text{ where } |\Psi_1| = |\Psi_2|}{x : A \vdash_L \Delta; \Psi_1 \cdot \mathtt{let}\ Jy = e\ \mathtt{in}\ \Psi_2}\end{array}} \tag{3.8}$$

If $\Delta = \overline{R} : \Delta$, $\Psi_1 = \overline{R'} : \Psi_1$, $\Psi_2 = \overline{S} : \Psi_2$, then the categorical interpretation of the rule is given by an operation of the form

$$\mathcal{L}(A, \Delta \bullet JT \bullet J\Psi_1) \times C(T, \Psi_2) \to \mathcal{L}(A, \Delta \bullet J\Psi_1 \bullet J\Psi_2)$$

given by the following compositions

$$\dfrac{A \xrightarrow{\overline{R} \bullet e \bullet J\overline{R}'} \Delta \bullet JT \bullet J\Psi_1 \qquad \dfrac{T \xrightarrow{\overline{S}} \Psi_2 \quad \text{in } C}{JT \xrightarrow{J\overline{S}} J\Psi_2 \quad \text{in } \mathcal{L}}}{\begin{array}{r}A \xrightarrow{\overline{R} \bullet e \bullet J(\overline{R}')} \Delta \bullet J(T) \bullet J(\Psi_1) \xrightarrow{id_\Delta \bullet J(\overline{S}) \bullet id_{J(\Psi_1)}} \Delta \bullet J(\Psi_1) \bullet J(\Psi_2) \xrightarrow{id_\Delta \bullet j^{-1}_{\Psi_1 \Psi_2}} \Delta \bullet J(\Psi_1 + \Psi_2)\\ \text{since } |\Psi_1| = |\Psi_2|, \qquad \xrightarrow{id_\Delta \bullet \nabla_{\Psi_1}} \Delta \bullet J(\Psi_1)\end{array}}$$

3.4.4.2 *rules for $H : \mathcal{L} \to C$.*

$$\boxed{\begin{array}{c}\textit{H intro}\\ \dfrac{x : A \vdash_L \overline{R} : \Delta, e : B; \Psi}{x : A \vdash_L \overline{R} : \Delta; He : HB, \Psi}\end{array}} \tag{3.9}$$

Let $\Delta = \overline{R} : \Delta$ and $\Psi = \overline{S} : \Psi$: then

$$\frac{A \xrightarrow{\overline{R}\wp e \wp J(\overline{S})} \Delta \bullet B \bullet J(\Psi)}{A \xrightarrow{\overline{R}\wp JH(e) \bullet J(\overline{S})} \Delta \bullet JH(B) \bullet J(\Psi)} \text{ using } \eta_B : B \to JHB$$

$$\boxed{\begin{array}{c} H\ elim_1 \\ \dfrac{x : B \vdash_L \Delta; t : HA, \Psi_1 \qquad y : A \vdash_L ; \Psi_2 \quad \text{where } |\Psi_1| = |\Psi_2|}{x : B \vdash_L \Delta; \Psi_1 \cdot (\texttt{let}\ Hy = t\ \texttt{in}\ \Psi_2)} \end{array}} \tag{3.10}$$

$$\boxed{\begin{array}{c} H\ elim_2 \\ \dfrac{x : S \vdash_C t : HA, \Psi_1 \qquad y : A \vdash_L ; \Psi_2 \quad \text{where } |\Psi_1| = |\Psi_2|}{x : S \vdash_C \Psi_1 \cdot (\texttt{let}\ Hy = t\ \texttt{in}\ \Psi_2)} \end{array}} \tag{3.11}$$

The categorical interpretation of H elim$_2$ is as follows: Let $\Psi_1 = \overline{R} : \Psi_1$ and $\Psi_2 = \overline{S} : \Psi_2$. Then we have the following compositions:

$$\dfrac{S \xrightarrow{t+\overline{R}} H(A) + \Psi \qquad \dfrac{A \xrightarrow{J(\overline{S})} J(\Psi) \quad \text{in } \mathcal{L}}{HA \xrightarrow{HJ(\overline{S})} HJ(\Psi) \xrightarrow{\epsilon_\Psi} \Psi \quad \text{in } C}}{S \xrightarrow{t+\overline{R}} H(A) + \Psi \xrightarrow{HJ(\overline{S})+id_\Psi} \Psi + \Psi \xrightarrow{\nabla_\Psi} \Psi}$$

4. Related Work

TODO

5. Conclusion

TODO

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Appendix A. Proofs

A.1. **Proof of Lemma 23.** We show that both of the maps:

$$\mathsf{j}^{-1}_{R,S} := \mathsf{J}R \oplus \mathsf{J}S \xrightarrow{\eta} \mathsf{JH}(\mathsf{J}R \oplus \mathsf{J}S) \xrightarrow{\mathsf{Jh}_{A,B}} \mathsf{J}(\mathsf{HJ}R + \mathsf{HJ}S) \xrightarrow{\mathsf{J}(\varepsilon_R + \varepsilon_S)} \mathsf{J}(R+S)$$

$$\mathsf{j}^{-1}_{0} := \perp \xrightarrow{\eta} \mathsf{JH}\perp \xrightarrow{\mathsf{Jh}_\perp} \mathsf{J}0$$

are mutual inverses with $\mathsf{j}_{R,S} : \mathsf{J}(R+S) \longrightarrow \mathsf{J}R \oplus \mathsf{J}S$ and $\mathsf{j}_0 : \perp \longrightarrow \mathsf{J}0$ respectively.

Case. The following diagram implies that $\mathsf{j}^{-1}_{R,S} ; \mathsf{j}_{R,S} = \mathsf{id}$:

$$\begin{tikzcd}
 & \mathsf{J}R \oplus \mathsf{J}S \arrow[r, "\eta"] \arrow[dl, equal] \arrow[d, "\eta\oplus\eta"] & \mathsf{JH}(\mathsf{J}R \oplus \mathsf{J}S) \arrow[d, "\mathsf{Jh}"] \\
\mathsf{J}R \oplus \mathsf{J}S & \mathsf{JHJ}R \oplus \mathsf{JHJ}S \arrow[l, "\mathsf{J}\varepsilon\oplus\mathsf{J}\varepsilon"'] & \mathsf{J}(\mathsf{HJ}R + \mathsf{HJ}S) \arrow[l, "\mathsf{j}"'] \arrow[d, "\mathsf{J}(\varepsilon+\varepsilon)"] \\
 & & \mathsf{J}(R+S) \arrow[ull, "\mathsf{j}"]
\end{tikzcd}$$

The two top diagrams both commute because η and ε are the unit and counit of the adjunction respectively, and the bottom diagram commutes by naturality of j.

Case. The following diagram implies that $\mathsf{j}_{R,S} ; \mathsf{j}^{-1}_{R,S} = \mathsf{id}$:

$$\begin{tikzcd}
 & \mathsf{J}(R+S) \arrow[r, "\mathsf{j}"] \arrow[dl, equal] \arrow[d, "\eta"] & \mathsf{J}R \oplus \mathsf{J}S \arrow[d, "\eta"] \\
\mathsf{J}(R+S) & \mathsf{JHJ}(R+S) \arrow[l, "\mathsf{J}\varepsilon"'] \arrow[r, "\mathsf{JHj}"] & \mathsf{JH}(\mathsf{J}R \oplus \mathsf{J}S) \arrow[d, "\mathsf{Jh}"] \\
 & & \mathsf{J}(\mathsf{HJ}R + \mathsf{HJ}S) \arrow[ull, "\mathsf{J}(\varepsilon+\varepsilon)"]
\end{tikzcd}$$

The top left and bottom diagrams both commute because η and ε are the unit and counit of the adjunction respectively, and the top right diagram commutes by naturality of η.

Case. The following diagram implies that $\mathsf{j}^{-1}_0 ; \mathsf{j}_0 = \mathsf{id}$:

$$\begin{tikzcd}
\perp \arrow[r, "\eta"] \arrow[d, equal] & \mathsf{JH}\perp \arrow[d, "\mathsf{Jh}_\perp"] \\
\perp & \mathsf{J}0 \arrow[l, "\mathsf{j}_0"']
\end{tikzcd}$$

This diagram holds because η is the unit of the adjunction.

Case. The following diagram implies that $\mathsf{j}_0 ; \mathsf{j}^{-1}_0 = \mathsf{id}$:

$$\begin{tikzcd}
\mathsf{J}0 \arrow[rr, "\mathsf{j}_0"] \arrow[dr, "\eta"] \arrow[dd, equal] & & \perp \arrow[dd, "\eta"] \\
& \mathsf{JHJ}0 \arrow[dl, "\mathsf{J}\varepsilon"'] \arrow[dr, "\mathsf{JHj}_0"] & \\
\mathsf{J}0 & & \mathsf{JH}\perp \arrow[ll, "\mathsf{Jh}_\perp"']
\end{tikzcd}$$

The top-left and bottom diagrams commute because η and ε are the unit and counit of the adjunction respectively, and the top-right digram commutes by naturality of η.

A.2. **Proof of Lemma 25.** Since ? is the composition of two symmetric comonoidal functors we know it is also symmetric comonoidal, and hence, the following diagrams all hold:

$$\begin{tikzcd}
?((A \oplus B) \oplus C) \arrow[r, "\mathsf{r}_{A\oplus B,C}"] \arrow[d, "?\,\alpha_{A,B,C}"] & ?(A \oplus B) \oplus ?\,C \arrow[d, "\mathsf{r}_{A,B}\oplus\mathsf{id}_{?\,C}"] \\
?(A \oplus (B \oplus C)) \arrow[d, "\mathsf{r}_{A,B\oplus C}"] & (?\,A \oplus ?\,B) \oplus ?\,C \arrow[d, "\alpha_{?\,A,?\,B,?\,C}"] \\
?\,A \oplus ?(B \oplus C)) \arrow[r, "\mathsf{id}_{?\,A}\oplus\mathsf{r}_{B,C}"] & ?\,A \oplus (?\,B \oplus ?\,C)
\end{tikzcd}$$

$$\begin{tikzcd}
?(\perp \oplus A) \arrow[r, "\mathsf{r}_{\perp,A}"] \arrow[d, "?\,\lambda_A"] & ?\perp \oplus ?\,A \arrow[d, "\mathsf{r}_\perp\oplus\mathsf{id}_{?\,A}"] \\
?\,A \arrow[r, "\lambda^{-1}{}_{?\,A}"] & \perp \oplus ?\,A
\end{tikzcd}
\qquad
\begin{tikzcd}
?(A \oplus \perp) \arrow[r, "\mathsf{r}_{A,\perp}"] \arrow[d, "?\,\rho_A"] & ?\,A \oplus ?\perp \arrow[d, "\mathsf{id}_{?\,A}\oplus\mathsf{r}_\perp"] \\
?\,A \arrow[r, "\rho^{-1}{}_{?\,A}"] & ?\,A \oplus \perp
\end{tikzcd}$$

$$\begin{tikzcd}
?(A \oplus B) \arrow[r, "\mathsf{r}_{A,B}"] \arrow[d, "?\,\beta_{A,B}"] & ?\,A \oplus ?\,B \arrow[d, "\beta_{?\,A,?\,B}"] \\
?(B \oplus A) \arrow[r, "\mathsf{r}_{B,A}"] & ?\,B \oplus ?\,A
\end{tikzcd}$$

Next we show that $(?, \eta, \mu)$ defines a monad where $\eta_A : A \longrightarrow ?A$ is the unit of the adjunction, and $\mu_A = \mathsf{J}\varepsilon_{\mathsf{H}A} : ?\,?\,A \longrightarrow ?\,A$. It suffices to show that every diagram of Definition 13 holds.

Case.

$$\begin{tikzcd}
?^3 A \arrow[r, "\mu_{?A}"] \arrow[d, "?\mu_A"] & ?^2 A \arrow[d, "\mu_A"] \\
?^2 A \arrow[r, "\mu_A"'] & ?A
\end{tikzcd}$$

It suffices to show that the following diagram commutes:

$$\begin{tikzcd}
\mathsf{J}(\mathsf{H}(?^2 A)) \arrow[r, "\mathsf{J}\varepsilon_{\mathsf{H}\,?A}"] \arrow[d, "\mathsf{J}(\mathsf{H}\mu_A)"] & \mathsf{J}(\mathsf{H}\ ?A) \arrow[d, "\mathsf{J}\varepsilon_{\mathsf{H}A}"] \\
\mathsf{J}(\mathsf{H}\ ?A) \arrow[r, "\mathsf{J}\varepsilon_{\mathsf{H}A}"'] & \mathsf{J}(\mathsf{H}A)
\end{tikzcd}$$

But this diagram is equivalent to the following:

$$\begin{tikzcd}
\mathsf{HJHJH}A \arrow[r, "\varepsilon_{\mathsf{HJH}A}"] \arrow[d, "\mathsf{HJ}\varepsilon_{\mathsf{H}A}"] & \mathsf{HJH}A \arrow[d, "\varepsilon_{\mathsf{H}A}"] \\
\mathsf{HJH}A \arrow[r, "\varepsilon_{\mathsf{H}A}"'] & \mathsf{H}A
\end{tikzcd}$$

The previous diagram commutes by naturality of ε.

Case.

$$\begin{tikzcd}
 & ?A & \\
?A \arrow[ur, equal] \arrow[r, "\eta_{?A}"'] & ?^2 A \arrow[u, "\mu_A"] & ?A \arrow[ul, equal] \arrow[l, "?\eta_A"]
\end{tikzcd}$$

It suffices to show that the following diagrams commutes:

$$\begin{tikzcd}
 & JHA & \\
JHA \arrow[ur, equal] \arrow[r, "\eta_{JHA}"'] & JHJHA \arrow[u, "J\varepsilon_{HA}"] & JHA \arrow[ul, equal] \arrow[l, "JH\eta_A"]
\end{tikzcd}$$

Both of these diagrams commute because η and ε are the unit and counit of an adjunction.

It remains to be shown that η and μ are both symmetric comonoidal natural transformations, but this easily follows from the fact that we know η is by assumption, and that μ is because it is defined in terms of ε which is a symmetric comonoidal natural transformation. Thus, all of the following diagrams commute:

$$\begin{tikzcd}
A \oplus B \arrow[r, "\eta_A \oplus \eta_B"] \arrow[d, "\eta_A"'] & ?A \oplus ?B \\
?(A \oplus B) \arrow[ur, "\mathsf{r}_{A,B}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\perp \arrow[r, "\eta_\perp"] \arrow[dr, equal] & ?\perp \arrow[d, "\mathsf{r}_\perp"] \\
 & \perp
\end{tikzcd}$$

$$\begin{tikzcd}
?^2(A \oplus B) \arrow[r, "?\mathsf{r}_{A,B}"] \arrow[d, "\mu_{A\oplus B}"'] & ?(?A \oplus ?B) \arrow[r, "\mathsf{r}_{?A,?B}"] & ?^2 A \oplus ?^2 B \arrow[d, "\mu_A \oplus \mu_B"] \\
?(A \oplus B) \arrow[rr, "\mathsf{r}_{A,B}"] & & ?A \oplus ?B
\end{tikzcd}
\qquad
\begin{tikzcd}
?^2 \perp \arrow[r, "?\mathsf{r}_\perp"] \arrow[d, "\mu_\perp"'] & ?\perp \arrow[d, "\mathsf{r}_\perp"] \\
?\perp \arrow[r, "\mathsf{r}_\perp"'] & \perp
\end{tikzcd}$$

A.3. **Proof of Lemma 26.** Suppose (H, h) and (J, j) are two symmetric comonoidal functors, such that, $\mathcal{L} : \mathsf{H} \dashv \mathsf{J} : C$ is a dual LNL model. Again, we know $?A = H; J : \mathcal{L} \longrightarrow \mathcal{L}$ is a symmetric comonoidal monad by Lemma 25.

We define the following morphisms:

$$\mathsf{w}_A := \perp \xrightarrow{\mathsf{j}_0^{-1}} \mathsf{J}0 \xrightarrow{\mathsf{J}\diamond_{\mathsf{H}A}} \mathsf{JH}A = ?A$$

$$\mathsf{c}_A := ?A \oplus ?A = \mathsf{JH}A \oplus \mathsf{JH}A \xrightarrow{\mathsf{j}^{-1}_{\mathsf{H}A,\mathsf{H}A}} \mathsf{J}(\mathsf{H}A + \mathsf{H}A) \xrightarrow{\mathsf{J}\nabla_{\mathsf{H}A}} \mathsf{JHA} = ?A$$

Next we show that both of these are symmetric comonoidal natural transformations, but for which functors? Define $\mathsf{W}(A) = \perp$ and $\mathsf{C}(A) = ?A \oplus ?A$ on objects of $\mathcal{L}$, and $\mathsf{W}(f : A \longrightarrow B) = \mathsf{id}_\perp$ and $\mathsf{C}(f : A \longrightarrow B) = ?f \oplus ?f$ on morphisms. So we must show that $\mathsf{w} : \mathsf{W} \longrightarrow ?$ and $\mathsf{c} : \mathsf{C} \longrightarrow ?$ are symmetric comonoidal natural transformations. We first show that w is and then we show that c is. Throughout the proof we drop subscripts on natural transformations for readability.

Case. To show w is a natural transformation we must show the following diagram commutes for any morphism $f : A \longrightarrow B$:

$$\begin{tikzcd}
W(A) \arrow[r, "\mathsf{w}_A"] \arrow[d, "W(f)"'] & ?A \arrow[d, "?f"] \\
W(B) \arrow[r, "\mathsf{w}_B"'] & ?B
\end{tikzcd}$$

This diagram is equivalent to the following:

$$\begin{tikzcd}
\perp \arrow[r, "\mathsf{w}_A"] \arrow[d, "\mathsf{id}_\perp"'] & ?A \arrow[d, "?f"] \\
\perp \arrow[r, "\mathsf{w}_B"'] & ?B
\end{tikzcd}$$

It further expands to the following:

$$\begin{tikzcd}
\perp \arrow[r, "\mathsf{j}_0^{-1}"] \arrow[d, "\mathsf{id}_\perp"'] & \mathsf{J}0 \arrow[r, "\mathsf{J}(\diamond_{\mathsf{H}A})"] & \mathsf{JH}A \arrow[d, "\mathsf{JH}f"] \\
\perp \arrow[r, "\mathsf{j}_0^{-1}"'] & \mathsf{J}0 \arrow[r, "\mathsf{J}(\diamond_{\mathsf{H}B})"'] & \mathsf{JH}B
\end{tikzcd}$$

This diagram commutes, because $\mathsf{J}(\diamond_{\mathsf{H}A}); \mathsf{J}f = \mathsf{J}(\diamond_{\mathsf{H}A}; f) = \mathsf{J}(\diamond_{\mathsf{H}B})$, by the uniqueness of the initial map.

Case. The functor W is comonoidal itself. To see this we must exhibit a map

$$\mathsf{s}_\perp := \mathsf{id}_\perp : \mathsf{W}\perp \longrightarrow \perp$$

and a natural transformation

$$\mathsf{s}_{A,B} := \rho_\perp^{-1} : \mathsf{W}(A \oplus B) \longrightarrow \mathsf{W}A \oplus \mathsf{W}B$$

subject to the coherence conditions in Definition 8. Clearly, the second map is a natural transformation, but we leave showing they respect the coherence conditions to the reader. Now we can show that w is indeed symmetric comonoidal.

Case.

$$\begin{tikzcd}
\mathsf{W}(A \oplus B) \arrow[r, "\mathsf{s}_{A,B}"] \arrow[d, "\mathsf{w}_{A\oplus B}"] & \mathsf{W}A \oplus \mathsf{W}B \arrow[d, "\mathsf{w}_A \oplus \mathsf{w}_B"] \\
?(A \oplus B) \arrow[r, "\mathsf{r}_{A,B}"] & ?A \oplus ?B
\end{tikzcd}$$

Expanding the objects of the previous diagram results in the following:

$$\begin{tikzcd}
\perp \arrow[r, "\mathsf{s}_{A,B}"] \arrow[d, "\mathsf{w}_{A\oplus B}"] & \perp \oplus \perp \arrow[d, "\mathsf{w}_A \oplus \mathsf{w}_B"] \\
?(A \oplus B) \arrow[r, "\mathsf{r}_{A,B}"] & ?A \oplus ?B
\end{tikzcd}$$

This diagram commutes, because the following fully expanded diagram commutes:

$$
\begin{tikzcd}
\perp \arrow[rrrr, "\rho^{-1}"] \arrow[ddd, "\mathsf{j}_0^{-1}"'] \arrow[drr, phantom, "(6)"] & & & & \perp \oplus \perp \arrow[dll, "\mathsf{j}_0^{-1}\oplus\mathsf{id}"'] \arrow[d, "\mathsf{j}_0^{-1}\oplus\mathsf{j}_0^{-1}"] \arrow[dl, phantom, "(5)"] \\
 & & \mathsf{J}0\oplus\perp \arrow[rr, "\mathsf{id}\oplus\mathsf{j}_0^{-1}"] \arrow[d, equal] \arrow[dr, phantom, "(4)"] & & \mathsf{J}0\oplus\mathsf{J}0 \arrow[dll, "\mathsf{id}\oplus\mathsf{j}_0"] \arrow[dd, equal] \\
 & & \mathsf{J}0\oplus\perp \arrow[d, phantom, "(3)"] & & \\
\mathsf{J}0 \arrow[uurr, "\rho^{-1}"] \arrow[d, "\mathsf{J}\diamond"] \arrow[drr, phantom, "(1)"] & & \mathsf{J}(0+0) \arrow[ll, "\mathsf{J}\rho"'] \arrow[rr, "\mathsf{j}"] \arrow[d, "\mathsf{J}(\diamond+\diamond)"] \arrow[drr, phantom, "(2)"] & & \mathsf{J}0\oplus\mathsf{J}0 \arrow[d, "\mathsf{J}\diamond\oplus\mathsf{J}\diamond"] \\
\mathsf{JH}(A\oplus B) \arrow[rr, "\mathsf{Jh}"] & & \mathsf{J}(\mathsf{H}A+\mathsf{H}B) \arrow[rr, "\mathsf{j}"] & & \mathsf{JH}A\oplus\mathsf{JH}B
\end{tikzcd}
$$

Diagram 1 commutes because 0 is the initial object, diagram 2 commutes by naturality of j, diagram 3 commutes because J is a symmetric comonoidal functor, diagram 4 commutes because j_0 is an isomorphism (Lemma 23), diagram 5 commutes by functorality of J, and diagram 6 commutes by naturality of ρ.

Case.

$$
\begin{tikzcd}
\perp & & ?\perp \arrow[ll, "\mathsf{r}_\perp"'] \\
 & \mathsf{W}\perp \arrow[ul, "\mathsf{s}_\perp"] \arrow[ur, "\mathsf{w}_\perp"'] &
\end{tikzcd}
$$

Expanding the objects in the previous diagram results in the following:

$$
\begin{tikzcd}
\perp & & ?\perp \arrow[ll, "\mathsf{r}_\perp"'] \\
 & \perp \arrow[ul, equal] \arrow[ur, "\mathsf{w}_\perp"'] &
\end{tikzcd}
$$

This diagram commutes because the following one does:

$$
\begin{tikzcd}
\perp \arrow[d, equal] & & \mathsf{J}0 \arrow[ll, "\mathsf{j}_0"'] \\
\perp \arrow[r, "\mathsf{j}_0^{-1}"] & \mathsf{J}0 \arrow[ur, equal] \arrow[r, "\mathsf{J}\diamond"] & \mathsf{JH}\perp \arrow[u, "\mathsf{Jh}_\perp"']
\end{tikzcd}
$$

The diagram on the left commutes because j_0 is an isomorphism (Lemma 23), and the diagram on the right commutes because 0 is the initial object.

Case. Now we show that $\mathsf{c}_A : {?}A \oplus {?}A \longrightarrow {?}A$ is a natural transformation. This requires the following diagram to commute (for any $f : A \longrightarrow B$):

$$\begin{tikzcd}
\mathsf{C}A \arrow[r, "\mathsf{c}_A"] \arrow[d, "\mathsf{C}f"'] & {?}A \arrow[d, "{?}f"] \\
\mathsf{C}B \arrow[r, "\mathsf{c}_B"'] & {?}B
\end{tikzcd}$$

This expands to the following diagram:

$$\begin{tikzcd}
{?}A \oplus {?}A \arrow[r, "\mathsf{c}_A"] \arrow[d, "{?}f \oplus {?}f"'] & {?}A \arrow[d, "{?}f"] \\
{?}B \oplus {?}B \arrow[r, "\mathsf{c}_B"'] & {?}B
\end{tikzcd}$$

This diagram commutes because the following diagram does:

$$\begin{tikzcd}
\mathsf{JH}A \oplus \mathsf{JH}A \arrow[r, "\mathsf{j}^{-1}_{\mathsf{H}A,\mathsf{H}A}"] \arrow[d, "\mathsf{JH}f \oplus \mathsf{JH}f"'] & \mathsf{J}(\mathsf{H}A + \mathsf{H}A) \arrow[r, "\mathsf{J}\nabla_{\mathsf{H}A}"] \arrow[d, "\mathsf{J}(\mathsf{H}f + \mathsf{H}f)"] & \mathsf{JH}A \arrow[d, "\mathsf{JH}f"] \\
\mathsf{JH}B \oplus \mathsf{JH}B \arrow[r, "\mathsf{j}^{-1}_{\mathsf{H}B,\mathsf{H}B}"'] & \mathsf{J}(\mathsf{H}B + \mathsf{H}B) \arrow[r, "\mathsf{J}\nabla_{\mathsf{H}B}"'] & \mathsf{JH}B
\end{tikzcd}$$

The left square commutes by naturality of j^{-1}, and the right square commutes by naturality of the codiagonal $\nabla_A : A + A \longrightarrow A$.

Case. The functor $\mathsf{C} : \mathcal{L} \longrightarrow \mathcal{L}$ is indeed symmetric comonoidal where the required maps are defined as follows:

$$\mathsf{t}_\perp := {?}\perp \oplus {?}\perp = \mathsf{JH}\perp \oplus \mathsf{JH}\perp \xrightarrow{\mathsf{j}^{-1}} \mathsf{J}(\mathsf{H}\perp + \mathsf{H}\perp) \xrightarrow{\mathsf{J}\nabla} \mathsf{JH}\perp \xrightarrow{\mathsf{Jh}_\perp} \mathsf{J}0 \xrightarrow{\mathsf{j}_0} \perp$$

$$\mathsf{t}_{A,B} := {?}(A \oplus B) \oplus {?}(A \oplus B) \xrightarrow{\mathsf{r}_{A,B} \oplus \mathsf{r}_{A,B}} ({?}A \oplus {?}B) \oplus ({?}A \oplus {?}B) \xrightarrow{\mathsf{iso}} ({?}A \oplus {?}A) \oplus ({?}B \oplus {?}B)$$

where iso is a natural isomorphism that can easily be defined using the symmetric monoidal structure of $\mathcal{L}$. Clearly, t is indeed a natural transformation, but we leave checking that the required diagrams in Definition 8 commute to the reader. We can now show that $\mathsf{c}_A : {?}A \oplus {?}A \longrightarrow {?}A$ is symmetric comonoidal. The following diagrams from Definition 10 must commute:

Case.

$$\begin{tikzcd}
\mathsf{C}(A \oplus B) \arrow[r, "\mathsf{t}_{A,B}"] \arrow[d, "\mathsf{c}_{A \oplus B}"'] & \mathsf{C}A \oplus \mathsf{C}B \arrow[d, "\mathsf{c}_A \oplus \mathsf{c}_B"] \\
{?}(A \oplus B) \arrow[r, "\mathsf{r}_{A,B}"'] & {?}A \oplus {?}B
\end{tikzcd}$$

Expanding the objects in the previous diagram results in the following:

$$\begin{tikzcd}
?(A \oplus B) \oplus ?(A \oplus B) \arrow[r, "\mathsf{t}_{A,B}"] \arrow[d, "\mathsf{c}_{A\oplus B}"'] & (?\,A \oplus ?\,A) \oplus (?\,B \oplus ?\,B) \arrow[d, "\mathsf{c}_A \oplus \mathsf{c}_B"] \\
?(A \oplus B) \arrow[r, "\mathsf{r}_{A,B}"'] & ?\,A \oplus ?\,B
\end{tikzcd}$$

This diagram commutes, because the following fully expanded one does:

$$
\begin{tikzcd}
\mathsf{JH}(A \oplus B) \oplus \mathsf{JH}(A \oplus B) \arrow[r, "\mathsf{Jh} \oplus \mathsf{Jh}"] \arrow[d, "\mathsf{j}^{-1}"] \arrow[dr, phantom, "(2)"] & \mathsf{J}(\mathsf{H}A + \mathsf{H}B) \oplus \mathsf{J}(\mathsf{H}A + \mathsf{H}B) \arrow[r, "\mathsf{j} \oplus \mathsf{j}"] \arrow[d, "\mathsf{j}^{-1}"] \arrow[dr, phantom, "(4)"] & (\mathsf{JH}A \oplus \mathsf{JH}B) \oplus (\mathsf{JH}A \oplus \mathsf{JH}B) \arrow[r, "\mathsf{iso}"] \arrow[dr, phantom, "(6)"] & (\mathsf{JH}A \oplus \mathsf{JH}A) \oplus (\mathsf{JH}B \oplus \mathsf{JH}B) \arrow[d, "\mathsf{j}^{-1} \oplus \mathsf{j}^{-1}"] \\
\mathsf{J}(\mathsf{H}(A \oplus B) + \mathsf{H}(A \oplus B)) \arrow[r, "\mathsf{J}(\mathsf{h}+\mathsf{h})"] \arrow[d, "\mathsf{J}\nabla"] \arrow[dr, phantom, "(1)"] & \mathsf{J}((\mathsf{H}A + \mathsf{H}B) + (\mathsf{H}A + \mathsf{H}B)) \arrow[r, "\mathsf{Jiso}"] \arrow[d, "\mathsf{J}\nabla"] \arrow[dr, phantom, "(3)"] & \mathsf{J}((\mathsf{H}A + \mathsf{H}A) + (\mathsf{H}B + \mathsf{H}B)) \arrow[r, "\mathsf{j}"] \arrow[ur, "\mathsf{j};(\mathsf{j} \oplus \mathsf{j})"] \arrow[d, "\mathsf{J}(\nabla + \nabla)"] \arrow[dr, phantom, "(5)"] & \mathsf{J}(\mathsf{H}A + \mathsf{H}A) \oplus \mathsf{J}(\mathsf{H}B + \mathsf{H}B) \arrow[d, "\mathsf{J}\nabla \oplus \mathsf{J}\nabla"] \\
\mathsf{JH}(A \oplus B) \arrow[r, "\mathsf{Jh}"] & \mathsf{J}(\mathsf{H}A + \mathsf{H}B) \arrow[r, equal] & \mathsf{J}(\mathsf{H}A + \mathsf{H}B) \arrow[r, "\mathsf{j}"] & \mathsf{JH}A \oplus \mathsf{JH}B
\end{tikzcd}
$$

Diagram 1 commutes by naturality of ∇, diagram 2 commutes by naturality of j^{-1}, diagram 3 commutes by straightforward reasoning on coproducts, diagram 4 commutes by straightforward reasoning on the symmetric monoidal structure of J after expanding the definition of the two isomorphisms – here Jiso is the corresponding isomorphisms on coproducts – diagram 5 commutes by naturality of j, and diagram 6 commutes because j is an isomorphism (Lemma 23).

Case.

$$\begin{tikzcd}
\perp & & ?\perp \arrow[ll, "\mathsf{r}_\perp"'] \\
& \mathsf{C}\perp \arrow[ul, "\mathsf{t}\perp"] \arrow[ur, "\mathsf{c}_\perp"'] &
\end{tikzcd}$$

Expanding the objects of this diagram results in the following:

$$\begin{tikzcd}
\perp & ?\perp \arrow[l, "\mathsf{r}_\perp"'] \\
?\perp \oplus ?\perp \arrow[u, "\mathsf{t}\perp"] \arrow[r, equal] & ?\perp \oplus ?\perp \arrow[u, "\mathsf{c}_\perp"']
\end{tikzcd}$$

Simply unfolding the morphisms in the previous diagram reveals the following:

$$\begin{tikzcd}
\perp \arrow[r, equal] & \perp \\
\mathsf{J}0 \arrow[u, "\mathsf{j}"] & \mathsf{J}0 \arrow[u, "\mathsf{j}"'] \\
\mathsf{JH}\perp \arrow[u, "\mathsf{Jh}_\perp"] & \mathsf{JH}\perp \arrow[u, "\mathsf{Jh}_\perp"'] \\
\mathsf{J}(\mathsf{H}\perp + \mathsf{H}\perp) \arrow[u, "\mathsf{J}\nabla"] & \mathsf{J}(\mathsf{H}\perp + \mathsf{H}\perp) \arrow[u, "\mathsf{J}\nabla"'] \\
\mathsf{JH}\perp \oplus \mathsf{JH}\perp \arrow[u, "\mathsf{j}^{-1}"] \arrow[r, equal] & \mathsf{JH}\perp \oplus \mathsf{JH}\perp \arrow[u, "\mathsf{j}^{-1}"']
\end{tikzcd}$$

Clearly, this diagram commutes.

At this point we have shown that $\mathsf{w}_A : \perp \longrightarrow ?A$ and $\mathsf{c}_A : ?A \oplus ?A \longrightarrow ?A$ are symmetric comonoidal naturality transformations. Now we show that for any $?A$ the triple $(?A, \mathsf{w}_A, \mathsf{c}_A)$ forms a commutative monoid. This means that the following diagrams must commute:

Case.

$$\begin{tikzcd}
(?A \oplus ?A) \oplus ?A \arrow[r, "\alpha_{?A,?A,?A}"] \arrow[d, "\mathsf{c}_A \oplus \mathsf{id}_A"'] & ?A \oplus (?A \oplus ?A) \arrow[r, "\mathsf{id}_{?A} \oplus \mathsf{c}_A"] & ?A \oplus ?A \arrow[d, "\mathsf{c}_A"] \\
?A \oplus ?A \arrow[rr, "\mathsf{c}_A"] & & ?A
\end{tikzcd}$$

The previous diagram commutes, because the following one does (we omit subscripts for readability):

$$\begin{tikzcd}
(\mathsf{JH}A \oplus \mathsf{JH}A) \oplus \mathsf{JH}A \arrow[r, "\alpha"] \arrow[d, "\mathsf{j}^{-1} \oplus \mathsf{id}"'] & \mathsf{JH}A \oplus (\mathsf{JH}A \oplus \mathsf{JH}A) \arrow[r, "\mathsf{id} \oplus \mathsf{j}^{-1}"] \arrow[d, phantom, "(1)"] & \mathsf{JH}A \oplus \mathsf{J}(\mathsf{H}A + \mathsf{H}A) \arrow[r, "\mathsf{id} \oplus \mathsf{J}\bigtriangledown"] \arrow[d, "\mathsf{j}^{-1}"] \arrow[dr, phantom, "(2)"] & \mathsf{JH}A \oplus \mathsf{JH}A \arrow[d, "\mathsf{j}^{-1}"] \\
\mathsf{J}(\mathsf{H}A + \mathsf{H}A) \oplus \mathsf{JH}A \arrow[r, "\mathsf{j}^{-1}"] \arrow[d, "\mathsf{J}\bigtriangledown \oplus \mathsf{id}"'] \arrow[dr, phantom, "(3)"] & \mathsf{J}((\mathsf{H}A + \mathsf{H}A) + \mathsf{H}A) \arrow[r, "\mathsf{J}\alpha"] \arrow[d, "\mathsf{J}(\bigtriangledown + \mathsf{id})"] & \mathsf{J}(\mathsf{H}A + (\mathsf{H}A + \mathsf{H}A)) \arrow[r, "\mathsf{J}(\mathsf{id} + \bigtriangledown)"] \arrow[d, phantom, "(4)"] & \mathsf{J}(\mathsf{H}A + \mathsf{H}A) \arrow[d, "\mathsf{J}\bigtriangledown"] \\
\mathsf{JH}A \oplus \mathsf{JH}A \arrow[r, "\mathsf{j}^{-1}"] & \mathsf{J}(\mathsf{H}A + \mathsf{H}A) \arrow[rr, "\mathsf{J}\bigtriangledown"] & & \mathsf{JH}A
\end{tikzcd}$$

Diagram 1 commutes because J is a symmetric monoidal functor (Corollary 24), diagrams 2 and 3 commute by naturality of j^{-1}, and diagram 4 commutes because $(\mathsf{H}A, \diamond, \bigtriangledown)$ is a commutative monoid in C, but we leave the proof of this to the reader.

Case.

$$\begin{tikzcd}
?A \oplus \perp \arrow[d, "\mathsf{id}_{?A} \oplus \mathsf{w}_A"'] \arrow[dr, "\rho_{?A}"] & \\
?A \oplus ?A \arrow[r, "\mathsf{c}_A"] & ?A
\end{tikzcd}$$

The previous diagram commutes, because the following one does:

$$\begin{tikzcd}
\mathsf{JH}A \oplus \perp \arrow[rr, "\rho"] \arrow[d, "\mathsf{id} \oplus \mathsf{j}_0^{-1}"'] & \arrow[d, phantom, "(1)"] & \mathsf{JH}A \arrow[d, equal] \\
\mathsf{JH}A \oplus \mathsf{J}0 \arrow[r, "\mathsf{j}^{-1}"] \arrow[d, "\mathsf{id} \oplus \mathsf{J}\diamond"'] \arrow[dr, phantom, "(2)"] & \mathsf{J}(\mathsf{H}A + 0) \arrow[r, "\mathsf{J}\rho"] \arrow[d, "\mathsf{J}(\mathsf{id} \oplus \diamond)"] \arrow[dr, phantom, "(3)"] & \mathsf{JH}A \arrow[d, equal] \\
\mathsf{JH}A \oplus \mathsf{JH}A \arrow[r, "\mathsf{j}^{-1}"] & \mathsf{J}(\mathsf{H}A + \mathsf{H}A) \arrow[r, "\mathsf{J}\bigtriangledown"] & \mathsf{JH}A
\end{tikzcd}$$

Diagram 1 commutes because J is a symmetric monoidal functor (Corollary 24), diagram 2 commutes by naturality of j^{-1}, and diagram 3 commutes because $(\mathsf{H}A, \diamond, \bigtriangledown)$ is a commutative monoid in C, but we leave the proof of this to the reader.

Case.

$$\begin{tikzcd}
?\,A \oplus ?\,A \arrow[d, "\beta_{?A,?A}"'] \arrow[dr, "\mathsf{c}_A"] & \\
?\,A \oplus ?\,A \arrow[r, "\mathsf{c}_A"] & ?\,A
\end{tikzcd}$$

This diagram commutes, because the following one does:

$$\begin{tikzcd}
\mathsf{JH}A \oplus \mathsf{JH}A \arrow[r, "\mathsf{j}^{-1}"] \arrow[d, "\beta"'] & \mathsf{J}(\mathsf{H}A + \mathsf{H}A) \arrow[r, "\mathsf{J}\,\nabla"] \arrow[d, "\mathsf{J}\beta"] & \mathsf{JH}A \arrow[d, equal] \\
\mathsf{JH}A \oplus \mathsf{JH}A \arrow[r, "\mathsf{j}^{-1}"'] & \mathsf{J}(\mathsf{H}A + \mathsf{H}A) \arrow[r, "\mathsf{J}\,\nabla"'] & \mathsf{JH}A
\end{tikzcd}$$

The left diagram commutes by naturality of j^{-1}, and the right diagram commutes because $(\mathsf{H}A, \diamond, \nabla)$ is a commutative monoid in C, but we leave the proof of this to the reader.

Finally, we must show that $\mathsf{w}_A : \perp \longrightarrow ?\,A$ and $\mathsf{c}_A : ?\,A \oplus ?\,A \longrightarrow ?\,A$ are $?$-algebra morphisms. The algebras in play here are $(?\,A, \mu : ?\,?\,A \longrightarrow ?\,A)$, $(\perp, \mathsf{r}_\perp : ?\perp \longrightarrow \perp)$, and $(?\,A \oplus ?\,A, u_A : ?(?\,A \oplus ?\,A) \longrightarrow ?\,A \oplus ?\,A)$, where $u_A := ?(?\,A \oplus ?\,A) \xrightarrow{\mathsf{r}_{?A,?A}} ?^2A \oplus ?^2A \xrightarrow{\mu_A \oplus \mu_A} ?\,A \oplus ?\,A$. It suffices to show that the following diagrams commute:

Case.

$$\begin{tikzcd}
?\perp \arrow[r, "\mathsf{r}_\perp"] \arrow[d, "?\,\mathsf{w}"'] & \perp \arrow[d, "\mathsf{w}"] \\
?\,?\,A \arrow[r, "\mu"'] & ?\,A
\end{tikzcd}$$

This diagram commutes, because the following fully expanded one does:

$$\begin{tikzcd}
\mathsf{JH}\perp \arrow[r, "\mathsf{Jh}_\perp"] \arrow[dd, "\mathsf{JHj}_0^{-1}"'] \arrow[dr, "\mathsf{JHj}_0^{-1}"] & \mathsf{J0} \arrow[rr, "\mathsf{j}_0"] & & \perp \arrow[dd, "\mathsf{j}_0^{-1}"] \\
\arrow[r, phantom, "(3)"] & \mathsf{JHJ0} \arrow[r, "\mathsf{JHj}_0"] \arrow[dl, equal] & \mathsf{JH}\perp \arrow[dr, "\mathsf{Jh}_\perp"] \arrow[u, phantom, "(4)"] & \\
\mathsf{JHJ0} \arrow[rrr, "\mathsf{J}\varepsilon_0"] \arrow[d, "\mathsf{JHJ}\diamond"'] & \arrow[ur, phantom, "(2)"] & & \mathsf{J0} \arrow[d, "\mathsf{J}\diamond"] \\
\mathsf{JHJH}A \arrow[rrr, "\mathsf{J}\varepsilon"] \arrow[u, phantom, "(1)"] & & & \mathsf{JH}A
\end{tikzcd}$$

Diagram 1 commutes by naturality of ε, diagram 2 commutes because ε is the counit of the symmetric comonoidal adjunction, diagram 3 clearly commutes, and diagram 4 commutes because j_0 is an isomorphism (Lemma 23).

Case.

$$\begin{tikzcd}
?(?A \oplus ?A) \arrow[r, "u"] \arrow[d, "?\mathsf{c}"'] & ?A \oplus ?A \arrow[d, "\mathsf{c}"] \\
??A \arrow[r, "\mu"] & ?A
\end{tikzcd}$$

This diagram commutes because the following fully expanded one does:

$$\begin{tikzcd}
\mathsf{JH}(\mathsf{JH}A \oplus \mathsf{JH}A) \arrow[r, "\mathsf{Jh}"] \arrow[d, "\mathsf{JHj}^{-1}"'] \arrow[dr, phantom, "(5)"] & \mathsf{J}(\mathsf{HJH}A + \mathsf{HJH}A) \arrow[r, "\mathsf{j}"] & \mathsf{JHJH}A \oplus \mathsf{JHJH}A \arrow[r, "\mathsf{J}\varepsilon \oplus \mathsf{J}\varepsilon"] \arrow[d, "\mathsf{j}^{-1}"'] \arrow[dr, phantom, "(6)"] & \mathsf{JH}A \oplus \mathsf{JH}A \arrow[d, "\mathsf{j}^{-1}"] \\
\mathsf{JHJ}(\mathsf{H}A + \mathsf{H}A) \arrow[r, "\mathsf{JHj}"] \arrow[dr, "\mathsf{J}\varepsilon"] \arrow[ddd, "\mathsf{JHJ}\triangledown"'] & \mathsf{JH}(\mathsf{JH}A \oplus \mathsf{JH}A) \arrow[r, "\mathsf{Jh}"] \arrow[d, phantom, "(4)"] & \mathsf{J}(\mathsf{HJH}A + \mathsf{HJH}A) \arrow[r, "\mathsf{J}(\varepsilon + \varepsilon)"] \arrow[dl, "\mathsf{J}(\varepsilon + \varepsilon)"'] \arrow[ddd, "\mathsf{J}\triangledown"] \arrow[dddr, phantom, "(7)"] & \mathsf{J}(\mathsf{H}A + \mathsf{H}A) \arrow[ddd, "\mathsf{J}\triangledown"] \\
 & \mathsf{J}(\mathsf{H}A + \mathsf{H}A) \arrow[d, "\mathsf{J}\triangledown"] \arrow[dl, phantom, "(2)"] \arrow[dr, phantom, "(3)"] & & \\
 & \mathsf{JH}A \arrow[d, phantom, "(1)"] & & \\
\mathsf{JHJH}A \arrow[ur, "\mathsf{J}\varepsilon"] \arrow[rr, equal] & & \mathsf{JHJH}A \arrow[ul, "\mathsf{J}\varepsilon"'] \arrow[r, "\mathsf{J}\varepsilon"] & \mathsf{JH}A
\end{tikzcd}$$

Diagram 1 clearly commutes, diagram 2 commutes by naturality of ε, diagram 3 commutes by naturality of $\bigtriangledown$, diagram 4 commutes because ε is the counit of the symmetric comonoidal adjunction, diagram 5 commutes because j is an isomorphism (Lemma 23), diagram 6 commutes by naturality of j^{-1}, and diagram 7 is the same diagram as 3, but this diagram is redundant for readability.

Appendix B. Proof of Lemma 27

Suppose $\mathcal{L} : \mathsf{H} \dashv \mathsf{J} : C$ is a dual LNL model. Then we know $?A = \mathsf{JH}A$ is a symmetric comonoidal monad by Lemma 25. Bellin [1] remarks that by Maietti, Maneggia de Paiva and Ritter's Proposition 25 [7], it suffices to show that $\mu_A : ??A \longrightarrow ?A$ is a monoid morphism. Thus, the following diagrams must commute:

Case.

$$\begin{tikzcd} ??A \oplus ??A \arrow[r, "\mathsf{c}_{?A}"] \arrow[d, "\mu_A \oplus \mu_A"'] & ??A \arrow[d, "\mu_A"] \\ ?A \oplus ?A \arrow[r, "\mathsf{c}_A"] & ?A \end{tikzcd}$$

This diagram commutes because the following fully expanded one does:

$$\begin{tikzcd} \mathsf{JHJH}A \oplus \mathsf{JHJH}A \arrow[r, "\mathsf{j}^{-1}"] \arrow[d, "\mathsf{J}\varepsilon \oplus \mathsf{J}\varepsilon"'] & \mathsf{J}(\mathsf{HJH}A + \mathsf{HJH}A) \arrow[r, "\mathsf{J}\bigtriangledown"] \arrow[d, "\mathsf{J}(\varepsilon + \varepsilon)"] & \mathsf{JHJH}A \arrow[d, "\mathsf{J}\varepsilon"] \\ \mathsf{JH}A \oplus \mathsf{JH}A \arrow[r, "\mathsf{j}^{-1}"] & \mathsf{J}(\mathsf{H}A + \mathsf{H}A) \arrow[r, "\mathsf{J}\bigtriangledown"] & \mathsf{JH}A \end{tikzcd}$$

The left square commutes by naturality of j^{-1} and the right square commutes by naturality of the codiagonal.

Case.

$$\begin{tikzcd} & \perp \arrow[dl, "\mathsf{w}_{?A}"'] \arrow[dr, "\mathsf{w}_A"] & \\ ??A \arrow[rr, "\mu_A"] & & ?A \end{tikzcd}$$

This diagram commutes because the following fully expanded one does:

$$\begin{tikzcd} \perp \arrow[r, equal] \arrow[d, "\mathsf{j}_0^{-1}"'] & \perp \arrow[d, "\mathsf{j}_0^{-1}"] \\ \mathsf{J}0 \arrow[r, equal] \arrow[d, "\mathsf{J}\diamond"'] & \mathsf{J}0 \arrow[d, "\mathsf{J}\diamond"] \\ \mathsf{JHJH}A \arrow[r, "\mathsf{J}\varepsilon"] & \mathsf{JH}A \end{tikzcd}$$

The top square trivially commutes, and the bottom square commutes by uniqueness of the initial map.