

$vars, a, x, y, z, w, m, o, v$

$ivar, n, i, k, j, l$

$R, S, T ::=$
 $| 0$
 $| S + T$
 $| S - T$
 $| HA$

$A, B, C ::=$
 $| \perp$
 $| A \oplus B$
 $| A \bullet B$
 $| JS$

$s, t, r ::=$ non-linear terms
 $| x$
 $| \text{connect}_w \text{ to } t$
 $| t_1 \cdot t_2$
 $| \text{false } t$
 $| x(t)$
 $| \text{mkc}(t, x)$
 $| \text{postp}(x \mapsto t_1, t_2)$
 $| \text{inl } t$
 $| \text{inr } t$
 $| \text{case } t_1 \text{ of } x.t_2, y.t_3$
 $| H e$
 $| \text{let } J x = e \text{ in } t$
 $| \text{let } H x = t_1 \text{ in } t_2$
 $| (t) \quad S$

$e, u, p ::=$ linear terms
 $| x$
 $| \text{connect}_\perp \text{ to } e$
 $| \text{postp}_\perp e$
 $| \text{postp}(x \mapsto e_1, e_2)$
 $| \text{mkc}(e, x)$
 $| x(e)$
 $| e_1 \oplus e_2$
 $| \text{casel } e$
 $| \text{caser } e$
 $| J t$
 $| (e) \quad S$

$\Psi, \Theta ::=$
 $| \cdot$

		T	
		$t : T$	
		(Ψ)	S
Γ, Δ	::=		
		\cdot	
		A	
		$e : A$	
		Γ, Γ'	
		(Γ)	S
		$\text{postp}(x \mapsto e_1, e_2)$	

$$\boxed{S \vdash_{\text{C}} \Psi}$$

$$\begin{array}{c}
\frac{}{S \vdash_{\text{C}} S} \text{C_ID} \\
\frac{S \vdash_{\text{C}} \Psi}{S \vdash_{\text{C}} T, \Psi} \text{C_WK} \\
\frac{S \vdash_{\text{C}} T, T, \Psi}{S \vdash_{\text{C}} T, \Psi} \text{C_CR} \\
\frac{R \vdash_{\text{C}} \Psi_1, S, T, \Psi_2}{R \vdash_{\text{C}} \Psi_1, T, S, \Psi_2} \text{C_EX} \\
\frac{}{0 \vdash_{\text{C}} \Psi} \text{C_FL} \\
\frac{T_1 \vdash_{\text{C}} \Psi_1 \quad T_2 \vdash_{\text{C}} \Psi_2}{T_1 + T_2 \vdash_{\text{C}} \Psi_1, \Psi_2} \text{C_DL} \\
\frac{R \vdash_{\text{C}} \Psi, T_1}{R \vdash_{\text{C}} \Psi, T_1 + T_2} \text{C_DR1} \\
\frac{R \vdash_{\text{C}} \Psi, T_2}{R \vdash_{\text{C}} \Psi, T_1 + T_2} \text{C_DR2} \\
\frac{T_1 \vdash_{\text{C}} T_2, \Psi}{T_1 - T_2 \vdash_{\text{C}} \Psi} \text{C_sL} \\
\frac{S \vdash_{\text{C}} \Psi_1, T_1 \quad T_2 \vdash_{\text{C}} \Psi_2}{S \vdash_{\text{C}} \Psi_1, \Psi_2, T_1 - T_2} \text{C_sR} \\
\frac{S \vdash_{\text{C}} \Psi_1, T \quad T \vdash_{\text{C}} \Psi_2}{S \vdash_{\text{C}} \Psi_1, \Psi_2} \text{C_CUT} \\
\frac{S \vdash_{\text{C}} \Psi, S'' \quad S \vdash_{\text{C}} \Psi'}{S \vdash_{\text{C}} \Psi, \Psi'} \text{C_MCUT} \\
\frac{A \vdash_{\text{L}} ; \Psi}{HA \vdash_{\text{C}} \Psi} \text{C_HL} \\
\frac{T_1 \vdash_{\text{C}} \Psi \quad T_2 \vdash_{\text{C}} \Psi}{T_1 + T_2 \vdash_{\text{C}} \Psi} \text{C_ADL}
\end{array}$$

$$\boxed{A \vdash_{\text{L}} \Delta; \Psi}$$

$$\begin{array}{c}
\frac{}{A \vdash_L A; \cdot} \text{L-ID} \\
\frac{A \vdash_L \Delta; \Psi}{A \vdash_L \Delta; T, \Psi} \text{L-WK} \\
\frac{A \vdash_L \Delta; T, T, \Psi}{A \vdash_L \Delta; T, \Psi} \text{L-CTR} \\
\frac{A \vdash_L \Delta_1, A, B, \Delta_2; \Psi}{A \vdash_L \Delta_1, B, A, \Delta_2; \Psi} \text{L-EX} \\
\frac{A \vdash_L \Delta; \Psi_1, S, T, \Psi_2}{A \vdash_L \Delta; \Psi_1, T, S, \Psi_2} \text{L-Cex} \\
\frac{A \vdash_L \Delta_1, B; \Psi_1 \quad B \vdash_L \Delta_2; \Psi_2}{A \vdash_L \Delta_1, \Delta_2; \Psi_1, \Psi_2} \text{L-CUT} \\
\frac{A \vdash_L \Delta; \Psi_1, T \quad T \vdash_C \Psi_2}{A \vdash_L \Delta; \Psi_1, \Psi_2} \text{L-Ccut} \\
\frac{}{\perp \vdash_L \cdot; \cdot} \text{L-FLl} \\
\frac{A \vdash_L \Delta; \Psi}{A \vdash_L \perp, \Delta; \Psi} \text{L-FLR} \\
\frac{A \vdash_L \Delta; \Psi, T_1}{A \vdash_L \Delta; \Psi, T_1 + T_2} \text{L-dR1} \\
\frac{A \vdash_L \Delta; \Psi, T_2}{A \vdash_L \Delta; \Psi, T_1 + T_2} \text{L-dR2} \\
\frac{B_1 \vdash_L \Delta_1; \Psi_1 \quad B_2 \vdash_L \Delta_2; \Psi_2}{B_1 \oplus B_2 \vdash_L \Delta_1, \Delta_2; \Psi_1, \Psi_2} \text{L-pL} \\
\frac{A \vdash_L \Delta, B, C; \Psi}{A \vdash_L \Delta, B \oplus C; \Psi} \text{L-pR} \\
\frac{B_1 \vdash_L B_2, \Delta; \Psi}{B_1 \bullet B_2 \vdash_L \Delta; \Psi} \text{L-sL} \\
\frac{A \vdash_L B_1, \Delta_1; \Psi_1 \quad B_2 \vdash_L \Delta_2; \Psi_2}{A \vdash_L B \bullet C, \Delta_1, \Delta_2; \Psi_1, \Psi_2} \text{L-sR} \\
\frac{A \vdash_L \Delta; \Psi_1, T_1 \quad T_2 \vdash_C \Psi_2}{A \vdash_L \Delta; \Psi_1, \Psi_2, T_1 - T_2} \text{L-CsR} \\
\frac{T \vdash_C \Psi}{JT \vdash_L \cdot; \Psi} \text{L-JL} \\
\frac{A \vdash_L \Delta; T, \Psi}{A \vdash_L \Delta, JT; \Psi} \text{L-JR} \\
\frac{A \vdash_L \Delta, B; \Psi}{A \vdash_L \Delta; HB, \Psi} \text{L-hR} \\
\frac{A \vdash_L \Delta; \Psi, S^n \quad S \vdash_C \Psi'}{A \vdash_L \Delta; \Psi, \Psi'} \text{L-Cmcut}
\end{array}$$

$S \vdash_C \Psi$ Non-linear Natural Deduction

$$\begin{array}{c}
\frac{}{S \vdash_C S} \text{NC_ID} \\
\frac{S \vdash_C 0, \Psi \quad S_1 \vdash_C \Psi_1, \dots, S_n \vdash_C \Psi_n}{S \vdash_C \Psi, \Psi_1, \dots, \Psi_n} \text{NC_zE} \\
\frac{S \vdash_C \Psi, T_1}{S \vdash_C \Psi, T_1 + T_2} \text{NC_dI1} \\
\frac{S \vdash_C \Psi, T_2}{S \vdash_C \Psi, T_1 + T_2} \text{NC_dI2} \\
\frac{S \vdash_C \Psi_1, T_1 + T_2 \quad T_1 \vdash_C \Psi_2 \quad T_2 \vdash_C \Psi_2}{S \vdash_C \Psi_1, \Psi_2} \text{NC_dE} \\
\frac{S \vdash_C \Psi_1, T_1 \quad T_2 \vdash_C \Psi_2}{S \vdash_C \Psi_1, \Psi_2, T_1 - T_2} \text{NC_subI} \\
\frac{S \vdash_C \Psi_1, T_1 - T_2 \quad T_1 \vdash_C T_2, \Psi_2}{S \vdash_C \Psi_1, \Psi_2} \text{NC_subE} \\
\frac{S \vdash_C \Psi_1, HA \quad A \vdash_L \cdot; \Psi_2}{S \vdash_C \Psi_1, \Psi_2} \text{NC_HE} \\
\frac{S \vdash_C \Psi}{S \vdash_C T, \Psi} \text{NC_WEAK} \\
\frac{S \vdash_C T, T, \Psi}{S \vdash_C T, \Psi} \text{NC_CONTR} \\
\frac{S \vdash_C \Psi_1, T \quad T \vdash_C \Psi_2}{S \vdash_C \Psi_1, \Psi_2} \text{NC_CUT}
\end{array}$$

$A \vdash_L \Delta; \Psi$ Linear Natural Deduction

$$\begin{array}{c}
\frac{}{A \vdash_L A; \cdot} \text{NL_ID} \\
\frac{A \vdash_L \Delta; \Psi}{A \vdash_L \Delta, \perp; \Psi} \text{NL_PI} \\
\frac{A \vdash_L \perp, \Delta; \cdot}{A \vdash_L \Delta; \cdot} \text{NL_PE} \\
\frac{A \vdash_L \Delta, B_1, B_2; \Psi}{A \vdash_L \Delta, B_1 \oplus B_2; \Psi} \text{NL_parI} \\
\frac{A \vdash_L \Delta, B_1 \oplus B_2; \Psi \quad B_1 \vdash_L \Delta_1; \Psi_1 \quad B_2 \vdash_L \Delta_2; \Psi_2}{A \vdash_L \Delta, \Delta_1, \Delta_2; \Psi, \Psi_1, \Psi_2} \text{NL_parE} \\
\frac{A \vdash_L \Delta_1, B_1; \Psi_1 \quad B_2 \vdash_L \Delta_2; \Psi_2}{A \vdash_L B_1 \multimap B_2, \Delta_1, \Delta_2; \Psi_1, \Psi_2} \text{NL_subI} \\
\frac{A \vdash_L \Delta_1, B_1 \multimap B_2; \Psi_1 \quad B_1 \vdash_L B_1, \Delta_2; \Psi_2}{A \vdash_L \Delta_1, \Delta_2; \Psi_1, \Psi_2} \text{NL_subE} \\
\frac{A \vdash_L \Delta; T, \Psi}{A \vdash_L \Delta, \text{J } T; \Psi} \text{NL_JI}
\end{array}$$

$$\begin{array}{c}
\frac{A \vdash_L \Delta, \mathbb{J}T; \Psi_1 \quad T \vdash_C \Psi_2}{A \vdash_L \Delta; \Psi_1, \Psi_2} \text{NL_JE} \\
\frac{A \vdash_L \Delta, B; \Psi}{A \vdash_L \Delta; \mathbb{H}B, \Psi} \text{NL_HI} \\
\frac{A \vdash_L \Delta; \Psi_1, \mathbb{H}A \quad A \vdash_L \cdot; \Psi_2}{A \vdash_L \Delta; \Psi_1, \Psi_2} \text{NL_HE} \\
\frac{A \vdash_L \Delta; \Psi}{A \vdash_L \Delta; T, \Psi} \text{NL_WEAK} \\
\frac{A \vdash_L \Delta; T, T, \Psi}{A \vdash_L \Delta; T, \Psi} \text{NL_CONTR} \\
\frac{A \vdash_L \Delta; \Psi_1, T \quad T \vdash_C \Psi_2}{A \vdash_L \Delta; \Psi_1, \Psi_2} \text{NL_CCUT} \\
\frac{A \vdash_L \Delta_1, B; \Psi_1 \quad B \vdash_L \Delta_2; \Psi_2}{A \vdash_L \Delta_1, \Delta_2; \Psi_1, \Psi_2} \text{NL_CUT}
\end{array}$$

$$\boxed{x : S \vdash_C \Psi}$$

$$\begin{array}{c}
\frac{}{x : S \vdash_C x : S} \text{TC_ID} \\
\frac{x : S \vdash_C t : 0, \Psi \quad x_1 : S_1 \vdash_C \Psi_1, \dots, x_n : S_n \vdash_C \Psi_n}{x : S \vdash_C \Psi, [\text{false } t/x_1]\Psi_1, \dots, [\text{false } t/x_n]\Psi_n} \text{TC_zI} \\
\frac{x : S \vdash_C \Psi, t : T_1}{x : S \vdash_C \Psi, \text{inl } t : T_1 + T_2} \text{TC_dI1} \\
\frac{x : S \vdash_C \Psi, t : T_2}{x : S \vdash_C \Psi, \text{inr } t : T_1 + T_2} \text{TC_dI2} \\
\frac{x : S \vdash_C \Psi_1, t : T_1 + T_2 \quad y : T_1 \vdash_C \Psi_2 \quad z : T_2 \vdash_C \Psi_3 \quad |\Psi_2| = |\Psi_3|}{x : S \vdash_C \Psi_1, \text{case } t \text{ of } y. \Psi_2, z. \Psi_3} \text{TC_dE} \\
\frac{x : S \vdash_C \Psi_1, t : T_1 \quad y : T_2 \vdash_C \Psi_2}{x : S \vdash_C \Psi_1, \text{mkc}(t, y) : T_1 - T_2, [y(t)/y]\Psi_2} \text{TC_SUBI} \\
\frac{x : S \vdash_C \Psi_1, s : T_1 - T_2 \quad y : T_1 \vdash_C t : T_2, \Psi_2}{x : S \vdash_C \Psi_1, \text{postp}(y \mapsto t, s), [y(s)/y]\Psi_2} \text{TC_SUBE} \\
\frac{x : S \vdash_C \Psi_1, t : \mathbb{H}A \quad y : A \vdash_L \cdot; \Psi_2 \quad |\Psi_1| = |\Psi_2|}{x : S \vdash_C \Psi_1 \cdot (\text{let } Hy = t \text{ in } \Psi_2)} \text{TC_HE} \\
\frac{x : S \vdash_C \Psi \quad s : T' \in \Psi}{x : S \vdash_C \text{connect}_w \text{ to } s : T, \Psi} \text{TC_WEAK} \\
\frac{x : S \vdash_C t_1 : T, t_2 : T, \Psi}{x : S \vdash_C (t_1 \cdot t_2) : T, \Psi} \text{TC_CONTR} \\
\frac{x : S \vdash_C \Psi_1, t : T \quad y : T \vdash_C \Psi_2}{x : S \vdash_C \Psi_1, [t/y]\Psi_2} \text{TC_CUT}
\end{array}$$

$$\boxed{x : A \vdash_L \Delta; \Psi}$$

$$\frac{}{x : A \vdash_L x : A; \cdot} \text{TL_ID}$$

$$\begin{array}{c}
\frac{x : A \vdash_{\mathsf{L}} \Delta; \Psi \quad e : B \in \Delta}{x : A \vdash_{\mathsf{L}} \Delta, \text{connect}_{\perp} \text{ to } e : \perp; \Psi} \quad \text{TL_PI} \\
\frac{x : A \vdash_{\mathsf{L}} e : \perp, \Delta; \cdot}{x : A \vdash_{\mathsf{L}} \text{postp}_{\perp} e, \Delta; \cdot} \quad \text{TL_PE} \\
\frac{x : A \vdash_{\mathsf{L}} \Delta, e_1 : B_1, e_2 : B_2; \Psi}{x : A \vdash_{\mathsf{L}} \Delta, e_1 \oplus e_2 : B_1 \oplus B_2; \Psi} \quad \text{TL_PARI} \\
\frac{x : A \vdash_{\mathsf{L}} \Delta, e : B_1 \oplus B_2; \Psi \quad y : B_1 \vdash_{\mathsf{L}} \Delta_1; \Psi_1 \quad z : B_2 \vdash_{\mathsf{L}} \Delta_2; \Psi_2}{x : A \vdash_{\mathsf{L}} \Delta, [\text{caseI}(e)/y]\Delta_1, [\text{caseI}(e)/z]\Delta_2; \Psi, [\text{caseI}(e)/y]\Psi_1, [\text{caseI}(e)/z]\Psi_2} \quad \text{TL_PARE} \\
\frac{x : A \vdash_{\mathsf{L}} \Delta_1, e : B_1; \Psi_1 \quad y : B_2 \vdash_{\mathsf{L}} \Delta_2; \Psi_2}{x : A \vdash_{\mathsf{L}} \text{mkc}(e, y) : B_1 \bullet B_2, \Delta_1, [y(e)/y]\Delta_2; \Psi_1, [y(e)/y]\Psi_2} \quad \text{TL_SUBI} \\
\frac{x : A \vdash_{\mathsf{L}} \Delta_1, e_1 : B_1 \bullet B_2; \Psi_1 \quad y : B_1 \vdash_{\mathsf{L}} e_2 : B_1, \Delta_2; \Psi_2}{x : A \vdash_{\mathsf{L}} \Delta_1, \text{postp}(y \mapsto e_2, e_1), \Delta_2; \Psi_1, \Psi_2} \quad \text{TL_SUBE} \\
\frac{x : A \vdash_{\mathsf{L}} \Delta; t : T, \Psi}{x : A \vdash_{\mathsf{L}} \Delta, \mathsf{J}t : \mathsf{J}T; \Psi} \quad \text{TL_JI} \\
\frac{x : A \vdash_{\mathsf{L}} \Delta, e : \mathsf{J}T; \Psi_1 \quad y : T \vdash_{\mathsf{C}} \Psi_2}{x : A \vdash_{\mathsf{L}} \Delta; \Psi_1, \text{let } \mathsf{J}y = e \text{ in } \Psi_2} \quad \text{TL_JE} \\
\frac{x : A \vdash_{\mathsf{L}} \Delta, e : B; \Psi}{x : A \vdash_{\mathsf{L}} \Delta; \mathsf{H}e : \mathsf{H}B, \Psi} \quad \text{TL_HI} \\
\frac{x : A \vdash_{\mathsf{L}} \Delta; \Psi_1, t : \mathsf{H}A \quad y : A \vdash_{\mathsf{L}} \cdot; \Psi_2}{x : A \vdash_{\mathsf{L}} \Delta; \Psi_1, \text{let } \mathsf{H}y = t \text{ in } \Psi_2} \quad \text{TL_HE} \\
\frac{x : A \vdash_{\mathsf{L}} \Delta; \Psi \quad r : T' \in \Psi \text{ or } r : B \in \Delta}{x : A \vdash_{\mathsf{L}} \Delta; \text{connect}_{\Psi} \text{ to } r : T, \Psi} \quad \text{TL_WEAK} \\
\frac{x : A \vdash_{\mathsf{L}} \Delta; t_1 : T, t_2 : T, \Psi}{x : A \vdash_{\mathsf{L}} \Delta; t_1 \cdot t_2 : T, \Psi} \quad \text{TL_CONTR} \\
\frac{x : A \vdash_{\mathsf{L}} \Delta; \Psi_1, t : T \quad y : T \vdash_{\mathsf{C}} \Psi_2}{x : A \vdash_{\mathsf{L}} \Delta; \Psi_1, [t/y]\Psi_2} \quad \text{TL_CCUT} \\
\frac{x : A \vdash_{\mathsf{L}} \Delta_1, e : B; \Psi_1 \quad y : B \vdash_{\mathsf{L}} \Delta_2; \Psi_2}{x : A \vdash_{\mathsf{L}} \Delta_1, [e/y]\Delta_2; \Psi_1, [e/y]\Psi_2} \quad \text{TL_CUT}
\end{array}$$