

A COINTUITIONISTIC ADJOINT LOGIC

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ABSTRACT. TODO

1. INTRODUCTION

TODO [?]

2. THE ADJOINT MODEL

Suppose $(\mathcal{I}, 1, \times, \rightarrow)$ is a cartesian closed category, and $(\mathcal{L}, \top, \otimes, -\circ)$ is a symmetric monoidal closed category. Then relate these two categories with a symmetric monoidal adjunction $\mathcal{I} : \mathbf{F} \dashv \mathbf{G} : \mathcal{L}$ (Definition ??), where \mathbf{F} and \mathbf{G} are symmetric monoidal functors. The later point implies that there are natural transformations $m_{X,Y} : \mathbf{F}X \otimes \mathbf{F}Y \longrightarrow \mathbf{F}(X \times Y)$ and $n_{A,B} : \mathbf{G}A \times \mathbf{G}B \longrightarrow \mathbf{G}(A \otimes B)$, and maps $m_\top : \top \longrightarrow \mathbf{F}1$ and $n_1 : 1 \longrightarrow \mathbf{G}\top$ subject to several coherence conditions; see Definition ??. Furthermore, the functor \mathbf{F} is strong which means that $m_{X,Y}$ and m_\top are isomorphisms. This setup turns out to be one of the most beautiful models of intuitionistic linear logic called an formally a LNL model due to Benton [2]. In fact, the linear modality of-course can be defined by $!A = \mathbf{F}(\mathbf{G}(A))$ which defines a symmetric monoidal comonad using the adjunction; see Section 2.2 of [2]. This model is much simpler than other known models, and resulted in a logic called LNL logic which supports mixing intuitionistic logic with linear logic.

Taking the dual of the previous model results in what we call dual LNL models. They consist of a cocartesian coclosed category, $(C, 0, +, -)$, a symmetric monoidal coclosed category, $(\mathcal{L}', \perp, \oplus, \bullet-)$, where $\bullet- : \mathcal{L}' \times \mathcal{L}' \longrightarrow \mathcal{L}'$ is left adjoint to parr , and a symmetric monoidal adjunction $\mathcal{L}' : H \dashv \perp : C$. We will show that dual LNL models are a simplification of dual linear categories as defined by Bellin [1] in much of the same way that adjoint models are a simplification of linear categories. In fact, we will define Girard's exponential why-not by $?A = J(H(A))$, and hence, is the monad induced by the adjunction.

2.1. Symmetric (co)Monoidal Categories. We now introduce the necessary definitions related to symmetric monoidal categories that our model will depend on. Most of these definitions are equivalent to the ones given by Benton [2], but we give a lesser well-known definition for symmetric co-monoidal functors due to Bellin [1]. In this section we also introduce distributive categories, the notion of cocloser, and finally, the definition of bilinear categories. The reader may wish to simply skim this section, but refer back to it when they encounter a definition or result they do not know.

Definition 1. A **symmetric monoidal category (SMC)** is a category, \mathcal{M} , with the following data:

- An object \top of \mathcal{M} ,
- A bi-functor $\otimes : \mathcal{M} \times \mathcal{M} \rightarrow \mathcal{M}$,
- The following natural isomorphisms:

$$\begin{aligned}\lambda_A &: \top \otimes A \rightarrow A \\ \rho_A &: A \otimes \top \rightarrow A \\ \alpha_{A,B,C} &: (A \otimes B) \otimes C \rightarrow A \otimes (B \otimes C)\end{aligned}$$

- A symmetry natural transformation:

$$\beta_{A,B} : A \otimes B \rightarrow B \otimes A$$

- Subject to the following coherence diagrams:

$$\begin{array}{ccc} ((A \otimes B) \otimes C) \otimes D & \xrightarrow{\alpha_{A,B,C} \otimes \text{id}_D} & (A \otimes (B \otimes C)) \otimes D \\ \downarrow \alpha_{A \otimes B, C, D} & & \downarrow \alpha_{A, B \otimes C, D} \\ (A \otimes B) \otimes (C \otimes D) & & \\ \downarrow \alpha_{A, B, C \otimes D} & & \\ A \otimes (B \otimes (C \otimes D)) & \xleftarrow{\text{id}_A \otimes \alpha_{B, C, D}} & A \otimes ((B \otimes C) \otimes D) \end{array}$$

$$\begin{array}{ccccc} (A \otimes B) \otimes C & \xrightarrow{\alpha_{A, B, C}} & A \otimes (B \otimes C) & \xrightarrow{\beta_{A, B \otimes C}} & (B \otimes C) \otimes A \\ \downarrow \beta_{A, B} \otimes \text{id}_C & & & & \downarrow \alpha_{B, C, A} \\ (B \otimes A) \otimes C & \xrightarrow{\alpha_{B, A, C}} & B \otimes (A \otimes C) & \xrightarrow{\text{id}_B \otimes \beta_{A, C}} & B \otimes (C \otimes A) \end{array}$$

$$\begin{array}{ccc}
(A \otimes \top) \otimes B & \xrightarrow{\alpha_{A,\top,B}} & A \otimes (\top \otimes B) \\
\rho_A \searrow & & \swarrow \lambda_B \\
& A \otimes B &
\end{array}
\qquad
\begin{array}{ccc}
A \otimes B & & \\
\beta_{A,B} \downarrow & \searrow \text{id}_{A \otimes B} & \\
B \otimes A & \xrightarrow{\beta_{B,A}} & A \otimes B
\end{array}$$

$$\begin{array}{ccc}
\top \otimes A & \xrightarrow{\beta_{\top,A}} & A \otimes \top \\
\lambda_A \searrow & & \swarrow \rho_A \\
& A &
\end{array}$$

Categorical modeling implication requires that the model be closed; which can be seen as an internalization of the notion of a morphism.

Definition 2. A **symmetric monoidal closed category (SMCC)** is a symmetric monoidal category, $(\mathcal{M}, \top, \otimes)$, such that, for any object B of \mathcal{M} , the functor $- \otimes B : \mathcal{M} \longrightarrow \mathcal{M}$ has a specified right adjoint. Hence, for any objects A and C of \mathcal{M} there is an object $B \multimap C$ of \mathcal{M} and a natural bijection:

$$\text{Hom}_{\mathcal{M}}(A \otimes B, C) \cong \text{Hom}_{\mathcal{M}}(A, B \multimap C)$$

We call the functor $\multimap : \mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{M}$ the internal hom of \mathcal{M} .

Symmetric monoidal closed categories can be seen as a model of intuitionistic linear logic with a tensor product and implication. What happens when we take the dual? First, we have the following result:

Lemma 3 (Dual of Symmetric Monoidal Categories). If $(\mathcal{M}, \top, \otimes)$ is a symmetric monoidal category, then \mathcal{M}^{op} is also a symmetric monoidal category.

The previous result follows from the fact that the structures making up symmetric monoidal categories are isomorphisms, and so naturally taking their opposite will yield another symmetric monoidal category. To emphasize when we are thinking about a symmetric monoidal category in the opposite we use the notion $(\mathcal{M}, \perp, \oplus)$ which gives the suggestion of \oplus corresponding to a disjunctive tensor product which we call the *cotensor* of \mathcal{M} . The next definition describes when a symmetric monoidal category is coclosed.

Definition 4. A **symmetric monoidal coclosed category (SMCCC)** is a symmetric monoidal category, $(\mathcal{M}, \perp, \oplus)$, such that, for any object B of \mathcal{M} , the functor $- \oplus B : \mathcal{M} \longrightarrow \mathcal{M}$ has a specified left adjoint. Hence, for any objects A and C of \mathcal{M} there is an object $B \multimap C$ of \mathcal{M} and a natural bijection:

$$\text{Hom}_{\mathcal{M}}(C, A \oplus B) \cong \text{Hom}_{\mathcal{M}}(B \multimap C, A)$$

We call the functor $\multimap : \mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{M}$ the internal cohom of \mathcal{M} .

A symmetric monoidal category is a category with additional structure subject to several coherence diagrams. Thus, an ordinary functor is not enough to capture this structure, and hence, the introduction of symmetric monoidal functors.

Definition 5. Suppose we are given two symmetric monoidal closed categories $(\mathcal{M}_1, \top_1, \otimes_1, \alpha_1, \lambda_1, \rho_1, \beta_1)$ and $(\mathcal{M}_2, \top_2, \otimes_2, \alpha_2, \lambda_2, \rho_2, \beta_2)$. Then a **symmetric monoidal functor**

is a functor $F : \mathcal{M}_1 \longrightarrow \mathcal{M}_2$, a map $m_{\top_1} : \top_2 \longrightarrow F\top_1$ and a natural transformation $m_{A,B} : FA \otimes_2 FB \longrightarrow F(A \otimes_1 B)$ subject to the following coherence conditions:

$$\begin{array}{ccc}
 (FA \otimes_2 FB) \otimes_2 FC & \xrightarrow{\alpha_{2FA,FB,FC}} & FA \otimes_2 (FB \otimes_2 FC) \\
 \downarrow m_{A,B} \otimes \text{id}_{FC} & & \downarrow \text{id}_{FA} \otimes m_{B,C} \\
 F(A \otimes_1 B) \otimes_2 FC & & FA \otimes_2 F(B \otimes_1 C) \\
 \downarrow m_{A \otimes_1 B, C} & & \downarrow m_{A, B \otimes_1 C} \\
 F((A \otimes_1 B) \otimes_1 C) & \xrightarrow{F\alpha_{1A,B,C}} & F(A \otimes_1 (B \otimes_1 C))
 \end{array}$$

$$\begin{array}{ccc}
 \top_2 \otimes_2 FA & \xrightarrow{\lambda_{2FA}} & FA \\
 \downarrow m_{\top_1} \otimes \text{id}_{FA} & & \uparrow F\lambda_{1A} \\
 F\top_1 \otimes_2 FA & \xrightarrow{m_{\top_1, A}} & F(\top_1 \otimes_1 A)
 \end{array}
 \qquad
 \begin{array}{ccc}
 FA \otimes_2 \top_2 & \xrightarrow{\rho_{2FA}} & FA \\
 \downarrow \text{id}_{FA} \otimes m_{\top_1} & & \uparrow F\rho_{1A} \\
 FA \otimes_2 F\top_1 & \xrightarrow{m_{A, \top_1}} & F(A \otimes_1 \top_1)
 \end{array}$$

$$\begin{array}{ccc}
 FA \otimes_2 FB & \xrightarrow{\beta_{2FA,FB}} & FB \otimes_2 FA \\
 \downarrow m_{A,B} & & \downarrow m_{B,A} \\
 F(A \otimes_1 B) & \xrightarrow{F\beta_{1A,B}} & F(B \otimes_1 A)
 \end{array}$$

The following is dual to the previous definition.

Definition 6. Suppose we are given two symmetric monoidal closed categories $(\mathcal{M}_1, \perp_1, \oplus_1, \alpha_1, \lambda_1, \rho_1, \beta_1)$ and $(\mathcal{M}_2, \perp_2, \oplus_2, \alpha_2, \lambda_2, \rho_2, \beta_2)$. Then a **symmetric comonoidal functor** is a functor $F : \mathcal{M}_1 \longrightarrow \mathcal{M}_2$, a map $m_{\perp_1} : F\perp_1 \longrightarrow \perp_2$ and a natural transformation $m_{A,B} : F(A \oplus_1 B) \longrightarrow FA \oplus_2 FB$ subject to the following coherence conditions:

$$\begin{array}{ccc}
 F((A \oplus_1 B) \oplus_1 C) & \xrightarrow{m_{A \oplus_1 B, C}} & F(A \oplus_1 B) \oplus_2 FC \\
 \downarrow F\alpha_{A,B,C} & & \downarrow m_{A,B} \oplus \text{id}_{FC} \\
 F(A \oplus_1 (B \oplus_1 C)) & & (FA \oplus_2 FB) \oplus_2 FC \\
 \downarrow m_{A, B \oplus_1 C} & & \downarrow \alpha_{FA,FB,FC} \\
 FA \oplus_2 F(B \oplus_1 C) & \xrightarrow{\text{id}_{FA} \oplus m_{B,C}} & FA \oplus_2 (FB \oplus_2 FC)
 \end{array}$$

$$\begin{array}{ccc}
 F(\perp_1 \oplus_1 A) & \xrightarrow{m_{\perp_1, A}} & F\perp_1 \oplus_2 FA \\
 \downarrow F\lambda_{1A} & & \downarrow m_{\perp_1} \oplus \text{id}_{FA} \\
 FA & \xrightarrow{\lambda_{2FA}^{-1}} & \perp_2 \oplus_2 FA
 \end{array}
 \qquad
 \begin{array}{ccc}
 F(A \oplus_1 \perp_1) & \xrightarrow{m_{A, \perp_1}} & FA \oplus_2 F\perp_1 \\
 \downarrow F\rho_{1A} & & \downarrow \text{id}_{FA} \oplus m_{\perp_1} \\
 FA & \xrightarrow{\rho_{2FA}^{-1}} & FA \oplus_2 \perp_2
 \end{array}$$

$$\begin{array}{ccc}
F(A \oplus_1 B) & \xrightarrow{m_{A,B}} & FA \oplus_2 FB \\
\downarrow F\beta_{1A,B} & & \downarrow \beta_{2FA,FB} \\
F(B \oplus_1 A) & \xrightarrow{m_{B,A}} & FB \oplus_2 FA
\end{array}$$

Naturally, since functors are enhanced to handle the additional structure found in a symmetric monoidal category we must also extend natural transformations, and adjunctions.

Definition 7. Suppose $(\mathcal{M}_1, \top_1, \otimes_1)$ and $(\mathcal{M}_2, \top_2, \otimes_2)$ are SMCs, and (F, m) and (G, n) are a symmetric monoidal functors between \mathcal{M}_1 and \mathcal{M}_2 . Then a **symmetric monoidal natural transformation** is a natural transformation, $f : F \rightarrow G$, subject to the following coherence diagrams:

$$\begin{array}{ccc}
FA \otimes_2 FB & \xrightarrow{m_{A,B}} & F(A \otimes_1 B) \\
\downarrow f_A \otimes_2 f_B & & \downarrow f_{A \otimes_1 B} \\
GA \otimes_2 GB & \xrightarrow{n_{A,B}} & G(A \otimes_1 B)
\end{array}
\quad
\begin{array}{ccc}
F\top_1 & \xrightarrow{f_{\top_1}} & G\top_1 \\
\swarrow m_{\top_1} & & \searrow n_{\top_1} \\
& \top_2 &
\end{array}$$

Definition 8. Suppose $(\mathcal{M}_1, \perp_1, \oplus_1)$ and $(\mathcal{M}_2, \perp_2, \oplus_2)$ are SMCs, and (F, m) and (G, n) are a symmetric comonoidal functors between \mathcal{M}_1 and \mathcal{M}_2 . Then a **symmetric comonoidal natural transformation** is a natural transformation, $f : F \rightarrow G$, subject to the following coherence diagrams:

$$\begin{array}{ccc}
F(A \oplus_1 B) & \xrightarrow{m_{A,B}} & FA \oplus_2 FB \\
\downarrow f_{A \oplus_1 B} & & \downarrow f_{A \oplus_2 B} \\
G(A \oplus_1 B) & \xrightarrow{n_{A,B}} & GA \oplus_2 GB
\end{array}
\quad
\begin{array}{ccc}
\perp_2 & \xleftarrow{n_{\perp_1}} & G\perp_1 \\
\swarrow m_{\perp_1} & & \searrow f_{\perp_1} \\
& F\perp_1 &
\end{array}$$

Definition 9. Suppose $(\mathcal{M}_1, \top_1, \otimes_1)$ and $(\mathcal{M}_2, \top_2, \otimes_2)$ are SMCs, and (F, m) is a symmetric monoidal functor between \mathcal{M}_1 and \mathcal{M}_2 and (G, n) is a symmetric monoidal functor between \mathcal{M}_2 and \mathcal{M}_1 . Then a **symmetric monoidal adjunction** is an ordinary adjunction $\mathcal{M}_1 : F \dashv G : \mathcal{M}_2$ such that the unit, $\eta_A : A \rightarrow GFA$, and the counit, $\varepsilon_A : FGA \rightarrow A$, are symmetric monoidal natural transformations. Thus, the following diagrams must commute:

$$\begin{array}{ccc}
FGA \otimes_2 FGB & \xrightarrow{m_{GA,GB}} & F(GA \otimes_1 GB) \\
\downarrow \varepsilon_A \otimes_1 \varepsilon_B & & \downarrow F n_{A,B} \\
A \otimes_2 B & \xleftarrow{\varepsilon_{A \otimes_1 B}} & FGA \otimes_2 FGB
\end{array}
\quad
\begin{array}{ccc}
F\top_1 & \xrightarrow{F n_{\top_2}} & FG\top_2 \\
\uparrow m_{\top_1} & & \downarrow \varepsilon_{\top_1} \\
\top_2 & \xlongequal{\quad} & \top_2
\end{array}$$

$$\begin{array}{ccc}
GFA \otimes_1 GFB & \xleftarrow{\eta_A \otimes_1 \eta_B} & A \otimes_1 B \\
\downarrow n_{FA,FB} & & \downarrow \eta_{A \otimes_1 B} \\
G(FA \otimes_2 FB) & \xrightarrow{m_{A,B}} & GF(A \otimes_1 B)
\end{array}
\quad
\begin{array}{ccc}
G\top_2 & \xrightarrow{G m_{\top_1}} & GF\top_1 \\
\uparrow n_{\top_2} & & \uparrow \eta_{\top_1} \\
\top_1 & \xlongequal{\quad} & \top_1
\end{array}$$

Definition 10. Suppose $(\mathcal{M}_1, \perp_1, \oplus_1)$ and $(\mathcal{M}_2, \perp_2, \oplus_2)$ are SMCs, and (F, m) is a symmetric comonoidal functor between \mathcal{M}_1 and \mathcal{M}_2 and (G, n) is a symmetric comonoidal functor between \mathcal{M}_2 and \mathcal{M}_1 . Then a **symmetric comonoidal adjunction** is an ordinary adjunction $\mathcal{M}_1 : F \dashv G : \mathcal{M}_2$ such that the unit, $\eta_A : A \rightarrow GFA$, and the counit, $\varepsilon_A : FGA \rightarrow A$, are symmetric comonoidal natural transformations. Thus, the following diagrams must commute:

$$\begin{array}{ccc}
 A \oplus_1 B & \xrightarrow{\eta_{A \oplus_1 B}} & GF(A \oplus_1 B) \\
 \downarrow \eta_A \oplus_1 \eta_B & & \downarrow Gm_{A,B} \\
 GFA \oplus_1 GFB & \xleftarrow{m_{FA,FB}} & G(FA \oplus_2 FB)
 \end{array}
 \qquad
 \begin{array}{ccc}
 GF \perp_1 & \xrightarrow{Gm_{\perp_1}} & G \perp_2 \\
 \uparrow \eta_{\perp_1} & & \downarrow n_{\perp_2} \\
 \perp_1 & \xlongequal{\quad} & \perp_1
 \end{array}$$

$$\begin{array}{ccc}
 FG(A \oplus_2 B) & \xrightarrow{Fn_{A,B}} & F(GA \oplus_1 GB) \\
 \downarrow \varepsilon_{A \oplus_2 B} & & \downarrow m_{GA,GB} \\
 A \oplus_2 B & \xleftarrow{\varepsilon_A \oplus_2 \varepsilon_B} & FGA \oplus_2 FGB
 \end{array}
 \qquad
 \begin{array}{ccc}
 FG \perp_2 & \xrightarrow{\varepsilon_{\perp_2}} & \perp_2 \\
 \parallel & & \uparrow m_{\perp_1} \\
 FG \perp_2 & \xrightarrow{Fn_{\perp_2}} & F \perp_1
 \end{array}$$

We will be defining, and making use of the why-not exponentials from linear logic, but these correspond to a symmetric comonoidal monad. In addition, whenever we have a symmetric comonoidal adjunction, we immediately obtain a symmetric comonoidal monad on the left, and a symmetric comonoidal monad on the right.

Definition 11. A **symmetric comonoidal monad** on a symmetric monoidal category \mathcal{C} is a triple (T, η, μ) , where (T, η) is a symmetric comonoidal endofunctor on \mathcal{C} , $\eta_A : A \rightarrow TA$ and $\mu_A : T^2A \rightarrow TA$ are symmetric comonoidal natural transformations, which make the following diagrams commute:

$$\begin{array}{ccc}
 T^3A & \xrightarrow{\mu_{TA}} & T^2A \\
 \downarrow T\mu_A & & \downarrow \mu_A \\
 T^2A & \xrightarrow{\mu_A} & TA
 \end{array}
 \qquad
 \begin{array}{ccccc}
 & & TA & & \\
 & \nearrow & \uparrow \mu_A & \nwarrow & \\
 TA & \xrightarrow{\eta_{TA}} & T^2A & \xleftarrow{T\eta_A} & TA
 \end{array}$$

The assumption that η and μ are symmetric comonoidal natural transformations amount to the following diagrams commuting:

$$\begin{array}{ccc}
 A \oplus B & \xrightarrow{\eta_A \oplus \eta_B} & TA \oplus TB \\
 \downarrow \eta_A & \nearrow \eta_{A,B} & \\
 T(A \oplus B) & &
 \end{array}
 \qquad
 \begin{array}{ccc}
 \perp & \xrightarrow{\eta_{\perp}} & T \perp \\
 \parallel & \searrow \eta_{\perp} & \\
 \perp & &
 \end{array}$$

$$\begin{array}{ccccc}
T^2(A \oplus B) & \xrightarrow{T\eta_{A,B}} & T(TA \oplus TB) & \xrightarrow{\eta_{TA,TB}} & T^2A \oplus T^2B \\
\downarrow \mu_{A \oplus B} & & & & \downarrow \mu_A \oplus \mu_B \\
T(A \oplus B) & \xrightarrow{\eta_{A,B}} & TA \oplus TB & &
\end{array}
\quad
\begin{array}{ccc}
T^2 \perp & \xrightarrow{T\eta_\perp} & T \perp \\
\downarrow \mu_\perp & & \downarrow \eta_\perp \\
T \perp & \xrightarrow{\eta_\perp} & \perp
\end{array}$$

Finally, the dual concept of a symmetric comonoidal comonad.

Definition 12. A **symmetric comonoidal comonad** on a symmetric monoidal category \mathcal{C} is a triple (T, ε, δ) , where (T, m) is a symmetric comonoidal endofunctor on \mathcal{C} , $\varepsilon_A : TA \rightarrow A$ and $\delta_A : TA \rightarrow T^2A$ are symmetric comonoidal natural transformations, which make the following diagrams commute:

$$\begin{array}{ccc}
TA & \xrightarrow{\delta_A} & T^2A \\
\downarrow \delta_A & & \downarrow T\delta_A \\
T^2A & \xrightarrow{\delta_{TA}} & T^3A
\end{array}
\quad
\begin{array}{ccccc}
& & TA & & \\
& \swarrow & \downarrow \delta_A & \searrow & \\
TA & \xleftarrow{\varepsilon_{TA}} & T^2A & \xrightarrow{T\varepsilon_A} & TA
\end{array}$$

The assumption that ε and δ are symmetric monoidal natural transformations amount to the following diagrams commuting:

$$\begin{array}{ccc}
T(A \oplus B) & \xrightarrow{m_{A,B}} & TA \oplus TB \\
& \searrow \varepsilon_{A \oplus B} & \downarrow \varepsilon_A \oplus \varepsilon_B \\
& & A \oplus B
\end{array}
\quad
\begin{array}{ccc}
T \perp & \xrightarrow{\varepsilon_\perp} & \perp \\
& \searrow & \uparrow m_\perp \\
& & T \perp
\end{array}$$

$$\begin{array}{ccccc}
T(A \oplus B) & \xrightarrow{m_{A,B}} & TA \oplus TB & & \\
\downarrow \delta_{A \oplus B} & & \downarrow \delta_A \oplus \delta_B & & \\
T^2(A \oplus B) & \xrightarrow{Tm_{A,B}} & T(TA \oplus TB) & \xrightarrow{m_{TA,TB}} & T^2A \oplus T^2B
\end{array}
\quad
\begin{array}{ccc}
T \perp & \xrightarrow{m_\perp} & \perp \\
\downarrow \delta_\perp & & \uparrow m_\perp \\
T^2 \perp & \xrightarrow{Tm_\perp} & T \perp
\end{array}$$

2.2. Cartesian Closed and Cocartesian Coclosed Categories. The notion of a cartesian closed category is well-known, but for completeness we define them here. However, their dual is lesser known, especially in computer science, and so we given their full definition. We also review some know results concerning cocartesian coclosed categories and categories that are both cartesian closed and cocartesian coclosed.

Definition 13. A **cartesian category** is a category, $(C, 1, \times)$, with an object, 1 , and a bi-functor, $\times : C \times C \rightarrow C$, such that for any object A there is exactly one morphism $\diamond : A \rightarrow 1$, and for any

morphisms $f : C \longrightarrow A$ and $g : C \longrightarrow B$ there is a morphism $\langle f, g \rangle : C \rightarrow A \times B$ subject to the following diagram:

$$\begin{array}{ccccc}
 & & C & & \\
 & f \swarrow & \downarrow \langle f, g \rangle & \searrow g & \\
 A & \xleftarrow{\pi_1} & A \times B & \xrightarrow{\pi_2} & B
 \end{array}$$

A cartesian category models conjunction by the product functor, $\times : C \times C \longrightarrow C$, and the unit of conjunction by the terminal object. As we mention above modeling implication requires closer, and since it is well-known that any cartesian category is also a symmetric monoidal category the definition of closer for a cartesian category is the same as the definition of closer for a symmetric monoidal category (Definition 4). We denote the internal hom for cartesian closed categories by $A \rightarrow B$.

The dual of a cartesian category is a cocartesian category. They are a model of intuitionistic logic with disjunction and its unit.

Definition 14. A **cocartesian category** is a category, $(C, 0, +)$, with an object, 0 , and a bi-functor, $+: C \times C \longrightarrow C$, such that for any object A there is exactly one morphism $\square : 0 \rightarrow A$, and for any morphisms $f : A \longrightarrow C$ and $g : B \longrightarrow C$ there is a morphism $[f, g] : A + B \longrightarrow C$ subject to the following diagram:

$$\begin{array}{ccccc}
 & & C & & \\
 & f \swarrow & \uparrow [f, g] & \nwarrow g & \\
 A & \xrightarrow{\iota_1} & A + B & \xleftarrow{\iota_2} & B
 \end{array}$$

Cocloser, just like closer for cartesian categories, is defined in the same way that cocloser is defined for symmetric monoidal categories, because cocartesian categories are also symmetric monoidal categories. Thus, a cocartesian category is coclosed if there is a specified left-adjoint, which we denote $S - T$, to the coproduct.

There are many examples of cocartesian coclosed categories. Basically, any interesting cartesian category has an interesting dual, and hence, induces an interesting cocartesian coclosed category. The opposite of the category of sets and functions between them is isomorphic to the category of complete atomic boolean algebras, and both of which, are examples of cocartesian coclosed categories. As we mentioned above bi-linear categories [3] are models of bi-linear logic where the left adjoint to the cotensor models coimplication. Similarly, cocartesian coclosed categories model intuitionistic logic with disjunction and intuitionistic coimplication [4, 1].

Put more examples in here.

We might now ask if we can a category can be both cartesian closed and cocartesian coclosed, but this turns out to be where the matter meets antimatter in such away that the category degenerates to a preorder. That is, every homspace contains at most one morphism. We recall this proof here, which is due to Crolard [4]. We need a couple basic facts about cartesian closed categories with initial objects.

Lemma 15. In any cartesian category C , if 0 is an initial object in C and $\text{Hom}_C(A, 0)$ is non-empty, then $A \cong A \times 0$.

Proof. This follows easily from the universal mapping property for products. □

Lemma 16. In any cartesian closed category C , if 0 is an initial object in C , then so is $0 \times A$ for any object A of C .

Proof. We know that the universal morphism for the initial object is unique, and hence, the homspace $\text{Hom}_C(0, A \Rightarrow B)$ for any object B of C contains exactly one morphism. Then using the right adjoint to the product functor we know that $\text{Hom}_C(0, A \Rightarrow B) \cong \text{Hom}_C(0 \times A, B)$, and hence, there is only one arrow between $0 \times A$ and B . \square

The following lemma is due to Joyal [4], and is key to the next theorem.

Lemma 17 (Joyal's). In any cartesian closed category C , if 0 is an initial object in C and $\text{Hom}_C(A, 0)$ is non-empty, then A is an initial object in C .

Proof. Suppose C is a cartesian closed category, such that, 0 is an initial object in C , and A is an arbitrary object in C . Furthermore, suppose $\text{Hom}_C(A, 0)$ is non-empty. By the first basic lemma above we know that $A \cong A \times 0$, and by the second $A \times 0$ is initial, thus A is initial. \square

Finally, the following theorem shows that any category that is both cartesian closed and co-cartesian coclosed is a preorder.

Theorem 18 ((co)Cartesian (co)Closed Categories are Preorders (Crolard[4])). If C is both cartesian closed and cocartesian coclosed, then for any two objects A and B of C , $\text{Hom}_C(A, B)$ has at most one element.

Proof. Suppose C is both cartesian closed and cocartesian coclosed, and A and B are objects of C . Then by using the basic fact that the initial object is the unit to the coproduct, and the coproducts left adjoint we know the following:

$$\text{Hom}_C(A, B) \cong \text{Hom}_C(A, 0 + B) \cong \text{Hom}_C(B - A, 0)$$

Therefore, by Joyal's theorem above $\text{Hom}_C(A, B)$ has at most one element. \square

Notice that the previous result hinges on the fact that there are initial and terminal objects, and thus, this result does not hold for bi-linear categories, because the units to the tensor and cotensor are not initial nor terminal.

The repercussions of this result are that if we do not want to work with preorders, but do want to work with all of the structure, then we must separate the two worlds. Thus, this result can be seen as the motivation for the current work. Our ultimate goal is to enforce the separation using linear logic, but through the power of linear logic will be able to show that the separation is not far.

2.3. A Mixed Linear/Non-Linear Model for Co-Intuitionistic Logic. Benton [2] showed that from a LNL model it is possible to construct a linear category, and vice versa. Bellin [1] showed that the dual to linear categories are sufficient to model co-intuitionistic linear logic. We show that from the dual to a LNL model we can construct the dual to a linear category, and vice versa, thus, carrying out the same program for co-intuitionistic linear logic as Benton did for intuitionistic linear logic.

Combining a symmetric monoidal coclosed category with a cocartesian coclosed category via a symmetric comonoidal adjunction defines a dual LNL model.

Definition 19. A mixed linear/non-linear model for co-intuitionistic logic (dual LNL model), $\mathcal{L} : \mathcal{H} \dashv \mathcal{J} : C$, consists of the following:

- i. a symmetric monoidal coclosed category $(\mathcal{L}, \perp, \oplus, \multimap)$,
- ii. a cocartesian coclosed category $(C, 0, +, -)$, and

- iv. a symmetric comonoidal adjunction $\mathcal{L} : \mathbf{H} \dashv \mathbf{J} : \mathbf{C}$, where $\eta_A : A \rightarrow \mathbf{J}HA$ and $\varepsilon_R : \mathbf{H}JR \rightarrow R$ are the unit and counit of the adjunction respectively.

It is well-known that an adjunction $\mathcal{L} : \mathbf{H} \dashv \mathbf{J} : \mathbf{C}$ induces a monad $\mathbf{H}; \mathbf{J} : \mathcal{L} \rightarrow \mathcal{L}$, but when the adjunction is symmetric comonoidal we obtain a symmetric comonoidal monad, in fact, $\mathbf{H}; \mathbf{J}$ defines the linear exponential why-not denoted $?A = \mathbf{J}(\mathbf{H}A)$. By the definition of dual LNL models we know that both \mathbf{H} and \mathbf{J} are symmetric comonoidal functors, and hence, are equipped with natural transformations $h_{A,B} : \mathbf{H}(A \oplus B) \rightarrow \mathbf{H}A + \mathbf{H}B$ and $j_{R,S} : \mathbf{J}(R + S) \rightarrow \mathbf{J}R \oplus \mathbf{J}S$, and maps $h_\perp : \mathbf{H} \perp \rightarrow 0$ and $j_0 : \mathbf{J}0 \rightarrow \perp$. We will make heavy use of these maps throughout the sequel.

Compare this definition with that of Bellin's dual category from [1], and we can easily see that the definition of dual LNL models – much like LNL models – is more succinct.

Definition 20. A dual linear category, \mathbf{C} , consists of the following data:

- i. A symmetric monoidal coclosed category $(\mathbf{C}, \oplus, \perp, \bullet)$ with
- ii. a symmetric co-monoidal monad $(?, \eta, \mu)$ on \mathbf{C} such that
 - a. each free $?$ -algebra carries naturally the structure of a commutative \oplus -monoid. This implies that there are distinguished symmetric monoidal natural transformations $w_A : \perp \rightarrow ?A$ and $c_A : ?A \oplus ?A \rightarrow ?A$ which form a commutative monoid and are $?$ -algebra morphisms.
 - b. whenever $f : (?A, \mu_A) \rightarrow (?B, \mu_B)$ is a morphism of free $?$ -algebras, then it is also a monoid morphism.

2.3.1. A Useful Isomorphism. One useful property of Benton's LNL model is that the maps associated with the symmetric monoidal left adjoint in the model are isomorphisms. Since dual LNL models are dual we obtain similar isomorphisms with respect to the right adjoint.

Lemma 21 (Symmetric Comonoidal Isomorphisms). Given any dual LNL model $\mathcal{L} : \mathbf{H} \dashv \mathbf{J} : \mathbf{C}$, then there are the following isomorphisms:

$$\mathbf{J}(R + S) \cong \mathbf{J}R \oplus \mathbf{J}S \quad \text{and} \quad \mathbf{J}0 \cong \perp$$

Furthermore, the former is natural in R and S .

Proof. Suppose $\mathcal{L} : \mathbf{H} \dashv \mathbf{J} : \mathbf{C}$ is a dual LNL model. Then we can define the following family of maps:

$$j_{R,S}^{-1} := \mathbf{J}R \oplus \mathbf{J}S \xrightarrow{\eta} \mathbf{J}H(\mathbf{J}R \oplus \mathbf{J}S) \xrightarrow{\mathbf{J}h_{A,B}} \mathbf{J}(H\mathbf{J}R + H\mathbf{J}S) \xrightarrow{\mathbf{J}(\varepsilon_R + \varepsilon_S)} \mathbf{J}(R + S)$$

$$j_0^{-1} := \perp \xrightarrow{\eta} \mathbf{J}H \perp \xrightarrow{\mathbf{J}h_\perp} \mathbf{J}0$$

It is easy to see that $j_{R,S}^{-1}$ is natural, because it is defined in terms of a composition of natural transformations. All that is left to be shown is that $j_{R,S}^{-1}$ and j_0^{-1} are mutual inverses with $j_{R,S}$ and j_0 ; for the details see Appendix A.1. \square

Just as Benton we also do not have similar isomorphisms with respect to the functor \mathbf{H} . One fact that we can point out, that Benton did not make explicit – because he did not use the notion of symmetric comonoidal functor – is that j^{-1} makes \mathbf{J} also a symmetric monoidal functor.

Corollary 22. Given any dual LNL model $\mathcal{L} : \mathbf{H} \dashv \mathbf{J} : \mathbf{C}$, the functor (\mathbf{J}, j^{-1}) is symmetric monoidal.

Proof. This holds by straightforwardly reducing the diagrams defining a symmetric monoidal functor, Definition 5, to the diagrams defining a symmetric comonoidal functor, Definition 6, using the fact that j^{-1} is an isomorphism. \square

2.3.2. Dual LNL Model Implies Dual Category. The next result shows that any dual LNL model induces a symmetric comonoidal monad.

Lemma 23 (Symmetric Comonoidal Monad). Given a dual LNL model $\mathcal{L} : H \dashv J : C$, the functor, $? = H; J$, defines a symmetric comonoidal monad.

Proof. Suppose (H, h) and (J, j) are two symmetric comonoidal functors, such that, $\mathcal{L} : H \dashv J : C$ is a dual LNL model. We can easily show that $?A = JHA$ is symmetric monoidal by defining the following maps:

$$\begin{aligned} r_{\perp} &:= ?\perp \equiv JH\perp \xrightarrow{j_{H\perp}} J0 \xrightarrow{j_{\perp}} \perp \\ r_{A,B} &:= ?(A \oplus B) \equiv JH(A \oplus B) \xrightarrow{j_{H(A \oplus B)}} J(HA + HB) \xrightarrow{j_{HA,HB}} JHA \oplus JHB \equiv ?A \oplus ?B \end{aligned}$$

The fact that these maps satisfy the appropriate symmetric comonoidal functor diagrams from Definition 6 is obvious, because symmetric comonoidal functors are closed under composition.

We have a dual LNL model, and hence, we have the symmetric comonoidal natural transformations $\eta_A : A \longrightarrow JHA$ and $\varepsilon_R : HJR \longrightarrow R$ which correspond to the unit and counit of the adjunction respectfully. Define $\mu_A := J\varepsilon_{HA} : JHJHA \longrightarrow JHA$. This implies that we have maps $\eta_A : A \longrightarrow ?A$ and $\mu_A : ??A \longrightarrow ?A$, and thus, we can show that $(?, \eta, \mu)$ is a symmetric comonoidal monad. All the diagrams defining a symmetric comonoidal monad hold by the structure given by the adjunction. For the complete proof see Appendix A.2. \square

A $?A$ -algebra is a pair $(?A, t_X : ?^2 A \longrightarrow ?A)$, and is called *free* if it is an object of the full subcategory of all $?A$ -algebra's that is in adjunction with the category \mathcal{L} , such that, the right adjoint is the forgetful functor. The why-not monad must allow for the right structural rules, weakening and contraction, to be defined.

Lemma 24 (Right Weakening and Contraction). Given a dual LNL model $\mathcal{L} : H \dashv J : C$, then for any $?A$ there are distinguished symmetric comonoidal natural transformations $w_A : \perp \longrightarrow ?A$ and $c_A : ?A \oplus ?A \longrightarrow ?A$ that form a commutative monoid, and are $?A$ -algebra morphisms with respect to the canonical definitions of the algebras $?A, \perp, ?A \oplus ?A$.

Proof. Suppose (H, h) and (J, j) are two symmetric comonoidal functors, such that, $\mathcal{L} : H \dashv J : C$ is a dual LNL model. Again, we know $?A = H; J : \mathcal{L} \longrightarrow \mathcal{L}$ is a symmetric comonoidal monad by Lemma 23.

We define the following morphisms:

$$\begin{aligned} w_A &:= \perp \xrightarrow{j_{\perp}^{-1}} J0 \xrightarrow{j_{\perp}} JHA \equiv ?A \\ c_A &:= ?A \oplus ?A \equiv JHA \oplus JHA \xrightarrow{j_{HA,HA}^{-1}} J(HA + HA) \xrightarrow{j_{HA}} JHA \equiv ?A \end{aligned}$$

The remainder of the proof is by carefully checking all of the required diagrams. Please see Appendix A.3 for the complete proof. \square

Lemma 25 ($?A$ -Monoid Morphisms). Suppose $\mathcal{L} : H \dashv J : C$ is a dual LNL model. Then if $f : (?A, \mu_A) \longrightarrow (?B, \mu_B)$ is a morphism of free $?A$ -algebras, then it is a monoid morphism.

Proof. Suppose $\mathcal{L} : H \dashv J : C$ is a dual LNL model. Then we know $?A = JHA$ is a symmetric comonoidal monad by Lemma 23. Bellin [1] remarks that by Maietti, Maneggia de Paiva and Ritter's Proposition 25 [5], it suffices to show that $\mu_A : ??A \longrightarrow ?A$ is a monoid morphism. For the details see the complete proof in Appendix B. \square

Finally, we may now conclude the following corollary.

Corollary 26. Every dual LNL model is a dual category.

2.3.3. Dual Category implies Dual LNL Model.

2.4. A Mixed Bi-Linear/Non-Linear Model.

Definition 27. A **mixed bi-linear/non-linear model** consists of the following:

- i. a bi-linear category $(\mathcal{L}, \top, \otimes, \multimap, \perp, \oplus, \multimap)$,
- ii. a cartesian closed category $(\mathcal{I}, 1, \times, \rightarrow)$,
- iii. a cocartesian coclosed category $(\mathcal{C}, 0, +, -)$, and
- iv. two symmetric monoidal adjunctions $\mathcal{I} : F \dashv G : \mathcal{L}$ and $\mathcal{L} : H \dashv J : \mathcal{C}$.

3. MIXED LINEAR/NON-LINEAR COINTUITIONISTIC LOGIC: DUAL LNL LOGIC

Following Benton's [2] lead we can define a mixed linear/non-linear cointuitionistic logic, called dual LNL logic, based on the categorical model given in the previous section. Dual LNL logic consists of two fragments: an cointuitionistic fragment and a linear cointuitionistic fragment. Each of the fragments are related through a syntactic formalization of the adjoint functors from the dual LNL model. First, we define the syntax of dual LNL logic, and then discuss the inference rules for each fragment.

Definition 28. The syntax for dual LNL logic is defined as follows:

- (Cointuitionistic Formulas) $R, S, T ::= 0 \mid S + T \mid S - T \mid H A$
- (Linear Cointuitionistic Formulas) $A, B, C ::= \perp \mid A \oplus B \mid A \multimap B \mid J S$
- (Cointuitionistic Contexts) $\Psi ::= \cdot \mid R \mid \Psi_1, \Psi_2$
- (Linear Cointuitionistic Contexts) $\Gamma, \Delta ::= \cdot \mid A \mid \Gamma_1, \Gamma_2$

Sequents have the following syntax:

- (Cointuitionistic Sequents) $R \vdash_{\mathcal{C}} \Psi$
- (Dual LNL Sequents) $A \vdash_{\mathcal{L}} \Delta \mid \Psi$

The syntax of cointuitionistic formulas are typical. We denote coimplication by $S - T$, but all the other connectives are the usual ones. Linear cointuitionistic formulas are denoted in somewhat of a non-traditional style. We denote par by $A \oplus B$, instead of $A \wp B$. Lastly, we denote linear coimplication by $A \multimap B$ to emphasize its duality with linear implication $A \multimap B$. Each syntactic category of formulas contains the respective functor from the dual LNL model, and thus, we should view H as the left adjoint to J .

Sequents for the linear fragment have the form $A \vdash_{\mathcal{L}} \Delta \mid \Psi$. Similarly to the sequents of Benton's LNL logic [2], each context is separated for readability, but should actually be understood as being able to be mixed, that is, the contexts Δ and Ψ could be a single context.

The inference rules for the cointuitionistic fragment can be found in Figure 1.

$\frac{}{S \vdash_C S}$ C-ID	$\frac{S \vdash_C T, \Psi_2 \quad T \vdash_C \Psi_1}{S \vdash_C \Psi_1, \Psi_2}$ C-CUT	$\frac{S \vdash_C \Psi_1, \Psi_2}{S \vdash_C \Psi_1, T, \Psi_2}$ C-WK
$\frac{S \vdash_C \Psi_1, T, T, \Psi_2}{S \vdash_C \Psi_1, T, \Psi_2}$ C-CR	$\frac{R \vdash_C \Psi_1, S, T, \Psi_2}{R \vdash_C \Psi_1, T, S, \Psi_2}$ C-EX	$\frac{}{0 \vdash_C \Psi}$ C-FL
$\frac{S \vdash_C \Psi_1, \Psi_2}{S \vdash_C \Psi_1, 0, \Psi_2}$ C-FR	$\frac{S \vdash_C \Psi_1 \quad T \vdash_C \Psi_2}{S + T \vdash_C \Psi_1, \Psi_2}$ C-DL	$\frac{R \vdash_C \Psi_1, S, T, \Psi_2}{R \vdash_C \Psi_1, S + T, \Psi_2}$ C-DR
$\frac{S \vdash_C T, \Psi}{S - T \vdash_C \Psi}$ C-sL	$\frac{R \vdash_C \Psi_1, S, \Psi_2 \quad T \vdash_C \Psi_3}{R \vdash_C \Psi_1, S - T, \Psi_2, \Psi_3}$ C-sR	$\frac{A \vdash_L \cdot \Psi}{HA \vdash_C \Psi}$ C-hL

Figure 1: Inference Rules for Dual LNL Logic: Cointuitionistic Fragment

$\frac{A \vdash_L \Delta \mid \Psi_1, \Psi_2}{A \vdash_L \Delta \mid \Psi_1, S, \Psi_2}$	L_WK	$\frac{A \vdash_L \Delta \mid \Psi_1, S, S, \Psi_2}{A \vdash_L \Delta \mid \Psi_1, S, \Psi_2}$	L_CTR
$\frac{A \vdash_L \Delta_1, A, B, \Delta_2 \mid \Psi}{A \vdash_L \Delta_1, B, A, \Delta_2 \mid \Psi}$	L_EX	$\frac{A \vdash_L \Delta \mid \Psi_1, S, T, \Psi_2}{A \vdash_L \Delta \mid \Psi_1, T, S, \Psi_2}$	L_CEX
		$\frac{}{A \vdash_L A \mid \cdot}$	L_ID
$\frac{A \vdash_L \Delta_1, B, \Delta_3 \mid \Psi_1 \quad B \vdash_L \Delta_2 \mid \Psi_2}{A \vdash_L \Delta_1, \Delta_2, \Delta_3 \mid \Psi_1, \Psi_2} \quad \text{L_CUT}$			
$\frac{A \vdash_L \Delta \mid \Psi_1, S, \Psi_3 \quad S \vdash_C \Psi_2}{A \vdash_L \Delta \mid \Psi_1, \Psi_2, \Psi_3} \quad \text{L_CCUT}$			

Figure 2: Inference Rules for Dual LNL Logic: Structural Rules, Identity, and Cut Rules

4. EMBEDDING COINTUITIONISTIC LOGIC IN DUAL LNL LOGIC

5. DUAL LNL TERM ASSIGNMENT

TODO

6. RELATED WORK

TODO

7. CONCLUSION

TODO

$\frac{}{\perp \vdash_L \cdot \cdot}$	$L_FL L$	$\frac{A \vdash_L \Delta_1, \Delta_2 \Psi}{A \vdash_L \Delta_1, \perp, \Delta_2 \Psi}$	$L_FL R$	$\frac{A \vdash_L \Delta \Psi_1, S, T, \Psi_2}{A \vdash_L \Delta \Psi_1, S + T, \Psi_2}$	L_DR
$\frac{A \vdash_L \Delta_1 \Psi_1 \quad B \vdash_L \Delta_2 \Psi_2}{A \oplus B \vdash_L \Delta_1, \Delta_2 \Psi_1, \Psi_2}$	L_PL	$\frac{A \vdash_L \Delta_1, B, C, \Delta_2 \Psi}{A \vdash_L \Delta_1, B \oplus C, \Delta_2 \Psi}$	L_PR		
$\frac{A \vdash_L B, \Delta \Psi}{A \bullet B \vdash_L \Delta \Psi}$	L_SL	$\frac{A \vdash_L \Delta_1, B, \Delta_2 \Psi_1 \quad C \vdash_L \Delta_3 \Psi_2}{A \vdash_L \Delta_1, B \bullet C, \Delta_2, \Delta_3 \Psi_1, \Psi_2}$	L_SR		
$\frac{A \vdash_L \Delta \Psi_1, S, \Psi_2 \quad T \vdash_C \Psi_3}{A \vdash_L \Delta \Psi_1, S - T, \Psi_2, \Psi_3}$	$L_CS R$	$\frac{S \vdash_C \Psi}{JS \vdash_L \cdot \Psi}$	L_JL		
$\frac{A \vdash_L \Delta S, \Psi}{A \vdash_L \Delta, JS \Psi}$	L_JR	$\frac{A \vdash_L \Delta, B \Psi}{A \vdash_L \Delta HB, \Psi}$	L_HR		

Figure 3: Inference Rules for Dual LNL Logic: Cotensor, Coimplication, and Functor Rules

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APPENDIX A. PROOFS

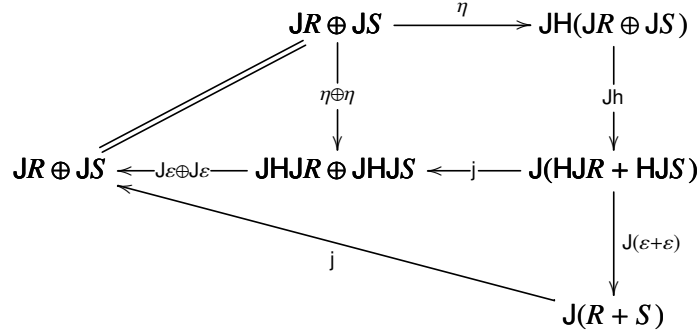
A.1. **Proof of Lemma 21.** We show that both of the maps:

$$j_{R,S}^{-1} := JR \oplus JS \xrightarrow{\eta} JH(JR \oplus JS) \xrightarrow{Jh_{A,B}} J(HJR + HJS) \xrightarrow{J(\varepsilon_R + \varepsilon_S)} J(R + S)$$

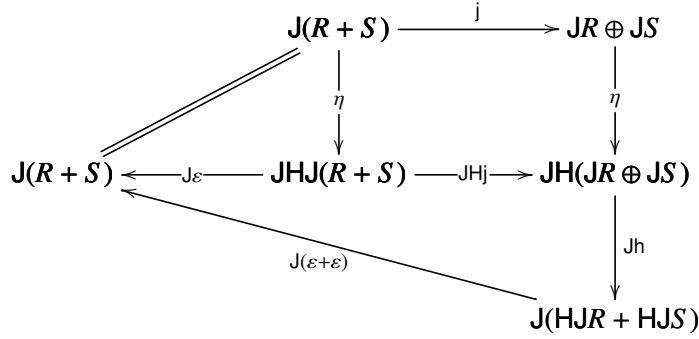
$$j_0^{-1} := \perp \xrightarrow{\eta} JH \perp \xrightarrow{Jh_{\perp}} J0$$

are mutual inverses with $j_{R,S} : J(R + S) \longrightarrow JR \oplus JS$ and $j_0 : \perp \longrightarrow J0$ respectively.

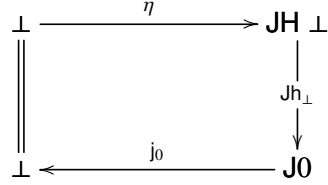
Case. The following diagram implies that $j_{R,S}^{-1}; j_{R,S} = \text{id}$:



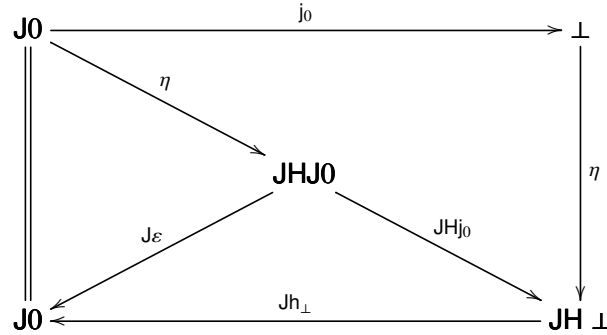
The two top diagrams both commute because η and ε are the unit and counit of the adjunction respectively, and the bottom diagram commutes by naturality of j .
Case. The following diagram implies that $j_{R,S}; j_{R,S}^{-1} = \text{id}$:



The top left and bottom diagrams both commute because η and ε are the unit and counit of the adjunction respectively, and the top right diagram commutes by naturality of η .
Case. The following diagram implies that $j_0^{-1}; j_0 = \text{id}$:



This diagram holds because η is the unit of the adjunction.
Case. The following diagram implies that $j_0; j_0^{-1} = \text{id}$:



The top-left and bottom diagrams commute because η and ε are the unit and counit of the adjunction respectively, and the top-right diagram commutes by naturality of η .

A.2. **Proof of Lemma 23.** Since $?$ is the composition of two symmetric comonoidal functors we know it is also symmetric comonoidal, and hence, the following diagrams all hold:

$$\begin{array}{ccc}
?(A \oplus B) \oplus C & \xrightarrow{r_{A \oplus B, C}} & ?(A \oplus B) \oplus ?C \\
\downarrow ?\alpha_{A, B, C} & & \downarrow r_{A, B} \oplus \text{id}_{?C} \\
?(A \oplus (B \oplus C)) & & (?A \oplus ?B) \oplus ?C \\
\downarrow r_{A, B \oplus C} & & \downarrow \alpha_{?A, ?B, ?C} \\
?A \oplus ?(B \oplus C) & \xrightarrow{\text{id}_{?A} \oplus r_{B, C}} & ?A \oplus (?B \oplus ?C)
\end{array}$$

$$\begin{array}{ccc}
?(\perp \oplus A) & \xrightarrow{r_{\perp, A}} & ?\perp \oplus ?A \\
\downarrow ?\lambda_A & & \downarrow r_{\perp} \oplus \text{id}_{?A} \\
?A & \xrightarrow{\lambda^{-1}_{?A}} & \perp \oplus ?A
\end{array}$$

$$\begin{array}{ccc}
?(A \oplus \perp) & \xrightarrow{r_{A, \perp}} & ?A \oplus ?\perp \\
\downarrow ?\rho_A & & \downarrow \text{id}_{?A} \oplus r_{\perp} \\
?A & \xrightarrow{\rho^{-1}_{?A}} & ?A \oplus \perp
\end{array}$$

$$\begin{array}{ccc}
?(A \oplus B) & \xrightarrow{r_{A, B}} & ?A \oplus ?B \\
\downarrow ?\beta_{A, B} & & \downarrow \beta_{?A, ?B} \\
?(B \oplus A) & \xrightarrow{r_{B, A}} & ?B \oplus ?A
\end{array}$$

Next we show that $(?, \eta, \mu)$ defines a monad where $\eta_A : A \rightarrow ?A$ is the unit of the adjunction, and $\mu_A = \text{J}\varepsilon_{HA} : ??A \rightarrow ?A$. It suffices to show that every diagram of Definition 11 holds.

Case.

$$\begin{array}{ccc}
?^3 A & \xrightarrow{\mu_{?A}} & ?^2 A \\
\downarrow ?\mu_A & & \downarrow \mu_A \\
?^2 A & \xrightarrow{\mu_A} & ?A
\end{array}$$

It suffices to show that the following diagram commutes:

$$\begin{array}{ccc}
\text{J}(\text{H}(?^2 A)) & \xrightarrow{\text{J}\varepsilon_{H ?A}} & \text{J}(\text{H } ?A) \\
\downarrow \text{J}(\text{H } \mu_A) & & \downarrow \text{J}\varepsilon_{HA} \\
\text{J}(\text{H } ?A) & \xrightarrow{\text{J}\varepsilon_{HA}} & \text{J}(HA)
\end{array}$$

But this diagram is equivalent to the following:

$$\begin{array}{ccc}
 \mathbf{HJHJHA} & \xrightarrow{\varepsilon_{\mathbf{HJHA}}} & \mathbf{HJHA} \\
 \downarrow \mathbf{HJ\varepsilon_{HA}} & & \downarrow \varepsilon_{\mathbf{HA}} \\
 \mathbf{HJHA} & \xrightarrow{\varepsilon_{\mathbf{HA}}} & \mathbf{HA}
 \end{array}$$

The previous diagram commutes by naturality of ε .
Case.

$$\begin{array}{ccccc}
 & & ?A & & \\
 & \swarrow & \uparrow \mu_A & \searrow & \\
 ?A & \xrightarrow{\eta_{?A}} & ?^2 A & \xleftarrow{? \eta_A} & ?A
 \end{array}$$

It suffices to show that the following diagrams commutes:

$$\begin{array}{ccccc}
 & & JHA & & \\
 & \swarrow & \uparrow J\varepsilon_{HA} & \searrow & \\
 JHA & \xrightarrow{\eta_{JHA}} & JHJHA & \xleftarrow{JH\eta_A} & JHA
 \end{array}$$

Both of these diagrams commute because η and ε are the unit and counit of an adjunction.

It remains to be shown that η and μ are both symmetric comonoidal natural transformations, but this easily follows from the fact that we know η is by assumption, and that μ is because it is defined in terms of ε which is a symmetric comonoidal natural transformation. Thus, all of the following diagrams commute:

$$\begin{array}{ccc}
 \begin{array}{ccc}
 A \oplus B & \xrightarrow{\eta_{A \oplus B}} & ?A \oplus ?B \\
 \downarrow \eta_A & \nearrow r_{A,B} & \\
 ?(A \oplus B) & &
 \end{array} & & \begin{array}{ccc}
 \perp & \xrightarrow{\eta_{\perp}} & ?\perp \\
 \downarrow & \nearrow r_{\perp} & \\
 \perp & &
 \end{array} \\
 \\
 \begin{array}{ccccc}
 ?^2(A \oplus B) & \xrightarrow{?r_{A,B}} & ?(?A \oplus ?B) & \xrightarrow{r_{?A, ?B}} & ?^2 A \oplus ?^2 B \\
 \downarrow \mu_{A \oplus B} & & & & \downarrow \mu_A \oplus \mu_B \\
 ?(A \oplus B) & \xrightarrow{r_{A,B}} & ?A \oplus ?B & &
 \end{array} & & \begin{array}{ccc}
 ?^2 \perp & \xrightarrow{?r_{\perp}} & ?\perp \\
 \downarrow \mu_{\perp} & & \downarrow r_{\perp} \\
 ?\perp & \xrightarrow{r_{\perp}} & \perp
 \end{array}
 \end{array}$$

A.3. Proof of Lemma 24. Suppose (H, h) and (J, j) are two symmetric comonoidal functors, such that, $\mathcal{L} : H \dashv J : \mathcal{C}$ is a dual LNL model. Again, we know $?A = H; J : \mathcal{L} \longrightarrow \mathcal{L}$ is a symmetric comonoidal monad by Lemma 23.

We define the following morphisms:

$$\begin{aligned} w_A &:= \perp \xrightarrow{j_0^{-1}} J0 \xrightarrow{J\phi_{HA}} JHA = ?A \\ c_A &:= ?A \oplus ?A = JHA \oplus JHA \xrightarrow{j_{HA, HA}^{-1}} J(HA + HA) \xrightarrow{J\nabla_{HA}} JHA = ?A \end{aligned}$$

Next we show that both of these are symmetric comonoidal natural transformations, but for which functors? Define $W(A) = \perp$ and $C(A) = ?A \oplus ?A$ on objects of \mathcal{L} , and $W(f : A \longrightarrow B) = \text{id}_{\perp}$ and $C(f : A \longrightarrow B) = ?f \oplus ?f$ on morphisms. So we must show that $w : W \longrightarrow ?$ and $c : C \longrightarrow ?$ are symmetric comonoidal natural transformations. We first show that w is and then we show that c is. Throughout the proof we drop subscripts on natural transformations for readability.

Case. To show w is a natural transformation we must show the following diagram commutes for any morphism $f : A \longrightarrow B$:

$$\begin{array}{ccc} W(A) & \xrightarrow{w_A} & ?A \\ W(f) \downarrow & & \downarrow ?f \\ W(B) & \xrightarrow{w_B} & ?B \end{array}$$

This diagram is equivalent to the following:

$$\begin{array}{ccc} \perp & \xrightarrow{w_A} & ?A \\ \text{id}_{\perp} \downarrow & & \downarrow ?f \\ \perp & \xrightarrow{w_B} & ?B \end{array}$$

It further expands to the following:

$$\begin{array}{ccccc} \perp & \xrightarrow{j_0^{-1}} & J0 & \xrightarrow{J(\phi_{HA})} & JHA \\ \text{id}_{\perp} \downarrow & & & & \downarrow JHf \\ \perp & \xrightarrow{j_0^{-1}} & J0 & \xrightarrow{J(\phi_{HB})} & JHB \end{array}$$

This diagram commutes, because $J(\phi_{HA}); Jf = J(\phi_{HA}; f) = J(\phi_{HB})$, by the uniqueness of the initial map.

Case. The functor W is comonoidal itself. To see this we must exhibit a map

$$s_{\perp} := \text{id}_{\perp} : W \perp \longrightarrow \perp$$

and a natural transformation

$$s_{A,B} := \rho_{\perp}^{-1} : W(A \oplus B) \longrightarrow WA \oplus WB$$

subject to the coherence conditions in Definition 6. Clearly, the second map is a natural transformation, but we leave showing they respect the coherence conditions to the reader. Now we can show that w is indeed symmetric comonoidal.

Case.

$$\begin{array}{ccc}
 W(A \oplus B) & \xrightarrow{s_{A,B}} & WA \oplus WB \\
 \downarrow w_{A \oplus B} & & \downarrow w_A \oplus w_B \\
 ?(A \oplus B) & \xrightarrow{r_{A,B}} & ?A \oplus ?B
 \end{array}$$

Expanding the objects of the previous diagram results in the following:

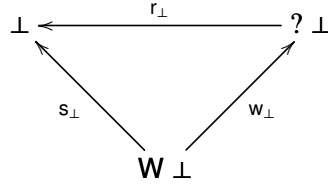
$$\begin{array}{ccc}
 \perp & \xrightarrow{s_{A,B}} & \perp \oplus \perp \\
 \downarrow w_{A \oplus B} & & \downarrow w_A \oplus w_B \\
 ?(A \oplus B) & \xrightarrow{r_{A,B}} & ?A \oplus ?B
 \end{array}$$

This diagram commutes, because the following fully expanded diagram commutes:

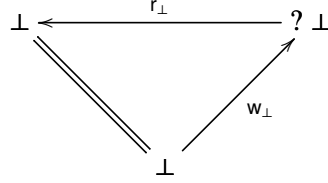
$$\begin{array}{ccccc}
 \perp & \xrightarrow{\rho^{-1}} & \perp \oplus \perp & & \\
 \downarrow j_0^{-1} & & \swarrow j_0^{-1} \oplus \text{id} & \searrow j_0^{-1} \oplus j_0^{-1} & \\
 & (6) & J0 \oplus \perp & \xrightarrow{\text{id} \oplus j_0^{-1}} & J0 \oplus J0 \\
 & & \parallel & (4) & \parallel \\
 & & J0 \oplus \perp & \xleftarrow{\text{id} \oplus j_0} & J0 \oplus J0 \\
 & \nearrow \rho^{-1} & (3) & \xleftarrow{j} & \\
 J0 & \xleftarrow{J\rho} & J(0 + 0) & \xrightarrow{j} & J0 \oplus J0 \\
 \downarrow J\circ & (1) & \downarrow J(\circ + \circ) & (2) & \downarrow J\circ \oplus J\circ \\
 JH(A \oplus B) & \xrightarrow{Jh} & J(HA + HB) & \xrightarrow{j} & JHA \oplus JHB
 \end{array}$$

Diagram 1 commutes because 0 is the initial object, diagram 2 commutes by naturality of j , diagram 3 commutes because J is a symmetric comonoidal functor, diagram 4 commutes because j_0 is an isomorphism (Lemma 21), diagram 5 commutes by functoriality of J , and diagram 6 commutes by naturality of ρ .

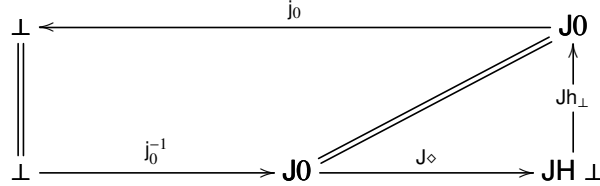
Case.



Expanding the objects in the previous diagram results in the following:

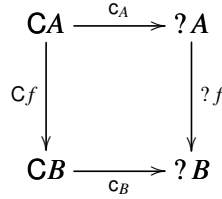


This diagram commutes because the following one does:

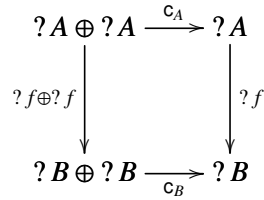


The diagram on the left commutes because j_0 is an isomorphism (Lemma 21), and the diagram on the right commutes because 0 is the initial object.

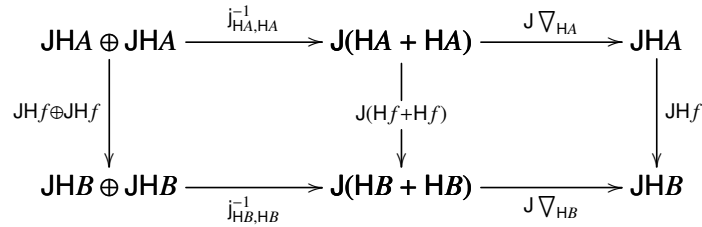
Case. Now we show that $c_A : ?A \oplus ?A \longrightarrow ?A$ is a natural transformation. This requires the following diagram to commute (for any $f : A \longrightarrow B$):



This expands to the following diagram:



This diagram commutes because the following diagram does:



The left square commutes by naturality of j^{-1} , and the right square commutes by naturality of the codiagonal $\nabla_A : A + A \longrightarrow A$.

Case. The functor $\mathbf{C} : \mathcal{L} \rightarrow \mathcal{L}$ is indeed symmetric comonoidal where the required maps are defined as follows:

$$t_{\perp} := ?\perp \oplus ?\perp = JH\perp \oplus JH\perp \xrightarrow{j^{-1}} J(H\perp + H\perp) \xrightarrow{J\nabla} JH\perp \xrightarrow{Jh_{\perp}} J0 \xrightarrow{j_0} \perp$$

$$t_{A,B} := ?(A \oplus B) \oplus ?(A \oplus B) \xrightarrow{r_{A,B} \oplus r_{A,B}} (?A \oplus ?B) \oplus (?A \oplus ?B) \xrightarrow{\text{iso}} (?A \oplus ?A) \oplus (?B \oplus ?B)$$

where iso is a natural isomorphism that can easily be defined using the symmetric monoidal structure of \mathcal{L} . Clearly, t is indeed a natural transformation, but we leave checking that the required diagrams in Definition 6 commute to the reader. We can now show that $c_A : ?A \oplus ?A \rightarrow ?A$ is symmetric comonoidal. The following diagrams from Definition 8 must commute:

Case.

$$\begin{array}{ccc} \mathbf{C}(A \oplus B) & \xrightarrow{t_{A,B}} & \mathbf{C}A \oplus \mathbf{C}B \\ \downarrow c_{A \oplus B} & & \downarrow c_A \oplus c_B \\ ?(A \oplus B) & \xrightarrow{r_{A,B}} & ?A \oplus ?B \end{array}$$

Expanding the objects in the previous diagram results in the following:

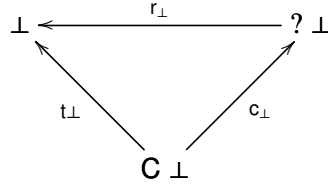
$$\begin{array}{ccc} ?(A \oplus B) \oplus ?(A \oplus B) & \xrightarrow{t_{A,B}} & (?A \oplus ?A) \oplus (?B \oplus ?B) \\ \downarrow c_{A \oplus B} & & \downarrow c_A \oplus c_B \\ ?(A \oplus B) & \xrightarrow{r_{A,B}} & ?A \oplus ?B \end{array}$$

This diagram commutes, because the following fully expanded one does:

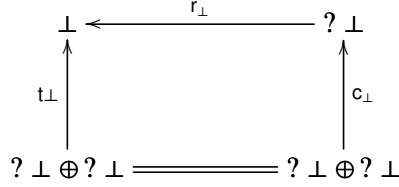
$$\begin{array}{c}
\begin{array}{c}
\text{JH}(A \oplus B) \oplus \text{JH}(A \oplus B) \xrightarrow{\text{Jh} \oplus \text{Jh}} \text{J}(HA + HB) \oplus \text{J}(HA + HB) \xrightarrow{\text{j} \oplus \text{j}} (\text{JHA} \oplus \text{JHB}) \oplus (\text{JHA} \oplus \text{JHB}) \xrightarrow{\text{iso}} (\text{JHA} \oplus \text{JHA}) \oplus (\text{JHB} \oplus \text{JHB}) \\
\downarrow \text{j}^{-1} \quad (2) \quad \downarrow \text{j}^{-1} \quad (4) \quad \downarrow \text{j}:(\oplus) \quad (6) \quad \downarrow \text{j}^{-1} \oplus \text{j}^{-1} \\
\text{J}(\text{H}(A \oplus B) + \text{H}(A \oplus B)) \xrightarrow{\text{J}(\text{h} + \text{h})} \text{J}((\text{HA} + \text{HB}) + (\text{HA} + \text{HB})) \xrightarrow{\text{Jiso}} \text{J}((\text{HA} + \text{HA}) + (\text{HB} + \text{HB})) \xrightarrow{\text{j}} \text{J}(\text{HA} + \text{HA}) \oplus \text{J}(\text{HB} + \text{HB}) \\
\downarrow \text{J}\nabla \quad (1) \quad \downarrow \text{J}\nabla \quad (3) \quad \downarrow \text{J}(\nabla + \nabla) \quad (5) \quad \downarrow \text{J}\nabla \oplus \nabla \\
\text{JH}(A \oplus B) \xrightarrow{\text{Jh}} \text{J}(HA + HB) \xlongequal{\quad} \text{J}(\text{HA} + \text{HB}) \xrightarrow{\text{j}} \text{JHA} \oplus \text{JHB}
\end{array}
\end{array}$$

Diagram 1 commutes by naturality of ∇ , diagram 2 commutes by naturality of j^{-1} , diagram 3 commutes by straightforward reasoning on coproducts, diagram 4 commutes by straightforward reasoning on the symmetric monoidal structure of J after expanding the definition of the two isomorphisms – here $J\text{iso}$ is the corresponding isomorphisms on coproducts – diagram 5 commutes by naturality of j , and diagram 6 commutes because j is an isomorphism (Lemma 21).

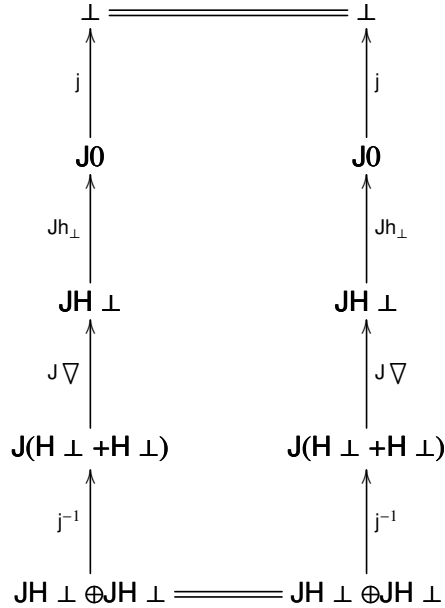
Case.



Expanding the objects of this diagram results in the following:



Simply unfolding the morphisms in the previous diagram reveals the following:



Clearly, this diagram commutes.

At this point we have shown that $w_A : \perp \longrightarrow ? A$ and $c_A : ? A \oplus ? A \longrightarrow ? A$ are symmetric comonoidal naturality transformations. Now we show that for any $? A$ the triple $(? A, w_A, c_A)$ forms a commutative monoid. This means that the following diagrams must commute:

Case.

$$\begin{array}{ccccc}
(?A \oplus ?A) \oplus ?A & \xrightarrow{\alpha_{?A, ?A, ?A}} & ?A \oplus (?A \oplus ?A) & \xrightarrow{\text{id}_{?A} \oplus c_A} & ?A \oplus ?A \\
\downarrow c_A \oplus \text{id}_A & & & & \downarrow c_A \\
?A \oplus ?A & \xrightarrow{c_A} & & & ?A
\end{array}$$

The previous diagram commutes, because the following one does (we omit subscripts for readability):

$$\begin{array}{ccccccc}
(JHA \oplus JHA) \oplus JHA & \xrightarrow{\alpha} & JHA \oplus (JHA \oplus JHA) & \xrightarrow{\text{id} \oplus j^{-1}} & JHA \oplus J(HA + HA) & \xrightarrow{\text{id} \oplus J \nabla} & JHA \oplus JHA \\
\downarrow j^{-1} \oplus \text{id} & & (1) & & \downarrow j^{-1} & (2) & \downarrow j^{-1} \\
J(HA + HA) \oplus JHA & \xrightarrow{j^{-1}} & J((HA + HA) + HA) & \xrightarrow{J\alpha} & J(HA + (HA + HA)) & \xrightarrow{J(\text{id} + \nabla)} & J(HA + HA) \\
\downarrow J \nabla \oplus \text{id} & (3) & \downarrow J(\nabla + \text{id}) & (4) & & & \downarrow J \nabla \\
JHA \oplus JHA & \xrightarrow{j^{-1}} & J(HA + HA) & \xrightarrow{J \nabla} & & & JHA
\end{array}$$

Diagram 1 commutes because J is a symmetric monoidal functor (Corollary 22), diagrams 2 and 3 commute by naturality of j^{-1} , and diagram 4 commutes because (HA, \diamond, ∇) is a commutative monoid in \mathcal{C} , but we leave the proof of this to the reader.

Case.

$$\begin{array}{ccc}
?A \oplus \perp & & \\
\downarrow \text{id}_{?A} \oplus w_A & \searrow \rho_{?A} & \\
?A \oplus ?A & \xrightarrow{c_A} & ?A
\end{array}$$

The previous diagram commutes, because the following one does:

$$\begin{array}{ccccc}
JHA \oplus \perp & \xrightarrow{\rho} & JHA & & \\
\downarrow \text{id} \oplus j_0^{-1} & & (1) & & \parallel \\
JHA \oplus J0 & \xrightarrow{j^{-1}} & J(HA + 0) & \xrightarrow{J\rho} & JHA \\
\downarrow \text{id} \oplus J\diamond & (2) & \downarrow J(\text{id} \oplus \diamond) & (3) & \parallel \\
JHA \oplus JHA & \xrightarrow{j^{-1}} & J(HA + HA) & \xrightarrow{J \nabla} & JHA
\end{array}$$

Diagram 1 commutes because J is a symmetric monoidal functor (Corollary 22), diagram 2 commutes by naturality of j^{-1} , and diagram 3 commutes because (HA, \diamond, ∇) is a commutative monoid in \mathcal{C} , but we leave the proof of this to the reader.

Case.

$$\begin{array}{ccc}
?A \oplus ?A & & \\
\downarrow \beta_{?A, ?A} & \searrow c_A & \\
?A \oplus ?A & \xrightarrow{c_A} & ?A
\end{array}$$

This diagram commutes, because the following one does:

$$\begin{array}{ccccc}
JHA \oplus JHA & \xrightarrow{j^{-1}} & J(HA + HA) & \xrightarrow{J\triangledown} & JHA \\
\downarrow \beta & & \downarrow J\beta & & \parallel \\
JHA \oplus JHA & \xrightarrow{j^{-1}} & J(HA + HA) & \xrightarrow{J\triangledown} & JHA
\end{array}$$

The left diagram commutes by naturality of j^{-1} , and the right diagram commutes because $(HA, \diamond, \triangledown)$ is a commutative monoid in \mathcal{C} , but we leave the proof of this to the reader.

Finally, we must show that $w_A : \perp \longrightarrow ?A$ and $c_A : ?A \oplus ?A \longrightarrow ?A$ are $?$ -algebra morphisms. The algebras in play here are $(?A, \mu : ??A \longrightarrow ?A)$, $(\perp, r_\perp : ?\perp \longrightarrow \perp)$, and $(?A \oplus ?A, u_A : ?(?A \oplus ?A) \longrightarrow ?A \oplus ?A)$, where $u_A := ?(?A \oplus ?A) \xrightarrow{r_{?A, ?A}} ?^2A \oplus ?^2A \xrightarrow{\mu_A \oplus \mu_A} ?A \oplus ?A$. It suffices to show that the following diagrams commute:

Case.

$$\begin{array}{ccc}
?\perp & \xrightarrow{r_\perp} & \perp \\
\downarrow ?w & & \downarrow w \\
??A & \xrightarrow{\mu} & ?A
\end{array}$$

This diagram commutes, because the following fully expanded one does:

$$\begin{array}{ccccc}
JH\perp & \xrightarrow{Jh_\perp} & J0 & \xrightarrow{j_0} & \perp \\
\downarrow JHj_0^{-1} & \searrow JHj_0^{-1} & & & \downarrow j_0^{-1} \\
& & JHJ0 & \xrightarrow{JHj_0} & JH\perp \\
& & & & \downarrow Jh_\perp \\
& & & & J0 \\
\downarrow JHJ_\diamond & & \downarrow J\varepsilon_0 & & \downarrow J_\diamond \\
JHJHA & \xrightarrow{J\varepsilon} & & & JHA
\end{array}$$

(1) (2) (3) (4)

Diagram 1 commutes by naturality of ε , diagram 2 commutes because ε is the counit of the symmetric comonoidal adjunction, diagram 3 clearly commutes, and diagram 4 commutes because j_0 is an isomorphism (Lemma 21).

Case.

$$\begin{array}{ccc}
?(A \oplus ?A) & \xrightarrow{u} & ?A \oplus ?A \\
\downarrow ?c & & \downarrow c \\
??A & \xrightarrow{\mu} & ?A
\end{array}$$

This diagram commutes because the following fully expanded one does:

[illegible]

Diagram 1 clearly commutes, diagram 2 commutes by naturality of ε , diagram 3 commutes by naturality of ∇ , diagram 4 commutes because ε is the counit of the symmetric comonoidal adjunction, diagram 5 commutes because j is an isomorphism (Lemma 21), diagram 6 commutes by naturality of j^{-1} , and diagram 7 is the same diagram as 3, but this diagram is redundant for readability.

APPENDIX B. PROOF OF LEMMA 25

Suppose $\mathcal{L} : \mathbf{H} \dashv \mathbf{J} : \mathbf{C}$ is a dual LNL model. Then we know $?A = \mathbf{J}HA$ is a symmetric comonoidal monad by Lemma 23. Bellin [1] remarks that by Maietti, Maneggia de Paiva and Ritter's Proposition 25 [5], it suffices to show that $\mu_A : ??A \longrightarrow ?A$ is a monoid morphism. Thus, the following diagrams must commute:

Case.

$$\begin{array}{ccc}
 ??A \oplus ??A & \xrightarrow{c_{?A}} & ??A \\
 \downarrow \mu_A \oplus \mu_A & & \downarrow \mu_A \\
 ?A \oplus ?A & \xrightarrow{c_A} & ?A
 \end{array}$$

This diagram commutes because the following fully expanded one does:

$$\begin{array}{ccccc}
 \mathbf{J}H\mathbf{J}HA \oplus \mathbf{J}H\mathbf{J}HA & \xrightarrow{j^{-1}} & \mathbf{J}(H\mathbf{J}HA + H\mathbf{J}HA) & \xrightarrow{J\nabla} & \mathbf{J}H\mathbf{J}HA \\
 \downarrow J\varepsilon \oplus J\varepsilon & & \downarrow J(\varepsilon + \varepsilon) & & \downarrow J\varepsilon \\
 \mathbf{J}HA \oplus \mathbf{J}HA & \xrightarrow{j^{-1}} & \mathbf{J}(HA + HA) & \xrightarrow{J\nabla} & \mathbf{J}HA
 \end{array}$$

The left square commutes by naturality of j^{-1} and the right square commutes by naturality of the codiagonal.

Case.

$$\begin{array}{ccc}
 & \perp & \\
 w_{?A} \swarrow & & \searrow w_A \\
 ??A & \xrightarrow{\mu_A} & ?A
 \end{array}$$

This diagram commutes because the following fully expanded one does:

$$\begin{array}{ccc}
 \perp & \xlongequal{\quad} & \perp \\
 \downarrow j_0^{-1} & & \downarrow j_0^{-1} \\
 \mathbf{J}0 & \xlongequal{\quad} & \mathbf{J}0 \\
 \downarrow J\circ & & \downarrow J\circ \\
 \mathbf{J}H\mathbf{J}HA & \xrightarrow{J\varepsilon} & \mathbf{J}HA
 \end{array}$$

The top square trivially commutes, and the bottom square commutes by uniqueness of the initial map.