Inductive Definitions

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A set of rules that are used to derive facts about judgments.

Inductive Definitions: Judgments

A judgment is an assertion about a syntactic object.

It states that one or more syntactic objects have a property or is related to another.

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Examples:

n nat	n is a natural number
$n = n_1 + n_2$	n is the sum of n_1 and n_2
t : type	t is a type
$e : t$	the expression e has type t

Inductive Definitions: Judgments

We denote an arbitrary judgment as J.

Inductive Definitions: Inference Rules

Judgements are defined by an inductive definition which consists of a set of inference rules.

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$$\frac{}{J}\ \text{Name} \qquad\qquad \frac{J_1 \quad \dots \quad J_n}{J}\ \text{Name}$$

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Premises $\longrightarrow$

$$\frac{J_1 \quad \cdots \quad J_n}{J}\,\text{Name}$$

Inductive Definitions: Inference Rules

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Premises ⟶

Conclusion ⟶

$$\frac{J_1 \quad \cdots \quad J_n}{J}\ \text{Name}$$

Inductive Definitions: Inference Rules

Judgements are defined by an inductive definition which consists of a set of <u>inference rules.</u>

$$\begin{array}{r} \text{Premises} \longrightarrow \\[2ex] \text{Conclusion} \longrightarrow \end{array} \quad \dfrac{J_1 \quad \cdots \quad J_n}{J}\ \text{Name}$$

Read as an if-then

$$(J_1 \wedge \cdots \wedge J_n) \implies J$$

Inductive Definitions: Examples

$$\frac{}{\text{true bool}}\ \text{True} \qquad \frac{}{\text{false bool}}\ \text{False}$$

$$\frac{b_1 \text{ bool} \qquad \begin{array}{c} b_2 \text{ bool} \\ b_3 \text{ bool} \end{array}}{\text{if}(b_1, b_2, b_3) \text{ bool}}\ \text{If}$$

Inductive Definitions: Examples

$$\frac{}{\mathsf{true} \rightsquigarrow \mathsf{true}}\ \textsf{True} \qquad \frac{}{\mathsf{false} \rightsquigarrow \mathsf{false}}\ \textsf{False}$$

$$\frac{b_1 \rightsquigarrow \mathsf{true}}{\mathsf{if}(b_1, b_2, b_3) \rightsquigarrow b_2}\ \textsf{IfTrue} \qquad \frac{b_1 \rightsquigarrow \mathsf{false}}{\mathsf{if}(b_1, b_2, b_3) \rightsquigarrow b_3}\ \textsf{IfFalse}$$

$$\frac{b_1 \rightsquigarrow b_1'}{\mathsf{if}(b_1, b_2, b_3) \rightsquigarrow \mathsf{if}(b_1', b_2, b_3)}\ \textsf{If}$$

Inductive Definitions: Examples

$$\frac{}{0 \ \mathsf{nat}}\ \textsf{Zero} \qquad \frac{n \ \mathsf{nat}}{\mathsf{succ}(n) \ \mathsf{nat}}\ \textsf{Succ}$$

$$\frac{}{\mathsf{empty} \ \mathsf{list}}\ \textsf{Empty} \qquad \frac{n \ \mathsf{nat} \qquad l \ \mathsf{list}}{\mathsf{cons}(n, l) \ \mathsf{list}}\ \textsf{Cons}$$

Inductive Definitions: Derivations

To prove that an inductively defined judgment, J, holds it is enough to exhibit a derivation of it.

Inductive Definitions: Derivations

Derivations are goal directed proofs written by stacking inference rules to form a derivation tree.

$$\frac{\Delta_1 \quad \cdots \quad \Delta_n}{J}\,\text{Name}$$

Inductive Definitions: Examples

cons(1,cons(2,cons(3,empty))) list

Inductive Definitions: Examples

$$\frac{1 \ \text{nat} \qquad \text{cons}(2,\text{cons}(3,\text{empty})) \ \text{list}}{\text{cons}(1,\text{cons}(2,\text{cons}(3,\text{empty}))) \ \text{list}} \ \text{Cons}$$

Inductive Definitions: Examples

$$\dfrac{\dfrac{0,\text{nat}}{1\ \text{nat}}\,\text{Succ} \qquad \text{cons}(2,\text{cons}(3,\text{empty}))\ \text{list}}{\text{cons}(1,\text{cons}(2,\text{cons}(3,\text{empty})))\ \text{list}}\,\text{Cons}$$

Inductive Definitions: Examples

$$\frac{\dfrac{\dfrac{}{0,\text{nat}}\,\text{Zero}}{1\ \text{nat}}\,\text{Succ} \qquad \text{cons}(2,\text{cons}(3,\text{empty}))\ \text{list}}{\text{cons}(1,\text{cons}(2,\text{cons}(3,\text{empty})))\ \text{list}}\,\text{Cons}$$

Inductive Definitions: Examples

$$\dfrac{\dfrac{\dfrac{}{0,\text{nat}}\,\text{Zero}}{1\ \text{nat}}\,\text{Succ} \qquad \dfrac{2\ \text{nat} \qquad \text{cons}(3,\text{empty})\ \text{list}}{\text{cons}(2,\text{cons}(3,\text{empty}))\ \text{list}}\,\text{Cons}}{\text{cons}(1,\text{cons}(2,\text{cons}(3,\text{empty})))\ \text{list}}\,\text{Cons}$$

Inductive Definitions: Examples

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Inductive Definitions: Examples

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\text{cons}(3,\text{empty})\ \text{list}
}{\text{cons}(2,\text{cons}(3,\text{empty}))\ \text{list}}\,\textsf{Cons}
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Inductive Definitions: Examples

$$\dfrac{\dfrac{\dfrac{}{0,\text{nat}}\,\text{Zero}}{1\ \text{nat}}\,\text{Succ} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{0,\text{nat}}\,\text{Zero}}{1\ \text{nat}}\,\text{Succ}}{2\ \text{nat}}\,\text{Succ} \qquad \dfrac{3\ \text{nat} \qquad \text{empty list}}{\text{cons}(3,\text{empty})\ \text{list}}\,\text{Cons}}{\text{cons}(2,\text{cons}(3,\text{empty}))\ \text{list}}\,\text{Cons}}{\text{cons}(1,\text{cons}(2,\text{cons}(3,\text{empty})))\ \text{list}}\,\text{Cons}$$

Inductive Definitions: Examples

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Inductive Definitions: Examples

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}{\text{cons}(2,\text{cons}(3,\text{empty}))\ \text{list}}\,\text{Cons}
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$$

Inductive Definitions: Examples

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Inductive Definitions: Examples

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\dfrac{\dfrac{\dfrac{\dfrac{}{0,\text{nat}}\,\text{Zero}}{1\ \text{nat}}\,\text{Succ}}{2\ \text{nat}}\,\text{Succ}}{3\ \text{nat}}\,\text{Succ}
\qquad
\text{empty list}
}{\text{cons}(3,\text{empty})\ \text{list}}\,\text{Cons}
}{\text{cons}(2,\text{cons}(3,\text{empty}))\ \text{list}}\,\text{Cons}
}{\text{cons}(1,\text{cons}(2,\text{cons}(3,\text{empty})))\ \text{list}}\,\text{Cons}
$$

Inductive Definitions: Examples

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}{\text{cons}(2,\text{cons}(3,\text{empty}))\ \text{list}}\,\text{Cons}
}{\text{cons}(1,\text{cons}(2,\text{cons}(3,\text{empty})))\ \text{list}}\,\text{Cons}
$$

Rule Induction

Suppose we want to prove a property P holds for judgment J.

Then for each rule defining J:

$$\frac{A_1 \quad \cdots \quad A_n}{C}\;\text{Name}$$

It is enough to assume $\mathsf{P}(A_1), \ldots, \mathsf{P}(A_n)$ hold and show $\mathsf{P}(C)$.

Rule Induction

$$\frac{}{0 = 0\ \mathsf{nat}}\ \mathsf{EqZero} \qquad \frac{a = b\ \mathsf{nat}}{\mathsf{succ}(a) = \mathsf{succ}(b)\ \mathsf{nat}}\ \mathsf{EqSucc}$$

1. $\mathsf{P}(0 = 0\ \mathsf{nat})$
2. **if** $a = b\ \mathsf{nat}$ **and** $\mathsf{P}(a = b\ \mathsf{nat})$, **then** $\mathsf{succ}(a) = \mathsf{succ}(b)\ \mathsf{nat}$ **and** $\mathsf{P}(\mathsf{succ}(a) = \mathsf{succ}(b)\ \mathsf{nat})$

Rule Induction

Lemma. If a nat, then $a = a$ nat

$$\frac{}{0 = 0 \text{ nat}}\ \text{EqZero} \qquad \frac{a = b \text{ nat}}{\text{succ}(a) = \text{succ}(b) \text{ nat}}\ \text{EqSucc}$$

Rule Induction

Lemma. If a nat, then $a = a$ nat

Proof. By rule induction on a nat.

Rule Zero: In this case, we know $a = 0$, and by rule EqZero we know $0 = 0$ nat.

Rule Succ: We know that $a = \text{succ}(b)$ and b nat by assumption. By the induction hypothesis we know $b = b$ nat, and by applying the EqSucc rule we know $\text{succ}(b) = \text{succ}(b)$ nat.

Rule Induction

Lemma. If $\mathsf{succ}(a) = \mathsf{succ}(b)$ nat, then $a = b$ nat

Rule Induction

Lemma. If $\mathsf{succ}(a) = \mathsf{succ}(b)\,\mathsf{nat}$, then $a = b\,\mathsf{nat}$

Proof. By rule induction on $\mathsf{succ}(a) = \mathsf{succ}(b)\,\mathsf{nat}$.

Rule EqZero: In this case we need $0 = \mathsf{succ}(a)$, but this is impossible.

Rule EqSucc: We know that $a = b\,\mathsf{nat}$ by the induction hypothesis.