Basic Set Theory
Mathematical Structures for CS (CSCI 3030)

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Read chapter 2.1 - 2.5

In computer science and mathematics we more often than not study collections of objects. For example, the collection of all real numbers or the collection of all computers on a network. In addition grouping objects together into collections we often need to relate one collection to another. In this lecture we mathematically define the notion of a collection and the notions need to relate collections to one another.

1 Mathematical Collections: Sets

In this section we give the formal definition of what we have been calling a "collection."

Definition 1. *A* ***set*** *is an unordered collection of objects called elements or members. Given elements* $a, b, c, d, e, f, \ldots$ *we denote the set containing these elements by* $\{a, b, c, d, e, f, \ldots\}$. *We say a set contains its elements, and when given a set,* S*, and an element,* x*, we write* $x \in S$ *if and only if the element* x *is a member of* S*, and we write* $x \notin S$ *otherwise. We often denote a set using a capital letter, and its elements using lowercase letters.*

There are many ways to define a set. The most basic way is to simply list all of its elements between curly braces. This is known as the roster method.

Example 2. *The following are several sets:*

- *The set of all integers is* $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$,
- *The set of all natural numbers is* $\mathbb{N} = \{0, 1, 2, 3, 4, 5, 6, \ldots\}$,
- *The set of boolean values is* $\mathbb{B} = \{True, False\}$,
- *A set of three colors is* $C = \{\mathit{Red}, \mathit{Green}, \mathit{Blue}\}$,
- *The set of propositional logical connectives is* $P_{op} = \{\land, \lor, \Rightarrow, \neg\}$, *and*
- *The set of four digit binary numbers is*

$$\mathsf{Bin}_4 = \{0000, 0001, 0010, 0011, 0100, 0101, 0110, 0111, 1000, 1001, 1010, 1011, 1100, 1101, 1110, 1111\}.$$

Definition 3. *There is a special set called the* ***empty set*** *denoted* $\emptyset$ *or* $\{\}$. *This set has no elements.*

A second, and more powerful way to define sets is by the **set builder** or **set comprehension** method. Consider an example. Suppose $\mathsf{isEven(x)}$ is a predicate that is true when its argument, x, is an even integer, then we can define the following set of even integers:

$$\mathsf{Even} = \{i \in \mathbb{Z} \mid \mathsf{isEven}(i)\}$$

This notation gives a pattern for the elements using variables, in the above this is i, and a predicate on i, in this case it is $i \in \mathbb{Z}$ and $\mathsf{isEven}(i)$, that when true for a particular i adds that integer to the set Even and leaves the integer out if not. Thus, the set

$$\mathsf{Even} = \{i \in \mathbb{Z} \mid \mathsf{isEven}(i)\}$$

can also be defined by

$$i \in \mathsf{Even} \text{ if and only if } i \in \mathbb{Z} \text{ and } \mathsf{isEven}(i).$$

Therefore, we know that $\mathsf{Even} = \{\ldots, -6, -4, -2, 0, 2, 4, 6, \ldots\}$ because all of its elements are integers and even. We have arrived at the following definition:

Definition 4. *A set defined by set builder notation is denoted as follows:*

$$S = \{x \in T \mid P(x)\}$$

where P is some predicate on the set T. Thus, $x \in S$ if and only if $x \in T$ and $P(x)$ holds.

Thus, set builder or set comprehensions combine the power of logic with the power of sets. The following are a few more examples:

- $\mathbb{N}_{<10} = \{n \in \mathbb{N} \mid n < 10\}$,
- $\mathbb{N}^{+} = \{n \in \mathbb{N} \mid n > 0\}$, and
- $\mathsf{Win}_7 = \{c \in \mathsf{GRUNet} \mid c \text{ runs Windows } 7\}$, where GruNet is the set of all computers on the GRU network.

Sets can be related to one another if they have elements in common.

Definition 5. *A set A is a* ***subset*** *of a set B if and only if $\forall x \in A.x \in B$. We denote this relationship by $A \subseteq B$. If $\exists y \in B.y \notin A$, then we say A is a* ***proper subset*** *of B and denote this by $A \subset B$.*

To prove that $A \subseteq B$ show if $x \in A$ then $x \in B$ for arbitrary x. To prove that $A \not\subseteq B$ show that there exists some element $x \in A$ such that $x \notin B$. The previous definition implies the following:

Lemma 6. *Suppose A is a set. Then $A \subseteq A$ and $A \not\subset A$.*

Proof. Suppose A is a set. Then we must show that $A \subseteq A$. By the definition of subset we must show that for any $x \in A$, $x \in A$, but this clearly follows.

We must now show that $A \not\subset A$, but this is equivalent to showing that $\neg((\forall x \in A.x \in A) \wedge (\exists y \in A.y \notin A))$. The latter is equivalent to $\neg(True \wedge \exists y.y \in A \wedge y \notin A)$ if and only if $\neg(\exists y.y \in A \wedge y \notin A)$ if and only if $\forall y.y \notin A \vee y \in A$ which is equivalent to LEM, and thus holds. □

Lemma 7. *Suppose A is a set. Then $\emptyset \subseteq A$.*

Proof. Suppose A is a set. Then we must show that $\emptyset \subseteq A$. Thus, we must show that for any $x \in \emptyset$, $x \in A$, but no such x exists, and thus, our result follows. □

Using the subset operation we can define when two sets are equivalent.

Definition 8. *We say two sets A and B are* ***equivalent*** *if and only if $A \subseteq B$ and $B \subseteq A$.*

The previous definition implies that two sets are equivalent if and only if they have exactly the same members. Thus, two sets A and B are equivalent if and only if $\forall x.(x \in A \Leftrightarrow x \in B)$ holds. Note that this says nothing about the order of the elements. Thus, the two sets $\{1,2,3,4\}$ and $\{4,2,3,1\}$ are equivalent, because they have the same elements. Furthermore, it does not matter if elements appear more than once, that is, the sets $\{1\}$ and $\{1,1,1,1,1,1\}$ are equivalent, and so are, $\{2,3,4,3,6\}$ and $\{2,3,4,6\}$. Therefore, sets cannot contain multiple elements, because equality ignores them, and so it is as if the copies are not even there.

Sets have a size called their cardinality.

Definition 9. *Suppose A is a set. If A has $n \in \mathbb{N}$ distinct elements, then we say A is finite, and its **cardinality** is n. We denote the cardinality of a set A by $|A|$.*

Some examples of cardinality are $|\mathbb{N}_{<10}| = 10$ and $|\{1,1,2,3,5,3,6\}| = 5$. However, what happens when the number of elements of a set is infinite? Then we say that the cardinality of the set is infinite.

Definition 10. *The cardinality of a set that is not finite, is called infinite.*

There is nothing stopping a set from having other sets as members. For example,

- $\{\emptyset, \{'a','b'\}, \{'c','d'\}\}$, and
- $\{\{\emptyset\}\}$.

The elements of a set are left abstract and can be anything at all. The following operation is ubiquitous in computer science.

Definition 11. *Given a set A, the **power set** of A, denoted $\mathcal{P}(A)$, is the set of all subsets of A.*

Suppose $A = \{1,2,3\}$, then $\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$. Later we will prove that if $|A| = n$, then $|\mathcal{P}(A)| = 2^n$. Exercises:

- What is $\mathcal{P}(\{\emptyset\})$? The set $\{\emptyset, \{\emptyset\}\}$.
- What is $\mathcal{P}(\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\})$? The set $\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\{\{\emptyset\}\}\}, \{\emptyset, \{\emptyset\}\}, \{\{\emptyset\}, \{\{\emptyset\}\}\}, \{\emptyset, \{\{\emptyset\}\}\}, \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\}\}$.

1.1 Operations on Sets

There are a number of set operations that allow us to construct other sets in terms of given sets.

Definition 12. *Suppose A and B are sets. Then the **union** of A and B is defined by $A \cup B = \{x \mid x \in A \vee x \in B\}$.*

Let $A = \{1,2,3\}$ and $B = \{4,5,6\}$, then $A \cup B = \{1,2,3,4,5,6\}$. Suppose $A = \{1,2,3\}$ and $B = \{1,5,3\}$, then $A \cup B = \{1,2,3,5\}$.

Note that union is defined in terms of disjunction, and this is important point to remember, because a lot of the same properties of disjunction extends to sets because of this.

Definition 13. *Suppose A and B are sets. Then the **intersection** of A and B is defined by $A \cap B = \{x \mid x \in A \wedge x \in B\}$.*

Let $A = \{1,2,3,4\}$ and $B = \{3,5,1,7\}$. Then $A \cap B = \{1,3\}$. The set $\mathbb{N} \cap \{x \in \mathbb{Z} \mid x^2 < 100\} = \{x \in \mathbb{Z} \mid 0 \leq x \wedge x^2 < 100\}$.

Note that intersection is defined in terms of conjunction, and this is important point to remember, because a lot of the same properties of conjunction extends to sets because of this.

Definition 14. *Suppose A and B are sets. Then the* ***difference*** *of A and B is defined by $A - B = \{x \mid x \in A \wedge x \notin B\}$. Note that the difference operation is sometimes denoted by $A \backslash B$. The difference of A and B is sometimes called the complement of B with respect to A.*

If we fix a set U where all of our sets draw their elements from, then we can define the complement of a set. The set U is called the universal set or the universe of our sets. Thus, if U has been chosen, and we have a set A, then we know $A \subseteq U$.

Definition 15. *Let U be the universal set. Then the complement of a set A is defined by $\overline{A} = U - A$. Thus, the complement of a set A is the set of all elements that are not in A.*

If there is no universal set, then one can only speak about complements with respect to other sets which is the same as the set difference.

1.2 Set Theoretic Equivalences

Just as we saw in propositional logic there are a number of equivalences one can prove about sets. Note that all equivalences given in this section are with respect to equality of sets given above (Definition 8). The following table lists a large number of set theoretic equivalences:

$A \cap U = A$ $A \cup \emptyset = A$	Identities
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associativity laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributivity laws
$\overline{A \cup B} = \overline{A} \cap \overline{B}$ $\overline{A \cap B} = \overline{A} \cup \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \emptyset$	Complement laws

We now prove a few of these. There are other example proofs in the book on page 130.

Lemma 16 (Complementation Law)**.** *Show that $\overline{(\overline{A})} = A$.*

Proof. Suppose $x \in \overline{(\overline{A})}$. Then we must show that $x \in A$. We know by definition that if $x \in \overline{(\overline{A})}$, then $x \notin \overline{A}$. The latter is equivalent to $\neg(x \in \overline{A})$ which is equivalent to $\neg(x \notin A)$, but this is also equivalent to $\neg(\neg(x \in A))$. Thus, by the law of double negation we know that $\neg(\neg(x \in A))$ is equivalent to $x \in A$.

Now suppose $x \in A$. Then we must show that $x \in \overline{(\overline{A})}$. We know by the law of double negation that $x \in A$ if and only if $\neg(\neg(x \in A))$. Now the latter yields:

$$\begin{array}{rll} \neg(\neg(x \in A)) & \text{iff} & \neg(x \notin A) \\ & \text{iff} & \neg(x \in \overline{A}) \\ & \text{iff} & x \notin \overline{A} \\ & \text{iff} & x \in \overline{(\overline{A})} \end{array}$$

Therefore, $\overline{(\overline{A})} = A$. ☐

1.3 Generalized Union and Intersection

Suppose we have a series of sets $A_1, \ldots, A_n$ for some $n \in \mathbb{N}$. How could we union together all of these sets? For small n,say $n = 3$, we could just write down $A_1 \cup A_2 \cup A_3$, but what if n is very large? There happens to be a special notation:

Definition 17. *Suppose I is a set, and for each $i \in I$ we have a set A_i. Then the **generalized union** of all the sets A_i is defined as follows:*

$$\textstyle\bigcup_{i \in I} A_i = \{x \mid \exists i \in I. x \in A_i\}$$

Note that this definition works for even an infinite series of sets. When we have a series of sets A_i for each $i \in I$ we say that sets A_i are **indexed** over the set I. We call I the **index set**.

We can do the same for intersection.

Definition 18. *Suppose I is a set, and for each $i \in I$ we have a set A_i. Then the **generalized intersection** of all the sets A_i is defined as follows:*

$$\textstyle\bigcap_{i \in I} A_i = \{x \mid \forall i \in I. x \in A_i\}$$

Suppose $A_i = \{i, i+1, \cdots\}$ for some $i \in \mathbb{N}$. Then consider the following:

i. $\bigcup_{i \in \mathbb{N}} A_i = \mathbb{N}$

ii. $\bigcap_{i \in \mathbb{N}} A_i = \emptyset$

We prove the following lemmata:

Lemma 19. *Suppose $I = \{1, 2, \ldots, n\}$ is a finite indexed set where $n \in \mathbb{N}$ and A_i is a series of sets for each $i \in I$. Then*

$$\bigcup_{i \in I} A_i = A_1 \cup \cdots \cup A_n$$

Proof. Suppose $I = \{1, 2, \ldots, n\}$ is a finite indexed set where $n \in \mathbb{N}$ and A_i is a series of sets for each $i \in I$. Then we know the following by the definition of union:

$$A_1 \cup \cdots \cup A_n = \{x \mid x \in A_1 \vee \cdots \vee x \in A_n\}$$

If the proposition $x \in A_1 \vee \cdots \vee x \in A_n$ is true, then there must exist at least one set A_i for some $i \in I$ such that $x \in A_i$. Thus, the proposition $x \in A_1 \vee \cdots \vee x \in A_n$ is equivalent to the proposition $\exists i \in I. x \in A_i$. Thus, we know

$$\begin{array}{rcl} A_1 \cup \cdots \cup A_n & = & \{x \mid x \in A_1 \vee \cdots \vee x \in A_n\} \\ & = & \{x \mid \exists i \in I. x \in A_i\} \\ & = & \bigcup_{i \in I} A_i. \end{array}$$

Therefore, $\bigcup_{i \in I} A_i = A_1 \cup \cdots \cup A_n$. ☐

Lemma 20. *Suppose $I = \{1, 2, \ldots, n\}$ is a finite indexed set where $n \in \mathbb{N}$ and A_i is a series of sets for each $i \in I$. Then*

$$\bigcap_{i \in I} A_i = A_1 \cap \cdots \cap A_n$$

Proof. Suppose $I = \{1, 2, \ldots, n\}$ is a finite indexed set where $n \in \mathbb{N}$ and A_i is a series of sets for each $i \in I$. Then we know the following by the definition of union:

$$A_1 \cap \cdots \cap A_n = \{x \mid x \in A_1 \wedge \cdots \wedge x \in A_n\}$$

If the proposition $x \in A_1 \wedge \cdots \wedge x \in A_n$ is true, then there for every set $i \in I$, $x \in A_i$. Thus, the proposition $x \in A_1 \wedge \cdots \wedge x \in A_n$ is equivalent to the proposition $\forall i \in I.x \in A_i$. Thus, we know

$$\begin{array}{lll} A_1 \cap \cdots \cap A_n & = & \{x \mid x \in A_1 \wedge \cdots \wedge x \in A_n\} \\ & = & \{x \mid \forall i \in I.x \in A_i\} \\ & = & \bigcap_{i \in I} A_i. \end{array}$$

Therefore, $\bigcap_{i \in I} A_i = A_1 \cap \cdots \cap A_n$. □

2 Tuples and Cartesian Product

In this section we define the notion of a tuple, and a new set theoretic operation called cartesian product which can be used to construct sets of tuples from other sets. Consider the following version of a truth table:

	1	2	3	4
p	*False*	*False*	*True*	*True*
q	*False*	*True*	*False*	*True*
$p \wedge q$	*False*	*False*	*False*	*True*

This truth table has each row labeled by a formula, instead of the columns, and then the columns consist of a particular value assignment to the variables. For example, column one represents assigning the value *False* to p, and *False* to q, and thus, $p \wedge q$ is assigned *False*. How would we model these types of truth tables using set theory?

We need a way to relate the rows to the columns and then those to the particular assignment. For example, we can see by the table that at row p and column 3 the assignment is *True*. Sets relate objects with a particular property and so one might first try cleverly constructing a set of sets to model the table where we model each coordinate in the table by a set. For example, the coordinate mentioned above would be modeled by the set $\{p, 3, True\}$, which means, choose row p, and then column 3, and the value at the coordinate is *True*. Another example might be $\{q, 2, False\}$.

Is this a good model? The answer is no, and the reason is that given a coordinate at a set we really do not have a way of determining which of the elements are the row, the column, and the value, because sets are unordered. That is, the coordinate $\{p, 3, True\} = \{3, p, False\}$. So we need a way to enforce an order on the set, and we can do this using a device called a tuple.

Definition 21. *A n-**tuple**, or sometimes just a tuple, is an ordered sequence of objects denoted $(a_1, \ldots, a_n)$. We call 2-tuples **pairs** or **coordinates**.*

The definition of n-tuple enforces an ordering on the objects. The object a_1 is considered first, and always first, and the object a_n is last. Thus, the pair $(1, 2) \neq (2, 1)$ which is not true about the sets $\{1, 2\} = \{2, 1\}$. Furthermore, sequences can have repeated objects, for example, $(1, 2, 3, 3) \neq (1, 2, 3)$.

Definition 22. *Suppose $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ are two n-tuples. Then we say they are equivalent if and only if $a_1 = b_1$, $a_2 = b_2$, $\ldots$, $a_n = b_n$. We denote this by $(a_1, \ldots, a_n) = (b_1, \ldots, b_n)$. This is known as* ***point wise*** *equivalence of tuples.*

Getting back to our example we now have a way to enforce a relationship between objects with an ordering. So we can model the coordinates of our example truth table from above using tuples. For example, $(p, 3, True)$ is the location in the table at row p, column 3, which has a value of $True$. Then we simply collect all of these coordinates into a tuple. The following tuple models the table given above:

$$\begin{array}{l} ((p,1,False),(p,2,False),(p,3,True),(p,4,True),\\ (q,1,False),(q,2,True),(q,3,False),(q,4,True),\\ (p \land q,1,False),(p \land q,2,False),(p \land q,3,False),\\ (p \land q,4,True)) \end{array}$$

Therefore, tuples are used to enforce a relationship between a sequence of objects that rely on a particular order. In fact, we can think of the above set as a primitive database.

Given two sets A and B it is possible to define all the possible pairs of objects from A and B.

Definition 23. *Suppose A and B are two sets. Then the* ***cartesian product*** *of A and B is defined as follows:*

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$$

We now give several examples.
Example 24.

i. $\{0, 1\} \times \{a, b\} = \{(0, a), (0, b), (1, a), (1, b)\}$

ii. $\emptyset \times \{1\} = \emptyset$

iii. $\{1\} \times \emptyset = \emptyset$

iv. $(\{1, 0\} \times \{a, b\}) \times \{c\} = \{(0, a), (0, b), (1, a), (1, b)\} \times \{c\} = \{((0, a), c), ((0, b), c), ((1, a), c), ((1, b), c)\}$