

Inductive Definitions

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A set of rules that are used to derive facts about judgments.

Inductive Definitions: Judgments

A judgment is an assertion about a syntactic object.

It states that one or more syntactic objects have a property or is related to another.

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Examples:

$n \text{ nat}$

n is a natural number

$n = n_1 + n_2$

n is the sum of n_1 and n_2

$t : \text{type}$

t is a type

$e : t$

the expression e has type t

Inductive Definitions: Judgments

We denote an arbitrary judgment as J .

Inductive Definitions: Inference Rules

Judgements are defined by an inductive definition which consists of a set of inference rules.

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$$\frac{}{J} \text{ Name}$$

$$\frac{J_1 \quad \dots \quad J_n}{J} \text{ Name}$$

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$$\text{Premises} \longrightarrow \frac{J_1 \quad \cdots \quad J_n}{J} \text{ Name}$$

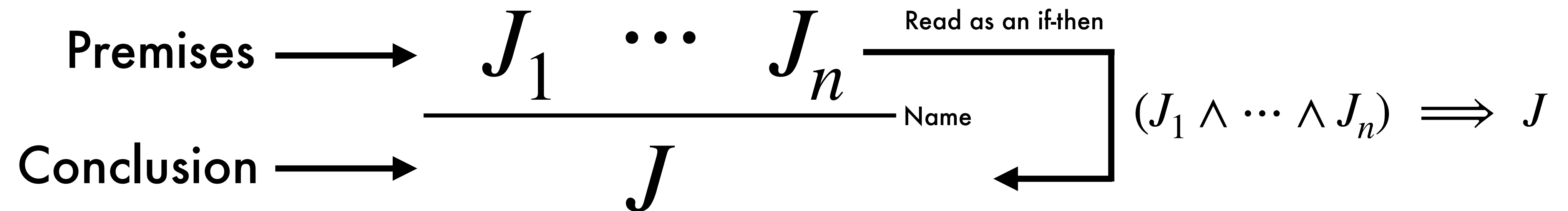
Inductive Definitions: Inference Rules

Judgements are defined by an inductive definition which consists of a set of inference rules.

$$\begin{array}{c} \text{Premises} \longrightarrow J_1 \quad \cdots \quad J_n \\ \hline \text{Conclusion} \longrightarrow J \end{array} \text{Name}$$

Inductive Definitions: Inference Rules

Judgements are defined by an inductive definition which consists of a set of inference rules.



Inductive Definitions: Examples

$$\begin{array}{c} \frac{}{\text{true bool}} \text{ True} \qquad \frac{}{\text{false bool}} \text{ False} \\[2em] \frac{b_1 \text{ bool} \quad b_2 \text{ bool} \quad b_3 \text{ bool}}{\text{if}(b_1, b_2, b_3) \text{ bool}} \text{ If} \end{array}$$

Inductive Definitions: Examples

$$\frac{}{\text{true} \rightsquigarrow \text{true}} \text{True}$$

$$\frac{}{\text{false} \rightsquigarrow \text{false}} \text{False}$$

$$\frac{b_1 \rightsquigarrow \text{true}}{\text{if}(b_1, b_2, b_3) \rightsquigarrow b_2} \text{IfTrue}$$

$$\frac{b_1 \rightsquigarrow \text{false}}{\text{if}(b_1, b_2, b_3) \rightsquigarrow b_3} \text{IfFalse}$$

$$\frac{b_1 \rightsquigarrow b'_1}{\text{if}(b_1, b_2, b_3) \rightsquigarrow \text{if}(b'_1, b_2, b_3)} \text{If}$$

Inductive Definitions: Examples

$$\frac{}{0 \text{ nat}} \text{ Zero}$$

$$\frac{n \text{ nat}}{\text{succ}(n) \text{ nat}} \text{ Succ}$$

$$\frac{}{\text{empty list}} \text{ Empty}$$

$$\frac{n \text{ nat} \quad l \text{ list}}{\text{cons}(n, l) \text{ list}} \text{ Cons}$$

Inductive Definitions: Derivations

To prove that an inductively defined judgment, J , holds it is enough to exhibit a derivation of it.

Inductive Definitions: Derivations

Derivations are goal directed proofs written by stacking inference rules to form a derivation tree.

$$\frac{\Delta_1 \quad \dots \quad \Delta_n}{J} \text{Name}$$

Inductive Definitions: Examples

`cons(1,cons(2,cons(3,empty))) list`

Inductive Definitions: Examples

1 nat

cons(2,cons(3,empty)) list

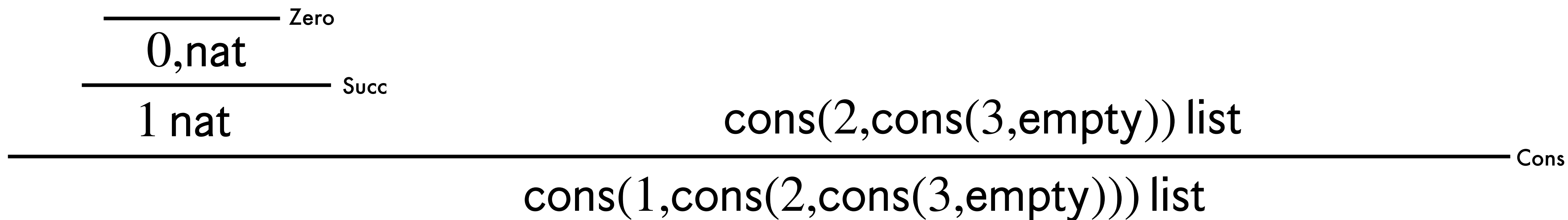
Cons

cons(1,cons(2,cons(3,empty))) list

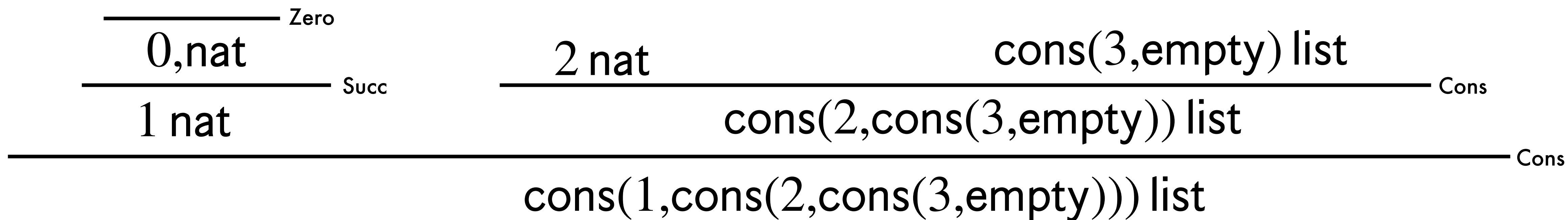
Inductive Definitions: Examples

$$\frac{\frac{0, \text{nat}}{1 \text{ nat}} \text{ Succ} \quad \text{cons}(2, \text{cons}(3, \text{empty})) \text{ list}}{\text{cons}(1, \text{cons}(2, \text{cons}(3, \text{empty}))) \text{ list}} \text{ Cons}$$

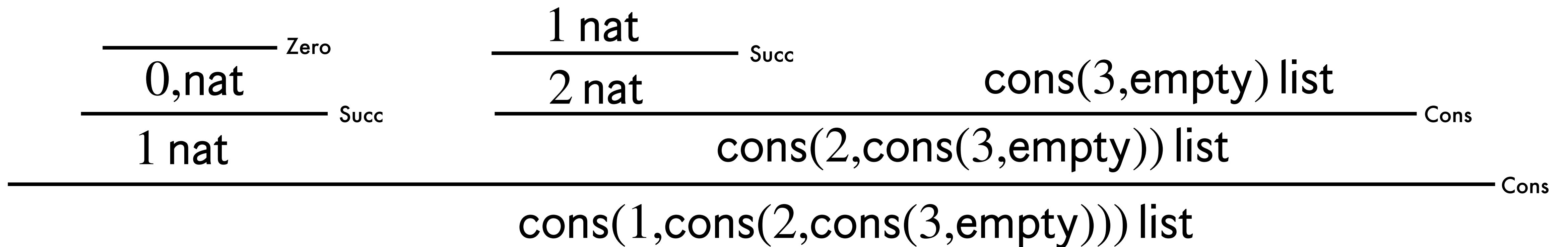
Inductive Definitions: Examples



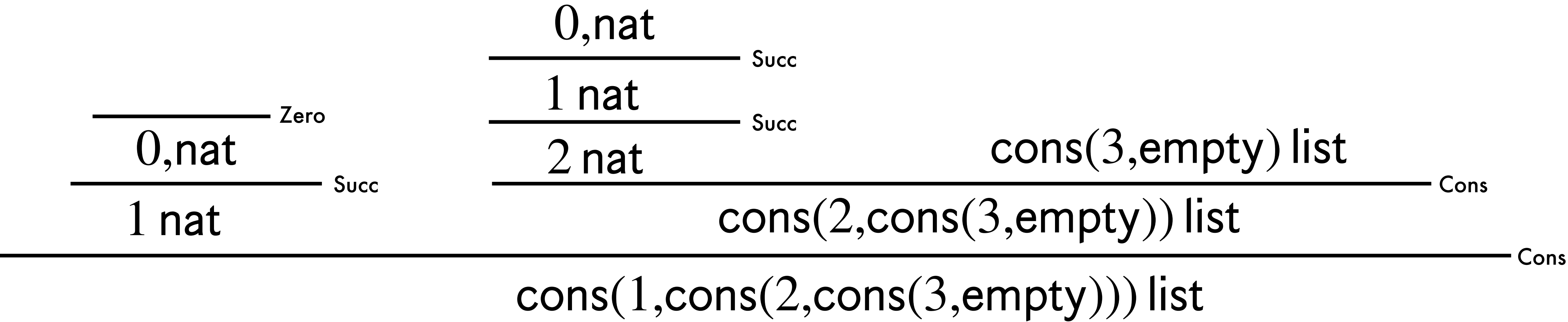
Inductive Definitions: Examples



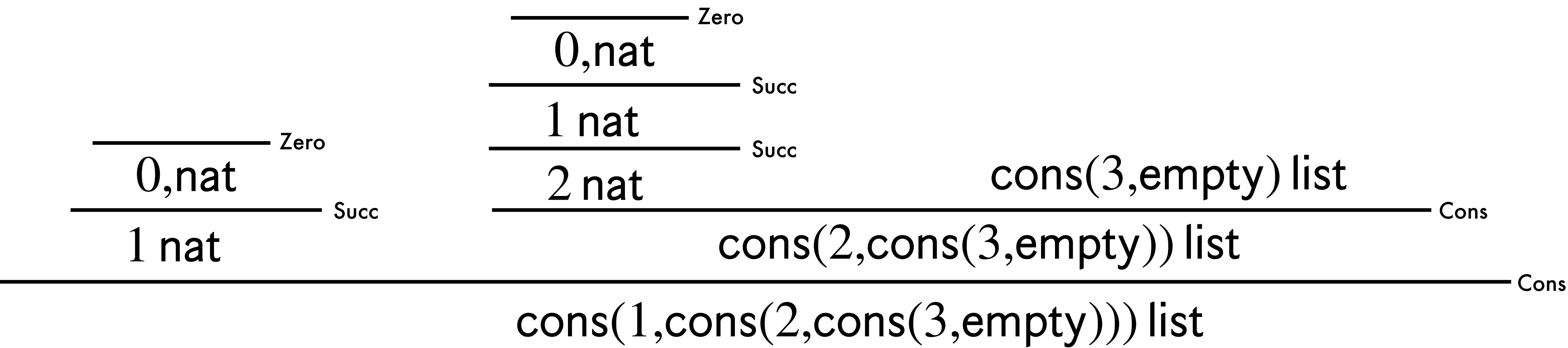
Inductive Definitions: Examples



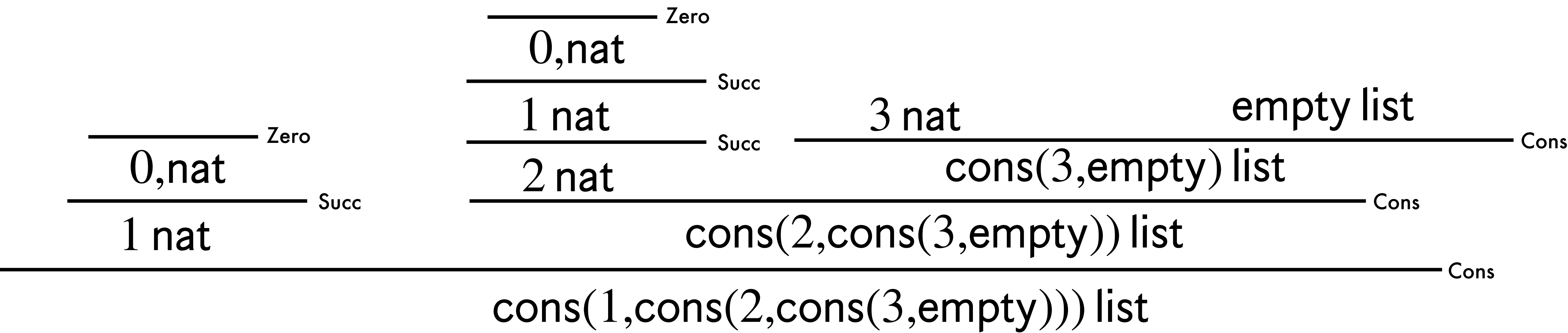
Inductive Definitions: Examples



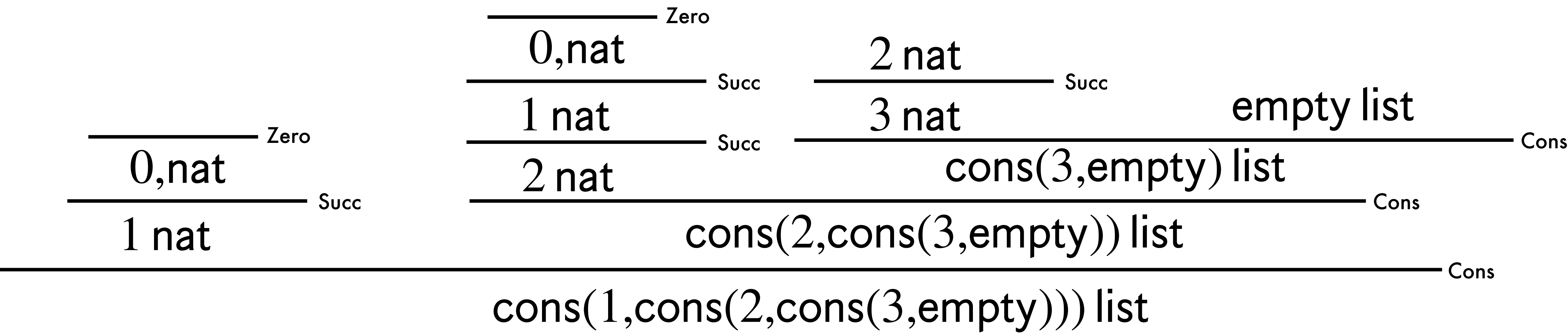
Inductive Definitions: Examples



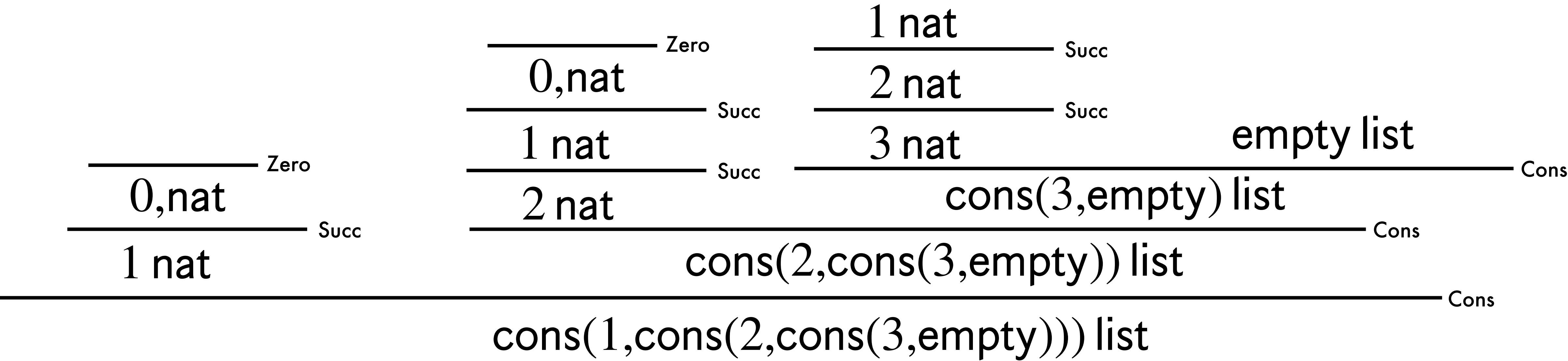
Inductive Definitions: Examples



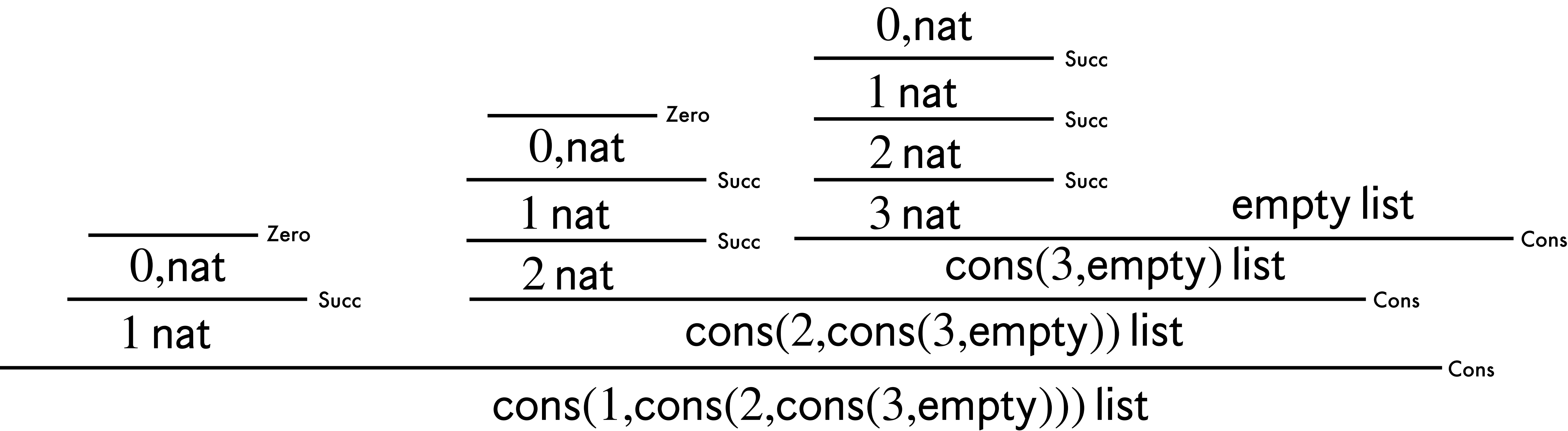
Inductive Definitions: Examples



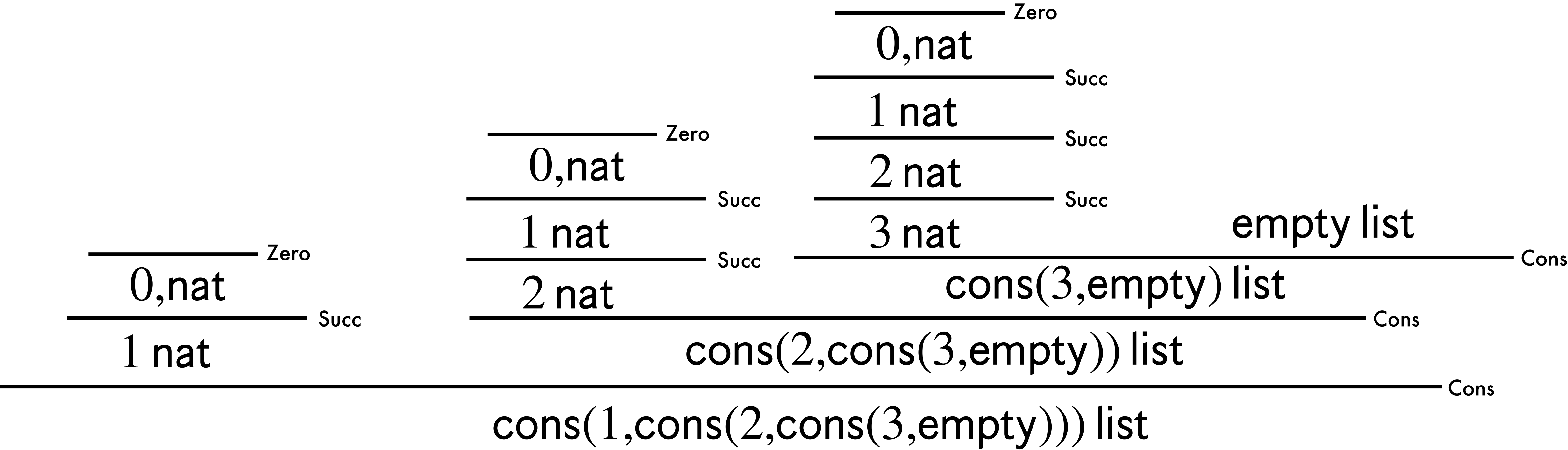
Inductive Definitions: Examples



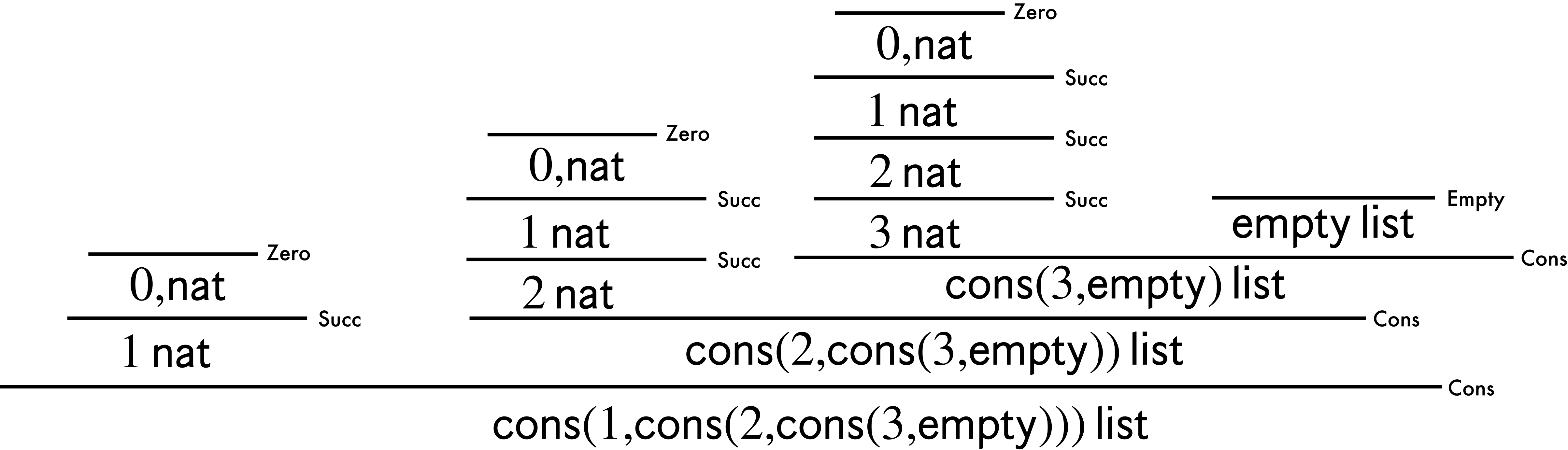
Inductive Definitions: Examples



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Rule Induction

Suppose we want to prove a property P holds for judgment J .

Then for each rule defining J :

$$\frac{A_1 \quad \dots \quad A_n}{C} \text{Name}$$

It is enough to assume $P(A_1), \dots, P(A_n)$ hold and show $P(C)$.

Rule Induction

$$\frac{}{0 = 0 \text{ nat}} \text{EqZero}$$

$$\frac{a = b \text{ nat}}{\text{succ}(a) = \text{succ}(b) \text{ nat}} \text{EqSucc}$$

1. $P(0 = 0 \text{ nat})$
2. if $a = b \text{ nat}$ and $P(a = b \text{ nat})$, then $\text{succ}(a) = \text{succ}(b) \text{ nat}$ and $P(\text{succ}(a) = \text{succ}(b) \text{ nat})$

Rule Induction

Lemma. If $a \text{ nat}$, then $a = a \text{ nat}$

$$\frac{}{0 = 0 \text{ nat}} \text{EqZero}$$

$$\frac{a = b \text{ nat}}{\text{succ}(a) = \text{succ}(b) \text{ nat}} \text{EqSucc}$$

Rule Induction

Lemma. If $a \text{ nat}$, then $a = a \text{ nat}$

Proof. By rule induction on $a \text{ nat}$.

Rule Zero: In this case, we know $a = 0$, and by rule EqZero we know $0 = 0 \text{ nat}$.

Rule Succ: We know that $a = \text{succ}(b)$ and $b \text{ nat}$ by assumption. By the induction hypothesis we know $b = b \text{ nat}$, and by applying the EqSucc rule we know $\text{succ}(b) = \text{succ}(b) \text{ nat}$.

Rule Induction

Lemma. If $\text{succ}(a) = \text{succ}(b) \text{ nat}$, then $a = b \text{ nat}$

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Proof. By rule induction on $\text{succ}(a) = \text{succ}(b) \text{ nat}$.

Rule EqZero: In this case we need $0 = \text{succ}(a)$, but this is impossible.

Rule EqSucc: We know that $a = b \text{ nat}$ by the induction hypothesis.