Inductive Definitions

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A set of rules that are used to derive facts about judgments.

Inductive Definitions: Judgments

A judgment is an assertion about a syntactic object.

It states that one or more syntactic objects have a property or is related to another.

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Examples:

n nat	n is a natural number
$n = n_1 + n_2$	n is the sum of n_1 and n_2
t : type	t is a type
$e : t$	the expression e has type t

Inductive Definitions: Judgments

We denote an arbitrary judgment as J.

Inductive Definitions: Inference Rules

Judgements are defined by an inductive definition which consists of a set of <u>inference rules.</u>

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$$\dfrac{}{J}\ \text{Name} \qquad\qquad \dfrac{J_1 \quad \dots \quad J_n}{J}\ \text{Name}$$

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$$\frac{J_1 \quad \cdots \quad J_n}{J}\,\text{Name}$$

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Premises ⟶

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Premises $\longrightarrow$ Conclusion $\longrightarrow$

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Premises

Conclusion

$$\frac{J_1 \quad \cdots \quad J_n}{J}\;\text{Name}$$

Read as an if-then

$$(J_1 \wedge \cdots \wedge J_n) \implies J$$

Inductive Definitions: Examples

$$\dfrac{}{\text{true bool}}\ \text{True}$$

$$\dfrac{}{\text{false bool}}\ \text{False}$$

$$\dfrac{b_1 \text{ bool} \qquad \begin{matrix} b_2 \text{ bool} \\ b_3 \text{ bool} \end{matrix}}{\text{if}(b_1, b_2, b_3) \text{ bool}}\ \text{If}$$

Inductive Definitions: Examples

$$\frac{}{\mathsf{true} \rightsquigarrow \mathsf{true}}\;\textsf{True} \qquad \frac{}{\mathsf{false} \rightsquigarrow \mathsf{false}}\;\textsf{False}$$

$$\frac{b_1 \rightsquigarrow \mathsf{true}}{\mathsf{if}(b_1, b_2, b_3) \rightsquigarrow b_2}\;\textsf{IfTrue} \qquad \frac{b_1 \rightsquigarrow \mathsf{false}}{\mathsf{if}(b_1, b_2, b_3) \rightsquigarrow b_3}\;\textsf{IfFalse}$$

$$\frac{b_1 \rightsquigarrow b_1'}{\mathsf{if}(b_1, b_2, b_3) \rightsquigarrow \mathsf{if}(b_1', b_2, b_3)}\;\textsf{If}$$

Inductive Definitions: Examples

$$\frac{}{0 \ \mathsf{nat}}\ \text{Zero} \qquad \frac{n \ \mathsf{nat}}{\mathsf{succ}(n) \ \mathsf{nat}}\ \text{Succ}$$

$$\frac{}{\mathsf{empty} \ \mathsf{list}}\ \text{Empty} \qquad \frac{n \ \mathsf{nat} \qquad l \ \mathsf{list}}{\mathsf{cons}(n, l) \ \mathsf{list}}\ \text{Cons}$$

Inductive Definitions: Derivations

To prove that an inductively defined judgment, J, holds it is enough to exhibit a derivation of it.

Inductive Definitions: Derivations

Derivations are goal directed proofs written by stacking inference rules to form a derivation tree.

$$\frac{\Delta_1 \quad \cdots \quad \Delta_n}{J}\,\text{Name}$$

Inductive Definitions: Examples

$$\dfrac{\dfrac{\dfrac{}{\text{true}}\,\text{True} \qquad \dfrac{}{\text{false}}\,\text{False} \qquad \dfrac{}{\text{true}}\,\text{True}}{\text{if(true, false, true)}}\,\text{If} \qquad \dfrac{}{\text{true}}\,\text{True} \qquad \dfrac{}{\text{false}}\,\text{False}}{\text{if(if(true, false, true), true, false)}}\,\text{If}$$