Sequences, Summations and Mathematical Induction
Mathematical Structures for CS (CSCI 3030)

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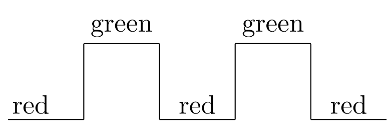
Read chapter 2.4 and 5.1

Sequences are ubiquitous in computer science. As an example consider modern day subway trains. Signals are placed a particular spots along the track to guide train conductors. If a signal is green then the train is free to pass, but if the signal is red then on coming traffic is coming and the train must halt. Suppose we need to model such a signaling network and verify that it always works, then how do model the signal? Let us assume that the signal is always on, and hence, electricity is always flowing. Furthermore, we assume that the signal is default red, and then when the electricity increases it switches to green. Thus, we can model the operational behavior by a constant flow of electricity. When the electricity is low then the signal will be red, and when the electricity is high, then the signal will be green. This situation is depicted by the following diagram:



We can model this situation by using what is called a sequence. A sequence can be thought of as an infinite list of objects written as follows:

$$a_1, a_2, \ldots, a_n, \ldots$$

The signal can now be modeled by a sequence of zeros and ones. When the signal is low the sequence is all zeros, and when the signal is high then the sequence is all ones:

$$\underbrace{0,0,0,0,0,0}_{red}, \underbrace{1,1,1,1,1,1,1}_{green}, \underbrace{0,0,0,0,0,0}_{red}, \underbrace{1,1,1}_{green}, \underbrace{0,0,0,0}_{red}, \ldots$$

Now we give the formal definition of a sequence.

Definition 1. *A **sequence** is a potentially infinite list of objects from some domain – domains are usually the natural numbers or the real numbers, but it could be anything.*

This sequence defines what is called the Fibonacci sequence and it amounts to the following:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots$$

The Fibonacci sequence has many applications and has even been found to arise in nature – see `http://en.wikipedia.org/wiki/Fibonacci_number`. Sequences such as the one above are sometimes written as follows:

$$\begin{array}{lcl} F_0 & = & 1 \\ F_1 & = & 1 \\ F_n & = & F_{n-1} + F_{n-2} \end{array}$$

We can think of this style of specifying a sequence as a pattern where F_{n-1} means first find the $(n-1)$th object of the sequence, and then find the $(n-2)$th object of the sequence, and finally, add those two objects together, and this computes the nth object of the sequence.

Consider a second example. Suppose we have the sequence defined by $b_n = (-1)^n$. This sequence looks like the following:

$$1, -1, 1, -1, 1, -1, 1, -1, 1, -1, \ldots$$

Given a prefix of an sequence sometimes one must find the general pattern. This is a very common logic problem. However, it can be exceedingly difficult. Suppose one is given the following prefix:

$$\begin{aligned} &F_0 = 2 \\ &F_1 = 2 \\ &F_2 = 5 \\ &F_3 = 19 \\ &F_4 = 85 \end{aligned}$$

Find F_5. There could be an infinite number of sequences with this prefix and so finding the general formula is very difficult. It turns out that the general formula is the following:

$$\begin{aligned} &F_0 = 2 \\ &F_n = nF_{n-1} + (n-1)^2 \end{aligned}$$

The rest of the sequence is then:

$$2, 2, 5, 19, 85, 441, 2671, 18733, 149913, \ldots$$

There two very common sequences that we define next.

Definition 2. *Given real numbers a and r, called the initial term and common ratio respectively, we define a* ***geometric progression*** *as follows:*

$$a, ar, ar^2, \ldots, ar^n, \ldots$$

As an example the sequence $b_n = (-1)^n$ is a geometric progression where $a = 1$ and $r = -1$. Another example is the sequence $b_n = 6 * (1/3)^n$ where $a = 6$ and $r = 1/3$.

Example 3. *Suppose you put $\$5,000$ in a bank account with 0.15% interest compound annually. Then how much do you have after 30 years?*

We can use a sequence to figure this out. First, we set $P_0 = \$5,000$, because we start with our first deposit with no interest. Then the nth element of the sequence can be computed by taking the amount in the account from the previous year, the $(n-1)$th element, and adding in the interest. Thus, $P_1 = P_0 + (0.15\%)P_0 = (1.0015)P_0$, and we have

$$\begin{aligned} P_2 &= P_1 + (0.0015)P_1 \\ &= (1.0015)P_0 + (0.0015)((1.0015)P_0) \\ &= (1.0015)P_0 + (0.0015 * 1.0015)P_0 \\ &= (1 + 0.0015)(1.0015)P_0 \\ &= (1.0015)(1.0015)P_0 \\ &= (1.0015)(1.0015)P_0 \\ &= (1.0015)^2 P_0 \end{aligned}$$

We can now see that $P_n = (1.0015)^n P_0$. Finally, $P_{30} = (1.0015)^{30} P_0 = (1.0015)^{30}(5000) = \$5,229.96$.

Definition 4. *An* ***arithmetic progression*** *is a sequence of the form*

$$a, a + d, a + 2d, \ldots, a + nd, \ldots$$

An example is the sequence $F_n = 42 + 7n$, and the sequence $F_n = 72 - 9n$.

0.1 Summations

It is very common to take the sum of a sequence. Suppose

$$a_m, \ldots, a_n, \ldots$$

is a sequence. Then its sum is denoted $\sum_{j=m}^{n} a_j = a_m + \cdots + a_n$. The variable j is called the **index** of the sum and the variable m is called the **upper bound**. The sum is said to start at the **lower limit** of the sequence, here m, and progress to the **upper limit** of the sequence, here n.

All of the usual laws of addition hold for summations and they can be given direct proofs. For example, the distributive law for addition holds: $\sum_{j=0}^{n}(ax_j + by_j) = a\sum_{j=0}^{n} x_j + b\sum_{j=0}^{n} y_j$. We will prove these below, but we first must introduce a new proof technique called mathematical induction.

Many summations are equivalent to what are called **closed formulas** which are formulas for the sum of the sequence without any summation symbols. Consider the summation $\sum_{j=1}^{n} j = 1 + 2 + 3 + \ldots + n$ what closed formula does this summation equal? First, consider the arbitrary sequence:

$$a_1, a_2, a_3, a_4, \ldots, a_{n-3}, a_{n-2}, a_{n-1}, a_n$$

there are exactly n elements in this sequence. How many elements are in the following sequence:

$$(a_1, a_n), (a_2, a_{n-1}), (a_3, a_{n-2}), (a_4, a_{n-3}), \ldots$$

There are exactly half of the number of elements in this sequence than in the original sequence. Thus, there are exactly $n/2$ elements in this sequence. Now using this intuition we can simplify the following equation:

$$\begin{array}{l}
\sum_{j=1}^{n} j \\
= 1 + 2 + 3 + 4 + \cdots + (n-3) + (n-2) + (n-1) + n \\
= (1+n) + (2 + (n-1)) + (3 + (n-2)) + (4 + (n-3)) + \ldots \\
= (n+1) + (n+1) + (n+1) + (n+1) + \ldots \\
= (n/2)(n+1)
\end{array}$$

Therefore, $\sum_{j=1}^{n} = \frac{n(n+1)}{2}$. There are lots of other common equations like these see Table 2 on page 166 of the book.

1 Mathematical Induction

The proof that $\sum_{j=1}^{n} j = \frac{n(n+1)}{2}$ was only a sketch and we would like it to be more rigorous. To make it more rigorous we need a new proof technique that is very powerful called mathematical induction. This proof technique allows one to prove formulas of the form $\forall n.P(n)$, for example, $n! \leq n^n$ for every natural number n.

Proofs using mathematical induction have two parts: i. the base case where we prove the formula holds for 1, and ii. the step case where we assume the formula holds for the case of n and prove that the formula holds for $n+1$ using the fact that we have assumed that it holds for n. Suppose we wanted to prove that $\forall n.P(n)$ holds by mathematical induction. Then we would first prove that $P(1)$ holds, we prove that $\forall k.P(k) \Rightarrow P(k+1)$ holds, and if both of these hold, then $\forall n.P(n)$ holds by mathematical induction. In fact, we can formulate an inference rule for mathematical induction as follows:

$$\frac{P(1) \qquad \forall k.(P(k) \Rightarrow P(k+1))}{\forall n.P(n)}\ \text{IN}$$

At this point we give an intuitive way of understanding mathematical induction. Suppose we have an infinite ladder and we want to prove that we can reach any step on the ladder where we only know these two facts:

1. We can reach the first step of the ladder.
2. If we can reach a particular step of the ladder, then we can reach the next step.

Can we now conclude that we can reach every step? Can we reach the first step? Yes, because by the first fact we know we can reach the first step. How about the second step? Well, since we know we can reach step one, we can use fact two to conclude that we can reach the second step. How about step three? Since we know we can reach step two, we can use the second fact to conclude that we can reach step three. Using reasoning like this we can prove that we can reach any step. The two facts above capture exactly the base case (fact one) and the step case (fact two) of a mathematical inductive argument.

We now give several examples using mathematical induction.

Theorem 5. *Show that for any* $n \in \mathbb{N}$, $\Sigma_{j=1}^{n} j = \frac{n(n+1)}{2}$.

Proof. This is a proof by mathematical induction on n.

Base Case. We must show the following:

$$\Sigma_{j=1}^{1} = \frac{1(1+1)}{2} = \frac{2}{2}$$

Clearly, it is the case that $\Sigma_{j=1}^{1} = 1 = \frac{2}{2}$.

Step Case. We now must state the induction hypothesis which is the assumption that $\Sigma_{j=1}^{n} j = \frac{n(n+1)}{2}$ holds for some $n \in \mathbb{N}$. Then it suffices to show that

$$\Sigma_{j=1}^{n+1} j = \frac{(n+1)((n+1)+1)}{2} = \frac{(n+1)(n+2)}{2}$$

holds using the induction hypothesis. This follows from the following reasoning:

$$\begin{array}{rcll}
\Sigma_{j=1}^{n+1} j & = & 1+2+3+\cdots+n+(n+1) & \text{(Def. of Summation)} \\
& = & (1+2+3+\cdots+n)+(n+1) & \text{(Associativity of Addition)} \\
& = & \Sigma_{j=1}^{n} j + (n+1) & \text{(Def. of Summation)} \\
& = & \frac{n(n+1)}{2} + (n+1) & \text{(Induction Hypothesis)} \\
& = & \frac{n(n+1)+2(n+1)}{2} & \text{(Basic Algebra)} \\
& = & \frac{(n+1)(n+2)}{2} & \text{(Basic Algebra)}
\end{array}$$

□

Theorem 6. *Show that for any* $n \in \mathbb{N}$, $\Sigma_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$.

Proof. This is a proof by mathematical induction on n.

Base Case. We must show the following:

$$\Sigma_{j=1}^{1} j^2 = \frac{1(1+1)(2(1)+1)}{6} = \frac{6}{6} = 1$$

Clearly, $\Sigma_{j=1}^{1} j^2 = 1^2 = 1$.

Step Case. We now must state the induction hypothesis which is the assumption that

$$\Sigma_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6} \quad \text{(IH)}$$

holds for some $n \in \mathbb{N}$. Then it suffices to show that

$$\Sigma_{j=1}^{n+1} j^2 = \frac{(n+1)((n+1)+1)(2(n+1)+1)}{6} = \frac{(n+1)(n+2)(2n+3)}{6}$$

holds using the induction hypothesis. This follows from the following reasoning:

$$\begin{array}{rcll}
\Sigma_{j=1}^{n+1} j^2 & = & 1^2 + 2^2 + 3^2 + \cdots + n^2 + (n+1)^2 & \text{(Def. of Summation)} \\
& = & \Sigma_{j=1}^{n} j^2 + (n+1)^2 & \text{(Def. of Summation)} \\
& = & \frac{n(n+1)(2n+1)}{6} + (n+1)^2 & \text{(Induction Hypothesis)} \\
& = & \frac{n(n+1)(2n+1)+6(n+1)^2}{6} & \text{(Basic Algebra)} \\
& = & \frac{(n+1)(n(2n+1)+6(n+1))}{6} & \text{(Basic Algebra)} \\
& = & \frac{(n+1)(2n^2+n+6n+6))}{6} & \text{(Basic Algebra)} \\
& = & \frac{(n+1)(2n^2+7n+6))}{6} & \text{(Basic Algebra)} \\
& = & \frac{(n+1)(n(2n)+3n+4n+2(3))}{6} & \text{(Basic Algebra)} \\
& = & \frac{(n+1)(n+2)(2n+3)}{6} & \text{(Basic Algebra)}
\end{array}$$

□