

COMONADIC MATTER MEETS MONADIC ANTI-MATTER: AN ADJOINT MODEL OF BI-INTUITIONISTIC LOGIC

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ABSTRACT. Bi-intuitionistic logic (BINT) is a conservative extension of intuitionistic logic with perfect duality. That is, BINT contains the usual intuitionistic logical connectives such as true, conjunction, and implication, but also their duals false, disjunction, and co-implication. One leading question with respect to BINT is, what does BINT look like across the three arcs – logic, typed λ -calculi, and category theory – of the Curry-Howard-Lambek correspondence? A non-trivial (does not degenerate to a poset) categorical model of BINT is currently an open problem. It is this open problem that this paper contributes to by providing the first fully developed categorical model of BINT. It is well-known that the linear counterpart, linear BINT, of BINT can be modeled in a symmetric monoidal closed category equipped with an additional monoidal structure that models par and a specified left adjoint to par called linear co-implication. We call this model a symmetric bi-monoidal bi-closed category. In addition, it is well-known that intuitionistic logic has a categorical model of cartesian closed categories, and their dual co-cartesian co-closed categories model co-intuitionistic logic. In this paper we exploit Benton’s beautiful LNL models of linear logic to show that these three models can be mixed by requiring a symmetric monoidal adjunction between a cartesian closed category and the symmetric bi-monoidal bi-closed category, in addition to a symmetric monoidal adjunction between a co-cartesian co-closed category and the symmetric bi-monoidal bi-closed category. As a result of this mixture we obtain two modalities the usual comonadic of-course modality of linear logic, but also a monadic modality allowing for the embedding of co-intuitionistic logic inside of linear BINT. Finally, using these modalities we show that BINT intuitionistic logic can be soundly modeled in this new categorical model. As a bi-product of this model we define BiLNL logic which can be seen as the mixture of intuitionistic logic with co-intuitionistic logic inside of linear BINT.

1. INTRODUCTION

TODO [?]

2. BiLNL LOGIC

3. EMBEDDING L IN BiLNL LOGIC

REFERENCES

$\frac{G, (w, w); \Theta \vdash_1 w : Y}{G; \Theta \vdash_1 w : Y} \text{ I}_{\text{RL}}$	$\frac{w_1 G w_2 \quad w_2 G w_3}{G, (w_1, w_3); \Theta \vdash_1 w : Y} \text{ I}_{\text{TS}}$	$\frac{}{G; w : Y \vdash_1 w : Y} \text{ I}_{\text{ID}}$
$\frac{G; \Theta_2 \vdash_1 w : X \quad G; \Theta_1, w : X \vdash_1 w : Z}{G; \Theta_1, \Theta_2 \vdash_1 w : Z} \text{ I}_{\text{CUT}}$	$\frac{G; \Theta \vdash_1 w : Y}{G; \Theta, w : X \vdash_1 w : Y} \text{ I}_{\text{WK}}$	
$\frac{G; \Theta, w : X, w : X \vdash_1 w : Y}{G; \Theta, w : X \vdash_1 w : Y} \text{ I}_{\text{CR}}$	$\frac{G; w : R \vdash_{\text{C}} \Psi_1, w_1 : S, w_2 : T, \Psi_2}{G; w : R \vdash_{\text{C}} \Psi_1, w_2 : T, w_1 : S, \Psi_2} \text{ C}_{\text{EX}}$	
$\frac{w_1 G w_2}{G; \Theta, w_1 : X, w_2 : X \vdash_1 w : Y} \text{ I}_{\text{ML}}$	$\frac{w_2 G w_1}{G; \Theta \vdash_1 w_2 : Y} \text{ I}_{\text{MR}}$	
$\frac{G; \Theta \vdash_1 w : Y}{G; \Theta, w : \top \vdash_1 w : Y} \text{ I}_{\text{TL}}$	$\frac{}{G; \Theta \vdash_1 w : \top} \text{ I}_{\text{TR}}$	
$\frac{G; \Theta, w_1 : X, w_1 : Y \vdash_1 w_2 : Z}{G; \Theta, w_1 : X \times Y \vdash_1 w_2 : Z} \text{ I}_{\text{AL}}$	$\frac{G; \Theta_1 \vdash_1 w : X \quad G; \Theta_2 \vdash_1 w : Y}{G; \Theta_1, \Theta_2 \vdash_1 w : X \times Y} \text{ I}_{\text{AR}}$	
$\frac{w_1 G w_2}{G; \Theta_2 \vdash_1 w_2 : X} \quad G; \Theta_1, w_2 : Y \vdash_1 w : Z$	$\frac{}{G; \Theta_1, \Theta_2, w_1 : X \rightarrow Y \vdash_1 w : Z} \text{ I}_{\text{IL}}$	
$\frac{w_2 \notin G , \Theta }{G, (w_1, w_2); \Theta, w_2 : X \vdash_1 w_2 : Y} \text{ I}_{\text{IR}}$	$\frac{G; \Theta \mid \cdot \vdash_{\text{LL}} w : A \mid \cdot}{G; \Theta \vdash_1 w : GA} \text{ I}_{\text{GR}}$	

Figure 1: Intuitionistic Fragment of L

$\frac{G, (w, w); w : S \vdash_C \Psi}{G; w : S \vdash_C \Psi} \quad C_RL$	$\frac{w_1 G w_2 \quad w_2 G w_3}{G, (w_1, w_3); w : S \vdash_C \Psi} \quad C_TS$
$\frac{}{G; w : S \vdash_C w : S} \quad C_ID$	$\frac{G; w : S \vdash_C w : T, \Psi_2 \quad G; w : T \vdash_C \Psi_1}{G; w : S \vdash_C \Psi_1, \Psi_2} \quad C_CUT$
$\frac{G; w : S \vdash_C \Psi}{G; w : S \vdash_C w : T, \Psi} \quad C_WK$	$\frac{G; w : S \vdash_C w : T, w : T, \Psi}{G; w : S \vdash_C w : T, \Psi} \quad C_CR$
$\frac{G; w : R \vdash_C \Psi_1, w_1 : S, w_2 : T, \Psi_2}{G; w : R \vdash_C \Psi_1, w_2 : T, w_1 : S, \Psi_2} \quad C_EX$	$\frac{w_1 G w_2}{G; w_2 : S \vdash_C \Psi} \quad C_ML$
$\frac{w_2 G w_1}{G; w : S \vdash_C w_2 : T, w_1 : T, \Psi} \quad C_MR$	$\frac{}{G; w : \perp \vdash_C \Psi} \quad C_FL$
$\frac{G; w : S \vdash_C \Psi}{G; w : S \vdash_C w : \perp, \Psi} \quad C_FR$	$\frac{G; w : S \vdash_C \Psi_1 \quad G; w : T \vdash_C \Psi_2}{G; w : S + T \vdash_C \Psi_1, \Psi_2} \quad C_DL$
$\frac{G; w : R \vdash_C w : S, w : T, \Psi}{G; w : R \vdash_C w : S + T, \Psi} \quad C_DR$	$\frac{w_2 \notin G , \Psi }{G, (w_2, w_1); w_2 : S \vdash_C w_2 : T, \Psi} \quad C_sL$
$\frac{w_2 G w_1}{G; w : R \vdash_C w_2 : S, \Psi_2} \quad G; w_2 : T \vdash_C \Psi_1 \quad C_sR$	$\frac{G; \cdot w : A \vdash_{LL} \cdot \Psi}{G; w : HA \vdash_C \Psi} \quad C_HL$
$\frac{}{G; w : R \vdash_C w_1 : S - T, \Psi_1, \Psi_2}$	

Figure 2: Co-intuitionistic Fragment of L

$\frac{G, (w, w); \Theta \mid \Gamma \vdash_{LL} \Delta \mid \Psi}{G; \Theta \mid \Gamma \vdash_{LL} \Delta \mid \Psi} \quad LL_RL$	$\frac{w_1 G w_2 \quad w_2 G w_3}{G, (w_1, w_3); \Theta \mid \Gamma \vdash_{LL} \Delta \mid \Psi} \quad LL_TS$
$\frac{w_1 G w_2}{G; \Theta \mid \Gamma, w_1 : A, w_2 : A \vdash_{LL} \Delta \mid \Psi} \quad LL_mL$	$\frac{w_2 G w_1}{G; \Theta \mid \Gamma \vdash_{LL} w_2 : A, w_1 : A, \Delta \mid \Psi} \quad LL_mR$
$\frac{w_1 G w_2}{G; \Theta, w_1 : X, w_2 : X \mid \Gamma \vdash_{LL} \Delta \mid \Psi} \quad LL_ImL$	
$\frac{w_2 G w_1}{G; \Theta \mid \Gamma \vdash_{LL} \Delta \mid w_2 : T, w_1 : T, \Psi} \quad LL_CmR$	

Figure 3: Inference Rules for BiLNL Logic: Abstract Kripke Graph Rules

$\frac{G; \Theta \mid \Gamma \vdash_{LL} \Delta \mid \Psi}{G; \Theta, w : X \mid \Gamma \vdash_{LL} \Delta \mid \Psi} \quad LL_wkL$	$\frac{G; \Theta \mid \Gamma \vdash_{LL} \Delta \mid \Psi}{G; \Theta \mid \Gamma \vdash_{LL} \Delta \mid \Psi, w : S} \quad LL_wkR$
$\frac{G; \Theta, w : X, w : X \mid \Gamma \vdash_{LL} \Delta \mid \Psi}{G; \Theta, w : X \mid \Gamma \vdash_{LL} \Delta \mid \Psi} \quad LL_CTR_L$	$\frac{G; \Theta \mid \Gamma \vdash_{LL} \Delta \mid \Psi, w : S, w : S}{G; \Theta \mid \Gamma \vdash_{LL} \Delta \mid \Psi, w : S} \quad LL_CTR_R$
$\frac{G; \Theta \mid \Gamma_1, w_1 : A, w_2 : B, \Gamma_2 \vdash_{LL} \Delta \mid \Psi}{G; \Theta \mid \Gamma_1, w_2 : B, w_1 : A, \Gamma_2 \vdash_{LL} \Delta \mid \Psi} \quad LL_exL$	
$\frac{G; \Theta \mid \Gamma \vdash_{LL} \Delta_1, w_1 : A, w_2 : B, \Delta_2 \mid \Psi}{G; \Theta \mid \Gamma \vdash_{LL} \Delta_1, w_2 : B, w_1 : A, \Delta_2 \mid \Psi} \quad LL_exR$	
$\frac{G; \Theta_1, w_1 : X, w_2 : Y, \Theta_2 \mid \Gamma \vdash_{LL} \Delta \mid \Psi}{G; \Theta_1, w_2 : Y, w_1 : X, \Theta_2 \mid \Gamma \vdash_{LL} \Delta \mid \Psi} \quad LL_ILexL$	
$\frac{G; \Theta \mid \Gamma \vdash_{LL} \Delta \mid \Psi_1, w_1 : S, w_2 : T, \Psi_2}{G; \Theta \mid \Gamma \vdash_{LL} \Delta \mid \Psi_1, w_2 : T, w_1 : S, \Psi_2} \quad LL_CLexL$	

Figure 4: Inference Rules for BiLNL Logic: Structural Rules

$$\begin{array}{c}
\frac{}{G; \cdot \mid w : A \vdash_{LL} w : A \mid \cdot} \text{LL_ID} \\
\\
\frac{G; \Theta_1 \mid \Gamma_1 \vdash_{LL} w : A, \Delta_2 \mid \Psi_1 \quad G; \Theta_2 \mid w : A, \Gamma_2 \vdash_{LL} \Delta_1 \mid \Psi_2}{G; \Theta_1, \Theta_2 \mid \Gamma_1, \Gamma_2 \vdash_{LL} \Delta_1, \Delta_2 \mid \Psi_1, \Psi_2} \text{LL_CUT} \\
\\
\frac{G; \Theta_2 \vdash_1 w : X \quad G; \Theta_1, w : X \mid \Gamma \vdash_{LL} \Delta \mid \Psi}{G; \Theta_1, \Theta_2 \mid \Gamma \vdash_{LL} \Delta \mid \Psi} \text{LL_ILCUT} \\
\\
\frac{G; \Theta \mid \Gamma \vdash_{LL} \Delta \mid \Psi_1, w : S \quad G; w : S \vdash_C \Psi_2}{G; \Theta \mid \Gamma \vdash_{LL} \Delta \mid \Psi_1, \Psi_2} \text{LL_CLCUT}
\end{array}$$

Figure 5: Inference Rules for BiLNL Logic: Identity and Cut Rules

$$\begin{array}{c}
\frac{G; \Theta \mid \Gamma \vdash_{LL} \Delta \mid \Psi}{G; \Theta \mid \Gamma, w : I \vdash_{LL} \Delta \mid \Psi} \text{LL_IL} \qquad \frac{}{G; \cdot \mid \cdot \vdash_{LL} w : I \mid \cdot} \text{LL_IR} \\
\\
\frac{G; \Theta_1, w : X, w : Y, \Theta_2 \mid \Gamma \vdash_{LL} \Delta \mid \Psi}{G; \Theta_1, w : X \times Y, \Theta_2 \mid \Gamma \vdash_{LL} \Delta \mid \Psi} \text{LL_cL} \\
\\
\frac{G; \Theta \mid \Gamma_1, w : A, w : B, \Gamma_2 \vdash_{LL} \Delta \mid \Psi}{G; \Theta \mid \Gamma_1, w : A \otimes B, \Gamma_2 \vdash_{LL} \Delta \mid \Psi} \text{LL_tL} \\
\\
\frac{G; \Theta_1 \mid \Gamma_1 \vdash_{LL} w : A, \Delta_1 \mid \Psi_1 \quad G; \Theta_2 \mid \Gamma_2 \vdash_{LL} w : B, \Delta_2 \mid \Psi_2}{G; \Theta_1, \Theta_2 \mid \Gamma_1, \Gamma_2 \vdash_{LL} w : A \otimes B, \Delta_1, \Delta_2 \mid \Psi_1, \Psi_2} \text{LL_tR}
\end{array}$$

Figure 6: Inference Rules for BiLNL Logic: Conjunction and Tensor Rules

$$\begin{array}{c}
\frac{}{G; \cdot \mid w : J \vdash_{LL} \cdot \mid \cdot} \text{LL_JL} \qquad \frac{G; \Theta \mid \Gamma \vdash_{LL} \Delta \mid \Psi}{G; \Theta \mid \Gamma \vdash_{LL} w : J, \Delta \mid \Psi} \text{LL_JR} \\
\\
\frac{G; \Theta \mid \Gamma \vdash_{LL} \Delta \mid \Psi_1, w : S, w : T, \Psi_2}{G; \Theta \mid \Gamma \vdash_{LL} \Delta \mid \Psi_1, w : S + T, \Psi_2} \text{LL_dR} \\
\\
\frac{G; \Theta_1 \mid \Gamma_1, w : A \vdash_{LL} \Delta_1 \mid \Psi_1 \quad G; \Theta_2 \mid \Gamma_2, w : B \vdash_{LL} \Delta_2 \mid \Psi_2}{G; \Theta_1, \Theta_2 \mid \Gamma_1, \Gamma_2, w : A \oplus B \vdash_{LL} \Delta_1, \Delta_2 \mid \Psi_1, \Psi_2} \text{LL_pL} \\
\\
\frac{G; \Theta \mid \Gamma \vdash_{LL} \Delta_1, w : A, w : B, \Delta_2 \mid \Psi}{G; \Theta \mid \Gamma \vdash_{LL} \Delta_1, w : A \oplus B, \Delta_2 \mid \Psi} \text{LL_pR}
\end{array}$$

Figure 7: Inference Rules for BiLNL Logic: Disjunction and Par Rules

$$\begin{array}{c}
\frac{w_1 G w_2 \quad G; \Theta_1 \mid \Gamma_1 \vdash_{LL} w_2 : A, \Delta_1 \mid \Psi_1 \quad G; \Theta_2 \mid \Gamma_2, w_2 : B \vdash_{LL} \Delta_2 \mid \Psi_2}{G; \Theta_1, \Theta_2 \mid \Gamma_1, \Gamma_2, w_1 : A \multimap B \vdash_{LL} \Delta_1, \Delta_2 \mid \Psi_1, \Psi_2} \text{LL}_{\multimap}L \\
\\
\frac{w_2 \notin |G|, |\Theta|, |\Gamma|, |\Delta|, |\Psi| \quad G, (w_1, w_2); \Theta \mid \Gamma, w_2 : A \vdash_{LL} w_2 : B, \Delta \mid \Psi}{G; \Theta \mid \Gamma \vdash_{LL} w_1 : A \multimap B, \Delta \mid \Psi} \text{LL}_{\multimap}R \\
\\
\frac{w_1 G w_2 \quad G; \Theta_1 \vdash_I w_2 : X \quad G; \Theta_2, w_2 : Y \mid \Gamma \vdash_{LL} \Delta \mid \Psi}{G; \Theta_1, \Theta_2, w_1 : X \rightarrow Y \mid \Gamma \vdash_{LL} \Delta \mid \Psi} \text{LL}_{\multimap}L_L
\end{array}$$

Figure 8: Inference Rules for BiLNL Logic: Implication Rules

$$\begin{array}{c}
\frac{w_2 \notin |G|, |\Theta|, |\Gamma|, |\Delta|, |\Psi| \quad G, (w_2, w_1); \Theta \mid \Gamma, w_2 : A \vdash_{LL} w_2 : B, \Delta \mid \Psi}{G; \Theta \mid \Gamma, w_1 : A \multimap B \vdash_{LL} \Delta \mid \Psi} \text{LL}_{\multimap}sL \\
\\
\frac{w_2 G w_1 \quad G; \Theta_1 \mid \Gamma_1 \vdash_{LL} w_2 : A, \Delta_1 \mid \Psi_1 \quad G; \Theta_2 \mid \Gamma_2, w_2 : B \vdash_{LL} \Delta_2 \mid \Psi_2}{G; \Theta_1, \Theta_2 \mid \Gamma_1, \Gamma_2 \vdash_{LL} w_2 : A \multimap B, \Delta_1, \Delta_2 \mid \Psi_1, \Psi_2} \text{LL}_{\multimap}sR \\
\\
\frac{w_2 G w_1 \quad G; \Theta \mid \Gamma \vdash_{LL} \Delta \mid w_2 : S, \Psi_1 \quad G; w_2 : T \vdash_C \Psi_2}{G; \Theta \mid \Gamma \vdash_{LL} \Delta \mid w_1 : S \multimap T, \Psi_1, \Psi_2} \text{LL}_{\multimap}CLsR
\end{array}$$

Figure 9: Inference Rules for BiLNL Logic: Co-implication Rules

$$\begin{array}{cc}
\frac{G; \Theta, w : X \mid \Gamma \vdash_{LL} \Delta \mid \Psi}{G; \Theta \mid \Gamma, w : F X \vdash_{LL} \Delta \mid \Psi} \text{LL}_{\multimap}fL & \frac{G; \Theta \vdash_I w : X}{G; \Theta \mid \cdot \vdash_{LL} w : F X \mid \cdot} \text{LL}_{\multimap}fR \\
\\
\frac{G; \Theta \mid \Gamma, w : A \vdash_{LL} \Delta \mid \Psi}{G; \Theta, w : G A \mid \Gamma \vdash_{LL} \Delta \mid \Psi} \text{LL}_{\multimap}gL & \frac{G; w : S \vdash_C \Psi}{G; \cdot \mid w : J S \vdash_{LL} \cdot \mid \Psi} \text{LL}_{\multimap}JL \\
\\
\frac{G; \Theta \mid \Gamma \vdash_{LL} \Delta \mid w : S, \Psi}{G; \Theta \mid \Gamma \vdash_{LL} \Delta, w : J S \mid \Psi} \text{LL}_{\multimap}JR & \frac{G; \Theta \mid \Gamma \vdash_{LL} \Delta, w : A \mid \Psi}{G; \Theta \mid \Gamma \vdash_{LL} \Delta \mid w : H A, \Psi} \text{LL}_{\multimap}HR
\end{array}$$

Figure 10: Inference Rules for BiLNL Logic: Adjoint Functors Rules

$\frac{G, (w, w); \Gamma \vdash_L \Delta}{G; \Gamma \vdash_L \Delta} \quad \text{RL}$	$\frac{w_1 G w_2 \quad w_2 G w_3}{G, (w_1, w_3); \Gamma \vdash_L \Delta} \quad \text{TS}$	
$\frac{G; \Gamma \vdash_L w : A, \Delta \quad G; \Gamma, w : A \vdash_L \Delta}{G; \Gamma \vdash_L \Delta} \quad \text{CUT}$	$\frac{}{G; \Gamma, w : A \vdash_L w : A, \Delta} \quad \text{ID}$	
$\frac{w_1 G w_2}{G; \Gamma, w_1 : A, w_2 : A \vdash_L \Delta} \quad \text{mL}$	$\frac{w_2 G w_1}{G; \Gamma \vdash_L w_2 : A, w_1 : A, \Delta} \quad \text{mR}$	
$\frac{G; \Gamma \vdash_L \Delta}{G; \Gamma, w : \top \vdash_L \Delta} \quad \text{tL}$	$\frac{}{G; \Gamma \vdash_L w : \top, \Delta} \quad \text{tR}$	$\frac{}{G; \Gamma, w : \perp \vdash_L \Delta} \quad \text{fL}$
$\frac{G; \Gamma \vdash_L \Delta}{G; \Gamma \vdash_L w : \perp, \Delta} \quad \text{fR}$	$\frac{G; \Gamma, w : A, w : B \vdash_L \Delta}{G; \Gamma, w : A \times B \vdash_L \Delta} \quad \text{AL}$	
$\frac{G; \Gamma \vdash_L w : A, \Delta \quad G; \Gamma \vdash_L w : B, \Delta}{G; \Gamma \vdash_L w : A \times B, \Delta} \quad \text{AR}$		
$\frac{G; \Gamma, w : A \vdash_L \Delta \quad G; \Gamma, w : B \vdash_L \Delta}{G; \Gamma, w : A + B \vdash_L \Delta} \quad \text{dL}$	$\frac{G; \Gamma \vdash_L w : A, w : B, \Delta}{G; \Gamma \vdash_L w : A + B, \Delta} \quad \text{dR}$	
$\frac{w_1 G w_2}{G; \Gamma \vdash_L w_2 : A, \Delta \quad G; \Gamma, w_2 : B \vdash_L \Delta} \quad \text{iL}$	$\frac{w_2 \notin G , \Gamma , \Delta }{G, (w_1, w_2); \Gamma, w_2 : A \vdash_L w_2 : B, \Delta} \quad \text{iR}$	
$\frac{w_2 \notin G , \Gamma , \Delta }{G, (w_2, w_1); \Gamma, w_2 : A \vdash_L w_2 : B, \Delta} \quad \text{sL}$	$\frac{w_2 G w_1}{G; \Gamma \vdash_L w_2 : A, \Delta \quad G; \Gamma, w_2 : B \vdash_L \Delta} \quad \text{sR}$	
$\frac{}{G; \Gamma, w_1 : A - B \vdash_L \Delta} \quad \text{RL}$		

Figure 11: Inference Rules for L