Comonadic Matter Meets Monadic Anti-Matter: An Adjoint Model of Dualized Simple Type Theory

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Abstract

Bi-intuitionistic logic is a conservative extension of intuitionistic logic with perfect duality; every logical operator of the logic has a corresponding dual operator also in the logic. This symmetry suggests that bi-intuitionistic logic may lead to a foundational theory of induction and co-induction. Dualized Simple Type Theory was proposed by Eades et al. as a bi-intuitionistic type theory that could be used to study the relationship between induction and coinduction. It is the purpose of this paper to complete this line of inquiry by giving a categorical model of Dualized Simple Type Theory using a combination of two LNL models of Benton, which give rise to a monadic/comonadic relationship between intuitionistic logic and co-intuitionistic logic. We then extend both Dualized Simple Type Theory and its corresponding semantics with induction and co-induction.

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1 Introduction

TODO [**?**]

2 Dualized Intuitionistic Linear Logic

In this section we introduce a linear version of Dualized Intuitionistic Logic we call Dualized Intuitionistic Linear Logic (ZILL)[1]. The novelty of the design of ZILL is its use of two different syntactic tools to enforce linearity and intuitionism. The former is enforced in the usual way found in linear logic, but the latter is enforced by the layering of the abstract Kripke graph used in DIL. However, unlike DIL there is no straightforward interpretation of ZILL into a relational model, but this does not concern us here. This formalization makes it easier to embedded DIL into ZILL. In addition, ZILL is a new formalization – in fact, an extension – of Full Intuitionistic Linear Logic (FILL) [2].

The syntax of ZILL is a straightforward adaption of the dualized syntax used in the design of DIL.

▶ **Definition 1.** The following defines the syntax of ZILL:

$$\begin{array}{ll}
\text{(Variance)} & p ::= + \mid - \\
\text{(Formulas)} & A, B, C ::= \langle p \rangle \mid A \otimes_p B \mid A \multimap_p B \\
\text{(Graphs)} & G ::= \cdot \mid n_1 \preccurlyeq_p n_2 \mid G_1, G_2 \\
\text{(Contexts)} & \Gamma ::= \cdot \mid p\,A \mathbin{@} n \mid \Gamma_1, \Gamma_2 \\
\text{(Sequents)} & Q ::= G; \Gamma \vdash_{\mathsf{L}} t : p\,A \mathbin{@} n
\end{array}$$

[1] I chose to not abbreviate Dualized Intuitionistic Linear Logic by DILL, because it is well-known to be used for Dual Intuitionistic Linear Logic of Barber [**?**]

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$$\frac{G \vdash n \preccurlyeq_p^* n'}{G; x : p\,A \;@\; n \vdash_{\mathsf{L}} x : p\,A \;@\; n'}\ \text{L\_AX} \qquad \frac{}{G; \cdot \vdash_{\mathsf{L}} \circ : p\,\langle p \rangle \;@\; n}\ \text{L\_UNIT}$$

$$\frac{G; \Gamma \vdash_{\mathsf{L}} t : p\,A \;@\; n}{G; \Gamma, x : p\,\langle \bar{p} \rangle \;@\; n \vdash_{\mathsf{L}} \mathsf{let}\, x\, \mathsf{be}\, \circ\, \mathsf{in}\, t : p\,A \;@\; n}\ \text{L\_UNITBAR}$$

$$\frac{G; \Gamma_1, x : p\,A \;@\; n, y : p\,B \;@\; n, \Gamma_2 \vdash_{\mathsf{L}} t : p\,C \;@\; n}{G; \Gamma_1, z : p\,(A \otimes_p B) \;@\; n, \Gamma_2 \vdash_{\mathsf{L}} \mathsf{let}\, z\, \mathsf{be}\, x \otimes_p y\, \mathsf{in}\, t : p\,C \;@\; n}\ \text{L\_MONOL}$$

$$\frac{G; \Gamma_1 \vdash_{\mathsf{L}} t_1 : p\,A \;@\; n \qquad G; \Gamma_2 \vdash_{\mathsf{L}} t_2 : p\,B \;@\; n}{G; \Gamma_1, \Gamma_2 \vdash_{\mathsf{L}} t_1 \otimes_p t_2 : p\,(A \otimes_p B) \;@\; n}\ \text{L\_MONOR}$$

$$\frac{\begin{array}{l} n' \notin |G|, |\Gamma| \\ (G, n \preccurlyeq_p n'); \Gamma, x : p\,A \;@\; n' \vdash_{\mathsf{L}} t : p\,B \;@\; n' \end{array}}{G; \Gamma \vdash_{\mathsf{L}} \lambda x.t : p\,(A \multimap_p B) \;@\; n}\ \text{L\_IMP}$$

$$\frac{\begin{array}{l} G \vdash n \preccurlyeq_{\bar{p}}^* n' \\ G; \Gamma \vdash_{\mathsf{L}} t_1 : \bar{p}\,A \;@\; n' \qquad G; \Gamma \vdash_{\mathsf{L}} t_2 : p\,B \;@\; n' \end{array}}{G; \Gamma \vdash_{\mathsf{L}} \langle t_1, t_2 \rangle : p\,(A \multimap_{\bar{p}} B) \;@\; n}\ \text{L\_IMPBAR}$$

$$\frac{G; \Gamma_1, x : \bar{p}\,A \;@\; n \vdash_{\mathsf{L}} t_1 : +B \;@\; n' \qquad G; \Gamma_2 \vdash_{\mathsf{L}} t_2 : -B \;@\; n'}{G; \Gamma_1, \Gamma_2 \vdash_{\mathsf{L}} \nu_{\mathsf{L}}\, x.t_1 \bullet t_2 : p\,A \;@\; n}\ \text{L\_CUTL}$$

$$\frac{G; \Gamma_1 \vdash_{\mathsf{L}} t_1 : +B \;@\; n' \qquad G; \Gamma_2, x : \bar{p}\,A \;@\; n \vdash_{\mathsf{L}} t_2 : -B \;@\; n'}{G; \Gamma_1, \Gamma_2 \vdash_{\mathsf{L}} \nu_{\mathsf{R}}\, x.t_1 \bullet t_2 : p\,A \;@\; n}\ \text{L\_CUTR}$$

$$\frac{G; \Gamma_1, x : p_1\,A \;@\; n_1, y : p_2\,B \;@\; n_2, \Gamma_2 \vdash_{\mathsf{L}} t : p\,C \;@\; n}{G; \Gamma_1, y : p_2\,B \;@\; n_2, x : p_1\,A \;@\; n_1, \Gamma_2 \vdash_{\mathsf{L}} t : p\,C \;@\; n}\ \text{L\_EX}$$

Figure 1 Typing Relation for ZILL

Just as with DIL variance is used to indicate when we are operating on the left of the sequent ($p = -$) or the right of the sequent ($p = +$). Thus, the unit of tensor is $\langle + \rangle$, the unit of par is $\langle - \rangle$, tensor product is denoted by $A \otimes_+ B$, par is denoted by $A \otimes_- B$, implication is denoted by $A \multimap_+ B$, and finally co-implication is denoted by $A \multimap_- B$. The sequents of ZILL are denoted by $G; \Gamma \vdash_{\mathsf{L}} t : p\,A @ n$ where A is called the active formula, and n is a node in the abstract Kripke model G.

The typing rules for ZILL can be found in Figure 1. The rules are straightforward and not very surprising. The only thing to note is that the cut rule found in DIL had to be split into two cut rules in ZILL in order to maintain linearity. The first rule binds a free variable occurring in the first term of the cut, and the second binds a free variable occurring in the second term of the cut. The completeness proof of DIL heavily depends on the ability to switch out the active formula using the left-to-right lemma (Lemma 14 of [**?**]) effectively modeling exchanging conclusions, but this derivable rule depends on weakening, and hence, is no longer derivable in ZILL, but this is expected. Since ZILL's role is a intermediate language between intuitionistic logic and co-intuitionistic logic this is not a barrier. In fact, when extending ZILL with the proper modalities we will regain this ability.

3 Symmetric Monoidal Co-closed Categories

In this section we recall the definition of a symmetric monoidal category [?], and perhaps the lesser known definition of a symmetric monoidal left-closed category [?].

▶ **Definition 2.** A **symmetric monoidal category (SMC)** is a category, $\mathcal{M}$, with the following data:

- An object I of $\mathcal{M}$,
- A bi-functor $\oplus : \mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{M}$,
- The following natural isomorphisms:

$$\begin{aligned} \lambda_A &: I \oplus A \longrightarrow A \\ \rho_A &: A \oplus I \longrightarrow A \\ \alpha_{A,B,C} &: (A \oplus B) \oplus C \longrightarrow A \oplus (B \oplus C) \end{aligned}$$

- A symmetry natural transformation:

$$\beta_{A,B} : A \oplus B \longrightarrow B \oplus A$$

- Subject to the following coherence diagrams:

$$\begin{tikzcd} ((A \oplus B) \oplus C) \oplus D \arrow[r, "\alpha_{A,B,C} \oplus \mathsf{id}_D"] \arrow[d, "\alpha_{A \oplus B,C,D}"] & (A \oplus (B \oplus C)) \oplus D \arrow[dd, "\alpha_{A,B \oplus C,D}"] \\ (A \oplus B) \oplus (C \oplus D) \arrow[d, "\alpha_{A,B,C \oplus D}"] & \\ A \oplus (B \oplus (C \oplus D)) & A \oplus ((B \oplus C) \oplus D) \arrow[l, "\mathsf{id}_A \oplus \alpha_{B,C,D}"'] \end{tikzcd}$$

$$\begin{tikzcd} (A \oplus B) \oplus C \arrow[r, "\alpha_{A,B,C}"] \arrow[d, "\beta_{A,B} \oplus \mathsf{id}_C"] & A \oplus (B \oplus C) \arrow[r, "\beta_{A,B \oplus C}"] & (B \oplus C) \oplus A \arrow[d, "\alpha_{B,C,A}"] \\ (B \oplus A) \oplus C \arrow[r, "\alpha_{B,A,C}"] & B \oplus (A \oplus C) \arrow[r, "\mathsf{id}_B \oplus \beta_{A,C}"] & B \oplus (C \oplus A) \end{tikzcd}$$

$$\begin{tikzcd} (A \oplus I) \oplus B \arrow[rr, "\alpha_{A,I,B}"] \arrow[dr, "\rho_A"'] & & A \oplus (I \oplus B) \arrow[dl, "\lambda_B"] \\ & A \oplus B & \end{tikzcd} \qquad \begin{tikzcd} A \oplus B \arrow[d, "\beta_{A,B}"'] \arrow[dr, "\mathsf{id}_{A \oplus B}"] & \\ B \oplus A \arrow[r, "\beta_{B,A}"'] & A \oplus B \end{tikzcd}$$

$$\begin{tikzcd} I \oplus A \arrow[rr, "\beta_{I,A}"] \arrow[dr, "\lambda_A"'] & & A \oplus I \arrow[dl, "\rho_A"] \\ & A & \end{tikzcd}$$

▶ **Definition 3.** A **symmetric monoidal left closed category (SMLCC)** is a symmetric monoidal category $(C, \oplus, I, \alpha_{A,B,C}, \lambda_A, \rho_A, \beta_{A,B})$ such that the for any object B of C the functor $- \oplus B : C \to C$ has a specified left adjoint $B \bullet\!\!- - : C \to C$. This means that for any objects A,B, and C of C, we have the following bijection:

$$\mathsf{Hom}_C(C, A \oplus B) \cong \mathsf{Hom}_C(B \bullet\!\!- C, A)$$

that is natural in all arguments.

As we can see from the definition a SMLCC is the categorical dual to symmetric monoidal closed categories. Just as SMCCs model intuitionistic linear logics [?] SMLCCs model co-intuitionistic linear logics [?]. There are many concrete examples of SMLCCs, in fact, the dual of any interesting concrete SMCCs must be an interesting SMLCC.

In this paper I make use of a specific SMLCC, $\mathsf{Set}^{\mathsf{op}}$, which is the dual of the category Set of all sets and functions between them. The definition of $\mathsf{Set}^{\mathsf{op}}$ I use here is the well-known one in power-set algebras $(\mathcal{P}(X), \cup, \cap, \overline{}, \emptyset, X)$ where X is a set. In fact, it is well-known that we can define the functor $\mathcal{P} : \mathsf{Set} \to \mathsf{Set}^{\mathsf{op}}$ on morphisms by $\mathcal{P}(f : X \to Y)(S \in \mathcal{P}(Y)) = \{x \in X \mid f(x) \in S\}$. Thus, objects of $\mathsf{Set}^{\mathsf{op}}$ are powersets and morphisms are set theoretic functions between powersets.

4 The category $\mathsf{Dial_L}(\mathsf{Sets}^{\mathsf{op}})$

Dialectica categories originate from de Paiva's thesis [1], and are one of the first models of intuitionistic linear logic. In fact, they are the first categorical model of full intuitionistic linear logic which is intuitionistic linear logic with every logical connective from linear logic; complete with multiple conclusions. In full generality, dialectica categories, denoted $\mathsf{Dial}_L(C)$, are symmetric monoidal closed categories constructed in terms of a lineale [?], L, and a symmetric monoidal closed category C. However, I take L to be a colineale (Definition **??**) and C to be a symmetric monoidal left-closed category (Definition 3), particularly, I take C to be $\mathsf{Set}^{\mathsf{op}}$, but I take care to insure that all constructions lift to the general case of an arbitrary symmetric monoidal left-closed category. This is the first time this construction has been given.

We begin by introducing colineales as the categorical dual to lineales. The following defines when a proset is symmetric monoidal.

▶ **Definition 4.** A **monoidal proset** is a proset, $(L, \leq)$, with a given symmetric monoidal structure $(L, \bullet, e)$. That is, a set L with a given binary relation $\leq : L \times L \to L$ satisfying the following:

- (reflexivity) $a \leq a$ for any $a \in L$
- (transitivity) If $a \leq b$ and $b \leq c$, then $a \leq c$

together with a monoidal structure $(\bullet, e)$ consisting of a binary operation, called multiplication, $\bullet : L \times L \to L$ and a distinguished element $e \in L$ called the unit such that the following hold:

- (associativity) $(a \bullet b) \bullet c = a \bullet (b \bullet c)$
- (identity) $a \bullet e = a = e \bullet a$
- (symmetry) $a \bullet b = b \bullet a$

Finally, the structures must be compatible, that is, if $a \leq b$, then $a \bullet c \leq b \bullet c$ for any $c \in L$.

A colineale is essentially a symmetric monoidal left-closed category in the category of prosets.

▶ **Definition 5.** A **colineale** is a monoidal proset, $(L, \leq, \bullet, e)$, with a given binary operation, called coimplication, $\bullet\!\!- : L \times L \to L$, such that the following hold:

- (relative complement) $b \leq a \bullet (a \bullet\!\!- b)$
- (adjunction) If $b \bullet\!\!- c \leq a$, then $c \leq a \bullet b$

An example of a concrete colineale is the three element set $\mathsf{3} = \{0, \perp, 1\}$ where $\perp$ stands for undefined[2]. However, one must be careful when defining the colineale $\mathsf{3}$ because it is possible to degenerate to classical logic.

[2] The full definition of the colineale $\mathsf{3}$ can be found in the formal development: ?

We now use colineales to define the category $\mathsf{Dial_L}(\mathsf{Sets}^{\mathsf{op}})$. This category is given as a construction similar to the Chu construction.

▶ **Definition 6.** Suppose $(L, \leq, \bullet, e, \bullet\!\!-)$ is a colineale. Then the category $\mathsf{Dial_L}(\mathsf{Sets}^{\mathsf{op}})$ consists of

- objects that are triples, $A = (\mathcal{P}(U), \mathcal{P}(X), \alpha)$, where U and X are sets, and $\alpha : \mathcal{P}(U) \times \mathcal{P}(X) \to L$ is a multi-relation, and
- maps that are pairs $(f, F) : (\mathcal{P}(U), \mathcal{P}(X), \alpha) \to (\mathcal{P}(V), \mathcal{P}(Y), \beta)$ where $f : \mathcal{P}(U) \to \mathcal{P}(V)$ and $F : \mathcal{P}(Y) \to \mathcal{P}(X)$ such that
  - For any $u \in \mathcal{P}(U)$ and $y \in \mathcal{P}(Y)$, $\alpha(u, F(y)) \leq \beta(f(u), y)$.

Suppose $A = (\mathcal{P}(U), \mathcal{P}(X), \alpha)$, $B = (\mathcal{P}(V), \mathcal{P}(Y), \beta)$, and $C = (\mathcal{P}(W), \mathcal{P}(Z), \gamma)$. Then identities are given by $(\mathsf{id}_U, \mathsf{id}_X) : A \to A$. The composition of the maps $(f, F) : A \to B$ and $(g, G) : B \to C$ is defined as $(f; g, G; F) : A \to C$.

References

1 Valeria de Paiva. Dialectica categories. In J. Gray and A. Scedrov, editors, *Categories in Computer Science and Logic*, volume 92, pages 47–62. Amerian Mathemaitcal Society, 1989.

2 Harley Eades and Valeria Paiva. *Logical Foundations of Computer Science: International Symposium, LFCS 2016, Deerfield Beach, FL, USA, January 4-7, 2016. Proceedings*, chapter Multiple Conclusion Linear Logic: Cut Elimination and More, pages 90–105. Springer International Publishing, Cham, 2016. URL: `http://dx.doi.org/10.1007/978-3-319-27683-0_7`, `doi:10.1007/978-3-319-27683-0_7`.