COMONADIC MATTER MEETS MONADIC ANTI-MATTER: AN ADJOINT MODEL OF BI-INTUITIONISTIC LOGIC

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Abstract. Bi-intuitionistic logic (BINT) is a conservative extension of intuitionistic logic with perfect duality. That is, BINT contains the usual intuitionistic logical connectives such as true, conjunction, and implication, but also their duals false, disjunction, and co-implication. One leading question with respect to BINT is, what does BINT look like across the three arcs – logic, typed λ-calculi, and category theory – of the Curry-Howard-Lambek correspondence? A non-trivial (does not degenerate to a poset) categorical model of BINT is currently an open problem. It is this open problem that this paper contributes to by providing the first fully developed categorical model of BINT. It is well-known that the linear counterpart, linear BINT, of BINT can be modeled in a symmetric monoidal closed category equipped with an additional monoidal structure that models par and a specified left adjoint to par called linear co-implication. We call this model a symmetric bi-monoidal bi-closed category. In addition, it is well-known that intuitionistic logic has a categorical model of cartesian closed categories, and their dual co-cartesian co-closed categories model co-intuitionistic logic. In this paper we exploit Benton's beautiful LNL models of linear logic to show that these three models can be mixed by requiring a symmetric monoidal adjunction between a cartesian closed category and the symmetric bi-monoidal bi-closed category, in addition to a symmetric monoidal adjunction between a co-cartesian co-closed category and the symmetric bi-monoidal bi-closed category. As a result of this mixture we obtain two modalities the usual comonadic of-course modality of linear logic, but also a monadic modality allowing for the embedding of co-intuitionistic logic inside of linear BINT. Finally, using these modalities we show that BINT intuitionistic logic can be soundly modeled in this new categorical model. As a bi-product of this model we define BiLNL logic which can be seen as the mixture of intuitionistic logic with co-intuitionistic logic inside of linear BINT.

1. Introduction

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2. Mixed Linear/Non-Linear Models of Bi-intuitionistic Logic: The Categorical Model

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3. Mixed Linear/Non-Linear Bi-intuitionistic Logic: BiLNL Logic

Following Benton's [1] lead we can define a mixed linear/non-linear bi-intuitionistic logic, called BiLNL logic, based on the categorical model given in the previous section. BiLNL logic consists of three fragments: an intuitionistic fragment, a co-intuitionistic fragment, and a linear bi-intuitionistic core fragment. This formalization allows us to have more control on the mixture of the intuitionistic and co-intuitionistic fragments so as to allow for a proper categorical model. Each of the fragments are related through a syntactic formalization of the adjoint functors from the BiLNL model. First, we define the syntax of BiLNL logic, and then discuss the inference rules for each fragment.

Definition 1. The syntax for BiLNL logic is defined as follows:

$$\begin{array}{rl}
\text{(Worlds)} & W ::= w_1 \mid \cdots \mid w_i \\
\text{(Graphs)} & G ::= w_1 \leq w_2 \mid G_1, G_2 \\
\text{(Intuitionistic Formulas)} & X, Y, Z ::= 1 \mid X \times Y \mid X \to Y \mid \mathsf{G}\,A \\
\text{(Co-intuitionistic Formulas)} & R, S, T ::= 0 \mid S + T \mid S - T \mid \mathsf{H}\,A \\
\text{(Linear Bi-intuitionistic Formulas)} & A, B, C ::= \top \mid \perp \mid A \otimes B \mid A \oplus B \mid A \multimap B \mid A \bullet\!\!- B \mid \mathsf{F}\,X \mid \mathsf{J}\,S \\
\text{(Intuitionistic Contexts)} & \Theta ::= \cdot \mid X@w \mid \Theta_1, \Theta_2 \\
\text{(Co-intuitionistic Contexts)} & \Psi ::= \cdot \mid R@w \mid \Psi_1, \Psi_2 \\
\text{(Linear Bi-intuitionistic Contexts)} & \Gamma, \Delta ::= \cdot \mid A@w \mid \Gamma_1, \Gamma_2
\end{array}$$

Worlds may also be denoted by (potentially subscripted) n, m, and o.

Sequents have the following syntax:

$$\begin{array}{rl}
\text{(Intuitionistic Sequents)} & G; \Theta \vdash_{\mathsf{I}} X@w \\
\text{(Co-intuitionistic Sequents)} & G; R@w \vdash_{\mathsf{C}} \Psi \\
\text{(LNL Bi-intuitionistic Sequents)} & G; \Theta \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi
\end{array}$$

The syntax of intuitionistic and co-intuitionistic formulas are typical. I denote co-implication by $S - T$, but all the other connectives are the usual ones. Linear bi-intuitionistic formulas are denoted in somewhat of a non-traditional style. I denote the unit of tensor by $\top$ instead of the usual 1, which is the unit of intuitionistic conjunction, in addition, I denote par by $A \oplus B$, instead of $A \parr B$. Lastly, I denote linear co-implication by $A \bullet\!\!- B$ to emphasize its duality with linear implication $A \multimap B$. Each syntactic category of formulas contains the respective functor from the BiLNL model, and thus, we should few F and H as the left adjoints to G and J respectively.

Formulas in each type of context are annotated with a world, and each sequent is annotated with a graph. These graphs are syntactic representations of Kripke models and are used to enforce intuitionism. They were first used in bi-intuitionistic logic by Pinto and Uustalu [4] to enforce intuitionism in their logic L. In fact, we can see the linear core of BiLNL logic as the linear version of L. The beauty of this type of formalization and the reason why this style of logic was used by Pinto and Uustalu is that the logic L, and as well as BiLNL logic, are complete for cut-free bi-intuitionistic proofs. This was a new result of Pinto and Uustalu, because earlier formalizations of bi-intuitionistic logic [2] used the Dragalin restriction [3] to enforce intuitionism, but this results in a failure of cut-elimination [5, 4].

The expert reader will notice that it is not necessary to annotate the sequents of intuitionistic and co-intuitionistic logic with graphs and worlds to enforce intuitionism and co-intuitionism respectively. It is well known that restricting the right and left contexts two a single formula enforces intuitionism and co-intuitionism respectively. However, when mixing these two fragments with the linear core, which requires the graphs to be intuitionistic, it is easier if they are annotated. If they were not, then a seemingly complex world inference system would need to be designed to add the world constraints before mixing with the linear core, and it is currently an open problem whether this can be done. Thus, with respect to intuitionistic and co-intuitionistic logic the graph and world annotations can be seen as book keeping.

A second fact an expert reader will notice is that Kripke models are a relational model of intuitionistic logic, but not intuitionistic linear logic. This is okay, because Kripke models enforce intuitionism and not linearity, but it is well known how to enforce linearity syntactically in the definition of the inference rules. It is also well-known that even in linear logic if sequents have multiple hypothesis and multiple conclusions the logic becomes classical. Thus, in BiLNL logic we combine both of these tools to enforce both intuitionism and linearity. We can simply few the graphs as an over approximation of the relational constraints necessary to enforce both intuitionism and linearity. This also makes it easier to embed Pinto and Uustalu's logic in BiLNL logic as we will do in Section 4.

Sequents for the linear core have the form $G;\Theta \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi$. Similarly to the sequents of Benton's LNL logic [1], each context is separated for readability, but should actually be understood as being able to be mixed, that is, the contexts Θ and Γ could be a single context, and so could Δ and Ψ. The sequent:

$$G;X_1@w_1,\dots,X_i@w_i \mid A_1@n_1,\dots,A_j@n_j \vdash_{\mathsf{L}} B_1@m_1,\dots,B_k@m_k \mid R_1@o_1,\dots,R_l@o_l$$

will be interpreted in a BiLNL model by a morphism of the following form:

$$\mathsf{F}\,X_1 \otimes \cdots \otimes \mathsf{F}\,X_i \otimes A_1 \otimes \cdots \otimes A_j \xrightarrow{f} B_1 \oplus \cdots \oplus B_k \oplus \mathsf{J}\,R_1 \oplus \cdots \oplus \mathsf{J}\,R_l$$

Thus, intuitionistic formulas in the linear core can be viewed as being under the left adjoint F, and co-intuitionistic formulas as being under the right adjoint J. This implies that even in the model all types of formulas can be freely mixed.

4. Embedding Bi-Intuitionistic Logic in BiLNL Logic

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5. BiLNL Term Assignment

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6. Related Work

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7. Conclusion

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$$\frac{G, w_1 \leq w_1; \Theta \vdash_{I} Y@w_2}{G; \Theta \vdash_{I} Y@w_2}\ \text{I\_RL} \qquad \frac{w_1 G w_2 \quad w_2 G w_3 \quad G, w_1 \leq w_3; \Theta \vdash_{I} Y@w}{G; \Theta \vdash_{I} Y@w}\ \text{I\_TS}$$

$$\frac{}{G; Y@w \vdash_{I} Y@w}\ \text{I\_ID} \qquad \frac{G; \Theta_2 \vdash_{I} X@w_2 \quad G; \Theta_1, X@w_2 \vdash_{I} Z@w_1}{G; \Theta_1, \Theta_2 \vdash_{I} Z@w_1}\ \text{I\_CUT}$$

$$\frac{G; \Theta \vdash_{I} Y@w_1}{G; \Theta, X@w_2 \vdash_{I} Y@w_1}\ \text{I\_WK} \qquad \frac{G; \Theta, X@w_2, X@w_2 \vdash_{I} Y@w_1}{G; \Theta, X@w_2 \vdash_{I} Y@w_1}\ \text{I\_CR}$$

$$\frac{G; \Theta_1, X@w_1, Y@w_2, \Theta_2 \vdash_{I} Z@w}{G; \Theta_1, Y@w_2, X@w_1, \Theta_2 \vdash_{I} Z@w}\ \text{I\_EX} \qquad \frac{w_1 G w_2 \quad G; \Theta, X@w_1, X@w_2 \vdash_{I} Y@w}{G; \Theta, X@w_1 \vdash_{I} Y@w}\ \text{I\_ML}$$

$$\frac{w_2 G w_1 \quad G; \Theta \vdash_{I} Y@w_2}{G; \Theta \vdash_{I} Y@w_1}\ \text{I\_MR} \qquad \frac{G; \Theta \vdash_{I} Y@w_1}{G; \Theta, 1@w_2 \vdash_{I} Y@w_1}\ \text{I\_TL} \qquad \frac{}{G; \Theta \vdash_{I} 1@w}\ \text{I\_TR}$$

$$\frac{G; \Theta, X@w_1, Y@w_1 \vdash_{I} Z@w_2}{G; \Theta, (X \times Y)@w_1 \vdash_{I} Z@w_2}\ \text{I\_PL} \qquad \frac{G; \Theta_1 \vdash_{I} X@w \quad G; \Theta_2 \vdash_{I} Y@w}{G; \Theta_1, \Theta_2 \vdash_{I} (X \times Y)@w}\ \text{I\_PR}$$

$$\frac{w_1 G w_2 \quad G; \Theta_2 \vdash_{I} X@w_2 \quad G; \Theta_1, Y@w_2 \vdash_{I} Z@w}{G; \Theta_1, \Theta_2, (X \to Y)@w_1 \vdash_{I} Z@w}\ \text{I\_IL}$$

$$\frac{w_2 \notin |G|, |\Theta| \quad G, w_1 \leq w_2; \Theta, X@w_2 \vdash_{I} Y@w_2}{G; \Theta \vdash_{I} (X \to Y)@w_1}\ \text{I\_IR} \qquad \frac{G; \Theta \mid \cdot \vdash_{L} A@w \mid \cdot}{G; \Theta \vdash_{I} \mathsf{G}A@w}\ \text{I\_GR}$$

Figure 1: Inference Rules for BiLNL Logic: Intuitionistic Fragment

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$$\frac{G, w_1 \leq w_1; S@w_2 \vdash_{\mathsf{C}} \Psi}{G; S@w_2 \vdash_{\mathsf{C}} \Psi}\ \mathrm{C\_RL} \qquad \frac{w_1 G w_2 \quad w_2 G w_3 \quad G, w_1 \leq w_3; S@w \vdash_{\mathsf{C}} \Psi}{G; S@w \vdash_{\mathsf{C}} \Psi}\ \mathrm{C\_TS}$$

$$\frac{}{G; S@w \vdash_{\mathsf{C}} S@w}\ \mathrm{C\_ID} \qquad \frac{G; S@w_1 \vdash_{\mathsf{C}} T@w_2, \Psi_2 \quad G; T@w_2 \vdash_{\mathsf{C}} \Psi_1}{G; S@w_1 \vdash_{\mathsf{C}} \Psi_1, \Psi_2}\ \mathrm{C\_CUT}$$

$$\frac{G; S@w_1 \vdash_{\mathsf{C}} \Psi}{G; S@w_1 \vdash_{\mathsf{C}} T@w_2, \Psi}\ \mathrm{C\_WK} \qquad \frac{G; S@w_1 \vdash_{\mathsf{C}} T@w_2, T@w_2, \Psi}{G; S@w_1 \vdash_{\mathsf{C}} T@w_2, \Psi}\ \mathrm{C\_CR}$$

$$\frac{G; R@w \vdash_{\mathsf{C}} \Psi_1, S@w_1, T@w_2, \Psi_2}{G; R@w \vdash_{\mathsf{C}} \Psi_1, T@w_2, S@w_1, \Psi_2}\ \mathrm{C\_EX} \qquad \frac{w_1 G w_2 \quad G; S@w_2 \vdash_{\mathsf{C}} \Psi}{G; S@w_1 \vdash_{\mathsf{C}} \Psi}\ \mathrm{C\_ML}$$

$$\frac{w_2 G w_1 \quad G; S@w \vdash_{\mathsf{C}} T@w_2, T@w_1, \Psi}{G; S@w \vdash_{\mathsf{C}} T@w_1, \Psi}\ \mathrm{C\_MR} \qquad \frac{}{G; 0@w \vdash_{\mathsf{C}} \Psi}\ \mathrm{C\_FL}$$

$$\frac{G; S@w_1 \vdash_{\mathsf{C}} \Psi}{G; S@w_1 \vdash_{\mathsf{C}} 0@w_2, \Psi}\ \mathrm{C\_FR} \qquad \frac{G; S@w \vdash_{\mathsf{C}} \Psi_1 \quad G; T@w \vdash_{\mathsf{C}} \Psi_2}{G; S + T@w \vdash_{\mathsf{C}} \Psi_1, \Psi_2}\ \mathrm{C\_DL}$$

$$\frac{G; R@w_1 \vdash_{\mathsf{C}} S@w_2, T@w_2, \Psi}{G; R@w_1 \vdash_{\mathsf{C}} S + T@w_2, \Psi}\ \mathrm{C\_DR} \qquad \frac{w_2 \notin |G|, |\Psi| \quad G, w_2 \leq w_1; S@w_2 \vdash_{\mathsf{C}} T@w_2, \Psi}{G; S - T@w_1 \vdash_{\mathsf{C}} \Psi}\ \mathrm{C\_sL}$$

$$\frac{w_2 G w_1 \quad G; R@w \vdash_{\mathsf{C}} S@w_2, \Psi_2 \quad G; T@w_2 \vdash_{\mathsf{C}} \Psi_1}{G; R@w \vdash_{\mathsf{C}} S - T@w_1, \Psi_1, \Psi_2}\ \mathrm{C\_sR} \qquad \frac{G; \cdot \mid A@w \vdash_{\mathsf{L}} \cdot \mid \Psi}{G; \mathsf{H}A@w \vdash_{\mathsf{C}} \Psi}\ \mathrm{C\_HL}$$

Figure 2: Inference Rules for BiLNL Logic: Co-intuitionistic Fragment

$$\frac{G, w \leq w; \Theta \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi}{G; \Theta \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi}\ \mathrm{L\_RL} \qquad \frac{w_1 G w_2 \quad w_2 G w_3 \quad G, w_1 \leq w_3; \Theta \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi}{G; \Theta \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi}\ \mathrm{L\_TS}$$

$$\frac{w_1 G w_2 \quad G; \Theta \mid \Gamma, A@w_1, A@w_2 \vdash_{\mathsf{L}} \Delta \mid \Psi}{G; \Theta \mid \Gamma, A@w_1 \vdash_{\mathsf{L}} \Delta \mid \Psi}\ \mathrm{L\_ML} \qquad \frac{w_2 G w_1 \quad G; \Theta \mid \Gamma \vdash_{\mathsf{L}} A@w_2, A@w_1, \Delta \mid \Psi}{G; \Theta \mid \Gamma \vdash_{\mathsf{L}} A@w_1, \Delta \mid \Psi}\ \mathrm{L\_MR}$$

$$\frac{w_1 G w_2 \quad G; \Theta, X@w_1, X@w_2 \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi}{G; \Theta, X@w_1 \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi}\ \mathrm{L\_ImL} \qquad \frac{w_2 G w_1 \quad G; \Theta \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid T@w_2, T@w_1, \Psi}{G; \Theta \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid T@w_1, \Psi}\ \mathrm{L\_CmR}$$

Figure 3: Inference Rules for BiLNL Logic: Abstract Kripke Graph Rules

$$\frac{G;\Theta \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi}{G;\Theta, X@w \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi}\ \text{L\_WKL} \qquad \frac{G;\Theta \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi}{G;\Theta \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid S@w, \Psi}\ \text{L\_WKR}$$

$$\frac{G;\Theta, X@w, X@w \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi}{G;\Theta, X@w \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi}\ \text{L\_CTRL} \qquad \frac{G;\Theta \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid S@w, S@w, \Psi}{G;\Theta \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid S@w, \Psi}\ \text{L\_CTRR}$$

$$\frac{G;\Theta \mid \Gamma_1, A@w_1, B@w_2, \Gamma_2 \vdash_{\mathsf{L}} \Delta \mid \Psi}{G;\Theta \mid \Gamma_1, B@w_2, A@w_1, \Gamma_2 \vdash_{\mathsf{L}} \Delta \mid \Psi}\ \text{L\_EXL}$$

$$\frac{G;\Theta \mid \Gamma \vdash_{\mathsf{L}} \Delta_1, A@w_1, B@w_2, \Delta_2 \mid \Psi}{G;\Theta \mid \Gamma \vdash_{\mathsf{L}} \Delta_1, B@w_2, A@w_1, \Delta_2 \mid \Psi}\ \text{L\_EXR}$$

$$\frac{G;\Theta_1, X@w_1, Y@w_2, \Theta_2 \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi}{G;\Theta_1, Y@w_2, X@w_1, \Theta_2 \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi}\ \text{L\_IEXL}$$

$$\frac{G;\Theta \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi_1, S@w_1, T@w_2, \Psi_2}{G;\Theta \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi_1, T@w_2, S@w_1, \Psi_2}\ \text{L\_CEXL}$$

Figure 4: Inference Rules for BiLNL Logic: Structural Rules

$$\frac{}{G;\cdot \mid A@w \vdash_{\mathsf{L}} A@w \mid \cdot}\ \text{L\_ID}$$

$$\frac{G;\Theta_1 \mid \Gamma_1 \vdash_{\mathsf{L}} A@w, \Delta_2 \mid \Psi_1 \qquad G;\Theta_2 \mid A@w, \Gamma_2 \vdash_{\mathsf{L}} \Delta_1 \mid \Psi_2}{G;\Theta_1, \Theta_2 \mid \Gamma_1, \Gamma_2 \vdash_{\mathsf{L}} \Delta_1, \Delta_2 \mid \Psi_1, \Psi_2}\ \text{L\_CUT}$$

$$\frac{G;\Theta_2 \vdash_{\mathsf{I}} X@w \qquad G;\Theta_1, X@w \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi}{G;\Theta_1, \Theta_2 \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi}\ \text{L\_ICUT}$$

$$\frac{G;\Theta \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi_1, S@w \qquad G; S@w \vdash_{\mathsf{C}} \Psi_2}{G;\Theta \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi_1, \Psi_2}\ \text{L\_CCUT}$$

Figure 5: Inference Rules for BiLNL Logic: Identity and Cut Rules

$$\frac{G;\Theta \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi}{G;\Theta \mid \Gamma, \top@w \vdash_{\mathsf{L}} \Delta \mid \Psi}\ \text{L\_IL} \qquad \frac{}{G;\cdot \mid \cdot \vdash_{\mathsf{L}} \top@w \mid \cdot}\ \text{L\_IR}$$

$$\frac{G;\Theta_1, X@w, Y@w, \Theta_2 \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi}{G;\Theta_1, X \times Y@w, \Theta_2 \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi}\ \text{L\_cL}$$

$$\frac{G;\Theta \mid \Gamma_1, A@w, B@w, \Gamma_2 \vdash_{\mathsf{L}} \Delta \mid \Psi}{G;\Theta \mid \Gamma_1, A \otimes B@w, \Gamma_2 \vdash_{\mathsf{L}} \Delta \mid \Psi}\ \text{L\_TL}$$

$$\frac{G;\Theta_1 \mid \Gamma_1 \vdash_{\mathsf{L}} A@w, \Delta_1 \mid \Psi_1 \qquad G;\Theta_2 \mid \Gamma_2 \vdash_{\mathsf{L}} B@w, \Delta_2 \mid \Psi_2}{G;\Theta_1, \Theta_2 \mid \Gamma_1, \Gamma_2 \vdash_{\mathsf{L}} A \otimes B@w, \Delta_1, \Delta_2 \mid \Psi_1, \Psi_2}\ \text{L\_TR}$$

Figure 6: Inference Rules for BiLNL Logic: Conjunction and Tensor Rules

$$\frac{}{G;\cdot \mid \perp @w \vdash_{\mathsf{L}} \cdot \mid \cdot}\ \text{L\_FLL} \qquad \frac{G;\Theta \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi}{G;\Theta \mid \Gamma \vdash_{\mathsf{L}} \perp @w, \Delta \mid \Psi}\ \text{L\_FLR}$$

$$\frac{G;\Theta \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi_1, S@w, T@w, \Psi_2}{G;\Theta \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi_1, S + T@w, \Psi_2}\ \text{L\_DR}$$

$$\frac{G;\Theta_1 \mid \Gamma_1, A@w \vdash_{\mathsf{L}} \Delta_1 \mid \Psi_1 \qquad G;\Theta_2 \mid \Gamma_2, B@w \vdash_{\mathsf{L}} \Delta_2 \mid \Psi_2}{G;\Theta_1, \Theta_2 \mid \Gamma_1, \Gamma_2, A \oplus B@w \vdash_{\mathsf{L}} \Delta_1, \Delta_2 \mid \Psi_1, \Psi_2}\ \text{L\_PL}$$

$$\frac{G;\Theta \mid \Gamma \vdash_{\mathsf{L}} \Delta_1, A@w, B@w, \Delta_2 \mid \Psi}{G;\Theta \mid \Gamma \vdash_{\mathsf{L}} \Delta_1, A \oplus B@w, \Delta_2 \mid \Psi}\ \text{L\_PR}$$

Figure 7: Inference Rules for BiLNL Logic: Disjunction and Par Rules

$$\frac{\begin{array}{l} w_1 G w_2 \\ G;\Theta_1 \mid \Gamma_1 \vdash_{\mathsf{L}} A@w_2, \Delta_1 \mid \Psi_1 \qquad G;\Theta_2 \mid \Gamma_2, B@w_2 \vdash_{\mathsf{L}} \Delta_2 \mid \Psi_2 \end{array}}{G;\Theta_1, \Theta_2 \mid \Gamma_1, \Gamma_2, A \multimap B@w_1 \vdash_{\mathsf{L}} \Delta_1, \Delta_2 \mid \Psi_1, \Psi_2}\ \text{L\_IMPL}$$

$$\frac{\begin{array}{l} w_2 \notin |G|, |\Theta|, |\Gamma|, |\Delta|, |\Psi| \\ G, w_1 \leq w_2;\Theta \mid \Gamma, A@w_2 \vdash_{\mathsf{L}} B@w_2, \Delta \mid \Psi \end{array}}{G;\Theta \mid \Gamma \vdash_{\mathsf{L}} A \multimap B@w_1, \Delta \mid \Psi}\ \text{L\_IMPR}$$

$$\frac{\begin{array}{l} w_1 G w_2 \\ G;\Theta_1 \vdash_{\mathsf{I}} X@w_2 \qquad G;\Theta_2, Y@w_2 \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi \end{array}}{G;\Theta_1, \Theta_2, X \to Y@w_1 \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi}\ \text{L\_IIMPL}$$

Figure 8: Inference Rules for BiLNL Logic: Implication Rules

$$\frac{\begin{array}{l} w_2 \notin |G|, |\Theta|, |\Gamma|, |\Delta|, |\Psi| \\ G, w_2 \leq w_1; \Theta \mid \Gamma, A@w_2 \vdash_{\mathsf{L}} B@w_2, \Delta \mid \Psi \end{array}}{G; \Theta \mid \Gamma, A \bullet\!\!- B@w_1 \vdash_{\mathsf{L}} \Delta \mid \Psi}\ \text{L\_sL}$$

$$\frac{\begin{array}{l} w_2 G w_1 \\ G; \Theta_1 \mid \Gamma_1 \vdash_{\mathsf{L}} A@w_2, \Delta_1 \mid \Psi_1 \qquad G; \Theta_2 \mid \Gamma_2, B@w_2 \vdash_{\mathsf{L}} \Delta_2 \mid \Psi_2 \end{array}}{G; \Theta_1, \Theta_2 \mid \Gamma_1, \Gamma_2 \vdash_{\mathsf{L}} A \bullet\!\!- B@w_1, \Delta_1, \Delta_2 \mid \Psi_1, \Psi_2}\ \text{L\_sR}$$

$$\frac{\begin{array}{l} w_2 G w_1 \\ G; \Theta \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid S@w_2, \Psi_1 \qquad G; T@w_2 \vdash_{\mathsf{C}} \Psi_2 \end{array}}{G; \Theta \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid S - T@w_1, \Psi_1, \Psi_2}\ \text{L\_CsR}$$

Figure 9: Inference Rules for BiLNL Logic: Co-implication Rules

$$\frac{G; \Theta, X@w \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi}{G; \Theta \mid \Gamma, \mathsf{F}X@w \vdash_{\mathsf{L}} \Delta \mid \Psi}\ \text{L\_{\scriptsize F}L} \qquad \frac{G; \Theta \vdash_{\mathsf{I}} X@w}{G; \Theta \mid \cdot \vdash_{\mathsf{L}} \mathsf{F}X@w \mid \cdot}\ \text{L\_{\scriptsize F}R}$$

$$\frac{G; S@w \vdash_{\mathsf{C}} \Psi}{G; \cdot \mid \mathsf{J}S@w \vdash_{\mathsf{L}} \cdot \mid \Psi}\ \text{L\_{\scriptsize J}L} \qquad \frac{G; \Theta \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid S@w, \Psi}{G; \Theta \mid \Gamma \vdash_{\mathsf{L}} \Delta, \mathsf{J}S@w \mid \Psi}\ \text{L\_{\scriptsize J}R}$$

$$\frac{G; \Theta \mid \Gamma, A@w \vdash_{\mathsf{L}} \Delta \mid \Psi}{G; \Theta, \mathsf{G}A@w \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi}\ \text{L\_{\scriptsize G}L} \qquad \frac{G; \Theta \mid \Gamma \vdash_{\mathsf{L}} \Delta, A@w \mid \Psi}{G; \Theta \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \mathsf{H}A@w, \Psi}\ \text{L\_{\scriptsize H}R}$$

Figure 10: Inference Rules for BiLNL Logic: Adjoint Functors Rules

$$\frac{G, w \leq w; \Gamma \vdash_{\mathsf{L}} \Delta}{G; \Gamma \vdash_{\mathsf{L}} \Delta}\ \textsc{rl} \qquad \frac{w_1 G w_2 \quad w_2 G w_3 \quad G, w_1 \leq w_3; \Gamma \vdash_{\mathsf{L}} \Delta}{G; \Gamma \vdash_{\mathsf{L}} \Delta}\ \textsc{ts} \qquad \frac{w_1 G w_2 \quad G; \Gamma, A@w_1, A@w_2 \vdash_{\mathsf{L}} \Delta}{G; \Gamma, A@w_1 \vdash_{\mathsf{L}} \Delta}\ \textsc{m}\mathrm{L}$$

$$\frac{w_2 G w_1 \quad G; \Gamma \vdash_{\mathsf{L}} A@w_2, A@w_1, \Delta}{G; \Gamma \vdash_{\mathsf{L}} A@w_1, \Delta}\ \textsc{m}\mathrm{R} \qquad \frac{G; \Gamma \vdash_{\mathsf{L}} \Delta}{G; \Gamma, A@w \vdash_{\mathsf{L}} \Delta}\ \textsc{wk}\mathrm{L} \qquad \frac{G; \Gamma \vdash_{\mathsf{L}} \Delta}{G; \Gamma \vdash_{\mathsf{L}} A@w, \Delta}\ \textsc{wk}\mathrm{R}$$

$$\frac{G; \Gamma, A@w, A@w \vdash_{\mathsf{L}} \Delta}{G; \Gamma, A@w \vdash_{\mathsf{L}} \Delta}\ \textsc{ctr}\mathrm{L} \qquad \frac{G; \Gamma \vdash_{\mathsf{L}} A@w, A@w, \Delta}{G; \Gamma \vdash_{\mathsf{L}} A@w, \Delta}\ \textsc{ctr}\mathrm{R}$$

$$\frac{G; \Gamma_1, A@w_1, B@w_2, \Gamma_2 \vdash_{\mathsf{L}} \Delta}{G; \Gamma_1, B@w_2, A@w_1, \Gamma_2 \vdash_{\mathsf{L}} \Delta}\ \textsc{ex}\mathrm{L} \qquad \frac{G; \Gamma \vdash_{\mathsf{L}} \Delta_1, A@w_1, B@w_2, \Delta_2}{G; \Gamma \vdash_{\mathsf{L}} \Delta_1, B@w_2, A@w_1, \Delta_2}\ \textsc{ex}\mathrm{R}$$

$$\frac{}{G; A@w \vdash_{\mathsf{L}} A@w}\ \textsc{id} \qquad \frac{G; \Gamma_1 \vdash_{\mathsf{L}} A@w, \Delta_2 \quad G; A@w, \Gamma_2 \vdash_{\mathsf{L}} \Delta_1}{G; \Gamma_1, \Gamma_2 \vdash_{\mathsf{L}} \Delta_1, \Delta_2}\ \textsc{cut}$$

$$\frac{G; \Gamma \vdash_{\mathsf{L}} \Delta}{G; \Gamma, \top@w \vdash_{\mathsf{L}} \Delta}\ \mathrm{IL} \qquad \frac{}{G; \cdot \vdash_{\mathsf{L}} \perp @w}\ \mathrm{IR} \qquad \frac{}{G; \top@w \vdash_{\mathsf{L}} \cdot}\ \textsc{fl}\mathrm{L}$$

$$\frac{G; \Gamma \vdash_{\mathsf{L}} \Delta}{G; \Gamma \vdash_{\mathsf{L}} \perp @w, \Delta}\ \textsc{fl}\mathrm{R} \qquad \frac{G; \Gamma_1, A@w, B@w, \Gamma_2 \vdash_{\mathsf{L}} \Delta}{G; \Gamma_1, A \times B@w, \Gamma_2 \vdash_{\mathsf{L}} \Delta}\ \mathrm{cL}$$

$$\frac{G; \Gamma_1 \vdash_{\mathsf{L}} A@w, \Delta_1 \quad G; \Gamma_2 \vdash_{\mathsf{L}} B@w, \Delta_2}{G; \Gamma_1, \Gamma_2 \vdash_{\mathsf{L}} A \times B@w}\ \mathrm{cR}$$

$$\frac{G; \Gamma_1, A@w \vdash_{\mathsf{L}} \Delta_1 \quad G; \Gamma_2, B@w \vdash_{\mathsf{L}} \Delta_2}{G; \Gamma_1, \Gamma_2, A + B@w \vdash_{\mathsf{L}} \Delta_1, \Delta_2}\ \textsc{d}\mathrm{L} \qquad \frac{G; \Gamma \vdash_{\mathsf{L}} \Delta_1, A@w, B@w, \Delta_2}{G; \Gamma \vdash_{\mathsf{L}} \Delta_1, A + B@w, \Delta_2}\ \textsc{d}\mathrm{R}$$

$$\frac{w_2 \notin |G|, |\Gamma|, |\Delta| \quad G, w_1 \leq w_2; \Gamma, A@w_2 \vdash_{\mathsf{L}} B@w_2, \Delta}{G; \Gamma \vdash_{\mathsf{L}} A \rightarrow B@w_1, \Delta}\ \mathrm{I}\textsc{mp}\mathrm{R}$$

$$\frac{w_1 G w_2 \quad G; \Gamma_1 \vdash_{\mathsf{L}} A@w_2, \Delta_1 \quad G; \Gamma_2, B@w_2 \vdash_{\mathsf{L}} \Delta_2}{G; \Gamma_1, \Gamma_2, A \rightarrow B@w_1 \vdash_{\mathsf{L}} \Delta_1, \Delta_2}\ \mathrm{I}\textsc{mp}\mathrm{L}$$

$$\frac{w_2 \notin |G|, |\Gamma|, |\Delta| \quad G, w_2 \leq w_1; \Gamma, A@w_2 \vdash_{\mathsf{L}} B@w_2, \Delta}{G; \Gamma, A - B@w_1 \vdash_{\mathsf{L}} \Delta}\ \mathrm{sL}$$

$$\frac{w_2 G w_1 \quad G; \Gamma_1 \vdash_{\mathsf{L}} A@w_2, \Delta_1 \quad G; \Gamma_2, B@w_2 \vdash_{\mathsf{L}} \Delta_2}{G; \Gamma_1, \Gamma_2 \vdash_{\mathsf{L}} A - B@w_1, \Delta_1, \Delta_2}\ \mathrm{sR}$$

Figure 11: Inference Rules for L