COMONADIC MATTER MEETS MONADIC ANTI-MATTER: AN ADJOINT MODEL OF BI-INTUITIONISTIC LOGIC

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ABSTRACT. Bi-intuitionistic logic (BINT) is a conservative extension of intuitionistic logic with perfect duality. That is, BINT contains the usual intuitionistic logical connectives such as true, conjunction, and implication, but also their duals false, disjunction, and co-implication. One leading question with respect to BINT is, what does BINT look like across the three arcs – logic, typed λ-calculi, and category theory – of the Curry-Howard-Lambek correspondence? A non-trivial (does not degenerate to a poset) categorical model of BINT is currently an open problem. It is this open problem that this paper contributes to by providing the first fully developed categorical model of BINT. It is well-known that the linear counterpart, linear BINT, of BINT can be modeled in a symmetric monoidal closed category equipped with an additional monoidal structure that models par and a specified left adjoint to par called linear co-implication. We call this model a symmetric bi-monoidal bi-closed category. In addition, it is well-known that intuitionistic logic has a categorical model of cartesian closed categories, and their dual co-cartesian co-closed categories model co-intuitionistic logic. In this paper we exploit Benton's beautiful LNL models of linear logic to show that these three models can be mixed by requiring a symmetric monoidal adjunction between a cartesian closed category and the symmetric bi-monoidal bi-closed category, in addition to a symmetric monoidal adjunction between a co-cartesian co-closed category and the symmetric bi-monoidal bi-closed category. As a result of this mixture we obtain two modalities the usual comonadic of-course modality of linear logic, but also a monadic modality allowing for the embedding of co-intuitionistic logic inside of linear BINT. Finally, using these modalities we show that BINT intuitionistic logic can be soundly modeled in this new categorical model. As a bi-product of this model we define BiLNL logic which can be seen as the mixture of intuitionistic logic with co-intuitionistic logic inside of linear BINT.

1. INTRODUCTION

TODO [**?**]

2. MIXED LINEAR/NON-LINEAR MODELS OF BI-INTUITIONISTIC LOGIC: THE CATEGORICAL MODEL

In this section we embark on the definition of a categorical model of bi-intuitionistic logic via a categorical model of bi-intuitionistic linear logic. First, we summarize our main results. Suppose $(\mathcal{I}, 1, \times, \rightarrow)$ is a cartesian closed category, and $(\mathcal{L}, \top, \otimes, \multimap)$ is a symmetric monoidal closed category. Then relate these two categories with a symmetric monoidal adjunction $\mathcal{I} : \mathsf{F} \dashv \mathsf{G} : \mathcal{L}$ (Definition **??**), where F and G are symmetric monoidal functors. The later point implies that there are natural transformations $\mathsf{m}_{X,Y} : \mathsf{F}X \otimes \mathsf{F}Y \longrightarrow \mathsf{F}(X \times Y)$ and $\mathsf{n}_{A,B} : \mathsf{G}A \times \mathsf{G}B \longrightarrow \mathsf{G}(A \otimes B)$, and maps $\mathsf{m}_\top : \top \longrightarrow \mathsf{F}1$ and $\mathsf{n}_1 : 1 \longrightarrow \mathsf{G}\top$ subject to several coherence conditions; see Definition **??**.

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Furthermore, the functor F is strong which means that $\mathsf{m}_{X,Y}$ and $\mathsf{m}_{\top}$ are isomorphisms. This setup turns out to be one of the most beautiful models of intuitionistic linear logic called an LNL model due to Benton [2]. In fact, the linear modality of-course can be defined by $!A = \mathsf{F}(\mathsf{G}(A))$ which defines a symmetric monoidal comonad using the adjunction; see Section 2.2 of [2]. This model is much simpler than other known models, and resulted in a logic called LNL logic which supports mixing intuitionistic logic with linear logic.

Taking the dual of the previous model results in what we call dual LNL models. They consist of a co-cartesian co-closed category, $(\mathcal{C}, 0, +, -)$, a symmetric monoidal co-closed category, $(\mathcal{L}', \perp, \oplus, \bullet\!\!-)$, where $\bullet\!\!- : \mathcal{L}' \times \mathcal{L}' \longrightarrow \mathcal{L}'$ is left adjoint to parr, and a symmetric monoidal adjunction $\mathcal{L}' : H \dashv \perp : \mathcal{C}$. We will show that dual LNL models are a simplification of dual linear categories as defined by Bellin [1] in much of the same way that LNL models are a simplification of linear categories. In fact, we will define Girard's exponential why-not by $?A = J(H(A))$, and hence, is the monad induced by the adjunction.

Now we combine the two previous models into one model of bi-intuitionistic linear logic. Extend $\mathcal{L}$ with a second symmetric monoidal structure, $(\perp, \oplus, \bullet\!\!-)$, such that $\otimes$ distributes over $\oplus$ and $\bullet\!\!- : \mathcal{L} \times \mathcal{L} \longrightarrow \mathcal{L}$ is left adjoint to $\oplus$, resulting in the category $(\mathcal{L}, \top, \otimes, \multimap, \perp, \oplus, \bullet\!\!-)$, called a symmetric monoidal bi-closed category. We extend LNL models into a new model of bi-intuitionistic logic called a mixed linear/non-linear bi-intuitionistic model or a BiLNL model. It consists of a cartesian closed category, $(\mathcal{I}, 1, \times, \rightarrow)$, a symmetric monoidal bi-closed category, $(\mathcal{L}, \top, \otimes, \multimap, \perp, \oplus, \bullet\!\!-)$, a co-cartesian co-closed category, $(\mathcal{C}, 0, +, -)$, and a pair of symmetric monoidal adjoint functors:

$$\begin{tikzcd} \mathcal{I} \arrow[r, bend left, "\mathsf{F}"] \arrow[r, phantom, "\dashv"] & \mathcal{L} \arrow[l, bend left, "\mathsf{G}"] \arrow[r, bend left, "\mathsf{H}"] \arrow[r, phantom, "\vdash"] & \mathsf{C} \arrow[l, bend left, "\mathsf{J}"] \end{tikzcd}$$

It is well known that $\mathcal{I}$ is a model of intuitionistic logic and $\mathcal{C}$ is a model of co-intuitionistic logic, and so, the adjoint situation $F \dashv G$ can be seen as a translation of intuitionistic logic into the intuitionistic fragment, $(\top, \otimes, \multimap)$, of $\mathcal{L}$, and the adjoint situation $H \dashv J$ can be seen as a translation of co-intuitionistic logic into the co-intuitionistic fragment, $(\perp, \oplus, \bullet\!\!-)$, of $\mathcal{L}$. This we will show turns out to be a model of bi-intuitionistic linear logic with modalities, but also as one of the first categorical models of bi-intuitionistic logic by exploiting the modalities of linear logic to embedded bi-intuitionistic logic inside of bi-intuitionistic linear logic.

The previous model also induces an adjunction:

$$\begin{tikzcd} \mathcal{I} \arrow[r, bend left, "\mathsf{F;H}"] \arrow[r, phantom, "\dashv"] & \mathcal{C} \arrow[l, bend left, "\mathsf{J;G}"] \end{tikzcd}$$

Q: Does this give us the double-negation monad/comonad?

2.1. **Symmetric Monoidal Categories.** We now introduce the necessary definitions related to symmetric monoidal categories that our model will depend on. Most of these definitions are equivalent to the ones given by Benton [2]. In this section we also introduce distributive categories, the notion of co-closer, and finally, the definition of bilinear categories. The reader may wish to simply skim this section, but refer back to it when they encounter a definition or result they do not know.

Definition 1. A **symmetric monoidal category (SMC)** is a category, $\mathcal{M}$, with the following data:

- An object $\top$ of $\mathcal{M}$,
- A bi-functor $\otimes : \mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{M}$,

- The following natural isomorphisms:

$$\begin{aligned}\lambda_A &: \top \otimes A \longrightarrow A\\ \rho_A &: A \otimes \top \longrightarrow A\\ \alpha_{A,B,C} &: (A \otimes B) \otimes C \longrightarrow A \otimes (B \otimes C)\end{aligned}$$

- A symmetry natural transformation:

$$\beta_{A,B} : A \otimes B \longrightarrow B \otimes A$$

- Subject to the following coherence diagrams:

$$\begin{tikzcd}
((A \otimes B) \otimes C) \otimes D \arrow[r, "\alpha_{A,B,C} \otimes \mathsf{id}_D"] \arrow[d, "\alpha_{A \otimes B,C,D}"] & (A \otimes (B \otimes C)) \otimes D \arrow[dd, "\alpha_{A,B \otimes C,D}"] \\
(A \otimes B) \otimes (C \otimes D) \arrow[d, "\alpha_{A,B,C \otimes D}"] & \\
A \otimes (B \otimes (C \otimes D)) & A \otimes ((B \otimes C) \otimes D) \arrow[l, "\mathsf{id}_A \otimes \alpha_{B,C,D}"']
\end{tikzcd}$$

$$\begin{tikzcd}
(A \otimes B) \otimes C \arrow[r, "\alpha_{A,B,C}"] \arrow[d, "\beta_{A,B} \otimes \mathsf{id}_C"] & A \otimes (B \otimes C) \arrow[r, "\beta_{A,B \otimes C}"] & (B \otimes C) \otimes A \arrow[d, "\alpha_{B,C,A}"] \\
(B \otimes A) \otimes C \arrow[r, "\alpha_{B,A,C}"] & B \otimes (A \otimes C) \arrow[r, "\mathsf{id}_B \otimes \beta_{A,C}"] & B \otimes (C \otimes A)
\end{tikzcd}$$

$$\begin{tikzcd}
(A \otimes \top) \otimes B \arrow[rr, "\alpha_{A,\top,B}"] \arrow[dr, "\rho_A"'] & & A \otimes (\top \otimes B) \arrow[dl, "\lambda_B"] \\
& A \otimes B &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes B \arrow[d, "\beta_{A,B}"'] \arrow[dr, "\mathsf{id}_{A \otimes B}"] & \\
B \otimes A \arrow[r, "\beta_{B,A}"'] & A \otimes B
\end{tikzcd}$$

$$\begin{tikzcd}
\top \otimes A \arrow[rr, "\beta_{\top,A}"] \arrow[dr, "\lambda_A"'] & & A \otimes \top \arrow[dl, "\rho_A"] \\
& A &
\end{tikzcd}$$

Categorical modeling implication requires that the model be closed; which can be seen as an internalization of the notion of a morphism.

Definition 2. A **symmetric monoidal closed category (SMCC)** is a symmetric monoidal category, $(\mathcal{M}, \top, \otimes)$, such that, for any object B of $\mathcal{M}$, the functor $- \otimes B : \mathcal{M} \longrightarrow \mathcal{M}$ has a specified right adjoint. Hence, for any objects A and C of $\mathcal{M}$ there is an object $B \multimap C$ of $\mathcal{M}$ and a natural bijection:

$$\mathsf{Hom}_{\mathcal{M}}(A \otimes B, C) \cong \mathsf{Hom}_{\mathcal{M}}(A, B \multimap C)$$

We call the functor $\multimap : \mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{M}$ the internal hom of $\mathcal{M}$.

Symmetric monoidal closed categories can be seen as a model of intuitionistic linear logic with a tensor product and implication. What happens when we take the dual? First, we have the following result:

Lemma 3 (Dual of Symmetric Monoidal Categories)**.** If $(\mathcal{M}, \top, \otimes)$ is a symmetric monoidal category, then $\mathcal{M}^{\mathsf{op}}$ is also a symmetric monoidal category.

The previous result follows from the fact that the structures making up symmetric monoidal categories are isomorphisms, and so naturally taking their opposite will yield another symmetric monoidal category. To emphasize when we are thinking about a symmetric monoidal category in the opposite we use the notion $(\mathcal{M}, \perp, \oplus)$ which gives the suggestion of $\oplus$ corresponding to a disjunctive tensor product which we call the *cotensor* of $\mathcal{M}$. The next definition describes when a symmetric monoidal category is co-closed.

Definition 4. A **symmetric monoidal co-closed category (SMCCC)** is a symmetric monoidal category, $(\mathcal{M}, \perp, \oplus)$, such that, for any object B of $\mathcal{M}$, the functor $- \oplus B : \mathcal{M} \longrightarrow \mathcal{M}$ has a specified left adjoint. Hence, for any objects A and C of $\mathcal{M}$ there is an object $B \bullet\!\!- C$ of $\mathcal{M}$ and a natural bijection:

$$\mathsf{Hom}_{\mathcal{M}}(C, A \oplus B) \cong \mathsf{Hom}_{\mathcal{M}}(B \bullet\!\!- C, A)$$

We call the functor $\bullet\!\!- : \mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{M}$ the internal co-hom of $\mathcal{M}$.

We combine the previous definitions into a single category. First, we define the notion of a distributive category due to Cockett and Seely [4].

Definition 5. We call a symmetric monoidal category, $(\mathcal{M}, \top, \otimes, \perp, \oplus)$, a **distributive category** if there are natural transformations:

$$\begin{array}{l} \delta^L_{A,B,C} : A \otimes (B \oplus C) \longrightarrow (A \otimes B) \oplus C \\ \delta^R_{A,B,C} : (B \oplus C) \otimes A \longrightarrow B \oplus (C \otimes A) \end{array}$$

subject to several coherence diagrams. Due to the large number of coherence diagrams we do not list them here, but they all can be found in Cockett and Seely's paper [4].

Requiring that the tensor and cotensor products have the corresponding right and left adjoints results in the following definition.

Definition 6. A **bilinear category** is a distributive category $(\mathcal{M}, \top, \otimes, \perp, \oplus)$ such that $(\mathcal{M}, \top, \otimes)$ is closed, and $(\mathcal{M}, \perp, \oplus)$ is co-closed. We will denote bi-linear categories by $(\mathcal{M}, \top, \otimes, \multimap, \perp, \oplus, \bullet\!\!-)$.

Originally, Lambek defined bilinear categories to be similar to the previous definition, but the tensor and cotensor were non-commutative [3], however, the bilinear categories given here are. We retain the name in homage to his original work. As we will see below bilinear categories form the core of the work given here, and are of crucial importance.

A symmetric monoidal category is a category with additional structure subject to several coherence diagrams. Thus, an ordinary functor is not enough to capture this structure, and hence, the introduction of symmetric monoidal functors.

Definition 7. Suppose we are given two symmetric monoidal closed categories $(\mathcal{M}_1, \top_1, \otimes_1, \alpha_1, \lambda_1, \rho_1, \beta_1)$ and $(\mathcal{M}_2, \top_2, \otimes_2, \alpha_2, \lambda_2, \rho_2, \beta_2)$. Then a **symmetric monoidal functor** is a functor $F : \mathcal{M}_1 \longrightarrow \mathcal{M}_2$, a map $m_\top : \top_2 \longrightarrow F\top_1$ and a natural transformation $m_{A,B} : FA \otimes_2$

$FB \longrightarrow F(A \otimes_1 B)$ subject to the following coherence conditions:

$$\begin{tikzcd}
(FA \otimes_2 FB) \otimes_2 FC \arrow[r, "\alpha_{2FA,FB,FC}"] \arrow[d, "m_{A,B} \otimes \mathsf{id}_{FC}"] & FA \otimes_2 (FB \otimes_2 FC) \arrow[d, "\mathsf{id}_{FA} \otimes m_{B,C}"] \\
F(A \otimes_1 B) \otimes_2 FC \arrow[d, "m_{A \otimes_1 B, C}"] & FA \otimes_2 F(B \otimes_1 C) \arrow[d, "m_{A, B \otimes_1 C}"] \\
F((A \otimes_1 B) \otimes_1 C) \arrow[r, "F\alpha_{1A,B,C}"] & F(A \otimes_1 (B \otimes_1 C))
\end{tikzcd}$$

$$\begin{tikzcd}
\top_2 \otimes_2 FA \arrow[r, "\lambda_{2FA}"] \arrow[d, "m_\top \otimes \mathsf{id}_{FA}"] & FA \\
F\top_1 \otimes_2 FA \arrow[r, "m_{\top_1, A}"] & F(\top_1 \otimes_1 A) \arrow[u, "F\lambda_{1A}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
FA \otimes_2 \top_2 \arrow[r, "\rho_{2FA}"] \arrow[d, "\mathsf{id}_{FA} \otimes m_\top"] & FA \\
FA \otimes_2 F\top_1 \arrow[r, "m_{A, \top_1}"] & F(A \otimes_1 \top_1) \arrow[u, "F\rho_{1A}"]
\end{tikzcd}$$

$$\begin{tikzcd}
FA \otimes_2 FB \arrow[r, "\beta_{2FA,FB}"] \arrow[d, "m_{A,B}"] & FB \otimes_2 FA \arrow[d, "m_{B,A}"] \\
F(A \otimes_1 B) \arrow[r, "F\beta_{1A,B}"] & F(B \otimes_1 A)
\end{tikzcd}$$

Naturally, since functors are enhanced to handle the additional structure found in a symmetric monoidal category we must also extend natural transformations, and adjunctions.

Definition 8. Suppose $(\mathcal{M}_1, \top_1, \otimes_1)$ and $(\mathcal{M}_2, \top_2, \otimes_2)$ are SMCs, and (F, m) and (G, n) are a symmetric monoidal functors between $\mathcal{M}_1$ and $\mathcal{M}_2$. Then a **symmetric monoidal natural transformation** is a natural transformation, $f : F \longrightarrow G$, subject to the following coherence diagrams:

$$\begin{tikzcd}
FA \otimes_2 FB \arrow[r, "m_{A,B}"] \arrow[d, "f_A \otimes_2 f_B"'] & F(A \otimes_1 B) \arrow[d, "f_{A \otimes_1 B}"] \\
GA \otimes_2 GB \arrow[r, "n_{A,B}"'] & G(A \otimes_1 B)
\end{tikzcd}
\qquad
\begin{tikzcd}
F\top_1 \arrow[rr, "f_{\top_1}"] & & G\top_1 \\
& \top_2 \arrow[ul, "m_{\top_1}"] \arrow[ur, "n_{\top_1}"'] &
\end{tikzcd}$$

Definition 9. Suppose $(\mathcal{M}_1, \top_1, \otimes_1)$ and $(\mathcal{M}_2, \top_2, \otimes_2)$ are SMCs, and (F, m) is a symmetric monoidal functor between $\mathcal{M}_1$ and $\mathcal{M}_2$ and (G, n) is a symmetric monoidal functor between $\mathcal{M}_2$ and $\mathcal{M}_1$. Then a **symmetric monoidal adjunction** is an ordinary adjunction $\mathcal{M}_1 : F \dashv G : \mathcal{M}_2$ such that the unit, $\varepsilon : A \to GFA$, and the counit, $\eta_A : FGA \to A$, are symmetric monoidal natural transformations.

Thus, the following diagrams must commute:

$$\begin{tikzcd} FGA \otimes_1 FGB \arrow[r, "\mathsf{q}_{A,B}"] \arrow[dr, "\eta_A \otimes_1 \eta_B"'] & FG(A \otimes_1 B) \arrow[d, "\eta_{A \otimes_1 B}"] \\ & A \otimes_1 B \end{tikzcd} \qquad \begin{tikzcd} FG\top_1 \arrow[rr, "\eta_{\top_1}"] & & \top_1 \\ & \top_1 \arrow[ul, "\mathsf{q}_{\top_1}"] \arrow[ur, equal] \end{tikzcd}$$

$$\begin{tikzcd} & A \otimes_2 B \arrow[dl, "\varepsilon_A \otimes_2 \varepsilon_B"'] \arrow[d, "\varepsilon_{A \otimes_2 B}"] \\ GFA \otimes_2 GFB \arrow[r, "\mathsf{p}_{A,B}"'] & GF(A \otimes_2 B) \end{tikzcd} \qquad \begin{tikzcd} \top_2 \arrow[rr, "\varepsilon_{\top_2}"] \arrow[dr, equal] & & GF\top_2 \\ & \top_2 \arrow[ur, "p_{\top_2}"'] \end{tikzcd}$$

Note that p and q exist because (FG, q) and (GF, p) are symmetric monoidal functors.

We will be defining, and making use of the of-course and why-not exponentials from linear logic, but these correspond to a symmetric monoidal comonad and a symmetric monoidal monad respectively, and so we define these concepts next. In addition, whenever we have a symmetric monoidal adjunction, we immediately obtain a symmetric monoidal comonad on the left, and a symmetric monoidal monad on the right.

Definition 10. A **symmetric monoidal monad** on a symmetric monoidal category C is a triple (T, η, μ), where (T, n) is a symmetric monoidal endofunctor on C, $\eta_A : A \longrightarrow TA$ and $\mu_A : T^2A \to TA$ are symmetric monoidal natural transformations, which make the following diagrams commute:

$$\begin{tikzcd} T^3A \arrow[r, "\mu_{TA}"] \arrow[d, "T\mu_A"'] & T^2A \arrow[d, "\mu_A"] \\ T^2A \arrow[r, "\mu_A"'] & TA \end{tikzcd} \qquad \begin{tikzcd} & TA & \\ TA \arrow[ur, equal] \arrow[r, "\eta_{TA}"'] & T^2A \arrow[u, "\mu_A"] & TA \arrow[ul, equal] \arrow[l, "T\eta_A"] \end{tikzcd}$$

The assumption that η and μ are symmetric monoidal natural transformations amount to the following diagrams commuting:

$$\begin{tikzcd} & A \otimes B \arrow[dl, "\eta_A \otimes \eta_B"'] \arrow[d, "\eta_A"] \\ TA \otimes TB \arrow[r, "\mathsf{n}_{A,B}"'] & T(A \otimes B) \end{tikzcd} \qquad \begin{tikzcd} \top \arrow[rr, "\eta_\top"] \arrow[dr, equal] & & T\top \\ & \top \arrow[ur, "\mathsf{n}_\top"'] \end{tikzcd}$$

$$
\begin{tikzcd}
T^2A \otimes T^2B \arrow[r, "\mathsf{n}_{TA,TB}"] \arrow[d, "\mu_A \otimes \mu_B"] & T(TA \otimes TB) \arrow[r, "T\mathsf{n}_{A,B}"] & T^2(A \otimes B) \arrow[d, "\mu_{A\otimes B}"] \\
TA \otimes TB \arrow[rr, "\mathsf{n}_{A,B}"] & & T(A \otimes B)
\end{tikzcd}
\qquad
\begin{tikzcd}
\top \arrow[r, "\mathsf{n}_\top"] \arrow[d, "\mathsf{n}_\top"] & T\top \arrow[d, "T\mathsf{n}_\top"] \\
T\top & T^2\top \arrow[l, "\mu_\top"]
\end{tikzcd}
$$

Finally the dual concept, of a symmetric monoidal comonad.

Definition 11. A **symmetric monoidal comonad** on a symmetric monoidal category C is a triple (T, ε, δ), where (T, m) is a symmetric monoidal endofunctor on C, $\varepsilon_A : TA \longrightarrow A$ and $\delta_A : TA \to T^2A$ are symmetric monoidal natural transformations, which make the following diagrams commute:

$$
\begin{tikzcd}
TA \arrow[r, "\delta_A"] \arrow[d, "\delta_A"] & T^2A \arrow[d, "T\delta_A"] \\
T^2A \arrow[r, "\delta_{TA}"] & T^3A
\end{tikzcd}
\qquad
\begin{tikzcd}
& TA \arrow[dl, equal] \arrow[d, "\delta_A"] \arrow[dr, equal] & \\
TA & T^2A \arrow[l, "\varepsilon_{TA}"] \arrow[r, "T\varepsilon_A"'] & TA
\end{tikzcd}
$$

The assumption that ε and δ are symmetric monoidal natural transformations amount to the following diagrams commuting:

$$
\begin{tikzcd}
TA \otimes TB \arrow[r, "\mathsf{m}_{A,B}"] \arrow[dr, "\varepsilon_A \otimes \varepsilon_B"'] & T(A \otimes B) \arrow[d, "\varepsilon_{A\otimes B}"] \\
& A \otimes B
\end{tikzcd}
\qquad
\begin{tikzcd}
T\top \arrow[rr, "\mathsf{m}_\top"] & & \top \arrow[dl, equal] \\
& \top \arrow[ul, "\varepsilon_\top"] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
TA \otimes TB \arrow[rr, "\mathsf{m}_{A,B}"] \arrow[d, "\delta_A \otimes \delta_B"] & & T(A \otimes B) \arrow[d, "\delta_{A\otimes B}"] \\
T^2A \otimes T^2B \arrow[r, "\mathsf{m}_{TA,TB}"'] & T(TA \otimes TB) \arrow[r, "T\mathsf{m}_{A,B}"'] & T^2(A \otimes B)
\end{tikzcd}
\qquad
\begin{tikzcd}
\top \arrow[r, "\mathsf{m}_\top"] \arrow[d, "\mathsf{m}_\top"] & T\top \arrow[d, "\delta_\top"] \\
T\top \arrow[r, "T\mathsf{m}_\top"'] & T^2\top
\end{tikzcd}
$$

2.2. **Cartesian Closed and Co-Cartesian Co-Closed Categories.** The notion of a cartesian closed category is well-known, but for completeness we define them here. However, their dual is lesser known, especially in computer science, and so we given their full definition. We also review some know results concerning co-cartesian co-closed categories and their union called bi-cartesian bi-closed categories.

Definition 12. A **cartesian category** is a category, $(C, 1, \times)$, with an object, 1, and a bi-functor, $\times : C \times C \longrightarrow C$, such that for any object A there is exactly one morphism $\diamond : A \to 1$, and for any morphisms $f : C \longrightarrow A$ and $g : C \longrightarrow B$ there is a morphism $\langle f, g\rangle : C \to A \times B$ subject to the following diagram:

$$\begin{tikzcd}
& C \arrow[dl, "f"'] \arrow[d, "\langle f,g\rangle"] \arrow[dr, "g"] & \\
A & A \times B \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & B
\end{tikzcd}$$

A cartesian category models conjunction by the product functor, $\times : C \times C \longrightarrow C$, and the unit of conjunction by the terminal object. As we mention above modeling implication requires closer, and since it is well-known that any cartesian category is also a symmetric monoidal category the definition of closer for a cartesian category is the same as the definition of closer for a symmetric monoidal category (Definition 4). We denote the internal hom for cartesian closed categories by $A \to B$.

The dual of a cartesian category is a co-cartesian category. They are a model of intuitionistic logic with disjunction and its unit.

Definition 13. A **co-cartesian category** is a category, $(C, 0, +)$, with an object, 0, and a bi-functor, $+ : C \times C \longrightarrow C$, such that for any object A there is exactly one morphism $\Box : 0 \to A$, and for any morphisms $f : A \longrightarrow C$ and $g : B \longrightarrow C$ there is a morphism $[f, g] : A + B \longrightarrow C$ subject to the following diagram:

$$\begin{tikzcd}
& C & \\
A \arrow[ur, "f"] \arrow[r, "\iota_1"'] & A + B \arrow[u, "{[f,g]}"] & B \arrow[ul, "g"'] \arrow[l, "\iota_2"]
\end{tikzcd}$$

Co-closer, just like closer for cartesian categories, is defined in the same way that co-closer is defined for symmetric monoidal categories, because co-cartesian categories are also symmetric monoidal categories. Thus, a co-cartesian category is co-closed if there is a specified left-adjoint, which we denote $S - T$, to the co-product.

There are many examples of co-cartesian co-closed categories. Basically, any interesting cartesian category has an interesting dual, and hence, induces an interesting co-cartesian co-closed category. The opposite of the category of sets and functions between them is isomorphic to the category of complete atomic boolean algebras, and both of which, are examples of co-cartesian co-closed categories. As we mentioned above bi-linear categories [3] are models of bi-linear logic where the left adjoint to the co-tensor models co-implication. Similarly, co-cartesian co-closed categories model co-intuitionistic logic with disjunction and intuitionistic co-implication [5, 1].

3. Mixed Linear/Non-Linear Bi-intuitionistic Logic: BiLNL Logic

Following Benton's [2] lead we can define a mixed linear/non-linear bi-intuitionistic logic, called BiLNL logic, based on the categorical model given in the previous section. BiLNL logic consists of three fragments: an intuitionistic fragment, a co-intuitionistic fragment, and a linear bi-intuitionistic core fragment. This formalization allows us to have more control on the mixture of the intuitionistic and co-intuitionistic fragments so as to allow for a proper categorical model. Each of the fragments are related through a syntactic formalization of the adjoint functors from the BiLNL model. First, we define the syntax of BiLNL logic, and then discuss the inference rules for each fragment.

Definition 14. The syntax for BiLNL logic is defined as follows:

$$
\begin{array}{rl}
\text{(Worlds)} & W ::= w_1 \mid \cdots \mid w_i \\
\text{(Graphs)} & G ::= w_1 \leq w_2 \mid G_1, G_2 \\
\text{(Intuitionistic Formulas)} & X, Y, Z ::= 1 \mid X \times Y \mid X \to Y \mid \mathsf{G} A \\
\text{(Co-intuitionistic Formulas)} & R, S, T ::= 0 \mid S + T \mid S - T \mid \mathsf{H} A \\
\text{(Linear Bi-intuitionistic Formulas)} & A, B, C ::= \top \mid \perp \mid A \otimes B \mid A \oplus B \mid A \multimap B \mid A \bullet\!\!- B \mid \mathsf{F} X \mid \mathsf{J} S \\
\text{(Intuitionistic Contexts)} & \Theta ::= \cdot \mid X@w \mid \Theta_1, \Theta_2 \\
\text{(Co-intuitionistic Contexts)} & \Psi ::= \cdot \mid R@w \mid \Psi_1, \Psi_2 \\
\text{(Linear Bi-intuitionistic Contexts)} & \Gamma, \Delta ::= \cdot \mid A@w \mid \Gamma_1, \Gamma_2
\end{array}
$$

Worlds may also be denoted by (potentially subscripted) n, m, and o.

Sequents have the following syntax:

$$
\begin{array}{rl}
\text{(Intuitionistic Sequents)} & G; \Theta \vdash_{\mathsf{I}} X@w \\
\text{(Co-intuitionistic Sequents)} & G; R@w \vdash_{\mathsf{C}} \Psi \\
\text{(LNL Bi-intuitionistic Sequents)} & G; \Theta \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi
\end{array}
$$

The syntax of intuitionistic and co-intuitionistic formulas are typical. I denote co-implication by $S - T$, but all the other connectives are the usual ones. Linear bi-intuitionistic formulas are denoted in somewhat of a non-traditional style. I denote the unit of tensor by $\top$ instead of the usual 1, which is the unit of intuitionistic conjunction, in addition, I denote par by $A \oplus B$, instead of $A \parr B$. Lastly, I denote linear co-implication by $A \bullet\!\!- B$ to emphasize its duality with linear implication $A \multimap B$. Each syntactic category of formulas contains the respective functor from the BiLNL model, and thus, we should view F and H as the left adjoints to G and J respectively.

Formulas in each type of context are annotated with a world, and each sequent is annotated with a graph. These graphs are syntactic representations of Kripke models and are used to enforce intuitionism. They were first used in bi-intuitionistic logic by Pinto and Uustalu [7] to enforce intuitionism in their logic L. In fact, we can see the linear core of BiLNL logic as the linear version of L. The beauty of this type of formalization and the reason why this style of logic was used by Pinto and Uustalu is that the logic L, and as well as BiLNL logic, are complete for cut-free bi-intuitionistic proofs. This was a new result of Pinto and Uustalu, because earlier formalizations of bi-intuitionistic logic [5] used the Dragalin restriction [6] to enforce intuitionism, but this results in a failure of cut-elimination [8, 7].

The expert reader will notice that it is not necessary to annotate the sequents of intuitionistic and co-intuitionistic logic with graphs and worlds to enforce intuitionism and co-intuitionism respectively. It is well known that restricting the right and left contexts two a single formula enforces intuitionism and co-intuitionism respectively. However, when mixing these two fragments with the linear core, which requires the graphs to be intuitionistic, it is easier if they are annotated. If they were not, then a seemingly complex world inference system would need to be designed to add the world constraints before mixing with the linear core, and it is currently an open problem whether this can be done. Thus, with respect to intuitionistic and co-intuitionistic logic the graph and world annotations can be seen as book keeping.

A second fact an expert reader will notice is that Kripke models are a relational model of intuitionistic logic, but not intuitionistic linear logic. This is okay, because Kripke models enforce intuitionism and not linearity, but it is well known how to enforce linearity syntactically in the definition of the inference rules. It is also well-known that even in linear logic if sequents have multiple hypothesis and multiple conclusions the logic becomes classical. Thus, in BiLNL logic we combine both of these tools to enforce both intuitionism and linearity. We can simply view the graphs as an over approximation of the relational constraints necessary to enforce both intuitionism and linearity. This also makes it easier to embed Pinto and Uustalu's logic in BiLNL logic as we will do in Section 4.

Sequents for the linear core have the form $G; \Theta \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi$. Similarly to the sequents of Benton's LNL logic [2], each context is separated for readability, but should actually be understood as being able to be mixed, that is, the contexts Θ and Γ could be a single context, and so could Δ and Ψ. The sequent:

$$G; X_1@w_1, \ldots, X_i@w_i \mid A_1@n_1, \ldots, A_j@n_j \vdash_{\mathsf{L}} B_1@m_1, \ldots, B_k@m_k \mid R_1@o_1, \ldots, R_l@o_l$$

will be interpreted in a BiLNL model by a morphism of the following form:

$$\mathsf{F}\,X_1 \otimes \cdots \otimes \mathsf{F}\,X_i \otimes A_1 \otimes \cdots \otimes A_j \xrightarrow{f} B_1 \oplus \cdots \oplus B_k \oplus \mathsf{J}\,R_1 \oplus \cdots \oplus \mathsf{J}\,R_l$$

Thus, intuitionistic formulas in the linear core can be viewed as being under the left adjoint F, and co-intuitionistic formulas as being under the right adjoint J. This implies that even in the model all types of formulas can be freely mixed.

As said above, BiLNL logic consists of three fragments: an intuitionistic fragment, a co-intuitionistic fragment, and a linear bi-intuitionistic fragment. Their relationships are captured by the following diagram:

$$\begin{tikzcd} \mathsf{INT} \arrow[r, bend left, "\mathsf{F}"] & \mathsf{BiL} \arrow[l, bend left, "\mathsf{G}"] \arrow[r, bend left, "\mathsf{H}"] & \mathsf{coINT} \arrow[l, bend left, "\mathsf{J}"] \end{tikzcd}$$

The inference rules for the intuitionistic fragment are defined in Figure 1, and the inference rules for the co-intuitionistic fragment can be found in Figure 2. Since the functor G translates the linear fragment over to intuitionistic logic, then the intuitionistic fragment must contain an inference rule (I_GR) for this functor, and the same goes for the co-intuitionistic fragment and the functor H (C_HL). Thus, these are the only interesting rules of those two fragments. Each fragment contains rules for working with the world constraints on sequents. For example, the rules I_RL and I_TS make the relation on worlds, $w_1 \leq w_2$, a preorder, and the rules I_ML and I_MR correspond to monotonicity on formulas. Similar rules exist for the other two fragments as well.

The linear core of BiLNL logic consists of a large number of rules, and for this reason they have been broken up into multiple figures. The inference rules for reflexivity, transitivity, and monotonicity are defined in Figure 3. Since the linear core is a mixed LNL logic there are monotonicity rules for intuitionistic hypothesis, and co-intuitionistic conclusions. The structural rules are defined in Figure 4. The most interesting rules are weakening and contraction for both intuitionistic hypothesis and co-intuitionistic conclusions. The identity and cut inference rules can be found in Figure 5. Similarly to Benton's LNL logic [2], we also have cut rules involving the different fragments:

$$\frac{G; \Theta_2 \vdash_{\mathsf{I}} X@w \qquad G; \Theta_1, X@w \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi}{G; \Theta_1, \Theta_2 \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi}\ \text{L\_ICUT}$$

$$\frac{G; \Theta \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi_1, S@w \qquad G; S@w \vdash_{\mathsf{C}} \Psi_2}{G; \Theta \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi_1, \Psi_2}\ \text{L\_CCUT}$$

The first rule is the same rule in LNL logic, but the second is new and cuts a co-intuitionistic conclusion in the linear core against the hypothesis proven in the co-intuitionistic fragment.

The rules for conjunction and tensor, and disjunction and par can be found in Figure 6 and Figure 7 respectively. Just as we have said above, the new rules are the inference rules for disjunction in the co-intuitionistic context. The inference rules for intuitionistic implication and linear

$$\frac{G, w_1 \leq w_1; \Theta \vdash_{\mathsf{I}} Y@w_2}{G; \Theta \vdash_{\mathsf{I}} Y@w_2}\ \text{I\_RL} \qquad \frac{w_1 G w_2 \quad w_2 G w_3 \quad G, w_1 \leq w_3; \Theta \vdash_{\mathsf{I}} Y@w}{G; \Theta \vdash_{\mathsf{I}} Y@w}\ \text{I\_TS}$$

$$\frac{w_1 G w_2 \quad G; \Theta, X@w_1, X@w_2 \vdash_{\mathsf{I}} Y@w}{G; \Theta, X@w_1 \vdash_{\mathsf{I}} Y@w}\ \text{I\_ML} \qquad \frac{w_2 G w_1 \quad G; \Theta \vdash_{\mathsf{I}} Y@w_2}{G; \Theta \vdash_{\mathsf{I}} Y@w_1}\ \text{I\_MR}$$

$$\frac{}{G; Y@w \vdash_{\mathsf{I}} Y@w}\ \text{I\_ID} \qquad \frac{G; \Theta_2 \vdash_{\mathsf{I}} X@w_2 \quad G; \Theta_1, X@w_2 \vdash_{\mathsf{I}} Z@w_1}{G; \Theta_1, \Theta_2 \vdash_{\mathsf{I}} Z@w_1}\ \text{I\_CUT}$$

$$\frac{G; \Theta \vdash_{\mathsf{I}} Y@w_1}{G; \Theta, X@w_2 \vdash_{\mathsf{I}} Y@w_1}\ \text{I\_WK} \qquad \frac{G; \Theta, X@w_2, X@w_2 \vdash_{\mathsf{I}} Y@w_1}{G; \Theta, X@w_2 \vdash_{\mathsf{I}} Y@w_1}\ \text{I\_CR}$$

$$\frac{G; \Theta_1, X@w_1, Y@w_2, \Theta_2 \vdash_{\mathsf{I}} Z@w}{G; \Theta_1, Y@w_2, X@w_1, \Theta_2 \vdash_{\mathsf{I}} Z@w}\ \text{I\_EX} \qquad \frac{G; \Theta \vdash_{\mathsf{I}} Y@w_1}{G; \Theta, 1@w_2 \vdash_{\mathsf{I}} Y@w_1}\ \text{I\_TL}$$

$$\frac{}{G; \Theta \vdash_{\mathsf{I}} 1@w}\ \text{I\_TR} \qquad \frac{G; \Theta, X@w_1, Y@w_1 \vdash_{\mathsf{I}} Z@w_2}{G; \Theta, (X \times Y)@w_1 \vdash_{\mathsf{I}} Z@w_2}\ \text{I\_PL}$$

$$\frac{G; \Theta_1 \vdash_{\mathsf{I}} X@w \quad G; \Theta_2 \vdash_{\mathsf{I}} Y@w}{G; \Theta_1, \Theta_2 \vdash_{\mathsf{I}} (X \times Y)@w}\ \text{I\_PR}$$

$$\frac{w_1 G w_2 \quad G; \Theta_2 \vdash_{\mathsf{I}} X@w_2 \quad G; \Theta_1, Y@w_2 \vdash_{\mathsf{I}} Z@w}{G; \Theta_1, \Theta_2, (X \rightarrow Y)@w_1 \vdash_{\mathsf{I}} Z@w}\ \text{I\_IL}$$

$$\frac{w_2 \notin |G|, |\Theta| \quad G, w_1 \leq w_2; \Theta, X@w_2 \vdash_{\mathsf{I}} Y@w_2}{G; \Theta \vdash_{\mathsf{I}} (X \rightarrow Y)@w_1}\ \text{I\_IR} \qquad \frac{G; \Theta \mid \cdot \vdash_{\mathsf{L}} A@w \mid \cdot}{G; \Theta \vdash_{\mathsf{I}} \mathsf{G}A@w}\ \text{I\_GR}$$

Figure 1: Inference Rules for BiLNL Logic: Intuitionistic Fragment

implication, and co-intuitionistic co-implication and linear co-implication are defined in Figure 8 and Figure 9. The most interesting rules between these are the rules for co-implication.

Lastly, the inference rules for each of the functors are defined in Figure 10. Using these rules it is possible to move between the fragments. The most restrictive rules are when coming from the intuitionistic or co-intuitionistic fragment into the linear core, or vice versa, because the sequents must only contain data that is movable or consistency could fail.

4. Embedding Bi-Intuitionistic Logic in BiLNL Logic

TODO

$$\frac{G, w_1 \leq w_1; S@w_2 \vdash_{\mathsf{C}} \Psi}{G; S@w_2 \vdash_{\mathsf{C}} \Psi}\ \text{C\_RL} \qquad \frac{w_1 G w_2 \quad w_2 G w_3 \quad G, w_1 \leq w_3; S@w \vdash_{\mathsf{C}} \Psi}{G; S@w \vdash_{\mathsf{C}} \Psi}\ \text{C\_TS}$$

$$\frac{w_1 G w_2 \quad G; S@w_2 \vdash_{\mathsf{C}} \Psi}{G; S@w_1 \vdash_{\mathsf{C}} \Psi}\ \text{C\_ML} \qquad \frac{w_2 G w_1 \quad G; S@w \vdash_{\mathsf{C}} T@w_2, T@w_1, \Psi}{G; S@w \vdash_{\mathsf{C}} T@w_1, \Psi}\ \text{C\_MR}$$

$$\frac{}{G; S@w \vdash_{\mathsf{C}} S@w}\ \text{C\_ID} \qquad \frac{G; S@w_1 \vdash_{\mathsf{C}} T@w_2, \Psi_2 \qquad G; T@w_2 \vdash_{\mathsf{C}} \Psi_1}{G; S@w_1 \vdash_{\mathsf{C}} \Psi_1, \Psi_2}\ \text{C\_CUT}$$

$$\frac{G; S@w_1 \vdash_{\mathsf{C}} \Psi}{G; S@w_1 \vdash_{\mathsf{C}} T@w_2, \Psi}\ \text{C\_WK} \qquad \frac{G; S@w_1 \vdash_{\mathsf{C}} T@w_2, T@w_2, \Psi}{G; S@w_1 \vdash_{\mathsf{C}} T@w_2, \Psi}\ \text{C\_CR}$$

$$\frac{G; R@w \vdash_{\mathsf{C}} \Psi_1, S@w_1, T@w_2, \Psi_2}{G; R@w \vdash_{\mathsf{C}} \Psi_1, T@w_2, S@w_1, \Psi_2}\ \text{C\_EX} \qquad \frac{}{G; 0@w \vdash_{\mathsf{C}} \Psi}\ \text{C\_FL}$$

$$\frac{G; S@w_1 \vdash_{\mathsf{C}} \Psi}{G; S@w_1 \vdash_{\mathsf{C}} 0@w_2, \Psi}\ \text{C\_FR} \qquad \frac{G; S@w \vdash_{\mathsf{C}} \Psi_1 \qquad G; T@w \vdash_{\mathsf{C}} \Psi_2}{G; S + T@w \vdash_{\mathsf{C}} \Psi_1, \Psi_2}\ \text{C\_DL}$$

$$\frac{G; R@w_1 \vdash_{\mathsf{C}} S@w_2, T@w_2, \Psi}{G; R@w_1 \vdash_{\mathsf{C}} S + T@w_2, \Psi}\ \text{C\_DR} \qquad \frac{w_2 \notin |G|, |\Psi| \quad G, w_2 \leq w_1; S@w_2 \vdash_{\mathsf{C}} T@w_2, \Psi}{G; S - T@w_1 \vdash_{\mathsf{C}} \Psi}\ \text{C\_SL}$$

$$\frac{w_2 G w_1 \quad G; R@w \vdash_{\mathsf{C}} S@w_2, \Psi_2 \qquad G; T@w_2 \vdash_{\mathsf{C}} \Psi_1}{G; R@w \vdash_{\mathsf{C}} S - T@w_1, \Psi_1, \Psi_2}\ \text{C\_SR} \qquad \frac{G; \cdot \mid A@w \vdash_{\mathsf{L}} \cdot \mid \Psi}{G; \mathsf{H} A@w \vdash_{\mathsf{C}} \Psi}\ \text{C\_HL}$$

Figure 2: Inference Rules for BiLNL Logic: Co-intuitionistic Fragment

$$\frac{G, w \leq w; \Theta \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi}{G; \Theta \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi}\ \text{L\_RL} \qquad \frac{w_1 G w_2 \quad w_2 G w_3 \quad G, w_1 \leq w_3; \Theta \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi}{G; \Theta \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi}\ \text{L\_TS}$$

$$\frac{w_1 G w_2 \quad G; \Theta \mid \Gamma, A@w_1, A@w_2 \vdash_{\mathsf{L}} \Delta \mid \Psi}{G; \Theta \mid \Gamma, A@w_1 \vdash_{\mathsf{L}} \Delta \mid \Psi}\ \text{L\_ML} \qquad \frac{w_2 G w_1 \quad G; \Theta \mid \Gamma \vdash_{\mathsf{L}} A@w_2, A@w_1, \Delta \mid \Psi}{G; \Theta \mid \Gamma \vdash_{\mathsf{L}} A@w_1, \Delta \mid \Psi}\ \text{L\_MR}$$

$$\frac{w_1 G w_2 \quad G; \Theta, X@w_1, X@w_2 \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi}{G; \Theta, X@w_1 \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi}\ \text{L\_IML} \qquad \frac{w_2 G w_1 \quad G; \Theta \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid T@w_2, T@w_1, \Psi}{G; \Theta \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid T@w_1, \Psi}\ \text{L\_CMR}$$

Figure 3: Inference Rules for BiLNL Logic: Abstract Kripke Graph Rules

$$\frac{G;\Theta \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi}{G;\Theta, X@w \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi}\ \text{L\_wkL} \qquad \frac{G;\Theta \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi}{G;\Theta \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid S@w, \Psi}\ \text{L\_wkR}$$

$$\frac{G;\Theta, X@w, X@w \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi}{G;\Theta, X@w \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi}\ \text{L\_ctrL} \qquad \frac{G;\Theta \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid S@w, S@w, \Psi}{G;\Theta \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid S@w, \Psi}\ \text{L\_ctrR}$$

$$\frac{G;\Theta \mid \Gamma_1, A@w_1, B@w_2, \Gamma_2 \vdash_{\mathsf{L}} \Delta \mid \Psi}{G;\Theta \mid \Gamma_1, B@w_2, A@w_1, \Gamma_2 \vdash_{\mathsf{L}} \Delta \mid \Psi}\ \text{L\_exL}$$

$$\frac{G;\Theta \mid \Gamma \vdash_{\mathsf{L}} \Delta_1, A@w_1, B@w_2, \Delta_2 \mid \Psi}{G;\Theta \mid \Gamma \vdash_{\mathsf{L}} \Delta_1, B@w_2, A@w_1, \Delta_2 \mid \Psi}\ \text{L\_exR}$$

$$\frac{G;\Theta_1, X@w_1, Y@w_2, \Theta_2 \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi}{G;\Theta_1, Y@w_2, X@w_1, \Theta_2 \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi}\ \text{L\_IexL}$$

$$\frac{G;\Theta \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi_1, S@w_1, T@w_2, \Psi_2}{G;\Theta \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi_1, T@w_2, S@w_1, \Psi_2}\ \text{L\_CexL}$$

Figure 4: Inference Rules for BiLNL Logic: Structural Rules

$$\frac{}{G;\cdot \mid A@w \vdash_{\mathsf{L}} A@w \mid \cdot}\ \text{L\_id}$$

$$\frac{G;\Theta_1 \mid \Gamma_1 \vdash_{\mathsf{L}} A@w, \Delta_2 \mid \Psi_1 \qquad G;\Theta_2 \mid A@w, \Gamma_2 \vdash_{\mathsf{L}} \Delta_1 \mid \Psi_2}{G;\Theta_1, \Theta_2 \mid \Gamma_1, \Gamma_2 \vdash_{\mathsf{L}} \Delta_1, \Delta_2 \mid \Psi_1, \Psi_2}\ \text{L\_cut}$$

$$\frac{G;\Theta_2 \vdash_{\mathsf{I}} X@w \qquad G;\Theta_1, X@w \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi}{G;\Theta_1, \Theta_2 \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi}\ \text{L\_Icut}$$

$$\frac{G;\Theta \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi_1, S@w \qquad G; S@w \vdash_{\mathsf{C}} \Psi_2}{G;\Theta \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi_1, \Psi_2}\ \text{L\_Ccut}$$

Figure 5: Inference Rules for BiLNL Logic: Identity and Cut Rules

5. BiLNL Term Assignment

TODO

6. Related Work

TODO

7. Conclusion

TODO

$$\dfrac{G;\Theta \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi}{G;\Theta \mid \Gamma, \top@w \vdash_{\mathsf{L}} \Delta \mid \Psi}\ \text{L\_IL} \qquad \dfrac{}{G;\cdot \mid \cdot \vdash_{\mathsf{L}} \top@w \mid \cdot}\ \text{L\_IR}$$

$$\dfrac{G;\Theta_1, X@w, Y@w, \Theta_2 \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi}{G;\Theta_1, X \times Y@w, \Theta_2 \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi}\ \text{L\_cL}$$

$$\dfrac{G;\Theta \mid \Gamma_1, A@w, B@w, \Gamma_2 \vdash_{\mathsf{L}} \Delta \mid \Psi}{G;\Theta \mid \Gamma_1, A \otimes B@w, \Gamma_2 \vdash_{\mathsf{L}} \Delta \mid \Psi}\ \text{L\_tL}$$

$$\dfrac{G;\Theta_1 \mid \Gamma_1 \vdash_{\mathsf{L}} A@w, \Delta_1 \mid \Psi_1 \qquad G;\Theta_2 \mid \Gamma_2 \vdash_{\mathsf{L}} B@w, \Delta_2 \mid \Psi_2}{G;\Theta_1, \Theta_2 \mid \Gamma_1, \Gamma_2 \vdash_{\mathsf{L}} A \otimes B@w, \Delta_1, \Delta_2 \mid \Psi_1, \Psi_2}\ \text{L\_tR}$$

Figure 6: Inference Rules for BiLNL Logic: Conjunction and Tensor Rules

$$\dfrac{}{G;\cdot \mid \perp @w \vdash_{\mathsf{L}} \cdot \mid \cdot}\ \text{L\_flL} \qquad \dfrac{G;\Theta \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi}{G;\Theta \mid \Gamma \vdash_{\mathsf{L}} \perp @w, \Delta \mid \Psi}\ \text{L\_flR}$$

$$\dfrac{G;\Theta \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi_1, S@w, T@w, \Psi_2}{G;\Theta \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi_1, S + T@w, \Psi_2}\ \text{L\_dR}$$

$$\dfrac{G;\Theta_1 \mid \Gamma_1, A@w \vdash_{\mathsf{L}} \Delta_1 \mid \Psi_1 \qquad G;\Theta_2 \mid \Gamma_2, B@w \vdash_{\mathsf{L}} \Delta_2 \mid \Psi_2}{G;\Theta_1, \Theta_2 \mid \Gamma_1, \Gamma_2, A \oplus B@w \vdash_{\mathsf{L}} \Delta_1, \Delta_2 \mid \Psi_1, \Psi_2}\ \text{L\_pL}$$

$$\dfrac{G;\Theta \mid \Gamma \vdash_{\mathsf{L}} \Delta_1, A@w, B@w, \Delta_2 \mid \Psi}{G;\Theta \mid \Gamma \vdash_{\mathsf{L}} \Delta_1, A \oplus B@w, \Delta_2 \mid \Psi}\ \text{L\_pR}$$

Figure 7: Inference Rules for BiLNL Logic: Disjunction and Par Rules

$$\dfrac{\begin{array}{l} w_1 G w_2 \\ G;\Theta_1 \mid \Gamma_1 \vdash_{\mathsf{L}} A@w_2, \Delta_1 \mid \Psi_1 \qquad G;\Theta_2 \mid \Gamma_2, B@w_2 \vdash_{\mathsf{L}} \Delta_2 \mid \Psi_2 \end{array}}{G;\Theta_1, \Theta_2 \mid \Gamma_1, \Gamma_2, A \multimap B@w_1 \vdash_{\mathsf{L}} \Delta_1, \Delta_2 \mid \Psi_1, \Psi_2}\ \text{L\_ImpL}$$

$$\dfrac{\begin{array}{l} w_2 \notin |G|, |\Theta|, |\Gamma|, |\Delta|, |\Psi| \\ G, w_1 \leq w_2; \Theta \mid \Gamma, A@w_2 \vdash_{\mathsf{L}} B@w_2, \Delta \mid \Psi \end{array}}{G;\Theta \mid \Gamma \vdash_{\mathsf{L}} A \multimap B@w_1, \Delta \mid \Psi}\ \text{L\_ImpR}$$

$$\dfrac{\begin{array}{l} w_1 G w_2 \\ G;\Theta_1 \vdash_{\mathsf{I}} X@w_2 \qquad G;\Theta_2, Y@w_2 \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi \end{array}}{G;\Theta_1, \Theta_2, X \to Y@w_1 \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi}\ \text{L\_IImpL}$$

Figure 8: Inference Rules for BiLNL Logic: Implication Rules

$$\frac{\begin{array}{l} w_2 \notin |G|, |\Theta|, |\Gamma|, |\Delta|, |\Psi| \\ G, w_2 \leq w_1; \Theta \mid \Gamma, A@w_2 \vdash_{\mathsf{L}} B@w_2, \Delta \mid \Psi \end{array}}{G; \Theta \mid \Gamma, A \bullet\!\!- B@w_1 \vdash_{\mathsf{L}} \Delta \mid \Psi} \text{ L\_sL}$$

$$\frac{\begin{array}{l} w_2 G w_1 \\ G; \Theta_1 \mid \Gamma_1 \vdash_{\mathsf{L}} A@w_2, \Delta_1 \mid \Psi_1 \qquad G; \Theta_2 \mid \Gamma_2, B@w_2 \vdash_{\mathsf{L}} \Delta_2 \mid \Psi_2 \end{array}}{G; \Theta_1, \Theta_2 \mid \Gamma_1, \Gamma_2 \vdash_{\mathsf{L}} A \bullet\!\!- B@w_1, \Delta_1, \Delta_2 \mid \Psi_1, \Psi_2} \text{ L\_sR}$$

$$\frac{\begin{array}{l} w_2 G w_1 \\ G; \Theta \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid S@w_2, \Psi_1 \qquad G; T@w_2 \vdash_{\mathsf{C}} \Psi_2 \end{array}}{G; \Theta \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid S - T@w_1, \Psi_1, \Psi_2} \text{ L\_CsR}$$

Figure 9: Inference Rules for BiLNL Logic: Co-implication Rules

$$\frac{G; \Theta, X@w \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi}{G; \Theta \mid \Gamma, \mathsf{F} X@w \vdash_{\mathsf{L}} \Delta \mid \Psi} \text{ L\_FL} \qquad \frac{G; \Theta \vdash_{\mathsf{I}} X@w}{G; \Theta \mid \cdot \vdash_{\mathsf{L}} \mathsf{F} X@w \mid \cdot} \text{ L\_FR}$$

$$\frac{G; S@w \vdash_{\mathsf{C}} \Psi}{G; \cdot \mid \mathsf{J} S@w \vdash_{\mathsf{L}} \cdot \mid \Psi} \text{ L\_JL} \qquad \frac{G; \Theta \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid S@w, \Psi}{G; \Theta \mid \Gamma \vdash_{\mathsf{L}} \Delta, \mathsf{J} S@w \mid \Psi} \text{ L\_JR}$$

$$\frac{G; \Theta \mid \Gamma, A@w \vdash_{\mathsf{L}} \Delta \mid \Psi}{G; \Theta, \mathsf{G} A@w \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi} \text{ L\_GL} \qquad \frac{G; \Theta \mid \Gamma \vdash_{\mathsf{L}} \Delta, A@w \mid \Psi}{G; \Theta \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \mathsf{H} A@w, \Psi} \text{ L\_HR}$$

Figure 10: Inference Rules for BiLNL Logic: Adjoint Functors Rules

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$$\frac{G, w \leq w; \Gamma \vdash_{\mathsf{L}} \Delta}{G; \Gamma \vdash_{\mathsf{L}} \Delta}\ \textsc{rl} \qquad \frac{\begin{array}{l} w_1 G w_2 \qquad w_2 G w_3 \\ G, w_1 \leq w_3; \Gamma \vdash_{\mathsf{L}} \Delta \end{array}}{G; \Gamma \vdash_{\mathsf{L}} \Delta}\ \textsc{ts} \qquad \frac{\begin{array}{l} w_1 G w_2 \\ G; \Gamma, A@w_1, A@w_2 \vdash_{\mathsf{L}} \Delta \end{array}}{G; \Gamma, A@w_1 \vdash_{\mathsf{L}} \Delta}\ \textsc{m}\mathrm{L}$$

$$\frac{\begin{array}{l} w_2 G w_1 \\ G; \Gamma \vdash_{\mathsf{L}} A@w_2, A@w_1, \Delta \end{array}}{G; \Gamma \vdash_{\mathsf{L}} A@w_1, \Delta}\ \textsc{m}\mathrm{R} \qquad \frac{G; \Gamma \vdash_{\mathsf{L}} \Delta}{G; \Gamma, A@w \vdash_{\mathsf{L}} \Delta}\ \textsc{wk}\mathrm{L} \qquad \frac{G; \Gamma \vdash_{\mathsf{L}} \Delta}{G; \Gamma \vdash_{\mathsf{L}} A@w, \Delta}\ \textsc{wk}\mathrm{R}$$

$$\frac{G; \Gamma, A@w, A@w \vdash_{\mathsf{L}} \Delta}{G; \Gamma, A@w \vdash_{\mathsf{L}} \Delta}\ \textsc{ctr}\mathrm{L} \qquad \frac{G; \Gamma \vdash_{\mathsf{L}} A@w, A@w, \Delta}{G; \Gamma \vdash_{\mathsf{L}} A@w, \Delta}\ \textsc{ctr}\mathrm{R}$$

$$\frac{G; \Gamma_1, A@w_1, B@w_2, \Gamma_2 \vdash_{\mathsf{L}} \Delta}{G; \Gamma_1, B@w_2, A@w_1, \Gamma_2 \vdash_{\mathsf{L}} \Delta}\ \textsc{ex}\mathrm{L} \qquad \frac{G; \Gamma \vdash_{\mathsf{L}} \Delta_1, A@w_1, B@w_2, \Delta_2}{G; \Gamma \vdash_{\mathsf{L}} \Delta_1, B@w_2, A@w_1, \Delta_2}\ \textsc{ex}\mathrm{R}$$

$$\frac{}{G; A@w \vdash_{\mathsf{L}} A@w}\ \textsc{id} \qquad \frac{G; \Gamma_1 \vdash_{\mathsf{L}} A@w, \Delta_2 \qquad G; A@w, \Gamma_2 \vdash_{\mathsf{L}} \Delta_1}{G; \Gamma_1, \Gamma_2 \vdash_{\mathsf{L}} \Delta_1, \Delta_2}\ \textsc{cut}$$

$$\frac{G; \Gamma \vdash_{\mathsf{L}} \Delta}{G; \Gamma, \top@w \vdash_{\mathsf{L}} \Delta}\ \mathrm{IL} \qquad \frac{}{G; \cdot \vdash_{\mathsf{L}} \perp @w}\ \mathrm{IR} \qquad \frac{}{G; \top@w \vdash_{\mathsf{L}} \cdot}\ \textsc{fl}\mathrm{L}$$

$$\frac{G; \Gamma \vdash_{\mathsf{L}} \Delta}{G; \Gamma \vdash_{\mathsf{L}} \perp @w, \Delta}\ \textsc{fl}\mathrm{R} \qquad \frac{G; \Gamma_1, A@w, B@w, \Gamma_2 \vdash_{\mathsf{L}} \Delta}{G; \Gamma_1, A \times B@w, \Gamma_2 \vdash_{\mathsf{L}} \Delta}\ \mathrm{cL}$$

$$\frac{G; \Gamma_1 \vdash_{\mathsf{L}} A@w, \Delta_1 \qquad G; \Gamma_2 \vdash_{\mathsf{L}} B@w, \Delta_2}{G; \Gamma_1, \Gamma_2 \vdash_{\mathsf{L}} A \times B@w}\ \mathrm{cR}$$

$$\frac{G; \Gamma_1, A@w \vdash_{\mathsf{L}} \Delta_1 \qquad G; \Gamma_2, B@w \vdash_{\mathsf{L}} \Delta_2}{G; \Gamma_1, \Gamma_2, A + B@w \vdash_{\mathsf{L}} \Delta_1, \Delta_2}\ \textsc{d}\mathrm{L} \qquad \frac{G; \Gamma \vdash_{\mathsf{L}} \Delta_1, A@w, B@w, \Delta_2}{G; \Gamma \vdash_{\mathsf{L}} \Delta_1, A + B@w, \Delta_2}\ \textsc{d}\mathrm{R}$$

$$\frac{\begin{array}{l} w_2 \notin |G|, |\Gamma|, |\Delta| \\ G, w_1 \leq w_2; \Gamma, A@w_2 \vdash_{\mathsf{L}} B@w_2, \Delta \end{array}}{G; \Gamma \vdash_{\mathsf{L}} A \to B@w_1, \Delta}\ \mathrm{I}\textsc{mp}\mathrm{R}$$

$$\frac{\begin{array}{l} w_1 G w_2 \\ G; \Gamma_1 \vdash_{\mathsf{L}} A@w_2, \Delta_1 \qquad G; \Gamma_2, B@w_2 \vdash_{\mathsf{L}} \Delta_2 \end{array}}{G; \Gamma_1, \Gamma_2, A \to B@w_1 \vdash_{\mathsf{L}} \Delta_1, \Delta_2}\ \mathrm{I}\textsc{mp}\mathrm{L}$$

$$\frac{\begin{array}{l} w_2 \notin |G|, |\Gamma|, |\Delta| \\ G, w_2 \leq w_1; \Gamma, A@w_2 \vdash_{\mathsf{L}} B@w_2, \Delta \end{array}}{G; \Gamma, A - B@w_1 \vdash_{\mathsf{L}} \Delta}\ \mathrm{sL}$$

$$\frac{\begin{array}{l} w_2 G w_1 \\ G; \Gamma_1 \vdash_{\mathsf{L}} A@w_2, \Delta_1 \qquad G; \Gamma_2, B@w_2 \vdash_{\mathsf{L}} \Delta_2 \end{array}}{G; \Gamma_1, \Gamma_2 \vdash_{\mathsf{L}} A - B@w_1, \Delta_1, \Delta_2}\ \mathrm{sR}$$

Figure 11: Inference Rules for L