COMONADIC MATTER MEETS MONADIC ANTI-MATTER: AN ADJOINT MODEL OF BI-INTUITIONISTIC LOGIC

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ABSTRACT. Bi-intuitionistic logic (BINT) is a conservative extension of intuitionistic logic with perfect duality. That is, BINT contains the usual intuitionistic logical connectives such as true, conjunction, and implication, but also their duals false, disjunction, and coimplication. One leading question with respect to BINT is, what does BINT look like across the three arcs – logic, typed λ-calculi, and category theory – of the Curry-Howard-Lambek correspondence? A non-trivial (does not degenerate to a poset) categorical model of BINT is currently an open problem. It is this open problem that this paper contributes to by providing the first fully developed categorical model of BINT. It is well-known that the linear counterpart, linear BINT, of BINT can be modeled in a symmetric monoidal closed category equipped with an additional monoidal structure that models par and a specified left adjoint to par called linear coimplication. We call this model a symmetric bi-monoidal bi-closed category. In addition, it is well-known that intuitionistic logic has a categorical model of cartesian closed categories, and their dual cocartesian coclosed categories model cointuitionistic logic. In this paper we exploit Benton's beautiful LNL models of linear logic to show that these three models can be mixed by requiring a symmetric monoidal adjunction between a cartesian closed category and the symmetric bi-monoidal bi-closed category, in addition to a symmetric monoidal adjunction between a cocartesian coclosed category and the symmetric bi-monoidal bi-closed category. As a result of this mixture we obtain two modalities the usual comonadic of-course modality of linear logic, but also a monadic modality allowing for the embedding of cointuitionistic logic inside of linear BINT. Finally, using these modalities we show that BINT intuitionistic logic can be soundly modeled in this new categorical model. As a bi-product of this model we define BiLNL logic which can be seen as the mixture of intuitionistic logic with cointuitionistic logic inside of linear BINT.

1. Introduction

TODO [**?**]

2. Mixed Linear/Non-Linear Models of Bi-intuitionistic Logic: The Categorical Model

In this section we embark on the definition of a categorical model of bi-intuitionistic logic via a categorical model of bi-intuitionistic linear logic. First, we summarize our main results. Suppose $(\mathcal{I}, 1, \times, \rightarrow)$ is a cartesian closed category, and $(\mathcal{L}, \top, \otimes, \multimap)$ is a symmetric monoidal closed category. Then relate these two categories with a symmetric monoidal adjunction $\mathcal{I} : \mathsf{F} \dashv \mathsf{G} : \mathcal{L}$ (Definition **??**), where F and G are symmetric monoidal functors. The later point implies that there are natural transformations $\mathsf{m}_{X,Y} : \mathsf{F}X \otimes \mathsf{F}Y \longrightarrow \mathsf{F}(X \times Y)$ and $\mathsf{n}_{A,B} : \mathsf{G}A \times \mathsf{G}B \longrightarrow \mathsf{G}(A \otimes B)$, and maps $\mathsf{m}_\top : \top \longrightarrow \mathsf{F}1$ and $\mathsf{n}_1 : 1 \longrightarrow \mathsf{G}\top$ subject to several coherence conditions; see Definition **??**.

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DOI:10.2168/LMCS-???

Furthermore, the functor F is strong which means that $\mathsf{m}_{X,Y}$ and $\mathsf{m}_\top$ are isomorphisms. This setup turns out to be one of the most beautiful models of intuitionistic linear logic called an LNL model due to Benton [2]. In fact, the linear modality of-course can be defined by $!A = \mathsf{F}(\mathsf{G}(A))$ which defines a symmetric monoidal comonad using the adjunction; see Section 2.2 of [2]. This model is much simpler than other known models, and resulted in a logic called LNL logic which supports mixing intuitionistic logic with linear logic.

Taking the dual of the previous model results in what we call dual LNL models. They consist of a cocartesian coclosed category, $(C, 0, +, -)$, a symmetric monoidal coclosed category, $(\mathcal{L}', \perp, \oplus, \bullet\!\!-)$, where $\bullet\!\!-\colon \mathcal{L}' \times \mathcal{L}' \longrightarrow \mathcal{L}'$ is left adjoint to parr, and a symmetric monoidal adjunction $\mathcal{L}' : H \dashv\perp: C$. We will show that dual LNL models are a simplification of dual linear categories as defined by Bellin [1] in much of the same way that LNL models are a simplification of linear categories. In fact, we will define Girard's exponential why-not by $?A = J(H(A))$, and hence, is the monad induced by the adjunction.

Now we combine the two previous models into one model of bi-intuitionistic linear logic. Extend $\mathcal{L}$ with a second symmetric monoidal structure, $(\perp, \oplus, \bullet\!\!-)$, such that $\otimes$ distributes over $\oplus$ and $\bullet\!\!-\colon \mathcal{L}\times\mathcal{L}\longrightarrow\mathcal{L}$ is left adjoint to $\oplus$, resulting in the category $(\mathcal{L}, \top, \otimes, \multimap, \perp, \oplus, \bullet\!\!-)$, called a symmetric monoidal bi-closed category. We extend LNL models into a new model of bi-intuitionistic logic called a mixed linear/non-linear bi-intuitionistic model or a BiLNL model. It consists of a cartesian closed category, $(\mathcal{I}, 1, \times, \rightarrow)$, a symmetric monoidal bi-closed category, $(\mathcal{L}, \top, \otimes, \multimap, \perp, \oplus, \bullet\!\!-)$, a cocartesian coclosed category, $(C, 0, +, -)$, and a pair of symmetric monoidal adjoint functors:

$$\begin{tikzcd}
\mathcal{I} \arrow[r, bend left, "\mathsf{F}"] \arrow[r, phantom, "\dashv"] & \mathcal{L} \arrow[l, bend left, "\mathsf{G}"] \arrow[r, bend left, "\mathsf{H}"] \arrow[r, phantom, "\vdash"] & \mathsf{C} \arrow[l, bend left, "\mathsf{J}"]
\end{tikzcd}$$

It is well known that $\mathcal{I}$ is a model of intuitionistic logic and C is a model of cointuitionistic logic, and so, the adjoint situation $F \dashv G$ can be seen as a translation of intuitionistic logic into the intuitionistic fragment, $(\top, \otimes, \multimap)$, of $\mathcal{L}$, and the adjoint situation $H \dashv J$ can be seen as a translation of cointuitionistic logic into the cointuitionistic fragment, $(\perp, \oplus, \bullet\!\!-)$, of $\mathcal{L}$. This we will show turns out to be a model of bi-intuitionistic linear logic with modalities, but also as one of the first categorical models of bi-intuitionistic logic by exploiting the modalities of linear logic to embedded bi-intuitionistic logic inside of bi-intuitionistic linear logic.

The previous model also induces an adjunction:

$$\begin{tikzcd}
\mathcal{I} \arrow[r, bend left, "\mathsf{F;H}"] \arrow[r, phantom, "\dashv"] & C \arrow[l, bend left, "\mathsf{J;G}"]
\end{tikzcd}$$

Does this give us the double-negation monad/comonad?

2.1. **Symmetric Monoidal Categories.** We now introduce the necessary definitions related to symmetric monoidal categories that our model will depend on. Most of these definitions are equivalent to the ones given by Benton [2], but we give a lesser well-known definition for symmetric comonoidal functors due to Bellin [1]. In this section we also introduce distributive categories, the notion of cocloser, and finally, the definition of bilinear categories. The reader may wish to simply skim this section, but refer back to it when they encounter a definition or result they do not know.

Definition 1. A **symmetric monoidal category (SMC)** is a category, $\mathcal{M}$, with the following data:

- An object $\top$ of $\mathcal{M}$,
- A bi-functor $\otimes : \mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{M}$,
- The following natural isomorphisms:

$$\begin{aligned}
&\lambda_A : \top \otimes A \longrightarrow A\\
&\rho_A : A \otimes \top \longrightarrow A\\
&\alpha_{A,B,C} : (A \otimes B) \otimes C \longrightarrow A \otimes (B \otimes C)
\end{aligned}$$

- A symmetry natural transformation:

$$\beta_{A,B} : A \otimes B \longrightarrow B \otimes A$$

- Subject to the following coherence diagrams:

$$\begin{tikzcd}
((A \otimes B) \otimes C) \otimes D \arrow[r, "\alpha_{A,B,C} \otimes \mathsf{id}_D"] \arrow[d, "\alpha_{A \otimes B,C,D}"] & (A \otimes (B \otimes C)) \otimes D \arrow[dd, "\alpha_{A,B \otimes C,D}"] \\
(A \otimes B) \otimes (C \otimes D) \arrow[d, "\alpha_{A,B,C \otimes D}"] & \\
A \otimes (B \otimes (C \otimes D)) & A \otimes ((B \otimes C) \otimes D) \arrow[l, "\mathsf{id}_A \otimes \alpha_{B,C,D}"']
\end{tikzcd}$$

$$\begin{tikzcd}
(A \otimes B) \otimes C \arrow[r, "\alpha_{A,B,C}"] \arrow[d, "\beta_{A,B} \otimes \mathsf{id}_C"] & A \otimes (B \otimes C) \arrow[r, "\beta_{A,B \otimes C}"] & (B \otimes C) \otimes A \arrow[d, "\alpha_{B,C,A}"] \\
(B \otimes A) \otimes C \arrow[r, "\alpha_{B,A,C}"] & B \otimes (A \otimes C) \arrow[r, "\mathsf{id}_B \otimes \beta_{A,C}"] & B \otimes (C \otimes A)
\end{tikzcd}$$

$$\begin{tikzcd}
(A \otimes \top) \otimes B \arrow[rr, "\alpha_{A,\top,B}"] \arrow[dr, "\rho_A"'] & & A \otimes (\top \otimes B) \arrow[dl, "\lambda_B"] \\
& A \otimes B &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes B \arrow[d, "\beta_{A,B}"'] \arrow[dr, "\mathsf{id}_{A \otimes B}"] & \\
B \otimes A \arrow[r, "\beta_{B,A}"'] & A \otimes B
\end{tikzcd}$$

$$\begin{tikzcd}
\top \otimes A \arrow[rr, "\beta_{\top,A}"] \arrow[dr, "\lambda_A"'] & & A \otimes \top \arrow[dl, "\rho_A"] \\
& A &
\end{tikzcd}$$

Categorical modeling implication requires that the model be closed; which can be seen as an internalization of the notion of a morphism.

Definition 2. A **symmetric monoidal closed category (SMCC)** is a symmetric monoidal category, $(\mathcal{M}, \top, \otimes)$, such that, for any object B of $\mathcal{M}$, the functor $- \otimes B : \mathcal{M} \longrightarrow \mathcal{M}$ has a specified right adjoint. Hence, for any objects A and C of $\mathcal{M}$ there is an object $B \multimap C$ of $\mathcal{M}$ and a natural bijection:

$$\mathsf{Hom}_{\mathcal{M}}(A \otimes B, C) \cong \mathsf{Hom}_{\mathcal{M}}(A, B \multimap C)$$

We call the functor $\multimap\colon \mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{M}$ the internal hom of $\mathcal{M}$.

Symmetric monoidal closed categories can be seen as a model of intuitionistic linear logic with a tensor product and implication. What happens when we take the dual? First, we have the following result:

Lemma 3 (Dual of Symmetric Monoidal Categories)**.** If $(\mathcal{M}, \top, \otimes)$ is a symmetric monoidal category, then $\mathcal{M}^{\mathsf{op}}$ is also a symmetric monoidal category.

The previous result follows from the fact that the structures making up symmetric monoidal categories are isomorphisms, and so naturally taking their opposite will yield another symmetric monoidal category. To emphasize when we are thinking about a symmetric monoidal category in the opposite we use the notion $(\mathcal{M}, \perp, \oplus)$ which gives the suggestion of $\oplus$ corresponding to a disjunctive tensor product which we call the *cotensor* of $\mathcal{M}$. The next definition describes when a symmetric monoidal category is coclosed.

Definition 4. A **symmetric monoidal coclosed category (SMCCC)** is a symmetric monoidal category, $(\mathcal{M}, \perp, \oplus)$, such that, for any object B of $\mathcal{M}$, the functor $- \oplus B : \mathcal{M} \longrightarrow \mathcal{M}$ has a specified left adjoint. Hence, for any objects A and C of $\mathcal{M}$ there is an object $B \bullet\!\!- C$ of $\mathcal{M}$ and a natural bijection:

$$\mathsf{Hom}_{\mathcal{M}}(C, A \oplus B) \cong \mathsf{Hom}_{\mathcal{M}}(B \bullet\!\!- C, A)$$

We call the functor $\bullet\!\!-\colon \mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{M}$ the internal cohom of $\mathcal{M}$.

We combine the previous definitions into a single category. First, we define the notion of a distributive category due to Cockett and Seely [4].

Definition 5. We call a symmetric monoidal category, $(\mathcal{M}, \top, \otimes, \perp, \oplus)$, a **distributive category** if there are natural transformations:

$$\begin{array}{l} \delta^L_{A,B,C} : A \otimes (B \oplus C) \longrightarrow (A \otimes B) \oplus C \\ \delta^R_{A,B,C} : (B \oplus C) \otimes A \longrightarrow B \oplus (C \otimes A) \end{array}$$

subject to several coherence diagrams. Due to the large number of coherence diagrams we do not list them here, but they all can be found in Cockett and Seely's paper [4].

Requiring that the tensor and cotensor products have the corresponding right and left adjoints results in the following definition.

Definition 6. A **bilinear category** is a distributive category $(\mathcal{M}, \top, \otimes, \perp, \oplus)$ such that $(\mathcal{M}, \top, \otimes)$ is closed, and $(\mathcal{M}, \perp, \oplus)$ is coclosed. We will denote bi-linear categories by $(\mathcal{M}, \top, \otimes, \multimap, \perp, \oplus, \bullet\!\!-)$.

Originally, Lambek defined bilinear categories to be similar to the previous definition, but the tensor and cotensor were non-commutative [3], however, the bilinear categories given here are. We retain the name in homage to his original work. As we will see below bilinear categories form the core of the work given here, and are of crucial importance.

A symmetric monoidal category is a category with additional structure subject to several coherence diagrams. Thus, an ordinary functor is not enough to capture this structure, and hence, the introduction of symmetric monoidal functors.

Definition 7. Suppose we are given two symmetric monoidal closed categories $(\mathcal{M}_1, \top_1, \otimes_1, \alpha_1, \lambda_1, \rho_1, \beta_1)$ and $(\mathcal{M}_2, \top_2, \otimes_2, \alpha_2, \lambda_2, \rho_2, \beta_2)$. Then a **symmetric monoidal functor** is a functor $F : \mathcal{M}_1 \longrightarrow \mathcal{M}_2$, a map $m_\top : \top_2 \longrightarrow F\top_1$ and a natural transformation $m_{A,B} : FA \otimes_2 FB \longrightarrow F(A \otimes_1 B)$ subject to the following coherence conditions:

$$\begin{tikzcd}
(FA \otimes_2 FB) \otimes_2 FC \arrow[r, "\alpha_{2FA,FB,FC}"] \arrow[d, "m_{A,B} \otimes \mathsf{id}_{FC}"] & FA \otimes_2 (FB \otimes_2 FC) \arrow[d, "\mathsf{id}_{FA} \otimes m_{B,C}"] \\
F(A \otimes_1 B) \otimes_2 FC \arrow[d, "m_{A \otimes_1 B,C}"] & FA \otimes_2 F(B \otimes_1 C) \arrow[d, "m_{A,B \otimes_1 C}"] \\
F((A \otimes_1 B) \otimes_1 C) \arrow[r, "F\alpha_{1A,B,C}"] & F(A \otimes_1 (B \otimes_1 C))
\end{tikzcd}$$

$$\begin{tikzcd}
\top_2 \otimes_2 FA \arrow[r, "\lambda_{2FA}"] \arrow[d, "m_\top \otimes \mathsf{id}_{FA}"] & FA \\
F\top_1 \otimes_2 FA \arrow[r, "m_{\top_1,A}"] & F(\top_1 \otimes_1 A) \arrow[u, "F\lambda_{1A}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
FA \otimes_2 \top_2 \arrow[r, "\rho_{2FA}"] \arrow[d, "\mathsf{id}_{FA} \otimes m_\top"] & FA \\
FA \otimes_2 F\top_1 \arrow[r, "m_{A,\top_1}"] & F(A \otimes_1 \top_1) \arrow[u, "F\rho_{1A}"]
\end{tikzcd}$$

$$\begin{tikzcd}
FA \otimes_2 FB \arrow[r, "\beta_{2FA,FB}"] \arrow[d, "m_{A,B}"] & FB \otimes_2 FA \arrow[d, "m_{B,A}"] \\
F(A \otimes_1 B) \arrow[r, "F\beta_{1A,B}"] & F(B \otimes_1 A)
\end{tikzcd}$$

The following is dual to the previous definition.

Definition 8. Suppose we are given two symmetric monoidal closed categories $(\mathcal{M}_1, \perp_1, \oplus_1, \alpha_1, \lambda_1, \rho_1, \beta_1)$ and $(\mathcal{M}_2, \perp_2, \oplus_2, \alpha_2, \lambda_2, \rho_2, \beta_2)$. Then a **symmetric comonoidal functor** is a functor $F : \mathcal{M}_1 \longrightarrow \mathcal{M}_2$, a map $m_\perp : F \perp_1 \longrightarrow \perp_2$ and a natural transformation $m_{A,B} : F(A \oplus_1 B) \longrightarrow FA \oplus_2 FB$ subject to the following coherence conditions:

$$\begin{tikzcd}
F((A \oplus_1 B) \oplus_1 C) \arrow[r, "m_{A \oplus_1 B,C}"] \arrow[d, "F\alpha_{A,B,C}"] & F(A \oplus_1 B) \oplus_2 FC \arrow[d, "m_{A,B} \oplus_2 \mathsf{id}_{FC}"] \\
F(A \oplus_1 (B \oplus_1 C)) \arrow[d, "m_{A,B \oplus_1 C}"] & (FA \oplus_2 FB) \oplus_2 FC \arrow[d, "\alpha_{FA,FB,FC}"] \\
FA \oplus_2 F(B \oplus_1 C)) \arrow[r, "\mathsf{id}_{FA} \oplus_2 m_{B,C}"] & FA \oplus_2 (FB \oplus_2 FC)
\end{tikzcd}$$

$$\begin{tikzcd}
F(\perp_1 \oplus_1 A) \arrow[r, "m_{\perp_1,A}"] \arrow[d, "F\lambda_{1A}"'] & F\perp_1 \oplus_2 FA \arrow[d, "m_\perp \oplus \mathsf{id}_{FA}"] \\
FA \arrow[r, "\lambda_2^{-1}{}_{FA}"] & \perp_2 \oplus_2 FA
\end{tikzcd}
\qquad
\begin{tikzcd}
F(A \oplus_1 \perp_1) \arrow[r, "m_{A,\perp_1}"] \arrow[d, "F\rho_{1A}"'] & FA \oplus_2 F\perp_1 \arrow[d, "\mathsf{id}_{FA} \oplus m_\perp"] \\
FA \arrow[r, "\rho_2^{-1}{}_{FA}"] & FA \oplus_2 \perp_2
\end{tikzcd}$$

$$\begin{tikzcd}
F(A \oplus_1 B) \arrow[r, "m_{A,B}"] \arrow[d, "F\beta_{1A,B}"'] & FA \oplus_2 FB \arrow[d, "\beta_{2FA,FB}"] \\
F(B \oplus_1 A) \arrow[r, "m_{B,A}"] & FB \oplus_2 FA
\end{tikzcd}$$

Naturally, since functors are enhanced to handle the additional structure found in a symmetric monoidal category we must also extend natural transformations, and adjunctions.

Definition 9. Suppose $(\mathcal{M}_1, \top_1, \otimes_1)$ and $(\mathcal{M}_2, \top_2, \otimes_2)$ are SMCs, and (F, m) and (G, n) are a symmetric monoidal functors between $\mathcal{M}_1$ and $\mathcal{M}_2$. Then a **symmetric monoidal natural transformation** is a natural transformation, $f : F \longrightarrow G$, subject to the following coherence diagrams:

$$\begin{tikzcd}
FA \otimes_2 FB \arrow[r, "m_{A,B}"] \arrow[d, "f_A \otimes_2 f_B"'] & F(A \otimes_1 B) \arrow[d, "f_{A \otimes_1 B}"] \\
GA \otimes_2 GB \arrow[r, "n_{A,B}"'] & G(A \otimes_1 B)
\end{tikzcd}
\qquad
\begin{tikzcd}
F\top_1 \arrow[rr, "f_{\top_1}"] & & G\top_1 \\
& \top_2 \arrow[ul, "m_{\top_1}"] \arrow[ur, "n_{\top_1}"'] &
\end{tikzcd}$$

Definition 10. Suppose $(\mathcal{M}_1, \perp_1, \oplus_1)$ and $(\mathcal{M}_2, \perp_2, \oplus_2)$ are SMCs, and (F, m) and (G, n) are a symmetric comonoidal functors between $\mathcal{M}_1$ and $\mathcal{M}_2$. Then a **symmetric comonoidal natural transformation** is a natural transformation, $f : F \longrightarrow G$, subject to the following coherence diagrams:

$$\begin{tikzcd}
F(A \oplus_1 B) \arrow[r, "m_{A,B}"] \arrow[d, "f_{A \oplus_1 B}"'] & FA \oplus_2 FB \arrow[d, "f_A \oplus_2 f_B"] \\
G(A \oplus_1 B) \arrow[r, "n_{A,B}"'] & GA \oplus_2 GB
\end{tikzcd}
\qquad
\begin{tikzcd}
\perp_2 & & G\perp_1 \arrow[ll, "n_{\perp_1}"'] \\
& F\perp_1 \arrow[ul, "m_{\perp_1}"] \arrow[ur, "f_{\perp_1}"'] &
\end{tikzcd}$$

Definition 11. Suppose $(\mathcal{M}_1, \top_1, \otimes_1)$ and $(\mathcal{M}_2, \top_2, \otimes_2)$ are SMCs, and (F, m) is a symmetric monoidal functor between $\mathcal{M}_1$ and $\mathcal{M}_2$ and (G, n) is a symmetric monoidal functor between $\mathcal{M}_2$ and $\mathcal{M}_1$. Then a **symmetric monoidal adjunction** is an ordinary adjunction $\mathcal{M}_1 : F \dashv G : \mathcal{M}_2$ such that the unit, $\eta_A : A \to GFA$, and the counit, $\varepsilon_A : FGA \to A$, are symmetric monoidal natural transformations.

Thus, the following diagrams must commute:

$$\begin{tikzcd}
FGA \otimes_1 FGB \arrow[r, "\mathsf{q}_{A,B}"] \arrow[dr, "\varepsilon_A \otimes_1 \varepsilon_B"'] & FG(A \otimes_1 B) \arrow[d, "\varepsilon_{A \otimes_1 B}"] \\
& A \otimes_1 B
\end{tikzcd}
\qquad
\begin{tikzcd}
FG\top_1 \arrow[rr, "\varepsilon_{\top_1}"] & & \top_1 \\
& \top_1 \arrow[ul, "\mathsf{q}_{\top_1}"] \arrow[ur, equal] &
\end{tikzcd}$$

$$\begin{tikzcd}
& A \otimes_2 B \arrow[dl, "\eta_A \otimes_2 \eta_B"'] \arrow[d, "\eta_{A \otimes_2 B}"] \\
GFA \otimes_2 GFB \arrow[r, "\mathsf{p}_{A,B}"'] & GF(A \otimes_2 B)
\end{tikzcd}
\qquad
\begin{tikzcd}
\top_2 \arrow[rr, "\eta_{\top_2}"] \arrow[dr, equal] & & GF\top_2 \\
& \top_2 \arrow[ur, "p_{\top_2}"'] &
\end{tikzcd}$$

Note that p and q exist because (FG, q) and (GF, p) are symmetric monoidal functors.

Definition 12. Suppose $(\mathcal{M}_1, \perp_1, \oplus_1)$ and $(\mathcal{M}_2, \perp_2, \oplus_2)$ are SMCs, and (F, m) is a symmetric comonoidal functor between $\mathcal{M}_1$ and $\mathcal{M}_2$ and (G, n) is a symmetric comonoidal functor between $\mathcal{M}_2$ and $\mathcal{M}_1$. Then a **symmetric comonoidal adjunction** is an ordinary adjunction $\mathcal{M}_1 : F \dashv G : \mathcal{M}_2$ such that the unit, $\eta_A : A \to GFA$, and the counit, $\varepsilon_A : FGA \to A$, are symmetric comonoidal natural transformations. Thus, the following diagrams must commute:

$$\begin{tikzcd}
A \oplus_1 B \arrow[r, "\eta_{A \oplus_1 B}"] \arrow[d, "\eta_A \oplus_1 \eta_B"'] & GF(A \oplus_1 B) \arrow[dl, "\mathsf{p}_{A,B}"] \\
GFA \oplus_1 GFB &
\end{tikzcd}
\qquad
\begin{tikzcd}
GF \perp_1 \arrow[rr, "\mathsf{p}_{\perp_1}"] & & \perp_1 \\
& \perp_1 \arrow[ul, "\eta_{\perp_1}"] \arrow[ur, equal] &
\end{tikzcd}$$

$$\begin{tikzcd}
FG(A \oplus_2 B) \arrow[r, "\mathsf{q}_{A,B}"] \arrow[dr, "\varepsilon_{A \oplus_2 B}"'] & FGA \oplus_2 FGB \arrow[d, "\varepsilon_A \oplus_2 \varepsilon_B"] \\
& A \oplus_2 B
\end{tikzcd}
\qquad
\begin{tikzcd}
FG \perp_2 \arrow[rr, "\varepsilon_{\perp_2}"] \arrow[dr, equal] & & \perp_2 \\
& FG \perp_2 \arrow[ur, "\mathsf{q}_{\perp_2}"'] &
\end{tikzcd}$$

Note that p and q exist because (FG, q) and (GF, p) are symmetric monoidal functors.

We will be defining, and making use of the of-course and why-not exponentials from linear logic, but these correspond to a symmetric monoidal comonad and a symmetric comonoidal monad respectively, and so we define these concepts next. In addition, whenever we have a symmetric monoidal adjunction, we immediately obtain a symmetric monoidal comonad on the left, and a symmetric monoidal monad on the right; similarly for symmetric comonad adjunctions.

Definition 13. A **symmetric monoidal monad** on a symmetric monoidal category C is a triple (T, η, μ), where (T, n) is a symmetric monoidal endofunctor on C, $\eta_A : A \longrightarrow TA$ and $\mu_A : T^2 A \to TA$

are symmetric monoidal natural transformations, which make the following diagrams commute:

$$
\begin{tikzcd}
T^3A \arrow[r, "\mu_{TA}"] \arrow[d, "T\mu_A"'] & T^2A \arrow[d, "\mu_A"] \\
T^2A \arrow[r, "\mu_A"'] & TA
\end{tikzcd}
\qquad
\begin{tikzcd}
 & TA \arrow[dl, equal] \arrow[dr, equal] & \\
TA \arrow[r, "\eta_{TA}"'] & T^2A \arrow[u, "\mu_A"] & TA \arrow[l, "T\eta_A"]
\end{tikzcd}
$$

The assumption that η and μ are symmetric monoidal natural transformations amount to the following diagrams commuting:

$$
\begin{tikzcd}
 & A \otimes B \arrow[dl, "\eta_A \otimes \eta_B"'] \arrow[d, "\eta_A"] \\
TA \otimes TB \arrow[r, "\mathsf{n}_{A,B}"'] & T(A \otimes B)
\end{tikzcd}
\qquad
\begin{tikzcd}
\top \arrow[rr, "\eta_\top"] \arrow[dr, equal] & & T\top \\
 & \top \arrow[ur, "\mathsf{n}_\top"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
T^2A \otimes T^2B \arrow[r, "\mathsf{n}_{TA,TB}"] \arrow[d, "\mu_A \otimes \mu_B"'] & T(TA \otimes TB) \arrow[r, "T\mathsf{n}_{A,B}"] & T^2(A \otimes B) \arrow[d, "\mu_{A \otimes B}"] \\
TA \otimes TB \arrow[rr, "\mathsf{n}_{A,B}"] & & T(A \otimes B)
\end{tikzcd}
\qquad
\begin{tikzcd}
\top \arrow[r, "\mathsf{n}_\top"] \arrow[d, "\mathsf{n}_\top"'] & T\top \arrow[d, "T\mathsf{n}_\top"] \\
T\top & T^2\top \arrow[l, "\mu_\top"]
\end{tikzcd}
$$

Definition 14. A **symmetric comonoidal monad** on a symmetric monoidal category C is a triple (T, η, μ), where (T, n) is a symmetric comonoidal endofunctor on C, $\eta_A : A \longrightarrow TA$ and $\mu_A : T^2A \to TA$ are symmetric comonoidal natural transformations, which make the following diagrams commute:

$$
\begin{tikzcd}
T^3A \arrow[r, "\mu_{TA}"] \arrow[d, "T\mu_A"'] & T^2A \arrow[d, "\mu_A"] \\
T^2A \arrow[r, "\mu_A"'] & TA
\end{tikzcd}
\qquad
\begin{tikzcd}
 & TA \arrow[dl, equal] \arrow[dr, equal] & \\
TA \arrow[r, "\eta_{TA}"'] & T^2A \arrow[u, "\mu_A"] & TA \arrow[l, "T\eta_A"]
\end{tikzcd}
$$

The assumption that η and μ are symmetric comonoidal natural transformations amount to the following diagrams commuting:

$$
\begin{tikzcd}
A \oplus B \arrow[r, "\eta_A \oplus \eta_B"] \arrow[d, "\eta_A"'] & TA \oplus TB \\
T(A \oplus B) \arrow[ur, "\mathsf{n}_{A,B}"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\perp \arrow[rr, "\eta_\perp"] \arrow[dr, equal] & & T\perp \arrow[dl, "\mathsf{n}_\perp"] \\
 & \perp &
\end{tikzcd}
$$

$$\begin{tikzcd}
T^2(A \oplus B) \arrow[r, "T\mathsf{n}_{A,B}"] \arrow[d, "\mu_{A\oplus B}"'] & T(TA \oplus TB) \arrow[r, "\mathsf{n}_{TA,TB}"] & T^2A \oplus T^2B \arrow[d, "\mu_A \oplus \mu_B"] \\
T(A \oplus B) \arrow[rr, "\mathsf{n}_{A,B}"] & & TA \oplus TB
\end{tikzcd}
\qquad
\begin{tikzcd}
T^2 \perp \arrow[r, "T\mathsf{n}_\perp"] \arrow[d, "\mu_\perp"'] & T \perp \arrow[d, "\mathsf{n}_\perp"] \\
T \perp \arrow[r, "\mathsf{n}_\perp"'] & \perp
\end{tikzcd}$$

Finally the dual concept, of a symmetric monoidal comonad.

Definition 15. A **symmetric monoidal comonad** on a symmetric monoidal category C is a triple (T, ε, δ), where (T, m) is a symmetric monoidal endofunctor on C, $\varepsilon_A : TA \longrightarrow A$ and $\delta_A : TA \to T^2A$ are symmetric monoidal natural transformations, which make the following diagrams commute:

$$\begin{tikzcd}
TA \arrow[r, "\delta_A"] \arrow[d, "\delta_A"'] & T^2A \arrow[d, "T\delta_A"] \\
T^2A \arrow[r, "\delta_{TA}"] & T^3A
\end{tikzcd}
\qquad
\begin{tikzcd}
& TA \arrow[dl, equal] \arrow[d, "\delta_A"] \arrow[dr, equal] & \\
TA & T^2A \arrow[l, "\varepsilon_{TA}"] \arrow[r, "T\varepsilon_A"'] & TA
\end{tikzcd}$$

The assumption that ε and δ are symmetric monoidal natural transformations amount to the following diagrams commuting:

$$\begin{tikzcd}
TA \otimes TB \arrow[r, "\mathsf{m}_{A,B}"] \arrow[dr, "\varepsilon_A \otimes \varepsilon_B"'] & T(A \otimes B) \arrow[d, "\varepsilon_{A\otimes B}"] \\
& A \otimes B
\end{tikzcd}
\qquad
\begin{tikzcd}
T\top \arrow[rr, "\mathsf{m}_\top"] & & \top \\
& \top \arrow[ul, "\varepsilon_\top"] \arrow[ur, equal] &
\end{tikzcd}$$

$$\begin{tikzcd}
TA \otimes TB \arrow[rr, "\mathsf{m}_{A,B}"] \arrow[d, "\delta_A \otimes \delta_B"'] & & T(A \otimes B) \arrow[d, "\delta_{A\otimes B}"] \\
T^2A \otimes T^2B \arrow[r, "\mathsf{m}_{TA,TB}"'] & T(TA \otimes TB) \arrow[r, "T\mathsf{m}_{A,B}"'] & T^2(A \otimes B)
\end{tikzcd}
\qquad
\begin{tikzcd}
\top \arrow[r, "\mathsf{m}_\top"] \arrow[d, "\mathsf{m}_\top"'] & T\top \arrow[d, "\delta_\top"] \\
T\top \arrow[r, "T\mathsf{m}_\top"'] & T^2\top
\end{tikzcd}$$

Definition 16. A **symmetric comonoidal comonad** on a symmetric monoidal category C is a triple (T, ε, δ), where (T, m) is a symmetric comonoidal endofunctor on C, $\varepsilon_A : TA \longrightarrow A$ and $\delta_A : TA \to T^2A$ are symmetric comonoidal natural transformations, which make the following diagrams commute:

$$\begin{tikzcd}
TA \arrow[r, "\delta_A"] \arrow[d, "\delta_A"'] & T^2A \arrow[d, "T\delta_A"] \\
T^2A \arrow[r, "\delta_{TA}"] & T^3A
\end{tikzcd}
\qquad
\begin{tikzcd}
& TA \arrow[dl, equal] \arrow[d, "\delta_A"] \arrow[dr, equal] & \\
TA & T^2A \arrow[l, "\varepsilon_{TA}"] \arrow[r, "T\varepsilon_A"'] & TA
\end{tikzcd}$$

The assumption that ε and δ are symmetric monoidal natural transformations amount to the following diagrams commuting:

$$\begin{tikzcd}
T(A \oplus B) \arrow[r, "\mathsf{m}_{A,B}"] \arrow[dr, "\varepsilon_{A\oplus B}"'] & TA \oplus TB \arrow[d, "\varepsilon_A \oplus \varepsilon_B"] \\
& A \oplus B
\end{tikzcd}
\qquad
\begin{tikzcd}
T\perp \arrow[rr, "\varepsilon_\perp"] \arrow[dr, equal] & & \perp \\
& T\perp \arrow[ur, "\mathsf{m}_\perp"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
T(A \oplus B) \arrow[rr, "\mathsf{m}_{A,B}"] \arrow[d, "\delta_{A\oplus B}"'] & & TA \oplus TB \arrow[d, "\delta_A \oplus \delta_B"] \\
T^2(A \oplus B) \arrow[r, "T\mathsf{m}_{A,B}"'] & T(TA \oplus TB) \arrow[r, "\mathsf{m}_{TA,TB}"'] & T^2A \oplus T^2B
\end{tikzcd}
\qquad
\begin{tikzcd}
T\perp \arrow[r, "\mathsf{m}_\perp"] \arrow[d, "\delta_\perp"'] & \perp \\
T^2\perp \arrow[r, "T\mathsf{m}_\perp"] & T\perp \arrow[u, "\mathsf{m}_\perp"']
\end{tikzcd}$$

2.2. **Cartesian Closed and Cocartesian Coclosed Categories.** The notion of a cartesian closed category is well-known, but for completeness we define them here. However, their dual is lesser known, especially in computer science, and so we given their full definition. We also review some know results concerning cocartesian coclosed categories and categories that are both cartesian closed and cocartesian coclosed.

Definition 17. A **cartesian category** is a category, $(C, 1, \times)$, with an object, 1, and a bi-functor, $\times : C \times C \longrightarrow C$, such that for any object A there is exactly one morphism $\diamond : A \to 1$, and for any morphisms $f : C \longrightarrow A$ and $g : C \longrightarrow B$ there is a morphism $\langle f, g\rangle : C \to A \times B$ subject to the following diagram:

$$\begin{tikzcd}
& C \arrow[dl, "f"'] \arrow[d, "\langle f,g\rangle"] \arrow[dr, "g"] & \\
A & A \times B \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & B
\end{tikzcd}$$

A cartesian category models conjunction by the product functor, $\times : C \times C \longrightarrow C$, and the unit of conjunction by the terminal object. As we mention above modeling implication requires closer, and since it is well-known that any cartesian category is also a symmetric monoidal category the definition of closer for a cartesian category is the same as the definition of closer for a symmetric monoidal category (Definition 4). We denote the internal hom for cartesian closed categories by $A \to B$.

The dual of a cartesian category is a cocartesian category. They are a model of intuitionistic logic with disjunction and its unit.

Definition 18. A **cocartesian category** is a category, $(C, 0, +)$, with an object, 0, and a bi-functor, $+ : C \times C \longrightarrow C$, such that for any object A there is exactly one morphism $\square : 0 \to A$, and for any morphisms $f : A \longrightarrow C$ and $g : B \longrightarrow C$ there is a morphism $[f, g] : A + B \longrightarrow C$ subject to the following diagram:

$$\begin{tikzcd}
 & C & \\
A \arrow[ur, "f"] \arrow[r, "\iota_1"'] & A + B \arrow[u, "{[f,g]}"] & B \arrow[ul, "g"'] \arrow[l, "\iota_2"]
\end{tikzcd}$$

Cocloser, just like closer for cartesian categories, is defined in the same way that cocloser is defined for symmetric monoidal categories, because cocartesian categories are also symmetric monoidal categories. Thus, a cocartesian category is coclosed if there is a specified left-adjoint, which we denote $S - T$, to the coproduct.

There are many examples of cocartesian coclosed categories. Basically, any interesting cartesian category has an interesting dual, and hence, induces an interesting cocartesian coclosed category. The opposite of the category of sets and functions between them is isomorphic to the category of complete atomic boolean algebras, and both of which, are examples of cocartesian coclosed categories. As we mentioned above bi-linear categories [3] are models of bi-linear logic where the left adjoint to the cotensor models coimplication. Similarly, cocartesian coclosed categories model cointuitionistic logic with disjunction and intuitionistic coimplication [5, 1].

Put more examples in here.

We might now ask if we can a category can be both cartesian closed and cocartesian coclosed just as bi-linear categories, but this turns out to be where the matter meets antimatter in such away that the category degenerates to a preorder. That is, every homspace contains at most one morphism. We recall this proof here, which is due to Crolard [5]. We need a couple basic facts about cartesian closed categories with initial objects.

Lemma 19. In any cartesian category C, if 0 is an initial object in C and $\mathsf{Hom}_C(A, 0)$ is non-empty, then $A \cong A \times 0$.

Proof. This follows easily from the universal mapping property for products. □

Lemma 20. In any cartesian closed category C, if 0 is an initial object in C, then so is $0 \times A$ for any object A of C.

Proof. We know that the universal morphism for the initial object is unique, and hence, the homspace $\mathsf{Hom}_C(0, A \Rightarrow B)$ for any object B of C contains exactly one morphism. Then using the right adjoint to the product functor we know that $\mathsf{Hom}_C(0, A \Rightarrow B) \cong \mathsf{Hom}_C(0 \times A, B)$, and hence, there is only one arrow between $0 \times A$ and B. □

The following lemma is due to Joyal [**?**], and is key to the next theorem.

Lemma 21 (Joyal's)**.** In any cartesian closed category C, if 0 is an initial object in C and $\mathsf{Hom}_C(A, 0)$ is non-empty, then A is an initial object in C.

Proof. Suppose C is a cartesian closed category, such that, 0 is an initial object in C, and A is an arbitrary object in C. Furthermore, suppose $\mathsf{Hom}_C(A, 0)$ is non-empty. By the first basic lemma above we know that $A \cong A \times 0$, and by the second $A \times 0$ is initial, thus A is initial. □

Finally, the following theorem shows that any category that is both cartesian closed and cocartesian coclosed is a preorder.

Theorem 22 ((co)Cartesian (co)Closed Categories are Preorders (Crolard[5]))**.** If C is both cartesian closed and cocartesian coclosed, then for any two objects A and B of C, $\mathsf{Hom}_C(A, B)$ has at most one element.

Proof. Suppose C is both cartesian closed and cocartesian coclosed, and A and B are objects of C. Then by using the basic fact that the initial object is the unit to the coproduct, and the coproducts left adjoint we know the following:

$$\mathsf{Hom}_C(A, B) \cong \mathsf{Hom}_C(A, 0 + B) \cong \mathsf{Hom}_C(B - A, 0)$$

Therefore, by Joyal's theorem above $\mathsf{Hom}_C(A, B)$ has at most one element. □

Notice that the previous result hinges on the fact that there are initial and terminal objects, and thus, this result does not hold for bi-linear categories, because the units to the tensor and cotensor are not initial nor terminal.

The repercussions of this result are that if we do not want to work with preorders, but do want to work with all of the structure, then we must separate the two worlds. Thus, this result can be seen as the motivation for the current work. We enforce the separation using linear logic, but through the power of linear logic we show that separation is not far.

2.3. **A Mixed Linear/Non-Linear Model for Co-Intuitionistic Logic.** Benton [2] showed that from a LNL model it is possible to construct a linear category, and vice versa. Bellin [1] showed that the dual to linear categories are sufficient to model co-intuitionistic linear logic. We show that from the dual to a LNL model we can construct the dual to a linear category, and vice versa, thus, carrying out the same program for co-intuitionistic linear logic as Benton did for intuitionistic linear logic.

Combining a symmetric monoidal coclosed category with a cocartesian coclosed category via a symmetric comonoidal adjunction defines a coLNL model.

Definition 23. A **mixed linear/non-linear model for co-intuitionistic logic (coLNL model)**, $\mathcal{L} : \mathsf{H} \dashv \mathsf{J} : C$, consists of the following:

i. a symmetric monoidal coclosed category $(\mathcal{L}, \perp, \oplus, \bullet\!\!-)$,
ii. a cocartesian coclosed category $(C, 0, +, -)$, and
iv. a symmetric comonoidal adjunction $\mathcal{L} : \mathsf{H} \dashv \mathsf{J} : C$, where $\eta_A : A \longrightarrow \mathsf{JH}A$ and $\varepsilon_R : \mathsf{HJ}R \longrightarrow R$ are the unit and counit of the adjunction respectively.

It is well-known that an adjunction $\mathcal{L} : \mathsf{H} \dashv \mathsf{J} : C$ induces a monad $\mathsf{H};\mathsf{J} : \mathcal{L} \longrightarrow \mathcal{L}$, but when the adjunction is symmetric comonoidal we obtain a symmetric comonoidal monad, in fact, $\mathsf{H};\mathsf{J}$ defines the linear exponential why-not denoted $?A = \mathsf{J}(\mathsf{H}A))$. By the definition of coLNL models we know that both H and J are symmetric comonoidal functors, and hence, are equipped with natural transformations $\mathsf{h}_{A,B} : \mathsf{H}(A \oplus B) \longrightarrow \mathsf{H}A + \mathsf{H}B$ and $\mathsf{j}_{R,S} : \mathsf{J}(R + S) \longrightarrow \mathsf{J}R \oplus \mathsf{J}S$, and maps $\mathsf{h}_\perp : \mathsf{H}\perp \longrightarrow 0$ and $\mathsf{j}_\perp : \mathsf{J}0 \longrightarrow \perp$. We will make heavy use of these maps throughout the sequel.

One useful property of Benton's LNL model is that the maps associated with the symmetric monoidal left adjoint in the model are isomorphisms. Since coLNL models are dual we obtain similar isomorphisms with respect to the right adjoint.

Lemma 24 (Symmetric Comonoidal Isomorphisms)**.** Given any coLNL model $\mathcal{L} : \mathsf{H} \dashv \mathsf{J} : C$, then there are the following isomorphisms:

$$\mathsf{J}(R + S) \cong \mathsf{J}R \oplus \mathsf{J}S \quad \text{and} \quad \mathsf{J}0 \cong \perp$$

Furthermore, the former is natural in R and S.

Proof. Suppose $\mathcal{L} : \mathsf{H} \dashv \mathsf{J} : C$ is a coLNL model. Then we can define the following family of maps:

$$\mathsf{j}^{-1}_{R,S} := \mathsf{J}R \oplus \mathsf{J}S \xrightarrow{\eta} \mathsf{JH}(\mathsf{J}R \oplus \mathsf{J}S) \xrightarrow{\mathsf{Jh}_{A,B}} \mathsf{J}(\mathsf{HJ}R + \mathsf{HJ}S) \xrightarrow{\mathsf{J}(\varepsilon_R + \varepsilon_S)} \mathsf{J}(R + S)$$

$$\mathsf{j}^{-1}_{\perp} := \perp \xrightarrow{\eta} \mathsf{JH}\perp \xrightarrow{\mathsf{Jh}_{\perp}} \mathsf{J}0$$

It is easy to see that $\mathsf{j}^{-1}_{R,S}$ is natural, because it is defined in terms of a composition of natural transformations. All that is left to be shown is that $\mathsf{j}^{-1}_{R,s}$ and $\mathsf{j}^{-1}_{\perp}$ are mutual inverses with $\mathsf{j}_{R,S}$ and $\mathsf{j}_{\perp}$; for the details see Appendix A.1. □

Just as Benton we also do not have similar isomorphisms with respect to the functor H. One fact that we can point out, that Benton did not make explicit – because he did not use the notion of symmetric comonoidal functor – is that j^{-1} is J also a symmetric monoidal functor.

Corollary 25. Given any coLNL model $\mathcal{L} : \mathsf{H} \dashv \mathsf{J} : C$, the functor $(\mathsf{J}, \mathsf{j}^{-1})$ is symmetric monoidal.

Proof. This holds by straightforwardly reducing the diagrams defining a symmetric monoidal functor, Definition 7, to the diagrams defining a symmetric comonoidal functor, Definition 8, using the fact that j^{-1} is an isomorphism. □

The next result shows that any coLNL model induces a symmetric comonoidal monad.

Lemma 26 (Symmetric Comonoidal Monad)**.** Given a coLNL model $\mathcal{L} : \mathsf{H} \dashv \mathsf{J} : C$, the functor, $? = H; J$, defines a symmetric comonoidal monad.

Proof. Suppose (H, h) and (J, j) are two symmetric comonoidal functors, such that, $\mathcal{L} : \mathsf{H} \dashv \mathsf{J} : C$ is a coLNL model. We can easily show that $?A = \mathsf{JH}A$ is symmetric monoidal by defining the following maps:

$$\mathsf{r}_{\perp} := ?\perp = \mathsf{JH}\perp \xrightarrow{\mathsf{Jh}_{\perp}} \mathsf{J}0 \xrightarrow{\mathsf{j}_{\perp}} \perp$$

$$\mathsf{r}_{A,B} := ?(A \oplus B) = \mathsf{JH}(A \oplus B) \xrightarrow{\mathsf{Jh}_{A,B}} \mathsf{J}(\mathsf{H}A + \mathsf{H}B) \xrightarrow{\mathsf{j}_{\mathsf{H}A,\mathsf{H}B}} \mathsf{JH}A \oplus \mathsf{JH}B = ?A \oplus ?B$$

The fact that these maps satisfy the appropriate symmetric comonoidal functor diagrams from Definition 8 is obvious, because symmetric comonoidal functors are closed under composition.

We have a coLNL model, and hence, we have the symmetric comonoidal natural transformations $\eta_A : A \longrightarrow \mathsf{JH}A$ and $\varepsilon_R : \mathsf{HJ}R \longrightarrow R$ which correspond to the unit and counit of the adjunction respectfully. Define $\mu_A := \mathsf{J}\varepsilon_{\mathsf{H}A} : \mathsf{JHJH}A \longrightarrow \mathsf{JH}A$. This implies that we have maps $\eta_A : A \longrightarrow ?A$ and $\mu_A : ??A \longrightarrow ?A$, and thus, we can show that $(?, \eta, \mu)$ is a symmetric comonoidal monad. All the diagrams defining a symmetric comonoidal monad hold by the structure given by the adjunction. For the complete proof see Appendix A.2. □

A ?-algebra is a pair $(?A, t_X : ?^2A \longrightarrow ?A)$, and is called *free* if it is an object of the full subcategory of all ?-algebra's that is in adjunction with the category $\mathcal{L}$, such that, the right adjoint is the forgetful functor. The why-not monad must allow for the right structural rules, weakening and contraction, to be defined.

Lemma 27 (Right Weakening and Contraction)**.** Given a coLNL model $\mathcal{L} : \mathsf{H} \dashv \mathsf{J} : C$, then for any $?A$ there are distinguished symmetric comonoidal natural transformations $\mathsf{w}_A : \perp \longrightarrow ?A$ and $\mathsf{c}_A : ?A \oplus ?A \longrightarrow ?A$ that form a commutative monoid, and are ?-algebra morphisms with respect to the canonical definitions of the algebras $?A$, $\perp$, $?A \oplus ?A$.

Proof. Suppose (H,h) and (J,j) are two symmetric comonoidal functors, such that, $\mathcal{L} : \mathsf{H} \dashv \mathsf{J} : C$ is a coLNL model. Again, we know $?A = H;J : \mathcal{L} \longrightarrow \mathcal{L}$ is a symmetric comonoidal monad by Lemma 26.

We define the following morphisms:

$$\mathsf{w}_A := \perp \xrightarrow{\mathsf{j}_\perp^{-1}} \mathsf{J}0 \xrightarrow{\mathsf{J}\diamond_{\mathsf{H}A}} \mathsf{JH}A = ?A$$

$$\mathsf{c}_A := ?A \oplus ?A = \mathsf{JH}A \oplus \mathsf{JH}A \xrightarrow{\mathsf{j}^{-1}_{\mathsf{H}A,\mathsf{H}A}} \mathsf{J}(\mathsf{H}A + \mathsf{H}A) \xrightarrow{\mathsf{J}\nabla_{\mathsf{H}A}} \mathsf{JHA} = ?A$$

The remainder of the proof is by carefully checking all of the required diagrams. Please see Appendix A.3 for the complete proof. □

2.4. **A Mixed Bi-Linear/Non-Linear Model.**

Definition 28. A **mixed bi-linear/non-linear model** consists of the following:

i. a bi-linear category $(\mathcal{L}, \top, \otimes, \multimap, \perp, \oplus, \bullet\!\!-)$,
ii. a cartesian closed category $(\mathcal{I}, 1, \times, \rightarrow)$,
iii. a cocartesian coclosed category $(C, 0, +, -)$, and
iv. two symmetric monoidal adjunctions $\mathcal{I} : F \dashv G : \mathcal{L}$ and $\mathcal{L} : H \dashv J : C$.

3. Mixed Linear/Non-Linear Bi-intuitionistic Logic: BiLNL Logic

Following Benton's [2] lead we can define a mixed linear/non-linear bi-intuitionistic logic, called BiLNL logic, based on the categorical model given in the previous section. BiLNL logic consists of three fragments: an intuitionistic fragment, a cointuitionistic fragment, and a linear bi-intuitionistic core fragment. This formalization allows us to have more control on the mixture of the intuitionistic and cointuitionistic fragments so as to allow for a proper categorical model. Each of the fragments are related through a syntactic formalization of the adjoint functors from the BiLNL model. First, we define the syntax of BiLNL logic, and then discuss the inference rules for each fragment.

Definition 29. The syntax for BiLNL logic is defined as follows:

(Worlds)	$W ::= w_1 \mid \cdots \mid w_i$
(Graphs)	$G ::= w_1 \leq w_2 \mid G_1, G_2$
(Intuitionistic Formulas)	$X, Y, Z ::= 1 \mid X \times Y \mid X \rightarrow Y \mid \mathsf{G}A$
(Cointuitionistic Formulas)	$R, S, T ::= 0 \mid S + T \mid S - T \mid \mathsf{H}A$
(Linear Bi-intuitionistic Formulas)	$A, B, C ::= \top \mid \perp \mid A \otimes B \mid A \oplus B \mid A \multimap B \mid A \bullet\!\!- B \mid \mathsf{F}X \mid \mathsf{J}S$
(Intuitionistic Contexts)	$\Theta ::= \cdot \mid X@w \mid \Theta_1, \Theta_2$
(Cointuitionistic Contexts)	$\Psi ::= \cdot \mid R@w \mid \Psi_1, \Psi_2$
(Linear Bi-intuitionistic Contexts)	$\Gamma, \Delta ::= \cdot \mid A@w \mid \Gamma_1, \Gamma_2$

Worlds may also be denoted by (potentially subscripted) n, m, and o.

Sequents have the following syntax:

(Intuitionistic Sequents)	$G; \Theta \vdash_{\mathsf{I}} X@w$
(Cointuitionistic Sequents)	$G; R@w \vdash_{\mathsf{C}} \Psi$
(LNL Bi-intuitionistic Sequents)	$G; \Theta \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi$

The syntax of intuitionistic and cointuitionistic formulas are typical. I denote coimplication by $S - T$, but all the other connectives are the usual ones. Linear bi-intuitionistic formulas are denoted in somewhat of a non-traditional style. I denote the unit of tensor by $\top$ instead of the usual 1, which is the unit of intuitionistic conjunction, in addition, I denote par by $A \oplus B$, instead of $A \parr B$. Lastly, I denote linear coimplication by $A \bullet\!\!- B$ to emphasize its duality with linear implication $A \multimap B$. Each syntactic category of formulas contains the respective functor from the BiLNL model, and thus, we should view F and H as the left adjoints to G and J respectively.

Formulas in each type of context are annotated with a world, and each sequent is annotated with a graph. These graphs are syntactic representations of Kripke models and are used to enforce intuitionism. They were first used in bi-intuitionistic logic by Pinto and Uustalu [7] to enforce intuitionism in their logic L. In fact, we can see the linear core of BiLNL logic as the linear version of L. The beauty of this type of formalization and the reason why this style of logic was used by Pinto and Uustalu is that the logic L, and as well as BiLNL logic, are complete for cut-free bi-intuitionistic proofs. This was a new result of Pinto and Uustalu, because earlier formalizations of bi-intuitionistic logic [5] used the Dragalin restriction [6] to enforce intuitionism, but this results in a failure of cut-elimination [8, 7].

The expert reader will notice that it is not necessary to annotate the sequents of intuitionistic and cointuitionistic logic with graphs and worlds to enforce intuitionism and cointuitionism respectively. It is well known that restricting the right and left contexts two a single formula enforces intuitionism and cointuitionism respectively. However, when mixing these two fragments with the linear core, which requires the graphs to be intuitionistic, it is easier if they are annotated. If they were not, then a seemingly complex world inference system would need to be designed to add the world constraints before mixing with the linear core, and it is currently an open problem whether this can be done. Thus, with respect to intuitionistic and cointuitionistic logic the graph and world annotations can be seen as book keeping.

A second fact an expert reader will notice is that Kripke models are a relational model of intuitionistic logic, but not intuitionistic linear logic. This is okay, because Kripke models enforce intuitionism and not linearity, but it is well known how to enforce linearity syntactically in the definition of the inference rules. It is also well-known that even in linear logic if sequents have multiple hypothesis and multiple conclusions the logic becomes classical. Thus, in BiLNL logic we combine both of these tools to enforce both intuitionism and linearity. We can simply view the graphs as an over approximation of the relational constraints necessary to enforce both intuitionism and linearity. This also makes it easier to embed Pinto and Uustalu's logic in BiLNL logic as we will do in Section 4.

Sequents for the linear core have the form $G; \Theta \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi$. Similarly to the sequents of Benton's LNL logic [2], each context is separated for readability, but should actually be understood as being able to be mixed, that is, the contexts Θ and Γ could be a single context, and so could Δ and Ψ. The sequent:

$$G; X_1@w_1, \ldots, X_i@w_i \mid A_1@n_1, \ldots, A_j@n_j \vdash_{\mathsf{L}} B_1@m_1, \ldots, B_k@m_k \mid R_1@o_1, \ldots, R_l@o_l$$

will be interpreted in a BiLNL model by a morphism of the following form:

$$\mathsf{F}X_1 \otimes \cdots \otimes \mathsf{F}X_i \otimes A_1 \otimes \cdots \otimes A_j \xrightarrow{f} B_1 \oplus \cdots \oplus B_k \oplus \mathsf{J}R_1 \oplus \cdots \oplus \mathsf{J}R_l$$

Thus, intuitionistic formulas in the linear core can be viewed as being under the left adjoint F, and cointuitionistic formulas as being under the right adjoint J. This implies that even in the model all types of formulas can be freely mixed.

As said above, BiLNL logic consists of three fragments: an intuitionistic fragment, a cointuitionistic fragment, and a linear bi-intuitionistic fragment. Their relationships are captured by the following diagram:

$$\begin{tikzcd} \mathsf{INT} \arrow[r, bend left, "\mathsf{F}"] & \mathsf{BiL} \arrow[l, bend left, "\mathsf{G}"] \arrow[r, bend left, "\mathsf{H}"] & \mathsf{coINT} \arrow[l, bend left, "\mathsf{J}"] \end{tikzcd}$$

The inference rules for the intuitionistic fragment are defined in Figure 1, and the inference rules for the cointuitionistic fragment can be found in Figure 2. Since the functor G translates the linear fragment over to intuitionistic logic, then the intuitionistic fragment must contain an inference rule (I_GR) for this functor, and the same goes for the cointuitionistic fragment and the functor H (C_HL). Thus, these are the only interesting rules of those two fragments. Each fragment contains rules for working with the world constraints on sequents. For example, the rules I_RL and I_TS make the relation on worlds, $w_1 \leq w_2$, a preorder, and the rules I_ML and I_MR correspond to monotonicity on formulas. Similar rules exist for the other two fragments as well.

The linear core of BiLNL logic consists of a large number of rules, and for this reason they have been broken up into multiple figures. The inference rules for reflexivity, transitivity, and monotonicity are defined in Figure 3. Since the linear core is a mixed LNL logic there are monotonicity rules for intuitionistic hypothesis, and cointuitionistic conclusions. The structural rules are defined in Figure 4. The most interesting rules are weakening and contraction for both intuitionistic hypothesis and cointuitionistic conclusions. The identity and cut inference rules can be found in Figure 5. Similarly to Benton's LNL logic [2], we also have cut rules involving the different fragments:

$$\frac{G;\Theta_2 \vdash_{\mathsf{I}} X@w \qquad G;\Theta_1, X@w \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi}{G;\Theta_1,\Theta_2 \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi}\ \text{L\_Icut}$$

$$\frac{G;\Theta \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi_1, S@w \qquad G; S@w \vdash_{\mathsf{C}} \Psi_2}{G;\Theta \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi_1,\Psi_2}\ \text{L\_Ccut}$$

The first rule is the same rule in LNL logic, but the second is new and cuts a cointuitionistic conclusion in the linear core against the hypothesis proven in the cointuitionistic fragment.

The rules for conjunction and tensor, and disjunction and par can be found in Figure 6 and Figure 7 respectively. Just as we have said above, the new rules are the inference rules for disjunction in the cointuitionistic context. The inference rules for intuitionistic implication and linear implication, and cointuitionistic coimplication and linear coimplication are defined in Figure 8 and Figure 9. The most interesting rules between these are the rules for coimplication.

Lastly, the inference rules for each of the functors are defined in Figure 10. Using these rules it is possible to move between the fragments. The most restrictive rules are when coming from the intuitionistic or cointuitionistic fragment into the linear core, or vice versa, because the sequents must only contain data that is movable or consistency could fail.

4. Embedding Bi-Intuitionistic Logic in BiLNL Logic

TODO

5. BiLNL Term Assignment

TODO

$$\frac{G, w_1 \leq w_1; \Theta \vdash_I Y@w_2}{G; \Theta \vdash_I Y@w_2}\ \mathrm{I\_RL} \qquad \frac{w_1 G w_2 \quad w_2 G w_3 \quad G, w_1 \leq w_3; \Theta \vdash_I Y@w}{G; \Theta \vdash_I Y@w}\ \mathrm{I\_TS}$$

$$\frac{w_1 G w_2 \quad G; \Theta, X@w_1, X@w_2 \vdash_I Y@w}{G; \Theta, X@w_1 \vdash_I Y@w}\ \mathrm{I\_ML} \qquad \frac{w_2 G w_1 \quad G; \Theta \vdash_I Y@w_2}{G; \Theta \vdash_I Y@w_1}\ \mathrm{I\_MR}$$

$$\frac{}{G; Y@w \vdash_I Y@w}\ \mathrm{I\_ID} \qquad \frac{G; \Theta_2 \vdash_I X@w_2 \quad G; \Theta_1, X@w_2 \vdash_I Z@w_1}{G; \Theta_1, \Theta_2 \vdash_I Z@w_1}\ \mathrm{I\_CUT}$$

$$\frac{G; \Theta \vdash_I Y@w_1}{G; \Theta, X@w_2 \vdash_I Y@w_1}\ \mathrm{I\_WK} \qquad \frac{G; \Theta, X@w_2, X@w_2 \vdash_I Y@w_1}{G; \Theta, X@w_2 \vdash_I Y@w_1}\ \mathrm{I\_CR}$$

$$\frac{G; \Theta_1, X@w_1, Y@w_2, \Theta_2 \vdash_I Z@w}{G; \Theta_1, Y@w_2, X@w_1, \Theta_2 \vdash_I Z@w}\ \mathrm{I\_EX} \qquad \frac{G; \Theta \vdash_I Y@w_1}{G; \Theta, 1@w_2 \vdash_I Y@w_1}\ \mathrm{I\_TL}$$

$$\frac{}{G; \Theta \vdash_I 1@w}\ \mathrm{I\_TR} \qquad \frac{G; \Theta, X@w_1, Y@w_1 \vdash_I Z@w_2}{G; \Theta, (X \times Y)@w_1 \vdash_I Z@w_2}\ \mathrm{I\_PL}$$

$$\frac{G; \Theta_1 \vdash_I X@w \quad G; \Theta_2 \vdash_I Y@w}{G; \Theta_1, \Theta_2 \vdash_I (X \times Y)@w}\ \mathrm{I\_PR}$$

$$\frac{w_1 G w_2 \quad G; \Theta_2 \vdash_I X@w_2 \quad G; \Theta_1, Y@w_2 \vdash_I Z@w}{G; \Theta_1, \Theta_2, (X \to Y)@w_1 \vdash_I Z@w}\ \mathrm{I\_IL}$$

$$\frac{w_2 \notin |G|, |\Theta| \quad G, w_1 \leq w_2; \Theta, X@w_2 \vdash_I Y@w_2}{G; \Theta \vdash_I (X \to Y)@w_1}\ \mathrm{I\_IR} \qquad \frac{G; \Theta \mid \cdot \vdash_L A@w \mid \cdot}{G; \Theta \vdash_I \mathsf{G}A@w}\ \mathrm{I\_GR}$$

Figure 1: Inference Rules for BiLNL Logic: Intuitionistic Fragment

6. Related Work

TODO

7. Conclusion

TODO

References

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$$\dfrac{G, w_1 \leq w_1; S@w_2 \vdash_{\mathsf{C}} \Psi}{G; S@w_2 \vdash_{\mathsf{C}} \Psi}\ \text{C\_RL} \qquad \dfrac{\begin{array}{l} w_1 G w_2 \qquad w_2 G w_3 \\ G, w_1 \leq w_3; S@w \vdash_{\mathsf{C}} \Psi \end{array}}{G; S@w \vdash_{\mathsf{C}} \Psi}\ \text{C\_TS}$$

$$\dfrac{\begin{array}{l} w_1 G w_2 \\ G; S@w_2 \vdash_{\mathsf{C}} \Psi \end{array}}{G; S@w_1 \vdash_{\mathsf{C}} \Psi}\ \text{C\_ML} \qquad \dfrac{\begin{array}{l} w_2 G w_1 \\ G; S@w \vdash_{\mathsf{C}} T@w_2, T@w_1, \Psi \end{array}}{G; S@w \vdash_{\mathsf{C}} T@w_1, \Psi}\ \text{C\_MR}$$

$$\dfrac{}{G; S@w \vdash_{\mathsf{C}} S@w}\ \text{C\_ID} \qquad \dfrac{G; S@w_1 \vdash_{\mathsf{C}} T@w_2, \Psi_2 \qquad G; T@w_2 \vdash_{\mathsf{C}} \Psi_1}{G; S@w_1 \vdash_{\mathsf{C}} \Psi_1, \Psi_2}\ \text{C\_CUT}$$

$$\dfrac{G; S@w_1 \vdash_{\mathsf{C}} \Psi}{G; S@w_1 \vdash_{\mathsf{C}} T@w_2, \Psi}\ \text{C\_WK} \qquad \dfrac{G; S@w_1 \vdash_{\mathsf{C}} T@w_2, T@w_2, \Psi}{G; S@w_1 \vdash_{\mathsf{C}} T@w_2, \Psi}\ \text{C\_CR}$$

$$\dfrac{G; R@w \vdash_{\mathsf{C}} \Psi_1, S@w_1, T@w_2, \Psi_2}{G; R@w \vdash_{\mathsf{C}} \Psi_1, T@w_2, S@w_1, \Psi_2}\ \text{C\_EX} \qquad \dfrac{}{G; 0@w \vdash_{\mathsf{C}} \Psi}\ \text{C\_FL}$$

$$\dfrac{G; S@w_1 \vdash_{\mathsf{C}} \Psi}{G; S@w_1 \vdash_{\mathsf{C}} 0@w_2, \Psi}\ \text{C\_FR} \qquad \dfrac{G; S@w \vdash_{\mathsf{C}} \Psi_1 \qquad G; T@w \vdash_{\mathsf{C}} \Psi_2}{G; S + T@w \vdash_{\mathsf{C}} \Psi_1, \Psi_2}\ \text{C\_DL}$$

$$\dfrac{G; R@w_1 \vdash_{\mathsf{C}} S@w_2, T@w_2, \Psi}{G; R@w_1 \vdash_{\mathsf{C}} S + T@w_2, \Psi}\ \text{C\_DR} \qquad \dfrac{\begin{array}{l} w_2 \notin |G|, |\Psi| \\ G, w_2 \leq w_1; S@w_2 \vdash_{\mathsf{C}} T@w_2, \Psi \end{array}}{G; S - T@w_1 \vdash_{\mathsf{C}} \Psi}\ \text{C\_SL}$$

$$\dfrac{\begin{array}{l} w_2 G w_1 \\ G; R@w \vdash_{\mathsf{C}} S@w_2, \Psi_2 \qquad G; T@w_2 \vdash_{\mathsf{C}} \Psi_1 \end{array}}{G; R@w \vdash_{\mathsf{C}} S - T@w_1, \Psi_1, \Psi_2}\ \text{C\_SR} \qquad \dfrac{G; \cdot \mid A@w \vdash_{\mathsf{L}} \cdot \mid \Psi}{G; \mathsf{H} A@w \vdash_{\mathsf{C}} \Psi}\ \text{C\_HL}$$

Figure 2: Inference Rules for BiLNL Logic: Cointuitionistic Fragment

$$\dfrac{G, w \leq w; \Theta \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi}{G; \Theta \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi}\ \text{L\_RL} \qquad \dfrac{\begin{array}{l} w_1 G w_2 \qquad w_2 G w_3 \\ G, w_1 \leq w_3; \Theta \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi \end{array}}{G; \Theta \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi}\ \text{L\_TS}$$

$$\dfrac{\begin{array}{l} w_1 G w_2 \\ G; \Theta \mid \Gamma, A@w_1, A@w_2 \vdash_{\mathsf{L}} \Delta \mid \Psi \end{array}}{G; \Theta \mid \Gamma, A@w_1 \vdash_{\mathsf{L}} \Delta \mid \Psi}\ \text{L\_ML} \qquad \dfrac{\begin{array}{l} w_2 G w_1 \\ G; \Theta \mid \Gamma \vdash_{\mathsf{L}} A@w_2, A@w_1, \Delta \mid \Psi \end{array}}{G; \Theta \mid \Gamma \vdash_{\mathsf{L}} A@w_1, \Delta \mid \Psi}\ \text{L\_MR}$$

$$\dfrac{\begin{array}{l} w_1 G w_2 \\ G; \Theta, X@w_1, X@w_2 \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi \end{array}}{G; \Theta, X@w_1 \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi}\ \text{L\_IML} \qquad \dfrac{\begin{array}{l} w_2 G w_1 \\ G; \Theta \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid T@w_2, T@w_1, \Psi \end{array}}{G; \Theta \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid T@w_1, \Psi}\ \text{L\_CMR}$$

Figure 3: Inference Rules for BiLNL Logic: Abstract Kripke Graph Rules

$$\frac{G;\Theta \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi}{G;\Theta, X@w \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi}\ \mathrm{L\_{\small WK}L} \qquad \frac{G;\Theta \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi}{G;\Theta \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid S@w, \Psi}\ \mathrm{L\_{\small WK}R}$$

$$\frac{G;\Theta, X@w, X@w \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi}{G;\Theta, X@w \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi}\ \mathrm{L\_{\small CTR}L} \qquad \frac{G;\Theta \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid S@w, S@w, \Psi}{G;\Theta \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid S@w, \Psi}\ \mathrm{L\_{\small CTR}R}$$

$$\frac{G;\Theta \mid \Gamma_1, A@w_1, B@w_2, \Gamma_2 \vdash_{\mathsf{L}} \Delta \mid \Psi}{G;\Theta \mid \Gamma_1, B@w_2, A@w_1, \Gamma_2 \vdash_{\mathsf{L}} \Delta \mid \Psi}\ \mathrm{L\_{\small EX}L}$$

$$\frac{G;\Theta \mid \Gamma \vdash_{\mathsf{L}} \Delta_1, A@w_1, B@w_2, \Delta_2 \mid \Psi}{G;\Theta \mid \Gamma \vdash_{\mathsf{L}} \Delta_1, B@w_2, A@w_1, \Delta_2 \mid \Psi}\ \mathrm{L\_{\small EX}R}$$

$$\frac{G;\Theta_1, X@w_1, Y@w_2, \Theta_2 \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi}{G;\Theta_1, Y@w_2, X@w_1, \Theta_2 \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi}\ \mathrm{L\_I{\small EX}L}$$

$$\frac{G;\Theta \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi_1, S@w_1, T@w_2, \Psi_2}{G;\Theta \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi_1, T@w_2, S@w_1, \Psi_2}\ \mathrm{L\_C{\small EX}L}$$

Figure 4: Inference Rules for BiLNL Logic: Structural Rules

$$\frac{}{G; \cdot \mid A@w \vdash_{\mathsf{L}} A@w \mid \cdot}\ \mathrm{L\_{\small ID}}$$

$$\frac{G;\Theta_1 \mid \Gamma_1 \vdash_{\mathsf{L}} A@w, \Delta_2 \mid \Psi_1 \qquad G;\Theta_2 \mid A@w, \Gamma_2 \vdash_{\mathsf{L}} \Delta_1 \mid \Psi_2}{G;\Theta_1, \Theta_2 \mid \Gamma_1, \Gamma_2 \vdash_{\mathsf{L}} \Delta_1, \Delta_2 \mid \Psi_1, \Psi_2}\ \mathrm{L\_{\small CUT}}$$

$$\frac{G;\Theta_2 \vdash_{\mathsf{I}} X@w \qquad G;\Theta_1, X@w \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi}{G;\Theta_1, \Theta_2 \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi}\ \mathrm{L\_I{\small CUT}}$$

$$\frac{G;\Theta \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi_1, S@w \qquad G; S@w \vdash_{\mathsf{C}} \Psi_2}{G;\Theta \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi_1, \Psi_2}\ \mathrm{L\_C{\small CUT}}$$

Figure 5: Inference Rules for BiLNL Logic: Identity and Cut Rules

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[3] J.R.B. Cockett and R.A.G. Seely. Proof theory for full intuitionistic linear logc, bilinear logic, and mix categories. *Theory and Applications of Categories*, 3(5):85–131, 1997.

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$$\frac{G;\Theta \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi}{G;\Theta \mid \Gamma, \top@w \vdash_{\mathsf{L}} \Delta \mid \Psi}\ \text{L\_IL} \qquad \frac{}{G;\cdot \mid \cdot \vdash_{\mathsf{L}} \top@w \mid \cdot}\ \text{L\_IR}$$

$$\frac{G;\Theta_1, X@w, Y@w, \Theta_2 \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi}{G;\Theta_1, X \times Y@w, \Theta_2 \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi}\ \text{L\_cL}$$

$$\frac{G;\Theta \mid \Gamma_1, A@w, B@w, \Gamma_2 \vdash_{\mathsf{L}} \Delta \mid \Psi}{G;\Theta \mid \Gamma_1, A \otimes B@w, \Gamma_2 \vdash_{\mathsf{L}} \Delta \mid \Psi}\ \text{L\_\textsc{t}L}$$

$$\frac{G;\Theta_1 \mid \Gamma_1 \vdash_{\mathsf{L}} A@w, \Delta_1 \mid \Psi_1 \qquad G;\Theta_2 \mid \Gamma_2 \vdash_{\mathsf{L}} B@w, \Delta_2 \mid \Psi_2}{G;\Theta_1, \Theta_2 \mid \Gamma_1, \Gamma_2 \vdash_{\mathsf{L}} A \otimes B@w, \Delta_1, \Delta_2 \mid \Psi_1, \Psi_2}\ \text{L\_\textsc{t}R}$$

Figure 6: Inference Rules for BiLNL Logic: Conjunction and Tensor Rules

$$\frac{}{G;\cdot \mid \perp @w \vdash_{\mathsf{L}} \cdot \mid \cdot}\ \text{L\_\textsc{fl}L} \qquad \frac{G;\Theta \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi}{G;\Theta \mid \Gamma \vdash_{\mathsf{L}} \perp @w, \Delta \mid \Psi}\ \text{L\_\textsc{fl}R}$$

$$\frac{G;\Theta \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi_1, S@w, T@w, \Psi_2}{G;\Theta \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi_1, S + T@w, \Psi_2}\ \text{L\_\textsc{d}R}$$

$$\frac{G;\Theta_1 \mid \Gamma_1, A@w \vdash_{\mathsf{L}} \Delta_1 \mid \Psi_1 \qquad G;\Theta_2 \mid \Gamma_2, B@w \vdash_{\mathsf{L}} \Delta_2 \mid \Psi_2}{G;\Theta_1, \Theta_2 \mid \Gamma_1, \Gamma_2, A \oplus B@w \vdash_{\mathsf{L}} \Delta_1, \Delta_2 \mid \Psi_1, \Psi_2}\ \text{L\_\textsc{p}L}$$

$$\frac{G;\Theta \mid \Gamma \vdash_{\mathsf{L}} \Delta_1, A@w, B@w, \Delta_2 \mid \Psi}{G;\Theta \mid \Gamma \vdash_{\mathsf{L}} \Delta_1, A \oplus B@w, \Delta_2 \mid \Psi}\ \text{L\_\textsc{p}R}$$

Figure 7: Inference Rules for BiLNL Logic: Disjunction and Par Rules

$$\frac{\begin{array}{l} w_1 G w_2 \\ G;\Theta_1 \mid \Gamma_1 \vdash_{\mathsf{L}} A@w_2, \Delta_1 \mid \Psi_1 \qquad G;\Theta_2 \mid \Gamma_2, B@w_2 \vdash_{\mathsf{L}} \Delta_2 \mid \Psi_2 \end{array}}{G;\Theta_1, \Theta_2 \mid \Gamma_1, \Gamma_2, A \multimap B@w_1 \vdash_{\mathsf{L}} \Delta_1, \Delta_2 \mid \Psi_1, \Psi_2}\ \text{L\_\textsc{Imp}L}$$

$$\frac{\begin{array}{l} w_2 \notin |G|, |\Theta|, |\Gamma|, |\Delta|, |\Psi| \\ G, w_1 \leq w_2;\Theta \mid \Gamma, A@w_2 \vdash_{\mathsf{L}} B@w_2, \Delta \mid \Psi \end{array}}{G;\Theta \mid \Gamma \vdash_{\mathsf{L}} A \multimap B@w_1, \Delta \mid \Psi}\ \text{L\_\textsc{Imp}R}$$

$$\frac{\begin{array}{l} w_1 G w_2 \\ G;\Theta_1 \vdash_{\mathsf{I}} X@w_2 \qquad G;\Theta_2, Y@w_2 \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi \end{array}}{G;\Theta_1, \Theta_2, X \to Y@w_1 \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi}\ \text{L\_I\textsc{Imp}L}$$

Figure 8: Inference Rules for BiLNL Logic: Implication Rules

$$\frac{\begin{array}{l} w_2 \notin |G|, |\Theta|, |\Gamma|, |\Delta|, |\Psi| \\ G, w_2 \leq w_1; \Theta \mid \Gamma, A@w_2 \vdash_{\mathsf{L}} B@w_2, \Delta \mid \Psi \end{array}}{G; \Theta \mid \Gamma, A \bullet\!\!- B@w_1 \vdash_{\mathsf{L}} \Delta \mid \Psi} \; \text{L\_sL}$$

$$\frac{\begin{array}{l} w_2 G w_1 \\ G; \Theta_1 \mid \Gamma_1 \vdash_{\mathsf{L}} A@w_2, \Delta_1 \mid \Psi_1 \qquad G; \Theta_2 \mid \Gamma_2, B@w_2 \vdash_{\mathsf{L}} \Delta_2 \mid \Psi_2 \end{array}}{G; \Theta_1, \Theta_2 \mid \Gamma_1, \Gamma_2 \vdash_{\mathsf{L}} A \bullet\!\!- B@w_1, \Delta_1, \Delta_2 \mid \Psi_1, \Psi_2} \; \text{L\_sR}$$

$$\frac{\begin{array}{l} w_2 G w_1 \\ G; \Theta \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid S@w_2, \Psi_1 \qquad G; T@w_2 \vdash_{\mathsf{C}} \Psi_2 \end{array}}{G; \Theta \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid S - T@w_1, \Psi_1, \Psi_2} \; \text{L\_CsR}$$

Figure 9: Inference Rules for BiLNL Logic: Coimplication Rules

$$\frac{G; \Theta, X@w \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi}{G; \Theta \mid \Gamma, \mathsf{F}X@w \vdash_{\mathsf{L}} \Delta \mid \Psi} \; \text{L\_FL} \qquad \frac{G; \Theta \vdash_{\mathsf{I}} X@w}{G; \Theta \mid \cdot \vdash_{\mathsf{L}} \mathsf{F}X@w \mid \cdot} \; \text{L\_FR}$$

$$\frac{G; S@w \vdash_{\mathsf{C}} \Psi}{G; \cdot \mid \mathsf{J}S@w \vdash_{\mathsf{L}} \cdot \mid \Psi} \; \text{L\_JL} \qquad \frac{G; \Theta \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid S@w, \Psi}{G; \Theta \mid \Gamma \vdash_{\mathsf{L}} \Delta, \mathsf{J}S@w \mid \Psi} \; \text{L\_JR}$$

$$\frac{G; \Theta \mid \Gamma, A@w \vdash_{\mathsf{L}} \Delta \mid \Psi}{G; \Theta, \mathsf{G}A@w \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \Psi} \; \text{L\_GL} \qquad \frac{G; \Theta \mid \Gamma \vdash_{\mathsf{L}} \Delta, A@w \mid \Psi}{G; \Theta \mid \Gamma \vdash_{\mathsf{L}} \Delta \mid \mathsf{H}A@w, \Psi} \; \text{L\_HR}$$

Figure 10: Inference Rules for BiLNL Logic: Adjoint Functors Rules

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Appendix A. Proofs

A.1. **Proof of Lemma 24.** We show that both of the maps:

$$\mathsf{j}^{-1}_{R,S} := \mathsf{J}R \oplus \mathsf{J}S \xrightarrow{\eta} \mathsf{JH}(\mathsf{J}R \oplus \mathsf{J}S) \xrightarrow{\mathsf{Jh}_{A,B}} \mathsf{J}(\mathsf{HJ}R + \mathsf{HJ}S) \xrightarrow{\mathsf{J}(\varepsilon_R + \varepsilon_S)} \mathsf{J}(R + S)$$

$$\mathsf{j}^{-1}_{\perp} := \perp \xrightarrow{\eta} \mathsf{JH} \perp \xrightarrow{\mathsf{Jh}_{\perp}} \mathsf{J}0$$

are mutual inverses with $\mathsf{j}_{R,S} : \mathsf{J}(R + S) \longrightarrow \mathsf{J}R \oplus \mathsf{J}S$ and $\mathsf{j}_{\perp} : \perp \longrightarrow \mathsf{J}0$ respectively.

Case. The following diagram implies that $\mathsf{j}^{-1}_{R,S}; \mathsf{j}_{R,S} = \mathsf{id}$:

$$\frac{G, w \leq w; \Gamma \vdash_{\mathsf{L}} \Delta}{G; \Gamma \vdash_{\mathsf{L}} \Delta}\ \textsc{rl} \qquad \frac{\begin{array}{l} w_1 G w_2 \qquad w_2 G w_3 \\ G, w_1 \leq w_3; \Gamma \vdash_{\mathsf{L}} \Delta \end{array}}{G; \Gamma \vdash_{\mathsf{L}} \Delta}\ \textsc{ts} \qquad \frac{\begin{array}{l} w_1 G w_2 \\ G; \Gamma, A@w_1, A@w_2 \vdash_{\mathsf{L}} \Delta \end{array}}{G; \Gamma, A@w_1 \vdash_{\mathsf{L}} \Delta}\ \textsc{m}\mathrm{L}$$

$$\frac{\begin{array}{l} w_2 G w_1 \\ G; \Gamma \vdash_{\mathsf{L}} A@w_2, A@w_1, \Delta \end{array}}{G; \Gamma \vdash_{\mathsf{L}} A@w_1, \Delta}\ \textsc{m}\mathrm{R} \qquad \frac{G; \Gamma \vdash_{\mathsf{L}} \Delta}{G; \Gamma, A@w \vdash_{\mathsf{L}} \Delta}\ \textsc{wk}\mathrm{L} \qquad \frac{G; \Gamma \vdash_{\mathsf{L}} \Delta}{G; \Gamma \vdash_{\mathsf{L}} A@w, \Delta}\ \textsc{wk}\mathrm{R}$$

$$\frac{G; \Gamma, A@w, A@w \vdash_{\mathsf{L}} \Delta}{G; \Gamma, A@w \vdash_{\mathsf{L}} \Delta}\ \textsc{ctr}\mathrm{L} \qquad \frac{G; \Gamma \vdash_{\mathsf{L}} A@w, A@w, \Delta}{G; \Gamma \vdash_{\mathsf{L}} A@w, \Delta}\ \textsc{ctr}\mathrm{R}$$

$$\frac{G; \Gamma_1, A@w_1, B@w_2, \Gamma_2 \vdash_{\mathsf{L}} \Delta}{G; \Gamma_1, B@w_2, A@w_1, \Gamma_2 \vdash_{\mathsf{L}} \Delta}\ \textsc{ex}\mathrm{L} \qquad \frac{G; \Gamma \vdash_{\mathsf{L}} \Delta_1, A@w_1, B@w_2, \Delta_2}{G; \Gamma \vdash_{\mathsf{L}} \Delta_1, B@w_2, A@w_1, \Delta_2}\ \textsc{ex}\mathrm{R}$$

$$\frac{}{G; A@w \vdash_{\mathsf{L}} A@w}\ \textsc{id} \qquad \frac{G; \Gamma_1 \vdash_{\mathsf{L}} A@w, \Delta_2 \qquad G; A@w, \Gamma_2 \vdash_{\mathsf{L}} \Delta_1}{G; \Gamma_1, \Gamma_2 \vdash_{\mathsf{L}} \Delta_1, \Delta_2}\ \textsc{cut}$$

$$\frac{G; \Gamma \vdash_{\mathsf{L}} \Delta}{G; \Gamma, \top@w \vdash_{\mathsf{L}} \Delta}\ \mathrm{IL} \qquad \frac{}{G; \cdot \vdash_{\mathsf{L}} \perp @w}\ \mathrm{IR} \qquad \frac{}{G; \top@w \vdash_{\mathsf{L}} \cdot}\ \textsc{fl}\mathrm{L}$$

$$\frac{G; \Gamma \vdash_{\mathsf{L}} \Delta}{G; \Gamma \vdash_{\mathsf{L}} \perp @w, \Delta}\ \textsc{fl}\mathrm{R} \qquad \frac{G; \Gamma_1, A@w, B@w, \Gamma_2 \vdash_{\mathsf{L}} \Delta}{G; \Gamma_1, A \times B@w, \Gamma_2 \vdash_{\mathsf{L}} \Delta}\ \mathrm{cL}$$

$$\frac{G; \Gamma_1 \vdash_{\mathsf{L}} A@w, \Delta_1 \qquad G; \Gamma_2 \vdash_{\mathsf{L}} B@w, \Delta_2}{G; \Gamma_1, \Gamma_2 \vdash_{\mathsf{L}} A \times B@w}\ \mathrm{cR}$$

$$\frac{G; \Gamma_1, A@w \vdash_{\mathsf{L}} \Delta_1 \qquad G; \Gamma_2, B@w \vdash_{\mathsf{L}} \Delta_2}{G; \Gamma_1, \Gamma_2, A + B@w \vdash_{\mathsf{L}} \Delta_1, \Delta_2}\ \textsc{d}\mathrm{L} \qquad \frac{G; \Gamma \vdash_{\mathsf{L}} \Delta_1, A@w, B@w, \Delta_2}{G; \Gamma \vdash_{\mathsf{L}} \Delta_1, A + B@w, \Delta_2}\ \textsc{d}\mathrm{R}$$

$$\frac{\begin{array}{l} w_2 \notin |G|, |\Gamma|, |\Delta| \\ G, w_1 \leq w_2; \Gamma, A@w_2 \vdash_{\mathsf{L}} B@w_2, \Delta \end{array}}{G; \Gamma \vdash_{\mathsf{L}} A \to B@w_1, \Delta}\ \mathrm{I}\textsc{mp}\mathrm{R}$$

$$\frac{\begin{array}{l} w_1 G w_2 \\ G; \Gamma_1 \vdash_{\mathsf{L}} A@w_2, \Delta_1 \qquad G; \Gamma_2, B@w_2 \vdash_{\mathsf{L}} \Delta_2 \end{array}}{G; \Gamma_1, \Gamma_2, A \to B@w_1 \vdash_{\mathsf{L}} \Delta_1, \Delta_2}\ \mathrm{I}\textsc{mp}\mathrm{L}$$

$$\frac{\begin{array}{l} w_2 \notin |G|, |\Gamma|, |\Delta| \\ G, w_2 \leq w_1; \Gamma, A@w_2 \vdash_{\mathsf{L}} B@w_2, \Delta \end{array}}{G; \Gamma, A - B@w_1 \vdash_{\mathsf{L}} \Delta}\ \mathrm{sL}$$

$$\frac{\begin{array}{l} w_2 G w_1 \\ G; \Gamma_1 \vdash_{\mathsf{L}} A@w_2, \Delta_1 \qquad G; \Gamma_2, B@w_2 \vdash_{\mathsf{L}} \Delta_2 \end{array}}{G; \Gamma_1, \Gamma_2 \vdash_{\mathsf{L}} A - B@w_1, \Delta_1, \Delta_2}\ \mathrm{sR}$$

Figure 11: Inference Rules for L

$$\begin{tikzcd}
& \mathsf{J}R \oplus \mathsf{J}S \arrow[r, "\eta"] \arrow[dl, equal] \arrow[d, "\eta\oplus\eta"] & \mathsf{JH}(\mathsf{J}R \oplus \mathsf{J}S) \arrow[d, "\mathsf{Jh}"] \\
\mathsf{J}R \oplus \mathsf{J}S & \mathsf{HJ}R \oplus \mathsf{JHJ}S \arrow[l, "\mathsf{J}\varepsilon\oplus\mathsf{J}\varepsilon"'] & \mathsf{J}(\mathsf{HJ}R + \mathsf{HJ}S) \arrow[l, "\mathsf{j}"'] \arrow[d, "\mathsf{J}(\varepsilon+\varepsilon)"] \\
& & \mathsf{J}(R+S) \arrow[ull, "\mathsf{j}"]
\end{tikzcd}$$

The two top diagrams both commute because η and ε are the unit and counit of the adjunction respectively, and the bottom diagram commutes by naturality of j.

Case. The following diagram implies that $\mathsf{j}_{R,S} ; \mathsf{j}^{-1}_{R,S} = \mathsf{id}$:

$$\begin{tikzcd}
& \mathsf{J}(R+S) \arrow[r, "\mathsf{j}"] \arrow[dl, equal] \arrow[d, "\eta"] & \mathsf{J}R \oplus \mathsf{J}S \arrow[d, "\eta"] \\
\mathsf{J}(R+S) & \mathsf{JHJ}(R+S) \arrow[l, "\mathsf{J}\varepsilon"'] \arrow[r, "\mathsf{JHj}"] & \mathsf{JH}(\mathsf{J}R \oplus \mathsf{J}S) \arrow[d, "\mathsf{Jh}"] \\
& & \mathsf{J}(\mathsf{HJ}R + \mathsf{HJ}S) \arrow[ull, "\mathsf{J}(\varepsilon+\varepsilon)"]
\end{tikzcd}$$

The top left and bottom diagrams both commute because η and ε are the unit and counit of the adjunction respectively, and the top right diagram commutes by naturality of η.

Case. The following diagram implies that $\mathsf{j}^{-1}_{\perp} ; \mathsf{j}_{\perp} = \mathsf{id}$:

$$\begin{tikzcd}
\perp \arrow[r, "\eta"] \arrow[d, equal] & \mathsf{JH}\perp \arrow[d, "\mathsf{Jh}_{\perp}"] \\
\perp & \mathsf{J}0 \arrow[l, "\mathsf{j}_{\perp}"']
\end{tikzcd}$$

This diagram holds because η is the unit of the adjunction.

Case. The following diagram implies that $\mathsf{j}_{\perp} ; \mathsf{j}^{-1}_{\perp} = \mathsf{id}$:

$$\begin{tikzcd}
\mathsf{J}0 \arrow[rr, "\mathsf{j}_{\perp}"] \arrow[dr, "\eta"] \arrow[dd, equal] & & \perp \arrow[dd, "\eta"] \\
& \mathsf{JHJ}0 \arrow[dl, "\mathsf{J}\varepsilon"'] \arrow[dr, "\mathsf{JHj}_{\perp}"] & \\
\mathsf{J}0 & & \mathsf{JH}\perp \arrow[ll, "\mathsf{Jh}_{\perp}"']
\end{tikzcd}$$

The top-left and bottom diagrams commute because η and ε are the unit and counit of the adjunction respectively, and the top-right digram commutes by naturality of η.

A.2. **Proof of Lemma 26.** Since ? is the composition of two symmetric comonoidal functors we know it is also symmetric comonoidal, and hence, the following diagrams all hold:

$$\begin{tikzcd}
?((A \oplus B) \oplus C) \arrow[r, "\mathsf{r}_{A \oplus B, C}"] \arrow[d, "?\,\alpha_{A,B,C}"] & ?(A \oplus B) \oplus ?\,C \arrow[d, "\mathsf{r}_{A,B} \oplus \mathsf{id}_{?\,C}"] \\
?(A \oplus (B \oplus C)) \arrow[d, "\mathsf{r}_{A, B \oplus C}"] & (?\,A \oplus ?\,B) \oplus ?\,C \arrow[d, "\alpha_{?\,A,?\,B,?\,C}"] \\
?\,A \oplus ?(B \oplus C)) \arrow[r, "\mathsf{id}_{?\,A} \oplus \mathsf{r}_{B,C}"] & ?\,A \oplus (?\,B \oplus ?\,C)
\end{tikzcd}$$

$$\begin{tikzcd}
?(\perp \oplus A) \arrow[r, "\mathsf{r}_{\perp,A}"] \arrow[d, "?\,\lambda_A"] & ?\perp \oplus ?\,A \arrow[d, "\mathsf{r}_{\perp} \oplus \mathsf{id}_{?\,A}"] \\
?\,A \arrow[r, "\lambda^{-1}{}_{?\,A}"] & \perp \oplus ?\,A
\end{tikzcd}
\qquad
\begin{tikzcd}
?(A \oplus \perp) \arrow[r, "\mathsf{r}_{A,\perp}"] \arrow[d, "?\,\rho_A"] & ?\,A \oplus ?\perp \arrow[d, "\mathsf{id}_{?\,A} \oplus \mathsf{r}_{\perp}"] \\
?\,A \arrow[r, "\rho^{-1}{}_{?\,A}"] & ?\,A \oplus \perp
\end{tikzcd}$$

$$\begin{tikzcd}
?(A \oplus B) \arrow[r, "\mathsf{r}_{A,B}"] \arrow[d, "?\,\beta_{A,B}"] & ?\,A \oplus ?\,B \arrow[d, "\beta_{?\,A,?\,B}"] \\
?(B \oplus A) \arrow[r, "\mathsf{r}_{B,A}"] & ?\,B \oplus ?\,A
\end{tikzcd}$$

Next we show that $(?, \eta, \mu)$ defines a monad where $\eta_A : A \longrightarrow ?A$ is the unit of the adjunction, and $\mu_A = \mathsf{J}\varepsilon_{\mathsf{H}\,A} : ?\,?\,A \longrightarrow ?\,A$. It suffices to show that every diagram of Definition 14 holds.

Case.

$$\begin{tikzcd}
?^3\,A \arrow[r, "\mu_{?\,A}"] \arrow[d, "?\,\mu_A"] & ?^2\,A \arrow[d, "\mu_A"] \\
?^2\,A \arrow[r, "\mu_A"'] & ?\,A
\end{tikzcd}$$

It suffices to show that the following diagram commutes:

$$\begin{tikzcd}
\mathsf{J}(\mathsf{H}(?^2\,A)) \arrow[r, "\mathsf{J}\varepsilon_{\mathsf{H}\,?\,A}"] \arrow[d, "\mathsf{J}(\mathsf{H}\mu_A)"] & \mathsf{J}(\mathsf{H}\ ?\,A) \arrow[d, "\mathsf{J}\varepsilon_{\mathsf{H}\,A}"] \\
\mathsf{J}(\mathsf{H}\ ?\,A) \arrow[r, "\mathsf{J}\varepsilon_{\mathsf{H}\,A}"'] & \mathsf{J}(\mathsf{H}\,A)
\end{tikzcd}$$

But this diagram is equivalent to the following:

$$\begin{tikzcd}
\mathsf{HJHJH}A \arrow[r, "\varepsilon_{\mathsf{HJH}A}"] \arrow[d, "\mathsf{HJ}\varepsilon_{\mathsf{H}A}"'] & \mathsf{HJH}A \arrow[d, "\varepsilon_{\mathsf{H}A}"] \\
\mathsf{HJH}A \arrow[r, "\varepsilon_{\mathsf{H}A}"'] & \mathsf{H}A
\end{tikzcd}$$

The previous diagram commutes by naturality of ε.

Case.

$$\begin{tikzcd}
 & ?\,A \arrow[dl, equal] \arrow[dr, equal] & \\
?\,A \arrow[r, "\eta_{?A}"'] & ?^2\,A \arrow[u, "\mu_A"] & ?\,A \arrow[l, "?\,\eta_A"]
\end{tikzcd}$$

It suffices to show that the following diagrams commutes:

$$\begin{tikzcd}
 & JHA \arrow[dl, equal] \arrow[dr, equal] & \\
JHA \arrow[r, "\eta_{JHA}"'] & JHJHA \arrow[u, "J\varepsilon_{HA}"] & JHA \arrow[l, "JH\eta_A"]
\end{tikzcd}$$

Both of these diagrams commute because η and ε are the unit and counit of an adjunction.

It remains to be shown that η and μ are both symmetric comonoidal natural transformations, but this easily follows from the fact that we know η is by assumption, and that μ is because it is defined in terms of ε which is a symmetric comonoidal natural transformation. Thus, all of the following diagrams commute:

$$\begin{tikzcd}
A \oplus B \arrow[r, "\eta_A \oplus \eta_B"] \arrow[d, "\eta_A"'] & ?\,A \oplus ?\,B \\
?(A \oplus B) \arrow[ur, "\mathsf{r}_{A,B}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\perp \arrow[r, "\eta_\perp"] \arrow[dr, equal] & ?\,\perp \arrow[d, "\mathsf{r}_\perp"] \\
 & \perp
\end{tikzcd}$$

$$\begin{tikzcd}
?^2(A \oplus B) \arrow[r, "?\,\mathsf{r}_{A,B}"] \arrow[d, "\mu_{A\oplus B}"'] & ?(?\,A \oplus ?\,B) \arrow[r, "\mathsf{r}_{?A,?B}"] & ?^2\,A \oplus ?^2\,B \arrow[d, "\mu_A \oplus \mu_B"] \\
?(A \oplus B) \arrow[rr, "\mathsf{r}_{A,B}"] & & ?\,A \oplus ?\,B
\end{tikzcd}
\qquad
\begin{tikzcd}
?^2\,\perp \arrow[r, "?\,\mathsf{r}_\perp"] \arrow[d, "\mu_\perp"'] & ?\,\perp \arrow[d, "\mathsf{r}_\perp"] \\
?\,\perp \arrow[r, "\mathsf{r}_\perp"'] & \perp
\end{tikzcd}$$

A.3. **Proof of Lemma 27.** Suppose (H,h) and (J,j) are two symmetric comonoidal functors, such that, $\mathcal{L} : \mathsf{H} \dashv \mathsf{J} : C$ is a coLNL model. Again, we know $?A = H;J : \mathcal{L} \longrightarrow \mathcal{L}$ is a symmetric comonoidal monad by Lemma 26.

We define the following morphisms:

$$\mathsf{w}_A := \perp \xrightarrow{\mathsf{j}_\perp^{-1}} \mathsf{J}0 \xrightarrow{\mathsf{J}\diamond_{\mathsf{H}A}} \mathsf{JH}A = ?A$$

$$\mathsf{c}_A := ?A \oplus ?A = \mathsf{JH}A \oplus \mathsf{JH}A \xrightarrow{\mathsf{j}^{-1}_{\mathsf{H}A,\mathsf{H}A}} \mathsf{J}(\mathsf{H}A + \mathsf{H}A) \xrightarrow{\mathsf{J}\nabla_{\mathsf{H}A}} \mathsf{JHA} = ?A$$

Next we show that both of these are symmetric comonoidal natural transformations, but for which functors? Define $\mathsf{W}(A) = \perp$ and $\mathsf{C}(A) = ?A \oplus ?A$ on objects of $\mathcal{L}$, and $\mathsf{W}(f : A \longrightarrow B) = \mathsf{id}_\perp$ and $\mathsf{C}(f : A \longrightarrow B) = ?f \oplus ?f$ on morphisms. So we must show that $\mathsf{w} : \mathsf{W} \longrightarrow\ ?$ and $\mathsf{c} : \mathsf{C} \longrightarrow\ ?$ are symmetric comonoidal natural transformations. We first show that w is and then we show that c is. Throughout the proof we drop subscripts on natural transformations for readability.

Case. To show w is a natural transformation we must show the following diagram commutes for any morphism $f : A \longrightarrow B$:

$$\begin{tikzcd}
W(A) \arrow[r, "\mathsf{w}_A"] \arrow[d, "W(f)"'] & ?A \arrow[d, "?f"] \\
W(B) \arrow[r, "\mathsf{w}_B"'] & ?B
\end{tikzcd}$$

This diagram is equivalent to the following:

$$\begin{tikzcd}
\perp \arrow[r, "\mathsf{w}_A"] \arrow[d, "\mathsf{id}_\perp"'] & ?A \arrow[d, "?f"] \\
\perp \arrow[r, "\mathsf{w}_B"'] & ?B
\end{tikzcd}$$

It further expands to the following:

$$\begin{tikzcd}
\perp \arrow[r, "\mathsf{j}_\perp^{-1}"] \arrow[d, "\mathsf{id}_\perp"'] & \mathsf{J}0 \arrow[r, "\mathsf{J}(\diamond_{\mathsf{H}A})"] & \mathsf{JH}A \arrow[d, "\mathsf{JH}f"] \\
\perp \arrow[r, "\mathsf{j}_\perp^{-1}"'] & \mathsf{J}0 \arrow[r, "\mathsf{J}(\diamond_{\mathsf{H}B})"'] & \mathsf{JH}B
\end{tikzcd}$$

This diagram commutes, because $\mathsf{J}(\diamond_{\mathsf{H}A});\mathsf{J}f = \mathsf{J}(\diamond_{\mathsf{H}A};f) = \mathsf{J}(\diamond_{\mathsf{H}B})$, by the uniqueness of the initial map.

Case. The functor W is comonoidal itself. To see this we must exhibit a map

$$\mathsf{s}_\perp := \mathsf{id}_\perp : \mathsf{W}\perp \longrightarrow \perp$$

and a natural transformation

$$\mathsf{s}_{A,B} := \rho_\perp^{-1} : \mathsf{W}(A \oplus B) \longrightarrow \mathsf{W}A \oplus \mathsf{W}B$$

subject to the coherence conditions in Definition 8. Clearly, the second map is a natural transformation, but we leave showing they respect the coherence conditions to the reader. Now we can show that w is indeed symmetric comonoidal.

Case.

$$\begin{tikzcd}
\mathsf{W}(A \oplus B) \arrow[r, "\mathsf{s}_{A,B}"] \arrow[d, "\mathsf{w}_{A\oplus B}"'] & \mathsf{W}A \oplus \mathsf{W}B \arrow[d, "\mathsf{w}_A \oplus \mathsf{w}_B"] \\
?(A \oplus B) \arrow[r, "\mathsf{r}_{A,B}"] & ?\,A \oplus ?\,B
\end{tikzcd}$$

Expanding the objects of the previous diagram results in the following:

$$\begin{tikzcd}
\perp \arrow[r, "\mathsf{s}_{A,B}"] \arrow[d, "\mathsf{w}_{A\oplus B}"'] & \perp \oplus \perp \arrow[d, "\mathsf{w}_A \oplus \mathsf{w}_B"] \\
?(A \oplus B) \arrow[r, "\mathsf{r}_{A,B}"] & ?\,A \oplus ?\,B
\end{tikzcd}$$

This diagram commutes, because the following fully expanded diagram commutes:

$$\begin{tikzcd}
\perp \arrow[rr, "\rho^{-1}"] \arrow[dddd, "\mathsf{j}_\perp^{-1}"'] \arrow[dr, phantom, "(6)"] & & \perp \oplus \perp \arrow[dl, "\mathsf{j}_\perp^{-1} \oplus \mathsf{id}"'] \arrow[d, "\mathsf{j}_\perp^{-1} \oplus \mathsf{j}_\perp^{-1}"] \\
& \mathsf{J}0 \oplus \perp \arrow[r, "\mathsf{id} \oplus \mathsf{j}_\perp^{-1}"] \arrow[d, equal] \arrow[ur, phantom, "(5)"] \arrow[dr, phantom, "(4)"] & \mathsf{J}0 \oplus \mathsf{J}0 \arrow[dl, "\mathsf{id} \oplus \mathsf{j}_\perp"] \arrow[dd, equal] \\
& \mathsf{J}0 \oplus \perp \arrow[d, phantom, "(3)"] & \\
\mathsf{J}0 \arrow[uur, "\rho^{-1}"] \arrow[d, "\mathsf{J}\diamond"'] \arrow[dr, phantom, "(1)"] & \mathsf{J}(0+0) \arrow[l, "\mathsf{J}\rho"'] \arrow[r, "\mathsf{j}"] \arrow[d, "\mathsf{J}(\diamond+\diamond)"] \arrow[dr, phantom, "(2)"] & \mathsf{J}0 \oplus \mathsf{J}0 \arrow[d, "\mathsf{J}\diamond \oplus \mathsf{J}\diamond"] \\
\mathsf{JH}(A \oplus B) \arrow[r, "\mathsf{Jh}"] & \mathsf{J}(\mathsf{H}A + \mathsf{H}B) \arrow[r, "\mathsf{j}"] & \mathsf{JH}A \oplus \mathsf{JH}B
\end{tikzcd}$$

Diagram 1 commutes because 0 is the initial object, diagram 2 commutes by naturality of j, diagram 3 commutes because J is a symmetric comonoidal functor, diagram 4 commutes because $\mathsf{j}_\perp$ is an isomorphism (Lemma 24), diagram 5 commutes by functorality of J, and diagram 6 commutes by naturality of ρ.

Case.

$$\begin{tikzcd}
\bot & & ?\,\bot \arrow[ll, "\mathsf{r}_\bot"'] \\
& \mathsf{W}\,\bot \arrow[ul, "\mathsf{s}_\bot"] \arrow[ur, "\mathsf{w}_\bot"'] &
\end{tikzcd}$$

Expanding the objects in the previous diagram results in the following:

$$\begin{tikzcd}
\bot & & ?\,\bot \arrow[ll, "\mathsf{r}_\bot"'] \\
& \bot \arrow[ul, equal] \arrow[ur, "\mathsf{w}_\bot"'] &
\end{tikzcd}$$

This diagram commutes because the following one does:

$$\begin{tikzcd}
\bot \arrow[d, equal] & & \mathsf{J}0 \arrow[ll, "\mathsf{j}_\bot"'] \\
\bot \arrow[r, "\mathsf{j}_\bot^{-1}"] & \mathsf{J}0 \arrow[r, "\mathsf{J}\diamond"] \arrow[ur, equal] & \mathsf{JH}\,\bot \arrow[u, "\mathsf{Jh}_\bot"']
\end{tikzcd}$$

The diagram on the left commutes because $\mathsf{j}_\bot$ is an isomorphism (Lemma 24), and the diagram on the right commutes because 0 is the initial object.

Case. Now we show that $\mathsf{c}_A : ?A \oplus ?A \longrightarrow ?A$ is a natural transformation. This requires the following diagram to commute (for any $f : A \longrightarrow B$):

$$\begin{tikzcd}
\mathsf{C}A \arrow[r, "\mathsf{c}_A"] \arrow[d, "\mathsf{C}f"'] & ?A \arrow[d, "?f"] \\
\mathsf{C}B \arrow[r, "\mathsf{c}_B"'] & ?B
\end{tikzcd}$$

This expands to the following diagram:

$$\begin{tikzcd}
?A \oplus ?A \arrow[r, "\mathsf{c}_A"] \arrow[d, "?f \oplus ?f"'] & ?A \arrow[d, "?f"] \\
?B \oplus ?B \arrow[r, "\mathsf{c}_B"'] & ?B
\end{tikzcd}$$

This diagram commutes because the following diagram does:

$$\begin{tikzcd}
\mathsf{JH}A \oplus \mathsf{JH}A \arrow[r, "\mathsf{j}^{-1}_{\mathsf{H}A,\mathsf{H}A}"] \arrow[d, "\mathsf{JH}f \oplus \mathsf{JH}f"'] & \mathsf{J}(\mathsf{H}A + \mathsf{H}A) \arrow[r, "\mathsf{J}\,\nabla_{\mathsf{H}A}"] \arrow[d, "\mathsf{J}(\mathsf{H}f + \mathsf{H}f)"] & \mathsf{JH}A \arrow[d, "\mathsf{JH}f"] \\
\mathsf{JH}B \oplus \mathsf{JH}B \arrow[r, "\mathsf{j}^{-1}_{\mathsf{H}B,\mathsf{H}B}"'] & \mathsf{J}(\mathsf{H}B + \mathsf{H}B) \arrow[r, "\mathsf{J}\,\nabla_{\mathsf{H}B}"'] & \mathsf{JH}B
\end{tikzcd}$$

The left square commutes by naturality of j^{-1}, and the right square commutes by naturality of the codiagonal $\nabla_A : A + A \longrightarrow A$.

Case. The functor $\mathsf{C} : \mathcal{L} \longrightarrow \mathcal{L}$ is indeed symmetric comonoidal where the required maps are defined as follows:

$$\mathsf{t}_\perp := ?\perp \oplus ?\perp = \mathsf{JH}\perp \oplus \mathsf{JH}\perp \xrightarrow{\mathsf{j}^{-1}} \mathsf{J}(\mathsf{H}\perp + \mathsf{H}\perp) \xrightarrow{\mathsf{J}\nabla} \mathsf{JH}\perp \xrightarrow{\mathsf{Jh}_\perp} \mathsf{J}0 \xrightarrow{\mathsf{j}_\perp} \perp$$

$$\mathsf{t}_{A,B} := ?(A \oplus B) \oplus ?(A \oplus B) \xrightarrow{\mathsf{r}_{A,B} \oplus \mathsf{r}_{A,B}} (?A \oplus ?B) \oplus (?A \oplus ?B) \xrightarrow{\mathsf{iso}} (?A \oplus ?A) \oplus (?B \oplus ?B)$$

where iso is a natural isomorphism that can easily be defined using the symmetric monoidal structure of $\mathcal{L}$. Clearly, t is indeed a natural transformation, but we leave checking that the required diagrams in Definition 8 commute to the reader. We can now show that $\mathsf{c}_A : ?A \oplus ?A \longrightarrow ?A$ is symmetric comonoidal. The following diagrams from Definition 10 must commute:

Case.

$$\begin{tikzcd}
\mathsf{C}(A \oplus B) \arrow[r, "\mathsf{t}_{A,B}"] \arrow[d, "\mathsf{c}_{A\oplus B}"'] & \mathsf{C}A \oplus \mathsf{C}B \arrow[d, "\mathsf{c}_A \oplus \mathsf{c}_B"] \\
?(A \oplus B) \arrow[r, "\mathsf{r}_{A,B}"'] & ?A \oplus ?B
\end{tikzcd}$$

Expanding the objects in the previous diagram results in the following:

$$\begin{tikzcd}
?(A \oplus B) \oplus ?(A \oplus B) \arrow[r, "\mathsf{t}_{A,B}"] \arrow[d, "\mathsf{c}_{A\oplus B}"'] & (?A \oplus ?A) \oplus (?B \oplus ?B) \arrow[d, "\mathsf{c}_A \oplus \mathsf{c}_B"] \\
?(A \oplus B) \arrow[r, "\mathsf{r}_{A,B}"'] & ?A \oplus ?B
\end{tikzcd}$$

This diagram commutes, because the following fully expanded one does:

$$
\begin{tikzcd}
\mathsf{JH}(A \oplus B) \oplus \mathsf{JH}(A \oplus B) \arrow[r, "\mathsf{Jh} \oplus \mathsf{Jh}"] \arrow[d, "\mathsf{j}^{-1}"] \arrow[dr, phantom, "(2)"] & \mathsf{J}(\mathsf{H}A + \mathsf{H}B) \oplus \mathsf{J}(\mathsf{H}A + \mathsf{H}B) \arrow[r, "\mathsf{j} \oplus \mathsf{j}"] \arrow[d, "\mathsf{j}^{-1}"] \arrow[dr, phantom, "(4)"] & (\mathsf{JH}A \oplus \mathsf{JH}B) \oplus (\mathsf{JH}A \oplus \mathsf{JH}B) \arrow[r, "\mathsf{iso}"] \arrow[dr, phantom, "(6)"] & (\mathsf{JH}A \oplus \mathsf{JH}A) \oplus (\mathsf{JH}B \oplus \mathsf{JH}B) \arrow[d, "\mathsf{j}^{-1} \oplus \mathsf{j}^{-1}"] \\
\mathsf{J}(\mathsf{H}(A \oplus B) + \mathsf{H}(A \oplus B)) \arrow[r, "\mathsf{J}(\mathsf{h}+\mathsf{h})"] \arrow[d, "\mathsf{J}\nabla"] \arrow[dr, phantom, "(1)"] & \mathsf{J}((\mathsf{H}A + \mathsf{H}B) + (\mathsf{H}A + \mathsf{H}B)) \arrow[r, "\mathsf{Jiso}"] \arrow[d, "\mathsf{J}\nabla"] \arrow[dr, phantom, "(3)"] & \mathsf{J}((\mathsf{H}A + \mathsf{H}A) + (\mathsf{H}B + \mathsf{H}B)) \arrow[r, "\mathsf{j}"] \arrow[d, "\mathsf{J}(\nabla + \nabla)"] \arrow[ur, "\mathsf{j};(\mathsf{j} \oplus \mathsf{j})"] \arrow[dr, phantom, "(5)"] & \mathsf{J}(\mathsf{H}A + \mathsf{H}A) \oplus \mathsf{J}(\mathsf{H}B + \mathsf{H}B) \arrow[d, "\mathsf{J}\nabla \oplus \mathsf{J}\nabla"] \\
\mathsf{JH}(A \oplus B) \arrow[r, "\mathsf{Jh}"] & \mathsf{J}(\mathsf{H}A + \mathsf{H}B) \arrow[r, equal] & \mathsf{J}(\mathsf{H}A + \mathsf{H}B) \arrow[r, "\mathsf{j}"] & \mathsf{JH}A \oplus \mathsf{JH}B
\end{tikzcd}
$$

Diagram 1 commutes by naturality of ∇, diagram 2 commutes by naturality of j^{-1}, diagram 3 commutes by straightforward reasoning on coproducts, diagram 4 commutes by straightforward reasoning on the symmetric monoidal structure of J after expanding the definition of the two isomorphisms – here Jiso is the corresponding isomorphisms on coproducts – diagram 5 commutes by naturality of j, and diagram 6 commutes because j is an isomorphism (Lemma 24).

Case.

$$\begin{tikzcd}
\perp & & ?\perp \arrow[ll, "\mathsf{r}_\perp"'] \\
& \mathsf{C}\perp \arrow[ul, "\mathsf{t}\perp"] \arrow[ur, "\mathsf{c}_\perp"'] &
\end{tikzcd}$$

Expanding the objects of this diagram results in the following:

$$\begin{tikzcd}
\perp & ?\perp \arrow[l, "\mathsf{r}_\perp"'] \\
?\perp \oplus ?\perp \arrow[u, "\mathsf{t}\perp"] \arrow[r, equal] & ?\perp \oplus ?\perp \arrow[u, "\mathsf{c}_\perp"']
\end{tikzcd}$$

Simply unfolding the morphisms in the previous diagram reveals the following:

$$\begin{tikzcd}
\perp \arrow[r, equal] & \perp \\
\mathsf{J}0 \arrow[u, "\mathsf{j}"] & \mathsf{J}0 \arrow[u, "\mathsf{j}"'] \\
\mathsf{JH}\perp \arrow[u, "\mathsf{Jh}_\perp"] & \mathsf{JH}\perp \arrow[u, "\mathsf{Jh}_\perp"'] \\
\mathsf{J}(\mathsf{H}\perp + \mathsf{H}\perp) \arrow[u, "\mathsf{J}\nabla"] & \mathsf{J}(\mathsf{H}\perp + \mathsf{H}\perp) \arrow[u, "\mathsf{J}\nabla"'] \\
\mathsf{JH}\perp \oplus \mathsf{JH}\perp \arrow[u, "\mathsf{j}^{-1}"] \arrow[r, equal] & \mathsf{JH}\perp \oplus \mathsf{JH}\perp \arrow[u, "\mathsf{j}^{-1}"']
\end{tikzcd}$$

Clearly, this diagram commutes.

At this point we have shown that $\mathsf{w}_A : \perp \longrightarrow ?A$ and $\mathsf{c}_A : ?A \oplus ?A \longrightarrow ?A$ are symmetric comonoidal naturality transformations. Now we show that for any $?A$ the triple $(?A, \mathsf{w}_A, \mathsf{c}_A)$ forms a commutative monoid. This means that the following diagrams must commute:

Case.

$$\begin{tikzcd}
(?A \oplus ?A) \oplus ?A \arrow[r, "\alpha_{?A,?A,?A}"] \arrow[d, "\mathsf{c}_A \oplus \mathsf{id}_A"'] & ?A \oplus (?A \oplus ?A) \arrow[r, "\mathsf{id}_{?A} \oplus \mathsf{c}_A"] & ?A \oplus ?A \arrow[d, "\mathsf{c}_A"] \\
?A \oplus ?A \arrow[rr, "\mathsf{c}_A"] & & ?A
\end{tikzcd}$$

The previous diagram commutes, because the following one does (we omit subscripts for readability):

$$\begin{tikzcd}
(\mathsf{JH}A \oplus \mathsf{JH}A) \oplus \mathsf{JH}A \arrow[r, "\alpha"] \arrow[d, "\mathsf{j}^{-1} \oplus \mathsf{id}"'] & \mathsf{JH}A \oplus (\mathsf{JH}A \oplus \mathsf{JH}A) \arrow[r, "\mathsf{id} \oplus \mathsf{j}^{-1}"] \arrow[d, phantom, "(1)"] & \mathsf{JH}A \oplus \mathsf{J}(\mathsf{H}A + \mathsf{H}A) \arrow[r, "\mathsf{id} \oplus \mathsf{J}\bigtriangledown"] \arrow[d, "\mathsf{j}^{-1}"] \arrow[dr, phantom, "(2)"] & \mathsf{JH}A \oplus \mathsf{JH}A \arrow[d, "\mathsf{j}^{-1}"] \\
\mathsf{J}(\mathsf{H}A + \mathsf{H}A) \oplus \mathsf{JH}A \arrow[r, "\mathsf{j}^{-1}"] \arrow[d, "\mathsf{J}\bigtriangledown \oplus \mathsf{id}"'] \arrow[dr, phantom, "(3)"] & \mathsf{J}((\mathsf{H}A + \mathsf{H}A) + \mathsf{H}A) \arrow[r, "\mathsf{J}\alpha"] \arrow[d, "\mathsf{J}(\bigtriangledown + \mathsf{id})"] & \mathsf{J}(\mathsf{H}A + (\mathsf{H}A + \mathsf{H}A)) \arrow[r, "\mathsf{J}(\mathsf{id} + \bigtriangledown)"] \arrow[d, phantom, "(4)"] & \mathsf{J}(\mathsf{H}A + \mathsf{H}A) \arrow[d, "\mathsf{J}\bigtriangledown"] \\
\mathsf{JH}A \oplus \mathsf{JH}A \arrow[r, "\mathsf{j}^{-1}"] & \mathsf{J}(\mathsf{H}A + \mathsf{H}A) \arrow[rr, "\mathsf{J}\bigtriangledown"] & & \mathsf{JH}A
\end{tikzcd}$$

Diagram 1 commutes because J is a symmetric monoidal functor (Corollary 25), diagrams 2 and 3 commute by naturality of j^{-1}, and diagram 4 commutes because $(\mathsf{H}A, \diamond, \bigtriangledown)$ is a commutative monoid in C, but we leave the proof of this to the reader.

Case.

$$\begin{tikzcd}
?A \oplus \perp \arrow[d, "\mathsf{id}_{?A} \oplus \mathsf{w}_A"'] \arrow[dr, "\rho_{?A}"] & \\
?A \oplus ?A \arrow[r, "\mathsf{c}_A"] & ?A
\end{tikzcd}$$

The previous diagram commutes, because the following one does:

$$\begin{tikzcd}
\mathsf{JH}A \oplus \perp \arrow[rr, "\rho"] \arrow[d, "\mathsf{id} \oplus \mathsf{j}_{\perp}^{-1}"'] \arrow[drr, phantom, "(1)"] & & \mathsf{JH}A \arrow[d, equal] \\
\mathsf{JH}A \oplus \mathsf{J}0 \arrow[r, "\mathsf{j}^{-1}"] \arrow[d, "\mathsf{id} \oplus \mathsf{J}\diamond"'] \arrow[dr, phantom, "(2)"] & \mathsf{J}(\mathsf{H}A + 0) \arrow[r, "\mathsf{J}\rho"] \arrow[d, "\mathsf{J}(\mathsf{id} \oplus \diamond)"] \arrow[dr, phantom, "(3)"] & \mathsf{JH}A \arrow[d, equal] \\
\mathsf{JH}A \oplus \mathsf{JH}A \arrow[r, "\mathsf{j}^{-1}"] & \mathsf{J}(\mathsf{H}A + \mathsf{H}A) \arrow[r, "\mathsf{J}\bigtriangledown"] & \mathsf{JH}A
\end{tikzcd}$$

Diagram 1 commutes because J is a symmetric monoidal functor (Corollary 25), diagram 2 commutes by naturality of j^{-1}, and diagram 3 commutes because $(\mathsf{H}A, \diamond, \bigtriangledown)$ is a commutative monoid in C, but we leave the proof of this to the reader.

Case.

$$\begin{tikzcd}
?A \oplus ?A \arrow[d, "\beta_{?A,?A}"'] \arrow[dr, "\mathsf{c}_A"] & \\
?A \oplus ?A \arrow[r, "\mathsf{c}_A"] & ?A
\end{tikzcd}$$

This diagram commutes, because the following one does:

$$\begin{tikzcd}
\mathsf{JH}A \oplus \mathsf{JH}A \arrow[r, "\mathsf{j}^{-1}"] \arrow[d, "\beta"'] & \mathsf{J}(\mathsf{H}A + \mathsf{H}A) \arrow[r, "\mathsf{J}\bigtriangledown"] \arrow[d, "\mathsf{J}\beta"] & \mathsf{JH}A \arrow[d, equal] \\
\mathsf{JH}A \oplus \mathsf{JH}A \arrow[r, "\mathsf{j}^{-1}"'] & \mathsf{J}(\mathsf{H}A + \mathsf{H}A) \arrow[r, "\mathsf{J}\bigtriangledown"'] & \mathsf{JH}A
\end{tikzcd}$$

The left diagram commutes by naturality of j^{-1}, and the right diagram commutes because $(\mathsf{H}A, \diamond, \bigtriangledown)$ is a commutative monoid in C, but we leave the proof of this to the reader.

Finally, we must show that $\mathsf{w}_A : \perp \longrightarrow ?A$ and $\mathsf{c}_A : ?A \oplus ?A \longrightarrow ?A$ are $?$-algebra morphisms. The algebras in play here are $(?A, \mu : ??A \longrightarrow ?A)$, $(\perp, \mathsf{r}_\perp : ?\perp \longrightarrow \perp)$, and $(?A \oplus ?A, u_A : ?(?A \oplus ?A) \longrightarrow ?A \oplus ?A)$, where $u_A := ?(?A \oplus ?A) \xrightarrow{\mathsf{r}_{?A,?A}} ?^2A \oplus ?^2A \xrightarrow{\mu_A \oplus \mu_A} ?A \oplus ?A$. It suffices to show that the following diagrams commute:

Case.

$$\begin{tikzcd}
?\perp \arrow[r, "\mathsf{r}_\perp"] \arrow[d, "?\mathsf{w}"'] & \perp \arrow[d, "\mathsf{w}"] \\
??A \arrow[r, "\mu"'] & ?A
\end{tikzcd}$$

This diagram commutes, because the following fully expanded one does:

$$\begin{tikzcd}
\mathsf{JH}\perp \arrow[rr, "\mathsf{Jh}_\perp"] \arrow[drr, "\mathsf{JHj}_\perp^{-1}"] \arrow[dd, "\mathsf{JHj}_\perp^{-1}"'] & & \mathsf{J}0 \arrow[rr, "\mathsf{j}_\perp"] \arrow[drr, phantom, "(4)"] & & \perp \arrow[dd, "\mathsf{j}_\perp^{-1}"] \\
& (3) & \mathsf{JHJ}0 \arrow[r, "\mathsf{JHj}_\perp"] \arrow[dll, equal] \arrow[d, phantom, "(2)"] & \mathsf{JH}\perp \arrow[dr, "\mathsf{Jh}_\perp"] & \\
\mathsf{JHJ}0 \arrow[rrrr, "\mathsf{J}\varepsilon_0"] \arrow[d, "\mathsf{JHJ}\diamond"'] \arrow[drrrr, phantom, "(1)"] & & & & \mathsf{J}0 \arrow[d, "\mathsf{J}\diamond"] \\
\mathsf{JHJH}A \arrow[rrrr, "\mathsf{J}\varepsilon"] & & & & \mathsf{JH}A
\end{tikzcd}$$

Diagram 1 commutes by naturality of ε, diagram 2 commutes because ε is the counit of the symmetric comonoidal adjunction, diagram 3 clearly commutes, and diagram 4 commutes because $\mathsf{j}_\perp$ is an isomorphism (Lemma 24).

Case.

$$\begin{tikzcd}
?(?A \oplus ?A) \arrow[r, "u"] \arrow[d, "?\mathsf{c}"'] & ?A \oplus ?A \arrow[d, "\mathsf{c}"] \\
??A \arrow[r, "\mu"] & ?A
\end{tikzcd}$$

This diagram commutes because the following fully expanded one does:

$$\begin{tikzcd}
\mathsf{JH}(\mathsf{JH}A \oplus \mathsf{JH}A) \arrow[r, "\mathsf{Jh}"] \arrow[d, "\mathsf{JHj}^{-1}"'] & \mathsf{J}(\mathsf{HJH}A + \mathsf{HJH}A) \arrow[r, "\mathsf{j}"] \arrow[d, phantom, "(5)"] & \mathsf{JHJH}A \oplus \mathsf{JHJH}A \arrow[r, "\mathsf{J}\varepsilon \oplus \mathsf{J}\varepsilon"] \arrow[d, "\mathsf{j}^{-1}"'] \arrow[dr, phantom, "(6)"] & \mathsf{JH}A \oplus \mathsf{JH}A \arrow[d, "\mathsf{j}^{-1}"] \\
\mathsf{JHJ}(\mathsf{H}A + \mathsf{H}A) \arrow[r, "\mathsf{JHj}"] \arrow[dr, "\mathsf{J}\varepsilon"] \arrow[dddd, "\mathsf{JHJ}\nabla"'] & \mathsf{JH}(\mathsf{JH}A \oplus \mathsf{JH}A) \arrow[r, "\mathsf{Jh}"] \arrow[d, phantom, "(4)"] & \mathsf{J}(\mathsf{HJH}A + \mathsf{HJH}A) \arrow[r, "\mathsf{J}(\varepsilon+\varepsilon)"] \arrow[dl, "\mathsf{J}(\varepsilon+\varepsilon)"'] \arrow[ddd, "\mathsf{J}\nabla"] \arrow[dddr, phantom, "(7)"] & \mathsf{J}(\mathsf{H}A + \mathsf{H}A) \arrow[ddd, "\mathsf{J}\nabla"] \\
 & \mathsf{J}(\mathsf{H}A + \mathsf{H}A) \arrow[d, "\mathsf{J}\nabla"] \arrow[r, phantom, "(3)"] & & \\
 & \mathsf{JH}A \arrow[d, phantom, "(1)"] \arrow[l, phantom, "(2)"] & & \\
\mathsf{JHJH}A \arrow[uur, "\mathsf{J}\varepsilon"'] \arrow[rr, equal] & & \mathsf{JHJH}A \arrow[uul, "\mathsf{J}\varepsilon"] \arrow[r, "\mathsf{J}\varepsilon"] & \mathsf{JH}A
\end{tikzcd}$$

Diagram 1 clearly commutes, diagram 2 commutes by naturality of ε, diagram 3 commutes by naturality of $\bigtriangledown$, diagram 4 commutes because ε is the counit of the symmetric comonoidal adjunction, diagram 5 commutes because j is an isomorphism (Lemma 24), diagram 6 commutes by naturality of j^{-1}, and diagram 7 is the same diagram as 3, but this diagram is redundant for readability.