COMONADIC MATTER MEETS MONADIC ANTI-MATTER: AN ADJOINT MODEL OF BI-INTUITIONISTIC LOGIC

HARLEY EADES III

e-mail address: heades@augusta.edu

Computer and Information Sciences, Augusta University, Augusta, GA

Abstract. Bi-intuitionistic logic (BINT) is a conservative extension of intuitionistic logic with perfect duality. That is, BINT contains the usual intuitionistic logical connectives such as true, conjunction, and implication, but also their duals false, disjunction, and co-implication. One leading question with respect to BINT is, what does BINT look like across the three arcs – logic, typed λ-calculi, and category theory – of the Curry-Howard-Lambek correspondence? A non-trivial (does not degenerate to a poset) categorical model of BINT is currently an open problem. It is this open problem that this paper contributes to by providing the first fully developed categorical model of BINT. It is well-known that the linear counterpart, linear BINT, of BINT can be modeled in a symmetric monoidal closed category equipped with an additional monoidal structure that models par and a specified left adjoint to par called linear co-implication. We call this model a symmetric bi-monoidal bi-closed category. In addition, it is well-known that intuitionistic logic has a categorical model of cartesian closed categories, and their dual co-cartesian co-closed categories model co-intuitionistic logic. In this paper we exploit Benton's beautiful LNL models of linear logic to show that these three models can be mixed by requiring a symmetric monoidal adjunction between a cartesian closed category and the symmetric bi-monoidal bi-closed category, in addition to a symmetric monoidal adjunction between a co-cartesian co-closed category and the symmetric bi-monoidal bi-closed category. As a result of this mixture we obtain two modalities the usual comonadic of-course modality of linear logic, but also a monadic modality allowing for the embedding of co-intuitionistic logic inside of linear BINT. Finally, using these modalities we show that BINT intuitionistic logic can be soundly modeled in this new categorical model. As a bi-product of this model we define BiLNL logic which can be seen as the mixture of intuitionistic logic with co-intuitionistic logic inside of linear BINT.

1. Introduction

TODO [**?**]

2. Mixed Linear/Non-Linear Models of Bi-intuitionistic Logic: The Categorical Model

TODO

3. Mixed Linear/Non-Linear Bi-intuitionistic Logic: BiLNL Logic

4. Embedding Bi-Intuitionistic Logic in BiLNL Logic

TODO

LOGICAL METHODS
IN COMPUTER SCIENCE

DOI:10.2168/LMCS-???

$$
\frac{G,(w,w);\Theta \vdash_{\mathsf{I}} w : Y}{G;\Theta \vdash_{\mathsf{I}} w : Y}\ \textsc{I\_rl}
\qquad
\frac{\begin{array}{c} w_1 G w_2 \qquad w_2 G w_3 \\ G,(w_1,w_3);\Theta \vdash_{\mathsf{I}} w : Y \end{array}}{G;\Theta \vdash_{\mathsf{I}} w : Y}\ \textsc{I\_ts}
\qquad
\frac{}{G; w : Y \vdash_{\mathsf{I}} w : Y}\ \textsc{I\_id}
$$

$$
\frac{G;\Theta_2 \vdash_{\mathsf{I}} w_2 : X \qquad G;\Theta_1, w_2 : X \vdash_{\mathsf{I}} w_1 : Z}{G;\Theta_1,\Theta_2 \vdash_{\mathsf{I}} w_1 : Z}\ \textsc{I\_cut}
\qquad
\frac{G;\Theta \vdash_{\mathsf{I}} w_1 : Y}{G;\Theta, w_2 : X \vdash_{\mathsf{I}} w_1 : Y}\ \textsc{I\_wk}
$$

$$
\frac{G;\Theta, w_2 : X, w_2 : X \vdash_{\mathsf{I}} w_1 : Y}{G;\Theta, w_2 : X \vdash_{\mathsf{I}} w_1 : Y}\ \textsc{I\_cr}
\qquad
\frac{G;\Theta_1, w_1 : X, w_2 : Y, \Theta_2 \vdash_{\mathsf{I}} w : Z}{G;\Theta_1, w_2 : Y, w_1 : X, \Theta_2 \vdash_{\mathsf{I}} w : Z}\ \textsc{I\_ex}
$$

$$
\frac{\begin{array}{l} w_1 G w_2 \\ G;\Theta, w_1 : X, w_2 : X \vdash_{\mathsf{I}} w : Y \end{array}}{G;\Theta, w_1 : X \vdash_{\mathsf{I}} w : Y}\ \textsc{I\_m}\mathrm{L}
\qquad
\frac{\begin{array}{l} w_2 G w_1 \\ G;\Theta \vdash_{\mathsf{I}} w_2 : Y \end{array}}{G;\Theta \vdash_{\mathsf{I}} w_1 : Y}\ \textsc{I\_m}\mathrm{R}
$$

$$
\frac{G;\Theta \vdash_{\mathsf{I}} w_1 : Y}{G;\Theta, w_2 : \top \vdash_{\mathsf{I}} w_1 : Y}\ \textsc{I\_t}\mathrm{L}
\qquad
\frac{}{G;\Theta \vdash_{\mathsf{I}} w : \top}\ \textsc{I\_t}\mathrm{R}
$$

$$
\frac{G;\Theta, w_1 : X, w_1 : Y \vdash_{\mathsf{I}} w_2 : Z}{G;\Theta, w_1 : X \times Y \vdash_{\mathsf{I}} w_2 : Z}\ \textsc{I\_p}\mathrm{L}
\qquad
\frac{G;\Theta_1 \vdash_{\mathsf{I}} w : X \qquad G;\Theta_2 \vdash_{\mathsf{I}} w : Y}{G;\Theta_1,\Theta_2 \vdash_{\mathsf{I}} w : X \times Y}\ \textsc{I\_p}\mathrm{R}
$$

$$
\frac{\begin{array}{l} w_1 G w_2 \\ G;\Theta_2 \vdash_{\mathsf{I}} w_2 : X \qquad G;\Theta_1, w_2 : Y \vdash_{\mathsf{I}} w : Z \end{array}}{G;\Theta_1,\Theta_2, w_1 : X \to Y \vdash_{\mathsf{I}} w : Z}\ \textsc{I\_i}\mathrm{L}
$$

$$
\frac{\begin{array}{l} w_2 \notin |G|, |\Theta| \\ G,(w_1,w_2);\Theta, w_2 : X \vdash_{\mathsf{I}} w_2 : Y \end{array}}{G;\Theta \vdash_{\mathsf{I}} w_1 : X \to Y}\ \textsc{I\_i}\mathrm{R}
\qquad
\frac{G;\Theta \mid \cdot \vdash_{\mathsf{LL}} w : A \mid \cdot}{G;\Theta \vdash_{\mathsf{I}} w : \mathsf{G}A}\ \textsc{I\_g}\mathrm{R}
$$

Figure 1: Intuitionistic Fragment of L

5. Related Work

TODO

6. Conclusion

TODO

References

$$\frac{G,(w,w);w:S\vdash_{\mathsf{C}}\Psi}{G;w:S\vdash_{\mathsf{C}}\Psi}\ \mathrm{C_RL}
\qquad
\frac{w_1Gw_2 \quad w_2Gw_3 \quad G,(w_1,w_3);w:S\vdash_{\mathsf{C}}\Psi}{G;w:S\vdash_{\mathsf{C}}\Psi}\ \mathrm{C_TS}$$

$$\frac{}{G;w:S\vdash_{\mathsf{C}}w:S}\ \mathrm{C_ID}
\qquad
\frac{G;w_1:S\vdash_{\mathsf{C}}w_2:T,\Psi_2 \qquad G;w_2:T\vdash_{\mathsf{C}}\Psi_1}{G;w_1:S\vdash_{\mathsf{C}}\Psi_1,\Psi_2}\ \mathrm{C_CUT}$$

$$\frac{G;w_1:S\vdash_{\mathsf{C}}\Psi}{G;w_1:S\vdash_{\mathsf{C}}w_2:T,\Psi}\ \mathrm{C_WK}
\qquad
\frac{G;w_1:S\vdash_{\mathsf{C}}w_2:T,w_2:T,\Psi}{G;w_1:S\vdash_{\mathsf{C}}w_2:T,\Psi}\ \mathrm{C_CR}$$

$$\frac{G;w:R\vdash_{\mathsf{C}}\Psi_1,w_1:S,w_2:T,\Psi_2}{G;w:R\vdash_{\mathsf{C}}\Psi_1,w_2:T,w_1:S,\Psi_2}\ \mathrm{C_EX}
\qquad
\frac{w_1Gw_2 \quad G;w_2:S\vdash_{\mathsf{C}}\Psi}{G;w_1:S\vdash_{\mathsf{C}}\Psi}\ \mathrm{C_ML}$$

$$\frac{w_2Gw_1 \quad G;w:S\vdash_{\mathsf{C}}w_2:T,w_1:T,\Psi}{G;w:S\vdash_{\mathsf{C}}w_1:T,\Psi}\ \mathrm{C_MR}
\qquad
\frac{}{G;w:\perp\vdash_{\mathsf{C}}\Psi}\ \mathrm{C_FL}$$

$$\frac{G;w_1:S\vdash_{\mathsf{C}}\Psi}{G;w_1:S\vdash_{\mathsf{C}}w_2:\perp,\Psi}\ \mathrm{C_FR}
\qquad
\frac{G;w:S\vdash_{\mathsf{C}}\Psi_1 \qquad G;w:T\vdash_{\mathsf{C}}\Psi_2}{G;w:S+T\vdash_{\mathsf{C}}\Psi_1,\Psi_2}\ \mathrm{C_DL}$$

$$\frac{G;w_1:R\vdash_{\mathsf{C}}w_2:S,w_2:T,\Psi}{G;w_1:R\vdash_{\mathsf{C}}w_2:S+T,\Psi}\ \mathrm{C_DR}
\qquad
\frac{w_2\notin|G|,|\Psi| \quad G,(w_2,w_1);w_2:S\vdash_{\mathsf{C}}w_2:T,\Psi}{G;w_1:S-T\vdash_{\mathsf{C}}\Psi}\ \mathrm{C_SL}$$

$$\frac{w_2Gw_1 \quad G;w:R\vdash_{\mathsf{C}}w_2:S,\Psi_2 \qquad G;w_2:T\vdash_{\mathsf{C}}\Psi_1}{G;w:R\vdash_{\mathsf{C}}w_1:S-T,\Psi_1,\Psi_2}\ \mathrm{C_SR}
\qquad
\frac{G;\cdot\mid w:A\vdash_{\mathsf{LL}}\cdot\mid\Psi}{G;w:\mathsf{H}A\vdash_{\mathsf{C}}\Psi}\ \mathrm{C_HL}$$

Figure 2: Co-intuitionistic Fragment of L

$$\frac{G,(w,w);\Theta \mid \Gamma \vdash_{\mathsf{LL}} \Delta \mid \Psi}{G;\Theta \mid \Gamma \vdash_{\mathsf{LL}} \Delta \mid \Psi}\ \text{LL\_RL} \qquad \frac{w_1 G w_2 \quad w_2 G w_3 \quad G,(w_1,w_3);\Theta \mid \Gamma \vdash_{\mathsf{LL}} \Delta \mid \Psi}{G;\Theta \mid \Gamma \vdash_{\mathsf{LL}} \Delta \mid \Psi}\ \text{LL\_TS}$$

$$\frac{w_1 G w_2 \quad G;\Theta \mid \Gamma, w_1 : A, w_2 : A \vdash_{\mathsf{LL}} \Delta \mid \Psi}{G;\Theta \mid \Gamma, w_1 : A \vdash_{\mathsf{LL}} \Delta \mid \Psi}\ \text{LL\_ML} \qquad \frac{w_2 G w_1 \quad G;\Theta \mid \Gamma \vdash_{\mathsf{LL}} w_2 : A, w_1 : A, \Delta \mid \Psi}{G;\Theta \mid \Gamma \vdash_{\mathsf{LL}} w_1 : A, \Delta \mid \Psi}\ \text{LL\_MR}$$

$$\frac{w_1 G w_2 \quad G;\Theta, w_1 : X, w_2 : X \mid \Gamma \vdash_{\mathsf{LL}} \Delta \mid \Psi}{G;\Theta, w_1 : X \mid \Gamma \vdash_{\mathsf{LL}} \Delta \mid \Psi}\ \text{LL\_ImL}$$

$$\frac{w_2 G w_1 \quad G;\Theta \mid \Gamma \vdash_{\mathsf{LL}} \Delta \mid w_2 : T, w_1 : T, \Psi}{G;\Theta \mid \Gamma \vdash_{\mathsf{LL}} \Delta \mid w_1 : T, \Psi}\ \text{LL\_CmR}$$

Figure 3: Inference Rules for BiLNL Logic: Abstract Kripke Graph Rules

$$\frac{G;\Theta \mid \Gamma \vdash_{\mathsf{LL}} \Delta \mid \Psi}{G;\Theta, w : X \mid \Gamma \vdash_{\mathsf{LL}} \Delta \mid \Psi}\ \text{LL\_WKL} \qquad \frac{G;\Theta \mid \Gamma \vdash_{\mathsf{LL}} \Delta \mid \Psi}{G;\Theta \mid \Gamma \vdash_{\mathsf{LL}} \Delta \mid \Psi, w : S}\ \text{LL\_WKR}$$

$$\frac{G;\Theta, w : X, w : X \mid \Gamma \vdash_{\mathsf{LL}} \Delta \mid \Psi}{G;\Theta, w : X \mid \Gamma \vdash_{\mathsf{LL}} \Delta \mid \Psi}\ \text{LL\_CTRL} \qquad \frac{G;\Theta \mid \Gamma \vdash_{\mathsf{LL}} \Delta \mid \Psi, w : S, w : S}{G;\Theta \mid \Gamma \vdash_{\mathsf{LL}} \Delta \mid \Psi, w : S}\ \text{LL\_CTRR}$$

$$\frac{G;\Theta \mid \Gamma_1, w_1 : A, w_2 : B, \Gamma_2 \vdash_{\mathsf{LL}} \Delta \mid \Psi}{G;\Theta \mid \Gamma_1, w_2 : B, w_1 : A, \Gamma_2 \vdash_{\mathsf{LL}} \Delta \mid \Psi}\ \text{LL\_EXL}$$

$$\frac{G;\Theta \mid \Gamma \vdash_{\mathsf{LL}} \Delta_1, w_1 : A, w_2 : B, \Delta_2 \mid \Psi}{G;\Theta \mid \Gamma \vdash_{\mathsf{LL}} \Delta_1, w_2 : B, w_1 : A, \Delta_2 \mid \Psi}\ \text{LL\_EXR}$$

$$\frac{G;\Theta_1, w_1 : X, w_2 : Y, \Theta_2 \mid \Gamma \vdash_{\mathsf{LL}} \Delta \mid \Psi}{G;\Theta_1, w_2 : Y, w_1 : X, \Theta_2 \mid \Gamma \vdash_{\mathsf{LL}} \Delta \mid \Psi}\ \text{LL\_ILEXL}$$

$$\frac{G;\Theta \mid \Gamma \vdash_{\mathsf{LL}} \Delta \mid \Psi_1, w_1 : S, w_2 : T, \Psi_2}{G;\Theta \mid \Gamma \vdash_{\mathsf{LL}} \Delta \mid \Psi_1, w_2 : T, w_1 : S, \Psi_2}\ \text{LL\_CLEXL}$$

Figure 4: Inference Rules for BiLNL Logic: Structural Rules

$$\frac{}{G;\cdot \mid w : A \vdash_{\mathsf{LL}} w : A \mid \cdot}\ \text{LL\_ID}$$

$$\frac{G;\Theta_1 \mid \Gamma_1 \vdash_{\mathsf{LL}} w : A, \Delta_2 \mid \Psi_1 \qquad G;\Theta_2 \mid w : A, \Gamma_2 \vdash_{\mathsf{LL}} \Delta_1 \mid \Psi_2}{G;\Theta_1,\Theta_2 \mid \Gamma_1,\Gamma_2 \vdash_{\mathsf{LL}} \Delta_1,\Delta_2 \mid \Psi_1,\Psi_2}\ \text{LL\_CUT}$$

$$\frac{G;\Theta_2 \vdash_{\mathsf{I}} w : X \qquad G;\Theta_1, w : X \mid \Gamma \vdash_{\mathsf{LL}} \Delta \mid \Psi}{G;\Theta_1,\Theta_2 \mid \Gamma \vdash_{\mathsf{LL}} \Delta \mid \Psi}\ \text{LL\_ILCUT}$$

$$\frac{G;\Theta \mid \Gamma \vdash_{\mathsf{LL}} \Delta \mid \Psi_1, w : S \qquad G; w : S \vdash_{\mathsf{C}} \Psi_2}{G;\Theta \mid \Gamma \vdash_{\mathsf{LL}} \Delta \mid \Psi_1,\Psi_2}\ \text{LL\_CLCUT}$$

Figure 5: Inference Rules for BiLNL Logic: Identity and Cut Rules

$$\frac{G;\Theta \mid \Gamma \vdash_{\mathsf{LL}} \Delta \mid \Psi}{G;\Theta \mid \Gamma, w : I \vdash_{\mathsf{LL}} \Delta \mid \Psi}\ \text{LL\_IL} \qquad \frac{}{G;\cdot \mid \cdot \vdash_{\mathsf{LL}} w : I \mid \cdot}\ \text{LL\_IR}$$

$$\frac{G;\Theta_1, w : X, w : Y, \Theta_2 \mid \Gamma \vdash_{\mathsf{LL}} \Delta \mid \Psi}{G;\Theta_1, w : X \times Y, \Theta_2 \mid \Gamma \vdash_{\mathsf{LL}} \Delta \mid \Psi}\ \text{LL\_cL}$$

$$\frac{G;\Theta \mid \Gamma_1, w : A, w : B, \Gamma_2 \vdash_{\mathsf{LL}} \Delta \mid \Psi}{G;\Theta \mid \Gamma_1, w : A \otimes B, \Gamma_2 \vdash_{\mathsf{LL}} \Delta \mid \Psi}\ \text{LL\_TL}$$

$$\frac{G;\Theta_1 \mid \Gamma_1 \vdash_{\mathsf{LL}} w : A, \Delta_1 \mid \Psi_1 \qquad G;\Theta_2 \mid \Gamma_2 \vdash_{\mathsf{LL}} w : B, \Delta_2 \mid \Psi_2}{G;\Theta_1,\Theta_2 \mid \Gamma_1,\Gamma_2 \vdash_{\mathsf{LL}} w : A \otimes B, \Delta_1,\Delta_2 \mid \Psi_1,\Psi_2}\ \text{LL\_TR}$$

Figure 6: Inference Rules for BiLNL Logic: Conjunction and Tensor Rules

$$\frac{}{G;\cdot \mid w : J \vdash_{\mathsf{LL}} \cdot \mid \cdot}\ \text{LL\_JL} \qquad \frac{G;\Theta \mid \Gamma \vdash_{\mathsf{LL}} \Delta \mid \Psi}{G;\Theta \mid \Gamma \vdash_{\mathsf{LL}} w : J, \Delta \mid \Psi}\ \text{LL\_JR}$$

$$\frac{G;\Theta \mid \Gamma \vdash_{\mathsf{LL}} \Delta \mid \Psi_1, w : S, w : T, \Psi_2}{G;\Theta \mid \Gamma \vdash_{\mathsf{LL}} \Delta \mid \Psi_1, w : S + T, \Psi_2}\ \text{LL\_DR}$$

$$\frac{G;\Theta_1 \mid \Gamma_1, w : A \vdash_{\mathsf{LL}} \Delta_1 \mid \Psi_1 \qquad G;\Theta_2 \mid \Gamma_2, w : B \vdash_{\mathsf{LL}} \Delta_2 \mid \Psi_2}{G;\Theta_1,\Theta_2 \mid \Gamma_1,\Gamma_2, w : A \oplus B \vdash_{\mathsf{LL}} \Delta_1,\Delta_2 \mid \Psi_1,\Psi_2}\ \text{LL\_PL}$$

$$\frac{G;\Theta \mid \Gamma \vdash_{\mathsf{LL}} \Delta_1, w : A, w : B, \Delta_2 \mid \Psi}{G;\Theta \mid \Gamma \vdash_{\mathsf{LL}} \Delta_1, w : A \oplus B, \Delta_2 \mid \Psi}\ \text{LL\_PR}$$

Figure 7: Inference Rules for BiLNL Logic: Disjunction and Par Rules

$$\frac{\begin{array}{l} w_1 G w_2 \\ G;\Theta_1 \mid \Gamma_1 \vdash_{\mathsf{LL}} w_2 : A, \Delta_1 \mid \Psi_1 \qquad G;\Theta_2 \mid \Gamma_2, w_2 : B \vdash_{\mathsf{LL}} \Delta_2 \mid \Psi_2 \end{array}}{G;\Theta_1,\Theta_2 \mid \Gamma_1,\Gamma_2, w_1 : A \multimap B \vdash_{\mathsf{LL}} \Delta_1,\Delta_2 \mid \Psi_1,\Psi_2}\ \text{LL\_IL}$$

$$\frac{\begin{array}{l} w_2 \notin |G|,|\Theta|,|\Gamma|,|\Delta|,|\Psi| \\ G,(w_1,w_2);\Theta \mid \Gamma, w_2 : A \vdash_{\mathsf{LL}} w_2 : B, \Delta \mid \Psi \end{array}}{G;\Theta \mid \Gamma \vdash_{\mathsf{LL}} w_1 : A \multimap B, \Delta \mid \Psi}\ \text{LL\_IR}$$

$$\frac{\begin{array}{l} w_1 G w_2 \\ G;\Theta_1 \vdash_{\mathsf{I}} w_2 : X \qquad G;\Theta_2, w_2 : Y \mid \Gamma \vdash_{\mathsf{LL}} \Delta \mid \Psi \end{array}}{G;\Theta_1,\Theta_2, w_1 : X \to Y \mid \Gamma \vdash_{\mathsf{LL}} \Delta \mid \Psi}\ \text{LL\_ILIL}$$

Figure 8: Inference Rules for BiLNL Logic: Implication Rules

$$\frac{\begin{array}{l} w_2 \notin |G|,|\Theta|,|\Gamma|,|\Delta|,|\Psi| \\ G,(w_2,w_1);\Theta \mid \Gamma, w_2 : A \vdash_{\mathsf{LL}} w_2 : B, \Delta \mid \Psi \end{array}}{G;\Theta \mid \Gamma, w_1 : A \bullet\!\!- B \vdash_{\mathsf{LL}} \Delta \mid \Psi}\ \text{LL\_sL}$$

$$\frac{\begin{array}{l} w_2 G w_1 \\ G;\Theta_1 \mid \Gamma_1 \vdash_{\mathsf{LL}} w_2 : A, \Delta_1 \mid \Psi_1 \qquad G;\Theta_2 \mid \Gamma_2, w_2 : B \vdash_{\mathsf{LL}} \Delta_2 \mid \Psi_2 \end{array}}{G;\Theta_1,\Theta_2 \mid \Gamma_1,\Gamma_2 \vdash_{\mathsf{LL}} w_2 : A \bullet\!\!- B, \Delta_1,\Delta_2 \mid \Psi_1,\Psi_2}\ \text{LL\_sR}$$

$$\frac{\begin{array}{l} w_2 G w_1 \\ G;\Theta \mid \Gamma \vdash_{\mathsf{LL}} \Delta \mid w_2 : S, \Psi_1 \qquad G; w_2 : T \vdash_{\mathsf{C}} \Psi_2 \end{array}}{G;\Theta \mid \Gamma \vdash_{\mathsf{LL}} \Delta \mid w_1 : S - T, \Psi_1,\Psi_2}\ \text{LL\_CLsR}$$

Figure 9: Inference Rules for BiLNL Logic: Co-implication Rules

$$\frac{G;\Theta, w : X \mid \Gamma \vdash_{\mathsf{LL}} \Delta \mid \Psi}{G;\Theta \mid \Gamma, w : \mathsf{F}X \vdash_{\mathsf{LL}} \Delta \mid \Psi}\ \text{LL\_FL} \qquad \frac{G;\Theta \vdash_{\mathsf{I}} w : X}{G;\Theta \mid \cdot \vdash_{\mathsf{LL}} w : \mathsf{F}X \mid \cdot}\ \text{LL\_FR}$$

$$\frac{G;\Theta \mid \Gamma, w : A \vdash_{\mathsf{LL}} \Delta \mid \Psi}{G;\Theta, w : \mathsf{G}A \mid \Gamma \vdash_{\mathsf{LL}} \Delta \mid \Psi}\ \text{LL\_GL} \qquad \frac{G; w : S \vdash_{\mathsf{C}} \Psi}{G;\cdot \mid w : \mathsf{J}S \vdash_{\mathsf{LL}} \cdot \mid \Psi}\ \text{LL\_JL}$$

$$\frac{G;\Theta \mid \Gamma \vdash_{\mathsf{LL}} \Delta \mid w : S, \Psi}{G;\Theta \mid \Gamma \vdash_{\mathsf{LL}} \Delta, w : \mathsf{J}S \mid \Psi}\ \text{LL\_JR} \qquad \frac{G;\Theta \mid \Gamma \vdash_{\mathsf{LL}} \Delta, w : A \mid \Psi}{G;\Theta \mid \Gamma \vdash_{\mathsf{LL}} \Delta \mid w : \mathsf{H}A, \Psi}\ \text{LL\_HR}$$

Figure 10: Inference Rules for BiLNL Logic: Adjoint Functors Rules

$$\frac{G,(w,w);\Gamma \vdash_{\mathsf{L}} \Delta}{G;\Gamma \vdash_{\mathsf{L}} \Delta}\ \textsc{rl} \qquad \frac{\begin{array}{l} w_1 G w_2 \quad w_2 G w_3 \\ G,(w_1,w_3);\Gamma \vdash_{\mathsf{L}} \Delta \end{array}}{G;\Gamma \vdash_{\mathsf{L}} \Delta}\ \textsc{ts}$$

$$\frac{G;\Gamma \vdash_{\mathsf{L}} w:A,\Delta \qquad G;\Gamma,w:A \vdash_{\mathsf{L}} \Delta}{G;\Gamma \vdash_{\mathsf{L}} \Delta}\ \textsc{cut} \qquad \frac{}{G;\Gamma,w:A \vdash_{\mathsf{L}} w:A,\Delta}\ \textsc{id}$$

$$\frac{\begin{array}{l} w_1 G w_2 \\ G;\Gamma,w_1:A,w_2:A \vdash_{\mathsf{L}} \Delta \end{array}}{G;\Gamma,w_1:A \vdash_{\mathsf{L}} \Delta}\ \textsc{m}\mathrm{L} \qquad \frac{\begin{array}{l} w_2 G w_1 \\ G;\Gamma \vdash_{\mathsf{L}} w_2:A,w_1:A,\Delta \end{array}}{G;\Gamma \vdash_{\mathsf{L}} w_1:A,\Delta}\ \textsc{m}\mathrm{R}$$

$$\frac{G;\Gamma \vdash_{\mathsf{L}} \Delta}{G;\Gamma,w:\top \vdash_{\mathsf{L}} \Delta}\ \textsc{t}\mathrm{L} \qquad \frac{}{G;\Gamma \vdash_{\mathsf{L}} w:\top,\Delta}\ \textsc{t}\mathrm{R} \qquad \frac{}{G;\Gamma,w:\perp \vdash_{\mathsf{L}} \Delta}\ \textsc{f}\mathrm{L}$$

$$\frac{G;\Gamma \vdash_{\mathsf{L}} \Delta}{G;\Gamma \vdash_{\mathsf{L}} w:\perp,\Delta}\ \textsc{f}\mathrm{R} \qquad \frac{G;\Gamma,w:A,w:B \vdash_{\mathsf{L}} \Delta}{G;\Gamma,w:A\times B \vdash_{\mathsf{L}} \Delta}\ \textsc{a}\mathrm{L}$$

$$\frac{G;\Gamma \vdash_{\mathsf{L}} w:A,\Delta \qquad G;\Gamma \vdash_{\mathsf{L}} w:B,\Delta}{G;\Gamma \vdash_{\mathsf{L}} w:A\times B,\Delta}\ \textsc{a}\mathrm{R}$$

$$\frac{G;\Gamma,w:A \vdash_{\mathsf{L}} \Delta \qquad G;\Gamma,w:B \vdash_{\mathsf{L}} \Delta}{G;\Gamma,w:A+B \vdash_{\mathsf{L}} \Delta}\ \textsc{d}\mathrm{L} \qquad \frac{G;\Gamma \vdash_{\mathsf{L}} w:A,w:B,\Delta}{G;\Gamma \vdash_{\mathsf{L}} w:A+B,\Delta}\ \textsc{d}\mathrm{R}$$

$$\frac{\begin{array}{l} w_1 G w_2 \\ G;\Gamma \vdash_{\mathsf{L}} w_2:A,\Delta \qquad G;\Gamma,w_2:B \vdash_{\mathsf{L}} \Delta \end{array}}{G;\Gamma,w_1:A\to B \vdash_{\mathsf{L}} \Delta}\ \textsc{i}\mathrm{L} \qquad \frac{\begin{array}{l} w_2 \notin |G|,|\Gamma|,|\Delta| \\ G,(w_1,w_2);\Gamma,w_2:A \vdash_{\mathsf{L}} w_2:B,\Delta \end{array}}{G;\Gamma \vdash_{\mathsf{L}} w_1:A\to B,\Delta}\ \textsc{i}\mathrm{R}$$

$$\frac{\begin{array}{l} w_2 \notin |G|,|\Gamma|,|\Delta| \\ G,(w_2,w_1);\Gamma,w_2:A \vdash_{\mathsf{L}} w_2:B,\Delta \end{array}}{G;\Gamma,w_1:A-B \vdash_{\mathsf{L}} \Delta}\ \textsc{s}\mathrm{L} \qquad \frac{\begin{array}{l} w_2 G w_1 \\ G;\Gamma \vdash_{\mathsf{L}} w_2:A,\Delta \qquad G;\Gamma,w_2:B \vdash_{\mathsf{L}} \Delta \end{array}}{G;\Gamma \vdash_{\mathsf{L}} w_1:A-B,\Delta}\ \textsc{s}\mathrm{R}$$

Figure 11: Inference Rules for L