# Monoidal-Annex

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## Contents



# 1 Monoidal Categories

## 1.1 Symmetric Monoidal Category

**Definition 1.** *A symmetric monoidal catergory (SMC), $(\mathbb{C}, \bullet, 1, \alpha, \lambda, \rho, \gamma)$, is a category $\mathbb{C}$ equippped with a bifunctor $\bullet : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ with a neutral element 1 and natural isomorphisms $\alpha, \lambda, \rho,$ and $\gamma$:*

1. $\alpha_{A,B,C} : A \bullet (B \bullet C) \xrightarrow{\sim} (A \bullet B) \bullet C$
2. $\lambda_A : 1 \bullet A \xrightarrow{\sim} A$
3. $\rho_A : A \bullet 1 \xrightarrow{\sim} A$
4. $\gamma_{A,B} : A \bullet B \xrightarrow{\sim} B \bullet A$

*which make the following 'coherence' diagrams commute.*

$$\begin{tikzcd}
A \bullet (B \bullet (C \bullet D)) \arrow[r, "\alpha_{A,B,C\bullet D}"] \arrow[d, "id_A \bullet \alpha_{B,C,D}"'] & (A \bullet B) \bullet (C \bullet D) \arrow[r, "\alpha_{A\bullet B,C,C}"] & ((A \bullet B) \bullet C) \bullet D \\
A \bullet ((B \bullet C) \bullet D) \arrow[rr, "\alpha_{A,B\bullet C,D}"'] & & (A \bullet (B \bullet C)) \bullet D \arrow[u, "\alpha_{A,B,C} \bullet id_D"']
\end{tikzcd}$$

$$\begin{tikzcd}
(A \bullet B) \bullet C \arrow[r, "\alpha_{A,B,C}"] \arrow[d, "\gamma_{A,B} \bullet id_C"'] & A \bullet (B \bullet C) \arrow[r, "\gamma_{A,B\bullet C}"] & (B \bullet C) \bullet A \arrow[d, "\alpha_{B,C,A}"'] \\
(B \bullet A) \bullet C \arrow[r, "\alpha_{B,A,C}"'] & B \bullet (A \bullet C) \arrow[r, "id_B \bullet \gamma_{A,C}"'] & B \bullet (C \bullet A)
\end{tikzcd}$$

$$\begin{tikzcd}
A \bullet (1 \bullet B) \arrow[r, "\alpha_{A,1,B}"] \arrow[d, "id_A \bullet \lambda_B"'] & (A \bullet 1) \bullet B \arrow[d, "\rho_A \bullet id_B"] \\
A \bullet B \arrow[r, equal] & A \bullet B
\end{tikzcd}
\qquad
\begin{tikzcd}
A \bullet B \arrow[r, "\gamma_{A,B}"] \arrow[dr, equal] & B \bullet A \arrow[d, "\gamma_{B,A}"] \\
& A \bullet B
\end{tikzcd}$$

$$\begin{tikzcd}
A \bullet 1 \arrow[r, "\gamma_{A,1}"] \arrow[d, "\rho_A"'] & 1 \bullet A \arrow[d, "\lambda_A"] \\
A \arrow[r, equal] & A
\end{tikzcd}$$

*The following equality is also require to hold:*

$$\lambda_1 \quad = \quad \rho_1 : 1 \bullet 1 \to 1$$

### 1.2 Symmetric Monoidal Closed Category

**Definition 2.** *A symmetric monoidal closed category (SMCC),* $(\mathbb{C}, \bullet, \multimap, 1, \alpha, \lambda, \rho, \gamma)$*, is a SMC such that for all objects* $A$ *in* $\mathbb{C}$*, the functor* $- \otimes A$ *has a specified right adjoint* $A \multimap -$*.*

## 2 Symmetric Monoidal Categories

**Definition 3.** *A symmetric monoidal catergory (SMC),* $(\mathbb{C}, \otimes, I, \alpha, \lambda, \rho, \gamma)$*, is a category* $\mathbb{C}$ *equippped with a bifunctor* $\otimes : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ *with a neutral element* $I$ *and natural isomorphisms* $\alpha, \lambda, \rho$*, and* $\gamma$*:*

1. $\alpha_{A,B,C} : A \otimes (B \otimes C) \xrightarrow{\sim} (A \otimes B) \otimes C$
2. $\lambda_A : I \otimes A \xrightarrow{\sim} A$
3. $\rho_A : A \otimes I \xrightarrow{\sim} A$
4. $\gamma_{A,B} : A \otimes B \xrightarrow{\sim} B \otimes A$

*which make the following 'coherence' diagrams commute.*

$$\begin{tikzcd}
A \otimes (B \otimes (C \otimes D)) \arrow[r, "\alpha_{A,B,C\otimes D}"] \arrow[d, "id_A \otimes \alpha_{B,C,D}"'] & (A \otimes B) \otimes (C \otimes D) \arrow[r, "\alpha_{A\otimes B,C,C}"] & ((A \otimes B) \otimes C) \otimes D \\
A \otimes ((B \otimes C) \otimes D) \arrow[rr, "\alpha_{A,B\otimes C,D}"'] & & (A \otimes (B \otimes C)) \otimes D \arrow[u, "\alpha_{A,B,C} \otimes id_D"']
\end{tikzcd}$$

$$\begin{tikzcd}
(A \otimes B) \otimes C \arrow[r, "\alpha_{A,B,C}"] \arrow[d, "\gamma_{A,B} \otimes id_C"'] & A \otimes (B \otimes C) \arrow[r, "\gamma_{A,B\otimes C}"] & (B \otimes C) \otimes A \arrow[d, "\alpha_{B,C,A}"'] \\
(B \otimes A) \otimes C \arrow[r, "\alpha_{B,A,C}"'] & B \otimes (A \otimes C) \arrow[r, "id_B \otimes \gamma_{A,C}"'] & B \otimes (C \otimes A)
\end{tikzcd}$$

$$\begin{tikzcd}
A \otimes (I \otimes B) \arrow[r, "\alpha_{A,I,B}"] \arrow[d, "id_A \otimes \lambda_B"'] & (A \otimes I) \otimes B \arrow[d, "\rho_A \otimes id_B"] \\
A \otimes B \arrow[r, equal] & A \otimes B
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes B \arrow[r, "\gamma_{A,B}"] \arrow[dr, equal] & B \otimes A \arrow[d, "\gamma_{B,A}"] \\
& A \otimes B
\end{tikzcd}$$

$$\begin{tikzcd}
A \otimes I \arrow[r, "\gamma_{A,I}"] \arrow[d, "\rho_A"'] & I \otimes A \arrow[d, "\lambda_A"] \\
A \arrow[r, equal] & A
\end{tikzcd}$$

*The following equality is also require to hold:*

$$\lambda_I \quad = \quad \rho_I : I \otimes I \to I$$

### 2.1 Symmetric Monoidal Closed Category

**Definition 4.** *A symmetric monoidal closed category (SMCC), $(\mathbb{C}, \otimes, \multimap, I, \alpha, \lambda, \rho, \gamma)$, is a SMC such that for all objects $A$ in $\mathbb{C}$, the functor $- \otimes A$ has a specified right adjoint $A \multimap -$.*