

# Monoidal-Annex

Preston Keel

Dr. Harley Eades

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## 1 Monoidal Categories

**Definition 1.** A monoidal category  $(\mathbb{C}, \otimes, I, \alpha, \lambda, \rho)$  is a category  $\mathbb{C}$ , a bifunctor  $\otimes : \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$ , an object  $I \in \mathbb{C}$ , and three natural isomorphisms  $\alpha, \lambda, \rho$ . Where,

$$\alpha = \alpha_{A,B,C} : A \otimes (B \otimes C) \cong (A \otimes B) \otimes C$$

is natural for all  $A, B, C \in \mathbb{C}$  and the diagram

$$\begin{array}{ccccc}
 A \otimes (B \otimes (C \otimes D)) & \xrightarrow{\alpha} & (A \otimes B) \otimes (C \otimes D) & \xrightarrow{\alpha} & ((A \otimes B) \otimes C) \otimes D \\
 \downarrow 1 \otimes \alpha & & & & \uparrow \alpha \otimes 1 \\
 A \otimes ((B \otimes C) \otimes D) & \xrightarrow{\alpha} & (A \otimes (B \otimes C)) \otimes D & & 
 \end{array}$$

commutes for all  $A, B, C, D \in \mathbb{C}$ .  $\gamma$  and  $\rho$  are natural

$$\gamma_A : I \otimes A \cong A \quad \rho_A : A \otimes I \cong A$$

for all objects  $A \in \mathbb{C}$ , the diagram

$$\begin{array}{ccc}
 A \otimes (I \otimes C) & \xrightarrow{\alpha} & (A \otimes I) \otimes C \\
 \downarrow 1 \otimes \lambda & & \downarrow \rho_A \otimes 1 \\
 A \otimes C & \xlongequal{\quad} & A \otimes C
 \end{array}$$

[3]

**Definition 2.** A monoidal functor or lax monoidal functor  $(F, m)$  between monoidal categories  $(\mathbb{C}, \otimes, I)$  and  $(\mathbb{D}, \otimes', I')$  is a functor  $F : \mathbb{C} \rightarrow \mathbb{D}$  equipped with natural transformations

$$m'_{A,B} : FA \otimes' FB \rightarrow F(A \otimes B) \quad m'' : I' \rightarrow FI$$

where the following diagrams commute in the category  $\mathbb{D}$  for all objects  $A, B, C \in \mathbb{C}$

$$\begin{array}{ccc}
(FA \otimes' FB) \otimes' FC & \xrightarrow{\alpha'} & FA \otimes' (FB \otimes' FC) \\
\downarrow m \otimes FC & & \downarrow FA \otimes m \\
F(A \otimes B) \otimes' FC & & FA \otimes' F(B \otimes C) \\
\downarrow m & & \downarrow m \\
F((A \otimes B) \otimes C) & \xrightarrow{F\alpha} & F(A \otimes (B \otimes C))
\end{array}$$
  

$$\begin{array}{ccc}
FA \otimes' I' & \xrightarrow{\rho'} & FA \\
\downarrow FA \otimes' m & & \uparrow F\rho \\
FA \otimes' FI & \xrightarrow{m} & F(A \otimes I)
\end{array}
\quad
\begin{array}{ccc}
I' \otimes' FB & \xrightarrow{\gamma'} & FB \\
\downarrow m \otimes' FB & & \uparrow F\gamma \\
FI \otimes' FB & \xrightarrow{m} & F(I \otimes B)
\end{array}$$

[4]

**Definition 3.** An oplax(colax/comonoidal) monoidal functor  $(F, n)$  between monoidal categories  $(\mathbb{C}, \otimes, I)$  and  $(\mathbb{D}, \otimes', I')$  is a functor  $F : \mathbb{C} \rightarrow \mathbb{D}$  and natural transformations

$$n'_{A,B} : F(A \otimes B) \rightarrow FA \otimes' FB \quad n'' : FI \rightarrow I'$$

where the following diagrams commute in the category  $\mathbb{D}$ , for all objects  $A, B, C \in \mathbb{C}$

$$\begin{array}{ccc}
F((A \otimes B) \otimes C) & \xrightarrow{F\alpha} & F(A \otimes (B \otimes C)) \\
\downarrow n & & \downarrow n \\
F(A \otimes B) \otimes' FC & & FA \otimes' F(B \otimes C) \\
\downarrow n \otimes' FC & & \downarrow FA \otimes' n \\
(FA \otimes' FB) \otimes' FC & \xrightarrow{\alpha'} & FA \otimes' (FB \otimes' FC)
\end{array}$$
  

$$\begin{array}{ccc}
F(A \otimes I) & \xrightarrow{F\rho} & FA \\
\downarrow n & & \uparrow \rho' \\
FA \otimes' FI & \xrightarrow{FA \otimes' n} & FA \otimes' I'
\end{array}
\quad
\begin{array}{ccc}
F(I \otimes B) & \xrightarrow{F\gamma} & FB \\
\downarrow n & & \uparrow \gamma' \\
FI \otimes' FB & \xrightarrow{n \otimes' FB} & I' \otimes' FB
\end{array}$$

[4]

**Definition 4.**  $(F, m)$  and  $(G, n)$  are (lax) monoidal functors between the monoidal categories:

$$(\mathbb{C}, \otimes, I) \rightarrow (\mathbb{D}, \otimes', I')$$

A monoidal natural transformation

$$\theta : (F, m) \Rightarrow (G, n) : (\mathbb{C}, \otimes, I) \rightarrow (\mathbb{D}, \otimes', I')$$

between the monoidal functors  $(F, m)$  and  $(G, n)$  is a natural transformation

$$\theta : F \Rightarrow G : \mathbb{C} \rightarrow \mathbb{D}$$

between the underlying functors, where the following diagrams commute, for all objects  $A, B \in \mathbb{C}$

$$\begin{array}{ccc} FA \otimes' FB & \xrightarrow{\theta_A \otimes' \theta_B} & GA \otimes' GB \\ \downarrow m & & \downarrow n \\ F(A \otimes B) & \xrightarrow{\theta_{A \otimes B}} & G(A \otimes B) \end{array} \quad \begin{array}{ccc} & I' & \\ m \swarrow & & \searrow n \\ FI & \xrightarrow{\theta_I} & GI \end{array}$$

[4]

**Definition 5.** A monoidal natural transformation

$$\theta : (F, m) \Rightarrow (G, n) : (\mathbb{C}, \otimes, I) \rightarrow (\mathbb{D}, \otimes', I')$$

between two oplax(colax/comonoidal) monoidal functors  $(F, m)$  and  $(G, n)$  is a natural transformation

$$\theta : F \Rightarrow G : \mathbb{C} \rightarrow \mathbb{D}$$

between the underlying functors, where the following diagrams commute, for all objects  $A, B \in \mathbb{C}$

$$\begin{array}{ccc} F(A \otimes B) & \xrightarrow{\theta_{A \otimes B}} & G(A \otimes B) \\ \downarrow m & & \downarrow n \\ FA \otimes' FB & \xrightarrow{\theta_A \otimes' \theta_B} & GA \otimes' GB \end{array} \quad \begin{array}{ccc} FI & \xrightarrow{\theta_I} & GI \\ m \searrow & & \swarrow n \\ & I' & \end{array}$$

[4]

**Definition 6.** A monoidal category  $\mathbb{C}$  is said to be biclosed if every  $- \otimes Y$  has a right adjoint  $[Y, -]$  and every  $X \otimes -$  has a right adjoint  $[[X, -]]$  [2]

## 2 Symmetric Monoidal Categories

**Definition 7.** A symmetric monoidal category (SMC),  $(\mathbb{C}, \otimes, I, \alpha, \lambda, \rho, \gamma)$ , is a category  $\mathbb{C}$  equipped with a bifunctor  $\otimes : \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$  with a neutral element  $I$  and natural isomorphisms  $\alpha, \lambda, \rho$ , and  $\gamma$ :

$$1. \alpha_{A,B,C} : A \otimes (B \otimes C) \xrightarrow{\sim} (A \otimes B) \otimes C$$

2.  $\lambda_A : I \otimes A \xrightarrow{\sim} A$
3.  $\rho_A : A \otimes I \xrightarrow{\sim} A$
4.  $\gamma_{A,B} : A \otimes B \xrightarrow{\sim} B \otimes A$

which make the following 'coherence' diagrams commute.

$$\begin{array}{ccccc}
A \otimes (B \otimes (C \otimes D)) & \xrightarrow{\alpha_{A,B,C \otimes D}} & (A \otimes B) \otimes (C \otimes D) & \xrightarrow{\alpha_{A \otimes B,C,C}} & ((A \otimes B) \otimes C) \otimes D \\
\downarrow id_A \otimes \alpha_{B,C,D} & & & & \uparrow \alpha_{A,B,C} \otimes id_D \\
A \otimes ((B \otimes C) \otimes D) & \xrightarrow{\alpha_{A,B \otimes C,D}} & & & (A \otimes (B \otimes C)) \otimes D
\end{array}$$
  

$$\begin{array}{ccccc}
(A \otimes B) \otimes C & \xrightarrow{\alpha_{A,B,C}} & A \otimes (B \otimes C) & \xrightarrow{\gamma_{A,B \otimes C}} & (B \otimes C) \otimes A \\
\downarrow \gamma_{A,B} \otimes id_C & & & & \downarrow \alpha_{B,C,A} \\
(B \otimes A) \otimes C & \xrightarrow{\alpha_{B,A,C}} & B \otimes (A \otimes C) & \xrightarrow{id_B \otimes \gamma_{A,C}} & B \otimes (C \otimes A)
\end{array}$$
  

$$\begin{array}{ccccc}
A \otimes (I \otimes B) & \xrightarrow{\alpha_{A,I,B}} & (A \otimes I) \otimes B & & A \otimes B \xrightarrow{\gamma_{A,B}} B \otimes A \\
\downarrow id_A \otimes \lambda_B & & \downarrow \rho_A \otimes id_B & & \downarrow \gamma_{B,A} \\
A \otimes B & \xlongequal{\quad} & A \otimes B & & A \otimes B
\end{array}$$
  

$$\begin{array}{ccc}
A \otimes I & \xrightarrow{\gamma_{A,I}} & I \otimes A \\
\downarrow \rho_A & & \downarrow \lambda_A \\
A & \xlongequal{\quad} & A
\end{array}$$

The following equality is also require to hold:

$$\lambda_I = \rho_I : I \otimes I \rightarrow I$$

[1]

**Definition 8.** A symmetric monoidal closed category (SMCC),  $(\mathbb{C}, \otimes, \multimap, I, \alpha, \lambda, \rho, \gamma)$ , is a SMC such that for all objects  $A$  in  $\mathbb{C}$ , the functor  $- \otimes A$  has a specified right adjoint  $A \multimap -$ .

**Definition 9.** A symmetric monoidal functor between SMCs  $(\mathbb{C}, \otimes, I, \alpha, \lambda, \rho, \gamma)$  and  $(\mathbb{C}', \otimes', I', \alpha', \lambda', \rho', \gamma')$  is a functor  $F : \mathbb{C} \rightarrow \mathbb{C}'$  equipped with

1. A morphism  $m_{I'} : I' \rightarrow FI$ .
2. For any two objects  $A$  and  $B$  in  $\mathbb{C}$ , a natural transformation  $m_{A,B} : F(A) \otimes' F(B) \rightarrow F(A \otimes B)$

These must satisfy the following diagrams:

$$\begin{array}{ccc}
FI \otimes' FA & \xrightarrow{m_{I,A}} & F(I \otimes A) \\
\uparrow m_I \otimes' id_{FA} & & \downarrow F(\lambda_A) \\
I' \otimes' FA & \xrightarrow{\lambda'_{FA}} & FA
\end{array}
\quad
\begin{array}{ccc}
FA \otimes' FI & \xrightarrow{m_{A,I}} & F(A \otimes I) \\
\uparrow id_{FA} \otimes' m_{I'} & & \downarrow F(\rho_A) \\
FA \otimes' I' & \xrightarrow{\rho'_{FA}} & FA
\end{array}$$
  

$$\begin{array}{ccccc}
(FA \otimes' FB) \otimes' FC & \xrightarrow{m_{A,B} \otimes' id_{FC}} & F(A \otimes B) \otimes' FC & \xrightarrow{m_{A \otimes B, C}} & F((A \otimes B) \otimes C) \\
\uparrow \alpha'_{FA, FB, FC} & & & & \uparrow F(\alpha_{A, B, C}) \\
FA \otimes' (FB \otimes' FC) & \xrightarrow{id_{FA} \otimes' m_{B, C}} & FA \otimes' F(B \otimes C) & \xrightarrow{m_{A, B \otimes C}} & F(A \otimes (B \otimes C))
\end{array}$$
  

$$\begin{array}{ccc}
FA \otimes' FB & \xrightarrow{m_{A,B}} & F(A \otimes B) \\
\downarrow \gamma'_{A,B} & & \downarrow F(\gamma_{A,B}) \\
FB \otimes' FA & \xrightarrow{m_{B,A}} & F(B \otimes A)
\end{array}$$

However in this particular case, assuming that  $!$  is a symmetric monoidal (endo) functor means that  $!$  comes equipped with a natural transformation

$$m_{A,B} : !A \otimes !B \rightarrow !(A \otimes B)$$

and a morphism

$$m_I : I \rightarrow !I$$

(where  $m_I$  is just the nullary version of the natural transformation.) The diagrams given in the above definition become the following:

$$\begin{array}{ccc}
!I \otimes !A & \xrightarrow{m_{I,A}} & !(I \otimes A) \\
\uparrow m_I \otimes id_{!A} & & \downarrow !(\lambda_A) \\
I \otimes !A & \xrightarrow{\lambda_{!A}} & !A
\end{array}
\quad
\begin{array}{ccc}
!A \otimes !I & \xrightarrow{m_{A,I}} & !(A \otimes I) \\
\uparrow id_{!A} \otimes m_I & & \downarrow !(\rho_A) \\
!A \otimes I & \xrightarrow{\rho_{!A}} & !A
\end{array}$$
  

$$\begin{array}{ccccc}
(!A \otimes !B) \otimes !C & \xrightarrow{m_{A,B} \otimes id_{!C}} & !(A \otimes B) \otimes !C & \xrightarrow{m_{A \otimes B, C}} & !((A \otimes B) \otimes C) \\
\uparrow \alpha_{!A, !B, !C} & & & & \uparrow !(\alpha_{A, B, C}) \\
!A \otimes (!B \otimes !C) & \xrightarrow{id_{!A} \otimes m_{B,C}} & !A \otimes !(B \otimes C) & \xrightarrow{m_{A, B \otimes C}} & !(A \otimes (B \otimes C))
\end{array}$$
  

$$\begin{array}{ccc}
!A \otimes !B & \xrightarrow{m_{A,B}} & !(A \otimes B) \\
\downarrow \gamma'_{A,B} & & \downarrow !(\gamma_{A,B}) \\
!B \otimes !A & \xrightarrow{m_{B,A}} & !(B \otimes A)
\end{array}$$

[1]

**Definition 10.** A symmetric monoidal functor,  $(F, m_{A,B}, m_{I'}) : \mathbb{C} \rightarrow \mathbb{C}'$ , is said to be

1. Strict if  $m_{A,B}$  and  $m_{I'}$  are identities.
2. Strong if  $m_{A,B}$  and  $m_{I'}$  are natural isomorphisms.

[1]

## References

- [1] G.M. Bierman. *On Intuitionistic Linear Logic*. PhD thesis, University of Cambridge, 1993.
- [2] Max Kelly. *Basic concepts of enriched category theory*, volume 64. CUP Archive, 1982.
- [3] Saunders Mac Lane. *Categories for the Working Mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1971.
- [4] Paul-André Mellies. Categorical semantics of linear logic. *Panoramas et synthèses*, 27:15–215, 2009.