Monoidal-Annex

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1 Monoidal Categories

Definition 1. *A monoidal category* $(\mathbb{C}, \otimes, I, \alpha, \lambda, \rho)$ *is a category* $\mathbb{C}$*, a bifunctor* $\otimes : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$*, an object* $I \in \mathbb{C}$*, and three natural isomorphisms* α, λ, ρ*. Where,*

$$\alpha = \alpha_{A,B,C} : A \otimes (B \otimes C) \cong (A \otimes B) \otimes C$$

is natural for all $A, B, C \in \mathbb{C}$ *and the diagram*

$$\begin{tikzcd}
A \otimes (B \otimes (C \otimes D)) \arrow[r, "\alpha"] \arrow[d, "1\otimes\alpha"'] & (A \otimes B) \otimes (C \otimes D) \arrow[r, "\alpha"] & ((A \otimes B) \otimes C) \otimes D \\
A \otimes ((B \otimes C) \otimes D) \arrow[rr, "\alpha"'] & & (A \otimes (B \otimes C)) \otimes D \arrow[u, "\alpha\otimes 1"']
\end{tikzcd}$$

commutes for all $A, B, C, D \in \mathbb{C}$*.* γ *and* ρ *are natural*

$$\gamma_A : I \otimes A \cong A \quad \rho_A : A \otimes I \cong A$$

for all objects $A \in \mathbb{C}$*, the diagram*

$$\begin{tikzcd}
A \otimes (I \otimes C) \arrow[r, "\alpha"] \arrow[d, "1\otimes\lambda"'] & (A \otimes I) \otimes C \arrow[d, "\rho_A\otimes 1"] \\
A \otimes C \arrow[r, equal] & A \otimes C
\end{tikzcd}$$

[3]

Definition 2. *A monoidal functor or lax monoidal functor* (F, m) *between monoidal categories* $(\mathbb{C}, \otimes, I)$ *and* $(\mathbb{D}, \otimes', I')$ *is a functor* $F : \mathbb{C} \to \mathbb{D}$ *equipped with nathral transformations*

$$m'_{A,B} : FA \otimes' FB \to F(A \otimes B) \quad m'' : I' \to FI$$

where the following diagrams commute in the category $\mathbb{D}$ *for all objects* $A, B, C \in \mathbb{C}$

$$\begin{tikzcd}
(FA \otimes' FB) \otimes' FC \arrow[rr, "\alpha'"] \arrow[d, "m \otimes FC"'] & & FA \otimes' (FB \otimes' FC) \arrow[d, "FA \otimes m"] \\
F(A \otimes B) \otimes' FC \arrow[d, "m"'] & & FA \otimes' F(B \otimes C) \arrow[d, "m"] \\
F((A \otimes B) \otimes C) \arrow[rr, "F\alpha"'] & & F(A \otimes (B \otimes C))
\end{tikzcd}$$

$$\begin{tikzcd}
FA \otimes' I' \arrow[r, "\rho'"] \arrow[d, "FA \otimes' m"'] & FA & I' \otimes' FB \arrow[r, "\gamma'"] \arrow[d, "m \otimes' FB"'] & FB \\
FA \otimes' FI \arrow[r, "m"] & F(A \otimes I) \arrow[u, "F\rho"'] & FI \otimes FB \arrow[r, "m"] & F(I \otimes B) \arrow[u, "F\gamma"']
\end{tikzcd}$$

[4]

Definition 3. *An oplax(colax/comonoidal) monoidal functor* (F, n) *between monoidal categories* $(\mathbb{C}, \otimes, I)$ *and* $(\mathbb{D}, \otimes', I')$ *is a functor* $F : \mathbb{C} \to \mathbb{D}$ *and natural transformations*

$$n'_{A,B} : F(A \otimes B) \to FA \otimes' FB \quad n'' : FI \to I'$$

where the following diagrams commute in the category $\mathbb{D}$*, for all objects* $A, B, C \in \mathbb{C}$

$$\begin{tikzcd}
F((A \otimes B) \otimes C) \arrow[rr, "F\alpha"] \arrow[d, "n"'] & & F(A \otimes (B \otimes C)) \arrow[d, "n"] \\
F(A \otimes B) \otimes' FC \arrow[d, "n \otimes' FC"'] & & FA \otimes' F(B \otimes C) \arrow[d, "FA \otimes' n"] \\
(FA \otimes' FB) \otimes' FC \arrow[rr, "\alpha'"'] & & FA \otimes' (FB \otimes' FC)
\end{tikzcd}$$

$$\begin{tikzcd}
F(A \otimes I) \arrow[r, "F\rho"] \arrow[d, "n"'] & FA & F(I \otimes B) \arrow[r, "F\gamma"] \arrow[d, "n"'] & FB \\
FA \otimes' FI \arrow[r, "FA \otimes' n"] & FA \otimes' I' \arrow[u, "\rho'"'] & FI \otimes' FB \arrow[r, "n \otimes' FB"] & I' \otimes' FB \arrow[u, "\gamma'"']
\end{tikzcd}$$

[4]

Definition 4. *(F, m) and (G, n) are (lax) monoidal functors between the monoidal categories:*

$$(\mathbb{C}, \otimes, I) \to (\mathbb{D}, \otimes', I')$$

A monoidal natural transformation

$$\theta : (F, m) \Rightarrow (G, n) : (\mathbb{C}, \otimes, I) \to (\mathbb{D}, \otimes', I')$$

between the monoidal functors (F, m) and (G, n) is a naural transformation

$$\theta : F \Rightarrow G : \mathbb{C} \to \mathbb{D}$$

between the underlying functors, where the following diamgrams commute, for all objects $A, B \in \mathbb{C}$

$$\begin{tikzcd} FA \otimes' FB \arrow[r, "\theta_A \otimes' \theta_B"] \arrow[d, "m"'] & GA \otimes' GB \arrow[d, "n"] \\ F(A \otimes B) \arrow[r, "\theta_{A \otimes B}"'] & G(A \otimes B) \end{tikzcd} \qquad \begin{tikzcd} & I' \arrow[dl, "m"'] \arrow[dr, "n"] & \\ FI \arrow[rr, "\theta_I"'] & & GI \end{tikzcd}$$

[4]

Definition 5. *A monoidal natural transformation*

$$\theta : (F, m) \Rightarrow (G, n) : (\mathbb{C}, \otimes, I) \to (\mathbb{D}, \otimes', I')$$

between two oplax(colax/comonoidal) monoidal functors (F, m) and (G, n) is a natural transformation

$$\theta : F \Rightarrow G : \mathbb{C} \to \mathbb{D}$$

between the underlying functors, where the following diagrmas commute, for all objects $A, B \in \mathbb{C}$

$$\begin{tikzcd} F(A \otimes B) \arrow[r, "\theta_{A \otimes B}"] \arrow[d, "m"'] & G(A \otimes B) \arrow[d, "n"] \\ FA \otimes' FB \arrow[r, "\theta_A \otimes' \theta_B"'] & GA \otimes' GB \end{tikzcd} \qquad \begin{tikzcd} FI \arrow[rr, "\theta_I"] \arrow[dr, "m"'] & & GI \arrow[dl, "n"] \\ & I' & \end{tikzcd}$$

[4]

Definition 6. *A monoidal category $\mathbb{C}$ is said to be biclosed if every $- \otimes Y$ has a right adjoint $[Y, -]$ and every $X \otimes -$ has a right adjoint $[\![X, -]\!]$ [2]*

Definition 7. *Given a pair of (lax) monoidal functors:*

$$(F_*, m) : (\mathbb{C}, \otimes, I) \to (\mathbb{D}, \otimes', I') \quad (F^*, n) : (\mathbb{D}, \otimes', I') \to (\mathbb{C}, \otimes, I).$$

A monoidal adjunction

$$(F_*, m) \dashv (F^*, n)$$

between the monoidal functors is defined as an adjunction $(F_*, F^*, \eta, \epsilon)$ between the underlying functors

$$F_* : \mathbb{C} \to \mathbb{D} \quad F^* : \mathbb{D} \to \mathbb{C}$$

whose natural transformations

$$\eta : id_C \Rightarrow F^* \circ F_* \quad \epsilon : F_* \circ F^* \Rightarrow id_D$$

are monoidal [4]

2 Symmetric Monoidal Categories

Definition 8. *A symmetric monoidal catergory (SMC), $(\mathbb{C}, \otimes, I, \alpha, \lambda, \rho, \gamma)$, is a category $\mathbb{C}$ equippped with a bifunctor $\otimes : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ with a neutral element I and natural isomorphisms α, λ, ρ, and γ:*

1. $\alpha_{A,B,C} : A \otimes (B \otimes C) \xrightarrow{\sim} (A \otimes B) \otimes C$
2. $\lambda_A : I \otimes A \xrightarrow{\sim} A$
3. $\rho_A : A \otimes I \xrightarrow{\sim} A$
4. $\gamma_{A,B} : A \otimes B \xrightarrow{\sim} B \otimes A$

which make the following 'coherence' diagrams commute.

$$\begin{tikzcd}
A \otimes (B \otimes (C \otimes D)) \arrow[r, "\alpha_{A,B,C\otimes D}"] \arrow[d, "id_A \otimes \alpha_{B,C,D}"'] & (A \otimes B) \otimes (C \otimes D) \arrow[r, "\alpha_{A\otimes B,C,C}"] & ((A \otimes B) \otimes C) \otimes D \\
A \otimes ((B \otimes C) \otimes D) \arrow[rr, "\alpha_{A,B\otimes C,D}"'] & & (A \otimes (B \otimes C)) \otimes D \arrow[u, "\alpha_{A,B,C} \otimes id_D"']
\end{tikzcd}$$

$$\begin{tikzcd}
(A \otimes B) \otimes C \arrow[r, "\alpha_{A,B,C}"] \arrow[d, "\gamma_{A,B} \otimes id_C"'] & A \otimes (B \otimes C) \arrow[r, "\gamma_{A,B\otimes C}"] & (B \otimes C) \otimes A \arrow[d, "\alpha_{B,C,A}"'] \\
(B \otimes A) \otimes C \arrow[r, "\alpha_{B,A,C}"'] & B \otimes (A \otimes C) \arrow[r, "id_B \otimes \gamma_{A,C}"'] & B \otimes (C \otimes A)
\end{tikzcd}$$

$$\begin{tikzcd}
A \otimes (I \otimes B) \arrow[r, "\alpha_{A,I,B}"] \arrow[d, "id_A \otimes \lambda_B"'] & (A \otimes I) \otimes B \arrow[d, "\rho_A \otimes id_B"] \\
A \otimes B \arrow[r, equal] & A \otimes B
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes B \arrow[r, "\gamma_{A,B}"] \arrow[dr, equal] & B \otimes A \arrow[d, "\gamma_{B,A}"] \\
& A \otimes B
\end{tikzcd}$$

$$\begin{tikzcd}
A \otimes I \arrow[r, "\gamma_{A,I}"] \arrow[d, "\rho_A"'] & I \otimes A \arrow[d, "\lambda_A"] \\
A \arrow[r, equal] & A
\end{tikzcd}$$

The following equality is also require to hold:

$$\lambda_I \quad = \quad \rho_I : I \otimes I \to I$$

[1]

Definition 9. *A symmetric monoidal closed category (SMCC), $(\mathbb{C}, \otimes, \multimap, I, \alpha, \lambda, \rho, \gamma)$, is a SMC such that for all objects A in $\mathbb{C}$, the functor $- \otimes A$ has a specified right adjoint $A \multimap -$.*

Definition 10. *A symmetric monoidal functor between SMCs $(\mathbb{C}, \otimes, I, \alpha, \lambda, \rho, \gamma)$ and $(\mathbb{C}', \otimes', I', \alpha', \lambda', \rho', \gamma')$ is a functor $F : \mathbb{C} \to \mathbb{C}'$ equipped with*

1. *A morphism $m_{I'} : I' \to FI$.*
2. *For any two objects A and B in $\mathbb{C}$, a natural transformation $m_{A,B} : F(A) \otimes' F(B) \to F(A \otimes B)$*

These must satisfy the following diagrams:

$$\begin{tikzcd}
FI \otimes' FA \arrow[r, "m_{I,A}"] & F(I \otimes A) \arrow[d, "F(\lambda_A)"] \\
I' \otimes' FA \arrow[u, "m_I \otimes' id_{FA}"] \arrow[r, "\lambda'_{FA}"'] & FA
\end{tikzcd}
\qquad
\begin{tikzcd}
FA \otimes' FI \arrow[r, "m_{A,I}"] & F(A \otimes I) \arrow[d, "F(\rho_A)"] \\
FA \otimes' I' \arrow[u, "id_{FA} \otimes' m_{I'}"] \arrow[r, "\rho'_{FA}"'] & FA
\end{tikzcd}$$

$$\begin{tikzcd}
(FA \otimes' FB) \otimes' FC \arrow[r, "m_{A,B} \otimes' id_{FC}"] & F(A \otimes B) \otimes' FC \arrow[r, "m_{A \otimes B, C}"] & F((A \otimes B) \otimes C) \\
FA \otimes' (FB \otimes' FC) \arrow[u, "\alpha'_{FA,FB,FC}"] \arrow[r, "id_{FA} \otimes' m_{B,C}"'] & FA \otimes' F(B \otimes C) \arrow[r, "m_{A, B \otimes C}"'] & F(A \otimes (B \otimes C)) \arrow[u, "F(\alpha_{A,B,C})"']
\end{tikzcd}$$

$$\begin{tikzcd}
FA \otimes' FB \arrow[r, "m_{A,B}"] \arrow[d, "\gamma'_{A,B}"'] & F(A \otimes B) \arrow[d, "F(\gamma_{A,B})"] \\
FB \otimes' FA \arrow[r, "m_{B,A}"'] & F(B \otimes A)
\end{tikzcd}$$

However in this particular case, assuming that ! is a symmetric monoidal (endo) functor means that ! comes equipped with a natural transformation

$$m_{A,B} : !A \otimes !B \to !(A \otimes B)$$

and a morphism

$$m_I : I \to !I$$

(where m_I is just the nullary version of the natural transformation.) The diagrmas given in the above definition become the following:

$$\begin{tikzcd}
!I \otimes !A \arrow[r, "m_{I,A}"] & !(I \otimes A) \arrow[d, "!(\lambda_A)"] \\
I \otimes !A \arrow[u, "m_I \otimes id_{!A}"] \arrow[r, "\lambda_{!A}"'] & !A
\end{tikzcd}
\qquad
\begin{tikzcd}
!A \otimes !I \arrow[r, "m_{A,I}"] & !(A \otimes I) \arrow[d, "!(\rho_A)"] \\
!A \otimes I \arrow[u, "id_{!A} \otimes m_I"] \arrow[r, "\rho_{!A}"'] & !A
\end{tikzcd}$$

$$\begin{tikzcd}
(!A \otimes !B) \otimes !C \arrow[r, "m_{A,B} \otimes id_{!C}"] & !(A \otimes B) \otimes !C \arrow[r, "m_{A \otimes B,C}"] & !((A \otimes B) \otimes C) \\
!A \otimes (!B \otimes !C) \arrow[u, "\alpha_{!A,!B,!C}"] \arrow[r, "id_{!A} \otimes m_{B,C}"'] & !A \otimes !(B \otimes C) \arrow[r, "m_{A,B \otimes C}"'] & !(A \otimes (B \otimes C)) \arrow[u, "!(\alpha_{A,B,C})"']
\end{tikzcd}$$

$$\begin{tikzcd}
!A \otimes !B \arrow[r, "m_{A,B}"] \arrow[d, "\gamma'_{A,B}"'] & !(A \otimes B) \arrow[d, "!(\gamma_{A,B})"] \\
!B \otimes !A \arrow[r, "m_{B,A}"'] & !(B \otimes A)
\end{tikzcd}$$

[1]

Definition 11. *A symmetric monoidal functor, $(F, m_{A,B}, m_{I'}) : \mathbb{C} \to \mathbb{C}'$, is said to be*

1. *Strict if $m_{A,B}$ and $m_{I'}$ are identities.*
2. *Strong if $m_{A,B}$ and $m_{I'}$ are natural isomorphisms.*

[1]

Definition 12. *An oplax monoidal functor*

$$(F, n) : (\mathbb{C}, \otimes, I) \to (\mathbb{D}, \otimes', I')$$

is symmetric when the following diagram commutes in the category $\mathbb{D}$ for all objects $A, B \in \mathbb{C}$

$$\begin{tikzcd}
F(A \otimes B) \arrow[r, "F_\gamma"] \arrow[d, "n"'] & F(B \otimes A) \arrow[d, "n"] \\
FA \otimes' FB \arrow[r, "\gamma'"'] & FB \otimes' FA
\end{tikzcd}$$

[4]

3 The Double Category of (co)Lax Monoidal Functors

Definition 13. *A* lax monoidal multifunctor *from $(\mathcal{C}_1, \ldots, \mathcal{C}_n)$ to $\mathcal{D}$ is a functor:*

$$F : \mathcal{C}_1 \times \cdots \times \mathcal{C}_n \longrightarrow \mathcal{D}$$

such that:

- *Each functor,* $(F^i, m^i(\overrightarrow{A_1}, -, \overrightarrow{A_{i+1}}), m^i_{I_i}(\overrightarrow{A_1}, -, \overrightarrow{A_{i+1}})$ *defined by* $F^i(X) = F(\overrightarrow{A_1}, X, \overrightarrow{A_{i+1}}) : \mathcal{C}_i \longrightarrow \mathcal{D}$*, for* $1 \leq i \leq n$ *is lax monoidal.*

- *The following equations hold for any* $1 \leq i < j \leq n$*:*

  – $m_{I_i}(\overrightarrow{I_1}, -, \overrightarrow{I_{i+1}}) = m_{I_j}(\overrightarrow{I_1}, -, \overrightarrow{I_{i+1}})$,

  – $(m^i_{I_i}(\overrightarrow{A_1}, -, \overrightarrow{A_{i+1}}, X, \overrightarrow{A_{j+1}}) \otimes m^i_{I_i}(\overrightarrow{A_1}, -, \overrightarrow{A_{i+1}}, Y, \overrightarrow{A_{j+1}})); m^j_{X,Y}(\overrightarrow{A_1}, I_i, \overrightarrow{A_{i+1}}, -, \overrightarrow{A_{j+1}})$
  $= \iota; m^i_{I_i}(\overrightarrow{A_1}, -, \overrightarrow{A_{i+1}}, X \otimes Y, \overrightarrow{A_{j+1}})$

  – $(m^j_{I_j}(\overrightarrow{A_1}, X, \overrightarrow{A_{i+1}}, -, \overrightarrow{A_{j+1}}) \otimes m^j_{I_j}(\overrightarrow{A_1}, Y, \overrightarrow{A_{i+1}}, -, \overrightarrow{A_{j+1}})); m^i_{X,Y}(\overrightarrow{A_1}, -, \overrightarrow{A_{i+1}}, I_j, \overrightarrow{A_{j+1}})$
  $= \iota; m^j_{I_j}(\overrightarrow{A_1}, X \otimes Y, \overrightarrow{A_{i+1}}, -, \overrightarrow{A_{j+1}})$

In the above definition ι is the isomorphism $\iota : I \otimes I \longrightarrow I$ which holds in any monoidal category.

Definition 14. *A* symmetric lax monoidal multifunctor *from* $(\mathcal{C}_1, \ldots, \mathcal{C}_n)$ *to* $\mathcal{D}$ *is a lax monoidal multifunctor*

$$F : (\mathcal{C}_1, \ldots, \mathcal{C}_n) \longrightarrow \mathcal{D}$$

such that:

- *Each functor,* $(F^i, m^i(\overrightarrow{A_1}, -, \overrightarrow{A_{i+1}}), m^i_{I_i}(\overrightarrow{A_1}, -, \overrightarrow{A_{i+1}})$ *defined by* $F^i(X) = F(\overrightarrow{A_1}, X, \overrightarrow{A_{i+1}}) : \mathcal{C}_i \longrightarrow \mathcal{D}$*, for* $1 \leq i \leq n$ *is symmetric lax monoidal.*

- *The following additional coherence axiom holds for any* $1 \leq i < j \leq n$*:*

  – $(m^i_{X,Y}(\overrightarrow{A_1}, -, \overrightarrow{A_{i+1}}, P, \overrightarrow{A_{j+1}}) \otimes m^i_{X,Y}(\overrightarrow{A_1}, -, \overrightarrow{A_{i+1}}, Q, \overrightarrow{A_{j+1}})); m^j_{P,Q}(\overrightarrow{A_1}, X \otimes Y, \overrightarrow{A_{i+1}}, -, \overrightarrow{A_{j+1}})$
  $= \tau; (m^j_{P,Q}(\overrightarrow{A_1}, X, \overrightarrow{A_{i+1}}, -, \overrightarrow{A_{j+1}}) \otimes m^j_{P,Q}(\overrightarrow{A_1}, Y, \overrightarrow{A_{i+1}}, -, \overrightarrow{A_{j+1}})); m^i_{X,Y}(\overrightarrow{A_1}, -, \overrightarrow{A_{i+1}}, P \otimes Q, \overrightarrow{A_{j+1}})$

In the above definition τ is the isomorphism $\tau : (A \otimes B) \otimes (C \otimes D) \longrightarrow (A \otimes C) \otimes (B \otimes D)$ which holds in any monoidal category.

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[4] Paul-André Mellies. Categorical semantics of linear logic. *Panoramas et syntheses*, 27:15–215, 2009.