## Monoidal-Annex

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### 1 Monoidal Categories

**Definition 1.** A monoidal category  $(\mathbb{C}, \otimes, I, \alpha, \lambda, \rho)$  is a category  $\mathbb{C}$ , a bifunctor  $\otimes : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ , an object  $I \in \mathbb{C}$ , and three natural isomorphisms  $\alpha, \lambda, \rho$ . Where,

$$\alpha = \alpha_{A,B,C} : A \otimes (B \otimes C) \cong (A \otimes B) \otimes C$$

is natural for all  $A, B, C \in \mathbb{C}$  and the diagram

$$A \otimes (B \otimes (C \otimes D)) \xrightarrow{\alpha} (A \otimes B) \otimes (C \otimes D) \xrightarrow{\alpha} ((A \otimes B) \otimes C) \otimes D$$

$$\uparrow_{0 \otimes 1}$$

$$A \otimes ((B \otimes C) \otimes D) \xrightarrow{\alpha} (A \otimes (B \otimes C)) \otimes D$$

commutes for all  $A, B, C, D \in \mathbb{C}$ .  $\gamma$  and  $\rho$  are natural

$$\gamma_A: I \otimes A \cong A \quad \rho_A: A \otimes I \cong A$$

for all objects  $A \in \mathbb{C}$ , the diagram

$$\begin{array}{c|c} A\otimes (I\otimes C) & \xrightarrow{\alpha} & (A\otimes I)\otimes C \\ & \downarrow^{\rho_A\otimes 1} & & \downarrow^{\rho_A\otimes 1} \\ & A\otimes C = & & A\otimes C \end{array}$$

[3]

**Definition 2.** A monoidal functor or lax monoidal functor (F, m) between monoidal categories  $(\mathbb{C}, \otimes, I)$  and  $(\mathbb{D}, \otimes', I')$  is a functor  $F : \mathbb{C} \to \mathbb{D}$  equipped with nathral transformations

$$m'_{A,B}: FA \otimes' FB \to F(A \otimes B) \quad m'': I' \to FI$$

where the following diagrams commute in the category  $\mathbb{D}$  for all objects  $A, B, C \in \mathbb{C}$ 

$$(FA \otimes' FB) \otimes' FC \xrightarrow{\alpha'} FA \otimes' (FB \otimes' FC)$$

$$\downarrow_{FA \otimes m}$$

$$F(A \otimes B) \otimes' FC \qquad FA \otimes' F(B \otimes C)$$

$$\downarrow_{m}$$

$$\downarrow_{m}$$

$$F((A \otimes B) \otimes C) \xrightarrow{F\alpha} F(A \otimes (B \otimes C))$$

$$FA \otimes' I' \xrightarrow{\rho'} FA \qquad I' \otimes' FB \xrightarrow{\gamma'} FB$$

$$\downarrow_{FA \otimes' m}$$

$$\downarrow_{FA \otimes' m}$$

$$\downarrow_{FA \otimes' FB}$$

$$\downarrow_{$$

[4]

**Definition 3.** An oplax(colax/comonoidal) monoidal functor (F, n) between monoidal categories  $(\mathbb{C}, \otimes, I)$  and  $(\mathbb{D}, \otimes', I')$  is a functor  $F : \mathbb{C} \to \mathbb{D}$  and natural transformations

$$n'_{A,B}: F(A\otimes B)\to FA\otimes' FB \quad n'':FI\to I'$$

where the following diagrams commute in the category  $\mathbb{D}$ , for all objects  $A, B, C \in \mathbb{C}$ 

$$F((A \otimes B) \otimes C) \xrightarrow{F\alpha} F(A \otimes (B \otimes C))$$

$$\downarrow n$$

$$\downarrow r$$

$$F(A \otimes B) \otimes' FC \qquad FA \otimes' F(B \otimes C)$$

$$\downarrow rA \otimes' FC \qquad \downarrow rA \otimes' r$$

$$(FA \otimes' FB) \otimes' FC \xrightarrow{\alpha'} FA \otimes' (FB \otimes' FC)$$

$$F(A \otimes I) \xrightarrow{F\rho} FA \qquad F(I \otimes B) \xrightarrow{F\gamma} FB$$

$$\downarrow n$$

$$\downarrow n$$

$$\uparrow r$$

$$\uparrow r$$

$$\uparrow r$$

$$\uparrow r$$

$$\downarrow r$$

$$\uparrow r$$

$$\uparrow r$$

$$\downarrow r$$

$$\uparrow r$$

$$\downarrow r$$

[4]

**Definition 4.** (F, m) and (G, n) are (lax) monoidal functors between the monoidal categories:

$$(\mathbb{C}, \otimes, I) \to (\mathbb{D}, \otimes', I')$$

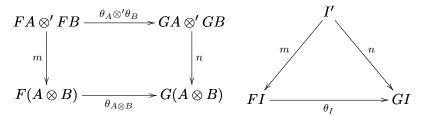
A monoidal natural transformation

$$\theta: (F, m) \Rightarrow (G, n): (\mathbb{C}, \otimes, I) \to (\mathbb{D}, \otimes', I')$$

between the monoidal functors (F, m) and (G, n) is a naural transformation

$$\theta: F \Rightarrow G: \mathbb{C} \to \mathbb{D}$$

between the underlying functors, where the following diamgrams commute, for all objects  $A, B \in \mathbb{C}$ 



[4]

**Definition 5.** A monoidal natural transformation

$$\theta: (F,m) \Rightarrow (G,n): (\mathbb{C},\otimes,I) \to (\mathbb{D},\otimes',I')$$

between two oplax(colax/comonoidal) monoidal functors (F, m) and (G, n) is a natural transformation

$$\theta: F \Rightarrow G: \mathbb{C} \to \mathbb{D}$$

between the underlying functors, where the following diagrams commute, for all objects  $A, B \in \mathbb{C}$ 

[4]

**Definition 6.** A monoidal category  $\mathbb{C}$  is said to be biclosed if every  $-\otimes Y$  has a right adjoint [Y, -] and every  $X \otimes -$  has a right adjoint [X, -] [2]

**Definition 7.** Given a pair of (lax) monoidal functors:

$$(F_*, m): (\mathbb{C}, \otimes, I) \to (\mathbb{D}, \otimes', I') \quad (F^*, n): (\mathbb{D}, \otimes', I') \to (\mathbb{C}, \otimes, I).$$

A monoidal adjunction

$$(F_*,m)\dashv (F^*,n)$$

between the monoidal functors is defined as an adjunction  $(F_*, F^*, \eta, \epsilon)$  between the underlying functors

$$F_*: \mathbb{C} \to \mathbb{D} \quad F^*: \mathbb{D} \to \mathbb{C}$$

whose natural transformations

$$\eta: id_C \Rightarrow F^* \circ F_* \quad \epsilon: F_* \circ F^* \Rightarrow id_D$$

are monoidal [4]

# 2 Symmetric Monoidal Categories

**Definition 8.** A symmetric monoidal catergory (SMC),  $(\mathbb{C}, \otimes, I, \alpha, \lambda, \rho, \gamma)$ , is a category  $\mathbb{C}$  equippped with a bifunctor  $\otimes : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$  with a neutral element I and natural isomorphisms  $\alpha, \lambda, \rho$ , and  $\gamma$ :

1. 
$$\alpha_{A,B,C}: A \otimes (B \otimes C) \xrightarrow{\sim} (A \otimes B) \otimes C$$

2. 
$$\lambda_A: I \otimes A \xrightarrow{\sim} A$$

3. 
$$\rho_A: A \otimes I \xrightarrow{\sim} A$$

4. 
$$\gamma_{A,B}: A \otimes B \xrightarrow{\sim} B \otimes A$$

which make the following 'coherence' diagrams commute.

$$A \otimes (B \otimes (C \otimes D)) \xrightarrow{\alpha_{A,B,C \otimes D}} (A \otimes B) \otimes (C \otimes D) \xrightarrow{\alpha_{A \otimes B,C,C}} ((A \otimes B) \otimes C) \otimes D$$

$$\downarrow^{id_{A} \otimes \alpha_{B,C,D}} \qquad \qquad \uparrow^{\alpha_{A,B,C} \otimes id_{D}}$$

$$A \otimes ((B \otimes C) \otimes D) \xrightarrow{\alpha_{A,B,C} \otimes C} A \otimes (B \otimes C) \xrightarrow{\alpha_{A,B \otimes C,D}} (A \otimes (B \otimes C)) \otimes D$$

$$(A \otimes B) \otimes C \xrightarrow{\alpha_{A,B,C}} A \otimes (B \otimes C) \xrightarrow{\gamma_{A,B \otimes C}} (B \otimes C) \otimes A$$

$$\uparrow^{\alpha_{A,B} \otimes id_{C}} \qquad \qquad \uparrow^{\alpha_{A,B,C}} A \otimes (A \otimes C) \xrightarrow{id_{B} \otimes \gamma_{A,C}} B \otimes (C \otimes A)$$

$$A \otimes (I \otimes B) \xrightarrow{\alpha_{A,I,B}} (A \otimes I) \otimes B \qquad A \otimes B \xrightarrow{\gamma_{A,B}} B \otimes A$$

$$A \otimes (I \otimes B) \xrightarrow{\alpha_{A,I,B}} (A \otimes I) \otimes B \qquad A \otimes B \xrightarrow{\gamma_{A,B}} B \otimes A$$

$$\downarrow id_A \otimes \lambda_B \qquad \qquad \downarrow \rho_A \otimes id_B \qquad \qquad \downarrow \gamma_{B,A}$$

$$A \otimes B = A \otimes B \qquad \qquad A \otimes B$$

$$\begin{array}{c|c}
A \otimes I \xrightarrow{\gamma_{A,I}} I \otimes A \\
 & \downarrow^{\lambda_A} \\
A = = A
\end{array}$$

The following equality is also require to hold:

$$\lambda_I = \rho_I : I \otimes I \to I$$

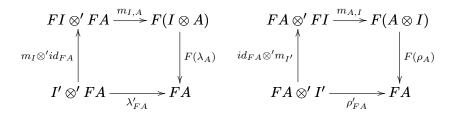
[1]

**Definition 9.** A symmetric monoidal closed category (SMCC),  $(\mathbb{C}, \otimes, \multimap, I, \alpha, \lambda, \rho, \gamma)$ , is a SMC such that for all objects A in  $\mathbb{C}$ , the functor  $-\otimes A$  has a specified right adjoint  $A \multimap -$ .

**Definition 10.** A symmetric monoidal functor between SMCs  $(\mathbb{C}, \otimes, I, \alpha, \lambda, \rho, \gamma)$  and  $(\mathbb{C}', \otimes', I', \alpha', \lambda', \rho', \gamma')$  is a functor  $F : \mathbb{C} \to \mathbb{C}'$  equipped with

- 1. A morphism  $m_{I'}: I' \to FI$ .
- 2. For any two objects A and B in  $\mathbb{C}$ , a natural transformation  $m_{A,B}: F(A) \otimes' F(B) \to F(A \otimes B)$

These must satisfy the following diagrams:



$$(FA \otimes' FB) \otimes' FC \xrightarrow{m_{A,B} \otimes' id_{FC}} F(A \otimes B) \otimes' FC \xrightarrow{m_{A \otimes B,C}} F((A \otimes B) \otimes C)$$

$$\alpha'_{FA,FB,FC} \qquad \qquad \uparrow \\ F(\alpha_{A,B,C})$$

$$FA \otimes' (FB \otimes' FC) \xrightarrow{id_{FA} \otimes' m_{B,C}} FA \otimes' F(B \otimes C) \xrightarrow{m_{A,B \otimes C}} F(A \otimes (B \otimes C))$$

$$FA \otimes' FB \xrightarrow{m_{A,B}} F(A \otimes B)$$

$$\downarrow^{\gamma'_{A,B}} \qquad \qquad \downarrow^{F(\gamma_{A,B})}$$

$$FB \otimes' FA \xrightarrow{m_{B,A}} F(B \otimes A)$$

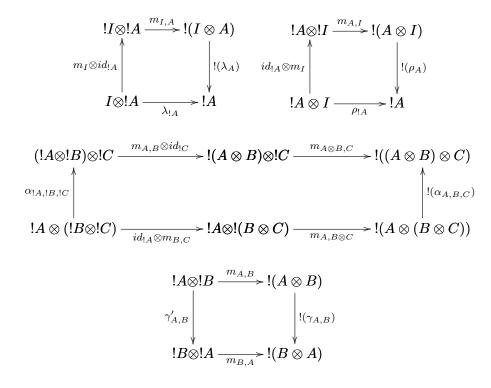
However in this particular case, assuming that ! is a symmetric monoidal (endo) functor means that ! comes equipped with a natural transformation

$$m_{A,B}: !A \otimes !B \rightarrow !(A \otimes B)$$

and a morphism

$$m_I:I\to !I$$

(where  $m_I$  is just the nullary version of the natural transformation.) The diagrams given in the above definition become the following:



[1]

**Definition 11.** A symmetric monoidal functor,  $(F, m_{A,B}, m_{I'}) : \mathbb{C} \to \mathbb{C}'$ , is said to be

- 1. Strict if  $m_{A,B}$  and  $m_{I'}$  are identities.
- 2. Strong if  $m_{A,B}$  and  $m_{I'}$  are natural isomorphisms.

[1]

**Definition 12.** An oplax monoidal functor

$$(F,n):(\mathbb{C},\otimes,I)\to(\mathbb{D},\otimes',I')$$

is symmetric when the following diagram commutes in the category  $\mathbb{D}$  for all objects  $A, B \in \mathbb{C}$ 

$$F(A \otimes B) \xrightarrow{F_{\gamma}} F(B \otimes A)$$

$$\downarrow n \qquad \qquad \downarrow n \qquad \qquad \downarrow n$$

$$FA \otimes' FB \xrightarrow{\gamma'} FB \otimes' FA$$

[4]

### References

[1] G.M. Bierman. On Intuitionistic Linear Logic. PhD thesis, University of Cambridge, 1993.

- [2] Max Kelly. Basic concepts of enriched category theory, volume 64. CUP Archive, 1982.
- [3] Saunders Mac Lane. Categories for the Working Mathematician, volume 5 of Graduate Texts in Mathematics. Springer-Verlag, New York, 1971.
- [4] Paul-André Mellies. Categorical semantics of linear logic. *Panoramas et syntheses*, 27:15–215, 2009.