Monoidal-Annex

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Contents

1 Monoidal Categories

Definition 1. *A monoidal category* $(\mathbb{C}, \otimes, I, \alpha, \lambda, \rho)$ *is a category* $\mathbb{C}$, *a bifunctor* $\otimes : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$, *an object* $I \in \mathbb{C}$, *and three natural isomorphisms* α, λ, ρ. *Where,*

$$\alpha = \alpha_{A,B,C} : A \otimes (B \otimes C) \cong (A \otimes B) \otimes C$$

is natural for all $A, B, C \in \mathbb{C}$ *and the diagram*

commutes for all $A, B, C, D \in \mathbb{C}$. γ *and* ρ *are natural*

$$\gamma_A : I \otimes A \cong A \quad \rho_A : A \otimes I \cong A$$

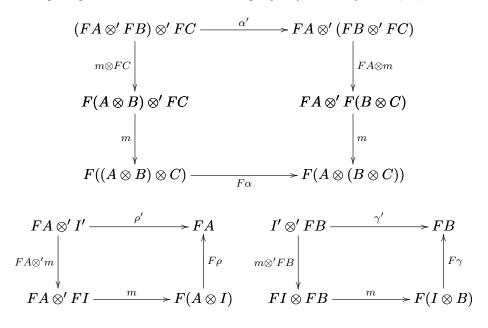*for all objects* $A \in \mathbb{C}$, *the diagram*

[3]

Definition 2. *A monoidal functor or lax monoidal functor* (F, m) *between monoidal categories* $(\mathbb{C}, \otimes, I)$ *and* $(\mathbb{D}, \otimes', I')$ *is a functor* $F : \mathbb{C} \to \mathbb{D}$ *equipped with a nathral transformation*

$$m_{A,B} : FA \otimes' FB \to F(A \otimes B)$$

and an isomorphism

$$m_I : I' \to FI$$

where the following diagrams commute in the category $\mathbb{D}$ for all objects $A, B, C \in \mathbb{C}$

$$
\begin{array}{ccc}
(FA \otimes' FB) \otimes' FC & \xrightarrow{\ \alpha'\ } & FA \otimes' (FB \otimes' FC) \\
{\scriptstyle m \otimes FC}\big\downarrow & & \big\downarrow {\scriptstyle FA \otimes m} \\
F(A \otimes B) \otimes' FC & & FA \otimes' F(B \otimes C) \\
{\scriptstyle m}\big\downarrow & & \big\downarrow {\scriptstyle m} \\
F((A \otimes B) \otimes C) & \xrightarrow[\ F\alpha\ ]{} & F(A \otimes (B \otimes C))
\end{array}
$$

$$
\begin{array}{ccc}
FA \otimes' I' & \xrightarrow{\ \rho'\ } & FA \\
{\scriptstyle FA \otimes' m}\big\downarrow & & \big\uparrow {\scriptstyle F\rho} \\
FA \otimes' FI & \xrightarrow[\ m\ ]{} & F(A \otimes I)
\end{array}
\qquad
\begin{array}{ccc}
I' \otimes' FB & \xrightarrow{\ \gamma'\ } & FB \\
{\scriptstyle m \otimes' FB}\big\downarrow & & \big\uparrow {\scriptstyle F\gamma} \\
FI \otimes FB & \xrightarrow[\ m\ ]{} & F(I \otimes B)
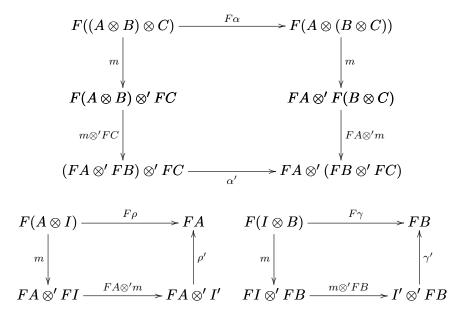\end{array}
$$

[4]

Definition 3. *An oplax(colax/comonoidal) monoidal functor (F, m) between monoidal categories $(\mathbb{C}, \otimes, I)$ and $(\mathbb{D}, \otimes', I')$ is a functor $F : \mathbb{C} \to \mathbb{D}$ with a natural transformation*

$$m_{A,B} : F(A \otimes B) \to FA \otimes' FB$$

and an isomorphism

$$m_I : FI \to I'$$

where the following diagrams commute in the category $\mathbb{D}$, for all objects $A, B, C \in \mathbb{C}$

$$
\begin{array}{ccc}
F((A \otimes B) \otimes C) & \xrightarrow{\;F\alpha\;} & F(A \otimes (B \otimes C)) \\
\Big\downarrow{m} & & \Big\downarrow{m} \\
F(A \otimes B) \otimes' FC & & FA \otimes' F(B \otimes C) \\
\Big\downarrow{m \otimes' FC} & & \Big\downarrow{FA \otimes' m} \\
(FA \otimes' FB) \otimes' FC & \xrightarrow{\;\alpha'\;} & FA \otimes' (FB \otimes' FC)
\end{array}
$$

$$
\begin{array}{ccc}
F(A \otimes I) & \xrightarrow{\;F\rho\;} & FA \\
\Big\downarrow{m} & & \Big\uparrow{\rho'} \\
FA \otimes' FI & \xrightarrow{\;FA \otimes' m\;} & FA \otimes' I'
\end{array}
\qquad
\begin{array}{ccc}
F(I \otimes B) & \xrightarrow{\;F\gamma\;} & FB \\
\Big\downarrow{m} & & \Big\uparrow{\gamma'} \\
FI \otimes' FB & \xrightarrow{\;m \otimes' FB\;} & I' \otimes' FB
\end{array}
$$

[4]

Definition 4. *Suppose (F, m) and (G, n) are (lax) monoidal functors between the monoidal categories:*

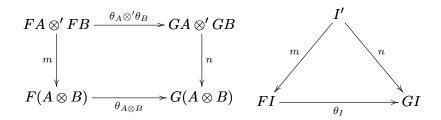$$(\mathbb{C}, \otimes, I) \to (\mathbb{D}, \otimes', I')$$

A monoidal natural transformation

$$\theta : (F, m) \Rightarrow (G, n) : (\mathbb{C}, \otimes, I) \to (\mathbb{D}, \otimes', I')$$

between the monoidal functors (F, m) and (G, n) is a naural transformation

$$\theta : F \Rightarrow G : \mathbb{C} \to \mathbb{D}$$

between the underlying functors, where the following diamgrams commute, for all objects $A, B \in \mathbb{C}$

$$
\begin{array}{ccc}
FA \otimes' FB & \xrightarrow{\;\theta_A \otimes' \theta_B\;} & GA \otimes' GB \\
\Big\downarrow{m} & & \Big\downarrow{n} \\
F(A \otimes B) & \xrightarrow{\;\theta_{A \otimes B}\;} & G(A \otimes B)
\end{array}
\qquad
\begin{array}{ccc}
 & I' & \\
 {}^{m}\swarrow & & \searrow^{n} \\
FI & \xrightarrow[\;\theta_I\;]{} & GI
\end{array}
$$

[4]

Definition 5. *A monoidal natural transformation*

$$\theta : (F, m) \Rightarrow (G, n) : (\mathbb{C}, \otimes, I) \to (\mathbb{D}, \otimes', I')$$

between two oplax(colax/comonoidal) monoidal functors (F, m) and (G, n) is a natural transformation

$$\theta : F \Rightarrow G : \mathbb{C} \to \mathbb{D}$$

3

between the underlying functors, where the following diagrmas commute, for all objects $A, B \in \mathbb{C}$

$$
\begin{array}{ccc}
F(A \otimes B) & \xrightarrow{\theta_{A \otimes B}} & G(A \otimes B) \\
\downarrow{\scriptstyle m} & & \downarrow{\scriptstyle n} \\
FA \otimes' FB & \xrightarrow{\theta_A \otimes' \theta_B} & GA \otimes' GB
\end{array}
\qquad
\begin{array}{ccc}
FI & \xrightarrow{\theta_I} & GI \\
& \searrow{\scriptstyle m} \quad \swarrow{\scriptstyle n} & \\
& I' &
\end{array}
$$

[4]

Definition 6. *A monoidal category $\mathbb{C}$ is said to be biclosed if every $- \otimes Y$ has a right adjoint $[Y, -]$ and every $X \otimes -$ has a right adjoint $[\![X, -]\!]$ [2]*

Definition 7. *Given a pair of (lax) monoidal functors:*

$$(F_*, m) : (\mathbb{C}, \otimes, I) \to (\mathbb{D}, \otimes', I') \quad (F^*, n) : (\mathbb{D}, \otimes', I') \to (\mathbb{C}, \otimes, I).$$

A monoidal adjunction

$$(F_*, m) \dashv (F^*, n)$$

between the monoidal functors is defined as an adjunction $(F_*, F^*, \eta, \epsilon)$ between the underlying functors

$$F_* : \mathbb{C} \to \mathbb{D} \quad F^* : \mathbb{D} \to \mathbb{C}$$

whose natural transformations

$$\eta : id_C \Rightarrow F^* \circ F_* \quad \epsilon : F_* \circ F^* \Rightarrow id_D$$

are monoidal [4]

2 Symmetric Monoidal Categories

Definition 8. *A symmetric monoidal catergory (SMC), is a monoidal category $\mathbb{C}$ with an additional natural isomorphism γ*
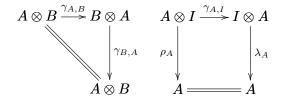
$$\gamma_{A,B} : A \otimes B \xrightarrow{\sim} B \otimes A$$

and three additional 'coherence' diagrams that commute.

$$
\begin{array}{ccccc}
(A \otimes B) \otimes C & \xrightarrow{\alpha_{A,B,C}} & A \otimes (B \otimes C) & \xrightarrow{\gamma_{A,B \otimes C}} & (B \otimes C) \otimes A \\
\downarrow{\scriptstyle \gamma_{A,B} \otimes id_C} & & & & \downarrow{\scriptstyle \alpha_{B,C,A}} \\
(B \otimes A) \otimes C & \xrightarrow{\alpha_{B,A,C}} & B \otimes (A \otimes C) & \xrightarrow{id_B \otimes \gamma_{A,C}} & B \otimes (C \otimes A)
\end{array}
$$

$$
\begin{array}{ccc}
A \otimes B & \xrightarrow{\gamma_{A,B}} & B \otimes A \\
\parallel & & \downarrow{\scriptstyle \gamma_{B,A}} \\
& & A \otimes B
\end{array}
\qquad
\begin{array}{ccc}
A \otimes I & \xrightarrow{\gamma_{A,I}} & I \otimes A \\
\downarrow{\scriptstyle \rho_A} & & \downarrow{\scriptstyle \lambda_A} \\
A & = & A
\end{array}
$$

4

The following equality is also require to hold:

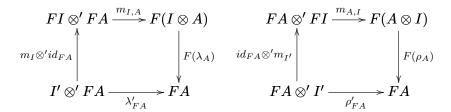$$\lambda_I \;\;=\;\; \rho_I : I \otimes I \to I$$

[1]

Definition 9. *A symmetric monoidal closed category (SMCC), $(\mathbb{C}, \otimes, \multimap, I, \alpha, \lambda, \rho, \gamma)$, is a SMC such that for all objects A in $\mathbb{C}$, the functor $- \otimes A$ has a specified right adjoint $A \multimap -$.*

Definition 10. *A symmetric monoidal functor between SMCs $(\mathbb{C}, \otimes, I, \alpha, \lambda, \rho, \gamma)$ and $(\mathbb{C}', \otimes', I', \alpha', \lambda', \rho', \gamma')$ is a functor $F : \mathbb{C} \to \mathbb{C}'$ equipped with*

1. *A morphism $m_{I'} : I' \to FI$.*

2. *For any two objects A and B in $\mathbb{C}$, a natural transformation $m_{A,B} : F(A) \otimes' F(B) \to F(A \otimes B)$*

These must satisfy the following diagrams:

$$
\begin{array}{ccc}
FI \otimes' FA & \xrightarrow{m_{I,A}} & F(I \otimes A) \\
{\scriptstyle m_I \otimes' id_{FA}}\uparrow & & \downarrow{\scriptstyle F(\lambda_A)} \\
I' \otimes' FA & \xrightarrow[\lambda'_{FA}]{} & FA
\end{array}
\qquad
\begin{array}{ccc}
FA \otimes' FI & \xrightarrow{m_{A,I}} & F(A \otimes I) \\
{\scriptstyle id_{FA} \otimes' m_{I'}}\uparrow & & \downarrow{\scriptstyle F(\rho_A)} \\
FA \otimes' I' & \xrightarrow[\rho'_{FA}]{} & FA
\end{array}
$$

$$
\begin{array}{ccccc}
(FA \otimes' FB) \otimes' FC & \xrightarrow{m_{A,B} \otimes' id_{FC}} & F(A \otimes B) \otimes' FC & \xrightarrow{m_{A \otimes B, C}} & F((A \otimes B) \otimes C) \\
{\scriptstyle \alpha'_{FA,FB,FC}}\uparrow & & & & \uparrow{\scriptstyle F(\alpha_{A,B,C})} \\
FA \otimes' (FB \otimes' FC) & \xrightarrow[id_{FA} \otimes' m_{B,C}]{} & FA \otimes' F(B \otimes C) & \xrightarrow[m_{A, B \otimes C}]{} & F(A \otimes (B \otimes C))
\end{array}
$$

$$
\begin{array}{ccc}
FA \otimes' FB & \xrightarrow{m_{A,B}} & F(A \otimes B) \\
{\scriptstyle \gamma'_{A,B}}\downarrow & & \downarrow{\scriptstyle F(\gamma_{A,B})} \\
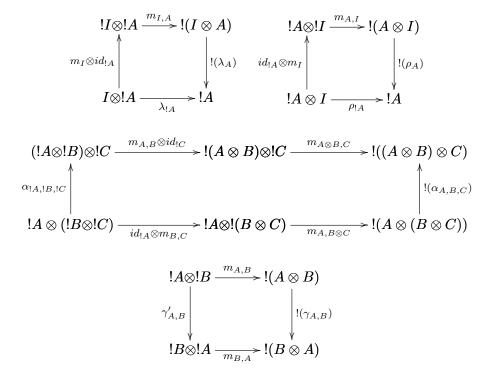FB \otimes' FA & \xrightarrow[m_{B,A}]{} & F(B \otimes A)
\end{array}
$$

However in this particular case, assuming that $!$ is a symmetric monoidal (endo) functor means that $!$ comes equipped with a natural transformation

$$m_{A,B} : \,!A \otimes \,!B \to \,!(A \otimes B)$$

and a morphism

$$m_I : I \to \,!I$$

(where m_I is just the nullary version of the natural transformation.) The diagrmas given in the above definition become the following:

$$
\begin{array}{ccc}
!I\otimes!A & \xrightarrow{\ m_{I,A}\ } & !(I\otimes A) \\[2pt]
{\scriptstyle m_I\otimes id_{!A}}\Big\uparrow & & \Big\downarrow{\scriptstyle !(\lambda_A)} \\[2pt]
I\otimes!A & \xrightarrow[\ \lambda_{!A}\ ]{} & !A
\end{array}
\qquad
\begin{array}{ccc}
!A\otimes!I & \xrightarrow{\ m_{A,I}\ } & !(A\otimes I) \\[2pt]
{\scriptstyle id_{!A}\otimes m_I}\Big\uparrow & & \Big\downarrow{\scriptstyle !(\rho_A)} \\[2pt]
!A\otimes I & \xrightarrow[\ \rho_{!A}\ ]{} & !A
\end{array}
$$

$$
\begin{array}{ccccc}
(!A\otimes!B)\otimes!C & \xrightarrow{\ m_{A,B}\otimes id_{!C}\ } & !(A\otimes B)\otimes!C & \xrightarrow{\ m_{A\otimes B,C}\ } & !((A\otimes B)\otimes C) \\[2pt]
{\scriptstyle \alpha_{!A,!B,!C}}\Big\uparrow & & & & \Big\uparrow{\scriptstyle !(\alpha_{A,B,C})} \\[2pt]
!A\otimes(!B\otimes!C) & \xrightarrow[\ id_{!A}\otimes m_{B,C}\ ]{} & !A\otimes!(B\otimes C) & \xrightarrow[\ m_{A,B\otimes C}\ ]{} & !(A\otimes(B\otimes C))
\end{array}
$$

$$
\begin{array}{ccc}
!A\otimes!B & \xrightarrow{\ m_{A,B}\ } & !(A\otimes B) \\[2pt]
{\scriptstyle \gamma'_{A,B}}\Big\downarrow & & \Big\downarrow{\scriptstyle !(\gamma_{A,B})} \\[2pt]
!B\otimes!A & \xrightarrow[\ m_{B,A}\ ]{} & !(B\otimes A)
\end{array}
$$

[1]

Definition 11. *A symmetric monoidal functor, $(F, m_{A,B}, m_{I'}) : \mathbb{C} \to \mathbb{C}'$, is said to be*

1. *Strict if $m_{A,B}$ and $m_{I'}$ are identities.*

2. *Strong if $m_{A,B}$ and $m_{I'}$ are natural isomorphisms.*

[1]

Definition 12. *An oplax monoidal functor*

$$(F, m) : (\mathbb{C}, \otimes, I) \to (\mathbb{D}, \otimes', I')$$

is symmetric when the following diagram commutes in the category $\mathbb{D}$ for all objects $A, B \in \mathbb{C}$

$$
\begin{array}{ccc}
F(A\otimes B) & \xrightarrow{\ F_\gamma\ } & F(B\otimes A) \\[2pt]
{\scriptstyle m}\Big\downarrow & & \Big\downarrow{\scriptstyle m} \\[2pt]
FA\otimes' FB & \xrightarrow[\ \gamma'\ ]{} & FB\otimes' FA
\end{array}
$$

[4]

3 The Double Category of (co)Lax Monoidal Functors

Definition 13. *A* lax monoidal multifunctor *from $(\mathcal{C}_1, \ldots, \mathcal{C}_n)$ to $\mathcal{D}$ is a functor:*

$$F : \mathcal{C}_1 \times \cdots \times \mathcal{C}_n \longrightarrow \mathcal{D}$$

such that:

- *Each functor, $(F^i, m^i(\overrightarrow{A_1}, -, \overrightarrow{A_{i+1}}), m^i_{I_i}(\overrightarrow{A_1}, -, \overrightarrow{A_{i+1}})$ defined by $F^i(X) = F(\overrightarrow{A_1}, X, \overrightarrow{A_{i+1}})$: $\mathcal{C}_i \longrightarrow \mathcal{D}$, for $1 \leq i \leq n$ is lax monoidal.*

- *The following equations hold for any $1 \leq i < j \leq n$:*

  - $m_{I_i}(\overrightarrow{I_1}, -, \overrightarrow{I_{i+1}}) = m_{I_j}(\overrightarrow{I_1}, -, \overrightarrow{I_{i+1}})$,

  - $(m^i_{I_i}(\overrightarrow{A_1}, -, \overrightarrow{A_{i+1}}, X, \overrightarrow{A_{j+1}}) \otimes m^i_{I_i}(\overrightarrow{A_1}, -, \overrightarrow{A_{i+1}}, Y, \overrightarrow{A_{j+1}})); m^j_{X,Y}(\overrightarrow{A_1}, I_i, \overrightarrow{A_{i+1}}, -, \overrightarrow{A_{j+1}})$
    $= \iota; m^i_{I_i}(\overrightarrow{A_1}, -, \overrightarrow{A_{i+1}}, X \otimes Y, \overrightarrow{A_{j+1}})$

  - $(m^j_{I_j}(\overrightarrow{A_1}, X, \overrightarrow{A_{i+1}}, -, \overrightarrow{A_{j+1}}) \otimes m^j_{I_j}(\overrightarrow{A_1}, Y, \overrightarrow{A_{i+1}}, -, \overrightarrow{A_{j+1}})); m^i_{X,Y}(\overrightarrow{A_1}, -, \overrightarrow{A_{i+1}}, I_j, \overrightarrow{A_{j+1}})$
    $= \iota; m^j_{I_j}(\overrightarrow{A_1}, X \otimes Y, \overrightarrow{A_{i+1}}, -, \overrightarrow{A_{j+1}})$

In the above definition ι is the isomorphism $\iota : I \otimes I \longrightarrow I$ which holds in any monoidal category.

Definition 14. *A* symmetric lax monoidal multifunctor *from $(\mathcal{C}_1, \ldots, \mathcal{C}_n)$ to $\mathcal{D}$ is a lax monoidal multifunctor*

$$F : (\mathcal{C}_1, \ldots, \mathcal{C}_n) \longrightarrow \mathcal{D}$$

such that:

- *Each functor, $(F^i, m^i(\overrightarrow{A_1}, -, \overrightarrow{A_{i+1}}), m^i_{I_i}(\overrightarrow{A_1}, -, \overrightarrow{A_{i+1}})$ defined by $F^i(X) = F(\overrightarrow{A_1}, X, \overrightarrow{A_{i+1}})$: $\mathcal{C}_i \longrightarrow \mathcal{D}$, for $1 \leq i \leq n$ is symmetric lax monoidal.*

- *The following additional coherence axiom holds for any $1 \leq i < j \leq n$:*

  - $(m^i_{X,Y}(\overrightarrow{A_1}, -, \overrightarrow{A_{i+1}}, P, \overrightarrow{A_{j+1}}) \otimes m^i_{X,Y}(\overrightarrow{A_1}, -, \overrightarrow{A_{i+1}}, Q, \overrightarrow{A_{j+1}})); m^j_{P,Q}(\overrightarrow{A_1}, X \otimes Y, \overrightarrow{A_{i+1}}, -, \overrightarrow{A_{j+1}})$
    $= \tau; (m^j_{P,Q}(\overrightarrow{A_1}, X, \overrightarrow{A_{i+1}}, -, \overrightarrow{A_{j+1}}) \otimes m^j_{P,Q}(\overrightarrow{A_1}, Y, \overrightarrow{A_{i+1}}, -, \overrightarrow{A_{j+1}})); m^i_{X,Y}(\overrightarrow{A_1}, -, \overrightarrow{A_{i+1}}, P \otimes Q, \overrightarrow{A_{j+1}})$

In the above definition τ is the isomorphism $\tau : (A \otimes B) \otimes (C \otimes D) \longrightarrow (A \otimes C) \otimes (B \otimes D)$ which holds in any monoidal category.

References

[1] G.M. Bierman. *On Intuitionistic Linear Logic*. PhD thesis, University of Cambridge, 1993.

[2] Max Kelly. *Basic concepts of enriched category theory*, volume 64. CUP Archive, 1982.

[3] Saunders Mac Lane. *Categories for the Working Mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1971.

[4] Paul-André Mellies. Categorical semantics of linear logic. *Panoramas et syntheses*, 27:15–215, 2009.