Monoidal-Annex

Preston Keel Dr. Harley Eades

February 27, 2019

Contents

1 Monoidal Categories **1**

2 Symmetric Monoidal Categories **3**

1 Monoidal Categories

Definition 1. *A monoidal category* $(\mathbb{C}, \otimes, I, \alpha, \lambda, \rho)$ *is a category* $\mathbb{C}$*, a bifunctor* $\otimes : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$*, an object* $I \in \mathbb{C}$*, and three natural isomorphisms* α, λ, ρ*. Where,*

$$\alpha = \alpha_{A,B,C} : A \otimes (B \otimes C) \cong (A \otimes B) \otimes C$$

is natural for all $A, B, C \in \mathbb{C}$ *and the diagram*

$$\begin{tikzcd}
A \otimes (B \otimes (C \otimes D)) \arrow[r, "\alpha"] \arrow[d, "1\otimes\alpha"'] & (A \otimes B) \otimes (C \otimes D) \arrow[r, "\alpha"] & ((A \otimes B) \otimes C) \otimes D \\
A \otimes ((B \otimes C) \otimes D) \arrow[rr, "\alpha"'] & & (A \otimes (B \otimes C)) \otimes D \arrow[u, "\alpha\otimes 1"']
\end{tikzcd}$$

commutes for all $A, B, C, D \in \mathbb{C}$*.* γ *and* ρ *are natural*

$$\gamma_A : I \otimes A \cong A \quad \rho_A : A \otimes I \cong A$$

for all objects $A \in \mathbb{C}$*, the diagram*

$$\begin{tikzcd}
A \otimes (I \otimes C) \arrow[r, "\alpha"] \arrow[d, "1\otimes\lambda"'] & (A \otimes I) \otimes C \arrow[d, "\rho_A\otimes 1"] \\
A \otimes C \arrow[r, equal] & A \otimes C
\end{tikzcd}$$

[2]

Definition 2. *A monoidal functor or lax monoidal functor* (F, m) *between monoidal categories* $(\mathbb{C}, \otimes, I)$ *and* $(\mathbb{D}, \otimes', I')$ *is a functor* $F : \mathbb{C} \to \mathbb{D}$ *equipped with nathral transformations*

$$m'_{A,B} : FA \otimes' FB \to F(A \otimes B) \quad m'' : I' \to FI$$

where the following diagrams commute in the category $\mathbb{D}$ for all objects $A, B, C \in \mathbb{C}$

$$
\begin{tikzcd}
(FA \otimes' FB) \otimes' FC \arrow[r, "\alpha'"] \arrow[d, swap, "m \otimes FC"] & FA \otimes' (FB \otimes' FC) \arrow[d, "FA \otimes m"] \\
F(A \otimes B) \otimes' FC \arrow[d, swap, "m"] & FA \otimes' F(B \otimes C) \arrow[d, "m"] \\
F((A \otimes B) \otimes C) \arrow[r, swap, "F\alpha"] & F(A \otimes (B \otimes C))
\end{tikzcd}
$$

$$
\begin{tikzcd}
FA \otimes' I' \arrow[r, "\rho'"] \arrow[d, swap, "FA \otimes' m"] & FA \\
FA \otimes' FI \arrow[r, "m"] & F(A \otimes I) \arrow[u, swap, "F\rho"]
\end{tikzcd}
\qquad
\begin{tikzcd}
I' \otimes' FB \arrow[r, "\gamma'"] \arrow[d, swap, "m \otimes' FB"] & FB \\
FI \otimes FB \arrow[r, "m"] & F(I \otimes B) \arrow[u, swap, "F\gamma"]
\end{tikzcd}
$$

[3]

Definition 3. *An oplax(colax/comonoidal) monoidal functor (F, n) between monoidal categories $(\mathbb{C}, \otimes, I)$ and $(\mathbb{D}, \otimes', I')$ is a functor $F : \mathbb{C} \to \mathbb{D}$ and natural transformations*

$$n'_{A,B} : F(A \otimes B) \to FA \otimes' FB \quad n'' : FI \to I'$$

where the following diagrams commute in the category $\mathbb{D}$, for all objects $A, B, C \in \mathbb{C}$

$$
\begin{tikzcd}
F((A \otimes B) \otimes C) \arrow[r, "F\alpha"] \arrow[d, swap, "n"] & F(A \otimes (B \otimes C)) \arrow[d, "n"] \\
F(A \otimes B) \otimes' FC \arrow[d, swap, "n \otimes' FC"] & FA \otimes' F(B \otimes C) \arrow[d, "FA \otimes' n"] \\
(FA \otimes' FB) \otimes' FC \arrow[r, swap, "\alpha'"] & FA \otimes' (FB \otimes' FC)
\end{tikzcd}
$$

$$
\begin{tikzcd}
F(A \otimes I) \arrow[r, "F\rho"] \arrow[d, swap, "n"] & FA \\
FA \otimes' FI \arrow[r, "FA \otimes' n"] & FA \otimes' I' \arrow[u, swap, "\rho'"]
\end{tikzcd}
\qquad
\begin{tikzcd}
F(I \otimes B) \arrow[r, "F\gamma"] \arrow[d, swap, "n"] & FB \\
FI \otimes' FB \arrow[r, "n \otimes' FB"] & I' \otimes' FB \arrow[u, swap, "\gamma'"]
\end{tikzcd}
$$

[3]

2 Symmetric Monoidal Categories

Definition 4. *A symmetric monoidal catergory (SMC), $(\mathbb{C}, \otimes, I, \alpha, \lambda, \rho, \gamma)$, is a category $\mathbb{C}$ equippped with a bifunctor $\otimes : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ with a neutral element I and natural isomorphisms α, λ, ρ, and γ:*

1. $\alpha_{A,B,C} : A \otimes (B \otimes C) \xrightarrow{\sim} (A \otimes B) \otimes C$
2. $\lambda_A : I \otimes A \xrightarrow{\sim} A$
3. $\rho_A : A \otimes I \xrightarrow{\sim} A$
4. $\gamma_{A,B} : A \otimes B \xrightarrow{\sim} B \otimes A$

which make the following 'coherence' diagrams commute.

$$\begin{tikzcd}
A \otimes (B \otimes (C \otimes D)) \arrow[r, "\alpha_{A,B,C\otimes D}"] \arrow[d, "id_A \otimes \alpha_{B,C,D}"'] & (A \otimes B) \otimes (C \otimes D) \arrow[r, "\alpha_{A\otimes B,C,C}"] & ((A \otimes B) \otimes C) \otimes D \\
A \otimes ((B \otimes C) \otimes D) \arrow[rr, "\alpha_{A,B\otimes C,D}"'] & & (A \otimes (B \otimes C)) \otimes D \arrow[u, "\alpha_{A,B,C} \otimes id_D"']
\end{tikzcd}$$

$$\begin{tikzcd}
(A \otimes B) \otimes C \arrow[r, "\alpha_{A,B,C}"] \arrow[d, "\gamma_{A,B} \otimes id_C"'] & A \otimes (B \otimes C) \arrow[r, "\gamma_{A,B\otimes C}"] & (B \otimes C) \otimes A \arrow[d, "\alpha_{B,C,A}"'] \\
(B \otimes A) \otimes C \arrow[r, "\alpha_{B,A,C}"'] & B \otimes (A \otimes C) \arrow[r, "id_B \otimes \gamma_{A,C}"'] & B \otimes (C \otimes A)
\end{tikzcd}$$

$$\begin{tikzcd}
A \otimes (I \otimes B) \arrow[r, "\alpha_{A,I,B}"] \arrow[d, "id_A \otimes \lambda_B"'] & (A \otimes I) \otimes B \arrow[d, "\rho_A \otimes id_B"] \\
A \otimes B \arrow[r, equal] & A \otimes B
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes B \arrow[r, "\gamma_{A,B}"] \arrow[dr, equal] & B \otimes A \arrow[d, "\gamma_{B,A}"] \\
& A \otimes B
\end{tikzcd}$$

$$\begin{tikzcd}
A \otimes I \arrow[r, "\gamma_{A,I}"] \arrow[d, "\rho_A"'] & I \otimes A \arrow[d, "\lambda_A"] \\
A \arrow[r, equal] & A
\end{tikzcd}$$

The following equality is also require to hold:

$$\lambda_I \quad = \quad \rho_I : I \otimes I \to I$$

[1]

Definition 5. *A symmetric monoidal closed category (SMCC), $(\mathbb{C}, \otimes, \multimap, I, \alpha, \lambda, \rho, \gamma)$, is a SMC such that for all objects A in $\mathbb{C}$, the functor $- \otimes A$ has a specified right adjoint $A \multimap -$.*

Definition 6. *A symmetric monoidal functor between SMCs* $(\mathbb{C}, \otimes, I, \alpha, \lambda, \rho, \gamma)$ *and* $(\mathbb{C}', \otimes', I', \alpha', \lambda', \rho', \gamma')$ *is a functor* $F : \mathbb{C} \to \mathbb{C}'$ *equipped with*

1. *A morphism* $m_{I'} : I' \to FI$.

2. *For any two objects* A *and* B *in* $\mathbb{C}$*, a natural transformation* $m_{A,B} : F(A) \otimes' F(B) \to F(A \otimes B)$

These must satisfy the following diagrams:

$$\begin{tikzcd}
FI \otimes' FA \arrow[r, "m_{I,A}"] & F(I \otimes A) \arrow[d, "F(\lambda_A)"] \\
I' \otimes' FA \arrow[u, "m_I \otimes' id_{FA}"] \arrow[r, "\lambda'_{FA}"'] & FA
\end{tikzcd}
\qquad
\begin{tikzcd}
FA \otimes' FI \arrow[r, "m_{A,I}"] & F(A \otimes I) \arrow[d, "F(\rho_A)"] \\
FA \otimes' I' \arrow[u, "id_{FA} \otimes' m_{I'}"] \arrow[r, "\rho'_{FA}"'] & FA
\end{tikzcd}$$

$$\begin{tikzcd}
(FA \otimes' FB) \otimes' FC \arrow[r, "m_{A,B} \otimes' id_{FC}"] & F(A \otimes B) \otimes' FC \arrow[r, "m_{A \otimes B, C}"] & F((A \otimes B) \otimes C) \\
FA \otimes' (FB \otimes' FC) \arrow[u, "\alpha'_{FA,FB,FC}"] \arrow[r, "id_{FA} \otimes' m_{B,C}"'] & FA \otimes' F(B \otimes C) \arrow[r, "m_{A, B \otimes C}"'] & F(A \otimes (B \otimes C)) \arrow[u, "F(\alpha_{A,B,C})"']
\end{tikzcd}$$

$$\begin{tikzcd}
FA \otimes' FB \arrow[r, "m_{A,B}"] \arrow[d, "\gamma'_{A,B}"'] & F(A \otimes B) \arrow[d, "F(\gamma_{A,B})"] \\
FB \otimes' FA \arrow[r, "m_{B,A}"'] & F(B \otimes A)
\end{tikzcd}$$

However in this particular case, assuming that ! is a symmetric monoidal (endo) functor means that ! comes equipped with a natural transformation

$$m_{A,B} : !A \otimes !B \to !(A \otimes B)$$

and a morphism

$$m_I : I \to !I$$

(where m_I *is just the nullary version of the natural transformation.) The diagrmas given in the above definition become the following:*

$$\begin{tikzcd}
!I\otimes !A \arrow[r, "m_{I,A}"] & !(I\otimes A) \arrow[d, "!(\lambda_A)"] \\
I\otimes !A \arrow[u, "m_I\otimes id_{!A}"] \arrow[r, "\lambda_{!A}"'] & !A
\end{tikzcd}
\qquad
\begin{tikzcd}
!A\otimes !I \arrow[r, "m_{A,I}"] & !(A\otimes I) \arrow[d, "!(\rho_A)"] \\
!A\otimes I \arrow[u, "id_{!A}\otimes m_I"] \arrow[r, "\rho_{!A}"'] & !A
\end{tikzcd}$$

$$\begin{tikzcd}
(!A\otimes !B)\otimes !C \arrow[r, "m_{A,B}\otimes id_{!C}"] & !(A\otimes B)\otimes !C \arrow[r, "m_{A\otimes B,C}"] & !((A\otimes B)\otimes C) \\
!A\otimes(!B\otimes !C) \arrow[u, "\alpha_{!A,!B,!C}"] \arrow[r, "id_{!A}\otimes m_{B,C}"'] & !A\otimes !(B\otimes C) \arrow[r, "m_{A,B\otimes C}"'] & !(A\otimes(B\otimes C)) \arrow[u, "!(\alpha_{A,B,C})"']
\end{tikzcd}$$

$$\begin{tikzcd}
!A\otimes !B \arrow[r, "m_{A,B}"] \arrow[d, "\gamma'_{A,B}"'] & !(A\otimes B) \arrow[d, "!(\gamma_{A,B})"] \\
!B\otimes !A \arrow[r, "m_{B,A}"'] & !(B\otimes A)
\end{tikzcd}$$

[1]

Definition 7. *A symmetric monoidal functor,* $(F, m_{A,B}, m_{I'}) : \mathbb{C} \to \mathbb{C}'$, *is said to be*

1. *Strict if* $m_{A,B}$ *and* $m_{I'}$ *are identities.*

2. *Strong if* $m_{A,B}$ *and* $m_{I'}$ *are natural isomorphisms.*

[1]

References

[1] G.M. Bierman. *On Intuitionistic Linear Logic*. PhD thesis, University of Cambridge, 1993.

[2] Saunders Mac Lane. *Categories for the Working Mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1971.

[3] Paul-André Mellies. Categorical semantics of linear logic. *Panoramas et syntheses*, 27:15–215, 2009.