Monoidal-Annex

Preston Keel   Dr. Harley Eades

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1   Monoidal Categories

Definition 1. *A monoidal category* $(\mathbb{C}, \otimes, I, \alpha, \lambda, \rho)$ *is a category* $\mathbb{C}$, *a bifunctor* $\otimes : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$, *an object* $I \in \mathbb{C}$, *and three natural isomorphisms* α, λ, ρ. *Where,*

$$\alpha = \alpha_{A,B,C} : A \otimes (B \otimes C) \cong (A \otimes B) \otimes C$$

is natural for all $A, B, C \in \mathbb{C}$ *and the diagram*

$$
\begin{array}{ccccc}
A \otimes (B \otimes (C \otimes D)) & \xrightarrow{\ \alpha\ } & (A \otimes B) \otimes (C \otimes D) & \xrightarrow{\ \alpha\ } & ((A \otimes B) \otimes C) \otimes D \\
{\scriptstyle 1 \otimes \alpha}\downarrow & & & & \uparrow{\scriptstyle \alpha \otimes 1} \\
A \otimes ((B \otimes C) \otimes D) & & \xrightarrow{\qquad\qquad \alpha \qquad\qquad} & & (A \otimes (B \otimes C)) \otimes D
\end{array}
$$

commutes for all $A, B, C, D \in \mathbb{C}$. γ *and* ρ *are natural*

$$\gamma_A : I \otimes A \cong A \quad \rho_A : A \otimes I \cong A$$

for all objects $A \in \mathbb{C}$, *the diagram*

$$
\begin{array}{ccc}
A \otimes (I \otimes C) & \xrightarrow{\ \alpha\ } & (A \otimes I) \otimes C \\
{\scriptstyle 1 \otimes \lambda}\downarrow & & \downarrow{\scriptstyle \rho_A \otimes 1} \\
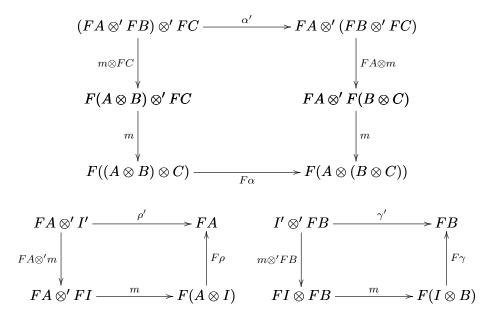A \otimes C & =\!=\!=\!=\!= & A \otimes C
\end{array}
$$

Definition 2. *A monoidal functor or lax monoidal functor (F, m) between monoidal categories $(\mathbb{C}, \otimes, I)$ and $(\mathbb{D}, \otimes', I')$ is a functor $F : \mathbb{C} \to \mathbb{D}$ equipped with a nathral transformation*

$$m_{A,B} : FA \otimes' FB \to F(A \otimes B)$$

and an isomorphism

$$m_I : I' \to FI$$

where the following diagrams commute in the category $\mathbb{D}$ for all objects $A, B, C \in \mathbb{C}$

Definition 3. *An oplax(colax/comonoidal) monoidal functor (F, m) between monoidal categories $(\mathbb{C}, \otimes, I)$ and $(\mathbb{D}, \otimes', I')$ is a functor $F : \mathbb{C} \to \mathbb{D}$ with a natural transformation*

$$m_{A,B} : F(A \otimes B) \to FA \otimes' FB$$

and an isomorphism

$$m_I : FI \to I'$$

where the following diagrams commute in the category $\mathbb{D}$, for all objects $A, B, C \in \mathbb{C}$
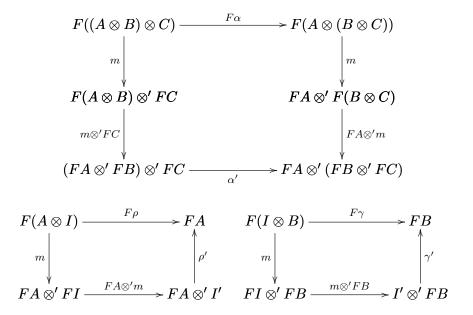
$$
\begin{array}{ccc}
F((A \otimes B) \otimes C) & \xrightarrow{F\alpha} & F(A \otimes (B \otimes C)) \\
\downarrow m & & \downarrow m \\
F(A \otimes B) \otimes' FC & & FA \otimes' F(B \otimes C) \\
\downarrow m \otimes' FC & & \downarrow FA \otimes' m \\
(FA \otimes' FB) \otimes' FC & \xrightarrow{\alpha'} & FA \otimes' (FB \otimes' FC)
\end{array}
$$

$$
\begin{array}{ccc}
F(A \otimes I) & \xrightarrow{F\rho} & FA \\
\downarrow m & & \uparrow \rho' \\
FA \otimes' FI & \xrightarrow{FA \otimes' m} & FA \otimes' I'
\end{array}
\qquad
\begin{array}{ccc}
F(I \otimes B) & \xrightarrow{F\gamma} & FB \\
\downarrow m & & \uparrow \gamma' \\
FI \otimes' FB & \xrightarrow{m \otimes' FB} & I' \otimes' FB
\end{array}
$$

[5]

Definition 4. *Suppose (F, m) and (G, n) are (lax) monoidal functors between the monoidal categories:*

$$(\mathbb{C}, \otimes, I) \to (\mathbb{D}, \otimes', I')$$

A monoidal natural transformation

$$\theta : (F, m) \Rightarrow (G, n) : (\mathbb{C}, \otimes, I) \to (\mathbb{D}, \otimes', I')$$

between the monoidal functors (F, m) and (G, n) is a naural transformation

$$\theta : F \Rightarrow G : \mathbb{C} \to \mathbb{D}$$

between the underlying functors, where the following diamgrams commute, for all objects $A, B \in \mathbb{C}$

$$
\begin{array}{ccc}
FA \otimes' FB & \xrightarrow{\theta_A \otimes' \theta_B} & GA \otimes' GB \\
\downarrow m & & \downarrow n \\
F(A \otimes B) & \xrightarrow{\theta_{A \otimes B}} & G(A \otimes B)
\end{array}
\qquad
\begin{array}{ccc}
& I' & \\
m \swarrow & & \searrow n \\
FI & \xrightarrow{\theta_I} & GI
\end{array}
$$

[5]

Definition 5. *A monoidal natural transformation*

$$\theta : (F, m) \Rightarrow (G, n) : (\mathbb{C}, \otimes, I) \to (\mathbb{D}, \otimes', I')$$

between two oplax(colax/comonoidal) monoidal functors (F, m) and (G, n) is a natural transformation

$$\theta : F \Rightarrow G : \mathbb{C} \to \mathbb{D}$$

between the underlying functors, where the following diagrmas commute, for all objects $A, B \in \mathbb{C}$

$$
\begin{array}{ccc}
F(A \otimes B) & \xrightarrow{\theta_{A \otimes B}} & G(A \otimes B) \\
\downarrow{\scriptstyle m} & & \downarrow{\scriptstyle n} \\
FA \otimes' FB & \xrightarrow{\theta_A \otimes' \theta_B} & GA \otimes' GB
\end{array}
\qquad
\begin{array}{ccc}
FI & \xrightarrow{\theta_I} & GI \\
& & \\
{\scriptstyle m} \searrow & & \swarrow {\scriptstyle n} \\
& I' &
\end{array}
$$

[5]

Definition 6. *A monoidal category $\mathbb{C}$ is said to be biclosed if every $- \otimes Y$ has a right adjoint $[Y, -]$ and every $X \otimes -$ has a right adjoint $[\![X, -]\!]$ [3]*

Definition 7. *Given a pair of (lax) monoidal functors:*

$$(F_*, m) : (\mathbb{C}, \otimes, I) \to (\mathbb{D}, \otimes', I') \quad (F^*, n) : (\mathbb{D}, \otimes', I') \to (\mathbb{C}, \otimes, I).$$

A monoidal adjunction

$$(F_*, m) \dashv (F^*, n)$$

between the monoidal functors is defined as an adjunction $(F_*, F^*, \eta, \epsilon)$ between the underlying functors

$$F_* : \mathbb{C} \to \mathbb{D} \quad F^* : \mathbb{D} \to \mathbb{C}$$

whose natural transformations

$$\eta : id_C \Rightarrow F^* \circ F_* \quad \epsilon : F_* \circ F^* \Rightarrow id_D$$

are monoidal [5]

2 Symmetric Monoidal Categories

Definition 8. *A symmetric monoidal catergory (SMC), is a monoidal category $\mathbb{C}$ with an additional natural isomorphism γ*
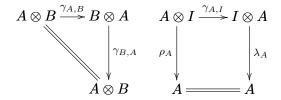
$$\gamma_{A,B} : A \otimes B \xrightarrow{\sim} B \otimes A$$

and three additional 'coherence' diagrams that commute.

$$
\begin{array}{ccccc}
(A \otimes B) \otimes C & \xrightarrow{\alpha_{A,B,C}} & A \otimes (B \otimes C) & \xrightarrow{\gamma_{A,B \otimes C}} & (B \otimes C) \otimes A \\
\downarrow{\scriptstyle \gamma_{A,B} \otimes id_C} & & & & \downarrow{\scriptstyle \alpha_{B,C,A}} \\
(B \otimes A) \otimes C & \xrightarrow{\alpha_{B,A,C}} & B \otimes (A \otimes C) & \xrightarrow{id_B \otimes \gamma_{A,C}} & B \otimes (C \otimes A)
\end{array}
$$

$$
\begin{array}{ccc}
A \otimes B & \xrightarrow{\gamma_{A,B}} & B \otimes A \\
\Vert & & \downarrow{\scriptstyle \gamma_{B,A}} \\
& A \otimes B &
\end{array}
\qquad
\begin{array}{ccc}
A \otimes I & \xrightarrow{\gamma_{A,I}} & I \otimes A \\
\downarrow{\scriptstyle \rho_A} & & \downarrow{\scriptstyle \lambda_A} \\
A & = & A
\end{array}
$$

The following equality is also require to hold:

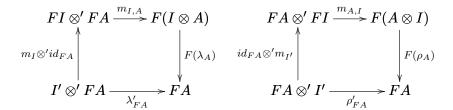$$\lambda_I \quad = \quad \rho_I : I \otimes I \to I$$

[1]

Definition 9. *A symmetric monoidal closed category (SMCC), $(\mathbb{C}, \otimes, \multimap, I, \alpha, \lambda, \rho, \gamma)$, is a SMC such that for all objects A in $\mathbb{C}$, the functor $- \otimes A$ has a specified right adjoint $A \multimap -$.*

Definition 10. *A symmetric monoidal functor between SMCs $(\mathbb{C}, \otimes, I, \alpha, \lambda, \rho, \gamma)$ and $(\mathbb{C}', \otimes', I', \alpha', \lambda', \rho', \gamma')$ is a functor $F : \mathbb{C} \to \mathbb{C}'$ equipped with*

1. *A morphism $m_{I'} : I' \to FI$.*

2. *For any two objects A and B in $\mathbb{C}$, a natural transformation $m_{A,B} : F(A) \otimes' F(B) \to F(A \otimes B)$*

These must satisfy the following diagrams:

$$
\begin{array}{ccc}
FI \otimes' FA & \xrightarrow{m_{I,A}} & F(I \otimes A) \\
\uparrow{\scriptstyle m_I \otimes' id_{FA}} & & \downarrow{\scriptstyle F(\lambda_A)} \\
I' \otimes' FA & \xrightarrow[\lambda'_{FA}]{} & FA
\end{array}
\qquad
\begin{array}{ccc}
FA \otimes' FI & \xrightarrow{m_{A,I}} & F(A \otimes I) \\
\uparrow{\scriptstyle id_{FA} \otimes' m_{I'}} & & \downarrow{\scriptstyle F(\rho_A)} \\
FA \otimes' I' & \xrightarrow[\rho'_{FA}]{} & FA
\end{array}
$$

$$
\begin{array}{ccccc}
(FA \otimes' FB) \otimes' FC & \xrightarrow{m_{A,B} \otimes' id_{FC}} & F(A \otimes B) \otimes' FC & \xrightarrow{m_{A \otimes B,C}} & F((A \otimes B) \otimes C) \\
\uparrow{\scriptstyle \alpha'_{FA,FB,FC}} & & & & \uparrow{\scriptstyle F(\alpha_{A,B,C})} \\
FA \otimes' (FB \otimes' FC) & \xrightarrow[id_{FA} \otimes' m_{B,C}]{} & FA \otimes' F(B \otimes C) & \xrightarrow[m_{A,B \otimes C}]{} & F(A \otimes (B \otimes C))
\end{array}
$$

$$
\begin{array}{ccc}
FA \otimes' FB & \xrightarrow{m_{A,B}} & F(A \otimes B) \\
\downarrow{\scriptstyle \gamma'_{A,B}} & & \downarrow{\scriptstyle F(\gamma_{A,B})} \\
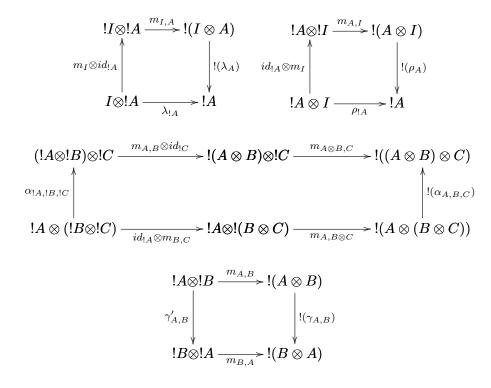FB \otimes' FA & \xrightarrow[m_{B,A}]{} & F(B \otimes A)
\end{array}
$$

However in this particular case, assuming that $!$ is a symmetric monoidal (endo) functor means that $!$ comes equipped with a natural transformation

$$m_{A,B} : !A \otimes !B \to !(A \otimes B)$$

and a morphism

$$m_I : I \to !I$$

(where m_I is just the nullary version of the natural transformation.) The diagrmas given in the above definition become the following:

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$$\begin{array}{ccc}
!I\otimes !A & \xrightarrow{m_{I,A}} & !(I \otimes A) \\
\uparrow{\scriptstyle m_I \otimes id_{!A}} & & \downarrow{\scriptstyle !(\lambda_A)} \\
I\otimes !A & \xrightarrow{\lambda_{!A}} & !A
\end{array}
\qquad
\begin{array}{ccc}
!A\otimes !I & \xrightarrow{m_{A,I}} & !(A \otimes I) \\
\uparrow{\scriptstyle id_{!A}\otimes m_I} & & \downarrow{\scriptstyle !(\rho_A)} \\
!A \otimes I & \xrightarrow{\rho_{!A}} & !A
\end{array}$$

$$\begin{array}{ccccc}
(!A\otimes !B)\otimes !C & \xrightarrow{m_{A,B}\otimes id_{!C}} & !(A \otimes B)\otimes !C & \xrightarrow{m_{A\otimes B,C}} & !((A \otimes B) \otimes C) \\
\uparrow{\scriptstyle \alpha_{!A,!B,!C}} & & & & \uparrow{\scriptstyle !(\alpha_{A,B,C})} \\
!A \otimes (!B\otimes !C) & \xrightarrow{id_{!A}\otimes m_{B,C}} & !A\otimes !(B \otimes C) & \xrightarrow{m_{A,B\otimes C}} & !(A \otimes (B \otimes C))
\end{array}$$

$$\begin{array}{ccc}
!A\otimes !B & \xrightarrow{m_{A,B}} & !(A \otimes B) \\
\downarrow{\scriptstyle \gamma'_{A,B}} & & \downarrow{\scriptstyle !(\gamma_{A,B})} \\
!B\otimes !A & \xrightarrow{m_{B,A}} & !(B \otimes A)
\end{array}$$

[1]

Definition 11. *A symmetric monoidal functor, $(F, m_{A,B}, m_{I'}) : \mathbb{C} \to \mathbb{C}'$, is said to be*

1. *Strict if $m_{A,B}$ and $m_{I'}$ are identities.*

2. *Strong if $m_{A,B}$ and $m_{I'}$ are natural isomorphisms.*

[1]

Definition 12. *An oplax monoidal functor*

$$(F, m) : (\mathbb{C}, \otimes, I) \to (\mathbb{D}, \otimes', I')$$

is symmetric when the following diagram commutes in the category $\mathbb{D}$ for all objects $A, B \in \mathbb{C}$

$$\begin{array}{ccc}
F(A \otimes B) & \xrightarrow{F_\gamma} & F(B \otimes A) \\
\downarrow{\scriptstyle m} & & \downarrow{\scriptstyle m} \\
FA \otimes' FB & \xrightarrow{\gamma'} & FB \otimes' FA
\end{array}$$

[5]

3 Monoidal Double Category

Definition 13. *The strict double category $\mathbb{D}$ has a full double subcategory $\mathbb{M}$ of monoidal categoreis. These are viewd as:*

- *vertical double categories on a formal object $*$*

- *vertical arrows $A : * \to *$*

- *cells $a : A \to A'$*

The horizontal arrows of $\mathbb{M}$ are monoidal functors(lax with respect to tensor product) and the vertical arrows are comonoidal functors(colax). A cell $\alpha : (FRSG)$ associates to every object A in $\mathbb{A}$ with an arrow $\alpha A : GRA \to SFA$ in $\mathbb{D}$ which satisfies the naturality condition
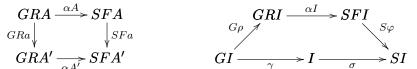
$$(GRa|\alpha v) = (\alpha u|SFa) \quad \text{(for } a : (ufgv) \text{ in } \mathbb{A})$$

and the coherence conditions

$$(C\rho A|\alpha I|S\varphi A) = (\gamma RA|I|\sigma FA) \qquad\qquad \text{(for } a \text{ in } \mathbb{A})$$

$$(G\rho(u,v)|\alpha w|S\varphi(u,v)) = (\gamma(Ru,Rv)|(\alpha u \otimes \alpha v)|\sigma(Fu,Fv)) \quad \text{(for } w = u \otimes v \text{ in } \mathbb{A})$$

and have the following diagrams

$$
\begin{array}{ccc}
GRA & \xrightarrow{\alpha A} & SFA \\
{\scriptstyle GRa}\downarrow & & \downarrow{\scriptstyle SFa} \\
GRA' & \xrightarrow{\alpha A'} & SFA'
\end{array}
\qquad\qquad
\begin{array}{ccc}
 & GRI \xrightarrow{\alpha I} SFI & \\
{\scriptstyle G\rho}\nearrow & & \searrow{\scriptstyle S\varphi} \\
GI \xrightarrow{\quad\gamma\quad} & I & \xrightarrow{\quad\sigma\quad} SI
\end{array}
$$

$$
\begin{array}{ccc}
GR(A \otimes A') & \xrightarrow{\alpha(A \otimes A')} & SF(A \otimes A') \\
{\scriptstyle G\rho}\uparrow & & \downarrow{\scriptstyle S\varphi} \\
G(RA \otimes RA') & & S(FA \otimes FA') \\
{\scriptstyle \gamma R}\downarrow & & \uparrow{\scriptstyle \sigma F} \\
GRA \otimes GRA' & \xrightarrow{\alpha A \otimes \alpha A'} & SFA \otimes SFA'
\end{array}
$$

where the lax monoidal functor R has comparison arrows

$$\rho = \rho(*) : I \to RI$$
$$\rho(A, A') : RA \otimes RA' \to R(A \otimes A')$$

[2]

4 The Double Category of (co)Lax Monoidal Functors

Definition 14. *A* lax monoidal multifunctor *from $(\mathcal{C}_1, \ldots, \mathcal{C}_n)$ to $\mathcal{D}$ is a functor:*

$$F : \mathcal{C}_1 \times \cdots \times \mathcal{C}_n \longrightarrow \mathcal{D}$$

such that:

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- *Each functor,* $(F^i, m^i(\overrightarrow{A_1}, -, \overrightarrow{A_{i+1}}), m^i_{I_i}(\overrightarrow{A_1}, -, \overrightarrow{A_{i+1}})$ *defined by* $F^i(X) = F(\overrightarrow{A_1}, X, \overrightarrow{A_{i+1}})$: $\mathcal{C}_i \longrightarrow \mathcal{D}$, *for* $1 \leq i \leq n$ *is lax monoidal.*

- *The following equations hold for any* $1 \leq i < j \leq n$:

  - $m_{I_i}(\overrightarrow{I_1}, -, \overrightarrow{I_{i+1}}) = m_{I_j}(\overrightarrow{I_1}, -, \overrightarrow{I_{i+1}})$,

  - $(m^i_{I_i}(\overrightarrow{A_1}, -, \overrightarrow{A_{i+1}}, X, \overrightarrow{A_{j+1}}) \otimes m^i_{I_i}(\overrightarrow{A_1}, -, \overrightarrow{A_{i+1}}, Y, \overrightarrow{A_{j+1}}))$; $m^j_{X,Y}(\overrightarrow{A_1}, I_i, \overrightarrow{A_{i+1}}, -, \overrightarrow{A_{j+1}})$
    $= \iota; m^i_{I_i}(\overrightarrow{A_1}, -, \overrightarrow{A_{i+1}}, X \otimes Y, \overrightarrow{A_{j+1}})$

  - $(m^j_{I_j}(\overrightarrow{A_1}, X, \overrightarrow{A_{i+1}}, -, \overrightarrow{A_{j+1}}) \otimes m^j_{I_j}(\overrightarrow{A_1}, Y, \overrightarrow{A_{i+1}}, -, \overrightarrow{A_{j+1}}))$; $m^i_{X,Y}(\overrightarrow{A_1}, -, \overrightarrow{A_{i+1}}, I_j, \overrightarrow{A_{j+1}})$
    $= \iota; m^j_{I_j}(\overrightarrow{A_1}, X \otimes Y, \overrightarrow{A_{i+1}}, -, \overrightarrow{A_{j+1}})$

In the above definition ι is the isomorphism $\iota : I \otimes I \longrightarrow I$ which holds in any monoidal category.

Definition 15. *A* symmetric lax monoidal multifunctor *from* $(\mathcal{C}_1, \ldots, \mathcal{C}_n)$ *to* $\mathcal{D}$ *is a lax monoidal multifunctor*

$$F : (\mathcal{C}_1, \ldots, \mathcal{C}_n) \longrightarrow \mathcal{D}$$

such that:

- *Each functor,* $(F^i, m^i(\overrightarrow{A_1}, -, \overrightarrow{A_{i+1}}), m^i_{I_i}(\overrightarrow{A_1}, -, \overrightarrow{A_{i+1}})$ *defined by* $F^i(X) = F(\overrightarrow{A_1}, X, \overrightarrow{A_{i+1}})$: $\mathcal{C}_i \longrightarrow \mathcal{D}$, *for* $1 \leq i \leq n$ *is symmetric lax monoidal.*

- *The following additional coherence axiom holds for any* $1 \leq i < j \leq n$:

  - $(m^i_{X,Y}(\overrightarrow{A_1}, -, \overrightarrow{A_{i+1}}, P, \overrightarrow{A_{j+1}}) \otimes m^i_{X,Y}(\overrightarrow{A_1}, -, \overrightarrow{A_{i+1}}, Q, \overrightarrow{A_{j+1}}))$; $m^j_{P,Q}(\overrightarrow{A_1}, X \otimes Y, \overrightarrow{A_{i+1}}, -, \overrightarrow{A_{j+1}})$
    $= \tau; (m^j_{P,Q}(\overrightarrow{A_1}, X, \overrightarrow{A_{i+1}}, -, \overrightarrow{A_{j+1}}) \otimes m^j_{P,Q}(\overrightarrow{A_1}, Y, \overrightarrow{A_{i+1}}, -, \overrightarrow{A_{j+1}}))$; $m^i_{X,Y}(\overrightarrow{A_1}, -, \overrightarrow{A_{i+1}}, P \otimes Q, \overrightarrow{A_{j+1}})$

In the above definition τ is the isomorphism $\tau : (A \otimes B) \otimes (C \otimes D) \longrightarrow (A \otimes C) \otimes (B \otimes D)$ which holds in any monoidal category.

5   Monoidal Monads

Definition 16. *Let* $\mathbb{C} = (C, \otimes, I)$ *be a monoidal category. A* monoidal monad $\mathbb{T} = (T, \mu, \eta)$ *is a monad in the 2-category of:*

1. *monoidal categories*

2. *monoidal functors*

3. *monoidal transformations*

where the functor $T : C \rightarrow C$ *comes with maps* $\kappa = (\kappa_{X,Y} : TX \otimes TY \rightarrow T(X \otimes Y))_{X,Y \in ob\, C}$ *that make* (T, κ, η_I) *a monoidal functor and* μ, η *monoidal natural transformations [6]*

References

[1] G.M. Bierman. *On Intuitionistic Linear Logic.* PhD thesis, University of Cambridge, 1993.

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[5] Paul-André Mellies. Categorical semantics of linear logic. *Panoramas et syntheses*, 27:15–215, 2009.

[6] Gavin J Seal. Tensors, monads and actions. *Theory and applications of categories*, 28(15):403–433, 2013.