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1 Monoidal Category

1.1 Symmetric Monoidal Category

Definition 1. A symmetric monoidal category (SMC), $(\mathbb{C}, \bullet, 1, \alpha, \lambda, \rho, \gamma)$, is a category \mathbb{C} equipped with a bifunctor $\bullet : \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$ with a neutral element 1 and natural isomorphisms α, λ, ρ , and γ :

1. $\alpha_{A,B,C} : A \bullet (B \bullet C) \xrightarrow{\sim} (A \bullet B) \bullet C$
2. $\lambda_A : 1 \bullet A \xrightarrow{\sim} A$
3. $\rho_A : A \bullet 1 \xrightarrow{\sim} A$
4. $\gamma_{A,B} : A \bullet B \xrightarrow{\sim} B \bullet A$

which make the following 'coherence' diagrams commute.

$$\begin{array}{c}
 \begin{array}{ccccc}
 A \bullet (B \bullet (C \bullet D)) & \xrightarrow{\alpha_{A,B,C \bullet D}} & (A \bullet B) \bullet (C \bullet D) & \xrightarrow{\alpha_{A \bullet B,C,C}} & ((A \bullet B) \bullet C) \bullet D \\
 \downarrow id_A \bullet \alpha_{B,C,D} & & & & \uparrow \alpha_{A,B,C} \bullet id_D \\
 A \bullet ((B \bullet C) \bullet D) & \xrightarrow{\alpha_{A,B \bullet C,D}} & & & (A \bullet (B \bullet C)) \bullet D
 \end{array} \\
 \\
 \begin{array}{ccccc}
 (A \bullet B) \bullet C & \xrightarrow{\alpha_{A,B,C}} & A \bullet (B \bullet C) & \xrightarrow{\gamma_{A,B \bullet C}} & (B \bullet C) \bullet A \\
 \downarrow \gamma_{A,B} \bullet id_C & & & & \downarrow \alpha_{B,C,A} \\
 (B \bullet A) \bullet C & \xrightarrow{\alpha_{B,A,C}} & B \bullet (A \bullet C) & \xrightarrow{id_B \bullet \gamma_{A,C}} & B \bullet (C \bullet A)
 \end{array} \\
 \\
 \begin{array}{ccccc}
 A \bullet (1 \bullet B) & \xrightarrow{\alpha_{A,1,B}} & (A \bullet 1) \bullet B & & A \bullet B \xrightarrow{\gamma_{A,B}} B \bullet A \\
 \downarrow id_A \bullet \lambda_B & & \downarrow \rho_A \bullet id_B & & \downarrow \gamma_{B,A} \\
 A \bullet B & \xlongequal{\quad} & A \bullet B & & A \bullet B
 \end{array} \\
 \\
 \begin{array}{ccc}
 A \bullet 1 & \xrightarrow{\gamma_{A,1}} & 1 \bullet A \\
 \downarrow \rho_A & & \downarrow \lambda_A \\
 A & \xlongequal{\quad} & A
 \end{array}
 \end{array}$$

The following equality is also require to hold:

$$\lambda_1 = \rho_1 : 1 \bullet 1 \rightarrow 1$$

1.2 Symmetric Monoidal Closed Category

Definition 2. A symmetric monoidal closed category (SMCC), $(\mathbb{C}, \bullet, \multimap, 1, \alpha, \lambda, \rho, \gamma)$, is a SMC such that for all objects A in \mathbb{C} , the functor $- \otimes A$ has a specified right adjoint $A \multimap -$.

Let \mathbb{C} be a SMC $(\mathbb{C}, \bullet, 1, \alpha, \lambda, \rho, \gamma)$. A structure $\langle M, \bullet \rangle$ in \mathbb{C} for a given signature S_g is specified by giving an object $\llbracket \sigma \rrbracket$ in \mathbb{C} for each type σ , and a morphism $\llbracket f \rrbracket : \llbracket \sigma_1 \rrbracket \bullet \dots \bullet \llbracket \sigma_n \rrbracket \rightarrow \llbracket \tau \rrbracket$ in \mathbb{C} for each function symbol $f : \sigma_1, \dots, \sigma_n \rightarrow \tau$. In the case where $n = 0$ then the structure assigns a morphism $\llbracket c \rrbracket : 1 \rightarrow \llbracket \tau \rrbracket$ to a constant $c : \tau$.

Given a context $\Gamma = [x_1 : \sigma_1, \dots, x_n : \sigma_n]$ we define $\llbracket \Gamma \rrbracket$ to be the product $\llbracket \sigma_1 \rrbracket \bullet \dots \bullet \llbracket \sigma_n \rrbracket$. We represent the empty context with the neutral element 1. We need to define the bracketing convention. It shall be assumed that the tensor product is left associative, i.e. $A_1 \bullet A_2 \bullet \dots \bullet A_n$ will be taken to mean $(\dots(A_1 \bullet A_2) \bullet \dots) \bullet A_n$. We find it useful to define two 'book-keeping' functions,

$$Split(\Gamma, \Delta) : \llbracket \Gamma, \Delta \rrbracket \rightarrow \llbracket \Gamma \rrbracket \bullet \llbracket \Delta \rrbracket$$

$$Split(\Gamma, \Delta) \stackrel{\text{def}}{=} \begin{cases} \lambda_{\Delta}^{-1} & \text{If } \Gamma \text{ empty} \\ \rho_{\Gamma}^{-1} & \text{If } \Delta \text{ empty} \\ id_{\Gamma \bullet A} & \text{If } \Delta = A \\ Split(\Gamma, \Delta') \bullet id_A; \alpha_{\Gamma, \Delta', A}^{-1} & \text{If } \Delta = \Delta', A \end{cases}$$

$$Join(\Gamma, \Delta) : \llbracket \Gamma, \Delta \rrbracket \rightarrow \llbracket \Gamma \rrbracket \bullet \llbracket \Delta \rrbracket$$

$$Join(\Gamma, \Delta) \stackrel{\text{def}}{=} \begin{cases} \lambda_{\Delta} & \text{If } \Gamma \text{ empty} \\ \rho_{\Gamma} & \text{If } \Delta \text{ empty} \\ id_{\Gamma \bullet A} & \text{If } \Delta = A \\ \alpha_{\Gamma, \Delta', A}; Join(\Gamma, \Delta') \bullet id_A & \text{If } \Delta = \Delta', A \end{cases}$$

We shall also refer to indexed variants of these; for example

$$Split_n(\Gamma_1, \dots, \Gamma_n) : \llbracket \Gamma_1, \dots, \Gamma_n \rrbracket \rightarrow \llbracket \Gamma_1 \rrbracket \bullet \dots \bullet \llbracket \Gamma_n \rrbracket$$

which is defined in the obvious way.

The semantics of a term in context is then specified by a structural induction on the term.

$$\llbracket x : \sigma \triangleright x : \sigma \rrbracket \stackrel{\text{def}}{=} id_{\sigma}$$

$$\llbracket \Gamma_1, \dots, \Gamma_n \triangleright f(M_1, \dots, M_n) : \tau \rrbracket \stackrel{\text{def}}{=} Split_n(\Gamma_1, \dots, \Gamma_n); \llbracket \Gamma_1 \triangleright M_1 : \sigma_1 \rrbracket \bullet \dots \bullet \llbracket \Gamma_n \triangleright M_n : \sigma_n \rrbracket; \llbracket f \rrbracket$$

2 Monodial Functor

2.1 Symmetric Monoidal Functor

Definition 3. A symmetric monoidal functor between SMCs $(\mathbb{C}, \bullet, 1, \alpha, \lambda, \rho, \gamma)$ and $(\mathbb{C}', \bullet', 1', \alpha', \lambda', \rho', \gamma')$ is a functor $F : \mathbb{C} \rightarrow \mathbb{C}'$ equipped with

1. A morphism $m_{1'} : 1' \rightarrow F1$.
2. For any two objects A and B in \mathbb{C} , a natural transformation $m_{A,B} : F(A) \bullet' F(B) \rightarrow F(A \bullet B)$

These must satisfy the following diagrams:

$$\begin{array}{ccc}
 F1 \bullet' FA & \xrightarrow{m_{1,A}} & F(1 \bullet A) \\
 \uparrow m_{1'} \bullet' id_{FA} & & \downarrow F(\lambda_A) \\
 1' \bullet' FA & \xrightarrow{\lambda'_{FA}} & FA \\
 \end{array}
 \qquad
 \begin{array}{ccc}
 FA \bullet' F1 & \xrightarrow{m_{A,1}} & F(A \bullet 1) \\
 \uparrow id_{FA} \bullet' m_{1'} & & \downarrow F(\rho_A) \\
 FA \bullet' 1' & \xrightarrow{\rho'_{FA}} & FA \\
 \end{array}$$

$$\begin{array}{ccccc}
 (FA \bullet' FB) \bullet' FC & \xrightarrow{m_{A,B} \bullet' id_{FC}} & F(A \bullet B) \bullet' FC & \xrightarrow{m_{A \bullet B, C}} & F((A \bullet B) \bullet C) \\
 \uparrow \alpha'_{FA, FB, FC} & & & & \uparrow F(\alpha_{A, B, C}) \\
 FA \bullet' (FB \bullet' FC) & \xrightarrow{id_{FA} \bullet' m_{B, C}} & FA \bullet' F(B \bullet C) & \xrightarrow{m_{A, B \bullet C}} & F(A \bullet (B \bullet C))
 \end{array}$$

$$\begin{array}{ccc}
 FA \bullet' FB & \xrightarrow{m_{A, B}} & F(A \bullet B) \\
 \downarrow \gamma'_{A, B} & & \downarrow F(\gamma_{A, B}) \\
 FB \bullet' FA & \xrightarrow{m_{B, A}} & F(B \bullet A)
 \end{array}$$

However in this particular case, assuming that $!$ is a symmetric monoidal (endo) functor means that $!$ comes equipped with a natural transformation

$$m_{A,B} : !A \otimes !B \rightarrow !(A \otimes B)$$

and a morphism

$$m_I : I \rightarrow !I$$

(where m_I is just the nullary version of the natural transformation.) The diagrams given in the above definition become the following:

$$\begin{array}{ccc}
!I \otimes !A & \xrightarrow{m_{I,A}} & !(I \otimes A) \\
\uparrow m_I \otimes id_{!A} & & \downarrow !(\lambda_A) \\
I \otimes !A & \xrightarrow{\lambda_{!A}} & !A
\end{array}
\quad
\begin{array}{ccc}
!A \otimes !I & \xrightarrow{m_{A,I}} & !(A \otimes I) \\
\uparrow id_{!A} \otimes m_I & & \downarrow !(\rho_A) \\
!A \otimes I & \xrightarrow{\rho_{!A}} & !A
\end{array}$$

$$\begin{array}{ccccc}
(!A \otimes !B) \otimes !C & \xrightarrow{m_{A,B} \otimes id_{!C}} & !(A \otimes B) \otimes !C & \xrightarrow{m_{A \otimes B, C}} & !((A \otimes B) \otimes C) \\
\uparrow \alpha_{!A, !B, !C} & & & & \uparrow !(\alpha_{A, B, C}) \\
!A \otimes (!B \otimes !C) & \xrightarrow{id_{!A} \otimes m_{B,C}} & !A \otimes !(B \otimes C) & \xrightarrow{m_{A, B \otimes C}} & !(A \otimes (B \otimes C))
\end{array}$$

$$\begin{array}{ccc}
!A \otimes !B & \xrightarrow{m_{A,B}} & !(A \otimes B) \\
\downarrow \gamma'_{A,B} & & \downarrow !(\gamma_{A,B}) \\
!B \otimes !A & \xrightarrow{m_{B,A}} & !(B \otimes A)
\end{array}$$

3 Monodial Adjunctions

4 Monodial Natural Transformations