Monoidal-Annex

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Contents

1 Monoidal Categories 1
2 Symmetric Monoidal Categories 4
3 Monoidal Double Category 6

4 The Double Category of (co)Lax Monoidal Functors 7

1 Monoidal Categories

Definition 1. A monoidal category $(\mathbb{C}, \otimes, I, \alpha, \lambda, \rho)$ is a category \mathbb{C} , a bifunctor $\otimes : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$, an object $I \in \mathbb{C}$, and three natural isomorphisms α, λ, ρ . Where,

$$\alpha = \alpha_{A,B,C} : A \otimes (B \otimes C) \cong (A \otimes B) \otimes C$$

is natural for all $A, B, C \in \mathbb{C}$ and the diagram

$$A \otimes (B \otimes (C \otimes D)) \xrightarrow{\alpha} (A \otimes B) \otimes (C \otimes D) \xrightarrow{\alpha} ((A \otimes B) \otimes C) \otimes D$$

$$\uparrow_{0 \otimes 1}$$

$$A \otimes ((B \otimes C) \otimes D) \xrightarrow{\alpha} (A \otimes (B \otimes C)) \otimes D$$

commutes for all $A, B, C, D \in \mathbb{C}$. γ and ρ are natural

$$\gamma_A: I \otimes A \cong A \quad \rho_A: A \otimes I \cong A$$

for all objects $A \in \mathbb{C}$, the diagram

$$A \otimes (I \otimes C) \xrightarrow{\alpha} (A \otimes I) \otimes C$$

$$\downarrow^{\rho_A \otimes 1}$$

$$A \otimes C = A \otimes C$$

[3]

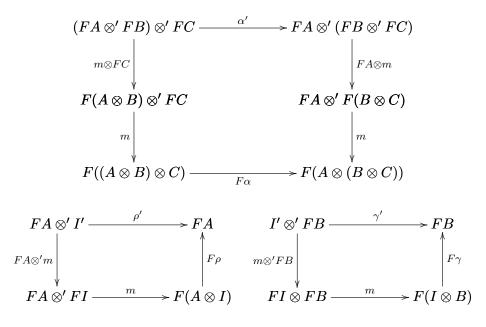
Definition 2. A monoidal functor or lax monoidal functor (F, m) between monoidal categories (\mathbb{C}, \otimes, I) and $(\mathbb{D}, \otimes', I')$ is a functor $F : \mathbb{C} \to \mathbb{D}$ equipped with a nathral transformation

$$m_{A,B}: FA \otimes' FB \to F(A \otimes B)$$

and an isomorphism

$$m_I: I' \to FI$$

where the following diagrams commute in the category $\mathbb D$ for all objects $A,B,C\in\mathbb C$



[4]

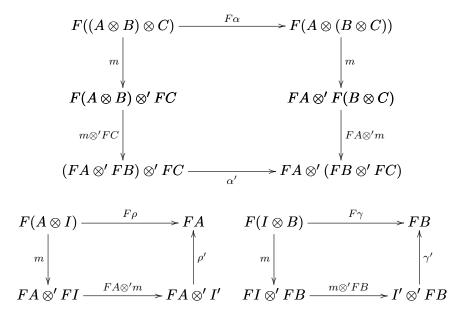
Definition 3. An oplax(colax/comonoidal) monoidal functor (F, m) between monoidal categories (\mathbb{C}, \otimes, I) and $(\mathbb{D}, \otimes', I')$ is a functor $F : \mathbb{C} \to \mathbb{D}$ with a natural transformation

$$m_{A,B}: F(A \otimes B) \to FA \otimes' FB$$

and an isomorphism

$$m_I: FI \to I'$$

where the following diagrams commute in the category \mathbb{D} , for all objects $A, B, C \in \mathbb{C}$



[4]

Definition 4. Suppose (F, m) and (G, n) are (lax) monoidal functors between the monoidal categories:

$$(\mathbb{C}, \otimes, I) \to (\mathbb{D}, \otimes', I')$$

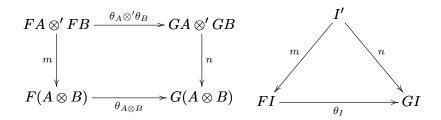
A monoidal natural transformation

$$\theta: (F, m) \Rightarrow (G, n): (\mathbb{C}, \otimes, I) \to (\mathbb{D}, \otimes', I')$$

between the monoidal functors (F, m) and (G, n) is a naural transformation

$$\theta: F \Rightarrow G: \mathbb{C} \to \mathbb{D}$$

between the underlying functors, where the following diamgrams commute, for all objects $A, B \in \mathbb{C}$



[4]

Definition 5. A monoidal natural transformation

$$\theta:(F,m)\Rightarrow(G,n):(\mathbb{C},\otimes,I)\to(\mathbb{D},\otimes',I')$$

between two oplax(colax/comonoidal) monoidal functors (F,m) and (G,n) is a natural transformation

$$\theta: F \Rightarrow G: \mathbb{C} \to \mathbb{D}$$

between the underlying functors, where the following diagrams commute, for all objects $A, B \in \mathbb{C}$

[4]

Definition 6. A monoidal category \mathbb{C} is said to be biclosed if every $-\otimes Y$ has a right adjoint [Y, -] and every $X \otimes -$ has a right adjoint [X, -] [2]

Definition 7. Given a pair of (lax) monoidal functors:

$$(F_*, m): (\mathbb{C}, \otimes, I) \to (\mathbb{D}, \otimes', I') \quad (F^*, n): (\mathbb{D}, \otimes', I') \to (\mathbb{C}, \otimes, I).$$

A monoidal adjunction

$$(F_*,m)\dashv (F^*,n)$$

between the monoidal functors is defined as an adjunction $(F_*, F^*, \eta, \epsilon)$ between the underlying functors

$$F_*: \mathbb{C} \to \mathbb{D} \quad F^*: \mathbb{D} \to \mathbb{C}$$

whose natural transformations

$$\eta: id_C \Rightarrow F^* \circ F_* \quad \epsilon: F_* \circ F^* \Rightarrow id_D$$

are monoidal [4]

2 Symmetric Monoidal Categories

Definition 8. A symmetric monoidal catergory (SMC), is a monoidal category \mathbb{C} with an additional natural isomorphism γ

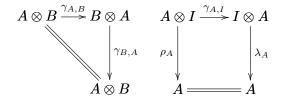
$$\gamma_{A,B}: A \otimes B \xrightarrow{\sim} B \otimes A$$

and three additional 'coherence' diagrams that commute.

$$(A \otimes B) \otimes C \xrightarrow{\alpha_{A,B,C}} A \otimes (B \otimes C) \xrightarrow{\gamma_{A,B \otimes C}} (B \otimes C) \otimes A$$

$$\uparrow_{A,B} \otimes id_{C} \downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$(B \otimes A) \otimes C \xrightarrow{\alpha_{B,A,C}} B \otimes (A \otimes C) \xrightarrow{id_{B} \otimes \gamma_{A,C}} B \otimes (C \otimes A)$$



The following equality is also require to hold:

$$\lambda_I = \rho_I : I \otimes I \to I$$

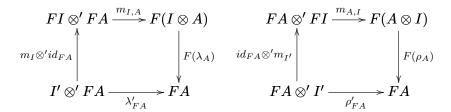
[1]

Definition 9. A symmetric monoidal closed category (SMCC), $(\mathbb{C}, \otimes, \neg, I, \alpha, \lambda, \rho, \gamma)$, is a SMC such that for all objects A in \mathbb{C} , the functor $-\otimes A$ has a specified right adjoint $A \multimap -$.

Definition 10. A symmetric monoidal functor between SMCs $(\mathbb{C}, \otimes, I, \alpha, \lambda, \rho, \gamma)$ and $(\mathbb{C}', \otimes', I', \alpha', \lambda', \rho', \gamma')$ is a functor $F : \mathbb{C} \to \mathbb{C}'$ equipped with

- 1. A morphism $m_{I'}: I' \to FI$.
- 2. For any two objects A and B in \mathbb{C} , a natural transformation $m_{A,B}: F(A) \otimes' F(B) \to F(A \otimes B)$

These must satisfy the following diagrams:



$$(FA \otimes' FB) \otimes' FC \xrightarrow{m_{A,B} \otimes' id_{FC}} F(A \otimes B) \otimes' FC \xrightarrow{m_{A \otimes B,C}} F((A \otimes B) \otimes C)$$

$$\uparrow^{F(\alpha_{A,B,C})}$$

$$FA \otimes' (FB \otimes' FC) \xrightarrow{id_{FA} \otimes' m_{B,C}} FA \otimes' F(B \otimes C) \xrightarrow{m_{A,B \otimes C}} F(A \otimes (B \otimes C))$$

$$FA \otimes' FB \xrightarrow{m_{A,B}} F(A \otimes B)$$

$$\downarrow^{F(\gamma_{A,B})}$$

$$FB \otimes' FA \xrightarrow{m_{A,B}} F(B \otimes A)$$

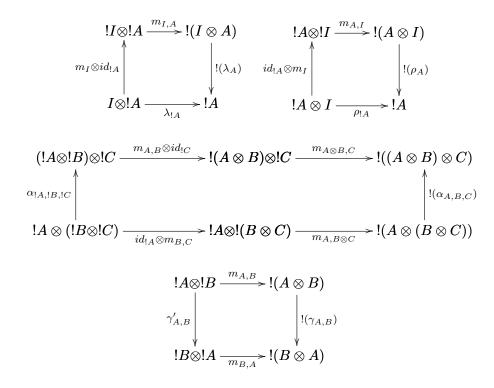
However in this particular case, assuming that ! is a symmetric monoidal (endo) functor means that ! comes equipped with a natural transformation

$$m_{A,B}: !A \otimes !B \rightarrow !(A \otimes B)$$

and a morphism

$$m_I:I\to !I$$

(where m_I is just the nullary version of the natural transformation.) The diagrams given in the above definition become the following:



[1]

Definition 11. A symmetric monoidal functor, $(F, m_{A,B}, m_{I'}) : \mathbb{C} \to \mathbb{C}'$, is said to be

- 1. Strict if $m_{A,B}$ and $m_{I'}$ are identities.
- 2. Strong if $m_{A,B}$ and $m_{I'}$ are natural isomorphisms.

[1]

Definition 12. An oplax monoidal functor

$$(F,m):(\mathbb{C},\otimes,I)\to(\mathbb{D},\otimes',I')$$

is symmetric when the following diagram commutes in the category \mathbb{D} for all objects $A, B \in \mathbb{C}$

$$F(A \otimes B) \xrightarrow{F_{\gamma}} F(B \otimes A)$$

$$\downarrow^{m} \qquad \qquad \downarrow^{m}$$

$$FA \otimes' FB \xrightarrow{\gamma'} FB \otimes' FA$$

[4]

3 Monoidal Double Category

Definition 13. The strict double category \mathbb{D} has a full double subcategory \mathbb{M} of monoidal categoreis. These are viewd as:

- vertical double categories on a formal object *
- vertical arrows $A: * \rightarrow *$
- $cells\ a:A\to A'$

The horizontal arrows of \mathbb{M} are monoidal functors(lax with respect to tensor product) and the vertical arrows are comonoidal functors(colax). A cell α : (FRSG) associates to every object A in \mathbb{A} with an arrow $\alpha A : GRA \to SFA$ in \mathbb{D} which satisfies the naturality condition

$$(GRa|\alpha v) = (\alpha u|SFa)$$
 (for $a: (ufgv)$ in \mathbb{A})

and the coherence conditions

$$(C\rho A|\alpha I|S\varphi A) = (\gamma RA|I|\sigma FA) \qquad (for \ a \ in \ \mathbb{A})$$

$$(G\rho(u,v)|\alpha w|S\varphi(u,v)) = (\gamma (Ru,Rv)|(\alpha u\otimes \alpha v)|\sigma(Fu,Fv)) \quad (for \ w=u\otimes v \ in \ \mathbb{A})$$

and have the following diagrmas

$$GRA \xrightarrow{\alpha A} SFA$$
 $GRI \xrightarrow{\alpha I} SFI$
 $GRa \downarrow \qquad \downarrow SFa$ $G\rho \nearrow I \xrightarrow{\sigma} SI$
 $GRA' \xrightarrow{\alpha A'} SFA'$ $GI \xrightarrow{\gamma} I \xrightarrow{\sigma} SI$

4 The Double Category of (co)Lax Monoidal Functors

Definition 14. A lax monoidal multifunctor from $(C_1, ..., C_n)$ to \mathcal{D} is a functor:

$$F: \mathcal{C}_1 \times \cdots \times \mathcal{C}_n \longrightarrow \mathcal{D}$$

such that:

- Each functor, $(F^i, m^i(\overrightarrow{A_1}, -, \overrightarrow{A_{i+1}}), m^i_{I_i}(\overrightarrow{A_1}, -, \overrightarrow{A_{i+1}})$ defined by $F^i(X) = F(\overrightarrow{A_1}, X, \overrightarrow{A_{i+1}}) : C_i \longrightarrow \mathcal{D}$, for $1 \leq i \leq n$ is lax monoidal.
- The following equations hold for any $1 \le i < j \le n$:

$$- m_{I_i}(\overrightarrow{I_1}, -, \overrightarrow{I_{i+1}}) = m_{I_j}(\overrightarrow{I_1}, -, \overrightarrow{I_{i+1}}),$$

$$- (m_{I_{i}}^{i}(\overrightarrow{A_{1}}, -, \overrightarrow{A_{i+1}}, X, \overrightarrow{A_{j+1}}) \otimes m_{I_{i}}^{i}(\overrightarrow{A_{1}}, -, \overrightarrow{A_{i+1}}, Y, \overrightarrow{A_{j+1}})); m_{X,Y}^{j}(\overrightarrow{A_{1}}, I_{i}, \overrightarrow{A_{i+1}}, -, \overrightarrow{A_{j+1}})$$

$$= \iota; m_{I_{i}}^{i}(\overrightarrow{A_{1}}, -, \overrightarrow{A_{i+1}}, X \otimes Y, \overrightarrow{A_{j+1}})$$

$$= (m_{I_j}^j(\overrightarrow{A_1}, X, \overrightarrow{A_{i+1}}, -, \overrightarrow{A_{j+1}}) \otimes m_{I_j}^j(\overrightarrow{A_1}, Y, \overrightarrow{A_{i+1}}, -, \overrightarrow{A_{j+1}})); m_{X,Y}^i(\overrightarrow{A_1}, -, \overrightarrow{A_{i+1}}, I_j, \overrightarrow{A_{j+1}})$$

$$= \iota; m_{I_j}^j(\overrightarrow{A_1}, X \otimes Y, \overrightarrow{A_{i+1}}, -, \overrightarrow{A_{j+1}})$$

In the above definition ι is the isomorphism $\iota: I \otimes I \longrightarrow I$ which holds in any monoidal category.

Definition 15. A symmetric lax monoidal multifunctor from (C_1, \ldots, C_n) to \mathcal{D} is a lax monoidal multifunctor

$$F: (\mathcal{C}_1, \ldots, \mathcal{C}_n) \longrightarrow \mathcal{D}$$

such that:

- Each functor, $(F^i, m^i(\overrightarrow{A_1}, -, \overrightarrow{A_{i+1}}), m^i_{I_i}(\overrightarrow{A_1}, -, \overrightarrow{A_{i+1}})$ defined by $F^i(X) = F(\overrightarrow{A_1}, X, \overrightarrow{A_{i+1}})$: $C_i \longrightarrow \mathcal{D}$, for $1 \le i \le n$ is symmetric lax monoidal.
- The following additional coherence axiom holds for any $1 \le i < j \le n$:

$$-\begin{array}{ll} & (m_{X,Y}^{i}(\overrightarrow{A_{1}},-,\overrightarrow{A_{i+1}},P,\overrightarrow{A_{j+1}})\otimes m_{X,Y}^{i}(\overrightarrow{A_{1}},-,\overrightarrow{A_{i+1}},Q,\overrightarrow{A_{j+1}})); m_{P,Q}^{j}(\overrightarrow{A_{1}},X\otimes Y,\overrightarrow{A_{i+1}},-,\overrightarrow{A_{j+1}})\\ & = \tau; (m_{P,Q}^{j}(\overrightarrow{A_{1}},X,\overrightarrow{A_{i+1}},-,\overrightarrow{A_{j+1}})\otimes m_{P,Q}^{j}(\overrightarrow{A_{1}},Y,\overrightarrow{A_{i+1}},-,\overrightarrow{A_{j+1}})); m_{X,Y}^{i}(\overrightarrow{A_{1}},-,\overrightarrow{A_{i+1}},P\otimes Q,\overrightarrow{A_{j+1}}) \end{array}$$

In the above definition τ is the isomorphism $\tau:(A\otimes B)\otimes(C\otimes D)\longrightarrow(A\otimes C)\otimes(B\otimes D)$ which holds in any monoidal category.

References

- [1] G.M. Bierman. On Intuitionistic Linear Logic. PhD thesis, University of Cambridge, 1993.
- [2] Max Kelly. Basic concepts of enriched category theory, volume 64. CUP Archive, 1982.
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- [4] Paul-André Mellies. Categorical semantics of linear logic. *Panoramas et syntheses*, 27:15–215, 2009.