Contents

1 Monoidal Category

1.1 Symmetric Monoidal Category

Definition 1. *A symmetric monoidal catergory (SMC), $(\mathbb{C}, \bullet, 1, \alpha, \lambda, \rho, \gamma)$, is a category $\mathbb{C}$ equippped with a bifunctor $\bullet : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ with a neutral element 1 and natural isomorphisms α, λ, ρ, and γ:*

1. $\alpha_{A,B,C} : A \bullet (B \bullet C) \xrightarrow{\sim} (A \bullet B) \bullet C$
2. $\lambda_A : 1 \bullet A \xrightarrow{\sim} A$
3. $\rho_A : A \bullet 1 \xrightarrow{\sim} A$
4. $\gamma_{A,B} : A \bullet B \xrightarrow{\sim} B \bullet A$

which make the following 'coherence' diagrams commute.

$$\begin{tikzcd}
A \bullet (B \bullet (C \bullet D)) \arrow[r, "\alpha_{A,B,C\bullet D}"] \arrow[d, "id_A \bullet \alpha_{B,C,D}"'] & (A \bullet B) \bullet (C \bullet D) \arrow[r, "\alpha_{A\bullet B,C,C}"] & ((A \bullet B) \bullet C) \bullet D \\
A \bullet ((B \bullet C) \bullet D) \arrow[rr, "\alpha_{A,B\bullet C,D}"'] & & (A \bullet (B \bullet C)) \bullet D \arrow[u, "\alpha_{A,B,C} \bullet id_D"']
\end{tikzcd}$$

$$\begin{tikzcd}
(A \bullet B) \bullet C \arrow[r, "\alpha_{A,B,C}"] \arrow[d, "\gamma_{A,B} \bullet id_C"'] & A \bullet (B \bullet C) \arrow[r, "\gamma_{A,B\bullet C}"] & (B \bullet C) \bullet A \arrow[d, "\alpha_{B,C,A}"'] \\
(B \bullet A) \bullet C \arrow[r, "\alpha_{B,A,C}"'] & B \bullet (A \bullet C) \arrow[r, "id_B \bullet \gamma_{A,C}"'] & B \bullet (C \bullet A)
\end{tikzcd}$$

$$\begin{tikzcd}
A \bullet (1 \bullet B) \arrow[r, "\alpha_{A,1,B}"] \arrow[d, "id_A \bullet \lambda_B"'] & (A \bullet 1) \bullet B \arrow[d, "\rho_A \bullet id_B"] \\
A \bullet B \arrow[r, equal] & A \bullet B
\end{tikzcd}
\qquad
\begin{tikzcd}
A \bullet B \arrow[r, "\gamma_{A,B}"] \arrow[dr, equal] & B \bullet A \arrow[d, "\gamma_{B,A}"] \\
& A \bullet B
\end{tikzcd}$$

$$\begin{tikzcd}
A \bullet 1 \arrow[r, "\gamma_{A,1}"] \arrow[d, "\rho_A"'] & 1 \bullet A \arrow[d, "\lambda_A"] \\
A \arrow[r, equal] & A
\end{tikzcd}$$

The following equality is also require to hold:

$$\lambda_1 \quad = \quad \rho_1 : 1 \bullet 1 \to 1$$

1.2 Symmetric Monoidal Closed Category

Definition 2. *A symmetric monoidal closed category (SMCC),* $(\mathbb{C}, \bullet, \multimap, 1, \alpha, \lambda, \rho, \gamma)$*, is a SMC such that for all objects* A *in* $\mathbb{C}$*, the functor* $- \otimes A$ *has a specified right adjoint* $A \multimap -$*.*

Let $\mathbb{C}$ *be a SMC* $(\mathbb{C}, \bullet, 1, \alpha, \lambda, \rho, \gamma)$*. A structure , M , in* $\mathbb{C}$ *for a given signature* S_g *is specified by giving an object* $[\![\sigma]\!]$ *in* $\mathbb{C}$ *for each type* σ*, and a morphism* $[\![f]\!] : [\![\sigma_1]\!] \bullet ... \bullet [\![\sigma_n]\!] \to [\![\tau]\!]$ *in* $\mathbb{C}$ *for each function symbol* $f : \sigma_1, ..., \sigma_n \to \tau$*. In the case where* $n = 0$ *then the structure assigns a morphism* $[\![c]\!] : 1 \to [\![\tau]\!]$ *to a constant* $c : \tau$*.*

Given a context $\Gamma = [x_1 : \sigma_1, ..., x_n : \sigma_n]$ *we define* $[\![\Gamma]\!]$ *to be the product* $[\![\sigma_1]\!] \bullet ... \bullet [\![\sigma_n]\!]$*. We represent the empyt context with the neutral element 1. We need to define the bracketing convention. It shall be assumed that the tensor product is left associative, i.e.* $A_1 \bullet A_2 \bullet ... \bullet A_n$ *will be taken to mean* $(...(A_1 \bullet A_2) \bullet ...) \bullet A_n$*. We find it useful to define two 'book-keeping' functions,*

$$Split(\Gamma, \Delta) : [\![\Gamma, \Delta]\!] \to [\![\Gamma]\!] \bullet [\![\Delta]\!]$$

$$Split(\Gamma, \Delta) \stackrel{\mathsf{def}}{=} \begin{cases} \lambda_{\Delta}^{-1} & \textit{If } \Gamma \textit{ empty} \\ \rho_{\Gamma}^{-1} & \textit{If } \Delta \textit{ empty} \\ id_{\Gamma \bullet A} & \textit{If } \Delta = A \\ Split(\Gamma, \Delta') \bullet id_A ; \alpha^{-1}_{\Gamma, \Delta', A} & \textit{If } \Delta = \Delta', A \end{cases}$$

$$Join(\Gamma, \Delta) : [\![\Gamma, \Delta]\!] \to [\![\Gamma]\!] \bullet [\![\Delta]\!]$$

$$Join(\Gamma, \Delta) \stackrel{\mathsf{def}}{=} \begin{cases} \lambda_{\Delta} & \textit{If } \Gamma \textit{ empty} \\ \rho_{\Gamma} & \textit{If } \Delta \textit{ empty} \\ id_{\Gamma \bullet A} & \textit{If } \Delta = A \\ \alpha_{\Gamma, \Delta', A} ; Join(\Gamma, \Delta') \bullet id_A & \textit{If } \Delta = \Delta', A \end{cases}$$

We shall also refer to indexed variants of these; for example

$$Split_n(\Gamma_1, ..., \Gamma_n) : [\![\Gamma_1, ..., \Gamma_n]\!] \to [\![\Gamma_1]\!] \bullet ... \bullet [\![\Gamma_n]\!]$$

which is defined in the obvious way.

The semantics of a term in context is then specified by a structural induction on the term.

$$[\![x : \sigma \rhd x : \sigma]\!] \stackrel{\mathsf{def}}{=} id_{\sigma}$$

$$[\![\Gamma_1, ..., \Gamma_n \rhd f(M_1, ..., M_n) : \tau]\!] \stackrel{\mathsf{def}}{=} Split_n(\Gamma_1, ..., \Gamma_n); [\![\Gamma_1 \rhd M_1 : \sigma_1]\!] \bullet ... \bullet [\![\Gamma_n \rhd M_1 : \sigma_n 0]\!]; [\![f]\!]$$

2 Monodial Functor

2.1 Symmetric Monoidal Functor

Definition 3. *A symmetric monoidal functor between SMCs* $(\mathbb{C}, \bullet, 1, \alpha, \lambda, \rho, \gamma)$ *and* $(\mathbb{C}', \bullet', 1', \alpha', \lambda', \rho', \gamma')$ *is a functor* $F : \mathbb{C} \to \mathbb{C}'$ *equipped with*

1. *A morphism* $m_{1'} : 1' \to F1$.

2. *For any two objects* A *and* B *in* $\mathbb{C}$, *a natural transformation* $m_{A,B} : F(A) \bullet' F(B) \to F(A \bullet B)$

These must satisfy the following diagrams:

$$\begin{tikzcd}
F1 \bullet' FA \arrow[r, "m_{1,A}"] & F(1 \bullet A) \arrow[d, "F(\lambda_A)"] \\
1' \bullet' FA \arrow[u, "m_1 \bullet' id_{FA}"] \arrow[r, "\lambda'_{FA}"'] & FA
\end{tikzcd}
\qquad
\begin{tikzcd}
FA \bullet' F1 \arrow[r, "m_{A,1}"] & F(A \bullet 1) \arrow[d, "F(\rho_A)"] \\
FA \bullet' 1' \arrow[u, "id_{FA} \bullet' m_{1'}"] \arrow[r, "\rho'_{FA}"'] & FA
\end{tikzcd}$$

$$\begin{tikzcd}
(FA \bullet' FB) \bullet' FC \arrow[r, "m_{A,B} \bullet' id_{FC}"] & F(A \bullet B) \bullet' FC \arrow[r, "m_{A \bullet B,C}"] & F((A \bullet B) \bullet C) \\
FA \bullet' (FB \bullet' FC) \arrow[u, "\alpha'_{FA,FB,FC}"] \arrow[r, "id_{FA} \bullet' m_{B,C}"'] & FA \bullet' F(B \bullet C) \arrow[r, "m_{A,B \bullet C}"'] & F(A \bullet (B \bullet C)) \arrow[u, "F(\alpha_{A,B,C})"']
\end{tikzcd}$$

$$\begin{tikzcd}
FA \bullet' FB \arrow[r, "m_{A,B}"] \arrow[d, "\gamma'_{A,B}"'] & F(A \bullet B) \arrow[d, "F(\gamma_{A,B})"] \\
FB \bullet' FA \arrow[r, "m_{B,A}"'] & F(B \bullet A)
\end{tikzcd}$$

However in this particular case, assuming that ! is a symmetric monoidal (endo) functor means that ! comes equipped with a natural transformation

$$m_{A,B} : !A \otimes !B \to !(A \otimes B)$$

and a morphism

$$m_I : I \to !I$$

(where m_I is just the nullary version of the natural transformation.) The diagrmas given in the above definition become the following:

$$
\begin{tikzcd}
!I \otimes !A \arrow[r, "m_{I,A}"] & !(I \otimes A) \arrow[d, "!(\lambda_A)"] \\
I \otimes !A \arrow[u, "m_I \otimes id_{!A}"] \arrow[r, "\lambda_{!A}"'] & !A
\end{tikzcd}
\qquad
\begin{tikzcd}
!A \otimes !I \arrow[r, "m_{A,I}"] & !(A \otimes I) \arrow[d, "!(\rho_A)"] \\
!A \otimes I \arrow[u, "id_{!A} \otimes m_I"] \arrow[r, "\rho_{!A}"'] & !A
\end{tikzcd}
$$

$$
\begin{tikzcd}
(!A \otimes !B) \otimes !C \arrow[r, "m_{A,B} \otimes id_{!C}"] & !(A \otimes B) \otimes !C \arrow[r, "m_{A \otimes B, C}"] & !((A \otimes B) \otimes C) \\
!A \otimes (!B \otimes !C) \arrow[u, "\alpha_{!A,!B,!C}"] \arrow[r, "id_{!A} \otimes m_{B,C}"'] & !A \otimes !(B \otimes C) \arrow[r, "m_{A, B \otimes C}"'] & !(A \otimes (B \otimes C)) \arrow[u, "!(\alpha_{A,B,C})"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
!A \otimes !B \arrow[r, "m_{A,B}"] \arrow[d, "\gamma'_{A,B}"'] & !(A \otimes B) \arrow[d, "!(\gamma_{A,B})"] \\
!B \otimes !A \arrow[r, "m_{B,A}"'] & !(B \otimes A)
\end{tikzcd}
$$

2.2 Symmetric Monoidal Functor (Strict and Strong)

Definition 4. *A symmetric monoidal functor, $(F, m_{A,B}, m_{1'}) : \mathbb{C} \to \mathbb{C}'$, is said to be*

1. *Strict if $m_{A,B}$ and $m_{1'}$ are identities.*
2. *Strong if $m_{A,B}$ and $m_{1'}$ are natural isomorphisms.*

The equation in context for Dereliction gives us that

$$
\begin{tikzcd}
!A_1 \arrow[r, "id_{!A_1}"] & !A_1 \arrow[r, "(\epsilon_A; f)^*"] \arrow[dr, "\epsilon_A; f"'] & !B \arrow[d, "\epsilon_B"] \\
 & & B
\end{tikzcd}
$$

commutes, or in other words

$$
\begin{tikzcd}
!A \arrow[r, "!f"] \arrow[d, "\epsilon_A"'] & !B \arrow[d, "\epsilon_B"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}
$$

commutes. Given that we have made the assumption that ! is a (symmetric) monoidal functor, this diagram suggests that ϵ is a monoidal natural transformation. We shall make this assumption and write ϵ for the monoidal natural transformation $\epsilon :! \to Id$.

3 Monodial Adjuctions

4 Monodial Natural Transformations