### Monoidal-Annex

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### 1 Monoidal Categories

**Definition 1.** A monoidal category  $(\mathbb{C}, \otimes, I, \alpha, \lambda, \rho)$  is a category  $\mathbb{C}$ , a bifunctor  $\otimes : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ , an object  $I \in \mathbb{C}$ , and three natural isomorphisms  $\alpha, \lambda, \rho$ . Where,

$$\alpha = \alpha_{A,B,C} : A \otimes (B \otimes C) \cong (A \otimes B) \otimes C$$

is natural for all  $A, B, C \in \mathbb{C}$  and the diagram

$$A \otimes (B \otimes (C \otimes D)) \xrightarrow{\alpha} (A \otimes B) \otimes (C \otimes D) \xrightarrow{\alpha} ((A \otimes B) \otimes C) \otimes D$$

$$\uparrow_{0 \otimes 1}$$

$$A \otimes ((B \otimes C) \otimes D) \xrightarrow{\alpha} (A \otimes (B \otimes C)) \otimes D$$

commutes for all  $A, B, C, D \in \mathbb{C}$ .  $\gamma$  and  $\rho$  are natural

$$\gamma_A: I \otimes A \cong A \quad \rho_A: A \otimes I \cong A$$

for all objects  $A \in \mathbb{C}$ , the diagram

$$A \otimes (I \otimes C) \xrightarrow{\alpha} (A \otimes I) \otimes C$$

$$\downarrow^{\rho_A \otimes 1}$$

$$A \otimes C = A \otimes C$$

[4]

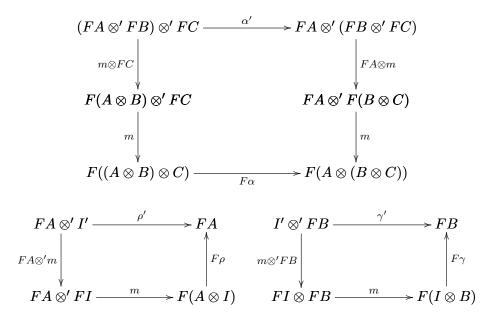
**Definition 2.** A monoidal functor or lax monoidal functor (F, m) between monoidal categories  $(\mathbb{C}, \otimes, I)$  and  $(\mathbb{D}, \otimes', I')$  is a functor  $F : \mathbb{C} \to \mathbb{D}$  equipped with a nathral transformation

$$m_{A,B}: FA \otimes' FB \to F(A \otimes B)$$

and an isomorphism

$$m_I: I' \to FI$$

where the following diagrams commute in the category  $\mathbb{D}$  for all objects  $A, B, C \in \mathbb{C}$ 



[5]

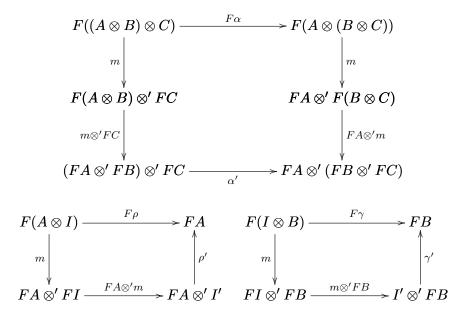
**Definition 3.** An oplax(colax/comonoidal) monoidal functor (F, m) between monoidal categories  $(\mathbb{C}, \otimes, I)$  and  $(\mathbb{D}, \otimes', I')$  is a functor  $F : \mathbb{C} \to \mathbb{D}$  with a natural transformation

$$m_{A,B}: F(A \otimes B) \to FA \otimes' FB$$

and an isomorphism

$$m_I: FI \to I'$$

where the following diagrams commute in the category  $\mathbb{D}$ , for all objects  $A, B, C \in \mathbb{C}$ 



[5]

**Definition 4.** Suppose (F, m) and (G, n) are (lax) monoidal functors between the monoidal categories:

$$(\mathbb{C}, \otimes, I) \to (\mathbb{D}, \otimes', I')$$

A monoidal natural transformation

$$\theta: (F,m) \Rightarrow (G,n): (\mathbb{C},\otimes,I) \to (\mathbb{D},\otimes',I')$$

between the monoidal functors (F, m) and (G, n) is a naural transformation

$$\theta: F \Rightarrow G: \mathbb{C} \to \mathbb{D}$$

between the underlying functors, where the following diamgrams commute, for all objects  $A, B \in \mathbb{C}$ 

$$FA \otimes' FB \xrightarrow{\theta_A \otimes' \theta_B} GA \otimes' GB$$

$$\downarrow n$$

$$\downarrow n$$

$$\downarrow n$$

$$\downarrow n$$

$$\downarrow n$$

$$\uparrow n$$

$$\downarrow n$$

$$\uparrow n$$

$$\downarrow n$$

$$\uparrow n$$

$$\downarrow n$$

$$\uparrow n$$

$$\downarrow n$$

$$\downarrow n$$

$$\uparrow n$$

$$\downarrow n$$

[5]

**Definition 5.** A monoidal natural transformation

$$\theta: (F, m) \Rightarrow (G, n): (\mathbb{C}, \otimes, I) \to (\mathbb{D}, \otimes', I')$$

between two oplax(colax/comonoidal) monoidal functors (F,m) and (G,n) is a natural transformation

$$\theta: F \Rightarrow G: \mathbb{C} \to \mathbb{D}$$

between the underlying functors, where the following diagrams commute, for all objects  $A, B \in \mathbb{C}$ 

|5|

**Definition 6.** A monoidal category  $\mathbb{C}$  is said to be biclosed if every  $-\otimes Y$  has a right adjoint [Y, -] and every  $X \otimes -$  has a right adjoint [X, -] [3]

**Definition 7.** Given a pair of (lax) monoidal functors:

$$(F_*, m): (\mathbb{C}, \otimes, I) \to (\mathbb{D}, \otimes', I') \quad (F^*, n): (\mathbb{D}, \otimes', I') \to (\mathbb{C}, \otimes, I).$$

A monoidal adjunction

$$(F_*,m)\dashv (F^*,n)$$

between the monoidal functors is defined as an adjunction  $(F_*, F^*, \eta, \epsilon)$  between the underlying functors

$$F_*: \mathbb{C} \to \mathbb{D} \quad F^*: \mathbb{D} \to \mathbb{C}$$

whose natural transformations

$$\eta: id_C \Rightarrow F^* \circ F_* \quad \epsilon: F_* \circ F^* \Rightarrow id_D$$

are monoidal [5]

# 2 Symmetric Monoidal Categories

**Definition 8.** A symmetric monoidal catergory (SMC), is a monoidal category  $\mathbb{C}$  with an additional natural isomorphism  $\gamma$ 

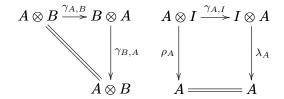
$$\gamma_{A,B}: A \otimes B \xrightarrow{\sim} B \otimes A$$

and three additional 'coherence' diagrams that commute.

$$(A \otimes B) \otimes C \xrightarrow{\alpha_{A,B,C}} A \otimes (B \otimes C) \xrightarrow{\gamma_{A,B \otimes C}} (B \otimes C) \otimes A$$

$$\uparrow_{A,B} \otimes id_{C} \downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$(B \otimes A) \otimes C \xrightarrow{\alpha_{B,A,C}} B \otimes (A \otimes C) \xrightarrow{id_{B} \otimes \gamma_{A,C}} B \otimes (C \otimes A)$$



The following equality is also require to hold:

$$\lambda_I = \rho_I : I \otimes I \to I$$

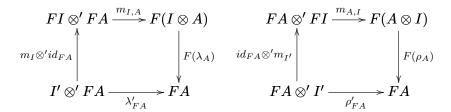
[1]

**Definition 9.** A symmetric monoidal closed category (SMCC),  $(\mathbb{C}, \otimes, \neg, I, \alpha, \lambda, \rho, \gamma)$ , is a SMC such that for all objects A in  $\mathbb{C}$ , the functor  $-\otimes A$  has a specified right adjoint  $A \multimap -$ .

**Definition 10.** A symmetric monoidal functor between SMCs  $(\mathbb{C}, \otimes, I, \alpha, \lambda, \rho, \gamma)$  and  $(\mathbb{C}', \otimes', I', \alpha', \lambda', \rho', \gamma')$  is a functor  $F : \mathbb{C} \to \mathbb{C}'$  equipped with

- 1. A morphism  $m_{I'}: I' \to FI$ .
- 2. For any two objects A and B in  $\mathbb{C}$ , a natural transformation  $m_{A,B}: F(A) \otimes' F(B) \to F(A \otimes B)$

These must satisfy the following diagrams:



$$(FA \otimes' FB) \otimes' FC \xrightarrow{m_{A,B} \otimes' id_{FC}} F(A \otimes B) \otimes' FC \xrightarrow{m_{A \otimes B,C}} F((A \otimes B) \otimes C)$$

$$\uparrow^{F(\alpha_{A,B,C})}$$

$$FA \otimes' (FB \otimes' FC) \xrightarrow{id_{FA} \otimes' m_{B,C}} FA \otimes' F(B \otimes C) \xrightarrow{m_{A,B \otimes C}} F(A \otimes (B \otimes C))$$

$$FA \otimes' FB \xrightarrow{m_{A,B}} F(A \otimes B)$$

$$\downarrow^{F(\gamma_{A,B})}$$

$$FB \otimes' FA \xrightarrow{m_{A,B}} F(B \otimes A)$$

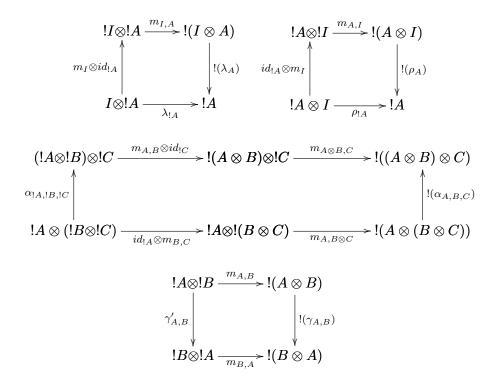
However in this particular case, assuming that ! is a symmetric monoidal (endo) functor means that ! comes equipped with a natural transformation

$$m_{A,B}: !A \otimes !B \rightarrow !(A \otimes B)$$

and a morphism

$$m_I:I\to !I$$

(where  $m_I$  is just the nullary version of the natural transformation.) The diagrams given in the above definition become the following:



[1]

**Definition 11.** A symmetric monoidal functor,  $(F, m_{A,B}, m_{I'}) : \mathbb{C} \to \mathbb{C}'$ , is said to be

- 1. Strict if  $m_{A,B}$  and  $m_{I'}$  are identities.
- 2. Strong if  $m_{A,B}$  and  $m_{I'}$  are natural isomorphisms.

[1]

**Definition 12.** An oplax monoidal functor

$$(F,m):(\mathbb{C},\otimes,I)\to(\mathbb{D},\otimes',I')$$

is symmetric when the following diagram commutes in the category  $\mathbb{D}$  for all objects  $A, B \in \mathbb{C}$ 

$$F(A \otimes B) \xrightarrow{F_{\gamma}} F(B \otimes A)$$

$$\downarrow^{m} \qquad \qquad \downarrow^{m}$$

$$FA \otimes' FB \xrightarrow{\gamma'} FB \otimes' FA$$

[5]

# 3 Monoidal Double Category

**Definition 13.** The strict double category  $\mathbb{D}$  has a full double subcategory  $\mathbb{M}$  of monoidal categoreis. These are viewd as:

- vertical double categories on a formal object \*
- vertical arrows  $A: * \rightarrow *$
- $cells\ a:A\to A'$

The horizontal arrows of  $\mathbb{M}$  are monoidal functors(lax with respect to tensor product) and the vertical arrows are comonoidal functors(colax). A cell  $\alpha$ : (FRSG) associates to every object A in  $\mathbb{A}$  with an arrow  $\alpha A : GRA \to SFA$  in  $\mathbb{D}$  which satisfies the naturality condition

$$(GRa|\alpha v) = (\alpha u|SFa) \quad (for \ a : (ufgv) \ in \ \mathbb{A})$$

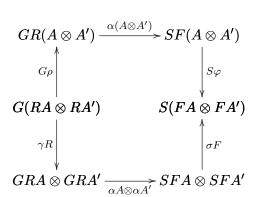
and the coherence conditions

$$(C\rho A|\alpha I|S\varphi A) = (\gamma RA|I|\sigma FA) \qquad (for a in A)$$

 $(G\rho(u,v)|\alpha w|S\varphi(u,v)) = (\gamma(Ru,Rv)|(\alpha u\otimes \alpha v)|\sigma(Fu,Fv)) \quad (\textit{for } w=u\otimes v \; \textit{in } \mathbb{A})$  and have the following diagrams

 $GRA \xrightarrow{\alpha A} SFA$ 

 $GRA \xrightarrow{\alpha A} SFA$   $GRa \downarrow \qquad \downarrow SFa$   $GRA' \xrightarrow{\alpha A'} SFA'$   $GI \xrightarrow{\gamma} I \xrightarrow{\sigma} SI$ 



where the lax monoidal functor R has comparison arrows

$$\rho = \rho(*): I \to RI$$
  
$$\rho(A, A'): RA \otimes RA' \to R(A \otimes A')$$

[2]

# 4 The Double Category of (co)Lax Monoidal Functors

**Definition 14.** A lax monoidal multifunctor from  $(C_1, \ldots, C_n)$  to  $\mathcal{D}$  is a functor:

$$F: \mathcal{C}_1 \times \cdots \times \mathcal{C}_n \longrightarrow \mathcal{D}$$

such that:

- Each functor,  $(F^i, m^i(\overrightarrow{A_1}, -, \overrightarrow{A_{i+1}}), m^i_{I_i}(\overrightarrow{A_1}, -, \overrightarrow{A_{i+1}})$  defined by  $F^i(X) = F(\overrightarrow{A_1}, X, \overrightarrow{A_{i+1}}) : C_i \longrightarrow \mathcal{D}$ , for  $1 \le i \le n$  is lax monoidal.
- The following equations hold for any  $1 \le i < j \le n$ :

$$- m_{I_i}(\overrightarrow{I_1}, -, \overrightarrow{I_{i+1}}) = m_{I_i}(\overrightarrow{I_1}, -, \overrightarrow{I_{i+1}}),$$

$$- (m_{I_{i}}^{i}(\overrightarrow{A_{1}}, -, \overrightarrow{A_{i+1}}, X, \overrightarrow{A_{j+1}}) \otimes m_{I_{i}}^{i}(\overrightarrow{A_{1}}, -, \overrightarrow{A_{i+1}}, Y, \overrightarrow{A_{j+1}})); m_{X,Y}^{j}(\overrightarrow{A_{1}}, I_{i}, \overrightarrow{A_{i+1}}, -, \overrightarrow{A_{j+1}})$$

$$= \iota; m_{I_{i}}^{i}(\overrightarrow{A_{1}}, -, \overrightarrow{A_{i+1}}, X \otimes Y, \overrightarrow{A_{j+1}})$$

$$= (m_{I_j}^j(\overrightarrow{A_1}, X, \overrightarrow{A_{i+1}}, -, \overrightarrow{A_{j+1}}) \otimes m_{I_j}^j(\overrightarrow{A_1}, Y, \overrightarrow{A_{i+1}}, -, \overrightarrow{A_{j+1}})); m_{X,Y}^i(\overrightarrow{A_1}, -, \overrightarrow{A_{i+1}}, I_j, \overrightarrow{A_{j+1}})$$

$$= \iota; m_{I_i}^j(\overrightarrow{A_1}, X \otimes Y, \overrightarrow{A_{i+1}}, -, \overrightarrow{A_{j+1}})$$

In the above definition  $\iota$  is the isomorphism  $\iota: I \otimes I \longrightarrow I$  which holds in any monoidal category.

**Definition 15.** A symmetric lax monoidal multifunctor from  $(C_1, \ldots, C_n)$  to  $\mathcal{D}$  is a lax monoidal multifunctor

$$F: (\mathcal{C}_1, \ldots, \mathcal{C}_n) \longrightarrow \mathcal{D}$$

such that:

- Each functor,  $(F^i, m^i(\overrightarrow{A_1}, -, \overrightarrow{A_{i+1}}), m^i_{I_i}(\overrightarrow{A_1}, -, \overrightarrow{A_{i+1}})$  defined by  $F^i(X) = F(\overrightarrow{A_1}, X, \overrightarrow{A_{i+1}})$ :  $C_i \longrightarrow \mathcal{D}$ , for  $1 \le i \le n$  is symmetric lax monoidal.
- The following additional coherence axiom holds for any  $1 \le i < j \le n$ :

$$= (m_{X,Y}^{i}(\overrightarrow{A_{1}}, -, \overrightarrow{A_{i+1}}, P, \overrightarrow{A_{j+1}}) \otimes m_{X,Y}^{i}(\overrightarrow{A_{1}}, -, \overrightarrow{A_{i+1}}, Q, \overrightarrow{A_{j+1}})); m_{P,Q}^{j}(\overrightarrow{A_{1}}, X \otimes Y, \overrightarrow{A_{i+1}}, -, \overrightarrow{A_{j+1}})$$

$$= \tau; (m_{P,Q}^{j}(\overrightarrow{A_{1}}, X, \overrightarrow{A_{i+1}}, -, \overrightarrow{A_{j+1}}) \otimes m_{P,Q}^{j}(\overrightarrow{A_{1}}, Y, \overrightarrow{A_{i+1}}, -, \overrightarrow{A_{j+1}})); m_{X,Y}^{i}(\overrightarrow{A_{1}}, -, \overrightarrow{A_{i+1}}, P \otimes Q, \overrightarrow{A_{j+1}})$$

In the above definition  $\tau$  is the isomorphism  $\tau: (A \otimes B) \otimes (C \otimes D) \longrightarrow (A \otimes C) \otimes (B \otimes D)$  which holds in any monoidal category.

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- [5] Paul-André Mellies. Categorical semantics of linear logic. *Panoramas et syntheses*, 27:15–215, 2009.