# Monoidal-Annex

Preston Keel Dr. Harley Eades

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## 1 Monoidal Categories

**Definition 1.** *A monoidal category $(\mathbb{C}, \otimes, I, \alpha, \lambda, \rho)$ is a category $\mathbb{C}$, a bifunctor $\otimes : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$, an object $I \in \mathbb{C}$, and three natural isomorphisms $\alpha, \lambda, \rho$. Where,*

$$\alpha = \alpha_{A,B,C} : A \otimes (B \otimes C) \cong (A \otimes B) \otimes C$$

*is natural for all $A, B, C \in \mathbb{C}$ and the diagram*

$$\begin{tikzcd}
A \otimes (B \otimes (C \otimes D)) \arrow[r, "\alpha"] \arrow[d, "1\otimes\alpha"'] & (A \otimes B) \otimes (C \otimes D) \arrow[r, "\alpha"] & ((A \otimes B) \otimes C) \otimes D \\
A \otimes ((B \otimes C) \otimes D) \arrow[rr, "\alpha"'] & & (A \otimes (B \otimes C)) \otimes D \arrow[u, "\alpha\otimes 1"']
\end{tikzcd}$$

*commutes for all $A, B, C, D \in \mathbb{C}$. $\gamma$ and $\rho$ are natural*

$$\gamma_A : I \otimes A \cong A \quad \rho_A : A \otimes I \cong A$$

*for all objects $A \in \mathbb{C}$, the diagram*

$$\begin{tikzcd}
A \otimes (I \otimes C) \arrow[r, "\alpha"] \arrow[d, "1\otimes\lambda"'] & (A \otimes I) \otimes C \arrow[d, "\rho_A\otimes 1"] \\
A \otimes C \arrow[r, equal] & A \otimes C
\end{tikzcd}$$

*[4]*

**Definition 2.** *A monoidal functor or lax monoidal functor $(F, m)$ between monoidal categories $(\mathbb{C}, \otimes, I)$ and $(\mathbb{D}, \otimes', I')$ is a functor $F : \mathbb{C} \to \mathbb{D}$ equipped with a nathral transformation*

$$m_{A,B} : FA \otimes' FB \to F(A \otimes B)$$

*and an isomorphism*

$$m_I : I' \to FI$$

*where the following diagrams commute in the category $\mathbb{D}$ for all objects $A, B, C \in \mathbb{C}$*

$$\begin{tikzcd}
(FA \otimes' FB) \otimes' FC \arrow[r, "\alpha'"] \arrow[d, "m \otimes FC"'] & FA \otimes' (FB \otimes' FC) \arrow[d, "FA \otimes m"] \\
F(A \otimes B) \otimes' FC \arrow[d, "m"'] & FA \otimes' F(B \otimes C) \arrow[d, "m"] \\
F((A \otimes B) \otimes C) \arrow[r, "F\alpha"'] & F(A \otimes (B \otimes C))
\end{tikzcd}$$

$$\begin{tikzcd}
FA \otimes' I' \arrow[r, "\rho'"] \arrow[d, "FA \otimes' m"'] & FA & I' \otimes' FB \arrow[r, "\gamma'"] \arrow[d, "m \otimes' FB"'] & FB \\
FA \otimes' FI \arrow[r, "m"] & F(A \otimes I) \arrow[u, "F\rho"'] & FI \otimes FB \arrow[r, "m"] & F(I \otimes B) \arrow[u, "F\gamma"']
\end{tikzcd}$$

*[5]*

**Definition 3.** *An oplax(colax/comonoidal) monoidal functor $(F, m)$ between monoidal categories $(\mathbb{C}, \otimes, I)$ and $(\mathbb{D}, \otimes', I')$ is a functor $F : \mathbb{C} \to \mathbb{D}$ with a natural transformation*

$$m_{A,B} : F(A \otimes B) \to FA \otimes' FB$$

*and an isomorphism*

$$m_I : FI \to I'$$

*where the following diagrams commute in the category $\mathbb{D}$, for all objects $A, B, C \in \mathbb{C}$*

$$\begin{tikzcd}
F((A \otimes B) \otimes C) \arrow[r, "F\alpha"] \arrow[d, "m"'] & F(A \otimes (B \otimes C)) \arrow[d, "m"] \\
F(A \otimes B) \otimes' FC \arrow[d, "m \otimes' FC"'] & FA \otimes' F(B \otimes C) \arrow[d, "FA \otimes' m"] \\
(FA \otimes' FB) \otimes' FC \arrow[r, "\alpha'"'] & FA \otimes' (FB \otimes' FC)
\end{tikzcd}$$

$$\begin{tikzcd}
F(A \otimes I) \arrow[r, "F\rho"] \arrow[d, "m"'] & FA & F(I \otimes B) \arrow[r, "F\gamma"] \arrow[d, "m"'] & FB \\
FA \otimes' FI \arrow[r, "FA \otimes' m"] & FA \otimes' I' \arrow[u, "\rho'"'] & FI \otimes' FB \arrow[r, "m \otimes' FB"] & I' \otimes' FB \arrow[u, "\gamma'"']
\end{tikzcd}$$

*[5]*

**Definition 4.** *Suppose* $(F, m)$ *and* $(G, n)$ *are (lax) monoidal functors between the monoidal categories:*

$$(\mathbb{C}, \otimes, I) \to (\mathbb{D}, \otimes', I')$$

*A monoidal natural transformation*

$$\theta : (F, m) \Rightarrow (G, n) : (\mathbb{C}, \otimes, I) \to (\mathbb{D}, \otimes', I')$$

*between the monoidal functors* $(F, m)$ *and* $(G, n)$ *is a naural transformation*

$$\theta : F \Rightarrow G : \mathbb{C} \to \mathbb{D}$$

*between the underlying functors, where the following diamgrams commute, for all objects* $A, B \in \mathbb{C}$

$$\begin{tikzcd}
FA \otimes' FB \arrow[r, "\theta_A \otimes' \theta_B"] \arrow[d, "m"'] & GA \otimes' GB \arrow[d, "n"] \\
F(A \otimes B) \arrow[r, "\theta_{A \otimes B}"'] & G(A \otimes B)
\end{tikzcd}
\qquad
\begin{tikzcd}
& I' \arrow[dl, "m"'] \arrow[dr, "n"] & \\
FI \arrow[rr, "\theta_I"'] & & GI
\end{tikzcd}$$

*[5]*

**Definition 5.** *A monoidal natural transformation*

$$\theta : (F, m) \Rightarrow (G, n) : (\mathbb{C}, \otimes, I) \to (\mathbb{D}, \otimes', I')$$

*between two oplax(colax/comonoidal) monoidal functors* $(F, m)$ *and* $(G, n)$ *is a natural transformation*

$$\theta : F \Rightarrow G : \mathbb{C} \to \mathbb{D}$$

*between the underlying functors, where the following diagrmas commute, for all objects $A, B \in \mathbb{C}$*

$$\begin{tikzcd} F(A \otimes B) \arrow[r, "\theta_{A\otimes B}"] \arrow[d, "m"'] & G(A \otimes B) \arrow[d, "n"] \\ FA \otimes' FB \arrow[r, "\theta_A \otimes' \theta_B"'] & GA \otimes' GB \end{tikzcd} \qquad \begin{tikzcd} FI \arrow[rr, "\theta_I"] \arrow[dr, "m"'] & & GI \arrow[dl, "n"] \\ & I' & \end{tikzcd}$$

*[5]*

**Definition 6.** *A monoidal category $\mathbb{C}$ is said to be biclosed if every $- \otimes Y$ has a right adjoint $[Y, -]$ and every $X \otimes -$ has a right adjoint $[\![X, -]\!]$ [3]*

**Definition 7.** *Given a pair of (lax) monoidal functors:*

$$(F_*, m) : (\mathbb{C}, \otimes, I) \to (\mathbb{D}, \otimes', I') \quad (F^*, n) : (\mathbb{D}, \otimes', I') \to (\mathbb{C}, \otimes, I).$$

*A monoidal adjunction*

$$(F_*, m) \dashv (F^*, n)$$

*between the monoidal functors is defined as an adjunction $(F_*, F^*, \eta, \epsilon)$ between the underlying functors*

$$F_* : \mathbb{C} \to \mathbb{D} \quad F^* : \mathbb{D} \to \mathbb{C}$$

*whose natural transformations*

$$\eta : id_C \Rightarrow F^* \circ F_* \quad \epsilon : F_* \circ F^* \Rightarrow id_D$$

*are monoidal [5]*

## 2 Symmetric Monoidal Categories

**Definition 8.** *A symmetric monoidal catergory (SMC), is a monoidal category $\mathbb{C}$ with an additional natural isomorphism $\gamma$*

$$\gamma_{A,B} : A \otimes B \xrightarrow{\sim} B \otimes A$$

*and three additional 'coherence' diagrams that commute.*

$$\begin{tikzcd} (A \otimes B) \otimes C \arrow[r, "\alpha_{A,B,C}"] \arrow[d, "\gamma_{A,B} \otimes id_C"'] & A \otimes (B \otimes C) \arrow[r, "\gamma_{A,B\otimes C}"] & (B \otimes C) \otimes A \arrow[d, "\alpha_{B,C,A}"'] \\ (B \otimes A) \otimes C \arrow[r, "\alpha_{B,A,C}"'] & B \otimes (A \otimes C) \arrow[r, "id_B \otimes \gamma_{A,C}"'] & B \otimes (C \otimes A) \end{tikzcd}$$

$$\begin{tikzcd} A \otimes B \arrow[r, "\gamma_{A,B}"] \arrow[dr, equal] & B \otimes A \arrow[d, "\gamma_{B,A}"] \\ & A \otimes B \end{tikzcd} \qquad \begin{tikzcd} A \otimes I \arrow[r, "\gamma_{A,I}"] \arrow[d, "\rho_A"'] & I \otimes A \arrow[d, "\lambda_A"] \\ A \arrow[r, equal] & A \end{tikzcd}$$

*The following equality is also require to hold:*

$$\lambda_I \quad = \quad \rho_I : I \otimes I \to I$$

*[1]*

**Definition 9.** *A symmetric monoidal closed category (SMCC),* $(\mathbb{C}, \otimes, \multimap, I, \alpha, \lambda, \rho, \gamma)$*, is a SMC such that for all objects* $A$ *in* $\mathbb{C}$*, the functor* $- \otimes A$ *has a specified right adjoint* $A \multimap -$*.*

**Definition 10.** *A symmetric monoidal functor between SMCs* $(\mathbb{C}, \otimes, I, \alpha, \lambda, \rho, \gamma)$ *and* $(\mathbb{C}', \otimes', I', \alpha', \lambda', \rho', \gamma')$ *is a functor* $F : \mathbb{C} \to \mathbb{C}'$ *equipped with*

1. *A morphism* $m_{I'} : I' \to FI$*.*
2. *For any two objects* $A$ *and* $B$ *in* $\mathbb{C}$*, a natural transformation* $m_{A,B} : F(A) \otimes' F(B) \to F(A \otimes B)$

*These must satisfy the following diagrams:*

$$\begin{tikzcd}
FI \otimes' FA \arrow[r, "m_{I,A}"] & F(I \otimes A) \arrow[d, "F(\lambda_A)"] \\
I' \otimes' FA \arrow[u, "m_I \otimes' id_{FA}"] \arrow[r, "\lambda'_{FA}"'] & FA
\end{tikzcd}
\qquad
\begin{tikzcd}
FA \otimes' FI \arrow[r, "m_{A,I}"] & F(A \otimes I) \arrow[d, "F(\rho_A)"] \\
FA \otimes' I' \arrow[u, "id_{FA} \otimes' m_{I'}"] \arrow[r, "\rho'_{FA}"'] & FA
\end{tikzcd}$$

$$\begin{tikzcd}
(FA \otimes' FB) \otimes' FC \arrow[r, "m_{A,B} \otimes' id_{FC}"] & F(A \otimes B) \otimes' FC \arrow[r, "m_{A \otimes B, C}"] & F((A \otimes B) \otimes C) \\
FA \otimes' (FB \otimes' FC) \arrow[u, "\alpha'_{FA,FB,FC}"] \arrow[r, "id_{FA} \otimes' m_{B,C}"'] & FA \otimes' F(B \otimes C) \arrow[r, "m_{A, B \otimes C}"'] & F(A \otimes (B \otimes C)) \arrow[u, "F(\alpha_{A,B,C})"']
\end{tikzcd}$$

$$\begin{tikzcd}
FA \otimes' FB \arrow[r, "m_{A,B}"] \arrow[d, "\gamma'_{A,B}"'] & F(A \otimes B) \arrow[d, "F(\gamma_{A,B})"] \\
FB \otimes' FA \arrow[r, "m_{B,A}"'] & F(B \otimes A)
\end{tikzcd}$$

*However in this particular case, assuming that ! is a symmetric monoidal (endo) functor means that ! comes equipped with a natural transformation*

$$m_{A,B} : !A \otimes !B \to !(A \otimes B)$$

*and a morphism*

$$m_I : I \to !I$$

*(where* $m_I$ *is just the nullary version of the natural transformation.) The diagrmas given in the above definition become the following:*

$$\begin{tikzcd} !I \otimes !A \arrow[r, "m_{I,A}"] & !(I \otimes A) \arrow[d, "!(\lambda_A)"] \\ I \otimes !A \arrow[u, "m_I \otimes id_{!A}"] \arrow[r, "\lambda_{!A}"'] & !A \end{tikzcd} \qquad \begin{tikzcd} !A \otimes !I \arrow[r, "m_{A,I}"] & !(A \otimes I) \arrow[d, "!(\rho_A)"] \\ !A \otimes I \arrow[u, "id_{!A} \otimes m_I"] \arrow[r, "\rho_{!A}"'] & !A \end{tikzcd}$$

$$\begin{tikzcd} (!A \otimes !B) \otimes !C \arrow[r, "m_{A,B} \otimes id_{!C}"] & !(A \otimes B) \otimes !C \arrow[r, "m_{A \otimes B, C}"] & !((A \otimes B) \otimes C) \\ !A \otimes (!B \otimes !C) \arrow[u, "\alpha_{!A,!B,!C}"] \arrow[r, "id_{!A} \otimes m_{B,C}"'] & !A \otimes !(B \otimes C) \arrow[r, "m_{A, B \otimes C}"'] & !(A \otimes (B \otimes C)) \arrow[u, "!(\alpha_{A,B,C})"'] \end{tikzcd}$$

$$\begin{tikzcd} !A \otimes !B \arrow[r, "m_{A,B}"] \arrow[d, "\gamma'_{A,B}"'] & !(A \otimes B) \arrow[d, "!(\gamma_{A,B})"] \\ !B \otimes !A \arrow[r, "m_{B,A}"'] & !(B \otimes A) \end{tikzcd}$$

*[1]*

**Definition 11.** *A symmetric monoidal functor, $(F, m_{A,B}, m_{I'}) : \mathbb{C} \to \mathbb{C}'$, is said to be*

1. *Strict if $m_{A,B}$ and $m_{I'}$ are identities.*
2. *Strong if $m_{A,B}$ and $m_{I'}$ are natural isomorphisms.*

*[1]*

**Definition 12.** *An oplax monoidal functor*

$$(F, m) : (\mathbb{C}, \otimes, I) \to (\mathbb{D}, \otimes', I')$$

*is symmetric when the following diagram commutes in the category $\mathbb{D}$ for all objects $A, B \in \mathbb{C}$*

$$\begin{tikzcd} F(A \otimes B) \arrow[r, "F_\gamma"] \arrow[d, "m"'] & F(B \otimes A) \arrow[d, "m"] \\ FA \otimes' FB \arrow[r, "\gamma'"'] & FB \otimes' FA \end{tikzcd}$$

*[5]*

# 3 Monoidal Double Category

**Definition 13.** *The strict double category $\mathbb{D}$ has a full double subcategory $\mathbb{M}$ of monoidal categoreis. These are viewd as:*

- *vertical double categories on a formal object* $*$
- *vertical arrows* $A : * \to *$
- *cells* $a : A \to A'$

*The horizontal arrows of* $\mathbb{M}$ *are monoidal functors(lax with respect to tensor product) and the vertical arrows are comonoidal functors(colax). A cell* $\alpha : (FRSG)$ *associates to every object* $A$ *in* $\mathbb{A}$ *with an arrow* $\alpha A : GRA \to SFA$ *in* $\mathbb{D}$ *which satisfies the naturality condition*

$$(GRa|\alpha v) = (\alpha u|SFa) \quad (\textit{for } a : (ufgv) \textit{ in } \mathbb{A})$$

*and the coherence conditions*

$$(C\rho A|\alpha I|S\varphi A) = (\gamma RA|I|\sigma FA) \qquad (\textit{for } a \textit{ in } \mathbb{A})$$

$$(G\rho(u,v)|\alpha w|S\varphi(u,v)) = (\gamma(Ru,Rv)|(\alpha u \otimes \alpha v)|\sigma(Fu,Fv)) \quad (\textit{for } w = u \otimes v \textit{ in } \mathbb{A})$$

*and have the following diagrams*

$$\begin{tikzcd} GRA \arrow[r, "\alpha A"] \arrow[d, "GRa"'] & SFA \arrow[d, "SFa"] \\ GRA' \arrow[r, "\alpha A'"'] & SFA' \end{tikzcd} \qquad \begin{tikzcd} & GRI \arrow[r, "\alpha I"] & SFI \arrow[dr, "S\varphi"] & \\ GI \arrow[ur, "G\rho"] \arrow[rr, "\gamma"'] & & I \arrow[r, "\sigma"'] & SI \end{tikzcd}$$

$$\begin{tikzcd} GR(A \otimes A') \arrow[r, "\alpha(A\otimes A')"] & SF(A \otimes A') \arrow[d, "S\varphi"] \\ G(RA \otimes RA') \arrow[u, "G\rho"] \arrow[d, "\gamma R"'] & S(FA \otimes FA') \\ GRA \otimes GRA' \arrow[r, "\alpha A \otimes \alpha A'"'] & SFA \otimes SFA' \arrow[u, "\sigma F"'] \end{tikzcd}$$

*where the lax monoidal functor* $R$ *has comparison arrows*

$$\rho = \rho(*) : I \to RI$$
$$\rho(A, A') : RA \otimes RA' \to R(A \otimes A')$$

*[2]*

## 4 The Double Category of (co)Lax Monoidal Functors

**Definition 14.** *A* lax monoidal multifunctor *from* $(\mathcal{C}_1, \dots, \mathcal{C}_n)$ *to* $\mathcal{D}$ *is a functor:*

$$F : \mathcal{C}_1 \times \cdots \times \mathcal{C}_n \longrightarrow \mathcal{D}$$

*such that:*

- *Each functor,* $(F^i, m^i(\overrightarrow{A_1}, -, \overrightarrow{A_{i+1}}), m^i_{I_i}(\overrightarrow{A_1}, -, \overrightarrow{A_{i+1}})$ *defined by* $F^i(X) = F(\overrightarrow{A_1}, X, \overrightarrow{A_{i+1}}) : \mathcal{C}_i \longrightarrow \mathcal{D}$*, for* $1 \leq i \leq n$ *is lax monoidal.*

- *The following equations hold for any* $1 \leq i < j \leq n$*:*

  - $m_{I_i}(\overrightarrow{I_1}, -, \overrightarrow{I_{i+1}}) = m_{I_j}(\overrightarrow{I_1}, -, \overrightarrow{I_{i+1}})$,

  - $(m^i_{I_i}(\overrightarrow{A_1}, -, \overrightarrow{A_{i+1}}, X, \overrightarrow{A_{j+1}}) \otimes m^i_{I_i}(\overrightarrow{A_1}, -, \overrightarrow{A_{i+1}}, Y, \overrightarrow{A_{j+1}})); m^j_{X,Y}(\overrightarrow{A_1}, I_i, \overrightarrow{A_{i+1}}, -, \overrightarrow{A_{j+1}})$
  $= \iota; m^i_{I_i}(\overrightarrow{A_1}, -, \overrightarrow{A_{i+1}}, X \otimes Y, \overrightarrow{A_{j+1}})$

  - $(m^j_{I_j}(\overrightarrow{A_1}, X, \overrightarrow{A_{i+1}}, -, \overrightarrow{A_{j+1}}) \otimes m^j_{I_j}(\overrightarrow{A_1}, Y, \overrightarrow{A_{i+1}}, -, \overrightarrow{A_{j+1}})); m^i_{X,Y}(\overrightarrow{A_1}, -, \overrightarrow{A_{i+1}}, I_j, \overrightarrow{A_{j+1}})$
  $= \iota; m^j_{I_j}(\overrightarrow{A_1}, X \otimes Y, \overrightarrow{A_{i+1}}, -, \overrightarrow{A_{j+1}})$

In the above definition $\iota$ is the isomorphism $\iota : I \otimes I \longrightarrow I$ which holds in any monoidal category.

**Definition 15.** *A* symmetric lax monoidal multifunctor *from* $(\mathcal{C}_1, \ldots, \mathcal{C}_n)$ *to* $\mathcal{D}$ *is a lax monoidal multifunctor*

$$F : (\mathcal{C}_1, \ldots, \mathcal{C}_n) \longrightarrow \mathcal{D}$$

*such that:*

- *Each functor,* $(F^i, m^i(\overrightarrow{A_1}, -, \overrightarrow{A_{i+1}}), m^i_{I_i}(\overrightarrow{A_1}, -, \overrightarrow{A_{i+1}})$ *defined by* $F^i(X) = F(\overrightarrow{A_1}, X, \overrightarrow{A_{i+1}}) : \mathcal{C}_i \longrightarrow \mathcal{D}$*, for* $1 \leq i \leq n$ *is symmetric lax monoidal.*

- *The following additional coherence axiom holds for any* $1 \leq i < j \leq n$*:*

  - $(m^i_{X,Y}(\overrightarrow{A_1}, -, \overrightarrow{A_{i+1}}, P, \overrightarrow{A_{j+1}}) \otimes m^i_{X,Y}(\overrightarrow{A_1}, -, \overrightarrow{A_{i+1}}, Q, \overrightarrow{A_{j+1}})); m^j_{P,Q}(\overrightarrow{A_1}, X \otimes Y, \overrightarrow{A_{i+1}}, -, \overrightarrow{A_{j+1}})$
  $= \tau; (m^j_{P,Q}(\overrightarrow{A_1}, X, \overrightarrow{A_{i+1}}, -, \overrightarrow{A_{j+1}}) \otimes m^j_{P,Q}(\overrightarrow{A_1}, Y, \overrightarrow{A_{i+1}}, -, \overrightarrow{A_{j+1}})); m^i_{X,Y}(\overrightarrow{A_1}, -, \overrightarrow{A_{i+1}}, P \otimes Q, \overrightarrow{A_{j+1}})$

In the above definition $\tau$ is the isomorphism $\tau : (A \otimes B) \otimes (C \otimes D) \longrightarrow (A \otimes C) \otimes (B \otimes D)$ which holds in any monoidal category.

## References


[1] G.M. Bierman. *On Intuitionistic Linear Logic*. PhD thesis, University of Cambridge, 1993.

[2] Marco Grandis and Robert Paré. Adjoint for double categories. *Cahiers de topologie et géométrie différentielle catégoriques*, 45(3):193–240, 2004.

[3] Max Kelly. *Basic concepts of enriched category theory*, volume 64. CUP Archive, 1982.

[4] Saunders Mac Lane. *Categories for the Working Mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1971.

[5] Paul-André Mellies. Categorical semantics of linear logic. *Panoramas et syntheses*, 27:15–215, 2009.