Notes on Fibrational Semantics of Simple, Polymorphic, and Dependent Type Theory

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1 The Simple Fibration

Definition 1.1. *A* CT-structure *is a pair* $(\mathbb{B}, T)$ *where* $\mathbb{B}$ *is a category with finite products, and* $T \subseteq \mathsf{Obj}(\mathbb{B})$ *is a collection of types.*

A CT-structure $(\mathbb{B}, T)$ should be thought of as a category of contexts $\mathbb{B}$ whose types draw their atomic elements from T. Given contexts $\Gamma, \Delta \in \mathsf{Obj}(\mathbb{B})$, their concatenation is defined as $(\Gamma, \Delta) = (\Gamma \times \Delta)$.

Definition 1.2. *...*

Definition 1.3. *...*

Definition 1.4. *The category* $\mathcal{L}_1$ *is defined as follows:*

Objects:	*Contexts* $\Gamma = x_1 : T_1, \ldots, x_n : T_n$.
Morphisms:	*Let* Γ *and* $\Delta = y_1 : T_1, \ldots, y_n : T_n$ *be contexts, then a morphism* $\Gamma \longrightarrow \Delta$ *is a n-tuple,* $([t_1], \ldots, [t_n])$*, such that* $[t_i] = \{t \mid t$ *differs from* t_i *only by the names of its free variables*$\}$ *is the equivalence class of terms such that* $\Gamma \vdash t_i : T_i$ *holds for each* $1 \leq i \leq n$.

Lemma 1.5 (Classifying Category for STLC)**.** $\mathcal{L}_1$ *is indeed a category.*

Proof. **(Identities)** Suppose $\Gamma = x_1 : T_1, \ldots, x_i : T_i$ is a context. Then $\mathsf{id} = (x_1, \ldots, x_i) : \Gamma \longrightarrow \Gamma$, because $\Gamma \vdash x_j : T_j$ for $1 \leq j \leq i$ each hold by the variable rules.

(Composition.) Suppose Γ, $\Delta = y_1 : T_1, \ldots, y_i : T_i$ and $\Phi = z_1 : T'_1, \ldots, z_j : T'_j$ are contexts, and $f = (t_1, \ldots, t_i) : \Gamma \longrightarrow \Delta$ and $g = (t'_1, \ldots, t'_j) : \Delta \longrightarrow \Phi$ are morphisms. Then define their composition $f; g = ([t_1/y_1] \cdots [t_i/y_i]t'_1, \ldots, [t_1/y_1] \cdots [t_i/y_i]t'_j) : \Gamma \longrightarrow \Phi$.

(Composition respects identities.). Suppose $\Gamma = x_1 : T_1, \ldots, x_i : T_i$ and $\Delta = y_1 : T_1, \ldots, y_j : T_j$ are contexts, and $f = (t_1, \ldots, t_j) : \Gamma \longrightarrow \Delta$ is a morphism. Then:

$$\begin{array}{rcl} id_\Gamma; f & = & ([x_1/x_1] \cdots [x_i/x_i]t_1, \ldots, [x_1/x_1] \cdots [x_i/x_i]t_j) \\ & = & (t_1, \ldots, t_j) \\ & = & f \end{array}$$

and

$$\begin{array}{rcl} f; id_\Delta & = & ([t_1/y_1] \cdots [t_j/y_j]y_1, \ldots, [t_1/y_1] \cdots [t_j/y_j]y_j) \\ & = & (t_1, \ldots, t_j) \\ & = & f \end{array}$$

□