Linear metatheory via linear algebra

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Introduction

- $\Gamma \vdash M : A$

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- $x_1 : A_1, \ldots, \quad x_n : A_n \vdash M : A$

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- Examples:
  1. $1x : A, 0y : B \vdash x : A$ (linearity; DILL from Bar96)
  2. $\uparrow x : \mathbb{Z}, \downarrow y : \mathbb{Z} \vdash x - y : \mathbb{Z}$ (monotonicity; Arntzenius)
  3. $\mathrm{prv}\, x : \mathrm{Secret} \nvdash \_ : !_{\mathrm{pub}}\mathrm{Secret}$ (privacy; DCC from ABHR99)
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Annotation posemiring

- Annotations form a partially ordered semiring.
- A *partially ordered semiring* $(\mathscr{R}, \trianglelefteq, 0, +, 1, \cdot)$ is:
  - A *partial order* $(\mathscr{R}, \trianglelefteq)$
  - A *semiring* $(\mathscr{R}, 0, +, 1, \cdot)$, which is:
    - A *commutative monoid* $(\mathscr{R}, 0, +)$
    - A *monoid* $(\mathscr{R}, 1, \cdot)$
    - Such that $\cdot$ distributes over 0 and $+$ on both sides
  - Such that $+$ and $\cdot$ are both monotonic in both arguments

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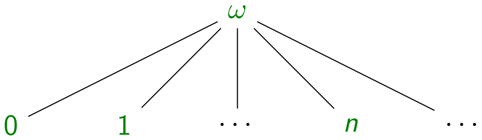
  - Such that $+$ and $\cdot$ are both monotonic in both arguments
- Ur-example: $(\mathbb{N} \cup \{\omega\}, \sqsupseteq, 0, +, 1, \times)$ "how many"



What are we dealing with?

$$\frac{\mathcal{R}\Gamma \vdash M : A \qquad \mathcal{R}\Gamma \vdash N : B}{\mathcal{R}\Gamma \vdash (M, N)_{\&} : A \mathbin{\&} B}\ \&\text{-I}$$

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Simultaneous substitution

$$\dfrac{\Gamma \vdash M : A \qquad \left(\begin{array}{c} \Delta \vdash N_1 : \Gamma_1 \\ \vdots \\ \Delta \vdash N_m : \Gamma_m \end{array}\right)}{\Delta \vdash M\{\vec{N}\} : A}\ \textsc{Subst}$$

Simultaneous substitution

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Simultaneous substitution

$$\dfrac{\begin{array}{l}\textsc{Subst} \\ \mathcal{P}\Gamma \vdash M : A \qquad \begin{pmatrix} \mathcal{Q}\Delta \vdash N_1 : \Gamma_1 \\ \vdots \\ \mathcal{Q}\Delta \vdash N_m : \Gamma_m \end{pmatrix}\end{array}}{\mathcal{Q}\Delta \vdash M\{\vec{N}\} : A}$$

- Wrong! Try identity substitution on $\mathcal{P}\Gamma = \mathcal{Q}\Delta = 1x : B, 1y : C$.
  $$\begin{pmatrix} 1x : B, 1y : C \nvdash x : B \\ 1x : B, 1y : C \nvdash y : C \end{pmatrix}$$

Simultaneous substitution

$$\frac{\mathcal{P}\Gamma \vdash M : A \qquad \begin{pmatrix} ?\Delta \vdash N_1 : \Gamma_1 \\ \vdots \\ ?\Delta \vdash N_m : \Gamma_m \end{pmatrix}}{\mathcal{Q}\Delta \vdash M\{\vec{N}\} : A}\ \textsc{Subst}$$

- We need to split $\mathcal{Q}$ up.

$$\begin{pmatrix} 1x : B, 0y : C \vdash x : B \\ 0x : B, 1y : C \vdash y : C \end{pmatrix}$$

Simultaneous substitution

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- We need to split $\mathcal{Q}$ up.

$$\begin{pmatrix} 1x : B, 0y : C \vdash x : B \\ 0x : B, 1y : C \vdash y : C \end{pmatrix}$$

- What is the relationship between $\mathcal{P}$, $\mathcal{Q}$, and S?

Substitution, stripped down

$$\textsc{Subst}\quad \frac{\mathcal{P}\Gamma \vdash M : A \qquad \begin{pmatrix} S_1\Delta \vdash N_1 : \Gamma_1 \\ \vdots \\ S_m\Delta \vdash N_m : \Gamma_m \end{pmatrix}}{\mathcal{Q}\Delta \vdash M\{\vec{N}\} : A}$$

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$$\textsc{Subst}\quad \frac{\pi_1 x_1, \ldots, \pi_m x_m \vdash M \qquad \begin{pmatrix} \sigma_{1,1} y_1, \ldots, \sigma_{1,n} y_n \vdash N_1 \\ \vdots \\ \sigma_{m,1} y_1, \ldots, \sigma_{m,n} y_n \vdash N_m \end{pmatrix}}{\rho_1 y_1, \ldots, \rho_n y_n \vdash M\{\vec{N}\}}$$

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- What is the relationship between $\mathcal{P}$, $\mathcal{Q}$, and $\mathcal{S}$?
- ρ_j is a weighted sum of $\sigma_{-,j}$ according to π_{-}.

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- What is the relationship between $\mathcal{P}$, $\mathcal{Q}$, and $\mathcal{S}$?
- ρ_j is a weighted sum of $\sigma_{-,j}$ according to π_{-}.
- $$\begin{pmatrix} \pi_1 & \cdots & \pi_m \end{pmatrix} \begin{pmatrix} \sigma_{1,1} & \cdots & \sigma_{1,n} \\ \vdots & \ddots & \vdots \\ \sigma_{m,1} & \cdots & \sigma_{m,n} \end{pmatrix} \trianglelefteq \begin{pmatrix} \rho_1 & \cdots & \rho_n \end{pmatrix}$$

What is a substitution?

$$\textsc{Subst}\quad \frac{\pi_1 x_1, \ldots, \pi_m x_m \vdash M \qquad \begin{pmatrix} \sigma_{1,1} y_1, \ldots, \sigma_{1,n} y_n \vdash N_1 \\ \vdots \\ \sigma_{m,1} y_1, \ldots, \sigma_{m,n} y_n \vdash N_m \end{pmatrix}}{\rho_1 y_1, \ldots, \rho_n y_n \vdash M\{\vec{N}\}}$$

- Notation: $\langle x|$ stands for the basis vector at x.
- A simultaneous substitution $\mathcal{Q}\Delta \Rightarrow \mathcal{P}\Gamma$ is:
  - A $|\Gamma|$-by-$|\Delta|$ matrix S, such that
  - For each $x : A$ in Γ, a term N_x such that $(\langle x|S)\Delta \vdash N_x : A$
  - $\mathcal{P}S \trianglelefteq \mathcal{Q}$

Substitution on &-I

$$\dfrac{\dfrac{\mathcal{P}\Gamma \vdash M : A \qquad \mathcal{P}\Gamma \vdash N : B}{\mathcal{P}\Gamma \vdash (M, N)_{\&} : A \,\&\, B}\,\&\text{-I} \qquad \mathcal{Q}\Delta \stackrel{\sigma}{\Rightarrow} \mathcal{P}\Gamma}{\mathcal{Q}\Delta \vdash (M\{\sigma\}, N\{\sigma\})_{\&} : A \,\&\, B}\,\textsc{Subst}$$

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$$\dfrac{\mathcal{P}'\Delta \vdash M\{\sigma\} : A \qquad \mathcal{Q}'\Delta \vdash N\{\sigma\} : B \qquad \mathcal{P}' + \mathcal{Q}' \trianglelefteq \mathcal{R}'}{\mathcal{R}'\Delta \vdash (M\{\sigma\}, N\{\sigma\})_{\otimes} : A \otimes B}\ \otimes\text{-I}$$

Substitution on $\otimes$-I

$$\dfrac{\dfrac{\mathcal{P}\Gamma \vdash M : A \qquad \mathcal{Q}\Gamma \vdash N : B \qquad \mathcal{P} + \mathcal{Q} \trianglelefteq \mathcal{R}}{\mathcal{R}\Gamma \vdash (M, N)_{\otimes} : A \otimes B}\ \otimes\text{-I} \qquad \mathcal{R}'\Delta \overset{\sigma}{\Rightarrow} \mathcal{R}\Gamma}{\mathcal{R}'\Delta \vdash (M\{\sigma\}, N\{\sigma\})_{\otimes} : A \otimes B}\ \textsc{Subst}$$

$$\mathcal{P}' := \mathcal{P}S \qquad \mathcal{Q}' := \mathcal{Q}S$$

$$\mathcal{P}'\Delta \overset{\sigma}{\Rightarrow} \mathcal{P}\Gamma \quad \mathcal{Q}'\Delta \overset{\sigma}{\Rightarrow} \mathcal{Q}\Gamma$$

$$\mathcal{P}' + \mathcal{Q}' = \mathcal{P}S + \mathcal{Q}S = (\mathcal{P} + \mathcal{Q})S \trianglelefteq \mathcal{R}S \trianglelefteq \mathcal{R}'$$

$$\dfrac{\mathcal{P}'\Delta \vdash M\{\sigma\} : A \qquad \mathcal{Q}'\Delta \vdash N\{\sigma\} : B \qquad \mathcal{P}' + \mathcal{Q}' \trianglelefteq \mathcal{R}'}{\mathcal{R}'\Delta \vdash (M\{\sigma\}, N\{\sigma\})_{\otimes} : A \otimes B}\ \otimes\text{-I}$$

Substitution on $\otimes$-I

$$\dfrac{\dfrac{\mathcal{P}\Gamma \vdash M : A \qquad \mathcal{Q}\Gamma \vdash N : B \qquad \mathcal{P} + \mathcal{Q} \trianglelefteq \mathcal{R}}{\mathcal{R}\Gamma \vdash (M, N)_\otimes : A \otimes B}\ \otimes\text{-I} \qquad \mathcal{R}'\Delta \overset{\sigma}{\Rightarrow} \mathcal{R}\Gamma}{\mathcal{R}'\Delta \vdash (M\{\sigma\}, N\{\sigma\})_\otimes : A \otimes B}\ \textsc{Subst}$$

$$\mathcal{P}' := \mathcal{P}S \qquad \mathcal{Q}' := \mathcal{Q}S$$

$$\mathcal{P}'\Delta \overset{\sigma}{\Rightarrow} \mathcal{P}\Gamma \quad \mathcal{Q}'\Delta \overset{\sigma}{\Rightarrow} \mathcal{Q}\Gamma$$

$$\mathcal{P}' + \mathcal{Q}' = \mathcal{P}S + \mathcal{Q}S = (\mathcal{P} + \mathcal{Q})S \trianglelefteq \mathcal{R}S \trianglelefteq \mathcal{R}'$$

$$\dfrac{\mathcal{P}'\Delta \vdash M\{\sigma\} : A \qquad \mathcal{Q}'\Delta \vdash N\{\sigma\} : B \qquad \mathcal{P}' + \mathcal{Q}' \trianglelefteq \mathcal{R}'}{\mathcal{R}'\Delta \vdash (M\{\sigma\}, N\{\sigma\})_\otimes : A \otimes B}\ \otimes\text{-I}$$

Substitution on $!_\rho$-I

$$\dfrac{\dfrac{\mathcal{P}\Gamma \vdash M : A \qquad \rho\mathcal{P} \trianglelefteq \mathcal{R}}{\mathcal{R}\Gamma \vdash [M] : !_\rho A}\ !_\rho\text{-I} \qquad \mathcal{R}'\Delta \overset{\sigma}{\Rightarrow} \mathcal{R}\Gamma}{\mathcal{R}'\Delta \vdash [M\{\sigma\}] : !_\rho A}\ \textsc{Subst}$$

$$\mathcal{P}' := \mathcal{P}S$$

$$\mathcal{P}'\Delta \overset{\sigma}{\Rightarrow} \mathcal{P}\Gamma$$

$$\rho\mathcal{P}' = \rho(\mathcal{P}S) = (\rho\mathcal{P})S \trianglelefteq \mathcal{R}S \trianglelefteq \mathcal{R}'$$

$$\dfrac{\mathcal{P}'\Delta \vdash M\{\sigma\} : A \qquad \rho\mathcal{P}' \trianglelefteq \mathcal{R}'}{\mathcal{R}'\Delta \vdash [M\{\sigma\}] : !_\rho A}\ !_\rho\text{-I}$$

Substitution on $!_\rho$-I

$$\dfrac{\dfrac{\mathcal{P}\Gamma \vdash M : A \qquad \rho\mathcal{P} \trianglelefteq \mathcal{R}}{\mathcal{R}\Gamma \vdash [M] : !_\rho A}\ !_\rho\text{-I} \qquad \mathcal{R}'\Delta \overset{\sigma}{\Rightarrow} \mathcal{R}\Gamma}{\mathcal{R}'\Delta \vdash [M\{\sigma\}] : !_\rho A}\ \textsc{Subst}$$

$$\mathcal{P}' := \mathcal{P}S$$

$$\mathcal{P}'\Delta \overset{\sigma}{\Rightarrow} \mathcal{P}\Gamma$$

$$\rho\mathcal{P}' = \rho(\mathcal{P}S) = (\rho\mathcal{P})S \trianglelefteq \mathcal{R}S \trianglelefteq \mathcal{R}'$$

$$\dfrac{\mathcal{P}'\Delta \vdash M\{\sigma\} : A \qquad \rho\mathcal{P}' \trianglelefteq \mathcal{R}'}{\mathcal{R}'\Delta \vdash [M\{\sigma\}] : !_\rho A}\ !_\rho\text{-I}$$

Substitution on $!_{\rho}$-I

$$\dfrac{\dfrac{\mathcal{P}\Gamma \vdash M : A \qquad \rho\mathcal{P} \trianglelefteq \mathcal{R}}{\mathcal{R}\Gamma \vdash [M] : !_{\rho}A}\ !_{\rho}\text{-I} \qquad \mathcal{R}'\Delta \overset{\sigma}{\Rightarrow} \mathcal{R}\Gamma}{\mathcal{R}'\Delta \vdash [M\{\sigma\}] : !_{\rho}A}\ \textsc{Subst}$$

$$\mathcal{P}' := \mathcal{P}S$$
$$\mathcal{P}'\Delta \overset{\sigma}{\Rightarrow} \mathcal{P}\Gamma$$

$$\rho\mathcal{P}' = \rho(\mathcal{P}S) = (\rho\mathcal{P})S \trianglelefteq \mathcal{R}S \trianglelefteq \mathcal{R}'$$

$$\dfrac{\mathcal{P}'\Delta \vdash M\{\sigma\} : A \qquad \rho\mathcal{P}' \trianglelefteq \mathcal{R}'}{\mathcal{R}'\Delta \vdash [M\{\sigma\}] : !_{\rho}A}\ !_{\rho}\text{-I}$$

Substitution on $\to$-I

$$\dfrac{\dfrac{\mathcal{R}\Gamma, \rho x : A \vdash M : B}{\mathcal{R}\Gamma \vdash \lambda x.\ M : \rho A \to B}\ \to\text{-I} \qquad \mathcal{R}'\Delta \stackrel{\sigma}{\Rightarrow} \mathcal{R}\Gamma}{\mathcal{R}'\Delta \vdash \lambda x.\ M\{\sigma, x \mapsto x\} : \rho A \to B}\ \textsc{Subst}$$

$$S' := \begin{pmatrix} & & & 0 \\ & S & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} \mathcal{R} & \rho \end{pmatrix} S' = \begin{pmatrix} \mathcal{R}' & \rho \end{pmatrix}$$

$$\forall (z : C) \in \Gamma, x : A.\ \exists N_z.\ (\langle z | S')\Delta \vdash N_z : C$$

Cases:

- $\forall (z : C) \in \Gamma.\ \exists N_z.\ (\langle z | S)\Delta, 0x : A \vdash N_z : C$
- $\begin{pmatrix} 0 & \cdots & 0 \end{pmatrix} \Delta, 1x : A \vdash x : A$

Substitution on $\to$-I

$$\dfrac{\dfrac{\mathcal{R}\Gamma, \rho x : A \vdash M : B}{\mathcal{R}\Gamma \vdash \lambda x.\ M : \rho A \to B}\ \to\text{-I} \qquad \mathcal{R}'\Delta \overset{\sigma}{\Rightarrow} \mathcal{R}\Gamma}{\mathcal{R}'\Delta \vdash \lambda x.\ M\{\sigma, x \mapsto x\} : \rho A \to B}\ \textsc{Subst}$$

$$S' := \begin{pmatrix} & & & 0 \\ & S & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix}\mathcal{R} & \rho\end{pmatrix} S' = \begin{pmatrix}\mathcal{R}' & \rho\end{pmatrix}$$

$$\forall (z : C) \in \Gamma, x : A.\ \exists N_z.\ (\langle z | S' \rangle)\Delta \vdash N_z : C$$

Cases:

- $\forall (z : C) \in \Gamma.\ \exists N_z.\ (\langle z | S \rangle)\Delta, 0x : A \vdash N_z : C$
- $\begin{pmatrix}0 & \cdots & 0\end{pmatrix} \Delta, 1x : A \vdash x : A$

Substitution on $\to$-I

$$\dfrac{\dfrac{\mathcal{R}\Gamma, \rho x : A \vdash M : B}{\mathcal{R}\Gamma \vdash \lambda x.\ M : \rho A \to B}\ \to\text{-I} \qquad \mathcal{R}'\Delta \overset{\sigma}{\Rightarrow} \mathcal{R}\Gamma}{\mathcal{R}'\Delta \vdash \lambda x.\ M\{\sigma, x \mapsto x\} : \rho A \to B}\ \textsc{Subst}$$

$$S' := \begin{pmatrix} & & & 0 \\ & S & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix}\mathcal{R} & \rho\end{pmatrix} S' = \begin{pmatrix}\mathcal{R}' & \rho\end{pmatrix}$$

$$\forall (z : C) \in \Gamma, x : A.\ \exists N_z.\ (\langle z | S')\Delta \vdash N_z : C$$

Cases:

- $\forall (z : C) \in \Gamma.\ \exists N_z.\ (\langle z | S)\Delta, 0x : A \vdash N_z : C$
- $\begin{pmatrix}0 & \cdots & 0\end{pmatrix} \Delta, 1x : A \vdash x : A$

Substitution on $\to$-I

$$\cfrac{\cfrac{\mathcal{R}\Gamma, \rho x : A \vdash M : B}{\mathcal{R}\Gamma \vdash \lambda x.\ M : \rho A \to B}\ \to\text{-I} \qquad \mathcal{R}'\Delta \overset{\sigma}{\Rightarrow} \mathcal{R}\Gamma}{\mathcal{R}'\Delta \vdash \lambda x.\ M\{\sigma, x \mapsto x\} : \rho A \to B}\ \textsc{Subst}$$

$$S' := \begin{pmatrix} & & & 0 \\ & S & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix}\mathcal{R} & \rho\end{pmatrix} S' = \begin{pmatrix}\mathcal{R}' & \rho\end{pmatrix}$$

$$\forall (z : C) \in \Gamma, x : A.\ \exists N_z.\ (\langle z | S' \rangle)\Delta \vdash N_z : C$$

Cases:

- $\forall (z : C) \in \Gamma.\ \exists N_z.\ (\langle z | S \rangle)\Delta, 0x : A \vdash N_z : C$
- $\begin{pmatrix}0 & \cdots & 0\end{pmatrix} \Delta, 1x : A \vdash x : A$

Conclusions

- A notion of substitution that remains constant, even though contexts change.
- Lesson: don't remove things from scope; mark them depleted (0).
- Agda code at `https://github.com/laMudri/quantitative`
- Translation table:

Syntax	Linear algebra
Annotation context	Vector
VAR annotation	Basis vector
Weakening	Embedding into a space of higher dimension
A substitution	A linear map

Substitution on Var

$$\dfrac{\dfrac{(x : A) \in \Gamma \qquad \langle x| \trianglelefteq \mathcal{R}}{\mathcal{R}\Gamma \vdash x : A}\ \textsc{Var} \qquad \mathcal{R}'\Delta \overset{\sigma}{\Rightarrow} \mathcal{R}\Gamma}{\mathcal{R}'\Delta \vdash \sigma_x : A}\ \textsc{Subst}$$

$$\langle x|S \trianglelefteq \mathcal{R}S \trianglelefteq \mathcal{R}'$$

From σ, we have $(\langle x|S)\Delta \vdash \sigma_x : A$.

Single substitution

$$\frac{\mathcal{P}\Gamma, \rho x : A \vdash M : B \qquad \mathcal{Q}\Gamma \vdash N : A}{(\mathcal{P} + \rho\mathcal{Q})\Gamma \vdash M[N/x]} \ \textsc{SingleSubst}$$

$$\mathcal{P} + \rho\mathcal{Q} \xRightarrow{[N/x]} \mathcal{P}, \rho x$$

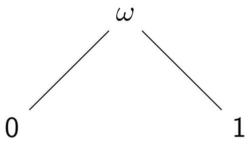
Need $\begin{pmatrix}\mathcal{P} & \rho\end{pmatrix} S \trianglelefteq \mathcal{P} + \rho\mathcal{Q}$

$$S := \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ \mathcal{Q}_1 & \mathcal{Q}_2 & \cdots & \mathcal{Q}_n \end{pmatrix}$$

DILL

$+$	0	1	ω
0	0	1	ω
1	1	ω	ω
ω	ω	ω	ω

$\cdot$	0	1	ω
0	0	0	0
1	0	1	ω
ω	0	ω	ω



$$\Gamma; \Delta \vdash M : A \leftrightsquigarrow \vec{\omega}\Gamma, \vec{1}\Delta, \vec{0}\Theta \vdash M : A$$

Pfenning, Davies

$0 =$ unused $\qquad\qquad 1 =$ true

$+$	u	t	v
u	u	t	v
t	t	t	v
v	v	v	v

$\cdot$	u	t	v
u	u	u	u
t	u	t	v
v	u	v	v