Linear metatheory via linear algebra

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SPLS, June 2019

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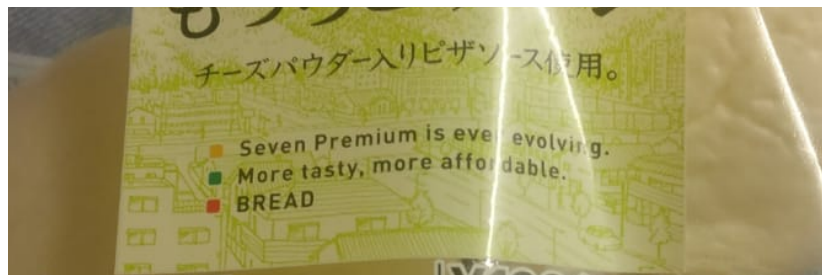
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- ▶ We should be able to replace variables by terms (substitution).
- ▶ $!A$ is like A, but can be discarded and duplicated.
- ▶ Substitution is really difficult!

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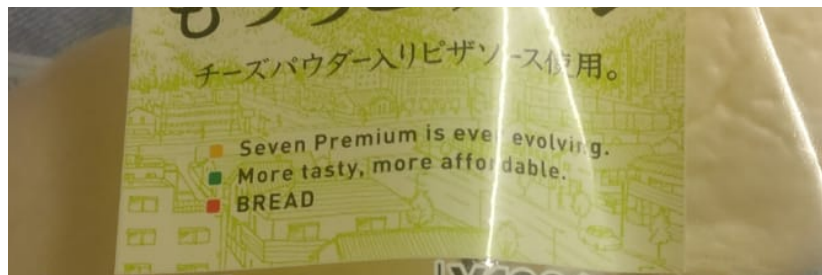
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- ▶ BREAD

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- Weakening admissible at 0 demand:
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- Examples:
    1. $1x : A, 0y : B \vdash x : A$ (linearity; DILL from Bar96)
    2. $\uparrow x : \mathbb{Z}, \downarrow y : \mathbb{Z} \vdash x - y : \mathbb{Z}$ (monotonicity; Arntzenius)
    3. $\text{prv} x : \text{Secret} \nvdash {}_- : {!}_{\text{pub}}\text{Secret}$ (privacy; DCC from ABHR99)

- Other related work: PD99; RP10; POM14; GS14; BGMZ14; the Granule project

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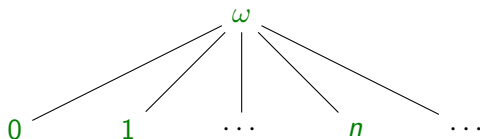 3. $\text{prv} x : \text{Secret} \nvdash \_ : !_{\text{pub}}\text{Secret}$ (privacy; DCC from ABHR99)

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Annotation posemiring

- ▶ **Annotations** form a partially ordered semiring.
- ▶ A *partially ordered semiring* $(\mathscr{R}, \trianglelefteq, 0, +, 1, \cdot)$ is:
    - ▶ A *partial order* $(\mathscr{R}, \trianglelefteq)$
    - ▶ A *semiring* $(\mathscr{R}, 0, +, 1, \cdot)$, which is:
        - ▶ A *commutative monoid* $(\mathscr{R}, 0, +)$
        - ▶ A *monoid* $(\mathscr{R}, 1, \cdot)$
        - ▶ Such that $\cdot$ distributes over 0 and $+$ on both sides
    - ▶ Such that $+$ and $\cdot$ are both monotonic in both arguments

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    - ► Such that $+$ and $\cdot$ are both monotonic in both arguments
- ► Ur-example: $(\mathbb{N} \cup \{\omega\}, \sqsupseteq, 0, +, 1, \times)$ "how many"

What are we dealing with?

$$\frac{\begin{array}{c}\mathcal{R}\Gamma \vdash M : A \\ \mathcal{R}\Gamma \vdash N : B\end{array}}{\mathcal{R}\Gamma \vdash (M, N)_{\&} : A \,\&\, B} \ \&\text{-I}$$

$$\frac{\begin{array}{c}\mathcal{P}\Gamma \vdash M : A \\ \mathcal{Q}\Gamma \vdash N : B \\ \mathcal{P} + \mathcal{Q} \preceq \mathcal{R}\end{array}}{\mathcal{R}\Gamma \vdash (M, N)_{\otimes} : A \otimes B} \ \otimes\text{-I}$$

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Simultaneous substitution

$$\text{Subst} \quad \frac{\Gamma \vdash M : A \qquad \begin{pmatrix} \Delta \vdash N_1 : \Gamma_1 \\ \vdots \\ \Delta \vdash N_m : \Gamma_m \end{pmatrix}}{\Delta \vdash M\{\vec{N}\} : A}$$

Simultaneous substitution

$$
\text{SUBST}
$$

$$
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- Wrong! Try identity substitution on
  $\mathcal{P}\Gamma = \mathcal{Q}\Delta = 1x : B, 1y : C.$
  $$\begin{pmatrix} 1x : B, 1y : C \nvdash x : B \\ 1x : B, 1y : C \nvdash y : C \end{pmatrix}$$

Simultaneous substitution

$$
\text{SUBST} \\
\dfrac{\mathcal{P}\Gamma \vdash M : A \qquad \begin{pmatrix} ?\Delta \vdash N_1 : \Gamma_1 \\ \vdots \\ ?\Delta \vdash N_m : \Gamma_m \end{pmatrix}}{\mathcal{Q}\Delta \vdash M\{\vec{N}\} : A}
$$

▶ We need to split $\mathcal{Q}$ up.

$$
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$$\begin{pmatrix} 1x : B, 0y : C \vdash x : B \\ 0x : B, 1y : C \vdash y : C \end{pmatrix}$$

▶ What is the relationship between $\mathcal{P}$, $\mathcal{Q}$, and S?

Substitution, stripped down

SUBST

$$\frac{\mathcal{P}\Gamma \vdash M : A \qquad \begin{pmatrix} S_1\Delta \vdash N_1 : \Gamma_1 \\ \vdots \\ S_m\Delta \vdash N_m : \Gamma_m \end{pmatrix}}{\mathcal{Q}\Delta \vdash M\{\vec{N}\} : A}$$

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$$\frac{\mathcal{P} \vdash M \quad \begin{pmatrix} S_1 \vdash N_1 \\ \vdots \\ S_m \vdash N_m \end{pmatrix}}{\mathcal{Q} \vdash M\{\vec{N}\}}$$

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$$\frac{\pi_1 x_1, \ldots, \pi_m x_m \vdash M \qquad \begin{pmatrix} \sigma_{1,1} y_1, \ldots, \sigma_{1,n} y_n \vdash N_1 \\ \vdots \\ \sigma_{m,1} y_1, \ldots, \sigma_{m,n} y_n \vdash N_m \end{pmatrix}}{\rho_1 y_1, \ldots, \rho_n y_n \vdash M\{\vec{N}\}}$$

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- What is the relationship between $\mathcal{P}$, $\mathcal{Q}$, and S?
- ρ_j is a weighted sum of $\sigma_{-,j}$ according to π_-.

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$$\begin{pmatrix} \pi_1 & \cdots & \pi_m \end{pmatrix} \begin{pmatrix} \sigma_{1,1} & \cdots & \sigma_{1,n} \\ \vdots & \ddots & \vdots \\ \sigma_{m,1} & \cdots & \sigma_{m,n} \end{pmatrix} \unlhd \begin{pmatrix} \rho_1 & \cdots & \rho_n \end{pmatrix}$$

What is a substitution?

$$\text{SUBST}$$

$$\frac{\pi_1 x_1, \ldots, \pi_m x_m \vdash M \qquad \begin{pmatrix} \sigma_{1,1} y_1, \ldots, \sigma_{1,n} y_n \vdash N_1 \\ \vdots \\ \sigma_{m,1} y_1, \ldots, \sigma_{m,n} y_n \vdash N_m \end{pmatrix}}{\rho_1 y_1, \ldots, \rho_n y_n \vdash M\{\vec{N}\}}$$

- Notation: $\langle x|$ stands for the basis vector at x.
- A simultaneous substitution $\mathcal{Q}\Delta \Rightarrow \mathcal{P}\Gamma$ is:
    - A $|\Gamma|$-by-$|\Delta|$ matrix S, such that
    - For each $x : A$ in Γ, a term N_x
      such that $(\langle x|S)\Delta \vdash N_x : A$
    - $\mathcal{P}S \trianglelefteq \mathcal{Q}$

Substitution on &-I

$$\dfrac{\dfrac{\mathcal{P}\Gamma \vdash M : A \qquad \mathcal{P}\Gamma \vdash N : B}{\mathcal{P}\Gamma \vdash (M, N)_{\&} : A \,\&\, B} \;\&\text{-I} \qquad \mathcal{Q}\Delta \xRightarrow{\sigma} \mathcal{P}\Gamma}{\mathcal{Q}\Delta \vdash (M\{\sigma\}, N\{\sigma\})_{\&} : A \,\&\, B} \;\text{Subst}$$

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Substitution on ⊗-I

$$\cfrac{\cfrac{\mathcal{P}\Gamma \vdash M : A \qquad \mathcal{Q}\Gamma \vdash N : B \qquad \mathcal{P} + \mathcal{Q} \trianglelefteq \mathcal{R}}{\mathcal{R}\Gamma \vdash (M, N)_{\otimes} : A \otimes B} \; \otimes\text{-I} \qquad \mathcal{R}'\Delta \overset{\sigma}{\Rightarrow} \mathcal{R}\Gamma}{\mathcal{R}'\Delta \vdash (M\{\sigma\}, N\{\sigma\})_{\otimes} : A \otimes B} \; \textsc{Subst}$$

$\mathcal{P}' := \mathcal{P}S \qquad \mathcal{Q}' := \mathcal{Q}S$

$\mathcal{P}'\Delta \overset{\sigma}{\Rightarrow} \mathcal{P}\Gamma \qquad \mathcal{Q}'\Delta \overset{\sigma}{\Rightarrow} \mathcal{Q}\Gamma$

$\mathcal{P}' + \mathcal{Q}' = \mathcal{P}S + \mathcal{Q}S = (\mathcal{P} + \mathcal{Q})S \trianglelefteq \mathcal{R}S \trianglelefteq \mathcal{R}'$

$$\cfrac{\mathcal{P}'\Delta \vdash M\{\sigma\} : A \qquad \mathcal{Q}'\Delta \vdash N\{\sigma\} : B \qquad \mathcal{P}' + \mathcal{Q}' \trianglelefteq \mathcal{R}'}{\mathcal{R}'\Delta \vdash (M\{\sigma\}, N\{\sigma\})_{\otimes} : A \otimes B} \; \otimes\text{-I}$$

Substitution on $\otimes$-I

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$\mathcal{P}'\Delta \overset{\sigma}{\Rightarrow} \mathcal{P}\Gamma \quad \mathcal{Q}'\Delta \overset{\sigma}{\Rightarrow} \mathcal{Q}\Gamma$

$\mathcal{P}' + \mathcal{Q}' = \mathcal{P}S + \mathcal{Q}S = (\mathcal{P} + \mathcal{Q})S \trianglelefteq \mathcal{R}S \trianglelefteq \mathcal{R}'$

$$\dfrac{\mathcal{P}'\Delta \vdash M\{\sigma\} : A \qquad \mathcal{Q}'\Delta \vdash N\{\sigma\} : B \qquad \mathcal{P}' + \mathcal{Q}' \trianglelefteq \mathcal{R}'}{\mathcal{R}'\Delta \vdash (M\{\sigma\}, N\{\sigma\})_{\otimes} : A \otimes B} \, {\otimes\text{-I}}$$

Substitution on ⊗-I

$$\dfrac{\dfrac{\begin{array}{cc}\mathcal{P}\Gamma \vdash M : A & \mathcal{Q}\Gamma \vdash N : B\\ \mathcal{P} + \mathcal{Q} \trianglelefteq \mathcal{R}\end{array}}{\mathcal{R}\Gamma \vdash (M, N)_{\otimes} : A \otimes B} \; \otimes\text{-I} \qquad \mathcal{R}'\Delta \overset{\sigma}{\Rightarrow} \mathcal{R}\Gamma}{\mathcal{R}'\Delta \vdash (M\{\sigma\}, N\{\sigma\})_{\otimes} : A \otimes B} \; \textsc{Subst}$$

$\mathcal{P}' := \mathcal{P}S \qquad \mathcal{Q}' := \mathcal{Q}S$
$\mathcal{P}'\Delta \overset{\sigma}{\Rightarrow} \mathcal{P}\Gamma \qquad \mathcal{Q}'\Delta \overset{\sigma}{\Rightarrow} \mathcal{Q}\Gamma$

$\mathcal{P}' + \mathcal{Q}' = \mathcal{P}S + \mathcal{Q}S = (\mathcal{P} + \mathcal{Q})S \trianglelefteq \mathcal{R}S \trianglelefteq \mathcal{R}'$

$$\dfrac{\begin{array}{cc}\mathcal{P}'\Delta \vdash M\{\sigma\} : A & \mathcal{Q}'\Delta \vdash N\{\sigma\} : B\\ \mathcal{P}' + \mathcal{Q}' \trianglelefteq \mathcal{R}'\end{array}}{\mathcal{R}'\Delta \vdash (M\{\sigma\}, N\{\sigma\})_{\otimes} : A \otimes B} \; \otimes\text{-I}$$

Substitution on $\otimes$-I

$$\dfrac{\dfrac{\mathcal{P}\Gamma \vdash M : A \qquad \mathcal{Q}\Gamma \vdash N : B \qquad \mathcal{P} + \mathcal{Q} \trianglelefteq \mathcal{R}}{\mathcal{R}\Gamma \vdash (M, N)_{\otimes} : A \otimes B} \ \otimes\text{-I} \qquad \mathcal{R}'\Delta \overset{\sigma}{\Rightarrow} \mathcal{R}\Gamma}{\mathcal{R}'\Delta \vdash (M\{\sigma\}, N\{\sigma\})_{\otimes} : A \otimes B} \ \textsc{Subst}$$

$\mathcal{P}' := \mathcal{P}S \qquad \mathcal{Q}' := \mathcal{Q}S$
$\mathcal{P}'\Delta \overset{\sigma}{\Rightarrow} \mathcal{P}\Gamma \qquad \mathcal{Q}'\Delta \overset{\sigma}{\Rightarrow} \mathcal{Q}\Gamma$

$\mathcal{P}' + \mathcal{Q}' = \mathcal{P}S + \mathcal{Q}S = (\mathcal{P} + \mathcal{Q})S \trianglelefteq \mathcal{R}S \trianglelefteq \mathcal{R}'$

$$\dfrac{\mathcal{P}'\Delta \vdash M\{\sigma\} : A \qquad \mathcal{Q}'\Delta \vdash N\{\sigma\} : B \qquad \mathcal{P}' + \mathcal{Q}' \trianglelefteq \mathcal{R}'}{\mathcal{R}'\Delta \vdash (M\{\sigma\}, N\{\sigma\})_{\otimes} : A \otimes B} \ \otimes\text{-I}$$

Substitution on $\otimes$-I

$$\dfrac{\dfrac{\mathcal{P}\Gamma \vdash M : A \qquad \mathcal{Q}\Gamma \vdash N : B \qquad \mathcal{P} + \mathcal{Q} \trianglelefteq \mathcal{R}}{\mathcal{R}\Gamma \vdash (M, N)_\otimes : A \otimes B} \otimes\text{-I} \qquad \mathcal{R}'\Delta \overset{\sigma}{\Rightarrow} \mathcal{R}\Gamma}{\mathcal{R}'\Delta \vdash (M\{\sigma\}, N\{\sigma\})_\otimes : A \otimes B} \;\text{Subst}$$

$$\mathcal{P}' := \mathcal{P}S \qquad \mathcal{Q}' := \mathcal{Q}S$$
$$\mathcal{P}'\Delta \overset{\sigma}{\Rightarrow} \mathcal{P}\Gamma \qquad \mathcal{Q}'\Delta \overset{\sigma}{\Rightarrow} \mathcal{Q}\Gamma$$

$$\mathcal{P}' + \mathcal{Q}' = \mathcal{P}S + \mathcal{Q}S = (\mathcal{P} + \mathcal{Q})S \trianglelefteq \mathcal{R}S \trianglelefteq \mathcal{R}'$$

$$\dfrac{\mathcal{P}'\Delta \vdash M\{\sigma\} : A \qquad \mathcal{Q}'\Delta \vdash N\{\sigma\} : B \qquad \mathcal{P}' + \mathcal{Q}' \trianglelefteq \mathcal{R}'}{\mathcal{R}'\Delta \vdash (M\{\sigma\}, N\{\sigma\})_\otimes : A \otimes B} \otimes\text{-I}$$

Substitution on $\otimes$-I

$$\dfrac{\dfrac{\mathcal{P}\Gamma \vdash M : A \qquad \mathcal{Q}\Gamma \vdash N : B \qquad \mathcal{P} + \mathcal{Q} \trianglelefteq \mathcal{R}}{\mathcal{R}\Gamma \vdash (M, N)_{\otimes} : A \otimes B} \otimes\text{-I} \qquad \mathcal{R}'\Delta \overset{\sigma}{\Rightarrow} \mathcal{R}\Gamma}{\mathcal{R}'\Delta \vdash (M\{\sigma\}, N\{\sigma\})_{\otimes} : A \otimes B} \text{ Subst}$$

$\mathcal{P}' := \mathcal{P}S \qquad \mathcal{Q}' := \mathcal{Q}S$
$\mathcal{P}'\Delta \overset{\sigma}{\Rightarrow} \mathcal{P}\Gamma \quad \mathcal{Q}'\Delta \overset{\sigma}{\Rightarrow} \mathcal{Q}\Gamma$

$\mathcal{P}' + \mathcal{Q}' = \mathcal{P}S + \mathcal{Q}S = (\mathcal{P} + \mathcal{Q})S \trianglelefteq \mathcal{R}S \trianglelefteq \mathcal{R}'$

$$\dfrac{\mathcal{P}'\Delta \vdash M\{\sigma\} : A \qquad \mathcal{Q}'\Delta \vdash N\{\sigma\} : B \qquad \mathcal{P}' + \mathcal{Q}' \trianglelefteq \mathcal{R}'}{\mathcal{R}'\Delta \vdash (M\{\sigma\}, N\{\sigma\})_{\otimes} : A \otimes B} \otimes\text{-I}$$

Substitution on ⊗-I

$$\dfrac{\begin{array}{c} \mathcal{P}\Gamma \vdash M : A \qquad \mathcal{Q}\Gamma \vdash N : B \\ \mathcal{P} + \mathcal{Q} \unlhd \mathcal{R} \\ \hline \mathcal{R}\Gamma \vdash (M,N)_{\otimes} : A \otimes B \end{array} \ \otimes\text{-I} \qquad \mathcal{R}'\Delta \overset{\sigma}{\Rightarrow} \mathcal{R}\Gamma}{\mathcal{R}'\Delta \vdash (M\{\sigma\}, N\{\sigma\})_{\otimes} : A \otimes B} \ \textsc{Subst}$$

$\mathcal{P}' := \mathcal{P}S \qquad \mathcal{Q}' := \mathcal{Q}S$

$\mathcal{P}'\Delta \overset{\sigma}{\Rightarrow} \mathcal{P}\Gamma \quad \mathcal{Q}'\Delta \overset{\sigma}{\Rightarrow} \mathcal{Q}\Gamma$

$\mathcal{P}' + \mathcal{Q}' = \mathcal{P}S + \mathcal{Q}S = (\mathcal{P}+\mathcal{Q})S \unlhd \mathcal{R}S \unlhd \mathcal{R}'$

$$\dfrac{\begin{array}{c} \mathcal{P}'\Delta \vdash M\{\sigma\} : A \qquad \mathcal{Q}'\Delta \vdash N\{\sigma\} : B \\ \mathcal{P}' + \mathcal{Q}' \unlhd \mathcal{R}' \end{array}}{\mathcal{R}'\Delta \vdash (M\{\sigma\}, N\{\sigma\})_{\otimes} : A \otimes B} \ \otimes\text{-I}$$

Substitution on $!_\rho$-I

$$\dfrac{\dfrac{\mathcal{P}\Gamma \vdash M : A \qquad \rho\mathcal{P} \trianglelefteq \mathcal{R}}{\mathcal{R}\Gamma \vdash [M] : !_\rho A} \; !_\rho\text{-I} \qquad \mathcal{R}'\Delta \xRightarrow{\sigma} \mathcal{R}\Gamma}{\mathcal{R}'\Delta \vdash [M\{\sigma\}] : !_\rho A} \; \text{SUBST}$$

$\mathcal{P}' := \mathcal{P}S$
$\mathcal{P}'\Delta \xRightarrow{\sigma} \mathcal{P}\Gamma$

$\rho\mathcal{P}' = \rho(\mathcal{P}S) = (\rho\mathcal{P})S \trianglelefteq \mathcal{R}S \trianglelefteq \mathcal{R}'$

$$\dfrac{\mathcal{P}'\Delta \vdash M\{\sigma\} : A \qquad \rho\mathcal{P}' \trianglelefteq \mathcal{R}'}{\mathcal{R}'\Delta \vdash [M\{\sigma\}] : !_\rho A} \; !_\rho\text{-I}$$

$$\dfrac{\dfrac{\mathcal{P}\Gamma \vdash M : A \qquad \rho\mathcal{P} \trianglelefteq \mathcal{R}}{\mathcal{R}\Gamma \vdash [M] : !_\rho A} \; !_\rho\text{-I} \qquad \mathcal{R}'\Delta \overset{\sigma}{\Rightarrow} \mathcal{R}\Gamma}{\mathcal{R}'\Delta \vdash [M\{\sigma\}] : !_\rho A} \; \text{Subst}$$

$\mathcal{P}' := \mathcal{P}S$
$\mathcal{P}'\Delta \overset{\sigma}{\Rightarrow} \mathcal{P}\Gamma$

$\rho\mathcal{P}' = \rho(\mathcal{P}S) = (\rho\mathcal{P})S \trianglelefteq \mathcal{R}S \trianglelefteq \mathcal{R}'$

$$\dfrac{\mathcal{P}'\Delta \vdash M\{\sigma\} : A \qquad \rho\mathcal{P}' \trianglelefteq \mathcal{R}'}{\mathcal{R}'\Delta \vdash [M\{\sigma\}] : !_\rho A} \; !_\rho\text{-I}$$

Substitution on $!_\rho$-I

$$\dfrac{\dfrac{\mathcal{P}\Gamma \vdash M : A \qquad \rho\mathcal{P} \trianglelefteq \mathcal{R}}{\mathcal{R}\Gamma \vdash [M] : !_\rho A} \ !_\rho\text{-I} \qquad \mathcal{R}'\Delta \overset{\sigma}{\Rightarrow} \mathcal{R}\Gamma}{\mathcal{R}'\Delta \vdash [M\{\sigma\}] : !_\rho A} \ \text{Subst}$$

$\mathcal{P}' := \mathcal{P}S$

$\mathcal{P}'\Delta \overset{\sigma}{\Rightarrow} \mathcal{P}\Gamma$

$\rho\mathcal{P}' = \rho(\mathcal{P}S) = (\rho\mathcal{P})S \trianglelefteq \mathcal{R}S \trianglelefteq \mathcal{R}'$

$$\dfrac{\mathcal{P}'\Delta \vdash M\{\sigma\} : A \qquad \rho\mathcal{P}' \trianglelefteq \mathcal{R}'}{\mathcal{R}'\Delta \vdash [M\{\sigma\}] : !_\rho A} \ !_\rho\text{-I}$$

Substitution on →-I

$$\frac{\dfrac{\mathcal{R}\Gamma, \rho x : A \vdash M : B}{\mathcal{R}\Gamma \vdash \lambda x.\ M : \rho A \to B} \text{ →-I} \qquad \mathcal{R}'\Delta \overset{\sigma}{\Rightarrow} \mathcal{R}\Gamma}{\mathcal{R}'\Delta \vdash \lambda x.\ M\{\sigma, x \mapsto x\} : \rho A \to B} \text{ SUBST}$$

$$S' := \begin{pmatrix} & & & 0 \\ & S & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix} \qquad (\mathcal{R} \quad \rho)\, S' = (\mathcal{R}' \quad \rho)$$

$$\forall (z : C) \in \Gamma, x : A.\ \exists N_z.\ (\langle z|S'\rangle \Delta \vdash N_z : C)$$

Cases:

▶ $\forall (z : C) \in \Gamma.\ \exists N_z.\ (\langle z|S\rangle \Delta, 0x : A \vdash N_z : C)$

▶ $(0 \quad \cdots \quad 0)\, \Delta, 1x : A \vdash x : A$

Substitution on →-I

$$\dfrac{\dfrac{\mathcal{R}\Gamma, \rho x : A \vdash M : B}{\mathcal{R}\Gamma \vdash \lambda x.\ M : \rho A \to B} \ \to\text{-I} \qquad \mathcal{R}'\Delta \overset{\sigma}{\Rightarrow} \mathcal{R}\Gamma}{\mathcal{R}'\Delta \vdash \lambda x.\ M\{\sigma, x \mapsto x\} : \rho A \to B} \ \textsc{Subst}$$

$$S' := \begin{pmatrix} & & & 0 \\ & S & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} \mathcal{R} & \rho \end{pmatrix} S' = \begin{pmatrix} \mathcal{R}' & \rho \end{pmatrix}$$

$$\forall (z : C) \in \Gamma, x : A.\ \exists N_z.\ (\langle z | S' \rangle \Delta \vdash N_z : C)$$

Cases:

- $\forall (z : C) \in \Gamma.\ \exists N_z.\ (\langle z | S \rangle \Delta, 0x : A \vdash N_z : C)$
- $\begin{pmatrix} 0 & \cdots & 0 \end{pmatrix} \Delta, 1x : A \vdash x : A$

Substitution on →-I

$$\dfrac{\dfrac{\mathcal{R}\Gamma, \rho x : A \vdash M : B}{\mathcal{R}\Gamma \vdash \lambda x.\ M : \rho A \to B} \to\text{-I} \qquad \mathcal{R}'\Delta \overset{\sigma}{\Rightarrow} \mathcal{R}\Gamma}{\mathcal{R}'\Delta \vdash \lambda x.\ M\{\sigma, x \mapsto x\} : \rho A \to B}\ \textsc{Subst}$$

$$S' := \begin{pmatrix} & & & 0 \\ & S & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} \mathcal{R} & \rho \end{pmatrix} S' = \begin{pmatrix} \mathcal{R}' & \rho \end{pmatrix}$$

$$\forall (z : C) \in \Gamma, x : A.\ \exists N_z.\ (\langle z|S'\rangle\Delta \vdash N_z : C)$$

Cases:

▶ $\forall (z : C) \in \Gamma.\ \exists N_z.\ (\langle z|S\rangle\Delta, 0x : A \vdash N_z : C$

▶ $\begin{pmatrix} 0 & \cdots & 0 \end{pmatrix} \Delta, 1x : A \vdash x : A$

Substitution on →-I

$$\frac{\dfrac{\mathcal{R}\Gamma, \rho x : A \vdash M : B}{\mathcal{R}\Gamma \vdash \lambda x.\ M : \rho A \to B} \to\text{-I} \qquad \mathcal{R}'\Delta \overset{\sigma}{\Rightarrow} \mathcal{R}\Gamma}{\mathcal{R}'\Delta \vdash \lambda x.\ M\{\sigma, x \mapsto x\} : \rho A \to B} \ \textsc{Subst}$$

$$S' := \begin{pmatrix} & & & 0 \\ & S & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} \mathcal{R} & \rho \end{pmatrix} S' = \begin{pmatrix} \mathcal{R}' & \rho \end{pmatrix}$$

$$\forall (z : C) \in \Gamma, x : A.\ \exists N_z.\ (\langle z | S' \rangle \Delta \vdash N_z : C)$$

Cases:

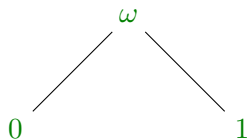- $\forall (z : C) \in \Gamma.\ \exists N_z.\ (\langle z | S \rangle \Delta, 0x : A \vdash N_z : C)$
- $\begin{pmatrix} 0 & \cdots & 0 \end{pmatrix} \Delta, 1x : A \vdash x : A$

DILL

+	0	1	ω
0	0	1	ω
1	1	ω	ω
ω	ω	ω	ω

$\cdot$	0	1	ω
0	0	0	0
1	0	1	ω
ω	0	ω	ω

$$\Gamma; \Delta \vdash M : A \iff \vec{\omega}\Gamma, \vec{1}\Delta, \vec{0}\Theta \vdash M : A$$

Pfenning, Davies

$$0 = \text{unused} \qquad\qquad 1 = \text{true}$$

+	u	t	v
u	u	t	v
t	t	t	v
v	v	v	v

·	u	t	v
u	u	u	u
t	u	t	v
v	u	v	v

valid

|

true

|

unused

Conclusions

- A notion of substitution that remains constant, even though contexts change.
- Lesson: don't remove things from scope; mark them depleted (0).
- Agda code at
  `https://github.com/laMudri/quantitative`
- Translation table:

Syntax	Linear algebra
Annotation context	Vector
VAR annotation	Basis vector
Weakening	Embedding into a space of higher dimension
A substitution	A linear map

Substitution on VAR

$$\dfrac{\dfrac{(x : A) \in \Gamma \qquad \langle x| \trianglelefteq \mathcal{R}}{\mathcal{R}\Gamma \vdash x : A} \; \textsc{Var} \qquad \mathcal{R}'\Delta \overset{\sigma}{\Rightarrow} \mathcal{R}\Gamma}{\mathcal{R}'\Delta \vdash \sigma_x : A} \; \textsc{Subst}$$

$\langle x|S \trianglelefteq \mathcal{R}S \trianglelefteq \mathcal{R}'$

From σ, we have $(\langle x|S)\Delta \vdash \sigma_x : A$.

Single substitution

$$\frac{\mathcal{P}\Gamma, \rho x : A \vdash M : B \qquad \mathcal{Q}\Gamma \vdash N : A}{(\mathcal{P} + \rho\mathcal{Q})\Gamma \vdash M[N/x]} \ \text{SingleSubst}$$

$$\mathcal{P} + \rho\mathcal{Q} \xrightarrow{[N/x]} \mathcal{P}, \rho x$$

Need $\begin{pmatrix} \mathcal{P} & \rho \end{pmatrix} S \unlhd \mathcal{P} + \rho\mathcal{Q}$

$$S := \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ \mathcal{Q}_1 & \mathcal{Q}_2 & \cdots & \mathcal{Q}_n \end{pmatrix}$$