Context Constrained Computation

via a linear-like lambda calculus

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Motivation

- Constrain how variables are used

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- Generalise the "how many" of linear typing
  - At what security level? – information flow
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  - In which direction? – monotonicity
- Formalised in Agda – the free theorems of the object language are available to Agda programs

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Free theorem

sort is a permutation

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Free theorem

f is monotonic

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- Bidirectional layered typing
  - Well scoped terms $e : \mathrm{Term}\ n\ \mathrm{syn}$ and $s : \mathrm{Term}\ n\ \mathrm{chk}$
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  - Either: $tt : (\Gamma \vdash t : S)$
  - Resourcing: $tr : (\Delta \vdash tt)$
  - Abbreviations $\Delta^{\Gamma} \vdash e \in S$, $\Delta^{\Gamma} \vdash S \ni s$, etc.

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With	Tensor
$A \mathbin{\&} B$	$A \otimes B$

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\mathit{curry} &: (A \multimap B \multimap C) \multimap (A \otimes B \multimap C)
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With product

Negative type

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Introduction

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    $\implies$ all functions are permutations
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    $\implies$ all functions are non-expansive
  - Worlds are sets of security levels, semiring same
    $\implies$ high security data do not interfere with low security data

Conclusion

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- Reed, Pierce 2010 – Distance Makes the Types Grow Stronger
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- Staged computation?
- More problems?