# Centered Categories: Categorical Models for Type Theories with Determinisitic Reduction

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## 1 Introduction

Consider how we interpret the application rule for STLC in a CCC $C$:

$$\frac{\Gamma \vdash t_2 : A \qquad \Gamma \vdash t_1 : A \Rightarrow B}{\Gamma \vdash t_1\, t_2 : B}\ \textsc{App}$$

First assume we have the following morphisms in $\mathbb{C}$:

$$[\![\Gamma]\!] \xrightarrow{t_1} A \Rightarrow B \qquad [\![\Gamma]\!] \xrightarrow{t_2} A$$

Then we must define a new morphism $t \in \mathbb{C}([\![\Gamma]\!], B)$. We can accomplish this in the following way:

$$[\![\Gamma]\!] \xrightarrow{(t_1, t_2)} (A \Rightarrow B) \times A \xrightarrow{\mathsf{app}\, A\, B} B$$

Where $\mathsf{app}$ is the evaluator for exponentials. Now suppose $t_1 \equiv \lambda x.t$ then the in the type theory $t_1\, t_2 \equiv [t_2/x]t$. We know the morphism for $t_1\, t_2$, but what is the morphism for $[t_2/x]t$? Substitution amounts to essentially composition. Recalling that $\lambda$-abstractions are modeled by curring the previous term is modled by the following:

$$[\![\Gamma]\!] \xrightarrow{(\mathsf{id}_{[\![\Gamma]\!]}, t_2)} [\![\Gamma]\!] \times A \xrightarrow{\mathsf{cur}^{-1} \circ \mathsf{cur}\, t} B$$

Since $\mathsf{cur}$ is an isomorphism the previous diagram is equivalent to the following:

$$[\![\Gamma]\!] \xrightarrow{(\mathsf{id}_{[\![\Gamma]\!]}, t_2)} [\![\Gamma]\!] \times A \xrightarrow{t} B$$

In the previous morphism there are no restrictions on what $t_2$ can be. Thus, this corresponds to full $\beta\eta$-reduction. In order for this to be the correct model the equality $\mathsf{app}\, A\, B \circ (\mathsf{cur}\, t, t_2) = t \circ (\mathsf{id}_{[\![\Gamma]\!]}, t_2)$ must hold. To prove this we must show that the following diagram commutes:

$$\begin{tikzcd}
{[\![\Gamma]\!]} \arrow[r, "{(\mathsf{cur}\, t, t_2)}"] \arrow[d, "{(\mathsf{id}_{[\![\Gamma]\!]}, t_2)}"'] & (A \Rightarrow B) \times A \arrow[d, "\mathsf{app}\, A\, B"] \\
{[\![\Gamma]\!] \times A} \arrow[r, "t"] \arrow[ur, "\mathsf{cur}\, t \times id_A"] & B
\end{tikzcd}$$

The UMP for exponentials gives us

$$\mathsf{app}\, A\, B \circ \mathsf{cur}\, t \times id_A \quad = \quad t$$

Now

$$(\mathsf{app}\, A\, B \circ \mathsf{cur}\, t \times id_A) \circ (\mathsf{id}_{[\![\Gamma]\!]}, t_2) \quad = \quad \mathsf{app}\, A\, B \circ (\mathsf{cur}\, t \times id_A \circ (\mathsf{id}_{[\![\Gamma]\!]}, t_2))$$

To finish this proof we must show that

$$\mathsf{cur}\, t \times id_A \circ (\mathsf{id}_{[\![\Gamma]\!]}, t_2) = (\mathsf{cur}\, t, t_2).$$

This follows from the following lemma.

**Lemma 1.** *Suppose $\mathbb{C}$ is a CCC. Then for any $f \in \mathbb{C}[A, C]$, $g \in \mathbb{C}[B, D]$, $h \in \mathbb{C}[J, A]$, and $j \in \mathbb{C}[J, B]$, we have, $f \times g \circ (h, j) = (f \circ h, g \circ j)$.*

*Proof.* Consider the following diagram:

$$\begin{tikzcd}
 & J \arrow[dl, "f \circ h"'] \arrow[d, dashed, "f \times g \circ (h,j)"] \arrow[dr, "g \circ j"] & \\
C & C \times D \arrow[l, "\mathsf{proj}_1\, C\, D"] \arrow[r, "\mathsf{proj}_2\, C\, D"'] & D
\end{tikzcd}$$

The UMP for products insures the previous diagram commutes. Furthermore, by uniqueness it is the case that $f \times g \circ (h, j) = (f \circ h, g \circ j)$. □

Therefore, the equality $\mathsf{app}\, A\, B \circ (\mathsf{cur}\, t, t_2) = t \circ (\mathsf{id}_{[\![\Gamma]\!]}, t_2)$ does indeed hold. This tells us that equality in the model is $\beta\eta$-equality. Thus, in order to model a different notion of equality we must modify the equality in the category.

## 2 Centered Category

**Definition 2.** *A centered category $\mathbb{C}^T$ is a pair $(\mathbb{C}, \mathbb{V}, \top, \bot, \times, +, \Rightarrow)$ such that*

*i.* $(\mathbb{C}, \top, \times)$ *and* $(\mathbb{C}, \bot, +)$ *are symmetric monoidal categories,*

*ii.* $\mathbb{C}$ *contains the following morphisms:*

*a. Units:*

$$\bot \xrightarrow{\mathsf{contr}\, A} A \qquad\qquad A \xrightarrow{\mathsf{unit}\, A} \top$$

*b. Products:*

$$A \times B \xrightarrow{\mathsf{proj}_1\, A\, B} A \qquad\qquad A \times B \xrightarrow{\mathsf{proj}_2\, A\, B} B$$

$$\begin{tikzcd}
 & C \arrow[dl, "f_1"'] \arrow[d, dashed, "(f_1, f_2)!"] \arrow[dr, "f_2"] & \\
A & A \times B \arrow[l, "\mathsf{proj}_1\, A\, B"] \arrow[r, "\mathsf{proj}_2\, A\, B"'] & B.
\end{tikzcd}$$

*In addition we define the functor $- \times - : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$:*

$$\begin{array}{rll}
\textit{Objects:} & (- \times -)(A, B) & := \quad A \times B \\
\textit{Morphisms:} & (- \times -)(f : A \to C, g : B \to D) & := \quad (f \circ \mathsf{proj}_1\, A\, B, g \circ \mathsf{proj}_2\, A\, B)
\end{array}$$

*In the sequel the above functor will always be applied infix.*

*c. Coproducts:*

$$A \xrightarrow{\mathsf{inj}_1\, A\, B} A + B \qquad\qquad B \xrightarrow{\mathsf{inj}_2\, A\, B} A + B$$

$$\begin{tikzcd}
 & C & \\
A \arrow[ur, "f_1"] \arrow[r, "\mathsf{inj}_1\, A\, B"'] & A + B \arrow[u, dashed, "{[f_1, f_2]!}"] & B. \arrow[ul, "f_2"'] \arrow[l, "\mathsf{inj}_2\, A\, B"]
\end{tikzcd}$$

*d. Exponentials:*

*For any two objects A and B in* $\mathbb{C}$ *there is an object* $A \Rightarrow B$ *and an arrow* $\mathsf{app}\, A\, B : (A \Rightarrow B) \times A \to B$ *called the evaluator. The evaluator must satisfy the universal property: for any object A and arrow* $f : A \times B \to C$*, there is a unique arrow,* $\mathsf{cur} f : A \to B \Rightarrow C$ *such that the following diagram commutes:*

$$\begin{tikzcd} & C & \\ A \times B \arrow[ur, "f"] \arrow[rr, dashed, "f^* \times id_B"] & & C^B \times B \arrow[ul, "\epsilon"'] \end{tikzcd}$$

# References


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