# Centered Categories: Categorical Models for Type Theories with Determinisitic Reduction

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#### 1 Introduction

#### 2 Equality in CCCs

Consider how we interpret the application rule for STLC in a CCC C:

$$\frac{\Gamma \vdash t_2 : A \qquad \Gamma \vdash t_1 : A \Rightarrow B}{\Gamma \vdash t_1 t_2 : B} \text{ App}$$

First assume we have the following morphisms in  $\mathbb{C}$ :

wing morphisms in 
$$\mathbb{C}$$
:
$$\llbracket \Gamma \rrbracket \xrightarrow{t_1} A \Rightarrow B \qquad \llbracket \Gamma \rrbracket \xrightarrow{t_2} A$$

Then we must define a new morphism  $t \in \mathbb{C}(\llbracket \Gamma \rrbracket, B)$ . We can accomplish this in the following way:

$$\llbracket \Gamma \rrbracket \xrightarrow{(t_1, t_2)} (A \Rightarrow B) \times A \xrightarrow{\mathsf{app} \, A \, B} B$$

Where app is the evaluator for exponentials. Now suppose  $t_1 \equiv \lambda x.t$  then the in the type theory  $t_1 t_2 \equiv [t_2/x]t$ . We know the morphism for  $t_1 t_2$ , but what is the morphism for  $[t_2/x]t$ ? Substitution amounts to essentially composition. Recalling that  $\lambda$ -abstractions are modeled by curring the previous term is modled by the following:

$$\llbracket \Gamma \rrbracket \xrightarrow{(\mathsf{id}_{\llbracket \Gamma \rrbracket}, t_2)} \llbracket \Gamma \rrbracket \times A \xrightarrow{\mathsf{cur}^{-1} \circ \mathsf{cur}\, t} B$$

Since cur is an isomorphism the previous diagram is equivalent to the following:

$$\llbracket \Gamma \rrbracket \xrightarrow{(\mathsf{id}_{\llbracket \Gamma \rrbracket}, t_2)} \llbracket \Gamma \rrbracket \times A \xrightarrow{t} B$$

In the previous morphism there are no restrictions on what  $t_2$  can be. Thus, this corresponds to full  $\beta\eta$ reduction. In order for this to be the correct model the equality  $\operatorname{app} AB \circ (\operatorname{cur} t, t_2) = t \circ (\operatorname{id}_{\llbracket\Gamma\rrbracket}, t_2)$ . To prove
this we must show that the following diagram commutes:

The UMP for exponentials gives us

$$\operatorname{app} A B \circ \operatorname{cur} t \times id_A = t$$

Now

$$(\operatorname{\mathsf{app}} AB \circ \operatorname{\mathsf{cur}} t \times id_A) \circ (\operatorname{\mathsf{id}}_{\llbracket \Gamma \rrbracket}, t_2) = \operatorname{\mathsf{app}} AB \circ (\operatorname{\mathsf{cur}} t \times id_A \circ (\operatorname{\mathsf{id}}_{\llbracket \Gamma \rrbracket}, t_2))$$

To finish this proof we must show that

$$\operatorname{cur} t \times id_A \circ (\operatorname{id}_{\llbracket \Gamma \rrbracket}, t_2) = (\operatorname{cur} t, t_2).$$

This follows from the following lemma.

## **3** Centered Category

**Definition 1.** A centered category  $\mathbb{C}^T$  is a pair  $(\mathbb{C}, \mathbb{V}, \top, \bot, \times, +, \Rightarrow)$  such that

i.  $(\mathbb{C}, \top, \times)$  and  $(\mathbb{C}, \bot, +)$  are symmetric monoidal categories,

ii.  $\mathbb{C}$  contains the following morphisms:

a. Units:

$$\downarrow \frac{\text{contr } A}{A}$$
  $A = \frac{\text{unit } A}{A}$ 

b. Products:

$$A \times B \xrightarrow{\mathsf{proj}_1 A B} A \qquad A \times B \xrightarrow{\mathsf{proj}_2 A B} B$$

$$A \times B \xrightarrow{\mathsf{proj}_1 A B} A \times B \xrightarrow{\mathsf{proj}_2 A B} B.$$

*In addition we define the functor*  $- \times - : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ *:* 

Objects: 
$$(-\times -)(A, B)$$
 :=  $A \times B$ 

*Morphisms*: 
$$(-\times -)(f:A\to C,g:B\to D)$$
 :=  $(f\circ \operatorname{proj}_1 AB,g\circ \operatorname{proj}_2 AB)$ 

In the sequel the above functor will always be applied infix.

c. Coproducts:

$$A \xrightarrow{\text{inj}_1 A B} A + B$$

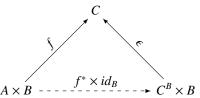
$$B \xrightarrow{\text{inj}_2 A B} A + B$$

$$[f_1, f_2]!$$

$$A \xrightarrow{\text{inj}_1 A B} A + B \xrightarrow{\text{inj}_2 A B} B.$$

d. Exponentials:

For any two objects A and B in  $\mathbb C$  there is an object  $A\Rightarrow B$  and an arrow  $\operatorname{app} AB:(A\Rightarrow B)\times A\to B$  called the evaluator. The evaluator must satisfy the universal property: for any object A and arrow  $f:A\times B\to C$ , there is a unique arrow,  $\operatorname{cur} f:A\to B\Rightarrow C$  such that the following diagram commutes:



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