Categorical Semantics of Type Theories

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Abstract

Category theory and its applications to type theory are well known and have been explored extensively. In this manuscript we present basic category theory and give a categorical semantics to a large class of type theories. In this document we emphasize rigor and give as much detail as possible with respect to the abilities of the authors. This document will also be self contained.

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0.1 Introduction

We give an amazingly rich and engaging introduction.

Chapter 1

Category Theory

1.0.1 Monads and Comonads

In the sequel $\mathcal{T}^n$, for $n \in \mathbb{N}$, is defined to be the n-fold composition of $\mathcal{T}$, e.g. $\mathcal{T}^2 = \mathcal{T} \circ \mathcal{T}$.

Definition 1. *A **monad** of a category $\mathcal{C}$ consists of an endofunctor $\mathcal{T} : \mathcal{C} \to \mathcal{C}$ and two natural transformations $\eta : id_{\mathcal{C}} \to \mathcal{T}$ and $\mu : \mathcal{T}^2 \to \mathcal{T}$ satisfying the following commutative diagrams:*

$$\begin{tikzcd}
\mathcal{T}^3 \arrow[r, "\mu_{\mathcal{T}}"] \arrow[d, "T\mu"'] & \mathcal{T}^2 \arrow[d, "\mu"] \\
\mathcal{T}^2 \arrow[r, "\mu"] & \mathcal{T}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{T} \arrow[r, "\eta_T"] \arrow[d, "T\eta"'] \arrow[dr, "id"] & \mathcal{T}^2 \arrow[d, "\mu"] \\
\mathcal{T}^2 \arrow[r, "\mu"] & \mathcal{T}
\end{tikzcd}$$

Definition 2. *A **comonad** of a category $\mathcal{C}$ consists of an endofunctor $\mathcal{T} : \mathcal{C} \to \mathcal{C}$ and two natural transformations $\tilde{\eta} : \mathcal{T} \to id_{\mathcal{C}}$ and $\tilde{\mu} : \mathcal{T} \to \mathcal{T}^2$ satisfying the following commutative diagrams:*

$$\begin{tikzcd}
\mathcal{T} \arrow[r, "\tilde{\mu}"] \arrow[d, "\mu"'] & \mathcal{T}^2 \arrow[d, "T\mu"] \\
\mathcal{T}^2 \arrow[r, "\tilde{\mu}_{\mathcal{T}}"] & \mathcal{T}^3
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{T} \arrow[r, "\tilde{\mu}"] \arrow[d, "\tilde{\mu}"'] \arrow[dr, "id"] & \mathcal{T}^2 \arrow[d, "T\tilde{\eta}"] \\
\mathcal{T}^2 \arrow[r, "\tilde{\eta}_T"] & \mathcal{T}
\end{tikzcd}$$

1.0.2 Initial Algebras

Definition 3 (Algebra)**.** *An **algebra** or **$\mathcal{F}$-algebra** is a pair $(\mathcal{F}, s_A)$ with respect to some category $\mathcal{C}$ consisting of an object A called the carrier object of the algebra, an endofunctor $\mathcal{F} : \mathcal{C} \to \mathcal{C}$, and a morphism $s_A : \mathcal{F}(A) \to A$ called the structure of the algebra.*

Definition 4 (Homomorphism)**.** *A **homomorphism** between two algebras $s_1 : \mathcal{F}(A) \to A$ and $s_2 : \mathcal{F}(B) \to B$ is a morphism $f : A \to B$ such that the following diagram commutes:*

$$\begin{tikzcd}
A \arrow[r, "f"] & B \\
\mathcal{F}(A) \arrow[u, "s_1"] \arrow[r, "\mathcal{F}(f)"'] & \mathcal{F}(B) \arrow[u, "s_2"']
\end{tikzcd}$$

The above diagram expresses that f commutes with the operations of the two algebras.

Definition 5 (Initial Algebras)**.** *A **initial algebra** $s_1 : \mathcal{F}(A) \to A$ is one such that for any other algebra $s_2 : \mathcal{F}(B) \to B$ there is a unique homomorphism between them.*

Lets consider a simple example. Suppose $\mathcal{C}$ is some category with an object Nat, and morphisms $z : 1 \to Nat$ and $s : Nat \to Nat$. Then it is possible to construct any natural number i using just these two morphisms. That is i is defined by $s^i(z)$, hence z is the natural number zero and s is the successor operator. Now coparing these two operations up into a single morphism results in $[z, s] : 1 + Nat \to Nat$. Defining the functor $\mathcal{F}(X) := 1 + X$ we then obtain $[z, s] : \mathcal{F}(Nat) \to Nat$. Clearly, this is an $\mathcal{F}$-algebra. Next we show that the algebra $[z, s] : \mathcal{F}(Nat) \to Nat$ is initial.

Lemma 6 (Initiality of the Natural Number Algebra)**.** *Let $\mathcal{F}(X) := 1 + X$. Then algebra $[z, s] : \mathcal{F}(Nat) \to Nat$ is initial.*

Proof. Suppose $\mathcal{F}(X) := 1 + X$ and $[z', s'] : \mathcal{F}(A) \to A$ is an arbitrary algebra. It suffices to show that there exists a unique morphism $f : Nat \to A$ such that the following diagram commutes:

$$\begin{tikzcd}
Nat \arrow[r, "f"] & A \\
\mathcal{F}(Nat) \arrow[u, "{[z,s]}"] \arrow[r, "\mathcal{F}(f)"'] & \mathcal{F}(A) \arrow[u, "{[z',s']}"']
\end{tikzcd}$$

This diagram is equivalent to the following:

$$\begin{tikzcd}
Nat \arrow[r, "f"] & A \\
1 + Nat \arrow[u, "{[z,s]}"] \arrow[r, "id + f"'] & 1 + A \arrow[u, "{[z',s']}"']
\end{tikzcd}$$

We know $\mathcal{F}(f) := id + f$, because the only morphisms from 1 to 1 are the identity morphisms and they are unique. Take $f(n) := s'^{(n)}(z'(*))$. Where (n) is a inductively defined by the following equations:

$$\begin{array}{lll}
(z(*)) & := & 0 \\
(s(a)) & := & 1 + (a).
\end{array}$$

At this point we can see that:

$$\begin{array}{lll}
(f \circ z)(*) & = & f(s(*)) \\
& = & s'^{(z(*))}(z'(*)) \\
& = & s'^{0}(z'(*)) \\
& = & z'(*) \\
& = & (z' \circ id)(*)
\end{array}
\quad \text{and} \quad
\begin{array}{lll}
(f \circ s)(n) & = & f(s(n)) \\
& = & s'^{1+(n)}(z'(*)) \\
& = & s'(s'^{(n)}(z'(*))) \\
& = & s'(f(n)) \\
& = & (s' \circ f)(n).
\end{array}$$

Thus,

$$f \circ [z, s] \quad = \quad [z', s'] \circ (id + f).$$

The previous results shows that the above diagram commutes. However, we still must show that f is unique in order for it to be a homomorphism of algebras. Suppose $g : Nat \to \mathcal{A}$ such that $g \circ [z, s] = [z', s'] \circ (id + g)$. Then it must be the case that $g(z(*)) = z'(*)$ and $g(s(a)) = id + g$, which can be easily shown to be equivalent to f by structural induction on the input to f and g. Therefore, f is a homomorphism of algebras, and the algebra $[z, s] : \mathcal{F}(Nat) \to Nat$ for functor $\mathcal{F}(X) := 1 + X$ is initial. $\square$

Lemma 7 (Uniqueness of Initial Algebras)**.** *If $s : \mathcal{F}(A) \to A$ is an initial algebra of the functor $\mathcal{F}$, then s is unique up to isomorphism.*

Proof. Suppose $s : \mathcal{F}(A) \to A$ is an initial algebra for the functor $\mathcal{F}$. It suffices to show that the mediating homomorphism between s and any other initial algebra of the functor $\mathcal{F}$ is an isomorphism. Suppose $t : \mathcal{F}(B) \to B$ is another initial algebra of the functor $\mathcal{F}$. By initiality of these two algebras we know there exists a unique algebra homomorphism $f : A \to B$ between s and t. Similarly we know there exists a unique algebra homomorphism $f' : B \to A$ between s and t. Hence we have the following diagrams:

$$\begin{tikzcd}
A \arrow[r, "f!"] & B \arrow[r, "f'!"] & A \\
\mathcal{F}(A) \arrow[u, "s"] \arrow[r, "\mathcal{F}(f)"'] & \mathcal{F}(B) \arrow[u, "t"] \arrow[r, "\mathcal{F}(f')"'] & \mathcal{F}(A) \arrow[u, "s"]
\end{tikzcd}$$

and

$$\begin{tikzcd}
B \arrow[r, "f'!"] & A \arrow[r, "f!"] & B \\
\mathcal{F}(B) \arrow[u, "t"] \arrow[r, "\mathcal{F}(f')"'] & \mathcal{F}(A) \arrow[u, "s"] \arrow[r, "\mathcal{F}(f)"'] & \mathcal{F}(B) \arrow[u, "t"]
\end{tikzcd}$$

Now we know by initiality that f and f' are unique and hence their compositions $f \circ f' : A \to A$ and $f' \circ f : B \to B$ are also unique. Thus, it must be the case that $f \circ f' = id_A$ and $f' \circ f = id_B$. So f and f' are mutual inverses. Therefore, f is an isomorphism. $\square$

Lemma 8 (Initial Algebras are Isomorphisms)**.** *If $s : \mathcal{F}(A) \to A$ is an initial algebra for some functor $\mathcal{F}$, then it has an inverse $s^{-1} : A \to \mathcal{F}(A)$.*

Proof. Suppose $s : \mathcal{F}(A) \to A$ is an initial algebra for the functor $\mathcal{F}$. We must show there exists an inverse $s^{-1} : A \to \mathcal{F}(A)$ of s. This can be shown by cleverly finding another algebra such that s^{-1} is the algebra homomorphism between s and this other algebra. To do this we must fill in the holes of the following diagram:

$$\begin{tikzcd}
A \arrow[r, "s'"] & \mathcal{F}(A) \\
\mathcal{F}(A) \arrow[u, "s"] \arrow[r, "\mathcal{F}(?)"'] & \mathcal{F}(?) \arrow[u, "?"']
\end{tikzcd}$$

So our goal is to find an algebra $t : \mathcal{F}(?) \to \mathcal{F}(A)$ such that there is an algebra homomorphism $s' : A \to \mathcal{F}(A)$, and the above diagram commutes. Choose $\mathcal{F}(s) : \mathcal{F}(\mathcal{F}(A)) \to \mathcal{F}(A)$. Hence, we obtain the following diagram:

$$\begin{tikzcd}
A \arrow[r, "s'"] & \mathcal{F}(A) \\
\mathcal{F}(A) \arrow[u, "s"] \arrow[r, "\mathcal{F}(s')"'] & \mathcal{F}(\mathcal{F}(A)) \arrow[u, "\mathcal{F}(s)"']
\end{tikzcd}$$

Now we know $s : \mathcal{F}(A) \to A$ is an initial algebra, and that $\mathcal{F}(s) : \mathcal{F}(\mathcal{F}(A)) \to \mathcal{F}(A)$ is clearly an algebra. Thus, by initiality (Definition 5) we know that $s' : A \to \mathcal{F}(A)$ must exist and is an algebra homomorphism. Therefore, it is unique by Definition 4. Hence, $s \circ s' : \mathcal{F}(A) \to \mathcal{F}(A)$ is unique. Furthermore, $s' \circ s = id_{\mathcal{F}(A)}$ and $s \circ s' = id_A$. Therefore, we obtain the following:

$$\begin{array}{rcl} s' \circ s & = & id_{\mathcal{F}(A)} \\ & = & \mathcal{F}(id_A) \\ & = & \mathcal{F}(s \circ s') \\ & = & \mathcal{F}(s) \circ \mathcal{F}(s'). \end{array}$$

Therefore, the inverse of s is s'. □

Since our main example has been the natural number initial algebra $[z, s] : 1 + Nat \to Nat$ we will continue developing this example by showing how to use initiality to define functions on the natural numbers by induction. Consider the doubling function, double, which takes a natural number as input, and if it is zero returns zero, otherwise, it returns the number summed with itself. We can define this function by induction using initiality as follows.

$$\begin{tikzcd} Nat \arrow[r, "\mathsf{double}"] & Nat \\ 1 + Nat \arrow[u, "{[z,s]}"] \arrow[r, "id + \mathsf{double}"'] & 1 + Nat \arrow[u, "{[z, s \circ s]}"'] \end{tikzcd}$$

Initiality tells us that double : $Nat \to Nat$ is unique and exists. Furthermore it tells us the following:

$$\begin{array}{rcl} \mathsf{double} \circ z & = & z \circ id = z \\ \mathsf{double} \circ s & = & s \circ s \circ \mathsf{double} \end{array}$$

This is equivalent to the following:

$$\begin{array}{rcl} \mathsf{double}(z(*)) & = & z(*) \\ \mathsf{double}(s(n)) & = & s(s(\mathsf{double}(n))) \end{array}$$

Therefore inductively defined functions are no more than algebra homomorphisms between initial algebras!

$$\begin{tikzcd} Nat \arrow[r, "\mathsf{plus}"] & Nat^{Nat} \\ 1 + Nat \arrow[u, "{[z,s]}"] \arrow[r, "\mathsf{plus}"'] & 1 + Nat^{Nat} \arrow[u, "{[\lambda x.x, \lambda f.\lambda x.s(f(x))]}"'] \end{tikzcd}$$

Initiality tells us that plus : $Nat \to Nat^{Nat}$ is unique and exists. Furthermore it tells us the following:

$$\begin{array}{rcl} \mathsf{plus} \circ z & = & \lambda f.\lambda x.x \circ \mathsf{plus} \\ \mathsf{plus} \circ s & = & \lambda f.\lambda x.s(f(x)) \circ \mathsf{plus} \end{array}$$

This is equivalent to the following:

$$\begin{array}{rcl} \mathsf{plus}(z(*)) & = & (\lambda f.\lambda x.x)(\mathsf{plus}(z)) = \lambda x.x \\ \mathsf{plus}(s(n)) & = & (\lambda f.\lambda x.s(f(x)))(\mathsf{plus}(s(n))) = \lambda x.s(\mathsf{plus}(s(n))(x)) \end{array}$$

1.0.3 Final Coalgebras

Lets consider the infinite stream of ones.

$$1 :: 1 :: 1 :: 1 \cdots$$

Now what can be said about such an object? We definitely can make observations about it. That is, we can observe that, say, the first element is a 1. In addition to that we can observe that the second element is also a 1. Furthermore, we can make the observation that what follows the second element is still an infinite streams of ones. We can define this stream using a few morphisms. That is $\mathsf{head} : \mathcal{A} \to \mathbb{N}$ and $\mathsf{next} : \mathcal{A} \to \mathcal{A}$ are the morphisms, where $\mathcal{A}$ is some fixed set. Now notice that for any $a \in \mathcal{A}$ we have $\mathsf{head}(a) = 1$, and $\mathsf{next}(a) = a' \in A$. Now if we compose these two we obtain $\mathsf{head}(\mathsf{next}(a)) = \mathsf{head}(a') = 1$. Thus, using these morphisms we can completely define our stream above by taking for each position $n \in \mathbb{N}$ in the stream $\mathsf{head}(\mathsf{next}^n(a))$ to be the value at that position for any $a \in \mathcal{A}$. These two morphisms head and next are the observations we can make about the element a. It just happens, in this simple example, that the observations are the same for all elements of $\mathcal{A}$.

At this point we can take the product of the two morphisms head and next to obtain the morphism $\langle \mathsf{head}, \mathsf{next} \rangle : \mathcal{A} \to \mathbb{N} \times \mathcal{A}$. If we define the functor $\mathcal{F}(X) := \mathbb{N} \times X$ then we can redefine the product as $\langle \mathsf{head}, \mathsf{next} \rangle : \mathcal{A} \to \mathcal{F}(\mathcal{A})$. Furthermore, if we make $\mathbb{N}$ arbitrary and call it say $\mathcal{B}$ then we obtain the functor $\mathcal{F}(X) := \mathcal{B} \times X$. Using this new definition of the functor $\mathcal{F}$ and $\langle \mathsf{head}, \mathsf{next} \rangle : \mathcal{A} \to \mathcal{F}(\mathcal{A})$ we can define any infinite stream of elements of $\mathcal{B}$. Taking the functor $\mathcal{F}$ and the pair $(\mathcal{A}, \langle \mathsf{head}, \mathsf{next} \rangle)$ we obtain what is called a coalgebra.

Definition 9 (Coalgebra)**.** *A **coalgebra** or $\mathcal{F}$**-coalgebra** is a pair $(\mathcal{F}, s_{\mathcal{A}})$ with respect to some category $\mathcal{C}$ consisting of an object $\mathcal{A}$ called the carrier object of the coalgebra, an endofunctor $\mathcal{F} : \mathcal{C} \to \mathcal{C}$, and a morphism $s_{\mathcal{A}} : \mathcal{A} \to \mathcal{F}(\mathcal{A})$ called the structure of the coalgebra.*

It is often the case that coalgebras are denoted simply by specifying the type of the structure of the coalgebra. That is $s_{\mathcal{A}} : \mathcal{A} \to \mathcal{F}(\mathcal{A})$. This is precise, because it gives all of the elements of a coalgebra, but the category, but this is usually determinable from the context. So, in our example above we can see that $\langle \mathsf{head}, \mathsf{next} \rangle : \mathcal{A} \to \mathcal{F}(\mathcal{A})$ is indeed a coalgebra with respect to the category Set.

Now consider the infinite stream of twos.

$$2 :: 2 :: 2 :: 2 \cdots$$

This is definable by a coalgebra $\langle \widehat{\mathsf{head}}, \widehat{\mathsf{next}} \rangle : \mathcal{B} \to \mathcal{F}(\mathcal{B})$ with respect to the category Set. Suppose further that there is a function $f : \mathcal{A} \to \mathcal{B}$ such that $\widehat{\mathsf{head}} \circ f = \mathsf{head}$ and $\widehat{\mathsf{next}} \circ f = f \circ \mathsf{next}$. In this case f is called a homomorphism between coalgebras.

Definition 10 (Homomorphism)**.** *A **homomorphism** between two coalgebras $s_1 : \mathcal{A} \to \mathcal{F}(\mathcal{A})$ and $s_2 : \mathcal{B} \to \mathcal{F}(\mathcal{B})$ is a morphism $f : \mathcal{A} \to \mathcal{B}$ such that the following diagram commutes:*

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "f"] \arrow[d, "s_1"'] & \mathcal{B} \arrow[d, "s_2"] \\
\mathcal{F}(\mathcal{A}) \arrow[r, "\mathcal{F}(f)"'] & \mathcal{F}(\mathcal{B})
\end{tikzcd}$$

The above diagram expresses that f commutes with the operations of the two coalgebras.

Definition 11 (Final Coalgebras)**.** *A **final coalgebra** $s_1 : A \to \mathcal{F}(\mathcal{A})$ is one such that for any other coalgebra $s_2 : \mathcal{B} \to \mathcal{F}(\mathcal{B})$ there is a unique homomorphism between them.*

Chapter 2

Simple Type Theories

2.1 A Metaframework

We will use Martin-Löf's Type Theory for our metaframework throughout this chapter. This will allow use to give precise types to all of the structures in our object languages. In fact all of the type theories discussed in this chapter can be rigorously defined within this framework where binding can be encoded using either de Bruijn indecies or using the locally nameless representation [1] [**?**].

2.2 The Theory of Constants

We begin our journey into the world of categorical semantics of type theories by first showing how to interpret a simple algebraic theory consisting of a countably infinite set of variables, a finite set of constant types, a finite set of i-ary function symbols, a typing judgment, and a definitional equality judgment. This theory is called the theory of constants. We first give a formal definition of this theory in the presentation we will adopt for the reminder of this document. The syntax for the theory of constants is defined in Figure 2.1.

The free variables of the theory can be defined at the metalevel as de Bruijn indices, but we will use mathematical notation to simplify the presentation. They have type Term and are classified by constant types of type Type. The i-ary function symbols have type $\mathbf{Term}^i \Rightarrow \mathbf{Term}$. The constant types of the theory of constants have meta-type Type. Judgments are metastatements describing what type a term can be assigned. All of the judgements we will define can be defined at the metalevel as an inductive datatype where each rule of the judgment is defined as a constructor. If we call the typing judgment has_type at the type level then its type is $(\Gamma : [\mathbf{Term} \times \mathbf{Type}]) \Rightarrow (t : \mathbf{Term}) \Rightarrow (U : \mathbf{Type}) \Rightarrow \mathbf{Type}$. We denote this judgment by $\Gamma \vdash t : U$. The type of the definitional equality judgment is similar. We define the type assignment judgment in Figure 2.2 and the definitional equality judgment in Figure 2.3.

[1] We prefer the latter.

$$
\begin{array}{rccl}
(Types) & T & ::= & S \mid U \\
(Terms) & t & ::= & x \mid c \mid f\, x_1 \ldots x_i \\
(Contexts) & \Gamma & ::= & x : T \mid \Gamma_1, \Gamma_2
\end{array}
$$

Figure 2.1: Syntax of the theory of constants

$$\frac{}{\Gamma, x : S, \Gamma' \vdash x : S}\ \textsc{Var} \qquad \frac{}{x_1 : S_1,\ \ldots, x_i : S_i \vdash f\ x_1 \ldots x_i : U}\ \textsc{Fun}$$

$$\frac{}{\Gamma \vdash c : U}\ \textsc{Const}$$

Figure 2.2: Type assignment for the theory of constants

$$\frac{}{\Gamma \vdash t = t : T}\ \textsc{Refl} \qquad \frac{\Gamma \vdash t_1 = t_2 : T}{\Gamma \vdash t_2 = t_1 : T}\ \textsc{Sym}$$

$$\frac{\begin{array}{c}\Gamma \vdash t_1 = t_2 : T \\ \Gamma \vdash t_2 = t_3 : T\end{array}}{\Gamma \vdash t_1 = t_3 : T}\ \textsc{Trans} \qquad \frac{\begin{array}{c}\Gamma \vdash t_1 = t_2 : T_1 \\ \Gamma, x : T_1 \vdash t = t' : T_2\end{array}}{\Gamma \vdash [t_1/x]t = [t_2/x]t' : T_2}\ \textsc{Subst}$$

Figure 2.3: Definitional equality for the theory of constants

Chapter 3

Data Types

3.1 Coalgebras and Analysis

It is widely known that the natural numbers and induction come hand in hand. That is induction is used to both define and prove predicates on the natural numbers. That is elementary arithmetic. So we have the following picture:

$$\frac{\text{Induction}}{\text{arithmetic (Natural Numbers)}}$$

We then dare pose the question, how do we fill in the hole in the following picture?

$$\frac{\text{Coinduction}}{?}$$

Well D. Pavlović, V. Pratt, and M. Escardo have come up with an answer for us. They answer with the following:

$$\frac{\text{Coinduction}}{\text{analysis (real numbers)}}$$

In this section we will slowly uncover how this works by closely examining their work. One thing to notice regarding the relationship between the above pictures is that induction and coinduction are duels in a very precise way. We pose yet another question, does this imply that arithmetic and analysis are duels? Are natural numbers and real numbers duals? These are unknown questions as of right now.

3.1.1 Real Numbers