

Centered Categories: Categorical Models for Type Theories with Determinisitic Reduction

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1 Introduction

Consider how we interpret the application rule for STLC in a CCC C :

$$\frac{\Gamma \vdash t_2 : A \quad \Gamma \vdash t_1 : A \Rightarrow B}{\Gamma \vdash t_1 t_2 : B} \text{App}$$

First assume we have the following morphisms in \mathbb{C} :

$$\llbracket \Gamma \rrbracket \xrightarrow{t_1} A \Rightarrow B \quad \llbracket \Gamma \rrbracket \xrightarrow{t_2} A$$

Then we must define a new morphism $t \in \mathbb{C}(\llbracket \Gamma \rrbracket, B)$. We can accomplish this in the following way:

$$\llbracket \Gamma \rrbracket \xrightarrow{(t_1, t_2)} (A \Rightarrow B) \times A \xrightarrow{\text{app } A B} B$$

Where app is the evaluator for exponentials. Now suppose $t_1 \equiv \lambda x.t$ then the in the type theory $t_1 t_2 \equiv [t_2/x]t$. We know the morphism for $t_1 t_2$, but what is the morphism for $[t_2/x]t$? Substitution amounts to essentially composition. Recalling that λ -abstractions are modeled by currying the previous term is modled by the following:

$$\llbracket \Gamma \rrbracket \xrightarrow{(\text{id}_{\llbracket \Gamma \rrbracket}, t_2)} \llbracket \Gamma \rrbracket \times A \xrightarrow{\text{cur}^{-1} \circ \text{cur } t} B$$

Since cur is an isomorphism the previous diagram is equivalent to the following:

$$\llbracket \Gamma \rrbracket \xrightarrow{(\text{id}_{\llbracket \Gamma \rrbracket}, t_2)} \llbracket \Gamma \rrbracket \times A \xrightarrow{t} B$$

In the previous morphism there are no restrictions on what t_2 can be. Thus, this corresponds to full $\beta\eta$ -reduction. In order for this to be the correct model the equality $\text{app } A B \circ (\text{cur } t, t_2) = t \circ (\text{id}_{\llbracket \Gamma \rrbracket}, t_2)$ must hold. To prove this we must show that the following diagram commutes:

$$\begin{array}{ccc} \llbracket \Gamma \rrbracket & \xrightarrow{(\text{cur } t, t_2)} & (A \Rightarrow B) \times A \\ \downarrow (\text{id}_{\llbracket \Gamma \rrbracket}, t_2) & \nearrow \text{cur } t \times \text{id}_A & \downarrow \text{app } A B \\ \llbracket \Gamma \rrbracket \times A & \xrightarrow{t} & B \end{array}$$

The UMP for exponentials gives us

$$\text{app } A B \circ \text{cur } t \times \text{id}_A = t$$

Now

$$(\text{app } A B \circ \text{cur } t \times \text{id}_A) \circ (\text{id}_{\llbracket \Gamma \rrbracket}, t_2) = \text{app } A B \circ (\text{cur } t \times \text{id}_A \circ (\text{id}_{\llbracket \Gamma \rrbracket}, t_2))$$

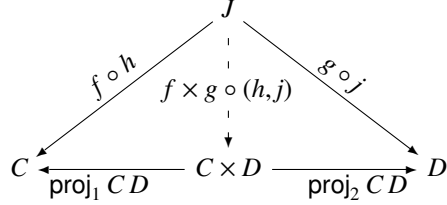
To finish this proof we must show that

$$\text{cur } t \times \text{id}_A \circ (\text{id}_{\llbracket \Gamma \rrbracket}, t_2) = (\text{cur } t, t_2).$$

This follows from the following lemma.

Lemma 1. Suppose \mathbb{C} is a CCC. Then for any $f \in \mathbb{C}[A, C]$, $g \in \mathbb{C}[B, D]$, $h \in \mathbb{C}[J, A]$, and $j \in \mathbb{C}[J, B]$, we have, $f \times g \circ (h, j) = (f \circ h, g \circ j)$.

Proof. Consider the following diagram:



The UMP for products insures the previous diagram commutes. Furthermore, by uniqueness it is the case that $f \times g \circ (h, j) = (f \circ h, g \circ j)$. \square

Therefore, the equality $\text{app } A B \circ (\text{cur } t, t_2) = t \circ (\text{id}_{\llbracket \Gamma \rrbracket}, t_2)$ does indeed hold. This tells us that equality in the model is $\beta\eta$ -equality. Thus, in order to model a different notion of equality we must modify the equality in the category.

2 Centered Category

Definition 2. A centered category \mathbb{C}^T is a pair $(\mathbb{C}, \mathbb{V}, \top, \perp, \times, +, \Rightarrow)$ such that

i. $(\mathbb{C}, \top, \times)$ and $(\mathbb{C}, \perp, +)$ are symmetric monoidal categories,

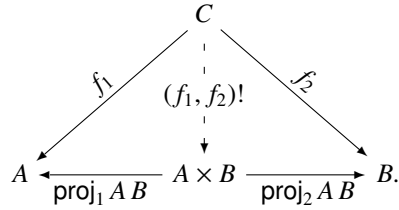
ii. \mathbb{C} contains the following morphisms:

a. Units:

$$\perp \xrightarrow{\text{contr } A} A \quad A \xrightarrow{\text{unit } A} \top$$

b. Products:

$$A \times B \xrightarrow{\text{proj}_1 A B} A \quad A \times B \xrightarrow{\text{proj}_2 A B} B$$



In addition we define the functor $- \times - : \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$:

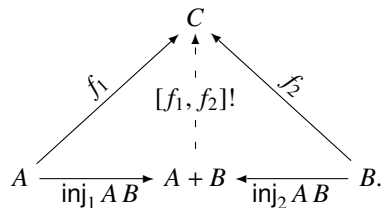
$$\text{Objects: } (- \times -)(A, B) := A \times B$$

$$\text{Morphisms: } (- \times -)(f : A \rightarrow C, g : B \rightarrow D) := (f \circ \text{proj}_1 A B, g \circ \text{proj}_2 A B)$$

In the sequel the above functor will always be applied infix.

c. Coproducts:

$$A \xrightarrow{\text{inj}_1 A B} A + B \quad B \xrightarrow{\text{inj}_2 A B} A + B$$



d. *Exponentials:*

For any two objects A and B in \mathbb{C} there is an object $A \Rightarrow B$ and an arrow $\text{app } A B : (A \Rightarrow B) \times A \rightarrow B$ called the evaluator. The evaluator must satisfy the universal property: for any object A and arrow $f : A \times B \rightarrow C$, there is a unique arrow, $\text{cur } f : A \rightarrow B \Rightarrow C$ such that the following diagram commutes:

$$\begin{array}{ccc}
 & C & \\
 \uparrow \text{§} & & \downarrow \epsilon \\
 A \times B & \xrightarrow{\text{---} f^* \times \text{id}_B \text{---}} & C^B \times B
 \end{array}$$

References

- [1] Tristan Crolard. Subtractive logic. *Theor. Comput. Sci.*, 254(1-2):151–185, 2001.
- [2] J. Lambek and P.J. Scott. *Introduction to Higher-Order Categorical Logic*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1988.
- [3] R. Seely. Linear logic, *-autonomous categories and cofree coalgebras. In *Computer Science Logic*, 1989.
- [4] P. Selinger. Control categories and duality: on the categorical semantics of the lambda-mu calculus. *Mathematical Structures in Computer Science*, 11(02):207–260, 2001.