Centered Categories: Categorical Models for Type Theories with Determinisitic Reduction

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1 Introduction

2 Equality in CCCs

Consider how we interpret the application rule for STLC in a CCC C:

$$\frac{\Gamma \vdash t_2 : A \qquad \Gamma \vdash t_1 : A \Rightarrow B}{\Gamma \vdash t_1\, t_2 : B}\ \textsc{App}$$

First assume we have the following morphisms in $\mathbb{C}$:

$$\llbracket\Gamma\rrbracket \xrightarrow{t_1} A \Rightarrow B \qquad \llbracket\Gamma\rrbracket \xrightarrow{t_2} A$$

Then we must define a new morphism $t \in \mathbb{C}(\llbracket\Gamma\rrbracket, B)$. We can accomplish this in the following way:

$$\llbracket\Gamma\rrbracket \xrightarrow{(t_1,t_2)} (A \Rightarrow B) \times A \xrightarrow{\mathsf{app}\,A\,B} B$$

Where app is the evaluator for exponentials. Now suppose $t_1 \equiv \lambda x.t$ then the in the type theory $t_1\,t_2 \equiv [t_2/x]t$. We know the morphism for $t_1\,t_2$, but what is the morphism for $[t_2/x]t$? Substitution amounts to essentially composition. Recalling that λ-abstractions are modeled by curring the previous term is modled by the following:

$$\llbracket\Gamma\rrbracket \xrightarrow{(\mathsf{id}_{\llbracket\Gamma\rrbracket},t_2)} \llbracket\Gamma\rrbracket \times A \xrightarrow{\mathsf{cur}^{-1} \circ \mathsf{cur}\,t} B$$

Since cur is an isomorphism the previous diagram is equivalent to the following:

$$\llbracket\Gamma\rrbracket \xrightarrow{(\mathsf{id}_{\llbracket\Gamma\rrbracket},t_2)} \llbracket\Gamma\rrbracket \times A \xrightarrow{t} B$$

In the previous morphism there are no restrictions on what t_2 can be. Thus, this corresponds to full $\beta\eta$-reduction. In order for this to be the correct model the equality $\mathsf{app}\,A\,B \circ (\mathsf{cur}\,t, t_2) = t \circ (\mathsf{id}_{\llbracket\Gamma\rrbracket}, t_2)$. To prove this we must show that the following diagram commutes:

$$\begin{tikzcd}
\llbracket\Gamma\rrbracket \arrow[r, "{(\mathsf{cur}\,t, t_2)}"] \arrow[d, "{(\mathsf{id}_{\llbracket\Gamma\rrbracket}, t_2)}"'] & (A \Rightarrow B) \times A \arrow[d, "\mathsf{app}\,A\,B"] \\
\llbracket\Gamma\rrbracket \times A \arrow[r, "t"'] \arrow[ur, "\mathsf{cur}\,t \times id_A"] & B
\end{tikzcd}$$

The UMP for exponentials gives us

$$\mathsf{app}\,A\,B \circ \mathsf{cur}\,t \times id_A \quad = \quad t$$

Now

$$(\mathsf{app}\,A\,B \circ \mathsf{cur}\,t \times id_A) \circ (\mathsf{id}_{\llbracket\Gamma\rrbracket}, t_2) \quad = \quad \mathsf{app}\,A\,B \circ (\mathsf{cur}\,t \times id_A \circ (\mathsf{id}_{\llbracket\Gamma\rrbracket}, t_2))$$

To finish this proof we must show that

$$\mathsf{cur}\,t \times id_A \circ (\mathsf{id}_{\llbracket\Gamma\rrbracket}, t_2) = (\mathsf{cur}\,t, t_2).$$

This follows from the following lemma.

3 Centered Category

Definition 1. *A centered category* $\mathbb{C}^T$ *is a pair* $(\mathbb{C}, \mathbb{V}, \top, \perp, \times, +, \Rightarrow)$ *such that*

i. $(\mathbb{C}, \top, \times)$ *and* $(\mathbb{C}, \perp, +)$ *are symmetric monoidal categories,*

ii. $\mathbb{C}$ *contains the following morphisms:*

a. Units:

$$\perp \xrightarrow{\mathsf{contr}\,A} A \qquad\qquad A \xrightarrow{\mathsf{unit}\,A} \top$$

b. Products:

$$A \times B \xrightarrow{\mathsf{proj}_1\,A\,B} A \qquad\qquad A \times B \xrightarrow{\mathsf{proj}_2\,A\,B} B$$

$$\begin{tikzcd} & C \arrow[dl, "f_1"'] \arrow[d, dashed, "(f_1, f_2)!"] \arrow[dr, "f_2"] & \\ A & A \times B \arrow[l, "\mathsf{proj}_1\,A\,B"] \arrow[r, "\mathsf{proj}_2\,A\,B"'] & B. \end{tikzcd}$$

In addition we define the functor $- \times - : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$*:*

Objects:	$(- \times -)(A, B)$	$:=$	$A \times B$
Morphisms:	$(- \times -)(f : A \to C, g : B \to D)$	$:=$	$(f \circ \mathsf{proj}_1\,A\,B, g \circ \mathsf{proj}_2\,A\,B)$

In the sequel the above functor will always be applied infix.

c. Coproducts:

$$A \xrightarrow{\mathsf{inj}_1\,A\,B} A + B \qquad\qquad B \xrightarrow{\mathsf{inj}_2\,A\,B} A + B$$

$$\begin{tikzcd} & C & \\ A \arrow[ur, "f_1"] \arrow[r, "\mathsf{inj}_1\,A\,B"'] & A + B \arrow[u, dashed, "{[f_1, f_2]!}"] & B. \arrow[ul, "f_2"'] \arrow[l, "\mathsf{inj}_2\,A\,B"] \end{tikzcd}$$

d. Exponentials:

For any two objects A and B in $\mathbb{C}$ *there is an object* $A \Rightarrow B$ *and an arrow* $\mathsf{app}\,A\,B : (A \Rightarrow B) \times A \to B$ *called the evaluator. The evaluator must satisfy the universal property: for any object A and arrow* $f : A \times B \to C$*, there is a unique arrow,* $\mathsf{cur}\,f : A \to B \Rightarrow C$ *such that the following diagram commutes:*

$$\begin{tikzcd} & C & \\ A \times B \arrow[ur, "f"] \arrow[rr, dashed, "f^* \times id_B"] & & C^B \times B \arrow[ul, "\epsilon"'] \end{tikzcd}$$

References

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