

odeint

Solving ordinary differential equations in C++

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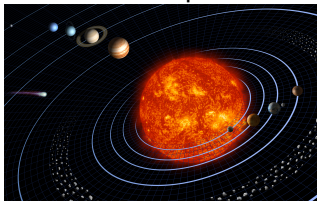


Outline

- 1 Introduction
- 2 Tutorial
- 3 Technical details
- 4 Discussion

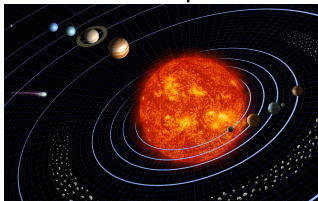
What is an ODE? – Examples

Newtons equations



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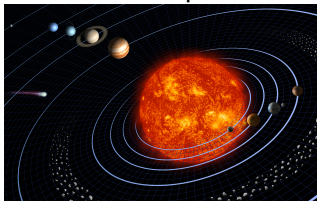
Newtons equations



Reaction and relaxation
equations (i.e. blood alcohol
content)

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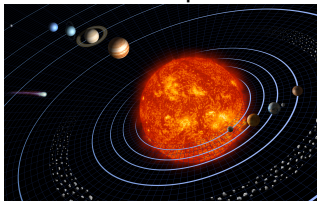
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Newtons equations

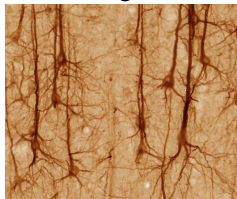


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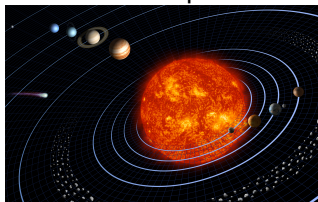


Interacting neurons



What is an ODE? – Examples

Newtons equations

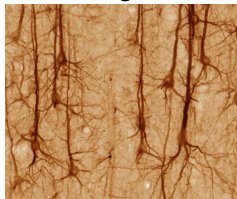


Reaction and relaxation equations (i.e. blood alcohol content)

Granular systems



Interacting neurons



- Many examples in physics, biology, chemistry, social sciences
- Fundamental in mathematical modelling

What is an ODE?

$$\frac{dx(t)}{dt} = f(x(t), t) \quad \text{short form} \quad \dot{x} = f(x, t)$$

- $x(t)$ – dependent variable
- t – independent variable (time)
- $f(x, t)$ – defines the ODE

Initial Value Problem (IVP):

$$\dot{x} = f(x, t), \quad x(t = 0) = x_0$$

Numerical integration of ODEs

Find a numerical solution of an ODE and its initial value problem

$$\dot{x} = f(x, t) , \quad x(t = 0) = x_0$$

Example: Explicit Euler

$$x(t + \Delta t) = x(t) + \Delta t f(x(t), t) + \mathcal{O}(\Delta t^2)$$

General scheme of order s

$$x(t) \mapsto x(t + \Delta t) \quad , \text{ or}$$

$$x(t + \Delta t) = \mathcal{F}_t x(t) + \mathcal{O}(\Delta t^{s+1})$$

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Solving ordinary differential equations in C++

Open source

- Boost license – do whatever you want do to with it

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Modern C++

- Generic programming, functional programming
- Fast, easy-to-use and extendable.
- Container independent
- Portable

Who uses odeint

NetEvo



OMPL – Open Motion
Planning Library

Motivation: The interface problem in C/C++

- Many frameworks exist to do numerical computations.
- Data has to be stored in containers or collections.
- GSL: `gsl_vector`, `gsl_matrix`
- NR: pointers with Fortran-style indexing
- Blitz++, MTL4, `boost::ublas`
- QT: `QVector`, wxWidgets: `wxArray`, MFC: `CArray`

But: All books on C++ recommend the use of the STL containers `std::vector`, `std::list`, ...

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Theoretical solution of the interface mess

GoF Design Pattern: Adaptor, also known as Wrapper

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Theoretical solution of the interface mess

GoF Design Pattern: Adaptor, also known as Wrapper

Alternative

Generic, container independent algorithms

Portability of your algorithm

How to run your algorithm?

- Single machine, single CPU
- Single machine, multiple CPU's (OpenMP, threads, ...)
- Multiple machines (MPI)
- GPU (Cuda, Thrust, OpenCL)

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Which data types are used by your algorithm?

- Build-in data types – `double`, `complex<double>`
- Arbitrary precision types – GMP, MPFR
- Vectorial data types `float2d`, `float3d`

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Theoretical solution

GoF Design Pattern: Strategy, also known as Policy

Alternative

Generic algorithms

Lets step into odeint

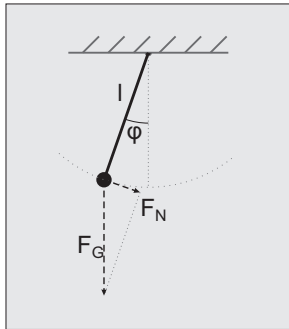
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Example – Pendulum



Newtons law: $ma = F$

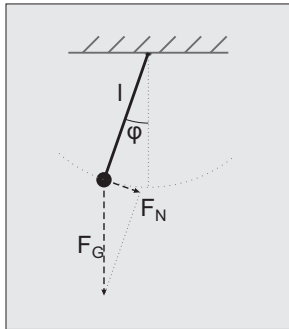
Acceleration: $a = l\ddot{\varphi}$

Force: $F = F_N = -mg \sin \varphi$

\Rightarrow **ODE for φ**

$$\ddot{\varphi} = -g/l \sin \varphi = -\omega_0^2 \sin \varphi$$

Example – Pendulum



$$\ddot{\varphi} = -\omega_0^2 \sin \varphi$$

Small angle: $\sin \varphi \approx \varphi$

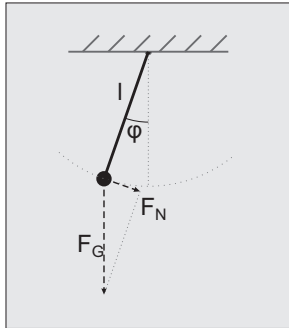
Harmonic oscillator $\ddot{\varphi} = -\omega_0^2 \varphi$

Analytic solution:

$$\varphi = A \cos \omega_0 t + B \sin \omega_0 t$$

Determine A and B from initial condition

Example – Pendulum



Full equation: $\ddot{\varphi} = -\omega_0^2 \sin \varphi$

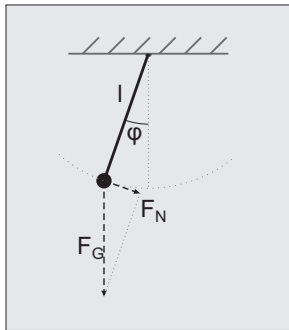
Pendulum with friction and external driving:

$$\ddot{\varphi} = -\omega_0^2 \sin \varphi - \mu \dot{\varphi} + \varepsilon \sin \omega_E t$$

No analytic solution is known

\Rightarrow **Solve this equation numerically.**

Example – Pendulum



$$\ddot{\varphi} = -\omega_0^2 \sin \varphi - \mu \dot{\varphi} + \varepsilon \sin \omega_E t$$

Create a first order ODE

$$x_1 = \varphi, \quad x_2 = \dot{\varphi}$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\omega_0^2 \sin x_1 - \mu x_2 + \varepsilon \sin \omega_E t$$

x_1 and x_2 are the state space variables

Let's solve the pendulum example numerically

```
#include <boost/numeric/odeint.hpp>

namespace odeint = boost::numeric::odeint;
```

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -\omega_0 \sin x_1 - \mu x_2 + \varepsilon \sin \omega_E t$$

```
typedef std::array<double,2> state_type;
```

Let's solve the pendulum example numerically

$$\dot{x}_1 = x_2, \dot{x}_2 = -\omega_0^2 \sin x_1 - \mu x_2 + \varepsilon \sin \omega_E t$$

```
struct pendulum
{
    double m_mu, m_omega, m_eps;

    pendulum(double mu, double omega, double eps)
        : m_mu(mu), m_omega(omega), m_eps(eps) { }

    void operator()(const state_type &x,
                    state_type &dxdt, double t) const
    {
        dxdt[0] = x[1];
        dxdt[1] = -sin(x[0]) - m_mu * x[1] +
                    m_eps * sin(m_omega*t);
    }
};
```

Let's solve the pendulum example numerically

$$\varphi(0) = 1, \quad \dot{\varphi}(0) = 0$$

```
odeint::rk4< state_type > rk4;  
pendulum p( 0.1 , 1.05 , 1.5 );  
  
state_type x = {{ 1.0 , 0.0 }};  
double t = 0.0;  
  
const double dt = 0.01;  
rk4.do_step( p , x , t , dt );  
t += dt;
```

$$x(0) \mapsto x(\Delta t)$$

Let's solve the pendulum example numerically

```
std::cout<<t<<" "<< x[0]<<" "<<x[1]<<"\n";  
for( size_t i=0 ; i<10 ; ++i )  
{  
    rk4.do_step( p , x , t , dt );  
    t += dt;  
    std::cout<<t<<" "<< x[0]<<" "<<x[1]<<"\n";  
}
```

$x(0) \mapsto x(\Delta t) \mapsto x(2\Delta t) \mapsto x(3\Delta) \mapsto \dots$

Grafik einfügen

Simulation

Oscillator: $\mu = 0$, $\omega_E = 0$, $\varepsilon = 0$

Damped oscillator: $\mu = 0.1$, $\omega_E = 0$, $\varepsilon = 0$

Damped, driven oscillator: $\mu = 0.1$, $\omega_E = 1.05$, $\varepsilon = 1.5$

Different Steppers

```
runge_kutta_fehlberg78< state_type > s;
```

```
runge_kutta_dopri5< state_type > s;
```

Symplectic steppers (for Hamiltonian systems)

```
symplectic_rkn_sb3a_mclachlan< state_type > s;
```

Implicit steppers (for stiff systems)

```
rosenbrock4< double > s;
```

These steppers perform one step with constant step size!

Controlled steppers – Step size control

insert graphic

Controlled steppers

```
auto s = make_controlled(1.0e-6, 1.0e6,  
    runge_kutta_fehlberg78<state_type>() );  
controlled_step_result r =  
    s.try_step(ode, x, t, dt);
```

Tries to perform the step and updates x , t , and dt !

It works because Runge-Kutta-Fehlberg has error estimation:

```
runge_kutta_fehlberg78<state_type> s;  
s.do_step(ode, x, t, dt, xerr);
```


Controlled steppers

```
auto s = make_controlled(1.0e-6,1.0e6,  
    runge_kutta_fehlberg78<state_type>() );  
while( t < t_end )  
{  
    controlled_step_result res  
        = s.try_step(ode,x,t,dt);  
    while( res != success )  
    {  
        res = s.try_step(ode,x,t,dt);  
    }  
}
```

Non-trivial time-stepping logic

Use integrate functions!

```
integrate_adaptive(s,ode,x,t_start,t_end,dt);  
integrate_adaptive(s,ode,x,t_start,t_end,dt,  
    observer);
```

Observer: Callable object `obs(x,t)`

Example (using Boost.Phoenix):

```
integrate_adaptive(s,ode,x,t_start,t_end,dt,  
    cout<< arg1[0] << " " << arg1[1] << "\n" );
```

More integrate versions:

`integrate_const`, `integrate_times`, ...

```
integrate_const(s,ode,x,t,dt,obs);
```

Grafik with problem and solution

Dense output

```
auto s = make_dense_output( 1.0e-6 , 1.0e-6 ,  
    runge_kutta_dopri5< state_type >() );  
integrate_const( s , p , x , t , dt );
```

Interpolation between two steps with same precision as the original stepper!

Grafik!

More steppers

Stepper Concepts: Stepper, ErrorStepper, ControlledStepper, DenseOutputStepper

Stepper types:

- Implicit – `implicit_euler`, `rosenbrock4`
- Symplectic – `symplectic_rkn_sb3a_mclachlan`
- Predictor-Corrector – `adams_bashforth_moulton`
- Extrapolation – `bulirsch_stoer`
- Multistep methods – `adams_bashforth_moulton`

Some of them have step-size control and dense-output!

Small summary

- Very easy example – harmonic oscillator
- Basic features of odeint
- Different stepper – Controlled steppers, Dense output steppers
- Integrate functions

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- Very easy example – harmonic oscillator
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Now, lets look at the advanced features!

Extended systems

Lattice systems

Extended systems

Lattice systems

Discretizations of PDEs

Extended systems

Lattice systems

Discretizations of PDEs

Granular systems



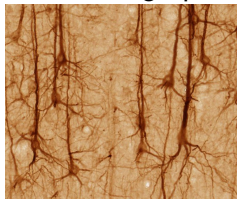
Extended systems

Lattice systems

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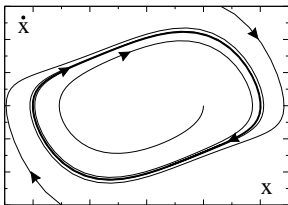
Granular systems

ODEs on graphs



High-Performance-Computing

Phase oscillator lattices



Any oscillator can be described by one variable, its phase.

Trivial dynamics: $\dot{\varphi} = \omega\varphi$

Coupled phase oscillators

Neurosciences

Heart dynamics

Synchronization

Any weakly perturbed oscillator system

$$\dot{\varphi}_k = \omega_k \varphi_k + q(\varphi_{k+1}, \varphi_k) + q(\varphi_k, \varphi_{k-1})$$

Phase compacton lattice

$$\dot{\varphi}_k = \cos \varphi_{k+1} - \cos \varphi_{k-1}$$

State space contains N variables

```
typedef std::vector<double> state_type;
```

Animation

Space-time plot for visualization of compactons and chaos

Ensemble of phase oscillators

$$\dot{\varphi}_k = \omega_k + \sum_l \sin(\varphi_l - \varphi_k)$$

Synchronization – all oscillator oscillates with the same frequency

Synchronized state $\varphi_k = \omega_S t + \varphi_{0,k}$

Ensemble of phase oscillators

```
typedef std::vector<double> state_type;

struct ensemble
{
    state_type m_omega,m_eps;

    ensemble(size_t n,double eps)
    : m_omega(n,0.0),m_eps(eps)
    {
        create_frequencies();
    }

    void create_frequencies() { ... }

    void operator()(const state_type &x,
                    state_type &dxdt,double t) const
    {
        ...
    }
};
```

Solving ODEs with CUDA using Thrust

Thrust is a parallel algorithms library which resembles the C++ Standard Template Library (STL). Thrust's high-level interface greatly enhances developer productivity while enabling performance portability between GPUs and multicore CPUs. Interoperability with established technologies (such as CUDA, TBB and OpenMP) facilitates integration with existing software. Develop high-performance applications rapidly with Thrust!



Solving ODEs with CUDA using thrust

Applications and use cases for GPUs:

- Large systems, discretizations of PDEs, lattice systems, granular systems, etc.
- Parameter studies, solve many ODEs in parallel with different parameters
- Initial value studies, solve the same ODE with many different initial conditions in parallel

Lorenz system – Parameter study

$$\dot{x} = \sigma(y - x) \quad \dot{y} = Rx - y - xz \quad \dot{z} = -bz + xy$$

Standard parameters $\sigma = 10$, $R = 28$, $b = 8/3$

Deterministic chaos – Butterfly effect

Picture of Lorenz system

Lorenz system – Parameter study

$$\dot{x} = \sigma(y - x) \quad \dot{y} = Rx - y - xz \quad \dot{z} = -bz + xy$$

Does one observe chaos over the whole parameter range?

Lyapunov exponents:

- Measure of chaos
- Perturbations of the original system

Vary R from 0 to 50 and calculate the Lyapunov exponents!

Use CUDA and Thrust!

odeint's algebras and operations

Euler method

$$x_i(t + \Delta t) = x_i(t) + \Delta t * f_i(x)$$

Algebras perform the iteration over i and operation the elementary addition.

Algebras and operations enter the stepper as template parameters

```
typedef runge_kutta4<state_type,value_type,  
    deriv_type,time_type,  
    algebra,operations,resize_policy> stepper;
```

- default_operations
- range_algebra – Boost.Ranges
- vector_space_algebra – Passes the state directly to the operations
- fusion_algebra – For compile time sequences, like *std :: tuple < double, double >*
- thrust_algebra and thrust_algebra – for thrust

Thrust example for Lorenz system,
Implementation of the system function

More advanced features, die themen können auch auf mehreren folien zusammengefasst werden

Boost::ref

boost::range

complex state types, vielleicht auch nicht

arbitrary precision types

matrices as state types

graph as state types

self expanding lattices

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Independent Algorithms

Goal

Container- and computation-independent implementation of the numerical algorithms.

Benefit

High flexibility and applicability, ODEINT can be used for virtually any formulation of an ODE.

Approach

Detach the algorithm from memory management and computation detail and make each part interchangeable.

Mathematical Algorithm

Typical mathematical computation to calculate the solution of an ODE ($\dot{\vec{x}} = \vec{f}(\vec{x}, t)$):

$$\vec{F}_1 = \vec{f}(\vec{x}_0, t_0)$$

$$\vec{x}' = \vec{x}_0 + a_{21} \cdot \Delta t \cdot \vec{F}_1$$

$$\vec{F}_2 = \vec{f}(\vec{x}', t_0 + c_1 \cdot \Delta t)$$

$$\vec{x}' = \vec{x}_0 + a_{31} \cdot \Delta t \cdot \vec{F}_1 + a_{32} \cdot \Delta t \cdot \vec{F}_2$$

$$\vdots$$

$$\vec{x}_1 = \vec{x}_0 + b_1 \cdot \Delta t \cdot \vec{F}_1 + \dots + b_s \cdot \Delta t \cdot \vec{F}_s$$

Structural Requirements

$$\vec{F}_1 = \vec{f}(\vec{x}_0, t_0)$$

$$\vec{x}' = \vec{x}_0 + a_{21} \cdot \Delta t \cdot \vec{F}_1$$

Types:

- **vector type**, mostly, but not necessarily, some container like `vector<double>` (actually we have `state_type` and `deriv_type`)
- **time type**, usually `double`, but might be a multi-precision type
- **value type**, most likely the same as time type

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Function Call:

```
void rhs( const vector_type &x , vector_type &
          dxdt , const time_type t )
{ /* user defined */ }

rhs( x0 , F1 , t ); //memory allocation for F1?
```

- Memory allocation for temporary results (F , x')

Computational Requirements

$$\vec{x}_1 = \vec{x}_0 + b_1 \cdot \Delta t \cdot \vec{F}_1 + \cdots + b_s \cdot \Delta t \cdot \vec{F}_s$$

- vector-vector addition
- scalar-scalar multiplication
- scalar-vector multiplication

(\longrightarrow vector space)

Type Declarations

Tell ODEINT which types you are working with:

```
/* define your types */  
typedef vector<double> state_type;  
typedef vector<double> deriv_type;  
typedef double value_type;  
typedef double time_type;  
  
/* define your stepper algorithm */  
typedef runge_kutta4< state_type , value_type ,  
    deriv_type , time_type > stepper_type;
```

Reasonable standard values for the template parameters allows for:

```
typedef runge_kutta4<state_type> stepper_type;
```

Memory Allocation / Resizing

Two possible situations: dynamic size / fixed size `vector_type`

dynamic size - memory allocation required

- e.g. `vector<double>`
- declare type as resizable
- specialize resize template
- use `initially_resizer` or `always_resizer` in stepper algorithm

fixed size - memory allocation not required

- e.g. `array<double, N>`
- declare type as not resizable
- that's it

Declare Resizability

```
/* by default any type is not resizable */
template< class Container >
struct is_resizable
{
    typedef boost::false_type type;
    const static bool value = type::value;
};

/* specialization for std::vector */
template< class T, class A >
struct is_resizable< std::vector< T , A  > >
{
    typedef boost::true_type type;
    const static bool value = type::value;
};
```

To use a new dynamic sized type, this has to be specialized by the user.

Tell ODEINT how to resize

Again: only required if

`is_resizable<state_type>::type == boost::true_type.`

Class Template responsible for resizing:

```
template< class StateOut , class StateIn >
struct resize_impl
{
    /* standard implementation */
    static void resize( StateOut &x1 , const
        StateIn &x2 )
    {
        x1.resize( boost::size( x2 ) );
    }
};
```

For anything that does not support `boost::size` or `resize` the user must provide a specialization.

Scalar Computations

For the scalar types we require the following:

Assume:

```
time_type t , dt;  
value_type a1 , a2 , c;
```

Valid Expressions:

- `a1 = static_cast< value_type >(1)`
- `a1*a2`
- `a1/a2`
- `t + c*dt`
- `t + dt/c`
- `t += dt`

Vector Computations

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