TMC: Linear Equations

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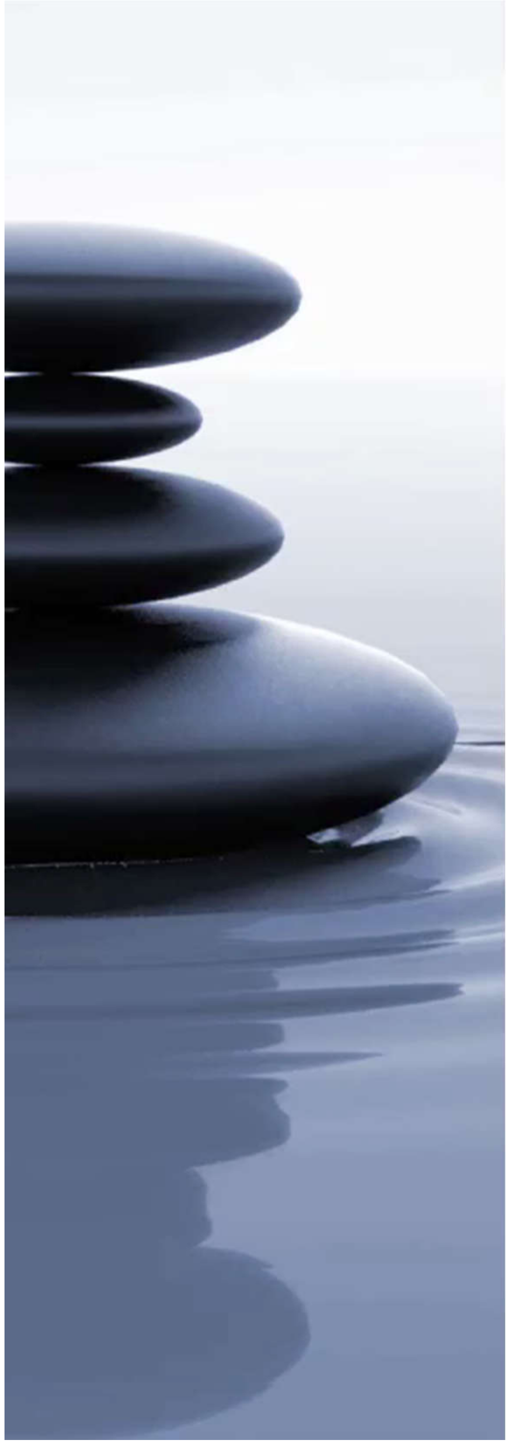
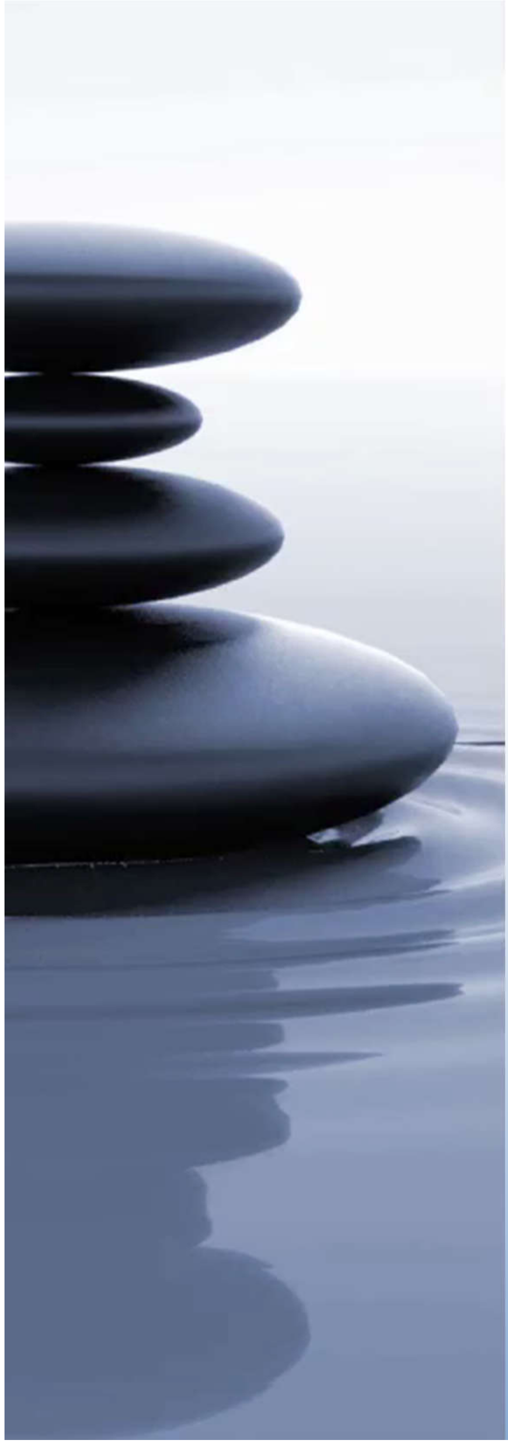


MOTIVATION

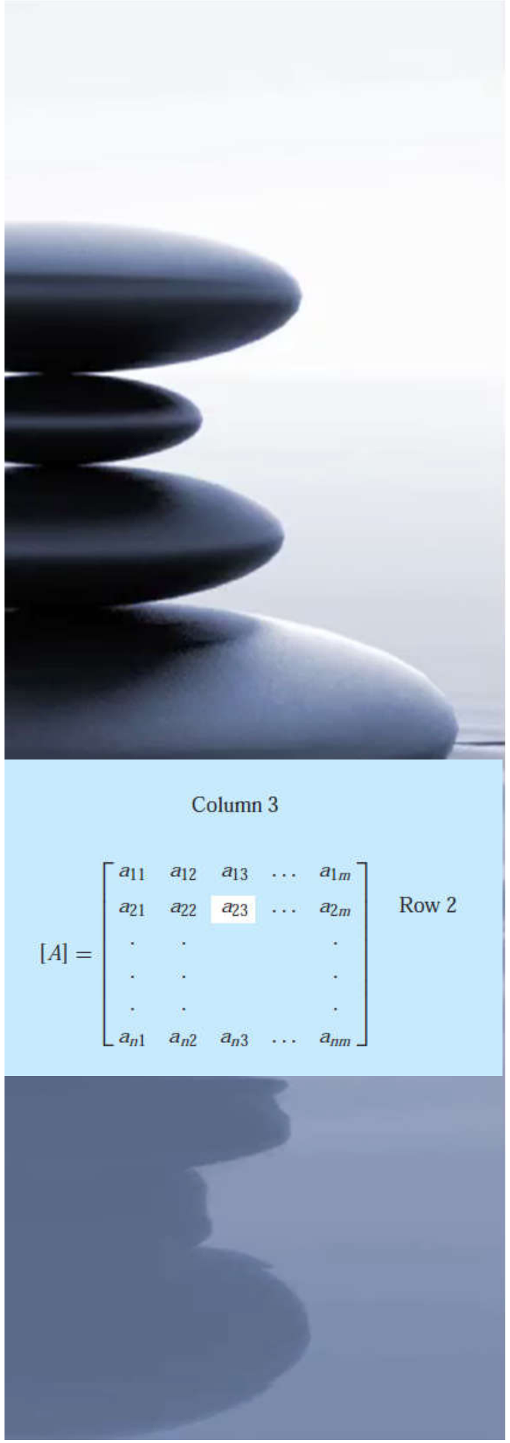
$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1N}x_N &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2N}x_N &= b_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3N}x_N &= b_3 \\
\cdots \qquad & \quad \cdots \\
a_{M1}x_1 + a_{M2}x_2 + a_{M3}x_3 + \cdots + a_{MN}x_N &= b_M
\end{aligned}
$$

In matrix form: *A x = b*

$$
A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}
$$



MATHEMATICAL BACKGROUND



$$
\begin{array}{cc}
 & \begin{array}{ccccc} & & \text{Column 3} & & \end{array} \\
[A] = & \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1m} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2m} \\ \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & & \cdot \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nm} \end{bmatrix} \begin{array}{l} \\ \text{Row 2} \\ \\ \\ \\ \\ \end{array}
\end{array}
$$

Mathematical Background

A *matrix* consists of a rectangular array of elements represented by a single symbol.

[A] is the shorthand notation for the matrix and a_{ij} designates an individual *element* of the matrix.

A horizontal set of elements is called a *row* and a vertical set is called a *column*. The first subscript i always designates the number of the row in which the element lies. The second subscript j designates the column.

A matrix has n rows and m columns and is said to have a dimension of n by m (or n × m). It is referred to as an *n by m matrix*

Matrices where n = m are called *square matrices*.

The diagonal consisting of the elements a_{ii} is termed the *principal or main diagonal* of the matrix.

Special Types of Square Matrices

There are a number of special forms of square matrices that are important and should be noted:

A *symmetric matrix* is one where $a_{ij} = a_{ji}$ for all i's and j's. For example,

$$[A] = \begin{bmatrix} 5 & 1 & 2 \\ 1 & 3 & 7 \\ 2 & 7 & 8 \end{bmatrix}$$

is a 3 by 3 symmetric matrix.

A *diagonal matrix* is a square matrix where all elements off the main diagonal are equal to zero, as in

$$[A] = \begin{bmatrix} a_{11} & & & \\ & a_{22} & & \\ & & a_{33} & \\ & & & a_{44} \end{bmatrix}$$

Note that where large blocks of elements are zero, they are left blank.

An *identity matrix* is a diagonal matrix where all elements on the main diagonal are equal to 1, as in

$$[I] = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

The symbol $[I]$ is used to denote the identity matrix. The identity matrix has properties similar to unity.

An *upper triangular matrix* is one where all the elements below the main diagonal are zero, as in

$$[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ & a_{22} & a_{23} & a_{24} \\ & & a_{33} & a_{34} \\ & & & a_{44} \end{bmatrix}$$

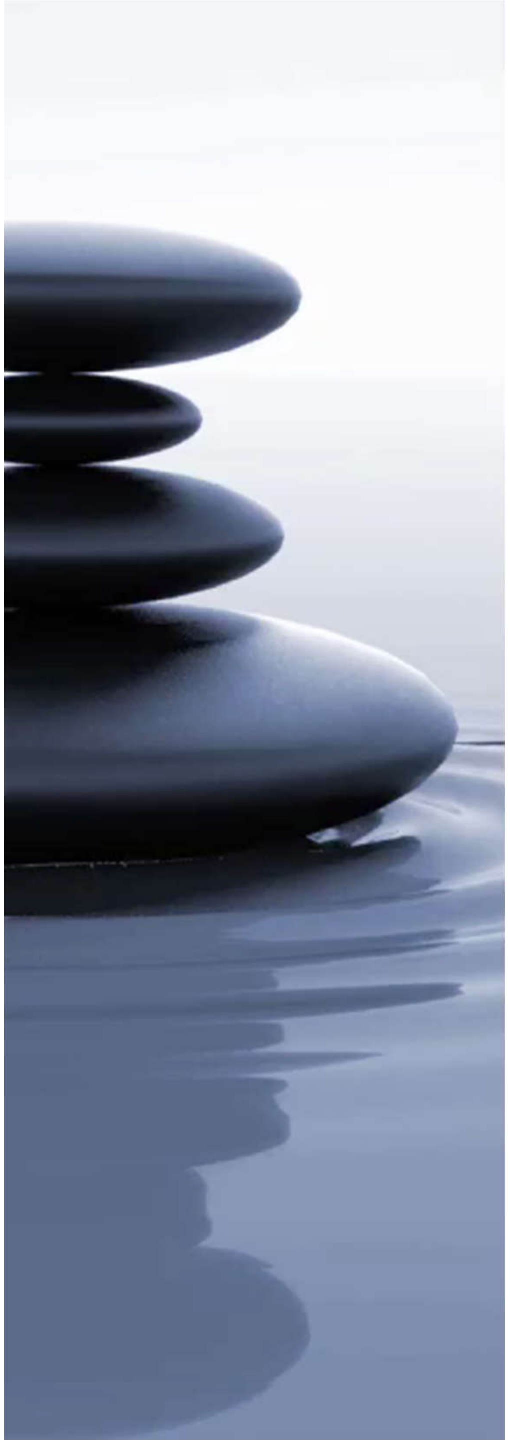
A *lower triangular matrix* is one where all elements above the main diagonal are zero, as in

$$[A] = \begin{bmatrix} a_{11} & & & \\ a_{21} & a_{22} & & \\ a_{31} & a_{32} & a_{33} & \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

A *banded matrix* has all elements equal to zero, with the exception of a band centered on the main diagonal:

$$[A] = \begin{bmatrix} a_{11} & a_{12} & & \\ a_{21} & a_{22} & a_{23} & \\ & a_{32} & a_{33} & a_{34} \\ & & a_{43} & a_{44} \end{bmatrix}$$

The above matrix has a bandwidth of 3 and is given a special name—the *tridiagonal matrix*.



Matrix Operating Rules

Addition

$cij = aij + bij$

Subtraction

$dij = eij - fij$

Both addition and subtraction are *commutative:*

$[A] + [B] = [B] + [A]$

Addition and subtraction are also *associative*, that is,

$([A] + [B]) + [C] = [A] + ([B] + [C])$

The *multiplication* of a matrix $[A]$ by a scalar g is obtained by multiplying every element of $[A]$ by g, as in

$$[D] = g[A] = \begin{bmatrix} ga_{11} & ga_{12} & \cdots & ga_{1m} \\ ga_{21} & ga_{22} & \cdots & ga_{2m} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ ga_{n1} & ga_{n2} & \cdots & ga_{nm} \end{bmatrix}$$

A Simple Method for Multiplying Two Matrices

Although Eq. (PT3.2) is well suited for implementation on a computer, it is not the simplest means for visualizing the mechanics of multiplying two matrices. What follows gives more tangible expression to the operation.

Suppose that we want to multiply $[X]$ by $[Y]$ to yield $[Z]$,

$$[Z] = [X][Y] = \begin{bmatrix} 3 & 1 \\ 8 & 6 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 5 & 9 \\ 7 & 2 \end{bmatrix}$$

A simple way to visualize the computation of $[Z]$ is to raise $[Y]$, as in

$$\begin{array}{ccc} & \Uparrow & \\ & \begin{bmatrix} 5 & 9 \\ 7 & 2 \end{bmatrix} & \leftarrow [Y] \\ [X] \rightarrow \begin{bmatrix} 3 & 1 \\ 8 & 6 \\ 0 & 4 \end{bmatrix} & \begin{bmatrix} \\ ? \\ \\ \end{bmatrix} & \leftarrow [Z] \end{array}$$

Now the answer $[Z]$ can be computed in the space vacated by $[Y]$. This format has utility because it aligns the appropriate rows and columns that are to be multiplied. For example, according to Eq. (PT3.2), the element z_{11} is obtained by multiplying the first row of $[X]$ by the first column of $[Y]$. This amounts to adding the product of x_{11} and y_{11} to the product of x_{12} and y_{21}, as in

$$\begin{array}{ccc} & & \begin{bmatrix} 5 & 9 \\ 7 & 2 \end{bmatrix} \\ & & \downarrow \\ \begin{bmatrix} \mathbf{3} & \mathbf{1} \\ 8 & 6 \\ 0 & 4 \end{bmatrix} & \rightarrow & \begin{bmatrix} \mathbf{3 \times 5 + 1 \times 7 = 22} & \\ & \\ & \end{bmatrix} \end{array}$$

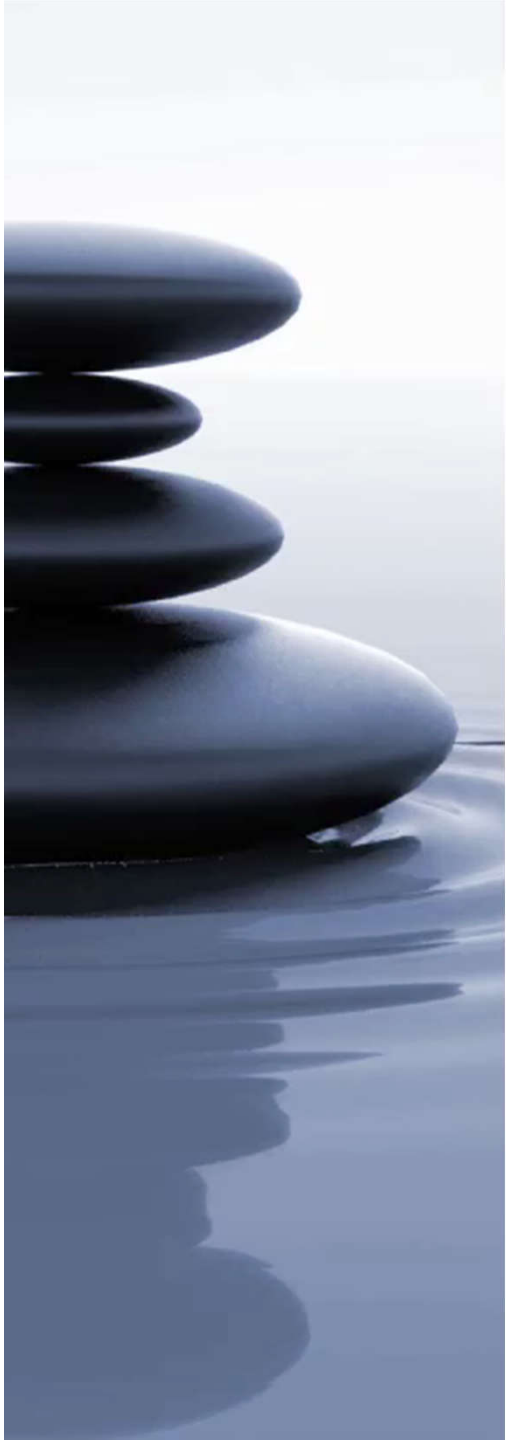
Thus, z_{11} is equal to 22. Element z_{21} can be computed in a similar fashion, as in

$$\begin{array}{ccc} & & \begin{bmatrix} 5 & 9 \\ 7 & 2 \end{bmatrix} \\ & & \downarrow \\ \begin{bmatrix} 3 & 1 \\ \mathbf{8} & \mathbf{6} \\ 0 & 4 \end{bmatrix} & \rightarrow & \begin{bmatrix} 22 & \\ \mathbf{8 \times 5 + 6 \times 7 = 82} & \\ & \end{bmatrix} \end{array}$$

The computation can be continued in this way, following the alignment of the rows and columns, to yield the result

$$[Z] = \begin{bmatrix} 22 & 29 \\ 82 & 84 \\ 28 & 8 \end{bmatrix}$$

Note how this simple method makes it clear why it is impossible to multiply two matrices if the number of columns of the first matrix does not equal the number of rows in the second matrix. Also, note how it demonstrates that the order of multiplication matters (that is, matrix multiplication is not commutative).



Matrix Operating Rules

The *transpose* of a matrix involves transforming its rows into columns and its columns into rows.

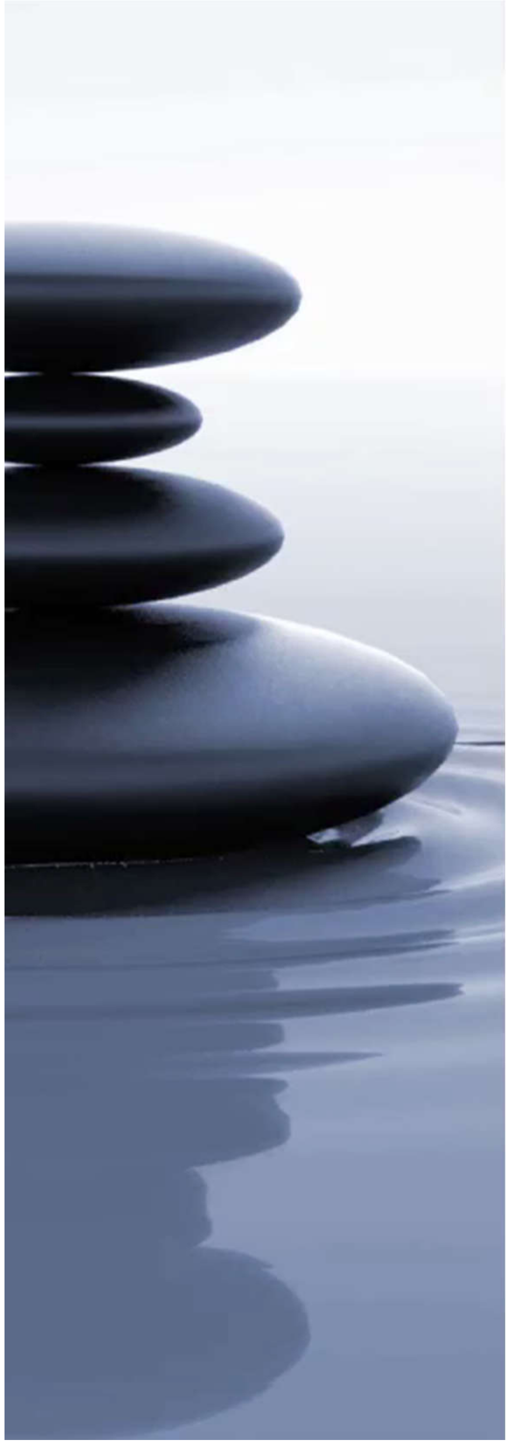
$$[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

the transpose, designated $[A]^T$, is defined as

$$[A]^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} & a_{41} \\ a_{12} & a_{22} & a_{32} & a_{42} \\ a_{13} & a_{23} & a_{33} & a_{43} \\ a_{14} & a_{24} & a_{34} & a_{44} \end{bmatrix}$$

The *trace* of a matrix is the sum of the elements on its principal diagonal. It is designated as tr[A]

$$tr[A] = \sum_{i=1}^{n} a_{ii}$$



Existence and Solutions

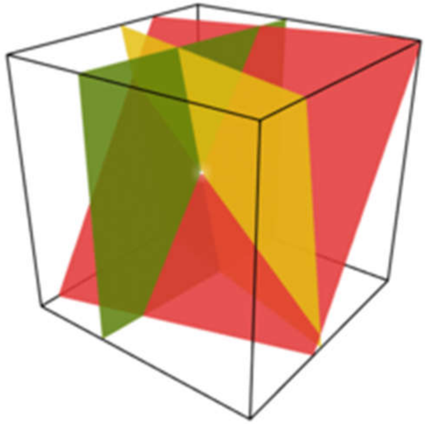
$M = N$ (square matrix) and nonsingular, a unique solution exists

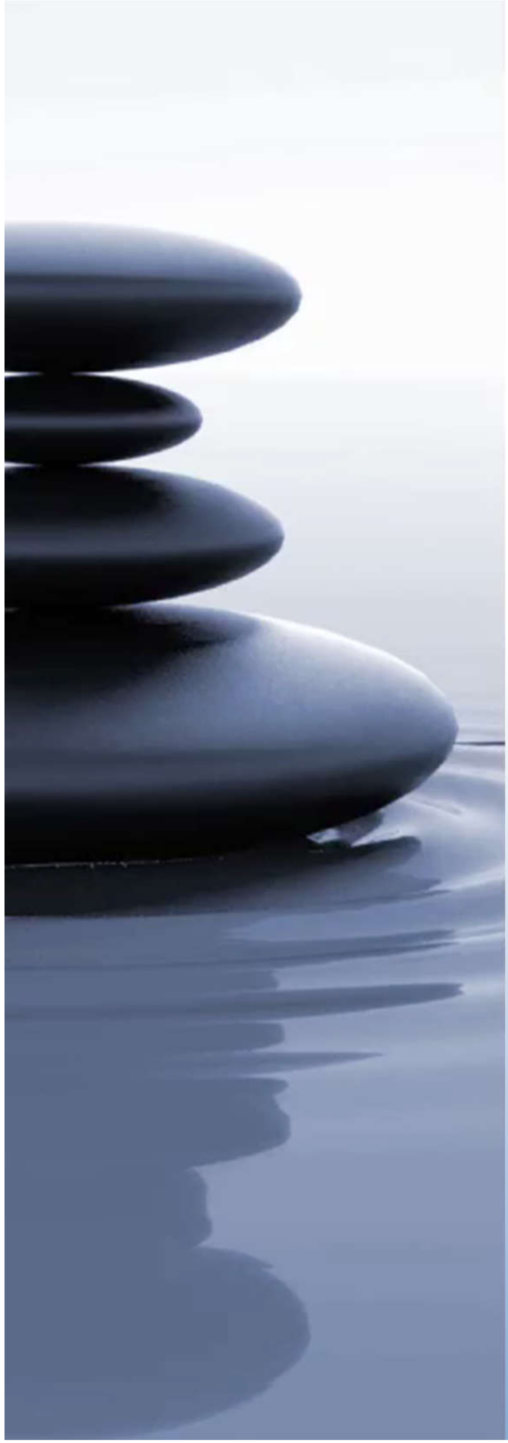
$M < N$, more variables than equations – need to find the null space of A, $A\,x = 0$ (e.g. SVD)

$M > N$, one can ask for least-square solution, e.g., from $(A^T A)\,x = (A^T b)$

Example:

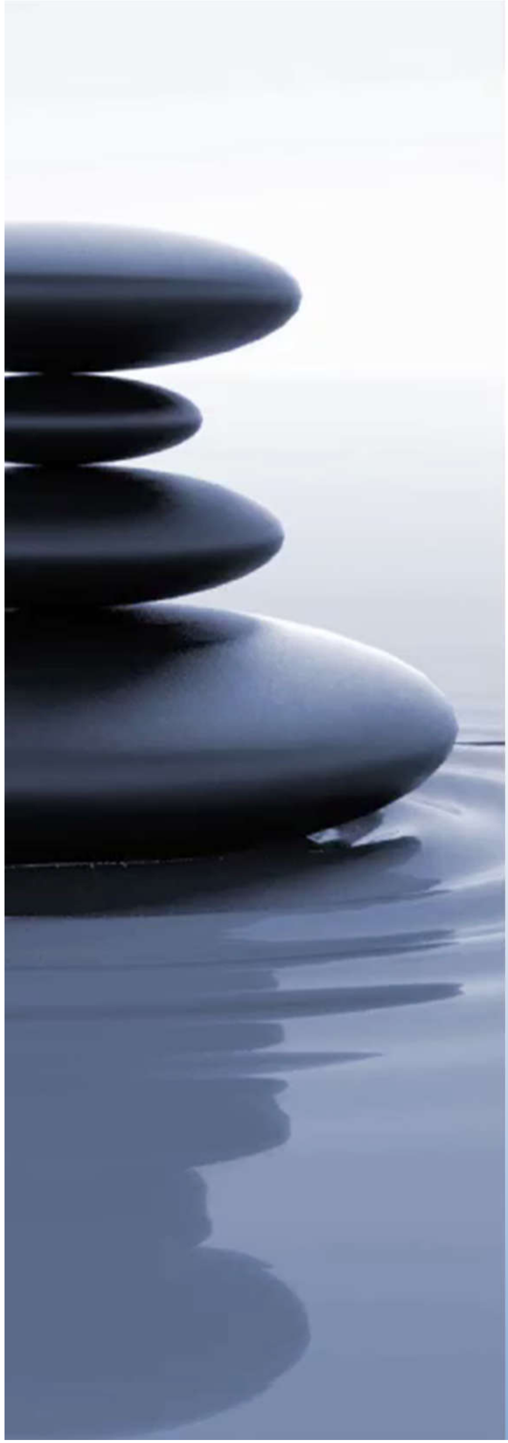
$$
\begin{aligned}
3x + 2y - z &= 1 \\
2x - 2y + 4z &= -2 \\
-x + \tfrac{1}{2}y - z &= 0
\end{aligned}
$$





Part A

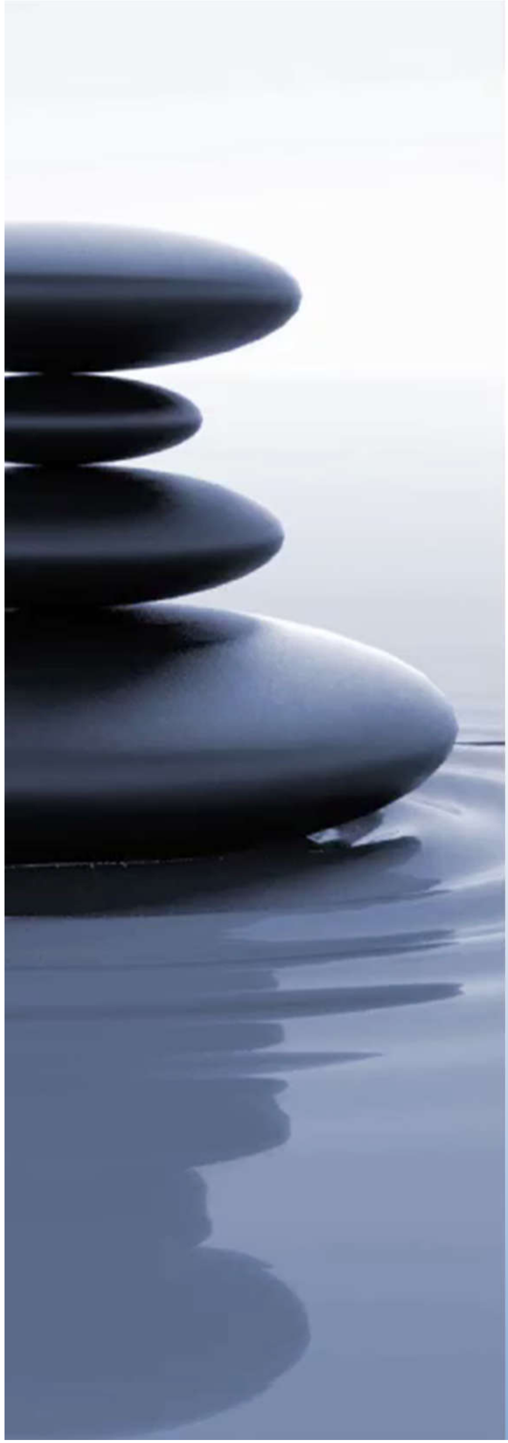
ELIMINATION METHODS



Method of Elimination

The basic strategy is to successively solve one of the equations of the set for one of the unknowns and to eliminate that variable from the remaining equations by substitution.

The elimination of unknowns can be extended to systems with more than two or three equations; however, the method becomes extremely tedious to solve by hand.



1. Graphical Method

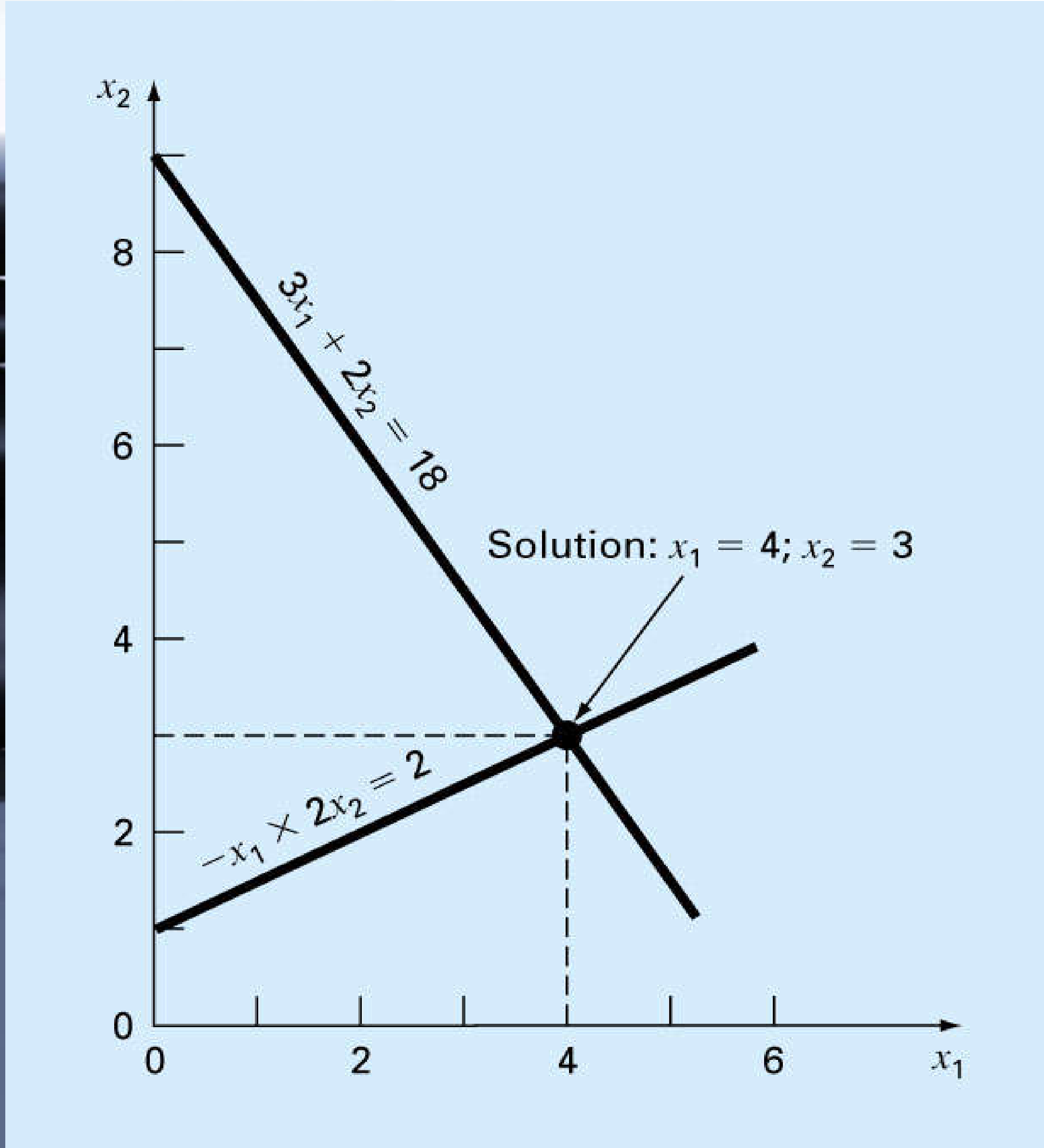
For two equations:

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2$$

Solve both equations for $x_{2:}$

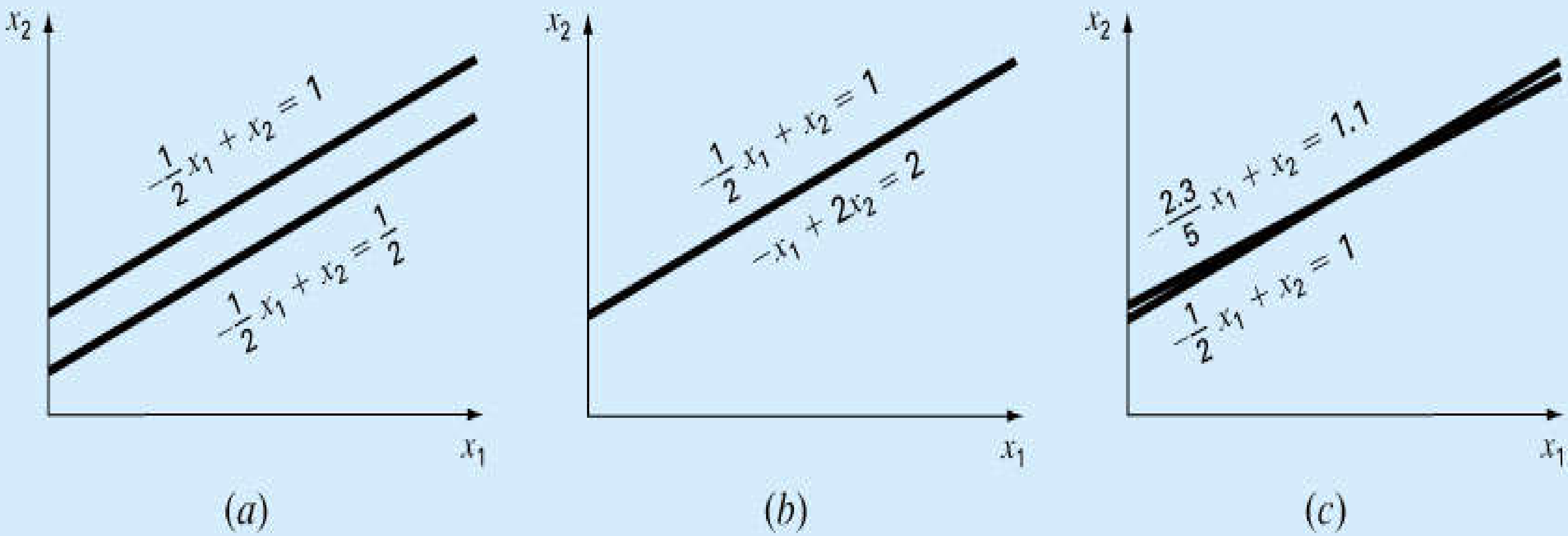
$$x_2 = -\left(\frac{a_{11}}{a_{12}}\right)x_1 + \frac{b_1}{a_{12}} \quad \Rightarrow \quad x_2 = (\text{slope})x_1 + \text{intercept}$$

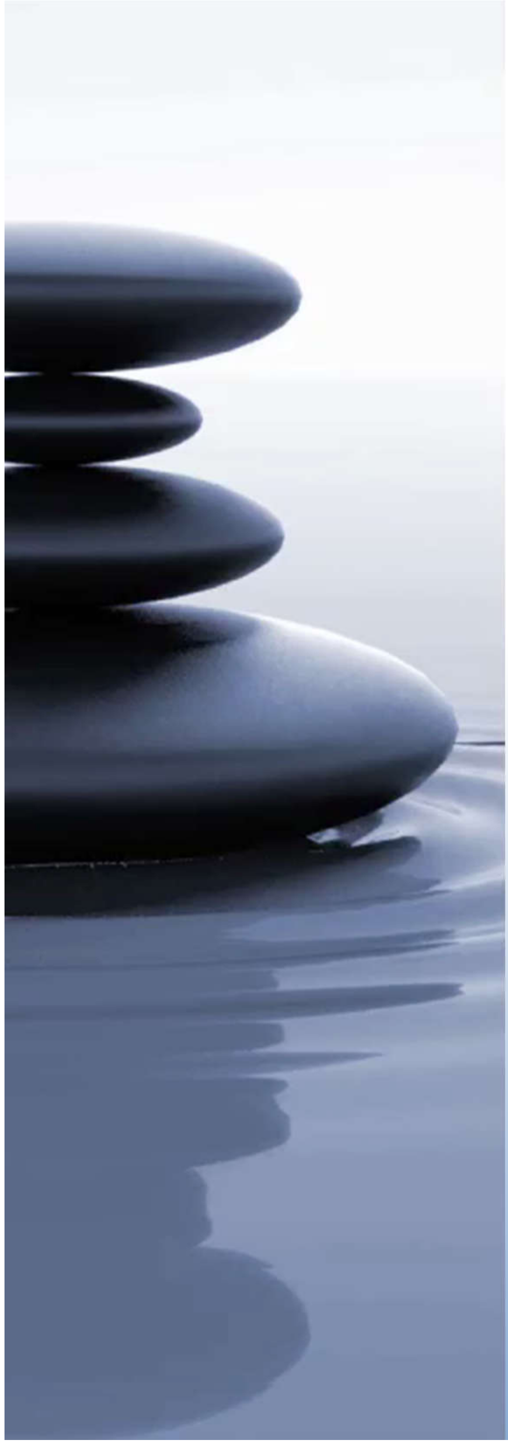
$$x_2 = -\left(\frac{a_{21}}{a_{22}}\right)x_1 + \frac{b_2}{a_{22}}$$



Graphical solution of a set of two simultaneous linear algebraic equations. The intersection of the lines represents the solution.

Graphical depiction of singular and ill-conditioned systems: (a) no solution, (b) infinite solutions, and (c) ill-conditioned system where the slopes are so close that the point of intersection is difficult to detect visually.





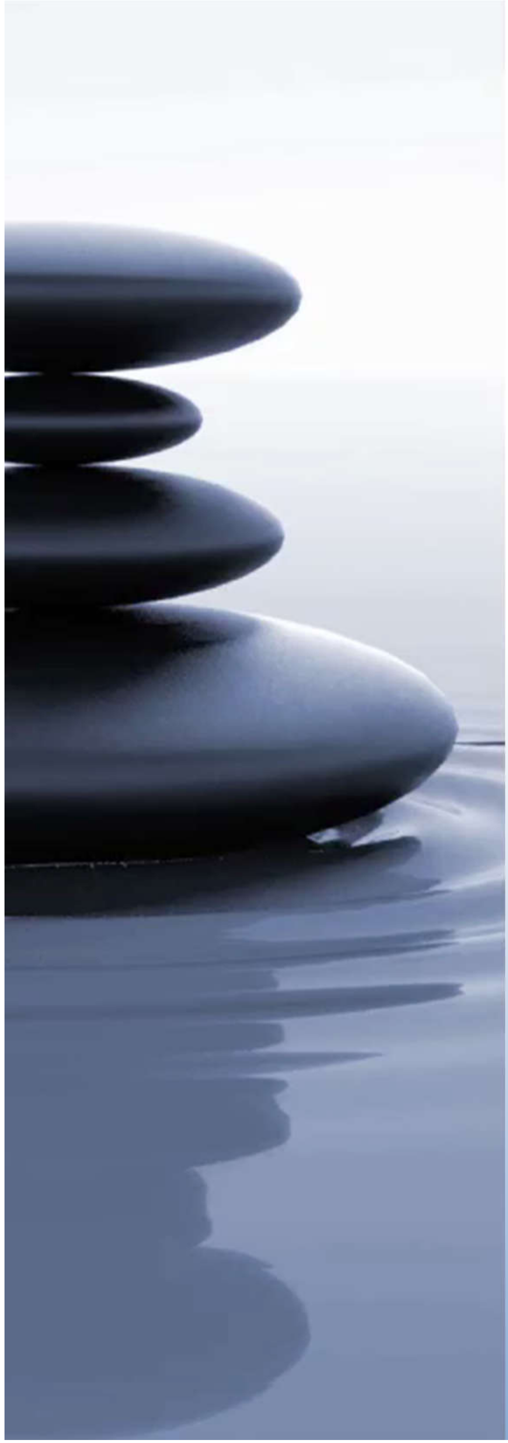
2. Determinants and Cramer’s Rule

Determinant can be illustrated for a set of three equations:

$$[A]\{x\} = \{B\}$$

Where [A] is the coefficient matrix:

$$[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$



Determinant

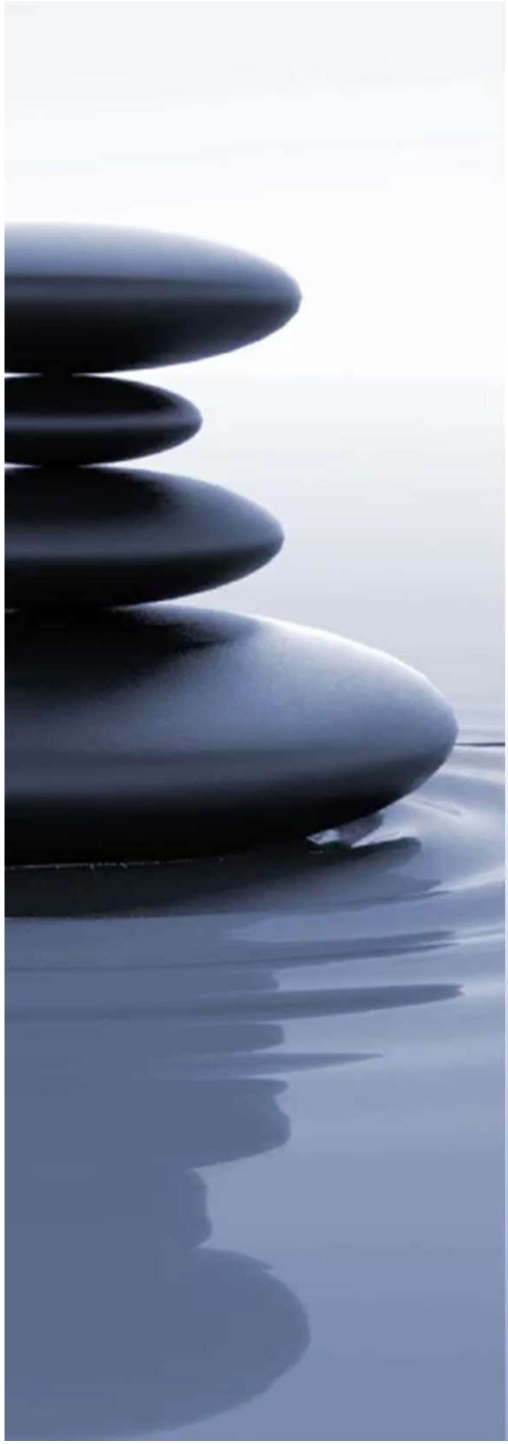
$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$D_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}\, a_{33} - a_{32}\, a_{23}$$

$$D_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = a_{21}\, a_{33} - a_{31}\, a_{23}$$

$$D_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = a_{21}\, a_{32} - a_{31}\, a_{22}$$

$$D = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

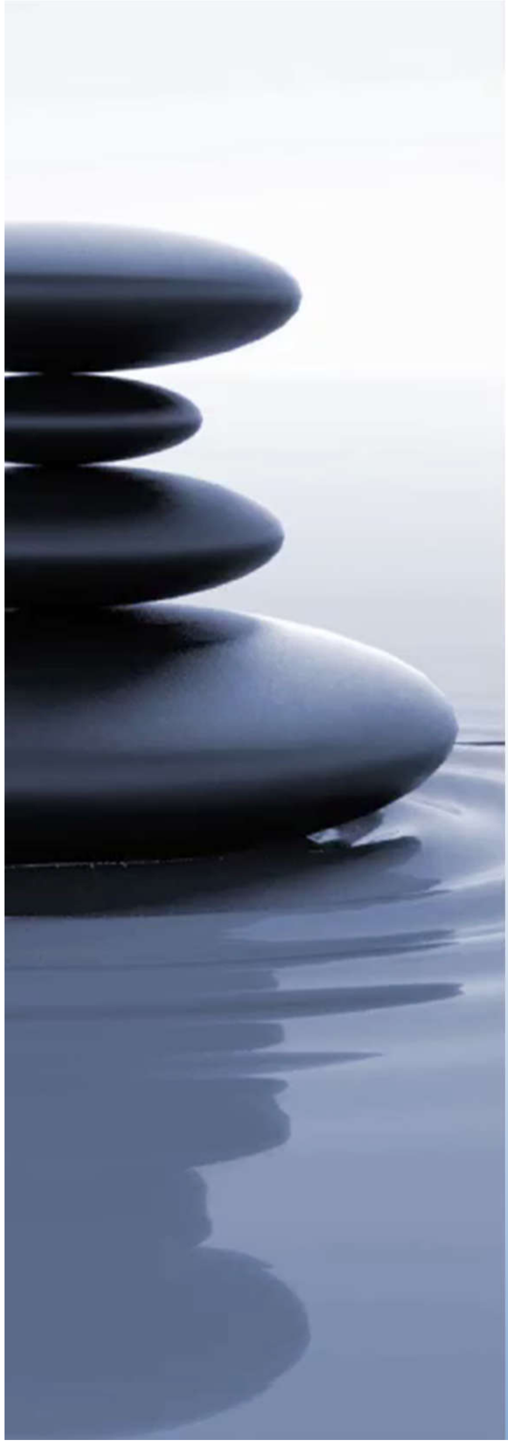


Cramer’s rule

Cramer’s rule expresses the solution of a systems of linear equations in terms of ratios of determinants of the array of coefficients of the equations.

For example, x_1 would be computed as:

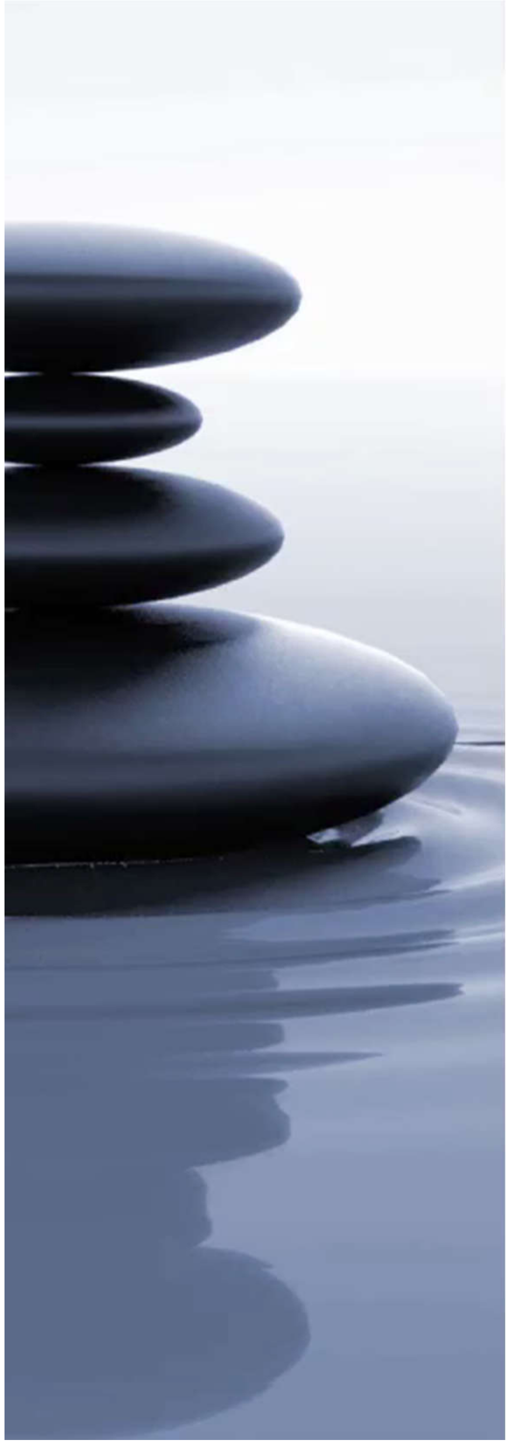
$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}}{D}$$



Example #1

$$\begin{aligned} x + 3y - 2z &= 5 \\ 3x + 5y + 6z &= 7 \\ 2x + 4y + 3z &= 8 \end{aligned}$$

$$x = \frac{\begin{vmatrix} 5 & 3 & -2 \\ 7 & 5 & 6 \\ 8 & 4 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & 3 & -2 \\ 3 & 5 & 6 \\ 2 & 4 & 3 \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} 1 & 5 & -2 \\ 3 & 7 & 6 \\ 2 & 8 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & 3 & -2 \\ 3 & 5 & 6 \\ 2 & 4 & 3 \end{vmatrix}}, \quad z = \frac{\begin{vmatrix} 1 & 3 & 5 \\ 3 & 5 & 7 \\ 2 & 4 & 8 \end{vmatrix}}{\begin{vmatrix} 1 & 3 & -2 \\ 3 & 5 & 6 \\ 2 & 4 & 3 \end{vmatrix}}.$$



Example #2

Use Cramer's rule to solve

0.3x1 + 0.52x2 + x3 = −0.01

0.5x1 + x2 + 1.9x3 = 0.67

0.1x1 + 0.3x2 + 0.5x3 = −0.44

Solution. The determinant D can be written as

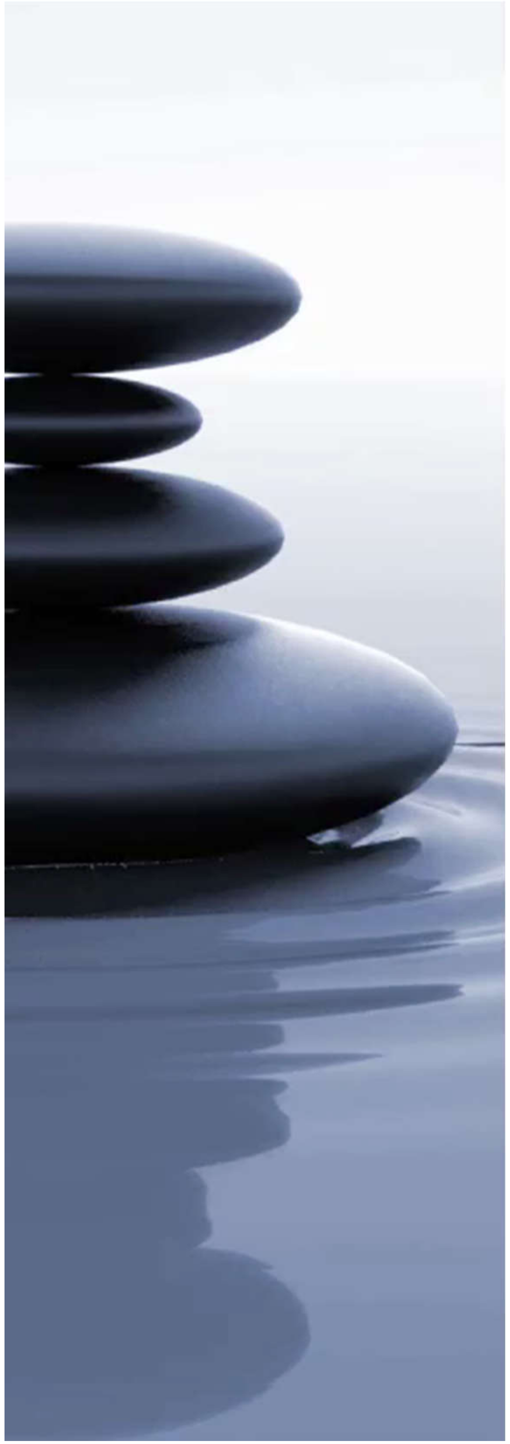
$$D = \begin{vmatrix} 0.3 & 0.52 & 1 \\ 0.5 & 1 & 1.9 \\ 0.1 & 0.3 & 0.5 \end{vmatrix}$$

The minors are [Eq. (9.3)]

$$A_1 = \begin{vmatrix} 1 & 1.9 \\ 0.3 & 0.5 \end{vmatrix} = 1(0.5) - 1.9(0.3) = -0.07$$

$$A_2 = \begin{vmatrix} 0.5 & 1.9 \\ 0.1 & 0.5 \end{vmatrix} = 0.5(0.5) - 1.9(0.1) = 0.06$$

$$A_3 = \begin{vmatrix} 0.5 & 1 \\ 0.1 & 0.3 \end{vmatrix} = 0.5(0.3) - 1(0.1) = 0.05$$



These can be used to evaluate the determinant, as in

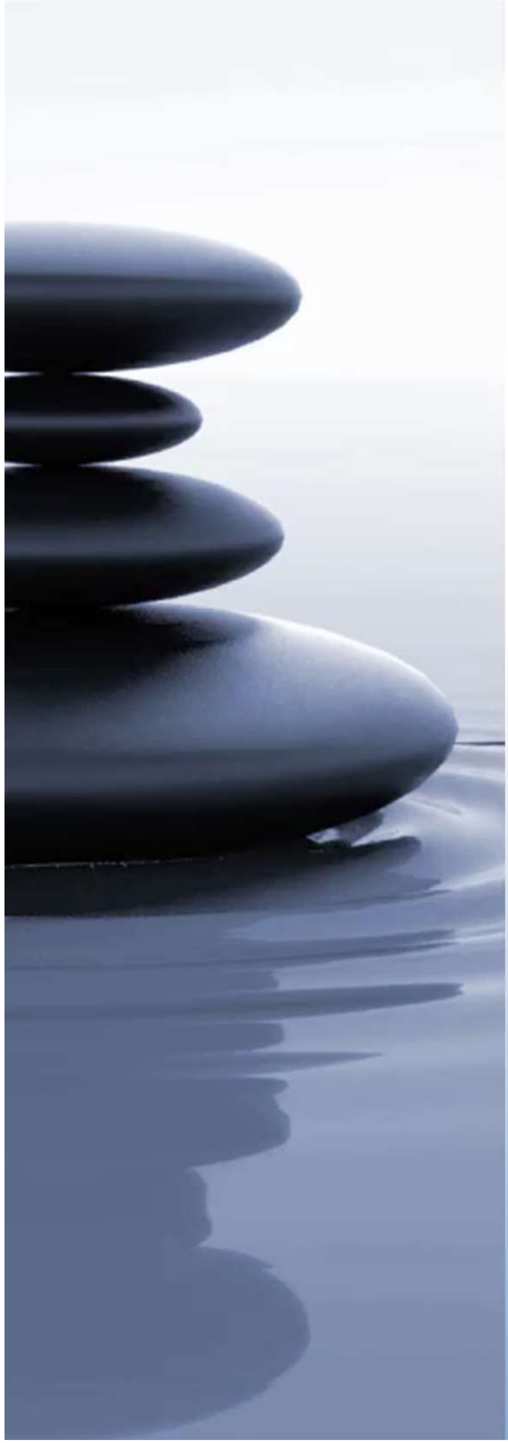
$$D = 0.3(-0.07) - 0.52(0.06) + 1(0.05) = -0.0022$$

Applying Eq. (9.5), the solution is

$$x_1 = \frac{\begin{vmatrix} -0.01 & 0.52 & 1 \\ 0.67 & 1 & 1.9 \\ -0.44 & 0.3 & 0.5 \end{vmatrix}}{-0.0022} = \frac{0.03278}{-0.0022} = -14.9$$

$$x_2 = \frac{\begin{vmatrix} 0.3 & -0.01 & 1 \\ 0.5 & 0.67 & 1.9 \\ 0.1 & -0.44 & 0.5 \end{vmatrix}}{-0.0022} = \frac{0.0649}{-0.0022} = -29.5$$

$$x_3 = \frac{\begin{vmatrix} 0.3 & 0.52 & -0.01 \\ 0.5 & 1 & 0.67 \\ 0.1 & 0.3 & -0.44 \end{vmatrix}}{-0.0022} = \frac{-0.04356}{-0.0022} = 19.8$$



2. Naive Gauss Elimination

Extension of *method of elimination* to large sets of equations by developing a systematic scheme or algorithm to eliminate unknowns and to back substitute.

As in the case of the solution of two equations, the technique for *n* equations consists of two phases:

- Forward elimination of unknowns
- Back substitution

$$\left[\begin{array}{ccc:c} a_{11} & a_{12} & a_{13} & c_1 \\ a_{21} & a_{22} & a_{23} & c_2 \\ a_{31} & a_{32} & a_{33} & c_3 \end{array}\right]$$

$$\Downarrow$$

$$\left[\begin{array}{ccc:c} a_{11} & a_{12} & a_{13} & c_1 \\ & a'_{22} & a'_{23} & c'_2 \\ & & a''_{33} & c''_3 \end{array}\right]$$

Forward elimination

$$\Downarrow$$

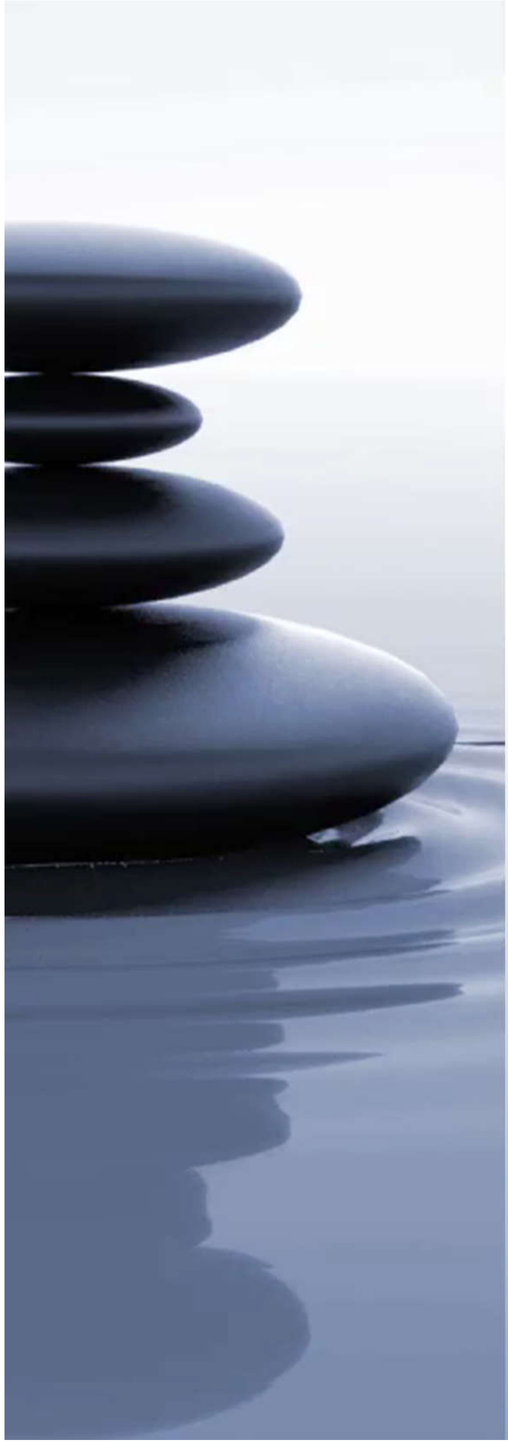
$$x_3 = c''_3/a''_{33}$$

$$x_2 = (c'_2 - a'_{23}x_3)/a'_{22}$$

$$x_1 = (c_1 - a_{12}x_2 - a_{13}x_3)/a_{11}$$

Back substitution

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & 9 \\ 1 & 1 & -1 & 1 \\ 3 & 11 & 5 & 35 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 3 & 1 & 9 \\ 0 & -2 & -2 & -8 \\ 0 & 2 & 2 & 8 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 3 & 1 & 9 \\ 0 & -2 & -2 & -8 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -2 & -3 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$



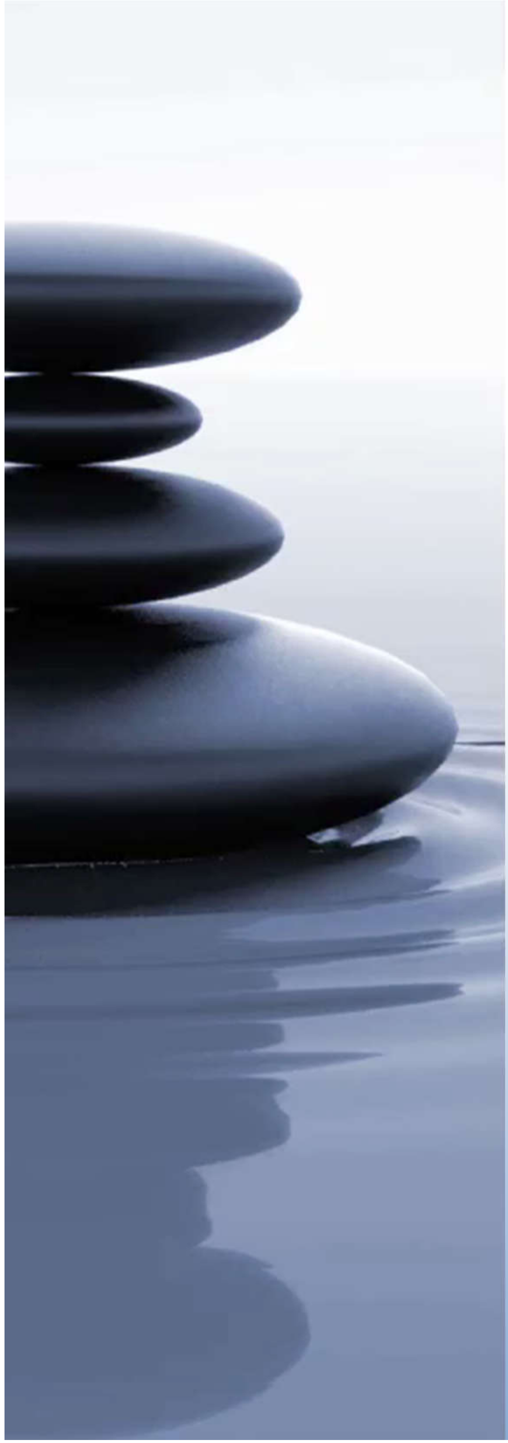
3. Gauss-Jordan Elimination

Basic facts about linear equations

Interchanging any two rows of A and b does not change the solution x

Replace a row by a linear combination of itself and any other row does not change x

Interchange column permutes the solution



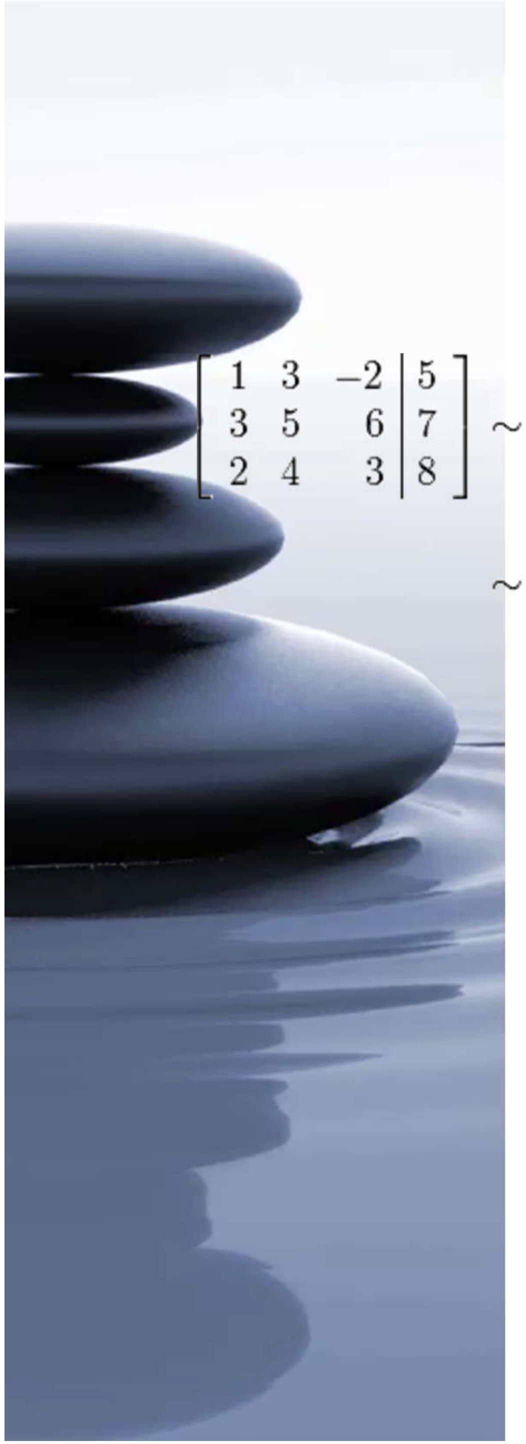
Gauss-Jordan Elimination

When an unknown is eliminated, it is eliminated from all other equations rather than just the subsequent ones.

All rows are normalized by dividing them by their pivot elements.

Elimination step results in an identity matrix.

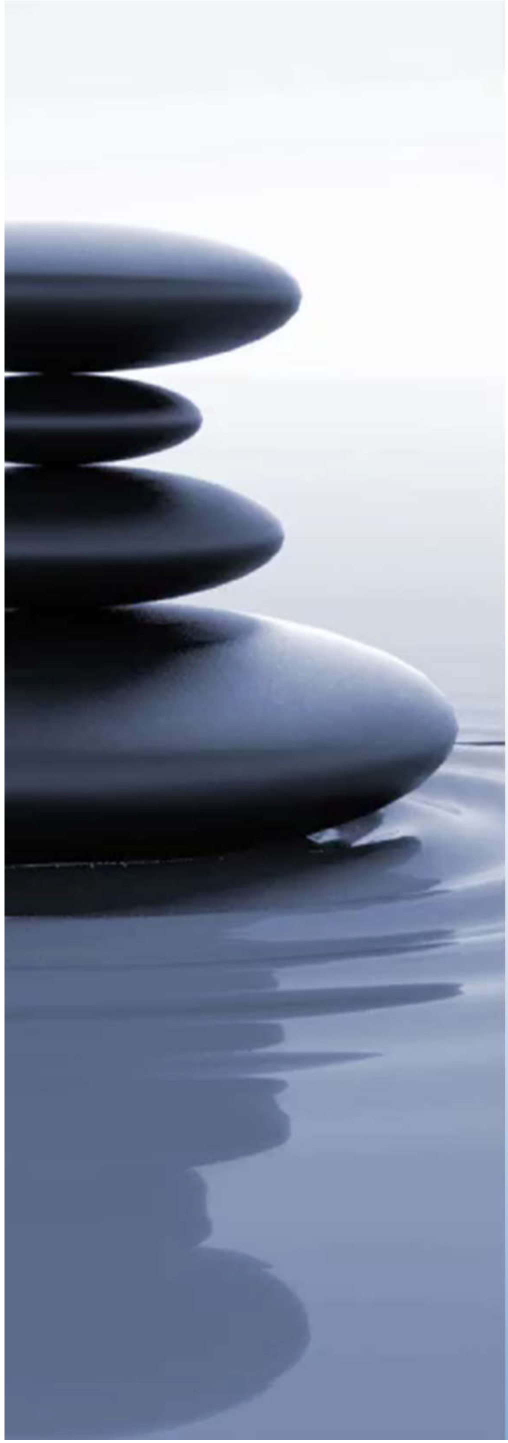
Consequently, it is not necessary to employ back substitution to obtain solution.



Example #3

$$\left[\begin{array}{ccc|c} 1 & 3 & -2 & 5 \\ 3 & 5 & 6 & 7 \\ 2 & 4 & 3 & 8 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & -2 & 5 \\ 0 & -4 & 12 & -8 \\ 2 & 4 & 3 & 8 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & -2 & 5 \\ 0 & -4 & 12 & -8 \\ 0 & -2 & 7 & -2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & -2 & 5 \\ 0 & 1 & -3 & 2 \\ 0 & -2 & 7 & -2 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 3 & -2 & 5 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & -2 & 5 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & 0 & 9 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -15 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & 2 \end{array}\right].$$

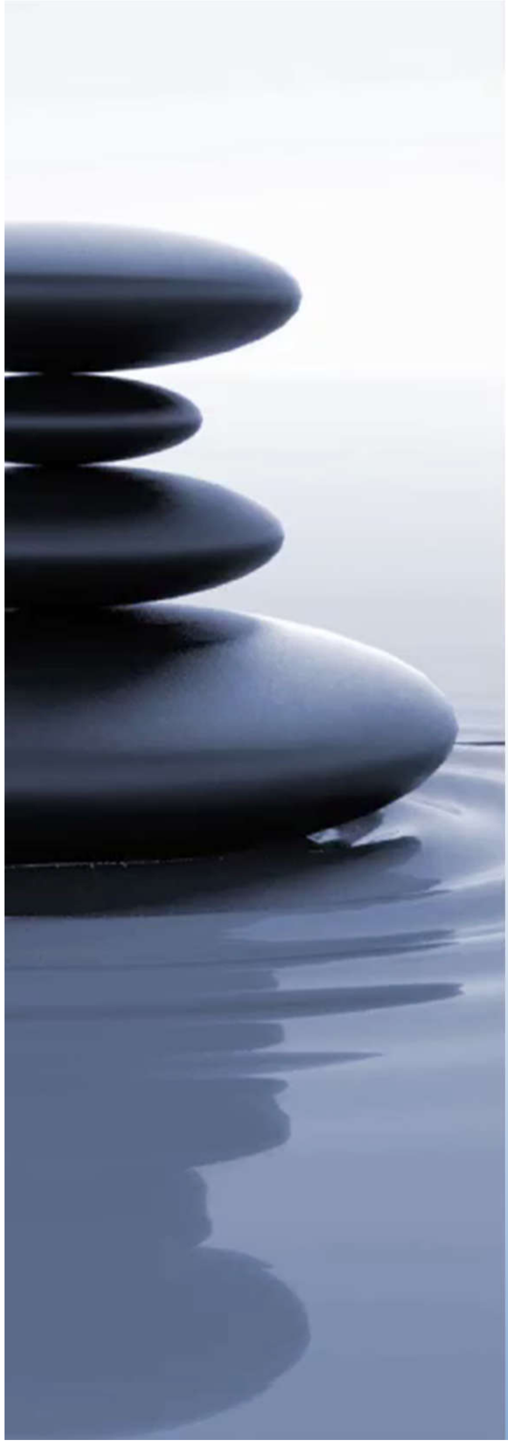


Pitfalls of Elimination Methods

Division by zero. It is possible that during both elimination and back-substitution phases a division by zero can occur.

Round-off errors. makes the system singular, or numerical instability makes the answer wrong

Ill-conditioned systems. Systems where small changes in coefficients result in large changes in the solution. Alternatively, it happens when two or more equations are nearly identical, resulting a wide ranges of answers to approximately satisfy the equations. Since round off errors can induce small changes in the coefficients, these changes can lead to large solution errors.



Techniques for Improving Solutions

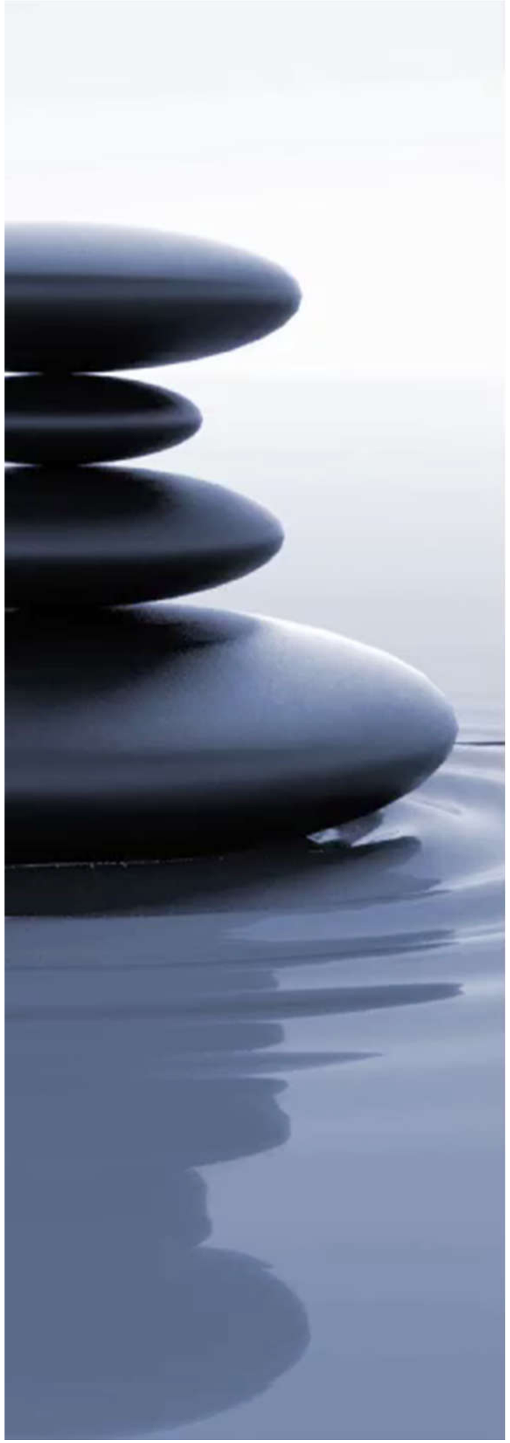
Use of more significant figures.

1. Pivoting. If a pivot element is zero, normalization step leads to division by zero. The same problem may arise, when the pivot element is close to zero. Problem can be avoided:

Partial pivoting. Switching the rows so that the largest element is the pivot element.

Complete pivoting. Searching for the largest element in all rows and columns then switching.

2. Scaling



Example #4

Use Gauss elimination to solve

$0.0003x1 + 3.0000x2 = 2.0001$

$1.0000x1 + 1.0000x2 = 1.0000$

Note that in this form the first pivot element, a11 = 0.0003, is very close to zero. Then repeat the computation, but partial pivot by reversing the order of the equations. The exact solution is x1 = 1/3 and x2 = 2/3.

Multiplying the first equation by 1/(0.0003) yields

$x1 + 10{,}000x2 = 6667$

which can be used to eliminate *x*1 from the second equation:

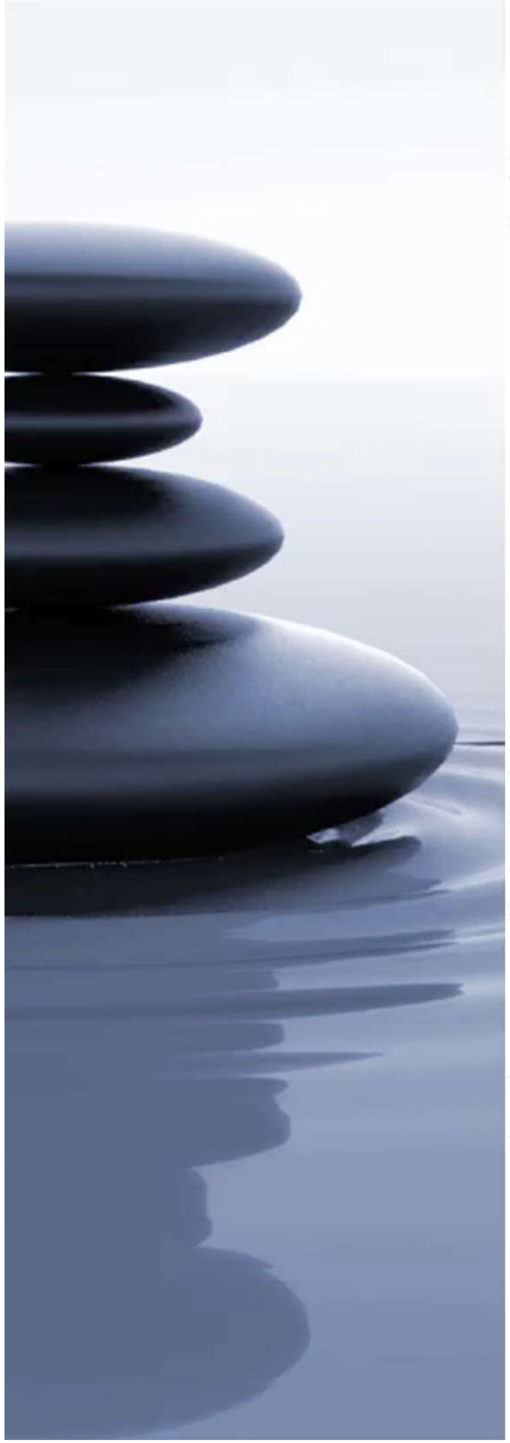
$-9999x2 = -6666$

which can be solved for

$$x_2 = \frac{2}{3}$$

This result can be substituted back into the first equation to evaluate x_1:

$$x_1 = \frac{2.0001 - 3(2/3)}{0.0003} \qquad \text{(E9.9.1}$$



However, due to subtractive cancellation, the result is very sensitive to the number of significant figures carried in the computation:

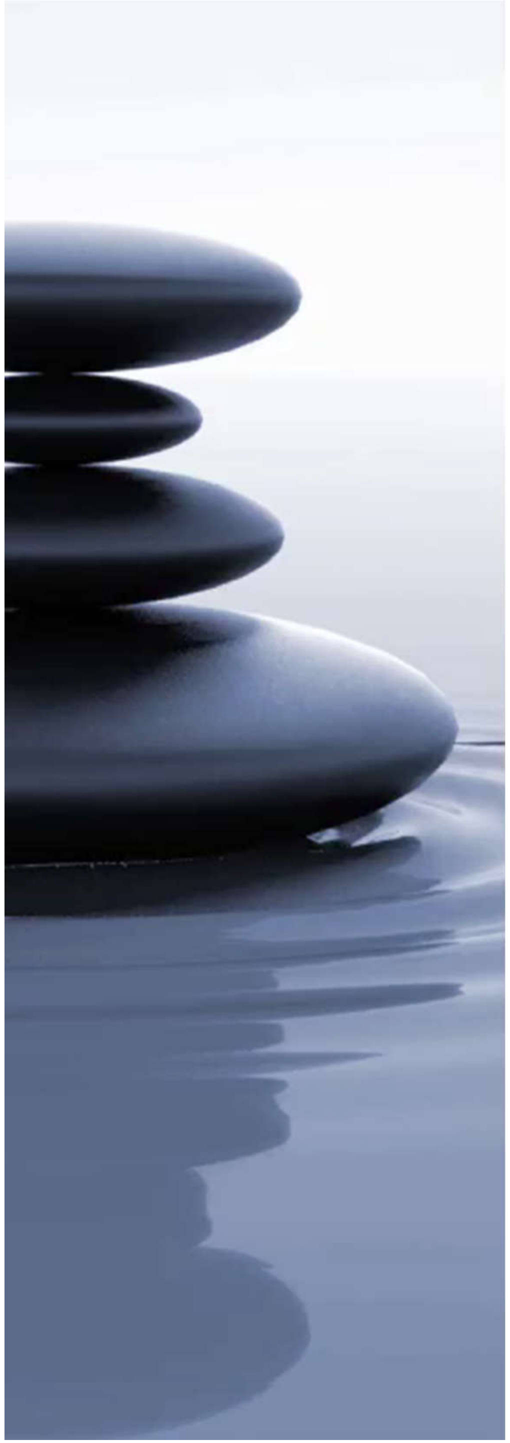
Significant Figures	x_2	x_1	Absolute Value of Percent Relative Error for x_1
3	0.667	−3.33	1099
4	0.6667	0.0000	100
5	0.66667	0.30000	10
6	0.666667	0.330000	1
7	0.6666667	0.3330000	0.1

Note how the solution for x_1 is highly dependent on the number of significant figures. This is because in Eq. (E9.9.1), we are subtracting two almost-equal numbers. On the other hand, if the equations are solved in reverse order, the row with the larger pivot element is normalized. The equations are

$$1.0000x_1 + 1.0000x_2 = 1.0000$$
$$0.0003x_1 + 3.0000x_2 = 2.0001$$

Elimination and substitution yield $x_2 = 2/3$. For different numbers of significant figures, x_1 can be computed from the first equation, as in

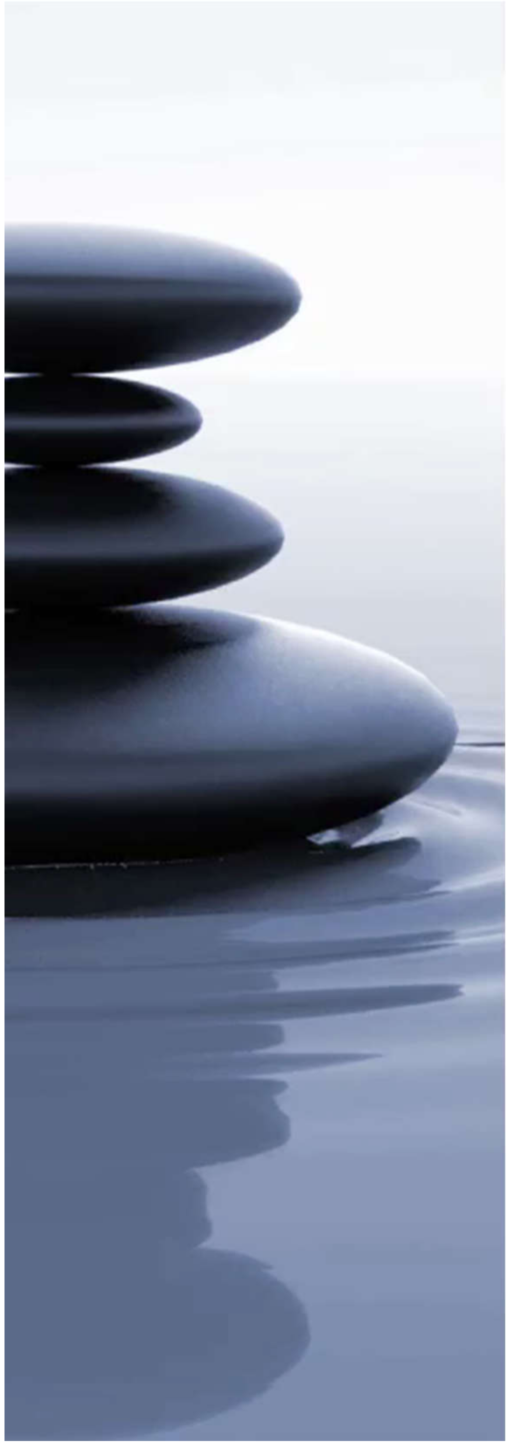
$$x_1 = \frac{1 - (2/3)}{1} \qquad \text{(E9.9.2)}$$



This case is much less sensitive to the number of significant figures in the computation:

Significant Figures	x_2	x_1	Absolute Value of Percent Relative Error for x_1
3	0.667	0.333	0.1
4	0.6667	0.3333	0.01
5	0.66667	0.33333	0.001
6	0.666667	0.333333	0.0001
7	0.6666667	0.3333333	0.00001

Thus, a pivot strategy is much more satisfactory.



Example #5

Problem Statement.

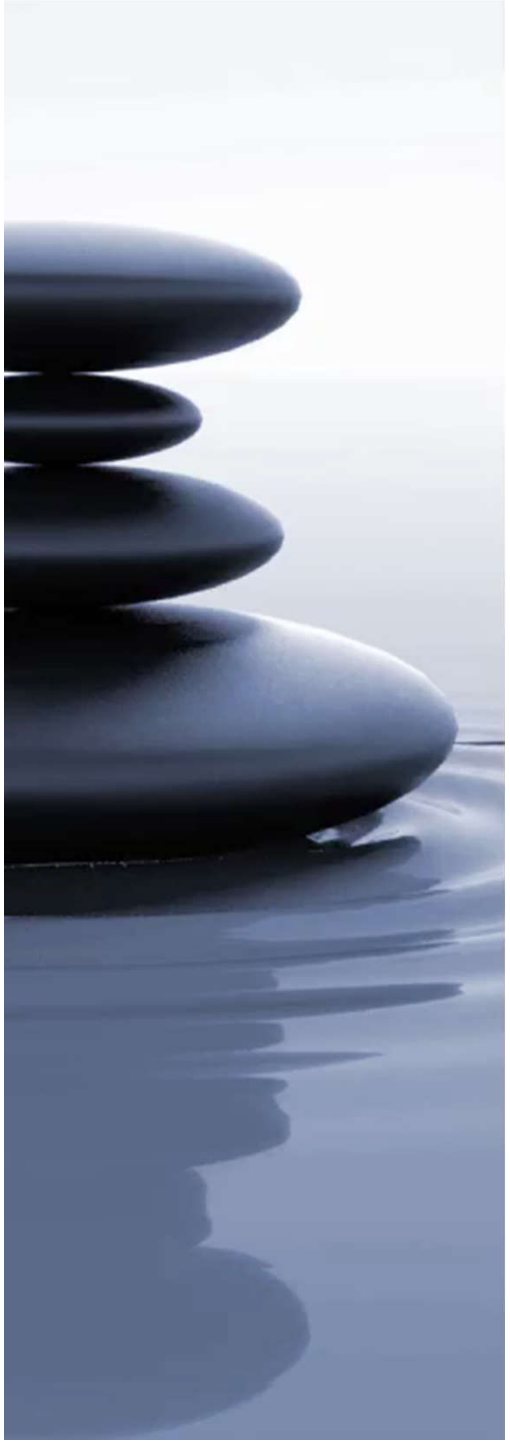
(a) Solve the following set of equations using Gauss elimination and a pivoting strategy:

$2x_1 + 100{,}000x_2 = 100{,}000$

$x_1 + x_2 = 2$

(b) Repeat the solution after scaling the equations so that the maximum coefficient in each row is 1.

(c) Finally, use the scaled coefficients to determine whether pivoting is necessary. However, actually solve the equations with the original coefficient values. For all cases, retain only three significant figures. Note that the correct answers are $x_1 = 1.00002$ and $x_2 = 0.99998$ or, for three significant figures, $x_1 = x_2 = 1.00$.



Solution.

(a) Without scaling, forward elimination is applied to give

$2x1 + 100{,}000x2 = 100{,}000$

$-50{,}000x2 = -50{,}000$

which can be solved by back substitution for

$x2 = 1.00$

$x1 = 0.00$

Although $x2$ is correct, $x1$ is 100 percent in error because of round-off.

(b) Scaling transforms the original equations to

$0.00002x1 + x2 = 1$

$x1 + x2 = 2$

Therefore, the rows should be pivoted to put the greatest value on the diagonal.

$x1 + x2 = 2$

$0.00002x1 + x2 = 1$

Forward elimination yields

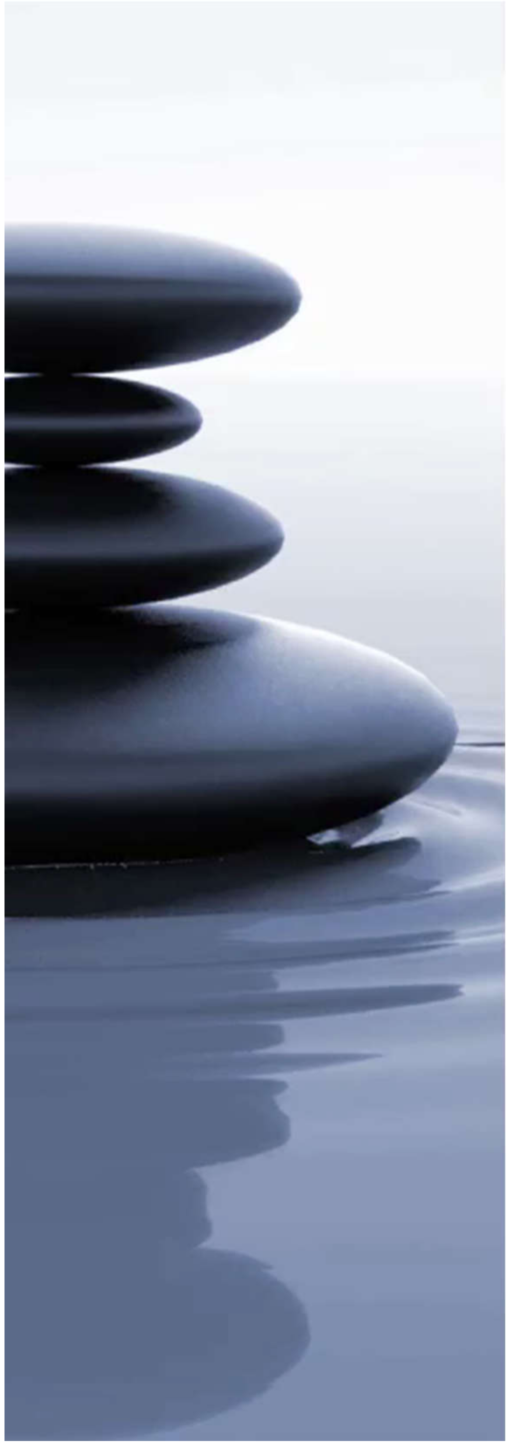
$x1 + x2 = 2$

$x2 = 1.00$

which can be solved for

$x1 = x2 = 1$

Thus, scaling leads to the correct answer.



(c) The scaled coefficients indicate that pivoting is necessary. We therefore pivot but retain the original coefficients to give

$x_1 + x_2 = 2$

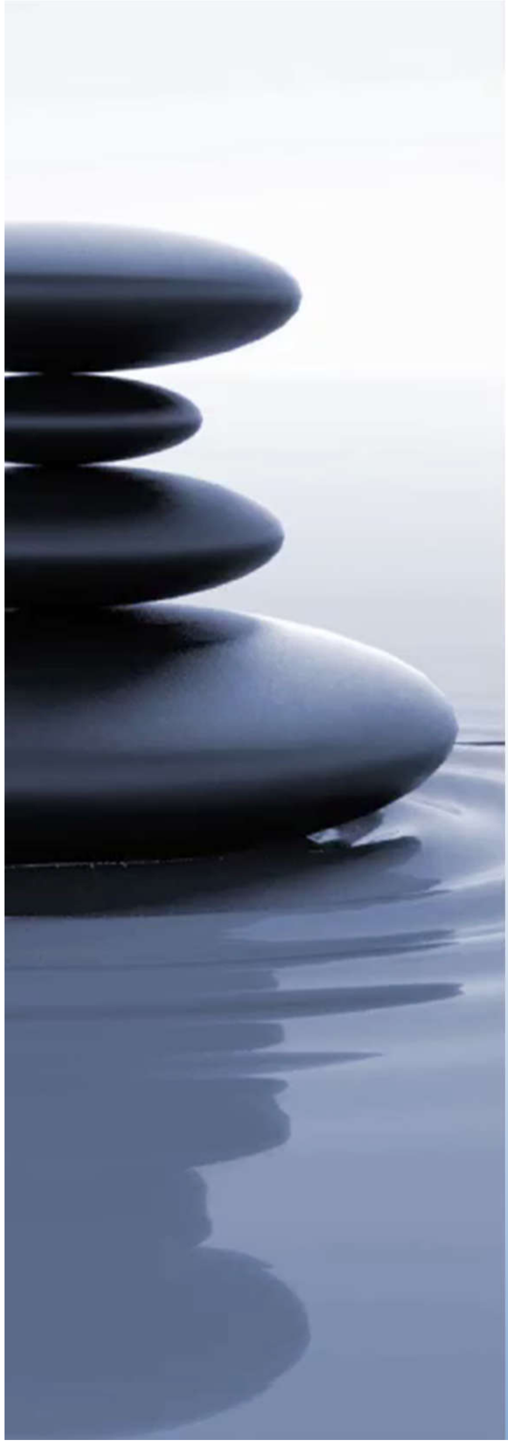
$2x_1 + 100{,}000x_2 = 100{,}000$

Forward elimination yields

$x_1 + x_2 = 2$

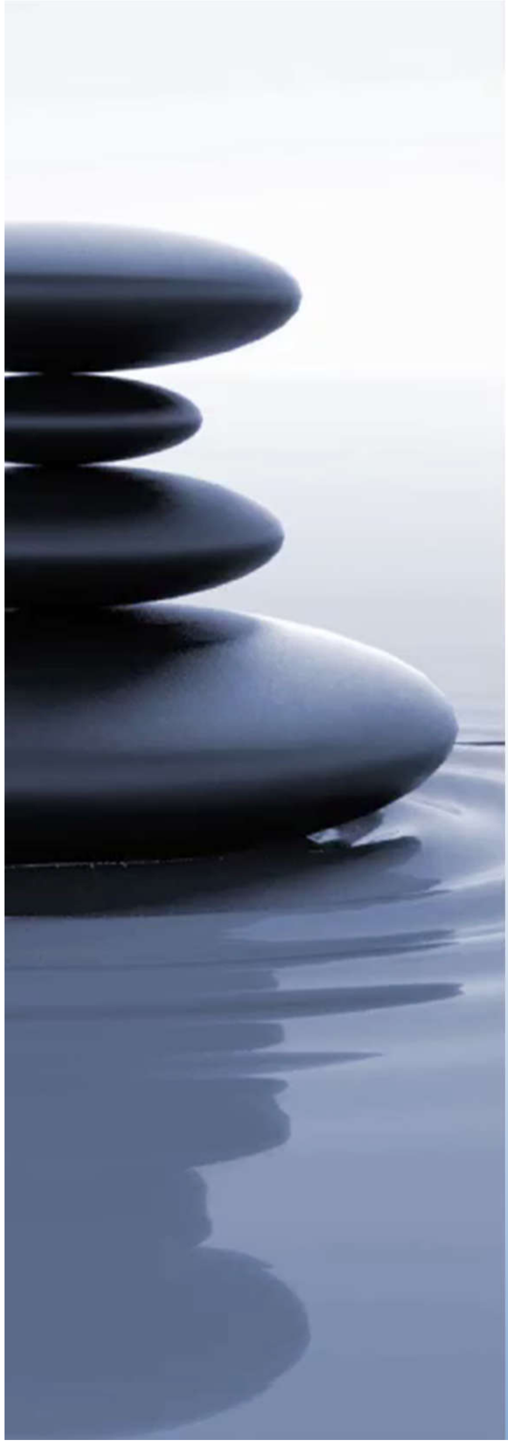
$100{,}000x_2 = 100{,}000$

which can be solved for the correct answer: $x_1 = x_2 = 1$. Thus, scaling was useful in determining whether pivoting was necessary, but the equations themselves did not require scaling to arrive at a correct result.



PART B

LU DECOMPOSITION



If

L- lower triangular matrix

U- upper triangular matrix

Then,

[A]{X}={B} can be decomposed into two matrices [L] and [U] such that

[L][U]=[A]

[L][U]{X}={B}

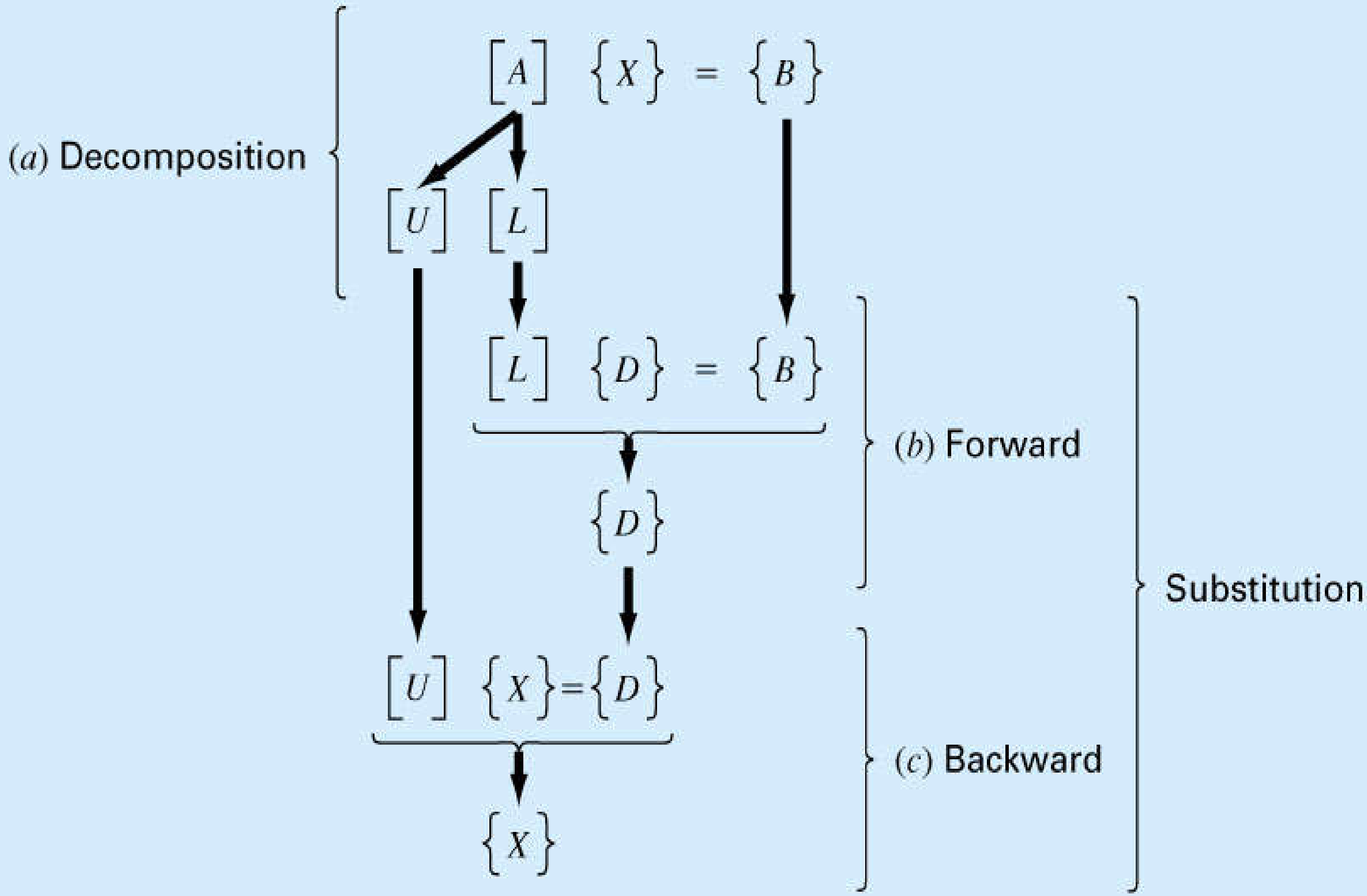
Similar to first phase of *Gauss elimination*, consider

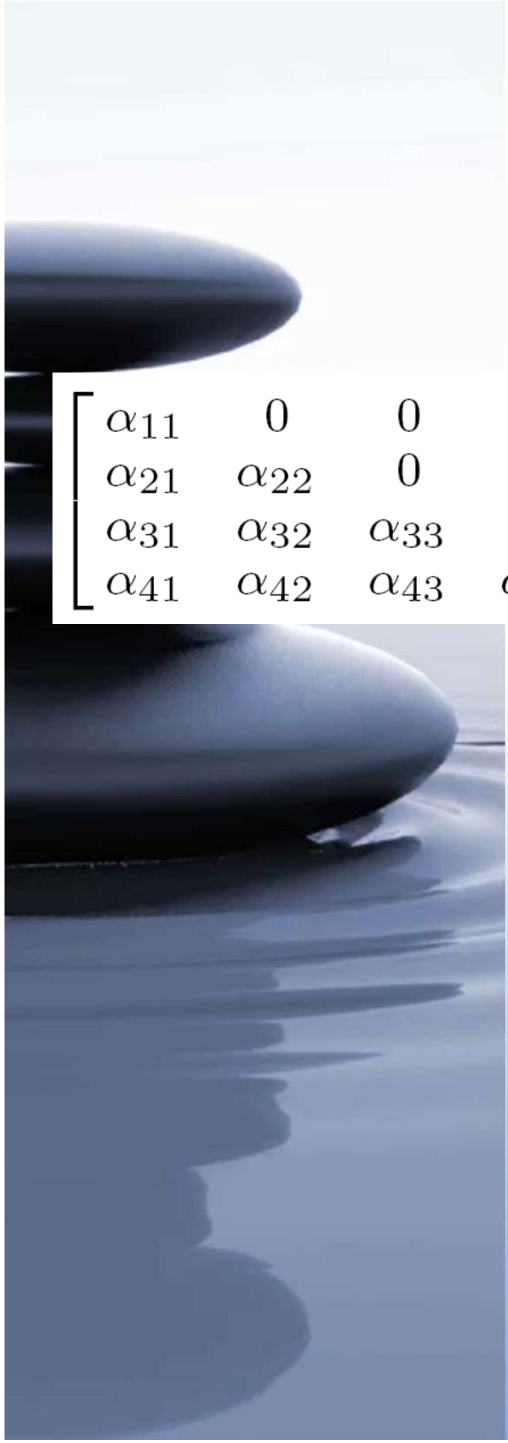
[U]{X}={D}

[L]{D}={B}

[L]{D}={B} is used to generate an intermediate vector {D} by forward substitution

Then, [U]{X}={D} is used to get {X} by backward substitution.

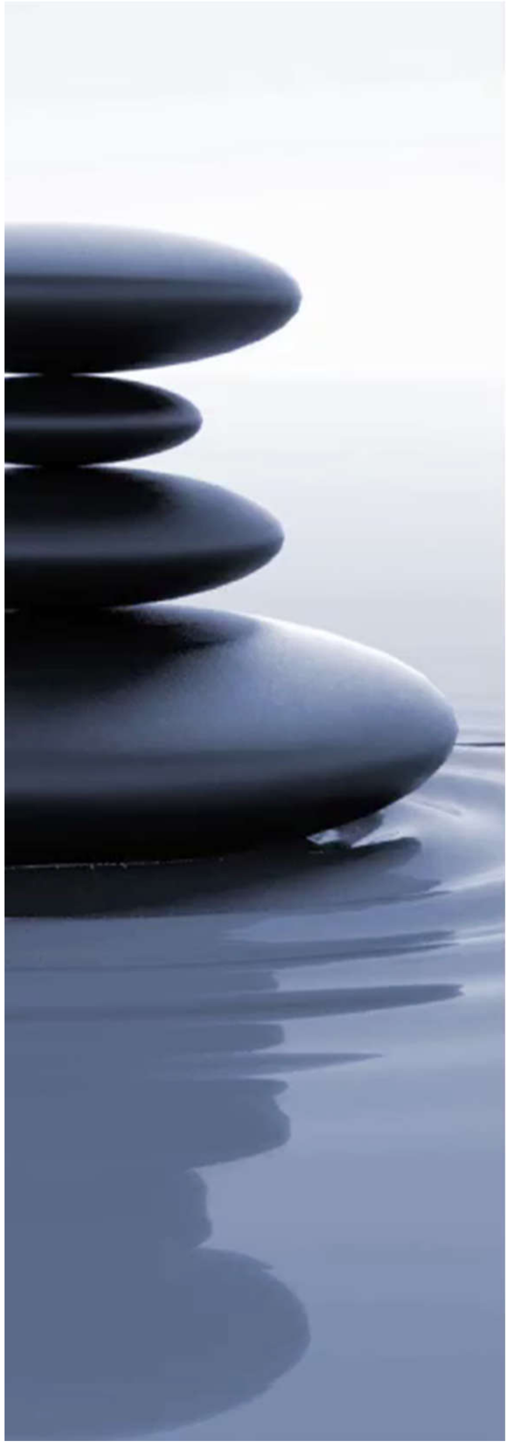

[A] {X} = {B}
(a) Decomposition
[U] [L]
[L] {D} = {B}
(b) Forward
{D}
Substitution
[U] {X}={D}
(c) Backward
{X}



LU Decomposition

$\alpha_{ii} = 1$

$$\begin{bmatrix} \alpha_{11} & 0 & 0 & 0 \\ \alpha_{21} & \alpha_{22} & 0 & 0 \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & 0 \\ \alpha_{41} & \alpha_{42} & \alpha_{43} & \alpha_{44} \end{bmatrix} \cdot \begin{bmatrix} \beta_{11} & \beta_{12} & \beta_{13} & \beta_{14} \\ 0 & \beta_{22} & \beta_{23} & \beta_{24} \\ 0 & 0 & \beta_{33} & \beta_{34} \\ 0 & 0 & 0 & \beta_{44} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$



Crout's Algorithm

Set $\alpha_{ii} = 1$ for all *i*

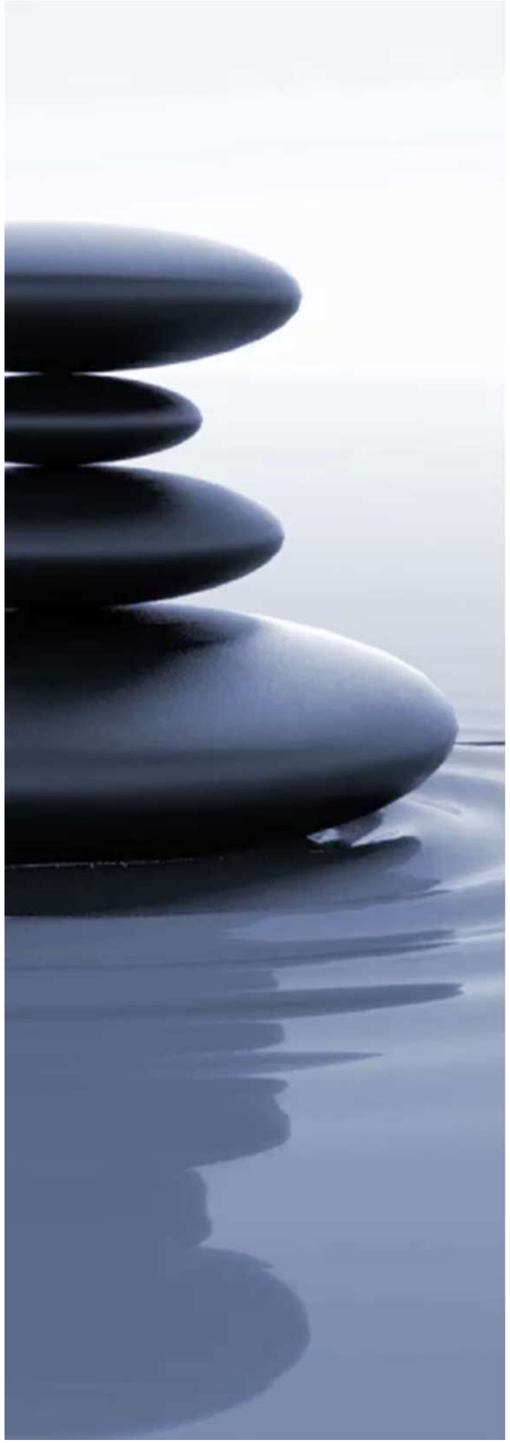
For each *j* = 1, 2, 3, ..., *N*

(a) for *i* = 1, 2, ..., *j*

$$\beta_{ij} = a_{ij} - \sum_{k=1}^{i-1} \alpha_{ik}\beta_{kj}$$

(b) for *i* = *j* +1, *j* +2, ..., *N*

$$\alpha_{ij} = \frac{1}{\beta_{jj}}\left(a_{ij} - \sum_{k=1}^{j-1} \alpha_{ik}\beta_{kj} \right)$$



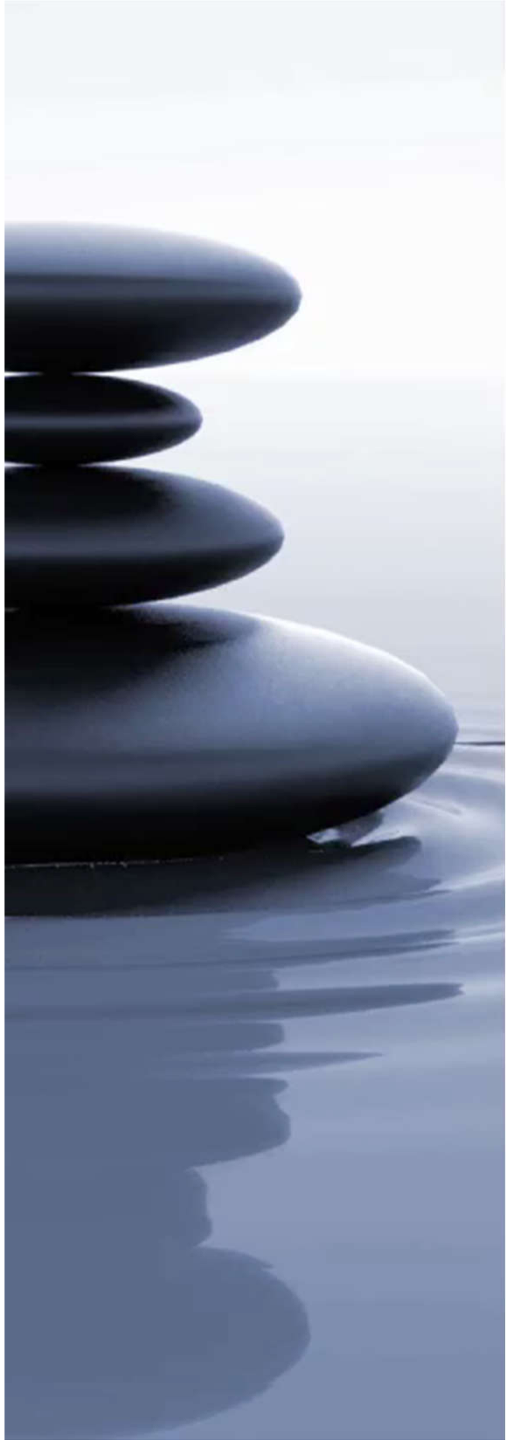
Solving linear equations

$LU = A$

Thus $A\,x = (LU)\,x = L\,(U\,x) = b$

Let $y = U\,x$, then solve y in $L\,y = b$ by forward substitution

Solve x in $U\,x = y$ by backward substitution



LU Decomposition

$$[A] \rightarrow [L][U]$$

where

$$[U] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a'_{22} & a'_{23} \\ 0 & 0 & a''_{33} \end{bmatrix}$$

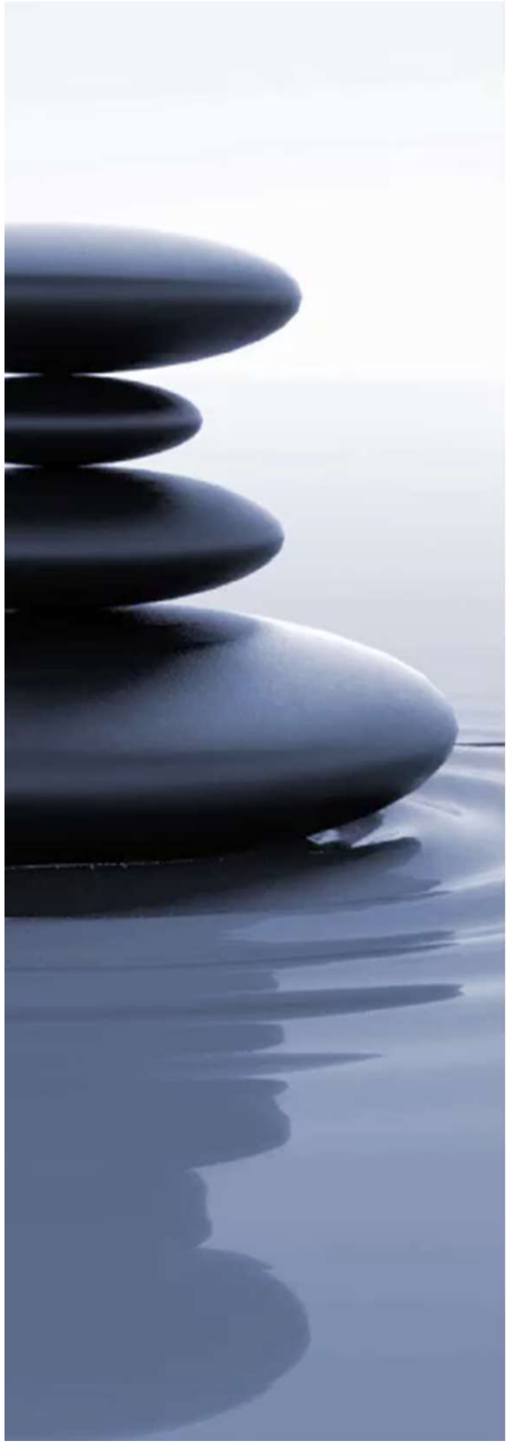
and

$$[L] = \begin{bmatrix} 1 & 0 & 0 \\ f_{21} & 1 & 0 \\ f_{31} & f_{32} & 1 \end{bmatrix}$$

$$f_{21} = \frac{a_{21}}{a_{11}} \qquad f_{31} = \frac{a_{31}}{a_{11}} \qquad a'_{22} = a_{22} - f_{21}a_{12}$$

$$f_{32} = \frac{1}{a'_{22}}\left(a_{32} - \frac{a_{31}a_{12}}{a_{11}}\right) \qquad a'_{23} = a_{23} - f_{21}a_{13}$$

$$a'_{33} = a_{33} - f_{31}a_{13} - f_{32}a'_{23}$$



Example #6

Use *LU* decomposition to solve

$3x_1 - 0.1x_2 - 0.2x_3 = 7.85$

$0.1x_1 + 7x_2 - 0.3x_3 = -19.3$

$0.3x_1 - 0.2x_2 + 10x_3 = 71.4$

We solve the matrix

$$[A] = \begin{bmatrix} 3 & -0.1 & -0.2 \\ 0.1 & 7 & -0.3 \\ 0.3 & -0.2 & 10 \end{bmatrix}$$

After forward elimination, the following upper triangular matrix was obtained:

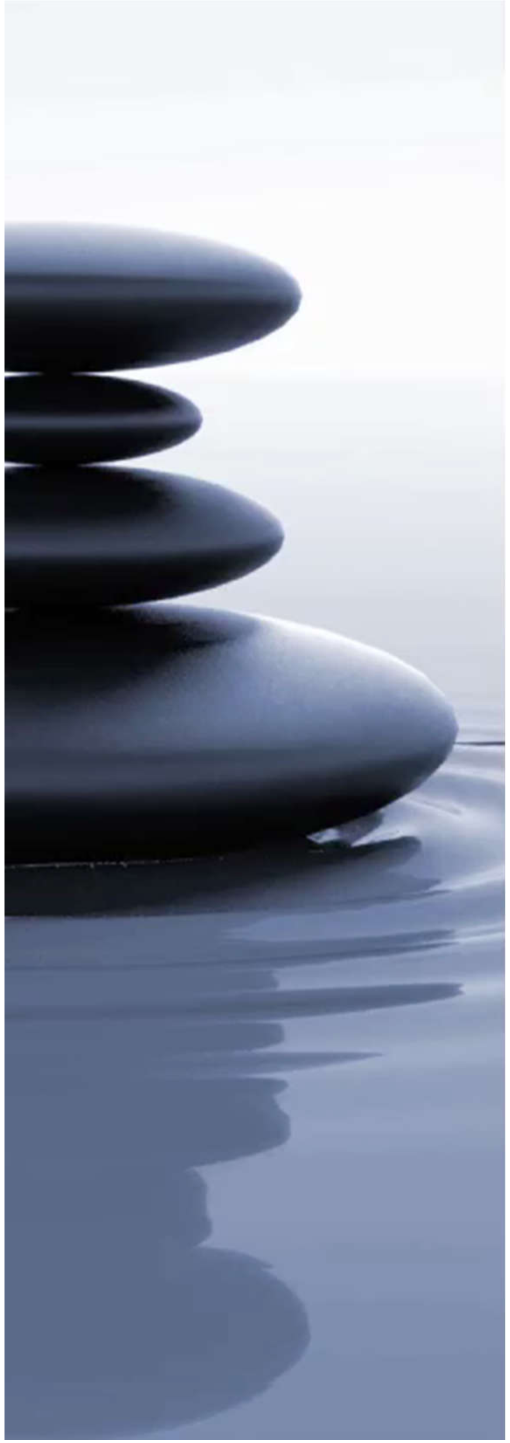
$$[U] = \begin{bmatrix} 3 & -0.1 & -0.2 \\ 0 & 7.00333 & -0.293333 \\ 0 & 0 & 10.0120 \end{bmatrix}$$

The factors employed to obtain the upper triangular matrix can be assembled into a lower triangular matrix. The elements a_{21} and a_{31} were eliminated by using the factors

$$f_{21} = \frac{0.1}{3} = 0.03333333 \qquad f_{31} = \frac{0.3}{3} = 0.1000000$$

and the element a'_{32} was eliminated by using the factor

$$f_{32} = \frac{-0.19}{7.00333} = -0.0271300$$



Example

Thus, the lower triangular matrix is

$$[L] = \begin{bmatrix} 1 & 0 & 0 \\ 0.0333333 & 1 & 0 \\ 0.100000 & -0.0271300 & 1 \end{bmatrix}$$

Consequently, the LU decomposition is

$$[A] = [L][U] = \begin{bmatrix} 1 & 0 & 0 \\ 0.0333333 & 1 & 0 \\ 0.100000 & -0.0271300 & 1 \end{bmatrix} \begin{bmatrix} 3 & -0.1 & -0.2 \\ 0 & 7.00333 & -0.293333 \\ 0 & 0 & 10.0120 \end{bmatrix}$$

This result can be verified by performing the multiplication of $[L][U]$ to give

$$[L][U] = \begin{bmatrix} 3 & -0.1 & -0.2 \\ 0.0999999 & 7 & -0.3 \\ 0.3 & -0.2 & 9.99996 \end{bmatrix}$$

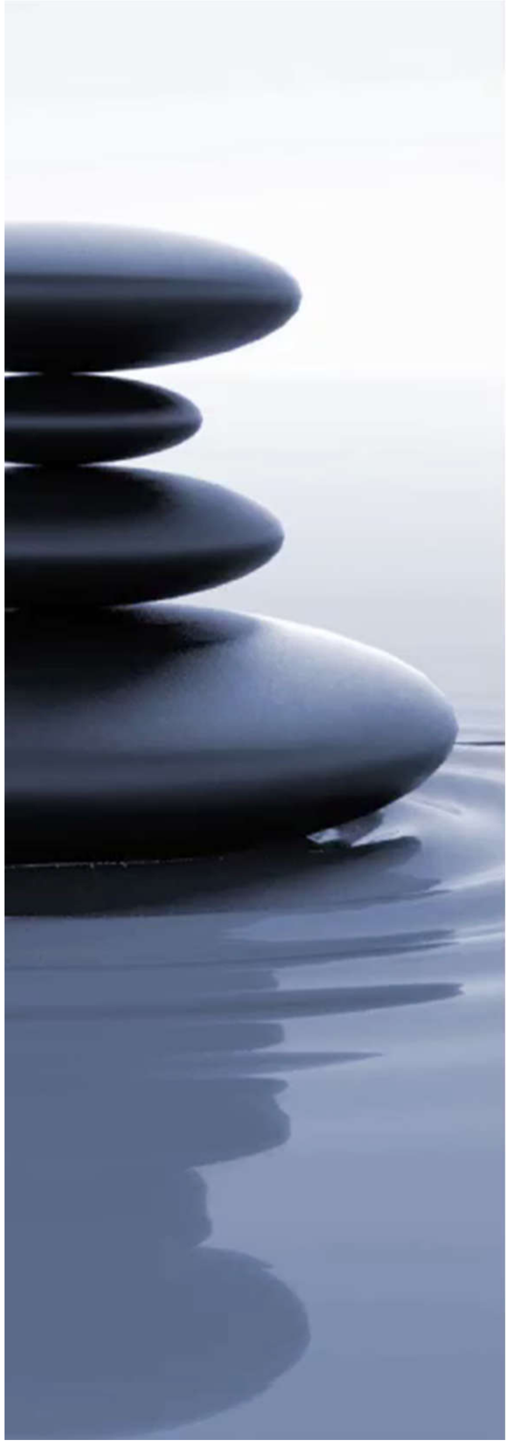
where the minor discrepancies are due to round-off.

Recall that the system being solved

$$\begin{bmatrix} 3 & -0.1 & -0.2 \\ 0.1 & 7 & -0.3 \\ 0.3 & -0.2 & 10 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 7.85 \\ -19.3 \\ 71.4 \end{Bmatrix}$$

and that the forward-elimination phase of conventional Gauss elimination resulted in

$$\begin{bmatrix} 3 & -0.1 & -0.2 \\ 0 & 7.00333 & -0.293333 \\ 0 & 0 & 10.0120 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 7.85 \\ -19.5617 \\ 70.0843 \end{Bmatrix}$$



Example

The forward-substitution phase is implemented

$$\begin{bmatrix} 1 & 0 & 0 \\ 0.0333333 & 1 & 0 \\ 0.100000 & -0.0271300 & 1 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix} = \begin{Bmatrix} 7.85 \\ -19.3 \\ 71.4 \end{Bmatrix}$$

or multiplying out the left-hand side,

$$\begin{aligned} d_1 \qquad\qquad &= 7.85 \\ 0.0333333 d_1 + \quad d_2 \qquad &= -19.3 \\ 0.1 d_1 - 0.02713 d_2 + d_3 &= 71.4 \end{aligned}$$

We can solve the first equation for d_1,

$$d_1 = 7.85$$

which can be substituted into the second equation to solve for

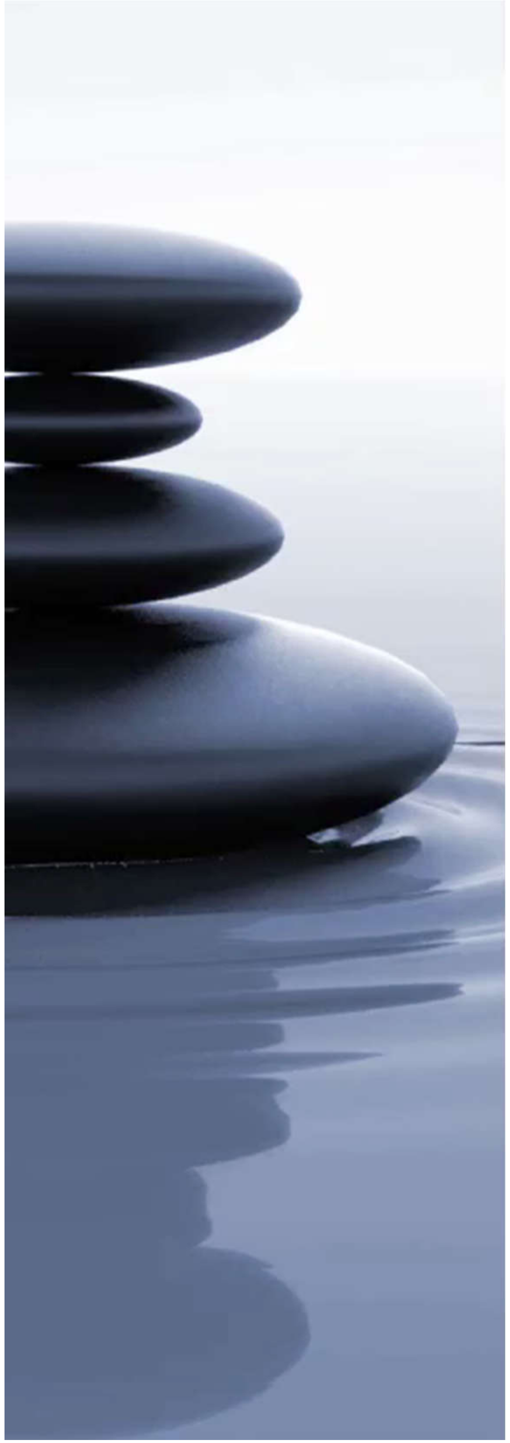
$$d_2 = -19.3 - 0.0333333(7.85) = -19.5617$$

Both d_1 and d_2 can be substituted into the third equation to give

$$d_3 = 71.4 - 0.1(7.85) + 0.02713(-19.5617) = 70.0843$$

Thus,

$$\{D\} = \begin{Bmatrix} 7.85 \\ -19.5617 \\ 70.0843 \end{Bmatrix} \qquad \{X\} = \begin{Bmatrix} 3 \\ -2.5 \\ 7.00003 \end{Bmatrix}$$

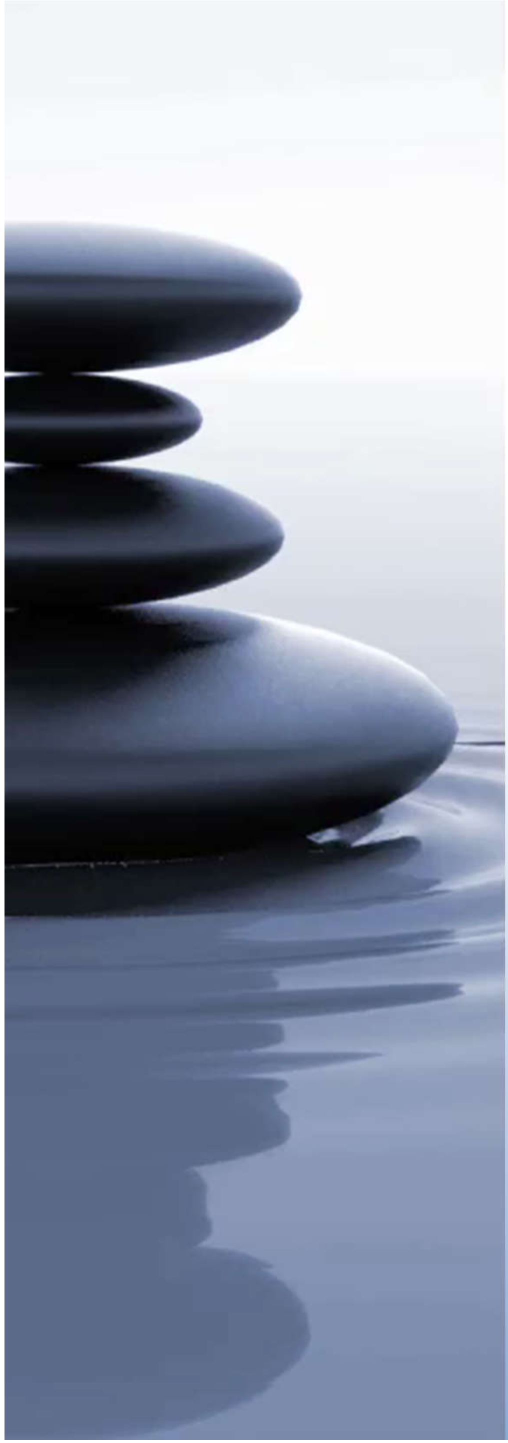


Inverting a matrix

When solving systems of equations, b is usually treated as a vector with a length equal to the height of matrix A. Instead of vector b, we have matrix B, where B is an n-by-p matrix, so that we are trying to find a matrix X (also a n-by-p matrix):

$$AX = LUX = \mathrm{B}$$

We can use the same algorithm presented earlier to solve for each column of matrix X. Now suppose that B is the identity matrix of size n. It would follow that the result X must be the inverse of A



Compute det(A)

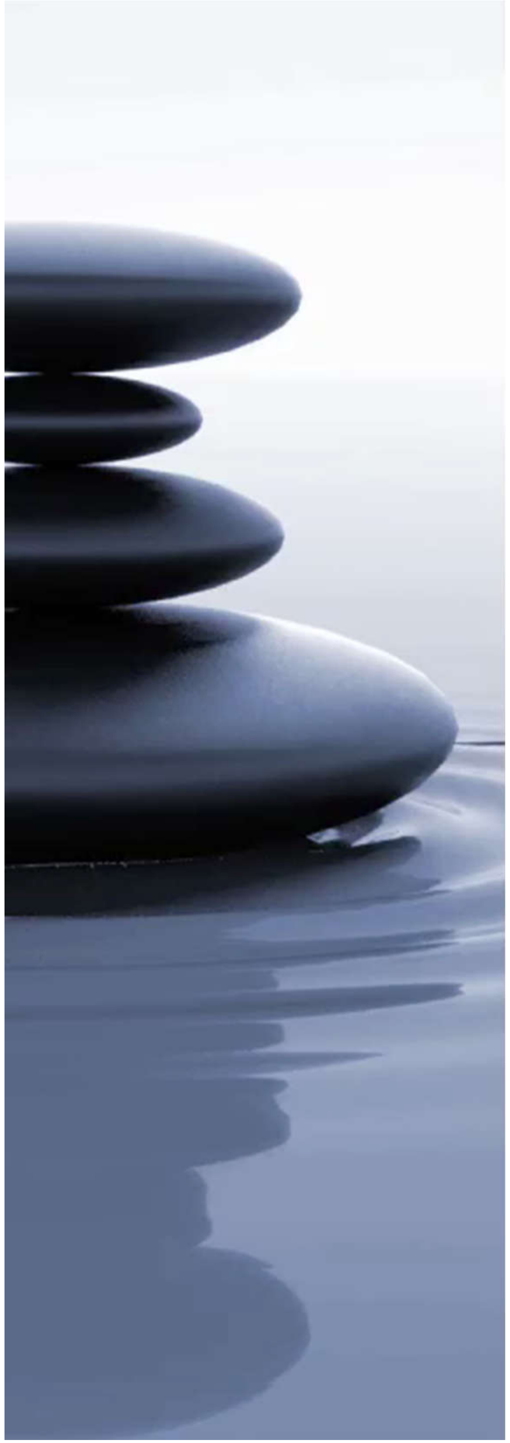
- Definition of determinant

$$\det(A) = \sum_{P} (-1)^{P} a_{1,i_1} a_{2,i_2} \cdots a_{n,i_n}$$

- Properties of determinant

Since det(LU) = det(L)det(U), thus

$$\det(A) = \det(U) = \prod_{j=1}^{N} \beta_{jj}$$



Example #7

Employ LU decomposition to determine the matrix inverse for the system from the previous example.

$$[A] = \begin{bmatrix} 3 & -0.1 & -0.2 \\ 0.1 & 7 & -0.3 \\ 0.3 & -0.2 & 10 \end{bmatrix}$$

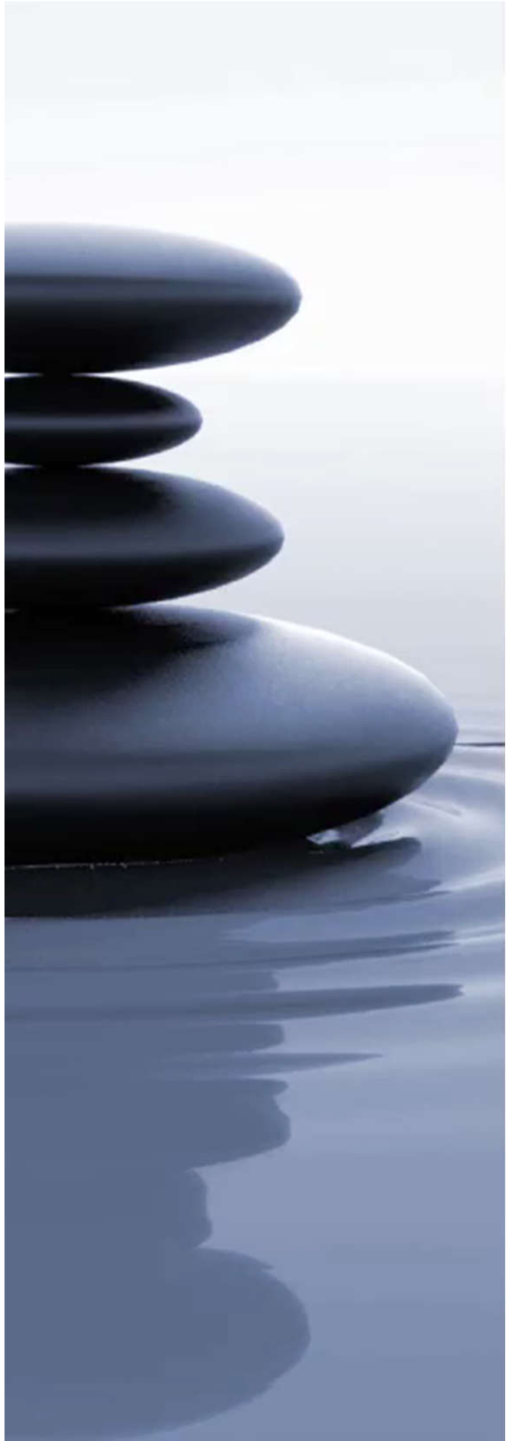
Recall that the decomposition resulted in the following lower and upper triangular matrices:

$$[U] = \begin{bmatrix} 3 & -0.1 & -0.2 \\ 0 & 7.00333 & -0.293333 \\ 0 & 0 & 10.0120 \end{bmatrix} \qquad [L] = \begin{bmatrix} 1 & 0 & 0 \\ 0.0333333 & 1 & 0 \\ 0.100000 & -0.0271300 & 1 \end{bmatrix}$$

Solution. The first column of the matrix inverse can be determined by performing the forward-substitution solution procedure with a unit vector (with 1 in the first row) as the right-hand-side vector.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0.0333333 & 1 & 0 \\ 0.100000 & -0.0271300 & 1 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$$

and solved with forward substitution for $\{D\}^T = \lfloor 1 \quad -0.03333 \quad -0.1009 \rfloor$.



$$\begin{bmatrix} 3 & -0.1 & -0.2 \\ 0 & 7.00333 & -0.293333 \\ 0 & 0 & 10.0120 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ -0.03333 \\ -0.1009 \end{Bmatrix}$$

which can be solved by back substitution for $\{X\}^T = \lfloor 0.33249 \quad -0.00518 \quad -0.01008 \rfloor$, which is the first column of the matrix,

$$[A]^{-1} = \begin{bmatrix} 0.33249 & 0 & 0 \\ -0.00518 & 0 & 0 \\ -0.01008 & 0 & 0 \end{bmatrix}$$

To determine the second column,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0.0333333 & 1 & 0 \\ 0.100000 & -0.0271300 & 1 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix}$$

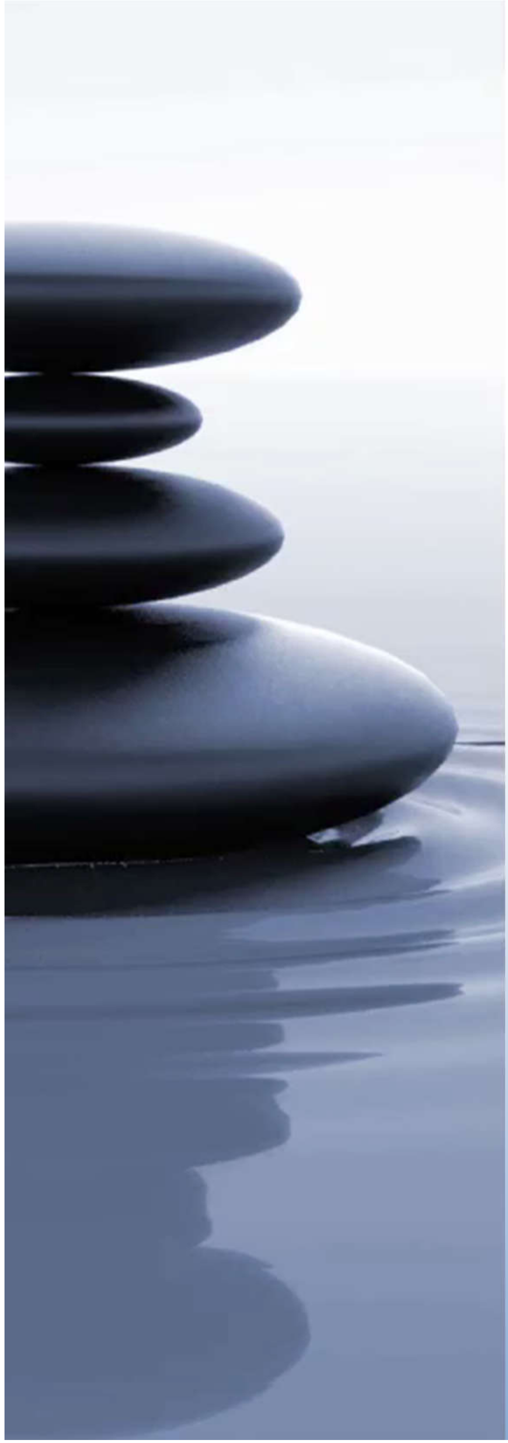
This can be solved for $\{D\}$, and the results are used with Eq. (10.3) to determine $\{X\}^T = \lfloor 0.004944 \quad 0.142903 \quad 0.00271 \rfloor$, which is the second column of the matrix,

$$[A]^{-1} = \begin{bmatrix} 0.33249 & 0.004944 & 0 \\ -0.00518 & 0.142903 & 0 \\ -0.01008 & 0.00271 & 0 \end{bmatrix}$$

Finally, the forward- and back-substitution procedures can be implemented with $\{B\}^T = \lfloor 0 \quad 0 \quad 1 \rfloor$ to solve for $\{X\}^T = \lfloor 0.006798 \quad 0.004183 \quad 0.09988 \rfloor$, which is the final column of the matrix,

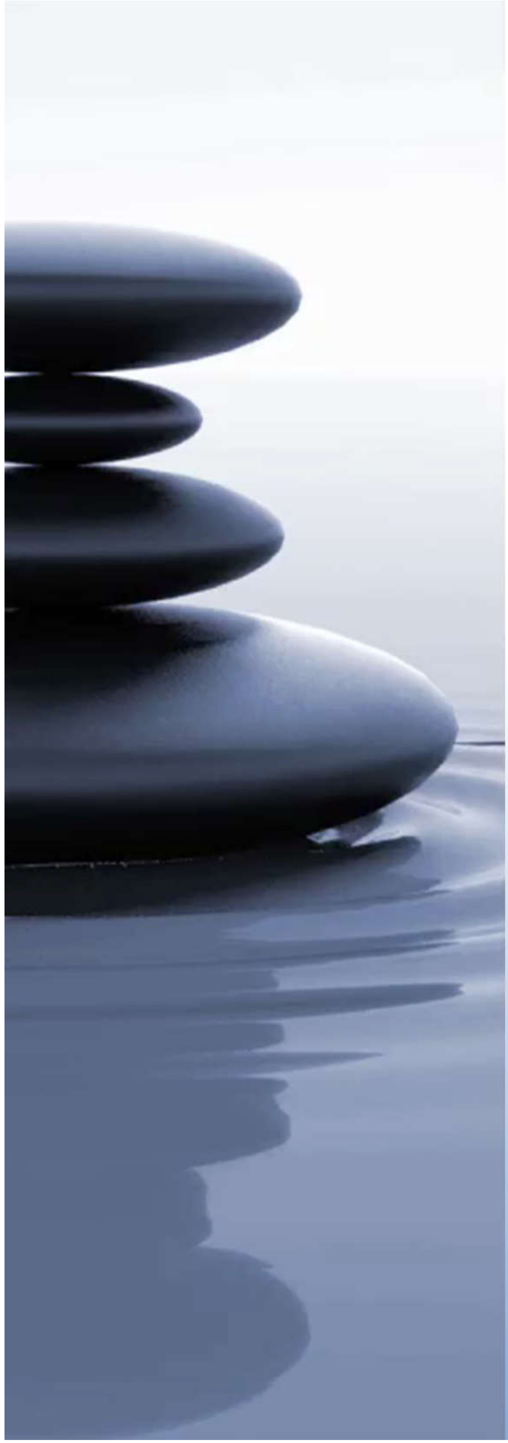
$$[A]^{-1} = \begin{bmatrix} 0.33249 & 0.004944 & 0.006798 \\ -0.00518 & 0.142903 & 0.004183 \\ -0.01008 & 0.00271 & 0.09988 \end{bmatrix}$$

The validity of this result can be checked by verifying that $[A][A]^{-1} = [I]$.



PART C

SPECIAL MATRICES & GAUSS-SEIDEL METHOD



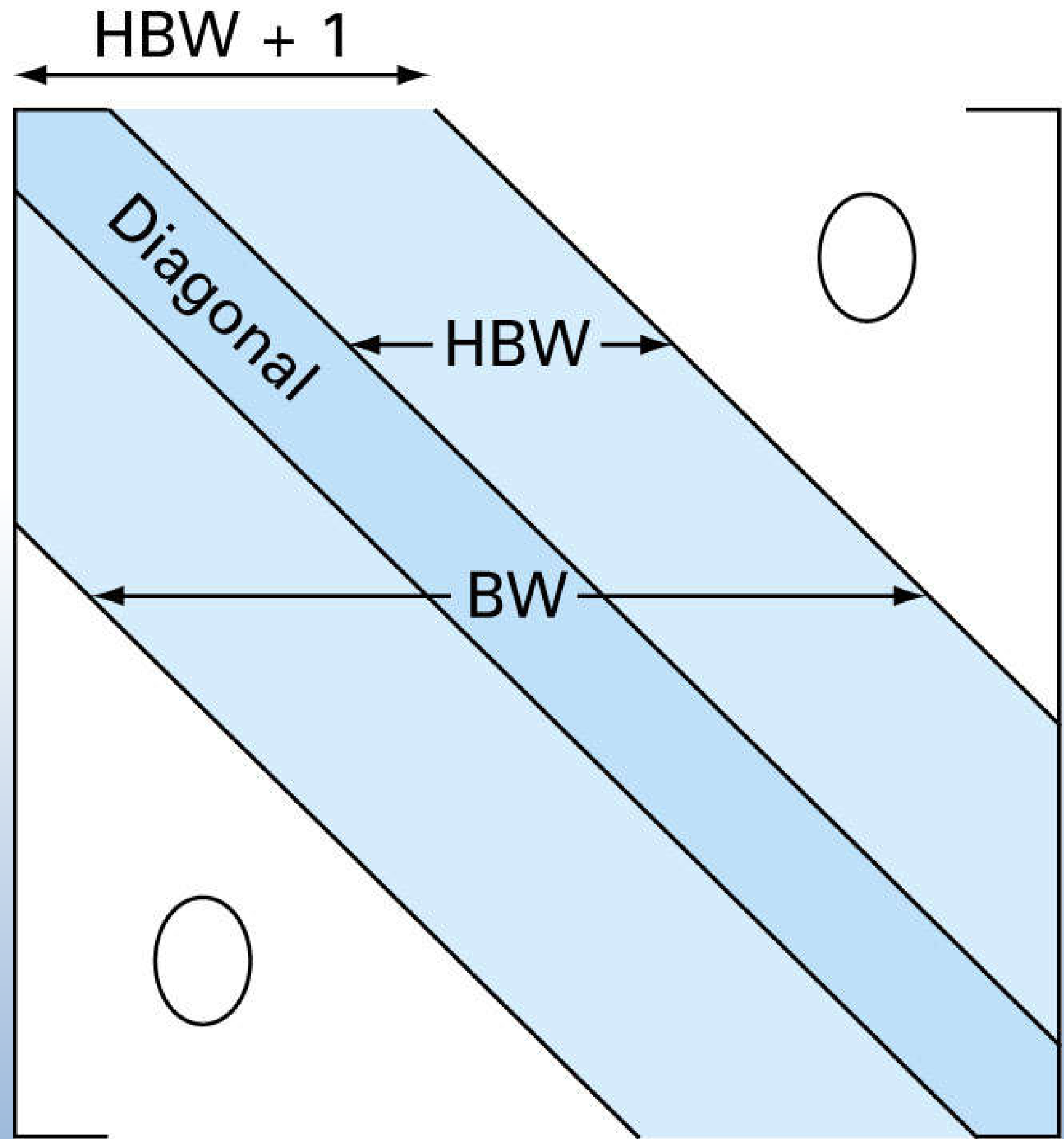
Special Matrices and Gauss-Seidel

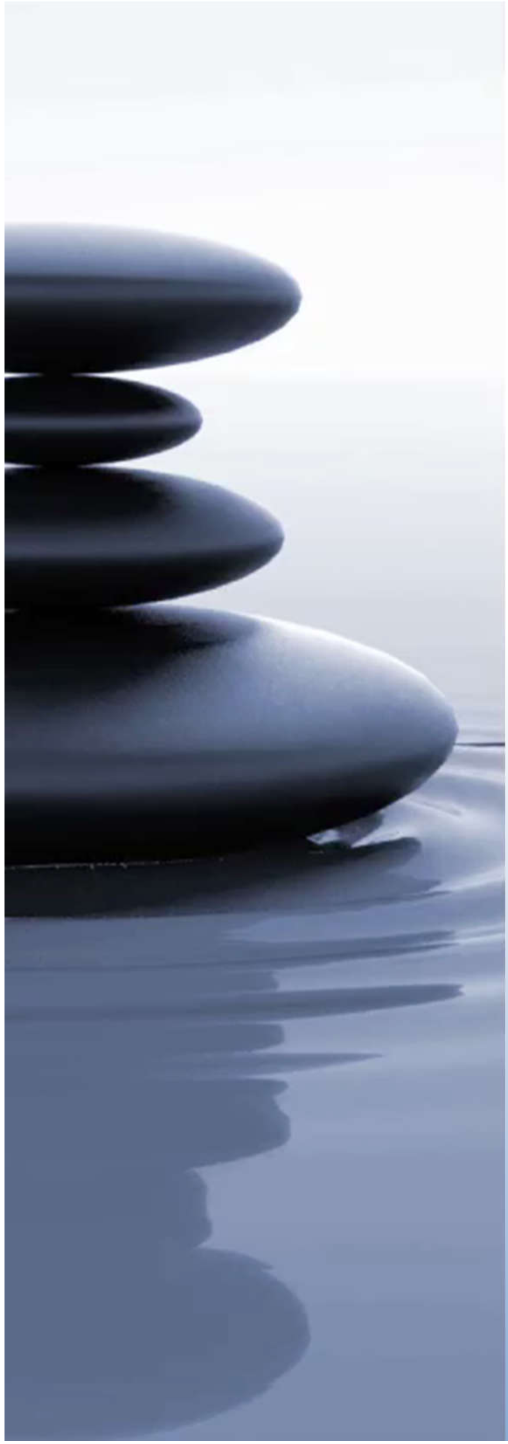
Certain matrices have particular structures that can be exploited to develop efficient solution schemes.

A *banded matrix* is a square matrix that has all elements equal to zero, with the exception of a band centered on the main diagonal. These matrices typically occur in solution of differential equations.

The dimensions of a banded system can be quantified by two parameters: the band width BW and half-bandwidth HBW. These two values are related by BW=2HBW+1.

Gauss elimination or conventional LU decomposition methods are inefficient in solving banded equations because pivoting becomes unnecessary.


HBW + 1
Diagonal
HBW
BW

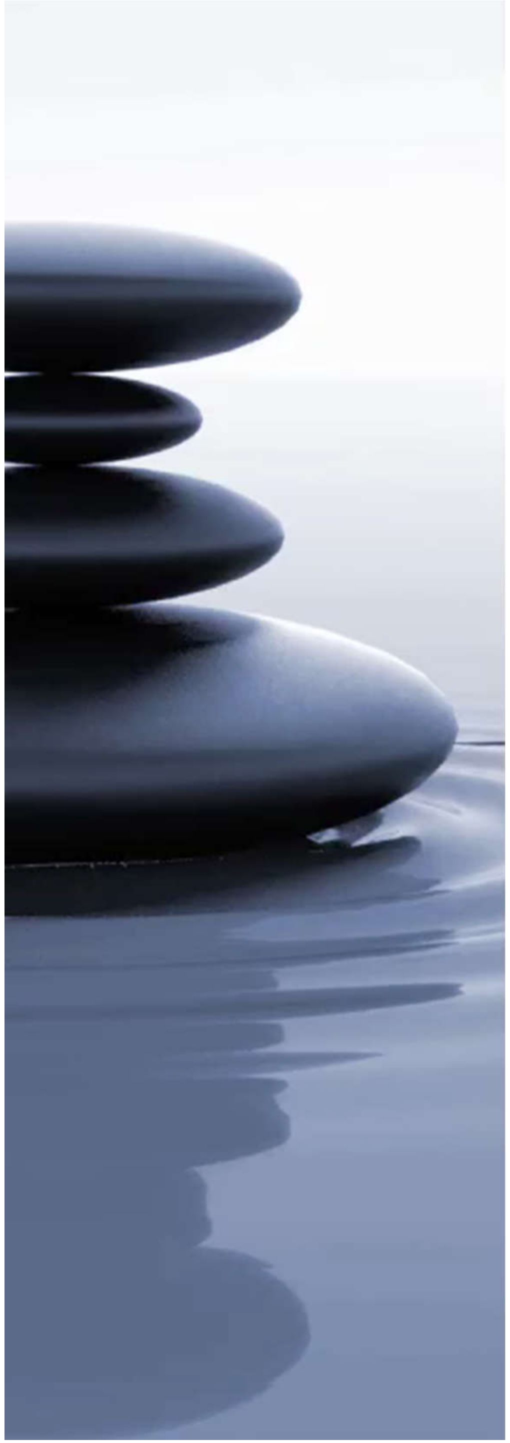


Tridiagonal Systems

A tridiagonal system has a bandwidth of 3:

$$\begin{bmatrix} f_1 & g_1 & & \\ e_2 & f_2 & g_2 & \\ & e_3 & f_3 & g_3 \\ & & e_4 & f_4 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} = \begin{Bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{Bmatrix}$$

- An efficient LU decomposition method, called *Thomas algorithm*, can be used to solve such an equation. The algorithm consists of three steps: decomposition, forward and back substitution, and has all the advantages of LU decomposition.



Example #8

Problem Statement. Solve the following tridiagonal system

$$\begin{bmatrix} 2.04 & -1 & & \\ -1 & 2.04 & -1 & \\ & -1 & 2.04 & -1 \\ & & -1 & 2.04 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} 40.8 \\ 0.8 \\ 0.8 \\ 200.8 \end{Bmatrix}$$

Solution. First, the decomposition is implemented as

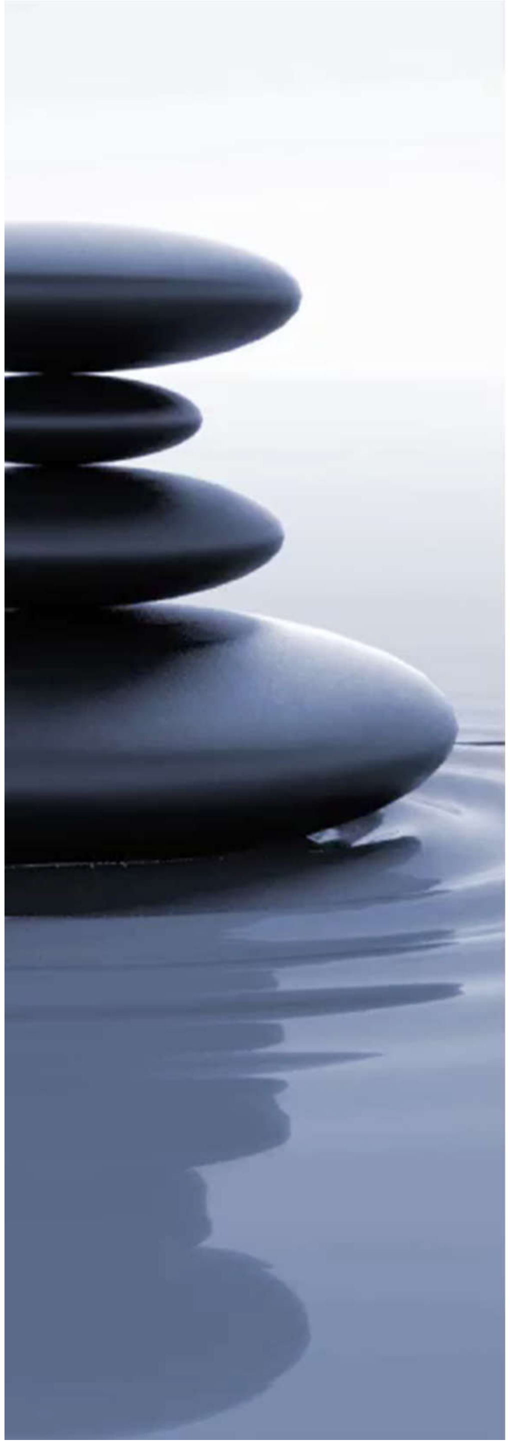
$$e_2 = -1/2.04 = -0.49$$
$$f_2 = 2.04 - (-0.49)(-1) = 1.550$$
$$e_3 = -1/1.550 = -0.645$$
$$f_3 = 2.04 - (-0.645)(-1) = 1.395$$
$$e_4 = -1/1.395 = -0.717$$
$$f_4 = 2.04 - (-0.717)(-1) = 1.323$$

Thus, the matrix has been transformed to

$$\begin{bmatrix} 2.04 & -1 & & \\ -0.49 & 1.550 & -1 & \\ & -0.645 & 1.395 & -1 \\ & & -0.717 & 1.323 \end{bmatrix}$$

and the LU decomposition is

$$[A] = [L][U] = \begin{bmatrix} 1 & & & \\ -0.49 & 1 & & \\ & -0.645 & 1 & \\ & & -0.717 & 1 \end{bmatrix} \begin{bmatrix} 2.04 & -1 & & \\ & 1.550 & -1 & \\ & & 1.395 & -1 \\ & & & 1.323 \end{bmatrix}$$



You can verify that this is correct by multiplying $[L][U]$ to yield $[A]$.

The forward substitution is implemented as

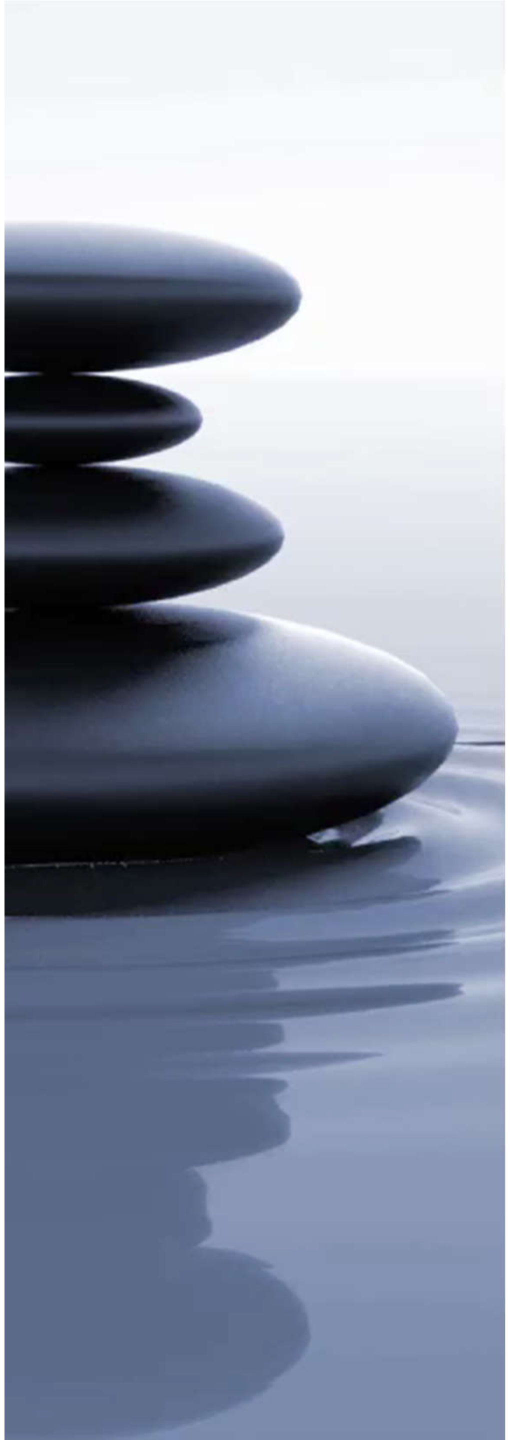
$$r_2 = 0.8 - (-0.49)40.8 = 20.8$$
$$r_3 = 0.8 - (-0.645)20.8 = 14.221$$
$$r_4 = 200.8 - (-0.717)14.221 = 210.996$$

Thus, the right-hand-side vector has been modified to

$$\left\{\begin{matrix} 40.8 \\ 20.8 \\ 14.221 \\ 210.996 \end{matrix}\right\}$$

which then can be used in conjunction with the $[U]$ matrix to perform back substitution and obtain the solution

$$T_4 = 210.996/1.323 = 159.480$$
$$T_3 = [14.221 - (-1)159.48]/1.395 = 124.538$$
$$T_2 = [20.800 - (-1)124.538]/1.550 = 93.778$$
$$T_1 = [40.800 - (-1)93.778]/2.040 = 65.970$$



Cholesky Decomposition

Cholesky decomposition is used for symmetric matrices.

This algorithm is based on the fact that a symmetric matrix can be decomposed, as in

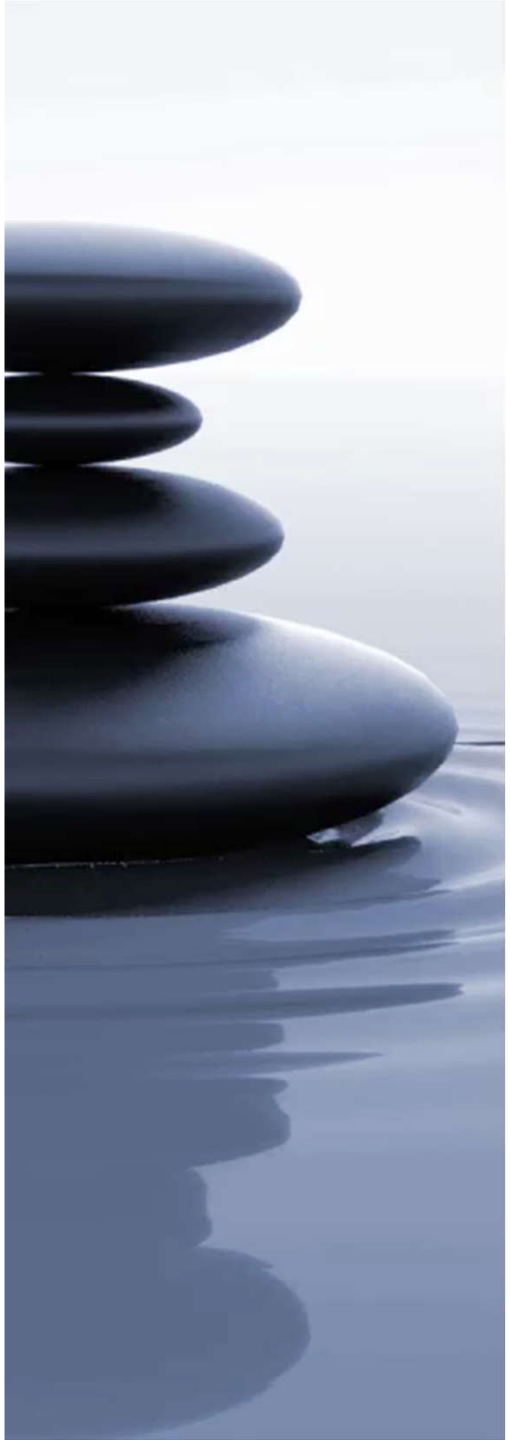
$$[A] = [L][L]^T$$

For the *k*th row

$$l_{ki} = \frac{a_{ki} - \sum_{j=1}^{i-1} l_{ij} l_{kj}}{l_{ii}} \qquad \text{for } i = 1, 2, \ldots, k-1$$

and

$$l_{kk} = \sqrt{a_{kk} - \sum_{j=1}^{k-1} l_{kj}^2}$$



Example #9

Problem Statement. Apply Cholesky decomposition to the symmetric matrix

$$[A] = \begin{bmatrix} 6 & 15 & 55 \\ 15 & 55 & 225 \\ 55 & 225 & 979 \end{bmatrix}$$

Solution. For the first row ($k = 1$), Eq. (11.3) is skipped and Eq. (11.4) is employed to compute

$$l_{11} = \sqrt{a_{11}} = \sqrt{6} = 2.4495$$

For the second row ($k = 2$), Eq. (11.3) gives

$$l_{21} = \frac{a_{21}}{l_{11}} = \frac{15}{2.4495} = 6.1237$$

and Eq. (11.4) yields

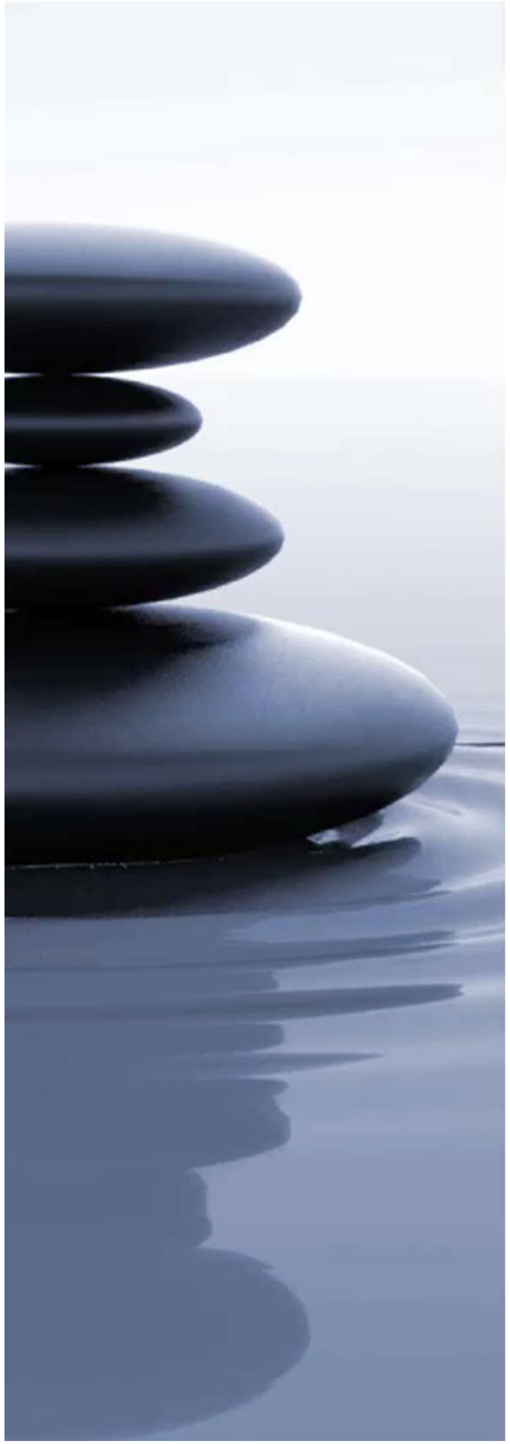
$$l_{22} = \sqrt{a_{22} - l_{21}^2} = \sqrt{55 - (6.1237)^2} = 4.1833$$

For the third row ($k = 3$), Eq. (11.3) gives ($i = 1$)

$$l_{31} = \frac{a_{31}}{l_{11}} = \frac{55}{2.4495} = 22.454$$

and ($i = 2$)

$$l_{32} = \frac{a_{32} - l_{21} l_{31}}{l_{22}} = \frac{225 - 6.1237(22.454)}{4.1833} = 20.917$$



Example

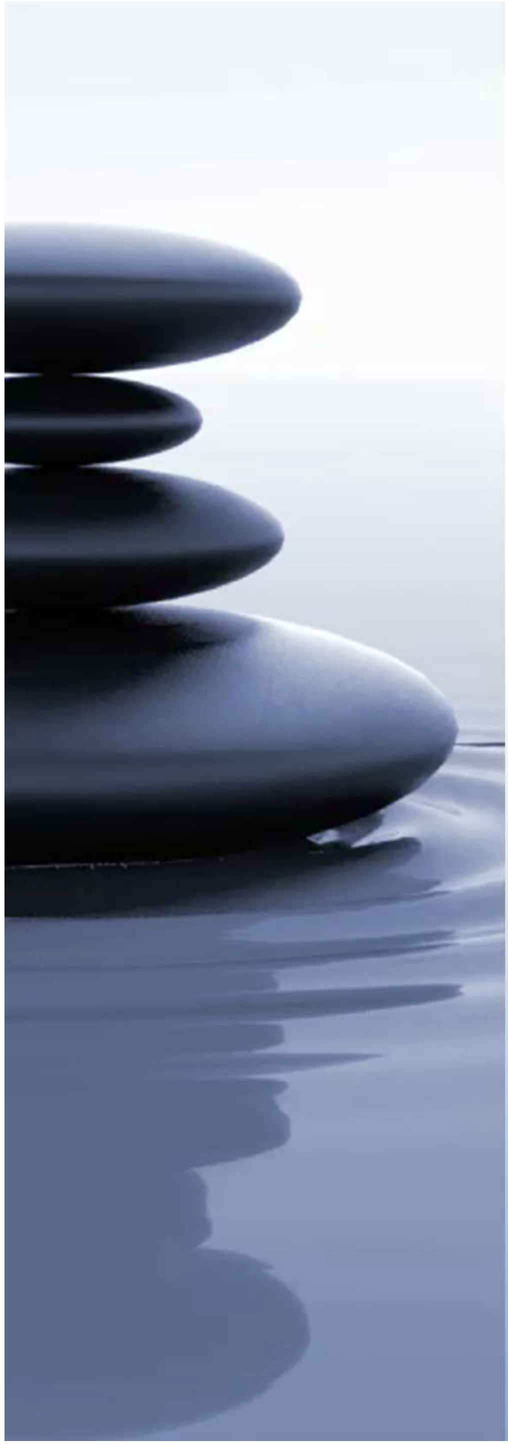
and Eq. (11.4) yields

$$l_{33} = \sqrt{a_{33} - l_{31}^2 - l_{32}^2} = \sqrt{979 - (22.454)^2 - (20.917)^2} = 6.1101$$

Thus, the Cholesky decomposition yields

$$[L] = \begin{bmatrix} 2.4495 & & \\ 6.1237 & 4.1833 & \\ 22.454 & 20.917 & 6.1101 \end{bmatrix}$$

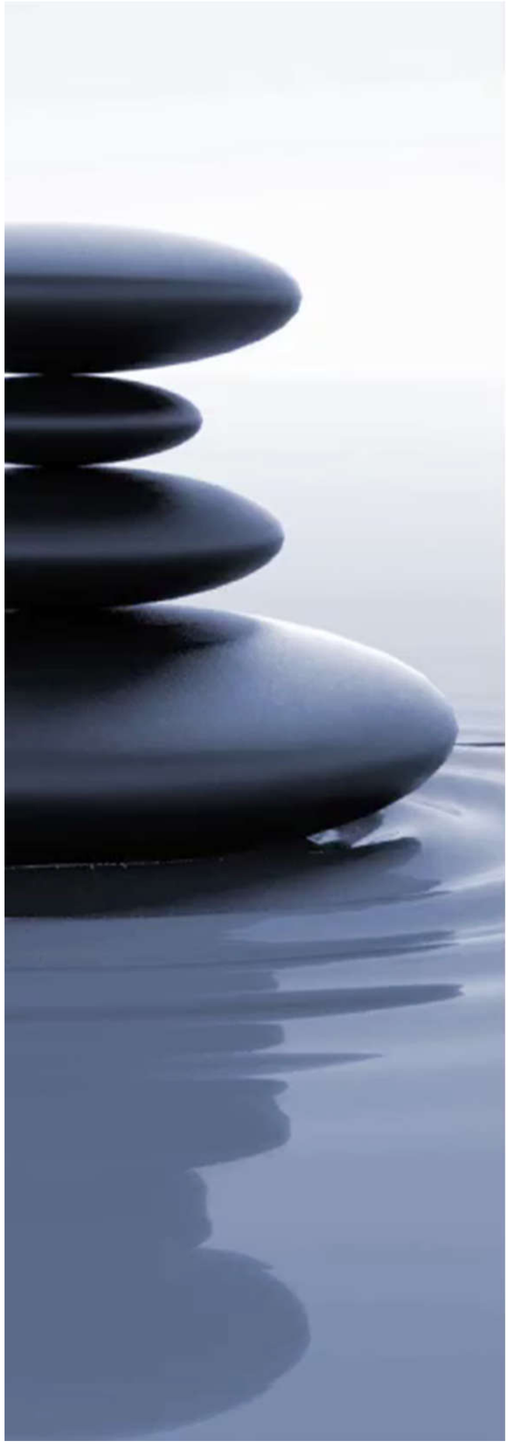
The validity of this decomposition can be verified by substituting it and its transpose into Eq. (11.2) to see if their product yields the original matrix $[A]$. This is left for an exercise.



Gauss-Seidel Decomposition

Iterative or approximate methods provide an alternative to the elimination methods. The Gauss-Seidel method is the most commonly used iterative method.

The system *[A]{X}={B}* is reshaped by solving the first equation for x_1, the second equation for x_2, and the third for x_3, ...and n^{th} equation for x_n. For conciseness, we will limit ourselves to a *3x3* set of equations.



Gauss–Seidel method

$$A\mathbf{x} = \mathbf{b}$$

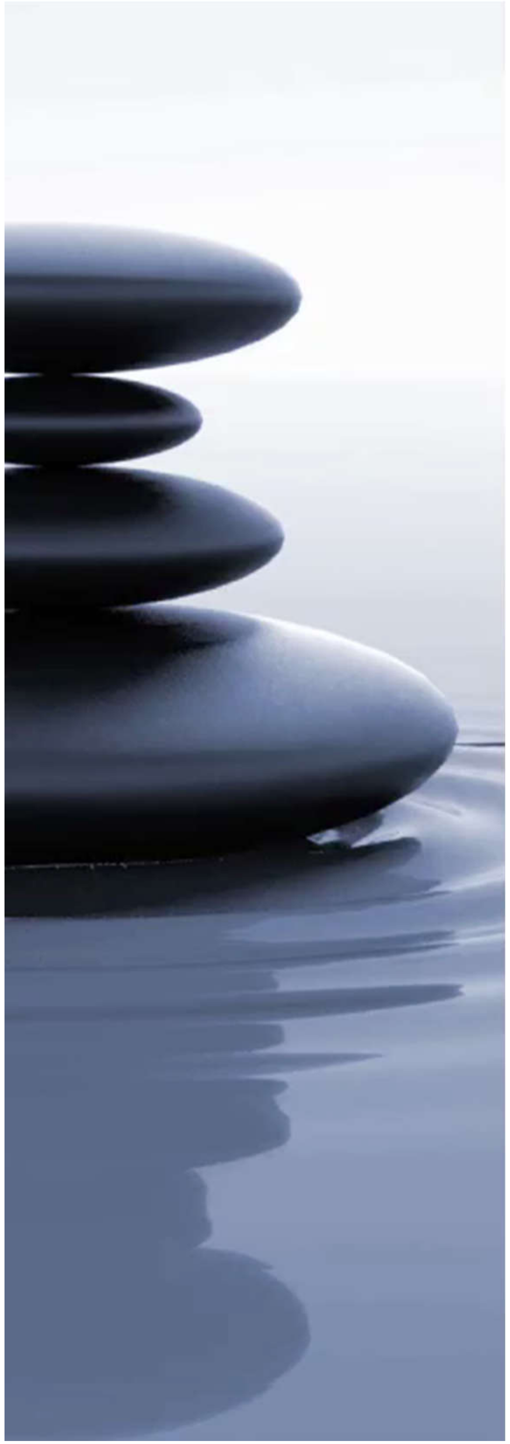
$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

$$A = L_* + U \qquad \text{where} \qquad L_* = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \quad U = \begin{bmatrix} 0 & a_{12} & \cdots & a_{1n} \\ 0 & 0 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}.$$

$$L_*\mathbf{x} = \mathbf{b} - U\mathbf{x}$$

$$\mathbf{x}^{(k+1)} = L_*^{-1}(\mathbf{b} - U\mathbf{x}^{(k)}).$$

$$x_i^{(k+1)} = \frac{1}{a_{ii}}\left(b_i - \sum_{j<i} a_{ij}x_j^{(k+1)} - \sum_{j>i} a_{ij}x_j^{(k)}\right), \quad i = 1, 2, \ldots, n.$$



Example #10

$$A = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix} \qquad b = \begin{bmatrix} 11 \\ 13 \end{bmatrix}.$$

$$\mathbf{x}^{(k+1)} = L_*^{-1}(\mathbf{b} - U\mathbf{x}^{(k)})$$

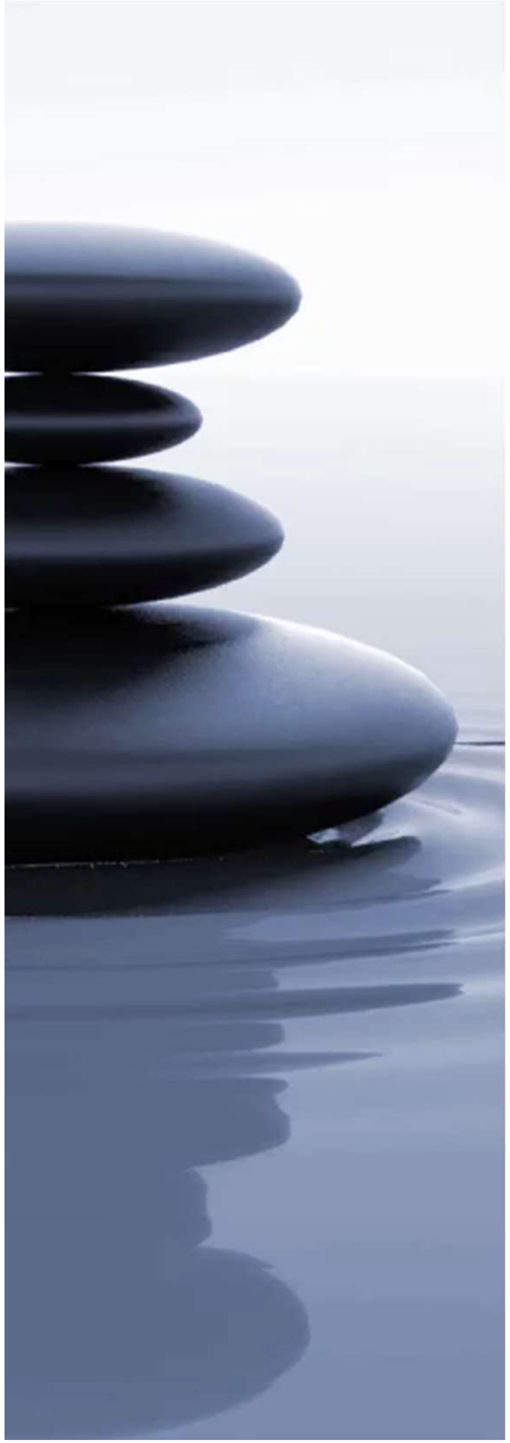
$$\mathbf{x}^{(k+1)} = T\mathbf{x}^{(k)} + C \qquad T = -L_*^{-1}U \qquad C = L_*^{-1}\mathbf{b}.$$

$$L_* = \begin{bmatrix} 2 & 0 \\ 5 & 7 \end{bmatrix} \qquad U = \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}.$$

$$L_*^{-1} = \begin{bmatrix} 2 & 0 \\ 5 & 7 \end{bmatrix}^{-1} = \begin{bmatrix} 0.500 & 0.000 \\ -0.357 & 0.143 \end{bmatrix}$$

$$T = -\begin{bmatrix} 0.500 & 0.000 \\ -0.357 & 0.143 \end{bmatrix} \times \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0.000 & -1.500 \\ 0.000 & 1.071 \end{bmatrix},$$

$$C = \begin{bmatrix} 0.500 & 0.000 \\ -0.357 & 0.143 \end{bmatrix} \times \begin{bmatrix} 11 \\ 13 \end{bmatrix} = \begin{bmatrix} 5.500 \\ -2.071 \end{bmatrix}.$$



Example

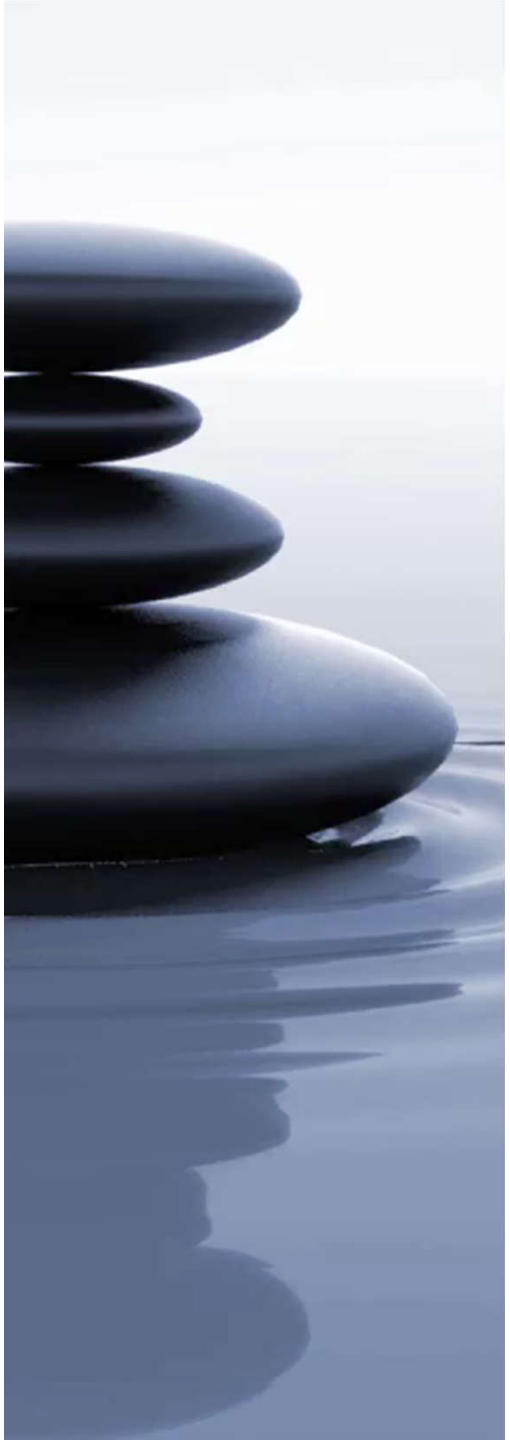
$$x^{(0)} = \begin{bmatrix} 1.1 \\ 2.3 \end{bmatrix}.$$

$$x^{(1)} = \begin{bmatrix} 0 & -1.500 \\ 0 & 1.071 \end{bmatrix} \times \begin{bmatrix} 1.1 \\ 2.3 \end{bmatrix} + \begin{bmatrix} 5.500 \\ -2.071 \end{bmatrix} = \begin{bmatrix} 2.050 \\ 0.393 \end{bmatrix}.$$

$$x^{(2)} = \begin{bmatrix} 0 & -1.500 \\ 0 & 1.071 \end{bmatrix} \times \begin{bmatrix} 2.050 \\ 0.393 \end{bmatrix} + \begin{bmatrix} 5.500 \\ -2.071 \end{bmatrix} = \begin{bmatrix} 4.911 \\ -1.651 \end{bmatrix}.$$

$$x^{(3)} = \cdots.$$

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} -38 \\ 29 \end{bmatrix}$$



Example

Problem Statement. Use the Gauss-Seidel method to obtain the solution of the same system used in Example 10.2:

$$
\begin{aligned}
3x_1 - 0.1x_2 - 0.2x_3 &= 7.85 \\
0.1x_1 + 7x_2 - 0.3x_3 &= -19.3 \\
0.3x_1 - 0.2x_2 + 10x_3 &= 71.4
\end{aligned}
$$

Recall that the true solution is $x_1 = 3$, $x_2 = -2.5$, and $x_3 = 7$.

Solution. First, solve each of the equations for its unknown on the diagonal.

$$x_1 = \frac{7.85 + 0.1x_2 + 0.2x_3}{3} \tag{E11.3.1}$$

$$x_2 = \frac{-19.3 - 0.1x_1 + 0.3x_3}{7} \tag{E11.3.2}$$

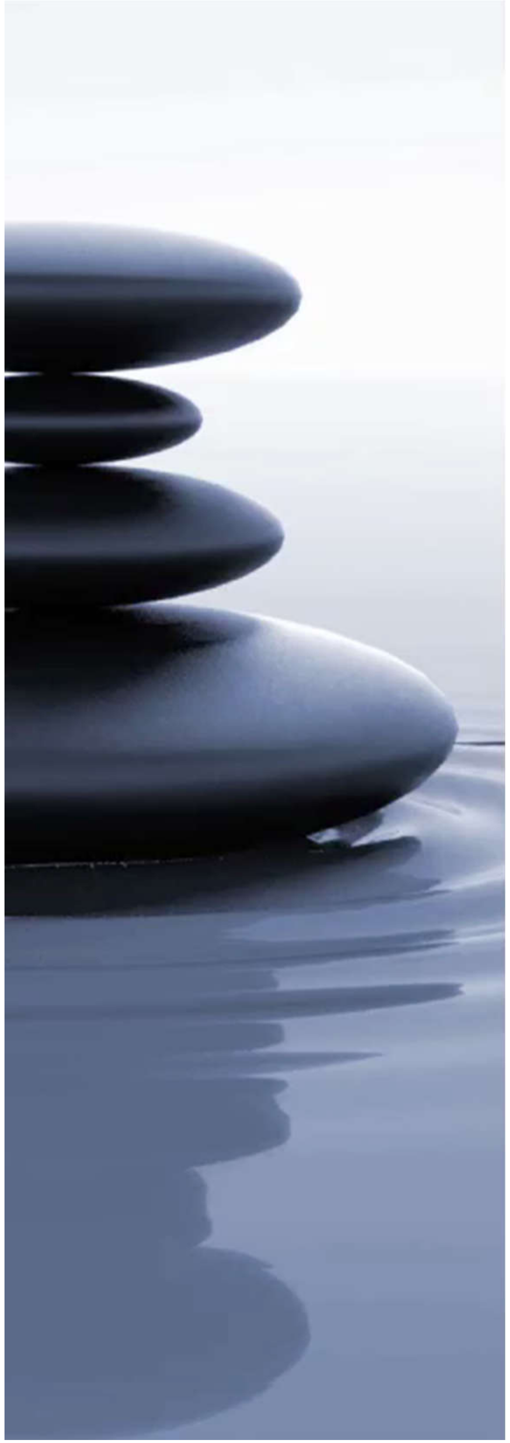
$$x_3 = \frac{71.4 - 0.3x_1 + 0.2x_2}{10} \tag{E11.3.3}$$

By assuming that x_2 and x_3 are zero, Eq. (E11.3.1) can be used to compute

$$x_1 = \frac{7.85 + 0 + 0}{3} = 2.616667$$

This value, along with the assumed value of $x_3 = 0$, can be substituted into Eq. (E11.3.2) to calculate

$$x_2 = \frac{-19.3 - 0.1(2.616667) + 0}{7} = -2.794524$$



The first iteration is completed by substituting the calculated values for x_1 and x_2 into Eq. (E11.3.3) to yield

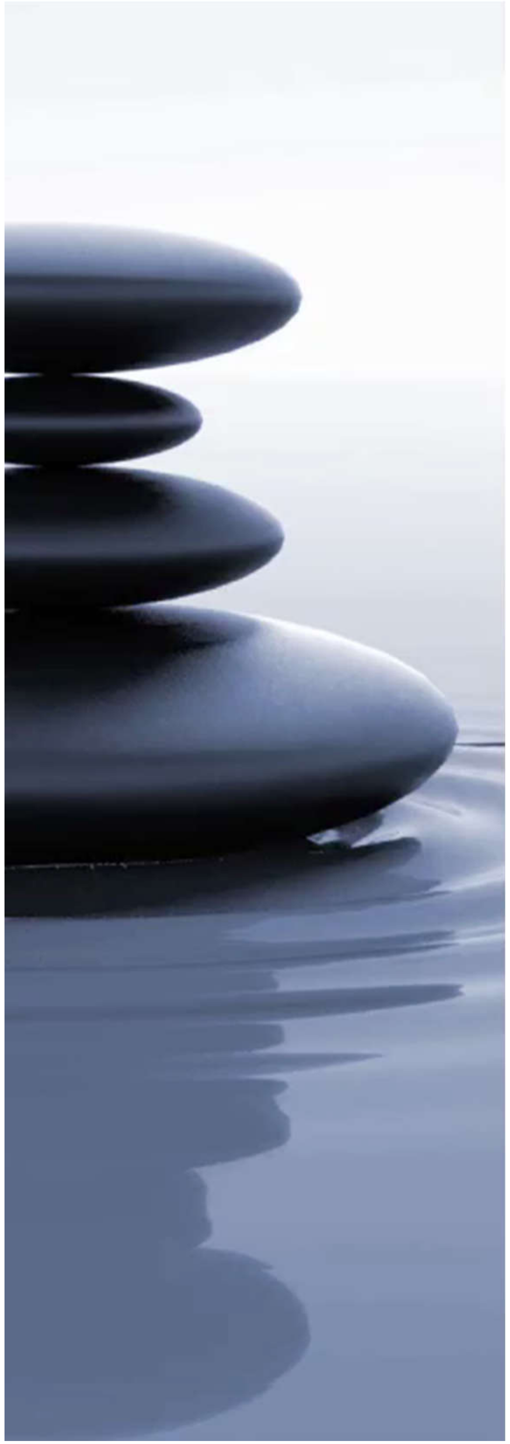
$$x_3 = \frac{71.4 - 0.3(2.616667) + 0.2(-2.794524)}{10} = 7.005610$$

For the second iteration, the same process is repeated to compute

$$x_1 = \frac{7.85 + 0.1(-2.794524) + 0.2(7.005610)}{3} = 2.990557 \qquad |\varepsilon_t| = 0.31\%$$

$$x_2 = \frac{-19.3 - 0.1(2.990557) + 0.3(7.005610)}{7} = -2.499625 \qquad |\varepsilon_t| = 0.015\%$$

$$x_3 = \frac{71.4 - 0.3(2.990557) + 0.2(-2.499625)}{10} = 7.000291 \qquad |\varepsilon_t| = 0.0042\%$$



Successive over-relaxation

$$A\mathbf{x} = \mathbf{b}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

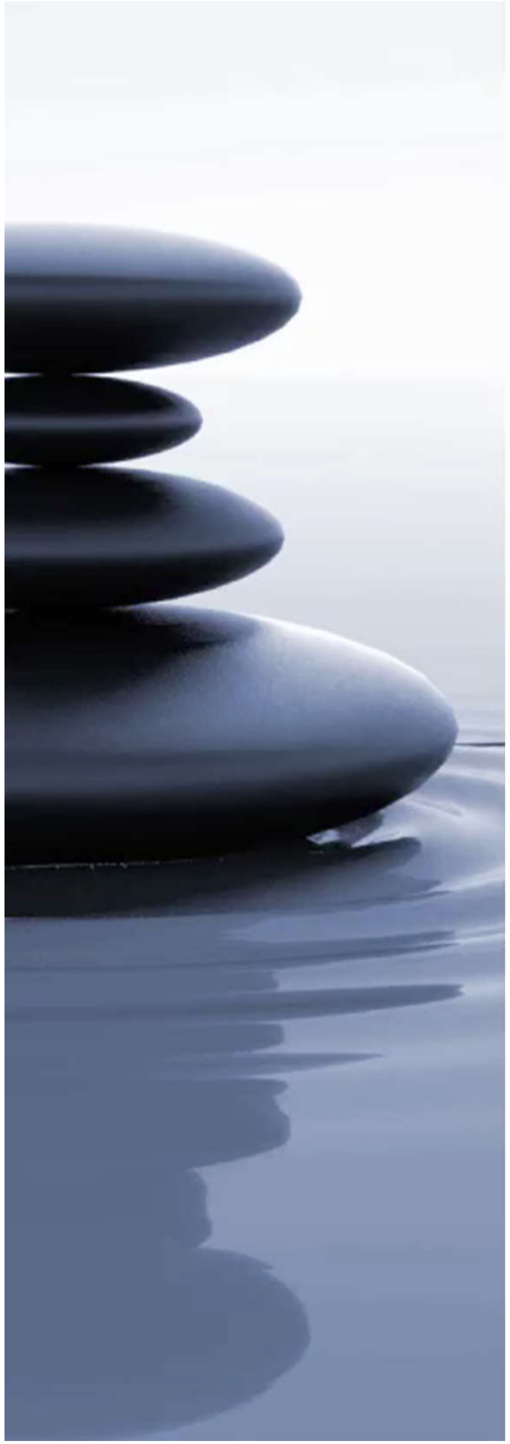
$$A = D + L + U,$$

$$D = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}, \quad L = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ a_{21} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & 0 \end{bmatrix}, \quad U = \begin{bmatrix} 0 & a_{12} & \cdots & a_{1n} \\ 0 & 0 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}.$$

$$(D + \omega L)\mathbf{x} = \omega\mathbf{b} - [\omega U + (\omega - 1)D]\mathbf{x}$$

$$\mathbf{x}^{(k+1)} = (D + \omega L)^{-1}\big(\omega\mathbf{b} - [\omega U + (\omega - 1)D]\mathbf{x}^{(k)}\big) = L_w\mathbf{x}^{(k)} + \mathbf{c}.$$

$$x_i^{(k+1)} = (1 - \omega)x_i^{(k)} + \frac{\omega}{a_{ii}}\left(b_i - \sum_{j<i} a_{ij}x_j^{(k+1)} - \sum_{j>i} a_{ij}x_j^{(k)}\right), \quad i = 1, 2, \ldots, n.$$

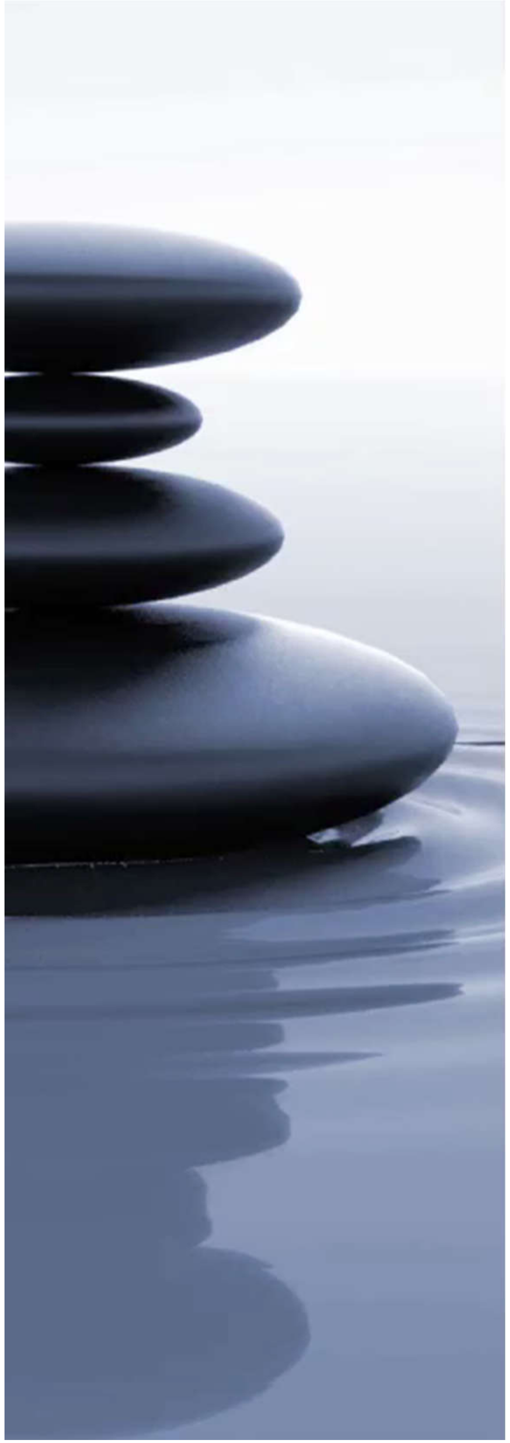


Symmetric successive over-relaxation

$$U = L^T,$$

$$P = \left(\frac{D}{\omega} + L\right) \frac{\omega}{2-\omega} D^{-1} \left(\frac{D}{\omega} + U\right),$$

$$\mathbf{x}^{k+1} = \mathbf{x}^k - \gamma^k P^{-1}(A\mathbf{x}^k - \mathbf{b}),\ k \geq 0.$$



Jacobi method

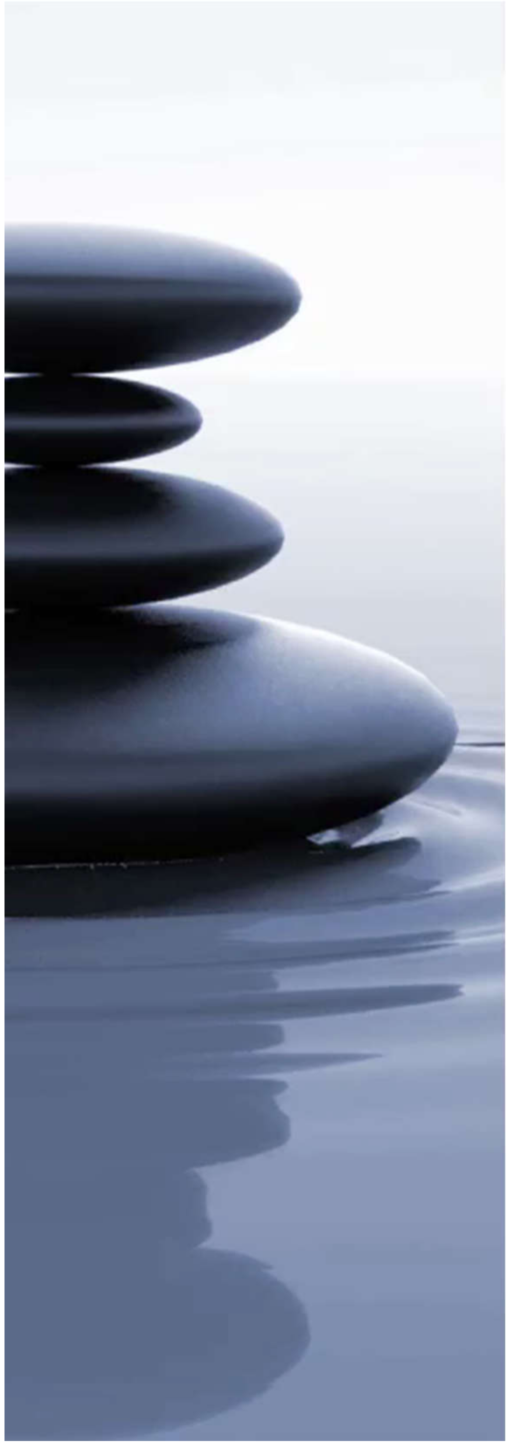
$$A\mathbf{x} = \mathbf{b}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

$$A = D + R \qquad \text{where} \qquad D = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} \text{ and } R = \begin{bmatrix} 0 & a_{12} & \cdots & a_{1n} \\ a_{21} & 0 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & 0 \end{bmatrix}$$

$$\mathbf{x}^{(k+1)} = D^{-1}(\mathbf{b} - R\mathbf{x}^{(k)}).$$

$$x_i^{(k+1)} = \frac{1}{a_{ii}} \left(b_i - \sum_{j \neq i} a_{ij} x_j^{(k)} \right), \quad i = 1, 2, \ldots, n.$$



Example #11

A linear system of the form $Ax = b$ with initial estimate $x^{(0)}$ is given by

$$A = \begin{bmatrix} 2 & 1 \\ 5 & 7 \end{bmatrix},\ b = \begin{bmatrix} 11 \\ 13 \end{bmatrix} \quad \text{and} \quad x^{(0)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

We use the equation $x^{(k+1)} = D^{-1}(b - Rx^{(k)})$, described above, to estimate x. First, we rewrite the equation in a more convenient form $D^{-1}(b - Rx^{(k)}) = Tx^{(k)} + C$, where $T = -D^{-1}R$ and $C = D^{-1}b$. Note that $R = L + U$ where L and U are the strictly lower and upper parts of A. From the known values

$$D^{-1} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/7 \end{bmatrix},\ L = \begin{bmatrix} 0 & 0 \\ 5 & 0 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

we determine $T = -D^{-1}(L + U)$ as

$$T = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/7 \end{bmatrix} \left\{ \begin{bmatrix} 0 & 0 \\ -5 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \right\} = \begin{bmatrix} 0 & -1/2 \\ -5/7 & 0 \end{bmatrix}.$$

Further, C is found as

$$C = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/7 \end{bmatrix} \begin{bmatrix} 11 \\ 13 \end{bmatrix} = \begin{bmatrix} 11/2 \\ 13/7 \end{bmatrix}.$$

With T and C calculated, we estimate x as $x^{(1)} = Tx^{(0)} + C$:

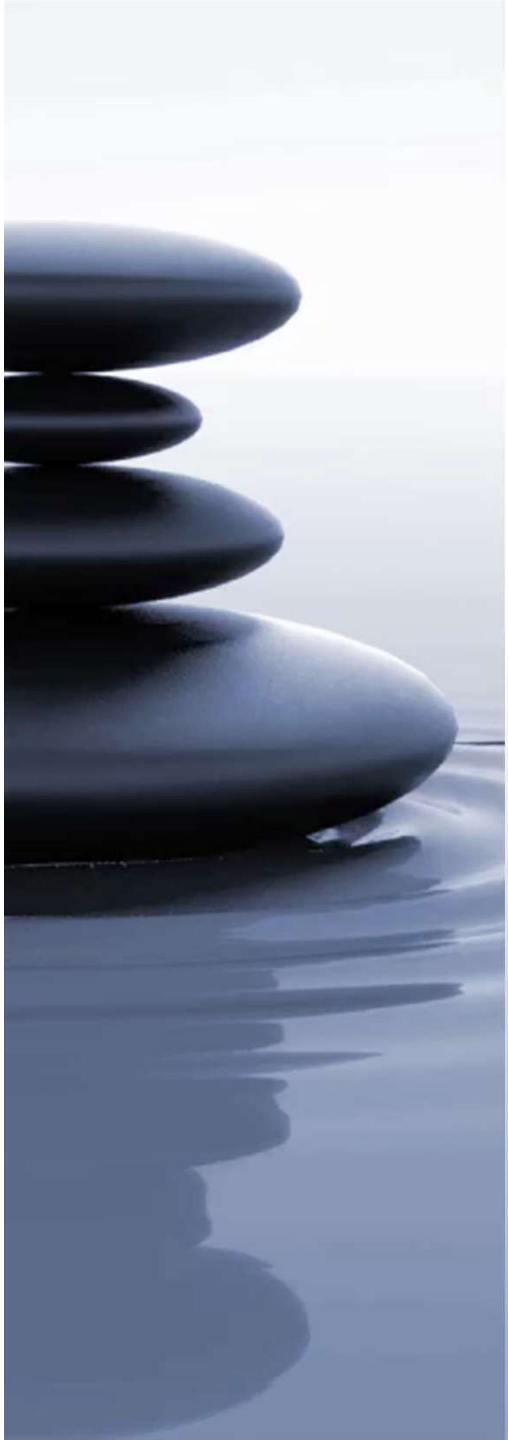
$$x^{(1)} = \begin{bmatrix} 0 & -1/2 \\ -5/7 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 11/2 \\ 13/7 \end{bmatrix} = \begin{bmatrix} 5.0 \\ 8/7 \end{bmatrix} \approx \begin{bmatrix} 5 \\ 1.143 \end{bmatrix}.$$

The next iteration yields

$$x^{(2)} = \begin{bmatrix} 0 & -1/2 \\ -5/7 & 0 \end{bmatrix} \begin{bmatrix} 5.0 \\ 8/7 \end{bmatrix} + \begin{bmatrix} 11/2 \\ 13/7 \end{bmatrix} = \begin{bmatrix} 69/14 \\ -12/7 \end{bmatrix} \approx \begin{bmatrix} 4.929 \\ -1.713 \end{bmatrix}.$$

This process is repeated until convergence (i.e., until $\|Ax^{(n)} - b\|$ is small). The solution after 25 iterations is

$$x = \begin{bmatrix} 7.111 \\ -3.222 \end{bmatrix}.$$



Any Questions?

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