TMC #2: Errors & Taylor Series

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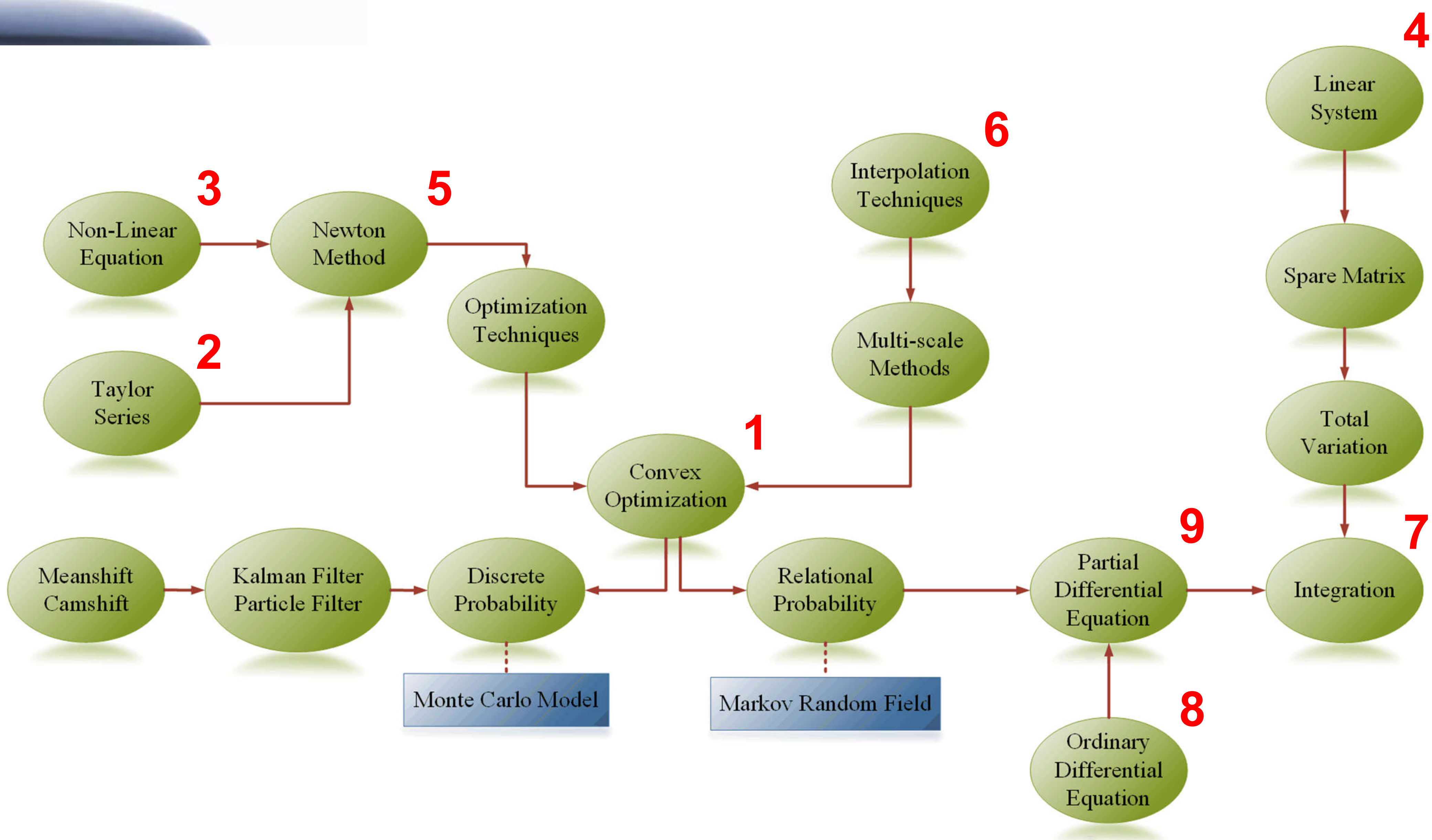
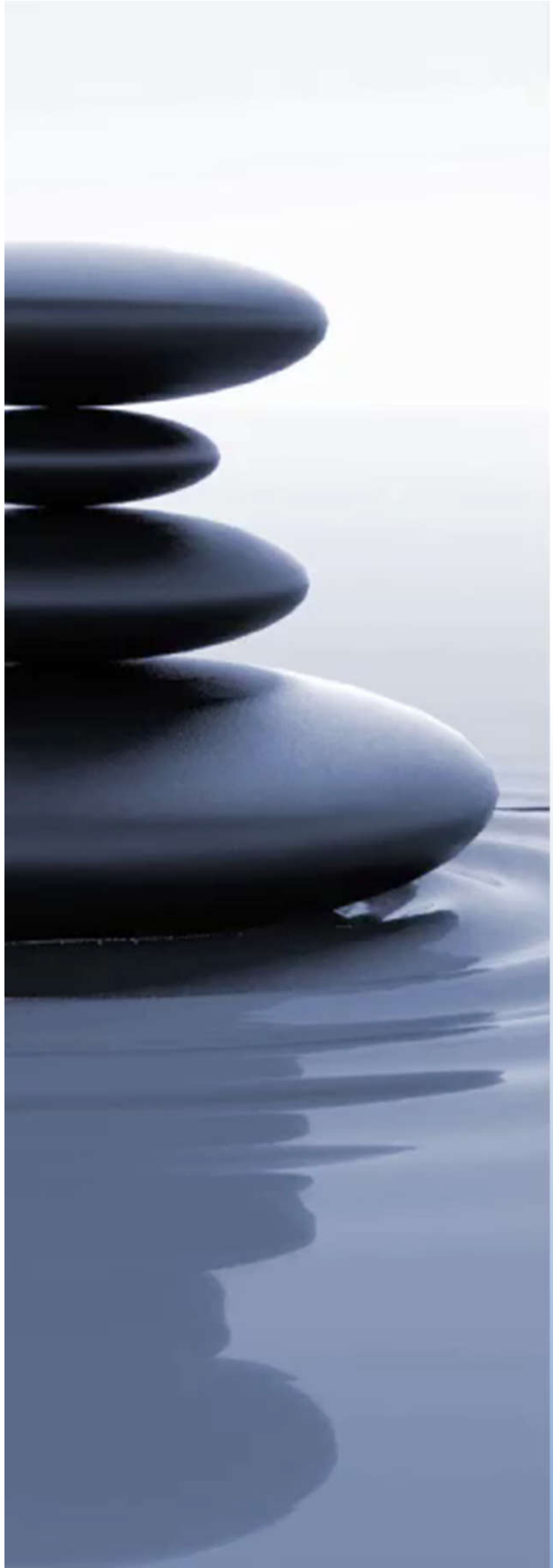
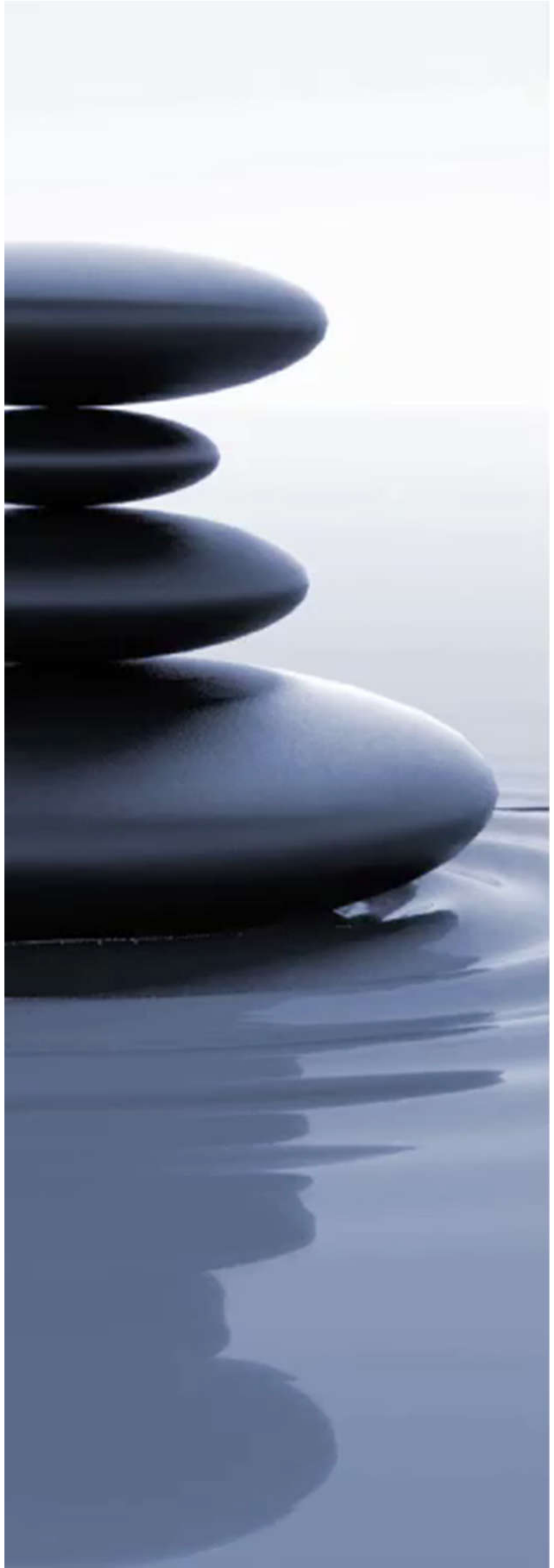
The Mind map of Engineering Mathematics





Part A

ACCURACY, ERROR & APPROXIMATE ERROR

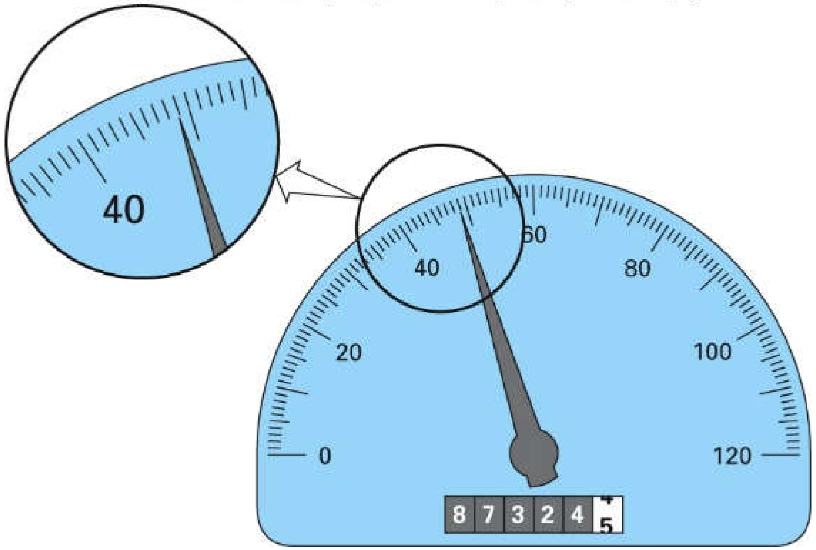


Motivations

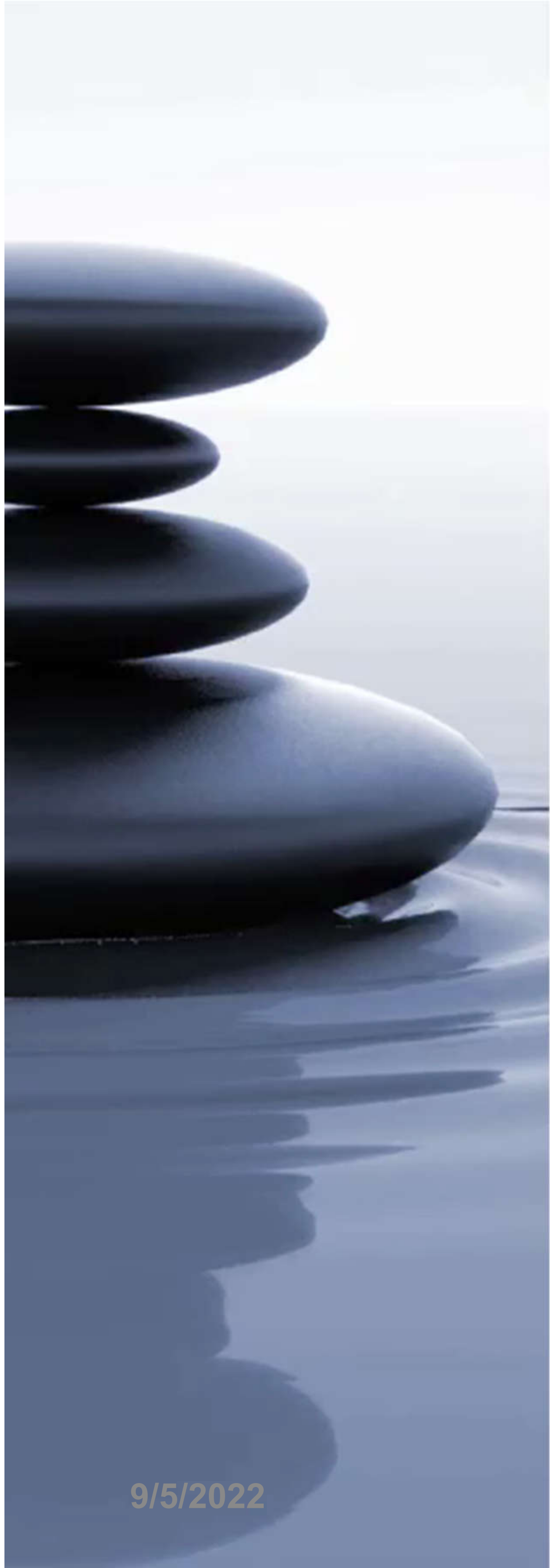
We ask for numerical methods since we cannot get exact solution !!

Numerical methods only provide approximate results, not exact ones.

So how we confident our results obtained from numerical methods ????

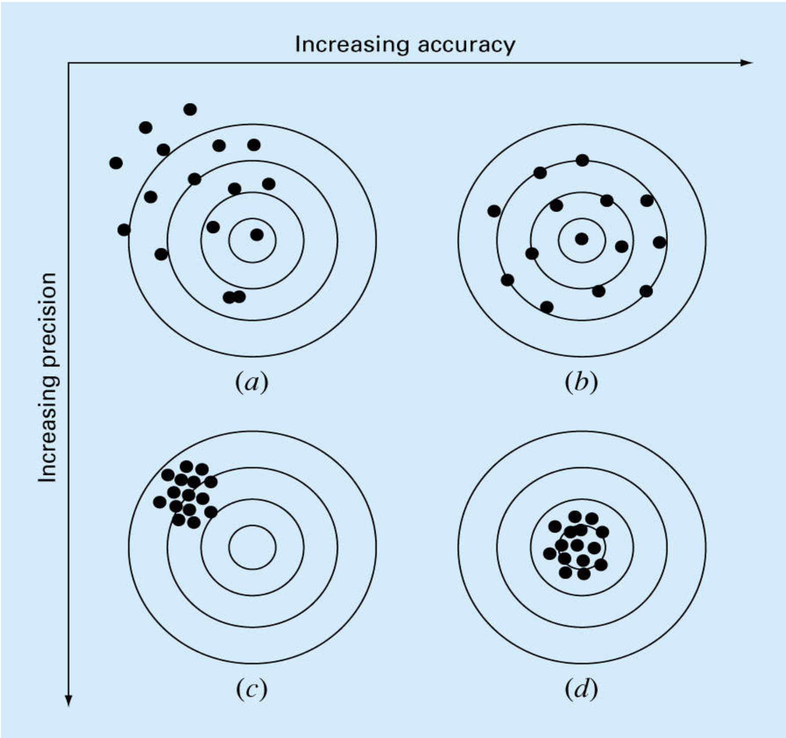


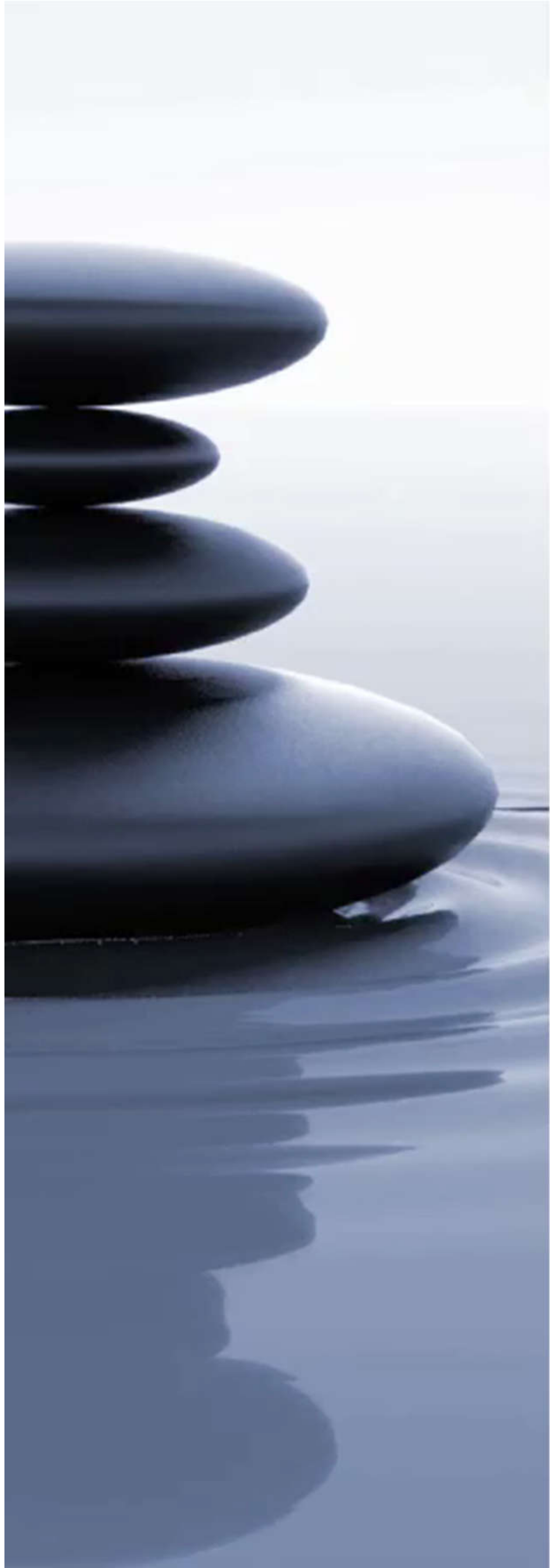
$$x = \sqrt{2} = 1.41421356237\ldots$$



Accuracy and Precision

- Errors associated with both calculations and measurements can be characterized with regard to their accuracy and precision
- Accuracy refers to how closely a computed or measured value agrees with the true value
- Precision refers to how closely individual computed or measured values agree with each other



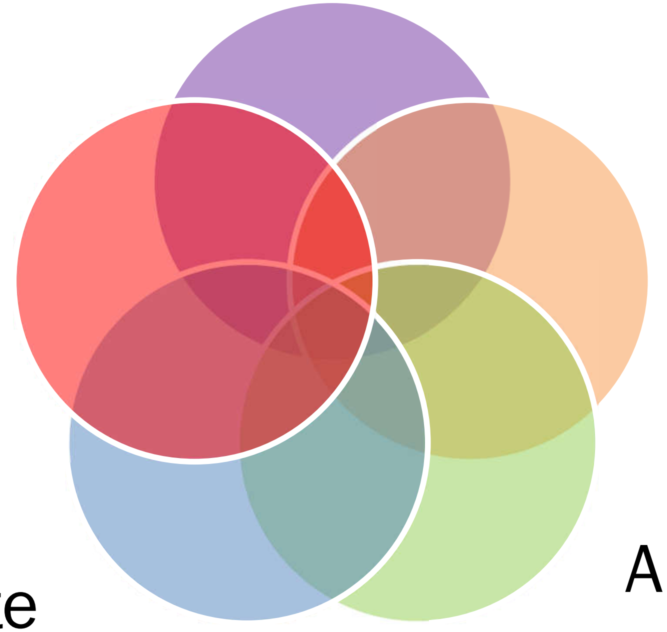


Objectives

True Error

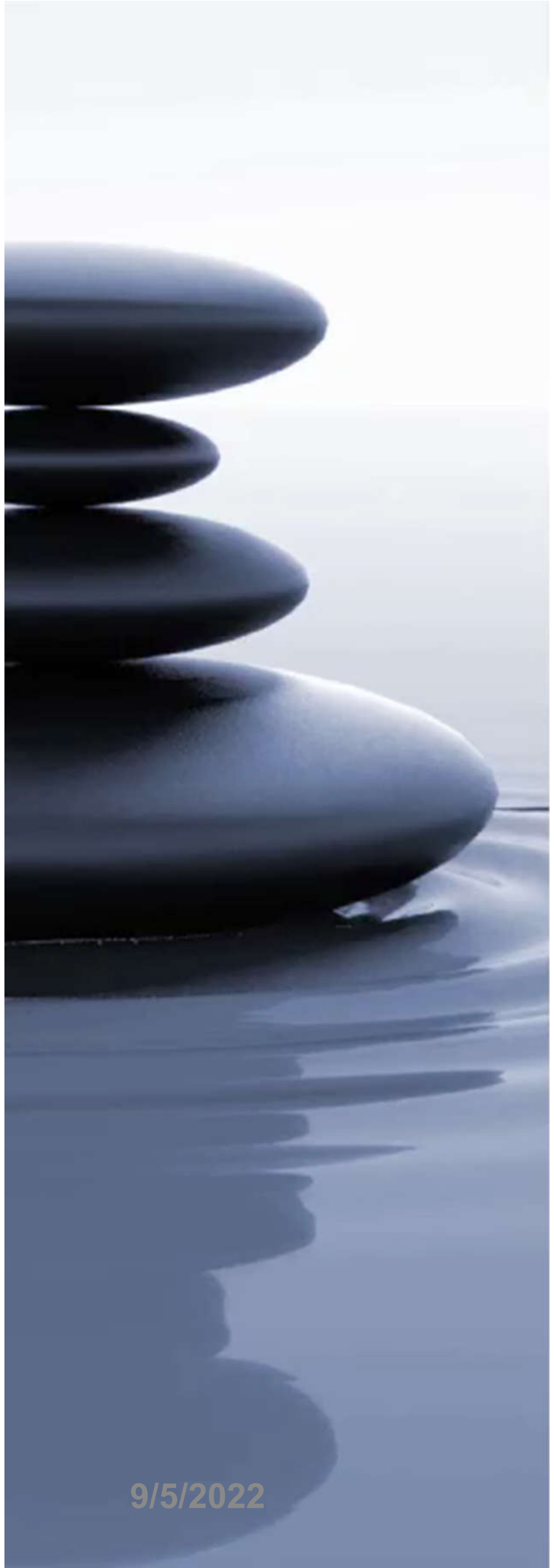
Significant Figures

Relative Error



Relative Approximate Error

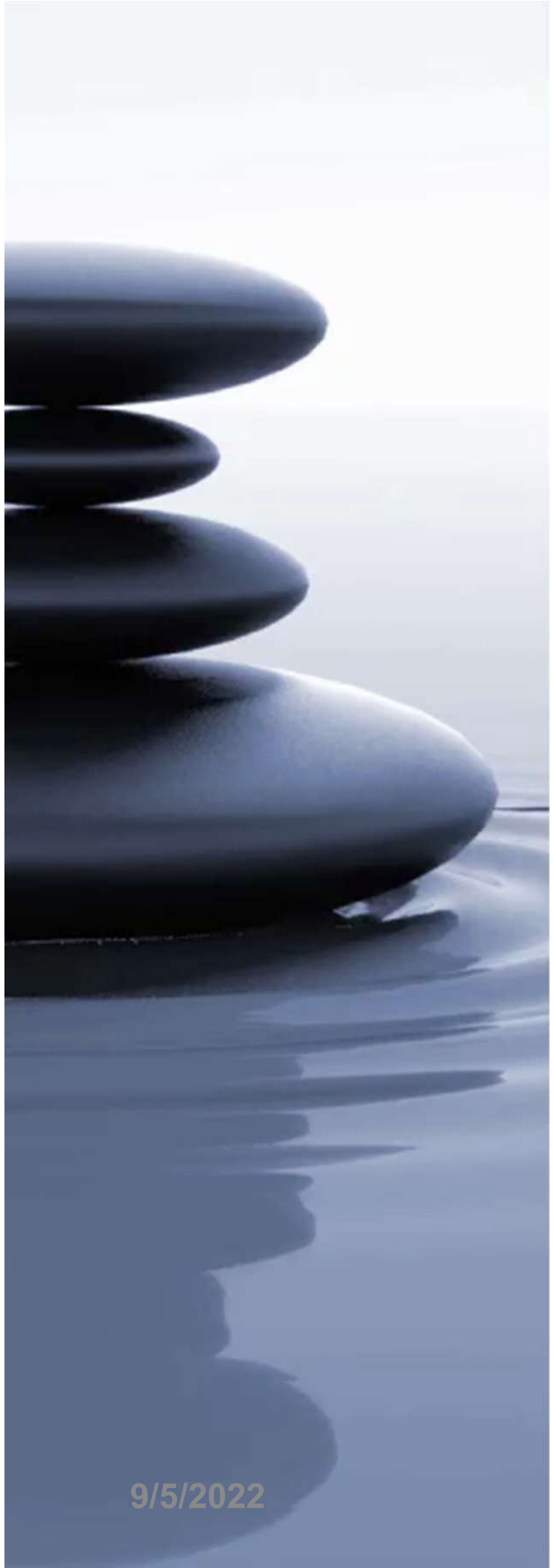
Approximate Error



1. True Error

Error, or true error E_t, is defined as the difference between the true value in a calculation and the approximate value found using a numerical method etc.

True Error E_t = True Value – Approximate Value



Example

The derivative, $f'(x)$ of a function $f(x)$ can be approximated by the equation,

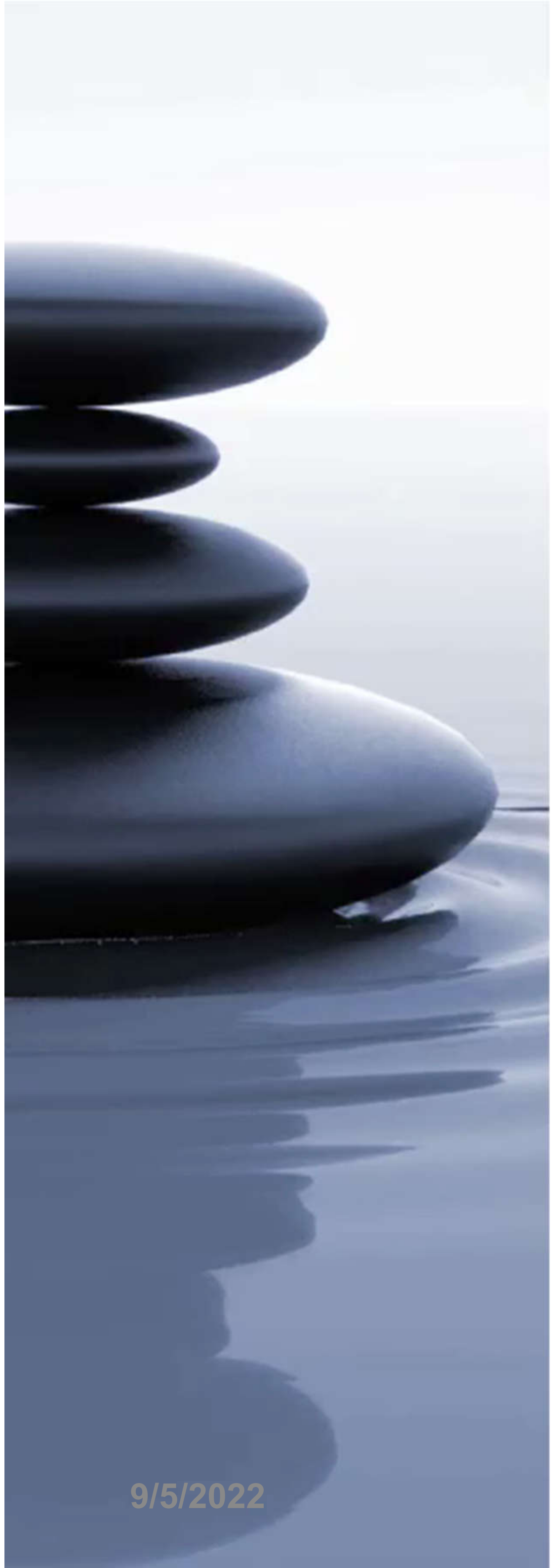
$$f'(x) \approx \frac{f(x+h) - f(x)}{h}$$

If $f(x) = 7e^{0.5x}$ and $h = 0.3$

a) Find the approximate value of $f'(2)$

b) True value of $f'(2)$

c) Error for part (a)



Example (cont.)

Solution:

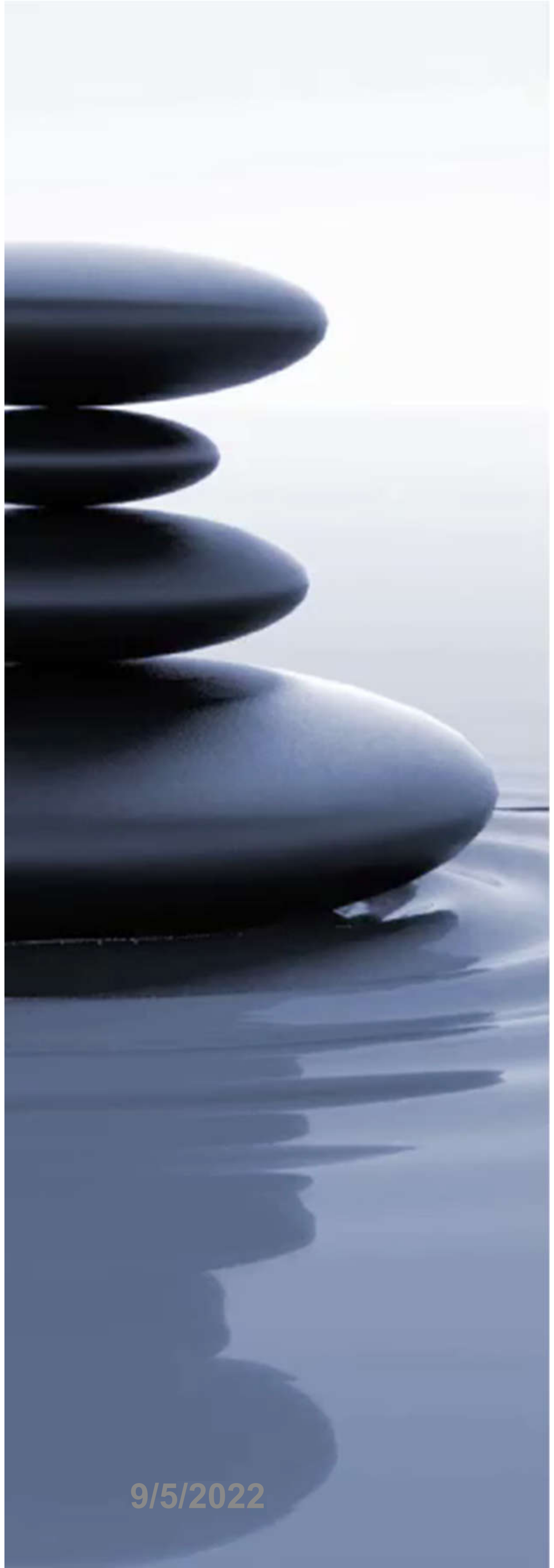
a) For $x = 2$ and $h = 0.3$

$$f'(2) \approx \frac{f(2+0.3) - f(2)}{0.3}$$

$$= \frac{f(2.3) - f(2)}{0.3}$$

$$= \frac{7e^{0.5(2.3)} - 7e^{0.5(2)}}{0.3}$$

$$= \frac{22.107 - 19.028}{0.3} = 10.263$$



Example (cont.)

Solution:

b) The exact value of $f'(2)$ can be found by using our knowledge of differential calculus.

$$f(x) = 7e^{0.5x}$$

$$f'(x) = 7 \times 0.5 \times e^{0.5x}$$

$$= 3.5e^{0.5x}$$

So the true value of $f'(2)$ is

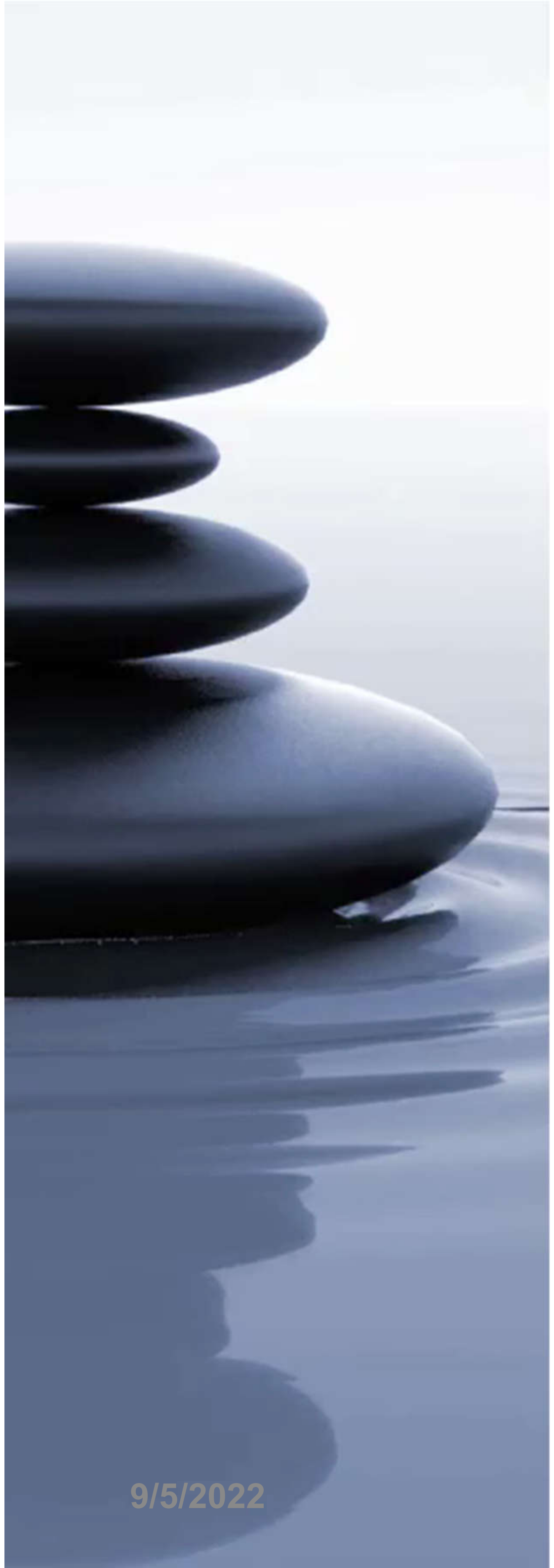
$$f'(2) = 3.5e^{0.5(2)}$$

$$= 9.5140$$

Error is calculated as

$E_t =$ True Value – Approximate Value

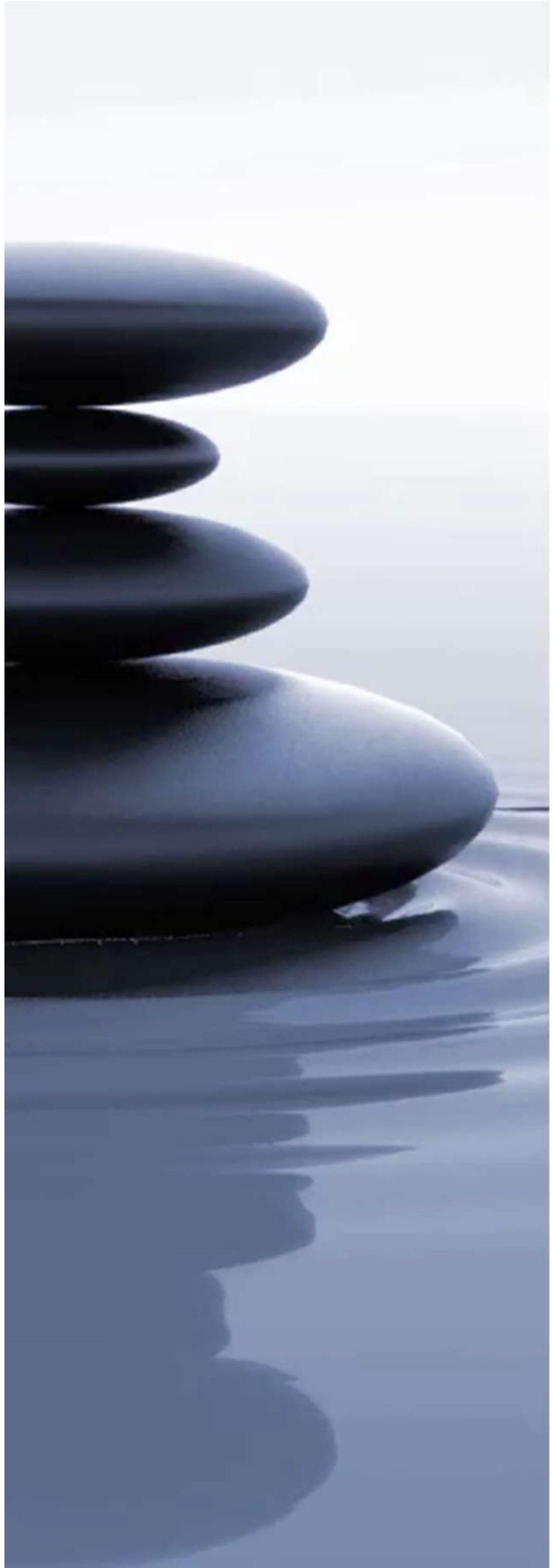
$$= 9.5140 - 10.263 = -0.722$$



2. Relative Error

Defined as the ratio between the true error, and the true value.

$$\text{Relative True Error } (\varepsilon_t) = \frac{\text{True Error}}{\text{True Value}}$$



Example - Relative True Error

Following from the previous example for true error, find the relative true error for $f(x) = 7e^{0.5x}$ at $f'(2)$ with $h = 0.3$

From the previous example,

$$E_t = -0.722$$

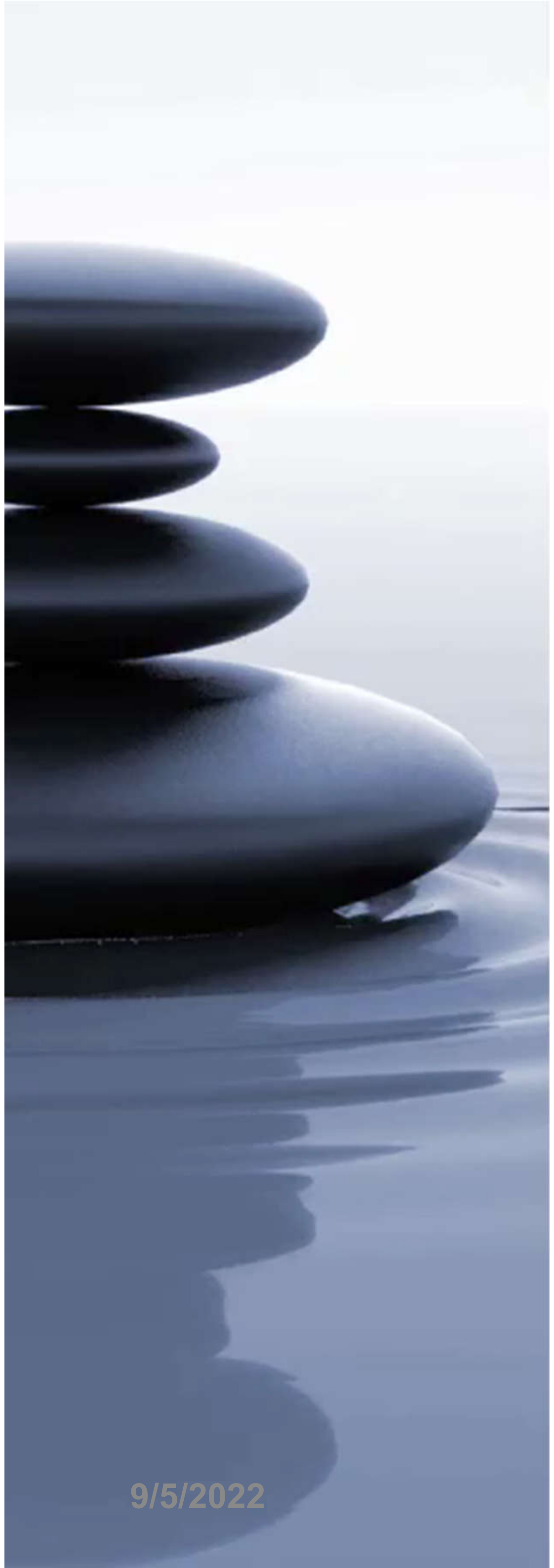
Relative True Error is defined as

$$\varepsilon_t = \frac{\text{True Error}}{\text{True Value}}$$

$$= \frac{-0.722}{9.5140} = -0.075888$$

as a percentage,

$$\varepsilon_t = -0.075888 \times 100\% = -7.5888\%$$



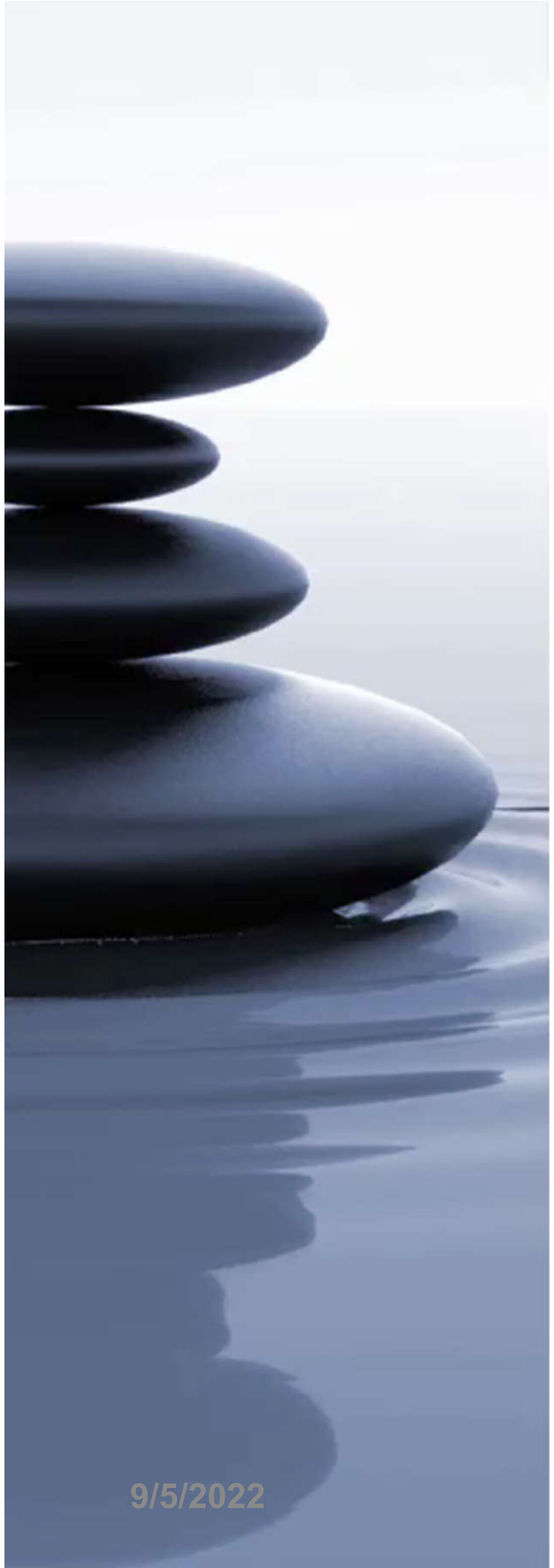
3. Approximate Error

What can be done if true values are not known or are very difficult to obtain?

Approximate error is defined as the difference between the present approximation and the previous approximation.

Approximate Error (E_a) = Present Approximation

– Previous Approximation



Example - Approximate Error

For $f(x) = 7e^{0.5x}$ at $x = 2$ find the following,

a) $f'(2)$ using $h = 0.3$

b) $f'(2)$ using $h = 0.15$

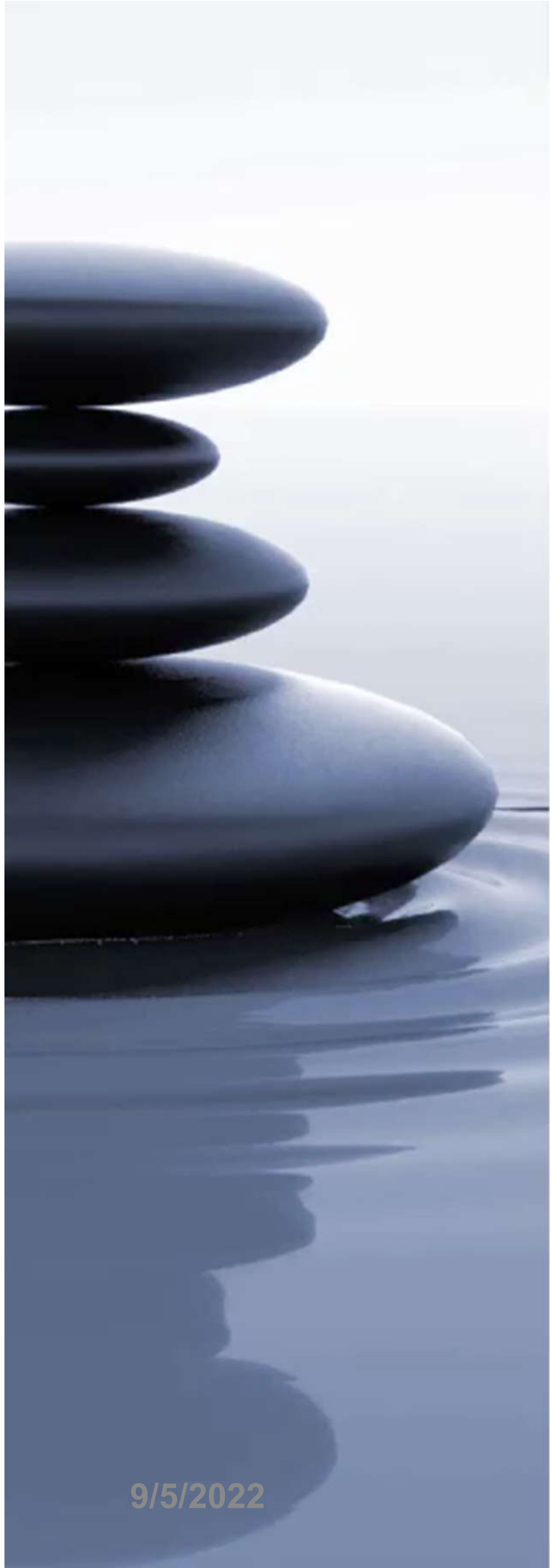
c) approximate error for the value of $f'(2)$ for part b)

Solution:

a) For $x = 2$ and $h = 0.3$

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}$$

$$f'(2) \approx \frac{f(2+0.3) - f(2)}{0.3}$$



Example (cont.)

Solution: (cont.)

$$= \frac{f(2.3) - f(2)}{0.3}$$

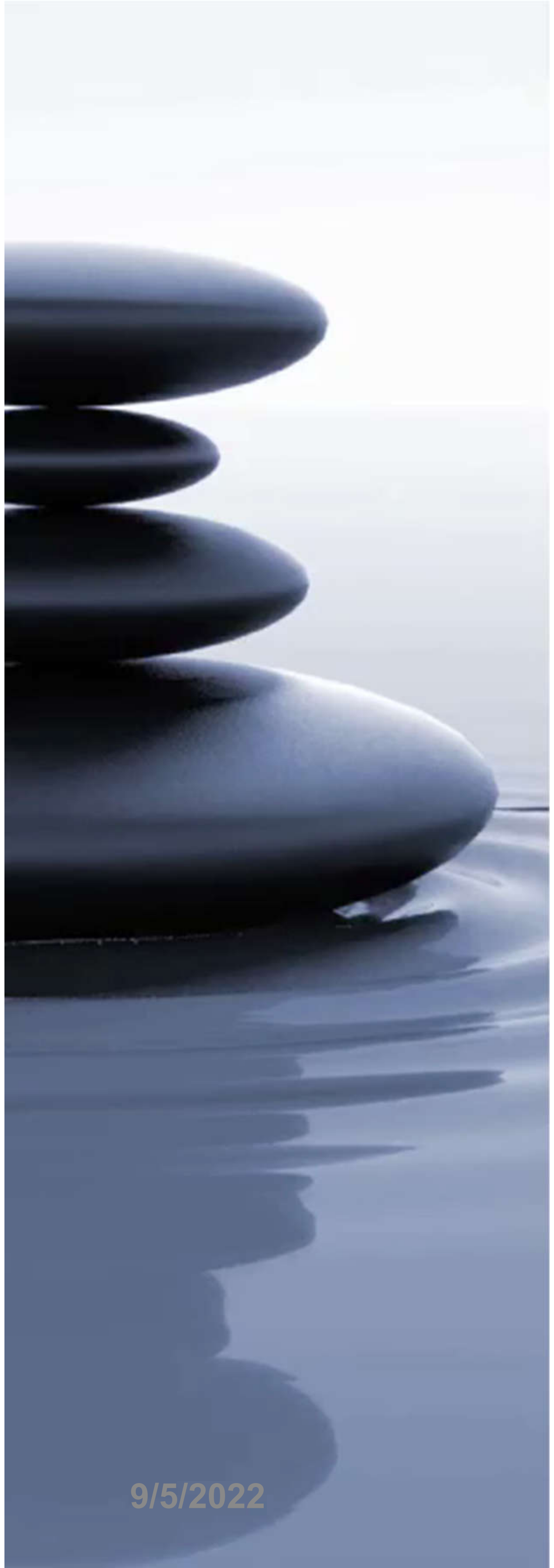
$$= \frac{7e^{0.5(2.3)} - 7e^{0.5(2)}}{0.3}$$

$$= \frac{22.107 - 19.028}{0.3} = 10.263$$

b) For $x = 2$ and $h = 0.15$

$$f'(2) \approx \frac{f(2 + 0.15) - f(2)}{0.15}$$

$$= \frac{f(2.15) - f(2)}{0.15}$$



Example (cont.)

Solution: (cont.)

$$= \frac{7e^{0.5(2.15)} - 7e^{0.5(2)}}{0.15}$$

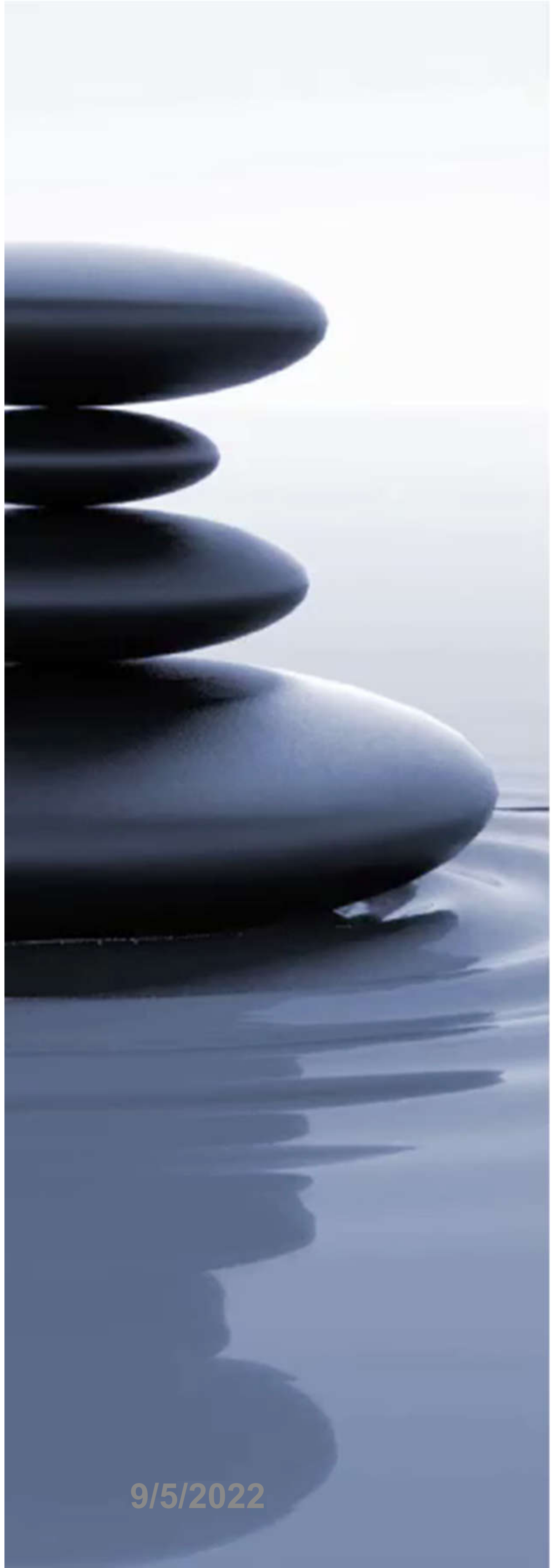
$$= \frac{20.50 - 19.028}{0.15} = 9.8800$$

c) So the approximate error, E_a is

$$E_a = \text{Present Approximation} – \text{Previous Approximation}$$

$$= 9.8800 - 10.263$$

$$= -0.38300$$



4. Relative Approximate Error

Defined as the ratio between the approximate error and the present approximation.

$$\text{Relative Approximate Error } (\varepsilon_a) = \frac{\text{Approximate Error}}{\text{Present Approximation}}$$

Example - Relative Approximate Error

For $f(x)=7e^{0.5x}$ at $x=2$, find the relative approximate error using values from $h=0.3$ and $h=0.15$

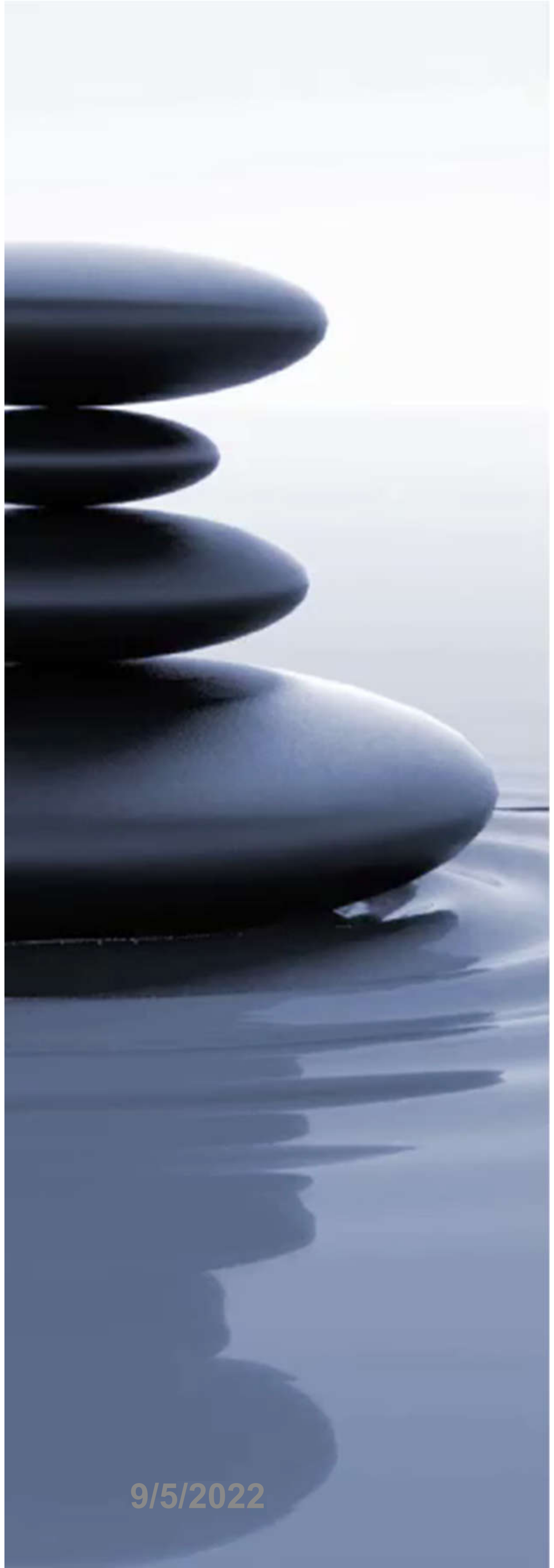
Solution:

From Example 3, the approximate value of $f'(2)=10.263$ using $h=0.3$ and $f'(2)=9.8800$ using $h=0.15$

$$E_a = \text{Present Approximation} - \text{Previous Approximation}$$

$$= 9.8800 - 10.263$$

$$= -0.38300$$



Example (cont.)

Solution: (cont.)

$$\varepsilon_a = \frac{\text{Approximate Error}}{\text{Present Approximation}}$$

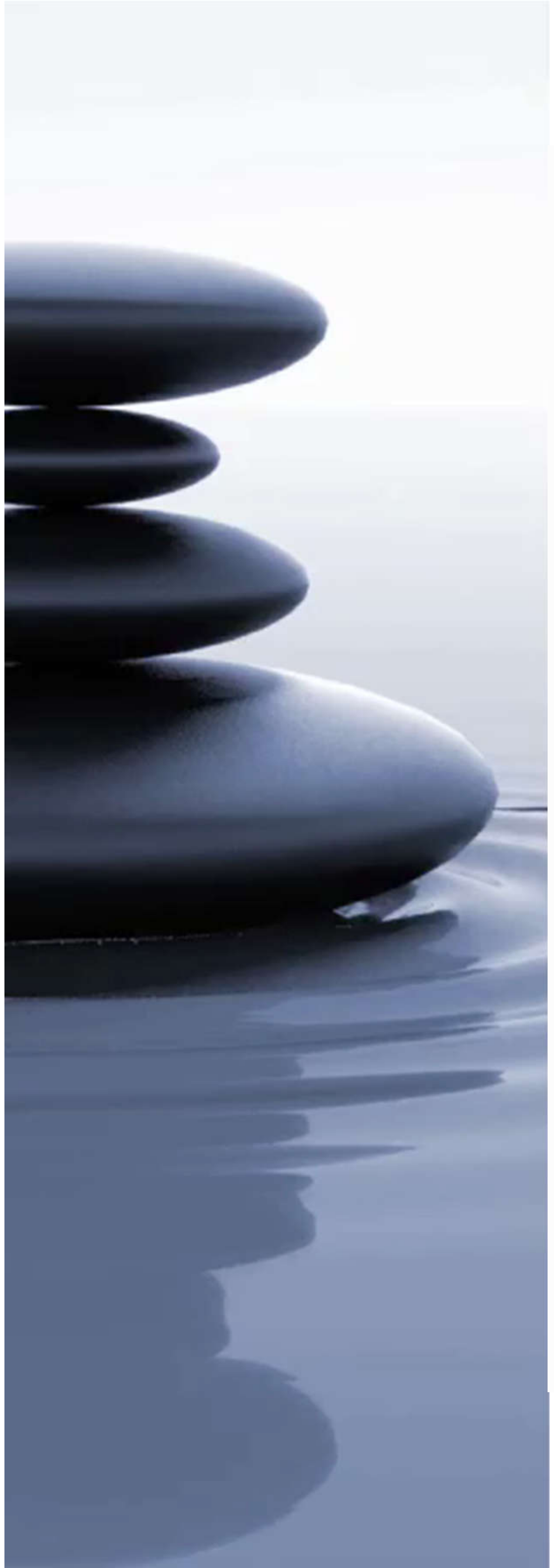
$$= \frac{-0.38300}{9.8800} = -0.038765$$

as a percentage,

$$\varepsilon_a = -0.038765 \times 100\,\% = -3.8765\,\%$$

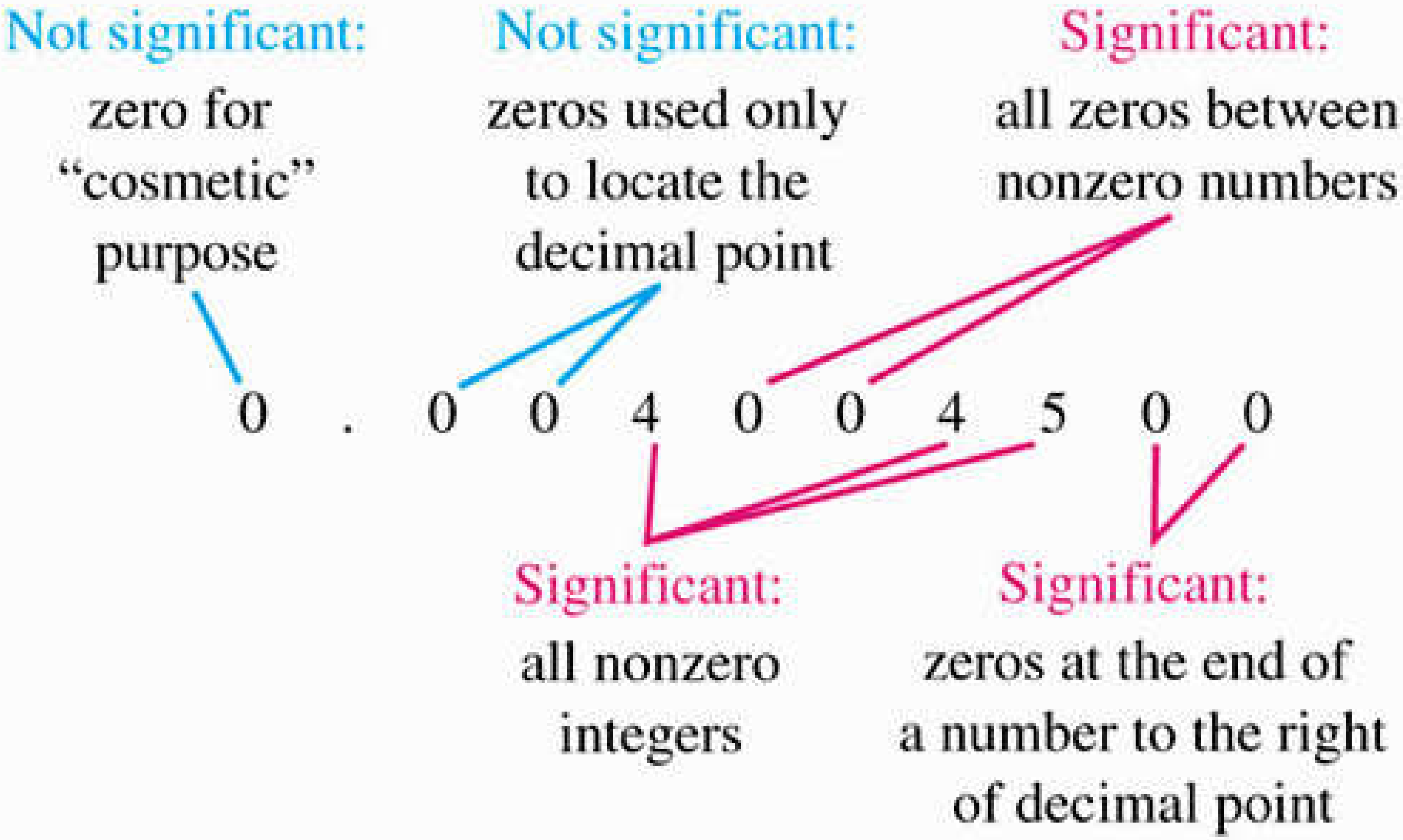
Absolute relative approximate errors may also need to be calculated,

$$|\varepsilon_a| = |-0.038765| = 0.038765 \text{ or } 3.8765\%$$



Significant Figures Rules

Significant Figures can be done using a set of around 5 rules, With a lot of complications for how to deal with zeroes.



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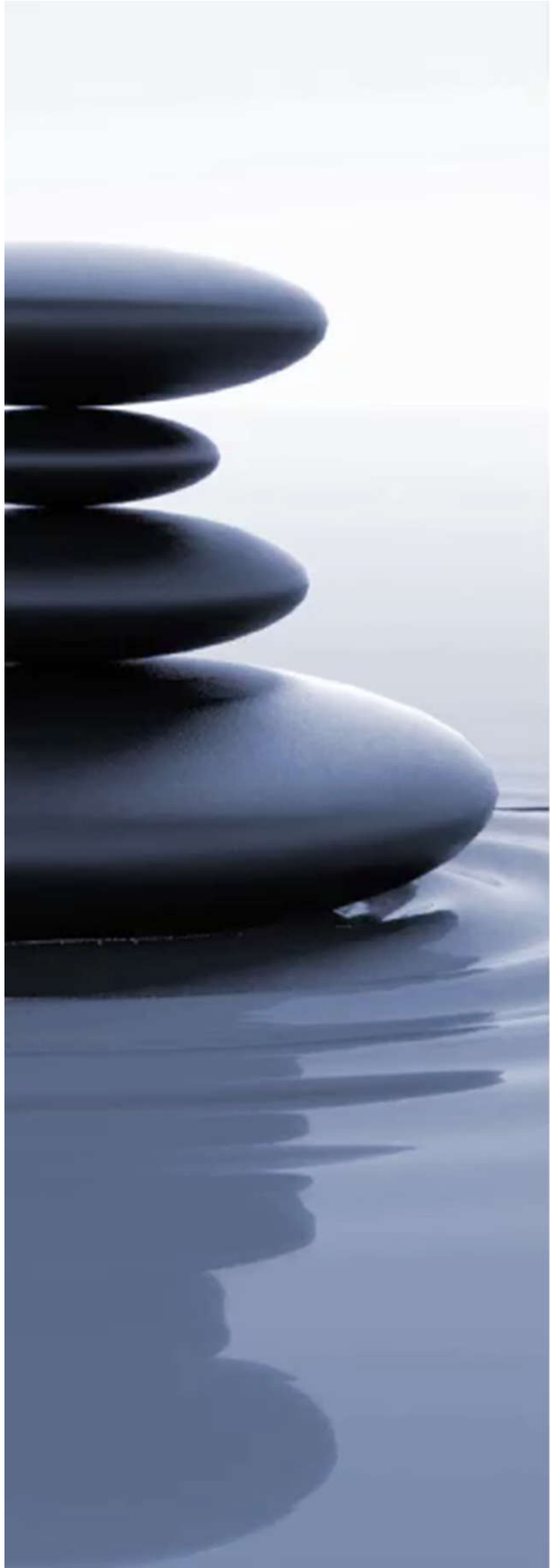


Important Helpful Hints

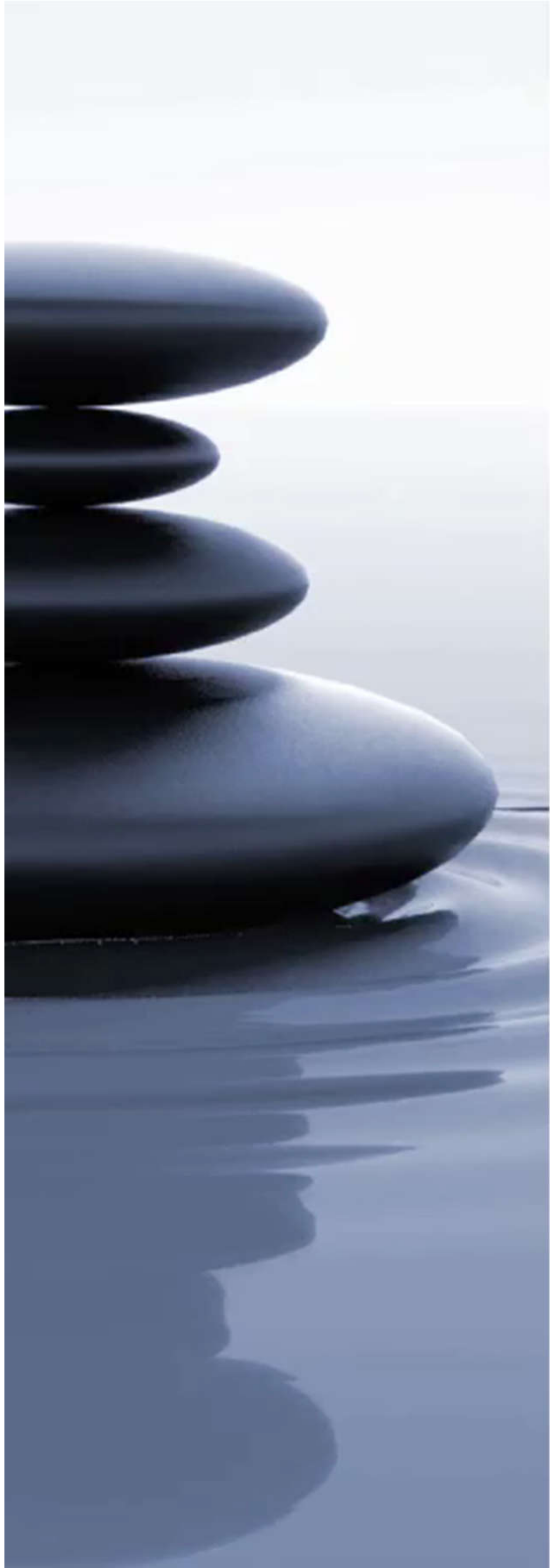
Normal Numbers bigger than 1, or large numbers,always have a POSITIVE Power of 10.

Values smaller than 1, usually decimal values, always have a NEGATIVE Power of 10.

The first part of Scientific Notation is always a number value that is between 1 and 10. (eg. 1, 2.345, 3.65, 6.310, 7.0, 8.5, 9.9999 etc)

The second part of Scientific Notation is a Power of 10 which tells us how many places the decimal point is moving.

The resulting number of digits in our 1 to 10 number is the number of Significant Figures.



"Handy" Helpful Tip

Keep in mind at all times the following:

Normal Numbers bigger than 1, or large numbers, always have a POSITIVE Power of 10.

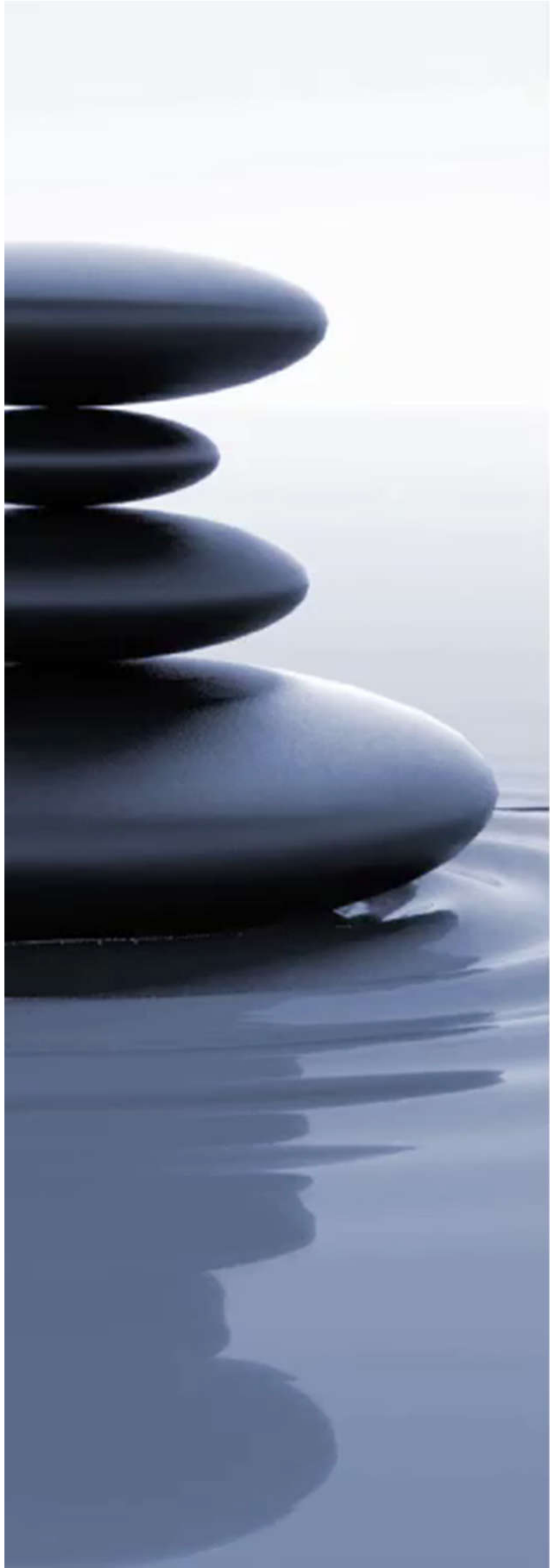
$$6.2 \times 10^{1} = 62$$

$$1.496 \times 10^{8} = 149\,600\,000$$

Values smaller than 1, usually decimal values, always have a NEGATIVE Power of 10.

$$2.31 \times 10^{-3} = 0.00231$$

$$6.234 \times 10^{-1} = 0.6234$$



Example

Determining Significant Figures

15 020

The Number is **Greater than 10**, so the **Exponent will be Positive**.

$= 1\,5\,0\,2\,0$

4 places

Move the Decimal point to the LEFT to create a number between 1 and 10.

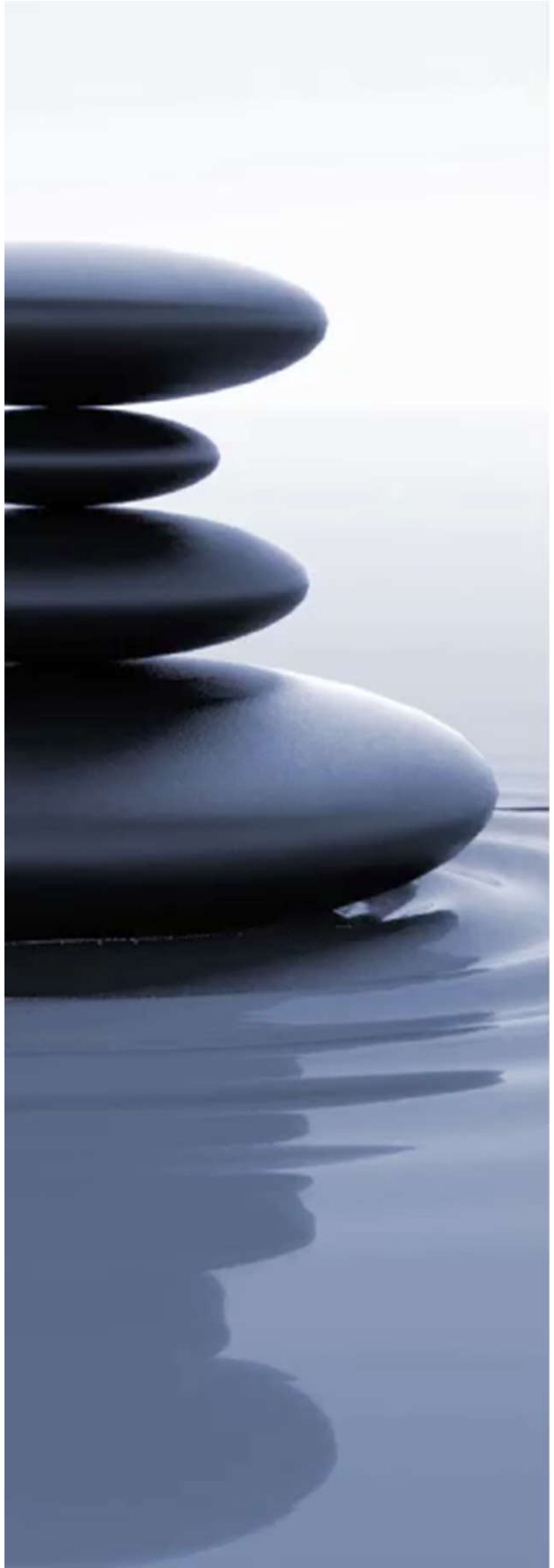
$= 1.5\,0\,2\,\cancel{0}$

Remove Zeroes that are not needed.

$= \underline{1.502} \times 10^4$

Count how many digits are present.

15 020 has FOUR Significant Figures ✓



Example

Determining Significant Figures

0.0043

The Number is a decimal **less than 1**, so the **Exponent will be Negative**.

$= 0\,.\underbrace{0\,0\,4}_{\text{3 places}}\,3$

Move the Decimal point to the RIGHT to create a number between 1 and 10.

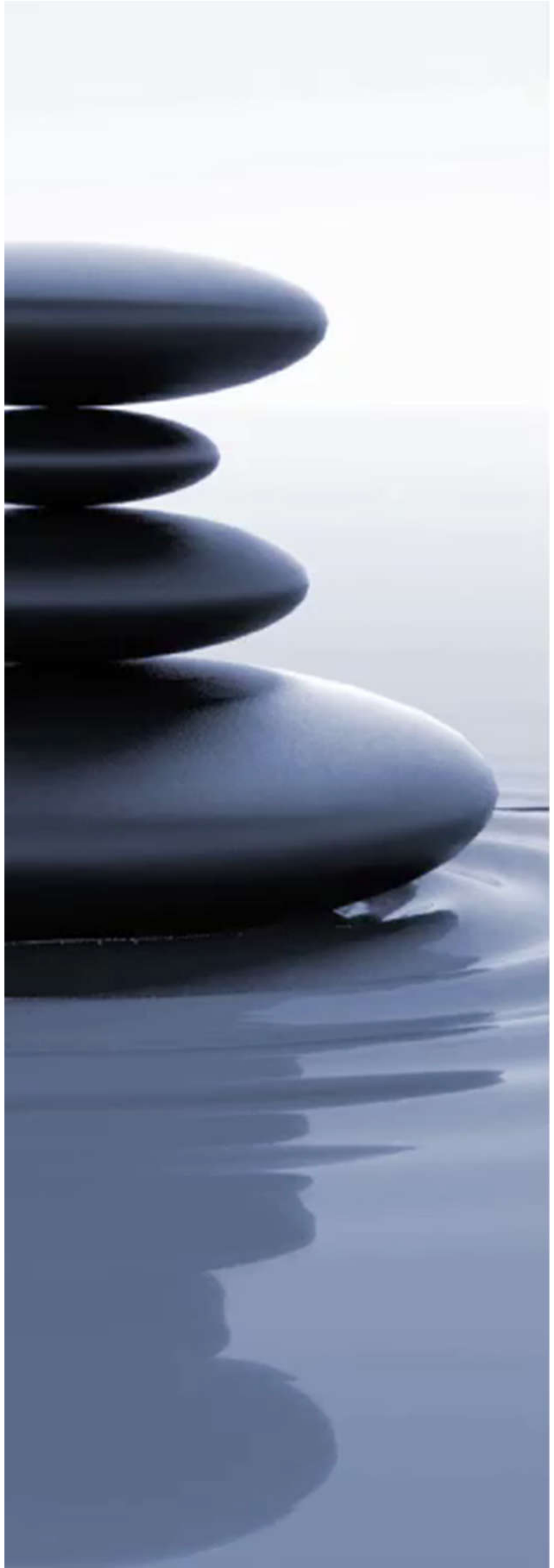
$= \cancel{0}\,\cancel{0}\,\cancel{0}\,4.3$

Remove Zeroes that are not needed.

$= \underline{4.3} \times 10^{-3}$

Count how many digits are there.

0.0043 has TWO Significant Figures ✓



Important Zeroes in Scientific Notation

0.0050

The Number is a decimal **less than 1**, so the **Exponent will be Negative**.

$= 0\,.\underbrace{0\,0\,5}_{\text{3 places}}\,0$

Move the Decimal point to the RIGHT to create a number between 1 and 10.

$= \cancel{0}\,\cancel{0}\,\cancel{0}\,5.0$

Remove Zeroes that are not needed.

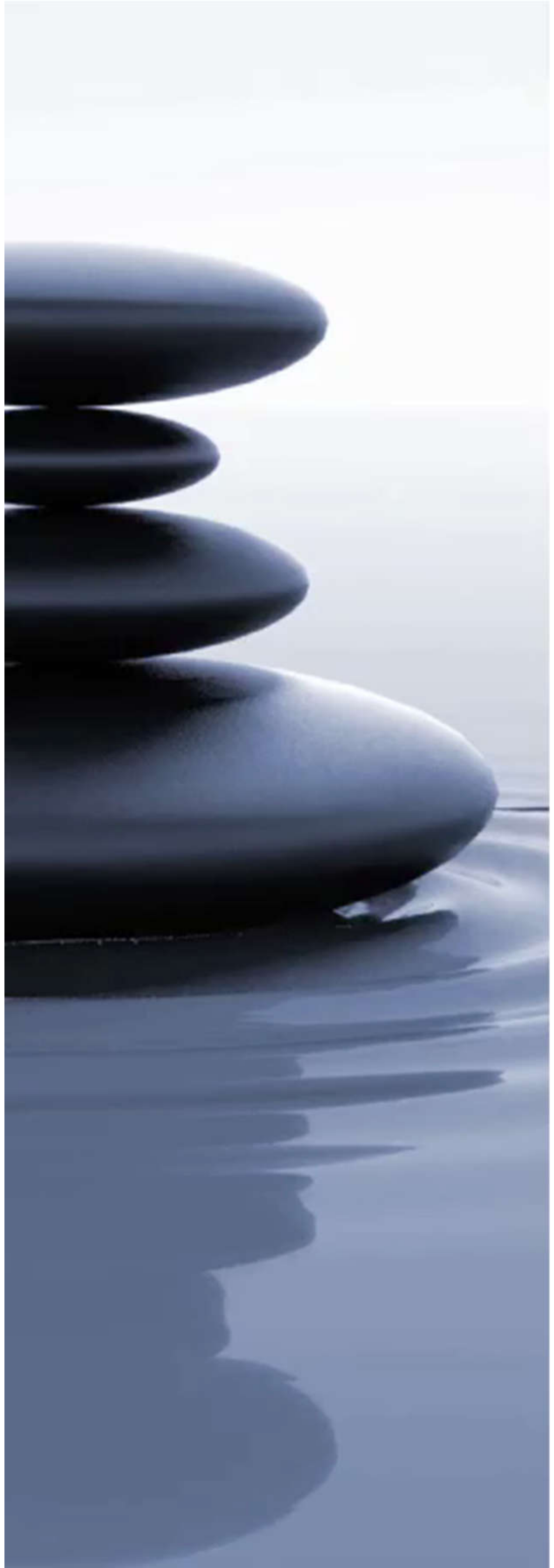
NEVER REMOVE ZEROES THAT CAME AFTER A DECIMAL POINT.

$= \underline{5.0} \times 10^{-3}$ ✓

2 Significant Figures

We moved **3 places** so Power of 10 is three : $\mathbf{10^{-3}}$

ANY ZERO THAT CAME AFTER THE DECIMAL POINT IN THE ORIGINAL STARTING DECIMAL NUMBER MUST NOT BE REMOVED.



Determining Significant Figures

0.0270

The Number is a decimal **less than 1**, so the **Exponent will be Negative**.

$= 0\ .0\ 2\ 7\ 0$

2 places

Move the Decimal point to the RIGHT to create a number between 1 and 10.

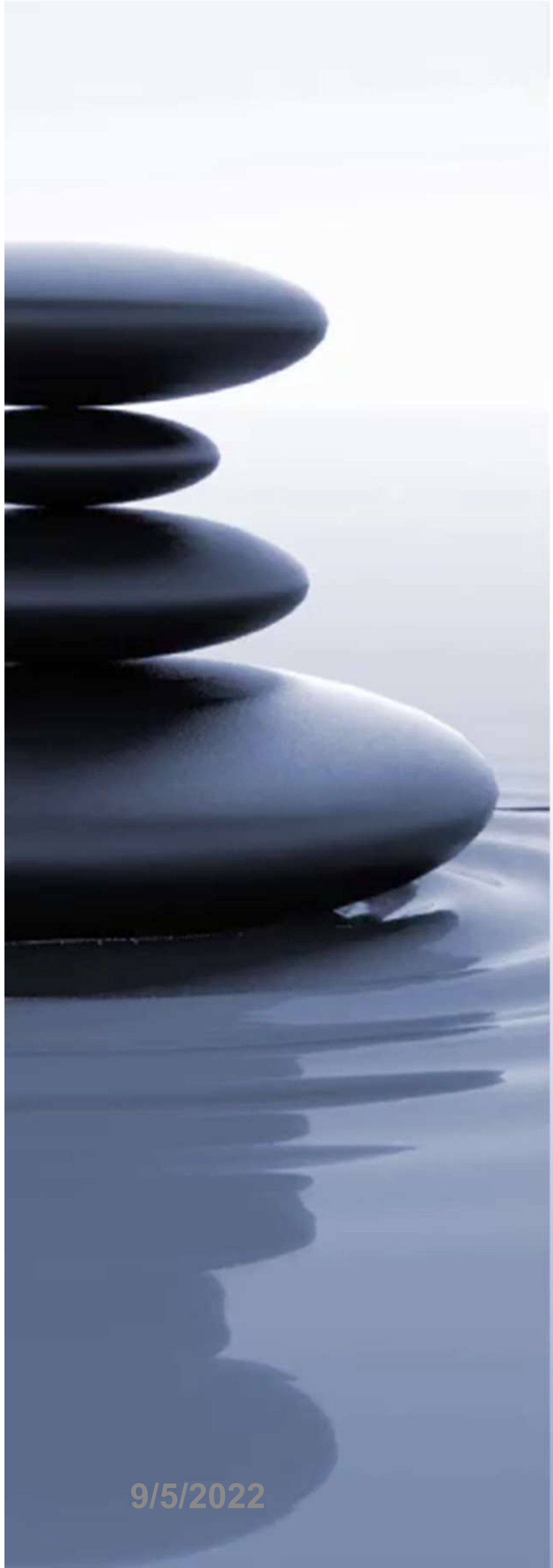
$= \cancel{0}\ \cancel{0}\ \cancel{0}\ 2.70$

Remove Zeroes that are not needed, **but not ones from after a Decimal Pt.**

$= \underline{2.70} \times 10^{-2}$

Count how many digits are there.

0.0270 has THREE Significant Figures ✓

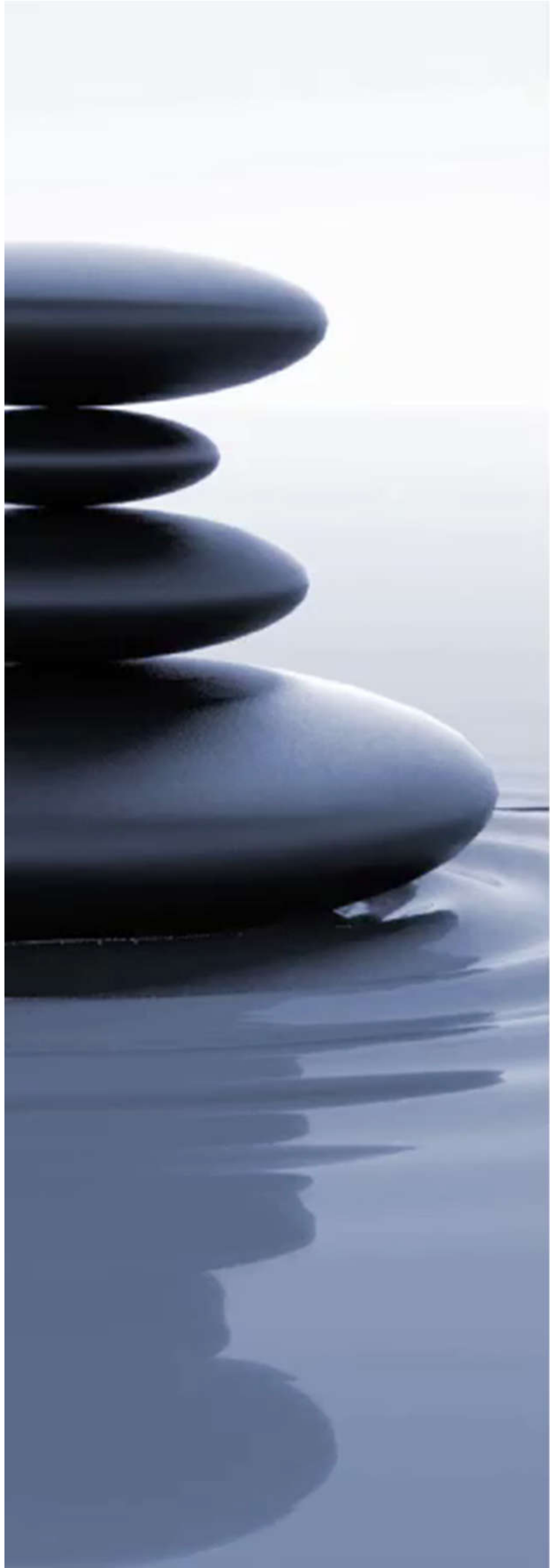


How is Absolute Relative Error used as a stopping criterion?

If $|\varepsilon_a| \langle \varepsilon_s$ where ε_s is a pre-specified tolerance, then no further iterations are necessary and the process is stopped.

If *at least n significant digits/figures* are required to be correct in the result, then

$$\varepsilon_s = (0.5 \times 10^{(2\text{-}n)})\%$$



5. Round-off Error vs Chopping Error

Example:

$\pi = 3.14159265358$

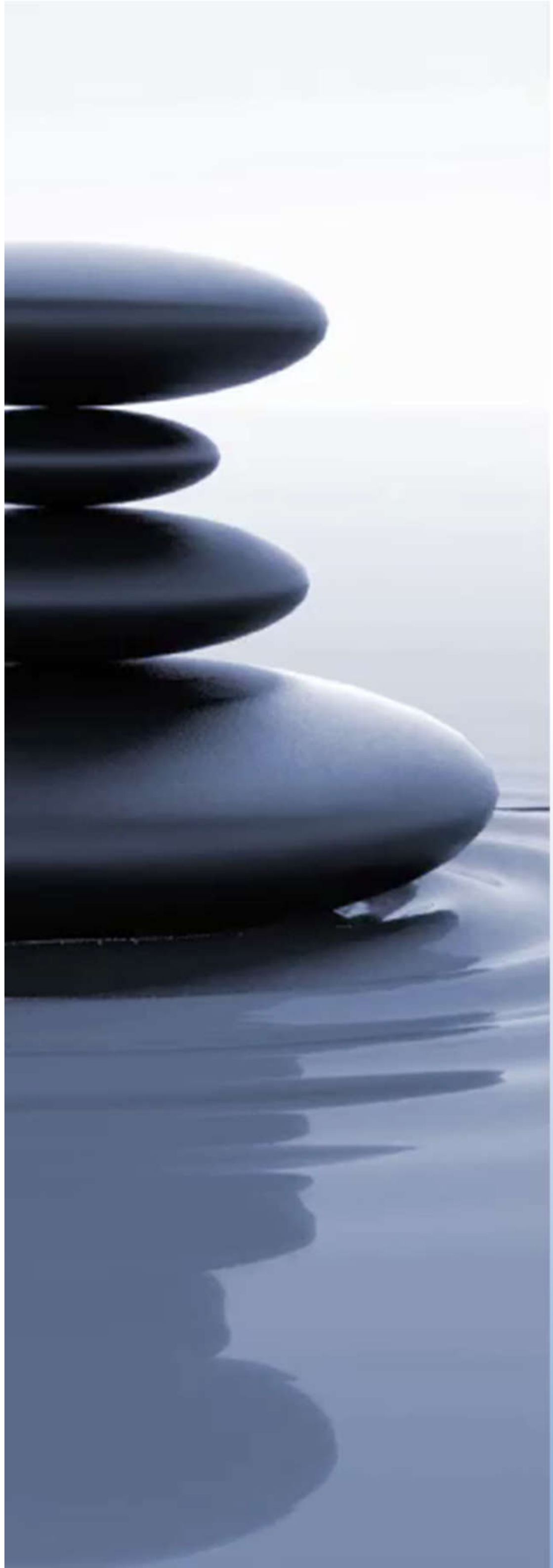
to be stored on a base-10 system carrying 7 significant digits $\pi = 3.141592$ => chopping error

$\varepsilon_t = 0.00000065$

If rounded $\pi = 3.141593$ => round-off error

$\varepsilon_t = 0.00000035$

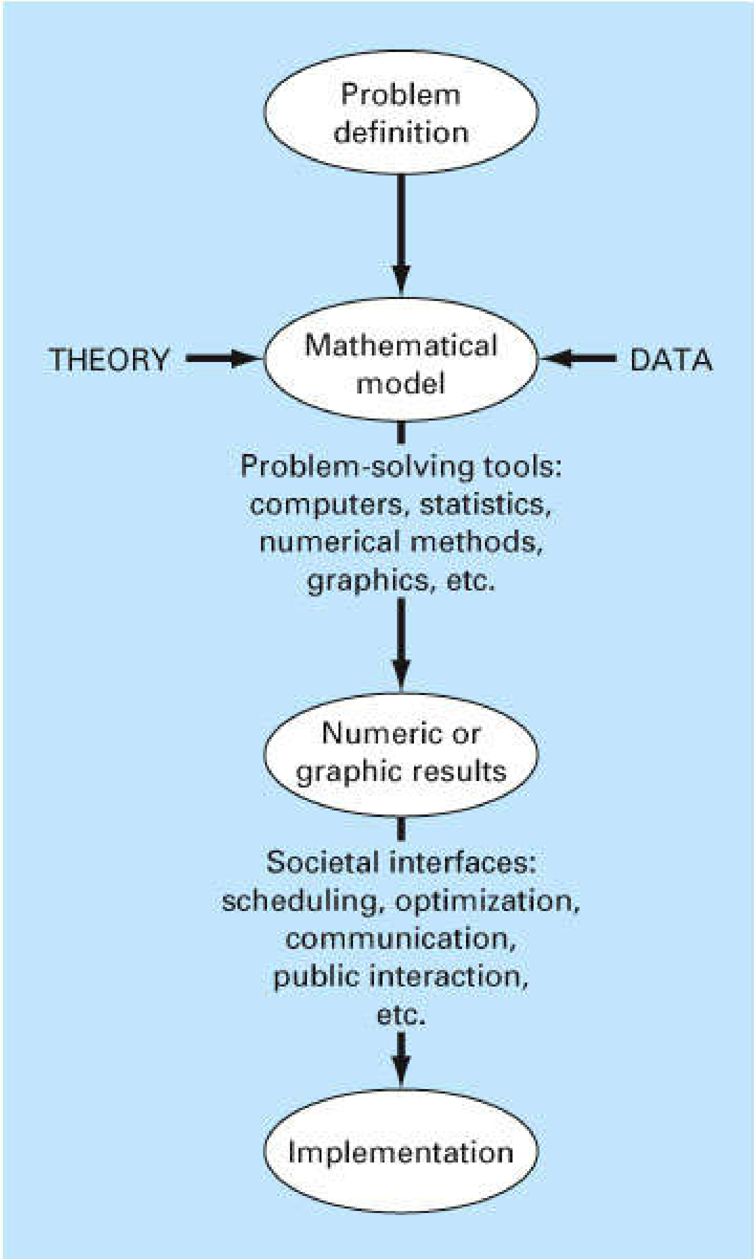
Some machines use chopping, because rounding adds to the computational overhead. Since number of significant figures is large enough, resulting chopping error is negligible.

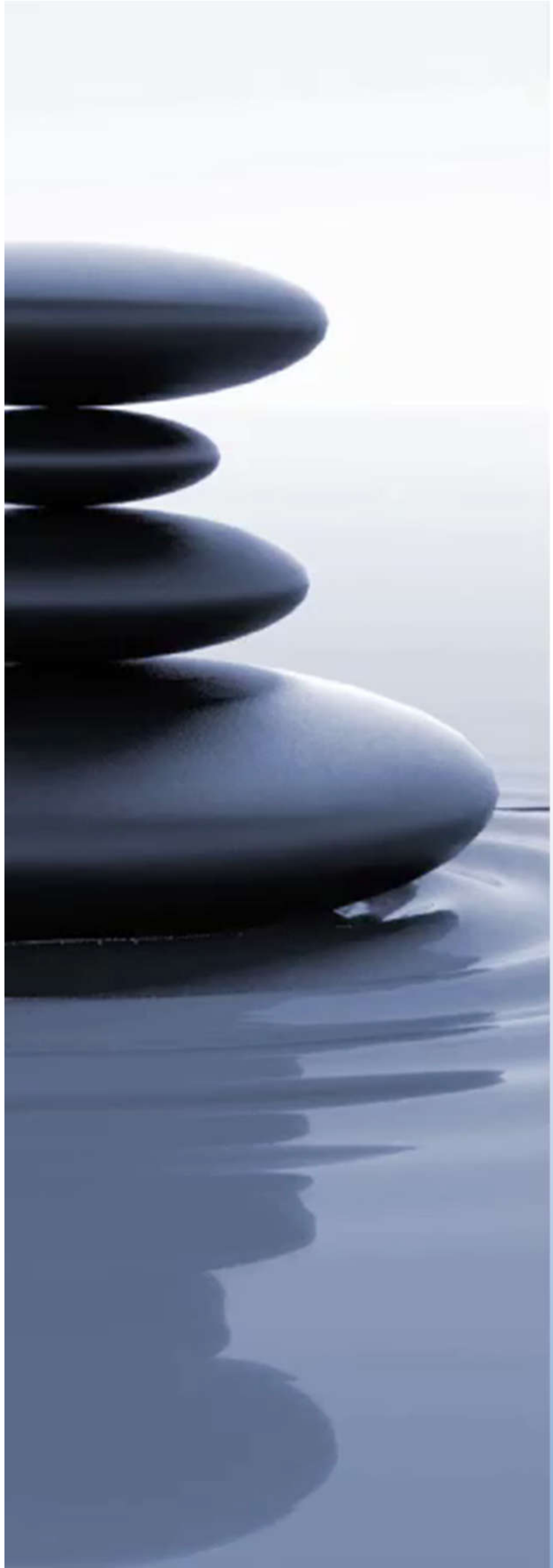


PART B

TAYLOR'S SERIES

Problem solving process





Taylor's Theorem

Suppose $f \in C^n[a, b]$ and $f^{(n+1)}$ exists on [*a*, *b*]. Let x_0 be a number in [*a*, *b*]. For every x in [*a*, *b*], there exists a number $\xi(x)$ between x_0 and x with

$$f(x) = Pn(x) + Rn_{(x)}$$

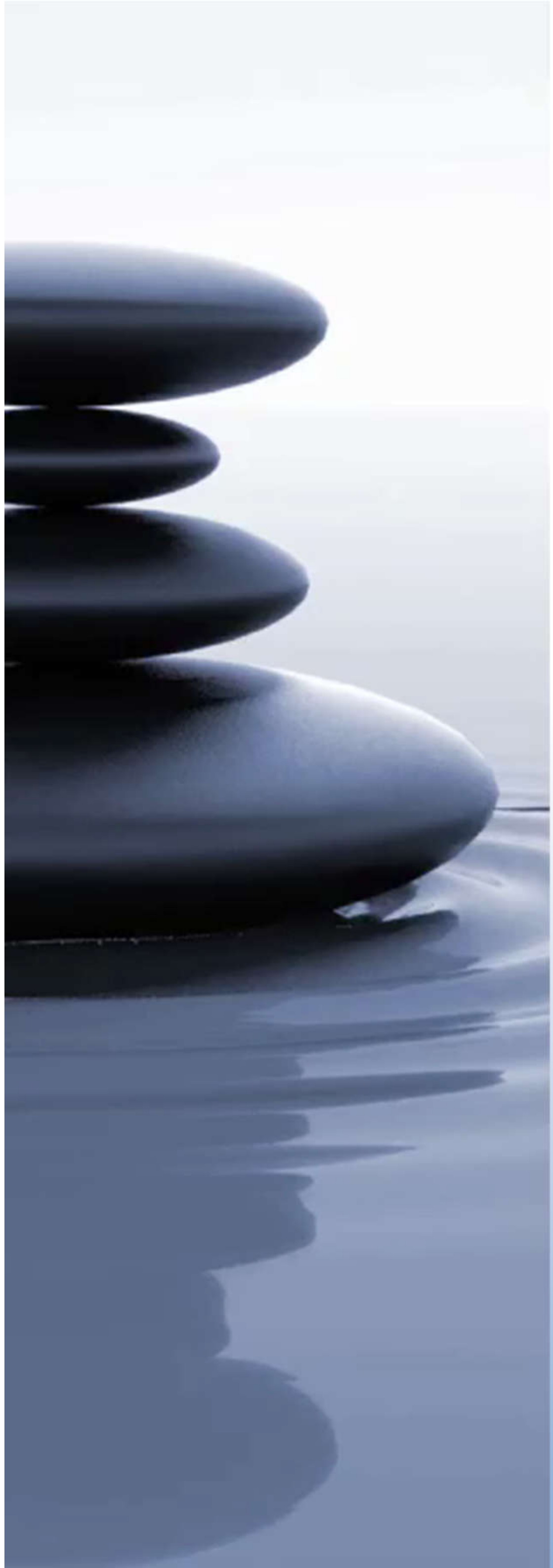
where

$$P_n(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2$$
$$+ \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

And

$$R_n(x) = \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x - x_0)^{(n+1)}$$

$P_n(x)$ _ the n^th Taylor polynomial for *f* about x_0.
$R_n(x)$ _ the **truncation error** (or *remainder term*) associated with $P_n(x)$.



n^{th} order approximation

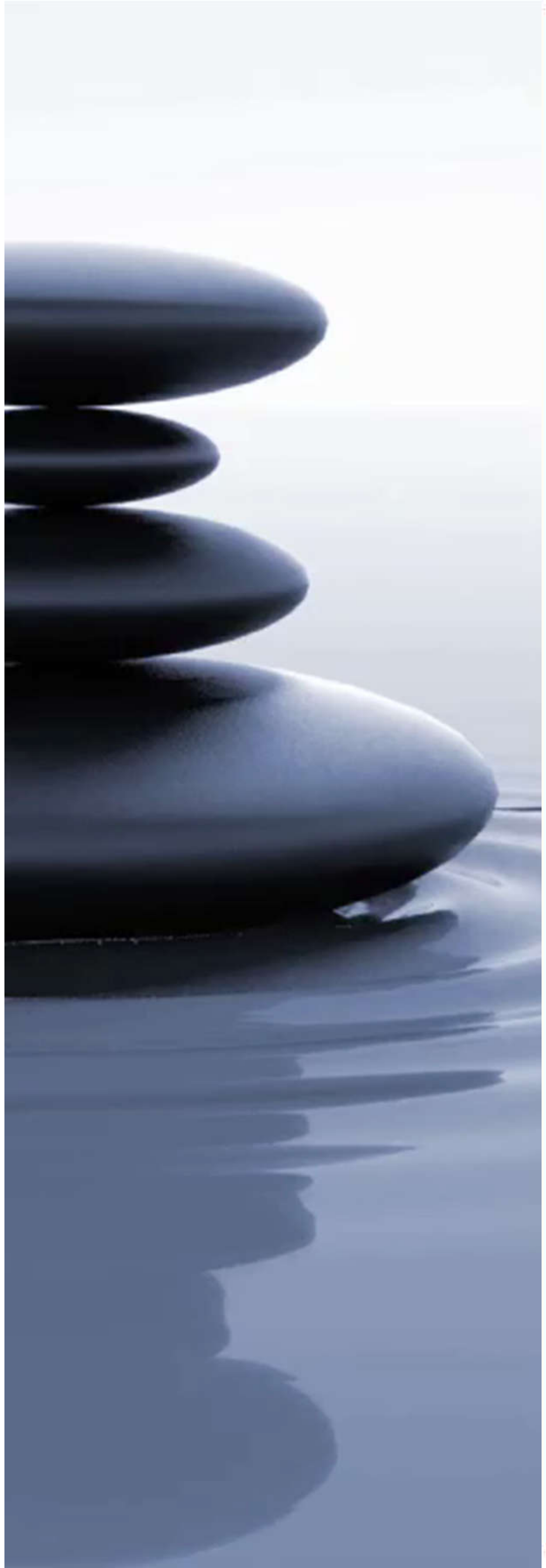
$$f(x_{i+1}) \cong f(x_i) + f'(x_i)(x_{i+1} - x_i) + \frac{f''}{2!}(x_{i+1} - x_i)^2 + \ldots$$

$$+ \frac{f^{(n)}}{n!}(x_{i+1} - x_i)^n + R_n$$

$(x_{i+1}-x_i)= h$ _ *step size* (define first)

$$R_n = \frac{f^{(n+1)}(\varepsilon)}{(n+1)!} h^{(n+1)}$$

Reminder term, R_n, accounts for all terms from (n+1) to infinity.



Taylor and MacLaurin Series

$$f(x)=f(a)+f'(a)\frac{(x-a)}{1!}+f''(a)\frac{(x-a)^2}{2!}+f'''(a)\frac{(x-a)^3}{3!}+\dots$$

$$=\sum_{n=0}^{\infty}f^{(n)}(a)\frac{(x-a)^n}{n!}$$

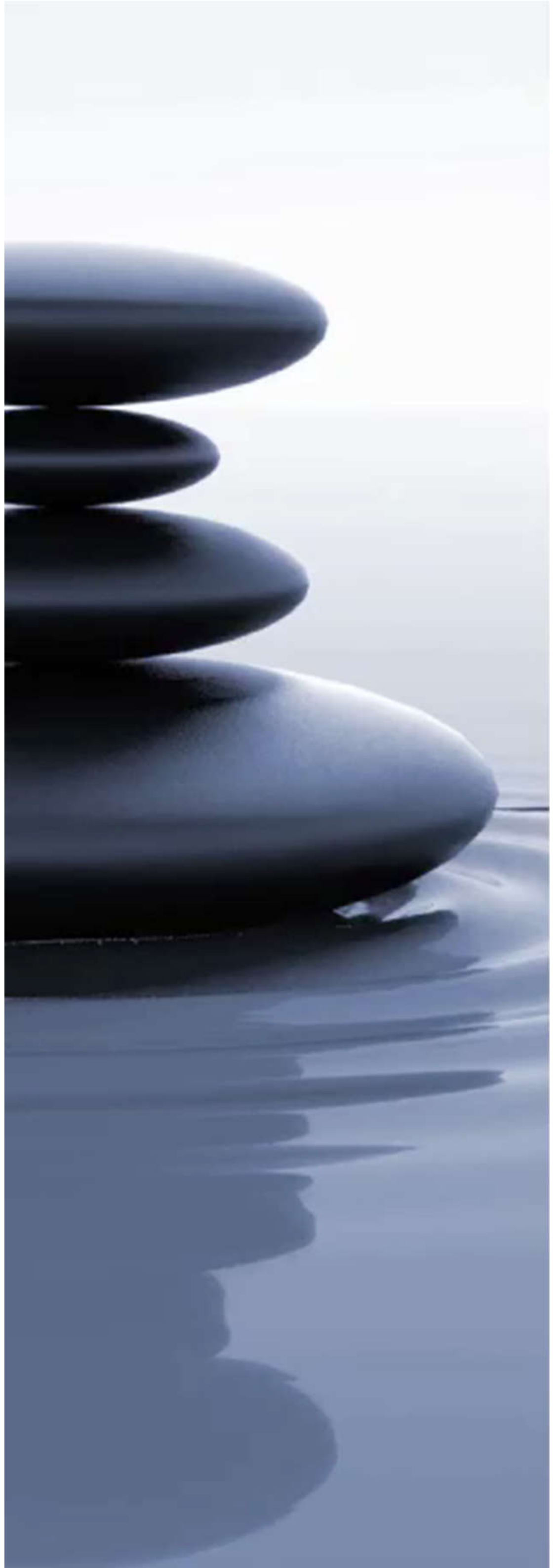
A **Maclaurin Series** is a Taylor Series where **a=0**

$$e^x=\sum_{k=0}^{\infty}\frac{x^k}{k!}=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\dots$$

$$\sin x=\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{(2k+1)!}=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\dots$$

$$\cos x=\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k}}{(2k)!}=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\dots$$

$$\frac{1}{1-x}=\sum_{k=0}^{\infty}x^k=1+x+x^2+x^3+\dots \quad \text{for } |x|<1$$



Partial Derivatives

Computing the partial derivative of simple functions is easy: simply treat every other variables in the equation as a constant and find the usual scalar derivative.

Ex: f(x,y) = $3x^2y$

Treating y as a constant, we can find partial of x:

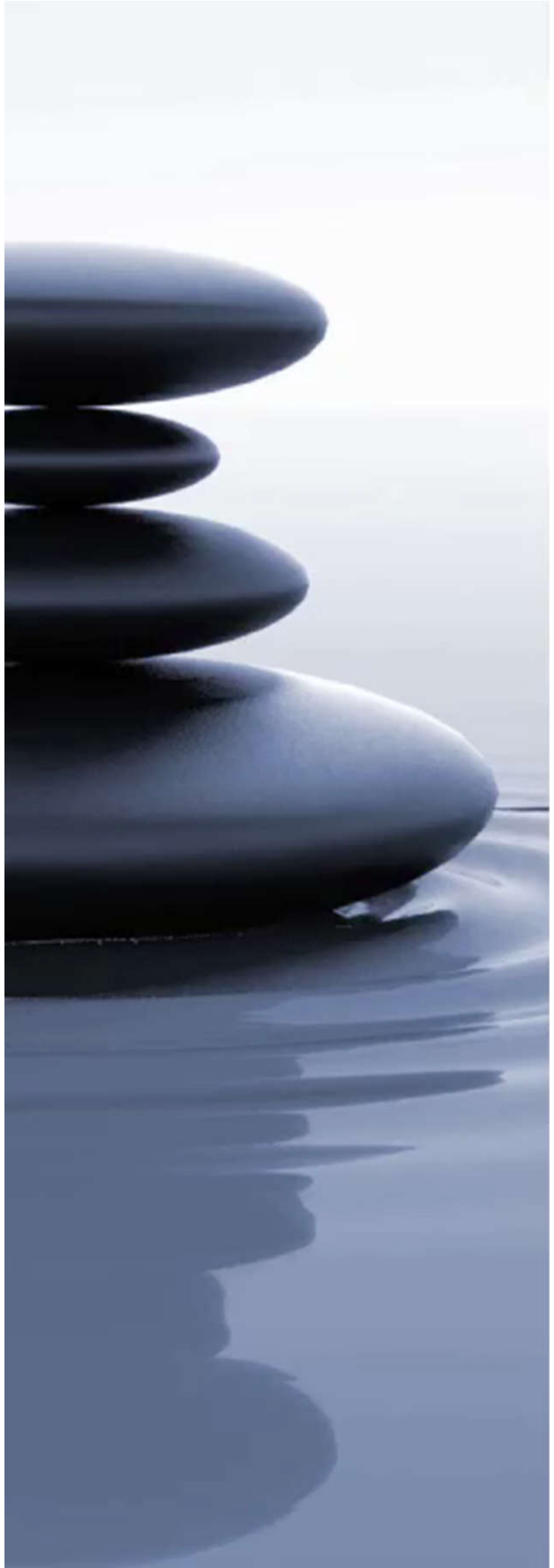
$$\frac{\partial}{\partial x}3yx^2 = 3y\frac{\partial}{\partial x}x^2 = 3y2x = 6yx$$

Similarly, we can find the partial of y:

$$\frac{\partial}{\partial y}3yx^2 = 3x^2\frac{\partial}{\partial y}y = 3x^2 \times 1 = 3x^2$$

The gradient of the function f(x,y) = $3x^2y$ is a horizontal vector, composed of the two partials:

$$\nabla f(x,y) = [\frac{\partial f(x,y)}{\partial x}, \frac{\partial f(x,y)}{\partial y}] = [6yx, 3x^2]$$



Example _ Taylor's Theorem

Determine

(a) the second and

(b) the third Taylor polynomials for f(x) = cos x about $x_0 = 0$, and use these polynomials to approximate cos(0.01).

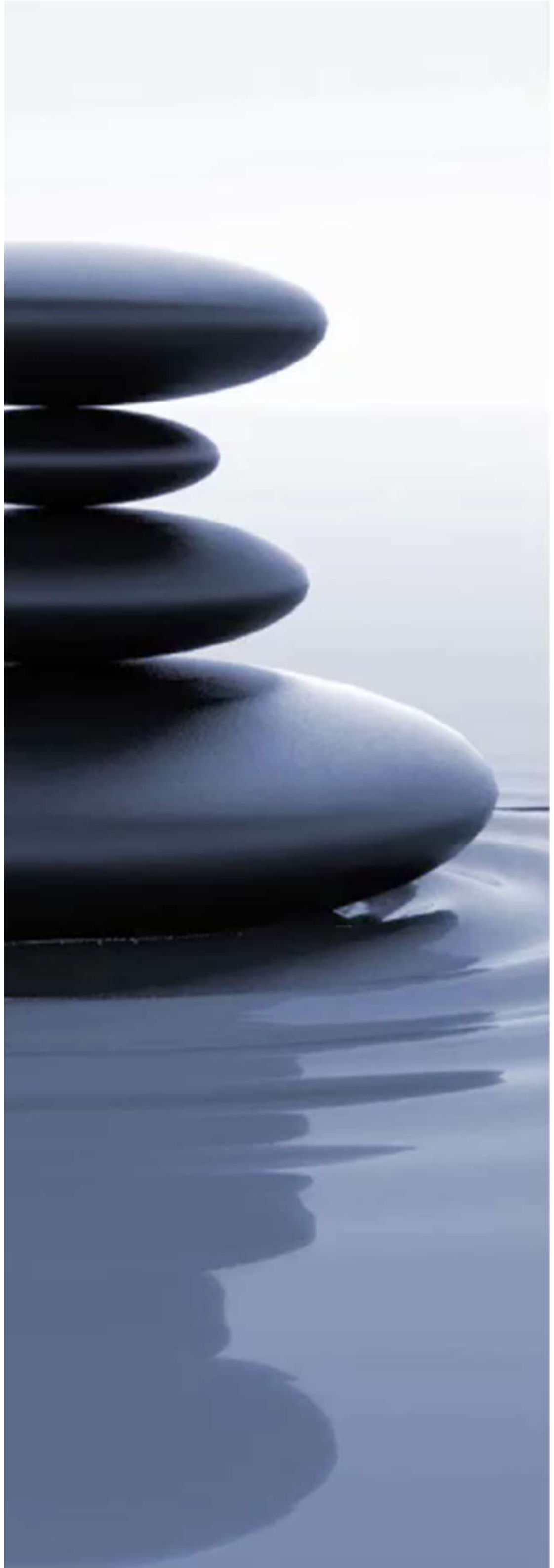
(c) Use the third Taylor polynomial and its remainder term to approximate $\int_0^{0.1} cosx\, dx$.

Since $f \in C^{\infty}(IR)$, Taylor's Theorem can be applied for any $n \geq 0$. Also,

$$f'(x) = -\sin x,$$
$$f''(x) = -\cos x,$$
$$f'''(x) = \sin x, \text{ and}$$
$$f^{(4)}(x) = \cos x,$$

so

$$f(0) = 1,$$
$$f'(0) = 0,$$
$$f''(0) = -1, \text{ and}$$
$$f'''(0) = 0.$$



Example (cont)

a. For n = 2 and $x_0 = 0$, we have

$$cosx = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(\xi(x))}{3!}x^3$$

$$= 1 - \frac{1}{2}x^2 + \frac{1}{6}x^3 \sin\xi(x)$$

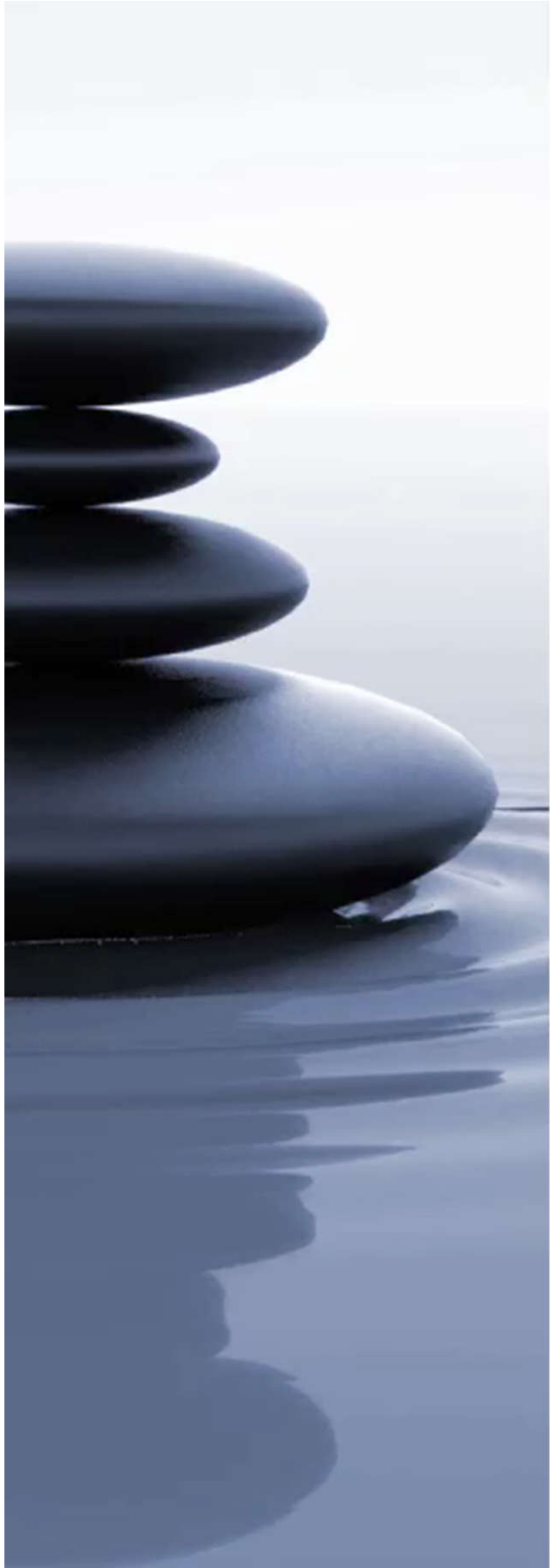
When x=0.01, this becomes

$$\cos(0.01) = 1 - \frac{1}{2}(0.01)^2 + \frac{1}{6}(0.01)^3 \sin\xi(0.01)$$

$$= 0.99995 + \frac{10^{-6}}{6}\sin\xi(0.01)$$

$$E_t = |\cos(0.01) - 0.99995|$$

$$= 0.16 \times 10^{-6} \sin\xi(x) \leq 0.16 \times 10^{-6}$$



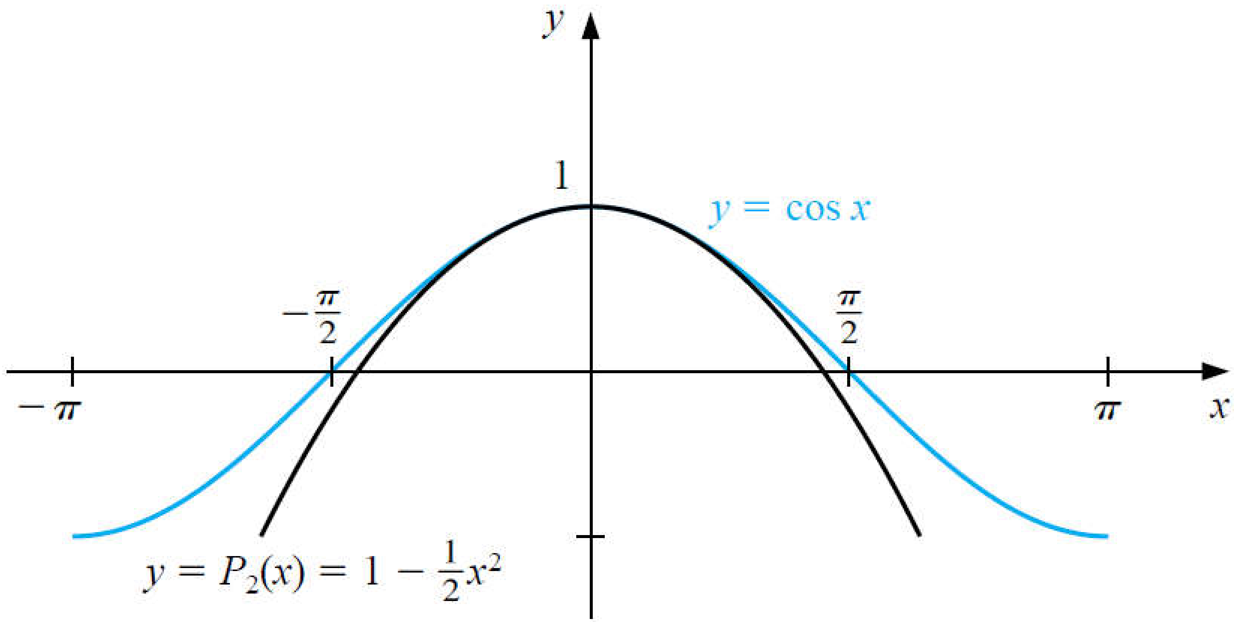
Example (cont)

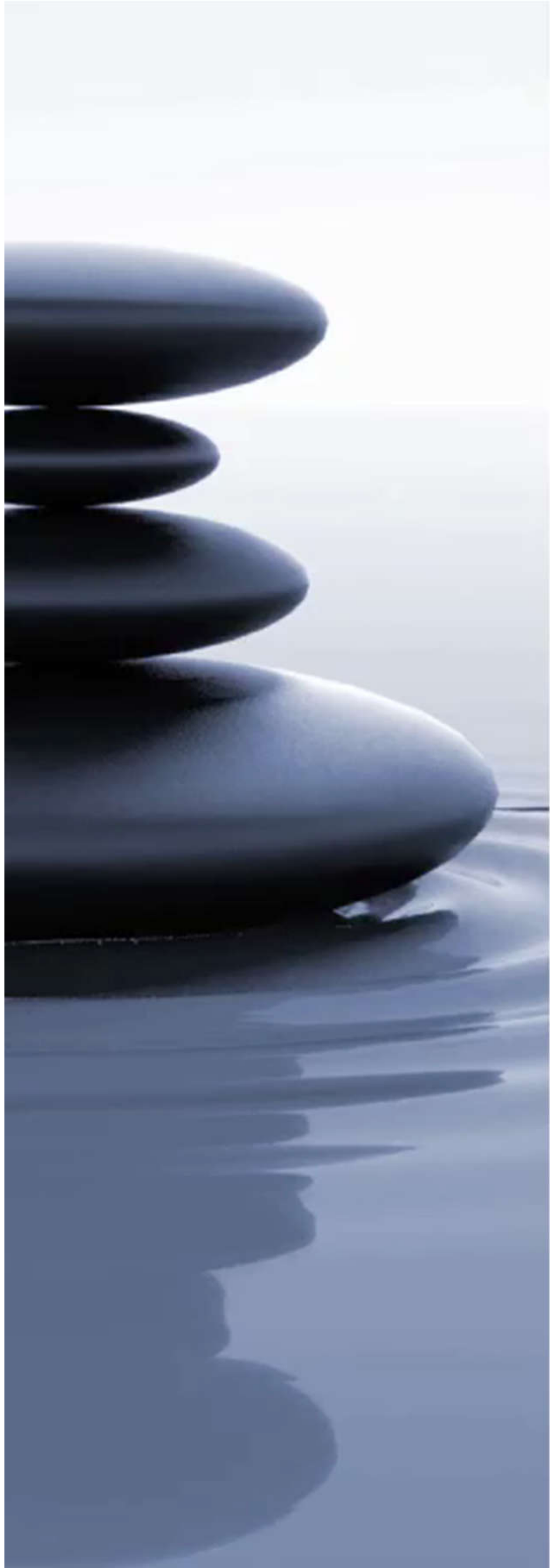
b. For n = 3 and x_0 = 0, we have f’’’(0)=0, the third Taylor polynomial and remainder term about x0 = 0 are

$$cos x = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 \cos\xi(x) = 0.99995$$

and

$$\left|\frac{1}{24}x^4 cos\,\xi(0.01)\right| \le \frac{1}{24}(0.01)^4(1) \approx 4.2 \times 10^{-10}$$





Example (cont)

c. Using the third Taylor polynomial gives

$$\int_0^{0.1} \cos x\, dx = \int_0^{0.1} \left(1 - \frac{1}{2}x^2\right) dx$$

$$+ \int_0^{0.1} \left(\frac{1}{24} x^4 \cos \xi(x)\right) dx$$

$$= \left[x - \frac{1}{6} x3\right]_0^{0.1} + \frac{1}{24} \int_0^{0.1} x^4 cosxdx$$

$$= 0.1 - \frac{1}{6}(0.1)^3 + \frac{1}{24} \int_0^{0.1} x^4 cosxdx$$

Therefore,

$$\int_0^{0.1} \cos x\, dx \approx 0.1 - \frac{1}{6}(0.1)^3 = 0.09983$$

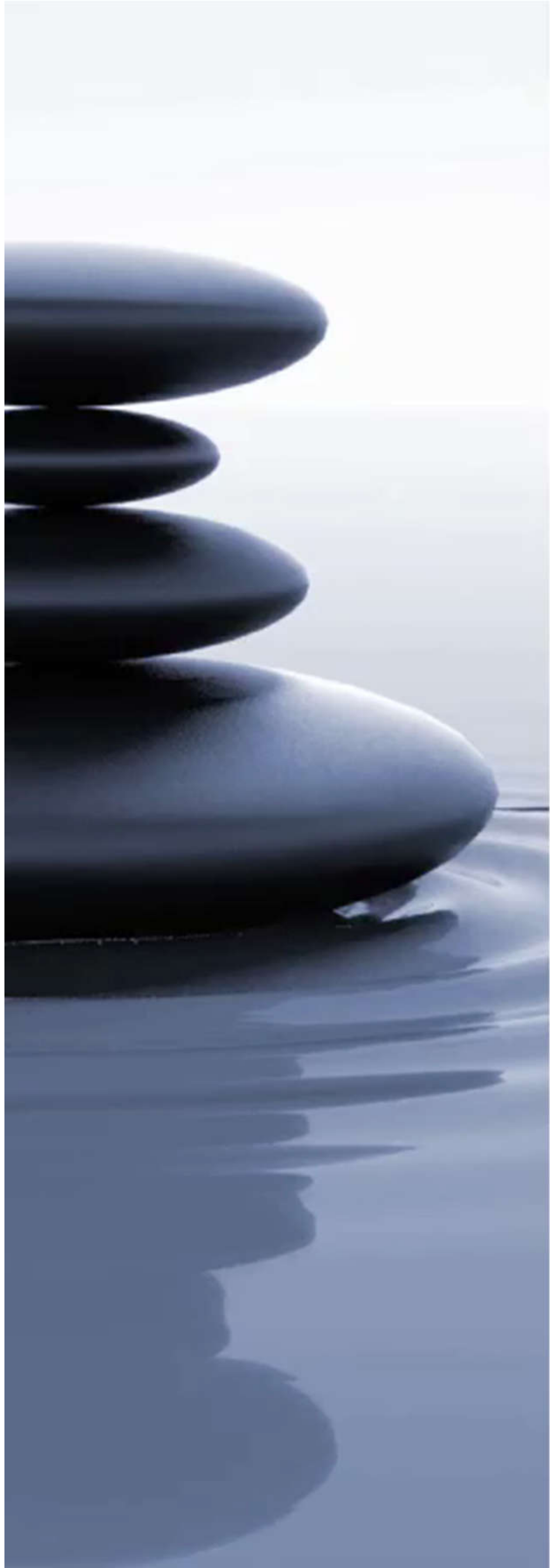
So,

$$Et = \left|\sin x_0^{0.1} - 0.09983\right| \approx 8.4 \times 10^{-8}$$

Theorem 9.10.2 Algebra of Power Series

Let $f(x)=\sum_{n=0}^{\infty} a_n x^n$ and $g(x)=\sum_{n=0}^{\infty} b_n x^n$ converge absolutely for $|x|<R$, and let $h(x)$ be continuous.

1. $f(x) \pm g(x) = \sum_{n=0}^{\infty} (a_n \pm b_n) x^n \quad$ for $|x|<R$.
2. $f(x)\, g(x) = \left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} (a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0)\, x^n$ for $|x|<R$.
3. $f\left(h(x)\right) = \sum_{n=0}^{\infty} a_n \left(h(x)\right)^n \quad$ for $|h(x)|<R$.



Combining Taylor series

Write out the first 3 terms of the Maclaurin Series for

f(x)= e^x cos x

Given that

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad \text{and} \qquad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots.$$

We have

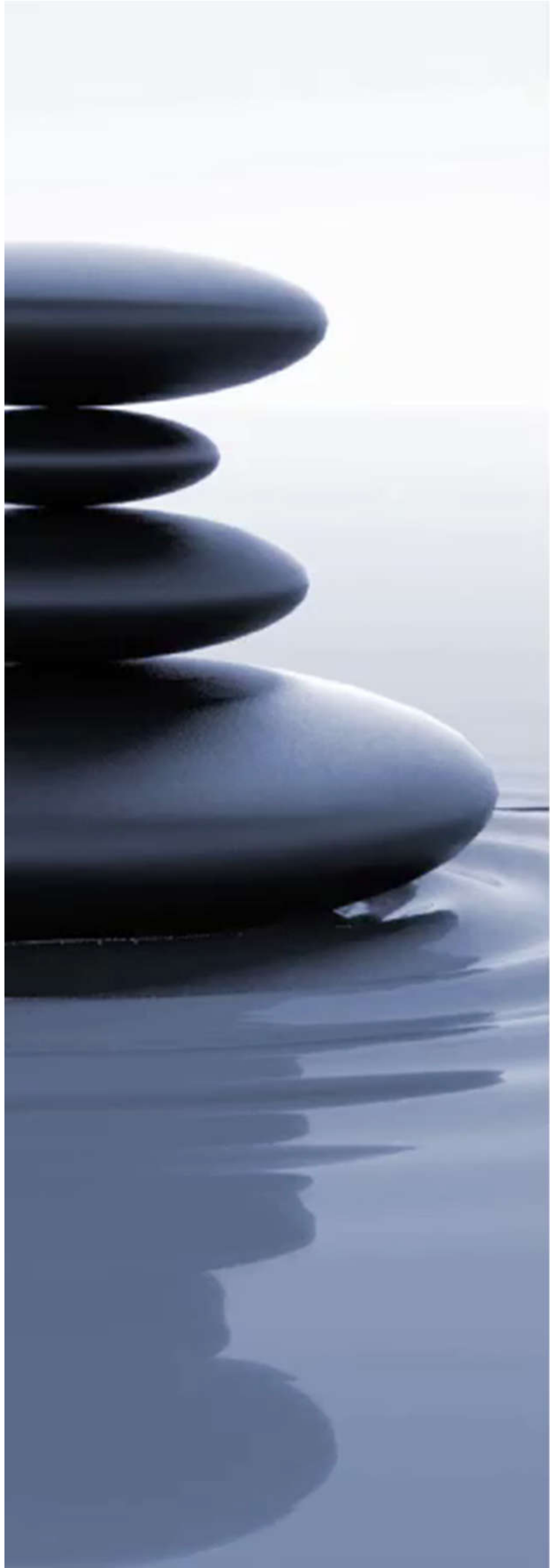
$$e^x \cos x = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right).$$

Distribute the right hand expression across the left:

$$\begin{aligned} &= 1\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) + x\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) \\ &\quad + \frac{x^2}{2!}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) + \frac{x^3}{3!}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) \\ &\quad + \frac{x^4}{4!}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) + \cdots \end{aligned}$$

Distribute again and collect like terms.

$$= 1 + x - \frac{x^3}{3} - \frac{x^4}{6} - \frac{x^5}{30} + \frac{x^7}{630} + \cdots$$



Creating new Taylor series

Create the Taylor series for *y=sin(x²)* centered at x=0

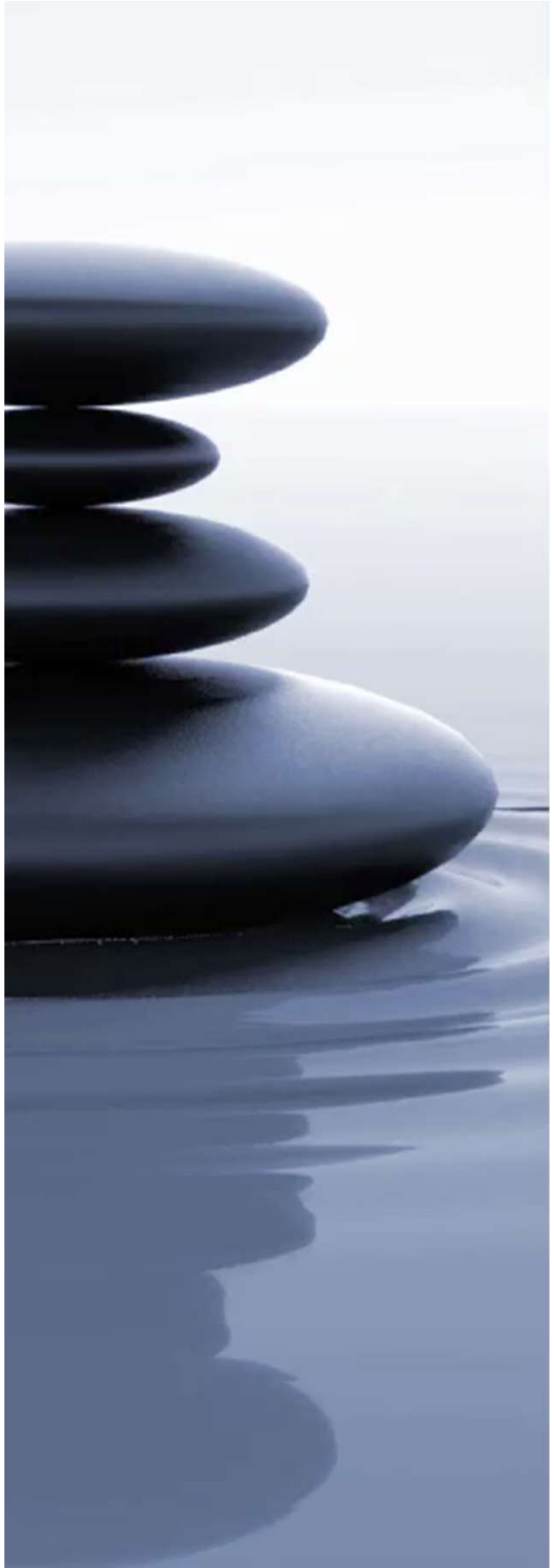
Given that

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots,$$

we simply substitute x^2 for x in the series, giving

$$\sin(x^2) = \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{(2n+1)!} = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} \cdots.$$

Since the Taylor series for $\sin x$ has an infinite radius of convergence, so does the Taylor series for $\sin(x^2)$.



Creating new Taylor series

Suppose we want the Taylor series at 0 of the function

$$g(x) = \frac{e^x}{\cos x}.$$

We have for the exponential function

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

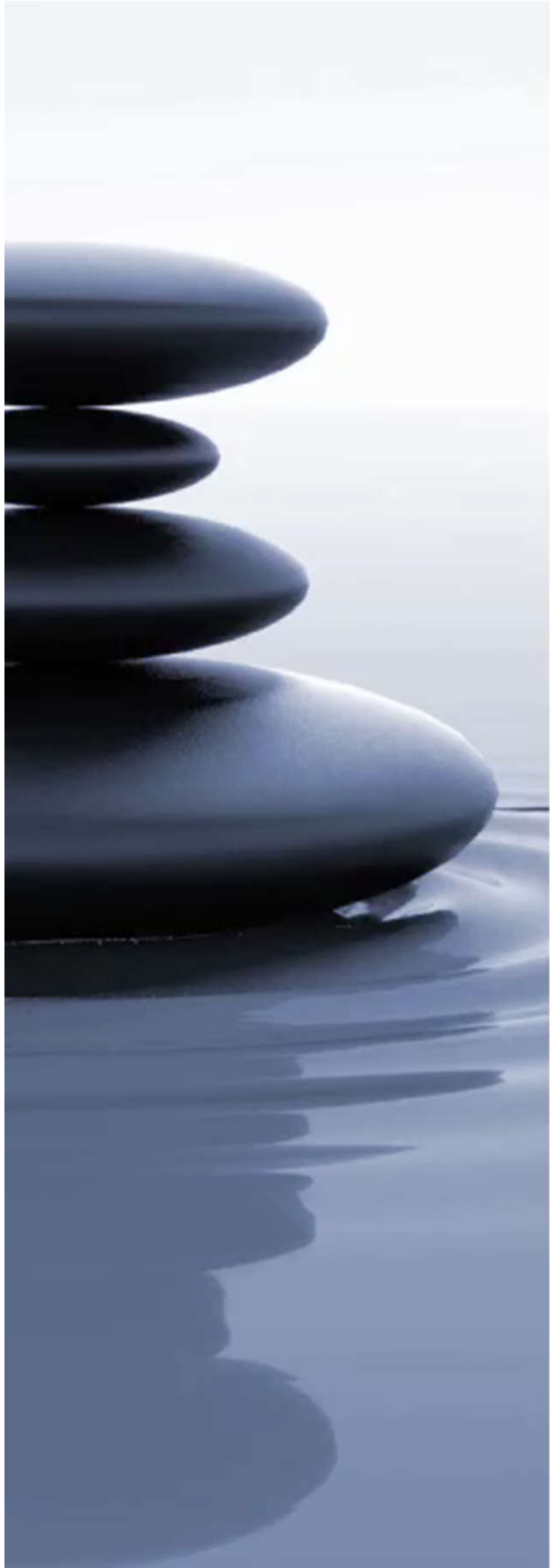
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

Assume the power series is

$$\frac{e^x}{\cos x} = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

Then multiplication with the denominator and substitution of the series of the cosine yields

$$\begin{aligned} e^x &= (c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots) \cos x \\ &= (c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots) \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) \\ &= c_0 - \frac{c_0}{2} x^2 + \frac{c_0}{4!} x^4 + c_1 x - \frac{c_1}{2} x^3 + \frac{c_1}{4!} x^5 + c_2 x^2 - \frac{c_2}{2} x^4 + \frac{c_2}{4!} x^6 + c_3 x^3 - \end{aligned}$$



Example (cont)

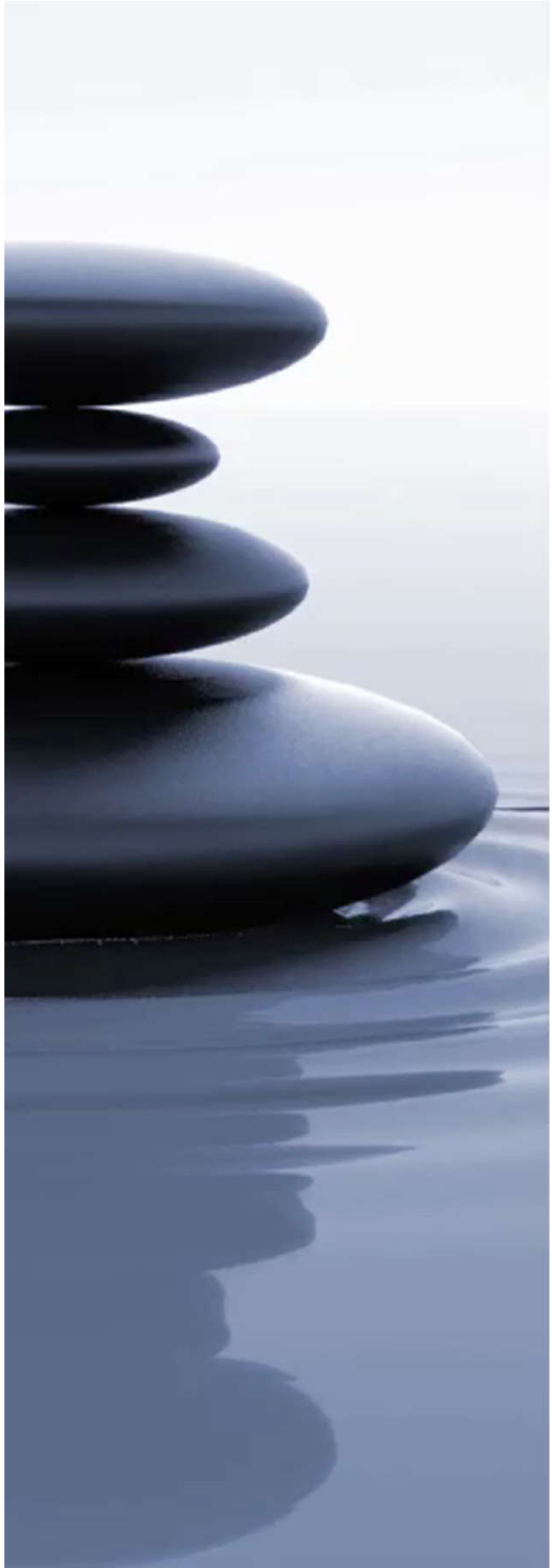
Collecting the terms up to fourth order yields

$$= c_0 + c_1 x + \left(c_2 - \frac{c_0}{2}\right) x^2 + \left(c_3 - \frac{c_1}{2}\right) x^3 + \left(c_4 - \frac{c_2}{2} + \frac{c_0}{4!}\right) x^4 + \cdots$$

Comparing coefficients with the above series of the exponential function yields the desired Taylor series

$$\frac{e^x}{\cos x} = 1 + x + x^2 + \frac{2x^3}{3} + \frac{x^4}{2} + \cdots .$$

$$f(x) = g(x) \quad \equiv \quad \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n \quad \equiv \quad an = bn\ \forall n$$

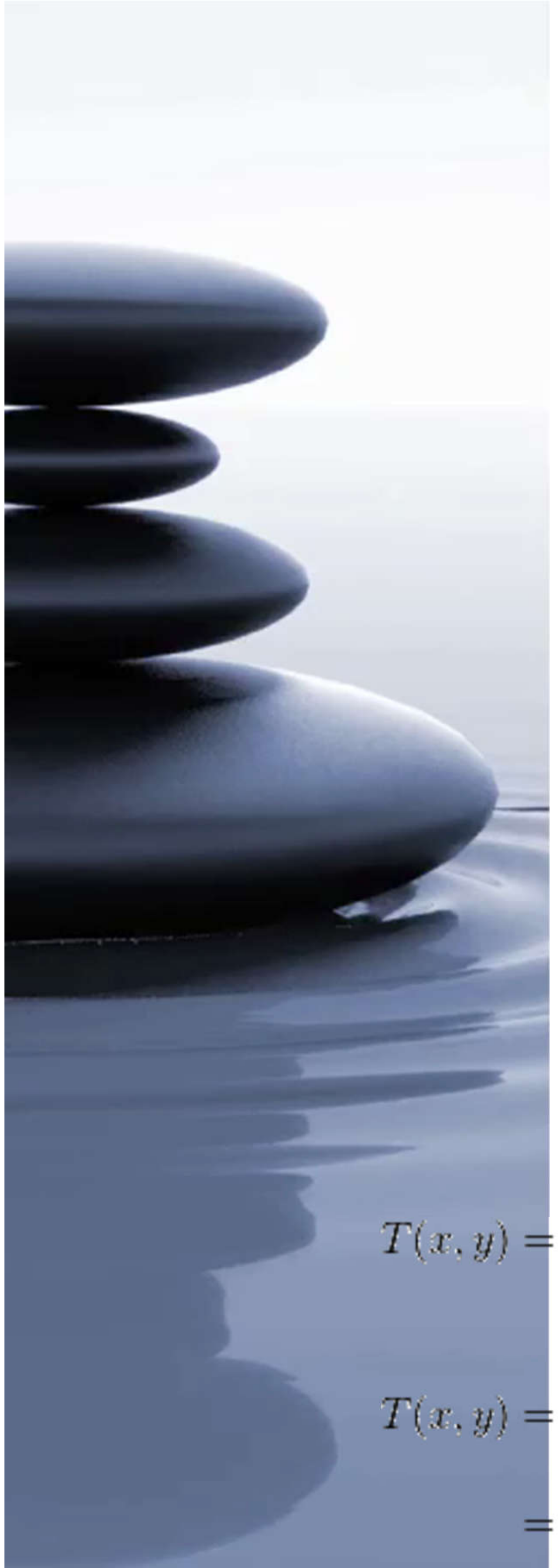


Taylor series in several variables

For a function that depends on two variables, x and y, the Taylor series to second order about the point (a, b) is

$$f(a,b) + (x-a)\,f_x(a,b) + (y-b)\,f_y(a,b)$$
$$+ \frac{1}{2!}\left[(x-a)^2\,f_{xx}(a,b) + 2(x-a)(y-b)\,f_{xy}(a,b) + (y-b)^2\,f_{yy}(a,b)\right]$$

where the subscripts denote the respective partial derivatives.



Example

Compute a second-order Taylor series expansion around point (a, b) = (0, 0) of a function

$$f(x,y) = e^x \log(1+y).$$

Firstly, we compute all partial derivatives we need

$$f_x(a,b) = e^x \log(1+y)\Big|_{(x,y)=(0,0)} = 0\,,$$

$$f_y(a,b) = \frac{e^x}{1+y}\Big|_{(x,y)=(0,0)} = 1\,,$$

$$f_{xx}(a,b) = e^x \log(1+y)\Big|_{(x,y)=(0,0)} = 0\,,$$

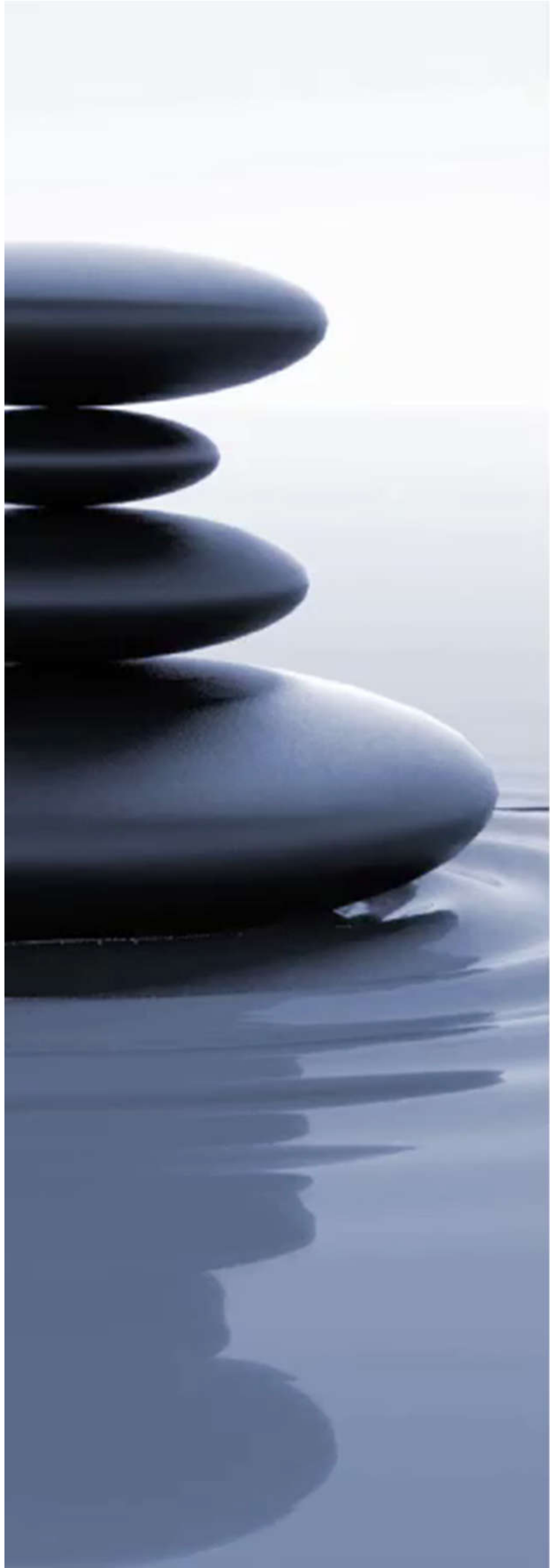
$$f_{yy}(a,b) = -\frac{e^x}{(1+y)^2}\Big|_{(x,y)=(0,0)} = -1\,,$$

$$f_{xy}(a,b) = f_{yx}(a,b) = \frac{e^x}{1+y}\Big|_{(x,y)=(0,0)} = 1.$$

The Taylor series is

$$\begin{aligned} T(x,y) = {} & f(a,b) + (x-a)\, f_x(a,b) + (y-b)\, f_y(a,b) \\ & + \frac{1}{2!}\left[(x-a)^2\, f_{xx}(a,b) + 2(x-a)(y-b)\, f_{xy}(a,b) + (y-b)^2\, f_{yy}(a,b)\right] + \cdots\,, \end{aligned}$$

$$\begin{aligned} T(x,y) &= 0 + 0(x-0) + 1(y-0) + \frac{1}{2}\left[0(x-0)^2 + 2(x-0)(y-0) + (-1)(y-0)^2\right] + \cdots \\ &= y + xy - \frac{y^2}{2} + \cdots\,. \end{aligned}$$



Taylor's series derivation

Let's assume function $f(x)$ can be expressed as power series:

$$f(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots \quad (*)$$
$$= \sum_{i=0}^{\infty} a_i x^i$$

To find coefficient a_0 lets place $x=0$ to equation $(*)$:

$$f(0) = a_0$$

If we take derivate of $f(x)$ we get:

$$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \dots$$

so $f'(0) = a_1$

Derive another time and we get

$$f''(x) = 2a_2 + 3\cdot 2\, a_3 x + \dots$$

let x be 0 and we find that $f''(0) = 2a_2$

so $a_2 = \frac{1}{2} f''(0)$ Let's keep taking derivatives

$$f'''(0) = 3\cdot 2\, a_3 = 3!\, a_3$$

$$f^{(4)}(0) = 4\cdot 3\cdot 2\, a_4 = 4!\, a_4$$

$$\Rightarrow a_4 = \frac{1}{4!} f^{(4)}(0)$$

so n'th coefficient a_n must be equal to $\frac{1}{n!} f^{(n)}(0)$

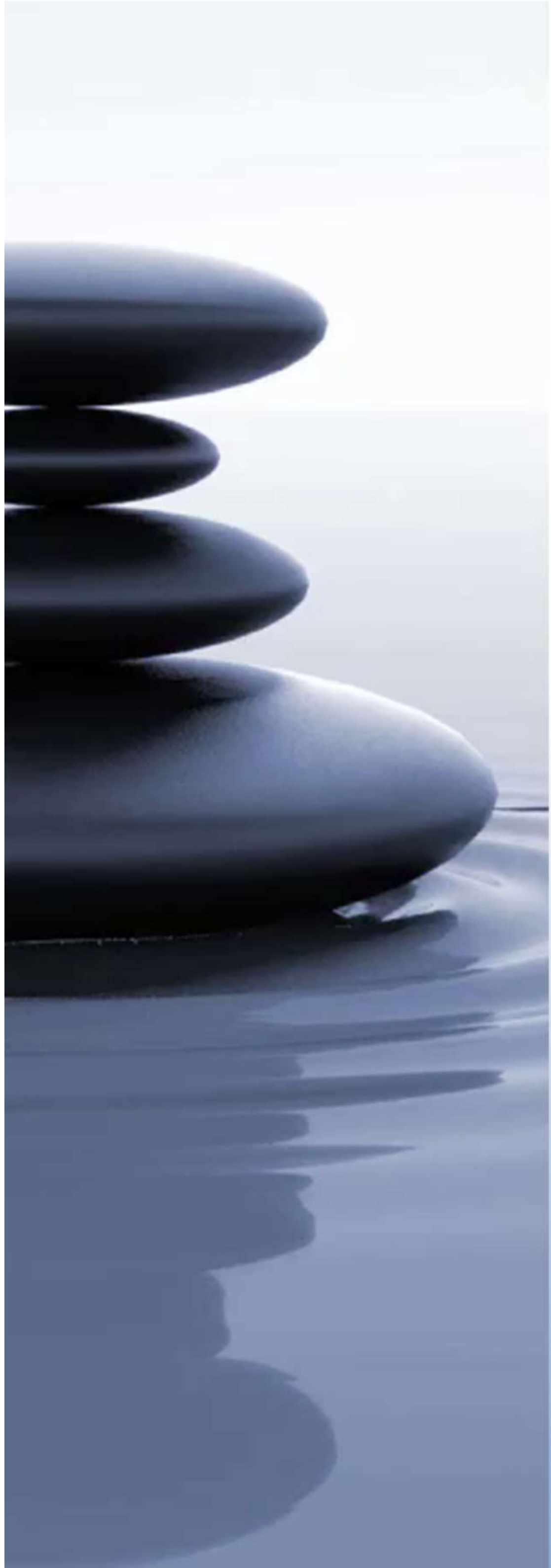
Let's put the coefficients back to equation $(*)$

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3 + \dots$$

$$\Rightarrow f(x) = \sum_{i=0}^{\infty} \frac{f^{(i)}(0)}{i!} x^i$$

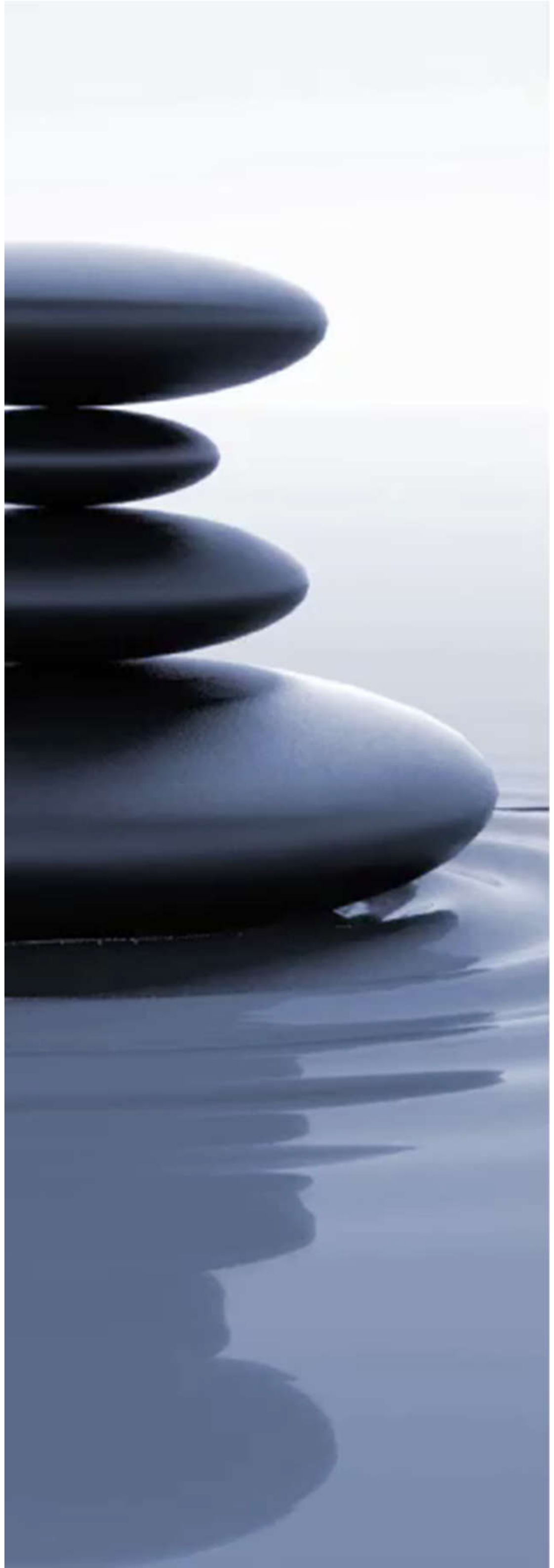
That's called Taylor's series. For example we can express function $\sin(x)$ as power series:

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$



PART C

OTHER APPLICATIONS OF TAYLOR'S SERIES



Numerical Differentiation

$$f(x_{i+1}) \cong f(x_i) + f'(x_i)(x_{i+1} - x_i) + \frac{f''}{2!}(x_{i+1} - x_i)^2 + \dots$$

$$+ \frac{f^{(n)}}{n!}(x_{i+1} - x_i)^n + R_n$$

- First Forward Difference

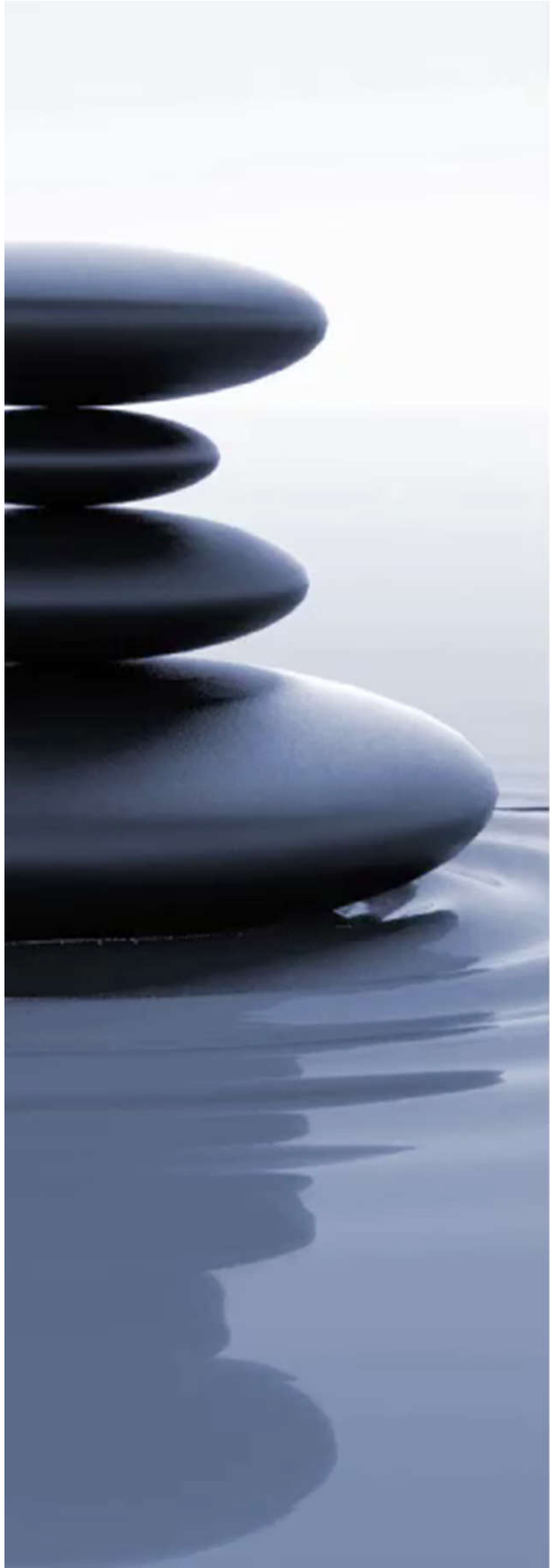
$$f(x_{i+1}) \cong f(x_i) + f'(x_i)(x_{i+1} - x_i) + R_1$$

$$\Rightarrow f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i} = \frac{f(x_{i+1}) - f(x_i)}{h}$$

- First Backward Difference

$$f(x_{i-1}) \cong f(x_i) - f'(x_i)(x_i - x_{i-1}) + R_1$$

$$\Rightarrow f'(x_i) = \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} = \frac{f(x_i) - f(x_{i-1})}{h}$$



Numerical Differentiation

- First Centered Difference

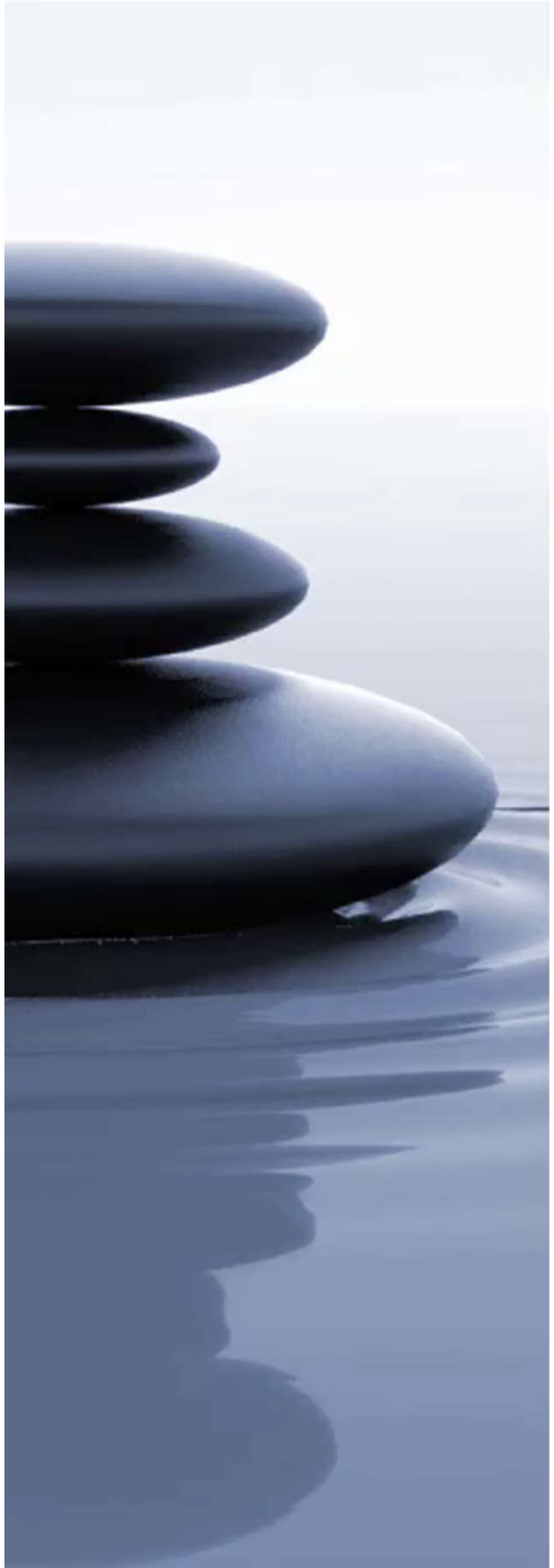
$$f(x_{i+1}) \cong f(x_i) + f'(x_i)h + R_1$$

$$f(x_{i-1}) \cong f(x_i) - f'(x_i)h + R_1$$

$$f(x_i) \cong f(x_{i-1}) + f'(x_i)h + R_1$$

$$=> f(x_{i+1}) \cong f(x_{i-1}) + 2f'(x_i)h + R_1$$

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_{i-1})}{2h}$$



Example

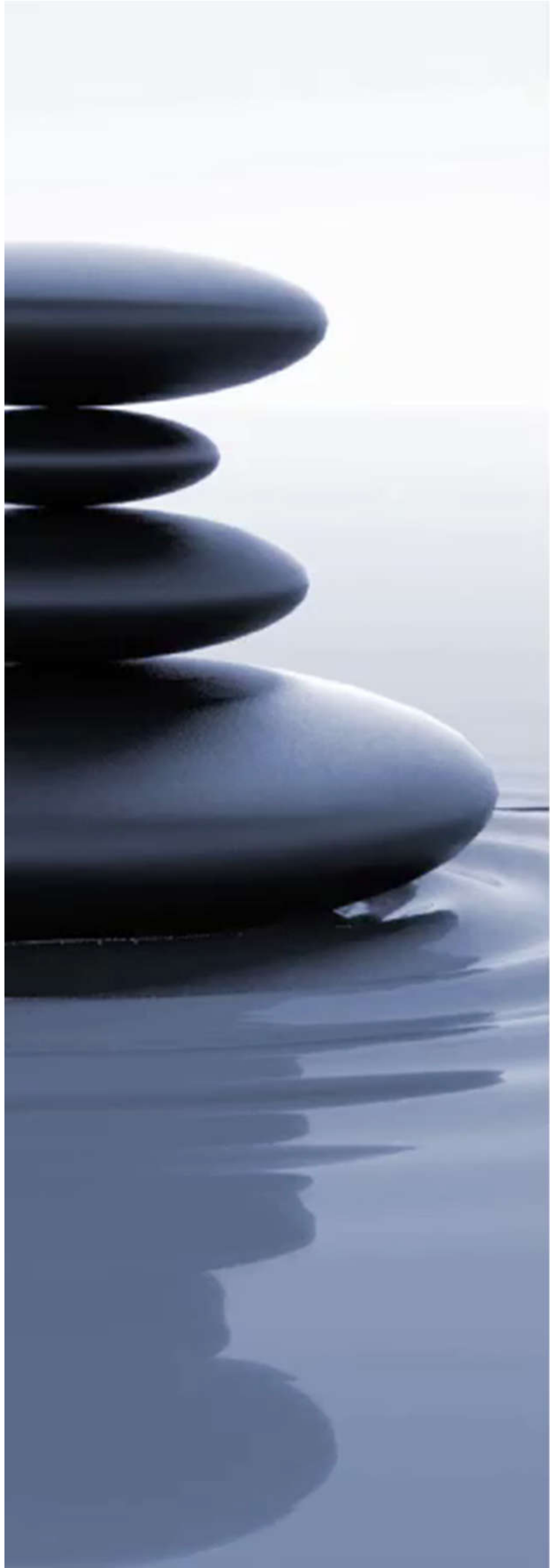
Use forward and backward difference approximations of $O(h)$ and a centered difference approximation of $O(h^2)$ to estimate the first derivative of

$$f(x) = -0.1x^4 - 0.15x^3 - 0.5x^2 - 0.25x + 1.2$$

at $x = 0.5$ using a step size $h = 0.5$. Repeat the computation using $h = 0.25$. Note that the derivative can be calculated directly as

$$f(x) = -0.4x^3 - 0.45x^2 - 1.0x - 0.25$$

and can be used to compute the true value as $f(0.5) = -0.9125$.



Example

1. For h = 0.5, the function can be employed to determine

$x_{i-1} = 0$

$x_i = 0.5$

$x_{i+1} = 1.0$

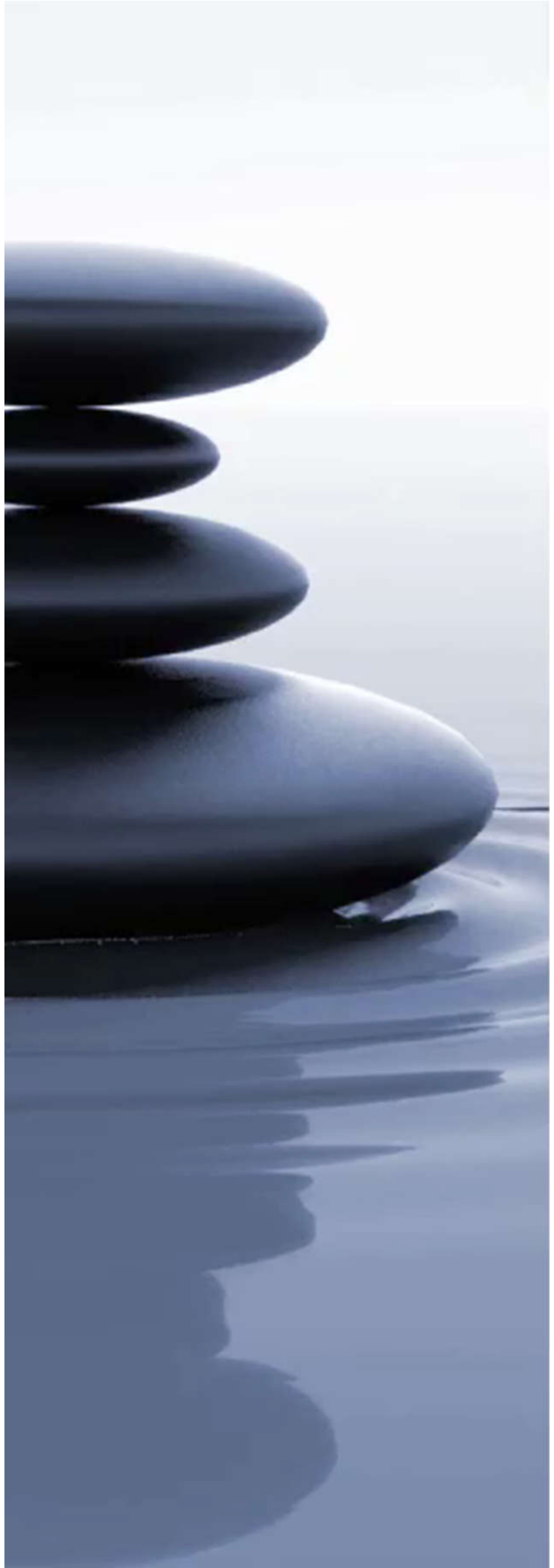
$f(x_{i-1}) = 1.2$

$f(x_i) = 0.925$

$f(x_{i+1}) = 0.2$

These values can be used to compute the forward divided difference

$$f'(0.5) = \frac{0.2 - 0.925}{0.5} = -1.45 \qquad |\varepsilon_t| = 58.9\%$$



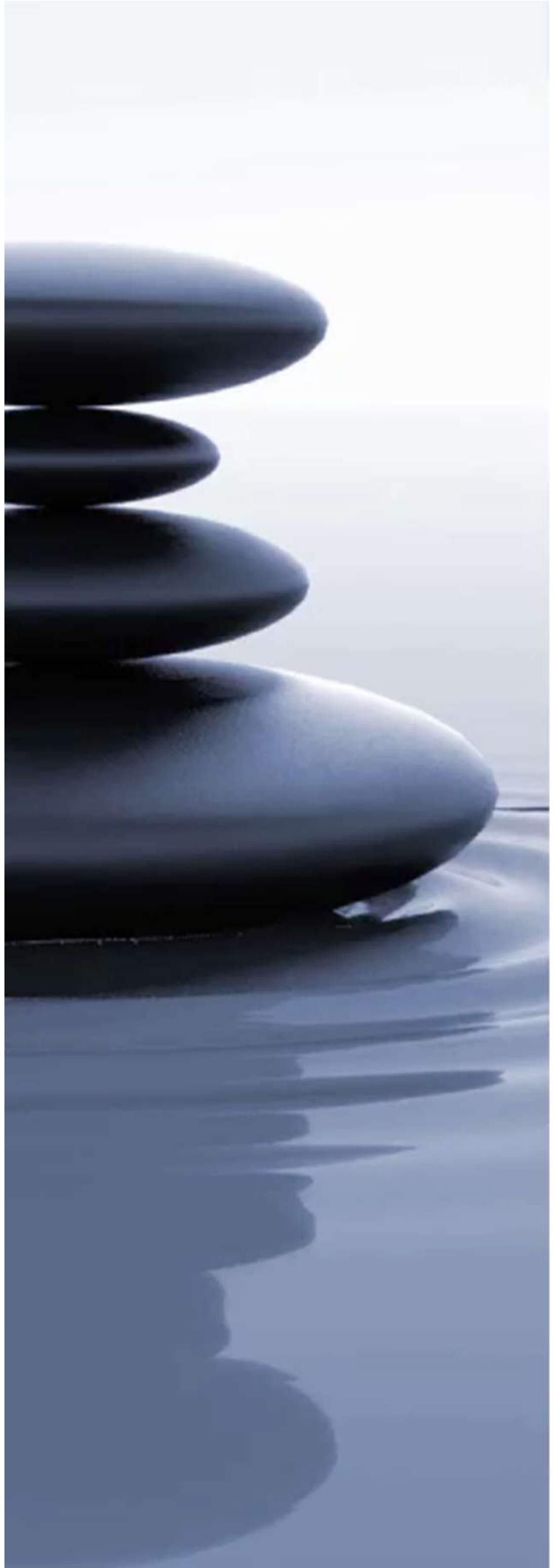
Example

The backward divided difference

$$f'(0.5) = \frac{0.925 - 1.2}{0.5} = -0.55 \qquad |\varepsilon_t| = 39.7\%$$

And the centered divided difference

$$f'(0.5) = \frac{0.2 - 1.2}{1.0} = -1.0 \qquad |\varepsilon_t| = 9.6\%$$



Example

2. For h = 0.25, the function can be employed to determine

$x_{i-1} = 0.25$

$x_i = 0.5$

$x_{i+1} = 0.75$

$f(x_{i-1}) = 1.10351563$

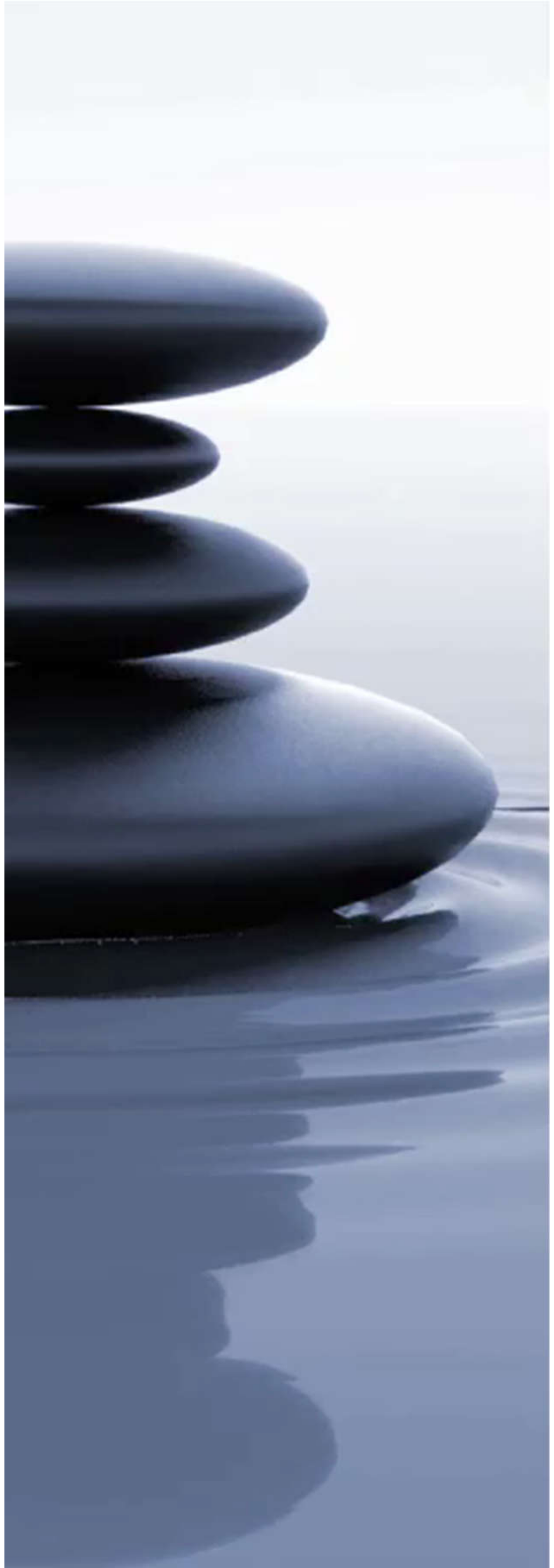
$f(x_i) = 0.925$

$f(x_{i+1}) = 0.63632813$

These values can be used to compute the forward divided difference

$$f'(0.5) = \frac{0.63632813 - 0.925}{0.25} = -1.155$$

$$|\varepsilon_t| = 26.5\%$$



Example

The backward divided difference

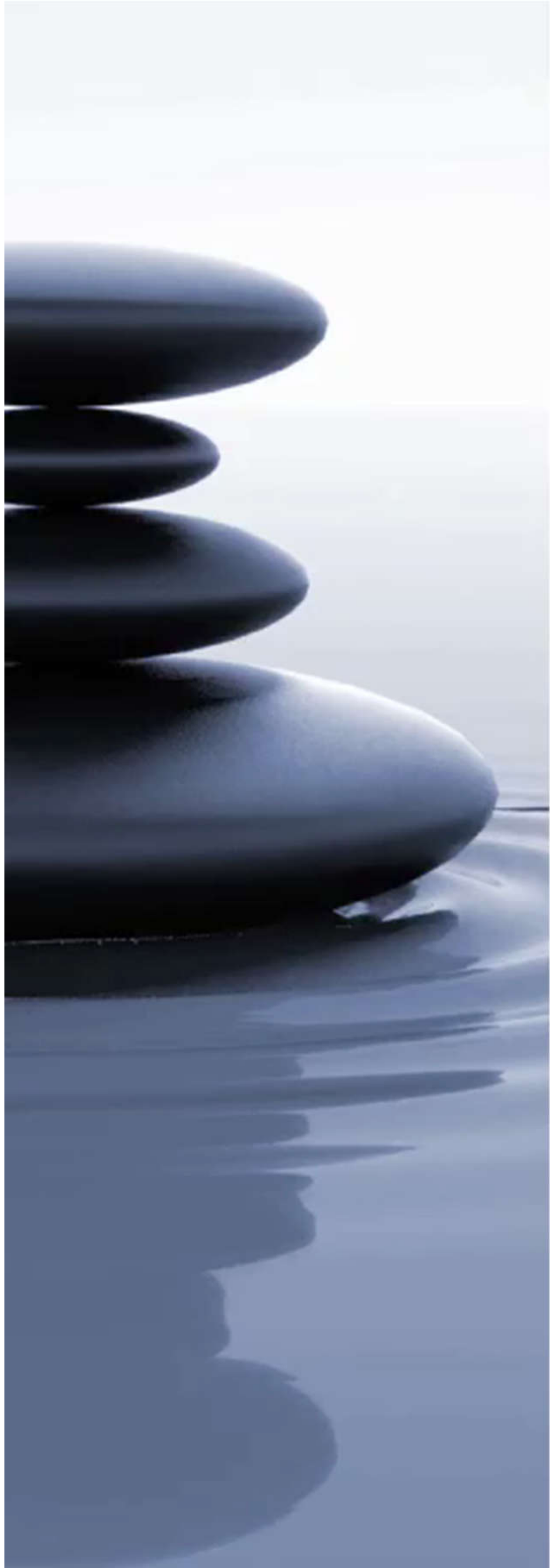
$$f'(0.5) = \frac{0.925 - 1.10351563}{0.25} = -0.714$$

$$|\varepsilon_t| = 21.7\%$$

And the centered divided difference

$$f'(0.5) = \frac{0.63632813 - 1.10351563}{0.5} = -0.934$$

$$|\varepsilon_t| = 2.4\%$$



Finite Difference Approximation of Higher Derivatives

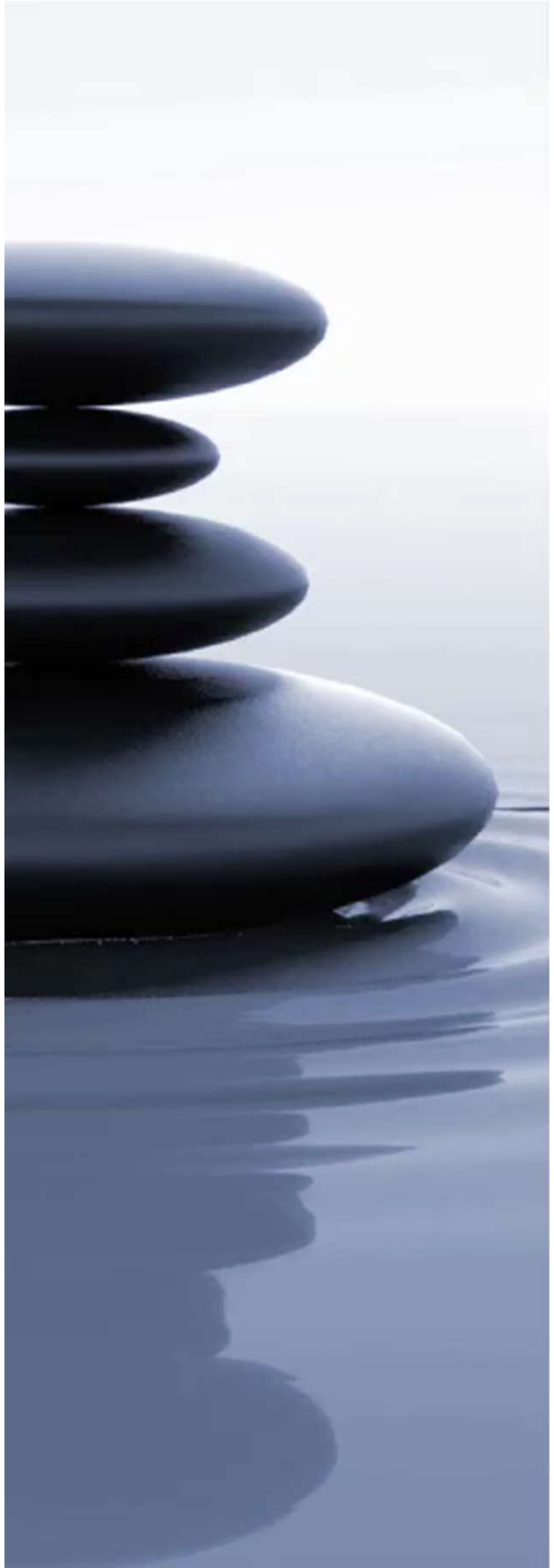
$$f(x_{i+2}) \cong f(x_i) + f'(x_i)(2h) + \frac{f''(x_i)}{2!}(2h)^2 + \ldots (1)$$

$$f(x_{i+1}) \cong f(x_i) + f'(x_i)(h) + \frac{f''(x_i)}{2!}h^2 + \ldots$$

$$\Leftrightarrow 2f(x_{i+1}) \cong 2f(x_i) + 2f'(x_i)(h) + 2\frac{f''(x_i)}{2!}h^2 + \ldots (2)$$

$$=> f(x_{i+2}) - 2f(x_{i+1}) \cong -f(x_i) + f''(x_i)h^2 + \ldots$$

$$=> f''(x_i) = \frac{f(x_{i+2}) - 2f(x_{i+1}) + f(x_i)}{h^2}$$



Summary

Error Definitions	
True error	$E_t = \text{true value} - \text{approximation}$
True percent relative error	$\varepsilon_t = \dfrac{\text{true value} - \text{approximation}}{\text{true value}}\, 100\%$
Approximate percent relative error	$\varepsilon_a = \dfrac{\text{present approximation} - \text{previous approximation}}{\text{present approximation}}\, 100\%$
Stopping criterion	Terminate computation when $\varepsilon_a < \varepsilon_s$ where ε_s is the desired percent relative error
Taylor Series	
Taylor series expansion	$f(x_{i+1}) = f(x_i) + f'(x_i)h + \dfrac{f''(x_i)}{2!}h^2 + \dfrac{f'''(x_i)}{3!}h^3 + \cdots + \dfrac{f^{(n)}(x_i)}{n!}h^n + R_n$
	where
Remainder	$R_n = \dfrac{f^{(n+1)}(\xi)}{(n+1)!}h^{n+1}$
	or
	$R_n = O(h^{n+1})$

Any Questions?

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Basic Derivatives Rules

Constant Rule: $\frac{d}{dx}(c) = 0$

Constant Multiple Rule: $\frac{d}{dx}[cf(x)] = cf'(x)$

Power Rule: $\frac{d}{dx}(x^n) = nx^{n-1}$

Sum Rule: $\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$

Difference Rule: $\frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x)$

Product Rule: $\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$

Quotient Rule: $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$

Chain Rule: $\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$

Key Idea 9.10.1 Important Maclaurin Series Expansions

Function and Series	First Few Terms	Interval of Convergence
$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$	$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$	$(-\infty, \infty)$
$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$	$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$	$(-\infty, \infty)$
$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$	$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$	$(-\infty, \infty)$
$\ln(x+1) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$	$x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$	$(-1, 1]$
$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$	$1 + x + x^2 + x^3 + \cdots$	$(-1, 1)$
$(1+x)^k = \sum_{n=0}^{\infty} \frac{k(k-1)\cdots(k-(n-1))}{n!} x^n$	$1 + kx + \frac{k(k-1)}{2!} x^2 + \cdots$	$\begin{cases} (-1,1) & k \le -1 \\ (-1,1] & -1 < k < 0 \\ [-1,1] & 0 < k \end{cases}$
$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$	$x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$	$[-1, 1]$

Derivative Rules

Exponential Functions

$$\frac{d}{dx}\left(e^{x}\right) = e^{x}$$

$$\frac{d}{dx}\left(a^{x}\right) = a^{x}\ln a$$

$$\frac{d}{dx}\left(e^{g(x)}\right) = e^{g(x)}g'(x)$$

$$\frac{d}{dx}\left(a^{g(x)}\right) = \ln(a)\,a^{g(x)}\,g'(x)$$

Logarithmic Functions

$$\frac{d}{dx}(\ln x) = \frac{1}{x},\ x > 0$$

$$\frac{d}{dx}\ln(g(x)) = \frac{g'(x)}{g(x)}$$

$$\frac{d}{dx}\left(\log_a x\right) = \frac{1}{x\ln a},\ x > 0$$

$$\frac{d}{dx}\left(\log_a g(x)\right) = \frac{g'(x)}{g(x)\ln a}$$

Trigonometric Functions

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\csc x) = -\csc x\cot x$$

$$\frac{d}{dx}(\sec x) = \sec x\tan x$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

Inverse Trigonometric Functions

$$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}},\ x \neq \pm 1$$

$$\frac{d}{dx}\left(\cos^{-1} x\right) = \frac{-1}{\sqrt{1-x^2}},\ x \neq \pm 1$$

$$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}\left(\cot^{-1} x\right) = \frac{-1}{1+x^2}$$

$$\frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{x\sqrt{x^2-1}},\ x \neq \pm 1, 0$$

$$\frac{d}{dx}\left(\csc^{-1} x\right) = \frac{-1}{x\sqrt{x^2-1}},\ x \neq \pm 1, 0$$

$$\tan(x) = \frac{\sin(x)}{\cos(x)},\quad \cot(x) = \frac{\cos(x)}{\sin(x)},\quad \sec(x) = \frac{1}{\cos(x)},\quad \csc(x) = \frac{1}{\sin(x)}$$