TMC: Partial Differential Equations

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MOTIVATION

Given a function u that depends on both x and y, the partial derivative of u with respect to x at an arbitrary point (x, y) is defined as

$$\frac{\partial u}{\partial x} = \lim_{\Delta x \to 0} \frac{u(x + \Delta x, y) - u(x, y)}{\Delta x}$$

Similarly, the partial derivative with respect to y is defined as

$$\frac{\partial u}{\partial y} = \lim_{\Delta y \to 0} \frac{u(x, y + \Delta y) - u(x, y)}{\Delta y}$$

An equation involving partial derivatives of an unknown function of two or more independent variables is called a *partial differential equation,* or *PDE.* For example,

$$\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial y^2} + u = 1$$

$$\frac{\partial^3 u}{\partial x^2 \partial y} + x\frac{\partial^2 u}{\partial y^2} + 8u = 5y$$

$$\left(\frac{\partial^2 u}{\partial x^2}\right)^3 + 6\frac{\partial^3 u}{\partial x \partial y^2} = x$$

$$\frac{\partial^2 u}{\partial x^2} + xu\frac{\partial u}{\partial y} = x$$

MOTIVATION

Because of their widespread application in engineering, our treatment of PDEs will focus on linear, second-order equations. For two independent variables, such equations can be expressed in the following general form:

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D = 0$$

where A, B, and C are functions of x and y and D is a function of x, y, u, $\partial u/\partial x$, and $\partial u/\partial y$.

$B^2 - 4AC$	Category	Example
< 0	Elliptic	Laplace equation (steady state with two spatial dimensions) $\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$
$= 0$	Parabolic	Heat conduction equation (time variable with one spatial dimension) $\frac{\partial T}{\partial t} = k' \frac{\partial^2 T}{\partial x^2}$
> 0	Hyperbolic	Wave equation (time variable with one spatial dimension) $\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 y}{\partial t^2}$

PART A

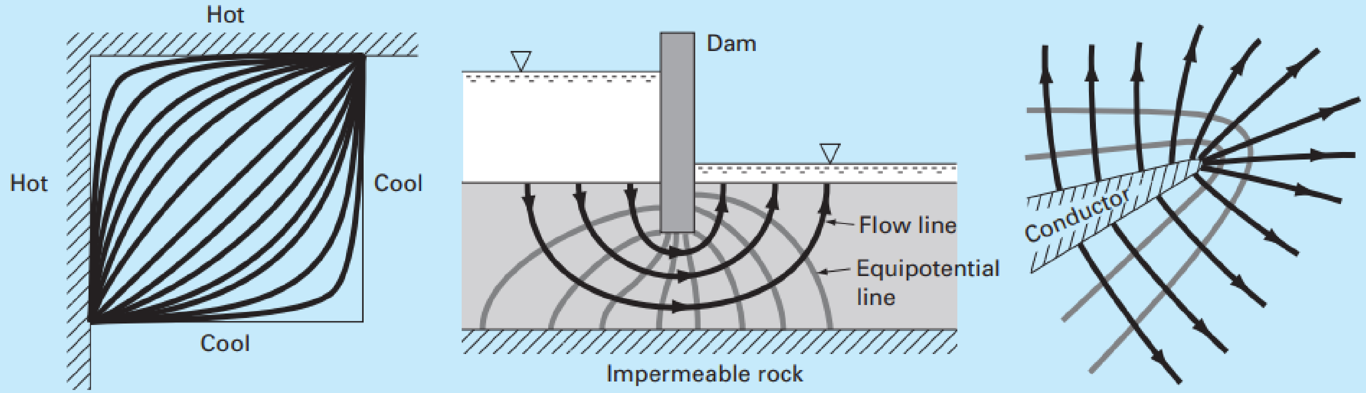
FINITE DIFFERENCE: ELLIPTIC EQUATIONS

Solution Technique

Elliptic equations in engineering are typically used to characterize steady-state, boundary value problems.

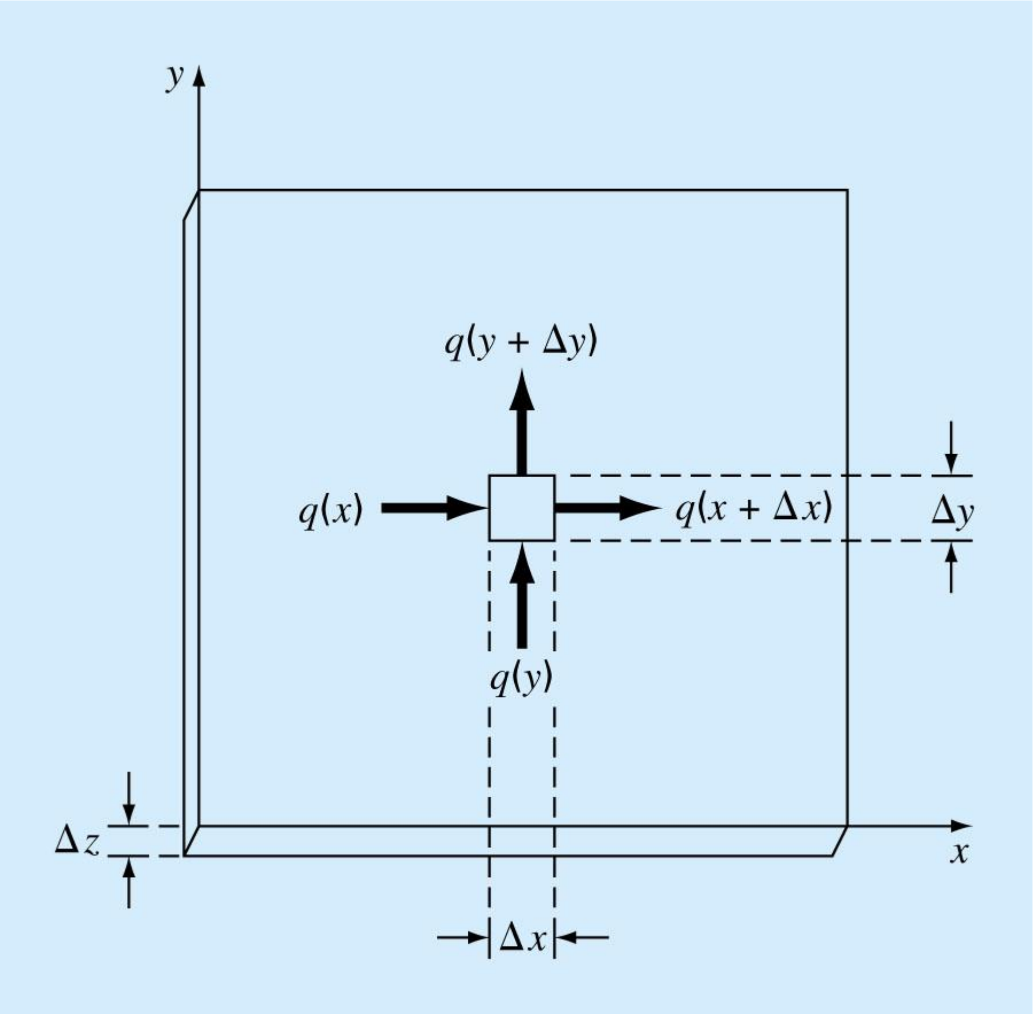
For numerical solution of elliptic PDEs, the PDE is transformed into an algebraic difference equation.

Because of its simplicity and general relevance to most areas of engineering, we will use a heated plate as an example for solving elliptic PDEs.



(a) (b) (c)

The Laplace Equation



A thin plate of thickness Δz. An element is shown about which a heat balance is taken.

The Laplace Equation

At steady state, the flow of heat into the element over a unit time period Δt must equal the flow out,

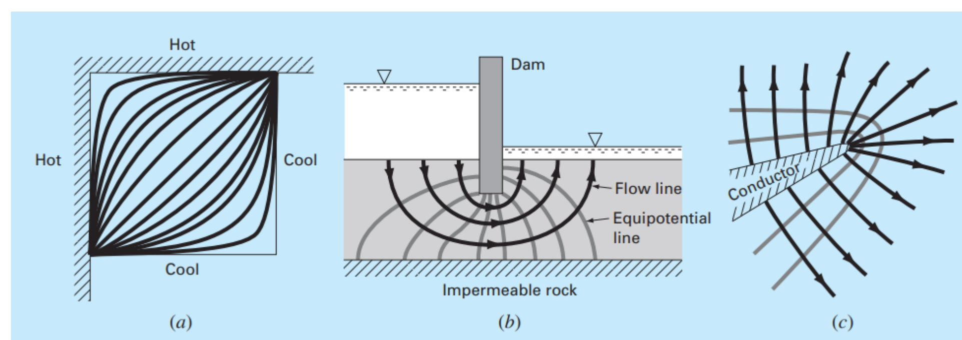
$$q(x)\Delta y\,\Delta z\,\Delta t + q(y)\,\Delta x\,\Delta z\,\Delta t = q(x+\Delta x)\Delta y\,\Delta z\,\Delta t + q(y+\Delta y)\Delta x\,\Delta z\,\Delta t$$

where q(x) and q(y) are the heat fluxes at x and y, respectively [cal/(cm^2.s)].

$$[q(x) - q(x+\Delta x)]\Delta y + [q(y) - q(y+\Delta y)]\Delta x = 0$$

$$\frac{q(x) - q(x+\Delta x)}{\Delta x}\Delta x\,\Delta y + \frac{q(y) - q(y+\Delta y)}{\Delta y}\Delta y\,\Delta x = 0$$

$$-\frac{\partial q}{\partial x} - \frac{\partial q}{\partial y} = 0$$



Poisson equation

The link between flux and temperature is provided by Fourier's law of heat conduction, which can be represented as

$$q_i = -k\rho C \frac{\partial T}{\partial i}$$

where q_i = heat flux in the direction of the i dimension [cal/(cm$^2 \cdot$ s)], k = coefficient of *thermal diffusivity* (cm^2/s), ρ = density of the material (g/cm^3), C = heat capacity of the material [cal/(g $\cdot$ °C)], and T = temperature (°C), which is defined as

$$T = \frac{H}{\rho C V}$$

where H = heat (cal) and V = volume (cm^3).

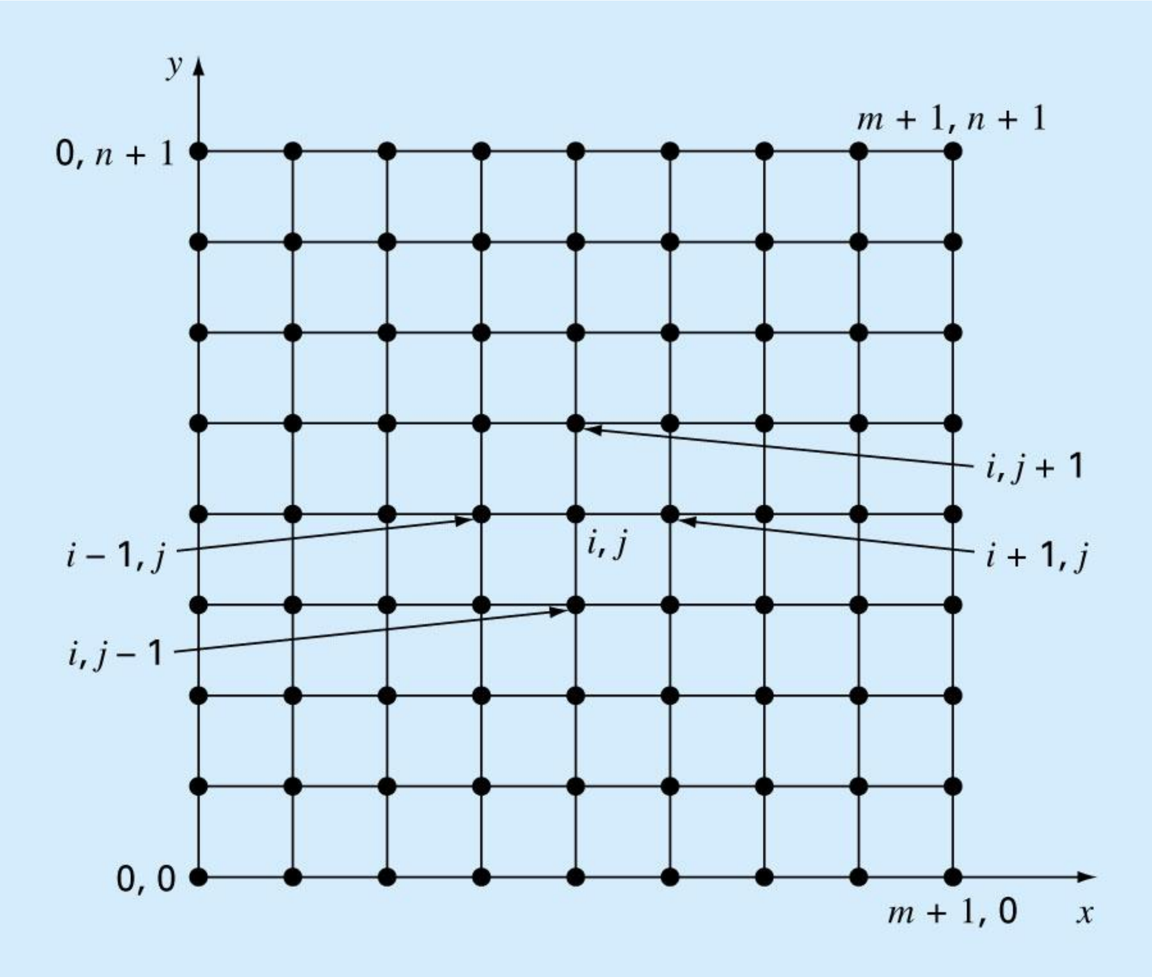
$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

which is the *Laplace equation.* Note that for the case where there are sources or sinks of heat within the two-dimensional domain, the equation can be represented as

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = f(x, y)$$

where $f(x, y)$ is a function describing the sources or sinks of heat.

Solution Technique



A grid used for the finite-difference solution of elliptic PDEs in two independent variables such as the Laplace equation.

The Laplacian Difference Equations

$$\boxed{\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}=0}$$

Laplace Equation

$$\frac{\partial^2 T}{\partial x^2}=\frac{T_{i+1,j}-2T_{i,j}+T_{i-1,j}}{\Delta x^2}$$

O[Δ(x)²]

$$\frac{\partial^2 T}{\partial y^2}=\frac{T_{i,j+1}-2T_{i,j}+T_{i,j-1}}{\Delta y^2}$$

O[Δ(y)²]

$$\frac{T_{i+1,j}-2T_{i,j}+T_{i-1,j}}{\Delta x^2}+\frac{T_{i,j+1}-2T_{i,j}+T_{i,j-1}}{\Delta y^2}=0$$

$$\Delta x=\Delta y$$

$$\boxed{T_{i+1,j}+T_{i-1,j}+T_{i,j+1}+T_{i,j-1}-4T_{i,j}=0}$$

<u>Laplacian difference equation.</u>

In addition, boundary conditions along the edges must be specified to obtain a unique solution.

The simplest case is where the temperature at the boundary is set at a fixed value, *Dirichlet boundary condition*.

A balance for node (1,1) is:

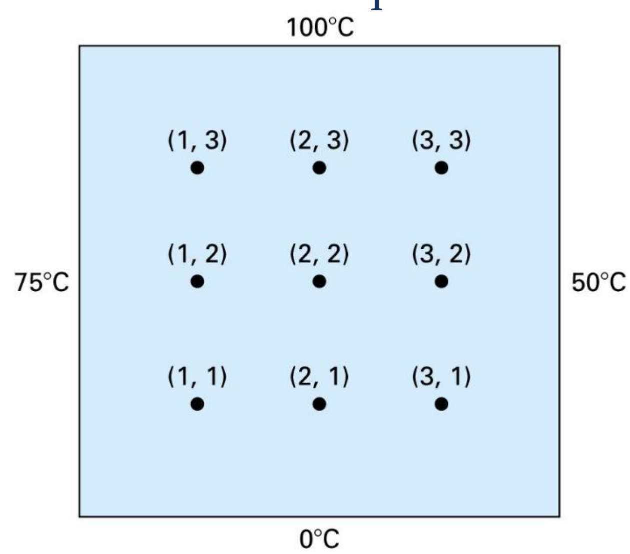
$$T_{21} + T_{01} + T_{12} + T_{10} - 4T_{11} = 0$$

$$T_{01} = 75$$

$$T_{10} = 0$$

$$-4T_{11} + T_{12} + T_{21} = -75$$

Similar equations can be developed for other interior points to result a set of simultaneous equations.



The result is a set of nine simultaneous equations with nine unknowns:

$$
\begin{array}{ccccccccccc}
4T_{11} & -T_{21} & & -T_{12} & & & & & & = 75 \\
-T_{11} & +4T_{21} & -T_{13} & & -T_{22} & & & & & = 0 \\
& -T_{21} & +4T_{31} & & & -T_{32} & & & & = 50 \\
-T_{11} & & & +4T_{12} & -T_{22} & & -T_{13} & & & = 75 \\
& -T_{21} & & -T_{12} & +4T_{22} & -T_{32} & & -T_{23} & & = 0 \\
& & -T_{31} & & -T_{22} & +4T_{32} & & & -T_{33} & = 50 \\
& & & -T_{12} & & & +4T_{13} & -T_{23} & & = 175 \\
& & & & -T_{22} & & -T_{13} & +4T_{23} & -T_{33} & = 100 \\
& & & & & -T_{32} & & -T_{23} & +4T_{33} & = 150
\end{array}
$$

The Liebmann Method

Most numerical solutions of Laplace equation involve systems that are very large.

For larger size grids, a significant number of terms will be zero.

For such sparse systems, most commonly employed approach is *Gauss-Seidel*, which when applied to PDEs is also referred as *Liebmann's method*.

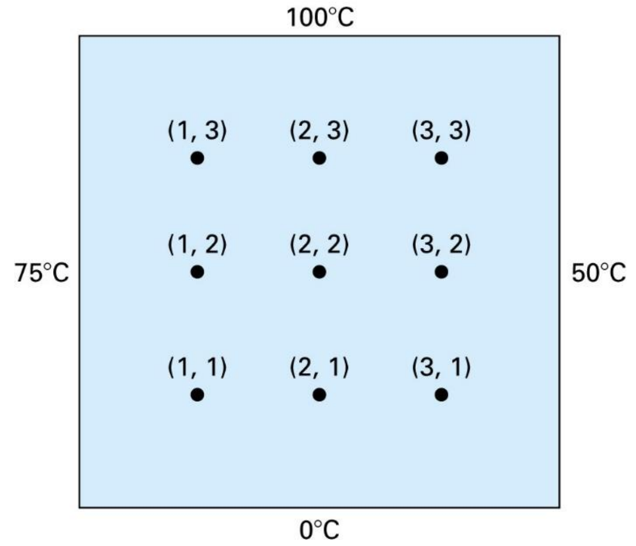
$$T_{i+1,j} + T_{i-1,j} + T_{i,j+1} + T_{i,j-1} - 4T_{i,j} = 0$$

$$T_{i,j} = \frac{T_{i+1,j} + T_{i-1,j} + T_{i,j+1} + T_{i,j-1}}{4}$$

$$T_{i,j}^{\text{new}} = \lambda T_{i,j}^{\text{new}} + (1 - \lambda) T_{i,j}^{\text{old}} \qquad |(\varepsilon_a)_{i,j}| = \left| \frac{T_{i,j}^{\text{new}} - T_{i,j}^{\text{old}}}{T_{i,j}^{\text{new}}} \right| 100\%$$

Example

Use Liebmann's method (Gauss-Seidel) to solve for the temperature of the heated plate in the following picture.



Employ over-relaxation with a value of 1.5 for the weighting factor and iterate to e_s = 1%.

Example

At i = 1, j = 1 is

$$T_{11} = \frac{0 + 75 + 0 + 0}{4} = 18.75$$

and applying overrelaxation yields

$$T_{11} = 1.5(18.75) + (1 - 1.5)0 = 28.125$$

For $i = 2, j = 1,$

$$T_{21} = \frac{0 + 28.125 + 0 + 0}{4} = 7.03125$$

$$T_{21} = 1.5(7.03125) + (1 - 1.5)0 = 10.54688$$

For $i = 3, j = 1,$

$$T_{31} = \frac{50 + 10.54688 + 0 + 0}{4} = 15.13672$$

$$T_{31} = 1.5(15.13672) + (1 - 1.5)0 = 22.70508$$

The computation is repeated for the other rows to give

$$T_{12} = 38.67188 \qquad T_{22} = 18.45703 \qquad T_{32} = 34.18579$$
$$T_{13} = 80.12696 \qquad T_{23} = 74.46900 \qquad T_{33} = 96.99554$$

Because all the $T_{i,j}$'s are initially zero, all ε_a's for the first iteration will be 100%.

For the second iteration the results are

$$T_{11} = 32.51953 \qquad T_{21} = 22.35718 \qquad T_{31} = 28.60108$$
$$T_{12} = 57.95288 \qquad T_{22} = 61.63333 \qquad T_{32} = 71.86833$$
$$T_{13} = 75.21973 \qquad T_{23} = 87.95872 \qquad T_{32} = 67.68736$$

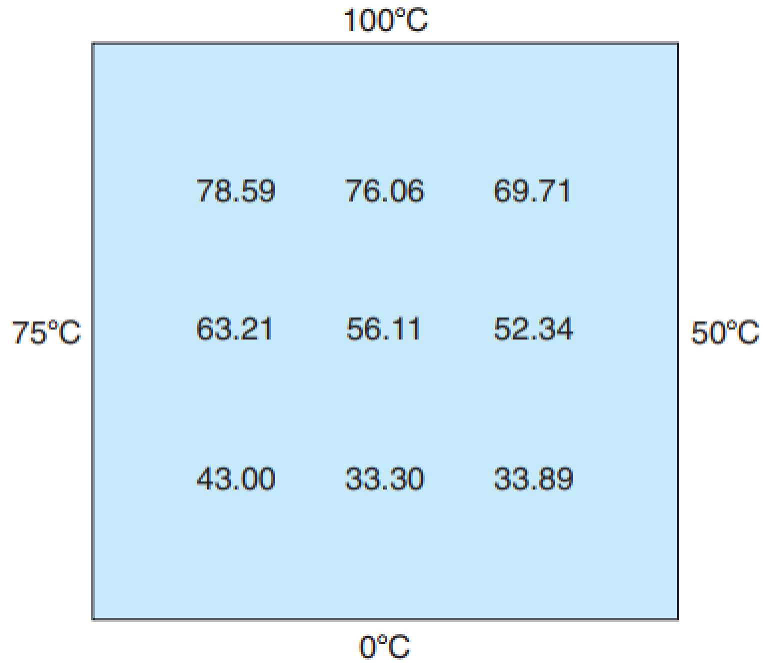
The error for $T_{1,1}$ can be estimated as [Eq. (29.13)]

$$|(\varepsilon_a)_{1,1}| = \left|\frac{32.51953 - 28.12500}{32.51953}\right| 100\% = 13.5\%$$

Because this value is above the stopping criterion of 1%, the computation is continued. The ninth iteration gives the result

$$T_{11} = 43.00061 \qquad T_{21} = 33.29755 \qquad T_{31} = 33.88506$$
$$T_{12} = 63.21152 \qquad T_{22} = 56.11238 \qquad T_{32} = 52.33999$$
$$T_{13} = 78.58718 \qquad T_{23} = 76.06402 \qquad T_{33} = 69.71050$$

where the maximum error is 0.71%.


100°C
78.59 76.06 69.71
75°C
63.21 56.11 52.34
50°C
43.00 33.30 33.89
0°C

Secondary Variables

For the heated plate, a secondary variable is the rate of heat flux across the plate's surface. This quantity can be computed from Fourier's law.

$$k' = k\rho C$$

where k' is referred to as the *coefficient of thermal conductivity* [cal/(s · cm · °C)].

$$q_x = -k' \frac{T_{i+1,j} - T_{i-1,j}}{2\,\Delta x}$$

and

$$q_y = -k' \frac{T_{i,j+1} - T_{i,j-1}}{2\,\Delta y}$$

The resultant heat flux can be computed from these two quantities by

$$q_n = \sqrt{q_x^2 + q_y^2}$$

where the direction of q_n is given by

$$\theta = \tan^{-1}\left(\frac{q_y}{q_x}\right)$$

for $q_x > 0$ and

$$\theta = \tan^{-1}\left(\frac{q_y}{q_x}\right) + \pi$$

for $q_x < 0$. Recall that the angle can be expressed in degrees by multiplying it by $180°/\pi$. If $q_x = 0$, θ is $\pi/2$ (90°) or $3\pi/2$ (270°), depending on whether q_y is positive or negative, respectively.

Example

Employ the results of the previous example to determine the distribution of heat flux for the heated plate from Fig. 29.4. Assume that the plate is 40x40 cm and is made out of aluminum [k'=0.49 cal/(s.cm.ºC)].

For i = j = 1

$$q_x = -0.49 \frac{\text{cal}}{\text{s} \cdot \text{cm} \cdot {}^\circ\text{C}} \frac{(33.29755 - 75)^\circ\text{C}}{2(10 \text{ cm})} = 1.022 \text{ cal/(cm}^2 \cdot \text{s)}$$

and

$$q_y = -0.49 \frac{\text{cal}}{\text{s} \cdot \text{cm} \cdot {}^\circ\text{C}} \frac{(63.21152 - 0)^\circ\text{C}}{2(10 \text{ cm})} = -1.549 \text{ cal/(cm}^2 \cdot \text{s)}$$

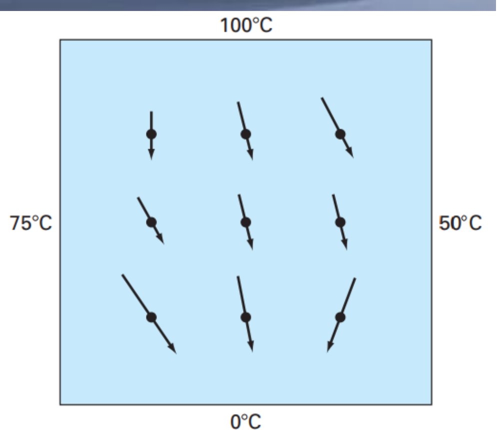
The resultant flux can be computed

$$q_n = \sqrt{(1.022)^2 + (-1.549)^2} = 1.856 \text{ cal/(cm}^2 \cdot \text{s)}$$

and the angle of its trajectory by Eq. (29.17)

$$\theta = \tan^{-1}\left(\frac{-1.549}{1.022}\right) = -0.98758 \times \frac{180^\circ}{\pi} = -56.584^\circ$$

Thus, at this point, the heat flux is directed down and to the right.



Derivative Boundary Conditions

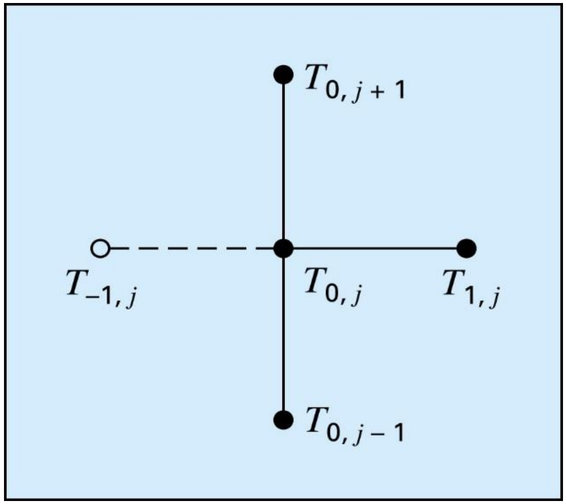
We will address problems that involve boundaries at which the derivative is specified and boundaries that are irregularly shaped.

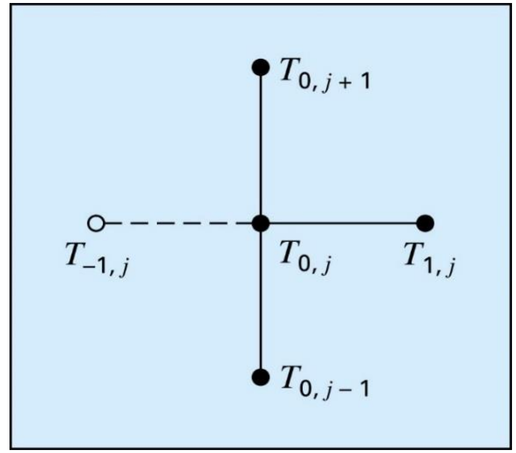
Derivative Boundary Conditions

Known as a *Neumann boundary condition*.

For the heated plate problem, heat flux is specified at the boundary, rather than the temperature.

If the edge is insulated, this derivative becomes zero.





$$T_{1,j} + T_{-1,j} + T_{0,j+1} + T_{0,j-1} - 4T_{0,j} = 0$$

$$\frac{\partial T}{\partial x} \cong \frac{T_{1,j} - T_{-1,j}}{2\Delta x}$$

$$T_{-1,j} = T_{1,j} - 2\Delta x \frac{\partial T}{\partial x}$$

$$2T_{1,j} - 2\Delta x \frac{\partial T}{\partial x} + T_{0,j+1} + T_{0,j-1} - 4T_{0,j} = 0$$

•Thus, the derivative has been incorporated into the balance.

•Similar relationships can be developed for derivative boundary conditions at the other edges.

Example

Repeat the same problem as the previous example, but with the lower edge insulated.

The general equation to characterize a derivative at the lower edge (that is, at j=0) of a heated plate is

$$T_{i+1,0} + T_{i-1,0} + 2T_{i,1} - 2\,\Delta y \frac{\partial T}{\partial y} - 4T_{i,0} = 0$$

For an insulated edge, the derivative is zero and the equation becomes

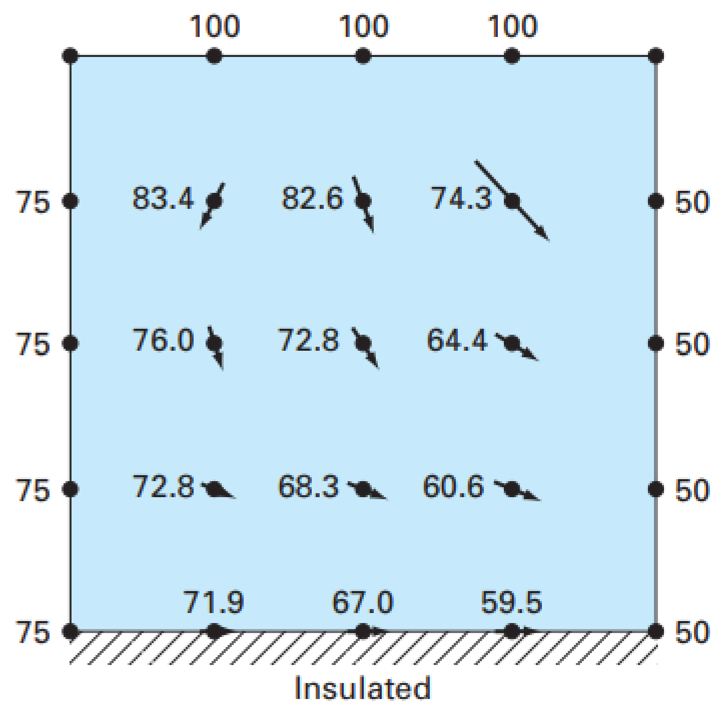
$$T_{i+1,0} + T_{i-1,0} + 2T_{i,1} - 4T_{i,0} = 0$$

The simultaneous equations for temperature distribution on the plate with an insulated lower edge can be written in matrix form as

$$\begin{bmatrix}
4 & -1 & & -2 & & & & & & & & \\
-1 & 4 & -1 & & -2 & & & & & & & \\
 & -1 & 4 & & & -2 & & & & & & \\
-1 & & & 4 & -1 & & -1 & & & & & \\
 & -1 & & -1 & 4 & -1 & & -1 & & & & \\
 & & -1 & & -1 & 4 & -1 & & -1 & & & \\
 & & & -1 & & & 4 & -1 & & -1 & & \\
 & & & & -1 & & -1 & 4 & -1 & & -1 & \\
 & & & & & -1 & & -1 & 4 & & & -1 \\
 & & & & & & -1 & & & 4 & -1 & \\
 & & & & & & & -1 & & -1 & 4 & -1 \\
 & & & & & & & & -1 & & -1 & 4
\end{bmatrix}
\begin{Bmatrix} T_{10} \\ T_{20} \\ T_{30} \\ T_{11} \\ T_{21} \\ T_{31} \\ T_{12} \\ T_{22} \\ T_{32} \\ T_{13} \\ T_{23} \\ T_{33} \end{Bmatrix}
=
\begin{Bmatrix} 75 \\ 0 \\ 50 \\ 75 \\ 0 \\ 50 \\ 75 \\ 0 \\ 50 \\ 175 \\ 100 \\ 150 \end{Bmatrix}$$

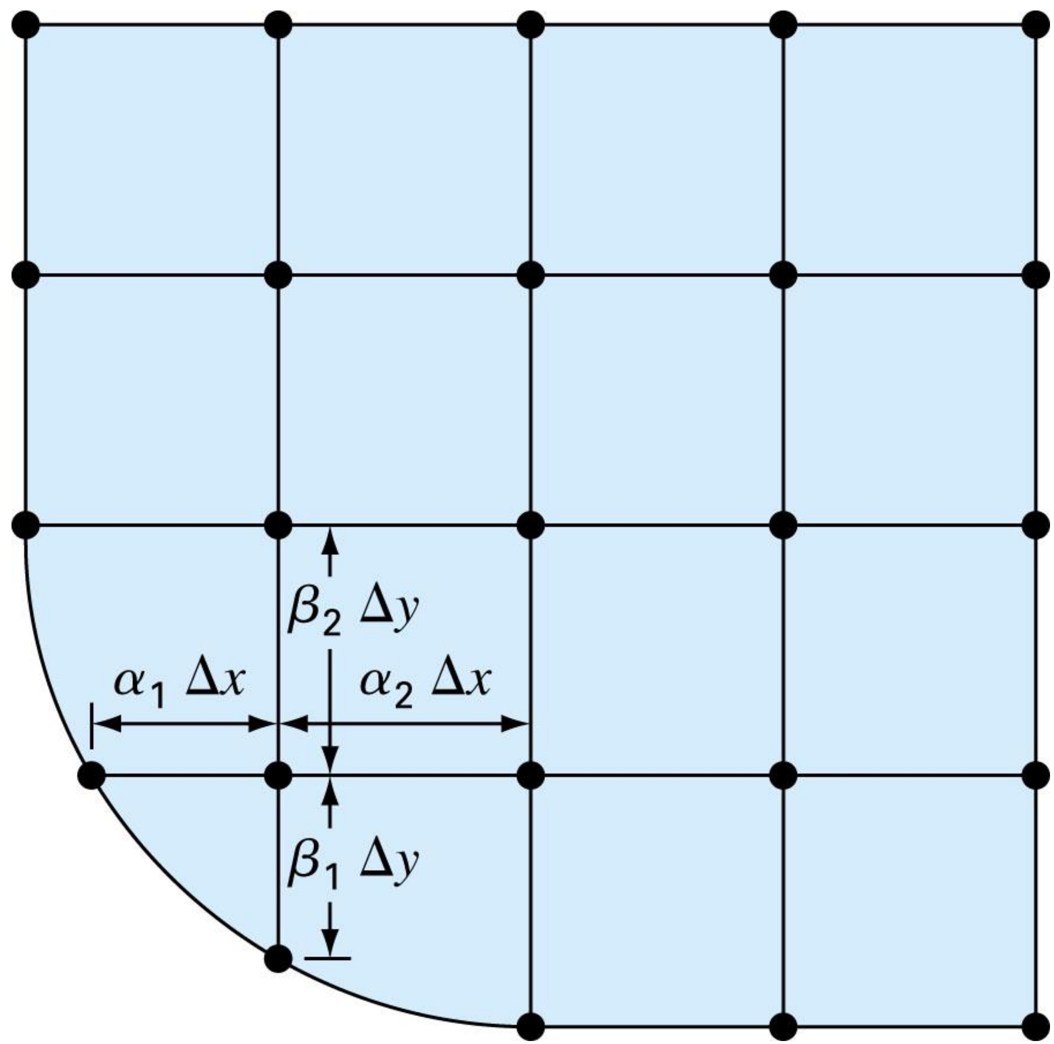
Note that because of the derivative boundary condition, the matrix is increased to 12x12 in contrast to the 9x9 system

$T_{10} = 71.91$	$T_{20} = 67.01$	$T_{30} = 59.54$
$T_{11} = 72.81$	$T_{21} = 68.31$	$T_{31} = 60.57$
$T_{12} = 76.01$	$T_{22} = 72.84$	$T_{32} = 64.42$
$T_{13} = 83.41$	$T_{23} = 82.63$	$T_{33} = 74.26$



Irregular Boundaries

Many engineering problems exhibit irregular boundaries.



First derivatives in the x direction can be approximated as:

$$\left(\frac{\partial T}{\partial x}\right)_{i-1,i} \cong \frac{T_{i,j} - T_{i-1,j}}{\alpha_1 \Delta x}$$

$$\left(\frac{\partial T}{\partial x}\right)_{i,i+1} \cong \frac{T_{i+1,j} - T_{i,j}}{\alpha_2 \Delta x}$$

$$\frac{\partial^2 T}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}\right) = \frac{\left(\frac{\partial T}{\partial x}\right)_{i,i+1} - \left(\frac{\partial T}{\partial x}\right)_{i-1,i}}{\frac{\alpha_1 \Delta x + \alpha_2 \Delta x}{2}}$$

$$\frac{\partial^2 T}{\partial x^2} = 2\,\frac{\frac{T_{i,j} - T_{i-1,j}}{\alpha_1 \Delta x} - \frac{T_{i+1,j} - T_{i,j}}{\alpha_2 \Delta x}}{\frac{\alpha_1 \Delta x + \alpha_2 \Delta x}{2}}$$

A similar equation can be developed in the y direction.

$$\frac{\partial^2 T}{\partial x^2} = \frac{2}{\Delta x^2}\left[\frac{T_{i-1,j} - T_{i,j}}{\alpha_1(\alpha_1 + \alpha_2)} + \frac{T_{i+1,j} - T_{i,j}}{\alpha_2(\alpha_1 + \alpha_2)}\right]$$

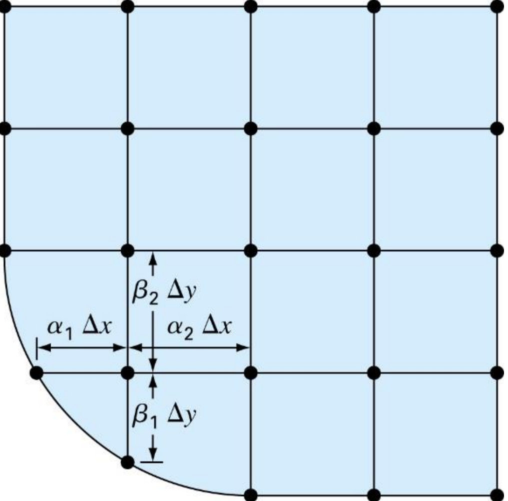
A similar equation can be developed in the y dimension:

$$\frac{\partial^2 T}{\partial y^2} = \frac{2}{\Delta y^2}\left[\frac{T_{i,j-1} - T_{i,j}}{\beta_1(\beta_1 + \beta_2)} + \frac{T_{i,j+1} - T_{i,j}}{\beta_2(\beta_1 + \beta_2)}\right]$$

So,

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

$$\frac{2}{\Delta x^2}\left[\frac{T_{i-1,j} - T_{i,j}}{\alpha_1(\alpha_1 + \alpha_2)} + \frac{T_{i+1,j} - T_{i,j}}{\alpha_2(\alpha_1 + \alpha_2)}\right] + \frac{2}{\Delta y^2}\left[\frac{T_{i,j-1} - T_{i,j}}{\beta_1(\beta_1 + \beta_2)} + \frac{T_{i,j+1} - T_{i,j}}{\beta_2(\beta_1 + \beta_2)}\right] = 0$$



Example

Repeat the same problem as in the previous example, but with the lower edge as depicted in the following picture.

For this case, $\Delta x = \Delta y$, $\alpha 1 = \beta 1 = 0.732$, and $\alpha 2 = \beta 2 = 1$.

At the node (1, 1):

$$0.788675(T_{01} - T_{11}) + 0.57735(T_{21} - T_{11}) + 0.788675(T_{10} - T_{11}) + 0.57735(T_{12} - T_{11}) = 0$$

Collecting terms, we can express this equation as

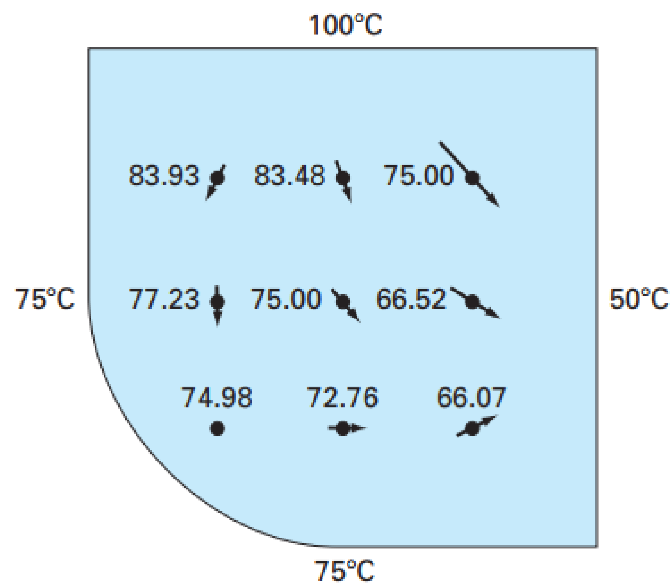
$$-4T_{11} + 0.8453T_{21} + 0.8453T_{12} = -1.1547T_{01} - 1.1547T_{10}$$

The simultaneous equations for temperature distribution on the plate with a lower-edge boundary temperature of 75 can be written in matrix form as

$$
\begin{bmatrix}
4 & -0.845 & & -0.845 & & & & & \\
-1 & 4 & -1 & & -1 & & & & \\
& -1 & 4 & & & -1 & & & \\
-1 & & & 4 & -1 & & -1 & & \\
& -1 & & -1 & 4 & -1 & & -1 & \\
& & -1 & & -1 & 4 & & & -1 \\
& & & -1 & & & 4 & -1 & \\
& & & & -1 & & -1 & 4 & -1 \\
& & & & & -1 & & -1 & -4
\end{bmatrix}
\begin{Bmatrix} T_{11} \\ T_{21} \\ T_{31} \\ T_{12} \\ T_{22} \\ T_{32} \\ T_{13} \\ T_{23} \\ T_{33} \end{Bmatrix}
=
\begin{Bmatrix} 173.2 \\ 75 \\ 125 \\ 75 \\ 0 \\ 50 \\ 175 \\ 100 \\ 150 \end{Bmatrix}
$$

These equations can be solved for

$$
\begin{aligned}
T_{11} &= 74.98 \qquad & T_{21} &= 72.76 \qquad & T_{31} &= 66.07 \\
T_{12} &= 74.23 \qquad & T_{22} &= 75.00 \qquad & T_{32} &= 66.52 \\
T_{13} &= 83.93 \qquad & T_{23} &= 83.48 \qquad & T_{33} &= 75.00
\end{aligned}
$$



PART B

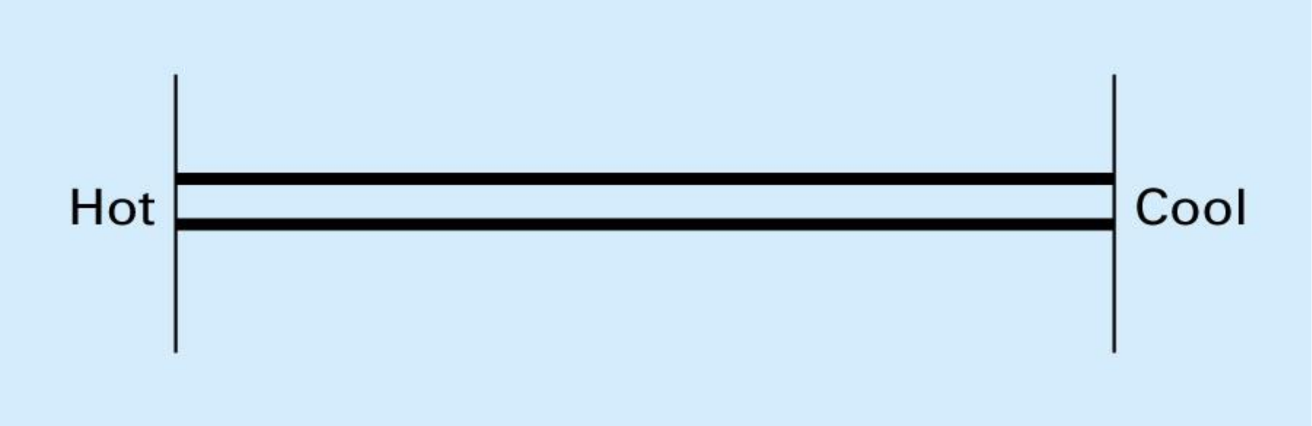
FINITE DIFFERENCE: PARABOLIC EQUATIONS

Parabolic equations

In contrast to the elliptic category, parabolic equations determine how an unknown varies in both space and time. Such cases are referred to as propagation problems because the solution "propagates," or changes, in time.

Parabolic equations are employed to characterize time-variable (*unsteady-state*) problems.

Conservation of energy can be used to develop an *unsteady-state* energy balance for the differential element in a long, thin insulated rod.



Parabolic equations

The present balance also considers the amount of heat stored in the element over a unit time period Δt. Thus, the balance is in the form,

inputs - outputs = storage,

$$q(x)\,\Delta y\,\Delta z\,\Delta t - q(x+\Delta x)\,\Delta y\,\Delta z\,\Delta t = \Delta x\,\Delta y\,\Delta z \rho C\,\Delta T$$

Dividing by the volume of the element ($= \Delta x\,\Delta y\,\Delta z$) and Δt gives

$$\frac{q(x) - q(x+\Delta x)}{\Delta x} = \rho C \frac{\Delta T}{\Delta t}$$

Taking the limit yields

$$-\frac{\partial q}{\partial x} = \rho C \frac{\partial T}{\partial t}$$

Substituting Fourier's law of heat conduction results in $\quad q_i = -k\rho C \dfrac{\partial T}{\partial i}$

$$k\frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t}$$

which is the heat-conduction equation.

$$k\frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t}$$

Explicit Methods

The heat conduction equation requires approximations for the second derivative in space and the first derivative in time:

$$\frac{\partial^2 T}{\partial x^2} = \frac{T_{i+1}^{l} - 2T_i^{l} + T_{i-1}^{l}}{\Delta x^2}$$

$$\frac{\partial T}{\partial t} = \frac{T_i^{l+1} - T_i^{l}}{\Delta t}$$

$$k\frac{T_{i+1}^{l} - 2T_i^{l} + T_{i-1}^{l}}{(\Delta x)^2} = \frac{T_i^{l+1} - T_i^{l}}{\Delta t}$$

$$T_i^{l+1} = T_i^{l} + \lambda\left(T_{i+1}^{l} - 2T_i^{l} + T_{i-1}^{l}\right) \qquad \lambda = \frac{k\Delta t}{(\Delta x)^2}$$

This equation can be written for all interior nodes on the rod.

It provides an explicit means to compute values at each node for a future time based on the present values at the node and its neighbors.

Example

Problem Statement. Use the explicit method to solve for the temperature distribution of a long, thin rod with a length of 10 cm and the following values: $k' = 0.49$ cal/(s · cm · °C), $\Delta x = 2$ cm, and $\Delta t = 0.1$ s. At $t = 0$, the temperature of the rod is zero and the boundary conditions are fixed for all times at $T(0) = 100$°C and $T(10) = 50$°C. Note that the rod is aluminum with $C = 0.2174$ cal/(g · °C) and $\rho = 2.7$ g/cm^3. Therefore, $k = 0.49/(2.7 \cdot 0.2174) = 0.835$ cm^2/s and $\lambda = 0.835(0.1)/(2)^2 = 0.020875$.

Solution:

At t=0.1 s for the node at x=2 cm

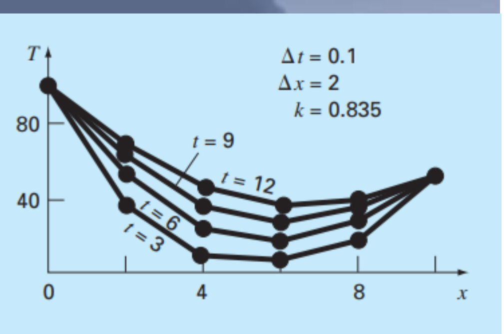
$$T_1^1 = 0 + 0.020875[0 - 2(0) + 100] = 2.0875$$

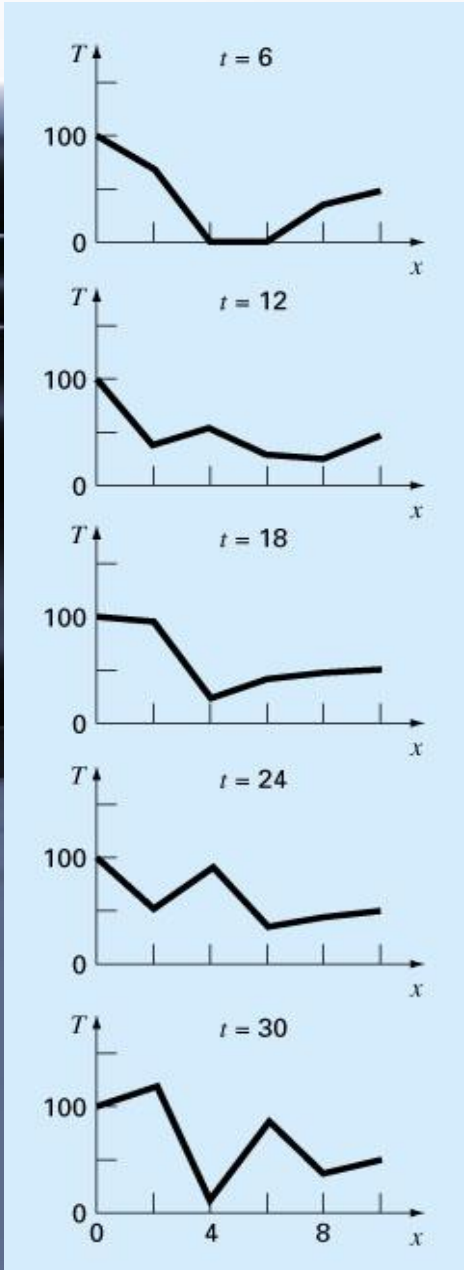
At the other interior points, $x = 4$, 6, and 8 cm, the results are

$$T_2^1 = 0 + 0.020875[0 - 2(0) + 0] = 0$$
$$T_3^1 = 0 + 0.020875[0 - 2(0) + 0] = 0$$
$$T_4^1 = 0 + 0.020875[50 - 2(0) + 0] = 1.0438$$

At $t = 0.2$ s, the values at the four interior nodes are computed as

$$T_1^2 = 2.0875 + 0.020875[0 - 2(2.0875) + 100] = 4.0878$$
$$T_2^2 = 0 + 0.020875[0 - 2(0) + 2.0875] = 0.043577$$
$$T_3^2 = 0 + 0.020875[1.0438 - 2(0) + 0] = 0.021788$$
$$T_4^2 = 1.0438 + 0.020875[50 - 2(1.0438) + 0] = 2.0439$$





Convergence and Stability

Convergence means that as Δx and Δt approach zero, the results of the finite difference method approach the true solution.

Stability means that errors at any stage of the computation are not amplified but are attenuated as the computation progresses.

The explicit method is both convergent and stable if

$$\lambda \leq 1/2$$

or

$$\Delta t \leq \frac{1}{2}\frac{\Delta x^2}{k}$$

Derivative Boundary Conditions

As was the case for elliptic PDEs, derivative boundary conditions can be readily incorporated into parabolic equations.

$$T_0^{l+1} = T_0^l + \lambda\left(T_1^l - 2T_0^l + T_{-1}^l\right)$$

Thus an imaginary point is introduced at *i= -1*, providing a vehicle for incorporating the derivative boundary condition into the analysis.

A simple Implicit Method

Implicit methods overcome difficulties associated with explicit methods at the expense of somewhat more complicated algorithms.

In implicit methods, the spatial derivative is approximated at an advanced time interval *l+1*:

$$\frac{\partial^2 T}{\partial x^2} \cong \frac{T_{i+1}^{l+1} - 2T_i^{l+1} + T_{i-1}^{l+1}}{(\Delta x)^2}$$

which is second-order accurate.

$$k\frac{T_{i+1}^{l+1}-2T_i^{l+1}+T_{i-1}^{l+1}}{(\Delta x)^2}=\frac{T_i^{l+1}-T_i^l}{\Delta t}$$

$$-\lambda T_{i-1}^{l+1}+(1+2\lambda)T_i^{l+1}-\lambda T_{i+1}^{l+1}=T_i^l$$

This eqn. applies to all but the first and the last interior nodes, which must be modified to reflect the boundary conditions:

$$T_0^{l+1}=f_0\left(t^{l+1}\right)$$

$$(1+2\lambda)T_1^{l+1}-\lambda T_2^{l+1}=T_1^l+f_0\left(t^{l+1}\right)$$

$$i=m$$

$$(1+2\lambda)T_m^{l+1}-\lambda T_{m-1}^{l+1}=T_m^l+f_{m+1}\left(t^{l+1}\right)$$

Resulting *m* unknowns and *m* linear algebraic equations

Example

Problem Statement. Use the simple implicit finite-difference approximation to solve the same problem as in the previous example.

Solution.

λ=0.020875

- At t=0,

$$1.04175T_1^1 - 0.020875T_2^1 = 0 + 0.020875(100)$$

or

$$1.04175T_1^1 - 0.020875T_2^1 = 2.0875$$

With,

$$-\lambda T_{i-1}^{l+1} + (1+2\lambda)T_i^{l+1} - \lambda T_{i+1}^{l+1} = T_i^l$$

This leads to the following set of simultaneous equations:

$$\begin{bmatrix} 1.04175 & -0.020875 & & \\ -0.020875 & 1.04175 & -0.020875 & \\ & -0.020875 & 1.04175 & -0.020875 \\ & & -0.020875 & 1.04175 \end{bmatrix} \begin{Bmatrix} T_1^1 \\ T_2^1 \\ T_3^1 \\ T_4^1 \end{Bmatrix} = \begin{Bmatrix} 2.0875 \\ 0 \\ 0 \\ 1.04375 \end{Bmatrix}$$

which can be solved for the temperature at $t = 0.1$ s:

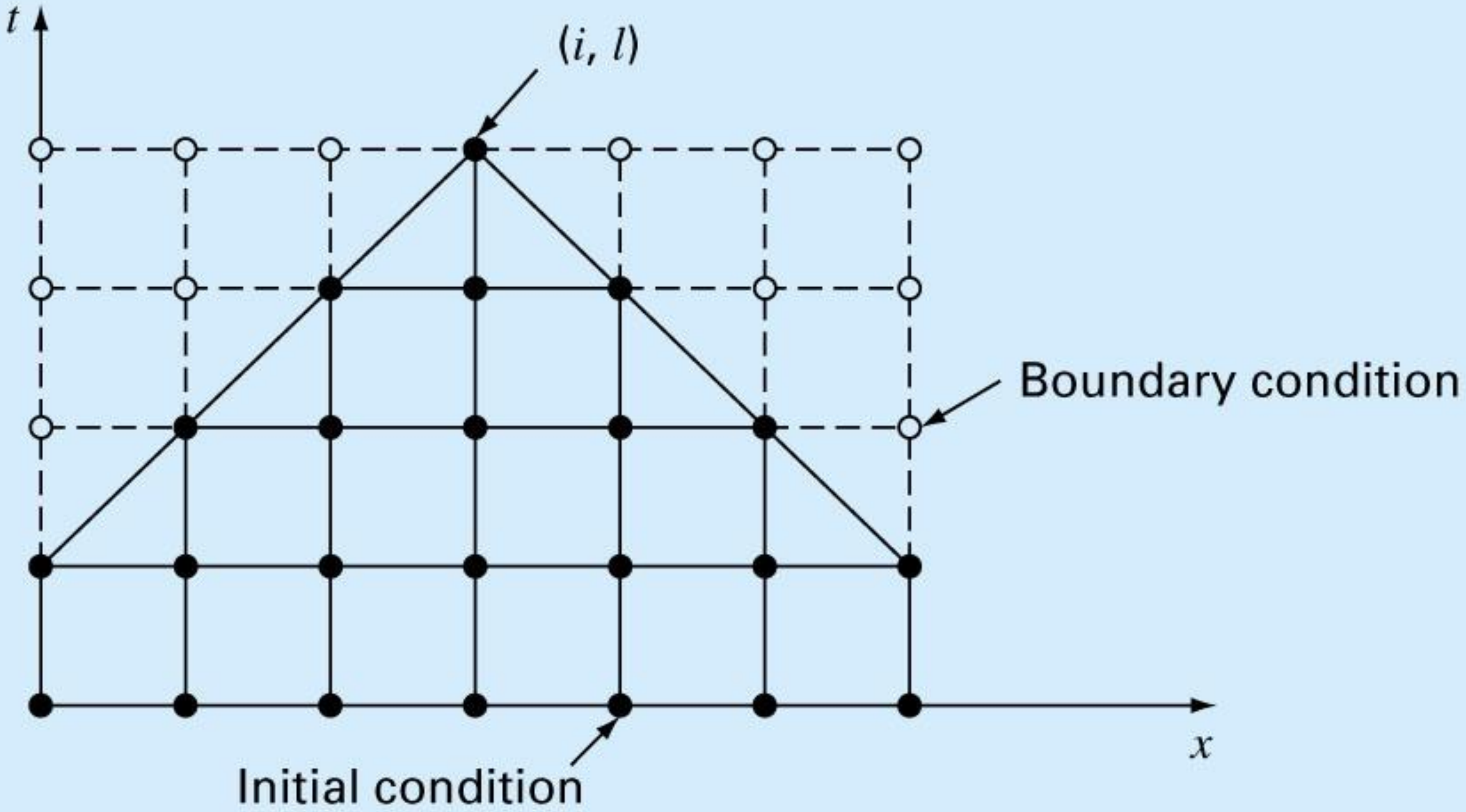
$$T_1^1 = 2.0047$$
$$T_2^1 = 0.0406$$
$$T_3^1 = 0.0209$$
$$T_4^1 = 1.0023$$

To solve for the temperatures at $t = 0.2$, the right-hand-side vector must be modified to account for the results of the first step, as in

$$\begin{Bmatrix} 4.09215 \\ 0.04059 \\ 0.02090 \\ 2.04069 \end{Bmatrix}$$

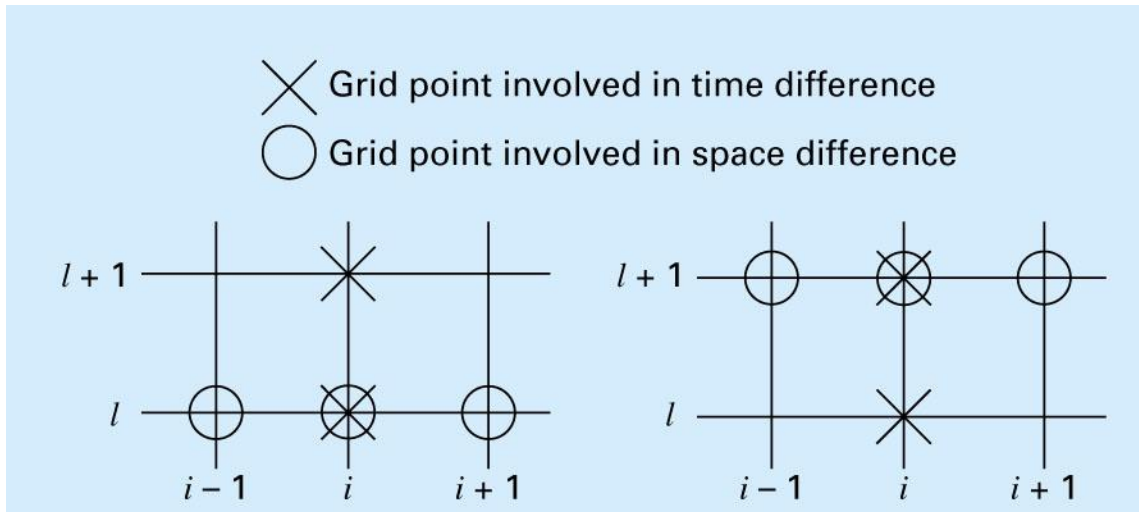
The simultaneous equations can then be solved for the temperatures at $t = 0.2$ s:

$$T_1^2 = 3.9305$$
$$T_2^2 = 0.1190$$
$$T_3^2 = 0.0618$$
$$T_4^2 = 1.9653$$



t
(i, l)
Boundary condition
x
Initial condition

Explicit Method vs Implicit Method



(a) Explicit (b) Implicit

The Crank-Nicolson Method

Provides an alternative implicit scheme that is second order accurate both in space and time.

To provide this accuracy, difference approximations are developed at the midpoint of the time increment:

$$\frac{\partial T}{\partial t} \cong \frac{T_i^{l+1} - T_i^l}{\Delta t}$$

$$\frac{\partial^2 T}{\partial x^2} \cong \frac{1}{2}\left[\frac{T_{i+1}^l - 2T_i^l + T_{i-1}^l}{(\Delta x)^2} + \frac{T_{i+1}^{l+1} - 2T_i^{l+1} + T_{i-1}^{l+1}}{(\Delta x)^2}\right]$$

$$-\lambda T_{i-1}^{l+1} + 2(1+\lambda)T_i^{l+1} - \lambda T_{i+1}^{l+1} = \lambda T_{i-1}^l + 2(1-\lambda)T_i^l + \lambda T_{i+1}^l$$

where $\lambda = k\,\Delta t/(\Delta x)^2$. As was the case with the simple implicit approach, boundary conditions of $T_0^{l+1} = f_0(t^{l+1})$ and $T_{m+1}^{l+1} = f_{m+1}(t^{l+1})$

$$2(1+\lambda)T_1^{l+1} - \lambda T_2^{l+1} = \lambda f_0(t^l) + 2(1-\lambda)T_1^l + \lambda T_2^l + \lambda f_0(t^{l+1})$$

and for the last interior node,

$$-\lambda T_{m-1}^{l+1} + 2(1+\lambda)T_m^{l+1} = \lambda f_{m+1}(t^l) + 2(1-\lambda)T_m^l + \lambda T_{m-1}^l + \lambda f_{m+1}(t^{l+1})$$

Parabolic Equations in Two Spatial Dimensions

For two dimensions the heat-conduction equation can be applied more than one spatial dimension:

$$\frac{\partial T}{\partial t} = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)$$

Standard Explicit and Implicit Schemes

An *explicit* solution can be obtained by substituting finite-differences approximations for the partial derivatives. However, this approach is limited by a stringent stability criterion, thus increases the required computational effort.

The direct application of implicit methods leads to solution of m x n simultaneous equations.

When written for two or three dimensions, these equations lose the valuable property of being tridiagonal and require very large matrix storage and computation time.

Example

Problem Statement. Use the Crank-Nicolson method to solve the same problem as in the previous examples.

$$\begin{bmatrix} 2.04175 & -0.020875 & & \\ -0.020875 & 2.04175 & -0.020875 & \\ & -0.020875 & 2.04175 & -0.020875 \\ & & -0.020875 & 2.04175 \end{bmatrix} \begin{Bmatrix} T_1^1 \\ T_2^1 \\ T_3^1 \\ T_4^1 \end{Bmatrix} = \begin{Bmatrix} 4.175 \\ 0 \\ 0 \\ 2.0875 \end{Bmatrix}$$

which can be solved for the temperatures at $t = 0.1$ s:

$$\begin{aligned} T_1^1 &= 2.0450 \\ T_2^1 &= 0.0210 \\ T_3^1 &= 0.0107 \\ T_4^1 &= 1.0225 \end{aligned}$$

To solve for the temperatures at $t = 0.2$ s, the right-hand-side vector must be changed to

$$\begin{Bmatrix} 8.1801 \\ 0.0841 \\ 0.0427 \\ 4.0901 \end{Bmatrix}$$

The simultaneous equations can then be solved for

$$\begin{aligned} T_1^2 &= 4.0073 \\ T_2^2 &= 0.0826 \\ T_3^2 &= 0.0422 \\ T_4^2 &= 2.0036 \end{aligned}$$

The ADI Scheme

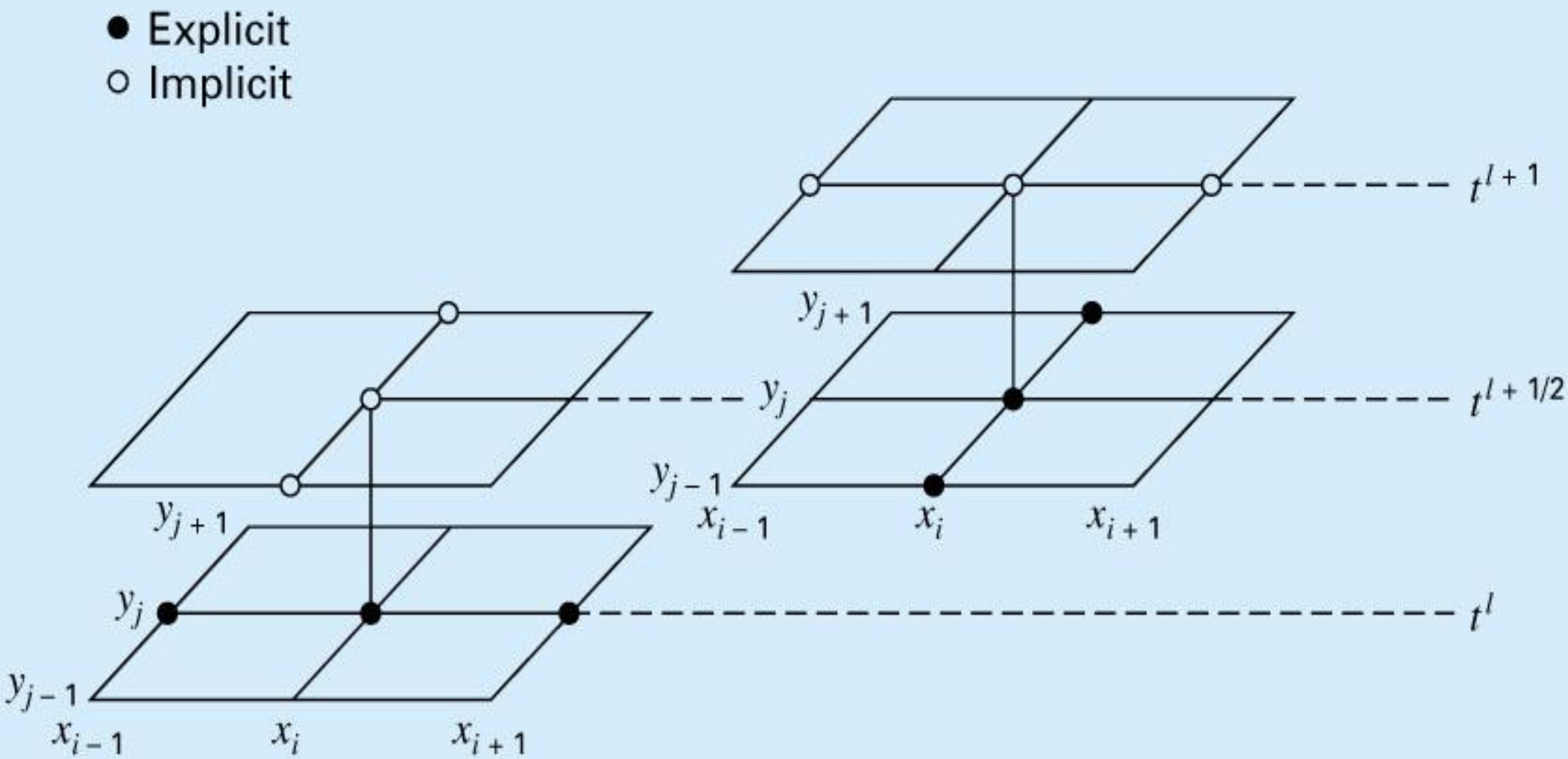
The alternating-direction implicit, or ADI, scheme provides a means for solving parabolic equations in two spatial dimensions using tridiagonal matrices.

Each time increment is executed in two steps.

For the first step, heat conduction equation is approximated by:

$$\frac{T_{i,j}^{l+1/2} - T_{i,j}^{l}}{\Delta t / 2} = k\left[\frac{T_{i+1,j}^{l} - 2T_{i,j}^{l} + T_{i-1,j}^{l}}{(\Delta x)^2} + \frac{T_{i,j+1}^{l+1/2} - 2T_{i,j}^{l+1/2} + T_{i,j-1}^{l+1/2}}{(\Delta y)^2}\right]$$

Thus the approximation of partial derivatives are written explicitly, that is at the base point t^l where temperatures are known. Consequently only the three temperature terms in each approximation are unknown.



(*a*) First half-step

(*b*) Second half-step

Any Questions?

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