Solutions to Midterm
14.12 Fall 2023

November 13, 2023

Problem 1

1. Note that $BR_M(s)$ is not the optimal solution given R's action s, but rather the **set** of all possible actions M can take as best responses. (The argument for $BR_R(w)$ is symmetric, so we just need to analyze the actions of player M.)

   First, we take the first order condition w.r.t. w of the utility function $\pi_M(w,s) = w(48 - 2(w+s))$ and get
   $$w^* = BR_M(s) = \frac{24-s}{2}.$$
   For $BR_M(s)$, player M just wants to find all the possible prices w it could set so that for some plausible s, $w = \frac{24-s}{2}$.

   The best response functions are
   $$BR_M(s) = \max\{\frac{24-s}{2}, 0\}, BR_M(w) = \max\{\frac{24-s}{w}, 0\}.$$

2. Note that the set of best responses at the beginning of the game is $BR_M(s) = [0,24], BR_R(w) = [0,24]$ (any value outside of this range would either set profit below 0 or is ifeasible.) Knowing that $s \in [0,24]$, we have
   $$\frac{24-12}{2} \le BR_M^1 \le \frac{24-0}{2},$$
   i.e. $\boxed{BR_M^1 = BR_R^1 = [0,12]}$.

   Using similar logic, $\boxed{BR_M^2 = BR_R^2 = [6,12]}$.

3. We calculate $BR_M^4 = BR_R^4 = [7.5, 9]$, so $\boxed{n=4}$ is the lowest n where 7 is not included in BR_R^n. (we gave full credit to $n=3$ as well if the derivation for iterated elimination was correct.)

4. We infer from part (1) that

$$w^* = BR_M(s) = \frac{24-s}{2}; s^* = BR_R(w) = \frac{24-w}{2}.$$

Plug $s = \frac{24-w}{2}$ into the expression for w^*, we get

$$\begin{aligned} w &= \frac{24 - \frac{24-w}{2}}{2} \\ w &= 6 - \frac{w}{4} \\ w &= 8 \end{aligned}$$

The calculation for s^* follows identically. The unique Nash equilibrium is $\boxed{(w^*, s^*) = (8, 8)}$.

5. The joint profit wrt p is

$$\pi(p) = p(48-2p);$$

taking the FOC wrt p gives

$$\pi'(p) = 48 - 4p = 0 \Rightarrow \boxed{p^I = 12}.$$

This is smaller than the total price $w + s = \boxed{16}$ that consumers face in part 4.

Problem 2

1. Having exactly one person calling the police is a Nash equilibrium because 1) the person who calls the police does not want to deviate to not calling; otherwise, her utility decreases from $v - c > 0$ to 0 and 2) people who do not call the police do not deviate to calling; otherwise, her utility decreases from v to $v - c$. Since there are N possible ways to select a person who calls the police, we have found N pure strategy Nash equilibria.

2. The probability that no witnesses 2 to N call is p^{N-1}. If witness 1 calls, her utility is $v - c$. If witness 1 does not call, her expected utility is $(1 - p^{N-1})v$. Thus, witness 1 is indifferent between call or not if and only if

$$v - c = (1 - p^{N-1})v.$$

3. Solving the above equation gives $p = \left(\frac{c}{v}\right)^{1/(N-1)}$. This means the probability that no one calls is $p^N = \left(\frac{c}{v}\right)^{N/(N-1)}$ which converges to $\frac{c}{v}$ as N goes to ∞. Thus, the probability that no one calls is not roughly zero when N is very large.

Problem 3

1. We apply backwards induction. Payoffs are denoted (buyer, seller).

   In the final period, the seller receives payoff 0 from rejecting the offer, and δp_1 from accepting the offer. So, the seller will accept any $p_1 \geq 0 \implies$ the buyer will offer $p_1 = 0$, giving payoffs of $(\delta, 0)$.

   Moving to the earlier period, the seller will accept any p_0 where $1 - p_0 \geq \delta$. So, the seller will offer $p_0 = 1 - \delta$, and the buyer will accept. (The seller has no incentive to under-offer, and force the game to move to period 1, where the seller would get a payoff of 0).

   So, the good will be sold in period 0, at price $p_0 = 1 - \delta$. Payoffs are $(\delta, 1 - \delta)$.

   Equilibrium strategies are:

   - Buyer: in period 0, accept if $p_0 \leq 1 - \delta$ and reject otherwise; in period 1, offer $p_1 = 0$.
   - Seller: in period 0, offer $p_0 = 1 - \delta$; in period 1, accept if $p_1 \geq 0$ and reject otherwise;

2. We again apply backwards induction.

   In period 2, if the seller rejects the offer, then both players get a payoff of $(0, 0)$. If the seller accepts the offer, the payoffs are $(\delta^2(1 - p_2), \delta^2 p_2)$. So, the seller will accept any $p_2 \geq 0$, and the buyer will offer $p_2 = 0$.

   Next, in period 1, if the buyer rejects the offer, their payoff will be δ^2. So, the buyer will accept any p_1 with $\delta(1 - p_1) \geq \delta^2 \implies$ the seller will offer $p_1 = 1 - \delta$, leaving payoffs $(\delta^2, \delta(1 - \delta))$.

   Finally, in period 0, the buyer will accept any p_0 with $1 - p_0 \geq \delta^2$. So, the seller will offer $p_0 = 1 - \delta^2$.

   Trade will happen in period 0, at price $p_0 = 1 - \delta^2$. Payoffs are $(\delta^2, 1 - \delta^2)$.

   Equilibrium strategies are:

   - Buyer: in period 0, accept if $p_0 \leq 1 - \delta^2$ and reject otherwise; in period 1, accept if $p_1 \leq 1 - \delta$ and reject otherwise; in period 2, offer $p_2 = 0$.
   - Seller: in period 0, offer $p_0 = 1 - \delta^2$; in period 1, offer $p_1 = 1 - \delta$; in period 2, accept if $p_2 \geq 0$ and reject otherwise;

   For both of these questions, note that if you were asked to write out the equilibrium strategies, you should then specify the receiver's action as a function of their offer.