Solutions to Problem Set 2
14.12 Fall 2023

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Problem 1

Part 1

One example is the following game:

	L	R
T	3, 1	2, 6
B	1, 5	2, 2

Notes:

- 2 strategies with the exact same payoffs for player X does not mean that they "weakly dominate each other".

Part 2

Let β_2 put strictly positive probability weights on all $s_2^i \in S_2$. We can write $\beta_2 = (\beta_2^1, \beta_2^2, ..., \beta_2^n)$ where all $\beta_2^i > 0$, and sum to 1. For s_1' to be a best response to β_2, the following inequality must hold:

$$u_1(s_1', \beta_2) \geq u_1(s_1^*, \beta_2) \forall s_1^*$$

Equivalently, the following must be true:

$$\sum_{s_2^i \in S_2} \beta_2^i u_1(s_1', s_2^i) \geq \sum_{s_2^i \in S_2} \beta_2^i u_1(s_1^*, s_2^i) \forall s_1^* \tag{1}$$

Additionally, we have that s_1' is weakly dominated. So, for some σ_1,

$$u_1(\sigma_1, s_2) \geq u_1(s_1', s_2) \forall s_2;$$

and, for some s_2, this inequality is strict. Summing over all $s_2^i \in S_2$, for arbitrary probability weights $\alpha = (\alpha^1, \alpha^2, ...)$ where $\alpha^i = \beta_2^i > 0$ as defined above, we have:

$$\sum_{s_2^i \in S_2} \alpha^i u_1(\sigma_1, s_2^i) > \sum_{s_2^i \in S_2} \alpha^i u_1(s_1', s_2^i)$$

But this contradicts equation (1), so s_1' cannot be a best response to any belief β_2 that puts positive weight on each of player 2's strategies.

Problem 2

We start by calculating the utility of player i

$$u_i(e_i, e_j) = \alpha \log(e_i) - \beta \log(e_j) - e_i \tag{2}$$

Note that u_i is strictly concave and $\lim_{e_i \to 0} u_i(e_i, e_j) = -\infty$ and $\lim_{e_i \to \infty} u_i(e_i, e_j) = -\infty$. Thus, we can characterize the unique optimum (which is a strict best response) by taking the FOC:

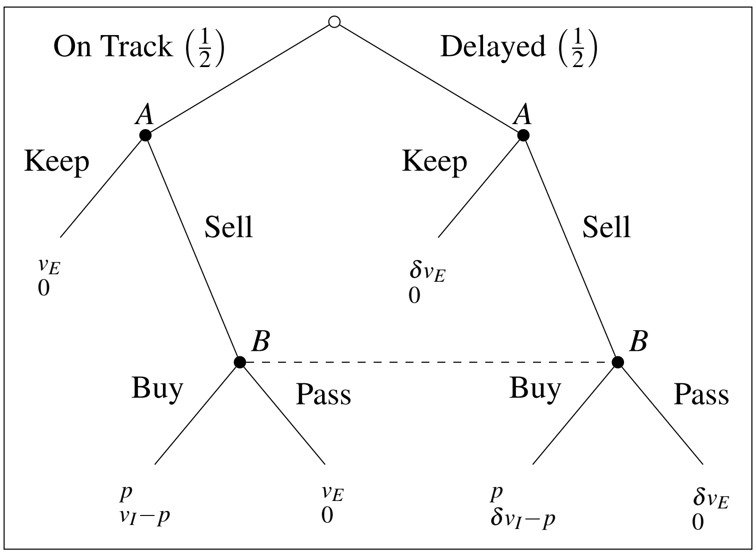
$$\frac{\partial u_i}{\partial e_i} = \alpha \frac{1}{e_i} - 1 = 0 \iff e_i^* = \alpha \tag{3}$$

Replicating argument above for player j, we find that (α, α) is the dominant strategy equilibrium of this game. We got a dominant strategy equilibrium in this game since the the best response of a player does not depend on the choice of the other player and is a constant function.

Problem 3

Part 1

There are two players, Executive and Investor, so the set of players is $\boxed{N = \{Executive, Investor\}}$. A tree, which contains all other relevant information, that captures this extensive-form game is given below. At each terminal node of the tree, the first payoff is that of Executive while the second is that of Investor.



Part 2

As above, there are two players, Executive and Investor, so the set of players is $\boxed{N = \{Executive, Investor\}}$.

Executive has four pure strategies: Sell-Sell represents Executive selling the stock when production is both On Track and Delayed, Sell-Keep represents Executive selling the stock when production is On Track but keeping the stock when production is Delayed, Keep-Sell represents Executive keeping the stock when production is On Track but selling the stock when production is Delayed, and Keep-Keep represents Executive keeping the stock when production is both On Track and Delayed. So the set of strategies for Executive is $\boxed{S_E = \{\text{Sell-Sell}, \text{Sell-Keep}, \text{Keep-Sell}, \text{Keep-Keep}\}}$.

Investor has two pure strategies: Buy represents Investor buying the stock when Executive sells it, and Pass represents Investor passing on the stock when Executive sells it. So the set of strategies for Investor is $\boxed{S_I = \{\text{Buy}, \text{Pass}\}}$.

We now determine the utilities of the players from the various strategy pairs. We first compute each player's expected payoff when Executive uses the Sell-Sell strategy and Investor uses the Buy strategy. In this case, Executive sells and Investor buys the stock both when production is On Track and Delayed. Thus, when production is On Track, Executive's payoff is p (the price Executive receives) while Investor's payoff is $v_I - p$ (Investor's value from the stock minus the price Investor pays). On the other hand, when production is Delayed, Executive's payoff is p (the price Executive receives) while Investor's payoff is $\delta v_I - p$ (Investor's value from the stock minus the price Investor pays). This means that, when Executive uses the Sell-Sell strategy and Investor uses the Buy strategy, Executive's expected payoff is $u_E(\text{sell-Sell}, \text{Buy}) = (1/2)p + (1/2)p = p$ while Investor's expected payoff is $u_I(\text{Sell-Sell}, \text{Buy}) = (1/2)(v_I - p) + (1/2)(\delta v_I - p) = (1+\delta)v_I/2 - p$.

We now compute each player's expected payoff when Executive uses the Sell-Keep strategy and Investor uses the Buy strategy. In this case, Executive sells and Investor buys the stock when production is On Track but Executive keeps the stock (and consequently Investor does not get the stock) when production is Delayed. Thus, when production is On Track, Executive's payoff is p (the price Executive receives) while Investor's payoff is $v_I - p$ (Investor's value from the stock minus the price Investor pays). On the other hand, when production is Delayed, Executive's payoff is δv_E (Executive's value from the stock) while Investor's payoff is 0 (Investor's default value). This means that, when Executive uses the Sell-Keep strategy and Investor uses the Buy strategy, Executive's expected payoff is $u_E(\text{sell-Keep}, \text{Buy}) = (1/2)p + (1/2)\delta v_E = (p + \delta v_E)/2$ while Investor's expected payoff is $u_I(\text{Sell-Keep}, \text{Buy}) = (1/2)(v_I - p) + (1/2)0 = (v_I - p)/2$.

We can perform similar computations for all the possible pure strategy pairs. The results of such computations are summarized in the game matrix below, where the row player represents Executive and the column player represents Investor.

	Buy	Pass
Sell-Sell	$p, \frac{1+\delta}{2}v_I - p$	$\frac{1+\delta}{2}v_E, 0$
Sell-Keep	$\frac{p+\delta v_E}{2}, \frac{v_I - p}{2}$	$\frac{1+\delta}{2}v_E, 0$
Keep-Sell	$\frac{v_E+p}{2}, \frac{\delta v_I - p}{2}$	$\frac{1+\delta}{2}v_E, 0$
Keep-Keep	$\frac{1+\delta}{2}v_E, 0$	$\frac{1+\delta}{2}v_E, 0$

Part 3

Neither player has a strictly dominant strategy. This is true for Executive because when Investor plays Pass, Executive's utility is the same for all Executive's strategies. Likewise, when Executive plays Keep-Keep, Investor's utility is the same for all Investor's strategies.

However, $\boxed{\text{Keep-Sell is a weakly dominant strategy for Executive}}$. This is because when Investor plays Buy, Keep-Sell gives Executive a strictly higher payoff than any of Executive's other strategies. Formally,

$$\begin{aligned}
&u_E(\text{Keep-Sell}, \text{Buy}) > u_E(\text{Sell-Sell}, \text{Buy}) \Leftrightarrow \frac{v_E + p}{2} > p \Leftrightarrow v_E > p, \\
&u_E(\text{Keep-Sell}, \text{Buy}) > u_E(\text{Sell-Keep}, \text{Buy}) \Leftrightarrow \frac{v_E + p}{2} > \frac{p + \delta v_E}{2} \Leftrightarrow v_E > p, \\
&u_E(\text{Keep-Sell}, \text{Buy}) > u_E(\text{Keep-Keep}, \text{Buy}) \Leftrightarrow \frac{v_E + p}{2} > \frac{1+\delta}{2} v_E \Leftrightarrow p > \delta v_E.
\end{aligned}$$

Moreover, when Investor plays Pass, Keep-Sell gives Executive the same payoff as (and so a weakly higher payoff than) any of Executive's other strategies.

In contrast to Executive, Investor also does not have a weakly dominant strategy. When Executive plays Sell-Keep, it is strictly better for Investor to play Buy rather than Pass since $(v_I - p)/2 > 0$. On the other hand, when Executive plays Keep-Sell, it is strictly worse for Investor to play Buy rather than Pass since $(\delta v_I - p)/2 < 0$.

Part 4

$\boxed{\text{All strategies are rationalizable}}$. We can express this notationally by writing

$\boxed{S_E^\infty = \{\text{Sell-Sell}, \text{Sell-Keep}, \text{Keep-Sell}, \text{Keep-Keep}\} \text{ and } S_I^\infty = \{\text{Buy}, \text{Pass}\}}$, or

$\boxed{S^\infty = \{\text{Sell-Sell}, \text{Sell-Keep}, \text{Keep-Sell}, \text{Keep-Keep}\} \times \{\text{Buy}, \text{Pass}\}}$.

To see why, note that all of Executive's strategies give Executive the same payoff of $(1+\delta)v_E/2$ (and are thus best responses) when Investor plays Pass. Similarly, all of Investor's strategies give Investor the same payoff of 0 (and are thus best responses) when Executive plays Keep-Keep. Thus, all the strategies of both players survive the iterated elimination of strictly dominated strategies.

Problem 4

The set of rationalizable strategy profiles is $\boxed{S^\infty = \{(2\theta_1/3+\theta_2/3, \theta_1/3+2\theta_2/3)\}}$, or, equivalently, the set of rationalizable strategies for Player 1 is $\boxed{S_1^\infty = \{2\theta_1/3+\theta_2/3\}}$ and the set of rationalizable strategies for Player 2 is $\boxed{S_2^\infty = \{\theta_1/3+2\theta_2/3\}}$.

Note that the utility functions of Players 1 and 2 can be respectively rewritten as

$$\begin{aligned} u_1(x_1,x_2) &= -\left(x_1 - \frac{\theta_1+x_2}{2}\right)^2 - \frac{1}{4}(\theta_1-x_2)^2, \\ u_2(x_1,x_2) &= -\left(x_2 - \frac{\theta_2+x_1}{2}\right)^2 - \frac{1}{4}(\theta_2-x_1)^2. \end{aligned} \tag{4}$$

We will use these expressions throughout our analysis.

As usual with the iterated elimination of strictly dominated strategies, initialize the sets S_1^0 and S_2^0 with the strategy spaces for the respective players, i.e. $S_1^0 = [0,100]$ and $S_2^0 = [0,100]$. Note that $(\theta_1+x_2)/2 \le \theta_1/2+50$ for all $x_2 \in [0,100]$. As can be seen from the expression for u_1 in (4), this means that $s_1 = \theta_1/2+50$ strictly dominates every strictly higher strategy $\tilde{s}_1 > \theta_1/2+50$. Moreover, $(\theta_1+x_2)/2 \ge \theta_1/2$ for all $x_2 \in [0,100]$, which means that $s_1 = \theta_1/2$ strictly dominates every strictly lower strategy $\tilde{s}_1 < \theta_1/2$. Since every strategy $s_1 \in [\theta_1/2, \theta_1/2+50]$ is a best response to the opponent strategy $s_2 = 2s_1 - \theta_1 \in [0,100]$, we conclude that the set of strategies that are not strictly dominated for Player 1 is $S_1^1 = [\theta_1/2, \theta_1/2+50]$. Identical arguments but with the player roles reversed show that the set of strategies that are not strictly dominated for Player 2 is $S_2^1 = [\theta_2/2, \theta_2/2+50]$.

Now, we compute the set of strategies that are not conditionally strictly dominated given that the opponent does not play a strictly dominated strategy. Note that $(\theta_1+x_2)/2 \le \theta_1/2 + \theta_2/2+25$ for all $x_2 \in [\theta_2/2, \theta_2/2+50] = S_2^1$. This means that $s_1 = \theta_1/2+\theta_2/2+25$ conditionally strictly dominates every strictly higher strategy $\tilde{s}_1 > \theta_1/2+\theta_2/2+25$. Moreover, $(\theta_1+x_2)/2 \ge \theta_1/2+\theta_2/2$ for all $x_2 \in [\theta_2/2, \theta_2/2+50] = S_2^1$, which means that $s_1 = \theta_1/2+\theta_2/2$ conditionally strictly dominates every strictly lower strategy $\tilde{s}_1 < \theta_1/2+\theta_2/2$. Since every strategy $s_1 \in [\theta_1/2+\theta_2/2, \theta_1/2+\theta_2/2+25]$ is a best response to the opponent strategy $s_2 = 2s_1 - \theta_1 \in [\theta_2/2, \theta_2/2+50] = S_2^1$, we conclude that the set of strategies that are not conditionally strictly dominated for Player 1 (given that Player 2 does not play strategies that are strictly dominated) is

$S_1^2 = [\theta_1/2 + \theta_2/2, \theta_1/2 + \theta_2/2 + 25]$. Identical arguments but with the player roles reversed show that the set of strategies that are not conditionally strictly dominated for Player 2 (given that Player 1 does not play strategies that are strictly dominated) is $S_2^1 = [\theta_2/2 + \theta_1/2, \theta_2/2 + \theta_1/2 + 25]$.

Moreover, inductively applying these same sorts of arguments shows that the set of strategies surviving the k-th round of elimination are $S_1^k = [\underline{s}_1^k, \overline{s}_1^k]$ (for Player 1) and $S_2^k = [\underline{s}_2^k, \overline{s}_2^k]$ (for Player 2), where the $\underline{s}_1^k$ and $\underline{s}_2^k$ are given by the initial conditions $\underline{s}_1^0 = \underline{s}_2^0 = 0$ and the recursive relation

$$\begin{aligned} \underline{s}_1^{k+1} &= \frac{1}{2}\theta_1 + \frac{1}{2}\underline{s}_2^k, \\ \underline{s}_2^{k+1} &= \frac{1}{2}\theta_2 + \frac{1}{2}\underline{s}_1^k, \end{aligned} \tag{5}$$

and the $\overline{s}_1^k$ and $\overline{s}_2^k$ are given by the initial conditions $\overline{s}_1^0 = \overline{s}_2^0 = 100$ and the recursive relation

$$\begin{aligned} \overline{s}_1^{k+1} &= \frac{1}{2}\theta_1 + \frac{1}{2}\overline{s}_2^k, \\ \overline{s}_2^{k+1} &= \frac{1}{2}\theta_2 + \frac{1}{2}\overline{s}_1^k. \end{aligned} \tag{6}$$

We can establish by induction that both $\underline{s}_1^k$ and $\underline{s}_2^k$ are weakly increasing sequences that are bounded above by 100. Thus, $\underline{s}_1^\infty \equiv \lim_{k\to\infty} \underline{s}_1^k$ and $\underline{s}_2^\infty \equiv \lim_{k\to\infty} \underline{s}_2^k$ both exist, and, by (5), must satisfy

$$\begin{aligned} \underline{s}_1^\infty &= \frac{1}{2}\theta_1 + \frac{1}{2}\underline{s}_2^\infty, \\ \underline{s}_2^\infty &= \frac{1}{2}\theta_2 + \frac{1}{2}\underline{s}_1^\infty. \end{aligned}$$

Solving these equalities gives $\underline{s}_1^\infty = 2\theta_1/3 + \theta_2/3$ and $\underline{s}_2^\infty = \theta_1/3 + 2\theta_2/3$. We can similarly establish by induction that both $\overline{s}_1^k$ and $\overline{s}_2^k$ are weakly decreasing sequences that are bounded below by 0. Thus, $\overline{s}_1^\infty \equiv \lim_{k\to\infty} \overline{s}_1^k$ and $\overline{s}_2^\infty \equiv \lim_{k\to\infty} \overline{s}_2^k$ both exist, and, by (6), must satisfy

$$\begin{aligned} \overline{s}_1^\infty &= \frac{1}{2}\theta_1 + \frac{1}{2}\overline{s}_2^\infty, \\ \overline{s}_2^\infty &= \frac{1}{2}\theta_2 + \frac{1}{2}\overline{s}_1^\infty. \end{aligned}$$

This is the same system of equations as above, just with $\underline{s}_1^\infty$ replaced with $\overline{s}_1^\infty$ and $\underline{s}_2^\infty$ replaced with $\overline{s}_2^\infty$. So $\overline{s}_1^\infty = 2\theta_1/3 + \theta_2/3$ and $\overline{s}_2^\infty = \theta_1/3 + 2\theta_2/3$. As $\underline{s}_1^\infty = \overline{s}_1^\infty = 2\theta_1/3 + \theta_2/3$ and $\underline{s}_2^\infty = \overline{s}_2^\infty = \theta_1/3 + 2\theta_2/3$, we can conclude that $S_1^\infty = \{2\theta_1/3 + \theta_2/3\}$ and $S_2^\infty = \{\theta_1/3 + 2\theta_2/3\}$.

An alternative approach to obtain this result would be to explicitly solve for the values of the various $\underline{s}_1^k$, $\underline{s}_2^k$, $\bar{s}_1^k$, and $\bar{s}_2^k$, and then take the $k \to \infty$ limit of these sequences. Proceeding in this way, it can be shown that, for all $n \in \mathbb{N}$,

$$\begin{aligned}
\underline{s}_1^{2n} &= \frac{2^{2n}-1}{3 \cdot 2^{2n-1}}\theta_1 + \frac{2^{2n}-1}{3 \cdot 2^{2n}}\theta_2, \\
\underline{s}_2^{2n} &= \frac{2^{2n}-1}{3 \cdot 2^{2n}}\theta_1 + \frac{2^{2n}-1}{3 \cdot 2^{2n-1}}\theta_2, \\
\bar{s}_1^{2n} &= \frac{2^{2n}-1}{3 \cdot 2^{2n-1}}\theta_1 + \frac{2^{2n}-1}{3 \cdot 2^{2n}}\theta_2 + \frac{1}{2^{2n}}100, \\
\bar{s}_1^{2n} &= \frac{2^{2n}-1}{3 \cdot 2^{2n}}\theta_1 + \frac{2^{2n}-1}{3 \cdot 2^{2n-1}}\theta_2 + \frac{1}{2^{2n}}100,
\end{aligned} \tag{7}$$

so that the set of strategies surviving the $2n$-th round of elimination are

$$\begin{aligned}
S_1^{2n} &= \left[\frac{2^{2n}-1}{3 \cdot 2^{2n-1}}\theta_1 + \frac{2^{2n}-1}{3 \cdot 2^{2n}}\theta_2, \frac{2^{2n}-1}{3 \cdot 2^{2n-1}}\theta_1 + \frac{2^{2n}-1}{3 \cdot 2^{2n}}\theta_2 + \frac{1}{2^{2n}}100\right] \text{ (for Player 1)}, \\
S_2^{2n} &= \left[\frac{2^{2n}-1}{3 \cdot 2^{2n}}\theta_1 + \frac{2^{2n}-1}{3 \cdot 2^{2n-1}}\theta_2, \frac{2^{2n}-1}{3 \cdot 2^{2n}}\theta_1 + \frac{2^{2n}-1}{3 \cdot 2^{2n-1}}\theta_2 + \frac{1}{2^{2n}}100\right] \text{ (for Player 2)},
\end{aligned}$$

and

$$\begin{aligned}
\underline{s}_1^{2n+1} &= \frac{2^{2n+2}-1}{3 \cdot 2^{2n+1}}\theta_1 + \frac{2^{2n}-1}{3 \cdot 2^{2n}}\theta_2, \\
\underline{s}_2^{2n+1} &= \frac{2^{2n}-1}{3 \cdot 2^{2n}}\theta_1 + \frac{2^{2n+2}-1}{3 \cdot 2^{2n+1}}\theta_2, \\
\bar{s}_1^{2n+1} &= \frac{2^{2n+2}-1}{3 \cdot 2^{2n+1}}\theta_1 + \frac{2^{2n}-1}{3 \cdot 2^{2n}}\theta_2 + \frac{1}{2^{2n+1}}100, \\
\bar{s}_1^{2n+1} &= \frac{2^{2n}-1}{3 \cdot 2^{2n}}\theta_1 + \frac{2^{2n+2}-1}{3 \cdot 2^{2n+1}}\theta_2 + \frac{1}{2^{2n+1}}100,
\end{aligned} \tag{8}$$

so that the set of strategies surviving the $2n+1$-th round of elimination are

$$\begin{aligned}
S_1^{2n+1} &= \left[\frac{2^{2n+2}-1}{3 \cdot 2^{2n+1}}\theta_1 + \frac{2^{2n}-1}{3 \cdot 2^{2n}}\theta_2, \frac{2^{2n+2}-1}{3 \cdot 2^{2n+1}}\theta_1 + \frac{2^{2n}-1}{3 \cdot 2^{2n}}\theta_2 + \frac{1}{2^{2n+1}}100\right] \text{ (for Player 1)}, \\
S_2^{2n+1} &= \left[\frac{2^{2n}-1}{3 \cdot 2^{2n}}\theta_1 + \frac{2^{2n+2}-1}{3 \cdot 2^{2n+1}}\theta_2, \frac{2^{2n}-1}{3 \cdot 2^{2n}}\theta_1 + \frac{2^{2n+2}-1}{3 \cdot 2^{2n+1}}\theta_2 + \frac{1}{2^{2n+1}}100\right] \text{ (for Player 2)}.
\end{aligned}$$

Since

$$\lim_{n\to\infty} \frac{2^{2n}-1}{3\cdot 2^{2n-1}}\theta_1 + \frac{2^{2n}-1}{3\cdot 2^{2n}}\theta_2 = \lim_{n\to\infty} \frac{2^{2n}-1}{3\cdot 2^{2n-1}}\theta_1 + \frac{2^{2n}-1}{3\cdot 2^{2n}}\theta_2 + \frac{1}{2^{2n}}100 = \frac{2}{3}\theta_1 + \frac{\theta_2}{3},$$
$$\lim_{n\to\infty} \frac{2^{2n}-1}{3\cdot 2^{2n}}\theta_1 + \frac{2^{2n}-1}{3\cdot 2^{2n-1}}\theta_2 = \lim_{n\to\infty} \frac{2^{2n}-1}{3\cdot 2^{2n}}\theta_1 + \frac{2^{2n}-1}{3\cdot 2^{2n-1}}\theta_2 + \frac{1}{2^{2n}}100 = \frac{1}{3}\theta_1 + \frac{2}{3}\theta_2,$$

it follows from (7) that $\lim_{n\to\infty} \underline{s}_1^{2n} = \lim_{n\to\infty} \overline{s}_1^{2n} = 2\theta_1/3 + \theta_2/3$ and $\lim_{n\to\infty} \underline{s}_2^{2n} = \lim_{n\to\infty} \overline{s}_2^{2n} = \theta_1/3 + 2\theta_2/3$. Likewise,

$$\lim_{n\to\infty} \frac{2^{2n+2}-1}{3\cdot 2^{2n+1}}\theta_1 + \frac{2^{2n}-1}{3\cdot 2^{2n}}\theta_2 = \lim_{n\to\infty} \frac{2^{2n+2}-1}{3\cdot 2^{2n+1}}\theta_1 + \frac{2^{2n}-1}{3\cdot 2^{2n}}\theta_2 + \frac{1}{2^{2n+1}}100 = \frac{2}{3}\theta_1 + \frac{1}{3}\theta_2,$$
$$\lim_{n\to\infty} \frac{2^{2n}-1}{3\cdot 2^{2n}}\theta_1 + \frac{2^{2n+2}-1}{3\cdot 2^{2n+1}}\theta_2 = \lim_{n\to\infty} \frac{2^{2n}-1}{3\cdot 2^{2n}}\theta_1 + \frac{2^{2n+2}-1}{3\cdot 2^{2n+1}}\theta_2 + \frac{1}{2^{2n+1}}100 = \frac{1}{3}\theta_1 + \frac{2}{3}\theta_2,$$

together with (8), gives $\lim_{n\to\infty} \underline{s}_1^{2n+1} = \lim_{n\to\infty} \overline{s}_1^{2n+1} = 2\theta_1/3 + \theta_2/3$ and $\lim_{n\to\infty} \underline{s}_2^{2n+1} = \lim_{n\to\infty} \overline{s}_2^{2n+1} = \theta_1/3 + 2\theta_2/3$. We can conclude that $\lim_{k\to\infty} \underline{s}_1^k = \lim_{k\to\infty} \overline{s}_1^k = 2\theta_1/3 + \theta_2/3$, which gives $S_1^\infty = \{2\theta_1/3 + \theta_2/3\}$, and we can also conclude that $\lim_{k\to\infty} \underline{s}_2^k = \lim_{k\to\infty} \overline{s}_2^k = \theta_1/3 + 2\theta_2/3$, which gives $S_2^\infty = \{\theta_1/3 + 2\theta_2/3\}$.