Fall 2023
14.12 Game Theory
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Online Ad Auctions and the VCG Mechanism

November 28, 2023

This note first discusses auctions for online advertisements. Then we formally define the general VCG mechanism.

1 Background

If you google "DoorDash," the first result will probably be a link to Uber Eats. This is an advertisement that appears above the organic search results. Advertisers pay for their ads to be shown to users who enter relevant search terms. When a user searches, there is room to display multiple ads. Users are more likely to click on ads in more prominent positions, so some slots are more valuable than others. Google runs an auction to allocate these ad slots to advertisers. Since there are multiple, differentiated items (each slot is a different item), the auctions we have studied so far cannot be used.

Recall that the single-item second-price auction solicits a bid from each bidder. The highest bidder wins the good and pays the second-highest bid. All other bidders win nothing and pay nothing. Ties are usually broken randomly. We showed that in a single-item second–price auction, truthful bidding is a dominant strategy equilibrium. That is, each player weakly prefers to bid truthfully than to make any other bid, no matter how her opponents bid.

2 Model

We study the auction for advertising slots for a given search term, e.g. "DoorDash" or "flights to Europe." There are n bidders (advertisers) and m advertising slots. Each advertising slot is distinguished by its click-through-rate (CTR), the probability that a user will click on an ad displayed in that slot. The m slots have CTRs $\alpha_1, \ldots, \alpha_m$. We label the slots in order of their CTRs, so

$$\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_m.$$

It is convenient to set $\alpha_j = 0$ for $j > m$. That is, we include null slots with zero CTR, so that every advertiser can be assigned a slot.

Each bidder i privately knows her valuation v_i for a click on her ad. Therefore, the expected value for bidder i of appearing in slot j is $v_i \alpha_j$. With probability α_j, the ad will be clicked, giving bidder i a value of v_i. With probability $1 - \alpha_j$, the ad will not be clicked.

This basic model makes a couple of important simplifying assumptions:

- CTR for a slot does not depend on which ad appears;
- value per click on a bidder's ad does not depend on the slot in which the ad is shown.

The payment structure for online advertisements come in three general forms:

- *pay-per-impression*: advertisers pay when their ad is displayed
- *pay-per-click*: advertisers pay when someone actually clicks on their ad
- *pay-per-transaction*: advertisers pay when someone clicks on their ad and then purchases a product

Here, we will focus on the pay-per-click format, which is common in search advertising.

3 Auction formats

We study two different generalizations of the single-item second-price auction.

Generalized second-price auction In the single-item second-price auction, the winning bidder pays the second highest bid. To extend this to ad auctions, it is natural for the highest bidder to be allocated the first ad slot and to pay the second-highest bid. Unlike in the single-item auction, there are still additional items to allocate. The second-highest bidder gets the second ad slot and pays the next highest bid, i.e., the third highest bid. In general, the j-th highest bidder gets the j-th slot and pays a per-click price equal to the $(j+1)$-th highest bid.

While this seems like a very natural generalization of the second-price auction, truthful bidding is no longer a dominant strategy. If a bidder shades his bid, he can get a worse position at a lower price-per-click. This results in fewer clicks, but more profit-per-click. If the CTR of the worse position is close enough to the CTR of the better position, the bidder strictly prefers to reduce his bid. The next generalization of the second-price auction remedies this.

Vickrey–Clarke–Groves (VCG) Mechanism For this other generalization, we have to look at the single-item second-price auction in a different way. The highest bidder pays the second highest bid. If we think of the bids as truthfully representing the bidders' valuations, then this second-highest bid is the *externality* that the highest bidder imposes on the other bidders. If the highest bidder were not present, the good would be allocated to the second highest bidder, yielding utility equal to the second-highest bid.

We can extend this idea in the ad context. A bidder imposes an externality on everyone who submits a lower bid. Specifically, the j-th highest bidder shifts down by one position the ad slots assigned to all lower bidders. The VCG mechanism charges the j-th highest bidder this externality. To state the mechanism formally, use the notation $b^{(k)}$ to denote the k-th highest bid. For example, if $(b_1, b_2, b_3) = (4, 6, 5)$, then $b^{(1)} = 6$, $b^{(2)} = 5$, $b^{(3)} = 4$.

The VCG auction solicits per-click bids $b_1, \ldots, b_n$ from the n bidders. The j-th highest bidder gets to display her add in the j-th slot (which has CTR α_j) and must pay a per-click-price of p_j, where p_j is chosen so that that the expected payment equals the expected externality imposed on other bidders:

$$\alpha_j p_j = \sum_{k=j+1}^{n} b^{(k)}(\alpha_{k-1} - \alpha_k). \tag{1}$$

For each lower bidder k, this sum captures the difference in value from being knocked down by one slot. This difference is the product of the value-per-click (captured by the bid $b^{(k)}$) and the difference in the CTRs of slot k (which bidder actually gets) and the slot $k-1$ (which the bidder would have gotten if the j-th highest bidder were absent). The difference $\alpha_{k-1} - \alpha_k$ is nonnegative. It is the *loss* experienced by the k-th highest bidder as a result of the presence of the j-th highest bidder.

If $\alpha_j > 0$, we can divide (1) by α_j to get

$$p_j = \frac{1}{\alpha_j} \sum_{k=j+1}^{n} b^{(k)}(\alpha_{k-1} - \alpha_k) = \sum_{k=j+1}^{n} b^{(k)} \left(\frac{\alpha_{k-1} - \alpha_k}{\alpha_j} \right).$$

Evaluating the telescoping sum

$$\sum_{k=j+1}^{n} (\alpha_{k-1} - \alpha_k) = \alpha_j - \alpha_n,$$

we see that

$$\sum_{k=j+1}^{n} \left(\frac{\alpha_{k-1} - \alpha_k}{\alpha_j} \right) = \frac{\alpha_j - \alpha_n}{\alpha_j} \leq 1,$$

with equality if $\alpha_n = 0$.

As long as there are more bidders than slots, we have $\alpha_n = 0$, so the price-per-click p_j for slot j is a weighted average of the bids $b^{(j+1)}, \ldots, b^{(n)}$.[1] In the generalized second price auction, $p_j = b^{(j+1)}$, so we see that the VCG mechanism charges a lower price (as a function of the bids). This is necessary to discourage the bidders from shading their bids. Why does this exact pricing formula ensure that bidding truthfully a dominant strategy? The key idea is that paying one's externality aligns each bidder's payoffs with social welfare (the sum of all bidders' consumption utilities), and bidding truthfully maximizes social welfare by allocating the slots efficiently. We now give the formal proof in a more general setting.

[1] If $\alpha_n > 0$, then $p_j \alpha_j / (\alpha_j - \alpha_n)$ is a weighted average of the bids $b^{(j+1)}, \ldots, b^{(n)}$. We can imagine that the worst slot is given away for free, and each bidder only has to pay per *additional* click above this baseline.

4 General VCG mechanism

The VCG mechanism can be used in a much more general setting. There is an abstract set X of allocations. There are n players, labeled $i = 1, \ldots, n$. Each player i privately knows her preferences over allocations, as captured by a function $v_i \colon X \to \mathbf{R}$. Player i's preferences are represented by a function, called a *valuation*, rather than a single *value*.

To define the VCG mechanism, we first define an efficient allocation. Given valuations $v_1, \ldots, v_n$, define the the social welfare function $W(\cdot; v_1, \ldots, v_n)$ by

$$W(x; v_1, \ldots, v_n) = v_1(x) + \cdots + v_n(x).$$

Let $x^*(v_1, \ldots, v_n)$ be a maximizer of the function $W(\cdot; v_1, \ldots, v_n)$.

The VCG mechanism is a direct mechanism. Each agent i is asked to report his valuation $\hat{v}_i \colon X \to \mathbf{R}$. (We denote reports by $\hat{v}_i$ and true valuations by v_i.) Given reports $\hat{v}_1, \ldots, \hat{v}_n$, the mechanism selects allocation

$$x^*(\hat{v}_1, \ldots, \hat{v}_n)$$

and transfers

$$t_i(\hat{v}_1, \ldots, \hat{v}_n) = -\sum_{j \neq i} \hat{v}_j(x^*(\hat{v}_i, \hat{v}_{-i})) + h_i(\hat{v}_{-i}),$$

for each player i.

Now we check that reporting truthfully is optimal for each player, no matter how the other players bid. Consider player i. Suppose her opponents report $\hat{v}_j$ for $j \neq i$. If player i has true type v_i, but reports $\hat{v}_i$, her utility is given by

$$\begin{aligned} U_i(\hat{v}_i, \hat{v}_{-i} | v_i) &= v_i(x^*(\hat{v}_i, \hat{v}_{-i})) - t_i(\hat{v}_i, \hat{v}_{-i}) \\ &= v_i(x^*(\hat{v}_i, \hat{v}_{-i})) + \sum_{j \neq i} \hat{v}_j(x^*(\hat{v}_i, \hat{v}_{-i}) - h_i(\hat{v}_{-i}) \\ &= W(x^*(\hat{v}_i, \hat{v}_{-i}); v_i, \hat{v}_{-i}) - h(\hat{v}_{-i}). \end{aligned}$$

In particular,

$$U_i(v_i, \hat{v}_{-i} | v_i) = W(x^*(v_i, \hat{v}_{-i}); v_i, \hat{v}_{-i}) - h(\hat{v}_{-i}).$$

Comparing these expressions, we see that

$$U_i(v_i, \hat{v}_{-i}|v_i) \geq U_i(\hat{v}_i, \hat{v}_{-i}|v_i),$$

for all $\hat{v}_i, \hat{v}_{-i}, v_i$.

We can choose $h_i(\hat{v}_i)$ arbitrarily. A natural choice is

$$h_i(\hat{v}_{-i}) = \max_{x \in X} \sum_{j \neq i} \hat{v}_j(x).$$

Then

$$t_i(\hat{v}_1, \ldots, \hat{v}_n) = -\sum_{j \neq i} \hat{v}_j(x^*(\hat{v}_i, \hat{v}_{-i})) + \max_{x \in X} \sum_{j \neq i} \hat{v}_j(x),$$

which is indeed the externality imposed on the other players. By construction, the transfers are always nonnegative. The transfer equals zero if bidder i's presence does not change the efficient allocation.