Fall 2022
14.12 Game Theory
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14.12 Midterm Exam

October 26, 2022

You have **80 minutes** to complete the exam.

This exam is **closed book**. You may not use any electronic devices or written material brought into the exam.

The exam has **four** questions worth a total of **70 points**:

- Problem 1: 10 points
- Problem 2: 20 points
- Problem 3: 20 points
- Problem 4: 20 points

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Problem 1 (10 points)**.** Consider a finite strategic-form game with n players, labeled $i = 1, \ldots, n$. Each player i has a strategy set S_i. Each player i has a utility function

$$u_i \colon S_1 \times \cdots \times S_n \to \mathbf{R}.$$

Using formal mathematical expressions, complete the following definitions (in your blue book):

1. For player i, strategy s_i *strictly dominates* strategy s'_i if ...

2. A (pure) strategy profile $(s^*_1, \ldots, s^*_n)$ is a Nash equilibrium if ...

Note: You will receive partial credit for informal, verbal definitions.

Solution

1. For player i, strategy s_i *strictly dominates* strategy s'_i if $u_i(s_i, s_{-i}) > u_i(s'_i, s_{-i})$ for each s_{-i} in S_{-i}.

2. A (pure) strategy profile $(s^*_1, \ldots, s^*_n)$ is a Nash equilibrium if for each player i and each s'_i in S_i, we have $u_i(s^*_i, s^*_{-i}) \geq u_i(s'_i, s^*_{-i})$.

Problem 2 (20 points)**.** Consider the following strategic-form game between two players:

		Player 2			
		a	b	c	d
	A	$0, 0$	$\underline{3}, 1$	$-2, 0$	$\underline{4}, \underline{2}$
Player 1	B	$0, \underline{3}$	$1, 1$	$\underline{4}, -3$	$0, 1$
	C	$\underline{1}, \underline{2}$	$2, 0$	$-1, -3$	$2, 1$
	D	$-5, \underline{5}$	$-5, \underline{5}$	$-5, \underline{5}$	$-5, \underline{5}$

1. Compute the set of rationalizable strategies for each player.

2. Find *every* (pure and mixed) Nash equilibrium.

Solution

1. We can iteratively eliminate conditionally dominated strategies in the following order:

   (i) D (strictly dominated by A,B, and C)

   (ii) c (strictly conditionally dominated by b and d)

   (iii) B (strictly conditionally dominated by C)

   (iv) b (strictly conditionally dominated by d).

   For player 1, strategies A and C are rationalizable. For player 2, strategies a and d are rationalizable.

2. By underlining best responses in the matrix game (see above), we find the two pure-strategy Nash equilibria: (C, a) and (A, d).

   Now we look for mixed-strategy Nash equilibria. Only rationalizable strategies can be played with positive probability. Each rationalizable strategy has a unique best response, so each player must put positive probability on at least two strategies. Therefore, a mixed-strategy Nash equilibrium must take the form $(p_1 A + (1 - p_1)C, p_2 a + (1 - p_2)d)$, for some p_1, p_2 in $(0, 1)$. To make player 2 indifferent between a and d, we must have $p_1 = 1/3$. In order for player 1 to be indifferent, we must have $p_2 = 2/3$.

Problem 3 (20 points)**.** Consider the following game between two players, called the seller and the buyer. The seller has a single good to trade. The seller values the good at v_S, and the buyer values the good at v_B, where

$$0 < v_S < v_B < 1.$$

These values are measured in monetary units. Each player's utility equals her value from the good (if she gets/keeps it) net any payments she pays/receives. The buyer and seller haggle over the price as follows. There is no discounting.

- In period 0, the proposer (who will be specified below) proposes that the good is sold at a price $p_0 \in [0, 1]$. If the responder accepts this offer, the trade is carried out at the specified price and the game ends. If the responder rejects the offer, play proceeds to period 1.
- In period 1, the proposer (who will be specified below) proposes that the good is sold at a price $p_1 \in [0, 1]$. If the responder accepts this offer, the trade is carried out at the specified price and the game ends. If the responder rejects the offer, there is no trade and the game ends (i.e., the seller keeps the good).

1. Suppose the seller is the proposer in period 0 and the buyer is the proposer in period 1. Find *one* subgame perfect Nash equilibrium and describe its outcome.
2. Suppose the buyer is the proposer in period 0 and the seller is the proposer in period 1. Find *one* subgame perfect Nash equilibrium and describe its outcome.
3. Based on your answers to parts 1–2, is it better to propose in period 0 or in period 1? Explain why.

Solution

1. We apply backward induction.

   - In period 1, no matter which offer was made (and rejected) in period 0, the seller accepts the buyer's offer if and only if $p_1 \geq v_S$. Therefore, the buyer offers $p_1 = v_S$.
   - In period 0, the buyer accepts the seller's offer if and only if $p_0 \leq v_S$, and hence the seller offers $p_0 = v_S$.

   Therefore, the good is sold in period 0 at price v_S.[1]

2. We apply backward induction.

   - In period 1, no matter which offer was made (and rejected) in period 0, the buyer accepts the seller's offer if and only if $p_1 \leq v_B$. Therefore the seller offers $p_1 = v_B$.
   - In period 0, the seller accepts the buyer's offer if and only if $p_0 \geq v_B$, and hence the buyer offers $p_0 = v_B$.

   Therefore, the good is sold in period 0 at price v_B.[2]

3. It is better to propose in period 1. The period-1 proposer gets ultimatum power, so she can hold the responder's utility to his outside option. This ultimatum power amounts to a commitment not to renegotiate if the responder rejects.

[1] By breaking indifference differently, we can find other SPNE, but the price will be the same in all SPNE.

[2] As in part 1, there are other SPNE, but the price is the same in all of them.

Problem 4 (20 points)**.** Firm 1 and firm 2 are engaged in Cournot quantity competition. Simultaneously, each firm i chooses a production quantity $q_i \in [0, 10]$. The market price is

$$P(Q) = \max\{10 - Q, 0\},$$

where Q is the total quantity produced by the two firms.

Each firm must pay a fixed cost $c_0 > 0$ to initiate production, and a per-unit marginal cost of 2 thereafter. That is, each firm i has production cost function

$$c(q_i) = \begin{cases} 0 & \text{if } q_i = 0, \\ c_0 + 2q_i & \text{if } 0 < q_i \leq 10. \end{cases}$$

Each firm's utility equals its profits.

1. For which values of c_0 is it an equilibrium for both firms to produce 0?

2. For which values of c_0 does there exist an equilibrium in which one firm produces 0 and the other produces a strictly positive quantity?

Hint: Your answers should be inequalities involving c_0.

Solution First we write down the payoff function for player 1 (player 2's payoff function is symmetric). We have:

$$\pi_1(q_1, q_2) = q_1 P(q_1 + q_2) - c(q_1).$$

If firm 1 does not produce, her profit is 0. If she does produce (and the price is nonnegative) her profit is

$$q_1(10 - q_1 - q_2) - 2q_1 - c_0 = q_1(8 - q_1 - q_2) - c_0,$$

which is maximized at $g(q_2) = (8 - q_2)_+/2$. Therefore, firm 1's best response to q_2 is 0 or $g(q_2)$, depending on which gives a higher payoff.

1. From $(q_1, q_2) = (0, 0)$, the best unilateral deviation, for either player, is to produce $g(0) = 4$, which gives payoff

$$4(8-4) - c_0 = 16 - c_0.$$

   We have an equlibrium if and only if $16 - c_0 \leq 0$, i.e., $c_0 \geq 16$.

2. By symmetry, it suffices to look for an equilibrium with $q_1 > 0$ and $q_2 = 0$. The only candidate to consider is $q_1 = g(0) = 4$ (otherwise, firm 1 can profitably deviate to $g(0) = 4$).

   We check whether there is a profitable deviation from $(q_1, q_2) = (4, 0)$.

   - Firm 1 cannot profit by deviating to a different positive quantity since $q_1 = g(0)$. Deviating to 0 is not strictly profitable if $16 - c_0 \geq 0$, i.e., $c_0 \leq 16$.
   - Firm 2's best deviation is to $g(4) = 2$, which yields a payoff of

$$2(4-2) - c_0 = 4 - c_0.$$

     This deviation is not strictly profitable if $4 - c_0 \leq 0$, i.e., $c_0 \geq 4$.

   Therefore, the desired equilibrium exists if and only if $4 \leq c_0 \leq 16$.