Chapter 9

Application: Negotiation

Negotiation is an essential aspect of social and economic interactions. States negotiate their borders with their neighbors; legislators negotiate the laws that they make; litigants negotiate to settle their disputes; workers negotiate their wages and the terms of their employment with their employers; investment bankers negotiate mergers and acquisitions of firms; financiers negotiate the price of over the counter assets; family members negotiate their spending and maintenance of the household with each other, and even children negotiate how to divide a treat or who will do which chore with each other or with their parents. Even when there is no explicit formal negotiation in the form of offers and counteroffers, the parties may negotiate in other ways, taking actions towards reaching an outcome that all parties agree to. For example, even a war can be viewed as a form of negotiation in this sense.

Despite their central importance, negotiations were presumed to be outside of the purview of economic analysis until the emergence of game theory. Today, there are many game theoretical models of bargaining. This chapter applies backward induction to several important bargaining games. The first model considers pretrial negotiation in law. The second model considers negotiation of the price in a bilateral trade example. The third model is a general model of bargaining that can be applied to many different settings in economics. The fourth model considers congressional bargaining. It abstracts away from the back-room deals that lead to the proposed bills and focus on the way legislators vote between various alternatives. In all these models, there is a deadline after which there is no room for negotiation. This allows one to apply backward induction to solve the game. Chapters 11 and 23 will present the analyses of infinite-horizon bargaining and bargaining with private information, respectively.

9.1 Pre-trial Negotiations

Only 5% of the civil cases filed in the United States go to trial. The rest of them are settled out of the court. Likewise many criminal cases are settled between the prosecutors and the defendants through plea bargaining.

There is a very good reason for this: litigation and settlement delays are very costly for both parties, and the parties would like to avoid these costs by settling immediately. Typically, the legal fees alone can easily lead to bankruptcy. For example, Princeton University spent more than 40 million dollars in its legal defense against the Robertson family before reaching a settlement in 2008. Likewise, in the well-known case of Pennzoil v. Texaco, the legal expenses were several hundreds of millions of dollars, and the case was settled for 3 billion dollars after a long litigation process. In commercial litigation, ongoing litigations also have indirect costs due to uncertainty, delayed decisions, missed business opportunities, and suppressed market valuation, and these costs may dwarf the legal expenses above. In general, in litigation, parties' private information may become public record, and the costs associated with such revelations may outweigh any benefit one gets. For example, a patent dispute may lead to revelation of highly guarded trade secrets if it is pursued in the court. Prosecutors may not want to reveal the evidence they have in a more important case even if that evidence would lead to a conviction in a less important case. Likewise, the disclosure of evidence may have national security implications, as it may reveal methods of intelligence gathering.

There are many impediments to reaching an agreement in negotiations, such as private information and excessive optimism. In order to illustrate the basic forces in negotiations, this section will rule out all such impediments and focuses on the most basic case in which all information is public.

There are two players: a Plaintiff and a Defendant. The Plaintiff suffers a loss due to the negligence of the Defendant, and he is suing her now. The court date is set at date $2n$. It is known that if they go to court, the Judge will order the Defendant to pay $J > 0$ to the Plaintiff. In order to avoid the legal costs, the Plaintiff and the Defendant are also negotiating an out of court settlement. The negotiation follows the following protocol.

- At each date $t \in \{0, 2, \ldots, 2n-2\}$, if they have not yet settled, the Plaintiff offers a settlement s_t,
- and the Defendant decides whether to accept or reject it. If she accepts, the game ends with the Defendant paying s_t to the Plaintiff; the game continues otherwise.
- At dates $t \in \{1, 3, \ldots, 2n-1\}$, the Defendant offers a settlement s_t,
- and the Plaintiff decides whether to accept the offer, ending the game with the Defendant paying s_t to the Plaintiff, or to reject it and continue.
- If they do not reach an agreement at the end of period $t = 2n-1$, they go to court, and the Judge orders the Defendant to pay $J > 0$ to the Plaintiff.

The Plaintiff pays his lawyer c_P for each day they negotiate and an extra C_P if they go to court. Similarly, the Defendant pays her lawyer c_D for each day they negotiate and an extra C_D if they go to court. Each party tries to maximize the expected amount of money he or she has at the end of the game.

The backward induction analysis of the game as follows. The payoff from going to court for the Plaintiff is

$$J - C_P - 2nc_P.$$

If he accepts the settlement offer s_{2n-1} of the Defendant at date $2n-1$, his payoff will be

$$s_{2n-1} - 2nc_P.$$

Hence, if $s_{2n-1} > J - C_P$, he must accept the offer, and if $s_{2n-1} < J - C_P$, he must reject the offer. If $s_{2n-1} = J - C_P$, he is indifferent between accepting and rejecting the offer. Assume that he accepts that offer, too.[1] To sum up, he accepts an offer s_{2n-1} if and only if $s_{2n-1} \geq J - C_P$.

[1] In fact, he must accept $s_{2n-1} = J - C_P$ in equilibrium. For, if he doesn't accept it, the best response of the Defendant will be empty, inconsistent with an equilibrium. (Any offer $s_{2n-1} = J - C_P + \epsilon$ with $\epsilon > 0$ will be accepted. But for any offer $s_{2n-1} = J - C_P + \epsilon$, there is a better offer $s_{2n-1} = J - C_P + \epsilon/2$, which will also be accepted.)

The Plaintiff accepts a settlement offer if it is at least what he will get in the court minus the cost of going to the court. The legal costs accumulated up to that period are sunk, in that the Plaintiff's actions cannot change them. Hence, the Plaintiff ignores those costs—as they appear on both sides of the inequality. This will be a general feature throughout.

What should the Defendant offer at date $2n-1$? Given the behavior of the Plaintiff, if she offers $s_{2n-1} \geq J - C_P$, the offer is accepted and she pays s_{2n-1} to the Plaintiff in addition to the costs $2nc_D$ she has incurred, obtaining payoff of $-s_{2n-1} - 2nc_D$. If she offers $s_{2n-1} < J - C_P$, the offer is rejected and she will be ordered to pay the Plaintiff J in the court in addition to the costs $C_D + 2nc_D$ she incurs, obtaining the payoff of $-J - C_D - 2nc_D$. All in all, her payoff from s_{2n-1} is

$$\begin{aligned} -s_{2n-1} - 2nc_D \text{ if } s_{2n-1} &\geq J - C_P \\ -J - C_D - 2nc_D \text{ if } s_{2n-1} &< J - C_P. \end{aligned}$$

Her payoff is plotted in Figure 9.1. Notice that when $s_{2n-1} = J - C_P$, her payoff is $-J + C_P - 2nc_D$, and offering anything less would cause her to lose $C_D + C_P$. Clearly, her payoff from offering s_{2n-1} is maximized at $s_{2n-1} = J - C_P$. Therefore, the Defendant offers

$$s_{2n-1} = J - C_P$$

at date $2n-1$.

Now at date $2n-2$, the Plaintiff offers a settlement s_{2n-2} and the Defendant accepts or rejects the offer. If the Defendant rejects the offer, she will get the payoff from settling for $s_{2n-1} = J - C_P$ at date $2n-1$, which is

$$-J + C_P - 2nc_D.$$

If she accepts the offer, she will get

$$-s_{2n-2} - (2n-1)\, c_D.$$

She accepts the offer if and only if the last expression is greater than or equal to the previous one, i.e.,

$$s_{2n-2} \leq J - C_P + c_D.$$

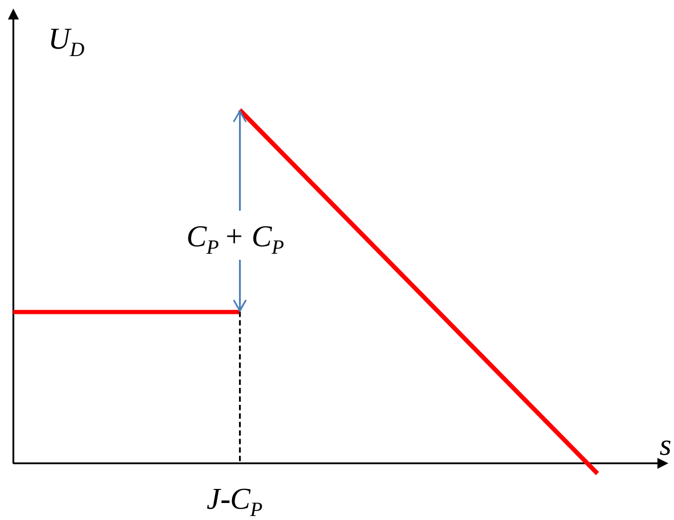

Figure 9.1: Payoff of Defendant from her offer at the last period

The Plaintiff offers the highest acceptable settlement to the Defendant:

$$s_{2n-2} = J - C_P + c_D.$$

In summary, since the Plaintiff is making an offer, he offers the settlement amount of next date plus the cost of negotiating one more day for the Defendant.

Let us apply the backward induction one more step. At date $2n-3$, the Defendant offers a settlement s_{2n-3} and the Plaintiff accepts or rejects the offer. If he rejects the offer, he gets the payoff from settling for $s_{2n-2} = J - C_P + c_D$ at date $2n-2$, which is

$$s_{2n-2} - (2n-1)\, c_P = J - C_P + c_D - (2n-1)\, c_P.$$

If he accepts the offer, he gets

$$s_{2n-3} - (2n-2)\, c_P.$$

He accepts the offer if and only if the last expression is greater than or equal to the previous one, i.e.,

$$s_{2n-3} \geq s_{2n-2} - c_P = J - C_P + c_D - c_P.$$

The Defendant offers the lowest acceptable settlement to the Plaintiff:

$$s_{2n-3} = s_{2n-2} - c_P = J - C_P + c_D - c_P.$$

In summary, since the Defendant is making an offer, she offers the settlement amount of next date *minus* the cost of negotiating one more day for the Plaintiff.

Now the pattern is clear. At any even date t, the Defendant accepts an offer s_t if and only if $s_t \leq s_{t+1} + c_D$, and the Plaintiff offers

$$s_t = s_{t+1} + c_D \qquad (t \text{ is even}).$$

At any odd date t, the Plaintiff accepts an offer s_t if and only if $s_t \geq s_{t+1} - c_P$, and the Defendant offers

$$s_t = s_{t+1} - c_P \qquad (t \text{ is odd}).$$

The solution to the above difference equation is

$$s_t = \begin{cases} J - C_P + (n - 1 - t/2)(c_D - c_P) + c_D & \text{if } t \text{ is even} \\ J - C_P + (n - (t+1)/2)(c_D - c_P) & \text{if } t \text{ is odd.} \end{cases}$$

In particular, at the beginning, the Plaintiff offers

$$s_0 = J - C_P + (n-1)(c_D - c_P) + c_D,$$

and the offer is accepted. The players reach an agreement immediately. However, the settlement amount reflects what players would have done at any given period if they have not reached an agreement by then. When the legal fees are equal (i.e., $c_P = c_D$), the settlement amount is simply

$$s_0 = J - C_P + c_D.$$

The plaintiff foregoes his cost of going to court to obtain J but demands the defendant's cost of retaining a lawyer for one period.

The solution is substantially different if the order of the proposers is changed. In that case, the plaintiff makes the last offer and offers $s_{2n-1} = J + C_D$. This is $C_P + C_D$ higher than the other case, and settlement in equilibrium also reflects this difference; it is $J + C_D - c_P$ when $c_P = c_D$. The difference stems from the fact that, at the last day, the cost of delaying the agreement is quite high (the cost of going to court), and the party who accepts or rejects the offer is willing to accept a wide range of offers. Hence, the last proposer has a great advantage.

9.2 Price Negotiation

A seller owns a good that is of no value to her. There is also a buyer for whom the value of the good is $v > 0$. They would like to trade the good, and they are negotiating the price at which they will trade the good. Both of them are risk-neutral. Hence, if they trade the good for price p, the payoff of the buyer and the seller would be

$$v - p \text{ and } p,$$

respectively. The parties also discount the future, and the value of the trade diminishes as time passes. In particular, the dates are $t = 0, 1, \ldots, 2n - 1$. The value of the good becomes zero for the buyer after $t = 2n - 1$. The payoff of trading the good at date t for price p is

$$\delta^t (v - p) \text{ and } \delta^t p$$

for the buyer and the seller, respectively, where $\delta \in (0, 1)$ is a known parameter that measures players' patience. The players negotiate using alternating-offer protocol as in the previous section. Until they reach an agreement, the seller offers a price at each even date $t = 0, 2, \ldots$, and the buyer offers a price at each odd date $t = 1, 3, \ldots$. If the offer is accepted by the other player, they trade the good at the offered price; they proceed to the next date otherwise. If the offer is rejected in the last period, the game ends with zero payoff for each player.

First consider the simpler case with only two rounds of negotiations, i.e., $n = 1$. The backward induction analysis of this case is as follows. At $t = 1$, if the seller rejects the offer, she gets 0. Hence, she accepts any price more than 0, and she is indifferent between accepting and rejecting zero price. As in the previous section, she accepts the zero price in equilibrium. Then, at $t = 1$, the buyer would offer price

$$p_1 = 0,$$

the best price for the buyer. Therefore, if they do not agree at $t = 0$, then the buyer buys the good for free. The value of having the good for free on the next day for the buyer is δv. In contrast, if he buys the good at price p_0 at $t = 0$, the buyer's payoff is $v - p_0$. This payoff is greater than δv if and only if $p_0 < (1 - \delta) v$. Hence, at $t = 0$, the buyer accepts any price less than $(1 - \delta) v$, rejects any price more than $(1 - \delta) v$, and he is indifferent between accepting and rejecting the price $(1 - \delta) v$. Now, the seller can either sell the good for price $(1-\delta)\,v$ or less, or ask a higher price and get rejected and sell for free next period. The choice seems obvious. She asks price

$$p_0 = (1-\delta)\,v,$$

and the buyer accepts it.

Now consider the general case $n > 1$. To solve the game using backward induction, take any date $t = 2k-1$, and assume that backward induction leads the players to trade the good at any given date $t' > t$ at price $p_{t'}$ if they have not agreed by then. In particular, at date $t+1$, the seller asks price p_{t+1} and the buyer accepts the price. Now, at t, the seller is indifferent between selling the good for price p_{t+1} at $t+1$ and selling it at t for price δp_{t+1}. (She gets $\delta^{t+1}p_{t+1}$ in both cases.) Hence, she accepts a price p if and only if $p \geq \delta p_{t+1}$. Since the seller accepts the price δp_{t+1}, the buyer would not offer any higher price. Moreover, if he offers less, the offer would be rejected, and he would buy the good at the higher price of p_{t+1} in the next period. Hence, the buyer offers price

$$p_t = \delta p_{t+1} \tag{9.1}$$

and buys the good at this price. Now, at $t-1$, the buyer is indifferent between buying the good at price p_{t-1} right away and waiting one more period and buying it at price p_t if and only if[2]

$$v - p_{t-1} = \delta\,(v - p_t)\,.$$

The solution to this equation is

$$p_{t-1} = (1-\delta)\,v + \delta p_t.$$

The buyer accepts a price p if and only if $p \leq p_{t-1}$. Now, if the seller asks a higher price than p_{t-1}, the buyer will reject it, and she will sell it for price p_t the next day, a price lower than p_{t-1}. Therefore, the seller asks price

$$p_{t-1} = (1-\delta)\,v + \delta p_t, \tag{9.2}$$

and sells the good at that price.

[2]Note that the indifference condition is $\delta^{t-1}\,(v-p) = \delta^{t}\,(v-p_t)$, and δ^{t-1} is canceled on both sides to get $v - p = \delta\,(v-p_t)$. This is another form of ignoring the sunk costs, this time in terms of the cost of time elapsed up to that point.

All in all, at any given date, the players trade the good if they have not done so already. The price can be explicitly computed using (9.1) and (9.2) as follows. Substituting (9.1) in (9.2), one obtains

$$p_{t-1} = (1-\delta)\, v + \delta^2 p_{t+1}$$

for any odd date t. This yields a difference equation for the prices at even dates:

$$p_{2k} = (1-\delta)\, v + \delta^2 p_{2(k+1)}.$$

Since the price is $p_{2(n-1)} = (1-\delta)\, v$ at period $2(n-1)$, the solution to this difference equation is[3]

$$p_t = \frac{1-\delta^{2n-t}}{1+\delta} v \tag{9.3}$$

at any even period t. For any odd period t, the seller buys the good at price

$$p_t = \delta p_{t+1} = \frac{\delta\left(1-\delta^{2n-t-1}\right)}{1+\delta} v = \frac{\delta-\delta^{2n-t}}{1+\delta} v. \tag{9.4}$$

Backward induction solution is as follows. At any even period t, the seller asks price p_t and the buyer accepts a price p if and only if $p \leq p_t$; at any odd period t, the buyer offers price p_t and the seller accepts a price p if and only if $p \geq p_t$ where price p_t is given as above.

In the limit $n \to \infty$, the buyer offers price

$$p_B = \frac{\delta}{1+\delta} v,$$

and the seller asks

$$p_S = \frac{1}{1+\delta} v$$

whenever it is their turn to make an offer. The seller asks more than half the value of the good for the buyer while the buyer offers less than half of its value for him, so that

[3] One can solve the difference equation forward by substituting $p_{2k} = (1-\delta)\, v + \delta^2 p_{2(k+1)}$ repeatedly until one reaches the period $2(n-1)$:

$$p_{2k} = (1-\delta)\, v \left(1+\delta^2+\cdots+\delta^{2(n-k-1)}\right) = (1-\delta)\, \frac{1-\delta^{2(n-k)}}{1-\delta^2} v = \frac{1-\delta^{2(n-k)}}{1+\delta} v,$$

where the first equality is by repeated substitution, the second equality is by the formula for geometric sums, and the last one is by $1-\delta^2 = (1-\delta)(1+\delta)$.

the responding party is indifferent between accepting the offer today or waiting next period to get the more favorable price. As $\delta \to 1$, the players become infinitely patient (or the real time difference between period becomes negligible), and the price is sold at nearly price $v/2$, half of the value of the good for the buyer.

9.3 A General Model of Bilateral Bargaining

The previous price negotiation can be viewed more abstractly as follows. The players have v units of gain from trade that they can realize only if they both agree on how to share it between them. The price p is the share of the seller and the remaining $v - p$ is the share of the buyer. Many negotiations in real life can be viewed abstractly in this way: the parties can make a decision that is better than disagreement for all parties if they can all agree on such a decision. This section presents such a general abstract bargaining model with two players. As in the previous examples, there is a deadline after which no agreement is possible. The deadline will be dispensed with in Chapter 11.

There are two players, namely 1 and 2, and there are finitely many periods $t = 0, 1, \ldots, \bar{t}$. The players are to make a joint decision. They both must agree on what decision to make in order to implement it. Clearly, each decision leads to a pair of expected utilities, one for Player 1 and one for Player 2. Each decision will be directly represented by the expected utility pair it leads to, although the actual decision itself may be highly complex in real life applications. Let $X \subset \mathbb{R}^2$ be the set of feasible decisions and write each decision $x \in X$ as a pair (x_1, x_2) where x_1 and x_2 are the payoffs of Players 1 and 2, respectively. The disagreement payoffs are normalized to zero. For simplicity, assume that

$$X = \left\{x \in [0,1]^2 \,|x_1 \leq f_1(x_2)\right\}$$

for some strictly decreasing function $f_1 : [0,1] \to [0,1]$ with $f_1(0) = 1$ and $f_1(1) = 0$. Writing f_2 for the inverse of f_1, one can also define the set by $x_2 \leq f_2(x_1)$. Here, $0 \in X$ assumes that disagreement is also a feasible decision, and the decisions that are worse than disagreement for some player are ignored, as they will not affect the solution.

Observe that, for any distinct i and j,

$$f_i(x_j) = \max_{\{y \in X | y_j \geq x_j\}} y_i$$

is the highest payoff player i can get if she needs to give a payoff level x_j to the other player for an agreement. As in the price-negotiation example, the players discount the future. The payoff of player i from agreeing on x at period t is $\delta_i^t x_i$ where $\delta_1, \delta_2 \in (0,1)$ are known parameters.

As in the previous examples, players make alternate in making offers Player 1 makes offers at periods $t = 0, 2, \ldots$, and Player 2 makes offers at periods $t = 1, 3, \ldots$. At each period t, a player offers a decision $x \in X$, and the other player accepts or rejects the offer. If the offer is accepted, the decision x is implemented, and the game ends with payoff vector $(\delta_1 x_1, \delta_2 x_2)$. If the offer is rejected, they proceed to the next period; at $\bar{t}$ the game ends with payoff vector $(0,0)$ if they do not reach an agreement.

Backward induction leads to a unique solution as follows. Assume $\bar{t}$ is odd so that player 2 makes an offer at $\bar{t}$. At $\bar{t}$, Player 1 accepts an offer x if and only if $x_1 \geq 0$. Player 2 offers $(0,1)$, and the offer is accepted. Hence, at $\bar{t} - 1$, Player 2 accepts an offer x if and only if $x_2 \geq \delta_2$. Since Player 1 will get 0 in the next period if the offer is rejected, she offers $(f_1(\delta_2), \delta_2)$, and the offer is accepted. Now, applying backward induction in this fashion, suppose that one determined that the players settle on $\left(x_{1(t+1)}, x_{2(t+1)}\right)$ at period $t+1$ if they have not agreed by then. Then, at t, the responding player j accepts an offer x if and only if $x_j \geq \delta_j x_{j(t+1)}$, and the proposing player i offers a decision that gives $f_i\left(\delta_j x_{j(t+1)}\right)$ and $\delta_j x_{j(t+1)}$ to players i and j, respectively. Therefore, the backward induction solution is as follows. At any period t, the responding player j accepts and offer x if and only if $x_j \geq \delta_j x_{j(t+1)}$, and the proposing player i offers a decision that gives $f_i\left(\delta_j x_{j(t+1)}\right)$ and $\delta_j x_{j(t+1)}$ to players i and j, respectively, where the sequence $x_t = (x_{1t}, x_{2t})$ of decisions is defined by $x_{\bar{t}} = (0,1)$ and by

$$x_{it} = \begin{cases} f_i\left(\delta_j x_{j(t+1)}\right) & \text{if } i \text{ is to make an offer at } t \\ \delta_i x_{i(t+1)} & \text{otherwise} \end{cases} \tag{9.5}$$

for each distinct i and j and for each $t < \bar{t}$. If $\bar{t}$ is even, Player 1 makes an offer in the last period, and the sequence is modified by setting $x_{\bar{t}} = (1,0)$.

9.4 Random Proposer Model

Two players $\{1,2\}$ collectively own one dollar which they can use only after they divide it. Given any $n>0$, consider the following n-period, random-proposer bargaining model. At any date $t \in \{0,1,\ldots,n-1\}$, one of the players is selected randomly as the proposer, where Players 1 and 2 are selected with probabilities p and $1-p$, respectively, for some known $p \in [0,1]$. The selected player makes an offer $x \in [0,1]^2$ such that $x_1 + x_2 \leq 1$. Then, the other player accepts or rejects the offer. If the offer x is accepted, the game ends, yielding payoff vector $\delta^t x$ for some known $\delta \in (0,1)$. If the offer is rejected, we proceed to the next date, when the same procedure is repeated, except for $t = n-1$, after which the game ends, yielding (0,0).

Consider the last period. If a player rejects an offer, she gets 0. Hence, she accepts any offer. (She is indifferent between accepting and rejecting an offer that gives her exactly 0, but she will accept that offer in any backward induction solution.) Hence, the selected player offers 0 to her opponent, taking the entire dollar for herself; and her offer will be accepted. Therefore, the outcome is (1,0) if Player 1 is selected, and (0, 1) if Player 2 is selected. The expected shares (before the proposer is selected) are

$$V = p\,(1,0) + (1-p)\,(0,1) = (p, 1-p)\,;$$

the payoffs are given by $\delta^{n-1}V$.

Now consider the period before the last one. Suppose Player 1 is selected and offered x. Player 2 gets $\delta^{n-2}x_2$ if she accepts the offer, and she gets the expected payoff of $\delta^{n-1}(1-p)$ if she rejects the offer. Hence, she accepts the offer x if an only if $x_2 \geq \delta(1-p)$. Player 1 offers $(1-\delta(1-p), \delta(1-p))$. Similarly, if Player 2 is selected, she offers $(\delta p, 1-\delta p)$, and Player 1 accepts an offer x if and only if $x_1 \geq \delta p$. Before the proposer is selected, the expected share of Player 1 is

$$p(1-\delta(1-p)) + (1-p)\,\delta p = p.$$

Likewise, the expected share of Player 2 is $1-p$.

The pattern is clear. At any given round the players reach an agreement if they have not agreed yet, and the expected shares of Players 1 and 2 are p and $1-p$, respectively. Expecting this to happen in the next period, Player 2 accepts an offer x by Player 1 if and only if $x_2 \geq \delta(1-p)$, and Player 1 offers $(1-\delta(1-p), \delta(1-p))$ when selected. If

Player 2 is selected, Player 1 accepts an offer x if $x_1 \geq \delta p$, and Player 2 offers $(\delta p, 1-\delta p)$. When δ is nearly one (e.g. when the parties make frequent offers), the players agree on a division near $(p, 1-p)$, where each player's share is nearly her probability of making an offer.

9.5 Congressional Bargaining—Voting with a Binary Agenda

In the US Congress, when a new bill is introduced, there are often other alternative proposals, such as amendments, amendments to amendments, substitute bills, amendments to substitute bills, and so on. There are rules of the Congress that determine the order in which these proposals, or "alternatives", are voted against each other, eventually leading to a final bill. In the final vote, the final bill, which may not be the original one, is voted. If the final bill passes, then it becomes the law. If the final bill fails, then the status quo prevails. For example, if there is a bill, an amendment, and the status quo, first they vote between the bill and the amendment, then they vote between the winner of the previous vote and the status quo. These rules and the available proposals lead to a "binary" agenda; it is *binary* because in any session *two* alternatives are voted against each other.

Let $\{1, \ldots, 2n+1\}$ be the set of players and $\{x_0, \ldots, x_m\}$ be the set of alternatives. Each player has a strict preference ordering for the set of alternatives. There is a fixed binary agenda according to which the alternatives come to the floor for voting. At any given voting session, the legislators vote between two alternatives. They vote sequentially according to a given order, each legislator seeing how the legislators before him or her voted before them. The alternative with the highest vote proceeds, and the other alternative is eliminated. Everything described here and the legislators' preferences are common knowledge. The sequential voting assumption here ensures that the game is of perfect information, and one can apply backward induction. Since there are only two alternatives at any given session, it is a best response for each legislator to vote for the alternative that he or she prefers proceeding to the next round — or being implemented if in the last round. This will be assumed throughout. If the legislators voted simultaneously, one could simply assume this behavior in voting and apply backward induction

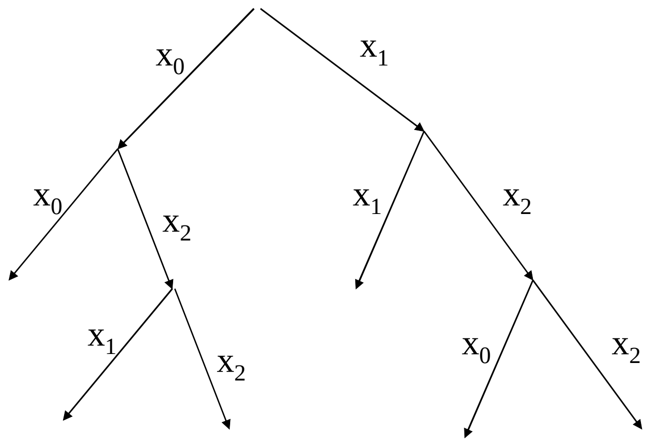

Figure 9.2: A binary agenda

between the rounds. Note that a player may vote for an inferior alternative when she is not pivotal, and her vote does not affect the outcome. This is a weakly dominated best response. Note also that a legislator may prefer an alternative that she likes less to proceed as that alternative may lead to a better outcome in the end.

To solve this game by backward induction, start from a last vote (a vote after which there is no further voting). Assume that each legislator votes according to her preference. The alternative that gets $n+1$ or more votes wins. Then, truncate the game by replacing the voting session with the winning alternative. Proceed in this way until there is only one alternative.

For example, consider three legislators, namely 1, 2, and 3, and three alternatives, namely x_0, x_1, and x_2. The agenda is as in Figure 9.2. According to the agenda, x_0 and x_1 are voted against each other first; the winner is voted against x_2 next. If the winner defeats x_2 as well, then it is implemented; otherwise x_2 (the winner of the second vote) is voted against the loser of the first vote and the winner of this vote is implemented.

The preference ordering of the three players is as follows:

1	2	3
x_0	x_2	x_1
x_1	x_0	x_2
x_2	x_1	x_0

where the higher-ranked alternatives are placed in the higher rows.

Consider the branch on the left first. In the last vote, which is between x_1 and x_2, every player votes for her better alternative according to the table. Legislators 1 and 3 vote for x_1, and Legislator 2 votes for x_2. In this vote x_1 beats x_2. Now consider the preceding vote, between x_0 and x_2. Now, everyone knows that if x_2 wins, in the next round x_1 will be implemented. Hence, a vote for x_2 is simply a vote for x_1. Hence, in the backward induction, the final vote is replaced with its winner, namely x_1. Those who prefer x_0 to x_1, who are Legislators 1 and 2, vote for x_0, and the other legislator, who prefers x_1 to x_0, votes for x_2. In this vote, x_0 wins.

Now consider the right branch. In the last round, between x_0 and x_2, Legislator 1 votes for x_0, and 2 and 3 vote for x_2, resulting in the winning of x_2. Hence, in the backward induction, the last round is replaced by x_2. In the previous vote between x_1 and x_2, if x_1 wins it is implemented, and if x_2 wins it will be implemented (after defeating x_0, which will happen). Then, each player votes according to his true preference: Legislators 1 and 3 for x_1, and Legislator 2 for x_2. Alternative x_1 wins. Therefore, on the right branch, x_1 wins.

Finally, at the very first vote, between x_0 and x_1, the legislators know that the winning alternative will be implemented in the future. Hence, everybody votes according to her original preferences and x_0 wins.

I next turn to an interesting phenomenon in legislative process: a *killer amendment* or a *poison pill*. Suppose that we have a bill x_1 that is preferred by a majority of the legislators to the status quo, x_0.[4] If the bill is voted against the status quo, it will pass. A poison pill or a killer amendment is an amendment x_2 that is worse than the status quo, x_0, according to a majority. Recall that the amendment x_2 is first voted against the bill x_1 and the winner is finally voted against the status quo x_0. If the amendment passes, then it will fail in the last round, and the status quo will be kept. Hence the term *killer amendment*.

According to backward induction, a killer amendment is defeated in the first round (assuming that a majority prefers x_1 to x_0). Indeed, if x_2 defeats x_1, in the next round x_0 will be implemented. Hence, in the first round a vote for x_2 is a vote for the status quo, x_0. Then, the players who prefer the status quo, x_0, to the bill will vote for the

[4]The alternatives x_0, x_1, and x_2 in this example are not related to the alternatives x_0, x_1, and x_2 in the previous example.

amendment, x_2, and the players who prefer the bill, x_1, to the status quo will vote for the bill. Since the latter group is a majority, x_1 defeats the amendment in the first round.

But poison pills and killer amendments are frequently introduced and sometimes they defeat the original bill (and eventually are defeated by the status quo). A famous example of this is Depew amendment to the "17th amendment to the constitution" in 1912. Here, the 17th amendment, x_1, requires the senators to be elected by the statewide popular vote. This bill was supported by the (Southern) Democrats and half of the Republicans, making up the two-thirds of the Congress. The Depew amendment, x_2, required that these elections be monitored by the federal government. Each Republican slightly prefers x_2 to x_1, so the proponent Republicans' ordering is $x_2 \succ x_1 \succ x_0$ and the opposing Republicans' ordering is $x_0 \succ x_2 \succ x_1$, where x_0 is the status quo. But the federal oversight of the state elections is unacceptable to the southern Democrats for obvious reasons: $x_1 \succ x_0 \succ x_2$. Notice that "opposing Republicans" and Democrats, which is about the two-thirds of the legislators, prefer the status quo to the Depew amendment. Hence, the Depew amendment is a killer amendment. According to the backward induction analysis, it should be defeated in the first round, and the original bill, the 17th amendment, should eventually pass. But this did *not* happen. The Depew amendment killed the bill.

Why does this happen? It would be too naive to think that a legislator is so myopic that he cannot see one step ahead and fails to recognize a killer amendment. Sometimes, legislators might not know the preferences of the other legislators. After all, these preferences are elicited in these votes. In that case, the backward induction analysis above is not valid and needs to be modified. Of course, in that case, an amendment may defeat the bill (because of the proponents who think that it has enough support for an eventual passage) but later be defeated in the final vote because of the lack of sufficient support (which was not known in the first vote). But mostly, the killer amendments are introduced intentionally, and the legislators have a clear idea about the preferences. Even in that case, a killer amendment can pass, not because of the stupidity or ignorance of the proponents of the original bill, but because their votes against the amendment can be exploited by their opponents in the upcoming elections when the voters are not informed about the details of these bills.

The moral of the story is that it is not enough that your analysis is correct. You must also be analyzing the correct game. You will learn the first task in the Game Theory class; for the second, and more important, task of considering the correct game, you need to look at the underlying facts of the situation.

9.6 Exercises with Solutions

Exercise 9.1. Apply backward induction to the following game. There are two players A and B, who own a firm and want to dissolve their partnership. Each owns half of the firm. The value of the firm for players A and B are v_A and v_B, respectively, where $v_A > v_B > 0$. Player A sets a price p for half of the firm. Player B then decides whether to sell his share or to buy A's share at this price, p. If B decides to sell his share, then A owns the firm and pays p to B, yielding playoffs $v_A - p$ and p for players A and B, respectively. If B decides to buy, then B owns the firm and pays p to A, yielding playoffs p and $v_B - p$ for players A and B, respectively. All these are common knowledge.

Solution. Given any price p, the best response of B is

$$\begin{cases} \text{buy} & \text{if } v_B - p > p, \text{ i.e., if } p < v_B/2; \\ \text{sell} & \text{if } p > v_B/2; \\ \{\text{buy, sell}\} & \text{if } p = v_B/2. \end{cases}$$

In equilibrium, B must be selling at price $p = v_B/2$. This is because, if he were buying, then the payoff of A as a function of p would be

$$\begin{cases} p & \text{if } p \le v_B/2; \\ v_A - p & \text{if } p > v_B/2. \end{cases}$$

Then, A will not have a best response in the previous node as no price could maximize the payoff of A. Hence, the equilibrium strategy of B must be

$$\begin{cases} \text{buy} & \text{if } p < v_B/2; \\ \text{sell} & \text{if } p \ge v_B/2. \end{cases}$$

In that case, the payoff of A as a function of p would be

$$\begin{cases} p & \text{if } p < v_B/2; \\ v_A - p & \text{if } p \ge v_B/2. \end{cases}$$

This function is maximized at $p = v_B/2$. Player A sets the price as $p = v_B/2$.

Exercise 9.2. Apply backward induction to the following pretrial negotiation model, where the plaintiff's lawyer gets a share from the settlement instead of fixed daily fee. Paul has lost his left arm due to complications in a surgery. He is suing the Doctor.

- The court date is set at date $2n+1$. It is known that if they go to court, the judge will order the Doctor to pay $J > 0$ to Paul.
- They negotiate for a settlement before the court. At each date $t \in \{1, 3, \ldots, 2n-1\}$, if they have not yet settled, Paul offers a settlement s_t, and the Doctor decides whether to accept or reject it. If she accepts, the game ends with the Doctor paying s_t to Paul; game continues otherwise. At dates $t \in \{2, 4, \ldots, 2n\}$, the Doctor offers a settlement s_t, and Paul decides whether to accept the offer, ending the game with Doctor paying s_t to Paul, or to reject it and continue.
- Paul pays his lawyer only a share of the money he gets from the Doctor. He pays $(1-\alpha)\, s_t$ if they settle at date t; $(1-\beta)\, J$ if they go to court, where $0 < \beta < \alpha < 1$. The Doctor pays her lawyer c for each day they negotiate and an extra C if they go to court.

Solution. At date $2n+1$, Paul gets J from the doctor and pays $(1-\beta)\, J$ to his lawyer, netting βJ. Now at date $2n$, if he accepts s_{2n}, he pays $(1-\alpha)\, s_{2n}$ to his lawyer, receiving αs_{2n}. Hence, he accept s_{2n} if and only if $s_{2n} \geq (\beta/\alpha)\, J$. The doctor offers

$$s_{2n} = (\beta/\alpha)\, J,$$

instead of going to court and paying $J > (\beta/\alpha)\, J$ to Paul and an extra C to her lawyer. Now, at $2n-1$, the Doctor accepts s_{2n-1} if and only if $s_{2n-1} \leq (\beta/\alpha)\, J + c$, as she would pay $(\beta/\alpha)\, J$ to Paul next day and an extra c to her lawyer. Paul then offers

$$s_{2n-1} = (\beta/\alpha)\, J + c,$$

as the settlement will be only $(\beta/\alpha)\, J$ next day. He nets $\alpha s_{2n-1} = \beta J + \alpha c$ for himself. Now, at $2n-2$, Paul will accept an offer s_{2n-2} if and only if $s_{2n-2} \geq s_{2n-1} = (\beta/\alpha)\, J + c$, for he could settle for s_{2n-1} next day. (Note that offer gives him αs_{2n-2} and rejection gives him αs_{2n-1}.) Therefore, the Doctor would offer him

$$s_{2n-2} = s_{2n-1}.$$

The pattern is now clear. At any odd date t, the Doctor accepts an offer if and only if $s_t \leq s_{t+1} + c$, and Paul offers

$$s_t = s_{t+1} + c \qquad (t \text{ is odd}).$$

At any even date t, Paul accepts an offer if and only if $s_t \geq s_{t+1}$, and the Doctor offers

$$s_t = s_{t+1} \qquad (t \text{ is even}).$$

The solution to the above equations is

$$s_t = \begin{cases} \frac{\beta}{\alpha}J + \frac{2n+1-t}{2}c & \text{if } t \text{ is odd} \\ \frac{\beta}{\alpha}J + \frac{2n-t}{2}c & \text{if } t \text{ is even} \end{cases}.$$

At the beginning, Paul offers

$$s_1 = \frac{\beta}{\alpha}J + nc,$$

and the Doctor accepts the offer.

Exercise 9.3. In the previous exercise, suppose that with probability 1/2 the Judge may become sick on the court date and a Substitute Judge decides the case in the court. The Substitute Judge is sympathetic to doctors and will dismiss the case. In that case, the Doctor does not pay anything to Paul. (With probability 1/2, the Judge will order the Doctor to pay J to Paul.)

Solution. The expected payment in the court is now

$$J' = \frac{1}{2} \cdot J + \frac{1}{2} \cdot 0 = J/2.$$

Hence, one can simply replace J with $J/2$. The settlement offer at any t is

$$s_t = \begin{cases} \frac{\beta}{2\alpha}J + \frac{2n+1-t}{2}c & \text{if } t \text{ is odd} \\ \frac{\beta}{2\alpha}J + \frac{2n-t}{2}c & \text{if } t \text{ is even} \end{cases}.$$

Exercise 9.4. Apply backward induction to the following game—about food subsidies. The players are a farmer, a consumer, and a government. There are three dates.

- At date $t = 0$, the farmer sets a price $p_0 \in [0, 1]$, and the consumer either agrees to the price or rejects it.

- At $t = 1$, the government sets a subsidy rate $s \in [0, 1]$.

- At $t = 2$, if the consumer has agreed to p_0, the price is p_0, and the consumer decides how much to buy from the farmer. Otherwise, farmer sets a new price $p_1 \in [0, 1]$, and the consumer decides how much to buy from the farmer (at price p_1). (The new price p_1 can be equal to p_0.)

At any instance all the previous moves are known. The payoffs of the farmer, the consumer, and the government are

$$\begin{aligned} u_F(p, q, s) &= pq \\ u_C(p, q, s) &= q - q^2/2 - (p - s)\, q \\ u_G(p, q, s) &= (1 - s)\, q, \end{aligned}$$

respectively, where p is the price, $q \geq 0$ is the amount the consumer buys, and s is the subsidy rate.

Solution. It may be useful to describe each player's strategy set first. The Farmer's strategies are pairs (p_0, p_1) where $p_0 \in [0, 1]$ and

$$p_1 : [0, 1]^2 \to [0, 1],$$

mapping (p_0, s) to a new price. The consumer's strategies are triplets (α, q_0, q_1) where

$$\alpha : [0, 1] \to \{\text{Accept, Reject}\}$$

determines which prices she accepts at $t = 0$,

$$q_0 : [0, 1]^2 \to [0, \infty)$$

determines the demand $q_0(p_0, s)$ at price p_0 after agreement, and

$$q_1 : [0, 1]^3 \to [0, \infty)$$

determines the demand $q_1(p_0, s, p_1)$ at price p_1 after rejection. The government's strategies are functions

$$s : [0, 1] \times \{\text{Accept, Reject}\} \to [0, 1].$$

To apply backward induction, first maximize $u_C(p,q,s)$ with respect to q to obtain

$$
\begin{aligned}
q_0(p_0,s) &= 1+s-p_0 && (9.6)\\
q_1(p_0,s,p_1) &= 1+s-p_1. && (9.7)
\end{aligned}
$$

Now consider any history (p_0, Reject, s). Then, the payoff of the farmer from p_1 is

$$u_F(p_1,s,q_1(p_0,s,p_1)) = p_1\cdot(1+s-p_1),$$

which is maximized at

$$p_1(p_0,s) = (1+s)/2. \quad (9.8)$$

Going backwards one more step, at $t=1$, given any history (p_0, Reject), the payoff of the government from subsidy level s is

$$u_G(p_1(p_0,s), q_1(p_0,s,p_1(p_0,s)), s) = (1-s)\cdot(1+s-p_1(p_0,s)) = (1-s)\cdot(1+s)/2.$$

Clearly, the final expression is strictly decreasing in s over $[0,1]$. Hence, at any history (p_0, Reject), the government chooses

$$s(p_0, \text{Reject}) = 0. \quad (9.9)$$

On the other hand, at any history (p_0, Accept), the payoff of the government from subsidy level s is

$$u_G(p_0, q_0(p_0,s), s) = (1-s)\cdot(1+s-p_0).$$

Maximizing this over s, the government chooses

$$s(p_0, \text{Accept}) = p_0/2. \quad (9.10)$$

Now, going back one more step, at $t=0$, consider any offered price p_0. If the consumer rejects p_0, the consumer faces subsidy level $s(p_0, \text{Reject}) = 0$ and price $p_1(p_0,0) = 1/2$. If she agrees to price p_0, then the subsidy level will be $p_0/2$, paying only half of the price herself. Hence, she agrees to p_0 if and only if $p_0 \le 1$. Now, the payoff of the farmer from a price $p_0 \in [0,1]$ is

$$u_F(p_0, q_0(p_0,p_0/2), p_0/2) = p_0\cdot(1+p_0/2-p_0) = p_0\cdot(1-p_0/2),$$

which is maximized at

$$p_0 = 1. \quad (9.11)$$

This completes the backward induction analysis; the solution is given by (9.6-9.11).

Exercise 9.5. Apply backward induction to the following game. Three siblings, namely Alice, Bob, and Carl, own a dollar. The dollar is worth 1 at date $t = 0$, $\delta \in (0,1)$ at date $t = 1$ and 0 afterwards. They are to divide the dollar according to the following protocol. At $t = 0$, Alice is to propose a division $x = (x_1, x_2, x_3)$, where x_1, x_2, x_3 are all non-negative real numbers that add up to 1 (denote the set of all divisions by X). In alphabetical order, Bob and Carl are to respond to the proposal, by saying Accept or Reject. If both brothers accept the proposal, then the dollar is divided according to x, and the game ends with payoff vector x. If any of them rejects the proposal, we proceed to date $t = 1$ (without waiting for Carl's response if Bob rejects). At $t = 1$, **the brother who rejected Alice's proposal** is to propose a division $y = (y_1, y_2, y_3)$, and the other two siblings are to respond in the alphabetical offer, by saying Accept or Reject. If both siblings accept the proposal, then the dollar is divided according to y, and the game ends with payoff vector $(\delta y_1, \delta y_2, \delta y_3)$. If any of them rejects the proposal, the game ends and each gets 0. The players observe all the previous moves (i.e. it is a perfect-information game).

Solution. At $t = 1$, both siblings accept any offer, as they get 0 when the offer is rejected. Hence, at that round, proposing brother offers 0 to the others and keeps 1 dollar to himself. Now consider $t = 0$. Consider any offer x of Alice and suppose that Bob accepts the offer. Now, Carl would get δ (i.e. 1 dollar at $t+1$) if he rejects the offer x. Hence, Carl accepts the offer x if and only if $x_3 \geq \delta$. Now consider Bob's response to any offer x of Alice. If $x_3 < \delta$, Bob would get 0 if he accepts x because Carl would reject it and offer 0 to Bob in the next round. Hence, Bob rejects x whenever $x_3 < \delta$. For any x with $x_3 \geq \delta$, Bob gets x_2 if he accepts the offer and δ if he rejects the offer. Hence, Bob accepts an offer x if and only if $x_2 \geq \delta$ and $x_3 \geq \delta$. Now, consider Alice's decision. If $\delta > 1/2$, all offers of Alice are rejected by Bob, yielding her 0, and hence she is indifferent between all offers. Any offer, which will be rejected, can be picked at the initial node. If $\delta < 1/2$, then Alice's best response is to offer $(1 - 2\delta, \delta, \delta)$.

9.7 Exercises

Exercise 9.6. Apply backward induction to the following special case of pre-trial negotiation. A plaintiff files a suit against a defendant. It is common knowledge that, when they go to court, the defendant will have to pay \$1000,000 to the plaintiff, and \$100,000 to the court. The court date is set 10 days from now. Before the court date plaintiff and the defendant can settle privately, in which case they do not have the court. Until the case is settled (whether in the court or privately) for each day, the plaintiff and the defendant pay \$2000 and \$1000, respectively, to their legal team. To avoid all these costs plaintiff and the defendant are negotiating in the following way. In the first day, the plaintiff demands an amount of money for the settlement. If the defendant accepts, then she pays the amount and they settle. If she rejects, then she offers a new amount. If the plaintiff accepts the offer, they settle for that amount; otherwise the next day the plaintiff demands a new amount; and they make offers alternatively in this fashion until the court day. Players are risk-neutral and do not discount the future.

Exercise 9.7. A Defendant is liable for a damage to a Plaintiff. If they go to court, then with probability 0.1 the Plaintiff will win and get a compensation of amount \$100,000 from the Defendant; if he does not win, there will be no compensation. Going to court is costly: if they go to court, each of the Plaintiff and Defendant will pay \$20,000 for the legal costs, independent of the outcome in the court. Both the Plaintiff and the Defendant are risk-neutral, i.e., each maximizes the expected value of his wealth. Apply backward induction to find a solution in each scenario below.

1. Consider the following scenario: The Plaintiff first decides whether or not to sue the defendant, by filing a case and paying a non-refundable filing fee of \$100. If he does not sue, the game ends and each gets 0. If he sues, then he is to decide whether or not to offer a settlement of amount $\$25,000$. If he offers a settlement, then the Defendant either accepts the offer, in which case the Defendant pays the settlement amount to the Plaintiff, or rejects the offer. If the Defendant rejects the offer, or the Plaintiff does not offer a settlement, the Plaintiff can either pursue the suit and go to court, or drop the suit.

2. Now imagine that the Plaintiff has already paid his lawyer \$20,000 for the legal costs, and the lawyer is to keep the money if they do not go to court. That is, independent of whether or not they go to court, the Plaintiff pays the \$20,000 of legal costs.

Exercise 9.8. Apply backward induction to the following TV game, called Deal or No Deal. There are two players: Banker and Contestant. There are n cash prizes, $v_1, \ldots, v_n$, which are randomly put in n cases, 1, ..., n. Each permutation is equally likely. The prizes $v_1, \ldots, v_n$ are known but neither player knows which prize is in which case. The contestant owns Case 1. There are $n-1$ periods: $t = 1, \ldots, n-1$. Until they reach an agreement, at each period t, Banker makes a cash offer p—to buy Case 1 from the Contestant. The Contestant is to accept ("Deal") or reject ("No Deal") the offer. If she accepts the offer, the Banker buys the case from the Contestant at price p and the game ends. (Banker gets the prize in Case 1 minus p, and the Contestant gets p.) If she rejects the offer, then Case $t+1$ is opened to reveal its content to the players, and we proceed to period $t+1$. When all the cases 2,..., n are opened, the game automatically ends; the Banker gets 0 and the Contestant gets the prize in Case 1. Assume that the utility of having x dollar is x for the Banker and $\sqrt{x}$ for the Contestant. Everything described is common knowledge. (It may be useful to solve first the special case: $n = 3$, $v_1 = 1$, $v_2 = 100$, and $v_3 = 10000$.)

Exercise 9.9. Apply backward induction to the following game. Alice sues a large corporation (defendant) for damages. At date $2n+1$, the judge will decide whether the defendant is guilty. If the judge decides that the defendant is guilty, then the defendant will be ordered to pay 1 to Alice; otherwise there will be no payment between the parties. (Here, the unit of money is million US dollars.) The probability that the judge decides guilty is $p \in (0,1)$. Before the court date, Alice and the defendant can settle out of court, in which case they do not go to court. The settlement negotiation is as follows. In each date $t \in \{1, \ldots, 2n\}$, one of them is to make a settlement offer s_t, and the other party is to decide whether to accept it. If the offer is accepted, the game ends and the defendant pays s_t to Alice. Alice makes the offers on odd dates $1, 3, \ldots, 2n-1$, and the defendant makes the offers on even dates $2, \ldots, 2n$. Alice's payoff from receiving payment x is $x^{1/\alpha}$ for some $\alpha > 1$. She does not discount the payoffs and does not pay any cost for negotiation or for going to court. On the other hand, the defendant is *risk-neutral* and it needs to pay a small fee $c > 0$ to the lawyers for every day the case has not settled (paying ct if they settle at date t and $c(2n+1)$ if they go to court).

Exercise 9.10. Apply backward induction to the following game about arbitration, a common dispute resolution method in the US. We have a Worker, an Employer, and an Arbitrator. They want to set the wage w. If they determine the wage w at date t, the payoffs of the Worker, the Employer and the Arbitrator will be $\delta^t w$, $\delta^t (1-w)$ and $w(1-w)$, respectively, where $\delta \in (0,1)$. The timeline is as follows:

- At $t = 0$,
  - the Worker offers a wage w_0;
  - the Employer accepts or rejects the offer;
  - if she accepts the offer, then the wage is set at w_0 and the game ends; otherwise we proceed to the next date;
- at $t = 1$,
  - the Employer offers a wage w_1;
  - the Worker accepts or rejects the offer;
  - if he accepts the offer, then the wage is set at w_1 and the game ends; otherwise we proceed to the next date;
- at $t = 2$, the Arbitrator sets a wage $w_2 \in [0,1]$ and the game ends.

Compute an equilibrium of this game using backward induction.

Exercise 9.11. Apply backward induction to the following variation of the previous problem. *Final Offer Arbitration:* At $t = 2$, the Arbitrator sets a wage $w_2 \in \{w_0, w_1\}$, i.e., the Arbitrator has to choose one of the offers made by the parties.

Exercise 9.12. Apply backward induction to the following game. The players are $2n+1$ congressmen, $i = 1, 2, \ldots, 2n+1$, for some $n \geq 1$. Each congressmen represents one district, where the distinct districts are represented by distinct congressmen. The government has a total budget of 1 unit to be allocated to the $2n+1$ districts the congressmen represent, as follows. The set of possible allocations are $(x_1, x_2, \ldots, x_{2n+1}) \geq (0,,0,\ldots,0)$ with $x_1 + x_2 + \cdots + x_{2n+1} \leq 1$. The possible dates are $t = 1, 2, \ldots, T$ where $T \gg 2n+1$. Fix also a number $\delta \in (0,1)$ and a sequence $i_1, i_2, \ldots, i_T$ of congressmen. At each date t, the congressman i_t offers an allocation $(x_1, x_2, \ldots, x_{2n+1})$, and all the remaining congressmen say Yes or No in the increasing order (i.e. 1 responds before 2; 2 responds before 3, and so on). If at least n congressmen say Yes, the game ends, and the proposed allocation $(x_1, x_2, \ldots, x_{2n+1})$ is implemented, yielding payoff vector $\left(\delta^t x_1, \delta^t x_2, \ldots, \delta^t x_{2n+1}\right)$. Otherwise, we proceed to the next day—until the date T. If the proposal at T also fails to get n Yes votes, then the game ends with payoff vector $(0, 0, \ldots, 0)$.

Exercise 9.13. Apply backward induction to the following game of coalition formation in a parliamentary system. There are three parties A, B, and C who just won 41, 35, and 25 seats, respectively, in a 101-seats parliament. In order to form a government, a coalition (a subset of $\{A, B, C\}$) needs 51 seats in total. The parties in the government enjoy a total 1 unit of perks, which they can share in any way they want. The parties outside the government get 0 units of perks, and each party tries to maximize the expected value of its own perks. The process of coalition formation is as follows. First A is given right to form a government. If it fails, then B is given right to form a government, and if B also fails then C is given to form a government. If C also fails, then the game ends and each gets 0. The party who is given right to form a government, say i, approaches one of the other two parties, say j, and offers some $x \in [0, 1]$. If j accepts, then they form the government and i gets $1 - x$ and j gets x units of perks. If j rejects the offer, then i fails to form a government (in which case, as described above, either another party is given right to form a government or game will and with 0 payoff).