Chapter 6

Application: Imperfect Competition

In many markets, there are only a couple of firms that supply most of the products. For example, there are only a couple of grocery stores in a typical suburban neighborhood, a couple of hospitals in a typical medium size city, and a couple of airlines flying directly between most cities.

When there are few firms, firms may have market power, and each of them may be able to affect the market prices. For example, consider the case of monopoly with a single divisible good. There is only one firm in the market. The firm can produce any quantity $Q \in [0, 1]$ of the good at zero cost and sell it at price $P = 1 - Q$. The monopolist unilaterally determines the market price, either directly by setting its own price P or indirectly by setting its supply Q. If the firm sets price P, it sells $1 - P$ units, obtaining profit level $P \times (1 - P)$. In order to maximize its profit, it sets the price at $P_{mon} = 1/2$ and sells $Q_{mon} = 1/2$ units, obtaining a profit of $1/4$. Alternatively, towards maximizing its profit, it can supply Q_{mon} and end up selling at price P_{mon}, obtaining the same profit.

When there are two or more firms, a firm's optimal strategy depends on the strategies employed by the other firms because the other firms can also affect the market price. This requires a non-trivial game theoretical analysis of the market. Equilibrium analysis was applied to such markets a century before John Nash introduced the concept of Nash equilibrium for general games. In 1838, Antoine Augustin Cournot developed a game theoretical model of oligopoly—with multiple firms serving a single market—and analyzed its Nash equilibrium. (Hence, sometimes Nash equilibrium is called Cournot-Nash equilibrium.) In this model, there are n firms. Each firm i produces q_i units, and the price is a function of total supply $Q = q_1 + \cdots + q_n$. When $P = 1 - Q$, there is a unique equilibrium, in which each firm produces $q_i = 1/(n+1)$, and the equilibrium price is $1/(n+1)$.

Firms' market power and profit maximization come in the expense of the consumers and overall social efficiency. For example, in the case of monopoly, there are customers who are willing to buy the product at price $p = 1/3$, which would also yield a positive profit for the firm. But the firm foregoes this profit opportunity and prices out those customers because lowering its price costs the monopolist as a price reduction to all the customers who are willing to pay higher prices. For each number n of firms, there is some social inefficiency in terms of foregone trade opportunities (below price $1/(n+1)$), but as n gets larger, the set of forgone trade opportunities shrink. In the limit, the price is zero, and all the gain from trade is realized. The limit corresponds to the competitive equilibrium, in which the demand is equal to supply. In this example, the demand is given by $P = 1 - Q$, and the supply is given by $P = 0$ as it is free to produce arbitrary amounts.

All in all, in small markets, firms exercise market power that causes social inefficiency, but the inefficiency shrinks as the market becomes competitive.

In the above oligopoly model, the firms choose quantities, and the price is determined by the market. But the market is just a metaphor here, and there is no actual mechanism that sets the price as a function of total supply. Instead we see firms setting their prices and the consumers choosing whether or where to buy from.[1] What would happen if each firm sets its own price and the customers choose whether to buy the good (from the cheapest firm)? It may seem that this should not change the analysis, as in the case of the monopolist. As it turns out, it does, and it changes the conclusions dramatically.

In 1883, Joseph Bertrand formulated exactly the above model. In this model, two is enough for competition. Even with two firms, each firm tries to undercut each other when the prices are positive, and in the unique Nash equilibrium, the prices are set to

[1] When the customers also have large market power, they may be negotiating the price. For example, it is hard to imagine Amazon or Walmart passively deciding whether to buy at a given price set by some small factory in Alabama.

zero, as in a competitive market.

These models are still central to microeconomics and industrial organization, and they will form the building blocks of more modern applications in later chapters. The two models have drastically different policy implications. Should we regulate the markets to gain efficiency and protect the consumers? Cournot oligopoly implies that regulation, such as price caps, may increase efficiency and lead to lower prices in markets with few firms, such as the airline industry. It also implies that regulation may not be effective or necessary in markets with many firms, such as the restaurant industry (with restaurants clustered in downtowns). Bertrand competition begs to differ. As long as there are two firms, competition over prices leads to perfectly competitive outcomes, and hence regulation is not necessary and cannot be effective. Would e-commerce increase the market efficiency by providing alternatives to customers who otherwise have only a few places to buy from? Cournot oligopoly would say Yes while Bertrand's model suggests otherwise.

Which model is more relevant? As it turns out, even if Bertrand competition is a more accurate description of competition, Cournot's predictions are the correct ones when it comes to capacity creation. For example, imagine two farmers selling their produce in a farmers' market. If they fill their carts with so much produce that either of them can satisfy the entire demand, then in the market they will engage price competition and they will undercut each others' prices and end up selling their produce at nearly zero price as in Bertrand competition. If they put small amounts in their carts, then price competition will not be effective and each will lower its price only to the level where they sell the produce they brought. Foreseeing this behavior, each will bring a certain amount to the market. Cournot's model corresponds to the stage at which the farmers decide how much to bring to the market.

Of course, the relevant model for policy analysis could be different from either of these models, as it may involve repeated interaction, incomplete information, and market frictions, such as search costs. All these will have substantial impact on the outcome and they will be incorporated in due course. Indeed, search cost will be incorporated in this chapter.

6.1 Principles of Demand and Supply

This chapter uses basic concepts in microeconomics, such as demand, supply, and competitive equilibrium. This section briefly introduces these concept for the sake of completeness. Consider n firms and a single divisible goods produced by these firms. There are also consumers who buy the good where the amount of the good they buy depends on the price.

Demand Given any price P, each consumer buys some amount of the good depending on the price. Let $Q(P)$ be the total amount bought by the consumers at price P. The amount $Q(P)$ is called the *demand at price* P. For example, if there are two consumers, namely Alice and Bob, and Alice buys 5 units and Bob buys 3 units at a given price P, the demand at price P is $Q(P) = 5 + 3 = 8$. The function Q that maps prices P to the demand $Q(P)$ at those prices is called *the demand function*, and the inverse of the demand function, which maps the quantities to the prices, is called *the inverse-demand function.*

Supply Imagine that the price is fixed at some P, and no firm can influence the price. Each firm i produces some amount $q_i(P)$ to maximize its profit under price P. The total amount, $q_1(P) + \cdots + q_n(P)$, produced by the firms is called the supply at price P, and the function that maps prices to the supplies at those prices is called the *supply function.* The inverse of the supply function is called the *inverse-supply function.*

Each firm's production level and the supply depend on the cost of production for the firms. For each firm i, let $C_i(q)$ be the total cost of producing q units for firm i. The derivative $C_i'(q)$ of the cost function is called the *marginal cost*, often denoted by c_i. I will often use a constant marginal cost c, so that the total cost is $C_i(q) = c \times q$. In general, the total cost is $C_i(q) = \int_0^q c_i(x)\,dx$, where c_i is the marginal cost function. Each firm chooses $q_i(P)$ to maximize its profit $Pq - C_i(q)$, and hence it produces an amount q_i at which the price is equal to its marginal cost:

$$P = C_i'(q_i).$$

For example, if the cost function is $C_i(q) = \frac{1}{2}\gamma q^2$ for each firm, then the marginal cost of each firm is $c_i(q) = \gamma q$, and the equation $P = C_i'(q_i(P)) = \gamma q_i(P)$ yields

$$q_i(P) = P/\gamma,$$

and the supply function is $S(P) = nP/\gamma$. For another example, consider the case in which each firm has constant marginal cost c for some $c > 0$. Then, any production level q_i is allowable at price $P = c$; each firm supplies 0 at any price $P < c$, and the supply is not well-defined at any price $P > c$ as each firm would like to produce an infinite amount of the good. The supply function is not well-defined, but the inverse-supply function is well defined, it is given by $P = c$.

Competitive Equilibrium The economy is in a competitive equilibrium when the demand is equal to supply. A competitive equilibrium is a vector of price P^* and the quantities $q_1^*, \ldots, q_n^*$ such that $q_i^* = q_i(P^*)$ for each firm (so that the firms choose their production levels optimally given the price), and the demand is equal to supply, i.e., $Q(P^*) = q_1(P^*) + \cdots + q_n(P^*)$. Graphically, the competitive equilibrium arises when the demand function intersects the supply function. For example, under a constant marginal cost c and decreasing demand function $Q(P)$, a competitive equilibrium is given by the price

$$P^* = c,$$

which is equal to the marginal cost, and production levels for the firms that add up to $Q(c)$.

6.2 Cournot (Quantity) Competition

Consider n firms. Each firm i produces $q_i \geq 0$ units of a good at marginal cost $c \geq 0$ and sells it at price

$$P = \max\{1 - Q, 0\} \tag{6.1}$$

where

$$Q = q_1 + \cdots + q_n \tag{6.2}$$

is the total supply; here $\max\{1 - Q, 0\}$ is the inverse-demand function. Each firm maximizes the expected profit. Hence, the payoff of firm i is

$$\pi_i = q_i(P - c). \tag{6.3}$$

Unlike in the previous section, where a firm cannot affect the price, here each firm can affect the price as long as the total production by the other firms is less than 1. The competition is imperfect in this sense. Such imperfect competition economy is called *Cournot oligopoly*; it is called *monopoly* when there is only one firm and *duopoly* when there are two firms. When the demand function is linear and the marginal costs are constant, it is called *linear Cournot oligopoly*. This section explores the linear Cournot oligopoly above.

Assuming all of the above is commonly known, one can write this as a game in normal form, by setting

- $N = \{1, 2, \ldots, n\}$ as the set of players
- $S_i = [0, \infty)$ as the strategy space of player i, where a typical strategy is the quantity q_i produced by firm i, and
- $\pi_i : S_1 \times \cdots \times S_n \to \mathbb{R}$ as the payoff function.

Best Response We first determine the best response of a firm i to the total production

$$Q_{-i} = \sum_{j \neq i} q_j \tag{6.4}$$

by the other firms. This best response will be used extensively throughout (see also Exercise 2.5). If $Q_{-i} > 1$, then the price is zero, and the best firm i can do is to produce zero and obtain zero profit. Now assume $Q_{-i} \leq 1$. For any $q_i \in [0, 1 - Q_{-i}]$, the profit of the firm i is

$$\pi_i \left(q_i, Q_{-i}\right) = q_i \left(1 - q_i - Q_{-i} - c\right). \tag{6.5}$$

This profit function is maximized at

$$q_i^B \left(Q_{-i}\right) = \frac{1 - Q_{-i} - c}{2}. \tag{6.6}$$

This is the best response function of a firm i. (The profit function is plotted in Figure 6.1. The best response function is plotted in Figure 6.2.)

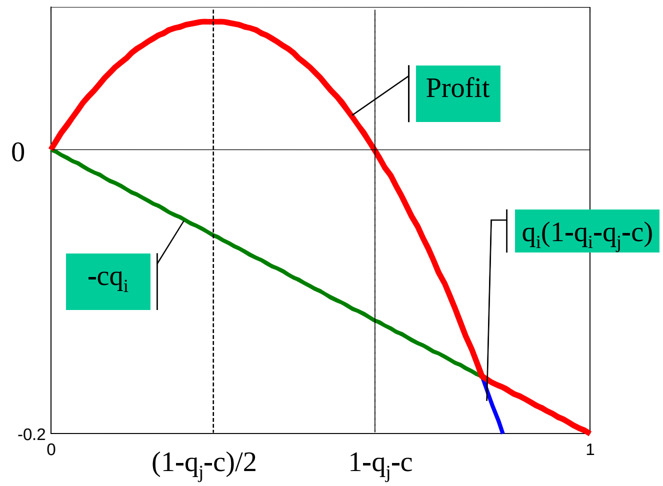

Figure 6.1: Profit function in linear Cournot dupopoly

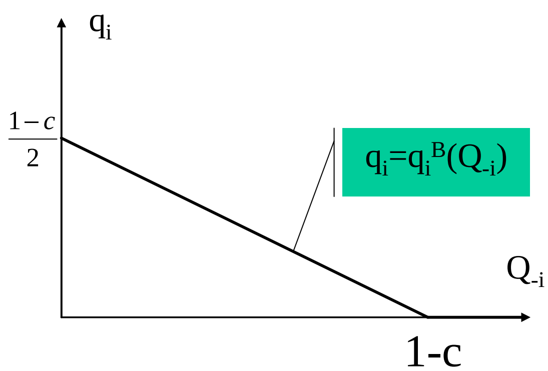

Figure 6.2: Best Response function in linear Cournot oligopoly

6.2.1 Cournot Duopoly

Now, consider the case of two firms, i.e., duopoly. In that case, a firm i faces only one other firm j, and hence $Q_{-i} = q_j$. This special case can be analyzed graphically with the aid of the best-response functions as in Figure 6.3. One applies Nash equilibrium and rationalizability to this case as follows.

Nash Equilibrium A Nash equilibrium is a pair (q_1, q_2) of production levels for the firms. Any Nash equilibrium (q_1, q_2) must satisfy

$$q_1 = q_1^B(q_2) \equiv \frac{1 - q_2 - c}{2}$$

and

$$q_2 = q_2^B(q_1) \equiv \frac{1 - q_1 - c}{2}.$$

Solving these two equations simultaneously, one can obtain

$$q_1^* = q_2^* = \frac{1-c}{3}$$

as the only Nash equilibrium. Graphically, as in Figure 6.3, one can plot the best response functions of each firm and identify the intersections of the graphs of these functions as Nash equilibria. In this case, there is a unique intersection, and therefore there is a unique Nash equilibrium.

Rationalizability The (linear) Cournot duopoly game considered here is dominance-solvable. That is, there is a unique rationalizable strategy. This can be seen intuitively from the first couple rounds of elimination. (It will be shown mathematically in the Appendix that this is indeed the case.)

Round 1 Notice that a strategy $\hat{q}_i > (1-c)/2$ is strictly dominated by $(1-c)/2$. To see this, consider any q_j. As in Figure 6.1, $\pi_i(q_i, q_j)$ is strictly increasing until $q_i = (1 - c - q_j)/2$ and strictly decreasing thereafter. In particular,

$$\pi_i((1-c-q_j)/2, q_j) \geq \pi_i((1-c)/2, q_j) > \pi_i(\hat{q}_i, q_j),$$

showing that $\hat{q}_i$ is strictly dominated by $(1-c)/2$. Therefore, eliminate every strategy $\hat{q}_i$ with $\hat{q}_i > (1-c)/2$ for each player i. The remaining strategies are as in Figure 6.4, where the shaded area is eliminated.

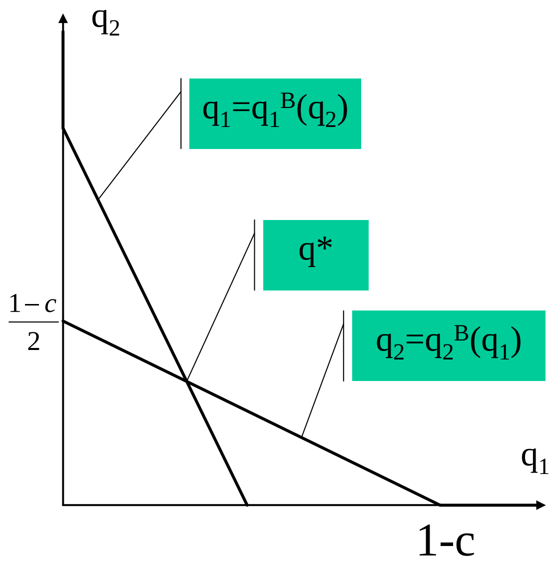

Figure 6.3: Best response functions and the Nash equilibrium in Cournot duopoly

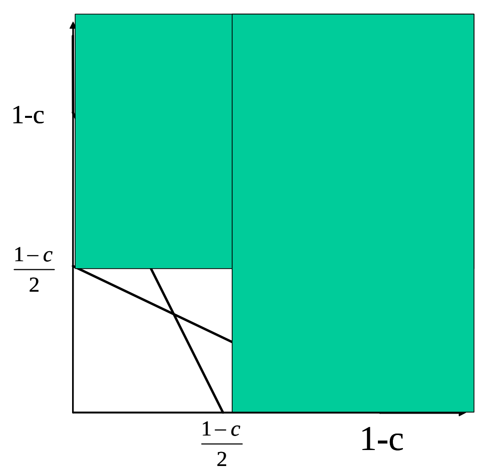

Figure 6.4: The remaining strategies after one round of elimination in linear Cournot duopoly

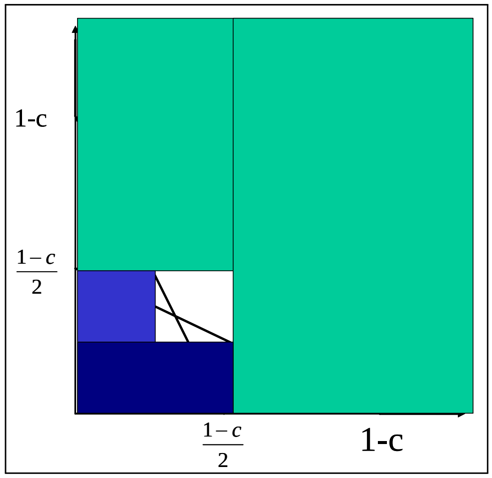

Figure 6.5: The strategies remaining after two rounds of elimination in linear Cournot duopoly

Round 2 In the remaining game $q_j \leq (1-c)/2$. Consequently, any strategy $\hat{q}_i < (1-c)/4$ is strictly dominated by $(1-c)/4$. To see this, take any $q_j \leq (1-c)/2$ and recall from Figure 6.1 that π_i is strictly increasing until $q_i = (1-c-q_j)/2$, which is greater than or equal to $(1-c)/4$. Hence,

$$\pi_i(\hat{q}_i, q_j) < \pi_i((1-c)/4, q_j) \leq \pi_i((1-c-q_j)/2, q_j),$$

showing that $\hat{q}_i$ is strictly dominated by $(1-c)/4$. Therefore, eliminate all $\hat{q}_i$ with $\hat{q}_i < (1-c)/4$. The remaining strategies are as in Figure 6.5.

Notice that the remaining game is a smaller replica of the original game. Applying the same procedure repeatedly, one can eliminate all strategies except for the Nash equilibrium. (After every two rounds, a smaller replica is obtained.) Therefore, the only rationalizable strategy is the unique Nash equilibrium strategy:

$$q_i^* = (1-c)/3.$$

6.2.2 Cournot Oligopoly

Now consider the case of three or more firms. When there are three or more firms, rationalizability does not help: one cannot eliminate any strategy less than the monopoly production $q^1 = (1-c)/2$.

Rationalizability In the first round, one can eliminate any strategy $q_i > (1-c)/2$, using the same argument in the case of duopoly. But in the second round, the maximum possible total supply by the other firms is

$$(n-1)(1-c)/2 \geq 1-c,$$

where n is the number of firms. The best response to this aggregate supply level is 0. Hence, one cannot eliminate any strategy in round 2. The elimination process stops, yielding $[0,(1-c)/2]$ as the set of rationalizable strategies. Since the set of rationalizable strategies is large, rationalizability has a weak predictive power in this game.

Nash Equilibrium Nash equilibrium remains to have a strong predictive power: there is a unique Nash equilibrium. Recall that $q^* = (q_1^*, q_2^*, \ldots, q_n^*)$ is a Nash equilibrium if and only if

$$q_i^* = q_i^B\left(\sum_{j\neq i} q_j^*\right) = \frac{1-\sum_{j\neq i} q_j^* - c}{2}$$

for all i, where the second equality by (6.6) and the fact that the firms cannot have negative profits in equilibrium (i.e. $\sum_{j\neq i} q_j^* \leq 1-c$). Rewrite this equation system more explicitly:

$$\begin{aligned}
2q_1^* + q_2^* + \cdots + q_n^* &= 1-c \\
q_1^* + 2q_2^* + \cdots + q_n^* &= 1-c \\
&\cdots \\
q_1^* + q_2^* + \cdots + 2q_n^* &= 1-c.
\end{aligned}$$

For any i and j, by subtracting jth equation from ith, one can obtain

$$q_i^* - q_j^* = 0.$$

That is, all firms supply the same amount in equilibrium:

$$q_1^* = q_2^* = \cdots = q_n^*.$$

Substituting this into the first equation, one then obtains

$$(n+1)\,q_1^* = 1-c;$$

i.e.,

$$q_1^* = q_2^* = \cdots = q_n^* = \frac{1-c}{n+1}.$$

Therefore, there is a unique Nash equilibrium, in which each firm produces $(1-c)\,/\,(n+1)$.

In the unique equilibrium, the total supply is

$$Q = \frac{n}{n+1}\,(1-c)$$

and the price is

$$P = c + \frac{1-c}{n+1}.$$

The profit level for each firm is

$$\pi = \left(\frac{1-c}{n+1}\right)^2.$$

Recall that, in competitive equilibrium, the price is equal to the marginal cost c and the total supply is $1-c$, as these are the values at which the demand ($P = \max\{1-Q, 0\}$) is equal to supply ($P = c$). In general for any n, the price is larger than the competitive price c, and the firms charge a mark up $\frac{1-c}{n+1}$. They do so by collectively supplying less than the competitive supply level $1-c$. Consumers are hurt by this non-competitive behavior, some by paying higher prices and some by foregoing the benefit of the good, when the benefit is in between the marginal cost c and the price P. When there are few firms, the price is significantly higher than the competitive price c, and the total supply is significantly lower than the competitive supply $1-c$. As n goes to infinity, the total supply Q converges to the competitive supply $1-c$, and price P converges to the competitive price c. In the limit, the economy is in its competitive equilibrium. The firms collectively produce the competitive supply $1-c$ and sell their marginal cost c, obtaining zero profit. Next section presents another model, in which two firms are enough for the competitive outcome.

6.3 Bertrand (Price) Competition

Consider two firms. Simultaneously, each firm i sets a price p_i. The firm i with the lower price $p_i < p_j$ sells $1-p_i$ units and the other firm cannot sell any. If the firms set the same price, the demand is divided between them equally. That is, the amount of sales for firm i is

$$Q_i(p_1, p_2) = \begin{cases} 1 - p_i & \text{if } p_i < p_j \\ \frac{1-p_i}{2} & \text{if } p_i = p_j \\ 0 & \text{otherwise.} \end{cases}$$

Assume that it costs nothing to produce the good (i.e. $c = 0$). Therefore, the profit of a firm i is

$$\pi_i(p_1, p_2) = p_i Q_i(p_1, p_2) = \begin{cases} (1 - p_i) p_i & \text{if } p_i < p_j \\ \frac{(1-p_i)p_i}{2} & \text{if } p_i = p_j \\ 0 & \text{otherwise.} \end{cases}$$

Assuming all of the above is commonly known, one can write this formally as a game in normal form by setting

- $N = \{1, 2\}$ as the set of players
- $S_i = [0, \infty)$ as the set of strategies for each i, with price p_i a typical strategy,
- π_i as the utility function.

Observe that when $p_j = 0$, $\pi_i(p_1, p_2) = 0$ for every p_i, and hence every p_i is a best response to $p_j = 0$. This has two important implications:

1. Every strategy is rationalizable (one cannot eliminate any strategy because each of them is a best reply to zero).
2. $p_1^* = p_2^* = 0$ is a Nash equilibrium.

In the sequel, I will first show that this is indeed the only Nash equilibrium. In other words, even with two firms, when the firms compete by setting prices, the competitive equilibrium will emerge. I will then show that if we modify the game slightly by discretizing the set of allowable prices and putting a minimum price, then the game becomes dominance-solvable, i.e., only one strategy remains rationalizable. In the modified game, the minimum price is the only rationalizable strategy, as in competitive equilibrium. Finally I will introduce small search costs on the part of consumers, who are not modeled as players, and illustrate that the equilibrium behavior is dramatically different from the equilibrium behavior in the original game and competitive equilibrium.

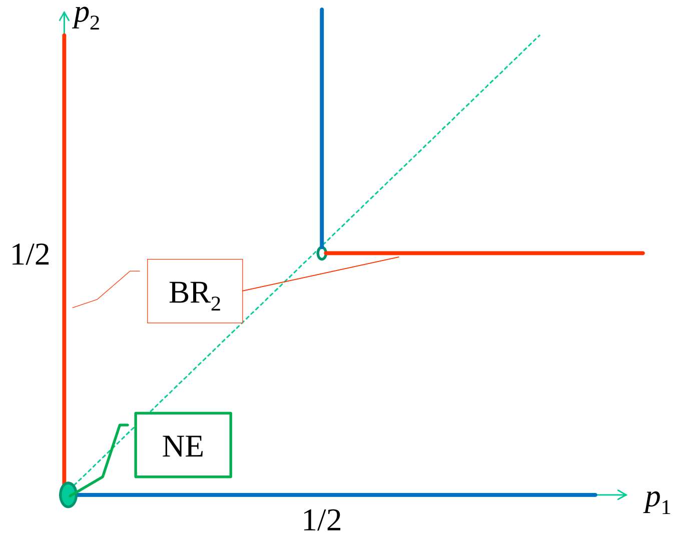

Figure 6.6: Best responses and Nash equilibrium in price competition

6.3.1 Rationalizability and Nash Equilibrium

Theorem 6.1. *In Bertrand competition, every strategy is rationalizable and the only Nash equilibrium is* $p^* = (0, 0)$.

A graphical proof for the above result is as follows. The best responses in this game have been derived in Exercise 2.5 and plotted in Figure 6.6. Observe that everything is a best response to $p_j = 0$; nothing is a best response to any p_j with $0 < p_j \leq 1/2$, and $1/2$ is the unique best response to any $p > 1/2$. Now, since everything is a best response to zero, no strategy is strictly dominated, and hence all strategies survive iterated elimination of strictly dominated strategies. Therefore, every strategy is rationalizable. Towards computing Nash equilibria recall that (p_1, p_2) is a Nash equilibrium if and only if it is in the intersection of the best responses. But, as shown in Figure 6.6, the best responses intersect each other only at $(0, 0)$, showing that $(0, 0)$ is the only Nash equilibrium.

6.3.2 Discrete prices*

Now suppose that the firms have to set prices as multiples of pennies, and they cannot charge zero price. That is, the set of allowable prices is

$$P = \{0.01, 0.02, 0.03, \ldots\}.$$

The key assumption here is that the minimum allowable price $P_{\min} = 0.01$ yields a positive profit. Under this assumption, the game is dominance-solvable. In particular $P_{\min}$ is the only rationalizable strategy, and it is the only Nash equilibrium strategy. The following lemma is the first step to establish this fact.

Lemma 6.1 (Step 1). *Any price p greater than the monopoly price $P_{mon} = 0.5$ is strictly dominated by a mixed strategy that assigns some probability $\epsilon > 0$ to the price $P_{\min} = 0.01$ and probability $1 - \epsilon$ to the price $P_{mon} = 0.5$.*

A detailed proof of this step is presented in the appendix. Here, I will assume ϵ is positive but negligible, without discussing how small it should be. In that case, the payoff from playing σ^ϵ will be approximately the payoff from playing monopoly price P_{mon} but it will always be strictly positive—because with positive probability ϵ, one charges the minimum price $P_{\min}$ and gets strictly positive profit against any price. Take any player i and any price $p_i > P_{mon}$. We want to show that the mixed strategy σ^ϵ strictly dominates p_i:

$$\pi_i(\sigma^\epsilon, p_j) > \pi_i(p_i, p_j) \tag{6.7}$$

for every p_j. First, consider $p_j > P_{mon}$. In that case, clearly,

$$\pi_i(p_i, p_j) \leq p_i Q(p_i) = p_i(1 - p_i) \leq 0.51 \cdot 0.49 = 0.2499,$$

where the first inequality is by definition and the last inequality is due to the fact that $p_i \geq 0.51$. On the other hand, if he played σ^ϵ, his payoff would be approximately payoff from charging monopoly price

$$\pi_i(P_{mon}, p_j) = P_{mon}(1 - P_{mon}) = 1/4 > \pi_i(p_i, p_j).$$

Therefore, (6.7) holds when ϵ is small. Now, consider $p_j \leq P_{mon}$. In that case, since $p_i > P_{mon} \geq p_j$, price p_i yields zero profit: $\pi_i(p_i, p_j) = 0$. In contrast, the profit from σ^ϵ is always positive. Therefore, (6.7) holds once again.

Step 1 yields the eliminations in the first round.

Round 1 By Lemma 6.1, all strategies p_i with $p_i > P_{mon} = 0.5$ are eliminated. Moreover, each $p_i \leq P_{mon}$ is a best reply to $p_j = p_i + 0.01$, and is not eliminated. Therefore, the set of remaining strategies is

$$P^2 = \{0.01, 0.02, \ldots, 0.5\}.$$

Round m Suppose that the set of remaining strategies to round m is

$$P^m = \{0.01, 0.02, \ldots, \bar{p}\}.$$

Then, the strategy $\bar{p}$ is strictly dominated by the mixed strategy σ^ϵ with $\sigma^\epsilon(\bar{p} - 0.01) = 1 - \epsilon$ and $\sigma^\epsilon(P_{\min}) = \epsilon$, as we will see momentarily. Hence, the strategy $\bar{p}$ is eliminated. There will be no more elimination because each $p_i < \bar{p}$ is a best reply to $p_j = p_i + 0.01$.

To prove that $\bar{p}$ is strictly dominated by σ^ϵ, note that the profit from $\bar{p}$ for player i is zero whenever $p_j < \bar{p}$. Since the payoff from σ^ϵ is always positive (as in step 1), this shows that $\pi_i\left(\sigma^\epsilon_{\bar{p}}, p_j\right) > \pi_i(\bar{p}, p_j) = 0$ for any $p_j < \bar{p}$. Now take $p_j = \bar{p}$. Then, $\bar{p}$ yields

$$\pi_i(\bar{p}, \bar{p}) = \bar{p}(1 - \bar{p})/2$$

while σ^ϵ yields approximately the payoff from charging $\bar{p} - 0.01$:

$$\begin{aligned} \pi_i(\bar{p} - 0.01, \bar{p}) &= (\bar{p} - 0.01)(1 - \bar{p} + 0.01) \\ &= \bar{p}(1 - \bar{p}) - 0.01(1 - 2\bar{p}) \\ &> \bar{p}(1 - \bar{p})/2 = \pi_i(\bar{p}, \bar{p}) \end{aligned}$$

where the inequality holds because $\bar{p} \geq 0.02$. Therefore, σ^ϵ strictly dominates $\bar{p}$.

The elimination process continues until the set of remaining strategies is $\{P_{\min}\}$ and it stops there. Therefore, $P_{\min}$ is the only rationalizable strategy. Since players can put positive probability only on rationalizable strategies in a Nash equilibrium, the only possible Nash equilibrium is $(P_{\min}, P_{\min})$, which is clearly a Nash equilibrium.

The main idea is that the firms can always profitably undercut other firms' prices if the others are charging more than the minimum. They do this either lowering the prices slightly or charging the monopoly price. The presence of a minimum price with positive profit is crucial. If the firms could charge zero price, then, as in the original Bertrand competition, every price would be a best response to zero, and every price would be rationalizable.

6.3.3 Price competition with search costs

This section illustrates that the equilibrium behavior is quite different when the consumers need to engage a costly search in order to learn the prices offered by the firms, regardless of how small these costs are.

For simplicity, allow only two prices: 3 and 5. Suppose that the demand for the good comes from a single buyer, for who the value of the good is 6. She needs only 1 unit of good. In the previous models, the prices were automatically revealed to her and she bought from the firm with the lower price when the prices are different. Now, the buyer needs to incur a small search cost $c_s \in (0, 1)$ in order to check the prices. If foregoes checking the prices, she will need to choose a firm to buy the product from without knowing the prices.

The game is as follows:

- There are two firms, namely 1 and 2, and a buyer.
- Firms 1 and 2 set prices $p_1 \in \{3, 5\}$ and $p_2 \in \{3, 5\}$, respectively, and the buyer decides whether to check the prices, all simultaneously.
- If she checks the prices, then she buys from the firm with the lower price. If she decides not to check or if $p_1 = p_2$, then she buys from either of the firms with equal probabilities. This behavior is set, so that the strategies of the buyer is only "check" and "no check".

Formally,

- $N = \{1, 2, B\}$ is the set of players;
- $S_1 = S_2 = \{3, 5\}$ and $S_B = \{\text{check, no check}\}$ are the strategy sets; and
- the payoffs are as in the following table:

check

	5	3
5	$5/2, 5/2, 1 - c_s$	$0, 3, 3 - c_s$
3	$3, 0, 3 - c_s$	$3/2, 3/2, 3 - c_s$

no check

	5	3
5	$5/2, 5/2, 1$	$5/2, 3/2, 2$
3	$3/2, 5/2, 2$	$3/2, 3/2, 3$

Here, the first entry is the payoff of firm 1; the second entry is the payoff of firm 2, and the final entry is the payoff of the buyer. Firm 1 chooses the row; firm 2 chooses the column, and the buyer chooses the matrix. The payoffs are computed according to the set behavior above. For example, if the buyer doesn't check the price, she buys from the either firm with probability 1/2. Hence, the payoff of firm i is $p_i/2$, independent of p_j. The payoff of the buyer is

$$0.5\,(6-p_1)+0.5\,(6-p_2)=6-\frac{p_1+p_2}{2}.$$

If the buyer checks and $p_1=p_2$, then the payoffs are: $p_1/2$ to each firm and $6-p_1-c_s$ to the buyer. If the buyer checks and $p_i<p_j$, then the buyer buys one unit from i, and the payoff of firm i is p_i; the payoff of firm j is 0, and the payoff of the buyer is $6-p_i-c_s$.

A quick glance at the above table reveals that there is a unique pure-strategy Nash equilibrium: both firms set price to 5 ($p_1=p_2=5$), and the buyer does not check the prices. This is clearly different from the previous games, where price competition pushes the prices to the minimum.

It is easy to check that ($p_1=p_2=5;$ no check) is a Nash equilibrium: Given "no check", $p_i=5$ dominates $p_i=3$. Given that prices are equal, the buyer saves c_s by not checking. It is also easy to check that this is the only Nash equilibrium in pure strategies. If $p_1=p_2$, the best response of the buyer is "no check". If buyer does not check, then the best reply of each firm is 5. Therefore, the only equilibrium with $p_1=p_2$ is ($p_1=p_2=5;$ no check). Moreover, there cannot be a Nash equilibrium with $p_1\neq p_2$. To see this, suppose that $p_i=5$ and $p_j=3$. Then, the buyer gets $3-c_s$ when she checks and 2 when she does not. The best reply is to check because $c_s<1$. That is, ($p_i=5$, $p_j=3$, no check) is not an equilibrium. In order to have an equilibrium, she must check. But ($p_i=5$, $p_j=3$, check) is not an equilibrium either. Now, firm i gets 0, and she has an incentive to deviate and charge price $p_i=3$, obtaining a higher payoff of 3/2.

There is also a symmetric Nash equilibrium in mixed strategies. To find the equilibrium, write q for the probability that a firm sets $p_i=5$ (the probabilities are equal by assumption) and r for the probability that the buyer checks, assuming $q,r\in(0,1)$. The expected payoff from checking for the buyer is

$$U_B\,(\text{check};q)=q^2+3\left(1-q^2\right)-c_s=3-2q^2-c_s.$$

To see this, observe that with probability q^2, both firm charges a high price of 5 and the buyer gets $1 - c_s$, with the remaining probability of $1 - q^2$, the buyer buys at price 3 and gets a payoff of from $3 - c_s$. If she does not check, her expected payoff is

$$U_B\left(\text{no check}; q\right) = q + 3\left(1 - q\right) = 3 - 2q.$$

(Since she chooses the firm randomly without knowing the prices, the probability that the price will be high is q. Hence, she gets 1 with probability q (when the price is high) and 3 with probability $1 - q$ (when the price is low). She does not incur a cost as she does not check the prices.) Since $0 < r < 1$, the buyer must be indifferent between checking and not checking. Hence, the two displayed payoffs above must be equal:

$$\begin{aligned} U_B\left(\text{check}; q\right) &= U_B\left(\text{no check}; q\right) \\ 2q(1-q) &= c_s. \end{aligned} \tag{6.8}$$

Solving this equation for q, one obtains

$$q = \frac{1 \mp \sqrt{1 - 2c_s}}{2}. \tag{6.9}$$

Now consider a firm's decision in equilibrium. Compute the seller's expected payoff from charging price $p_i = 5$. The firm cannot sell at this price if the buyer checks (with probability r) and the other firm charges a low price (with probability q). The probability of this jointly happening is $r\left(1 - q\right)$. In all other cases, the firm sells with probability 1/2. Hence, the probability of sale is $\frac{1}{2} \times \left(1 - r\left(1 - q\right)\right)$. Since the firm gets 5 from sale, its expected payoff from charging price $p_i = 5$ is

$$U_i\left(5; q, r\right) = \frac{5}{2}\left(1 - r\left(1 - q\right)\right).$$

On the other hand, if she charges price $p_i = 3$, she sells with probability 1 if the other firm charges a high price and the buyer check (which has probability qr) and sells with probability 1/2 in all other cases (which have probability $1 - qr$ in total). The probability of sale is $qr + \frac{1}{2} \times \left(1 - qr\right)$, which is equal to $\frac{1}{2} \times \left(1 + qr\right)$. The payoff from sale is only 3 now, and hence the expected payoff charging price $p_i = 3$ is

$$U_i\left(3; q, r\right) = \frac{3}{2}\left(1 + qr\right).$$

Since $q \in (0, 1)$, the firm must be indifferent between these two prices, i.e., $U_i\left(5; q, r\right) = U_i\left(3; q, r\right)$:

$$\frac{5}{2}\left(1 - r\left(1 - q\right)\right) = \frac{3}{2}\left(1 + qr\right).$$

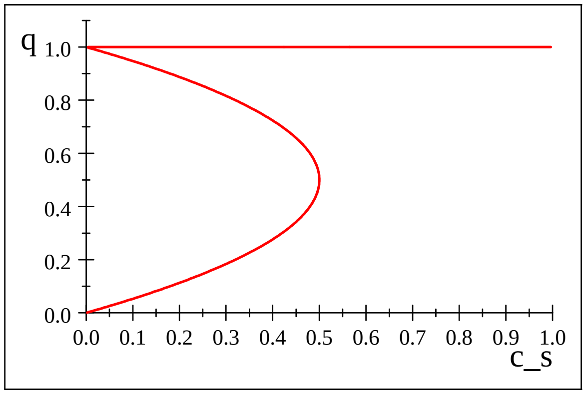

Figure 6.7: Equilibrium probability q of high price as a function of search cost c_s.

Solving this equation for r, one obtains

$$r = \frac{2}{5 - 2q}. \tag{6.10}$$

The symmetric Nash equilibrium probabilities q and r are the solutions to the equation system (6.9-6.10) with $q, r \in (0, 1)$; the solution is simply obtained by substituting the value of q in (6.9) to (6.10). There are two solutions, leading to two symmetric mixed strategy equilibria:

$$\left(q = \frac{1 + \sqrt{1 - 2c_s}}{2}, r = \frac{2}{4 - \sqrt{1 - 2c_s}}\right)$$

and

$$\left(q = \frac{1 - \sqrt{1 - 2c_s}}{2}, r = \frac{2}{4 + \sqrt{1 - 2c_s}}\right).$$

Figure 6.7 plots the possible values of q as a function of c_s, including the pure strategy Nash equilibrium where $q = 1$. When $c_s > 1/2$, there is a unique Nash equilibrium, in which the firms charge high prices. When $c_s = 1/2$, the there is also a mixed strategy Nash equilibrium, in which the firms charge high and low prices with equal probabilities. For lower costs c_s, there are two mixed strategy equilibria and a pure strategy equilibrium with high price. The probability of charging a high price reacts to the changes in c_s differently in the two mixed-strategy equilibria. In one equilibrium, as the cost c_s decreases to zero, the probability of charging a high price also decreases to zero, when the firms charge low prices. In the other equilibrium, that probability increases to 1, when the firms charge high prices.

6.4 Other models of Imperfect Competition

This section presents some other environments of imperfect competition that are important for microeconomics. Analyses of some other environments are left as exercises.

6.4.1 Cournot Competition with Entry Costs

The first model introduces entry costs that are necessary to start up the firm and uses this model to investigate the market size (in terms of number of firms that are actively producing positive amounts in the market).

In the Cournot oligopoly model with n firms, imagine that the firms must incur a fixed cost $F > 0$ to produce a positive amount. Specifically, the cost of producing q_i unit for a firm i is

$$C(q_i) = \begin{cases} cq_i + F & \text{if } q_i > 0 \\ 0 & \text{if } q_i = 0 \end{cases}$$

where $c \in (0, 1)$ is the marginal cost and F is a fixed cost. The rest of the model is as before: the firms simultaneously produce $q_1, \ldots, q_n$ and sell at price $P(Q) = \max\{1 - Q, 0\}$, where $Q = q_1 + \cdots + q_n$ is the total supply. The Cournot oligopoly model considered the limit case $F = 0$.

Once it is determined how many firms are active in the market and produce positive amounts, the Nash equilibrium behavior is as in the baseline case. Suppose that $m \leq n$ firms produce some positive quantity and the remaining firms produce 0. For any i with positive production q_i^*,

$$q_i^* = \frac{1 - c - \sum_{j \neq i} q_j^*}{2}.$$

As in the usual Cournot model above, the unique solution to this equation system is

$$q_i^* = \frac{1 - c}{m + 1}$$

for each firm with positive production. The payoff is

$$\pi^* = \left(\frac{1 - c}{m + 1}\right)^2 - F.$$

The equilibrium behavior is as in the usual Cournot model where the inactive firms are ignored. The active firms pay an entry cost, and hence their payoffs are smaller than the case without the entry cost, but since the entry cost does not depend on the production level, it does not affect the production amount directly in equilibrium, given the number of the active firms.

The number of the active firms (i.e., the market size) does depend on the entry cost. First, any active firm has the option of not producing at all and obtaining zero profit. Hence, its profit cannot be negative: $\pi^* \geq 0$. This leads to an upper bound on the number of active firms

$$m \leq \frac{1-c}{\sqrt{F}} - 1 \equiv m^*.$$

Moreover, if $m < n$, the firms that produce 0 must not profit from deviation to positive production. If an inactive firm produces a positive amount, then it produces the best response to the total supply mq_i^* of the other active firms, obtaining the payoff of $\left(\frac{1-c-mq_i^*}{2}\right)^2 - F$. Since an inactive firm gets zero profit in equilibrium, this deviation payoff cannot be strictly positive:

$$\left(\frac{1-c-mq_i^*}{2}\right)^2 - F \leq 0$$

Since $1 - c - mq_i^* = (1-c)/(m+1)$, this inequality simplifies to

$$m \geq \frac{m^* - 1}{2},$$

providing a lower bound on the market size. Typically there are multiple integers in between the lower and upper bounds, leading to multiple Nash equilibria with varying market sizes. For any integer $m \in [(m^* - 1)/2, \ldots, m^*]$, m firms produce $(1-c)/(m+1)$ each, and the remaining firms produce 0.

The bounds on the market size have intuitive implications for economic policy analysis. The upper bound m^* is decreasing with the marginal cost as well as the entry cost. The upper bound m^* is less than one for large entry costs with $F > (1-c)^2/4$. In that case all (potential) firms produce zero and there is no market for the good. As entry cost falls, both upper and the lower bound ($(m^* - 1)/2$) increase, increasing the number of active firms—up to n. All (potential) firms become active, and the equilibrium outcome is close to the competitive equilibrium. This suggests that a government who wants to increase competition and make the outcome more active would seek to remove the barriers to entry.

6.4.2 Differentiated Bertrand Oligopoly—Monopolistic Price Competition

Bertrand competition assumes that the goods sold by the firms are perfect substitutes for each other, leading the buyer to buy from the firm that charges the lowest price. This leads firms to compete on prices and to charge the marginal cost in equilibrium. In real-world applications, competition often occurs among firms that sell goods that are imperfect substitutes to each other. For example, the consumers may have differing tastes with respect to the food made by various restaurants in a neighborhood, but they will also take into account the prices when they decide which restaurant to dine in. This section presents a variation of Bertrand's model with such imperfect substitutes.

There are n firms. Simultaneously, each firm i sets price $p_i \geq 0$ for its own product and sells

$$Q_i\left(p_1, \ldots, p_n\right) = 1 - p_i + b\frac{\sum_{j \neq i} p_j}{n-1}$$

units, where $a_i > 0$ and $b \in (0, 2)$. Here, the goods are imperfect substitutes because the demand for a good sold by firm i is increasing in the other firm's prices; the substitution is imperfect because the demand for a firm may be positive even when another firm charges a lower price. The demand is allowed to be negative to simplify the analysis. The marginal cost is assumed to be zero, so that the profit of each firm i is

$$u_i\left(p_1, \ldots, p_n\right) = p_i Q_i\left(p_1, \ldots, p_n\right).$$

The best response function for each player i is

$$B_i\left(p_{-i}\right) = \left(1 + b\frac{\sum_{j \neq i} p_j}{n-1}\right)/2. \tag{6.11}$$

It is also useful to define the best response when all other firms charge the same price p:

$$B(p) \equiv B_i(p, \ldots, p) = (1 + bp)/2.$$

A Nash equilibrium is a vector of prices $(p_1, \ldots, p_n)$ such that $p_i = B_i(p_{-i})$ for each player i. There is a unique Nash equilibrium, and it is symmetric: $(p^*, p^*, \ldots, p^*)$, where $p^* = B(p^*)$. Solving $p^* = B(p^*)$, one obtains

$$p^* = 1/(2-b).$$

The Nash equilibrium price is higher than the marginal cost, zero, and it can be arbitrarily high when b gets close to 2.

Any price $p \geq p^*$ turns out to be rationalizable, and no price below the equilibrium price p^* is rationalizable. To compute the set of rationalizable strategies, write $\underline{p}^0 = 0$. For any $m \geq 1$, assume that the strategies that survive $m-1$ round of iteration is $[\underline{p}^{m-1}, \infty)$. Then, the set of strategies that survive m rounds of iteration is $[\underline{p}^m, \infty)$ where

$$\underline{p}^m = B\left(\underline{p}^{m-1}\right).$$

To see this, observe that each $p \geq \underline{p}^m$ is a best response to a symmetric strategy profile $(p', \ldots, p')$ with $p' \geq \underline{p}^{m-1}$ and survives the round m. Moreover, any strategy $p < \underline{p}^m$ is strictly dominated by $\underline{p}^m$. As $m \to \infty$, price $\underline{p}^m$ converges to p^*. Therefore, every price lower than the equilibrium price p^* is eliminated at some round and no price above the equilibrium price is eliminated. Therefore, a price p is rationalizable if and only if $p \geq p^*$.

Interestingly, when the set of possible prices is bounded from above (for example when the consumers do not demand any good when the prices are too high), the game is dominance-solvable, and the Nash equilibrium price p^* is uniquely rationalizable. To establish this, suppose that the set of available prices is $[0, \bar{p}]$ for some price $\bar{p} > p^*$. Write $\bar{p}^0 = \bar{p}$. For any $m \geq 1$, assume that the strategies that survive $m-1$ round of iteration is $\left[\underline{p}^{m-1}, \bar{p}^{m-1}\right]$. Then, the set of strategies that survive m rounds of iteration is $\left[\underline{p}^m, \bar{p}^m\right]$ where $\bar{p}^m = B\left(\bar{p}^{m-1}\right)$. This is because any strategy $p > \bar{p}^m$ is strictly dominated by $\bar{p}^m$. As $m \to \infty$, both $\underline{p}^m$ and $\bar{p}^m$ converge to p^*. Therefore, p^* is the unique rationalizable strategy.

The resulting behavior under price competition with imperfect substitution starkly differs from the one with perfect substitutes. Under perfect substitutes, the firms charge the marginal cost, which is zero in this case. When the prices are imperfect substitutes, the equilibrium price is far from the marginal cost, and no lower price can be rationalizable. Under imperfect substitution, the firms charge higher prices when they expect the other firms to charge a higher price. Iterative expectations of such high prices leads to an equilibrium in which the firms charge a very high price.

6.4.3 Spacial Competition

An instance of monopolistic competition arises when the firms are differentiated spacially. For example, two restaurants located in two distinct locations may be competing for customers. More absractly, two firms with distinct product specifications may compete just like two restaurants, except that their locations are differetiated in an abstract product space. This section presents such a model, using an example of two ice-cream parlors.

Consider a street, denoted by the unit interval $[0,1]$. There are two ice-cream parlors, namely 1 and 2, located at a_1 and $1-a_2$, respectively, for some $a_1, a_2 \in [0,1/2]$. There is also a unit mass of kids, uniformly located on $[0,1]$. Each ice-cream parlor i simultaneously sets a price $p_i \geq 0$ for its own ice cream. Each kid buys a unit of ice cream, choosing the store to buy from according to the following mechanical rule. For some $c > 0$, he is to pay a transportation cost $c\,(w-y)^2$ to buy from a store located at y where w is his own location, and he buys from the store with the lowest total cost, which is the sum of the price and the transportation cost. (If the total cost is the same, he flips a coin between the stores.)

The players in this game are the ice-cream parlors, as the kids' behavior is mechanically set by the model. When the players are located at the same point, the game reduces to the usual Bertrand duopoly, and the unique equilibrium price is zero. The rest of the section assumes that the stores are located at distinct locations, i.e., $a_1 + a_2 < 1$.

Start with determining the demand for each ice-cream parlor as a function of their locations and prices using the kids' behavior. The demand depends on the distance

$$\Delta = 1 - (a_1 + a_2)$$

between the two stores and the mid-point

$$x_m = \frac{1}{2} + \frac{a_1 - a_2}{2}.$$

The total costs of buying from ice cream parlors 1 and 2 are

$$C_1 = p_1 + c\,(w - a_1)^2 \text{ and } C_2 = p_2 + c\,(w - 1 + a_2)^2\,,$$

respectively. Clearly, the cost differential,

$$C_1 - C_2 = p_1 - p_2 + 2c\Delta\,(w - x_m)\,,$$

is increasing in the location w. Hence, there exists a cutoff $\bar{w}$ such that the kids below the cutoff buy from ice-cream parlor 1 and the remaining kids buy from the ice-cream parlor 2, where the cutoff may also be 0 or 1. When the cutoff is strictly between 0 and 1, the cost differential must be zero, and the cutoff is determined by setting $C_1 - C_2 = 0$:

$$\bar{w} = x_m + \frac{p_2 - p_1}{2c\Delta}.$$

The cutoff is 0 when this value is negative, and it is 1 when the value is greater than 1. Assuming that $\bar{w} \in (0,1)$, the demand for ice cream parlors 1 and 2 are $\bar{w}$ and $1 - \bar{w}$, respectively. This is an instance of the monopolistic-price competition model in Section 6.4.2, where the demand for ice-cream parlor i is

$$Q_i(p_1, p_2) = \frac{1}{2} + \frac{a_i - a_j}{2} + \frac{p_j - p_i}{2c\Delta} \tag{6.12}$$

when $\bar{w} \in (0,1)$; it is 0 when the above expression is negative and 1 when the above expression exceeds 1. As in Section 6.4.2, the payoff of firm i is

$$U_i(p_1, p_2) = p_i Q_i(p_1, p_2),$$

defining a game between the ice-cream parlors, who choose their prices as their strategies.

As in Section 6.4.2, there is a unique Nash equilibrium. Towards computing the equilibrium, observe that the demand for each parlor must be positive in equilibrium. Indeed, if the other parlor charges a positive price p_j^* in equilibrium, a parlor i can obtain a positive payoff by setting her price equal to p_j^*, obtaining a strictly positive demand and a strictly positive payoff. If the other parlor charges zero price, parlor i can obtain a positive payoff by charging a small but positive price. Hence, the equilibrium payoff of each parlor must be positive, showing that the demand for each prlor is positive. Hence, the demand for each parlor is in $(0,1)$, and the demand is given by (6.12) in equilibrium. Thus, the first-order condition for optimization of ice cream parlor i is

$$U_i' = \frac{1}{2} + \frac{a_i - a_j}{2} + \frac{p_j - 2p_i}{2c\Delta} = 0.$$

This condition can be written as

$$2p_i - p_j = c\Delta(1 + a_i - a_j).$$

Solving this equation ststem for p_1 and p_2, one obtains the Nash equilibrium prices as

$$p_i^* = c\Delta\left(1 + \frac{a_i - a_j}{3}\right). \tag{6.13}$$

Observe that the equilibrium prices are increasing with the cost c of traveling, which decreases the competiton between the firms. The prices also depend on the locations of the firms, and the prices are increasing in the distance Δ between the ice-cream shops (for any fixed midpoint x_m), as it also decreases the competition. As the distance goes to zero, the prices and the equilibrium profits approach zero. The prices are zero in the limit, replicating the equilibrium behavior in the Bertrand competition. Finally, fixing the distance Δ between the stores, a player's profit increases as she moves away from the end-point and captures a bigger share of the demand.

6.5 Exercises with Solutions

6.6 Exercises

Exercise 6.1. There are two countries, namely 1 and 2. There are also two profit-maximizing firms, namely 1 and 2, located in Countries 1 and 2, respectively. Simultaneously, each firm i produces quantities q_{i1} and q_{i2} at marginal cost c_i and sells quantities q_{i1} and q_{i2} in Countries 1 and 2 at prices

$$P_1 = \theta_1 - q_{11} - q_{21} \text{ and } P_2 = \theta_2 - q_{12} - q_{22},$$

respectively, where c_1, $c_2 \in (0,1)$ and θ_1, $\theta_2 > 1$ are known parameters. Each country i charges the "foreign" firm $j \neq i$ a tariff $\tau_i P_i q_{ji}$ proportional to its revenue from the sales in that country, where $\tau_i \geq 0$ is a known parameter.

1. Write this formally as a normal-form game (where the players are the firms, and the payoffs are given by profits).

2. Compute a Nash equilibrium of this game.

3. Briefly discuss how tariff τ_1 affects the consumer surplus (i.e., $(\theta_1 - P_1)^2/2$), the profit of the domestic firm (Firm 1), and the tax revenue from charging the tariff to Firm 2.

Exercise 6.2. In Section 6.4.3, assume that the firms cannot charge a price more than 1; e.g., each cannot afford a price more than 1. Compute the set of rationalizable strategies.

Exercise 6.3. There are two profit-maximizing firms, A and B, which produce applesauce and banana puree, respectively. (Applesauce and banana puree are imperfect substitutes.) Simultaneously, A and B sets the prices p_A and p_B of applesauce and banana puree, respectively. Firms A and B sell

$$q_A(p_A, p_B) = 1 - p_A - \gamma(1 - p_B) \text{ and } q_B(p_A, p_B) = 1 - p_B - \gamma(1 - p_A)$$

units of applesauce and banana puree, respectively, where $0 < \gamma < 1$. (The amounts can be negative.) The marginal cost is $c \in (0, \gamma)$ for each firm.

1. Write this formally as a game in normal form.

2. Compute the set of Nash equilibria.

3. Assuming that the prices must be in $[0, 1]$, compute the set of rationalizable strategies. (What would be your answer if any nonnegative price were allowed?)

4. How does the equilibrium change as one varies the parameter γ and the marginal cost c? Briefly discuss your finding.

Exercise 6.4. Consider the Cournot duopoly with linear demand function $P = 1 - Q$, where P is the price and $Q = q_1 + q_2$ is the total supply.[2] Firm 1 has zero marginal cost. Firm 2 has marginal cost $c(q_2) = q_2$, so that the total cost of producing q_2 is $q_2^2/2$.

1. Compute all the Nash equilibria.

2. Compute the set of all rationalizable strategies. Explain your steps.

Exercise 6.5. Show that all Nash equilibria of Cournot oligopoly game above are in pure strategies (i.e., one does not need to check for mixed strategy equilibria). (See Exercise 5.19.)

Exercise 6.6. Can you find a mixed-strategy Nash equilibrium in Bertrand competition (Section 6.3)? What if the demand function is $Q(P) = P^{-\alpha}$ for some $\alpha \in (0, 1)$?

[2] Recall that in Cournot duopoly Firms 1 and 2 simultaneously produce q_1 and q_2, and they sell at price P.

Exercise 6.7. There are two firms. Simultaneously, each firm i sets price $p_i \in [0,1]$ for its own product and sells

$$Q_i(p_1, p_2) = 1 - p_i + b\sqrt{p_j}$$

units, where $b \in \{-1, 1\}$ and $j \neq i$.[3] The payoff of each firm i is

$$u_i(p_1, p_2) = p_i Q_i(p_1, p_2).$$

1. Write this formally as a game in normal form.

2. Compute the best response function for each player (and plot it).

3. Compute the set of Nash equilibria for $b = 1$.

4. Compute the set of rationalizable strategies for $b = 1$. (Show your work.)

5. How would your answer to parts 3 and 4 change if $b = -1$?

Exercise 6.8. In section 6.4.2, compute the sets of rationalizable strategies and Nash equilibria, by assuming instead that p_i is picked from $[0, 2]$ and that the cost of producing Q_i units is $cQ_i^2/2$ for some $c \in (0, 1)$, so that the payoff of i is $p_i Q_i - cQ_i^2/2$.

Exercise 6.9. This question asks you to analyze Cournot oligopoly under demand uncertainty. Consider a Cournot oligopoly with n firms, and assume that the inverse-demand function is

$$P(Q, \theta) = \theta - Q$$

where θ is unknown and can take values 1 and 2, each with probability 1/2. (The price and the quantities are allowed to be negative for simplicity.)

1. First consider the case without uncertainty. In this case, Nature chooses θ and its value becomes publicly observable. Then, observing θ, simultaneously, each firm i chooses q_i and sells at price $P(Q, \theta)$ where $Q = q_1 + \cdots + q_n$. Assuming that each firm's payoff is its profit π_i, compute a Nash equilibrium.

2. Now consider the case with uncertainty. In this case, Nature chooses θ and its value is not observable. Without observing θ, simultaneously, each firm i chooses q_i and sells at price $P(Q, \theta)$ where $Q = q_1 + \cdots + q_n$. Assuming that each firm's payoff is its profit π_i, compute a Nash equilibrium.

[3] The products are imperfect substitutes when $b = 1$ and complements when $b = -1$.

3. Redo the previous parts, assuming that the firms are risk-averse. In particular, assume that the payoff of each firm i is π_i^α for some $\alpha \in (0,1)$.

4. Comparing your answers to the previous parts, briefly discuss the role of uncertainty and risk aversion in equilibrium.

6.7 Appendix

6.7.1 Rationalizability in Cournot duopoly, more formally

One can prove that linear Cournot duopoly is dominance solvable more formally by invoking the following lemma repeatedly:

Lemma 6.2. *Given that $q_j \leq \bar{q}$, every strategy $\hat{q}_i$ with $\hat{q}_i < q_i^B(\bar{q})$ is strictly dominated by $q_i^B(\bar{q}) \equiv (1-\bar{q}-c)/2$. Given that $q_j \geq \bar{q}$, every strategy $\hat{q}_i$ with $\hat{q}_i > q_i^B(\bar{q})$ is strictly dominated by $q_i^B(\bar{q}) \equiv (1-\bar{q}-c)/2$.*

Proof. To prove the first statement, take any $q_j \leq \bar{q}$. Note that $\pi_i(q_i; q_j)$ is strictly increasing in q_i at any $q_i < q_i^B(q_j)$. Since $\hat{q}_i < q_i^B(\bar{q}) \leq q_i^B(q_j)$,[4] this implies that

$$\pi_i(\hat{q}_i, q_j) < \pi_i\left(q_i^B(\bar{q}), q_j\right).$$

That is, $\hat{q}_i$ is strictly dominated by $q_i^B(\bar{q})$.

To prove the second statement, take any $q_j \leq \bar{q}$. Note that $\pi_i(q_i; q_j)$ is strictly decreasing in q_i at any $q_i > q_i^B(q_j)$. Since $q_i^B(q_j) \leq q_i^B(\bar{q}) < \hat{q}_i$, this implies that

$$\pi_i(\hat{q}_i, q_j) < \pi_i\left(q_i^B(\bar{q}), q_j\right).$$

That is, $\hat{q}_i$ is strictly dominated by $q_i^B(\bar{q})$. □

Now, define a sequence $q^0, q^1, q^2, \ldots$ by $q^0 = 0$ and

$$q^m = q_i^B\left(q^{m-1}\right) \equiv \left(1 - q^{m-1} - c\right)/2 = (1-c)/2 - q^{m-1}/2$$

[4] This is because q_i^B is decreasing.

for all $m > 0$. That is,

$$\begin{aligned}
q^0 &= 0 \\
q^1 &= \frac{1-c}{2} \\
q^2 &= \frac{1-c}{2} - \frac{1-c}{4} \\
q^3 &= \frac{1-c}{2} - \frac{1-c}{4} + \frac{1-c}{8} \\
&\ldots \\
q^m &= \frac{1-c}{2} - \frac{1-c}{4} + \frac{1-c}{8} - \cdots - (-1)^m \frac{1-c}{2^m} \\
&\ldots
\end{aligned}$$

Theorem 6.2. *The set of remaining strategies after any odd round m ($m = 1, 3, \ldots$) is $[q^{m-1}, q^m]$. The set of remaining strategies after any even round m ($m = 2, 4, \ldots$) is $[q^m, q^{m-1}]$. The set of rationalizable strategies is $\{(1-c)/3\}$.*

Proof. We use mathematical induction on m. For $m = 1$, we have already proven the statement. Assume that the statement is true for some odd m. Then, for any q_j available at even round $m+1$, we have $q^{m-1} \leq q_j \leq q^m$. Hence, by Lemma 6.2, any $\hat{q}_i < q_i^B(q^m) = q^{m+1}$ is strictly dominated by q^{m+1} and eliminated. That is, if q_i survives round $m+1$, then $q^{m+1} \leq q_i \leq q^m$. On the other hand, every $q_i \in [q^{m+1}, q^m] = \left[q_i^B(q^m), q_i^B(q^{m-1})\right]$ is a best response to some q_j with $q^{m-1} \leq q_j \leq q^m$, and it is not eliminated. Therefore, the set of strategies that survive the even round $m+1$ is $[q^{m+1}, q^m]$.

Now, assume that the statement is true for some even m. Then, for any q_j available at odd round $m+1$, we have $q^m \leq q_j \leq q^{m-1}$. Hence, by Lemma 6.2, any $\hat{q}_i > q_i^B(q^m) = q^{m+1}$ is strictly dominated by q^{m+1} and eliminated. Moreover, every $q_i \in [q^m, q^{m+1}] = \left[q_i^B(q^{m-1}), q_i^B(q^m)\right]$ is a best response to some q_j with $q^m \leq q_j \leq q^{m-1}$, and it is not eliminated. Therefore, the set of strategies that survive the odd round $m+1$ is $[q^m, q^{m+1}]$.

Finally, notice that

$$\lim_{m\to\infty} q^m = (1-c)/3.$$

Therefore, the intersection of the above intervals is $\{(1-c)/3\}$, which is the set of rationalizable strategies. $\square$

6.7.2 Price competition, more formally

This section presents a more detailed analysis of the price competition, flashing out the intuitive analyses in the text. It starts with a more detailed proof of Theorem 6.1, showing that price competition leads to competitive equilibrium. It then goes on to presents the details of the key steps in the model with discrete prices.

Proof of Theorem 6.1. We have seen already that $p^* = (0, 0)$ is a Nash equilibrium. I will here show that if (p_1, p_2) is a Nash equilibrium, then $p_1 = p_2 = 0$. To do this, take any Nash equilibrium (p_1, p_2). I first show that $p_1 = p_2$. Towards a contradiction, suppose that $p_i > p_j$. If $p_j = 0$, then $\pi_j (p_i, p_j) = 0$, while $\pi_j (p_i, p_i) = (1 - p_i) p_i/2 > 0$. That is, choosing p_i is a profitable deviation for firm j, showing that $p_i > p_j = 0$ is not a Nash equilibrium. Therefore, in order $p_i > p_j$ to be an equilibrium, it must be that $p_j > 0$. But then, firm i has a profitable deviation: $\pi_i (p_i, p_j) = 0$ while $\pi_i (p_j, p_j) = (1 - p_j) p_j/2 > 0$. All in all, this shows that one cannot have $p_i > p_j$ in equilibrium. Therefore, $p_1 = p_2$. But if $p_1 = p_2$ in a Nash equilibrium, then it must be that $p_1 = p_2 = 0$. This is because if $p_1 = p_2 > 0$, then Firm 1 would have a profitable deviation: $\pi_1 (p_1, p_2) = (1 - p_1) p_1/2$ while $\pi_1 (p_1 - \varepsilon, p_2) = (1 - p_1 + \varepsilon) (p_1 - \varepsilon)$, which is close to $(1 - p_1) p_1$ when ε is close to zero. $\square$

Proof of Lemma 6.1. Take any player i and any price $p_i > p^{mon}$. We want to show that the mixed strategy σ^ϵ with $\sigma^\epsilon (p^{mon}) = 1 - \epsilon$ and $\sigma^\epsilon \left(p^{\min}\right) = \epsilon$ strictly dominates p_i for some $\epsilon > 0$.

Take any strategy $p_j > p^{mon}$ of the other player j. We have

$$\pi_i (p_i, p_j) \leq p_i Q (p_i) = p_i (1 - p_i) \leq 0.51 \cdot 0.49 = 0.2499,$$

where the first inequality is by definition, and the last inequality is due to the fact that $p_i \geq 0.51$. On the other hand,

$$\begin{aligned}
\pi_i (\sigma^\epsilon, p_j) &= (1 - \epsilon) p^{mon} (1 - p^{mon}) + \epsilon p^{\min} \left(1 - p^{\min}\right) \\
&> (1 - \epsilon) p^{mon} (1 - p^{mon}) \\
&= 0.25 (1 - \epsilon) .
\end{aligned}$$

Thus, $\pi_i (\sigma^\epsilon, p_j) > 0.2499 \geq \pi_i (p_i, p_j)$ whenever $0 < \epsilon \leq 0.0004$. Choose $\epsilon = 0.0004$.

Now, pick any $p_j \leq p^{mon}$. Since $p_i > p^{mon}$, we now have $\pi_i(p_i, p_j) = 0$. On the other hand,[5]

$$\pi_i(\sigma^\epsilon, p_j) \geq \epsilon p^{\min}\left(1 - p^{\min}\right)/2 > 0 = \pi_i(p_i, p_j).$$

Therefore, σ^ϵ strictly dominates p_i. □

Proof of eliminations in Round m. To prove that $\bar{p}$ is strictly dominated by σ^ϵ, note that the profit from $\bar{p}$ for player i is

$$\pi_i(\bar{p}, p_j) = \begin{cases} \bar{p}(1-\bar{p})/2 & \text{if } p_j = \bar{p}, \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand,

$$\begin{aligned} \pi_i(\sigma^\epsilon, \bar{p}) &= (1-\epsilon)(\bar{p} - 0.01)(1 - \bar{p} + 0.01) + \epsilon p^{\min}\left(1 - p^{\min}\right) \\ &> (1-\epsilon)(\bar{p} - 0.01)(1 - \bar{p} + 0.01) \\ &= (1-\epsilon)\left[\bar{p}(1-\bar{p}) - 0.01(1 - 2\bar{p})\right]. \end{aligned}$$

Then, $\pi_i(\sigma^\epsilon, \bar{p}) > \pi_i(\bar{p}, \bar{p})$ whenever

$$\epsilon \leq 1 - \frac{\bar{p}(1-\bar{p})/2}{\bar{p}(1-\bar{p}) - 0.01(1-2\bar{p})}.$$

But $\bar{p} \geq 0.02$, hence $0.01(1-2\bar{p}) < \bar{p}(1-\bar{p})/2$, thus the right hand side is greater than 0. Choose

$$\epsilon = 1 - \frac{\bar{p}(1-\bar{p})/2}{\bar{p}(1-\bar{p}) - 0.01(1-2\bar{p})} > 0$$

so that $\pi_i(\sigma^\epsilon, \bar{p}) > \pi_i(\bar{p}, \bar{p})$. Moreover, for any $p_j < \bar{p}$,

$$\pi_i(\sigma^\epsilon, p_j) \geq \epsilon p^{\min}\left(1 - p^{\min}\right)/2 > 0 = \pi_i(\bar{p}, p_j),$$

showing that σ^ϵ strictly dominates $\bar{p}$. □

[5]

$$\pi_i(\sigma^\epsilon, p_j) = \begin{cases} (1-\epsilon)p^{mon}(1-p^{mon})/2 + \epsilon p^{\min}\left(1-p^{\min}\right) & \text{if } p_j = p^{mon} \\ \epsilon p^{\min}\left(1-p^{\min}\right) & \text{if } p^{\min} < p_j < p^{mon} \\ \epsilon p^{\min}\left(1-p^{\min}\right)/2 & \text{if } p_j = p^{\min}. \end{cases}$$