Fall 2021
14.12 Game Theory
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14.12 Midterm Exam

October 21, 2021

You have **80 minutes** to complete the exam.

This exam is **open book**. You may consult any physical materials you brought with you into the exam, any saved documents on your computer, and any 14.12 materials posted on Canvas. **Other than going to Canvas, you may not access the internet during the exam.**

The exam has **three** questions worth a total of **44 points**:

- Problem 1: 12 points
- Problem 2: 16 points
- Problem 3: 16 points

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Problem 1 (12 points)**.** Recall the following "matching pennies" game between Alice and Bob:

		Bob L	Bob R
Alice	L	$1, -1$	$-1, 1$
Alice	R	$-1, 1$	$1, -1$

Consider a dynamic version of this game. Alice first chooses L or R. Then a *biased* coin is flipped to determine whether Bob observes Alice's move. The coin comes up Heads with probability 0.75 and Tails with probability 0.25.

- If the coin is Heads, Bob observes Alice's move and then chooses L or R.
- If the coin is Tails, Bob chooses L or R without observing Alice's move.

After Bob's move, the game ends. Payoffs are determined by the matrix above.

(a) Write out this game in extensive form.

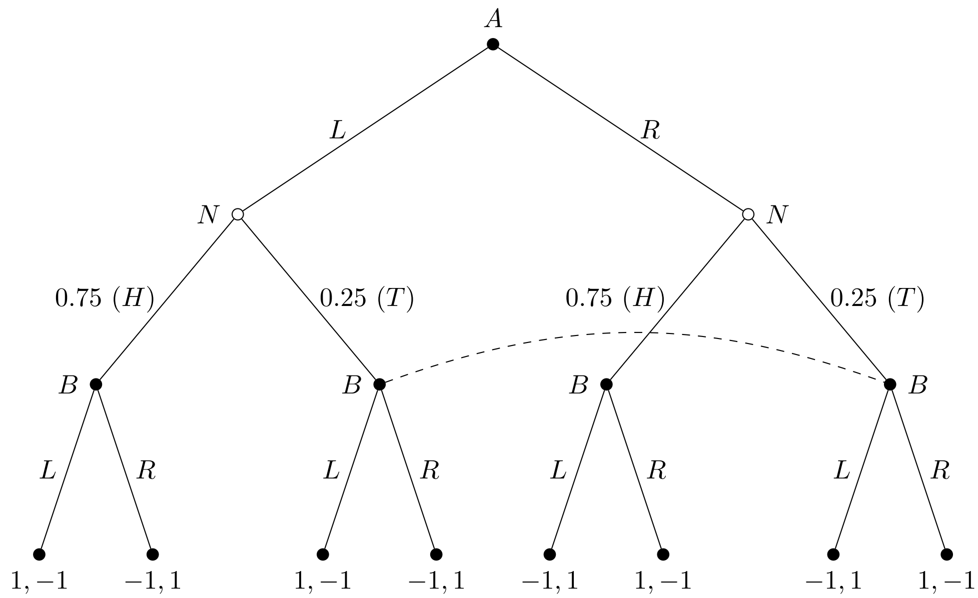
(b) How many pure strategies does Alice have?

(c) How many pure strategies does Bob have?

(d) Express the extensive form game from part (a) in strategic form. (Be sure to explain the notation you use for Bob's strategies.) To save time, pick one of Alice's strategies and **fill in the utilities only for the cells in which Alice uses that strategy**. You can leave all other cells blank.

Problem 1 Solution

(a) Here is the game in extensive form:



(b) Alice has 2 strategies. She has a single information set with two moves.

(c) Bob has $2^3 = 8$ strategies. He has three information sets, each with two moves.

(d) We denote Bob's strategies as triples, where the first component denotes Bob's move at information set $\{LH\}$, the second component denotes Bob's move at the information set $\{LT, RT\}$, and the third componet denotes Bob's move at the information set $\{RH\}$. We fill in the payoffs for the entire matrix, but you are asked only to fill in one row.

	LLL	LLR	LRL	LRR	RLL	RLR	RRL	RRR
L	$1, -1$	$1, -1$	$0.5, -0.5$	$0.5, -0.5$	$-0.5, 0.5$	$-0.5, 0.5$	$-1, 1$	$-1, 1$
R	$-1, 1$	$0.5, -0.5$	$-0.5, 0.5$	$1, -1$	$-1, 1$	$0.5, -0.5$	$-0.5, 0.5$	$1, -1$

This matrix can be filled in from the following general formula. Letting $u\colon \{L, R\} \times \{L, R\} \to \mathbf{R}^2$ denote the utility vector function for "matching pennies," the utility vector function U for the extensive form game is given by

$$U(L, xyz) = 0.75u(L, x) + 0.25u(L, y),$$
$$U(R, xyz) = 0.75u(R, z) + 0.25u(R, y).$$

Problem 2 (16 points)**.** Two players (player 1 and player 2) are working together on a project. Simultaneously, each player i chooses an effort level e_i in $[0, 1]$. The players' effort choices jointly determine the quality q of the project:

$$q = \theta e_1 e_2,$$

where θ in $(0, 1]$ is a known parameter. Each player i's utility is

$$q - \frac{e_i^2}{2},$$

where q is the quality of the project and e_i is player i's effort level.

(a) Are the players' effort levels strategic complements? Show why or why not.

(b) Compute each player's best response function.

(c) First suppose $\theta = 1$.

- Compute the set of rationalizable strategies for each player.
- Find every pure Nash equilibrium.
- Is every rationalizable strategy profile a Nash equlibrium?

(d) Now suppose $0 < \theta < 1$. Answer the same questions from part (c):

- Compute the set of rationalizable strategies for each player.
- Find every pure Nash equilibrium.
- Is every rationalizable strategy profile a Nash equlibrium?

Problem 2 Solution

(a) Yes, their effort levels are strategic complements. Player i's utility function for the strategic form game is

$$u_i(e_1, e_2) = \theta e_1 e_2 - \frac{e_i^2}{2}.$$

Therefore, for each $i = 1, 2$,

$$\frac{\partial u_i(e_1, e_2)}{\partial e_1 \partial e_2} = \theta \geq 0.$$

(b) First, we compute player 1's best response function. The first order condition is

$$0 = \frac{\partial u_1(e_1, e_2)}{\partial e_1} = \theta e_2 - e_1.$$

Solving for e_1, we see that Player 1's best response function $\mathrm{BR}_1 \colon [0,1] \to [0,1]$ is given by

$$\mathrm{BR}_1(e_2) = \theta e_2.$$

Symmetrically, Player 2's best response function $\mathrm{BR}_2 \colon [0,1] \to [0,1]$ is given by

$$\mathrm{BR}_2(e_1) = \theta e_1.$$

(c) Suppose $\theta = 1$. From the best response functions, we see that

$$\mathrm{BR}_1([0,1]) = \mathrm{BR}_2([0,1]) = [0,1].$$

For each player, every strategy is rationalizable:

$$S_1^\infty = S_2^\infty = [0,1].$$

A strategy profile (e_1^*, e_2^*) is a Nash equilibrium if and only if the following best-response conditions hold:

$$\begin{aligned} e_1^* &= \mathrm{BR}_1(e_2^*) = e_2^* \\ e_2^* &= \mathrm{BR}_2(e_1^*) = e_1^*. \end{aligned}$$

Therefore, the Nash equilibria are $(e_1^*, e_2^*) = (e, e)$ for all e in $[0,1]$.

Not every rationalizable strategy profile is a Nash equlibrium. Every strategy profile in $[0,1]^2$ is rationalizable. Only strategy profiles on the diagonal are Nash equilibria.

(d) Now suppose that $\theta < 1$. For any e in $[0,1]$, we have

$$\mathrm{BR}_1([0,e]) = \mathrm{BR}_2([0,e]) = [0,\theta e].$$

It follows that for each $k = 0, 1, \ldots$, we have

$$S_1^k = S_2^k = [0,\theta^k].$$

Hence

$$S_1^\infty = S_2^\infty = \bigcap_{k=1}^{\infty} [0,\theta^k] = \{0\}.$$

The only rationalizable strategy profile is $(0,0)$. This strategy profile $(0,0)$ is a Nash equilibrium and it is the only Nash equilibrium (since a Nash equilibrium must be rationalizable).

In this case, every rationalizable strategy profile is a Nash equilibrium.

Problem 3 (16 points)**.** The government is considering a per-gallon gas tax t. Under such a tax t, when a firm sells q gallons of gas, it must pay tq to the government. There are two firms (firm 1 and firm 2) that can sell gas. Each firm can produce up to 100 gallons of gas at a cost of c per gallon, where c in $(0, 10)$ is known. The market price of gas (per gallon) is

$$P(Q) = \max\{100 - Q, 0\},$$

where Q is the total quantity of gas (in gallons) produced by the two firms.

The timing is as follows. First, the government chooses a per-gallon tax t in $[0, 50]$. Both firms observe this tax level. Then, simultaneously, each firm i chooses a gas production level (in gallons) q_i in $[0, 100]$. Each firm then sells all the gas it produces at the market price, and pays the government the tax it owes.

The government's utility is the tax revenue. Each firm's utility is its after-tax profits.

(a) For each τ in $[0, 50]$, find a Nash equilibrium in which the government chooses $t = \tau$. For each equilbirium, compute the government's revenue.

(b) Find a subgame perfect Nash equilibrium. Compute the government's revenue in this subgame perfect equilibrium.

(c) Compare the revenues computed in part (a) to the revenue computed in part (b). Discuss your finding.

Problem 3 Solution

(a) This game has three players—the government, firm 1, and firm 2. For each τ in $[0, 50]$ consider the following strategy profile. The government's strategy is $t^* = \tau$. For $i = 1, 2$, firm i's strategy is the function $q_i^* \colon [0, 50] \to \mathbf{R}$ defined by

$$q_i^*(t) = \begin{cases} \frac{100-\tau-c}{3} & \text{if } t = \tau, \\ 0 & \text{if } t \neq \tau. \end{cases}$$

We check that each player is playing a best response to her opponents' strategies. Clearly, the government is playing a best response since any deviation away from $t = \tau$ results in revenue of 0. The firms are symmetric, so it suffices to check that firm 1 is playing a best response. Given her opponents' strategies, firm 1's payoff as a function of her own strategy $q_1(\cdot)$ is

$$\begin{aligned} \Pi_1\big(\tau, q_1(\cdot), q_2^*(\cdot)\big) &= q_1(\tau)(100 - q_1(\tau) - q_2^*(\tau) - \tau - c) \\ &= q_1(\tau)\left(100 - q_1(\tau) - \frac{100 - \tau - c}{3} - \tau - c\right) \\ &= q_1(\tau)\left(\frac{2}{3}(100 - \tau - c) - q_1(\tau)\right), \end{aligned}$$

provided that $q_1(\tau) \leq 100 - (100 - \tau - c)/3$. (Any larger $q_1(\tau)$ will result in negative profits and hence won't be a best response.)

This expression is maximized when

$$q_1(\tau) = \frac{100 - \tau - c}{3},$$

so firm 1 is playing a best response.

The government's revenue in this equilibrium is

$$\tau(q_1^*(\tau) + q_2^*(\tau)) = \frac{2}{3}\tau(100 - \tau - c). \tag{1}$$

(b) Now we find a subgame perfect Nash equilibrium. Each tax t determines a subgame in which payoffs are the same as standard Cournot competition with demand $P(Q) = \max\{100 - Q, 0\}$ and marginal cost $\tau + c$. This subgame has a unique Nash equilibrium given by

$$q_1^*(t) = q_2^*(t) = \frac{100 - t - c}{3}.$$

Now we consider the government's choice. Taking as given the strategies of the firms, the government's revenue as a function of t is

$$\frac{2}{3}t(100 - t - c), \tag{2}$$

which is maximized at

$$t = \frac{100 - c}{2}.$$

Formally, the equilibrium consists of the government's strategy $t^* = (100 - c)/2$ and the firms' strategies $q_1^*, q_2^* \colon [0, 50] \to \mathbf{R}$ with $q_1^*(t) = q_2^*(t) = (100 - t - c)/3$.

From (2), the government's revenue is

$$\frac{2}{3}\left(\frac{100 - c}{2}\right)\left(1 - \frac{100 - c}{2} - c\right) = \frac{(100 - c)^2}{6}. \tag{3}$$

(c) The subgame perfect equilibrium revenue in (3) is the maximum value, over τ in $[0, 50]$, of the Nash equilibrium revenue in (1)

In part (a), for each τ in $[0, 50]$, we found an equilibrium in which firms threaten to shut down production if the government chooses a tax $t \neq \tau$. The firms are playing an equilibrium in the $t = \tau$ subgame, but not in other "off-path" subgames. These off-path threats make it a best response for the government to choose $t = \tau$.

In part (b), subgame perfection rules out these non-credible threats by the firms. Whatever tax t is chosen, the firms must play an equilibrium in the resulting subgame, yielding the revenue in (2). Effectively, the government selects the subgame in which equilibrium production generates the greatest revenue.