18.650. Fall 2023
Prof. Rigollet

Date: 9/27/2023

Name:______________________________

Test 1

PLEASE DO NOT TURN THIS PAGE OR START ANSWERING QUESTIONS UNTIL YOU ARE INSTRUCTED TO DO SO.

1. This is an open book, closed notes test.
2. You are also allowed a two-sided letter-sized cheat sheet.
3. Calculators are permitted but the test can be done without.
4. Connected devices like phones, laptops, or tablets are strictly forbidden.
5. The test starts at 1:05
6. The test ends at 1:55 regardless of your time of arrival.
7. All questions should be answered on the present exam sheet
8. Make sure to mark your name on the first page and **do not remove the staple**

Problem1	50	
Problem2	50	
TOTAL	100	

Problem 1. (50 pts) Check the (unique) correct answer. No justification is required. 10 points each.

1. Convergence in distribution always implies convergence in probability

   ◯ TRUE

   $\surd$ **FALSE**

2. The model $\left\{\mathcal{N}_d(0, \Sigma) : \Sigma \in \mathbb{R}^{d \times d}\right\}$ is

   $\surd$ **parametric**

   ◯ nonparametric

3. Let $X \sim N(-1, 4)$. What is $\mathbb{P}(|X| \leq 1)$?

   ◯ 0.1343

   $\surd$ **0.3413**

   ◯ 0.4131

   ◯ 0.8413

4. Let $X_1, \ldots, X_n$ be i.i.d Bernoulli $p \in (0, 1)$. We have that

$$\sqrt{n}\left(\frac{1}{\bar{X}_n} - \frac{1}{p}\right) \to \mathcal{N}(0, \sigma^2)$$

   where σ^2 is equal to

   ◯ 1

   ◯ $p(1-p)$

   $\surd$ $(1-p)/p^3$

   ◯ $p^2/(1-p)$

5. An estimator of θ whose MSE is equal to 0 is...

   ◯ unbiased

   ◯ deterministic

◯ equal to θ almost surely

$\surd$ **All of the above**

Problem 2. (50 pts)

Let $X_1, \ldots, X_n$ be n iid random vectors in $\mathbb{R}^k$ and denote with distribution $\mathcal{N}_k(\mu, I_k)$ where $\mu = (\mu_1, \ldots, \mu_k)^\top$ is an unknown vector. We do known that μ_j can only take two values: 1 or -1 and we are interested in estimating the parity of μ, which is defined as

$$p = \prod_{j=1}^{k} \mu_j$$

using the estimator

$$\hat{p} = \prod_{j=1}^{k} \bar{X}^{(j)}$$

that is, $\hat{p}$ is the product of the entries of the average vector

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i = \begin{pmatrix} \bar{X}^{(1)} \\ \vdots \\ \bar{X}^{(k)} \end{pmatrix}$$

1. (10 points) What values can p take?

   We know that p is the product of μ_j's, each of which can either by 1 or -1. Thus their product must be either -1 or 1.

2. (10 points) Compute the bias of $\hat{p}$.

   We can compute $\text{bias}(\hat{p}) = \mathbb{E}_p[\hat{p}] - p = \prod_{j=1}^{k} \mathbb{E}[\bar{X}^{(j)}] - p = \prod_{j=1}^{k} \mathbb{E}[\frac{1}{n}\sum_{i=1}^{n} X_i^{(j)}] - p = \prod_{j=1}^{k} \mu_j - p = p - p = 0$.

3. (10 points) Write a multivariate central limit theorem for $\bar{X}_n$.

   Since we know that $\mathbb{E}[X_1] = \mu$ and $\mathbb{V}[X_1] = I_k$, we can apply the multivariate central limit theorem to find that

$$\sqrt{n}(\bar{X}_n - \mu) \rightsquigarrow \mathcal{N}(0, I_k).$$

4. (10 points) Show that $\hat{p}$ is asymptotically normal with asymptotic variance equal to k.

   We can use the multivariate delta method and take $g(X^{(1)}, X^{(2)}, \ldots, X^{(k)}) = \prod_{j=1}^{k} X^{(j)}$. We find its gradient to simply be

$$\nabla g(x) = \begin{pmatrix} X^{(2)}X^{(3)} \ldots X^{(k)} \\ X^{(1)}X^{(3)} \ldots X^{(k)} \\ \vdots \\ X^{(1)}X^{(2)} \ldots X^{(k-1)} \end{pmatrix}.$$

   Plugging in μ, we find that

$$\nabla g(\mu) = \begin{pmatrix} \mu_2\mu_3 \ldots \mu_k \\ \mu_1\mu_3 \ldots \mu_k \\ \vdots \\ \mu_1\mu_2 \ldots \mu_{k-1} \end{pmatrix} = \begin{pmatrix} p/\mu_1 \\ p/\mu_2 \\ \vdots \\ p/\mu_k \end{pmatrix}.$$

   Since $\Sigma = I_k$, we find that

$$\begin{aligned} &\sqrt{n}(g(\bar{X}_n) - g(\mu)) \rightsquigarrow \mathcal{N}(0, \nabla g(\mu)^T \Sigma \nabla g(\mu)) \\ \implies &\sqrt{n}(\hat{p} - p) \rightsquigarrow \mathcal{N}(0, \nabla g(\mu)^T \Sigma \nabla g(\mu)) \\ \implies &\sqrt{n}(\hat{p} - p) \rightsquigarrow \mathcal{N}(0, \sum_{j=1}^{k} \frac{p^2}{\mu_j^2}). \end{aligned}$$

   But since p is either 1 or -1 and so is μ_j for all j, we know that $\frac{p^2}{\mu_j^2} = 1$ for all j. Thus, we have that

$$\sqrt{n}(\hat{p} - p) \rightsquigarrow \mathcal{N}(0, k).$$

5. (10 points) Use the above questions to build a confidence interval for p with asymptotic coverage 95%.

We know from the previous problem that

$$\frac{\sqrt{n}(\hat{p}-p)}{\sqrt{k}} \rightsquigarrow \mathcal{N}(0,1).$$

With $\alpha = 1 - 0.95 = 0.05, z_{\alpha/2} = \Phi^{-1}(1-\frac{\alpha}{2}) \approx 1.96$. Thus, we can compute a confidence interval of

$$(\hat{p} - z_{\alpha/2} \cdot \hat{\mathrm{se}}, \hat{p} + z_{\alpha/2} \cdot \hat{\mathrm{se}}) = (\hat{p} - 1.96\sqrt{\frac{k}{n}}, \hat{\theta} + 1.96\sqrt{\frac{k}{n}}).$$

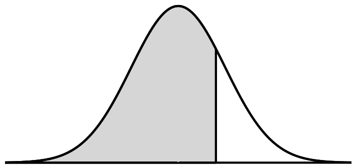

Table 1: The table lists $P(Z \leq z)$ where $Z \sim N(0,1)$ for positive values of z.

	Second decimal place of Z									
Z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

*For $Z \geq 3.50$, the probability is greater than or equal to 0.9998.