# 14.12 Recitation 8

## Bayesian NE and Auction Application

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# Bayesian Nash Equilibrium

- Up until this past week, we have had games with complete information!
  - Note that complete information is not equal $\neq$ perfect information
  - Complete information: you know all the parameters of the game
  - Perfect information: at each stage, you know the exact history of the game
- What happens when you don't know what the parameters are?
- Welcome to... Bayesian games!

## BNE Setup

A bayesian game includes:

- a set $N = \{1, \ldots, n\}$ of players, with generic members $i$ and $j$;
- a set $\Theta$ of payoff parameters (aka states) with generic member $\theta$;
- a set $T_i$ of types for each player $i$ with generic member $t_i$
- a probability distribution $p$ on $\Theta \times T$ where $T = T_1 \times \ldots \times T_n$ is the set of type profiles $t = (t_1, \ldots, t_n)$;
- a set $A_i$ of actions for each player $i$ with generic member $a_i$; and
- a utility function $u_i : A \times \Theta \times T \to \mathbb{R}$ for each player $i$, where $A = A_1 \times \ldots \times A_n$ is the set of action profiles $a = (a_1, \ldots, a_n)$.

# BNE Setup

- In most settings, nature decides $\theta$ for everyone
- There could be infinitely many $\theta$'s (aka states)!
- Each player $i$ observes their own $\theta_i$ but not player $j$'s parameter, $\theta_j$, when $i \neq j$.
    - However - player $i$ knows the distribution of $\theta_j$'s!
    - Player $i$ gets to calculate their <u>expected</u> payoff/utility and tries to optimize it

## Symmetric BNE

- A symmetric Bayesian Nash Equilibrium is a BNE where every player's strategy is identical.
- Example: for ALL players $i$, play A if $\theta_i$=0.5 and play B if $\theta_i$=0.75

# Symmetric BNE Problem Solving Strategies

For questions asking for symmetric BNEs with a finite number of states:

1. Start by assuming that for all player $i$, the strategy is to play $s_a$ when $\theta_i = a$, $s_b$ when $\theta_i = b$, ...
   - $s_a, s_b, \ldots$ represent actions and $a, b, \ldots$ represent a value that $\theta_i$ can take on
2. Casework!
   - WLOG: assume $\theta_i =$ a
   - *** calculate $u_{ia}(x_i)$, the utility that player $i$ gets when $\theta_i = a$ AND player $i$ plays some action $x_i$ (NOT necessarily $a$!)
   - Take first order condition to get an equation describing what the utility-maximizing $x_i$ should be.
   - now let $x_i = s_a$!
3. After doing this for all states, one should get some equations that relate all the $s_a, s_b, \ldots$ together. Solve for $s_a$ so that your final answer is not dependent on $s_b, s_c, \ldots$ (and vice versa for all others).

## Symmetric BNE Problem Solving Strategies

For questions asking for symmetric BNEs with an infinite number of states:

1. Assume some function, $s(\theta_i)$, is the action player $i$ will play when they see $\theta_i$.
   - E.g. if $s(\theta_i) = 2\theta_i^2$, then this means player $i$ will play $2 \times 0.5^2 = 0.5$ if $\theta_i = 0.5$ and $2 \times 1^2 = 2$ if $\theta_i = 1$.
   - Note that at this moment, we have no idea what form $s(\theta_j)$ will take!
2. Find the expected utility of player $i$ given $\theta_i$ in terms of $\mathbb{E}[s(\theta_j)]$; optimize using FOC.
3. The step above should give us a good sense of what $s(\theta_i)$ is going to look like (e.g., linear, logarithmic, quadratic, square root,... wrt $\theta_i$). Plug in your "guessed" form into $s(\theta_i)$ and solve for it.

## Non-symmetric BNE Problem Solving Strategies

- There will most likey be a payoff matrix
- Conditional probability is your friend!
    - $$P(A|B) = P(A) \times \frac{P(A \cap B)}{P(B)}$$
    - E.g. $P(\theta = 1|\gamma = 2) = ...$
- For each player $i$, fine expected utility for each available action of $i$
- For visual convenience, one can also draw a larger payoff matrix where the choices for each player aren't their actions, but their strategies

## Notes

- VERY importantly: generally, $\mathbb{E}[u_i(\theta_i)] \neq u_i(\mathbb{E}[\theta_i])$!!
  - In other words: the expected utility for player $i$ is NOT necessarily equal to the utility that player $i$ gets assuming player $j$ gets $\theta_j = \mathbb{E}[\theta_j] = \mathbb{E}[\theta_i]$.
- When solving for FOC, do not plug in assumed values/actions until you have found a utility-maximizing action.
  - plugging in values too early will almost certainly cause arithmetic errors later
- Many BNE questions are simpler in concept than computation - it may be helpful to write out every step!

1. Bayesian Nash Equilibrium
2. BNE Example Question
3. BNE Application: Auctions

# Problem 14.4

Two partners simultaneously invest in a project, where the level of investment can bt any non-negative real number. If partner $i$ invests $x_i$ and the other partner $j$ invests $x_j$, then the payoff of partner $i$ is

$$t_i x_i x_j - x_i^3.$$

Here, $t_i$ is privately known by partner $i$, and $t_1, t_2$ are independently distributed with the uniform distribution on $[0, 1]$. Find all symmetric BNEs.

# Problem 14.4 Walk-through

This is a symmetric BNE question with infinitely many states (i.e., choices of parameters), where $t_i$ is the parameter.

1. Assume that our symmetric BNE will be: player $i$ invests value $s(t_i)$ when they observe parameter $t_i$.

## Problem 14.4 Walk-through

1. Assume that our symmetric BNE will be: player $i$ invests value $x_i = s(t_i)$ when they observe parameter $t_i$.
2. Then, player $i$'s expected utility from playing $x_i$ is:

$$\begin{aligned}\mathbb{E}[u_i(x_i, t_i)] &= \mathbb{E}[t_i x_i x_j - x_i^3] \\ &= \mathbb{E}[t_i x_i s(t_j) - x_i^3] \\ &= t_i x_i \mathbb{E}[s(t_j)] - x_i^3;\end{aligned}$$

Taking the FOC of the above with respect to $x_i$ gives us

$$\begin{aligned}t_i \mathbb{E}[s(t_j)] - 3x_i^2 &= 0 \\ x_i &= \sqrt{t_i \mathbb{E}[s(t_j)]/3}\end{aligned}$$

# Problem 14.4 Walk-through

2. We know that the utility-maximizing $x_i = \sqrt{\mathbb{E}[t_i s(t_j)]/3}$.
3. Note that $s(t_i)$ is defined to be the strategy that maximizes utility (otherwise it won't be an NE). So, we let

$$s(t_i) = x_i = \sqrt{\mathbb{E}[t_i s(t_j)]/3} = \sqrt{t_i(\mathbb{E}[s(t_j)]/3)}.$$

Now, notice that $s(t_i)$ is in the form of $\alpha\sqrt{t_i}$, where $\alpha$ is some constant that we can solve for!

- (to see this, further note that $\mathbb{E}[s(t_j)]/3$ is a constant.)

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Now, notice that $s(t_i)$ is in the form of $\alpha\sqrt{t_i}$, where $\alpha = \sqrt{\mathbb{E}[s(t_j)]/3}$ is some constant that we can solve for!

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## Problem 14.4 Walk-through

3. We calculate

$$\mathbb{E}[s(t_i)] = \mathbb{E}[\alpha\sqrt{t_i}] = \alpha\mathbb{E}[\sqrt{t_i}]$$
$$= \alpha \int_0^1 \sqrt{t_i} dt_i = \frac{2\alpha}{3}$$

Substituting into $\alpha = \sqrt{\mathbb{E}[s(t_j)]/3}$, we get

$$\alpha = \sqrt{2\alpha/9}.$$

4. Finally, we see that there are 2 BNE solutions to this question:
    1. $\alpha = 0$, in which case no one invests.
    2. $\alpha = \frac{2}{9}$, in which case player $i$ invests $\frac{2\sqrt{t_i}}{9}$.

1. Bayesian Nash Equilibrium
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## What are Auctions?

- Auctions are processes of trading goods through some type of bidding process
- Each potential buyer $i$ has a true valuation of the good $v_i$, and they will make a bid $b_i$
- There are many types of auctions
    - The (arguably) most ubiquitously known auction form is first-price auction, where the highest bidder gets to trade
    - In general, there could be many buyers, many sellers, many goods, and trades do not necessarily have to happen for the highest/lowest bidders
- Examples of IRL auctions: Advertisement slots on search engines, art/artifact markets, construction companies competing for a project (lowest-price bidding)

## How is this a Bayesian game?

- In an auction, no buyer knows what other buyers' true valuation $v_i$ are; they only observe the bid $b_i$, which is usually some function of $v_i$. In auctions, NEs can be interpreted as "given that I have true valuation $v_i$, I won't be better off bidding any other amount than $b_i$".

# First-price auction with independent private values

- $n$ players bidding for 1 good; everyone announces their bid simultaneously
- each player $v$ has true valuation of the good $v_i \in [0, 1]$ that is uniformly distributed
- Each player also bids $b_i \in [0, \infty)$
- Highest bidder wins. If there is a tie, the winning bidder is determined randomly.
- The utility function is

$$u_i(b_1, ..., b_n, v_1, ..., v_n) = \begin{cases} v_i - b_i \text{ if } b_i \text{ is the unique highest bid} \\ \frac{v_i - b_i}{n} \text{ if } b_i \text{ is one of the highest bids} \\ 0 \text{ if } b_i \text{ is not one of the highest bids} \end{cases}$$

# Finding a linear symmetric BNE - 2 bidder case

1. Assume that the equilibrium is for each player $i$ with valuation $v_i$ to play
$$b(v_i) = a + cv_i.$$
   - Importantly, $a$ and $c$ do NOT depend on the player!
   - Note that $c > 0$ - and thus $b(v_i)$ is a strictly increasing function of $v_i$.

# Finding a linear symmetric BNE - 2 bidder case

2. Let's first assume that player $i$ placed a bid of value $b_i$. Assume that player $j$'s bid is $b_(v_j)$. The expected utility of player $i$ is

$$\begin{aligned}\mathbb{E}[u_i(b_i, b_j^*, v_1, v_2)|v_i] &= (v_i - b_i) \times P(b(v_j) \leq b_i) \\ &+ 0 \times P(b(v_j) > b_i \\ &= (v_i - b_i)P(a + cv_j \leq b_i) \\ &= (v_i - b_i)P(v_j \leq \frac{b_i - a}{c}) \\ &= (v_i - b_i)\frac{b_i - a}{c}.\end{aligned}$$

Taking FOC of this w.r.t. $b_i$ gives best response $b_i = \frac{v_i + a}{2}$. We see that this is indeed in the linear of $a + cv_i$.

## Finding a linear symmetric BNE - 2 bidder case

3. Compute constants $a$ and $c$.

- We must have $a + cv_i = \frac{a}{2} + \frac{v_i}{2}$ for all values of $v_i$
- Therefore, $a = 0, c = \frac{1}{2}$
- the linear symmetric BNE is $b(v_i) = \frac{v_i}{2}$
- for the general case, the linear symmetric BNE is $b(v_i) = \frac{(n-1)v_i}{n}$