18.650. Fall 2023
Prof. Rigollet

Date: 10/27/23

Name:________________________________

ID#:________________________________

Test 2

PLEASE DO NOT TURN THIS PAGE OR START ANSWERING QUESTIONS UNTIL YOU ARE INSTRUCTED TO DO SO.

1. This is an open book, closed notes test.
2. You are also allowed a two-sided letter-sized cheat sheet.
3. Calculators are permitted but the test can be done without.
4. Connected devices like phones, laptops, or tablets are strictly forbidden.
5. The test starts at 1:05
6. The test ends at 1:55 regardless of your time of arrival.
7. All questions should be answered on the present exam sheet
8. Make sure to mark your **name and ID** on the first page in a legible way (a computer will have to read them) and **do not remove the staple**.

Problem1	50	
Problem2	50	
TOTAL	100	

Problem 1. (50 pts) 5 points each. Check only one box per question.

1. The bootstrap method always provides accurate confidence intervals regardless of the original sample size.

    ◯ TRUE

    ◯ FALSE **Solution:** False.

2. The maximum likelihood estimator is always unbiased

    ◯ TRUE

    ◯ FALSE **Solution:** False.

3. The maximum likelihood estimator is always asymptotically normal

    ◯ TRUE

    ◯ FALSE **Solution:** False.

4. The asymptotic variance of the MLE is equal to the Fisher information

    ◯ TRUE

    ◯ FALSE **Solution:** False.

5. The p-value is the probability that H_0 is correct given the data

    ◯ TRUE

    ◯ FALSE **Solution:** False.

6. In hypothesis testing, the null hypothesis represents a specific claim or status quo that we aim to challenge or retain based on the data.

    ◯ TRUE **Solution:** True.

    ◯ FALSE

7. A Type I error occurs when we fail to reject the null hypothesis when it is actually true.

    ◯ TRUE

    ◯ FALSE **Solution:** False.

8. The E-M algorithm is used to compute a maximum likelihood estimator
   - ◯ TRUE **Solution:** True.
   - ◯ FALSE

9. Let $X_1, \ldots, X_n$ be i.i.d $\mathcal{N}(\mu, \sigma^2)$. We want to test

$$H_0 : \mu \geq 0 \quad \text{vs} \quad H_1 : \mu < 0$$

   We observe $\bar{X}_n = 0.1$ and use the Wald test.
   - ◯ The p-value is less than 1%
   - ◯ The p-value is at least 50% **Solution:** Correct
   - ◯ The test is rejected at level 5%
   - ◯ We cannot decide whether the test is rejected at level 5% without knowing the sample standard deviation.

10. When might the bootstrap method be particularly useful?
   - ◯ When the sample size is very small and the population distribution is known.
   - ◯ When the sample size is large and the sampling distribution of the statistic of interest is difficult to derive mathematically. **Solution:** Correct
   - ◯ When the population distribution is uniform and the sample size is small.
   - ◯ When computational resources are limited and quick approximations are unnecessary.

Problem 2.
For all $\theta > 0$ and $x \in (0,1)$, let $f_\theta(x) = \theta x^{\theta-1}$.

Let $X_1, \ldots, X_n$ be i.i.d. random variables with density f_θ, for some unknown $\theta > 0$.

1. (10 points) Compute the maximum likelihood estimator $\hat{\theta}$ of θ. **Solution:** The log-likelihood is $\ell(\theta) = \sum_{i=1}^n (\log(\theta) + (\theta-1)\log(X_i))$. Solving $\ell'(\theta) = 0$ gives $\hat{\theta} = -1/(\frac{1}{n}\sum_{i=1}^n \log(X_i))$

2. (10 points) Show that $\hat{\theta}$ is asymptotically normal and show that its asymptotic variance is θ^2. **Solution:** Let $g(x) = 1/x$. By the CLT we have $\frac{1}{n}\sum_{i=1}^n(-\log(X_i)) \approx N(\frac{1}{\theta}, \frac{\mathbb{V}(\log(X_1))}{n})$. Applying the delta-method to $g(\frac{1}{n}\sum_{i=1}^n \log(X_i))$ we see that $\hat{\theta}$ is asymptotically normal. To compute its asymptotic variance we use the fact that the asymptotic variance of the MLE is given by 1 over the Fisher information in regular models. The Fisher information is given by $I(\theta) = -\mathbb{E}_\theta[\frac{d^2}{d\theta^2}\log f_\theta(X_!)] = \frac{1}{\theta^2}$ and the result follows.

3. (10 points) Using $\hat{\theta}$, find a 95 percent confidence interval for θ. **Solution:** Using the Gaussian approximation from the previous part, the interval $\hat{\theta}(1 \pm 1.96\frac{1}{\sqrt{n}})$ has 95% coverage asymptotically.

We want to test

$$H_0 : \theta = 1 \quad \text{vs.} \quad H_1 : \theta \neq 1$$

To that end, we collect $n = 100$ observation and compute that $\hat{\theta} = 0.82$.

4. (10 points) What is the name of the distribution with density f_1? **Solution:** f_1 is the density of the uniform distribution.

5. (10 points) What is the p-value of the Wald test? How strong is the evidence against H_0? **Solution:** From part 2, we know that the asymptotic variance of $\hat{\theta}$ is $\theta^2 = 1$ under the null hypothesis that $\theta = 1$. Thus, the level-α Wald's test rejects the null hypothesis whenever

$$\left| \frac{\hat{\theta} - \theta}{\sqrt{\theta^2/n}} \right| = \left| \frac{0.82 - 1}{\sqrt{1/100}} \right| = 1.8 \geq \Phi^{-1}(1 - \alpha/2),$$

where Φ is the cdf of the standard normal. Looking at the z-table we see that Wald's test wouldn't reject for $\alpha < 2(1 - \Phi(1.8)) \approx 7.2\%$, which is the p-value. There is weak evidence against H_0.

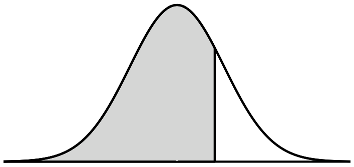

Table 1: The table lists $P(Z \leq z)$ where $Z \sim N(0,1)$ for positive values of z.

Z	Second decimal place of Z									
	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

*For $Z \geq 3.50$, the probability is greater than or equal to 0.9998.