14.12 Pset 9 Solutions

December 2023

Problem 1

We have conjectured that bidding function is of the following form

$$b(v_i) = \begin{cases} v_i & \text{if } v_i \le r \\ \beta v_i + (1-\beta)r & \text{if } v_i > r \end{cases}$$

Since it's symmetric BNE, it must be that

$$b(v_i) = \arg\max_b EU(v_i, b, b(v_j))$$

where

$$\begin{aligned} EU(v_i, x, b(v_j)) &= \begin{cases} 0 & \text{if } x < r \\ (v_i - r)P(b(v_j) < r) + (v_i - x)P(r \le b(v_j) < x) & \text{if } x \ge r \end{cases} \\ &= \begin{cases} 0 & \text{if } x < r \\ (v_i - r)P(v_j < r) + (v_i - x)P(r \le v_j < b^{-1}(x)) & \text{if } x \ge r \end{cases} \end{aligned}$$

where

$$b^{-1}(x) = \frac{x - (1-\beta)r}{\beta}$$

Lets write down the pay-off matrix for different cases for v_i and b

	$b < r$	$b \ge r$
$v_i \le r$	0	$(v_i - r)r + (v_i - b)\frac{b-r}{\beta}$ (-ve)
$v_i > r$	0	$(v_i - r)r + (v_i - b)\frac{b-r}{\beta}$ (+ve)

When $v_i > r$, taking FOC of the above with respect to b, we find that the pay-off is maximized at $b(v_i) = \frac{v_i + r}{2}$, which implies $\beta = \frac{1}{2}$

Expected Revenue of the seller is thus given by

$$2(r)(1-r)r + (1-r)^2 \frac{1}{2}\left(E(v^{max}|\text{both bids are above } r) + r\right)$$

where

$$
\begin{aligned}
E(v^{max}|\text{both bids are above } r) &= \int_r^1 vd\left(\frac{v-r}{1-r}\right)^2 \\
&= \frac{2}{(1-r)^2}\left(\int_0^{1-r}(v-r)^2 d(v-r) + r\int_0^{1-r}(v-r)d(v-r)\right) \\
&= \frac{2}{(1-r)^2}\left[\frac{x^3}{3} + r\frac{x^2}{2}\right]_0^{1-r} \\
&= 2\left(\frac{1-r}{3} + \frac{r}{2}\right) = \frac{2+r}{3}
\end{aligned}
$$

The expected revenue is given by

$$
ER(r) = 2r^2(1-r) + \frac{1+2r}{3}(1-r)^2
$$

This is maximised at $\boxed{r = 0.5}$ and the maximum revenue is $\boxed{\dfrac{5}{12}}$.

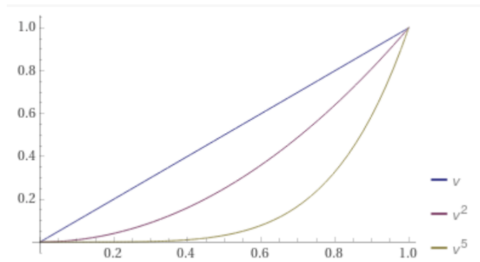

Figure 1: F(v)

Problem 2

The equilibrium bids in a first price auction are given by

$$b(v_i) = v_i - \frac{\int_{\underline{v}}^{v_i} F^{n-1}(v)dv}{F^{n-1}(v_i)}$$

(See lecture notes Equation 20.6)

In this question, it is given that $F(v) = v^{\alpha}$. Substituting this into above, we get

$$b(v_i) = v_i - \frac{v_i}{\alpha(n-1)+1}$$

As $\alpha \to \infty$ or $n \to \infty$, $b(v_i) \to v_i$.

Economic Intuition:

- In a first price auction, as bidders bid less, the probability of their winning decreases but their payoff conditional on winning goes up. The amount that bidders shade $(v_i - b(v_i))$ depends on this trade-off.
- As α increases, the cumulative distribution of the private values skews to the right. (see figure 1).
- This means that as α goes up, players are likely to lose more on the margin in terms of probability of winning as they shade their bid, which reduces their incentive to lower their bid.

The same intuition holds the case where the number of players increases. As n becomes very large, the probability of losing increases at a given level of bid.

Problem 3

We hypothesize a symmetric Bayesian Nash equilibrium in which each buyer bids $b^*(v)$ when their value is v, where the bidding function $b^* : [0,1] \to [0,1]$ is differentiable with a strictly positive derivative everywhere. Assuming their opponent bids thusly, the expected utility of a buyer with value v from bidding some $b \in [b^*(0), b^*(1)]$ is

$$\pi_{b^*}(v,b) = \int_0^{(b^*)^{-1}(b)} \left[v - \frac{b + b^*(\tilde{v})}{2} \right] d\tilde{v},$$

where $(b^*)^{-1} : [b^*(0), b^*(1)] \to [0,1]$ is the inverse of b^*. Differentiating this expression with respect to b gives

$$\frac{\partial \pi_{b^*}}{\partial b}(v,b) = \frac{\partial}{\partial b} \int_0^{(b^*)^{-1}(b)} \left[v - \frac{b + b^*(\tilde{v})}{2} \right] d\tilde{v} = \frac{v-b}{\frac{db^*}{dv}((b^*)^{-1}(b))} - \frac{1}{2}(b^*)^{-1}(b). \tag{1}$$

Substituting $b = b^*(v)$ into this expression and setting it equal to 0 gives, we obtain

$$\frac{\partial \pi_{b^*}}{\partial b}(v, b^*(v)) = \frac{v - b^*(v)}{\frac{db^*}{dv}(v)} - \frac{v}{2}.$$

For $\frac{\partial \pi_{b^*}}{\partial b}(v, b^*(v)) = 0$ to hold, it must be that

$$v^2 \frac{db^*}{dv}(v) + 2v b^*(v) = 2v^2.$$

Solving this differential equation gives $b^*(v) = \frac{2}{3}v$.

We now verify that it is a Bayesian Nash equilibrium for each buyer to bid $\boxed{b^*(v) = \frac{2}{3}v}$ when their value is v. When an opponent bids according to this b^*, the expression in (1) for the derivative of the expected utility of a buyer with value v from bidding some $b \in \left[0, \frac{2}{3}\right]$ amounts to

$$\frac{\partial \pi_{b^*}}{\partial b}(v,b) = \frac{3}{2}(v-b) - \frac{3}{4}b = \frac{9}{4}\left(\frac{2}{3}v - b\right).$$

Since this expression is strictly positive for $b < \frac{2}{3}v$ and strictly negative for $b > \frac{2}{3}v$, we conclude that it is optimal for a buyer with value v to bid $b^*(v) = \frac{2}{3}v$ when their opponent is also using this bidding rule.

Problem 3: Alt. Solution

Assume that we have a linear symmetric BNE: that is, $b^*(v_i) = a + cv_i$ where $c > 0$ and a, c hold constant for all i. (The intuition is that the payment is in linear relationship with the bids b_1 and b_2). In this case, $b^{*-1}(b) = \frac{b-a}{c}$ is the true valuation that is implied by a player's bid at b.

WLOG, the expected utility of player 1 with valuation v and bid b is:

$$\begin{aligned}
\pi_{b*}(v, b) &= \int_0^{b^{*-1}(b)} \left(v - \frac{b + b^*(v_2)}{2}\right) dv_2 \\
&= \int_0^{(b-a)/c} \left(v - \frac{b + b^*(v_2)}{2}\right) dv_2 \\
&= \left(v - \frac{b+a}{2}\right)\left(\frac{b-a}{c}\right) + \frac{c}{2}\int_0^{(b-a)/c} v_2 dv_2 \\
&= \left(v - \frac{b+a}{2}\right)\left(\frac{b-a}{c}\right) + \frac{c}{4}\left(\frac{b-a}{c}\right)^2 \\
&= \frac{v}{c}(b-a) - \frac{b^2 - a^2}{2c} + \frac{b^2 - 2ab + a^2}{4c}.
\end{aligned}$$

Taking the first order condition of the above with respect to b, we get

$$\frac{\partial \pi_{b*}(v, b)}{\partial b} = \frac{v-b}{c} + \frac{b-a}{2c} = 0;$$

Solving for b, we get $a + cv = b = \frac{2v}{3} + \frac{a}{3}$, which implies $a = 0, c = \frac{2}{3}$. Thus our solution is $\boxed{b^*(v_i) = \frac{2}{3}v_i}$.

Problem 4

First, we can compute the VCG $\alpha_j p_j$ in each of the 3 cases:

1. $\alpha_j p_j = \sum_{k=2}^{3} b^{(k)}(\alpha_{k-1} - \alpha_k) = b_1(\alpha_1 - \alpha_2) + b_2(\alpha_2 - \alpha_3)$
   So, in this case,
   $$U_1 = v\alpha_1 - b_1(\alpha_1 - \alpha_2) - b_2(\alpha_2 - \alpha_3)$$
2. $\alpha_j p_j = \sum_{k=3}^{3} b^{(k)}(\alpha_{k-1} - \alpha_k) = b_2(\alpha_2 - \alpha_3)$
   So, here:
   $$U_2 = v\alpha_2 - (b_2(\alpha_2 - \alpha_3))$$
3. Here, the VCG payment cancels to 0, so,
   $$U_3 = v\alpha_3$$

So, to see when each one is highest, we consider the differences:

$$U_1 - U_2 = \alpha_1(v - b_1) + \alpha_2(b_1 - v) = (v - b_1)(\alpha_1 - \alpha_2)$$

$$U_2 - U_3 = (v - b_2)(\alpha_2 - \alpha_3)$$

$$U_1 - U_3 = v(\alpha_1 - \alpha_3) - b_1(\alpha_1 - \alpha_2) - b_2(\alpha_2 - \alpha_3)$$

$$= \alpha_1(v - b_1) + \alpha_2(b_1 - b_2) + \alpha_3(b_2 - v)$$

- If $v > b_1$, then $U_1 - U_2$ is clearly positive. Then, from the second equation, and the fact that $v > b_1 > b_2$, we can say that $U_2 > U_3$, meaning that U_1 is the highest in this case.
- Next, if $b_1 > v > b_2$, then $U_1 - U_2 < 0$, and $U_2 - U_3 > 0$, so U_2 is highest in this case.
- Finally, if $b_2 > v$, then clearly $U_2 - U_3 < 0$. Note that $b_1 > b_2 > v$ implies that $U_2 > U_1$, so $U_3 > U_2 > U_1$, meaning that U_3 is highest in this case.