Part II

Dynamic Games

Chapter 8

Backward Induction

In dynamic games, players learn about other players' actions and possibly the payoffs as the game proceeds. Some plans that appear to be rational may turn out to be irrational when some unanticipated contingencies arise. This leads to more stringent rationality requirements on plans, leading to sharper solution concepts.

For a concrete example, consider the following game. Hari and Sita are two siblings. Their parents have promised them 5 pieces of chocolates if they agree on how to share them. Hari offers k chocolates to Sita, and Sita accepts or rejects the offer. If she accepts the offer, their parents give $5 - k$ and k chocolates to Hari and Sita, respectively; they give zero chocolates to each of them if she rejects it. When this Ultimatum Game is played in the class, the results were as in Figure 8.1. All but one proposers offered one chocolate to the other player and kept four for themselves; the exceptional proposer offered two. The probability of acceptance was 77%. Of course, it is a Nash equilibrium that Hari offers 1 chocolate to Sita expecting that she will accept an offer if and only if she gets at least 1 chocolates. But there are also many other Nash equilibria. For example, it is a Nash equilibrium that Alice only accepts offer 5 and rejects all the other offers, and Hari offers 5 chocolates to her. To see that this is a Nash equilibrium, observe that Hari gets 0 no matter what he offers: if he offers anything less than 5 chocolates to his sister, she would reject the offer and they both would get zero chocolates. Hence, Hari does not have an incentive to deviate. On the other side, Sita could not possibly raise her payoff by playing another strategy because 5 is the highest possible payoff. In particular, if she planned to accept the offer of 4 chocolates, she would still get 5

# of Groups	Total Pie	Avg. Offer	Avg. Accepted Offer	Avg. Rejected Offer	Mode Offer	Rejection %
11	5	1.09	1.13	1.00	1	27.27

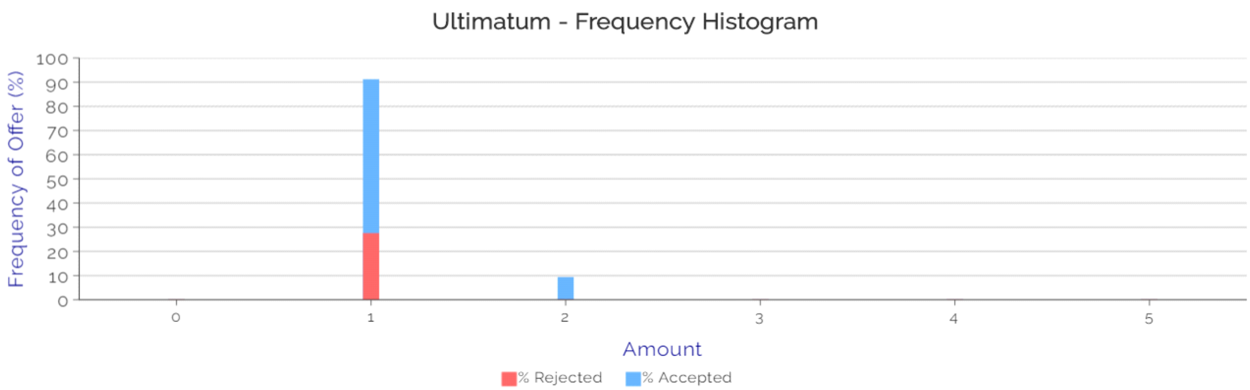


Figure 8.1: Frequency of offers in ultimatum game in a classroom experiment.

chocolates because Hari does not offer 4 chocolates in this equilibrium. More generally, for any $k \in \{0, 1, 2, 3, 5\}$, it is a Nash equilibrium that Hari offers k chocolates and Sita accepts only offer k. In the classroom, none of these equilibria are played other than $k = 1$ (and a single offer of $k = 2$).

Not all Nash equilibria have equal appeal. The equilibrium where one offers only one chocolate appears to be special. As an intelligent seven year old boy, Hari could have explained this as follows. "If I offer zero chocolates to my sister, she would probably reject it—although she is indifferent. But if I offer her any positive number of chocolates, she would accept it because she would prefer having any number of chocolates to having no chocolate at all. So, I offered her one chocolate, the smallest number of chocolates that is acceptable to her."

The other equilibria miss this nuance. For example, in the Nash equilibrium where Hari offers 5 chocolates, Sita is supposed to reject it if Hari gave her 4 chocolates, keeping only one for himself. At the planning stage this would be rational if (and only if) Sita did not anticipate such an offer because what she does in that contingency does not affect her payoff. However, Sita cannot maintain this belief if she is offered 4 chocolates. In that case, she *knows* that Hari offered 4 chocolates and her original assumption has been proven false. In that case, Sita should update her belief and accept the offer as the only best response to new belief.

A rational player can plan on taking an irrational action in a contingency if she assigns zero probability to that contingency. This detail can be ignored when in a single-person decision problem, as one can ignore such zero-probability events. The case of dynamic games is an entirely different story. Typically, under any solution, many information sets are off the path, and thus the solution itself rules out those contingencies. According to the solution concept, the players may assign zero probability to those contingencies and may plan on taking suboptimal actions in those contingencies. This is highly problematic for at least two reasons. First, consider an information set where a player i is supposed to take a suboptimal action a, as in the case of Sita rejecting 4 chocolates. At the beginning of the game, player i can rationally make such a plan because she believes that the information set will not be reached. When the information set is reached, however, she can no longer make that assumption because she knows that she is at that information set. Hence, she will not follow through the plan when the information set is reached. Therefore, the original plan is somewhat inconsistent with players' rationality and the informational assumptions embedded in the extensive form game. Second, as in the case of equilibrium where Hari offers 5 chocolates, some solutions may rely on self-fulfilling theories based on irrational behavior: a player i plans to take a suboptimal action a at an information set that will not be reached, and the information set is not reached precisely because player i is expected to play a at that information set. Such self-fulfilling theories will be unraveled once it is recognized that the rational players will not follow thorough if the contingencies are reached. Ruling out such solutions is a main objective in the analyses of dynamic games.

This section formalizes this idea in the context of perfect-information games with finite horizon, where each information set is singleton and there can only be finitely many moves in any history of moves. For these games, it introduces a powerful solution concept: backward induction.

8.1 Definition

The concept of backward induction corresponds to the assumption that it is common knowledge that each player will act rationally at each future node where she moves – even if her rationality would imply that such a node will not be reached. (The assumption that the player moves rationally at each information set she moves is called *sequential rationality*.)

Mechanically, backward induction corresponds to the following procedure, depicted in Figure 8.2. Consider any decision node at which each move leads to a terminal node. If the player who moves at this node acts rationally, she chooses the best move for herself at that node. Hence, select one of the moves that give this player the highest payoff. Assign the payoff vector associated with this move to the node at hand and delete all the moves stemming from this node, obtaining a shorter game, where the above node is a terminal node. Repeat this procedure until the origin is the only remaining node. During the procedure, a move is picked at each information set, resulting in a strategy profile. This strategy profile is a backward induction solution. If there were only one best move at each information set, the solution is unique. Otherwise, there are multiple solutions, stemming from the choice of best moves in information sets with multiple best

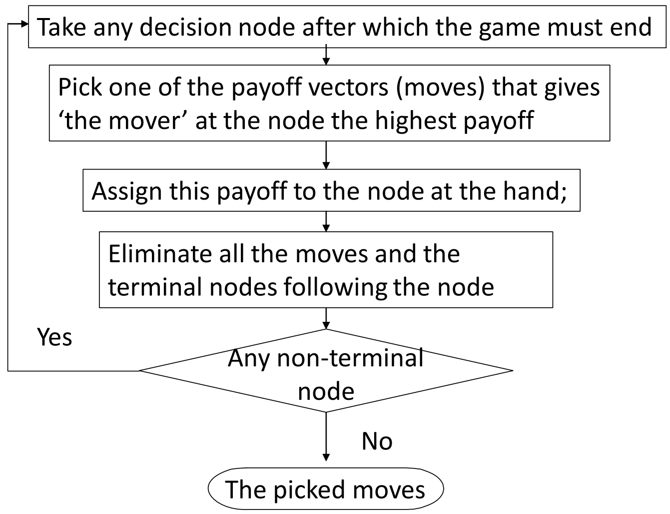


Figure 8.2: Algorithm for backward induction

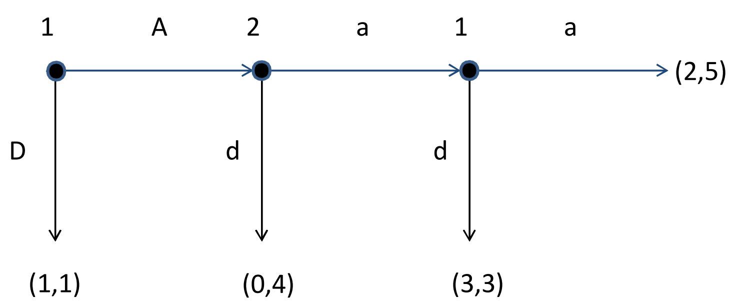


Figure 8.3: A centipede game.

moves.

For an illustration of the procedure, consider the game in Figure 8.3. This game is a version of a well-known game called the Centipede Game. In this game, it is mutually beneficial for all players to stay in a partnership, but each player would like to exit the partnership if she knows that the other player will exit in the next day. To apply backward induction to this game, start from the last day. This is the node in which Player 1 chooses between going across (a) and going down (d), where across means staying in the partnership and down means exiting the partnership. If Player 1 goes across, he would get 2; if he goes down, he would get the higher payoff of 3. Hence,

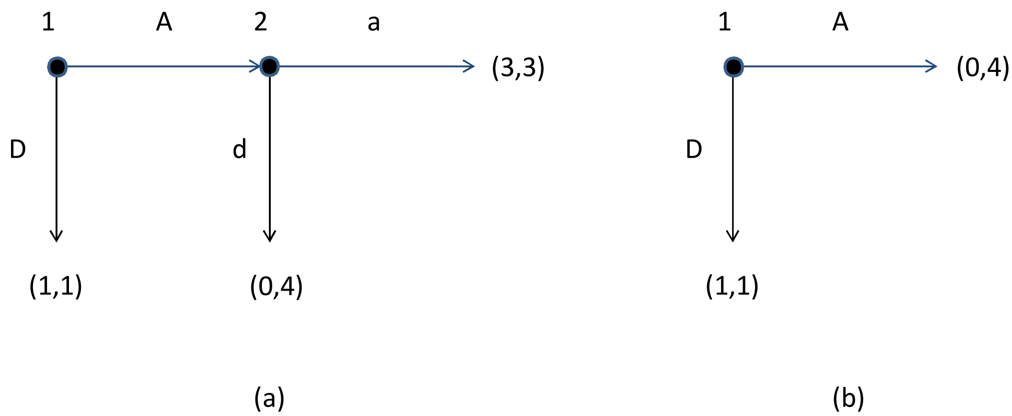


Figure 8.4: Reduced forms of Centipede game during backward induction: (a) after one round of elimination, (b) after two rounds of elimination.

according to the procedure, he goes down. One then selects the move d and replaces the moves at this node with the payoff $(3,3)$ associated with the selected move. This results in the reduced game in panel (a) of Figure 8.4.

In the reduced game, the game ends on the second day, when Player 2 chooses between going across (a) and going down (d). If she goes across, she get 3; if she goes down, she gets the higher payoff of 4. Hence, according to the procedure, she goes down. One then selects the move d and replaces the moves at this node with the payoff $(0,4)$ associated with the selected move. This results in the reduced game in panel (b) of Figure 8.4. In this reduced game, Player 1 gets 0 if he goes across (A), and he gets 1 if he goes down (D). Therefore, he goes down. Since this was the initial node, the backward induction stops here. The procedure results in the strategy profile (Dd, d) according to which, at each node, the player who is to move goes down, exiting the partnership.

It is useful to go over the assumptions made in the construction of the above strategy profile. In the first round, when d is selected for Player 1 in the last node, it is assumed that Player 1 will act rationally at the last date. In the second round, when d is selected for Player 2, it is assumed that Player 2 is rational and that upon observing that Player 1 went across she keeps believing that Player 1 will act rationally in the last node—and his payoffs are as described in the game. Finally, when D is selected for Player 1 in the last round of elimination, it is assumed that Player 1 is rational and anticipates all these. That is, he is assumed to know that Player 2 is rational, and that she will keep believing that Player 1 will act rationally on the third day.

It is important to emphasize that a backward induction solution is a strategy profile. It tells what each player will do at each contingency in which she is to make a move. For example, in the Centipede game above, the unique solution states that each player will go down whenever she is to make a move. On the other hand, the outcome of the strategy profile is often a simple sequence of events. For example, in the Centipede game, the outcome is simply that Player 1 goes down (i.e., he exits the partnership possibly before it starts). This is what an outside observer, such as a journalist or statistician, sees. But this outcome is just the tip of an iceberg; the iceberg is the strategy profile that describes what players would have done in all hypothetical situations that never materialize. Game theory is about the iceberg not merely its tip. A game theorist is interested in not only what players do but why they do it, by analyzing what players would have done if the players had taken some other actions. In the above example, a game theorist is not only interested in the fact that the partnership ends before it starts but also why Player 1 exits the partnership as soon as possible, foregoing high payoffs at the end of the partnership. The simple answer given by backward induction is that those high payoffs are not viable; in the last instance Player 1 would have exited in order to get the higher payoff at the expense of Player 2, and Player 2 would have exited in order to avoid this contingency if Player 1 stayed in the partnership.

8.2 Examples

This section applies backward induction to several simple games. Each of these applications illustrates a subtlety of backward induction procedure.

Multiple Solutions In the Centipede game above, in each round the player who moved at the node considered had a unique best move, and this resulted in a unique solution. This is generally the case when there are finitely many moves at each node except for knife-edge cases for payoffs in which a player is indifferent between two outcomes. When a player is indifferent between multiple outcomes, there may be multiple best moves for a player. In that case, when there are finite moves at each node, there will be multiple solutions, each solution corresponding to a choice from multiple best

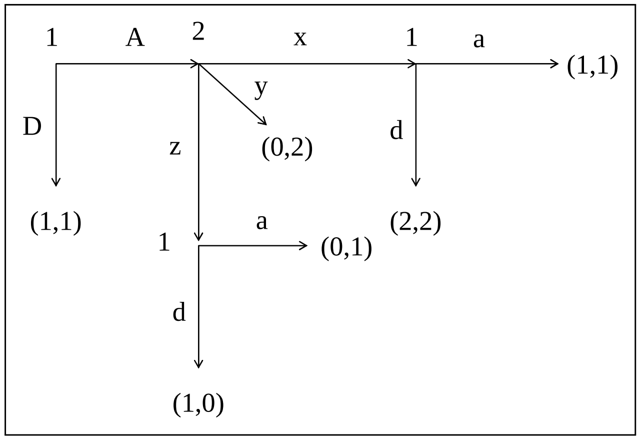


Figure 8.5: A game with multiple backward induction solutions.

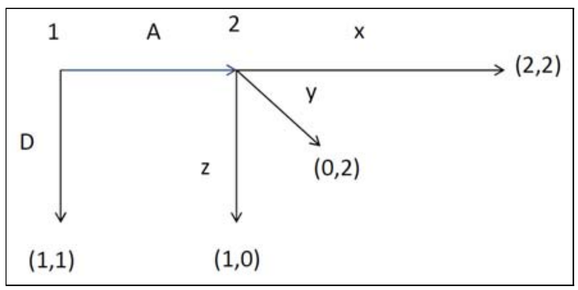
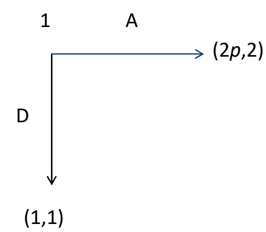


Figure 8.6:

responses to a player at some node. This is illustrated in the next example.

Consider the game in Figure 8.5. According to backward induction, in his nodes on the right and at the bottom, Player 1 goes down, choosing d at both nodes. This leads to the reduced game in Figure 8.6. Clearly, in the reduced game, both x and y yield 2 for Player 2, while z only yields 1. Hence, she must choose either x or y or any randomization between the two. In other words, for any $p \in [0,1]$, the mixed strategy that puts p on x, $1-p$ on y and 0 on z can be selected by the backward induction. Select such a strategy. Then, the payoff vector associated with the decision of Player 2 is $(2p, 2)$. The game reduces to



The strategy selected for Player 1 depends on the choice of p. If some $p > 1/2$ is selected for Player 2, Player 1 must choose A. This results in the equilibrium in which Player 1 plays strategy Add, according to which he chooses A in his first node and plays d in his subsequent decision nodes and Player 2 plays x with probability p and y with probability $1-p$. If $p < 1/2$, Player 1 must choose D. In the resulting equilibrium, Player 1 plays Ddd and Player 2 plays x with probability p and y with probability $1-p$. Finally, if $p = 1/2$ is selected, then Player 1 is indifferent, and we can select any randomization between A and D, each resulting in a different equilibrium.

When there is a unique solution, the solution is in pure strategies. The above example illustrates that when there are multiple solutions, some of these solutions may be in mixed strategies.

For another example of multiplicity, consider the Ultimatum game in the introduction. In this game Hari offers some $k \in \{0, 1, \ldots, 5\}$, and Sita accepts or rejects the offer, leading to the payoff vectors $(m-k, k)$ or $(0, 0)$, respectively, where the first entry is Hari's payoff. For each $k > 0$, there is a unique best move for Sita: accept. One then selects accept for Sita at these node in the application of backward induction. For $k = 0$, Sita is indifferent between accept and reject. One can select either move in backward induction. In one solution, one selects accept $k = 0$. Then, anticipating that all of his offers will be accepted, Hari will offer $k = 0$. In another solution, one selects reject for $k = 0$. Under that solution, anticipating that his offer will be accepted if and only if he offers at least one to Sita, Hari will offer $k = 1$. As in the previous example, there are also solutions in mixed strategies where Sita mixed between accepting and rejecting the offer when Hari offers $k = 0$.

Indifference with Continuum of Moves When there is a continuum of moves at some nodes, indifference between two moves is not a knife-edge case, and more importantly, backward induction may force a particular tie breaking rule in such indifferences. For example, consider the ultimatum game in which the set of possible offers is $[0, 5]$. That is, Hari can offer any real number to Sita, who still accepts or rejects the offer as above. As in the previous case, for any offer $k > 0$, accept is the only best response for Sita and one must select accept for Sita at these nodes. For $k = 0$, Sita is again indifferent between accept and reject. Unlike in the previous case, however, there is a unique solution, in which Sita accepts $k = 0$. To see this, suppose one selects reject for Sita in response to $k = 0$. Then, in the reduced game, Hari selects $k \in [0, 5]$, and his payoff is

$$U_i(k) = \begin{cases} 5 - k & \text{if } k > 0 \\ 0 & \text{if } k = 0. \end{cases}$$

The best response set is empty for this function: for any $k > 0$, Hari can get be better off by offering $k/2$, and no such k can be a best response. The offer $k = 0$ cannot be a best response either as it leads to zero payoff. In that case, the backward induction procedure cannot proceed, and one aborts the solution, going back to Sita's decision at node $k = 0$. The same happens for mixed best responses of Sita where she rejects the offer $k = 0$ with positive probability. One select accept for Sita at $k = 0$. Then, in the reduced game, Hari selects $k \in [0, 5]$, and his payoff is

$$U_i(k) = 5 - k,$$

which is maximized at $k = 0$. Then, one selects $k = 0$ at this node, leading to the unique solution: Hari offers $k = 0$ and Sita accepts all offers.

Games with Nature Randomness is incorporated in backward induction analysis in a straightforward manner. Recall that in a perfect information game with randomness, some nodes are used by nature where the nature's probabilities are given. When applying backward induction, if a normal player is to move at a given node, one applies backward induction as usual; if nature is to move, her mixed action at that node is given in the description of the game already and one uses that distribution to compute the (expected) continuation payoffs for the node—in the same manner that one would do if she were computing a mixed solution to backward induction (as in the example above).

For a concrete example, consider the game in Figure 8.7. In this game, after Player 1 plays L, Nature delegates the decision to one of the players, randomizing between

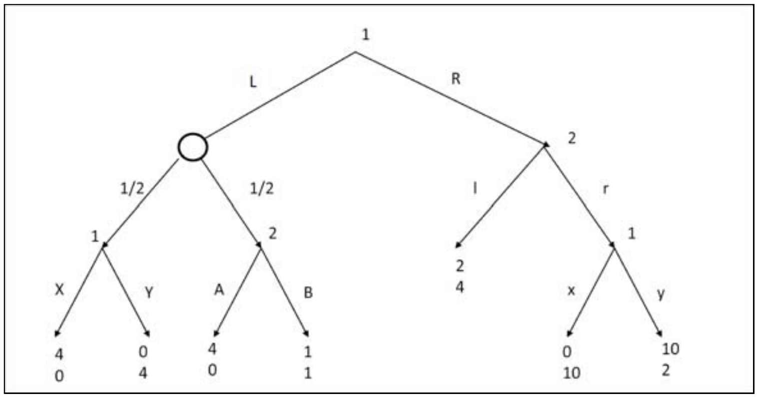


Figure 8.7:

them with equal probabilities. One applies backward induction as follows. In the node at which Player 1 chooses between X and Y, one chooses X and replaces the decision with the associated payoff of $(4,0)$. Likewise, for the node at which Player 2 makes the decision, one chooses B and replaces the node with the associated payoff of $(1,1)$. This brings us to the node at which Nature moves. Nature moves to the left and the payoff vector is $(4,0)$ with probability $1/2$, and Nature moves to the right and the payoff vector is $(1,1)$ with the remaining probability. Hence, the expected payoffs associated with Nature's node is

$$\frac{1}{2}(4,0)+\frac{1}{2}(1,1)=\left(\frac{5}{2},\frac{1}{2}\right).$$

One replaces the Nature's node with payoff vector $(5/2,1/2)$, so that in the reduced game one reaches to the terminal node with payoff vector $(5/2,1/2)$ when Player 1 plays L. On the other branch, one first chooses y for Player 1, yielding $(10,2)$, and then l for Player 2, replacing the tree after R with payoff $(2,4)$. Now, in the reduced game, Player 1 chooses between L and R, with payoff vectors $(5/2,1/2)$ and $(2,4)$, respectively. One chooses L for this node, resulting in the solution (Lxy,bl).

8.3 Backward Induction and Nash Equilibrium

Careful readers must have noticed that the strategy profile resulting from the backward induction above is a Nash equilibrium. (If you have not noticed that, check that it is indeed a Nash equilibrium). This is not a coincidence:

Proposition 8.1. *In a finite game of perfect information, every backward induction solution is a Nash equilibrium.*

Proof. Let $s^* = (s_1^*, \ldots, s_n^*)$ be a backward induction solution. Consider any player i and any strategy s_i. To show that s^* is a Nash equilibrium, it suffices to show that

$$u_i(s^*) \geq u_i\left(s_i, s_{-i}^*\right),$$

where $s_{-i}^* = \left(s_j^*\right)_{j \neq i}$. Take any node

- at which player i moves, and
- s_i^* and s_i prescribe the same moves for player i at every node that comes after this node.

(There is always such a node; for example, the last node player i moves.) Consider a new strategy s_i^1 according to which i plays everywhere according to s_i except for the above node, where she plays according to s_i^*. According to both $\left(s_i^1, s_{-i}^*\right)$ and $\left(s_i, s_{-i}^*\right)$, after this node, the play is as in $\left(s_i^*, s_{-i}^*\right)$, as in the backward induction solution. Moreover, in the construction of s^*, player i plays a best move at this node given this continuation play. Therefore, the change from s_i to s_i^1, which follows the backward induction recommendation, can only increase the payoff of i:

$$u_i\left(s_i^1, s_{-i}^*\right) \geq u_i\left(s_i, s_{-i}^*\right).$$

Applying the same procedure to s_i^1, now construct a new strategy s_i^2 that differs from s_i^1 only at one node, and player i plays according to s_i^* at that node. As in the previous case,

$$u_i\left(s_i^2, s_{-i}^*\right) \geq u_i\left(s_i^1, s_{-i}^*\right).$$

Repeat this procedure, producing a sequence of strategies $s_i \neq s_i^1 \neq s_i^2 \neq \cdots \neq s_i^m \neq \cdots$. Since the game has finitely many nodes, and we are always changing the moves to those of s_i^*, there is some M such that $s_i^M = s_i^*$. By construction, we have

$$u_i(s^*) = u_i\left(s_i^M, s_{-i}^*\right) \geq u_i\left(s_i^{M-1}, s_{-i}^*\right) \geq \cdots \geq u_i\left(s_i^1, s_{-i}^*\right) \geq u_i\left(s_i, s_{-i}^*\right),$$

completing the proof. $\square$

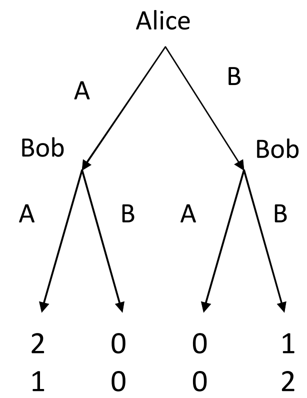


Figure 8.8:

It is tempting to conclude that backward induction results in Nash equilibrium because one plays a best response at every node, finding the above proof unnecessarily long. Since one takes her future moves given and picks only a move for the node at hand, choosing the best moves at the given nodes does not necessarily lead to a best response among all contingent plans in general.

Example 8.1. Consider a single player, who chooses between good and bad everyday forever. His payoff is 1 if he chooses good at everyday, and his payoff is 0 otherwise. Clearly, the optimal plan is to play good everyday, yielding 1. Now consider the strategy according to which he plays bad everyday at all nodes. This gives him 0. But he plays a best response at each node, when he chooses bad action; according to the moves selected in the future, he gets zero regardless of what he does at the current node.

The above pathological case is a counterexample to the idea that if one is playing a best move at every node, her plan is a best response. The latter idea is a major principle of dynamic optimization, called the One-Shot Deviation Principle. It applies in most cases except for the pathological cases as above. The above proof shows that the principle applies in games with finitely many moves. One-Shot Deviation Principle will be the main tool in the analyses of the infinite-horizon games in upcoming chapters.

But not all Nash equilibria can be obtained by backward induction. As discussed in the introduction, in the Ultimatum game, there is a multitude of Nash equilibria that are not a backward induction solution. For another example, consider the version of the Battle of the Sexes game in Figure 8.8. In this version, Alice moves first, and Bob observes her move before he makes his move, so that there is perfect information. In this game, backward induction leads to a unique solution, according to which Bob plays the action played by Alice, and Alice plays A. There is another Nash equilibrium: Alice plays B, and Bob plays B at both of his decision nodes, a strategy denoted by BB. To see that this is a Nash equilibrium, observe that Alice plays a best response to the strategy of Bob: if she plays B she gets 1, and if she plays A she gets 0. Bob's strategy (BB) is also a best response to Alice's strategy: under this strategy he gets 2, which is the highest he can get in this game.

One can, however, discredit the latter Nash equilibrium because it relies on an irrational move at the node after Alice plays A. This contingency does not arise according to Alice's strategy, and it is therefore ignored in Nash equilibrium. Nevertheless, if Alice plays A, playing B would be irrational for Bob, and he would rationally play A as well. And Alice should foresee this and play A.

This example illustrates a shortcoming of the usual rationality condition, which requires that one must play a best response (as a complete contingent plan) at the beginning of the game. While this requires that the player plays a best response at the nodes that he assigns a positive probability, it leaves the player free to choose any move at the nodes that he puts zero probability—because all the payoffs after those nodes are multiplied by zero in the expected utility calculation. Since the likelihoods of the nodes are determined as part of the solution, this may lead to somewhat erroneous solutions in which a node is not reached because a player plays irrationally at the node, anticipating that the node will not be reached, as in (B, BB) equilibrium. Of course, this is erroneous in that when that node is reached the player cannot pretend that the node will not be reached as he will know that the node is reached by the definition of information set. Then, he must play a best response taking it given that the node is reached.

8.4 Interpretations and Limitations

This section is devoted to a discussion of how backward induction relates to some other central concepts, such as commitment, communication and credibility, and some of its main limitations, in particular, its reliance of "common knowledge assumptions" that

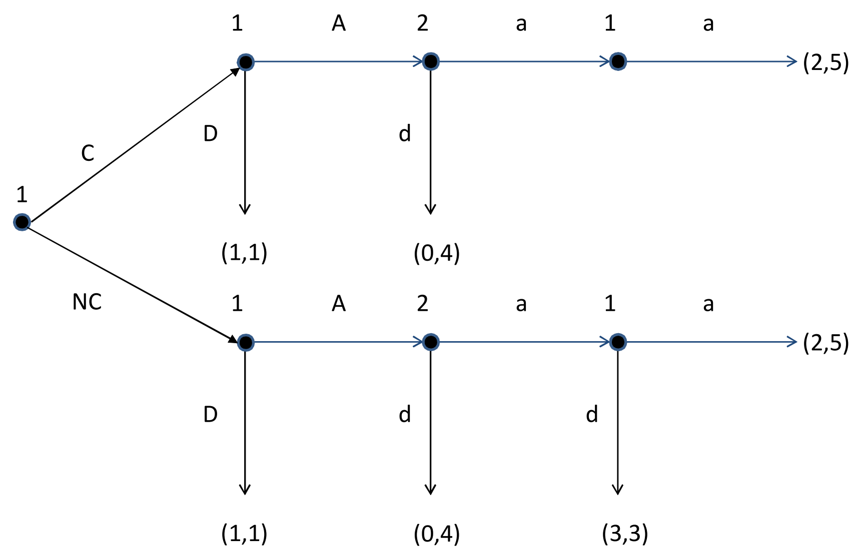


Figure 8.9: Centipe game in which Player 1 can commit to going across in the future

become onerous in long games.

Backward induction assumes that players cannot commit to their planned strategies, and they must choose their moves as they become available. This lack of commitment leads to a specific solution that could be quite different from if a player could commit. For example, in the Centipede game, the result could be quite different if players could commit to a strategy. In that game, the outcomes on the third day (i.e., $(3,3)$ and $(2,5)$) are both strictly better than the outcome $(1,0)$ of the unique backward induction solution. But they cannot reach these outcomes, because Player 2 cannot commit to going across, and anticipating that Player 2 will go down, Player 1 exits the partnership in the first day. There is also a further commitment problem in this example. If Player 1 where able to commit to go across on the third day, then Player 2 would definitely go across on the second day. In that case, Player 1 would go across on the first. Of course, Player 1 cannot commit to going across on the third day, and the game ends in the first day, yielding the low payoffs $(1,0)$.

If players can commit to certain future moves, such a commitment will be included in the description of the game. Backward induction assumes that all of players' commitment abilities are included in the description of the game, and applies the idea of rationality to those games without adding extra assumptions about commitments.

For example, if Player 1 had the option of choosing whether to commit to playing across in his last node, then the game would be as depicted in Figure 8.9. In this game, Player 1 chooses between C (meaning commitment) and NC (meaning no commitment) at the beginning, and he does not have the option of going down in his last move if he chooses C. The backward induction solution on the lower branch is as before: all players go down when they are asked to move. On the upper branch, since Player 1 must go across, Player 2 goes across in order to get 5, and Player 1 also goes across in his initial node; all players go across when they are asked to move. Now, at the initial node, Player 1 chooses between commitment, which yields $(2, 5)$, and no commitment, which yields $(1, 1)$. He chooses to commit.

Similarly, one can view the Battle of the Sexes game in Figure 8.8 depicting a situation in which Alice can commit to playing a move, but Bob cannot. According to the backward induction solution, this commitment helps Alice and hurts Bob. (Although the game is symmetric Alice gets a higher payoff.) Another interpretation is that Alice can communicate to Bob, while Bob cannot communicate to Alice. This enables Alice to commit to her actions, and it is the power of commitment that gives her the high payoff (see Exercise 8.13).

It is tempting to conclude that the ability to commit is always good. While this is true in many games, in some games it is not the case. For example, consider a version of the Matching Penny game in which Player 2 chooses her action after observing Player 1's action; i.e., Player 1 commits to a strategy. Recall that the payoffs are as follows:

	H	T
H	$1, -1$	$-1, 1$
T	$-1, 1$	$1, -1$

In this game, if Player 1 chooses H, Player 2 will play H; and if Player 1 chooses T, Player 2 will play T, too. Hence, the game is reduced to Player 1 choosing between H and T, but both moves yield $(-1, 1)$. In that case, Player 1 will be indifferent between H and T, choosing any of these two actions or any randomization between them will be a backward induction solution. In all solutions, Player 2 beats Player 1. If players chose their actions simultaneously (as in the usual Matching Penny game), they would play each strategy with probability $1/2$ in the unique Nash equilibrium, yielding a payoff of zero to each player in expectation. Commitment hurts Player 1.

In summary, backward induction formalizes the idea that a player should maximize her expected payoff at any given information set knowing that she is at that information set, a piece of information that is included in the definition of an information set. Plans that do not satisfy this more stringent rationality requirement will unravel when players play the game.

8.5 Example—Stackelberg duopoly

In the Cournot duopoly, it is assumed that the firms set the quantities simultaneously. This reflects the assumption that no firm can commit to a quantity level. Sometimes a firm may be able to commit to a quantity level. For example, a firm may be already in the market and constructed its factory and warehouses etc., and its production level is fixed. The other firm enters the market later knowing the production level of the first firm. Such a situation is modeled by Stackelberg duopoly. There are two firms. The first firm is called the *Leader*, and the second firm is called the *Follower*. The marginal cost of each firm is a constant c, as in the Cournot duopoly case.

The timeline is as follows:

- The Leader first chooses its production level q_1.
- Then, knowing q_1, the Follower chooses its own production level q_2.
- Each firm i sells its quantity q_i at the realized market price

  $$P(q_1 + q_2) = \max\{1 - (q_1 + q_2), 0\},$$

  yielding the payoff of

  $$\pi_i(q_1, q_2) = q_i(P(q_1 + q_2) - c).$$

One can represent this situation as an extensive-form game as follows:

- At the initial node, Firm 1 chooses an action q_1; the set of allowable actions is $[0, \infty)$.
- After each action of Firm 1, Firm 2 moves and chooses action q_2; the set of allowable actions now is again $[0, \infty)$.

- Each of these actions leads to a terminal node, at which the payoff vector is $(\pi_1(q_1,q_2),\pi_2(q_1,q_2))$.

Notice that a strategy of Firm 1 is a non-negative real number q_1, and a strategy of Firm 2 is a function from $[0,\infty)$ to $[0,\infty)$, assigning a production level $q_2(q_1)$ to each q_1. In the normal-form representation, the utility function is given by $u_i(q_1,q_2) = \pi_i(q_1,q_2(q_1))$.

Backward induction analysis of this game is as follows. Given $q_1 \leq 1-c$, the best production level for Firm 2 is

$$q_2^*(q_1) = \frac{1-q_1-c}{2}; \tag{8.1}$$

the best production level of Firm 2 is $q_2^*(q_1) = 0$ when $q_1 > 1-c$. Under strategy q_2^*, for each strategy q_1 of Firm 1, the production level of Firm 2 is $q_2^*(q_1)$. One can then write the payoffs of players as a function of q_1 as[1]

$$\begin{pmatrix} \pi_1(q_1,q_2^*(q_1)) \\ \pi_2(q_1,q_2^*(q_1)) \end{pmatrix} = \begin{pmatrix} \frac{1}{2}q_1(1-q_1-c) \\ \frac{1}{4}(1-q_1-c)^2 \end{pmatrix}. \tag{8.2}$$

This corresponds to deleting the moves of firm 2 and replacing them with the payoffs associated with $q_2^*(q_1)$. In the reduced game, Firm 1 chooses q_1, leading to the payoff vector in (8.2). The payoff $\frac{1}{2}q_1(1-q_1-c)$ of Firm 1 is maximized at

$$q_1^* = (1-c)/2, \tag{8.3}$$

the quantity level produced by a monopoly. The backward induction solution is the strategy profile (q_1^*, q_2^*) defined by (8.1) and (8.3). The solution prescribes a production level for Firm 1 and a contingent plan for Firm 2, prescribing what to produce in response to each production level of Firm 1. The outcome is that Firm 1 produces $q_1^* = (1-c)/2$, and Firm 2 produces $q_2^*(q_1^*) = (1-c)/4$ in response. This is the unique backward induction solution.

There are many other Nash equilibria. For example, it is a Nash equilibrium that Firm 1 produces the Cournot quantity $(1-c)/3$, and Firm 2 also produces the Cournot quantity irrespective of what Firm 1 produces. Indeed, Firm 1 does not have any incentive to deviate because Firm 2 blindly produces Cournot quantity regardless of

[1]Note that $q_1\left(1-q_1-\frac{1-q_1-c}{2}-c\right) = \frac{1}{2}q_1(1-q_1-c)$.

what Firm 1 produces, and Cournot quantity is the unique best response to this constant strategy. Firm 2 does not have an incentive to deviate because Firm 2 assumes that Firm 1 will produce Cournot quantity against which producing Cournot quantity is a best response, and adjusting its responses to other production levels of Firm 1 does not affect its payoff in the normal form. One can see this clearly, by observing that the payoff of Firm 2 is

$$u_2\left(\frac{1-c}{3}, q_2\right) = \pi_2\left(\frac{1-c}{3}, q_2\left(\frac{1-c}{3}\right)\right) = \left(\frac{1-c}{3}\right)^2$$

for any strategy q_2 that produces Cournot quantity in response to Cournot quantity. Of course, this is not consistent with backward induction. For example, when Firm 1 produces the Stackelberg quantity, Firm 2 will know this, knowing that the above assumption is falsified. As a result, Firm 2 will produce a lower quantity, the quantity that is computed during the backward induction.

8.6 Exercises with Solutions

Exercise 8.1. Apply backward induction to the game in Figure 2.12.

Solution. In the last nodes, Player 1 will select l in response to M and ρ in response to R, yielding payoff vector $(3,1)$ in both cases. Then, Player 2 will play L and obtain the higher payoff of 2. The payoff vector associated with this move is $(1,2)$. Then, for Player 1, the choice between X and E is a choice between $(2,1)$ and $(1,2)$, and Player 1 plays X.

Exercise 8.2. Apply backward induction to the following game. There are two players Alice and Bob.

- First, Alice chooses a price $p \in \{0,1,2,3\}$.
- Then, a random number $x \in \{0,1,3,5\}$ is drawn; each number in $\{0,1,3,5\}$ can be drawn with probability $1/4$.
- Then, observing p and x, Bob says Yes or No.
- If Bob says Yes, the payoff of Alice is $x-p$, while the payoff of Bob is p. If Bob says No, the payoff of Alice is 0 while the payoff of Bob is $x/2$.

Solution. This is an example of a game with private information, modeled as a game of perfect information. Bob, the seller in this case, knows the value of the object that he is selling, but Alice, the buyer in this case, does not. Backward induction is applied to this game as follows. For any x and p, Bob says Yes if and only if

$$p \geq x/2.$$

This is Bob's strategy. Then, for any x and p, the payoff of Alice is $x-p$ if $p \geq x/2$ and 0 otherwise. For any offer p, her expected payoff is the expected value of the random variable that takes value $x-p$ when $x \leq 2p$ and 0 when $x > 2p$. If she offers $p = 0$, Bob accepts only when $x = 0$, and she gets 0. If she offers $p = 1$, then Bob accepts when x is 0 or 1. In that case, her expected payoff is negative:

$$-1 \cdot \frac{1}{4} + 0 \cdot \frac{1}{4} = -1/4.$$

If she offers $p = 2$, then Bob accepts when x is 0, 1, or 3. Her expected payoff is negative again:

$$-2 \cdot \frac{1}{4} - 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{4} = -1/4.$$

Finally, if she offers $p = 3$, then Bob accepts it for all values of x. Since the expected value of x is 9/4, her expected payoff is $9/4 - 3 = -3/4$, negative again. Therefore, Alice offers $p = 0$, and there is no trade. One might have naively thought that Alice would want to buy the good for a price less than 9/4, but since Bob knows the value of the good, offering price of 1 or 2 will actually lead to a negative payoff for Alice. This is a version of market for Lemons problem that will be studied in great detail later.

Exercise 8.3. Apply backward induction to the following game. There are two profit-maximizing firms, Firm 1 and Firm 2. Each firm has zero marginal cost. First, Firm 1 chooses a price $p_1 \geq 0$. Then, after observing p_1, Firm 2 chooses p_2. Each firm i sells

$$Q_i = A - ap_i + bp_j$$

where A, a, and b are known positive real numbers with $a \geq b > 0$.

Solution. The profit of Firm 2 is

$$u_2 = Q_2 p_2 = (A - ap_2 + bp_1)p_2.$$

For any given p_1, this quadratic function is maximized at[2]

$$p_2^*(p_1) = \frac{A + bp_1}{2a}.$$

When making its decision, Firm 1 reckons that Firm 2 will play this strategy, and hence its profit from choosing p_1 is

$$\begin{aligned} u_1(p_1, p_2^*(p_1)) &= (A - ap_1 + bp_2^*(p_1))p_1 = (A - ap_1 + \frac{b}{2a}(A + bp_1))p_1 \\ &= \left(A + \frac{Ab}{2a} - \left(a - \frac{b^2}{2a}\right)p_1\right)p_1. \end{aligned}$$

This quadratic function is maximized at[3]

$$p_1^* = \frac{A + \frac{Ab}{2a}}{2\left(a - \frac{b^2}{2a}\right)} = \frac{2a + b}{4a^2 - 2b^2}A.$$

The backward induction solution is the strategy profile (p_1^*, p_2^*) where the strategy p_2^* is a function that maps each price p_1 for Firm 1 to a price for Firm 2.

It is useful to compare the backward induction solution to the symmetric linear differentiated Bertrand duopoly in which the firms set the prices simultaneously as in Examples 7.2 and 7.5. In that case, the game is dominance solvable, and the unique rationalizable strategy is

$$p^* = \frac{A}{2a - b}$$

by (7.1). For a direct comparison, one can also write the price of Firm 1 in the backward induction solution as

$$p_1^* = \frac{A}{2a - b - \frac{b^2}{2a+b}}.$$

Clearly, $p_1^* > p^*$. As a leader in price competition, Firm 1 sets a higher price in order to encourage the other firm to raise its price too. In response, Firm 2 also sets a price

[2] One can see this from the first-order condition

$$\frac{\partial u_2}{\partial p_2} \equiv A - ap_2 + bp_1 - ap_2 = 0.$$

[3] If you do not see this immediately, the first-order condition is

$$A(1 + \frac{b}{2a}) - 2(a - \frac{b^2}{2a})p_1 = 0.$$

higher than what it would have charged in the simultaneous case: $p_2^*(p_1^*) > p_2^*(p^*) = p^*$. Both firms end up charging higher prices and obtain higher profits when there is a leader.

Exercise 8.4. A committee of three members, namely $m = 1, 2, 3$, is to decide on a new bill that would make file sharing more difficult. The value of the bill to member m is v_m where $v_3 > v_2 > v_1 > 0$. The music industry, represented by a lobbyist named Alice, stands to gain W from the passage of the bill, and the file-sharing industry, represented by a lobbyist named Bob, stands to lose L from the passage of the bill where $W > L > 0$. Consider the following game.

- First, Alice promises non-negative contributions a_1, a_2, and a_3 to the members $1, 2$, and 3, respectively, where a_m is to be paid to member m by Alice if the bill passes.

- Then, observing (a_1, a_2, a_3), Bob promises non-negative contributions b_1, b_2, and b_3 to the members $1, 2$, and 3, respectively, where b_m is to be paid to member m by Bob if the bill does not pass.

- Finally, each member m votes, voting for the bill if $v_m + a_m > b_m$ and against the bill otherwise. The bill passes if and only if at least two members vote for it.

- The payoff of Alice is $W - (a_1 + a_2 + a_3)$ if the bill passes and zero otherwise. The payoff of Bob is $-L$ if the bill passes and $-(b_1 + b_2 + b_3)$ otherwise.

Assuming that $2v_3 > L > 2v_2$, apply backward induction to this game. (Note that Alice and Bob are the only players here because the actions of the committee members are fixed already.) [**Hint**: Bob chooses not to contribute when he is indifferent between contribution and not contributing at all.]

Solution. Given any (a_1, a_2, a_3) by Alice, for each m, write $p_m(a_m) = v_m + a_m$ for the "price" of member m for Bob. If the total price of the cheapest two members exceeds L (i.e., $\sum_m p_m(a_m) - \max_m p_m(a_m) \geq L$), then Bob needs to pay at least L to stop the bill, in which case, he contributes 0 to each member. If the total price of the cheapest two members is lower than L, then the only best response for Bob is to pay exactly the cheapest two members their price and pay nothing to the remaining member, stopping the bill, which would have cost him L. In sum, Bob's strategy is given by

$$b_m^*(a_1, a_2, a_3) = \begin{cases} v_m + a_m & \text{if } \sum_{m'} p_{m'}(a_{m'}) - \max_{m'} p_{m'}(a_{m'}) < L \text{ and } m \neq m^* \\ 0 & \text{otherwise,} \end{cases}$$

where m^* is the most expensive member, which is chosen randomly when there is a tie.

Given b^*, as a function of (a_1, a_2, a_3), Alice's payoff is

$$U_A(a_1, a_2, a_3) = \begin{cases} W - (a_1 + a_2 + a_3) & \text{if } \sum_m p_m(a_m) - \max_m p_m(a_m) \geq L \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, this is maximized either at some (a_1, a_2, a_3) with $\sum_m p_m(a_m) - \max_m p_m(a_m) = L$ (i.e., the cheapest two members costs exactly L to Bob) or at $(0, 0, 0)$. Since $2v_3 > L > 2v_2$, Alice can set the prices of 1 and 2 to $L/2$ by contributing $(L/2 - v_1, L/2 - v_2, 0)$, which yields her $W - L + v_1 + v_2 > 0$ as $W > L$. Her strategy is

$$a^* = (L/2 - v_1, L/2 - v_2, 0).$$

Exercise 8.5. Apply backward induction to the game in the previous exercise without assuming $2v_3 > L > 2v_2$.

Solution. First, consider the case $L \leq 2v_3$. Then, Alice chooses a contribution vector $(a_1, a_2, 0)$ such that $a_1 + a_2 + v_1 + v_2 = L$, $a_1 + v_1 \leq v_3$, and $a_2 + v_2 \leq v_3$. Such a vector is feasible because $L < 2v_3$ and $v_3 > v_2 > v_1 > 0$. Optimality of this contribution is as before. Now consider the case $L > 2v_3$. Now, Alice must contribute to all members in order to pass the bill, and the optimality requires that the prices of all members are $L/2$ (as Bob buys the cheapest two). That is, she must contribute

$$a^{**} = (L/2 - v_1, L/2 - v_2, L/2 - v_3).$$

Since this costs Alice $3L/2 - (v_1 + v_2 + v_3)$, she makes such a contribution to pass the bill if and only if $3L/2 \leq W + (v_1 + v_2 + v_3)$. Otherwise, she contributes $(0, 0, 0)$ and the bill fails.

Exercise 8.6. In Exercise 7.3, take $n = 2$, and assume that the partners choose their effort levels sequentially. First, partner 1 chooses e_1, and then partner 2 chooses e_2 after observing e_1. Apply backward induction to this game, and briefly discuss your results comparing the outcome to the equilibria in Exercise 7.3.

Solution. Player 2 exerts effort level

$$e_2^* = \alpha^\beta e_1^\beta$$

where $\alpha = \theta/\gamma$ and $\beta = 1/(\gamma - 1)$. Then, as a function of e_1, the payoff of player 1 is

$$u_1 = \theta e_1 e_2^* - e_1^\gamma = \theta\alpha^\beta e_1^{1+\beta} - e_1^\gamma.$$

If $\gamma \leq 2$, then $1 + \beta = \gamma/(\gamma - 1) > \gamma$ and hence u_1 is maximized at 0 or 1 depending on whether $\theta\alpha^\beta$ is smaller or greater than 1. When $\theta > \gamma^{1/\gamma}$, we have $\theta\alpha^\beta > 1$ and $e_1^* = 1$. Otherwise, $e_1^* = 0$. When $\gamma > 2$, the function $\theta\alpha^\beta e_1^{1+\beta} - e_1^\gamma$ is concave, and the unique solution is

$$e_1^* = \left(\theta\alpha^{1+\beta}\right)^{1/(\gamma-1-\beta)}.$$

Exercise 8.7. Use backward induction to a solution to the following game, which is a simplified version of a game called Weakest Link. There are 4 risk-neutral contestants, 1,2, 3, and 4, with "values" $v_1, \ldots, v_4$ where $v_1 > v_2 > v_3 > v_4 > 0$. The game has 3 rounds. At each round, an outside party adds the value of each "surviving" contestant to a common account,[4] and at the end of third round one of the contestants wins and gets the amount collected in the common account. (A contestant is said to be surviving at a round if she was not eliminated at a previous round.) At the end of rounds 1 and 2, the surviving contestants vote out one of the contestants. The contestants vote sequentially in the order of their indices (i.e., 1 votes before 2; 2 votes before 3, and so on), observing the previous votes. The contestant who gets the highest vote is eliminated; the ties are broken at random. At the end of the third round, a contestant i wins the contest with probability $v_i/(v_i + v_j)$, where i and j are the surviving contestants at the third round. (Be sure to specify which player will be eliminated for each combination of surviving contestants, but you need not necessarily specify how every contestant will vote at all contingencies.)

8.7 Exercises

Exercise 8.8. Apply backward induction to the games in Figures 8.7, 8.10 and 8.11.

[4] For example, if contestant 2 is eliminated in the first round and contestant 4 is eliminated in the second round, the total amount in the account is $(v_1 + v_2 + v_3 + v_4) + (v_1 + v_3 + v_4) + (v_1 + v_3)$ at the end of the game.

Exercise 8.9. Consider the extensive form game in Figure 8.12.

1. Apply backward induction to find an equilibrium.

2. Write this game in normal form.

3. Find a Nash equilibrium that leads to a different outcome than that of the solution in Part 1.

Exercise 8.10. Find all pure-strategy Nash equilibria in Figure 8.13. Which of these equilibria can be obtained by backward induction?

Exercise 8.11. In Stackelberg duopoly example, for every $q_1 \in (0,1)$, find a Nash equilibrium in which Firm 1 plays q_1.

Exercise 8.12. Apply backward induction to the "sequential Stackelberg oligopoly" with n firms: Firm 1 chooses q_1 first, firm 2 chooses q_2 second, firm 3 chooses q_3 third, ..., and firm n chooses q_n last.

Exercise 8.13. Apply backward induction to the following version of the sequential Battle of the Sexes game in Figure 8.8: after knowing what Bob does, Alice gets a chance to change her action; then, the game ends. In other words, Alice chooses between A and B; knowing Alice's choice, Bob chooses between A and B; knowing Bob's choice, Alice decides whether to keep her original choice or to change it. What if changing her action would cost her c utils?

Exercise 8.14. Apply backward induction to the following version of the Ultimatum game discussed in the introduction. In this version, Sita cares about fairness, in that she does not like it when her brother gets more chocolates than she does. In particular, her payoff is

$$u_S(m_H, m_S) = m_S - \alpha \max\{m_H - m_S, 0\}$$

where m_H and m_S are the number of chocolates Hari and Sita get, respectively, and $\alpha \in [0,1]$ is a known parameter.

Exercise 8.15. Now imagine that Sita knows how much she cares about fairness but Hari does not. In particular, after Hari offers k chocolates to Sita, a random number $\alpha \in [0,1]$ is drawn—with uniform distribution on $[0,1]$. After observing k and α, Sita accepts or rejects the offer, and the payoffs of Hari and Sita are $m-k$ and $u_S(m-k,k)$, respectively, if she accepts the offer and the payoffs are zero otherwise. Apply backward induction to this game.

Exercise 8.16. Apply backward induction to the following 2-person game. First, player 1 picks an integer x_0 with $1 \leq x_0 \leq 10$. Then, Player 2 picks an integer y_1 with $x_0+1 \leq y_1 \leq x_0+10$. Then, player 1 picks an integer x_2 with $y_1+1 \leq x_2 \leq y_1+10$. In this fashion, they pick integers, alternatively. At each time, the player picks an integer, by adding an integer between 1 and 10 to the number picked by the other player last time. Whoever picks 100 wins the game and gets 100; the other loses the game and gets zero.

Exercise 8.17. Apply backward induction to the following game. There are two monkeys, named Alex and Bob, in adjacent cells, each with a lever. Each day one of the monkeys gets a chance to pull the lever in his cell; he can either pull or pass. If he pulls the lever, a banana drops to the other cell, resulting in 2 utils for the neighbor (from consuming the banana), and costing him 1 util. If he passes, nothing happens that day. Alex gets to pull the lever at odd dates $t = 1, 3, 5, \ldots, 9$, and Bob gets to pull the lever at even dates $t = 2, 4, \ldots, 10$. The game ends at the end of day 10. The payoff of a monkey is 2 times the number of times the other monkey pulls the lever minus the number of times he himself pulls the lever.

Exercise 8.18. Consider the following game, which depicts a story of intergenerational conflict under altruism. There are n players: 1, 2, $\ldots$, n, where player i is an offspring of player $i-1$ for each $i > 1$. Player 1 starts with wealth level $w_1 > 0$, chooses a consumption amount $c_1 \in [0, w_1]$ for herself, leaving $w_2 = w_1 - c_1$ to the next player. For any $i \in \{2, \ldots, n-1\}$, player i observes the previous consumption choices $c_1, \ldots,$ c_{i-1}, and chooses a consumption level $c_i \in [0, w_i]$ for herself and leaves $w_{i+1} = w_i - c_i$ to player $i+1$. Player n consumes $c_n = w_n$. The payoff of each player i is

$$u_i = \sqrt{c_i} + \beta \sum_{j \neq i} \sqrt{c_j}$$

for some $\beta \in (0, 1)$.

1. Suppose player 1 could choose a consumption c_i for each player i subject to the constraint that $c_1 + \cdots + c_n = w_1$. What would he choose?

2. For $n = 3$, apply backward induction to the above game, and compare the resulting consumptions to the answer in part (a).

3. Redo part (2) for arbitrary $n \geq 3$.

Exercise 8.19. Redo Exercise 8.6 for $n > 2$ assuming that the partners choose their effort levels in order $1, 2, \ldots, n$, each observing the effort levels of the previous partners.

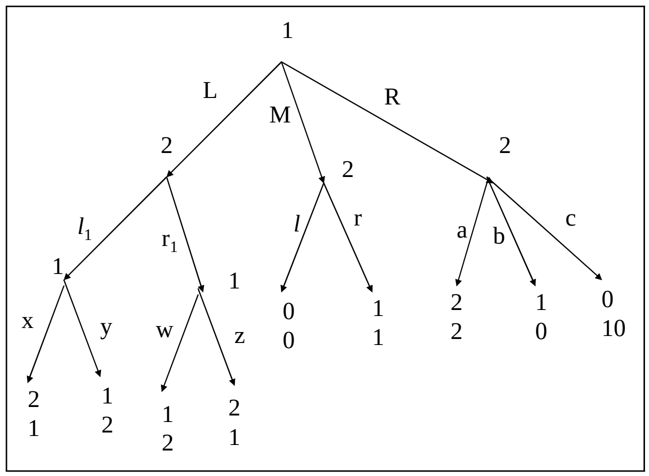


Figure 8.10:

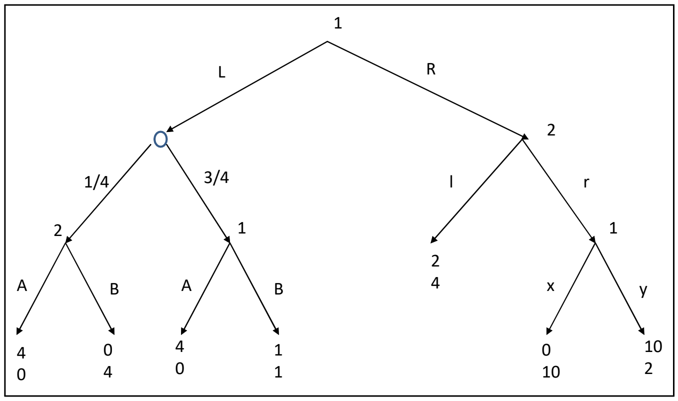


Figure 8.11:

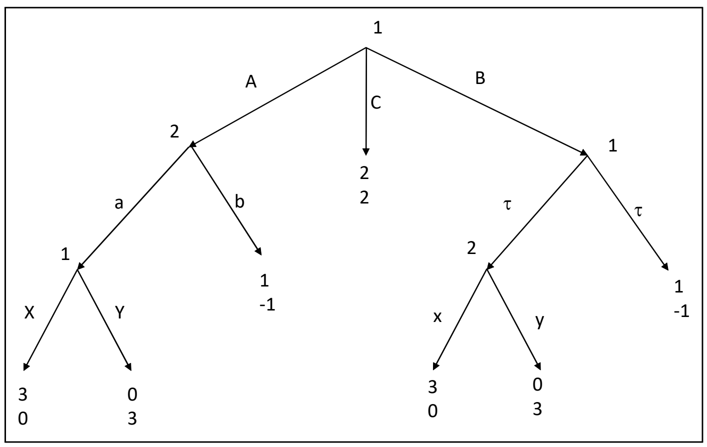


Figure 8.12:

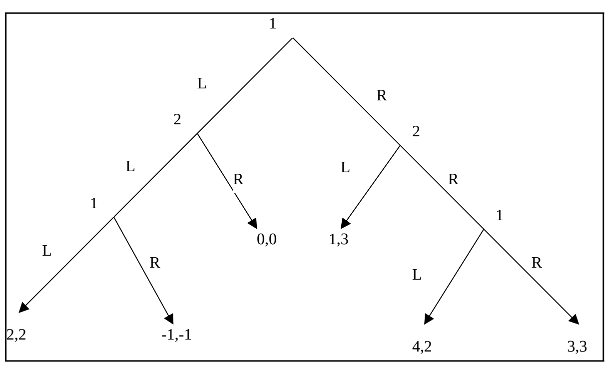


Figure 8.13: