18.650. Fall 2023
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Test 1: REVIEW SHEET (Solution)

Exercise

Let X be a random vector such that $\mathbb{E}[X] = \mu \in \mathbb{R}^k$ and $\mathbb{V}[X] = I_k$. We are interested in estimating the squared norm of μ given by

$$\theta := \|\mu\|^2 = \mu^\top \mu = \sum_{j=1}^{k} \mu_j^2 \,.$$

To that end, we have n independent copies of X whose average is the vector $\bar{X} = (\bar{X}_1, \ldots, \bar{X}_k)^\top$. We propose to use the estimator

$$\hat{\theta} = \|\bar{X}\|^2$$

1. Show that $\hat{\theta}$ is consistent.

   By the (multivariate) LLN, we know that

   $$\bar{X} \xrightarrow{\mathbb{P}} \mu$$

   Moreover, the function $g(x) = \|x\|^2$ is continuous so by the Continuous Mapping Theorem, we have

   $$\|\bar{X}\|^2 \xrightarrow{\mathbb{P}} \|\mu\|^2$$

2. Compute the bias of $\hat{\theta}$ and show that it is asymptotically unbiased.

Note that

$$\begin{aligned}
\mathbb{E}[\hat{\theta}] = \mathbb{E}[\bar{X}^T \bar{X}] &= \sum_{j=1}^{k} \mathbb{E}[\bar{X}_j^2] \\
&= \sum_{j=1}^{k} (\mathbb{E}[\bar{X}_j]^2 + \mathbb{V}[\bar{X}_j]) \\
&= \sum_{j=1}^{k} (\mu_j^2 + \frac{1}{n}) \\
&= \frac{k}{n} + \sum_{j=1}^{k} \mu_j^2 \\
&= \frac{k}{n} + \theta \, .
\end{aligned}$$

Hence:

$$\text{bias}(\hat{\theta}) = \mathbb{E}_\theta[\hat{\theta}] - \theta = \frac{k}{n} + \theta - \theta = \frac{k}{n} \to 0$$

as $n \to \infty$. Therefore $\hat{\theta}$ is asymptotically unbiased.

3. Show that $\hat{\theta}$ is asymptotically normal and compute its asymptotic variance.

   From the (multivariate) central limit theorem we get that

   $$\sqrt{n}(\bar{X} - \mu) \rightsquigarrow \mathcal{N}_k(0, I_k) \, .$$

   We now apply the (multivariate) Delta method to the function $g(x) = \|x\|^2 = x^\top x$ so that its gradient is given by $\nabla g(x) = 2x$. It yields

   $$\sqrt{n}(\|\bar{X}\|^2 - \|\mu\|^2) \rightsquigarrow \mathcal{N}(0, \nabla g(\mu) I_k \nabla g(\mu)) \, .$$

   To compute the asymptotic variance, observe that

   $$\nabla g(\mu) I_k \nabla g(\mu) = \|\nabla g(\mu)\|^2 = \|2\mu\|^2 = 4\theta.$$

   Hence we have

   $$\sqrt{n}(\hat{\theta} - \theta) \rightsquigarrow \mathcal{N}(0, 4\theta) \, .$$

4. Use the previous question to compute a confidence interval for θ with asymptotic coverage 95%.

We know from Problem 1 that $\hat{\theta} \xrightarrow{\mathbb{P}} \theta$, so by Slutsky's and the previous problem we have that

$$\frac{\sqrt{n}(\hat{\theta} - \theta)}{\sqrt{4\hat{\theta}}} \rightsquigarrow \mathcal{N}(0, 1).$$

With $\alpha = 1 - 0.95 = 0.05, z_{\alpha/2} = \Phi^{-1}(1 - \frac{\alpha}{2}) \approx 1.96$. Thus, we can compute a confidence interval of

$$(\hat{\theta} - z_{\alpha/2} \cdot \hat{\text{se}}, \hat{\theta} + z_{\alpha/2} \cdot \hat{\text{se}}) = (\hat{\theta} - 1.96\sqrt{\frac{4\hat{\theta}}{n}}, \hat{\theta} + 1.96\sqrt{\frac{4\hat{\theta}}{n}}) = (\hat{\theta} - 3.92\sqrt{\frac{\hat{\theta}}{n}}, \hat{\theta} + 3.92\sqrt{\frac{\hat{\theta}}{n}}).$$

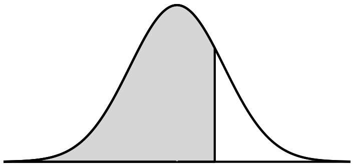

Table 1: The table lists $P(Z \leq z)$ where $Z \sim N(0,1)$ for positive values of z.

	Second decimal place of Z									
Z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

*For $Z \geq 3.50$, the probability is greater than or equal to 0.9998.