Fall 2021
14.12 Game Theory
Ian Ball

# 14.12 Final Exam

You have **3 hours** to complete the exam.

This exam is **open book**. You may consult any physical materials you brought with you into the exam, any saved documents on your computer, and any 14.12 materials posted on Canvas. **Other than going to Canvas, you may not access the internet during the exam.**

No calculators or computing software.

The exam has **six** questions worth a total of **72** points:

- Problem 1: 10 points
- Problem 2: 10 points
- Problem 3: 13 points
- Problem 4: 10 points
- Problem 5: 14 points
- Problem 6: 15 points

[Intentionally blank]

**Problem 1** (10 points)**.** Consider the following strategic-form game between two players:

Player 2

| | | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|---|
| Player 1 | $A$ | $0, 0$ | $3, 1$ | $-2, 0$ | $4, 2$ |
| | $B$ | $0, 3$ | $1, 1$ | $4, -3$ | $0, 1$ |
| | $C$ | $1, 2$ | $2, 0$ | $-1, -3$ | $2, 1$ |
| | $D$ | $-10, 10$ | $-10, 10$ | $-10, 10$ | $-10, 10$ |

1. Compute the set of rationalizable strategies for each player.
2. Find every (pure and mixed) Nash equilibrium.

**Problem 2** (10 points)**.** Two players play a two-period game. In the first period, they play the following stage game:

| | | | |
|---|---|---|---|
| | | Player 2 | |
| | | $L$ | $R$ |
| Player 1 | $T$ | $2, 1$ | $0, 0$ |
| | $B$ | $0, 0$ | $1, 2$ |

In the second period, after observing the action profile chosen in the first period, they play the classical prisoner's dilemma:

| | | | |
|---|---|---|---|
| | | Player 2 | |
| | | $C$ | $D$ |
| Player 1 | $C$ | $2, 2$ | $-1, 3$ |
| | $D$ | $3, -1$ | $0, 0$ |

Each player's payoff in the dynamic game is the sum of her payoffs in the two stage games.

1. How many pure strategies does each player have?
2. Find every pure subgame-perfect Nash equilibrium.

**Problem 3** (13 points)**.** This problem considers an infinitely repeated game.

The stage game is the following partnership game. Simultaneously, each player chooses effort $e_i$ in $[0, 1]$. The quality of the project is the sum of the players' effort levels. For $i = 1, 2$, player $i$'s payoff equals the quality of the project minus her quadratic cost of effort:

$$u_i(e_1, e_2) = e_1 + e_2 - e_i^2.$$

In the infinitely repeated game, the stage game is played in each period $t = 0, 1, 2, \ldots$. The players observe all actions chosen in periods $0, 1, \ldots, t-1$ before playing the stage game in period $t$. They both use discount factor $\delta$ in $(0, 1)$. Hence, for $i = 1, 2$, player $i$'s utility is

$$U_i(e^0, e^1, \ldots) = \sum_{t=0}^{\infty} \delta^t u_i(e^t),$$

where $e^t$ denotes the effort profile chosen in period $t$.

Consider the following carrot–stick strategy profile, which depends on parameters $e_L$ and $e_H$, satisfying

$$0 \le e_L < e_H \le 1.$$

There are two states, $L$ and $H$. In state $L$, both players play $e_L$; in state $H$, both players play $e_H$. The state evolves as follows. In period 0, the stage is $H$. For each $t > 0$, the state in period $t$ is $H$ if no one deviated from the prescribed play in period $t-1$; otherwise, the state in period $t$ is $L$.

Questions (1)–(2) are about the stage game:

1. Compute each player $i$'s best response function. Then find a Nash equilibrium of the stage game.

2. Which effort profile $(e_1, e_2)$ maximizes the sum of the players' payoffs?

Questions (3)–(5) are about the repeated game:

3. Find the range of $\delta$ for which neither player has a profitable one-shot deviation **in state** $H$ from the carrot-stick strategy profile . (Your answer should depend on $e_L$ and $e_H$.)

4. Find the range of $\delta$ for which neither player has a profitable one-shot deviation

**in state** $L$ from the carrot-stick strategy profile. (Your answer should depend on $e_L$ and $e_H$.)

5. Consider the carrot-stick strategy profile with $e_H = 1$ and $e_L = 0$. Find the range of $\delta$ for which this strategy profile is a subgame-perfect Nash equilbrium. Justify your answer.

**Problem 4** (10 points)**.** Consider the following Bayesian game between Alice and Bob. Alice privately observes her type $\alpha$ in $\{1, 3\}$ and Bob privately observes his type $\beta$ in $\{0, 2, 4\}$. The payoff matrix is

| | | Bob | | |
|---|---|---|---|---|
| | | $L$ | $M$ | $R$ |
| Alice | $T$ | $0, \beta + 1$ | $2, 2\beta$ | $4, 3\beta - 3$ |
| | $B$ | $\alpha, \beta + 1$ | $\alpha, 2\beta$ | $\alpha, 3\beta - 3$ |

The prior distribution $p$ over $\{1, 3\} \times \{0, 2, 4\}$ is

$$\begin{array}{lll} p(1,0) = 1/3, & p(1,2) = 1/6, & p(1,4) = 0, \\ p(3,0) = 0, & p(3,2) = 1/6 & p(3,4) = 1/3. \end{array}$$

1. How many strategies does Alice have?
2. How many strategies does Bob have?
3. Find a Bayes–Nash equilibrium.

**Problem 5** (14 points)**.** In a mixed-price auction with two bidders, each bidder $i$ is asked to submit a bid $b_i$ in $\mathbf{R}_+$. The highest bidder wins the good and pays pay $\lambda$ times her own bid plus $(1-\lambda)$ times the other player's bid, where $\lambda$ in $[0,1]$ is a known parameter of the auction. The other bidder pays nothing. Ties are broken with a fair coin flip.

Each bidder's valuation is uniformly distributed over $[0,1]$, independent of the other bidder's valuation.

1. Suppose bidder 2 uses the bidding strategy $b_2(v_2) = \alpha v_2$ for some fixed $\alpha > 0$. Compute bidder 1's expected utility, when her valuation is $v_1$ and she bids $b_1$. (You may assume $0 \leq b_1 \leq \alpha$).

2. Find a symmetric Bayes-Nash equilibrium of this auction. (Your answer should depend on $\lambda$.)

3. Find a direct auction

$$(\hat{x}_1, \hat{x}_2, \hat{t}_1, \hat{t}_2) \colon [0,1] \times [0,1] \to \Delta \times \mathbf{R}^2$$

   that *replicates* the equilibrium from part 2. That is, if each player reports her valuation truthfully in the direct auction, then for every value profile $(v_1, v_2)$, the outcome coincides with outcome of the mixed-price auction equilibrium from part 2. (Here, $\Delta$ denotes the space of nonnegative vectors $(x_1, x_2)$ satisfying $x_1 + x_2 \leq 1$.)

**Problem 6** (15 points)**.** Consider the following Cournot quantity competition game with demand uncertainty. There are two firms (firm 1 and firm 2) that can each supply up to 10 units of a good, at zero cost. The market price is

$$P = \max\{\theta - Q, 0\},$$

where $\theta$ is the demand intensity and $Q$ is the total quantity produced by the two firms. The demand intensity $\theta$ is either low ($\theta = 2$) or high ($\theta = 3$), each with probability 1/2. Firm 1 privately observes the demand intensity $\theta$ before choosing its quantity. Firm 2 does not observe the demand intensity $\theta$ (but it understands the distribution of $\theta$).

The timing is as follows. First, firm 1 chooses quantity $q_1$ in $[0, 10]$. Firm 2 observes the quantity $q_1$ and then chooses quantity $q_2$ in $[0, 10]$. Each firm then sells its quantity at the market price. Each firm's utility is its profit (which equals revenue since production is costless).

1. Find a quantity $q_2$ that is sequentially optimal for firm 2 when firm 2 observes quantity $q_1$ and believes that demand intensity $\theta$ is high with probability $\mu$.

2. Construct a *separating* perfect Bayesian equilibrium in which firm 1 chooses $q_1^L = 1/2$ when demand intensity is low. (Remember to specify every component of the equilibrium.) Verify that your answer is indeed a perfect Bayesian equilibrium.

3. Construct a *pooling* perfect Bayesian equilibrium in which firm 1 always chooses quantity $q_1 = 1$. (Remember to specify every component of the equilibrium.) Verify that your answer is indeed a perfect Bayesian equilibrium.