14.12 Recitation 10

Signaling and Perfect Bayesian Equilibrium

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Logistics

- PSET 10 is optional and not to be turned in - solutions to be posted before Monday
- Final Exam Information
  - 50-340 (Walker Memorial 3^{rd} floor
  - Wednesday, December 20^{th}
  - 9:00am - 12:00pm
  - closed books and notes

The idea of signaling

The idea: Player i may have an unobserved quality (for example, aptitude, ability, skill, opinion, utility function...) that can be inferred from the way they behave/choose.

Example 1

- High school students A and B are applying to MIT. Their high school allows each student to put 3 classes on pass/fail. Both students have all A's showing on their transcript; however, student A puts no class on pass/fail and student B puts 3 classes on p/f.
- Are we inclined to believe that student B earned all A's on all 3 classes for which he chose the p/f grading scheme?

Example 2

- Farmer C and farmer D are selling apples at a farmers' market. You know that there exists 2 types of apples - good apples and bad apples - that look identical but have different grades of qualities. You also know that a farmer can either produce good apples or bad apples, but not both.
- You observe that Farmer A sells apples at $\$3.99$ per pound and farmer B sells apples at $\$0.50$ per pound.
- Who do you think sells good apples?

Takeaways from Examples

- It is not known to us if student B earned all A's or not on the 3 classes he chose to p/f.
- Similarly, it is not known if farmer C or farmer D (or both, or neither) sells good apples (unless we bought them to try).
- However, the difference in the players' behaviors makes the message receiver inclined to believe that the players are of different *types*.

The Signaling Framework

In a signaling game, we have:

- An *informed* player, called the Sender, who sends a message to the other player
- An *uninformed* player, called the Receiver, who takes an action that affects the utility of both players.

In the previous examples, the message could be having 3 pass/fails on transcript, or setting the price of apple unreasonably lower than market.

Screening

The idea of *screening* is similar to that of signaling, but with 1 more step.

- There are 2 players, 1 and 2.
- Player 1 is clueless about player 2's type and she wants to gain information about it.
- Thus, player 1 sends a "message" to player 2 to elicit a response. based on this response, player 1 can make an educated guess of what player 2's type is and choose an action that maximizes her payoff.
- *Signaling vs. screening: in signaling, the uninformed player is the receiver. in screening, the uninformed player is the (initial) message sender.*

Screening in a Market Context

- There are 2 players: buyer and seller
- Seller does not know what the buyer's true valuation of the good is. However, seller's value of the good is c and this information is known to both players.
- Seller sets a price p and see if buyer will buy the good.
- Based on the buyer's response (to buy or to not buy), seller sets p to maximize profit.

Dynamic Bayesian Games

A dynamic Bayesian game consists of:

- a set of players $N = \{1, ..., n\}$,
- a set of type profiles $T = \{T_1, ..., T_n\}$,
- a probability distribution p_i on T_i for each player i,
- a game tree (where all past actions are known to everyone),
- an assignment of each non-terminal history to either a player or Nature, and
- a payoff function $u_i : T \times Z \to \mathbb{R}$ for each player where Z is the set of terminal nodes.

Nature picks a type profile for each player at the beginning of the game; player i sees their own profile t_i but not any other's.

Why do we need another equilibrium?

- In games with complete information (i.e., each player knows everyone's payoffs and actions at every stage), we can find subgame-perfect NEs
    - When we don't have those information, there aren't generally SPNEs!
- BNEs can help with incomplete information games! So why not BNE?
    - BNE does not specify how players' beliefs *evolve* with how the game proceeds, but in signaling/screening games, we explicitly want players to play based on the message they receive
    - BNE allows players to play suboptimal actions at information sets they may never arrive, and we don't want that :(
    - instead, we want some solution that allows us to implement *sequential rationality* directly!

1. Signaling/Screening
2. Dynamic Bayesian Games
3. Perfect Bayesian Equilibrium

PBE?

...jk.

Perfect Bayesian Equilibrium - a simple idea sketch

- The idea: the Receiver knows that the Sender has 2 types, h and l.
- Based on the Sender's message, the Receiver believes that the Sender is either type h or type l.
- The Receiver will play action H if she thinks Sender is of type h and action L if she thinks Sender is of type l. In other words, there is an action specified for each type.
- The work we have left to do is to figure out the Sender's actions (i.e. messages) and the Receiver's actions (based on a belief) that maximize their payoff!

Assessment and Beliefs

In order to formally introduce the concept of perfect Bayesian equilibrium, we need some definitions:

- An **assessment** is a pair (σ, b) of a strategy profile σ and a belief system b.
- An assessment (σ, b) is said to be **sequentially rational** if at each history h, the player i is to move at h maximizes her expected utility
  - given her type t_i and her beliefs $b(\cdot|h)$ about the other players' types at history h, and
  - given that the players will play according to σ in the continuation game.

Perfect Bayesian Equilibrium

An assessment (σ, b) is said to be a **perfect Bayesian equilibrium** (PBE) if it is sequentially rational and satisfies the following 2 conditions throughout:

1. $$b_j(t_j|h, a) = b_j(t_j|h) \quad \forall i \neq j, t_i \in T_j$$

   That is, $i's$ beliefs about other players' types should not be affected by what player i chooses;

2. $$b_i(t_i|h, a) = \frac{\sigma_i(a|t_i, h) b_i(t_i|h)}{Pr(a|\sigma_i, h)} \quad \text{if } Pr(a|\sigma_i, h) > 0.$$

Types of PBE

There may be several types of PBE that we want to consider:

- **Pooling equilibrium**: the sender decides to send the same message no matter of her type
- **Separating equilibrium**: each type of sender sends a different message
- **Semi-separating equilibrium**: in-between the 2 types above, where more than 1 type of sender chooses to send message s and some other type of sender chooses to send message s'

There could be questions on either *existence* of those PBE (e.g., for which values of certain parameters does a pooling PBE exist?) or explicitly finding them.

Example Walk-through: PSET 10.3

Consider a generalized setting where the sender (worker) has 3 types $t \in \{l, m, h\}$, where $0 < l < m < h$. The probability that the sender is of type i is π_i (note that $\pi_l + \pi_m + \pi_h = 1$). The sender knows her type and chooses an education level $e \in [0, 1]$. The receiver (firm) does not know sender's type but observes her education level and chooses a non-negative wage w. The payoffs for the sender and receiver are

$$u_S(e, w, t) = w - \frac{e^2}{t}, \qquad u_R(e, w, t) = -(w - t)^2.$$

Part 1

1. For which education levels e^* does there exist a pooling PBE in which every sender type chooses education e^*?

Solution.

For a pooling equilibrium to exist, we need 2 things:

1. $w \geq l$. (Otherwise, there is no point in getting education!)
2. every sender has utility greater than 0. That is, $w - \frac{e^2}{t} \geq 0$ for all t.

In addition, let e_t be the level of education gained by type t. Intuitively, $e_l \leq e_m \leq e_h$ because the lower types get more disutility from more education.

Part 1

Define $t^* = l\pi_l + m\pi_m + h\pi_h$. This can be roughly interpreted as "the average productivity of all types". Since the firm only observes 1 type of education e^*, they will set the wage $w = t^*$. Thus, in accordance of the 2 things we listed, we must have:

1. $u_S(e, w = t^*, l) = t^* - \frac{(e^*)^2}{l} > l$; and
2. $u_S(e, w = t^*, l) = t^* - \frac{(e^*)^2}{l} > 0$.

(note that (1) implies (2).) We conclude from (1) that the pooling PBE needs to satisfy

$$\boxed{e^* < \sqrt{l(t^* - l)}.}$$

Part 2

2. For which education levels $\underline{e}$ and $\bar{e}$ does there exist a semi-separating PBE in which type l chooses $\underline{e}$ and types m and h choose $\bar{e}$?

Solution.

Define $\bar{t} = \frac{m\pi_m + h\pi_h}{\pi_m + \pi_h}$ - this could be seen as the expected productivity out of a pool of workers who are of either type m or h. Notice that $\underline{e} < \bar{e}$ so that type l has no incentive to gain as much education as type m and h. Naturally, $\boxed{\underline{e} = 0}$ and $w = l$: Since the firm can automatically identify type l, they will assign $w = l$ for them to maximize u_R, leaving no incentive for type l to gain non-zero education.

Part 2

Now, for the pooled type m and h, their wage would be $w = \bar{t}$.
We must make sure that:

1. Type l workers do not have an incentive to get as much education as type m/h so to signal themselves as a higher type worker.
2. Similarly, type m (and thus h) workers have an incentive to get lower education (i.e., as much education as type l).

Part 2

The constraints on the precious slide gives us the setup below:

1. $\bar{t} - \frac{(\bar{e})^2}{l} < l - \frac{0^2}{l}$
2. $\bar{t} - \frac{(\bar{e})^2}{m} < l - \frac{0^2}{m}$

Those 2 constraints yield

$$\boxed{\sqrt{l(\bar{t} - l)} < \bar{e} < \sqrt{m(\bar{t} - l)}}.$$

Part 3

3. For which education levels $\underline{e}'$ and $\bar{e}'$ does there exist a semi-separating PBE in which type l and m choose $\underline{e}'$ and types h chooses $\bar{e}'$?

Solution. Define $\underline{t} = \frac{m\pi_m + l\pi_l}{\pi_m + \pi_l}$. The firm will assign wage t^* for the lower types and h for high type. Again, to make sure that types h and m do not want to mimic each other, $\bar{e}'$ and $\underline{e}'$ should satisfy:

1. $\underline{t} - \frac{(\underline{e}')^2}{l} > l$ (so every worker has positive payoff)
2. $h - \frac{(\bar{e}')^2}{m} < \underline{t} - \frac{(\underline{e}')^2}{m}$ (so type m won't get higher education)
3. $h - \frac{(\bar{e}')^2}{h} < \underline{t} - \frac{(\underline{e}')^2}{h}$ (so type h won't get lower education)
4. $\underline{e}' < \bar{e}'$ (by construction)

Part 4

4. For which education levels e_l, e_m, e_h does there exist a separating equilibrium in which type t chooses e_t?

Solution. Using the same logic as part 2, we know that $e_l = 0$. By intuition we also know that $e_l < e_m < e_h$ since they are now all different. The e_m and e_h we seek should thus also satisfy:

1. $m - \frac{e_m^2}{m} > h - \frac{e_h^2}{m}$ (type m does not mimic type h)
2. $h - \frac{e_h^2}{h} > m - \frac{e_m^2}{h}$ (type h does not mimic type m)
3. $m - \frac{e_m^2}{m} > l$ (type m does not mimic type l)
4. $m - \frac{e_m^2}{l} < l$ (type l does not mimic type m)

Part 4

All in all, we have:

$$\boxed{\sqrt{m(m-l)} > e_m > \sqrt{l(m-l)}}, \text{ and}$$

$$\boxed{\sqrt{hm - m^2 + e_m^2} < e_h < \sqrt{h^2 - mh + e_m^2}}.$$

Takeaway!

- Solve for the receiver's action first (since they don't actually know the types of senders) by using expectations (as we have seen above, you may need to group types of senders together)
- Based on the receiver's action, write down the payoffs of senders
- Make sure that no types of sender has the incentive to deviate to any message that other types of sender are sending!
    - Note: we do not have to check if senders want to deviate to any other message. (Why? because the sender can just assume they are any type!)

Happy End of the Semester!

- We still have OHs next week!
- The Economics department has a holiday party Wednesday 5-7:30pm - RSVP here!