Proposition 3 *A strategy profile s is subgame perfect if and only if no player i has a profitable one shot deviation, i.e. there is a strategy s_i' such that there is a unique history h^t with $s_{i,t}\left(h^t\right) \neq s_{i,t}'\left(h^t\right)$, and conditional on this history h^t s_i' is a profitable deviation from s_i.*

Proof. (We will only prove the proposition for pure stratregies - general case is analogous, only more notationally cumbersome). That the condition is necessary is obvious. To show sufficiency, suppose that s is not a subgame equilibrium and that i can profitably deviate by using some alternative s_i' after history h^t. By moving to a subgame, we can assume that the deviation occurs at the begining of the game. We will show that i has a one shot deviation from s_i.

Let $\varepsilon = u_i\left(s_i', s_{-i}\right) - u_i\left(s\right) > 0$, and $M := \max_a u_i\left(a\right) - \min_a u_i\left(a\right)$. Fix T such that $\frac{\delta^T}{1-\delta} M < \varepsilon$. It follows that a strategy s_i'' that plays as s_i' in the first T periods and as s_i afterwards is a profitable deviation from s_i. Now, lets look at a history h^T induced by the strategy profile (s_i'', s_{-i}). Lets look at the play in period T (i.e. the last one in which s_i'' plays differently than s_i). There are two possibilities:

1) Conditional on history h^T the strategy s_i'' is strictly better than s_i. In this case a strategy that plays like s_i'' after h^T and else plays as s_i is a profitable deviation.

2) Conditional on history h^T the strategy s_i'' is not strictly better than s_i. Then define a strategy s_i''' that plays like s_i'' in first $T-1$ periods and plays like s_i otherwise. s_i''' is a profitable deviation from s_i - and so we can repeat the argument again with $T-1$ instead of T. This concludes the proof.

This one shot deviation principle is hugely useful, since it decreases considerably the set of possible deviations that we must consider - no need to analyze deviations that change play in a sequence of 18 consecutive periods...

Theorem 2 *(Folk Theorem) Let V^* be the set of feasible and strictly individually rational payoffs. Assume that $\dim V^* = I$. Then for any $(v_1, ..., v_I) \in$ $intV^*$, there exists a $\underline{\delta} < 1$, such that for any $\delta > \underline{\delta}$, there is a subgame perfect equilibrium of $G^\infty(\delta)$ with average payoffs $(v_1, ..., v_I)$.*

Proof. Fixing a payoff vector $(v_1, ..., v_I) \in V^*$, we construct a SPE that achieves it. For convenience, let's assume that there is some profile $(a_1, ..., a_I)$ such that $g_i(a) = v_i$ for all i. The key to the proof is find payoffs that allow us to "reward" all agents $j \neq i$ in the event that i deviates and has to be min-maxed for some length of time.

- Choose $v' \in Int(V^*)$ such that $v_i' < v_i$ for all i.

- Choose T such that $\max_a g_i(a) + T\underline{v}_i < \min_a g_i(a) + Tv_i'$
- Choose $\varepsilon > 0$ such that for each i,

$$v^i(\varepsilon) = \left(v_1' + \varepsilon, ..., v_{i-1}' + \varepsilon, v_i', v_{i+1}' + \varepsilon, ..., v_I' + \varepsilon\right)$$

  is feasible and $v_i' + \varepsilon < v_i$.
- Let a^i be the profile with $g(a^i) = v^i(\varepsilon)$
- Let m^i be the profile that min-maxes i, so $g_i(m^i) = \underline{v}^i$.

Consider the following strategies for $i = 1, 2, ..., I$.

I. Play a_i so long as no player deviates from $(a_1, ..., a_I)$. If j alone deviates, go to II_j. (If two or more players simultaneously deviate, play stays in I.)

II_j. Play m_i^j for T periods, then go to III_j if no one deviates. If $k \neq j$ deviates, re-start II_k. If j deviates, re-start II_j at the end of this phase.

III_j. Play a_i^j so long as no one deviates. If k deviates, go to II_k.

Note that strategies involve both punishments (the stick) and rewards (the carrot). Let's check that they are indeed a subgame perfect equilibrium using the one-shot deviation principle. We need to check for each of the different subgames.

Subgame I. Consider i's payoff to playing the strategy and deviating:

$$\begin{aligned} i \text{ follows strategy} &: \quad (1-\delta)\left[v_i + \delta v_i + ...\right] = v_i \\ i \text{ deviates} &: \quad (1-\delta)\left[\overline{v}_i + \delta \underline{v}_i + ... + \delta^T \underline{v}_i + \delta^{T+1} v_i' + ...\right] \end{aligned}$$

Subgame II_i. (suppose there are $T' \leq T$ periods left)

$$\begin{aligned} i \text{ follows strategy} &: \quad (1-\delta^{T'})\underline{v}_i + \delta^{T'} v_i' \\ i \text{ deviates} &: \quad (1-\delta)\underline{v}_i + \delta(1-\delta^{T+T'})\underline{v}_i + \delta^{T+T'+1} v_i' \end{aligned}$$

Subgame II_j. (suppose there are $T' \leq T$ periods left)

$$\begin{aligned} i \text{ follows strategy} &: \quad (1-\delta^{T'}) g_i(m^j) + \delta^{T'}(v_i' + \varepsilon) \\ i \text{ deviates} &: \quad (1-\delta)\overline{v}_i + \delta(1-\delta^T)\underline{v}_i + \delta^{T+1} v_i' \end{aligned}$$

Subgame III_i,III_j. Consider i's payoff to playing the strateg and deviating:

$$\begin{aligned} i \text{ follows strategy} \quad &: \quad v_i' \\ i \text{ deviates} \quad &: \quad (1-\delta)\overline{v}_i + \delta(1-\delta^T)\underline{v}_i + \delta^{T+1}v_i' \end{aligned}$$

The payoffs here are the least i could get if he follows the strategy and the most he could get if he deviates. A small amount of algebra shows that for $\delta \approx 1$, it is best not to deviate. *Q.E.D.*

Remark 1 *The equilibrium constructed in the above proof involves both the "stick" (Phase II) and the "carrot" (Phase III). Often, however, only the stick is necessary. The carrot phase is needed only if the parties punishing in Phase II get less than their min-max payoffs.*

Note that the (perfect) Folk Theorem requires an extra (though relatively mild) assumption, namely that $\dim V^* = I$. The assumption ensures that each player i can, in the event of a deviation, be singled out for punishment. It rules out special games like the one in the following example.

