Chapter 5

Nash Equilibrium

For a given strategic situation, a game formalizes the assumptions about who the players are, what moves are available to them, what their payoffs are, what they know at any given point where they make a decision, what their beliefs about each others' payoffs and information are, and so on. The strategic implications of these assumptions are characterized by rationalizability. Rationalizability may have powerful implications in some games. For example, it predicts a unique outcome in the beauty contest game in Chapter 1. However, rationalizability has weak predictive power in many applications, where a wide range of outcomes can be rationalized. This is because the assumptions made in the description of the game need not restrict the range of strategic uncertainty the players may face, and the players' behavior may vary as their beliefs about the others' strategy vary.

Luckily, games are often played in a context, in which one may have a good understanding of what kind of beliefs the players may entertain about the others' strategies. By imposing further restrictions on those beliefs, one may be able to make sharper predictions. In particular, in some environments, a player may guess accurately what other players play. For example, when I drive in the United States, I assume that the other drivers will drive on the right side of the road, and I make life and death decisions based on that assumption. Most other drivers also make the same assumptions. Similarly, the drivers in the United Kingdom drive on the left side of the road, expecting that the others will also do so. Although this is mandated by law in these countries, it is not necessarily the law that entices them to use the expected side of the road. After all, if

I thought that the other drivers are all going to drive on the wrong side of the road, I would also drive on the wrong side of the road and get a traffic ticket in order to avoid a deadly accident. One can imagine that even if there was no fine for driving on the wrong side of the road, the drivers would overwhelmingly drive on the correct side once it is established which side of the road they are supposed to drive. Hence, in this particular example, the law mainly functions as a social convention that tells drivers what they ought to do. Like in this example, there may be social norms and conventions that lead players to hold a certain belief about the other players' behavior.

Nash equilibrium corresponds to such situations. We are in a Nash equilibrium when all players correctly guess what strategy the other players play and no player has an incentive to deviate. For example, consider the following coordination game that represents the driving example above:

	Left	Right
Left	$1,1$	$0,0$
Right	$0,0$	$1,1$

Each player gets 1 if they play the same strategy and 0 otherwise. In the driving example, this models two players who drive in opposite directions to each other. If each of them drives on her left side of the road (denoted by Left), then they reach home safely obtaining the payoff of 1. If one drives on her left side and the other player drives on his right side, they will cause an accident and get 0. The strategy profile (Left, Left) is a Nash equilibrium. Player 1 plays Left, knowing that the other player also plays Left. She does not have an incentive to deviate because she would get 0 instead of 1 if she deviated and played Right. This corresponds to the equilibrium that appears to be played in the United Kingdom. Anticipating that the other players will follow the law and drive on the left side, they also follow the law and drive on the left side of the road. Likewise, (Right, Right) is also equilibrium, which appears to be played in the United States. But (Left, Right) is not a Nash equilibrium because if Player 1 knew that player 2 would play Right, she would also play Right, deviating from her purported strategy. For example, if for some bizarre reason, the drivers were asked to drive on the left side when driving to the east and on the right side when driving to the west, then the players would deviate from this plan if they thought that the others would follow the plan. Sometimes Nash equilibrium is in mixed strategies. For example, if it is expected that players will choose their strategies randomly with equal probability, players would be indifferent between Left and Right and would not have an incentive to deviate from such randomization.

Nash equilibrium may be a plausible solution concept in stable environments where players can reasonably guess what the other players do.[1] Formally, Nash equilibrium corresponds to the steady-state of adjustment processes where the players adjust their behavior to obtain a better payoff in the environment they are in. Hence, Nash equilibrium is used to study long-run outcomes of such processes, such as population dynamics, and social and biological evolution.

In many environments, players often develop social convention about how to play a game, e.g., which side of the road to drive on, even if not conforming to a social convention does not have an inherent payoff implication. In that case, Nash equilibrium corresponds to social conventions that are self-enforcing. Likewise, Nash equilibrium corresponds to self-enforcing agreements on how to play a game. In order to be viable, agreements must be self-enforcing when they cannot be enforced by third parties, such as international agreements between sovereign nations and illegal agreements between cartel members. One can use Nash equilibrium to understand the structure of such agreements. Since such agreements are made within a broader context, such as a repeated interaction, one studies some suitable Nash equilibria of the broader game—as Chapters 12 and 13.

5.1 Definition

Consider a game $G = (N, S, u)$. A Nash equilibrium is defined as a strategy profile in which no player has an incentive to deviate if all the other players follow the strategies that are prescribed for them:

Definition 5.1. A strategy profile $s^* = (s_1^*, ...s_n^*)$ is said to be a *Nash equilibrium* if s_i^*

[1] In such environments, players may have broader motives that may be in conflict with the basic incentives in the game as discussed in Chapter 1. In order for Nash equilibrium to be relevant, it must also be that such broader motives are weak, e.g., players cannot affect other players' future behavior substantially, as in environments in which players are relatively anonymous and have a small impact on strategic outcomes, as in the driving example above.

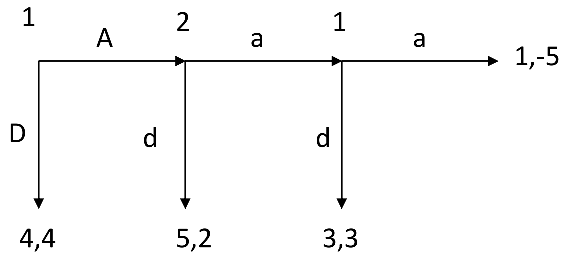


Figure 5.1: A centipede-like game

is a best response to $s^*_{-i} = (s^*_1, ...s^*_{i-1}, s^*_{i+1}, ...s^*_n)$ for each i. That is, for each i,

$$u_i(s^*_i, s^*_{-i}) \geq u_i(s_i, s^*_{-i}) \quad \forall s_i \in S_i.$$

For example, consider the Battle of the Sexes game

$$\begin{array}{c|c|c|} & A & B \\ \hline A & 2,1 & 0,0 \\ \hline B & 0,0 & 1,2 \\ \hline \end{array} \tag{5.1}$$

where the raw player is named Alice and the column player is named Bob. In this game, there is no dominant strategy, and everything is rationalizable. In this game, (A, A) is a Nash equilibrium because A is a best response to A for each player:

$$u_{Alice}(A, A) = 2 > 0 = u_{Alice}(B, A)$$

and

$$u_{Bob}(A, A) = 1 > 0 = u_{Bob}(A, B).$$

In that case, no player has an incentive to deviate. For example, if Alice correctly guesses that Bob plays A, she would also play A and get 2 instead of deviating and getting 0. Similarly, (B, B) is also a Nash equilibrium. On the other hand, (A, B) is not a Nash equilibrium because Bob would like to play A instead:

$$u_{Bob}(A, A) = 1 > 0 = u_{Bob}(A, B).$$

One must pay attention to a couple of fine points in the definition of equilibrium. First, a Nash equilibrium is a strategy profile: it identifies a strategy for each player. It is not an outcome or a payoff vector; e.g., (A, A) is a Nash equilibrium while $(2, 1)$ is only the payoff vector associated with this equilibrium. In particular, in an extensive-form game, an equilibrium prescribes a move for every contingency. For example, consider the extensive-form game in Figure 5.1. In order to specify an equilibrium one must assign a move for each of the two information sets of Player 1 and a move for Player 2. Second, in order for a strategy profile to be a Nash equilibrium, each player must be playing a best response. Hence, in order to show that s^* is a Nash equilibrium, one must verify that $u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*)$ for each player i and for each deviation s_i. For example, in the Battle of the Sexes game we have checked two inequalities, one for each player, as each player had only one possible deviation. On the other hand, in order to show that a strategy is not a Nash equilibrium, it suffices to find a profitable deviation for one player. For example, one inequality was enough to show that (A, B) is not a Nash equilibrium. Finally, note that equilibrium requires only that a player does not have a strict incentive to deviate. For example, the game

	A	B
A	$1, 1$	$0, 0$
B	$0, 0$	$0, 0$

(5.2)

has two Nash equilibria: (A, A) and (B, B). Here, (B, B) is an equilibrium because B is a best response to B. When a player is certain that the other player plays B, she is indifferent between her strategies and can play B as well.

Nash equilibrium is a strategy profile in which each player plays a best response. Hence, Nash equilibria correspond to the intersections of best responses. When feasible, one can identify the Nash equilibria by looking at such intersections. For example, the normal-form representation of the game in Figure 5.1 has been derived in Exercise 2.1 already:

	a	d
Aa	$1, -5$	$\mathbf{5}, \mathbf{2}$
Ad	$3, \mathbf{3}$	$\mathbf{5}, 2$
Da	$\mathbf{4}, \mathbf{4}$	$4, \mathbf{4}$
Dd	$\mathbf{4}, \mathbf{4}$	$4, \mathbf{4}$

Here, in order to make the equilibrium visible, the payoff of a player is written in bold when she plays a best response. Nash equilibria are those strategy profiles for which all players' payoffs are written in bold. One can see that there are three equilibria (Aa, d), (Da, a), and (Dd, a). In the first equilibrium, Player 1 plays A, and Player 2 plays d; if she played a, then Player 1 would have played a. In the second equilibrium, player 1 plays D expecting that Player 2 would play a, and he would have played a if both went across.

Again, an equilibrium describes the play at all contingencies. It is not enough to describe the outcome. For example, in equilibria (Da, a) and (Dd, a), the outcome is simply that Player 1 plays D and the game ends. But identifying the outcome is not enough; one must assign a move for each contingency. This is because we are not interested only in what happens. We want to understand why Player 1 goes down at the beginning and whether this is consistent with an equilibrium.

5.2 Relation to Dominance and Rationalizability

Nash equilibrium is weaker than dominant-strategy equilibrium but stronger than rationalizability.

Nash Equilibrium v. Dominant-strategy Equilibrium Every dominant-strategy equilibrium is also a Nash equilibrium. Indeed, dominant-strategy equilibrium requires that s_i^* is a best response to all s_{-i}, including s_{-i}^*. Therefore, no player has an incentive to deviate at dominant-strategy equilibrium. This is formally stated next.

Theorem 5.1. *If s^* is a dominant-strategy equilibrium, then s^* is a Nash equilibrium.*

Proof. Let s^* be a dominant strategy equilibrium. Take any player i. Since s_i^* is a dominant strategy for i, for any given s_i,

$$u_i(s_i^*, s_{-i}) \geq u_i(s_i, s_{-i}), \forall s_{-i} \in S_{-i}.$$

In particular,

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*).$$

Since i and s_i are arbitrary, this shows that s^* is a Nash equilibrium. □

The converse is not true: some Nash equilibria may not be a dominant-strategy equilibrium. For example, in the Battle of the Sexes game above, there is no dominant-strategy equilibrium, but there are multiple Nash equilibria. Moreover, there can also be other Nash equilibria when there is a dominant-strategy equilibrium. For an example, in game (5.2) above, (A, A) is a dominant-strategy equilibrium but (B, B) is also a Nash equilibrium.

The last example also illustrates that a Nash equilibrium can be in weakly dominated strategies. One may find such equilibria unreasonable and be willing to rule out such equilibria. Nevertheless, the next example shows that all Nash equilibria may need to be in dominated strategies in some games.[2]

Example 5.1. Consider a two-player game in which each player i selects a natural number $s_i \in \mathbb{N} = \{0, 1, 2, \ldots\}$, and the payoff of each player is $s_1 s_2$. It is easy to check that $(0, 0)$ is a Nash equilibrium, and there is no other Nash equilibrium. Nevertheless, all strategies, including 0, are weakly dominated.

Nash Equilibrium v. Rationalizability If a strategy is played in a Nash equilibrium, then it is rationalizable, but there may be rationalizable strategies that are not played in any Nash equilibrium.[3]

Theorem 5.2. *If s^* is a Nash equilibrium, then s_i^* is rationalizable for every player i.*

Proof. It suffices to show that none of the strategies $s_1^*, s_2^*, ..., s_n^*$ is eliminated at any round of the iterated elimination of strictly dominated strategies. Towards a mathematical induction, assume that none of the strategies $s_1^*, s_2^*, ..., s_n^*$ is eliminated before round k. Then, at round k, for any i, the profile s_{-i}^* is available, and s_i^* is a best response to s_{-i}^*. Therefore it is not eliminated in round k. □

The converse is not true: there can be a rationalizable strategy that is not played in any Nash equilibrium, as the next example illustrates.

Example 5.2. Consider the following game:

	a	b	c
a	$1, -2$	$-2, 1$	$0, 0$
b	$-2, 1$	$1, -2$	$0, 0$
c	$0, 0$	$0, 0$	$0, 0$

[2] This is also the case for some important games in Economics, such as the price-competition game that will be studied in Section 6.3.

[3] This result immediately follows from Exercise 4.6, by setting $Z = \{s^*\}$.

In this game, (c, c) is the only Nash equilibrium but all strategies are rationalizable. (Proof is left as an exercise.)

5.3 Mixed-strategy Nash equilibrium

In some games, there may not be any Nash equilibrium in pure strategies. For example, consider the following well-known game, called the *Matching-Penny Game*,

	H	T
H	$\mathbf{1}, -1$	$-1, \mathbf{1}$
T	$-1, \mathbf{1}$	$\mathbf{1}, -1$

(5.3)

where each player's payoff is written in bold when the player plays a best response. In this game, each player chooses between "head", denoted by H, and "tail", denoted by T. Player 1 wins if both choose the same strategy, and Player 2 wins if they choose different strategies. As shown in the table, there is no strategy profile at which both players play a best response. Hence, there is no Nash equilibrium in pure strategies.

Imagine that each player can choose her strategy randomly, choosing any probability of head they want. Now in this extended game, each player i chooses a probability p_i from $[0, 1]$ and plays strategies H and T with probabilities p_i and $1 - p_i$, respectively. The expected payoff of player 1 can be computed as

$$u_1(p_1, p_2) = [p_1 p_2 + (1 - p_1)(1 - p_2)] - [p_1(1 - p_2) + (1 - p_1) p_2] = (2p_1 - 1)(2p_2 - 1).$$

This is also equal to $-u_2(p_1, p_2)$. Formally, in the extended game, the strategy sets are $\Sigma_1 = \Sigma_2 = [0, 1]$, and the payoff functions are u_1 and u_2. One can see that the best responses to p_2 and p_1 are

$$BR_1(p_2) = \begin{cases} \{1\} & \text{if } p_2 > 1/2 \\ [0,1] & \text{if } p_2 = 1/2 \\ \{0\} & \text{if } p_2 < 1/2 \end{cases} \text{ and } BR_2(p_1) = \begin{cases} \{1\} & \text{if } p_1 < 1/2 \\ [0,1] & \text{if } p_1 = 1/2 \\ \{0\} & \text{if } p_1 > 1/2 \end{cases}$$

respectively. The best responses are plotted in Figure 5.2. As shown in the figure, the best responses have a unique intersection, and hence the extended game has unique Nash equilibrium:

$$p_1^* = p_2^* = 1/2.$$

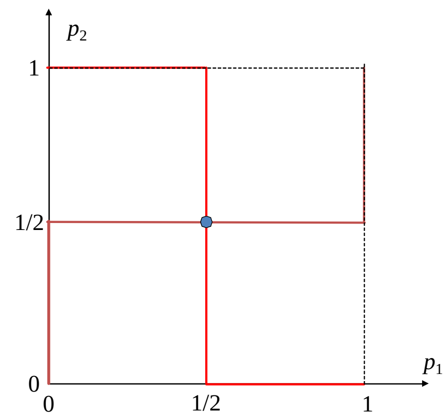


Figure 5.2: Best responses in mixed strategies in Matching Penny game

Although the original matching-penny game does not have a Nash equilibrium, the extended game in which players choose mixed strategies (or probability of their pure strategies) has a Nash equilibrium.

More generally, one can define the Nash equilibrium for mixed strategies as the Nash equilibrium of the extended game in which players choose their mixed strategies (as though they have access to arbitrary randomization devices on their own strategies that are independent of the randomization devices for other players). Formally, this is done by changing the pure strategies with the mixed strategies in the definition of Nash equilibrium. Again given the mixed strategy of the others, each agent maximizes her expected payoff over her own (mixed) strategies.

Definition 5.2. A mixed-strategy profile $\sigma^* = (\sigma_1^*, \ldots, \sigma_n^*)$ is said to be a Nash equilibrium if for every player i, σ_i^* is a best response to σ_{-i}^*, i.e., $u_i\left(\sigma_i^*, \sigma_{-i}^*\right) \geq u_i\left(\sigma_i, \sigma_{-i}^*\right)$ for every mixed strategy σ_i.

Nash equilibria in mixed strategies are computed as in the example of the matching-penny game above. One assumes that each player can choose any mixed strategy and finds a profile of mixed strategies in which each mixed strategy is a best response to others, and no player has an incentive to deviate.

For a further illustration of how to compute Nash equilibria in pure and mixed strategies, consider the Battle of Sexes game in (5.1) again. We have identified two pure-strategy equilibria: (A, A) and (B, B). In addition, there is a mixed-strategy equilibrium.

To compute the mixed-strategy equilibrium, write p for the probability that Alice plays A; she plays B with probability $1-p$. Write also q for the probability that Bob plays A. For Alice, the expected payoff from A is

$$u_A(A,q) = qu_A(A,A) + (1-q)\,u_A(A,B) = 2q,$$

and the expected payoff from B is

$$u_A(B,q) = qu_A(B,A) + (1-q)\,u_A(B,B) = 1-q.$$

Her expected payoff from mixing with probability p is

$$\begin{aligned} u_A(p,q) &= pu_A(A,q) + (1-p)\,u_A(B,q) \\ &= p\,(u_A(A,q) - u_A(B,q)) + u_A(B,q) \\ &= p\,(3q-1) + 1 - q. \end{aligned}$$

The payoff function $u_A(p;q)$ is linear in probabilities p and q. In particular, it is strictly increasing in p if the expected payoff $u_A(A,q)$ from A is strictly higher than the expected payoff $u_A(B,q)$ from B, i.e., $2q > 1-q$. This is the case when $q > 1/3$. In that case, the unique best response for Alice is $p=1$, and she plays A for sure. Likewise, when $q < 1/3$, we have $u_A(A,q) < u_A(B,q)$, and her expected payoff $u_A(p,q)$ is strictly decreasing in p. In that case, Alice's best response is $p=0$, i.e., playing B for sure. Finally, when $q = 1/3$, her expected payoff $u_A(p,q)$ does not depend on p, and any $p \in [0,1]$ is a best response. Alice chooses A if her expected utility from A is higher, B if her expected utility from B is higher, and can choose either strategy or any randomization between them if she is indifferent between the two. Similarly, one can compute that $q=1$ is a best response if $p > 2/3$; $q=0$ is best response if $p < 2/3$; and any q can be best response if $p = 2/3$. The best responses are plotted in Figure 5.3. The Nash equilibria are where these best responses intersect. There is one at $(0,0)$, when they both play B, one at $(1,1)$, when they both play A, and there is one at $(2/3, 1/3)$, when Alice plays A with probability $2/3$, and Bob plays A with probability $1/3$.

This example illustrates a key property of expected utility maximization. The player's expected payoffs are linear in probabilities (when they are viewed as vectors of probabilities), and a player mixes between any two strategies only when she is indifferent between them. Otherwise, she puts zero probability on the one with a lower expected value.

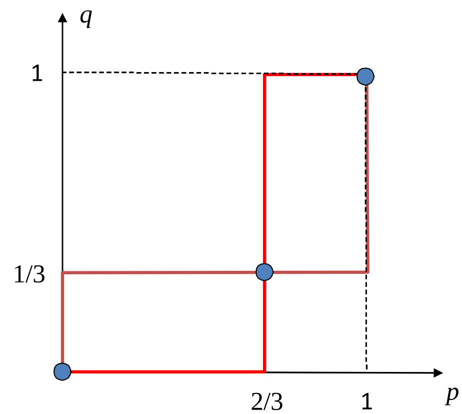


Figure 5.3: The best responses and the Nash equilibria in the Battle of the Sexes game

Remark 5.1. The above example illustrates ways to compute the mixed-strategy equilibrium. First, in general, one can compute all Nash equilibria, including those in mixed strategies, by computing mixed best responses and finding their intersections. This general technique can be challenging. It also illustrates a simpler technique for 2×2 games. Choose the mixed strategy of Player 1 in order to make Player 2 indifferent between her strategies, and choose the mixed strategy of Player 2 in order to make Player 1 indifferent. This is a valid technique to compute a mixed-strategy equilibrium, provided that it is known which strategies are played with positive probabilities in equilibrium.

More generally, with a finite set of strategies, the expected payoff from σ^* is

$$u_i(\sigma^*) = \sum_{s \in S} u_i(s)\, \sigma^*(s) = \sum_{s_i \in S_i} u_i\left(s_i, \sigma^*_{-i}\right) \sigma^*_i(s_i)$$

where $\sigma^*(s) = \sigma^*_1(s_1) \times \cdots \times \sigma^*_n(s_n)$ is the probability assigned on s, which is in product form, and $u_i\left(s_i, \sigma^*_{-i}\right)$ is the expected payoff from playing s_i against mixed strategy profile σ^*_{-i} of other players. The expression on the right-hand side establishes that the expected payoff from σ^* is a weighted average of the expected payoffs $u_i\left(s_i, \sigma^*_{-i}\right)$ from pure strategies s_i with weights $\sigma^*_i(s_i)$. Nash equilibrium requires that this expected payoff does not increase when one replaces σ^*_i with another mixed strategy σ_i. Since $u_i(\sigma^*)$ is a weighted average of payoffs $u_i\left(s_i, \sigma^*_{-i}\right)$, this is the case if and only if $u_i(\sigma^*)$ is equal to the largest payoff $u_i\left(s_i, \sigma^*_{-i}\right)$, i.e., player i does not have an incentive to deviate to any pure strategy s_i. This leads to a relatively simple condition to check whether σ^*

is a Nash equilibrium: for every i, every s_i with $\sigma_i^*(s_i) > 0$ is a best response to σ_{-i}^*. That is,

$$u_i\left(s_i, \sigma_{-i}^*\right) \geq u_i\left(s_i', \sigma_{-i}^*\right) \qquad \forall i, \forall s_i \text{ with } \sigma_i^*(s_i) > 0, \forall s_i'.$$

Remark 5.2. The above condition leads to the following simple conditions to check whether a given σ^* is a Nash equilibrium. For each player i,

1. she must be indifferent between all strategies that she plays with positive probability, i.e., $u_i\left(s_i, \sigma_{-i}^*\right) = u_i\left(s_i', \sigma_{-i}^*\right)$ whenever $\sigma_i^*(s_i) \neq 0 \neq \sigma_i^*(s_i')$;

2. no other pure strategy must give a higher payoff, i.e., $u_i\left(s_i, \sigma_{-i}^*\right) \geq u_i\left(s_i', \sigma_{-i}^*\right)$ whenever $\sigma_i^*(s_i) > 0$.

For example, Alice plays both strategies with positive probability in the above mixed-strategy equilibrium. The first condition requires that she is indifferent between them, which is indeed the case. Since there is no other strategy, the second condition is vacuously satisfied. This proves, once again, that the above profile is indeed a mixed-strategy Nash equilibrium.

5.4 General Properties of Nash Equilibria*

Existence Once we allow Nash equilibria in mixed strategies, Nash equilibria have many useful properties, and some of these technical properties are presented in the appendix. First, there is always a Nash equilibrium in games with finite strategy sets. As in the matching-penny game, the equilibria may be in mixed strategies.[4] Existence holds more generally. When the strategy sets are convex and utility functions are strictly concave, there is always a unique best response. In that case, there cannot be a Nash equilibrium in mixed strategies—because all the strategies a player plays with positive probability must be a best response to her belief in a mixed strategy Nash equilibrium. In that case, there is a Nash equilibrium in pure strategies when the strategy sets are closed and bounded and the utility functions are continuous.

[4] As in the examples above, the number of Nash equilibria is odd under "generic" payoff functions, but in some "knife-edge cases" there can be an even number of equilibria or a continuum of them (see Example 5.3 below). It is difficult to check when this property hold.

Rationalizability As in the pure strategy Nash equilibria, players play only rationalizable strategies in any Nash equilibrium including one in mixed strategies:

Proposition 5.1. *If σ^* is a mixed strategy Nash equilibrium and $\sigma_i^*(s_i) > 0$, then s_i is rationalizable. There exists a game in which some rationalizable strategies are not played in any Nash equilibrium.*

Proof. In Exercise 4.6, define $Z_i = \{s_i | \sigma_i^*(s_i) > 0\}$ for each i. Then, since σ^* is a Nash equilibrium, each $s_i \in Z_i$ is a best response to belief σ_{-i}^* on Z_{-i}, i.e., Z satisfies the property in Exercise 4.6. Therefore, each $s_i \in Z_i$ is rationalizable. For the second part, one can check that, in the game in Exercise 5.2, every strategy is rationalizable but (c, c) is the only Nash equilibrium. □

Proposition 5.1 establishes that Nash equilibrium is more restrictive than rationalizability. This is because Nash equilibrium imposes further restrictions on players' conjectures about each others' strategies in addition to the assumptions made in the description of the game.

Finding a Nash equilibrium is computationally hard. Even with the aid of computers, finding a Nash equilibrium can be highly time-consuming, and there is no known technique that would circumvent this problem. Therefore, in computing Nash equilibria, one uses the structure of the game to search for an equilibrium. In most applications, the game has considerable structure, and one may be able to use the structure to compute the best responses and identify where they intersect. In that case, one can use this to compute the set of Nash equilibria. In general, this may not be practical. One can first simplify the problem by iteratively eliminating strictly dominated strategies and search for equilibria within rationalizable strategies, as mixed strategy Nash equilibria can put positive probability only on rationalizable strategies.

Example 5.3. Consider the game

	a	b	c	d
A	$\mathbf{3}, 1$	$\mathbf{2}, \mathbf{2}$	$1, 0$	$0, 0$
B	$1, 0$	$1, 1$	$1, 2$	$1, \mathbf{4}$
C	$0, \mathbf{1}$	$1, 0$	$\mathbf{2}, \mathbf{1}$	$3, 0$
D	$0, \mathbf{3}$	$1, 0$	$0, 2$	$\mathbf{4}, 2$

where the payoff is written in bold when a player plays a best response. Clearly, there are two Nash equilibria in pure strategies: (A, b) and (C, c). To check if there is a Nash equilibrium in mixed strategies, compute the rationalizable strategies first. Observe that B is strictly dominated, and one can eliminate B, then d, and then D iteratively to obtain the set of rationalizable strategies as

	a	b	c
A	$\mathbf{3}, 1$	$\mathbf{2}, \mathbf{2}$	$1, 0$
C	$0, \mathbf{1}$	$1, 0$	$\mathbf{2}, \mathbf{1}$

In the reduced game c is weakly dominated, and hence c can be played in a mixed-strategy equilibrium only if player 1 plays C. Since both a and c are best responses to C, Player 2 can mix between them in such a Nash equilibrium. Player 1 plays C as a best response to such a σ_2 if and only if $\sigma_2(c) \geq 3/4$. There is a continuum of such Nash equilibria in mixed strategies

$$\{(C, \sigma_2) \,|\sigma_2(a) + \sigma_2(c) = 1, \sigma_2(c) \geq 3/4\}.$$

These are the mixed strategy equilibria in which Player 2 plays c with positive probability. There is no other Nash equilibrium. Indeed, if Player 2 does not play c with positive probability, then A is strictly better than C, and Player 1 must be playing A for sure in any such purported equilibrium. But b is the only best response to A, and the Nash equilibrium (A, b) above is the only such equilibrium.

Interpretation of Mixed Strategy Nash equilibrium Nash equilibrium in mixed strategies is formally defined as a Nash equilibrium in an extended game in which the players choose mixed strategies. There are other interpretations of such equilibria. One prominent interpretation is that a mixed strategy Nash equilibrium is the distributions of strategies when the game is played by randomly selected anonymous players. The distributions of strategies form a Nash equilibrium when these distributions are known (and stochastically independent of each other), and no player has an incentive to deviate.

For an illustration, imagine that the matching-penny game above is being played by some anonymous players, where each player is randomly drawn from a distinct large population, Player 1 from population 1 and Player 2 from population 2. When would players do not have an incentive to deviate when they know the distribution of strategies played by the members of each population? To answer this question, write p_1 and p_2 for the proportions of the players who play strategy H in populations 1 and 2, respectively. Given p_2, the expected payoff from playing strategy H for a player in group 1 is

$$u_1(H, p_2) = p_2 \times 1 + (1 - p_2) \times (-1) = 2p_2 - 1.$$

This is because, from population 2, we select a player who plays H with probability p_2 and select a player who plays T with probability $1 - p_2$. Likewise, the expected payoff from playing strategy T is

$$u_1(T, p_2) = 1 - 2p_2.$$

Using the two displayed equations, one can show that players do not have an incentive if and only if $p_1 = p_2 = 1/2$, so that (p_1, p_2) is the unique Nash equilibrium of the matching-penny game. Indeed, suppose the probability p_2 is strictly higher than $1/2$. Then, $u_1(H, p_2) > u_1(T, p_2)$, and each member of population 1 strictly prefers H. In that case, some players from population 1 deviate and play H unless $p_1 = 1$ already. Now suppose further that $p_1 = 1$. Then, the members of group 2 know that their opponents all play H, and the unique best response to this is playing T. Since $p_2 > 0$, somebody will switch in population 2. Therefore, it must be that $p_2 \leq 1/2$. Using the same argument, one can also show that $p_2 \geq 1/2$, concluding that $p_2 = 1/2$. Similarly, $p_1 = 1/2$. That is, if the distribution of the strategies in each group is known and no player has an incentive to deviate, then it must be that each strategy is played with probability $1/2$ in each group—as in the mixed strategy Nash equilibrium.

5.5 Application: Mutual Investment

In many real-world problems, players often need to coordinate their strategies as discussed in Chapter 1. Such coordination motives may lead to multiple equilibria. This section illustrates this on the following game, called the *Investment Game*:

	Invest	Stay Out
Invest	θ, θ	$\theta - c$, 0
Stay Out	0,$\theta - c$	0,0

where θ and $c > 0$ are known parameters. Here, two parties simultaneously decide whether to invest. If they both invest, the project succeeds and each gets θ; if only one of them invests, she incurs a cost $c > 0$ and gets only $\theta - c$. The investment is more valuable if the other party also invests. In real world, "invest" may correspond to employees investing in human capital and employers investing in a skill-intensive technology, speculators short-selling a currency, citizens protesting a repressive government, or partners putting high effort in a joint project.

When $\theta > c$, Invest strictly dominates Stay Out, and there is a unique Nash equilibrium: (Invest, Invest). When $\theta < 0$, Stay Out strictly dominates Invest, and there is a unique Nash equilibrium: (Stay Out, Stay Out). When $c > \theta > 0$, there are multiple Nash equilibria, one in which both parties invest, one in which nobody invests, and one in mixed strategies. Towards computing the mixed strategy Nash equilibrium, observe that the expected payoff from Invest is

$$u_i\,(\text{Invest},\, q) = \theta - c\,(1 - q)$$

where q is the probability that the other player invests. A player is indifferent if and only if this amount is equal to zero, yielding

$$q = 1 - \theta/c.$$

This yields the mixed strategy Nash equilibrium, where each player invests with probability $1 - \theta/c$.

In many economic environments, strategic outcomes differ, although underlying fundamentals seem similar. For example, a country may be in poverty where the firms do not invest in highly productive skill-intensive production technologies and skilled labor is scarce. Another country with similar natural resources and population may be thriving, where firms invest skill-intensive production technologies and skilled labor is abundant. Similarly, in one country, the government may successfully peg the local currency at a high exchange rate, and the speculators may be not attacking the currency. In another country with seemingly similar economic fundamentals, they attack the currency when the government pegs the local currency. Similarly, in a country with the local currency pegged, the speculators do not attack currency until some seemingly irrelevant event triggers all of them to attack the currency although there does not seem to have a significant change in fundamentals. Likewise, the protests erupt suddenly after a long tranquil period with almost no protest without a fundamental change in the country in between.

Such differences in strategic outcomes are sometimes explained by multiple equilibria, using a version of the investment game with $c > \theta > 0$. Here, mutual investment is strictly better for both players, as it yields strictly positive payoff θ for each player. This is a "good" equilibrium. And yet there is another "bad" equilibrium with strict incentives in which no player invests, each getting zero. Although mutual investment is beneficial for both players, in "bad" equilibrium, players stay out anticipating that the other player will not invest. The bad equilibrium corresponds to what is called a "poverty trap", in which the workers do not invest in their human capital anticipating that there will not be a demand for such skills, and the firms do not invest skill-intensive technologies with the anticipation that the skilled labor is scarce.

Existence of such multiple equilibria leads to many intriguing questions: why do players end up coordinating on such different outcomes? Is it simply a historical accident that traps one country in poverty while the other thrives? Or, are there some less obvious differences between those countries, such as informational differences, that lead to such dramatically different outcomes? What are the events that trigger such sudden dramatic shifts? Can one break such vicious cycles and how? Some of these questions will be answered later in Chapter 18 under incomplete information.

Some may argue that, since "good" equilibrium is better for everybody, the players ought to coordinate on that equilibrium or they will find a way to coordinate on that outcome. They may conclude that the bad equilibrium is an artifact of the model, and it is irrelevant for the real world. This is not true with incomplete information. Indeed, under some information structures, the bad equilibrium corresponds to the unique rationalizable outcome, as it has been illustrated in Chapter 1 and as it will be established more generally in Chapter 18.

5.6 Evolution of Hawks and Doves*

This section illustrates a general relation between the limits of evolution and the Nash equilibria of a game in which the payoffs reflect the fitness of the species. Consider the game

	Hawk	Dove
Hawk	$\frac{V-c}{2}, \frac{V-c}{2}$	V, 0
Dove	0,V	$\frac{V}{2}, \frac{V}{2}$

for some known positive values V and c. This is an important biological game, but is also quite similar to many games in Economics and Political Science. It represents situations in which the parties compete for resources, but competition is costly to the players. Such problems arise, for example, when the firms compete for market dominance or political parties and candidates with similar bases compete for votes. In this specific game, V is the value of a resource that one of the players will enjoy. If they share the resource, each receives value $V/2$. Hawk stands for a "tough" strategy, whereby the player does not give up the resource. However, if the other player is also playing Hawk, they end up fighting, and incur the cost $c/2$ each. On the other hand, a Hawk player gets the whole resource for herself when playing against a Dove. When $V > c$, this is a Prisoners' Dilemma game, yielding a fight.

Suppose that the cost of fighting exceeds the value of the resource, i.e., $V < c$. In this case, each player would like to play Hawk if her opponent plays Dove, and play Dove if her opponent plays Hawk. Such a situation is called a *war of attrition*, where a player would rather prefer the other party to concede (by playing Dove), but she would have conceded if she knew that the other party would not concede. This is common in the real world, such as in international relations and in competition for market dominance. This game is also similar to another well-known game, named "Chicken", in which two players driving towards a cliff have to decide whether to stop or continue.

When $V < c$, there are three equilibria, two asymmetric Nash equilibria in pure strategies, (Hawk, Dove) and (Dove, Hawk), and a symmetric equilibrium in mixed strategies. In the asymmetric equilibrium, one of the players plays Dove, while the other player plays Hawk and enjoys all of the resources without incurring any cost. In the mixed-strategy equilibrium, each player is indifferent between the two strategies as they play both strategies with positive probability. Let h be the probability of Player 2 playing Hawk, and $d = 1 - h$ be the probability that she plays Dove. Since Player 1 plays both strategies with positive probability, she must be indifferent between them:

$$\frac{V-c}{2} \cdot h + V \cdot d = \frac{V}{2} \cdot d,$$

where the left-hand side is the expected payoff from Hawk and the right-hand side is the expected payoff from Dove. The solution to this equation is

$$h = V/c.$$

That is, Player 2 must play both Hawk and Dove with positive probabilities V/c and $1-V/c$, respectively. In order for this to be a best response for Player 2, it must be that Player 1 also plays Hawk with probability V/c. Therefore, in the mixed-strategy Nash equilibrium, each player plays Hawk with probability V/c and Dove with probability $1 - V/c$.

Now imagine that this game is played by the genes. Assume that $V < c$, so that the payoffs are negative when two hawks meet. Imagine an island where hawks and doves live together. Let there be H_0 hawks and D_0 doves at the beginning where both H_0 and D_0 are very large. Suppose that, at each season, the birds are randomly matched and the number of offsprings of a bird is given by the payoff matrix above. That is, if a dove is matched to a dove as the neighbor, then it will have $V/2$ offsprings, and in the next generation, there will be $1 + V/2$ doves in its family. If a dove is matched with a hawk, then it will have zero offsprings and its family will have only 1 member, itself, in the next season. If two hawks are matched, then each will have $(V - c)/2$ offsprings, which is negative, reflecting the situation that the number of hawks from such matches will decrease when we go to the next season. Finally, if a hawk meets dove, it will have V offsprings, and there will $1 + V$ hawks in its family in the next season. We want to know the ratio of hawks and doves in this island millions of seasons later.

Let H_t and D_t be the number of hawks and doves, respectively, at season t. Define

$$h_t = \frac{H_t}{H_t + D_t} \text{ and } d_t = \frac{D_t}{H_t + D_t}$$

as the ratios of hawks and doves at t. In accordance with the strong law of large numbers, assume that the number of hawks that are matched to hawks is $H_t h_t$, and the number of hawks that are matched to doves is $H_t d_t$.[5] Each hawk in the first group multiplies to $1 + (V - c)/2$, and each hawk in the second group multiplies to $1 + V$. The number of

[5] The probabilities of matching to a hawk and dove are h_t and d_t, respectively. And there are H_t hawks.

hawks in the next season will be then

$$\begin{aligned} H_{t+1} &= \left(1+(V-c)/2\right) H_t h_t + (1+V) H_t d_t \\ &= (1+(V-c) h_t/2 + V d_t) H_t. \end{aligned} \tag{5.4}$$

Number of doves who are matched to hawks is $D_t h_t$, and number of doves that are matched to doves is $D_t d_t$. Each dove in the first and the second group multiplies to 1 and $1+V/2$, respectively. Hence, the number of doves in the next season will be

$$D_{t+1} = (1+0) D_t h_t + (1+V/2) D_t d_t = (1+V d_t/2) D_t. \tag{5.5}$$

It is easy to find the *steady states* of the ratio h_t (and d_t), defined by

$$h_{t+1} = h_t \text{ and } d_{t+1} = d_t.$$

From (5.4) and (5.5) it is clear that

$$h_t = 0 \text{ and } d_t = 1$$

is a steady state, which can be reached if there are only doves at the beginning. In that case, by (5.4), it will continue as "doves only." Similarly, another steady state is

$$h_t = 1 \text{ and } d_t = 0,$$

which can be reached if there are only hawks at the beginning. In the present case, since there are both hawks and doves at the beginning, both D_t and D_{t+1} are positive. Hence, the steady states can be computed by

$$\frac{H_t}{D_t} = \frac{H_{t+1}}{D_{t+1}} = \frac{H_t}{D_t} \frac{1+(V-c) h_t/2 + V d_t}{1+V d_t/2}, \tag{5.6}$$

where the last equality is due to (5.4) and (5.5). The equality holds if and only if

$$(V-c) h_t/2 + V d_t = V d_t/2,$$

or equivalently

$$h_t = V/c.$$

This is the only steady-state reached from a distribution with hawks and doves. Notice that it is the mixed-strategy Nash equilibrium of the underlying game. This is a general fact: if a population dynamic is as described in this section, then the steady states reachable from a completely mixed distribution are symmetric Nash equilibria.

If both hawks and doves are present at the beginning, the population ratios of hawks and doves will necessarily approach to the last steady-state, which is the mixed-strategy Nash equilibrium. For example, if h_t exceeds the equilibrium value, then it decreases towards the equilibrium value. To see this formally, assume

$$h_t > V/c.$$

Then,

$$\frac{1 + (V - c)\, h_t/2 + V d_t}{1 + V d_t/2} < 1,$$

yielding

$$\frac{H_{t+1}}{D_{t+1}} < \frac{H_t}{D_t}$$

by (5.6). By (5.4) and (5.5), this further yields $h_{t+1} < h_t$. Similarly, if $h_t < V/c$, then $h_{t+1} > h_t$, and h_t will increase towards the equilibrium.

5.7 Exercises with Solutions

Exercise 5.1. Compute the set of Nash equilibria in Exercise 4.1.

Solution. Since Nash equilibrium strategies put positive probability only on rationalizable strategies, it suffices to consider the rationalizable set. But there is only one rationalizable strategy profile (z, c). Therefore, (z, c) is the only Nash equilibrium.

Exercise 5.2. Compute the set of Nash equilibria in Exercise 4.2.

Solution. Since Nash equilibrium strategies put positive probability only on rationalizable strategies, it suffices to compute the Nash equilibria in the game with only rationalizable strategies. Recall also from the solution to Exercise 4.2 that, after the elimination of non-rationalizable strategies, the game reduces to

	w	y
a	$0, \mathbf{3}$	$\mathbf{3}, 0$
b	$\mathbf{3}, 0$	$2, \mathbf{4}$

Here, the best responses (to the pure strategies) are written in bold. Since the best responses do not intersect, there is no Nash equilibrium in pure strategies. There is a unique mixed strategy Nash equilibrium σ^*. In order for Player 1 to play a mixed strategy, she must be indifferent between a and b against σ_2^*:

$$3\sigma_2^*(y) = 3 - \sigma_2^*(y).$$

Here, $\sigma_2^*(y)$ is the probability Player 2's mixed strategy σ_2^* assigns to y; the left-hand side is the expected payoff from playing a against σ_2^*, and the right-hand side is the expected payoff from playing b against σ_2^*. To see the expected payoffs, observe that when Player 1 plays a, she gets 3 with probability $\sigma_2^*(y)$ and 0 with probability $\sigma_2^*(w) = 1 - \sigma_2^*(y)$, yielding $3\sigma_2^*(y)$; when she plays b she gets 3 with probability $\sigma_2^*(w) = 1 - \sigma_2^*(y)$ and 2 with probability $\sigma_2^*(y)$, yielding $3 - \sigma_2^*(y)$. The indifference condition yields

$$\sigma_2^*(y) = 3/4.$$

Of course, $\sigma_2^*(w) = 1/4$. Since Player 2 is playing a mixed strategy, she must be indifferent between playing w and y against σ_1^*:

$$3\sigma_1^*(a) = 4(1 - \sigma_1^*(a)).$$

Here the left-hand side is the expected payoff from w, and the right-hand side is the expected payoff from y. The indifference condition yields

$$\sigma_1^*(a) = 4/7,$$

and $\sigma_1^*(b) = 3/7$.

Exercise 5.3. Compute the set of Nash equilibria in the following game that represents the child game Rock-Paper-Scissors:

	R	P	S
R	$0,0$	$-1,1$	$1,-1$
P	$1,-1$	$0,0$	$-1,1$
S	$-1,1$	$1,-1$	$0,0$

Solution. A quick inspection shows that there is no Nash equilibrium in pure strategies. Moreover, all strategies must be played with positive probability in any Nash equilibrium.

For example, if S is not played by Player 2, R is strictly dominated by P, and Player 1 does not play R. Then, P is strictly dominated by S, and Player 2 must play R for sure. Since there is no equilibrium in pure strategies, this is not possible in Nash equilibrium. Then, all strategies are played with positive probability in any Nash equilibrium. Players must be indifferent between all of their strategies. By symmetry, these indifference conditions lead to the unique Nash equilibrium:

$$\sigma_i^*(R) = \sigma_i^*(P) = \sigma_i^*(S) = 1/3.$$

Exercise 5.4. In Exercise 2.3, for each $c > 0$, compute a Nash equilibrium.

Solution. It is easier to compute a Nash equilibrium from the normal-form representation. Recall from the solution to Exercise 2.3 that the normal-form representation of the game is

	same	new
RR	$1, 0$	$1, 0$
RM	$3, 1/2$	$3/2, (1-c)/2$
MR	$2, -1$	$1/2, -(1+c)/2$
MM	$4, -1/2$	$1, -c$

where the student is the row player, and the professor is the column player. It is useful to find the rationalizable strategies first. Observe that RR and MR are strictly dominated by RM and MM, respectively, as taking the regular exam is strictly worse when the student is sick. Hence, RR and MR are eliminated, leading to the following reduced game:

	same	new
RM	$3, \mathbf{1/2}$	$\mathbf{3/2}, (1-c)/2$
MM	$\mathbf{4}, -1/2$	$1, -c$

Moreover, when $c > 1/2$, strategy "new" is now strictly dominated by "same", and it is eliminated in the second round, leading to elimination of RM in the next round. In that case, the game is dominance-solvable, and the solution $(MM, same)$ is the unique Nash equilibrium. When $c = 1/2$, $(MM, same)$ remains to be the unique Nash equilibrium.

When $c < 1/2$, "new" is the unique best response to MM. In that case, all the strategies in the reduced game are rationalizable, and there is no Nash equilibrium in pure strategies. The only Nash equilibrium is in mixed strategies, denoted by σ^*.

Towards computing σ^*, the indifference condition for the student yields

$$3/2 + (3/2)\,\sigma_2^*\,(\text{same}) = 1 + 3\sigma_2^*\,(\text{same})\,,$$

where the payoffs from the strategies RM and MM are on the left and the right-hand sides of the equation, respectively. Therefore,

$$\sigma_2^*\,(\text{same}) = 1/3 \text{ and } \sigma_2^*\,(\text{new}) = 2/3.$$

The indifference condition for the professor yields

$$\sigma_1^*\,(RM) - 1/2 = \sigma_1^*\,(RM)\,(1+c)\,/2 - c,$$

yielding

$$\sigma_1^*\,(RM) = \frac{1-2c}{1-c} \text{ and } \sigma_1^*\,(MM) = \frac{c}{1-c}.$$

Note that, in equilibrium, the student takes the regular exam when he is healthy and mixes between regular exam and make up when he is sick.

5.8 Exercises

Exercise 5.5. Compute the set of all Nash equilibria in the following games (consider computing the set of rationalizable strategies first).

	w	x	y	z
a	$3,0$	$0,2$	$2,3$	$1,1$
b	$1,4$	$0,0$	$1,0$	$2,1$
c	$4,0$	$0,0$	$1,0$	$2,1$
d	$1,1$	$1,3$	$1,0$	$4,2$

	w	x	y	z
a	$3,3$	$2,1$	$0,2$	$2,1$
b	$1,1$	$1,2$	$1,0$	$1,4$
c	$0,0$	$1,0$	$3,2$	$1,1$
d	$0,0$	$0,5$	$0,2$	$3,1$

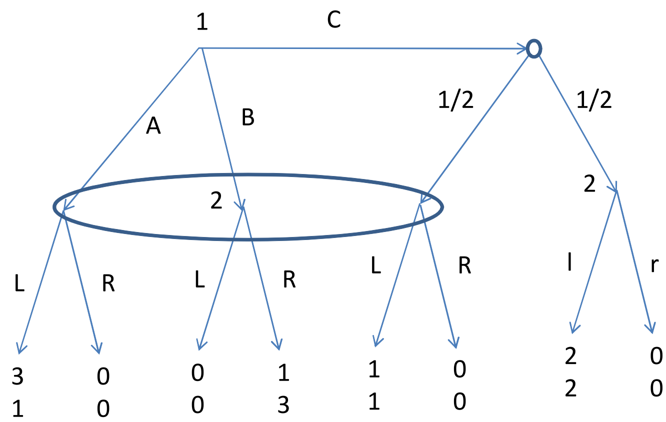
	a	b	c	d
w	$2,0$	$0,5$	$1,0$	$0,4$
x	$4,1$	$2,1$	$0,2$	$1,0$
y	$2,1$	$5,0$	$0,0$	$0,3$
z	$0,0$	$1,0$	$4,1$	$0,0$

	L	M	N	R
A	$4,2$	$0,0$	$5,0$	$0,0$
B	$1,4$	$1,4$	$0,5$	$-1,0$
C	$0,0$	$2,4$	$1,2$	$0,0$
D	$0,0$	$0,0$	$0,-1$	$0,0$

	L	M	R
A	$1,0$	$4,1$	$1,0$
B	$2,1$	$3,2$	$0,1$
C	$3,-1$	$2,0$	$2,2$

	L	M	R
A	$3,1$	$0,0$	$1,0$
B	$0,0$	$1,3$	$1,1$
C	$1,1$	$0,1$	$0,10$

Exercise 5.6. Consider the following game.



1. Find all Nash equilibria in pure strategies.

2. Find a Nash equilibrium in which Player 1 plays a mixed strategy (without putting probability 1 on any of her strategies).

Exercise 5.7. Consider the following game:

	A	B	C
a	$3, 0$	$0, 3$	$0, x$
b	$0, 3$	$3, 0$	$0, x$
v	$x, 0$	$x, 0$	x, x

1. Compute two Nash equilibria for $x = 1$.

2. For each equilibrium in Part 1, check if it remains a Nash equilibrium when $x = 2$.

Exercise 5.8. Compute a Nash equilibrium of the following version of the Rock-Paper-Scissors—with a preference for Paper:

	R	P	S
R	$0, 0$	$-2, 3$	$2, -2$
P	$3, -2$	$1, 1$	$-1, 2$
S	$-2, 2$	$2, -1$	$0, 0$

Exercise 5.9. Find all the Nash equilibria in the following game:

	L	M	R
T	$\mathbf{1},0$	$0,\mathbf{1}$	$\mathbf{5},0$
B	$0,\mathbf{2}$	$\mathbf{2},1$	$1,0$

Exercise* 5.10. In Example 5.2, show that (c, c) is the only Nash equilibrium, i.e., there is no Nash equilibrium in mixed strategies. Show also that no strategy is strictly dominated, and hence all strategies are rationalizable.

Exercise 5.11. There are n players, $1, 2, \ldots, n$, who bid for a painting in a second-price auction. Each player i bids b_i, and the bidder who bids highest buys the painting at the highest price bid by the players other than himself. (If two or more players bid the highest bid, the winner is decided by a coin toss.) The value of the art is v_i for each player i where $v_1 > v_2 > \cdots > v_n > 0$. Find a Nash equilibrium of this game in which player n, who values the painting least, buys the object for free (at price zero). Briefly discuss this result and compare it to the answer of Exercise 3.8.

Exercise 5.12. Compute the set of Nash equilibria in Exercise 4.14.

Exercise 5.13. Consider the following variation of the game in Exercise 4.14. Al does not know Bill's payoffs over action pairs. In particular, he assigns probabilities 1/2, 1/4, 1/4 on Bill being self-interested, vindictive or having high criminal integrity (HCI), respectively, where payoff function for each payoff type is as follows:

Self-interested

	C	D
C	4	6
D	0	2

Vindictive

	C	D
C	2	0
D	0	2

HCI

	C	D
C	1	0
D	1	0

Bill knows her own payoffs. Moreover, Al is vindictive, with payoffs

	C	D
C	4	0
D	0	4

and all of these are common knowledge.

1. Model this as an extensive-form game.

2. Write the extensive-form game in normal form.

3. Compute the set of all rationalizable strategies.

4. Compute the set of all Nash equilibria in pure strategies.

Exercise 5.14. Compute the set of all Nash equilibria in the following generalization of the matching-penny game, played by Anita and Beto. Simultaneously, each player chooses a number from $1, 2, \ldots, m$. If they choose the same number, Beto has to do something embarrassing, and Anita and Beto get payoffs 1 and -1, respectively. Otherwise, they each get zero.

Exercise* 5.15. Two drivers, named Ann and Beata, are at an intersection, where each of them chooses between "go" and "stop". The payoff of a player is as in the following table:

	other player goes	other player stops
go	$-c$	1
stop	0	0

where $c > 0$ is the cost of an accident that occurs when both players go.

1. Assuming that the players simultaneously choose between "go" and "stop", compute the set of Nash equilibria.

2. Now imagine that traffic lights are introduced at the intersection. There is a light for each player, which can be seen by only that player. The light can be "green" or "red". The color of each light is chosen probabilistically according to the following table

	Green	Red
Green	p	r
Red	r	q

   where $p, q, r \in [0, 1]$ and $r = (1 - p - q)/2$; the rows and the columns indicate the color of the light for Ann and Beata, respectively. Compute the set of (p, q, r) for which there is a Nash equilibrium in which each player goes if her light is green and stops if her light is red. What (p, q, r) would you choose if you were designing traffic lights towards maximizing the sum of the expected payoffs of the players? Briefly discuss.

Exercise 5.16. Consider a game in which the following prisoners' dilemma game is played twice:

	C	D
C	$5,5$	$0,6$
D	$6,0$	$1,1$

where C stands for Cooperate and D stands for Defect. They first simultaneously choose between C and D; then after each observing what each player played in the first round, they play the prisoners' dilemma game again. The payoff of a player is the sum of the payoffs he gets in two rounds; e.g., if they play (C,D) in the first round and (D,D) in the second round, the payoff vector is $(1,7)$.

1. Write this formally as a game in extensive form.

2. Find two Nash equilibria and verify that each of them is indeed a Nash equilibrium. (The equilibria you find may have the same outcome.)

3. Find a rationalizable strategy for Player 1 in which he plays C in the first round.

4. Can there be a Nash equilibrium in which a player plays C in the first round?

Exercise* 5.17. In the previous problem, consider that the prisoners' dilemma game is repeated n times, as opposed to twice. Suppose that players can make mistakes, in that the implemented action may be switched with small but positive probability ε. For example, at a given round, if a player chooses C, then nature moves and chooses C with probability $1-\varepsilon$ and switches to D with probability ε. The other player can see only what nature chooses but not what player has chosen; e.g., in the above example, he sees whether nature chose C or D without knowing that the player chose C. Find the set of all rationalizable strategies.

Exercise† 5.18. Assuming that the sets of strategies are finite, show that in a dominance-solvable game, the unique rationalizable strategy is the only Nash equilibrium.

Exercise[†] **5.19.** Assume that each strategy set S_i is convex and each utility function u_i is strictly concave in own strategy s_i.[6] Show that all Nash equilibria are in pure strategies.

5.9 Technical Appendix: Existence and Continuity of Nash Equilibrium

This technical appendix provides broad conditions under which a Nash equilibrium exists and the set of Nash equilibria is upperhemicontinuous, i.e., a drastically different solution does not appear when one perturbs the payoff (although some of the solutions may disappear). It can be skipped without loss of continuity.

It is useful to recall a couple of basic mathematical concepts to follow the appendix. Consider $\mathbb{R}^m$ with the usual metric that measures the distances between the vectors (this is called a Euclidean space). A set S is said to be *convex* if for any two elements $s, s' \in S$ and any $\alpha \in [0,1]$, the convex combination $\alpha s + (1-\alpha) s'$ remains in S. For example, the unit interval $[0,1]$ is a convex set. A set S is said to be *compact* if it is closed and bounded, as in the case of $[0,1]$.

For any strategy profile $s \in S$ and any player i, write $B_i(s)$ for the set of best responses to s_{-i}, and write $B(s) = B_1(s) \times \cdots \times B_n(s)$. Note that there can be multiple best responses, and B is a set-valued function.

Set-valued functions are called correspondences. Formally, a *correspondence* F from X to Y is an arbitrary subset of $X \times Y$. Conventionally, the correspondence F is denoted as $F : X \rightrightarrows Y$, writing $F(x)$ for the set $\{y : (x,y) \in F\}$. The defining subset of $X \times Y$ is called the *graph* of the correspondence and denoted by $G(F) = \{(x,y) : y \in F(x)\}$ for clarity. A correspondence F is said to be *non-empty* if $F(x)$ is always non-empty, *closed-valued* if $F(x)$ is always closed, *convex-valued* if $F(x)$ is always convex, and so on.

A key continuity concept for correspondences is called upperhemicontinuity. When

[6] A set S is convex if $\lambda a + (1-\lambda) b \in S$ for all $a, b \in S$ and all $\lambda \in [0,1]$. A function $f : S \to R$ is strictly concave if

$$f(\lambda a + (1-\lambda) b) > \lambda f(a) + (1-\lambda) f(b)$$

for all $a, b \in S$ and $\lambda \in (0,1)$.

Y is compact, a correspondence $F : X \rightrightarrows Y$ is said to be *upperhemicontinuous* if the graph of F is a closed set. This means that for any $x \in X$ and $\varepsilon > 0$, if x' is sufficiently close to x, every solution $y' \in F(x')$ at x is within ε distance of a solution $y \in F(x)$ at x. That is, a drastically different solution does not appear when x is perturbed.

Existence results often build on a fixed-point theorem. The following well-known fixed-point theorem will be used for the existence of Nash equilibria.

Theorem 5.3 (Kakutani's Fixed-Point Theorem)**.** *Let X be a non-empty, compact and convex subset of some Euclidean space. Let also $F : X \rightrightarrows X$ be a non-empty, convex-valued correspondence with a closed-graph. Then, there exists $x \in X$ with $x \in F(x)$.*

5.9.1 Existence of Nash Equilibrium

Under broad continuity assumptions for utility functions and compactness and convexity assumptions for strategy sets, one can easily show that a Nash equilibrium exists. Here, the continuity and compactness assumptions are indispensable because they are needed for the existence of a solution to optimization problems, which are Nash equilibria in single-player games. Convexity assumption is used for satisfying the conditions of fixed-point theorems, such as Kakutani's Fixed-Point Theorem (above). The following general existence theorem builds on Kakutani's Fixed-Point Theorem.

Theorem 5.4. *Let $G = (N, S, u)$ be a game where each S_i is a convex, compact subset of a Euclidean space and each $u_i : S \to \mathbb{R}$ is continuous in s and quasi-concave in s_i. Then, there exists a Nash equilibrium $s^* \in S$ of game G.*

Proof. In the proof, I will construct a correspondence $F : S \rightrightarrows S$ that satisfies the conditions of Kakutani's Fixed-Point Theorem and whose fixed-points are all Nash equilibria of G. One can then conclude that F has a fixed point, which is a Nash equilibrium. Let $F : S \rightrightarrows S$ be the "best reply" correspondence:

$$F_i(s) = B_i(s_{-i}) \qquad (\forall s \in S, i \in N).$$

Since S is compact and the utility functions are continuous, by the Maximum Theorem, F is non-empty and has closed graph. Moreover, by quasi-concavity, F is also convex-valued. Hence, F satisfies the conditions of Kakutani's fixed-point theorem. Therefore,

F has a fixed point:

$$s^* \in F(s^*).$$

Since $s_i^* \in B_i\left(s_{-i}^*\right)$ for each i by definition of F, s^* is a Nash equilibrium. □

For games with convex strategy sets and quasiconcave utility functions, Theorem 5.4 proves the existence of a *pure* strategy Nash equilibrium. One can use this result to establish the existence of equilibrium in classical economic models, such as the Cournot competition presented in the next chapter. Theorem 5.4 has another less obvious application:

Corollary 5.1. *Every finite game $G = (N, S, u)$ has a (possibly mixed) Nash equilibrium σ^*.*

Proof. Since S is finite, the set Σ_i of mixed strategies for player i is a simplex in a Euclidean space; in particular, it is convex and compact. Moreover, $u_i(\sigma) = \sum_s u_i(s)\,\sigma_1(s_1) \cdot \cdots \cdot \sigma_n(s_n)$ is continuous in σ and linear in σ_i. Hence, $G' = (N, \Sigma_1, \ldots, \Sigma_n, u)$ satisfies the assumptions of Theorem 5.4. Therefore, there exists a Nash equilibrium $\sigma^* \in \Sigma_1 \times \cdots \times \Sigma_n$, which is also a (mixed) Nash equilibrium of G. □

While continuity and compactness assumptions are often made in theory (as they are used for the existence of best response), the games in many applications, such as auctions, have discontinuous utility functions. Fortunately, continuity assumption can be substantially relaxed for the existence of equilibrium. The following classical result, due to Dasgupta and Maskin, establishes the existence of Nash equilibrium for a broad class of discontinuous games.

Theorem 5.5. *Consider a game $G = (N, S, u)$ where each $S_i \subset R$ is compact and each u_i is continuous except on a single one-dimensional submanifold of S. Suppose $u_1(s) + \cdots + u_n(s)$ is upper-semicontinuous and $u_i(s_i, s_{-i})$ is bounded and weakly lower-semicontinuous in s_i for each s_{-i}.*[7] *Then, G has a Nash Equilibrium (possibly in mixed strategies).*

[7] A function f on real numbers is upper-semicontinuous if for any x and $\varepsilon > 0$, there exists $\delta > 0$ such that $f(x') < f(x) + \varepsilon$ whenever $|x' - x| < \delta$. It is called lower-semicontinuous, if we use $f(x') > f(x) - \varepsilon$ instead.

5.9.2 Upperhemicontinuity of Nash Equilibrium

The best-response correspondence is upperhemicontinuous in parameters when the payoffs are continuous and the domain is compact. This is called the *Maximum Theorem*. In that case, in optimization problems, the limits of the solutions is a solution to the optimization problem in the limit. One can then find *a* solution by considering approximate problems and taking the limit. There can be other solutions in the limit. Nash equilibrium (like many other solution concepts) inherits these properties of the best-response correspondence. I will next establish this result.

Consider a compact metric space X of some payoff-relevant parameters. Fix a set N of players and set S of strategy profiles, where S is again a compact metric space. The utility function of each player i depends on x as well as s. That is, $u_i : S \times X \to \mathbb{R}$. The utility functions are assumed to be continuous (both in strategies and the parameters). It is also assumed that the parameter value x is known. Write $NE(x)$ and $PNE(x)$ for the sets of all Nash equilibria and all pure Nash equilibria, respectively, of game $(N, S, u(\cdot; x))$ in which it is common knowledge that the parameter value is x.

Theorem 5.6. *If S is compact and each u_i is continuous (in (s, x)), then the correspondences NE and PNE are compact-valued, and upperhemicontinuous.*

Proof. I will prove the result for NE; the result for PNE is straightforward. It suffices to show that the graph of NE is closed. To this end, take any sequence $(x^m, \sigma^m) \to (x, \sigma)$ with $\sigma^m \in NE(x^m)$ for each m. Suppose that $\sigma \notin NE(x)$. Then,

$$u_i(s_i, \sigma_{-i}, x) > u_i(\sigma_i, \sigma_{-i}, x)$$

for some $i \in N$ and $s_i \in S_i$. One can also show that $u(\sigma, x)$ is continuous in (σ, x). Since $(x^m, \sigma^m) \to (x, \sigma)$, this implies that

$$u_i\left(s_i, \sigma_{-i}^m, x^m\right) > u_i\left(\sigma_i^m, \sigma_{-i}^m, x^m\right)$$

for some large m, showing that $\sigma^m \notin NE(x^m)$—a contradiction. $\square$