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Auctions: Revenue Equivalence, The Revelation Principle, and VCG

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Review: Types of Auctions

- First price auction: The 1^{st} highest bidder gets the good
- Second price auction: The 2^{nd} highest bidder gets the good (the BNE is to bid your own valuation!)
- All-pay auction: everyone must pay to become a player (bidder) in the auction
- Forward auction: multiple buyers, single seller
- Reverse auction: multiple sellers, single buyer
- Wallet auction: "sealed-box auction"
  - no bidder knows other bidders' bids; only the auctioneer knows
  - auctioneer will distribute good so to maximize social utility

A More Formal Definition of General Auction

A general auction can be represented as a tuple $(B_1, ..., B_n, x, t)$, where there are 3 main components:

- bidding sets: $B_1, ..., B_n$
- allocation rule $x = (x_1, ..., x_n) : B_1 \times ... \times B_n \to \Delta$
  - Δ is a vector that sums to *at most* 1
  - We can think of Δ as recording the probability for each player to get the good
- transfer/payment rule $t = (t_1, ..., t_n) : B_1 \times ... \times B_n \to \mathbb{R}^n_+$

If bidder i bids $b_i \in B_i$, then she gets the good with probability $x_i(b_1, ..., b_n)$ and pays $t_i(b_1, ..., b_n)$.

Truthful bidding?

- Under certain conditions, there may be an incentive to NOT bid truthful values
  - First-price auction: underbid $b_i = \frac{n-1}{n} v_i$
  - Second-price auction: bid truthful value $b_i = v_i$
- Which of these 2 formats will generate more revenue for the auctioneer?

Truthful bidding?

- Both of those auction formats generate the same expected revenue for the auctioneer! (this result is known as the revenue equivalence theorem)
- Hacks from probability theory: when n values are uniformly randomly distributed on [0,1], the expected value of the k^{th} largest value is
$$E[v_{k^{th}}] = \frac{n+1-k}{n+1}$$
- So: in a first-price auction, the highest bidder pays $\frac{n-1}{n}\mathbb{E}(v_i) = \frac{n-1}{n}E[v_{1^{st}}] = \frac{n-1}{n} \times \frac{n}{n+1} = \frac{n-1}{n+1}$
- In a second-price auction, the highest bidder pays $E[v_{2^{nd}}] = \frac{n-1}{n}$

The Revenue Equivalence Theorem

The Expected Revenue (of the auctioneer) in symmetric Bayesian Nash Equilibria of the first- and second- price auction are equal to

$$ER = n \int \mathbb{E}[v_{-i,\max} | v_{-i,\max} \leq v_i] F^{n-1}(v_i) f(v_i) dv_i,$$

where $v_{-i,\max} = \max_{j \neq i} v_j$.

Motivation

- So far, we already know that different auction formats can generate the same revenue
- the central question: **How do we design the auction format so that it gives the auctioneer maximum revenue?** This seems very hard!
    - There are infinitely many ways to design auctions...
    - ...many of which also have lots of equilibria!

The Revelation Principle

- **For auction and an equilibrium in that auction, there exists an equivalent auction format where people bid truthfully and the expected utilities of bidders and the auctioneer remain the same as the old auction.**
- To show this, we need some definitions...

Some Definitions

- As a reminder: for player i, B_i is the set of all possible bids i can cast and v_i is the set of possible valuations player i can have.
- A general auction is *direct* if $B_i = V_i$ for each i; that is, each player i needs to report their valuation.
- A *strategy* in a direct auction is a function $b_i : V_i \to V_i$, where $b_i(v_i)$ is the valuation that player i reports when her true valuation is v_i.
- b_i is called a *truthful strategy* if $b_i(v_i) = v_i$.

Proof Idea

- Consider arbitrary auction $(B_1, ..., B_n; x, t)$ and an arbitrary BNE $(b_1^*, ..., b_n^*)$ of that auction.
- Assume that the bidders have valuations $v_1, ..., v_n$.
- The allocation and transfer will be, respectively:

$$\text{allocation:} x(b_1^*(v_1), ..., b_n^*(v_n))$$
$$\text{transfers:} t(b_1^*(v_1), ..., b_n^*(v_n))$$

We refer those as the old auction and the old equilibrium.

Proof Idea

- We would now like to get the same outcome with a new auction and a new equilibrium: specifically, this new auction has to be a direct auction where truthful bidding is a BNE.
- Here is the proof idea:
  1. The auctioneer just asks each player to report her valuation.
  2. Based on $v_1', \ldots, v_n'$, i.e. the valuations players reported, the auctioneer calculates the bids $b_1^*(v_1'), \ldots, b_n^*(v_n')$ using the b^* from the old equilibrium function.
  3. Now the new allocation and transfers are $\hat{x}(v_1', \ldots, v_n') = x(b_1^*(v_1'), \ldots, b_n^*(v_n'))$ and $\hat{t}(v_1', \ldots, v_n') = t(b_1^*(v_1'), \ldots, b_n^*(v_n'))$.

Proof Idea

- Now we have to show that for each player i, it is actually the best choice to report their true valuation, i.e., $v_i' = v_i$ for all i.
- Suppose bidder i is considering to report any $v_i' \neq v_i$.
- Then she gets the same allocation and transfer as she does in the old auction as if she has placed a bid $b_i^*(v_i')$.
- However, this can't be profitable, since it was already established that b^* is a BNE in the old auction.
- QED!

Vickrey-Clarke-Groves Auctions

In the advertisement auction context:

- There are m slots, with click-through rates $\alpha_1 > \alpha_2 > \alpha_3 > ... > \alpha_n$
- There are n bidders, each bid b_n simultaneously
- The highest bidder gets slot with click-through rate α_1 and pays the *externality* it exerts on other people
- As it turns out (see posted notes): bidding truthfully is the optimal strategy for each player

Externality

- Bidder i's externality is the difference in result it has caused others
- For example: WLOG, let's say $b_1 > b_2 > ... > b_n$, i.e. the first bidder's bid b_1 is the highest. Now bidder 1 gets slot with α_1, bidder 2 gets slot with α_2, ..., so on.
- What if bidder 1 didn't participate in this auction?

Externality

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- What if bidder 1 didn't participate in this auction?
- Answer: bidder 2 will get the slot with α_1 click-through rate, bidder 3 will get the slot with α_2 click-through rate, ..., so on.

Externality

bidder	1	2	3	...	n
slot value (no bidder 1)	n/a	α_1	α_2	...	α_{n-1}
slot value (has bidder 1)	α_1	α_2	α_3	...	α_n
difference	n/a	$\alpha_1 - \alpha_2$	$\alpha_2 - \alpha_3$	...	$\alpha_{n-1} - \alpha_n$

The externality bidder 1 has to pay is:

$$b_2(\alpha_1 - \alpha_2) + b_3(\alpha_2 - \alpha_3) + ... + b_n(\alpha_{n-1} - \alpha_n).$$

Written more compactly: the j^{th} player has to pay

$$\sum_{k=j+1}^{n} b^{(k)}(\alpha_{k-1} - \alpha_k)$$

Exercise 20.3, simplified

There are 2 bidders, i and j, trying to decide who should get a good. They have valuation $v_i, v_j \in [0,1]$ (uniformly and independently distributed) over the good, and each person only knows his/her own valuation.
They simultaneously bid b_i and b_j and the highest bidder wins the good; then, the highest bidder i will get the good and pay the lowest bidder j his/her bid b_j. If there is a tie, the allocation is decided by a fair coin toss.
Find all strictly increasing and continuously differentiable symmetric BNEs of this game.

Solution

Step-by-step walk-through.

1. We assume a BNE b that is strictly increasing and differentiable. WLOG, we simulate on player i's experience.
2. Assume that player i's true valuation of the good is v_i, but player i bids some $b(w)$.
   - A bid $b(w)$ implies that i's valuation of the good is w. We will later plug in $v_i = w$, but not until we have written down the FOC.

Solution

3. Let's write down player i's payoffs in winning/losing bid scenarios:
   - If $b(w) \geq b_j = b(v_j)$ (this implies $0 \leq v_j \leq w$), player i wins the bid, then they pay player j $b(v_j)$ and their utility would be $v_i - b(v_j)$.
   - If $b(w) < b_j$, player i loses the bid, then player j pays them $b(w)$. So their utility would be $b(w)$.
4. Therefore, the expected utility of player i bidding $b(w)$ is

$$U(w) = \int_0^w v_i - b(v_j)dv_j + \int_w^1 b(w)dv_j$$

$$= v_i w + b(w)(1-w) - \int_0^w b(v_j)dv_j$$

Solution

5. Now, take the derivative of $U(w)$ with respect to w, we get

$$\begin{aligned} U'(w) &= v_i + b'(w)(1-w) + (-1)b(w) - b(w) \\ &= v_i - 2b(w) + b'(w)(1-w) \end{aligned}$$

6. since player i plays $b(v_i)$ as best response in this BNE assumption, we substitute $b(v_i) = b(w)$ and get the following FOC:

$$\begin{aligned} U'(v_i) = v_i - 2b(v_i) + b'(v_i)(1-v_i) = 0 \\ b'(v_i)(v_i - 1) + 2b(v_i) = v_i \end{aligned}$$

Solution

$$b'(v_i)(v_i - 1) + 2b(v_i) = v_i$$

This differential equation above has solution form $Ax + B$. so we solve instead

$$A(v_i - 1) + 2(Av_i + B) = v_i$$

Rearranging terms, we get

$$(3A)v_i + (2B - A) = 1 \times v_i + 0$$

since the coefficients of each of those terms must match, we have $3A = 1$ and $2B - A = 0$. In conclusion, the BNE we seek is

$$\boxed{b^*(v_i) = \frac{1}{3}v_i + \frac{1}{6}}.$$