18.650. Fundamentals of Statistics
Fall 2023. Problem Set 1

Due Monday, Sep. 18

Solve all problems. No justification necessary

> The goal of this PSet is to help you refresh on probability. You should be able to answer all these questions. If you think there are multiple correct answers to one question, pick a single one. Grading is "all or nothing". Each question is 3 points (there are 33 questions so 1 point will be added to your total).
>
> New instructions

Let X be a random variable with pmf given by

$$\mathbb{P}(X = k) = \frac{c\lambda^k}{k!}, k = 0, 1, 2, \dots$$

for some $\lambda > 0$.

1. What is the value of c? A. 1 B. λ **C. $e^{-\lambda}$** D. e^{λ}
2. What is $\mathbb{E}[X]$? A. 1 **B. λ** C. $e^{-\lambda}$ D. e^{λ}
3. What is $\mathsf{Var}[X]$? A. 1 **B. λ** C. $e^{-\lambda}$ D. e^{λ}

Let X be a Gaussian[1] random variable with mean $\mu > 0$ and variance μ^2.

4. What is $\mathbb{E}[X]$?
   A. 0 **B. μ** C. μ^2 D. 2μ
5. What is $\mathbb{E}[X^2]$?
   A. μ^2 B. 2μ **C. $2\mu^2$** D. $\mu^2 + \sigma^2$
6. What is $\mathbb{E}[X^3]$?
   A. μ^3 B. $3\mu^3$ **C. $4\mu^3$** D. $8\mu^3$
7. What is $\mathsf{Var}[X^2]$?
   A. $6\mu^2$ **B. $6\mu^4$** C. $10\mu^2$ D. $10\mu^4$ E. $6\mu^2 + 10\mu^4$
8. What is $\mathbb{P}(X > 0)$ in terms of the CDF Φ of the standard Gaussian distribution?
   A. $1 - \Phi(1)$ **B. $\Phi(1)$** C. $\Phi(-\mu)$ D. $1 - \Phi(1/\mu)$

Let X be a random variable such that

$$X = \begin{cases} 1 & \text{with probability } p \\ -1 & \text{with probability } 1 - p \end{cases}$$

for some $p \in [0, 1]$.

[1] Gaussian=normal

9. What is $\mathbb{E}[X]$?
   A. $-p$ B. p C. $1-2p$ **D.** $2p-1$
10. What is $\mathsf{Var}[X]$?
    A. $p(1-p)$ B. $4p-p^2$ **C.** $4p(1-p)$ D. $4p^2(1-p)$
11. For what p is $\mathsf{Var}[X]$ maximized?
    A. 1 B. 0 **C.** 0.5 D. $1/\sqrt{2}$
12. What is $\mathbb{E}[X^k]$?
    A. p^k B. $p^k-(1-p)^k$ C. $p(-1)^k+(1-p)$ **D.** $p+(1-p)(-1)^k$

Let X be a uniform random variable in the interval $[2,8]$.

13. What is $\mathbb{E}[X]$?
    A. 2 B. 3 **C.** 5 D. 8
14. What is $\mathsf{Var}[X]$?
    A. 2 **B.** 3 C. 5 D. 8
15. What[2] is $\mathbb{P}[\log(X)\leq 1]$ approximately?
    A. .12 B. 0.8 C. $-.1$ D. 0

Let X be an exponential[3] random variable with parameter 3 and Y be a Poisson random variable with parameter 2. Assume that X and Y are independent.

16. What is $\mathbb{E}[X^2+Y^2]$?
    A. 12 B. 23 **C.** 24 D. 36
17. What is $\mathbb{E}[X^2Y]$?
    A. 12 B. 23 C. 24 **D.** 36
18. What is $\mathsf{Var}(2X+3Y)$?
    A. 24 B. 34 C. 44 **D.** 54

Let X,Y be two independent standard Gaussian random variables.

19. What is $\mathbb{E}[X^2Y]$?
    A. 0 B. 1 C. 2 D. 3
20. What is $\mathsf{Var}(X+Y)$?
    A. 0 B. 1 **C.** 2 D. 3
21. What is $\mathsf{Var}(XY)$?
    A. 0 **B.** 1 C. 2 D. 3
22. What is $\mathsf{cov}(X,X+Y)$?
    A. 0 **B.** 1 C. 2 D. 3
23. What is $\mathsf{cov}(X,XY)$?
    A. 0 B. 1 C. 2 D. 3

[2] all logs are natural unless specified otherwise
[3] We use the convention from AoS for the parameter of an exponential distribution.

Let $X \geq 0$ be a positive random variable such that $\mathbb{E}[X] = \lambda$.

24. Which is correct?
    A. $\mathbb{E}[1/X] = 1/\lambda$ **B.** $\mathbb{E}[1/X] \geq 1/\lambda$ C. $\mathbb{E}[1/X] \leq 1/\lambda$

Let X be an exponential random variable with parameter $1/2$ that models the lifetime (in years) of a lightbulb.

25. What is (approximately) the probability that the lightbulb will last at least 2 years?
    A. 0.002 **B.** 0.018 C. 0.180 D. 0.810
26. Given that the lightbulb has already lasted for at least 3 years, what is (approximately) the probability that it will last for two more years?
    A. 0.002 **B.** 0.018 C. 0.180 D. 0.810

Let X and Y be two random variables such that X is a Bernoulli random variable with parameter $p \in (0, 1)$, and $Y^2 + 2XY = 3X^2$ almost surely.

27. What is $\mathbb{E}[Y]$?
    A. 0 B. $-3p$ C. X D. $-3X$ **E. Some number in** $[-3, 1]$

Let $X_1, \ldots, X_n$ be i.i.d with mean μ and variance σ^2.

28. What is $\mathbb{E}[\sum_{i=1}^n X_i]$? A. μ B. $n\sigma$ **C.** $n\mu$ D. σ
29. What is $\mathbb{E}\big[\big(\sum_{i=1}^n X_i\big)^2\big]$? A. $n^2\mu^2$ B. $n\sigma^2$ C. $n\mu$ **D.** $n\sigma^2 + n^2\mu^2$
30. What is $\mathsf{Var}[\sum_{i=1}^n X_i]$? A. $n^2\sigma^2$ **B.** $n\sigma^2$ C. $n\sigma^2 + n^2\mu^2$ D. $n\mu$
31. What is $\mathbb{E}[\frac{1}{n}\sum_{i=1}^n X_i]$? A. σ B. $n\sigma^2$ C. $n\mu$ **D.** μ
32. What is $\mathsf{Var}[\frac{1}{n}\sum_{i=1}^n X_i]$? A. μ **B.** σ^2/n C. σ^2 D. $n\mu$

Let $X_1, \ldots, X_n$ be i.i.d with mean μ and variance σ^2.

33. Which sequence a_n and b_n is such that

$$a_n \sum_{i=1}^n X_i - b_n \to N(0, 1)$$

A. $a_n = \sigma/\sqrt{n}, b_n = n\mu$
B. $a_n = \sigma/\sqrt{n}, b_n = \sqrt{n}\mu$
C. $a_n = 1/(\sigma\sqrt{n}), b_n = (\mu/\sigma)\sqrt{n}$
D. $a_n = 1/(\sigma\sqrt{n}), b_n = \mu/\sqrt{n}$
E. $a_n = \sqrt{n}/\sigma, b_n = \mu\sigma\sqrt{n}$