Solutions to Problem Set 3
14.12 Fall 2023

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Problem 1

Part 1

Here is the normal form where best responses are underlined.

	Ll	Lr	Rl	Rr
A	$\underline{3},\underline{1}$	$\underline{3},\underline{1}$	0,0	0,0
B	0,0	0,0	$\underline{1},\underline{3}$	$\underline{1},\underline{3}$
C	$3/2,\underline{3/2}$	1/2,1/2	$\underline{1}$,1	0,0

The pure strategy Nash Equilibria are {A,Ll},{A,Lr}, {B,Rl} and {B,Rr}.

Part 2

First, let's go over how to approach this problem and determine what kind of MSNE we are looking for. The outlined solution below looks for one where both players play a mixed strategy, but you can also look for one that where only player 1 plays a mixed strategy putting positive weight on more than 1 strategy.

First, observe that if P1 does not play C, then the payoffs from Ll and Lr are same for P2. The payoffs from Rl and Rr are also same. Lets look for an equilibrium where P1 does not play C. It is easy to see that randomizing between Lr and Rr makes C a worse strategy for P1.

Thus, we check for a MSNE where P1 plays A with probability p_A, plays B with probability $1 - p_A$, P1 plays Lr with probability p_L, plays Rr with probability $1 - p_L$. The indifference conditions for players to mix are

$$p_A = 3(1 - p_A)$$
$$3p_L = 1 - p_L$$

Thus $p_A = 3/4$ and $p_L = 1/4$ makes both players indifferent under the conjectured strategies. Lastly, observe that payoff of P1 from playing A or B is $3(1/4)$ which is greater than payoff of playing C, which is $1/2(1/4)$. Checking this for P2 is trivial since the payoffs of Lr and Ll and Rr and Rl are equal.

Note: there are many solutions to this question - as long as your calculations justify that it is a MSNE, we give full credits.

Problem 2

We have the following normal form (best responses are underlined):

	L	C	R
T	1,$\underline{3}$	$\underline{1}$,0	$\underline{3}$, 2
M	$\underline{3}$,0	$\underline{1}$, $\underline{3}$	$\underline{3}$,2
B	0,$\underline{4}$	-1,0	2, 3

This game has one PSNE: $\boxed{\{M, C\}}$.

Next, we want to find the MSNE. The full answer contains 3 continuums:

- $\{\frac{1}{3}T + \frac{2}{3}M, p_cC + (1 - p_c)R : 0 \leq p_c \leq 1\}$;
- $\{p_tT + (1 - p_t)M : \frac{1}{3} \leq p_t \leq \frac{2}{3}, R\}$; and
- $\{p_tT + (1 - p_t)M : 0 \leq p_t \leq \frac{1}{3}, C\}$.

The first step is to eliminate any strictly dominated strategies for any player so to simplify our calculations. We see that for player 1, B is strictly dominated, so we can rule this out.

Now, to find all MSNE, we must consider all potential subsets of strategies for each player where it would be rational to put positive weights on them. We consider 7 cases and highlight every case where a viable MSNE is found.

Case 1: Player 1 plays T and M with positive probability, and Player 2 puts positive weight on L, C, R. To make player 2 indifferent, p_t must satisfy the following equations:

$$3p_t = 3(1 - p_t) = 2.$$

This is clearly not possible. So, there is not a MSNE where player 2 is indifferent between playing L, C, and R all with strictly positive probability.

Case 2: Player 1 plays T and M with positive probability, and Player 2 mixes between L and C (putting weight zero on R). We have the following indifference conditions:

$$3p_t = 3(1-p_t) \implies p_t = \frac{1}{2}$$

$$1 = 3p_l + 1(1-p_l) \implies p_1 = 0$$

This means that Player 2 cannot put any positive weight on L: otherwise, Player 1 will have a higher payoff from playing M.

Now, we must check that this is a Nash equilibrium, and that no players have any incentive to deviate. Playing according to these strategies gives the following payoffs:

$$u_1(\{\frac{1}{2}, \frac{1}{2}, 0\}, \{0, 1, 0\}) = 1$$
$$u_2(\{\frac{1}{2}, \frac{1}{2}, 0\}, \{0, 1, 0\}) = 1.5$$

We know that B is strictly dominated for Player 1, so switching to B cannot be a profitable deviation.

For Player 2, note that if Player 2 switches to R, this yields a guaranteed payoff of 2 - this is better than the payoff to Player 2 under the strategy that we have found, so, this is not a MSNE.

Case 3: Player 1 plays T and M with positive probability, and Player 2 mixes between C and R (putting weight zero on L). We have the following indifference conditions:

$$3(1-p_t) = 2 \implies p_t = \frac{1}{3}$$

$$p_c + 3(1-p_c) = p_c + 3(1-p_c) \implies p_c \in [0, 1]$$

Note that the payoffs to player 1 from T and M are the same no matter how player 2 mixes between C and R. Playing according to these strategies gives the following payoffs:

$$u_1(\{\frac{1}{3}, \frac{2}{3}, 0\}, \{0, p_c, 1-p_c\}) = p_c + 3(1-p_c)$$

$$u_2(\{\frac{1}{3}, \frac{2}{3}, 0\}, \{0, p_c, 1-p_c\}) = 2$$

Again, Player 1 will have no profitable deviation. We check that switching to L is not a profitable deviation for player 2:

$$u_2(\{\frac{1}{3}, \frac{2}{3}, 0\}, \{1, 0, 0\}) = 1 < 2$$

So, neither player has a profitable deviation, and we found a MSNE.

Case 4: Player 1 plays T and M with positive probability, and Player 2 mixes between L and R (putting weight zero on C). We have the following indifference conditions:

$$3(p_t) = 2 \implies p_t = \frac{2}{3}$$

Also note that for any p between $\frac{1}{3}$ and $\frac{2}{3}$, Player 2 prefers to play R over L (i.e., $3p_t < 2$ in this range).

$$p_l + 3(1-p_l) = 3p_l + 3(1-p_l) \implies p_l = 0$$

So, to make Player 1 indifferent between U and M, Player 2 must always play R. Playing according to these strategies gives the following payoffs:

$$u_1(\{\frac{2}{3}, \frac{1}{3}, 0\}, \{0, 0, 1\}) = 3$$
$$u_2(\{\frac{2}{3}, \frac{1}{3}, 0\}, \{0, 0, 1\}) = 2$$

Again, Player 1 has no incentive to deviate, because

$$u_1(\{0, 0, 1\}\{0, 0, 1\}) = 2 < 3.$$

For Player 2, we check the following condition, and see that Player 2 also has no incentive to deviate, because

$$u_2(\{\frac{2}{3}, \frac{1}{3}, 0\}, \{0, 1, 0\}) = 1 < 2.$$

Case 5: Player 1 plays T and M with positive probability, and Player 2 plays L with probability 1, and other strategies with probability 0.

Here, we cannot make Player 1 indifferent if Player 2 only plays L (Player 1 will always prefer to play M). Thus, we cannot find an MSNE this way.

Case 6: Player 1 plays T and M with positive probability, and Player 2 plays C with probability 1 and other strategies with probability 0.

Here, Player 1 is indifferent between T and M (either way will achieve a payoff of 1). To make Player 2 indifferent between deviating to L and R, we need the following to be true:

$$\begin{aligned} 3p_t \leq 3(1-p_t) &\implies p_t \leq \frac{1}{2} \\ 2 \leq 3(1-p_t) &\implies p_t \leq \frac{1}{3} \\ &\implies 0 \leq p_t \leq \frac{1}{3} \end{aligned}$$

For p_t in this range, deviating to L will give a payoff of ≤ 2, and deviating to R will give a payoff of 2 - so, the indifference conditions hold, and player 2 has no incentive to deviate.

Case 7: Player 1 plays T and M with positive probability, and Player 2 plays R with probability 1 and other strategies with probability 0.

Again, Player 1 is indifference between player T and M here. So, for player 2 to be willing to always play R, the following conditions must hold:

$$\begin{aligned} 2 \geq p_t(3) &\implies p_t \leq \frac{2}{3} \\ 2 \geq (1-p_t)3 &\implies p_t \geq \frac{1}{3} \end{aligned}$$

Note that for some of these conditions, we found equilibria (or potential equilibria) where Player 2 plays a strategy with probability 1. We observe that there can be no MSNE where Player 1 plays U with probability 1, and Player 2 mixes between $\{L, C\}$, $\{L, R\}$,or $\{C, R\}$ - one way to see this is that Player 2 has a unique best response to U and any mixing would be strictly dominated by playing L with probability 1. A similar logic follows for M.

Problem 3

Part 1

Remember, from the last problem set, we have the following strategic form where the best responses are red.

	Buy	Pass
Sell-Sell	$p, \frac{1+\delta}{2}v_I - p$	$\frac{1+\delta}{2}v_E, 0$
Sell-Keep	$\frac{p+\delta v_E}{2}, \frac{v_I - p}{2}$	$\frac{1+\delta}{2}v_E, 0$
Keep-Sell	$\frac{v_E+p}{2}, \frac{\delta v_I - p}{2}$	$\frac{1+\delta}{2}v_E, 0$
Keep-Keep	$\frac{1+\delta}{2}v_E, 0$	$\frac{1+\delta}{2}v_E, 0$

Thus, PSNE are $\boxed{\text{(Keep-Sell,Pass) and (Keep-Keep,Pass)}}$. Note that conditional on Buy, the executive will always prefer to sell when the production is delayed and keep when it is on track, which makes Pass the optimal choice of the investor.

Part 2

We will look for an equilibrium value p^* where the investor Buys and the executive plays any strategy that is not Keep-Keep. (We shorthand them as SS, SK, KS.) Specifically:

- for (SS, B): If (SS, B) is a PSNE, then $p > \max\{\frac{p+\delta v_E}{2}, \frac{p+v_E}{2}, \frac{v_E+\delta v_E}{2}\}$. Note that since $\delta < 1$ and $p > \delta v_E$, we have

$$\frac{p + \delta v_E}{2} < \frac{\delta v_E + v_E}{2} < \frac{p + v_E}{2}.$$

  We now need $\frac{p+v_E}{2} < p \implies v_E < p$, which imposes a contradiction because we assumed $v_E > \frac{1+\delta}{2} v_I > p$. Thus no p exists such that (SS,B) is a PSNE.

- for (SK, B): If (SK, B) is a PSNE, then $\frac{p+\delta v_E}{2} > \max\{p, \frac{p+v_E}{2}, \frac{v_E+\delta v_E}{2}\}$. This is not possible because $\delta < 1 \implies \frac{p+\delta v_E}{2} < \frac{p+v_E}{2}$.

- for (KS, B): If (KS, B) is a PSNE, then $\frac{p+v_E}{2} > \max\{p, \frac{p+\delta v_E}{2}, \frac{v_E+\delta v_E}{2}\}$. We can derive that $\delta v_E < p < v_E$. Furthermore, since $\delta v_I > p$, we further deduce that

$$\begin{aligned} p &> \delta v_E > \frac{\delta + \delta^2}{2} v_I \\ p &< \delta v_I = \frac{2\delta}{2} v_I < \frac{1 + \delta}{2} v_I < v_E. \end{aligned}$$

  Thus, our desired range is $p^* \in (\delta v_E, \delta v_I)$. With price p^*, (Keep-Sell, Buy) is a PSNE and trade may happen.

Problem 4

Parts 1 & 2

Players are $N = \{1, 2\}$. The strategy set for player i is $q_i = (q_{i1}, q_{i2}) \in \mathbb{R}^2_+$. The utility function for firms are

$$u_1(q_1, q_2) = q_{11}(\theta_1 - q_{11} - q_{21}) + q_{12}(1-\tau_2)(\theta_2 - q_{12} - q_{22}) - (q_{11} + q_{12})c_1 \tag{1}$$

$$u_2(q_1, q_2) = q_{21}(1-\tau_1)(\theta_1 - q_{11} - q_{21}) + q_{22}(\theta_2 - q_{12} - q_{22}) - (q_{21} + q_{22})c_2 \tag{2}$$

Assuming an interior solution, to calculate the best responses, we calculate the FOC and equate them to 0.

$$\begin{aligned}
\frac{\partial u_1}{\partial q_{11}} &= \theta_1 - 2q_{11} - q_{21} - c_1 = 0 \implies q_{11} = \frac{\theta_1 - q_{21} - c_1}{2} \\
\frac{\partial u_1}{\partial q_{12}} &= (1-\tau_2)(\theta_2 - 2q_{12} - q_{22}) - c_1 = 0 \implies q_{12} = \frac{(1-\tau_2)(\theta_2 - q_{22}) - c_1}{2(1-\tau_2)} \\
\frac{\partial u_2}{\partial q_{21}} &= (1-\tau_1)(\theta_1 - 2q_{21} - q_{11}) - c_2 = 0 \implies q_{21} = \frac{(1-\tau_1)(\theta_1 - q_{11}) - c_2}{2(1-\tau_1)} \\
\frac{\partial u_2}{\partial q_{22}} &= \theta_2 - 2q_{22} - q_{12} - c_2 = 0 \implies q_{22} = \frac{\theta_1 - q_{12} - c_1}{2}
\end{aligned} \tag{3}$$

We aren't done yet - notice that the expression for q_{11} contains q_{22}, but we want the expressions to only contain constants, i.e. $c_1, c_2, \tau_1, \tau_2, \theta_1, \theta_2$. We will plug in the value of q_{22} into the expression of q_{11} (similar for q_{21} and q_{12}. After some long arithmetic derivations, we get

$$\begin{aligned}
q^*_{11} &= \frac{\theta_1 - 2c_1}{3} + \frac{c_2}{3(1-\tau_1)} \\
q^*_{12} &= \frac{\theta_2 + c_2}{3} - \frac{2c_1}{3(1-\tau_2)} \\
q^*_{21} &= \frac{\theta_1 + c_1}{3} - \frac{2c_2}{3(1-\tau_1)} \\
q^*_{22} &= \frac{\theta_2 - 2c_2}{3} + \frac{c_1}{3(1-\tau_2)}.
\end{aligned} \tag{4}$$

Part 3

Note that the consumer surplus at country 1 is

$$\frac{(\theta_1 - P_1)^2}{2} = \frac{\left(2\theta_1 - c_1 - \frac{c_2}{1-\tau_1}\right)^2}{18} \tag{5}$$

Consumer surplus is decreasing in τ_1 - this is very intuitive since it makes firm 1 act like a monopolist by removing the competition.

To analyze firm 1's profit, one can derive it and show explicitly that it increases with τ_1. But here is a quicker argument: We have already derived that q^*_{21} is decreasing in τ_1, thus, after an increase in τ_1, if firm 1 does not change its production, it will get a strictly higher profit (since they sell same quantity with higher price, due to decreased production of firm 2). In equilibrium they are best-responding, so they must get an even higher profit in the case they also adjust their production, proving the fact that their profit is increasing in τ_1.

Tax revenue is given by

$$\tau_1 P_1 q^*_{21} = \tau_1 \left(\frac{\theta}{3} + \frac{c_1}{3} + \frac{c_2}{3(1-\tau_1)}\right)\left(\frac{\theta_1}{3} + \frac{c_1}{3} - \frac{2c_2}{3(1-\tau_1)}\right) \tag{6}$$

Taking the second derivative, one can show that this is a strictly concave function, though this is not necessary. Moreover, note that if $\tau_1 = 0$, then there is no tax, hence no tax revenue. If τ_1 is high enough that firm 2 does not produce, then again there is no tax collected and thus no tax revenue. [1] However, in the middle, there is positive tax and positive production, so the government collects tax revenue. The concavity of the revenue function above than gives us the fact that tax is maximized at a point. Google Laffer Curve if you are want to learn more.

[1] Apart from the knife-edge case where firm 2 just stops the production, this will violate our assumptions, but for any higher τ_1, the solution is equal to the solution at the knife-edge case.