Chapter 4

Rationalizability

A player is said to be rational if she maximizes the expected value of her utility function, as described in the game. The previous chapter explored the implications of rationality. This was captured by dominance. In natural strategic environments, this often yields weak predictions. Moreover the games in which dominance alone leads to a sharp prediction (e.g. the games with a dominant-strategy equilibrium) are not interesting for game theory because in such a game each player's decision can be analyzed separately without requiring a game theoretical analysis.

Nevertheless, in the definition of a game, one assumes much more than the rationality of the players. One further assumes that it is common knowledge that the players are rational. That is, everybody is rational; everybody knows that everybody is rational; everybody knows that everybody knows that everybody is rational ... up to infinity. If some of these assumptions fail, then one must consider a different game, the game that reflects the failure of those assumptions. This chapter explores the implications of the common knowledge of rationality. These implications are precisely captured by a solution concept called *rationalizability*, which is equivalent to iterative elimination of strictly dominated strategies. In this way, rationalizability precisely captures the implications of the assumptions embedded in the definition of the game.

4.1 Definition and Illustration

It is useful to illustrate the solution concept on the leading example of the previous chapter (3.2):

$$\begin{array}{c|c|c|} & L & R \\ \hline T & 2,0 & -1,1 \\ \hline M & 0,10 & 0,0 \\ \hline B & -1,-6 & 2,0 \\ \hline \end{array} \tag{4.1}$$

As it has been shown there, strategy M is strictly dominated (by a mixture of T and B), and hence it cannot be a best response to any belief. Hence, rationality of Player 1 implies that Player 1 does not play M. No other strategy is strictly dominated. For example, for Player 2, his both strategies can be a best reply. If he thinks that Player 1 is not likely to play M, then he must play R, and if he thinks that it is very likely that Player 1 will play M, then he must play L. Hence, rationality of Player 2 does not put any restriction on his behavior. But, what if he thinks that it is very likely that Player 1 is rational (and that her payoff are as in (4.1))? In that case, since a rational Player 1 does not play M, he must assign a very small probability to Player 1 playing M. In fact, if he knows that Player 1 is rational, then he must be sure that she will not play M. In that case, being rational, he must play R. In summary, *if Player 2 is rational and he knows that Player 1 is rational, then he must play R.*

Notice that we first eliminated all of the strategies that are strictly dominated (namely M), then taking the resulting game, we eliminated again all of the strategies that are strictly dominated (namely L). This is called *twice iterated elimination of strictly dominated strategies*. In general, if a player is rational and knows that the other players are also rational (and the payoffs are as given), then she must play a strategy that survives twice iterated elimination of strictly dominated strategies.

Under further rationality assumptions, one can further iteratively eliminate strictly dominated strategies (if there remains any). In example (4.1), recall that rationality of Player 1 requires her to play T or B, and knowledge of the fact that Player 2 is also rational does not put any restriction on her behavior—as rationality itself does not restrict Player 2's behavior. Now, in addition, assume that Player 1 not only knows Player 2 is rational but also knows that Player 2 knows that Player 1 is rational. Then, as the above analysis shows, Player 1 must know that Player 2 will play R. In that case, being rational, she must play B.

This analysis yields a mechanical procedure to analyze games, *k-times Iterated Elimination of Strictly Dominated Strategies*: eliminate all the strictly dominated strategies and iterate this *k times*. This procedure is formally defined as follows.

Iterated Elimination of Strictly Dominated Strategies:

Initialization set $S_i^0 = S_i$ for each player i.

Elimination Round k For each $k = 1, 2, \ldots$ and for each player j, let S_j^{k-1} be the set of strategies of player j that survived the first $k-1$ rounds of elimination. For each player i and each strategy $s_i \in S_i$, eliminate strategy s_i if and only if there exists a mixed strategy σ_i (in the original game) with

$$u_i(\sigma_i, s_{-i}) > u_i(s_i, s_{-i}) \qquad \text{for all } s_{-i} \in S_{-i}^{k-1},$$

where $S_{-i}^{k-1} = \prod_{j \neq i} S_j^{k-1}$. Set

$$S_i^k = S_i^{k-1} \setminus \{s_i | s_i \text{ is eliminated in round } k\}$$

as the set of strategies that survives the first k rounds. Iterate.

Note that one considers only the strategies s_{-i} of other players that are available at any given round. In contrast, in looking for dominating strategies, one considers all possible strategies σ_i of player i, even those that are eliminated previously. When S_i is finite, one can ignore the previously eliminated strategies of player i, too.

Caution: Two points are crucial for the elimination procedure:

1. One must eliminate only *strictly* dominated strategies. One cannot eliminate a strategy if it is weakly dominated but not strictly dominated. For example, in the game

	A	B
A	$1,1$	$0,0$
B	$0,0$	$0,0$

(A, A) is a dominant-strategy equilibrium, but no strategy is eliminated because A does not strictly dominate B. After all, if it were known that (B, B) is being played, no player would have any incentive to deviate; Player 1 would play B knowing that Player 2 plays B, and Player 2 would play B knowing that Player 1 plays B. Then, one cannot rule out strategy B for any player using rationality assumptions as above.

2. One must eliminate the strategies that are strictly dominated by pure or mixed strategies. For example, in the game in (4.1), M must be eliminated although neither T nor B dominates M.

When there are only finitely many strategies, this elimination process must stop at some k. That is, at some k, there will be no dominated strategy to eliminate. In that case, iterating the elimination further would not have any effect.

Definition 4.1. The above elimination process that keeps iteratively eliminating all strictly dominated strategies until there are no strictly dominated strategies is called *Iterated Elimination of Strictly Dominated Strategies*; one eliminates indefinitely if the process does not stop. A strategy is said to be *rationalizable* if it survives iterated elimination of strictly dominated strategies. The set of all rationalizable strategy profiles is

$$S^\infty = \bigcap_{k=0}^{\infty} S^k.$$

A game is said to be *dominance-solvable* if each player has a unique rationalizable strategy.

When the set of strategies is finite, the procedure is as follows. Eliminate all the strictly dominated strategies. In the resulting smaller game, some of the strategies may become strictly dominated. Check for those strategies. If there is one, apply the procedure one more time to the smaller game. This continues until there is no strictly dominated strategy. Since there are only finitely many strategies, the process stops at some point. The remaining strategies are called rationalizable. The order of eliminations does not matter for the resulting outcome. For example, even if one does not eliminate a strictly dominated strategy at a given round, the eventual outcome is not affected by such an omission. In that case, it is also okay to eliminate a strategy whenever it is deemed to be strictly dominated. (When there are infinitely many strategies, the process does not necessarily stop at any round. In that case, the elimination continues indefinitely. Moreover, with infinitely many strategies, one must also consider domination by previously eliminated strategies.)

Iterated elimination of strictly dominated strategies is important for the following reason. *If* (1) *every player is rational,* (2) *every player knows that every player is rational,* (3) *every player knows that every player knows that every player is rational, ... and* (k) *every player knows that every player knows that ... every player is rational, then every player must play a strategy that survives k-times iterated elimination of strictly dominated strategies.* Rationalizability is an important solution concept because it characterizes the strategic implications of the assumptions embedded in the game. *If it is common knowledge that every player is rational (and the game is as described), then every player must play a rationalizable strategy. Moreover, any rationalizable strategy is consistent with the common knowledge of rationality.*

A general problem with rationalizability is that there may be too many rationalizable strategies, leading to a weak predictive power. For example, in the Battle of the Sexes game

	A	B
A	$2,1$	$0,0$
B	$0,0$	$1,2$

every strategy is rationalizable, and rationalizability does not rule out any outcome. (This is true in applications that make many common knowledge assumptions about the payoffs. If one replaces these assumptions with informational assumptions under which players face uncertainty about other players' payoffs and information, the set of rationalizable strategies tend to shrink, and this may lead to a unique rationalizable strategy in some games—see Chapter 18.)

4.2 Example: Beauty Contest

Consider an n-player game in which each player i has strategies $x_i \in [0, 100]$, and payoff

$$u_i(x_1, \ldots, x_n) = -\left(x_i - \frac{2}{3}\frac{x_1 + \cdots + x_n}{n}\right)^2.$$

Notice that, in this game, each player tries to play a strategy that is equal to the two-thirds of the average strategy, which is also affected by her own strategy. Each person

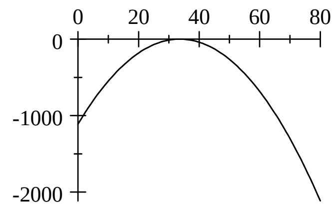

Figure 4.1: Utility as a function of x_i in beauty contest for fixed $X_{-i} = 50$ and $n = 80$.

is therefore interested in guessing the other players' average strategies, which depend on the other players' estimate of the average strategy.

First compute the best response function. Write $X_{-i} = \sum_{j \neq i} x_j / (n-1)$ for the average of the other players' strategies. The utility function of i can be written as

$$u_i(x_1, \ldots, x_n) = -\left(\frac{3n-2}{3n}\right)^2 \left(x_i - \frac{2n-2}{3n-2} X_{-i}\right)^2 \equiv f(x_i, X_{-i}).$$

It depends only on x_i and X_{-i}. Clearly, u_i is a quadratic function of x_i and is maximized at $x_i = BR(X_{-i})$ where

$$BR(X_{-i}) = \frac{2n-2}{3n-2} X_{-i}.$$

The function BR is called the best response function.[1] The utility function is plotted in Figure 4.1, as a function of x_i for a fixed X_{-i}: it increases up to $x_i = BR(X_{-i})$, which is around 33 in this case, and decreases thereafter.

One iteratively eliminates strictly dominated strategies as follows. First, since each strategy must be less than or equal to 100, the average X_{-i} cannot exceed 100. Conse-

[1] More generally, since u_i is concave in x_i, one can compute the best response by taking the derivative of $u_i(x_1, \ldots, x_n)$ with respect to x_i and set it equal to zero. (Equivalently, one can take the derivative of f and set it equal to zero.) That is, one can write

$$\frac{\partial u_i}{\partial x_i} = -2\left(\frac{3n-2}{3n}\right)^2 \left(x_i - \frac{2n-2}{3n-2} X_{-i}\right) = 0.$$

$BR(X_{-i})$ is the unique solution to this equation.

quently, any strategy $x_i > x^1$ is strictly dominated by x^1 where[2]

$$x^1 = BR(100) = \frac{2n-2}{3n-2}100.$$

To show that x_i is strictly dominated by x^1, fix any $x_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$ and show that

$$u_i(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n) < u_i\left(x_1, \ldots, x_{i-1}, x^1, x_{i+1}, \ldots, x_n\right). \tag{4.2}$$

Indeed, as shown in the previous paragraph, u_i is strictly decreasing in x_i when $x_i > BR(X_{-i})$. But, since $X_{-i} \leq 100$,

$$BR(X_{-i}) \leq BR(100) = x^1 < x_i.$$

Therefore, u_i is strictly decreasing between x^1 and x_i, and inequality (4.2) holds. Since x_{-i} is arbitrary, this further shows that every strategy $x_i > x^1$ is strictly dominated by x^1. Therefore, every strategy $x_i > x^1$ is eliminated in the first round.

On the other hand, each $x_i \leq x^1$ is a best response to some x_{-i} with

$$x_j = \frac{3n-2}{2n-2}x_i \qquad \text{for all } j \neq i.$$

Therefore, at the end of the first round the set of surviving strategies is $[0, x^1]$.

Now, suppose that at the end of round k, the set of surviving strategies is $\left[0, x^k\right]$ for some number x^k. By repeating the same analysis above with x^k instead of 100, we can conclude that at the end of round $k+1$, the set of surviving strategies is $\left[0, x^{k+1}\right]$ where

$$x^{k+1} = BR\left(x^k\right) = \frac{2(n-1)}{3n-2}x^k.$$

The solution to this equation with $x^0 = 100$ is

$$x^k = \left[\frac{2(n-1)}{3n-2}\right]^k 100.$$

Therefore, for each k, at the end of round k, a strategy x_i survives if and only if

$$0 \leq x_i \leq \left[\frac{2(n-1)}{3n-2}\right]^k 100.$$

[2] Here x^1 is just a real number, where superscript 1 indicates that we are in Round 1. I will sometimes use such superscripts to avoid confusion with player indices, which are denoted as subscripts. They should not be confused with power.

Since

$$\lim_{k\to\infty}\left[\frac{2(n-1)}{3n-2}\right]^k 100 = 0,$$

the only rationalizable strategy is $x_i = 0$.

Notice that the speed at which x^k goes to zero determines how fast we eliminate the strategies. If the elimination is slow (e.g., when $2(n-1)/(3n-2)$ is large), then many strategies are eliminated at very high iterations. In that case, predictions based on rationalizability will heavily rely on strong assumptions about rationality, i.e., everybody knows that everybody knows that ... everybody is rational. For example, if the n is large or the ratio $2/3$ is replaced by a number close to 1, the elimination is slow, and the predictions of rationalizability are less reliable. On the other hand, if n is small or the ratio $2/3$ is replaced by a small number, the elimination is fast, and the predictions of rationalizability are more reliable. In particular, the predictions of rationalizability for this game are more robust in a small group than a larger group.

It is important that one analyzes the game that describes the actual situation. For example, when the above game is played in the classroom, there are often some students who would rather move the mean in an unexpected direction and surprise the other students than get the prize of being closest to the two-thirds of the average. Those students bid 100 instead. In such experiments, the resulting outcome is often different from the rationalizable solution of 0 for the above game, which does not take into account the existence of such students. In fact, some students bid 0 in the first time they play the game and switch to relatively higher bids in the follow-up games. To analyze that situation, consider the following variation.

For example, in the beauty contest game suppose that there are m mischievous students, and the utility function of these students is

$$u_i(x_1,\ldots,x_n) = \left(x_i - \frac{x_1+\cdots+x_n}{n}\right)^2.$$

The remaining $n-m$ students are as before. The best response of a mischievous student is 0 if the expected value of X_{-i} is greater than 50, and it is 100 otherwise. Hence at the first round all strategies other than 0 and 100 are eliminated for the mischievous students.

For each round k, there are strategies $\underline{x}^k$ and $\bar{x}^k$ such that x_i survives k rounds of iterated elimination for a regular student if and only if $\underline{x}^k \leq x_i \leq \bar{x}^k$. Note that

$\underline{x}^0 = 0$ and $\bar{x}^0 = 100$. In the earlier rounds, both 0 and 100 are available for mischievous students, and in that case the lower bound remains $\underline{x}^k = 0$ because 0 is a best response to 0 for regular students. To compute the upper bound, fix a regular student i. The expected value of $(n-1)\,X_{-i}$ can take any value in $[0, 100m + (n-m-1)\bar{x}^{k-1}]$, where $100m + (n-m-1)\bar{x}^{k-1}$ is obtained by taking the highest possible bid for each remaining students, m mischievous students playing 100 and $(n-m-1)$ regular students playing $\bar{x}^{k-1}$. The best reply to this value (i.e. to $X_{-i} = \frac{100m+(n-m-1)\bar{x}^{k-1}}{n-1}$) yields the upper bound:

$$\bar{x}^k = \frac{2}{3n-2}[100m + (n-m-1)\bar{x}^{k-1}]. \tag{4.3}$$

Every strategy x_i with $x_i > \bar{x}^k$ is eliminated. As $k \to \infty$, $\bar{x}^k$ converges to

$$\bar{x}^\infty = \frac{200m}{n+2m} \tag{4.4}$$

The limit $\bar{x}^\infty$ is computed by substituting $\bar{x}^\infty$ for $\bar{x}^k$ and $\bar{x}^{k-1}$ in 4.3.

The lower bound $\underline{x}^k$ depends on whether 0 remains a best response for a mischievous student. This is the case when

$$\frac{\bar{x}^k(n-m) + 100(m-1)}{n-1} \geq 50$$

If $m \geq n/4$, then $\bar{x}^\infty$ satisfies the above inequality.[3] In that case, all $\bar{x}^k$ satisfy the inequality, and neither 0 nor 100 is eliminated for the mischievous students. In that case, the set of rationalizable strategies is $\{0, 100\}$ for mischievous students and $[0, 200m/(n+2m)]$ for the regular students. If $m < n/4$, then $\bar{x}^\infty$ fails the above inequality. Then, there exists k^* such that $\bar{x}^k$ fails the inequality for every $k \geq k^*$, and $\bar{x}^k$ satisfies the inequality for all $k < k^*$. In that case at round $k^* + 1$, 0 is eliminated for mischievous students. Consequently, at round $k = k^* + 2$ and after, for any regular student i, the lowest value for $(n-1)\,X_{-i}$ is $100m + (n-m-1)\underline{x}^{k-1}$. As in the above analysis, the best response to this yields the lower bound at k:

$$\underline{x}^k = \frac{2}{3n-2}[100m + (n-m-1)\underline{x}^{k-1}] \tag{4.5}$$

Of course, as $k \to \infty$, $\underline{x}^k$ converges to

$$\underline{x}^\infty = \bar{x}^\infty = \frac{200m}{n+2m}.$$

[3] The exact condition for this is $m \geq \left(n + n^2\right) / (4n-2)$.

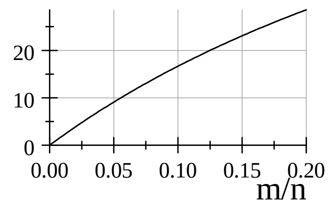

Figure 4.2: Rationalizable strategy of a regular student as a function of the fraction of the mischievous students

In that case, the unique rationalizable strategy is $200m/(n+2m)$ for regular students and 100 for the mischievous students. The rationalizable strategy is plotted in Figure 4.2. Note that the mischievous students have a large impact. For example, when 10% of the students are mischievous, the rationalizable strategy for regular students is $20/1.2 \cong 16.667$, and the average rationalizable bid is 25.

4.3 Iterated Elimination of Weakly Dominated Strategies

Rationalizability corresponds to iterative elimination of strictly dominated strategies, as this is the procedure that computes the set of strategies that are consistent with the common knowledge of rationality. One could also delete the weakly dominated strategies as well as the strictly dominated ones. This procedure is sometimes called *iterated admissibility*, and it leads to much sharper predictions in dynamic games, where there may be many weakly dominated strategies and few strictly dominated ones.

Unlike rationalizability, iterated elimination of weakly dominated strategies is typically order dependent. For example, consider the game

	L	R
A	$1,0$	$1,0$
B	$0,1$	$0,0$

Note that (A, L) is a dominant-strategy equilibrium. However, if one eliminates the dominated strategy B for Player 1 first, the remaining game is

	L	R
A	$1,0$	$1,0$

in which no strategy is weakly dominated. Both strategies L and R survive. If one started with eliminating the strategies of Player 2, then she would eliminate R first, and in the remaining game

	L
A	$1,0$
B	$0,1$

B would be dominated, and this would result in the dominant-strategy equilibrium (A, L) as the unique solution.

In iterative elimination of weakly dominated strategies, *at each round, all weakly dominated strategies for all players are eliminated simultaneously.* In the above game, both strategies B and R are eliminated in the first round, resulting in dominant-strategy equilibrium as the unique solution. Of course, this is the case whenever there is a dominant-strategy equilibrium.

For an illustration of the procedure, consider the extensive-form game in Figure 2.9. In the solution of Exercise 2.1, the normal form of this game is computed as

	a	d
Aa	$1,-5$	$5,2$
Ad	$3,3$	$5,2$
Da	$4,4$	$4,4$
Dd	$4,4$	$4,4$

In this game, there is no strictly dominated strategy, and hence all strategies are rationalizable. One applies the iterative elimination of weakly dominated strategies as follows. In the first round, Aa is weakly dominated by Ad and eliminated. There is no other dominated strategy in the first round. After the elimination, the game reduces to

	a	d
Ad	$3,3$	$5,2$
Da	$4,4$	$4,4$
Dd	$4,4$	$4,4$

In this game, d is weakly dominated by a, and d eliminated in the second round. In the third round, Ad becomes dominated and eliminated. The remaining strategies are Da and Dd for Player 1 and a for Player 2. That is, Player 1 goes down in the first period, and Player 2 would go across if Player 1 went across.

As illustrated in this example, iterated elimination of weakly dominated strategies can result in sharp predictions. Note, however, that this procedure makes much stronger assumptions than common knowledge of rationality, and these assumptions do not necessarily have reasonable foundations.[4] Hence, in computing rationalizable strategies, one eliminates only strictly dominated strategies.

4.4 Exercises with Solution

Exercise 4.1. Compute the set of rationalizable strategies in the following game.

	a	b	c	d
w	$3,1$	$1,0$	$0,2$	$1,1$
x	$1,0$	$0,10$	$1,0$	$0,10$
y	$2,1$	$1,0$	$0,0$	$0,0$
z	$0,0$	$1/2,0$	$3,1$	$0,0$

Solution. Strategy x is dominated by a mixed strategy that puts probability 1/2 on w and probability 1/2 on z. No other strategy is dominated. After elimination of x, strategies b and d become dominated; both b and d are dominated by any strategy that puts positive probabilities on a and c and zero probability on b and d. Strategies b and d are eliminated in the second round. In the next round, y is eliminated because it becomes dominated by a mixed strategy that puts probability 3/4 on w and probability 1/4 on z. The eliminations so far leave the following strategies:

	a	c
w	$3,1$	$0,2$
z	$0,0$	$3,1$

[4] There is a somewhat weaker solution concept that has substantially stronger foundations: eliminate all the weakly dominated strategies in the first round, and eliminate only strictly dominated strategies in the subsequent rounds. This solution concept also leads to sharp predictions in some dynamic games, and it characterizes the strategic implications of common knowledge of rationality and "cautiousness".

One can easily see that the strategy a and then w are eliminated next, yielding (z, c) as the only rationalizable strategies. The games with a unique rationalizable strategy are called dominance-solvable. We got one of them here.

Exercise 4.2. Compute the set of all rationalizable strategies in the following game.

	w	x	y	z
a	$0,3$	$0,1$	$3,0$	$0,1$
b	$3,0$	$0,2$	$2,4$	$1,1$
c	$2,4$	$3,2$	$1,2$	$10,1$
d	$0,5$	$5,3$	$1,2$	$0,10$

Solution. Strategy x is strictly dominated by the mixed strategy σ_2 with $\sigma_2(w) \in (1/3, 1/2)$ and $\sigma_2(y) = 1 - \sigma_2(w)$. In the first round, x is therefore eliminated. (No other strategy is eliminated in that round.) In the second round, d is strictly dominated by b and eliminated. In the third round, z is strictly dominated by σ_2 above and eliminated. In the fourth round, c is strictly dominated by b and eliminated. There is no other elimination, and the set of rationalizable strategies is $\{a, b\} \times \{w, y\}$.

Exercise 4.3. Compute the set of all rationalizable strategies in the following game. Simultaneously, Alice and Bob select arrival times t_A and t_B, respectively, for their meeting, where $t_A, t_B \in \{0, 1, 2, \ldots, 100\}$. The payoffs of Alice and Bob are

$$u_A(t_A, t_B) = \begin{cases} 2 - (t_A - t_B)^2 & \text{if } t_A < t_B \\ -(t_A - t_B)^2 & \text{otherwise} \end{cases}$$

$$u_B(t_A, t_B) = \begin{cases} 2 - (t_A - t_B)^2 & \text{if } t_B < t_A \\ -(t_A - t_B)^2 & \text{otherwise,} \end{cases}$$

respectively. [Note that t_A and t_B are integers between 0 and 100.]

Solution. If the set of remaining strategies from the earlier rounds is $\{0, \ldots, t_{\max}\}$ for some $t_{\max} > 0$, then the $t_{\max}$ is strictly dominated by $t_{\max} - 1$ and is eliminated. (Proof: For $t_B = t_{\max}$,

$$u_A(t_{\max} - 1, t_{\max}) = 1 > 0 = u_A(t_{\max}, t_{\max}),$$

and for any $t_B < t_{\max}$,

$$u_A(t_{\max} - 1, t_B) = -(t_{\max} - 1 - t_B)^2 > -(t_{\max} - t_B)^2 = u_A(t_{\max}, t_B),$$

showing that $t_{\max} - 1$ strictly dominates $t_{\max}$ for Alice. The same argument applies for Bob.) Therefore, one eliminates 100 in round 1, 99 in round 2, ..., and 1 in round 100. The set of rationalizable strategies is $\{0\}$ for both players.

The next exercise illustrates that rationalizability may be sensitive to the possibility of trembling, depending on the relative magnitude of trembling probabilities and the payoff differences.

Exercise 4.4. Consider the following game:

	L	R
T	$1,1$	$1,0$
B	$0,1$	$0,10000$

1. Compute the rationalizable strategies.

2. Now assume that players can tremble: when a player intends to play a strategy s, with probability $\epsilon = 0.001$, Nature switches it to the other strategy s'. For instance, if Player 2 plays L (or intends to play L), with probability ϵ, R is played, with probability $1-\epsilon$, L is played. Assume that the trembling probabilities are independent. Compute the rationalizable strategies for this new game.

Solution. In Part 1, first B and then R are eliminated. The rationalizable strategies are T for Player 1 and L for Player 2. In Part 2, taking into account the Nature's move, one can write the new game in normal form as:

	L	R
T	$1-\epsilon, 1-\epsilon+10000\epsilon^2$	$1-\epsilon, \epsilon+10000(1-\epsilon)\epsilon$
B	$\epsilon, 1-\epsilon+10000\epsilon(1-\epsilon)$	$\epsilon, 10000(1-\epsilon)^2+\epsilon$

To see how the payoffs are computed consider (T, L). If this strategy profile is intended, the outcome is (T, L) with probability $(1-\epsilon)^2$ [nobody trembles], (T, R) with probability $(1-\epsilon)\epsilon$ [only Player 2 trembles], (B, L) with probability $(1-\epsilon)\epsilon$ [only Player 1 trembles], and (B, R) with probability ϵ^2 [everybody trembles]. We mix the payoff vectors with the above probabilities to obtain the table. One can use the structure of payoffs to shorten the calculations. For example, Player 1 gets 1 if she does not tremble and gets 0 otherwise, yielding $1-\epsilon$. For $\epsilon = 0.001$, the payoffs are approximately as follows

	L	R
T	$1, 1$	$1, 10$
B	$0.001, 11$	$0.001, 9980$

To compute the rationalizable strategies, note that B is still dominated by T and is eliminated in the first round. In the second round, one cannot eliminate R, however. Indeed, the payoffs from L and R are approximately 1 and 10, respectively. Hence, L is eliminated in the second round, yielding (T, R) as the only rationalizable strategy profile.

The next exercise illustrates how to incorporate psychological concerns and private information about the players' motivations in a game-theoretical analysis.

Exercise* 4.5. Consider the Prisoners Dilemma game, played by Al and Bill as in Chapter 1. Imagine that Al is vindictive, in that he pays a cost 2 if he plays D against C and a cost 5 if he plays C against D, yielding the payoff function in (4.6) below. Al does not know whether Bill is self-interested—as in (3.1)—or vindictive. In particular, at the beginning, Nature chooses $t_B = s$ (self-interested) or $t_B = v$ (vindictive), each with probability 1/2, and Bill privately observes t_B; Al does not observe Nature's move. The payoffs are as follows:

$t_B = s$

	C	D
C	$5, 5$	$-5, 6$
D	$4, 0$	$1, 1$

$t_B = v$

	C	D
C	$5, 5$	$-5, 4$
D	$4, -5$	$1, 1$

(4.6)

Write this game in normal form and compute the set of rationalizable strategies.

Solution. The extensive-form representation of this game is plotted in Figure 4.3. Observe that Al has only one information set, and hence he has two strategies, C and D. Bill has two information sets, one for $t_B = s$, and one for $t_B = v$. Hence, he has four strategies, CC, CD, DC, and DD, where the first and second entries indicate what Bill would play when he observes s and v, respectively. The normal-form representation is

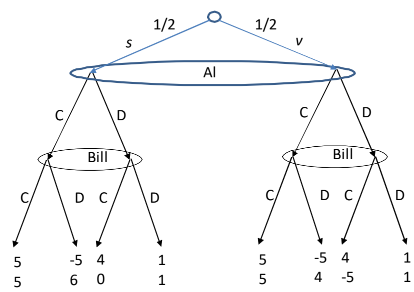

Figure 4.3: Extensive-form representation of the game in Exercise 4.5

as follows:

	CC	CD	DC	DD
C	$5,5$	$0,9/2$	$0,11/2$	$-5,5$
D	$4,-5/2$	$5/2,1/2$	$5/2,-2$	$1,1$

For example, under strategy profile (C, CD), the outcome is (s, C, C), yielding payoff vector $(5, 5)$, with probability 1/2, and it is (v, C, D), yielding payoff vector $(-5, 4)$, with probability 1/2. The expected value is $(0, 9/2)$. Observe that there is no dominant strategy. However, CC and CD are strictly dominated by DC and DD, respectively, and hence CC and CD are eliminated in the first round. (This is because C is dominated by D for a self-interested player.) In the second round, C is dominated by D, and eliminated. Finally, DC is eliminated in the third round, and the game is dominance-solvable with unique rationalizable strategy profile $\{(D, DD)\}$. Observe that, if it were known that both players are vindictive, there would have multiple solutions, but the uncertainty about Bill's motivations leads to a unique solution.

Observe that, in both extensive-form and normal-form representations, Al knows Bill's payoff function. For example, in the extensive-form, he knows that Bill gets 6 if Nature chooses s, Al plays C and Bill plays D, or Bill gets 5 if Nature chooses v and they both play C. As a result, in normal-form, he knows that Bill's expected payoff from (C, DC) is 11/2. Al's uncertainty about Bill's payoff is modeled by Nature's move. Since he does not observe Nature's move and Bill does observe it, he does not know Bill's payoff when he plays C and Bill plays D. The second half of the book will focus on games with such payoff uncertainty.

The next two exercises provide a useful characterization of rationalizable strategies:

a strategy is rationalizable if and only if it is a best response to a belief that puts positive probability only on rationalizable strategies of other players; in other words, it is not strictly dominated when all non-rationalizable strategies of other players are eliminated.

Exercise† **4.6.** Consider any collection of sets $Z_1 \subseteq S_1, \ldots, Z_n \subseteq S_n$ such that, for each player i, each $z_i \in Z_i$ is a best response to a belief that puts probability one on the set Z_{-i} (i.e., it puts zero probability on each $s_{-i} \notin Z_i$).[5] Show that each $z_i \in Z_i$ is rationalizable.

Solution. It suffices to show that

$$Z \subseteq S^k$$

for each k. To do this, I will use mathematical induction on k. It is clearly true for $k = 0$. Towards a mathematical induction, assume that

$$Z \subseteq S^{k-1}$$

for some $k \geq 1$. Take any $i \in N$ and $z_i \in Z_i$. Then, by the assumption in the exercise, z_i is a best response to a belief β_{-i} that puts probability one on Z_{-i}. But, since $Z_{-i} \subseteq S_{-i}^{k-1}$ by the inductive hypothesis, β_{-i} puts probability one on S_{-i}^{k-1}. Therefore, $z_i \in S_i^k$. Since z_i and i are arbitrary, this shows that $Z \subseteq S^k$.

Exercise† **4.7.** Assuming that the set of strategies is finite, show that the set of rationalizable strategies satisfies the property in Exercise 4.6.

Solution. Since there are only finitely many strategies, the elimination process stops at some round k:

$$S^k = S^{k-1}.$$

By definition, each strategy $s_i \in S_i^k$ is a best response to some belief that puts probability one on S_{-i}^{k-1}, which is equal to S_{-i}^k, showing that S^k satisfies the property in Exercise 4.6. Of course, S^k is the set of rationalizable strategies.

[5] By Theorem 3.1, this is equivalent to the condition that there exists no $z_i \in Z_i$ that is strictly dominated when the others' strategies are restricted to be in Z_{-i}. That is, for every $z_i \in Z_i$ and every mixed strategy σ_i of player i, there exists a strategy profile z_{-i} of other players such that $z_j \in Z_j$ for every $j \neq i$ and $u_i(z_i, z_{-i}) \geq u_i(\sigma_i, z_{-i})$.

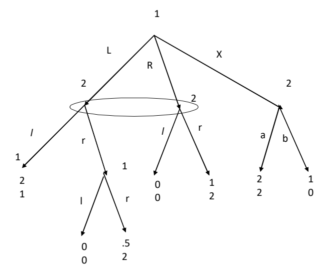

Figure 4.4:

4.5 Exercises

Exercise 4.8. Compute the rationalizable strategies in the following games:

	A	B	C	D
a	$0,-1$	$4,4$	$0,0$	$2,0$
b	$0,3$	$0,0$	$4,4$	$1,0$
c	$5,2$	$2,0$	$1,3$	$1,3$
d	$4,4$	$1,0$	$0,1$	$0,5$

	A	B	C	D
a	$2,0$	$2,4$	$0,0$	$0,-1$
b	$1,-2$	$-2,-2$	$4,2$	$0,1$
c	$1,3$	$0,0$	$1,3$	$5,2$
d	$0,5$	$-1,0$	$0,1$	$4,4$

.

Exercise 4.9. Write the extensive-form game in Figure 4.4 in normal form. Compute the set of rationalizable strategies, and iteratively eliminate weakly-dominated strategies.

Exercise 4.10. Compute the set of all rationalizable strategies in Exercise 2.6.

Exercise 4.11. Compute the set of rationalizable strategies for each of the following versions of the beauty contest game in Section 4.2 (without any mischievous students):

1. For each player, the strategy set is $\{0,1,\ldots,100\}$, and the utility function is

$$u_i(x_1,\ldots,x_n) = -\left(x_i - \frac{2}{3}\frac{x_1+\cdots+x_n}{n}\right)^2. \tag{4.7}$$

2. The strategy set is $[0,100]$, and the utility function is as in (4.7), but the number of players is unknown. First, an integer n is drawn from $\{3,4,\ldots,80\}$ randomly (where each number is equally likely) and only n students play the game (players do not know n).

3. The strategy set is $[0, 100]$, and the utility function is

$$u_i(x_1, \ldots, x_n) = 100 - \left(x_i - \frac{2}{3} med(x_1, \ldots, x_n)\right)^2,$$

where med finds the median.

Exercise† 4.12. Redo the previous exercise assuming that there are $m < n$ mischievous students. In Part 2, assume that first n is randomly drawn, and then m is randomly drawn from $\{0, 1, \ldots, n-1\}$. In Part 3, assume that the mischievous students have payoff $(x_i - med(x_1, \ldots, x_n))^2$, and briefly discuss your finding in comparing it to the result in Section 4.2.

Exercise 4.13. In Exercise 2.13, iteratively eliminate all weakly dominated strategies.

Exercise* 4.14. In Exercise 4.5, assume that Al has some imperfect private information about whether Bill is self-interested or vindictive. In particular, he observes a random variable t_A, with values s and v, which tell that Bill is self-interested and vindictive, respectively, but this information may be wrong. Formally, Nature randomly chooses a pair (t_A, t_B) from $\{s, v\}^2$ and privately reveals t_A to Al and t_B to Bill where

$$\Pr(t_A = s, t_B = s) = \Pr(t_A = v, t_B = v) = p \text{ and } \Pr(t_A = v, t_B = s) = \Pr(t_A = s, t_B = v) = 1/2 - p$$

for some $p \in [1/4, 1/2]$. Then, each player chooses between Cooperate, denoted by C, and Defect, denoted by D. The payoff functions are as in (4.6); Al is vindictive regardless of t_A or t_B, Bill is self-interested if $t_B = s$ and vindictive if $t_B = v$; t_A does not affect the payoffs. Write this game in extensive-form and in normal form. Compute the set of all rationalizable strategies for each p. Briefly describe the rationalizable behavior for $p = 1/4$ and $p = 1/2$.