Part I

Strategic Analysis

Chapter 1

Introduction

Game Theory is the study of strategic interactions. It develops tools, methods, and language for the analysis of decision-making processes with multiple decision-makers, aka *players*, whose payoffs may depend on other players' decisions. This chapter illustrates some of these methods and some main challenges one faces in strategic analysis on simple examples.

Consider the following game known as the *Beauty Contest* game. Each player simultaneously submits a real number between 0 and 100, inclusive of 0 and 100. The player whose number is closest to the two-thirds of the average of all the numbers submitted by the players wins a prize. Assume that each player wants to get the prize. Note that the outcome, i.e., the winner of the prize, is determined by the numbers submitted by all players. When deciding what number to submit, a player does not know whether she will win the prize because it depends not only on her number but also the numbers submitted by the other players, which she does not know in principle. The best number for her depends on her belief about the numbers submitted by the other players. When there are a lot of players, the best number for her is approximately two-thirds of the average of the numbers submitted by the other players.

What is that average? To answer that question, she would need to understand the other players' points of view and their motivations. Assuming that the others also want to win the prize, she then realizes that the others will also submit (approximately) two-thirds of the average of the numbers they expect to be submitted by others. She then asks: what do others expect the average number submitted by the other players to be?

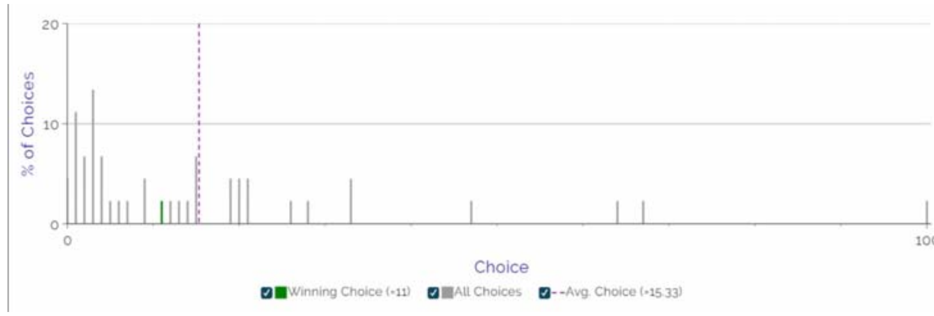

Figure 1.1: Frequency of numbers submitted when the Beauty Contest game has been played in the class

To answer this question, she would need to go one more step and understand what other players think about the other players' motivations and points of view, asking further the others' expectations of the others' expectations of the numbers submitted by others. Continuing in this fashion, in order to determine what number to submit, she asks about higher and higher order expectations, the others' expectation of the others' expectation of . . . the others' expectation of the numbers submitted by the others.

In this game, one can find a clear way to play the game thinking through the other players' expectations as above. Since the numbers must be between 0 and 100, the average of them must also be between 0 and 100. Hence, a player would not submit more than the 2/3 of 100 no matter what they think about the others. Knowing this, players expect the average of the numbers to be between 0 and $(2/3) \times 100$. Knowing that the other players know this and choose 2/3 of their expected average, the players expect the numbers to be between 0 and $(2/3)^2 \times 100$. Iterating this reasoning, players expect the other players to submit 0 and submit 0 in return.

This is theory. How do actual players play this game? Luckily, this game has been played by students in game theory classes and other classes about strategic interaction throughout the world, and we do know how they play. The numbers submitted by the students in a Game Theory class at MIT are plotted in Figure 1.1. In this version of the game, the students could only submit integers, and hence, both 0 and 1 are possible solutions according to the method described above. There were 46 participants. The average of the numbers submitted was 15.33, and the winning number was 11. There were two participants who entered 0, and many others submitted small numbers. The most frequent number submitted was 3 (by 7 participants), and the median was 9.

In my experience with MIT students in the last two decades, when they play this game for the first time, a substantial number of them submit 33, 22, or 11, as 33 is the best response to 50, 22 is the best response to 33, and 11 is the best response to 22. A very large number of them submit 0, the prediction above, only to learn that they are mistaken. The winning number is substantially larger. When the game is repeated, as the participants see how the game is played in earlier rounds, these patterns disappear, and students typically submit small but positive numbers. The result of the game with experienced players is close to the solution obtained above but somewhat different. This is because the average can be moved substantially by a large number, such as 100. For example, in Figure 1.1, one participant submitted 100. Without that submission, the average would have been 13.45, and the winning bid would have been 9 instead. Some participants would rather move the mean in an unexpected direction and surprise the other participants than get the prize of being closest to the two-thirds of the average. Those participants bid 100 instead, as the lone participant above. The solution of 0 above does not take into account the existence of such students. The game-theoretical solution with such students will also be different, and it will be closer to the realized outcome.

This illustrates one of the most important challenges a game theorist faces: identifying the game that reflects the actual strategic situation that is meant to be modeled, reflecting the decision makers' motivations, concerns, fears, and uncertainty. This book will teach you how to analyze a given game, but it cannot teach you how to write down the relevant game for the specific situation that you are interested in. That is an art that can be perfected only by experience. Such experience can be gained by both theoretically analyzing various strategic situations to get insights into what features of strategic environments are important and also by observing how players behave in real life. With such experience, one may know which features of the strategic situations she should pay attention to in the modeling stage.

The Beauty Contest game above may appear a made-up example, but it captures a main problem in many real-world applications. It takes its name from John Maynard Keynes (1936), who argued that financial markets are like beauty contests where the winners are those who pick the most popular faces with the other judges. One must not simply act based on her understanding of how much dividend a given asset pays. She must also take into account what other market participants think about the dividends, taking into account what they think about the other market participants' understanding of how much dividend to be paid, and so on. This has always been true for most markets. It is especially true in financial markets with high-frequency trading, as high-frequency traders mainly speculate on short-term price fluctuations.

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The method used above does not necessarily yield a clear prediction in all situations. If one wants to make a sharper prediction, she may use other methods by making stronger assumptions about the players' reasoning. There are many such methods, called *solution concepts*. A prominent solution concept is called *Nash equilibrium*, named after John Nash. It prescribes a *strategy* for each player such that no player has an incentive to deviate if the player knows that the other players play their prescribed strategies.

To illustrate this idea, consider the following game, called Stag Hunt game, based on a situation described by Jean-Jacques Rousseau. Imagine two hunters, named Artemis and Beatrice, who want to hunt a stag in a clearing with two exits. In order to hunt the stag, each must block one exit. There are also hares around, and a hunter can hunt a hare by herself, in which case the stag runs away from the exit she is supposed to block. This is formally represented as a game as follows. The players are Artemis and Beatrice as they are the decision-makers. Each player has two strategies (i.e., choices): *Stag*, meaning that blocking her exit towards hunting the stag, and *Hare*, meaning that hunting a hare and keeping her exit open. They pick the strategies simultaneously. Every pair of strategies leads to a payoff to each player, a payoff measured by a real number. If a player plays Hare, she will have a hare regardless of what the other player does. If she plays Stag, she will have half of a stag if the other player also plays Stag; she gets nothing if the other player plays Hare. Her payoffs are as follows:

	Other player plays Stag	Other player plays Hare
Stag	3	0
Hare	2	2

Here, 0 is the payoff from getting nothing; 2 is the payoff from a hare, and 3 is the payoff from half a stag. What would a player, say Artemis, do? Observe that, although

Artemis knows what she would get when she plays Hare, she does not know her payoff from playing Stag. Her payoff from Stag depends also on what Beatrice does. She gets 3 if Beatrice also plays Stag and 0 if Beatrice plays Hare. Since she does not know which strategy Beatrice plays, she does not know her own payoff. Her best strategy depends on what she thinks about what Beatrice will do. In particular, Artemis will play Stag if she thinks that Beatrice will play Stag and will play Hare if she thinks that Beatrice will play Hare. Artemis needs to ask: what does Beatrice do? But the same analysis applies for Beatrice: she will play Stag if she thinks Artemis will play Stag, and she will play Hare if she thinks Artemis will play Hare. Hence, Artemis needs to ask: what does Beatrice think about what I will do? To answer this question, she would imagine Beatrice putting herself in Artemis' shoes. But, of course, Artemis' decision depends on what she thinks about what Beatrice will do, and Artemis now would need to answer the question: what does Beatrice think about what I think about what Beatrice will do? Proceeding in this manner, Artemis would find herself in an infinite regress, and even infinite rounds of thinking would not give a clear answer in this problem.

In a Nash equilibrium, one prescribes a strategy for each player. We are in an equilibrium if each player is willing to play the prescribed strategy assuming the other players will also play their own prescribed strategies. In this game, there are two Nash equilibria: (Stag, Stag) and (Hare, Hare). Indeed, when one prescribes Stag for both players, no player has an incentive to deviate: the prescribed strategy Stag gives 3 (against the other player also playing Stag), and one would get only 2 if she deviated and played Hare. Similarly, if both players are supposed to play Hare, there is no incentive to deviate and play Stag, which now gives only 0.

Nash equilibrium is relevant when it is reasonable to assume that players have correct conjectures about the other players' strategies and this is known among the players. For example, drivers in the United States and continental Europe drive on the right side of the road, assuming that the others will also do so, while the drivers in the United Kingdom and Japan drive on the left side of the road, assuming that the other drivers will also do so. If we were to take some drivers from all these countries and put them in an otherwise uninhabited island, one would expect chaos on the roads in the early days of driving as they would not know what other drivers would do. More generally, the concept of Nash equilibrium is relevant in the analyses of stable institutions and social norms and conventions, but it may not be relevant in the analyses of novel strategic situations.

The Stag Hunt game illustrates an important feature: *coordination.* Each player would like to match what the other player does. In many real-life situations, coordination motives have fundamental economic roots. Some obvious examples are: drivers coordinate which side of the road they drive to avoid accidents; friends coordinate on what social network platform to use or where to meet; buyers and sellers coordinate on where to search for each other—choosing among the fairs in various distant cities in medieval times or choosing between online auction platforms in modern electronic commerce. Coordination motives may also be present in less obvious strategic environments. For example, potential employees may invest in their own human capital by attaining high levels of education and developing their skills if they expect to get better jobs where those skills will be used. At the same time, potential employers may invest in highly productive skill-intensive technologies if they expect to find highly skilled employees and invest in less productive technologies that do not require skilled labor if they do not expect to find such employees. In a similar fashion, speculators may sell a currency or a stock if they expect the other speculators will do so; the citizens in an oppressive regime may protest if they think that there will be enough protestors in the streets to topple the government and they may stay home otherwise, and partners in a joint project may work hard if they think that the others will also do so.

In the Stag Hunt game, observe that both players get a better payoff in (Stag, Stag) equilibrium than the one in (Hare, Hare) equilibrium. The latter is an equilibrium because switching to (Stag, Stag) requires both players to switch their strategies. Since each player chooses her own strategy, taking the other's strategy is given, the payoff 3 from (Stag, Stag) is out of reach for each player. In real world, the (Hare, Hare) equilibrium corresponds to a "bad" equilibrium situation, such as a "poverty trap" in which the workers do not invest in human capital anticipating that they will not find a skill-intensive job, while the employers do not invest in skill-intensive technology anticipating that they will not be able to find a skilled worker. One may argue that wildly different outcomes in similar environments in real world result from players' playing two different equilibria in two similar situations, as in this example. One may also dismiss the "bad" equilibrium as an artifact of the equilibrium concept. The book will show that both arguments are misguided, the latter argument being more misguided than the former one.

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In general, in strategic situations, individual optimization by each player does not necessarily lead to a social optimum, setting Game Theory apart from the traditional economic models in which an "invisible hand" leads to social optimum when everybody follows their own self-interest. This is illustrated by the following well-known game, called the *Prisoners' Dilemma* game. This game represents many real-life situations in which players' pursuit of self-interest has a negative impact on each other. Two prisoners, namely Al and Bill, are arrested for a crime for which there is no firm evidence, and they are being interrogated in separate rooms. Each prisoner can either "cooperate" with the other and stay mum or "defect" and confess their crime. If they both cooperate, then they walk free after a trial for insufficient evidence. If they both defect and confess their crime, then they both get a short jail sentence. However, if one cooperates while the other defects, then the one who defects walks free without waiting for the trial, while the other one gets a long jail sentence. What would a prisoner do in such a situation?

Formally, the players are Al and Bill. Each player has two strategies: Cooperate and Defect. They pick the strategies simultaneously. The payoffs of each player are as in the following table:

	Other player plays Cooperate	Other player plays Defect
Cooperate	5	0
Defect	6	1

For instance, if they both cooperate, then each gets the payoff of 5 associated with walking free after a trial. If Al cooperates while Bill defects, then Al gets 0, the payoff associated with a long jail sentence, and Bill gets 6, the payoff associated with walking free right away.

As in the Stag Hunt game, players do not know their own payoff from playing a strategy, as the payoff also depends on the other player's strategy. In order to determine which strategy is best for him, he may want to know what the other prisoner does. As it turns out, in this game, he does not need to know the other prisoner's strategy to make his decision. Take Al. He wants to know what is better for him: Cooperate or Defect. Although his payoff depends on what Bill does, Defect is better for him regardless of Bill's strategy—as vividly tabulated in the table above. Hence, no matter what he believes about Bill's strategy, Al chooses to Defect. The same is, of course, true for Bill, and they both play Defect and end up getting a jail sentence.

The Prisoners' Dilemma game represents a trivial situation for strategic analysis. Although the payoffs depend on both players' strategies, each player has a "dominant" strategy, which is better than any other strategy no matter what other players do. In that case, the choice is clear, and each should play his dominant strategy.

The Prisoners' Dilemma game does illustrate an important feature of strategic analysis: individual optimization does not necessarily lead to socially optimal outcomes. In this example, they both would have been better off if they played their dominated strategies and stayed mum. This is because, in game-theoretical applications, players impose externalities on each other, as their choices affect the other players' welfare. Thus, as they try to maximize their own payoffs, they affect each others' welfare, and this usually leads to socially suboptimal outcomes. In contrast, in a perfectly competitive market with no externality, individual optimization leads to a socially optimal outcome.

The prisoners' dilemma game represents situations with externalities. Such externalities are ubiquitous in the real world. They are present, for example, when fishermen decide how many fishes to catch in a lake; when community members decide how much to contribute to public-good production; when competing firms set their prices; when governments decide how much greenhouse gas emission they should allow for their industries, or how much tariff they should charge for their imports. In these examples, "cooperate" corresponds to the strategy that is socially beneficial for the players considered but costly individually, such as a sustainable level of fishing, a high (collusive) price, a low emission level, and free trade, while "defect" corresponds to strategy that is better for individuals but bad for the whole, such overfishing, undercutting, imposing no cap on emissions or charging high tariffs.

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The games are often played in a broader context. In particular, the players may interact repeatedly. For example, in the applications of the Prisoners' Dilemma game, the fishermen may be fishing repeatedly, seeing how much fish the others have caught in the past; the community members may need to contribute the public expenditures repeatedly; the firms may keep competing repeatedly in the same market, adjusting their production and price levels as they see others' past production and prices, and governments can adjust their environmental and trade policies as life goes on—often in response to other governments' policies.

In such interactions, the players' actions may have broader payoff implications, and players would take into account these broader implications as well as the narrow payoff implications to the specific game. Their behavior may be driven by their concerns in the broader game and may be in contradiction with the basic motives in the specific game.

For example, imagine that Al and Bill play the Prisoners' Dilemma game repeatedly, getting payoffs in each round according to the table above depending on what they do at that round. They remember what happened so far. A strategy in such a dynamic game is a contingent plan that prescribes what to do at any given period as a function of what happened until then. Consider the following strategy:

Play Cooperate until somebody plays Defect; play Defect thereafter.

According to this strategy, a player checks if Defect has ever been played. If the answer is Yes, he plays Defect, and if the answer is No, then he plays Cooperate. If both players play this strategy, they will play Cooperate at every round. Would they have an incentive to deviate if both players are supposed to play this strategy? The answer is No if there is no known last round and the players are sufficiently patient. To see this, take Al. If Defect has been played in the past, he expects that Bill will play Defect today and in the future regardless of how Al plays the game from this point on. Not being able to affect Bill's future behavior, Al must play Defect in every period thereafter, as prescribed by his strategy. Now suppose that Defect has not been played so far. Al expects that Bill will play Cooperate today and keep doing so until some plays Defect, and he will play Defect thereafter. Al can envision two reasonable responses to this: stick with the above strategy or play Defect forever. If he sticks with the above strategy, they both play Cooperate at every round, yielding the payoff of 5 at every round for Al. If he plays Defect forever, he expects to get 6 today (as Bill plays Cooperate in this round) and get 1 at every round from tomorrow on (as Bill will also switch to Defect). Defect results in 1 util gain for today and 4 utils losses for every period starting tomorrow. If Al is sufficiently patient, he would not want that, and he would stick with his strategy.

In accordance with this solution, in real life, the firms may charge high collusive prices in order to avoid a price war. Countries may sign multilateral agreements that commit them to limit their greenhouse gas emissions or form free-trade areas, not because those agreements can be enforced by the courts, but because they provide an understanding of how the sovereign nations will behave as long as the agreement is in effect, with the additional understanding that they will switch to high greenhouse gas emissions or protective trade policies if a party breaks the agreement. The fact that we see such cooperative behavior throughout the world does not mean that the predictions of game theory is not valid but because the game is repeated (or more generally played in a broader context). Indeed, a major part of this book will be devoted to the analyses of repeated games where we will study self enforcing solutions as in the above example. There will be a limit to how much cooperation can be obtained and we will have intuitive mathematical formulas that will express the amount of cooperation possible in such solutions as a function of the parameters of the strategic environment.

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In the previous examples it was implicitly assumed that each player's payoff from each outcome is common knowledge, i.e., everybody knows it; everybody knows that everybody knows it, and so on, ad infinitum. For example, in the Stag Hunt game it is common knowledge that a player gets 2 from a hare and 3 from sharing a stag. In real-world applications, players will have private information about their payoffs, knowing aspects of it that is not known by the other players. For example, Artemis may privately know how much she likes a hare although it is known how much she likes a stag. Her choice may depend on how much she likes a hare. Moreover, this private information may also affect Beatrice's choice as Beatrice would take into account how Artemis would behave depending on her private information. In general, players' private information will impact the solution—sometimes dramatically. In game theory, strategic environments with private information will be modeled by games of incomplete information (aka Bayesian games).

As a simple illustration of how information affects the solutions, now assume in the Stag Hunt game that it is not known how much Artemis likes a hare—while it is known that all the other payoffs—including Beatrice's payoffs—are as in the previous case.

Artemis is either a "hare-fanatic", getting a payoff of 4 from a hare, or a "regular" hunter with payoff 2 as above. Artemis knows whether she is a hare-fanatic or a regular hunter, but Beatrice does not. Beatrice thinks that either situation is equally likely, believing that Artemis is regular with 50% probability and hare-fanatic with the remaining 50% probability. Now, clearly, if Artemis is a hare-fanatic, she will play Hare, as Hare gives her a higher payoff regardless of what Beatrice does. What should she play if she is a regular hunter? To answer this question, she must ask: what strategy does Beatrice play? Now Beatrice believes that with 50% chance Artemis is a hare-fanatic and will play Hare, giving 0 to Beatrice if Beatrice plays Stag. Beatrice does not know what Artemis would play if she were regular, but that does not matter for her decision. If Artemis played Hare when she is regular, too, then Beatrice would get 0 for sure from playing Stag. If Artemis played Stag when she is regular instead, then Beatrice would get

$$\frac{1}{2} \times 3 + \frac{1}{2} \times 0 = \frac{3}{2}$$

in expectation from playing Stag. (She gets 3 with 50% chance—when Artemis is regular—and 0 with 50% chance—when Artemis is a hare fanatic.) In either case, her expected payoff is less than her payoff from Hare, 2. Hence, now, Beatrice will play Hare. Then, Artemis should play Hare even when she is regular. This is the only solution: both players play Hare regardless even when they are both regular. When it was known that both players were regular, there were two solutions: a "good" equilibrium in which both played Stag, and a "bad" equilibrium in which both played Hare. With private information, the "good" equilibrium seizes to exist, and "bad" equilibrium is the only solution.

In real life, private information is ubiquitous, as it is hard to know other people's motivations and beliefs. Such private information impacts strategic outcomes, and this impact may be dramatic. For example, private information inhibits trade opportunities and may make it impossible for the parties with private information to trade in some markets. For a concrete example, take insurance markets. Potential insurees have typically private information about their risk exposure as they have superior information about their own behavior. As a result, insurance companies exclude many groups of potential insurees (e.g., those with checkered insurance record in car insurance) or charge so high premiums that many potential insurees choose not to get insured. Individuals may take costly actions to signal their private information in order to influence others' behavior. For example, students may take hard courses or enroll in advanced degree programs not because those courses and programs will develop their skills but because they will allow them to signal that they are hard-working or talented. Buyers may delay their purchase decisions to signal that they do not value the good in the hopes that the seller will lower the price in return. More generally, private information may lead to long, costly delays in reaching an agreement in negotiations. A substantial part of this book will be devoted to the analyses of games of incomplete information where the impact of private information will be examined in such applications carefully and rigorously.