Chapter 10

Subgame-Perfect Nash Equilibrium

In many dynamic games, there may be contingencies that can be viewed as a game by itself. There would be node h at which the player who moves knows the entire history and players who move after that all know that node h has been reached. These are the contingencies at which entire history is common knowledge, and players start a fresh game. They remember the entire history, and the history may provide a context in which the players play the continuation game.

For an instance of such a contingency, consider the game in Figure 10.1. In this game, after Player 1 plays E, simultaneously, Player 1 chooses between T and B and Player 2 chooses between L and R, knowing that Player 1 has played E. This contingency can be viewed as a game in itself. If one cuts the tree at the node after move E, she would get a legitimate game: there is an initial node, and the information set of Player 2 is contained in the new tree. Such embedded games are called subgames.

A solution to a game also prescribe solutions to these subgames embedded in it, as it prescribes how players play at each contingency in these subgames as well. Sometimes, a valid solution to the larger game may be problematic when these subgames are considered separate games. For example, as it will be seen in detail in the next section, in Figure 10.1, it is a Nash equilibrium that Player 1 plays X at the beginning anticipating that they will play (T, L) if she plays E. But when the subgame is viewed separately, each player has a strictly dominant strategy, with dominant-strategy equilibrium (T, R). It is irrational for Player 2 to play L in this game. Player 2 played L in the original game believing that Player 1 would play X, but he cannot maintain that assumption in the

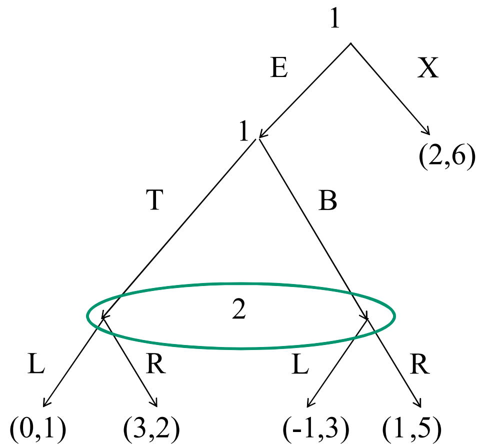


Figure 10.1: A game with a subgame

subgame when it is viewed as a separate game. He cannot maintain that assumption in the original game either—as his information set rules out X—but Nash equilibrium allows him to play L as a contingent plan made at the beginning.

One would want to restrict the solutions requiring that the solutions prescribed for the embedded subgames remain valid solutions to these subgames when they are viewed as separate games. In particular one may require that the prescribed solution to each subgame is also a Nash equilibrium of the subgame—when it is viewed as a separate game. Such Nash equilibria are called *subgame-perfect*.

Subgame-Perfect Nash Equilibrium (henceforth SPNE) is an important solution concepts in dynamic games and it will be the main solution concepts for the complete information games in the remainder of this book. This chapter is devoted to a careful introduction and illustration of this solution concept.

10.1 Motivating Example

For a concrete example, consider the game in Figure 10.1. In this game, Player 1 first chooses between actions E and X, which correspond to entry and exit decisions, respectively. If she chooses X, the game ends. If she chooses E, then she plays the following game with Player 2:

$$\begin{array}{c|c|c|}
 & L & R \\ \hline
T & 0,1 & 3,2 \\ \hline
B & -1,3 & 1,5 \\ \hline
\end{array} \qquad (10.1)$$

In this way, the game in (10.1) is embedded in the dynamic game in Figure 10.1, as part of the tree that follows the action E. The embedded game (10.1) will be referred to as the subgame. Note that one cannot apply backward induction to the dynamic game here because it is not a perfect information game.

The dynamic game here has many Nash equilibria. To see this, write it in normal form:

$$\begin{array}{c|c|c|}
 & L & R \\ \hline
XT & \mathbf{2},\mathbf{6} & 2,\mathbf{6} \\ \hline
XB & \mathbf{2},\mathbf{6} & 2,\mathbf{6} \\ \hline
ET & 0,1 & \mathbf{3},\mathbf{2} \\ \hline
EB & -1,3 & 1,\mathbf{5} \\ \hline
\end{array}$$

where a player's payoff is highlighted when she plays a best response. As one can see from the table, the pure-strategy Nash equilibria are (ET, R), (XT, L) and (XB, L).[1] There is also a continuum of Nash equilibria in mixed strategies, where Player 1 plays any mixed strategy that mixes between XT and XB and Player 2 plays L with probability $p \geq 1/3$. In (ET, R), Player 1 enters and they play (T, R) in the subgame. In the remaining equilibria, Player 1 exits the game anticipating that Player 2 would play L with high probability if she entered.

In the latter equilibria, Player 2 plays L only because he is certain that Player 1 plays X. Such a conviction can be reasonable before the game starts—and that is why (XT, L) and (XB, L) are Nash equilibria, but he cannot maintain that assumption at his information set. At his information set, Player 2 knows that Player 1 did not play X; she played E. He does not know if she will follow with T or B, but that uncertainty is not relevant for his decision: R is strictly better than L no matter how she follows. Indeed, R strictly dominates L in the subgame—as seen in (10.1).

[1] Throughout the chapter, unless otherwise stated, I will describe the strategies by the actions they prescribe in the relevant information sets. For example, here ET means that Player 1 chooses E at the first node and T after playing E; XT would have mean that she chooses X at the beginning but would have chosen T if she played E.

Although one cannot apply backward induction here, one can easily use the logic of backward induction to rule out such equilibria. In the subgame, each player has a strict dominant strategy, leading to (T, R). Thus, one can assume that each will play their dominant action if Player 1 plays E, yielding payoff vector $(3, 2)$. On the other hand, exit decision X leads to payoff of only 2 to Player 1. Hence, foreseeing that they will play (T, R), Player 1 plays E at the beginning. This leads to the strategy profile (ET, R) as the solution.

Subgame-perfect Nash equilibrium accomplishes this by requiring that a Nash equilibrium will be played in each embedded subgame, including the entire game. Since (T, R) is the unique Nash equilibrium of the subgame in (10.1), SPNE requires that (T, R) is played if Player 1 plays E. Given this behavior, Player 1 must then play E at the beginning, as a best response. (The latter comes from the requirement that the solution is a Nash equilibrium of the entire game.)

10.2 Definition and Examples

An extensive-form game can contain a part that could be considered a smaller game in itself; such a smaller game that is embedded in a larger game is called a *subgame*. A main property of backward induction is that, when restricted to a subgame of the game, the equilibrium computed using backward induction remains an equilibrium (computed again via backward induction) of the subgame. Subgame-perfect Nash equilibrium generalizes this notion to general dynamic games.

A subgame is defined as part of a game that is a well-defined game when it is considered separately:

- it must contain a unique "initial node" and
- all the moves and information sets from that node on must remain in the subgame.

Formally, a *sub-tree* is defined as a part of a tree that consists of a unique initial node and all the nodes that come after that initial tree. A *subgame* of an extensive-form game is a sub-tree—endowed with the original assignment of players, informations sets and payoffs—such that if an information set contains a node from the sub-tree then it is contained in the sub-tree. That is, we cannot cut an information set to form a sub-game.

For example, in Figure 10.1, the nodes that follow the move E form a subgame. On the other hand, the node ET and its successors do not form a subgame because half of the information set of Player 2 is in while the other is out. That is, he does not know whether he is in the purported subgame, making the purported subgame not a well-defined game as a separate entity. If we were to ignore the part of the information set that lies outside, then we would have altered the information of Player 2. For another example, the information set of Player 2 and the nodes that follow do not form a subgame because they do not form a sub-tree. Those nodes correspond to the decision problem of Player 2 where he does not know what Player 1 does but the choices of Player 1 is not specified as part of the purported game. Likewise, the initial node and the terminal node after X do not form a subgame because some of the moves that follow the initial node are not included. Indeed, the purported subgame describes a very different situation in which player 1 must choose X.

Note that each terminal node is a trivial subgame in which no decision is made, and they will be ignored as subgames. Note also that the entire game is also a subgame. Any subgame other than the entire game itself is called *proper*.

Subgame-perfect Nash equilibrium simply requires that solution induces a Nash equilibrium in every subgame:

Definition 10.1. A Nash equilibrium is said to be *subgame-perfect* if it is a Nash equilibrium in every subgame of the game.

For example, in Figure 10.1, the unique subgame-perfect Nash equilibrium is (ET, R). Indeed, this game has only one proper subgame. And (ET, R) is a Nash equilibrium of the entire game and specifies a Nash equilibrium (T, R) for the proper subgame. On the other hand, (XT, L) is not a subgame-perfect Nash equilibrium because it prescribes (T, L) as the solution to the proper subgame and (T, L) is not a Nash equilibrium of that subgame.

In order to illustrate how to compute subgame-perfect Nash equilibria in finite-horizon games, consider the game in Figure 10.2. The only proper subgame starts after E. This subgame can be written as

	l	r
L	$3,3$	$0,2$
R	$2,0$	$2,2$

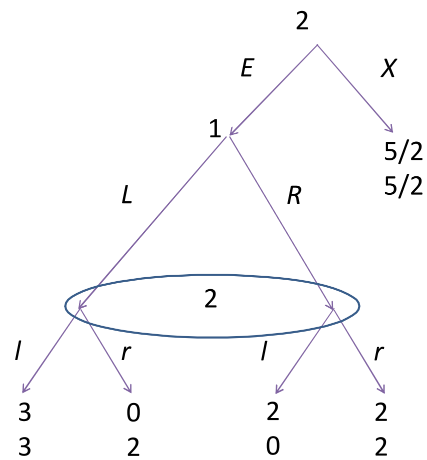
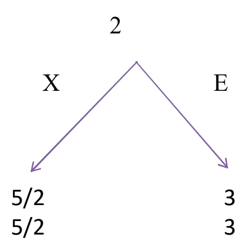
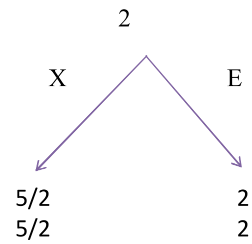


Figure 10.2: An entry game with multiple subgame-perfect Nash equilibria

in normal form. The subgame has three Nash equilibria: (L, l), (R, r), and the mixed strategy Nash equilibrium σ with $\sigma_1(L) = \sigma_2(l) = 2/3$. Using each Nash equilibrium, one can construct a subgame-perfect Nash equilibrium for the original game. First, fix the equilibrium (L, l) in the subgame. Given this behavior in the subgame, Player 2 expects that he will get 3 if he plays E. Since he gets only 2 from playing X, he plays E at the initial node. Mechanically, one can replace the subgame with the payoff vector $(3, 3)$ associated with equilibrium (L, l). The game then reduces to



Note that E is the only Nash equilibrium of the reduced game. Hence, the only Nash equilibrium of the original game in which players play (L, l) in the subgame is (L, El). This is the first subgame-perfect Nash equilibrium. Similarly, if one picks Nash equilibrium (R, r) in the subgame, then the game reduces to



for which X is the unique Nash equilibrium. This results in SPNE (R, Xr). Finally, if one picks σ in the subgame, the expected payoff vector for the subgame is $(2, 2)$, and Player 2 plays X. In the third SPNE, Player 2 plays X, and σ would have been played in the subgame otherwise.

The above example illustrates a technique to compute the subgame-perfect equilibria in finite-horizon games:

- Pick a subgame that does not contain any other subgame.
- Compute a Nash equilibrium of this game.
- Assign the payoff vector associated with this equilibrium to the starting node, and eliminate the subgame.
- Iterate this procedure until a move is assigned at every contingency, when there remains no subgame to eliminate.

As in backward induction, when there are multiple equilibria in the picked subgame, one can choose any of the Nash equilibrium, including one in a mixed strategy. Every choice of equilibrium leads to a different subgame-perfect Nash equilibrium in the original game. By varying the Nash equilibrium for the subgames at hand, one can compute all subgame perfect Nash equilibria.

A subgame-perfect Nash equilibrium is a Nash equilibrium because the entire game is also a subgame. The converse is not true. There can be a Nash Equilibrium that is not subgame-perfect. For example, as we have seen above, the game in Figure 10.1 has the following equilibrium: Player 1 plays X in the beginning, and they would have played (B, L) in the proper subgame. This equilibrium is not subgame perfect: Player 2 plays a strictly dominated strategy in the proper subgame.

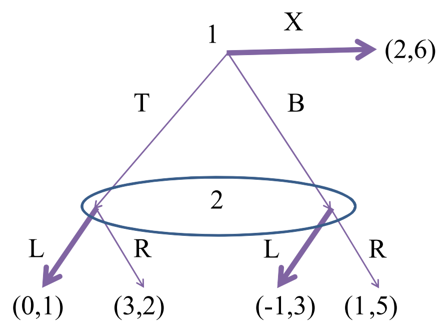


Figure 10.3: A subgame-perfect Nash equilibrium

Subgame-perfect Nash equilibrium can be highly sensitive to the formal representation of a strategic situation. For example, consider the game in Figure 10.3. This game is essentially identical to the one in Figure 10.1. The only difference is that Player 1 makes her choices here at once. One would have thought that such a modeling choice should not make a difference in the solution of the game. It does make a huge difference for subgame-perfect Nash equilibrium nonetheless. In the new game, the only subgame of this game is itself, hence any Nash equilibrium is subgame perfect. In particular, the non-subgame-perfect Nash equilibrium of the game above is subgame perfect. In the new game, it is formally written as the strategy profile (X, L) and takes the form that is indicated by the thicker arrows in Figure 10.3. Clearly, one could have used the idea of sequential rationality to solve this game. That is, by sequential rationality of Player 2 at her information set, she must choose R. Knowing this, Player 1 must choose T. Therefore, subgame-perfect Nash equilibrium does not fully formalize the idea of sequential rationality in general. Subgame-perfect Nash equilibrium does capture the idea of sequential rationality in a large class of games, which will be considered next.

10.3 Multi-stage Games

In a game there may be histories where all the previous actions are known but the players may move simultaneously. Such histories are called *stages*. For example, suppose that every day players play the Battle of the Sexes game, knowing what each player has played in each previous day. In that case, at each day, after any history of play in the previous days, we have a stage at which players move simultaneously, and a new subgame starts. Likewise, in Figure 10.1, there are two stages. The first stage is where Player 1 chooses between E and X, and the second stage is when they simultaneously play the 2×2 game. It is not a coincidence that there are two subgames because each stage is the beginning of a subgame.

For another example, consider alternating-offer bargaining. At each round, at the beginning of the round, the proposer knows all the previous offers, which have all been rejected, and makes an offer. Hence, at the beginning we have a stage, where only the proposer moves. Then, after the offer is made, the responder knows all the previous offers, which have all been rejected, and the current offer that has just been made. This is another stage, where only the responder moves. Therefore, in this game, each round has two stages.

A *multi-stage game* is defined as a game in which players move in stages throughout the game. That is, an information set I of a player i is either singleton, so that the player knows what happened so far, or player i is moving simultaneously with some other players, and in this information set simply reflects this uncertainty. More formally, there exists players $j_1, \ldots, j_m$ such that j_1 moves knowing the past—from an information set with single node—leading to nodes $\{n_{11}, \ldots n_{1k_1}\}$, then j_2 moves from information set $\{n_1, \ldots n_{k_1}\}$ leading to nodes $\{n_{21}, \ldots n_{2k_2}\}$, $\ldots$, then j_m moves from information set $\{n_{m1}, \ldots n_{mk_m}\}$.

For example the game in Figure 10.1 is a multistage game. Indeed, the only non-trivial information set is the one at which player 2 moves, and at that information set he moves simultaneously with player 1: Player 1 chooses between T and B knowing she played E, and the information set of player has two elements, one corresponding to T and one corresponding to B. In contrast, the game in Figure 10.3 is not a multi-stage game. Indeed, the information set of Player 2 consists of two nodes, corresponding to the moves T and B of Player 1 at the initial node, but Player 1 chooses from T, B, and X. Since the information set of Player 2 does not include the node associated with move X, this is not a multistage game. Intuitively, Player 2 knows that Player 1 did not choose X and hence they do not move simultaneously, and yet he does not know whether Player 1 played T or B.

In a multi-stage game, under any subgame-perfect Nash equilibrium, player act rationally throughout. In particular, for each player i and each information set I of player, the strategy of player i is a best response the other player's strategies starting from the stage that contains I. This is simply because a subgame starts at every stage and SPNE requires that we have Nash equilibrium in that stage. That is, each player plays a best response knowing that the stage is reached. Since the information set is reached at that stage, player i acts rationally at that information set (and onwards). That is why the equilibria involving irrational move L are ruled out in the multistage game in Figure 10.1—but not in the game in Figure 10.3.

10.4 Economic Applications

10.4.1 Entry Decisions in Cournot Competition

An important aspects of doing business is to decide in which markets to enter or whether to compete in a given market. This is a complex task that involve market research and understanding of ones own core competence and competitive edge in those markets as well as understanding how the other players would react to a firm's entrance in the market. Abstracting away from the other aspects of the decision, this section analyzes how the others in the market react to entry decisions of the firms and how this affects the market size in the context of Cournot competition.

There are n potential firms for a large n. First, simultaneously each firm decides whether to Enter or Exit the market. The potential firms who exit get zero and the game ends for them (i.e. the do net have any further move). The firms who enters each incur an entry cost $C \in (0,1)$ and proceed to the second stage. In the second stage, the firms who entered play the linear Cournot oligopoly. Each such firm i simultaneously produces q_i units of a good at zero marginal cost and sells it at price

$$P = 1 - Q$$

where $Q = q_1 + \cdots + q_m$ is the sum of the quantities produced by these firms.

The subgames in the second stage have unique Nash equilibria. If m firms entered, each produce

$$q(m) = \frac{1}{m+1}$$

and obtain the net profit of

$$U(m) = \frac{1}{(m+1)^2} - C.$$

Since the entry cost is sunk, it does not affect the equilibrium behavior but it enters into the profits as an additional cost.

Going back to the entry/exit stage, the firms enter if and only if they expect a non-negative profit from entry, as they get 0 from exit. The size of the market is determined by the entry cost C (which is defined relative to the demand size). If the entry cost is greater than 1/4, then no firm enters the market. Even if other firms do not enter and a firm would be monopoly if it entered, it will not enter because the monopoly profit does not recover the entry cost: $U(1) = 1/4 - C < 0$. Such markets never come to existence we do not see them in reality. On the other extreme, if the entry cost C is lower than $1/(n+1)^2$, all of the firms enter the market. Even when all others enter, a firm makes a net profit of $U(n) = 1/(n+1)^2 - C > 0$ from entering the market. In the intermediate case $1/(n+1)^2 \leq C \leq 1/4$, some m firms enter and the others exit. The market size m is determined as follows. First, each entering firm must make non-negative profit (i.e. $U(m) \geq 0$):

$$m \leq 1/\sqrt{C} - 1.$$

Second, a firm that exits would get a non-positive payoff if it entered (i.e. $U(m-1) \leq 0$):

$$m \geq 1/\sqrt{C} - 2.$$

Note that in the knife-edge case that $1/\sqrt{C}$ is an integer, there are two integers m that satisfy the above inequalities, and there are two sets of SPNE one with market size $m = 1/\sqrt{C} - 2$ and one with market size $m = 1/\sqrt{C} - 1$. In all other cases, there is a unique integer m^* between $1/\sqrt{C} - 2$ and $1/\sqrt{C} - 1$. In any pure-strategy subgame-perfect Nash equilibrium, m^* firms enter the market and the others exit. There are multiple equilibria that differ in the identity of entering firms but they are equivalent in terms of economic implications.

10.4.2 Spacial Competition and Product Differentiation

In many economic settings, the players choose the degree of competition before they engage each other. For example, the demands for two restaurants partly depend on their locations, and the restaurants choose their locations as they enter the market. Likewise, product developers choose the specifications of their products, and these specifications determine the degree of substitutability and complementarity of these product, affecting the equilibrium behavior in their interactions. This section presents a couple of simple examples that illustrate the equilibrium analyses of such markets. The first example introduces a location choice stage, at the beginning of the spacial competition model in Section 6.4.3. The second example, introduces product specification stage prior to differentiated price competition in Section 6.4.2.

Location Choice A unit mass of kids are uniformly located on a street, denoted by the $[0,1]$ interval. There are two ice cream parlors, namely 1 and 2. First, each ice-cream parlor i simultaneously selects her location, Player 1 choosing a_1 and Player 2 choosing $1-a_2$ for some $a_1, a_2 \in [0,1/2]$. Then, observing (a_1, a_2), each ice cream parlor i sets a price $p_i \geq 0$ for its own ice cream, simultaneously. A kid located in w is to pay a transportation cost $c\,(w-y)^2$ to buy from a store located at y, where $c>0$. Given the locations a_1 and a_2 and prices p_1 and p_2, each kid buys one unit of ice cream from the store with the lowest total cost, which is the sum of the price and the cost to go to the store. (If the total cost is the same, he flips a coin to choose the store to buy.)

As shown in Section 6.4.3, given any (a_1, a_2) the subgame has a unique Nash equilibrium. The equilibrium price is

$$p_i^* = c\Delta\left(1+\frac{a_i-a_j}{3}\right),$$

where $\Delta = 1-(a_1+a_2)$ is the distance between the stores. The equilibrium profit of each player i is

$$U_i\,(a_1, a_2) = \frac{1}{2}c\Delta\left(1+\frac{a_i-a_j}{3}\right)^2.$$

Given the equilibrium behavior above, the game at the location choice stage reduces to the normal-form game in which each player i chooses her location parameter a_i towards maximizing U_i. The function U_i is a strictly decreasing function of a_i, and hence, this game also has a unique Nash equilibrium:

$$a_1^* = a_2^* = 0.$$

Each ice-cream parlor chooses an endpoint of the street.

In choosing her location, each ice-cream parlor has two conflicting motives. On the one hand, she would like to increase her own market share by coming closer to the other ice-cream parlor, as the demand Q_i is increasing in a_i. On the other hand, she would like to decrease the competition and thereby charge a higher price by moving away from the other ice-cream parlor, as the prices are proportional to the distance Δ. When the customers have a quadratic cost function for transportation, as in this example, the latter motive dominates, and the firms end up choosing the maximum differentiation.

Product Differentiation There are two profit-maximizing firms, A and B. First, firms A and B simultaneously choose their own product specifications $\alpha \in [0,1]$ and $\beta \in [0,1]$, respectively, yielding a substitution parameter $\gamma = \alpha\beta$. After observing α and β, the firms A and B simultaneously set the prices p_A and p_B of their products, respectively. Firms A and B sell

$$Q_A(p_A, p_B) = 1 - p_A - \gamma(1 - p_B) \text{ and } Q_B(p_A, p_B) = 1 - p_B - \gamma(1 - p_A)$$

units of their products, respectively. The marginal cost is 0 for each firm.

Given any γ, as in Section 6.4.2, there exists a unique Nash equilibrium, in which each firm charges the price

$$p_i^*(\gamma) = \frac{1-\gamma}{2-\gamma},$$

obtaining equilibrium profit

$$U(\gamma) = \left(\frac{1-\gamma}{2-\gamma}\right)^2.$$

At product specification stage, this leads to a normal-form game in which firms A and B choose α and β, respectively, yielding a payoff of $u_i(\alpha, \beta) = U(\alpha\beta)$ for each player i. Since U is strictly decreasing in the substitutability parameter γ, the game has a unique Nash equilibrium:

$$\alpha^* = \beta^* = 0.$$

The substitution between the products leads to competition and suppresses prices. Hence, the firms differentiate their products and make them not substitutable.

10.4.3 Vertical Competition

Markets often have multiple layers. For example, bringing a simple smart phone involves numerous distinct firms competing at various stages of the production. Some firms produce raw materials; some others make semiconductors; multiple chipmakers produce components that will be used in the phone; a phone maker uses all these products to make a phone, and a retail firm sells it in its stores to the consumers. This section presents a couple of simple examples that illustrate how to apply subgame-perfect Nash equilibrium to study such markets.

In the first example, there are two producers, namely 1 and 2, and two retailers, namely 1 and 2, again. There are four players altogether. There is a divisible good. Each producer can produce the good at zero marginal cost, and each retailer can sell it in the market. First, simultaneously, Producers 1 and 2 set prices p_1 and p_2, respectively. Then, observing p_1 and p_2, each retailer i simultaneously buys $q_i \geq 0$ units of the good from producer i at price p_i and sells it on the market at price $P = 1-Q$, where $Q = q_1+q_2$ is the total supply. (The price P can be negative.)

This game has a unique subgame-perfect Nash equilibrium. Given the prices p_1 and p_2, the retailers play a Cournot duopoly game where each retailer i has marginal cost p_i, as it pays p_i for each unit it supplies to the market. The unique Nash equilibrium of such a subgame is $(q_1^* (p_1, p_2), q_2^* (p_1, p_2))$ where[2]

$$q_i^* (p_1, p_2) = \frac{1 + p_j - 2p_i}{3}$$

for distinct i and j. Given this equilibrium strategy profile for the retailers, where strategies are functions of the prices p_1 and p_2, the producers play a linear differentiated price competition game as in Example 7.2; producers 1 and 2 simultaneously set prices p_1 and p_2, respectively, and sell quantities $q_1^* (p_1, p_2)$ and $q_i^* (p_1, p_2)$, respectively. The profit of producer i is

$$p_i q_i^* (p_1, p_2) = p_i (1 + p_j - 2p_i)/3.$$

The unique Nash equilibrium is

$$p_1^* = p_2^* = 1/3.$$

[2]The profit of retailer i is $q_i (1 - q_1 - q_2 - p_i)$, and hence its best response function is $q_i = (1 - q_j - p_i)/2$. Solving the two best-response equations simultaneously, one obtains the unique Nash equilibrium.

The subgame-perfect Nash equilibrium is $(p_1^*, p_2^*, q_1^*, q_2^*)$, where the retailers' strategies are functions of the producers' prices. The equilibrium outcome is as follows. Each producer sets its price at 1/3, and each retailer supplies 2/9 units to the market, selling at price 5/9. Note that the price is even higher than the monopoly price that would emerge if only one of the producers produced and sold to the market directly.

10.4.4 Tariff Setting

Tariff setting and trade policies are among the central functions of governments. They are also the perennial topics of political debates. Free trade may be efficient in the sense of maximizing the total payoffs, but there are winners and losers of every trade policy. As long as the losers of free trade are not compensated for their loss, they would seek for protective policies, resulting in shifts in trade policies as the balance of political power shifts between proponents of different trade policies. This section introduces a very basic model of tariff setting that illustrates some of the main issues. We will return to this model and add new dimensions as we develop the tools necessary to analyze these more complex models.

Consider two countries A and B. In each country, there is one domestic firm, denoted by the country it is from. Each firm can sell in both countries but in order to export to the other country it has to pay a tariff. Formally, first the government in each country i simultaneously sets tariff rate x_i for that country. Then, observing x_A and x_B, each firm i simultaneously determines quantity levels $q_{ii} \geq 0$ and $q_{ij} \geq 0$, to be sold in countries i and j at prices P_i and P_j, respectively, where

$$P_A = 1 - q_{AA} - q_{BA} \text{ and } P_B = 1 - q_{AB} - q_{BB}.$$

Each firm i must pay a tariff x_j for each unit it exports to country $j \neq i$, and hence its profit is

$$u_i = P_i q_{ii} + (P_j - x_j)\, q_{ij}.$$

The payoff of the government in country i is

$$U_i = \frac{1}{2}(1 - P_i)^2 + u_i + x_i q_{ji}.$$

Since the marginal costs are constant (zero), a firm's profit can be separated additively according to which market they come from, and the profit from one market is not affected by what happens in the other market. Hence, given any (x_A, x_B), each market can be analyzed separately. Moreover, the tariff rate enters the profit of the foreign firm as a constant marginal cost. Hence, when $x_i \leq 1/2$, the unique Nash equilibrium is given by

$$q_{ii} = \frac{1+x_i}{3} \text{ and } q_{ji} = \frac{1-2x_i}{3}. \tag{10.2}$$

In each country i, the price, the consumer surplus, the profits of domestic and foreign firms are given by

$$\begin{aligned}
P_i &= \frac{1+x_i}{3} \\
CS_i &= \frac{(2-x_i)}{2\times 9} \\
u_{ii} &= \frac{(1+x_i)^2}{9} \\
u_{ji} &= \frac{(1-2x_i)^2}{9}
\end{aligned}$$

respectively. When $x_i \geq 1/2$, the foreign firm is shut out of the market in country i, and the domestic firm becomes a monopoly:

$$q_{ii} = \frac{1}{2} \text{ and } q_{ji} = 0. \tag{10.3}$$

When the government charges such a high tariff, it effectively prohibits imports.

Going back to the tariff-setting stage, given the above behavior in the second stage, the payoff of government i is

$$U_i = \frac{(2-x_i)}{2\times 9} + \frac{(1+x_i)^2}{9} + \frac{(1-2x_j)^2}{9} + x_i\frac{1-2x_i}{3}.$$

Note that the tariffs enter the payoff of a government in an additively separable way, and hence the best response does not depend on what the other government does. They have "dominant strategies give the firms' behavior". The above function is maximized at

$$x_i = 1/3. \tag{10.4}$$

The unique SPNE of the tariff setting game is given by (10.2 - 10.4). Note that the equilibrium tariff is very high. If the government banned the imports and made the domestic firm a monopoly, the firm would charge the price 1/2. Under the equilibrium tariff the price is 4/9, and the price reduction is merely 11%. Under free trade, the price would be 1/3, corresponding to a 33% price reduction.

At the tariff setting stage, the governments face a prisoners' dilemma situation. It is a "dominant" strategy to charge a high tariff but the sum of the payoffs is maximized by free trade. Indeed, the sum of the payoffs of the governments is

$$U_A + U_B = \frac{8}{9} - \frac{1}{18}x_A^2 - \frac{1}{9}x_A - \frac{1}{18}x_A^2 - \frac{1}{9}x_A,$$

decreasing in the tariff rates. In that case, governments would like to have a trade agreement that ensures low tariffs and high payoffs for each government. In general, trade agreements will be negotiated between the governments, and even without any trade agreement repeated interactions may lead to lower tariffs. Such dynamic issues will be tackled in later chapters after necessary tools are developed.

For a static analysis of trade agreements, define a *trade agreement* as a triplet (x_A, x_B, y) where x_A and x_B are tariff rates for countries A and B, respectively, and y is a monetary transfer from country B to A, where the transfer can be negative. A trade agreement (x_A, x_B, y) is said to be *Pareto optimal* if there is no trade agreement that gives higher payoff for each government given the firms' strategies in the subgame-perfect Nash equilibrium; it is said to be *individually rational* if each government weakly prefers the trade agreement to the subgame-perfect Nash equilibrium above. Since the governments can transfer money, the Pareto-optimal trade agreements maximize $U_A + U_B$, and as it has been shown above, this sum is uniquely maximized by free trade. Therefore, a trade agreement is Pareto-optimal if and only if it establishes free trade:

$$x_A = x_B = 0.$$

The governments achieve optimality by free trade and use monetary transfers for welfare distribution between the countries. In that case, when the trade agreement is negotiated, the negotiations will be about the transfer rather than the tariff rates. In general, monetary transfers may not be feasible—or more broadly the utilities may not be perfectly transferable. In that case, the governments may negotiate also on tariff rates, each trying to lower the tariffs set by the other government.

Tariff rates and more broadly trade policy enter the political discourse because of the tension between the consumers and the domestic firms; consumers want lower tariffs and more competition so they can buy the goods more cheaply, while the domestic firms want tariff protection in order to thwart competition from imports so they can charge high prices. This can be modeled using the government's payoff function as follows. Let the utility function of the government be

$$U_i = \gamma CS_i + (2-\gamma)\, u_i + x_i q_{ji}$$

where $\gamma \in [0,2]$ measures the weight the government puts on the consumer surplus and $2-\gamma$ is the weight it puts on the profit of the domestic firm. Observe that, given the behavior in the second stage,

$$\frac{\partial^2 U_i}{\partial x_i \partial \gamma} = \frac{\partial CS_i}{\partial x_i} - \frac{\partial u_i}{\partial x_i} = -\frac{1}{18} - \frac{2\,(1+x_i)}{9} < 0.$$

Hence, the more weight the government puts on the consumer welfare, the lower tariff it charges in the unique subgame-perfect Nash equilibrium. A government based on consumer interest would charge lower tariff than a government based on corporate interests.

10.5 Money Burning and Forward Induction

In dynamic games, an action can be part of a rational plan when it is combined by a continuation play, but it is not rational when it is combined with another continuation play. For example, consider the game depicted in Figure 10.4. In this game Bob has an outside option of 2. Bob can rationally choose to Play the Battle of the Sexes game and continue to play B in the game if he thinks that Alice will also play B; the strategy "Play, B" is a best response to B. On the other hand, the strategy "Play, A" is not a best response to any belief because Bob can get at most 1 by choosing "Play" and then choosing A, while he could get 2 by playing "Exit". Hence, the move "Play" is rational when it is combined with B, but it is not rational when it is combined with A. In multistage games, in any subgame-perfect Nash equilibrium, players assume that the other players will act rationally in the future but they do not necessarily assume that past actions are part of a rational plan. In particular, Alice can assume that Bob will play A after seeing him choosing "Play" although this could not be a rational plan. She attributes Bob's choosing "Play" to a past mistake, or to an unintentional trembling.

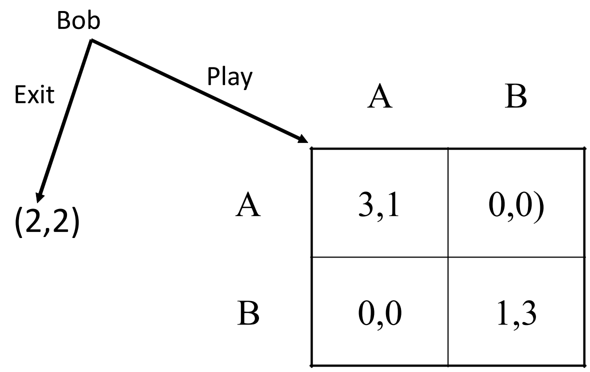


Figure 10.4: Battle of Sexes with exit option

Indeed, this game has two subgame-perfect Nash equilibria in pure strategies (and one in mixed strategies). In one equilibrium, Bob exits, thinking that, if he plays the Battle of Sexes, they will play the (A, A) equilibrium of the Battle of Sexes, yielding only 1 for him. In the second equilibrium, Bob chooses to Play the Battle of Sexes, and in the Battle of Sexes they play (B, B) equilibrium. In the former equilibrium, when asked to play, Alice knows that Bob has chosen to play the Battle of Sexes although Bob was not supposed to choose Play. In this equilibrium, she attributes Bob's unexpected move to a tremble (or a temporary mistake) and assumes that he will continue to play A.

Is it reasonable for Alice to attribute the move to a mistake while she could interpret this part of a rational plan and infer that Bob intends to play B? Some may argue that it is not. Because, when asked to play, Alice knows that Bob has chosen to play the Battle of Sexes, forgoing the payoff of 2. She must therefore realize that Bob cannot possibly be planning to play A, which can yield at most 1 for Bob. That is, when asked to play, she should understand that Bob is planning to play B, and thus she should play B. Anticipating this, Bob should choose to play the Battle of Sexes game, in which they play (B, B). Therefore, the second equilibrium is the only reasonable one.

This kind of reasoning is called *Forward Induction.* Roughly speaking, forward induction requires that when a player can attribute a move to a rational plan, she should interpret it as part of a rational plan and try to infer the intentions of the player who takes the action instead of blindly attributing it to a mistake. In other words, the players should maintain the rationality assumption until it is proven otherwise. One can further require that the players expect the other players to maintain rationality assumption until

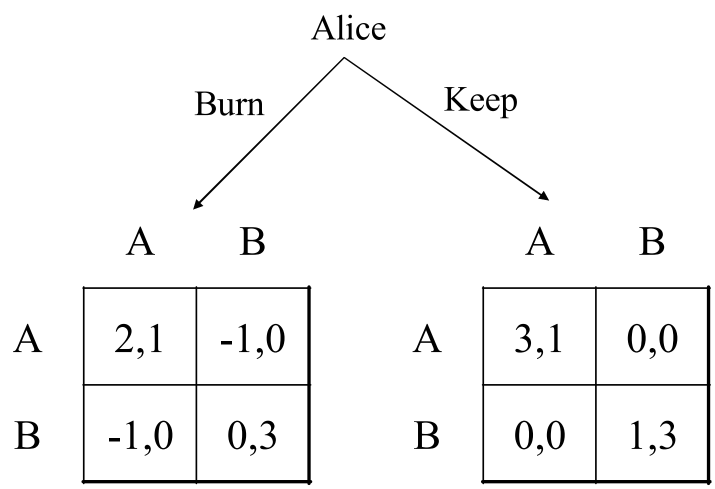


Figure 10.5: Money Burning game

it is proven otherwise, and so on.

For an example of multiple layers of such reasoning, consider the Money Burning game in Figure 10.5. In this game, before playing the Battle of the Sexes game, Alice gets to choose between burning a dollar, which is worth one util, or keeping it. First observe that Burning a dollar and playing B after that is not a best response to any belief for Alice. In doing so Alice gets at most zero, but if she keeps the dollar and plays a best response to her belief about what Bob would do after seeing that Alice kept the dollar, her expected payoff will be at least 3/4 (why?). Observe also that burning the dollar can be part of a rational plan. If Alice believes that Bob would play A after seeing that Alice burns $A$the dollar and B after seeing that she keeps the dollar, Alice would burn the dollar and play A (and would have played B if she kept the dollar) as a best response. Hence, according to forward induction reasoning, when he sees that Alice burns the dollar, Bob must conclude that Alice intends to play A and burning the dollar is part of that rational plan. Then, Bob plays A when he sees that Alice burns the dollar. If both players have this understanding, Alice can ensure a payoff of 2 by burning the dollar. Now imagine that Alice keeps the dollar. How should Bob interpret this? Now Alice can keep the dollar and play A rationally if she believes that Bob will play A when he sees that she keeps the dollar (and if she burns the dollar they play A). On the other hand, keeping the dollar and playing B can give at most 1 while she could get 2 by burning the dollar, and hence that course of action cannot be part of a best response—to a belief according to which Bob plays A after money burning. Hence, if Alice keeps the dollar, Bob should interpret that as a sign of Alice playing A and should play A. If Alice further anticipates all of these, she keeps the dollar and plays A in the Battle of the Sexes—regardless of money burning.

Forward induction reasoning has two weaknesses. First, the argument loses its intuitive appeal when it is applied in multiple layers as in the money burning example. The rationale of interpreting keeping the dollar as a sign of intention to play A is not as straightforward as interpreting burning the dollar as a sign of such an intention. Second, and perhaps more importantly, the reasoning heavily relies on the common knowledge assumptions made in the game. It seems to lose all of its power when these assumptions are slightly relaxed. For example, in the Battle of the Sexes game with outside option, imagine that Bob's payoff from exiting is not known. For some arbitrarily small but positive ϵ, his payoff from exiting is 2 with probability $1-\epsilon$ and 0 with probability ϵ. After seeing that Bob chooses to play the Battle of the Sexes, using forward induction, Alice concludes that it is not the case that Bob has a high outside option and intends to play A. She can conclude either that Bob has a high outside option and intends to play B or that Bob has a low outside option. If she interprets as a low outside option, she can also believe that Bob will play A, and play A in response. If Bob foresees this behavior, then he would exit when his outside option is high and play A when his outside option is low. This leads to an equilibrium similar to the SPNE of the original game where Bob exits; this time Alice takes Bob's playing the game as a sure sign of having a low outside option rather than a mistake.

10.6 One-Shot Deviation Principle

It may be difficult to check whether a strategy profile is a subgame-perfect Nash equilibrium in infinite-horizon games, where some paths in the game can go forever without ending the game. There is however a simple technique that can be used to check whether a strategy profile is subgame-perfect in most games. The technique is called *One-Shot Deviation Principle.* One-Shot Deviation Principle applies to multistage games with an additional mild continuity property. (In the case of repeated games, I will also present applications to games that are not multi-stage games.)

The continuity property is as follows. In a multistage game, if two strategies prescribe the same behavior at all stages, then they are identical strategies and yield the same payoff vector. Suppose that two strategies are different, but they prescribe the same behavior for very, very long successive stages, e.g., in bargaining they differ only after a billion rounds. Then, we would expect that the two strategies yield very similar payoffs. If this is indeed the case, then we call such games "continuous at infinity". (In this course, we will only consider games that are continuous at infinity. For an example of a game that is not continuous at infinity see Example 8.1.) The One-Shot Deviation Principle applies to multi-stage games that are continuous at infinity.

One-Shot Deviation Test Consider a strategy profile s^*. Pick any stage (after any history of moves). Assume that we are at that stage. Pick also a player i who moves at that stage. Fix all the other players' moves as prescribed by the strategy profile s^* at the current stage as well as in the following game. Fix also the moves of player i at all the future dates, but let his moves at the current stage vary. Can we find a move at the current stage that gives a higher payoff than s^*, given all the moves that we have fixed? If the answer is Yes, then s^* fails the one-shot deviation test at that stage for player i.

Clearly, if s^* fails the one-shot deviation test at any stage for any player i, then s^* cannot be a subgame-perfect Nash equilibrium. This is because s^* does not lead to a Nash equilibrium at the subgame that starts at that stage, as player i has an incentive to deviate to the strategy according to which i plays the better move at the current stage but follows s_i^* in the remainder of the subgame. It turns out that in a multistage game that is continuous at infinity, the converse is also true. If s^* passes the One-Shot Deviation Test at every stage (after every history of previous moves) for every player, then it is a subgame-perfect Nash equilibrium.

Theorem 10.1 (One-Shot Deviation Principle). *In a multi-stage game that is continuous at infinity, a strategy profile is a subgame-perfect Nash equilibrium if and only if it passes the one-shot deviation test at every stage for every player.*

This is a generalization of the fact that backward induction results in a Nash equilibrium, as established in Proposition 8.1. For an illustration of the proof, see the proof of Proposition 8.1. The proof in general case considered in the theorem here is similar.

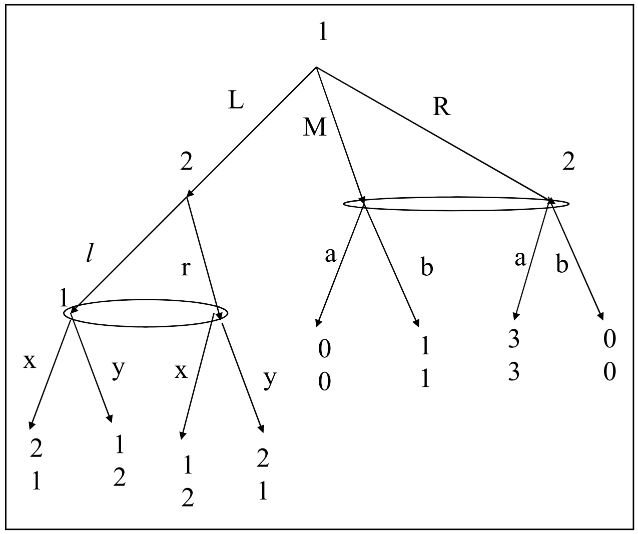


Figure 10.6: A game

Example 8.1 illustrates that the One-Shot Deviation Principle need not apply when the game is not continuous at infinity. Since all the games considered in this game are continuous at infinity, you do not need to worry about that possibility. I will illustrate the One-Shot Deviation Principle on infinite-horizon bargaining game in the next Chapter.

10.7 Exercises with Solutions

Exercise 10.1. Compute two subgame-perfect equilibria in Figure 10.6.

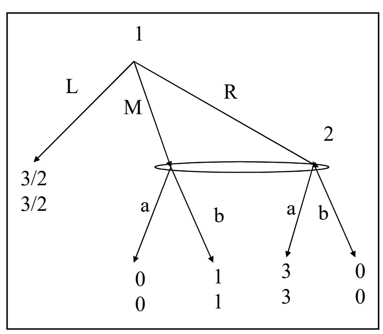


Figure 10.7: Reduced game for Figure 10.6

Solution. The only proper subgame starts after Player 1 plays L. The subgame is a matching-penny game. It has a unique Nash equilibrium, in which the each player puts equal weights on his moves. The expected payoff vector in equilibrium is $(3/2, 3/2)$. After fixing the payoffs of the subgame this way, the game reduces to the game in Figure 10.7, which can be written as

	a	b
L	$3/2, 3/2$	$3/2, 3/2$
M	$0, 0$	$1, 1$
R	$3, 3$	$0, 0$

in normal form. This game does not have a proper subgame, and each Nash equilibrium of this reduced game leads to a subgame-perfect Nash equilibrium. The pure strategy Nash equilibria are (R, a) and (L, b). These result in subgame-perfect Nash equilibria $\left(\frac{1}{2}Rx + \frac{1}{2}Ry, \frac{1}{2}la + \frac{1}{2}ra\right)$ and $\left(\frac{1}{2}Lx + \frac{1}{2}Ly, \frac{1}{2}lb + \frac{1}{2}rb\right)$ in mixed strategies.[3] There is also a continuum of Nash equilibrium, in which Player 1 plays L and Player 2 mixes between a and b putting probability $p \leq 1/2$ on a. This leads to a continuum of subgame-perfect Nash equilibria.

In order to see the equilibria more clearly, observe that M is strictly dominated by L in the reduced game, and hence Nash equilibria are contained within

	a	b
L	$3/2, 3/2$	$3/2, 3/2$
R	$3, 3$	$0, 0$

Once M is deleted, strategy a weakly dominates b. Hence, b is played with positive probability only when Player 1 plays L, in which case Player 2 is indifferent. Strategy L is a best response when the probability of a is less than or equal to $1/2$.

Exercise 10.2. Ashok and Beatrice would like to go on a date. They have two options: a quick dinner at Wendy's, or dancing at Pravda. Ashok first chooses where to go, and knowing where Ashok went Beatrice also decide where to go. Ashok prefers Wendy's, and Beatrice prefers Pravda. A player gets 3 out his/her preferred date, 1 out of his/her

[3] Here, I use $ps + (1-p)\,s'$ to denote the mixed strategy that puts probability p on s and probability $1-p$ on s'. For example, $\frac{1}{2}Rx + \frac{1}{2}Ry$ is the mixed strategy that puts equal weights on strategies Rx and Ry.

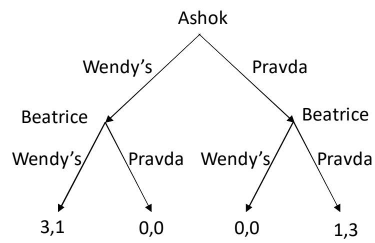


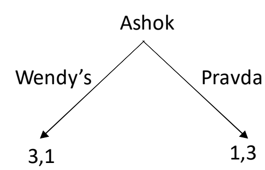
Figure 10.8: Game in Exercise 10.2, Part 1.

un-preferred date, and 0 if they end up at different places. All these are common knowledge.

1. Find a subgame-perfect Nash equilibrium. Find also a non-subgame-perfect Nash equilibrium with a different outcome.

2. Modify the game a little bit: Beatrice does not automatically know where Ashok went, but she can learn without any cost. (That is, now, without knowing where Ashok went, Beatrice first chooses between Learn and Not-Learn; if she chooses Learn, then she knows where Ashok went and then decides where to go; otherwise she chooses where to go without learning where Ashok went. The payoffs depend only on where each player goes —as before.) Find a subgame-perfect Nash equilibrium of this new game in which the outcome is the same as the outcome of the non-subgame-perfect Nash equilibrium in Part 1. (That is, for each player, he/she goes to the same place in these two equilibria.)

Solution. Part 1: This is a version of the Battle of The Sexes game in Chapter 1. In this version, the players move sequentially, as depicted in Figure 10.8. The game has two proper subgames, each corresponding to Beatrice's decision after Ashok makes his move. In each proper subgame there is a unique Nash equilibrium. Beatrice goes to Wendy's when she observes that Ashok goes to Wendy's, and she goes to Pravda when she observes that he goes to Pravda. After replacing the proper subgames with associated Nash equilibrium payoffs, the game reduces to



This game also has a unique Nash equilibrium: Ashok goes to Wendy's. Therefore, there is a unique *SPNE*: Beatrice goes wherever Ashok goes, and Ashok goes to Wendy's. The outcome is both go to Wendy's. There are also other Nash equilibria, which are not subgame-perfect. For example, in one equilibrium, Beatrice goes to Pravda at both proper subgames, and Ashok goes to Pravda. The outcome is each goes to Pravda. This is a Nash equilibrium because Ashok would have gotten 0 instead of 1 if he deviated and went to Wendy's, and Beatrice gets her maximum payoff in the game and she clearly could not gain by deviating. However, this is not subgame-perfect because it is not a Nash equilibrium in the subgame after Ashok goes to Wendy's.

Part 2: The extensive form game is as in Figure 10.9. Consider the strategy profile plotted in thicker arrows: Ashok plays Pravda, and Alice plays Don't and goes to Pravda; if she played Learn, then she would have played Wendy's if Ashok played Wendy's and Pravda if Ashok played Pravda. As in the non-subgame-perfect Nash equilibrium, they both go to Pravda at the end. This is a subgame-perfect Nash equilibrium in the new game however. The only proper subgames are the two decision nodes where Beatrice moves after learning where Ashok went, and she plays best response at these nodes, yielding a Nash equilibrium in these little subgames. As in the original game, the strategy profile is a Nash equilibrium of the whole game. Therefore, it is a subgame-perfect Nash equilibrium.

One can compute the set of all subgame-perfect Nash equilibria as follows. Pick the unique Nash equilibrium in the proper subgames as before, and replace these subgames with the associated Nash equilibrium payoffs. Then, in the reduced game, we have $(3, 1)$ after Ashok goes to Wendy's and Beatrice learns and $(1, 3)$ after Ashok goes to Wendy's and Beatrice learns, as she mimics Ashok's strategy. The normal form for the reduced

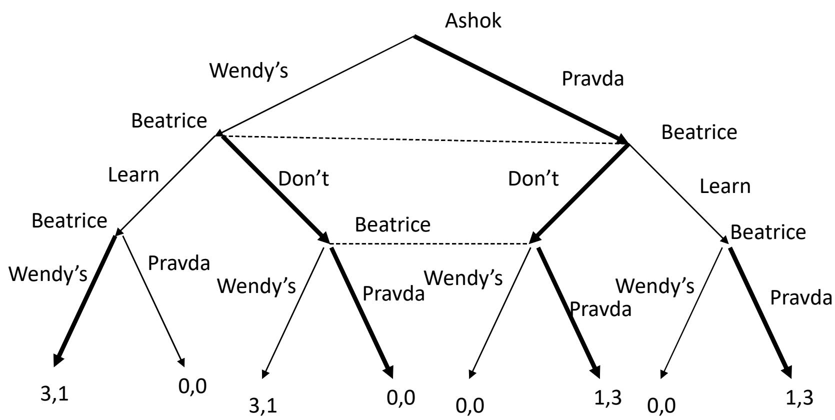


Figure 10.9:

game is as follows:

	Learn	Wendy's	Pravda
Wendy's	$3,1$	$3,1$	$0,0$
Pravda	$1,3$	$0,0$	$1,3$

Note that Beatrice has three strategies: Learn (and mimic Ashok's choice), go to Wendy's without learning, and go to Pravda without learning. Clearly, there are three subgame-perfect Nash equilibria in pure strategies: (Wendy's, Learn), (Wendy's, Wendy's), and (Pravda, Pravda). There is no subgame-perfect Nash equilibrium in mixed strategies. To see this, observe that Learn is a dominant strategy for Beatrice. Hence, if Ashok plays both strategies with positive probability, Beatrice must choose Learn. But Ashok must then choose Wendy's for sure.

Exercise 10.3. In the previous exercise, assume that Beatrice learns Ashok's choice through a noisy signal. After Ashok decides where to go, she gets a text message with a single letter $m \in \{\text{W, P}\}$, where message m is the first letter of where Ashok went with probability 0.99 and it is the first letter of the other place with probability 0.01, as depicted in Figure 10.10. Compute the subgame-perfect Nash equilibria in pure strategies.

Solution. Note that there is no proper subgame. Hence, all Nash equilibria are subgame

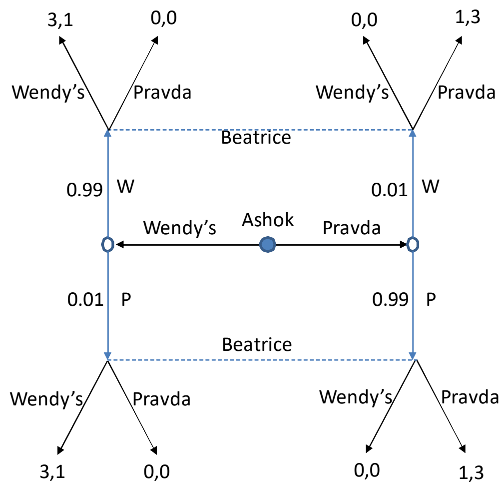


Figure 10.10: Game in Exercise 10.3.

perfect. To compute the Nash equilibria, write the game in normal form as follows:

	WW	WP	PW	PP
Wendy's	$3,1$	$0.99\,(3,1)$	$0.01\,(3,1)$	$0,0$
Pravda	$0,0$	$0.99\,(1,3)$	$0.01\,(1,3)$	$1,3$

Note that Ashok has two strategies, Wendy's and Pravda, while Beatrice has four strategies. Each strategy of Beatrice prescribes where she goes after message W and after message P. For example, WP means that she goes to Wendy's if she sees W and Pravda if she sees P. Clearly, there are two Nash equilibria in pure strategies: (Wendy's, WW) and (Pravda, PP).

Note that these equilibria corresponds to the Nash equilibria of the original Battle of Sexes game where Ashok and Beatrice choose their actions simultaneously, so that Beatrice has no information about Ashok's choice. In the new game, although Beatrice gets a highly informative signal about Ashok's choice, she ignores the signal and goes to where Ashok was supposed to go in equilibrium. For example, in equilibrium (Pravda, PP), she expects to see signal P. When she sees W instead, she attributes the aberration to mistake in signal, rather than Ashok's unexpected choice, as she is certain that Ashok will go to Pravda. She goes to Pravda as a result. Since Beatrice does not use the information in the signals, the presence of informative signals does not affect equilibria of the game. Surprisingly there is no pure-strategy equilibrium in which informative signals are used. (This is because in a pure strategy equilibrium, Beatrice will always attribute the variations in the signal to the noise and will not use them.) Note that this will remain true regardless of the probability of mistake in the signal, as long as it is positive.

Exercise 10.4. Alice is an art dealer, and Bob and Carroll are two art collectors. There is a painting that is worth v_A, v_B, and v_C for Alice, Bob, and Carroll, respectively, where $0 < v_A < v_B < v_C$. First, Alice sets a reserve price $r \geq 0$. Then, observing r, Bob and Carroll simultaneously submit bids $b_B \in [0, v_B]$ and $b_C \in [0, v_C]$, respectively. If the highest bid is less than r, then Alice keeps the painting, and each player gets 0. If the highest bid is at least r, then the highest bidder wins the auction; *if* $b_B = b_C \geq r$, *then Carroll wins the auction.* The winner pays his own bid to Alice and gets the painting. (Writing i for the winner and j for the other art collector, the payoffs are $b_i - v_A$ for Alice, $v_i - b_i$ for i, and 0 for j.)

1. Find all the subgame-perfect Nash equilibria in pure strategies.

2. How would your answer change if a bidder is allowed to bid above his own value?

Solution. Part 1: For any $r \leq v_B$, the only Nash equilibrium in the auction is (v_B, v_B). For any $r > v_B$, in any Nash equilibrium, $b_C = r$, and of course $b_B \leq v_B$. Any such pair is a Nash equilibrium. Then, Alice chooses $r = v_C$. In sum, Alice sets $r = v_C$ and Bob chooses any $b_B \in [0, v_B]$ if $r > v_B, b_B = v_B$ if $r \leq v_B$. For Carroll, he bids any $b_C \in [0, v_C]$ if $r > v_C$, $b_C = v_B$ if $r \leq v_B$, and r if $v_B < r \leq v_C$. Note that you should write a complete strategy profile for your answer.

Part 2: When $b_B > v_B$ is allowed, any pair (b, b) with $b \geq \max\{r, v_B\}$ is a Nash equilibrium in the subgame. Any $(\hat{r}, b(\cdot), b(\cdot))$ with $b(r) \geq \max\{r, v_B\}$ for all r and $b(\hat{r}) = v_C$ is a SPNE. All lead to same outcome as the one in part (a).

10.8 Exercises

Exercise 10.5. Compute the subgame-perfect Nash equilibria in Figure 10.11.

Exercise 10.6. Compute all the subgame-perfect equilibria in pure strategies in Figure 10.12.

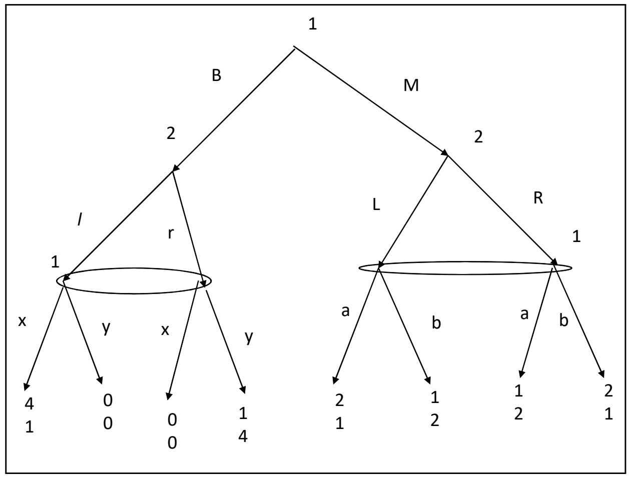


Figure 10.11: A game

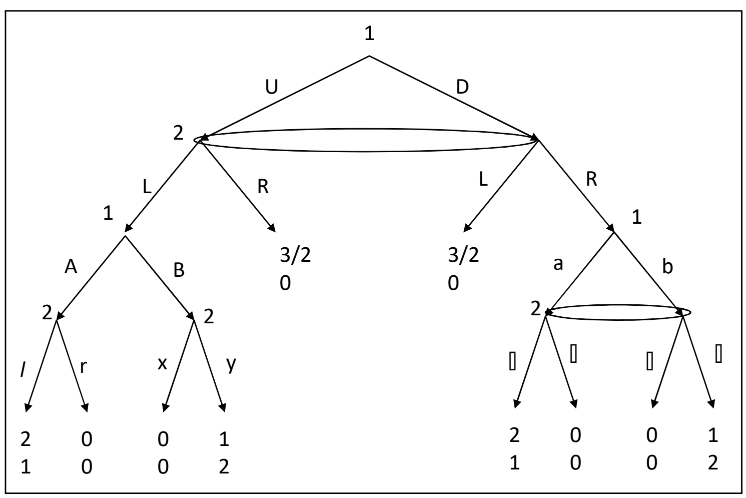


Figure 10.12: A game

Exercise 10.7. In part 2 of Exercise 10.2, assume that it costs Beatrice c to learn Ashok's choice where $c \in (0,1)$ is a small positive number. For example, in Figure 10.9, the payoff of Beatrice is $1-c$ in the leftmost branch and $-c$ in the next branch. Compute the set of subgame-perfect Nash equilibria.

Exercise 10.8. Find all subgame-perfect equilibria in the following game. Consider an employer and a worker. The employer provides the capital K (in terms of investment in technology, etc.) and the worker provides the labor L (in terms of the investment in the human capital) to produce $f(K,L) = \sqrt{KL}$, which they share equally. The parties determine their investment level (the employer's capital K and the worker's labor L) simultaneously. The worker cannot invest more than $\bar{L}$, where $\bar{L}$ is a very large number. Both capital and labor are costly, so that the payoffs for the employer and the worker are

$$\frac{1}{2}f(K,L) - rK$$

and

$$\frac{1}{2}f(K,L) - cL^2,$$

respectively. So far the problem is same as in Section 7.5.1. The present problem differs as follows. Before the worker joins the firm (in which they simultaneously choose K and L), the worker is to choose between working for this employer or working for another employer who pays the worker a constant wage $\tilde{w} > 0$ makes him work as much as $\tilde{L} = \sqrt{\frac{\tilde{w}}{2c}}$. (If he works for the other employer, the current employer gets 0.) Everything described up to here is common knowledge.

Exercise 10.9. Alice and Bob are competing to play a game against Casey. Alice and Bob simultaneously bid p_A and p_B, respectively. The one who bids higher wins; if $p_A = p_B$, the winner is determined by a coin toss. The winner pays his/her bid to Casey and play the following game with Casey:

Winner\Casey	L	R
T	3,1	0,0
B	0,0	1,3

Find two pure strategy subgame-perfect equilibria of this game. Which of the equilibria makes more sense to you?

Exercise 10.10. Find all the subgame-perfect Nash equilibria in the following game. The players are a Supplier and n (distribution) Firms. First the supplier sets a price $c \geq 0$ at which to sell to the distributors. Then, observing c, each Firm i simultaneously chooses a quantity $q_i \geq 0$, buying from the supplier at price c and selling (to consumers) at price

$$P = \max\{1 - Q, 0\}$$

where $Q = q_1 + \cdots + q_n$. Note that the payoff of the supplier is Qc, and the payoff of each firm i is $(P - c)\, q_i$.

Exercise 10.11. Consider the following game between two firms. Firm 1 either stays out, in which case Firm 1 gets 2 and Firm 2 gets 3, or enters the market where Firm 2 operates. If it enters, then the firms simultaneously choose between two strategies: Hawk (an aggressive strategy) and Dove (a peaceful strategy). In this subgame, if a firm plays Hawk and the other plays Dove, then Hawk gets 3 Dove gets 0; if both choose Hawk, then each gets -1, and if both play Dove, then each gets 1.

1. Compute the set of subgame-perfect Nash equilibria.

2. Which of the above equilibria is consistent with the assumption that Firm 2 remains to believe that Firm 1 is rational in the information set of Firm 2.

Exercise 10.12. Consider the following game, in which n firms advertise to generate demand for the good they sell before they play Cournot duopoly. First, each firm i simultaneously chooses an advertisement level $a_i \in [0, 1]$. Then, observing $(a_1, \ldots, a_n)$, each firm i simultaneously produces q_i units of the good and sells at price

$$P = a_1 + \cdots + a_n - (q_1 + \cdots + q_n),$$

obtaining payoff

$$u_i = Pq_i - \frac{1}{2}a_i^2.$$

1. Compute a subgame-perfect Nash equilibrium.

2. Imagine that the firms merge to form a new company that chooses an advertisement level a and a production level q for each firm to maximize $u_1 + \cdots + u_n$, where the advertisement and production levels are the same for all firms. Compute the advertisement level $\hat{a}$ and the production level $\hat{q}$ chosen by the company.

Exercise 10.13. Consider a two-player game with finite strategy sets S_i. Suppose that player 1 first chooses his strategy s_1. Player 2 does not see s_1 directly but observes a message $m \in S_1$ where the probabilities of messages depend on the strategy of player 1. Assume that probability $p(m|s_1)$ of message m given strategy s_1 is fixed and positive for all m and s_1. Compute the set of all subgame-perfect Nash equilibria in pure strategies. (Hint: See Exercise 10.3.)