Solutions to Problem Set 8
14.12 Fall 2023

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1 Problem 1

- Set of players $N = \{1,2\}$
- State space $\Theta = \{\frac{1}{2}, 1\}^2$
- Type space $T_i = \{t_i^{\frac{1}{2}}, t_i^1\}$
- Action space $A_i = R^+$
- Strategy space $S_i : \{\frac{1}{2}, 1\} \to R^+$
- $u(x_i, x_j; \theta) = \theta_i \sqrt{x_i}(1 - \theta_j \sqrt{x_j}) - x_i$

$$
\begin{aligned}
BR_{t_i^{\theta_i}}(s_j) &= \arg\max_x E_{\theta_j} u_{t_i^{\theta_i}}(x, s_j(\theta_j); \theta_i, \theta_j) \\
&= \arg\max_x \theta_i \sqrt{x}(1 - E_{\theta_j} \theta_j \sqrt{s_j(\theta_j)}) - x \\
&= \left(\frac{(1 - E_{\theta_j} \theta_j \sqrt{s_j(\theta_j)})\theta_i}{2} \right)^2 \\
&= \left(\frac{(1 - \frac{1}{2}(\frac{1}{2}\sqrt{s_j(\frac{1}{2})} + \sqrt{s_j(1)})\theta_i}{2} \right)^2
\end{aligned}
$$

We can focus on the symmetric BNE where $s_i^*(\theta_i = \frac{1}{2}) = a$ and $s_i^*(\theta_i = 1) = b$. a and b must satisfy the following system of equations -

$$
\begin{aligned}
a &= \left(\frac{(1 - \frac{1}{2}(\frac{1}{2}\sqrt{a}) + \sqrt{b})\frac{1}{2}}{2} \right)^2 \\
b &= \left(\frac{(1 - \frac{1}{2}(\frac{1}{2}\sqrt{a}) + \sqrt{b})}{2} \right)^2
\end{aligned}
$$

Solving the system we get $a = (\frac{4}{21})^2, b = (\frac{8}{21})^2$

2 Problem 2

- Set of players $N = \{1, 2\}$
- State space $\Theta = \{-1, 1\}$
- Type space $T_1 = \{t_1\}, T_2 = \{t_2^{-1}, t_2^1\}$
- $P(\theta, t_1, t_2) = \frac{1}{2}1(t_2 = t_2^\theta)$
- Action space $A_1 = \{X, Y, Z\}, A_2 = \{L, R\}$
- Strategy space $S_1 = A_1, S_2 = A_2 \times T_2$

$BR_{t_1}(s_2)$

	LL	RL	LR	RR
X	3	$\frac{3}{2}$	$\frac{3}{2}$	0
Y	2	2	2	2
Z	0	$\frac{3}{2}$	$\frac{3}{2}$	3

Table 1: $u_{t_1}(a_1 \mid s_2)$

$BR_{t_2^{-1}}(s_1)$

	L	R
X	-1	0
Y	-2	-1
Z	0	1

Table 2: $u_{t_2^{-1}}(a_2 \mid s_1)$

$BR_{t_2^1}(s_1)$

	L	R
X	1	0
Y	2	1
Z	0	-1

Table 3: $u_{t_2^1}(a_2 \mid s_1)$

Bayesian Nash Equilibrium is (Y, RL).

	L	R
X	3,-1	0,0
Y	2,-2	2,-1
Z	0,0	3,1

Table 4: $u(a_1, a_2 \mid \theta = -1)$

	L	R
X	3,1	0,0
Y	2,2	2,1
Z	0,0	3,-1

Table 5: $u(a_1, a_2 \mid \theta = 1)$

When it is common knowledge that $\theta = -1$, PSNE is (Z,R) (see table 4). When it is common knowledge that $\theta = 1$, PSNE is (X,L) (see table 5).

3 Problem 3

- Set of players $N = \{1, 2\}$
- $\Theta = \{1, 3\}$, $\Gamma = \{-2, 2\}$ State space is $\Theta \times \Gamma$
- Type space $T_1 = \{t_1^1, t_1^3\}, T_2 = \{t_2^{-2}, t_2^2\}$
- $P(\theta, \gamma, t_1, t_2) = P(\theta, \gamma)1(t_1 = t_2^\gamma, t_2 = t_2^\theta)$ where $P(\theta, \gamma)$ is given by the following matrix

$\theta\gamma$	-2	2
1	$\frac{1}{3}$	$\frac{1}{6}$
3	$\frac{1}{6}$	$\frac{1}{3}$

Table 6: $P(\theta, \gamma)$

- Action space $A_1 = \{a, b\}, A_2 = \{L, R\}$
- Strategy space $S_1 = A_1 \times T_1, S_2 = A_2 \times T_2$

$BR_{t_1^1}(s_2)$

	LL	RL	LR	RR
a	1	$\frac{1}{3}$	$\frac{2}{3}$	0
b	0	$\frac{2}{3}$	$\frac{1}{3}$	1

Table 7: $u_{t_1^1}(a_1 \mid s_2)$

$BR_{t_1^3}(s_2)$

	LL	RL	LR	RR
a	3	2	1	0
b	0	1	2	3

Table 8: $u_{t_1^3}(a_1 \mid s_2)$

$BR_{t_2^{-2}}(s_1)$

	L	R
aa	$-\frac{1}{3}$	0
ab	$-\frac{4}{3}$	1
ba	-1	$\frac{2}{3}$
bb	-2	$\frac{5}{3}$

Table 9: $u_{t_2^{-2}}(a_2 \mid s_1)$

$BR_{t_2^2}(s_1)$

	L	R
aa	$\frac{11}{3}$	0
ab	$\frac{7}{3}$	2
ba	4	$\frac{1}{3}$
bb	2	$\frac{7}{3}$

Table 10: $u_{t_2^2}(a_2 \mid s_1)$

So, a BNE is (bb, RR), as well as (ba, RL).

4 Problem 4

- $N = \{1, 2, ..., n\}$
- $\Theta = \{1, 2\}^n$
- $P(\theta) = \prod_i P_i(\theta_i) = \frac{1}{2^n}$
- $T_i = \{t_i^1, t_i^2\}$
- $A_i = R_+$
- $S_i : \{1, 2\} \to A_i$
- $u_{t_i^{\theta_i}}(x_i, x_{-i}) = (\theta_i - x_i - \sum_{j \neq i} x_j)x_i$.

$$
\begin{aligned}
BR_{t_i^{\theta_i}}(s_{-i}) &= \arg\max_{x\geq 0} E_{\theta_{-i}|\theta_i}[u_{t_i^{\theta_i}}(x, s_{-i}(\theta_{-i}))] \\
&= \arg\max_{x\geq 0} (\theta_i - x - E\sum_{j\neq i} x_j)x \\
&= \arg\max_{x\geq 0} (\theta_i - x - \sum_{j\neq i} \frac{s_j(1)+s_j(2)}{2})x \\
&= \frac{1}{2}(\theta_i - \sum_{j\neq i} \frac{s_j(1)+s_j(2)}{2})1(\theta_i > \sum_{j\neq i} \frac{s_j(1)+s_j(2)}{2})
\end{aligned}
$$

Denote the symmeteric BNE by (a, b) where agents choose a when their private signal is $\theta_i = 1$ and choose b when their private signal is $\theta_i = 2$

Case 1: Non-negativity constraint doesn't bind.

a and b should solve the following system of equations -

$$
\begin{aligned}
a &= \frac{1}{2}\left(1 - (n-1)\frac{a+b}{2}\right) \\
b &= \frac{1}{2}\left(2 - (n-1)\frac{a+b}{2}\right)
\end{aligned}
$$

which yields that when $n \leq 5$

$$
a = \frac{5-n}{4(n+1)}, \quad b = \frac{n+7}{4(n+1)}
$$

Case 2: Non-negativity constraint binds on choice of $s_i(\theta_i = 1)$.

$a = 0$ and b should solve

$$
b = \frac{1}{2}\left(2 - (n-1)\frac{b}{2}\right)
$$

which yields that when $n > 5$

$$a = 0, \quad b = \frac{4}{3+n}$$