Chapter 20

Auctions

Many economic transactions occur in formal centralized markets. In those markets, players trade according to strict rules that determine whether two parties will trade and the terms of trade if they trade. Some form of auction is used in many of those markets.

There are many auction forms. For example, in a typical art auction, the players raise their bids publicly until only one buyer remains, and the last standing bidder buys the piece of art paying her last bid. This *ascending-bid auction* is called *English auction*. For another example, governments buy goods and services through procurement auctions, where each seller submits a bid in a sealed envelope, and the seller with the lowest bid sells the good or the service for the price she bids. This is called *first-price procurement auction*. Similarly, many goods are sold via *first-price auction*, in which the buyers submit their bids in sealed envelopes, and the highest bidder gets the good and pays her own bid. For strategic reasons that will be clear momentarily, sometimes sellers sell the good to the highest bidder at a lower price than she bids, often charging the highest bid that did not get the good, as in the second-price auction in Chapter 3. For example, the US treasury bills are sold through a *uniform-price auction*, in which all winners pay the highest rejected bid. Finally financial markets often use a *double auction*. In these auctions, buyers submit bids and the sellers submit asking prices. A market clearing price is determined so that all buyers who bid higher than the price buy and all sellers who asked less than the price sell.

This chapter is devoted to the analyses of static auction forms, mainly focusing on the first- and the second-price auction formats. It starts with analyzing the Bayesian Nash equilibria of the first-price auction, when the players' valuation of the good auctioned is privately known and independently and identically distributed. It then goes on to comparing the first- and the second-price auctions in terms of the revenue they generate. In general, the bidders enjoy some informational rent, as they have some relevant private information that cannot be elicited without giving them incentive to reveal. Under the first-price auction, this is exhibited as shading of bids, where the buyers' bids are lower than their valuations, as bidding their valuations is weakly dominated. Under the second-price auction, it is a dominant strategy to bid their true valuations, but they pay only the second highest bid if they win. Either of the auctions can generate higher revenue depending on the realized valuations. As it turns out, the expected revenue is equal under the two auction forms, a result known as the Revenue Equivalence Theorem.

In many auctions, a buyer may have information that is relevant to another buyer's valuation. For example, in a mineral-rights auction for an oil field, the value of the field for a buyer comes from the oil reserve in the field, the cost of extraction in that field and general market conditions, and each buyer may have private information about all these aspects. When the buyers do not have any comparative advantage in extracting and selling the oil, the value will be the same for all buyers. In such common-value environments, when a buyer wins the good, she realizes that she won because the other buyers had lower assessments of the value of the good, leading them to bid lower. Thus, after learning that she won the good, a player lowers her valuation of the good. This is known as the *buyers' remorse* or the *winner's curse*. Of course, in equilibrium, the players anticipate all these and bid accordingly, taking into account the information revealed when they win the auction. In real life the players may not be that sophisticated, and naive inexperienced players may lose money when they win an auction.

20.1 First-price Auction with Independent Private Values

This section analyzes the symmetric Bayesian Nash equilibrium in first-price auctions under independently and identically distributed private values. There is an object to be sold. There are n bidders. Simultaneously, each bidder i submits a bid $b_i \geq 0$. Then, the highest bidder wins the object and pays her bid. If there are multiple players who bid the highest bid, then one of those players are selected randomly as the winner; this tie-breaking will not be relevant in the analysis. The value of the object for bidder i is v_i, which is privately known by bidder i. That is, v_i is the type of bidder i. Assume that v_1, v_2, ..., v_n are independently and identically distributed. Recall that the beliefs of a player about the other players' types may depend on the player's own type. Independence assumes that it doesn't.

20.1.1 Example: Uniformly Distributed Values

Consider the following simple case. There are only two bidders, and the values v_1 and v_2 are independently and identically distributed with uniform distribution over $[0,1]$. That is, for any $v \in [0,1]$,

$$\Pr(v_i \leq v) = v. \tag{20.1}$$

This simple formula will play a main role in computing the Bayesian Nash equilibria.

Formally, the Bayesian game is as follows. The players are the bidders. A player's type is her value, v_i. The set of types for each player is $[0,1]$. Knowing her own value v_i, player i believes that the other player's value v_j is distributed with uniform distribution over $[0,1]$. Actions are bids b_i, coming from the action spaces $[0,\infty)$. The utility functions are given by

$$u_i(b_1, b_2, v_1, v_2) = \begin{cases} v_i - b_i & \text{if } b_i > b_j, \\ \frac{v_i - b_i}{2} & \text{if } b_i = b_j, \\ 0 & \text{if } b_i < b_j. \end{cases}$$

The payoff function here reflects the fact that when player i bids higher than the other bidder, she gets the object of value v_i and pays b_i for it; when she bids lower than the other bidder, she does not get anything and does not pay anything, getting zero payoff. When the bids are equal, the payoff function assumes that player i wins with probability 1/2.

The payoff function also makes an important assumption: the bidders are risk-neutral so that the payoff from money is equal to the amount of money. This assumption will be maintained throughout the chapter, and it is crucial for the qualitative results.

In a Bayesian Nash equilibrium, each type v_i chooses some $b_i(v_i)$ in order to maximize the expected payoff

$$E\left[u_i(b_1,b_2,v_1,v_2)|v_i\right] = (v_i - b_i)\Pr\{b_i > b_j(v_j)\} + \frac{1}{2}(v_i - b_i)\Pr\{b_i = b_j(v_j)\} \qquad (20.2)$$

over b_i. In such an equilibrium, $b_i\left(v_i\right)$ must be weakly increasing in v_i.[1]

In this game, there is a unique Bayesian Nash equilibrium. The equilibrium is symmetric, and the symmetric bidding strategy is a linear function of the player's value. Section 14.5 defines symmetry and linearity and provides a step-by step recipe to compute such equilibria. The next section derives the symmetric linear equilibrium for the auction example here following those steps.

Symmetric, linear equilibrium

This section is devoted to the computation of a symmetric, linear equilibrium. Symmetric means that equilibrium action $b_i\left(v_i\right)$ of each type v_i is given by

$$b_i\left(v_i\right) = b\left(v_i\right)$$

for some function b from type space to action space, where b is the same function for all players. Linear means that b is an affine function of v_i, i.e.,

$$b_i\left(v_i\right) = a + cv_i.$$

To compute symmetric, linear equilibrium, one follows the following steps.

Step 1 *Assume a symmetric linear equilibrium:*

$$\begin{aligned} b_1^*\left(v_1\right) &= a + cv_1 \\ b_2^*\left(v_2\right) &= a + cv_2 \end{aligned}$$

for all types v_1 and v_2 for some constants a and c, that will be determined later. The key feature here is that the constants do not depend on the players or their types.

[1] Observe that the derivative of $E\left[u_i(b_1,b_2,v_1,v_2)|v_i\right]$ with respect to v_i is

$$\Pr\{b_i > b_j(v_j)\} + \frac{1}{2}\Pr\{b_i = b_j(v_j)\}.$$

Observe also that this expression is increasing in b_i. Then, by Monotonicity Theorem (Theorem 7.1), the maximizer $b_i\left(v_i\right)$ is weakly increasing in v_i.

Step 2 *Compute the best reply function of each type.* This is the main step. First, compute the expected payoff from bidding b_i for a type v_i for arbitrarily given type v_i and b_i with $a \leq b_i \leq v_i$; one can check that the best response for type v_i cannot be outside of that range. To this end, note that $c > 0$.[2] Then, $b_j^*(v_j)$ is strictly increasing in v_j and hence the other player's bid is equal to b_i with zero probability, i.e., $\Pr\{b_i = b_j^*(v_j)\} = 0$. Hence, the expected payoff from bidding b_i is

$$\begin{aligned}
E\left[u_i(b_i, b_j^*, v_1, v_2)|v_i\right] &= (v_i - b_i)\Pr\{b_j^*(v_j) \leq b_i\} \\
&= (v_i - b_i)\Pr\{a + cv_j \leq b_i\} \\
&= (v_i - b_i)\Pr\{v_j \leq \frac{b_i - a}{c}\} \\
&= (v_i - b_i) \cdot \frac{b_i - a}{c}.
\end{aligned}$$

The derivation can be spelled as follows. First, since the payoff from winning, $v_i - b_i$, does not depend on the other player's bid, the expected payoff is simply this payoff multiplied by the probability of winning (as the tie occurs with zero probability). This gives the first equality. Since b_j^* is strictly increasing, the probability of winning is equal to the probability that the other player's type is less than or equal to the type

$$v_j(b_i) = \frac{b_i - a}{c}$$

that bids b_i. This is expressed in the next two inequalities, which are obtained by substituting $b_j^*(v_j) = a + cv_j$ and simple algebra. The last equality is due to (20.1), the fact that v_j is distributed by uniform distribution on $[0, 1]$.

For a graphical derivation, consider Figure 20.1. The payoff of i is $v_i - b_i$ when $v_j \leq v_j(b_i)$ and is zero otherwise. Hence, her expected payoff is the integral

$$\int_0^{v_j(b_i)} (v_i - b_i)\, dv_j.$$

This is the shaded area of the rectangle that is between 0 and $v_j(b)$ horizontally and between b_i and v_i vertically: $(v_i - b_i)\, v_j(b_i)$.

[2] Since b is weakly increasing, $c \geq 0$. If $c = 0$, both bidders bid a independent of their type. Then, bidding 0 is a better response for a type $v_i < a$; a type $v_i > a$ also has an incentive to deviate by increasing her bid slightly. Thus, bidding $b_i(v_i) = a$ for each v_i cannot be a Bayesian Nash equilibrium. Therefore, $c > 0$.

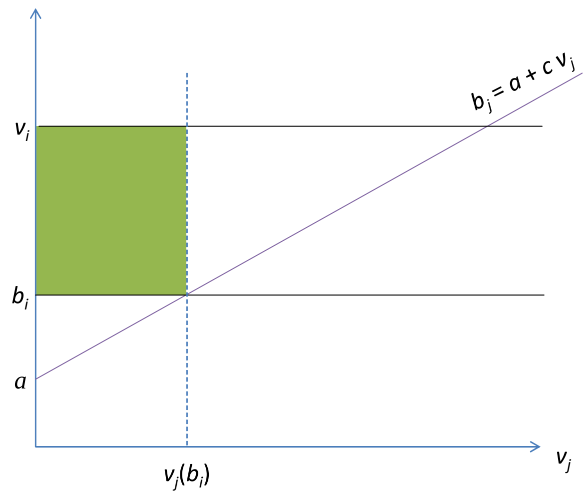

Figure 20.1: Payoff for a given bid b_i in the first-price auction

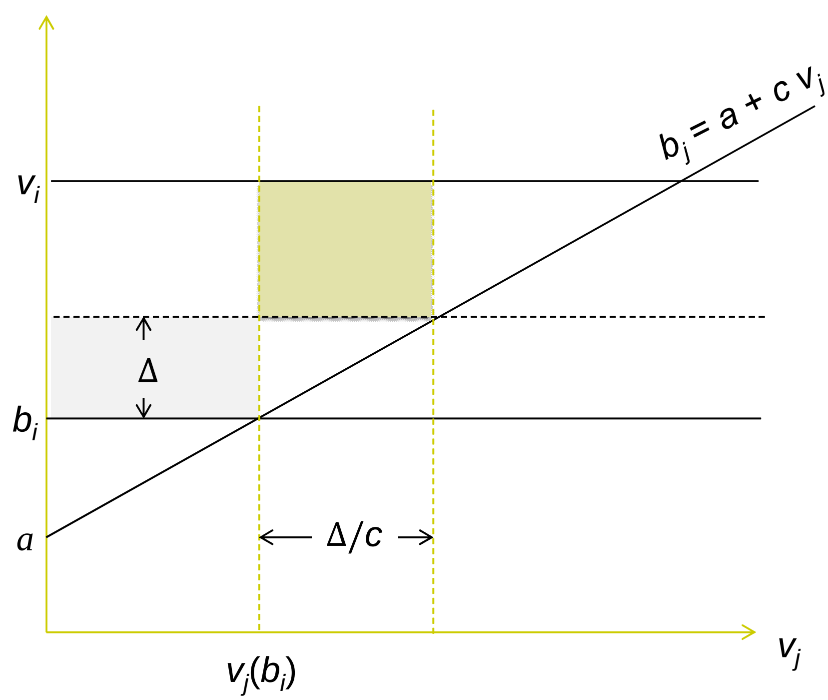

Figure 20.2: Payoff as a function of bid in first-price auction

To compute the best reply, maximize the expected payoff, $(v_i - b_i) \cdot \frac{b_i - a}{c}$, over b_i. Towards this end, take the derivative of the above expression with respect to b_i and set it equal to zero. This yields

$$b_i = \frac{v_i + a}{2}. \tag{20.3}$$

Since $(v_i - b_i)(b_i - a)/c$ is a concave function of b_i, this yields the unique best response. Graphically, as plotted in Figure 20.2, when b_i is increased by an amount of Δ, $v_j(b_i)$ increases by an amount of Δ/c. This alters the expected payoff as follows. On the one hand, it adds a rectangle of size $(v_i - b_i - \Delta)\Delta/c$, which is approximately $(v_i - b_i)\Delta/c$ when Δ is small. This is the *gain* from increasing the bid. On the other hand, it subtracts a rectangle of size $v_j(b_i)\Delta$. This is the *loss* from increasing the bid. At the optimum the gain must be equal to loss:

$$(v_i - b_i)\Delta/c = v_j(b_i)\Delta.$$

The bid in (20.3) above solves this equality.

Remark 20.1. Note that one takes an integral to compute the expected payoff and takes a derivative to compute the best response. Since the derivative is an inverse of integral, this involves unnecessary calculations in general. In this particular example, the calculations were simple. In general those unnecessary calculations may be the hardest step. Hence, it is advisable that one leaves the integral as is and use Leibniz's rule[3] to differentiate it to obtain the first-order condition. Indeed, the graphical derivation above does this.

Step 3 *Verify that best-reply functions are indeed affine*, i.e., b_i is of the form $b_i = a + cv_i$. To do this, rewrite (20.3) as

$$b_i = \frac{1}{2}v_i + \frac{a}{2}.$$

Check that both $1/2$ and $a/2$ are constant, i.e., they do not depend on v_i, and they are same for both players.

[3] Leibniz's Rule:

$$\frac{\partial}{\partial x}\int_{t=L(x,y)}^{U(x,y)} f(x,y,t)\,dt = \frac{\partial U}{\partial x}\cdot f(x,y,U(x,y)) - \frac{\partial L}{\partial x}\cdot f(x,y,L(x,y)) + \int_{t=L(x,y)}^{U(x,y)} \frac{\partial}{\partial x} f(x,y,t)\,dt. \tag{20.4}$$

Step 4 *Compute the constants a and c.* To do this, observe that in order to have an equilibrium, the best reply b_i in (20.3) must be equal to $b_i^*(v_i)$ for each v_i. That is,

$$\frac{1}{2}v_i + \frac{a}{2} = cv_i + a.$$

must be an identity, i.e., it must remain true for all values of v_i. Hence, the coefficient of v_i must be equal in both sides:

$$c = \frac{1}{2}.$$

The intercept must be same in both sides, too:

$$a = \frac{a}{2}.$$

Thus,

$$a = 0.$$

This yields the symmetric, linear Bayesian Nash equilibrium:

$$b_i\left(v_i\right) = \frac{1}{2}v_i.$$

This is the only symmetric linear equilibrium. Indeed, as we will see in the next section for the general case, there is a unique equilibrium, and hence the symmetric linear equilibrium above is the only equilibrium. In this equilibrium, each bidder bids only the half of her value. She bids strictly lower than her value because this is the only way she can get a positive payoff. Indeed, her payoff is the difference between her value and her bid multiplied by probability of winning.

Players face a fundamental trade off when they prepare their bids, as can be seen in Step 2 above. They want to increase the probability of winning and want to decrease the price they pay at the same time. When a player raises her bid, the probability of winning increases, but the player ends up paying more when she wins. The gain from raising the bid Δ amount is

$$Gain = \left(v_i - b_i\right) \times \Delta / c,$$

where $v_i - b_i$ is the value of winning the object at the current bid, and $1/c$ is the rate at which her bid affects her probability of winning. The loss from increasing the bid is

$$Loss = \Delta \Pr\left(b_j\left(v_j\right) \leq b_i\right),$$

where $\Pr(b_j(v_j) \leq b_i)$ is the probability of winning at the current bid. As seen in the main step, the loss is equal to the gain at the optimum, so that the player does not have an incentive to raise or lower her bid. What makes this cost-benefit analysis non-trivial is that the amounts of gain and the loss are endogenously determined in equilibrium. In particular, both the rate $1/c$ and the probability $\Pr(b_j(v_j) \leq b_i)$ depend on the equilibrium strategy employed by the other player. Analysis of a first-price auction boils down to determining the equilibrium amount of gain and loss in this trade off.

20.1.2 General Case

Now consider the general case: the types $v_1, \ldots, v_n$ are independently and identically distributed with a probability density function f and cumulative distribution function F. That is, for each type v,

$$\Pr(v_i \leq v) = F(v), \tag{20.5}$$

where F is a weakly increasing differentiable function and $f(v) = F'(v)$. (In the case of uniform, f is 1 and F is identity on $[0,1]$.) Assuming that $F(\bar{v}) = 1$ for some $\bar{v} > 0$ and $f(v)$ is bounded away from zero on $[0, \bar{v}]$, there exists a unique Bayesian Nash equilibrium. The equilibrium is symmetric, and the bidding function is strictly increasing and differentiable. This section computes the equilibrium assuming it satisfies the above properties. First, consider the two player case $n = 2$; the solution to this case will be used to obtain the solution for arbitrary n.

Equilibrium with two bidders There is an intuitive way to compute a symmetric Bayesian Nash equilibrium—as follows. Consider a symmetric Bayesian Nash equilibrium in which each player plays the same bidding strategy b where b is a strictly increasing and has a derivative everywhere.[4] Imagine that player i is offered *proxy bidding*. She would only need to submit her value v_i privately to a computer, and the computer computes the equilibrium bid $b(v_i)$ for her and submits $b(v_i)$ for her. The computer does not know her value and would bid $b(v_i')$ if she had submitted another bid v_i' instead. If b is a symmetric Bayesian Nash equilibrium, then player i would not have an incentive

[4] The bidding strategy must be weakly increasing by supermodularity as described above, and any weakly increasing function is differentiable almost everywhere. Here, we assume for simplicity that it is strictly increasing and differentiable everywhere.

to submit a false value $v_i' \neq v_i$. If she did, then she would have an incentive to deviate to the bid $b(v_i')$ in the actual auction. One uses this fact to compute the symmetric equilibrium strategy as follows. The payoff of player i from submitting value w to the computer is

$$U(w) = (v_i - b(w)) F(w).$$

To see this, observe that, when computer submits the bid $b(w)$, by symmetry, she wins the object if and only if the other player's equilibrium bid $b(v_j)$ is less than or equal to $b(w)$. Since b is strictly increasing, this is equivalent to $v_j \leq w$, which has probability $F(w)$ by definition. Since player i does not have an incentive to deviate, $U(w)$ must be maximized at $w = v_i$, and the necessary first-order condition for this maximization is $U'(v_i) = 0$. Towards computing the first order condition, compute

$$U'(w) = (v_i - b(w)) f(w) - b'(w) F(w).$$

The first-order condition is

$$(v_i - b(v_i)) f(v_i) - b'(v_i) F(v_i) = 0.$$

The equilibrium bidding strategy b must satisfy this differential equation.

In general, using the above technique, one can obtain a differential equation like above and a boundary condition (e.g. the bidder must bid zero when her value is zero in this case), but solving the differential equation may be challenging. In this particular case, it is straightforward. One can write the above differential equation as

$$\frac{d}{dv_i}[b(v_i) F(v_i)] = v_i f(v_i).$$

By integrating both sides,[5] one then obtains the solution

$$b(v_i) = \frac{\int_0^{v_i} v f(v)\, dv}{F(v_i)}. \tag{20.6}$$

The equilibrium bid will be spelled out momentarily. An important remark on this derivation is in order.

[5] Integration yields

$$b(v_i) F(v_i) = \int_0^{v_i} v f(v)\, dv + c$$

for some constant c. But $c = b(0) F(0) = 0$.

The equilibrium bid is derived from the first-order condition for the player submitting her true value to the computer in the proxy bidding. There are two loose ends in this derivation. First, this method checks that player i does not have an incentive to deviate to another bid $b(w) \neq b(v_i)$, and it does not guarantee that she has no incentive to a bid that is not used in equilibrium. For example, if the computer always bid $b(w) \equiv 0$, then she would be indifferent between all values, but she would not use the proxy bidding as she would like to raise her bid a little bid and win for sure. Second, the first-order condition is necessary but not sufficient. If one knows that there is a symmetric equilibrium with increasing differentiable strategies, the bid in (20.6) is the unique equilibrium. Otherwise, one would need to verify that $b(v_i)$ is indeed a best response to the strategy b for each v_i. This last step will be omitted throughout.

The equilibrium bid in (20.6) has a useful economic interpretation. It states that each player bids the expected value of the other player's value conditional on winning the auction:

$$b(v_i) = E[v_j | v_j \leq v_i, v_i]. \tag{20.7}$$

Indeed, in equilibrium, player i wins when the other player's bid is blow her bid, equivalently the other player's type is below her own type. The probability of this event is $F(v_i)$, which is the denominator of the expression in (20.6). The numerator in that expression is the integral of the other player's value on this event. Therefore, the expression computes the conditional expectation of the other player's value for this event. As players bid the conditional expectation of the other player's value given that that value is lower than their own, their bids are lower than their value.

To compute the amount of shading, using integration by parts, write

$$\int_0^{v_i} v f(v)\, dv = v_i F(v_i) - \int_0^{v_i} F(v)\, dv.$$

Then, by (20.6),

$$b(v_i) = v_i - \frac{\int_0^{v_i} F(v)\, dv}{F(v_i)}. \tag{20.8}$$

That is, in equilibrium, a bidder shades her bid down by an amount of

$$R(v_i) = \frac{\int_0^{v_i} F(v)\, dv}{F(v_i)}.$$

Note that the winner gains this amount from the transaction. This is the informational rent she extracts by virtue of knowing her own value privately.

Equilibrium with n bidders For general $n \geq 2$, observe that, if player i submits w to the computer for proxy bidding, the probability of winning is

$$\Pr\{b(v_j) \leq b(w) \quad \forall j \neq i\} = F^{n-1}(w).$$

In order for player i to win, bid $b(w)$ must exceed $b(v_j)$ for $n-1$ other players, and the probability of this event is $F^{n-1}(w)$ because $b(v_j)$ are independently and identically distributed and b is increasing. When $n = 2$, this number was simply $F(w)$. This is the only change in the above analysis when $n > 2$. The solution for general n is then obtained by substituting F^{n-1} for F. Therefore, the symmetric Bayesian equilibrium strategy is

$$b(v_i) = v_i - \frac{\int_0^{v_i} F^{n-1}(v)\,dv}{F^{n-1}(v_i)}. \tag{20.9}$$

Players shade their bids by the amount of

$$\frac{\int_0^{v_i} F^{n-1}(v)\,dv}{F^{n-1}(v_i)}.$$

The latter formula gives the difference between the value and the bid of a player, measuring the informational rent the player gets when she wins. This formula plays a central role in the economic analysis of first-price auctions. In order to shed some light on this abstract formula, the next example applies it to the case of uniform distribution with n bidders.

Example 20.1. Consider n-bidder first price auction where the values v_i are independently and identically distributed with uniform distribution on $[0,1]$. One can use the formula (20.9) to compute the symmetric Bayesian Nash equilibrium strategy as:

$$b(v_i) = v_i - \frac{\int_0^{v_i} F^{n-1}(v)\,dv}{F^{n-1}(v_i)} = v_i - \frac{\int_0^{v_i} v^{n-1}dv}{v_i^{n-1}} = v_i - \frac{1}{n}v_i = \frac{n-1}{n}v_i. \tag{20.10}$$

Here the second equality uses the fact that $F(v) = v$ for the uniform distribution, and the rest is simple algebra. The amount of shading is v_i/n, inversely proportional to the number of bidders. As the number of bidders grows, the shading goes to zero, and everybody bids approximately her own bid.

This is more generally true. One can write the shading for type v_i as

$$R(v_i) = \int_0^{v_i} \left(\frac{F(v)}{F(v_i)}\right)^{n-1} dv.$$

The function $(F(v)/F(v_i))^{n-1}$ is decreasing in n, rendering its integral $R(v_i)$ also decreasing in n. That is, as the number of bidders grows, the amount of shading decreases, increasing the bids for each realization of values. This makes the outcome more competitive and raises the revenue for the seller. As $n \to \infty$, the function $(F(v)/F(v_i))^{n-1}$ goes to zero. Hence, the amount $R(v_i)$ of shading also goes to zero, each player bidding her own value. The limiting outcome is fully competitive: the bidder with the highest value buys the good paying her own value.

As in the two-bidder case, in equilibrium, the players bid the expected value of the largest value lower than their own value. Indeed, write

$$v_{-i,\max} = \max_{j \neq i} v_j$$

for the largest value among the other players. When she submits the value w, she wins when $v_{-i,\max} \leq w$, which has probability $F^{n-1}(w)$. Thus, as in the two-player case, the equilibrium bid can be written as

$$b(v_i) = E[v_{-i,\max} | v_{-i,\max} \leq v_i]. \tag{20.11}$$

That is, equilibrium bid is the conditional expectation of the maximum of the other players' values given that it is less than her own value.

20.2 Revenue Equivalence

In a first-price auction, the seller charges the winner her bid, but the buyers have a strong incentive to under-bid, resulting in lower bid levels for all valuations. In contrast, in a second-price auction, the seller charges only the second-highest bid, but this time it is a dominant strategy to bid one's own value, resulting in higher bid levels. Which one of these auction formats generates a higher revenue to the seller?

Depending on the realization of the values, either auction formats can generate a higher-revenue. To see this, assume that the values are independently and identically distributed with uniform distribution on $[0, 1]$. Write

$$v_{1st} = \max\{v_1, \ldots v_n\}$$

for the highest value and

$$v_{2nd} = \max_{j \neq i^*} v_j$$

for the second-highest value, where i^* is one of the buyers with the highest value. Under the first price auction, the winner pays

$$b_{1st} = \frac{n-1}{n} v_{1st},$$

keeping v_{1st}/n as the informational rent. Under the second-price auction the winner pays

$$b_{2nd} = v_{2nd},$$

keeping $v_{1st} - v_{2nd}$ as the informational rent this time. When the difference between the highest and the second highest bid is large, the first-price auction generates more revenue. More precisely, when

$$v_{1st} - v_{2nd} > v_{1st}/n,$$

the first-price auction generates more revenue to the seller, as it leaves a smaller informational rent to the buyer. Conversely, when

$$v_{1st} - v_{2nd} < v_{1st}/n,$$

the second-price auction generates more revenue to the seller, as it leaves a smaller informational rent to the buyer.

Which auction format a seller should choose if he is risk-neutral and wants to maximize the expected revenue from the auction? To answer this question, it is useful to observe a basic fact about the order statistics from the probability theory: the expected value of the kth-highest value is

$$E[v_{kth}] = \frac{n+1-k}{n+1}$$

when n values are uniformly and independently distributed on $[0,1]$. Now, if the seller chooses the first-price auction, the expected revenue is

$$E[b_{1st}] = \frac{n-1}{n} E[v_{1st}] = \frac{n-1}{n+1}.$$

If he chooses the second-price auction, the expected revenue is again

$$E[b_{2nd}] = E[v_{2nd}] = \frac{n-1}{n+1}.$$

Therefore, the seller gets the same expected revenue regardless of which auction format he chooses.

This is an instance of the Revenue Equivalence Theorem that states under general distributions that the expected revenue under the symmetric Bayesian Nash equilibrium of the first-price auction is equal to the expected revenue under the dominant strategy of the second-price auction:

Theorem 20.1 (Revenue EquivalenceTheorem)**.** *Expected revenue in the symmetric Bayesian Nash equilibria of the first- and the second- price auction are equal to*

$$ER = n \int E\left[v_{-i,\max} | v_{-i,\max} \leq v_i\right] F^{n-1}(v_i) f(v_i) \, dv_i,$$

where $v_{-i,\max} = \max_{j \neq i} v_j$.

Proof. First consider the second-price auction, and take any player i. We want to compute how much a player i is expected to pay, which she does only when she wins. The expected revenue is the sum of the payments, and, by symmetry, it is equal to n times the expected payment by player i. Now, conditional on any v_i, in the second-price auction, player i knows that she will win the auction if and only if $v_{-i,\max} \leq v_i$, which has probability $F^{n-1}(v_i)$, and when she wins she will pay $v_{-i,\max}$, and the conditional expectation is $E\left[v_{-i,\max} | v_{-i,\max} \leq v_i\right]$. Therefore, player i with value v_i expects to pay

$$E\left[v_{-i,\max} | v_{-i,\max} \leq v_i\right] F^{n-1}(v_i).$$

The expected payment by player i is the expected value of this conditional expectation—for the unknown value v_i. The expected payment is computed by integrating over v_i according to the density $f(v_i)$:

$$\int E\left[v_{-i,\max} | v_{-i,\max} \leq v_i\right] F^{n-1}(v_i) f(v_i) \, dv_i.$$

The expected revenue is n times this expected payment as there are n players. This proves the theorem for the second-price auction.

Now consider the first-price auction. Once again, take any player i with value v_i. By (20.11), she bids $E\left[v_{-i,\max} | v_{-i,\max} \leq v_i\right]$ and knows that she will win the auction and pay $E\left[v_{-i,\max} | v_{-i,\max} \leq v_i\right]$ if and only if $v_{-i,\max} \leq v_i$, which has probability $F^{n-1}(v_i)$. Thus, she expects to pay

$$E\left[v_{-i,\max} | v_{-i,\max} \leq v_i\right] F^{n-1}(v_i),$$

exactly as in the case of second-price auction. By integrating this value and multiplying it by n, one then computes that the expected revenue is again ER as in the second-price auction. $\square$

In the first-price auction, a player pays the expected amount that she would have paid in the second price auction, conditional on having the highest value. Therefore, her expected payment must be equal in the two auctions. The equality is in terms of expectations only. For any given profile of values, the player i with the highest value v_i pays the second-highest value $v_{-i,\max}$ in the second-price auction while she pays its conditional expectation $E\left[v_{-i,\max}|v_{-i,\max} \leq v_i\right]$ in the first-price auction. Consequently, the realized prices are more dispersed under the second-price auction.

20.3 Interdependence of Values in Auctions

The previous sections assumed that each player (privately) knows the value of the auctioned object for her (and the values are independently distributed). While this assumption may hold in some environments, such as art auctions where the buyers are buying the art for personal consumption, it fails in many real-world applications. In those applications, players do not fully know the value of the object for themselves, and other players may have private information about this value.

For example, in an art auction, often the buyers are art collectors who buy the art not only for its consumption value but also as an investment to be sold later in life. Likewise, some other buyers may be art dealers who buy the art for resale. The value of the art for such buyers come not only from their private enjoyment of the art, which they may know, but also from its resale value, about which each buyer is privately informed. In this example, the players do not know the value of the piece of art for themselves and are aware that the other players have a relevant information about the piece of art. For another example, in a typical auction for the rights to extract minerals, the value of the rights is the amount of profits one can generate from those rights, which may be common for all buyers. Before bidding in such a high-stake auction, the buyers presumably carry out private research, examining the size of the reserve, difficulty of extraction on the site, market condition for the minerals, and so on. Each player has some information about the value, and the information a player has is relevant for the other players' valuations as well as her own. For yet another example, in a typical real-estate auction, the value of the real estate comes from many factors, such as the location, the quality of the school district and difficulty of the commute. Of course, different players may value each of these aspects differently (if they are buying the real estate for personal use), but the other buyers' information about these aspects are also relevant for a player's own valuation. In this example, the value of the object is not common but the values are interdependent.

In these environments, players would like to use the information that the other parties have for their valuation of the good. When they submit their bids, they take into account the private information revealed by other players' bid. Since the information revealed depends on the bidding strategies employed by the other players, this is a highly sophisticated inference problem that requires the knowledge of the (game theoretical) solution. In general, the game theoretical analysis of such strategic environments is substantially more complex than the one in the previous sections. The next section introduces a tractable environment with such an information structure to illustrate some of the main issues in such environments. The general case is analyzed in Section 20.5.

20.4 Wallet Auction

This section introduces a tractable environment with interdependent values. In this environment, the value of the object is common, and the common value depends on certain components. Each player knows some of the components but not all of them. For an artificial example, envision that a wallet with multiple pockets is being auctioned. There is some money in each pocket, and each player knows the amount of money in one pocket. (A less artificial application would be the sale of a firm that operates in many markets, where the buyers are local firms that operate in some of those markets.)

Formally, an object is being auctioned. There are two players, 1 and 2. The value of the object is

$$v = t_1 + t_2$$

for either player where each player i knows the value of t_i, and t_1 and t_2 are independently and identically distributed by uniform distribution on $[0, 1]$. This is a common-value environment, where the value of the object is the same for all players. Each player knows a separate component of this value. The expected value of the object for a type t_i is $E[v|t_i] = t_i + 1/2$. Substituting naively for the value of the object in the previous analyses leads to misleading conclusions, as it does not take other players' information into account.

Second-Price Wallet Auction To see this clearly, first imagine that the object is sold using second-price auction. When players knew their own value, under the second-price auction, there was a dominant-strategy equilibrium, where each player submits their own true value as her bid. Now, the players do not know their own value, v. Each player i knows only t_i, and has expected value $E[v|t_i] = t_i + 1/2$. Imagine that each player i naively submits

$$b_{naive,2nd}(t_i) = t_i + 1/2,$$

extrapolating from the independent-value case. Now, the higher value bidder i wins the object and pays the other player's bid $t_j + 1/2$. Then, observing that she won the object, the winning player i realizes that the other player's type t_j is below her own value t_i, and the expected value of the object under this information is only

$$E[v|t_i, t_j \leq t_i] = t_i + E[t_j|t_j \leq t_i] = t_i + t_i/2 = 3t_i/2.$$

Here the first equality is by the fact that $v = t_i + t_j$. Instead of naively putting the expected value of t_j, one puts $E[t_j|t_j \leq t_i] = t_i/2$. This value is $t_i/2$, yielding the second equality. This conditional expectation is always smaller than the interim expected value $t_i + 1/2$. Therefore, as soon as she wins the object, the winner lowers her expectation of the good. When she makes the payment $t_j + 1/2$, the winner also learns the true value of the object, $v = t_1 + t_2$. At this point the winner may lose money, and she may rather want to cancel the trade. This is the case when $t_i + t_j < t_j + 1/2$, i.e.,

$$t_i < 1/2.$$

When such a low-value bidder wins the auction despite her low bid, she is in for a rude awakening that she won only because the object has low value. She would now wish that she did not win the auction. This is sometimes referred to as *buyer's remorse* or *the winner's curse*.

The players in this book are much more sophisticated than that. When they submit their bid, they take the information that will be revealed as the outcome of the bids into

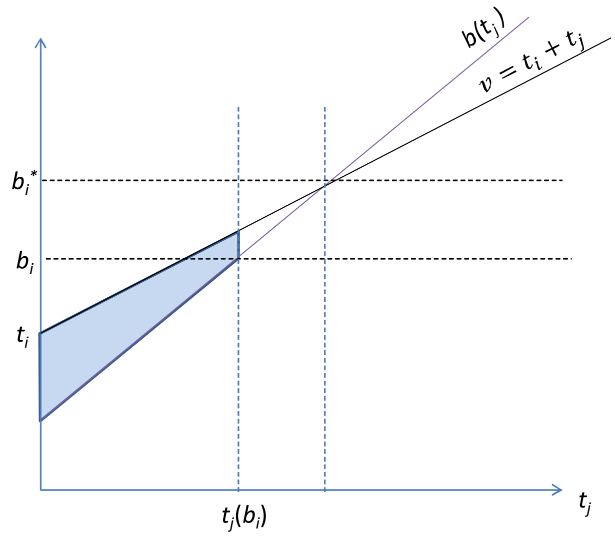

Figure 20.3: Expected payoff in second-price wallet auction

account. Since the revealed information depends on the bidding strategy the other player uses, the inference from these events depends on the player's belief about the bidding strategy of the other player, and she does not have a dominant strategy anymore.

To see this clearly and compute her best response, suppose that player j uses a bidding strategy b with slope larger than 1, where b is a function that maps each type t_j to a bid $b(t_j)$ for player j. As illustrated in Figure 20.3, for any t_j, the value of the object is $t_i + t_j$, and player i pays $b(t_j)$ if she ends up buying it. Imagine that she bids b_i as in the figure. When t_j is less than or equal to the type $t_j(b_i) = b^{-1}(b_i)$ who also bids b_i, player i wins the object and receives the value $v = t_i + t_j$ for price $b(t_j)$, obtaining the net benefit of

$$t_i + t_j - b(t_j).$$

This net benefit remains positive until the bid function $b(t_j)$ intersects the value v; the bid in the intersection is denoted by b^*, as plotted in the figure. For the types $t_j > t_j(b_i)$, player j wins the auction and player i gets zero. The expected payoff from bidding b_i is[6]

$$U(b_i) = \int_0^{t_j(b_i)} (t_i + t_j - b(t_j))\, dt_j.$$

When $b_i < b^*$, the expected payoff is the size of the shaded area in the figure. As one can see clearly from the figure, the area gets larger as b_i increases, and player i has an incentive to raise her bid. Formally,

$$U'(b_i) = t_i + t_j(b_i) - b_i,$$

and increasing the bid a little bit will add a positive area of a size proportional to $t_i + t_j(b_i) - b_i$. The expected payoff is maximized when $U'(b_i) = 0$. The solution b_i^* to this equation is given by

$$b_i^* = t_i + t_j(b_i^*), \tag{20.12}$$

as plotted in the figure.[7] When choosing her best response, player i focuses on the event that her bid is pivotal, in that she can change the winner by slightly adjusting her bid. This happens when her bid is equal to the other player's bid. At this event, player i infers from the other player's bidding strategy that the other player's type is $t_j(b_i)$, and the value of the object is $t_i + t_j(b_i)$. She bids this value as her best response. This formalizes the idea that one should bid her true value in a second-price auction. Unlike in the case that she knows her value, the value inferred here is *endogenously* determined both by her own bid b_i and the bidding strategy b of the other player. The best response varies as the bidding strategy of the other player varies. Since the best response depends on the bidding strategy of the other player, there is no dominant strategy.

Note that when a player plays a best response, she will not feel a buyer's remorse. Even if she lowered her expectation after observing the other player's bid, she does not regret that she won. Indeed, by definition of b_i^*, her net payoff is always non-negative:

$$t_i + t_j - b(t_j) \geq 0$$

[6]Notice that the expected payoff is an integral, as the return depends on the other player's type t_j when player i wins. As noted earlier, it is best to leave this integral as is, as one takes its derivative to compute the best response.

[7]This is true when the bidding function intersects the value as in the figure. If the bidding function does not intersect the value or its slope is less than 1, then there is no interior solution. Player i either bids a very high number to ensure winning with probability 1 or bids a very low number and loses with probability 1.

for any $t_j \leq t_j(b_i^*)$ under which she wins the object. This is because she has taken the possibility into account when she prepared her bid. Likewise, if she learns that the other player has bid a higher number and won the object, she will not wish that she bid higher and won the object. Although she now thinks that the value of the object is higher, i.e., the value is $t_i + t_j$ for some $t_j > t_j(b_i^*)$, she is not willing to pay the higher price to get the object because now the net benefit from winning is negative: $t_i + t_j - b(t_j) < 0$ as shown in the figure. She has prepared for this contingency, too, when she prepared her bid.

Using the above analysis, one can easily compute the symmetric Bayesian Nash equilibrium for the second-price wallet auction as follows. Assume that, in a Bayesian Nash equilibrium, both players play a strategy b, where the slope is greater than 1. Since type i plays $b(t_i)$ as a best response, the equality (20.12) holds for $b_i^* = b(t_i)$:

$$b(t_i) = t_i + t_j(b(t_i)) = t_i + b^{-1}(b(t_i)) = t_i + t_i = 2t_i.$$

Here, the first equality is by (20.12), the next equality is by definition $t_j(b_i) = b^{-1}(b_i)$, and the rest is simple algebra. In the symmetric Bayesian Nash equilibrium, each type t_i bids twice her own type, bidding for the value when the other player's type is equal to her own type. After all, her bid is pivotal only when their bids are equal, and under symmetry, this happens when the types are equal. (This simple characterization does not depend on uniformity assumption; it is valid for all distributions.)

First-Price Wallet Auction Now, consider the case that the object is sold using first-price auction. Simultaneously, each player i submits a bid b_i and the one who submits a higher bid wins; the winner is determined by a coin toss in case of a tie. The winner buys the object and pays her own bid.

Towards computing a symmetric Bayesian Nash equilibrium, assume that each player uses a strictly increasing bidding function b. Consider a bid b_i for type t_i. If the other player's type t_j is less than the cutoff type $t_j(b_i) = b^{-1}(b_i)$ who also bids b_i, player i wins the auction and gets the object of value $v = t_i + t_j$ at price b_i, obtaining a net payoff of

$$t_i + t_j - b_i.$$

If the other player's type is above $t_j(b_i)$, the other player wins, and player i gets zero.

Hence, the expected payoff from bidding b_i is

$$U(b_i) = \int_0^{t_j(b_i)} (t_i + t_j - b_i)\, dt_j.$$

It is useful to compare this expected payoff with the expected payoff when the value v_i was privately known. In that case, the expected payoff simplified to $(v_i - b_i)\Pr(t_j \leq t_j(b_i))$ because player i knew that she would get $(v_i - b_i)$ if she won, regardless of the other player's type. Now, the value of the object *does* depend on the other player's type, and hence the payoff of player i is also a function of type t_j. The expected payoff does not simplify to the above formula. One can still compute the integral easily, but as it has been noted earlier it is best to leave the integral as is because one takes the derivative of U next in order to compute the best response.

The best response is obtained when $U'(b_i) = 0$. Taking the derivative of the integral above, one obtains[8]

$$U'(b_i) = \frac{1}{b'(t_j(b_i))}(t_i + t_j(b_i) - b_i) - t_j(b_i).$$

Since type i plays $b(t_i)$ as a best response, we have $U'(b(t_i)) = 0$. Substituting $b_i = b(t_i)$—and thus $t_j(b_i) = t_i$—in the above expression and setting it equal to zero, one obtains

$$t_i b'(t_i) + b(t_i) = 2t_i.$$

As in the case of the first price auction with known values, the left-hand side is the derivative of $t_i b(t_i)$ with respect to t_i, and hence one can solve this differential equation as

$$t_i b(t_i) = t_i^2,$$

yielding

$$b(t_i) = t_i.$$

In the symmetric Bayesian Nash equilibrium, each type bids her own type. Intuitively, at the event in which her bid is pivotal, the other player's bid is equal to her own, and

[8] Here one first takes the derivative of $t_j(b_i)$, which is $1/b'(t_j(b_i))$, and multiplies it with the value of the expression in the integral at the upper limit of the integral, which is $t_i + t_j(b_i) - b_i$. Since the lower limit is constant, one does not need to consider the lower limit. One then adds the integral of the derivative of the expression, which is $\int_0^{t_j(b_i)} -1 dt_j = -t_j(b_i)$.

this happens when the other player's type is equal to her type. Thus, the inferred value of the object is $2t_i$. She bids half of this value—as in the equilibrium of the first-price auction with privately known value v_i, where she bids $v_i/2$.

Once again, learning that she won the object, the winner lowers her valuation of object from $E[v|t_i] = t_i + 1/2$ to $E[v|t_i, t_j \leq t_i] = 3t_i/2$. But she has foreseen this while she submitted her bid. For example, in this particular case, she does not lose money from winning because her bid, t_i, cannot exceed the total value, $t_i + t_j$. She is happy that she won. A naive solution would be bidding half of her expected value $t_i + 1/2$. If both player bid half of their expected value, then the winner would lose money when her type is low. Indeed, for the winner with type t_i,

$$v = t_i + t_j \leq 2t_i,$$

as $t_j \leq t_i$, and $2t_i$ is below her bid $(t_i + 1/2)/2$ whenever $t_i \leq 1/6$. If a player with such a low type wins the auction under this naive solution, she will lose money regardless of the other player's type, feeling a buyer's remorse.

One may think that a naive player—as described here—overbids by failing to take into the negative information that will be revealed when she wins. That is not the case. Under both the first- and the second-price auction formats, equilibrium strategy is more sensitive to the type—with a larger slope—as higher types imply better information to be revealed when their bids pivotal. In particular, naive players overbid for low values with $t_i < 1/2$ and underbid for high values with $t_i > 1/2$. Indeed, under the first price auction, the naive bid $(t_i + 1/2)/2$ is more than the equilibrium bid t_i if and only if $t_i < 1/2$, and these bids are doubled under the second price auction.

Remark 20.2. One can also use the proxy-bidding technique to compute the equilibrium as follows. Under the bidding function b, the expected payoff from submitting type w (instead of t_i) is

$$U(w) = \int_0^w (t_i + t_j - b(w))\, dt_j.$$

If b is a symmetric equilibrium bidding function, U must be maximized at $w = t_i$, implying the first-order condition that $U'(t_i) = 0$ for any $t_i \in (0, 1)$. Since $U'(w) = t_i + w - b(w) - b'(w)\, w$, the first-order condition is $t_i b'(t_i) + b(t_i) = 2t_i$, the differential equation above.

20.5 Interdependent Values†

The previous sections illustrated the main insights of auction theory and some useful techniques to analyze auctions under some simplifying assumption, such as independence of players' types (and the private values in most parts). This section presents the general theory of the first- and the second-price auctions, a theory that subsumes the previous case. It focuses on the two-bidder case for clarity, but the theory can be extended to many bidder case as in the independent-value case.

An object is to be sold through an auction. There are $n = 2$ bidders. Each bidder i has a type t_i that reflects player's information about the value of the object. The types (t_1, t_2) are symmetrically distributed with joint distribution function $F : [0, \bar{t}]^2 \to [0, 1]$ and joint density $f : [0, \bar{t}]^2 \to \mathbb{R}$ for some maximum type $\bar{t}$. The conditional distribution and the density are denoted by $F(t_j|t_i)$ and $f(t_j|t_i)$, respectively. It will be assumed throughout that $f(t_i|t_i) > 0$ (i.e., there is positive density everywhere on the diagonal). The value of the object for a player i may depend on both players' types; it is

$$v_i = v(t_i, t_j)$$

for some weakly increasing continuous function v, where the first entry is player's own type and the second entry is the type of the other player.

This model subsumes many special cases:

Private Value The player's value depends only on her own type: $v(t_i, t_j)$ does not depend on t_j. When types are independently distributed we have independent private value case.

Common Value The player's value depends both types symmetrically

$$v(t_i, t_j) = v(t_j, t_i).$$

Wallet Auction Wallet auction is a special case of common value with

$$v(t_i, t_j) = t_i + t_j$$

and independently distributed types.

The previous sections also assumed that the types are independently distributed. This assumption will be relaxed: it is assumed that the types are "affiliated", i.e.,

$$f(t_1, t_2) f(t_1', t_2') \leq f(\max\{t_1, t_1'\}, \max\{t_2, t_2'\}) f(\min\{t_1, t_1'\}, \min\{t_2, t_2'\}).$$

This condition is satisfied by equality when the types are independently distributed, and weak inequality expresses a strong form of positive correlation. For example, this condition is satisfied when $t_i = \theta + \varepsilon_i$ for some independently distributed random variables $(\theta, \varepsilon_1, \varepsilon_2)$ drawn from common distributions such as uniform and normal distributions. An important implication of this assumption for this section is that the conditional hazard rate

$$\frac{f(t_j|t_i)}{F(t_j|t_i)}$$

is weakly increasing in t_i.

With this generalization, the model has another special case:

Correlated Private Values The player's value depends only on her own type, i.e., $v(t_i, t_j)$ does not depend on t_j, while t_1 and t_2 are correlated.

I will next present the symmetric Bayesian Nash equilibrium for the first- and the second-price auctions. I will start with the second-price auction.

Second-Price Auction As we have seen in the case of the wallet auction, the players do not know their own value and there is no longer a dominant-strategy equilibrium. As in the case of wallet auction, there is a unique symmetric equilibrium, where each bidder bids

$$b(t_i) = v(t_i, t_i). \tag{20.13}$$

The intuition and the derivation for this result is as in the wallet auction. Here, I present the derivation using the proxy bidding technique; see Figure 20.3 for illustration (for the case of the wallet auction). Suppose that each player i submits a type w and a computer bids $b(w)$ for the player as a proxy. If b is a symmetric equilibrium bid, player should not have an incentive to submit a false type, i.e., submitting $w = t_i$ must be a best response. The payoff from submitting w is

$$U(w) = \int_0^w (v(t_i, t_j) - b(t_j)) f(t_j|t_i) \, dt_j.$$

This is because she wins when $t_j \leq w$, and for each such t_j, she obtains the payoff of $v(t_i, t_j) - b(t_j)$. Since she knows that her type is t_i, one enters $v(t_i, t_j)$ for the value and t_j has density $f(t_j|t_i)$. Observe that

$$U'(w) = (v(t_i, w) - b(w)) f(w|t_i).$$

Hence, $U(w)$ is strictly increasing when $v(t_i, w) > b(w)$ and strictly decreasing when $v(t_i, w) < b(w)$, as long as $f(w|t_i) > 0$. Therefore, in order for $w = t_i$ to be a best response, it must be that $v(t_i, t_i) = b(t_i)$, proving (20.13).

Intuitively, in choosing her bid, a player zeroes in the case that her bid is pivotal, i.e., her bid will affect whether she will get the object. Under the symmetric equilibrium, her bid is pivotal when the other player's type is equal to her own. In that case, the value of the object is $v(t_i, t_i)$ and she bids this value.

First-Price Auction Consider the first-price auction. Towards computing the symmetric Bayesian Nash equilibrium, once again, imagine that each player submits a type w, and a computer bids $b(w)$ for that player. For a type t_i, the expected payoff from submitting w is

$$U(w) = \int_0^w (v(t_i, t_j) - b(w)) f(t_j|t_i)\, dt_j.$$

This time she pays $b(w)$ when she wins, and hence the payment is $b(w)$; the rest is as in the second-price auction. The first-order condition for best response is that $U'(t_i) = 0$, so that it is optimal for type t_i to submit her true type. This first-order condition can be computed as follows. By Leibniz's formula,

$$\begin{aligned} U'(w) &= (v(t_i, w) - b(w)) f(w|t_i) - b'(w) \int_0^w f(t_j|t_i)\, dt_j \\ &= (v(t_i, w) - b(w)) f(w|t_i) - b'(w) F(w|t_i). \end{aligned}$$

Here, the first term is the marginal gain from submitting a higher type:

$$G(w) = (v(t_i, w) - b(w)) f(w|t_i).$$

By submitting a slightly higher type $w + \Delta$ instead of w, she increases her probability of winning by the amount of $f(w|t_i)\Delta$, and this value of trade at that pivotal event is $v(t_i, w) - b(w)$. The gain is $G(w)\Delta$. The second term is the marginal loss from raising her bid:

$$L(w) = b'(w) F(w|t_i).$$

By submitting $w+\Delta$ instead of w, she raises her bid by an amount of $b'(w)\Delta$, and she pays that extra amount whenever she wins, regardless of what the other player's type is. The probability of that extra payment is $F(w|t_i)$. The loss is $L(w)\Delta$. At the optimum the gain must be equal to the loss. In particular, in order for submitting the true type to be a best response, the gain must be equal to the loss at $w=t_i$:

$$G(t_i) = (v(t_i,t_i) - b(t_i)) f(t_i|t_i) = b'(t_i) F(t_i|t_i) = L(t_i).$$

This is the first-order condition above. One can re-write this as

$$b'(t_i) = (v(t_i,t_i) - b(t_i)) \frac{f(t_i|t_i)}{F(t_i|t_i)}. \tag{20.14}$$

This is the differential equation that any symmetric equilibrium must satisfy.

The differential equation for the bidding function reflects the players' focus on the event that their bids are pivotal. By raising her bid b_i by an amount of Δ, a player raises the probability of winning by an amount of $\Delta f(t_i|t_i)/b'(t_i)$. But, at the pivotal event, the types and thus the bids are equal, and the value of the object is $v(t_i,t_i) - b(t_i)$. Hence, her gain from raising her bid is $(v(t_i,t_i) - b(t_i)) \times \Delta f(t_i|t_i)/b'(t_i)$. Raising her bid also costs her. It increases her expected payment by an amount of $\Delta F(t_i|t_i)$ as she makes the extra payment at all cases that she buys the object. This is the loss. The differential equation makes the gain equal to the loss. In particular, the ratio $\frac{f(t_i|t_i)}{F(t_i|t_i)}$ reflects the fact that the gain is obtained only at the pivotal event $t_j = t_i$, while the loss is incurred for all types $t_j \le t_i$.

The important properties of equilibrium can be directly glanced from the differential equation. For example, one can see that the bid function is increasing. Indeed, the expected value of the object for the player is $E[v(t_i,t_j)|t_i, t_j \le t_i]$. Her bid cannot be more than this amount. Since $v(t_i,t_j)$ is weakly increasing in t_j, this implies that $(v(t_i,t_i) - b(t_i)) \ge 0$, showing that the bidding function is increasing.

The solution to the differential equation is as follows. Rewrite the differential equation in a cleaner form as

$$b'(t) + h(t) b(t) = v(t,t) h(t)$$

where

$$h(t) = \frac{f(t|t)}{F(t|t)}.$$

This is a standard differential equation, and its solution is

$$\mu(t)\, b(t) = \int \mu(t)\, v(t,t)\, h(t)\, dt + c$$

where c is a constant and

$$\mu(t) = \exp\left(\int h(t)\, dt\right)$$

is the integrating factor. Since $b(0) = 0$, the solution can be written as

$$b(t) = \int_0^t \frac{\mu(x)}{\mu(t)} v(x,x)\, h(x)\, dx,$$

where the ratio can be simplified as $\frac{\mu(x)}{\mu(t)} = \exp\left(\int_t^x h(y)\, dy\right)$.

20.6 All-Pay Auctions

In the above auction formats only the winner pays and buys the object, and the other bidders simply get zero. Although this is generally true in many applications, in some real-world applications all players end up paying regardless of whether they win or lose. Such "all-pay" auctions are relevant especially in the analysis of real-world contests that may not appear to be an auction. For example, in an athletic competition, the athletes exert effort and only the winner gets the prize, where presumably the athlete who puts the most effort wins. In product development, often multiple firms invest large sums of money in developing a product, such as a cure for a disease or a self-driving car, and the firm who develops the product first gets a patent and enjoys monopoly profits. (In this example, losing firms can also get some lower profit from developing other products later or recuperate some of their investments by dissolving the remaining assets.) This section illustrates the analyses of such all-pay auctions.

There are two buyers who bid for an object. The value of the object for each buyer i is $v_i \geq 0$, where v_1 and v_2 are identically and independently distributed with probability density function f and cumulative distribution function F. Each bidder simultaneously bid $b_i \geq 0$; the bidder who bids the highest bid gets the object, and each bidder i pays his own bid b_i. (If $b_1 = b_2$, then each gets the object with probability $1/2$.) The payoff of player i is

$$u_i = \begin{cases} v_i - b_i & \text{if } b_i > b_j, \\ v_i/2 - b_i & \text{if } b_i = b_j, \\ -b_i & \text{if } b_i < b_j. \end{cases}$$

This can be written as a Bayesian game as follows. The set of players is $N = \{1, 2\}$; the types are v_1 and v_2 where set of types is $[0, \infty)$ for each player; actions are b_1 and b_2, coming from action set $[0, \infty)$ for each player, and the payoffs are as above.

One can compute a symmetric Bayesian Nash equilibrium, where each player uses a strictly increasing differentiable bidding strategy b, as follows. Since the bidding strategy b of the other player is strictly increasing, when a player i with type v_i bids b_i, the ties occur with zero probability, and the expected player of player i is

$$U(b_i) = \Pr\left(v_j \leq b^{-1}(b_i)\right) v_i - b_i = F\left(b^{-1}(b_i)\right) v_i - b_i.$$

When the other player's type v_j is below $b^{-1}(b_i)$, player i wins the object and enjoys the value v_i, and this happens with probability $\Pr(v_j \leq b^{-1}(b_i)) = F(b^{-1}(b_i))$. With the remaining probability, player i loses the object. Regardless of what the other player bids, she pays b_i, and this yields the expected payoff above. The key feature in the formula is that the bid is paid regardless, and hence it is simply subtracted from expected payoff from winning. In contrast, the payoff in a first-price auction would be $F(b^{-1}(b_i))(v_i - b_i)$, as the winner pays b_i only when she wins. Towards computing the equilibrium strategy, compute that

$$U'(b_i) = \frac{f(b^{-1}(b_i))}{b'(b^{-1}(b_i))} v_i - 1.$$

Since type v_i plays $b(v_i)$ as a best response, it must be that $U'(b(v_i)) = 0$. Then substituting $b^{-1}(b_i) = v_i$ in the above expression and setting it equal to zero, one obtains

$$b'(v_i) = f(v_i) v_i.$$

By taking the integral of both sides, one then obtains the equilibrium strategy:

$$b(v_i) = \int_0^{v_i} v f(v)\, dv. \tag{20.16}$$

For example, when the values are uniformly distributed on $[0, 1]$, the equilibrium bidding strategy is

$$b(v_i) = \int_0^{v_i} v dv = \frac{1}{2} v_i^2.$$

This is substantially lower than the equilibrium bid for the first-price auction, $\frac{1}{2} v_i$. This is because the players now pay regardless of whether they win or lose, and hence they are stingier in their bids.

As in first-price auctions, one can use the above solution to obtain the solution for the case of n players. In that case, the expected payoff is

$$U(b_i) = F^{n-1}\left(b^{-1}(b_i)\right) v_i - b_i,$$

and one simply uses the distribution F^{n-1} in two player case. The density function for this distribution is $(n-1) f(v) F^{n-2}(v)$. Substituting this for f in (20.16), one obtains

$$b(v_i) = (n-1) \int_0^{v_i} v f(v) F^{n-2}(v)\, dv \tag{20.17}$$

as the unique symmetric Bayesian Nash equilibrium strategy for all-pay auction with n players.

For example, when the values are uniformly distributed on $[0,1]$, the equilibrium bidding strategy is

$$b(v_i) = (n-1) \int_0^{v_i} v \times v^{n-2} dv = (n-1) \int_0^{v_i} v^{n-1} dv = \frac{n-1}{n} v_i^n. \tag{20.18}$$

Once again, the bids under all-pay auction are substantially lower than the equilibrium bid for the first-price auction, $\frac{n-1}{n} v_i$. Moreover, the difference between the two auctions becomes more pronounced as the number of players gets larger. Indeed, as $n \to \infty$, the equilibrium bid goes to v_i under the first price auction, and it goes to 0 under the all-price auction. Intuitively, when there are many competitors, player i understands that it is likely that there are other players who have similar values to her own. Under first-price auction, this prevents her from shading her bid. Under all-price auction, this implies that she will pay the bid but win the object with small probability, and she bids a small amount in return.

20.7 Double Auction

In many trading mechanisms, both buyers and the sellers submit bids (although the price submitted by the seller is often referred to as "ask" rather than "bid"). Such mechanisms are called *double auction*, where the name emphasizes that both sides of the market are competing. This section is devoted to the case when there is only one buyer and one seller. (This case is clearly about bilateral bargaining, rather than general auctions.)

Consider a seller, who owns an object, and a buyer. They want to trade the object through the following mechanism. Simultaneously, the seller names p_S and the buyer names p_B.

- If $p_B < p_S$, then there is no trade;
- if $p_B \geq p_S$, then they trade at price

$$p = \frac{p_B + p_S}{2}.$$

The value of the object is v_S for the seller and v_B for the buyer. Each player knows her own valuation privately. Assume that v_S and v_B are independently and identically distributed with uniform distribution on $[0, 1]$. Then, the payoffs are

$$\begin{aligned} u_B &= \begin{cases} v_B - \frac{p_B + p_S}{2} & \text{if } p_B \geq p_S \\ 0 & \text{otherwise} \end{cases} \\ u_S &= \begin{cases} \frac{p_B + p_S}{2} - v_S & \text{if } p_B \geq p_S \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

In a Bayesian Nash equilibrium, one must compute an ask price $p_S(v_S)$ for each type v_S of the seller and a bid price $p_B(v_B)$ for each type v_B of the buyer. In a Bayesian Nash equilibrium, the bid price $p_B(v_B)$ maximizes

$$U_B(p_B) = \int_{p_B \geq p_S(v_S)} \left[v_B - \frac{p_B + p_S(v_S)}{2} \right] dv_S,$$

where the integral is taken over the range at which $p_B \geq p_S(v_S)$, and the ask price $p_S(v_S)$ maximizes

$$U_S(p_S) = \int_{p_B(v_B) \geq p_S} \left[\frac{p_S + p_B(v_B)}{2} - v_S \right] dv_B,$$

where the integral is taken over the range at which $p_B(v_B) \geq p_S$.

In this game, there are many Bayesian Nash equilibria. For example, one equilibrium is given by

$$\begin{aligned} p_B &= \begin{cases} X & \text{if } v_B \geq X \\ 0 & \text{otherwise} \end{cases}, \\ p_S &= \begin{cases} X & \text{if } v_S \leq X \\ 1 & \text{otherwise} \end{cases} \end{aligned}$$

for some any fixed number $X \in [0,1]$. The next section computes the Bayesian Nash equilibrium with linear strategies.

Equilibrium with linear strategies Consider an equilibrium where the strategies are affine functions of valuation, but they are not necessarily symmetric.

Step 1 *Assume that there is an equilibrium with linear strategies:*

$$\begin{aligned} p_B(v_B) &= a_B + c_B v_B \\ p_S(v_S) &= a_S + c_S v_S \end{aligned}$$

for some constants a_B, c_B, a_S, and c_S. Assume also that $c_B > 0$ and $c_S > 0$. Notice that the bid and ask functions, p_B and p_S, respectively, may differ from each other.

Step 2 *Compute the best responses for all types.* To do this, first note that

$$p_B \geq p_S(v_S) = a_S + c_S v_S \iff v_S \leq \frac{p_B - a_S}{c_S} \tag{20.19}$$

and

$$p_S \leq p_B(v_B) = a_B + c_B v_B \iff v_B \geq \frac{p_S - a_B}{c_B}. \tag{20.20}$$

To compute the best response for a type v_B, one first computes her expected payoff from her bid (leaving it in an integral form). By (20.19), her expected payoff from p_B is

$$U_B(p_B) = \int_0^{\frac{p_B - a_S}{c_S}} \left[v_B - \frac{p_B + p_S(v_S)}{2} \right] dv_S.$$

By substituting $p_S(v_S) = a_S + c_S v_S$ in this expression, one can obtain

$$U_B(p_B) = \int_0^{\frac{p_B - a_S}{c_S}} \left[v_B - \frac{p_B + a_S + c_S v_S}{2} \right] dv_S.$$

Visually, this is the shaded area in Figure 20.4.[9]

[9] The area is

$$U_B(p_B) = \frac{p_B - a_S}{c_S} \left(v_B - \frac{3p_B + a_S}{4} \right),$$

but it is not needed for the final result.

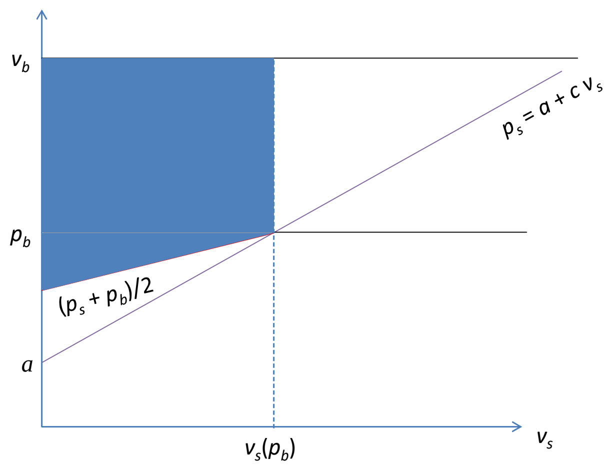

Figure 20.4: Payoff of of a buyer in double auction for a given bid; the expected payoff is the shaded area.

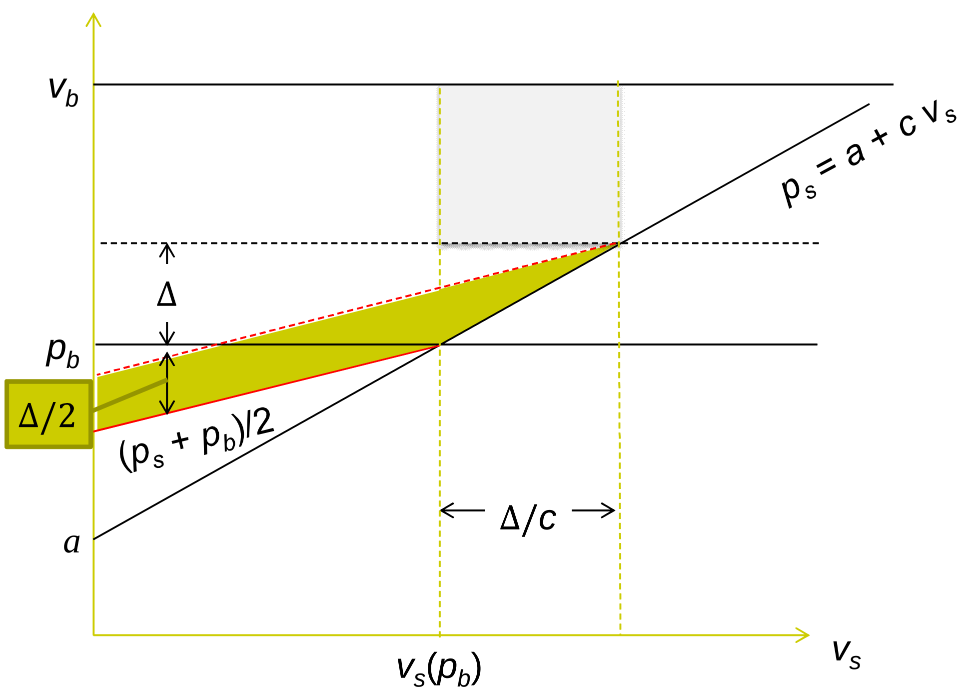

Figure 20.5: The change in the payoff of of a buyer in double auction as his bid changes

To compute the best response, compute the derivative $U_B'(p_B)$ using the Leibniz rule:

$$U_B'(p_B) = \frac{v_B - p_B}{c_S} - \int_0^{\frac{p_B - a_S}{c_S}} \frac{1}{2} dv_S = \frac{v_B - p_B}{c_S} - \frac{1}{2}\frac{p_B - a_S}{c_S}.$$

Setting $U_B'(p_B) = 0$ and solving the equation for p_B, obtain

$$p_B = \frac{2}{3} v_B + \frac{1}{3} a_S. \tag{20.21}$$

Graphically, a Δ amount of increase in p_B has two impacts on the expected payoff. First it causes a Δ/c_S amount of increase in $v_S(p_B)$, adding the shaded rectangular area of size $(v_B - p_B)\Delta/c_S$ in Figure 20.5. It also increases the price by an amount of $\Delta/2$, subtracting the shaded trapezoidal area of approximate size $v_S(b)\Delta/2$. At the optimum the two amounts must be equal, yielding the above equality.

Now compute the best reply of a type v_S. As in before, the expected payoff from playing p_S in equilibrium is

$$U_S(p_S) = \int_{\frac{p_S - a_B}{c_B}}^{1} \left[\frac{p_S + a_B + c_B v_B}{2} - v_S\right] dv_B,$$

where the last equality is by (20.20) and $p_B(v_B) = a_B + c_B v_B$. Once again, in order to compute the best response, compute the derivative $U_S'(p_S)$ and set it equal to zero:[10]

$$-\frac{1}{c_B}(p_S - v_S) + \int_{\frac{p_S - a_B}{c_B}}^{1} \frac{1}{2} dv_B = \frac{1}{2}\left(1 - \frac{p_S - a_B}{c_B}\right) - \frac{1}{c_B}(p_S - v_S) = 0.$$

Once again, a Δ increase in p_S leads to a $\Delta/2$ increase in the price, resulting in a gain of $\left(1 - \frac{p_S - a_B}{c_B}\right)\Delta/2$. It also leads to a Δ/c_B decrease in the types of buyers who trade, leading to a loss of $(p_S - v_S)\Delta/c_B$. At the optimum, the gain and the loss must be equal, yielding the above equality. Solving for p_S, one can then obtain

$$p_S = \frac{2}{3} v_S + \frac{a_B + c_B}{3}. \tag{20.22}$$

[10] One uses the Leibniz rule. The derivative of upper bound is zero, contributing zero to the derivative. The derivative of the lower bound is $1/c_S$, and this is multiplied by the expression in the integral at the lower bound, which is simply $p_S - v_S$. (Note that at the lower bound $p_B = p_S$, and hence the price is simply p_S.) Finally, one adds the integral of the derivative of the expression inside the integral, which is simply $1/2$.

Step 3 *Verify that best replies are of the form that is assumed in Step 1.* Inspecting (20.21) and (20.22), one concludes that this is indeed the case. The important point here is to check that in (20.21) the coefficient 2/3 and the intercept $\frac{1}{3}a_S$ are constants, independent of v_B. Similarly for the coefficient and the intercept in (20.22).

Step 4 *Compute the constants.* To do this, identify the coefficients and the intercepts in the best replies with the relevant constants in the functional form in Step 1. Firstly, (20.21) and $p_B(v_B) = p_B$ yield the identity

$$a_B + c_B v_B = \frac{1}{3}a_S + \frac{2}{3}v_B.$$

That is,

$$a_B = \frac{1}{3}a_S \tag{20.23}$$

and

$$c_B = \frac{2}{3}. \tag{20.24}$$

Similarly, (20.22) and $p_S(v_S) = p_S$ yield the identity

$$a_S + c_S v_S = \frac{a_B + c_B}{3} + \frac{2}{3}v_S.$$

That is,

$$a_S = \frac{a_B + c_B}{3} \tag{20.25}$$

and

$$c_S = \frac{2}{3}. \tag{20.26}$$

Solving (20.23), (20.24), (20.25), and (20.26), one can obtain $a_B = 1/12$ and $a_S = 1/4$.

Therefore, the linear Bayesian Nash equilibrium is given by

$$p_B(v_B) = \frac{2}{3}v_B + \frac{1}{12} \tag{20.27}$$

$$p_S(v_S) = \frac{2}{3}v_S + \frac{1}{4}. \tag{20.28}$$

In this equilibrium, the parties trade if and only if

$$p_B(v_B) \geq p_S(v_S)$$

i.e.,

$$\frac{2}{3}v_B + \frac{1}{12} \geq \frac{2}{3}v_S + \frac{1}{4},$$

which can be written as

$$v_B - v_S \geq \frac{1}{4}.$$

Whenever $v_B > v_S$ there is a positive gain form trade. When the gain from trade is lower than 1/4, the parties leave this gain from trade on the table. This is because of the incomplete information. The parties do not know that there is a positive gain from trade. Even if they tried to find ingenious mechanisms to elicit the values, buyer would have an incentive to understate v_B and seller would have an incentive to overstate v_S, and some gains from trade would not be realized.

20.8 Exercises with Solution

Exercise 20.1. A consumer needs 1 unit of a good. There are n firms who can supply the good. The cost of producing the good for firm i is c_i, which is privately known by i, and $(c_1, c_2, \ldots, c_n)$ are independently and uniformly distributed on $[0, 1]$. Simultaneously, each firm i sets a price p_i, and the consumer buys from the firm with the lowest price. (If $k > 1$ firms charge the lowest price, he buys from one of those firms randomly, each selling with probability $1/k$.) The payoff of i is $p_i - c_i$ if it sells and 0 otherwise. Find all symmetric Bayesian Nash equilibria in strictly increasing and differentiable strategies. What happens as $n \to \infty$? (**Hint:** Given any $\bar{c} \in (0, 1)$, the probability that $c_j \geq \bar{c}$ for all $j \neq i$ is $(1 - \bar{c})^{n-1}$.)

Solution. The problem here can be viewed as a *procurement auction*, in which the lowest bidder wins. This is closely related to the problem in which n buyers with privately known values bid in a first-price auction. One can convert this to regular first-price auction by changing the variables as $v_i = 1 - c_i$, which are uniformly distributed, and $b_i = 1 - p_i$. One can then use formula (20.10) to obtain $b_i = v_i - \frac{1}{n} v_i$. Therefore, the unique symmetric Bayesian Nash equilibrium strategy is

$$p_i = 1 - b_i = \frac{1}{n} + \frac{n-1}{n} c_i.$$

Exercise 20.2. Consider a game between two software developers, who sell operating systems (OS) for personal computers. (There are also a PC maker and the consumers, but their strategies are already fixed.) Each software developer i, simultaneously offers "bribe" b_i to the PC maker. (The bribes are in the form of contracts.) Looking at the offered bribes b_1 and b_2, the PC maker accepts the highest bribe (and tosses a coin between them if they happen to be equal), and he rejects the other. If a firm's offer is rejected, it goes out of business, and gets 0. Let i^* denote the software developer whose bribe is accepted. Then, i^* pays the bribe b_{i^*}, and the PC maker develops its PC compatible only with the operating system of i^*. Then in the next stage, i^* becomes the monopolist in the market for operating systems. In this market the inverse demand function is given by

$$P = 1 - Q,$$

where P is the price of OS and Q is the demand for OS. The marginal cost of producing the operating system for each software developer i is c_i. The costs c_1 and c_2 are independently and identically distributed with the uniform distribution on $[0, 1]$. Each software developer i privately knows its own marginal cost c_i and wants to maximize its own expected profit.

1. What quantity a software developer i would produce if it becomes monopolist? What would be its profit?

2. Compute a symmetric Bayesian Nash equilibrium.

3. Considering that the demand for PCs and the demand of OSs must be the same, should PC maker accept the highest bribe? (Assume that PC maker also tries to maximize its own profit. Explain your answer.)

Solution. Part 1: Since the software developer who wins becomes a monopoly, it produces the

$$q_i = \frac{1 - c_i}{2}$$

and obtains the profit

$$v_i = \left(\frac{1 - c_i}{2}\right)^2,$$

excluding the sunk cost of the bribe that it paid to the PC maker.

Part 2: This is a first-price auction in which the buyers are software developers and the value of the object for each buyer i is $v_i = (1 - c_i)^2 / 4$. Since c_i is uniformly distributed, the cumulative distribution function of v_i is given by

$$F(v) = \Pr\left((1 - c_i)^2 / 4 \le v\right) = \Pr\left(1 - c_i \le 2\sqrt{v}\right) = 2\sqrt{v}.$$

Substituting this in (20.8), one obtains

$$b(v_i) = v_i - \frac{\int_0^{v_i} 2\sqrt{v}dv}{2\sqrt{v_i}} = \frac{1}{3}v_i.$$

Each software developer offers one third of its profit as a bribe.

Part 3: A low-cost monopolist will charge a lower price, increasing the profit for the PC maker. Since low-cost software developers pay higher bribes, it is in the PC maker's interest to accept the higher bribe. In that case, he will get higher bribe now and higher profits later.

Exercise 20.3. Alice and Bob have inherited a factory from their parents. The value of the factory is v_A for Alice and v_B for Bob, where v_A and v_B are independently and uniformly distributed over $[0,1]$, and each of them knows his or her own value. Simultaneously, Alice and Bob bid b_A and b_B, respectively, and the highest bidder wins the factory and pays the other sibling the other sibling's bid. (The winner is determined by a coin toss if the bids are equal.) Write this game as a Bayesian game. Find all symmetric Bayesian Nash equilibria of this game in strictly increasing differentiable strategies.

Solution. Consider a symmetric Bayesian Nash equilibrium, in which both players play a strategy b where b is strictly increasing and differentiable. Use "proxy bidding" technique to compute b. For a type v_i, the payoff from bidding $b(w)$ is

$$U(w) = \int_0^w (v_i - b(v_j))\, dv_j + \int_w^1 b(w)\, dv_j.$$

When $v_j < w$, player i's bid $b(w)$ is higher than player j's bid $b(v_j)$, and player i wins the auction obtaining payoff $v_i - b(v_j)$, as player i gets the factory with value i and pays $b(v_j)$. This gives the first integral. When $v_j > w$, the other player wins the factory and pays $b(w)$ to player i, yielding the second integral. Towards computing the first-order condition for the best response, using the Leibniz's rule in (20.4), compute the derivative of the expected value as

$$U'(w) = v_i - 2b(w) + b'(w)(1-w).$$

Since type v_i plays $b(v_i)$ as a best response, this yields the first-order condition

$$U'(v_i) = v_i - 2b(v_i) + b'(v_i)(1-v_i) = 0.$$

This condition can be rewritten as

$$b'(v_i)(v_i - 1) + 2b(v_i) = v_i.$$

The solution to this differential equation must be of the form $b(v) = Av + B$ for some constants A and B. Towards computing the constants A and B, one substitutes $b'(v_i) = A$ and $b(v_i) = Av_i + B$ in the differential equation, obtaining

$$A(v_i - 1) + 2Av_i + 2B = v_i.$$

Since this must be an identity, one equalizes the slopes and the intercepts of the two sides of the equation as $3A = 1$ and $2B - A = 0$. Solving these equations, one obtains $A = 1/3$ and $B = 1/6$. Therefore, the unique equilibrium strategy is given by

$$b(v_i) = \frac{1}{3}v_i + \frac{1}{6}.$$

Observe that a player submits a bid of $1/6$ when the value of the factory is zero for that player. This is because in that instance, the player is selling the factory and asks such a positive price for it.

Exercise 20.4. Faraway is a small kingdom with two regions, Northern Faraway and Southern Faraway. Nazli and Sita have stardust businesses in Northern and Southern Faraway, respectively. (Nazli and Sita are the only players in this game.) Faraqhan, the king of Faraway, decided that one needs to buy a licence from him to be able to sell stardust in Faraway, in the hopes that he can raise enough money for the new palace that he wants to build. He issues one license that makes the owner of the license a monopoly in the entire Faraway and sells it using first-price auction, where the ties are broken by a coin toss. The value of the license is $\theta_N + \alpha\theta_S$ for Nazli and $\theta_S + \alpha\theta_N$ for Sita where $\alpha \in (0, 1)$ is a known parameter and θ_N and θ_S are demand parameters for stardust in Northern and Southern Faraway, respectively; θ_N and θ_S are independently and identically distributed with uniform distribution on $[0, 1]$. Nazli privately knows θ_N, and Sita privately knows θ_S. Compute a symmetric Bayesian Nash equilibrium. (Hint: there is a symmetric linear equilibrium.)

Solution. Assume that each player bid according to a strictly increasing differentiable function b. Then, the expected payoff for type θ_i from bidding b_i is

$$U(b_i) = \int_0^{b^{-1}(b_i)} (\theta_i + \alpha\theta_j - b_i)\, d\theta_j.$$

Towards computing the first-order condition, using the Leibniz rule and the fact that $\frac{db^{-1}(b_i)}{db_i} = 1/b'\left(b^{-1}\left(b_i\right)\right)$, compute that

$$U'\left(b_i\right) = \frac{\theta_i + \alpha b^{-1}\left(b_i\right) - b_i}{b'\left(b^{-1}\left(b_i\right)\right)} - b^{-1}\left(b_i\right).$$

Since type θ_i plays $b_i = b\left(\theta_i\right)$, the first-order condition is $U'\left(b_i\right) = 0$ for $b_i = b\left(\theta_i\right)$:

$$U'\left(b\left(\theta_i\right)\right) = \frac{\theta_i + \alpha\theta_i - b\left(\theta_i\right)}{b'\left(\theta_i\right)} - \theta_i = 0.$$

This differential equation can be re-arranged as

$$b'\left(\theta_i\right)\theta_i + b\left(\theta_i\right) = \left(1+\alpha\right)\theta_i.$$

Since the left hand side is the derivative of $b\left(\theta_i\right)\theta_i$, the solution to this equation is

$$b\left(\theta_i\right) = \frac{1+\alpha}{2}\theta_i.$$

Each player submits a half of what the value of the license would be if the other player's type were equal to hers.

Exercise* 20.5. In the previous exercise, imagine that Faraqhan issues two local licenses, one for Northern Faraway and one for Southern Faraway, and sells them in separate auctions using first-price auction. The value of a local license for Northern Faraway is θ_N for Nazli and $\alpha\theta_N$ for Sita, while the value of a local license for Southern Faraway is θ_S for Sita and $\alpha\theta_S$ for Nazli. Compute a Bayesian Nash equilibrium for each auction for $\alpha > 0$. What would raise more revenue for Faraqhan, a license for the entire Faraway or two local licenses?

Solution. Compute an equilibrium for the auction for Northern Faraway; one simply swaps the strategies of Nazli and Sita in the auction for the Southern Faraway. In this auction, Nazli privately knows the value of the license, while Sita does not have private information. Sita plays a mixed strategy in any Bayesian Nash equilibrium.[11] In particular, Nazli uses a strictly increasing bidding strategy β and Sita uses a mixed

[11] Since Sita does not have private information, if she plays a pure strategy in equilibrium, her bid is known, and Nazli can undercut her by bidding slightly above Sita, rendering the best response set empty for Nazli's types who value the license more than Sita's bid.

strategy, represented by cumulative distribution function F that is increasing over an interval $[0, \bar{v}]$ with some positive density f, where the upper limit is to be determined. Given Nazli's strategy β, the expected payoff of Sita from bidding b_S is

$$U_S(b_S) = \int_0^{\beta^{-1}(b_S)} (\alpha\theta_N - b_S)\, d\theta_N.$$

This is because, when $\theta_N < \beta^{-1}(b_S)$, Nazli's bid $\beta(\theta_N)$ is less than b_S, and Sita buys the license for b_S, generating a payoff of $\alpha\theta_N - b_S$; Sita's payoff is zero when $\theta_N > \beta^{-1}(b_S)$. Since Sita is playing a mixed strategy with positive density over $[0, \bar{v}]$, she must be indifferent between her bids on this interval, i.e., U_S must be constant on $[0, \bar{v}]$. Therefore,

$$U_S'(b_S) = \frac{\alpha\beta^{-1}(b_S) - b_S}{\beta'\left(\beta^{-1}(b_S)\right)} - \beta^{-1}(b_S) = 0$$

for every b_S in the range. Moreover, the range of Nazli's bids $\beta_N(\theta_N)$ coincides with this range, and one can change the variable b_S with variable $\theta_N = \beta^{-1}(b_S)$, rewriting the above condition as a differential equation for Nazli's bidding strategy β:

$$\frac{\alpha\theta_N - \beta(\theta_N)}{\beta'(\theta_N)} - \theta_N = 0.$$

One can re-arrange this differential equation as

$$\beta'(\theta_N)\,\theta_N + \beta(\theta_N) = \alpha\theta_N.$$

Since the left-hand side is the derivative of $\beta(\theta_N)\,\theta_N$, the solution to this differential equation is[12]

$$\beta(\theta_N) = \frac{\alpha}{2}\theta_N.$$

Having derived the equilibrium strategy of Nazli, observe that $\bar{v} = \alpha/2$ as $\theta_N \in [0, 1]$. One obtains Sita's mixed strategy F from the first-order condition for Nazli's optimization problem at her equilibrium bid $\frac{\alpha}{2}\theta_N$. Now, Nazli's expected payoff from bidding b_N for type θ_N is

$$U_N(b_N) = (\theta_N - b_N)\, F(b_N).$$

[12] By integrating both sides, one obtains $\beta(\theta_N)\,\theta_N = \frac{1}{2}\alpha\theta_N^2 + \mathit{const}$, where the *const* must be zero for equality at $\theta_N = 0$.

This is because she wins the license and gets the payoff $\theta_N - b_N$ when Sita's bid is lower than b_N, which happens with probability $F(b_N)$. The derivative of U_N is

$$U_N'(b_N) = (\theta_N - b_N) f(b_N) - F(b_N).$$

The first-order condition for best response is that $U_N'(b_N) = 0$ at her equilibrium bid $b_N = \frac{\alpha}{2}\theta_N$:

$$\left(\theta_N - \frac{\alpha}{2}\theta_N\right) f\left(\frac{\alpha}{2}\theta_N\right) - F\left(\frac{\alpha}{2}\theta_N\right) = 0.$$

This can be written in term's of Sita's bid $b_S = \frac{\alpha}{2}\theta_N$ as

$$\frac{2-\alpha}{\alpha} b_S f(b_S) = F(b_S).$$

Since $F'(b_S) = f(b_S)$, this is a differential equation for F and its solution is[13]

$$F(b_S) = A b_S^{\frac{\alpha}{2-\alpha}}$$

where A is a constant. To compute the constant A, observe that $F(\bar{v}) = 1$, where the upper limit $\bar{v}$ is $\alpha/2$ as determined above. Setting $F(\alpha/2) = 1$, one obtains

$$A = (2/\alpha)^{\frac{\alpha}{2-\alpha}}.$$

One can re-write the distribution function for Sita's mixed strategy as

$$F(b_S) = \left(\frac{2}{\alpha} b_S\right)^{\frac{\alpha}{2-\alpha}}.$$

Since $\alpha \leq 1$, the function F is concave, steep at the beginning and flat at the end. Hence, in comparison to the uniform distribution, it puts more weight on the lower values of bids.

In order to compare the expected revenues in the two auction formats, observe that

$$F(b_S) \geq \bar{F}(b_S) = \frac{2}{\alpha} b_S,$$

formally showing that F puts more wait on lower values of bids than the uniform distribution on $[0, \bar{v}]$. In that case, one says F first-order stochastically dominates $\bar{F}$, and

[13] This is by inspection, but one can also solve the differential equation as follows. One can write the equation as $F'(b_S)/F(b_S) = \frac{\alpha}{2-\alpha}\frac{1}{b_S}$. By integrating both sides, one obtains $\log(F(b_S)) = \frac{\alpha}{2-\alpha}\log(b_S) + const$ for some constant $const$. The solution is obtained by using exponential function on both sides.

the expected value of any increasing function of bids is higher under F. Observe that the uniform distribution can be artificially implemented if Sita used her type θ_S as a randomization device and bid $b_S = \frac{\alpha}{2}\theta_S$. Thus, the bids under uniform distribution for the uninformed players can be implemented as each player i submitting $\alpha\theta_i/2$ for each local license and the one who bids higher getting both licenses for a total price of $\alpha\theta_i$. In comparison, when there was one license for the entire Faraway, each player i submitted $(1+\alpha)\,\theta_i/2$ and the highest bidder received the license, which is equivalent to two local licenses. Since $(1+\alpha)\,/2 \geq \alpha$, this shows that selling a single license for entire Faraway yields higher bids and higher revenue than selling two separate local licenses in which uninformed player uses uniform distribution, bidding more than her equilibrium bid in the sense discussed above. Therefore, a single license generates more expected revenue than two separate licenses when players play equilibrium strategies in each auction. Note that selling two separate local licenses is more efficient because that allocates the license in each region to the firm with higher value in each region with higher probability. However, when there are two separate auctions, there is less competition for getting the license in each region because of asymmetric information, giving little incentive to the uninformed player to compete in an auction the other party knows the value of the license and values it more, leaving the license to the uninformed only when its value is very low. This leads to lower bids for all players, reducing the revenue from the auction.

Exercise 20.6. There are two bidders and an object to be auctioned. Each bidder i observes a private signal $t_i \in [0,1]$ where (t_1, t_2) is uniformly distributed on the set defined by $t_i \leq \sqrt{t_j}$ for all $i \neq j$. The value of the object for each bidder i is

$$v_i\,(t_i, t_j) = \lambda t_i + (1-\lambda)\,t_j$$

for some $\lambda \in [0,1]$. Compute the symmetric Bayesian Nash equilibrium and the expected revenue in equilibrium for the first- and the second-price auctions.

Solution. Under the second-price auction, each player uses the bidding function

$$b\,(t_i) = v_i\,(t_i, t_i) = t_i$$

in symmetric Bayesian Nash equilibrium. The revenue is $\min\{t_1, t_2\}$; the expected revenue is approximately 0.35. Computation of the equilibrium for the first-price auction is more complicated. One first computes the hazard function as

$$h(t) = \frac{f(t|t)}{F(t|t)} = \frac{1/\left(\sqrt{t} - t^2\right)}{\left(t - t^2\right)/\left(\sqrt{t} - t^2\right)} = \frac{1}{(t - t^2)}.$$

Then, the integrating factor is computed as

$$\mu(t) = \exp\left(\int h(t)\, dt\right) = \exp\left(\int \frac{1}{t - t^2} dt\right) = \frac{t}{1 - t}.$$

The equilibrium strategy is then computed as

$$\begin{aligned} b(t) &= \int_0^t \frac{\mu(x)}{\mu(t)} v_i(x, x)\, h(x)\, dx \\ &= \frac{1 - t}{t} \int_0^t \frac{x}{(1 - x)^2} dx \\ &= \frac{1 - t}{t} \ln(1 - t) + 1. \end{aligned}$$

The revenue is $b(\max\{t_1, t_2\})$. The expected revenue is approximately 0.46.

The first-price auction yields a higher expected revenue than the second-price auction does, exhibiting a failure of the Revenue Equivalence Theorem. The Revenue Equivalence Theorem established that the two auctions yield the same expected revenue in an independent private value environment. Here, the interdependence of the values is measured by λ, which is irrelevant for the equilibrium behavior in both auction environments. In particular, the same results hold in the private value case, which corresponds to $\lambda = 0$. The failure is due to failure of independence of types. When types are positively correlated, the first-price auction yields a higher expected revenue, as an application of a general result known as the Linkage Principle.

20.9 Exercises

Exercise 20.7. Consider an n-player first price auction as in Section 20.1, where the value of the object auctioned for player i is v_i, privately known by player i. Assume that the values $v_1, \ldots, v_n$ are independently and identically distributed on $[0, 1]$ with distribution function F where $F(v) = v^\alpha$ for some $\alpha > 0$. Compute the symmetric Bayesian Nash equilibrium, and compare it to the symmetric Bayesian Nash equilibrium under uniform distribution. What happens as $n \to \infty$, or as $\alpha \to \infty$? Give an economic explanation for each limit.

Exercise 20.8. A certificate is to be sold to the students in a class; there are n students in the class. The value of the certificate for each student i is v_i, where v_i is privately known by student i and $(v_1, \ldots, v_n)$ are independently and identically distributed with uniform distribution on $[0, 100]$.

1. Find a symmetric, linear Bayesian Nash equilibrium, and compute the equilibrium payoff of a student with value v_i.

2. Assume that $n = 80$. How much would a student with value v_i be willing to pay (in terms of lost opportunities and pain of sitting in the class) in order to play this game? What is the payoff difference between the luckiest student and the least lucky student?

Exercise 20.9. There is a house on the market. There are $n \geq 2$ buyers. The value of the house for buyer i is v_i (measured in million dollars) where v_1, v_2, $\ldots$, v_n are independently and identically distributed with uniform distribution on $[0, 1]$. The house is to be sold via first-price auction. This question explores whether various "incentives" can be effective in improving participation.

1. Suppose that seller gives a discount to the winner, so that winner pays only λb_i for some $\lambda \in (0, 1)$, where b_i is his own bid. Compute the symmetric Bayesian Nash equilibrium. (Throughout the question, you can assume linearity if you want.) Compute the expected revenue of the seller in that equilibrium.

2. Suppose that seller gives a prize $\alpha > 0$ to the winner. Compute the symmetric Bayesian Nash equilibrium. Compute the expected revenue of the seller in that equilibrium.

3. Consider three different scenarios:

   - the seller does not give any incentive;
   - the seller gives 20% discount ($\lambda = 0.8$);
   - the seller gives $100,000 to the winner.

   For each scenarios, determine how much a buyer with value v_i is willing to pay in order to participate the auction. Briefly discuss whether such incentives can facilitate the sale of the house.

Exercise 20.10. Consider an auction with two buyers where the value of the object auctioned is v_i for player i, where (v_1, v_2) are independently and identically distributed with uniform distribution on $[0,1]$. The value of v_i is privately known by player i. In the auction, the buyers simultaneously bid b_1 and b_2 and the highest bidder wins the object and pays the average bid $(b_1 + b_2)/2$ as the price. The ties are broken with a coin toss. Compute a symmetric Bayesian Nash equilibrium.

Exercise 20.11. In a state, there are two counties, A and B. The state is to dump the waste in one of the two counties. For a county i, the cost of having the wasteland is c_i, where c_A and c_B are independently and uniformly distributed on $[0,1]$. They decide where to dump the waste as follows. Simultaneously counties A and B bid b_A and b_B, respectively. The waste is dumped in the county i who bids lower, and the other county j pays b_j to i. (The county is determined by a coin toss if the bids are equal. The payoff of a county is the amount of money it has minus the cost—if it contains the wasteland.) Write this as a Bayesian game. Find all the symmetric equilibria where the bid is a strictly increasing differentiable function of the cost.

Exercise 20.12. There are $n \geq 2$ siblings, who have inherited a factory from their parents. The value of the factory is v_i for sibling i, where $(v_1, \ldots, v_n)$ are independently and uniformly distributed over $[0,1]$, and each of them knows his or her own value. Simultaneously, each i bids b_i, and the highest bidder wins the factory and pays his own bid to his siblings, who share it equally among themselves. (If the bids are equal, the winner is determined by a lottery with equal probabilities on the highest bidders.) Note that if i wins, i gets $v_i - b_i$ and any other j gets $b_i/(n-1)$. Write this as a Bayesian game. Compute all symmetric Bayesian Nash equilibria. What happens as $n \to \infty$? Briefly interpret.

Exercise 20.13. There are two identical objects and three potential buyers, named 1, 2, and 3. Each buyer only needs one object and does not care which of the identical objects he gets. The value of the object for buyer i is v_i where (v_1, v_2, v_3) are independently and uniformly distributed on $[0,1]$. The objects are sold to two of the buyers through the following auction. Simultaneously, each buyer i submits a bid b_i, and the buyers who bid one of the two highest bids buy the object and pay their own bid. (The ties are broken by a coin toss.) That is, if $b_i > b_j$ for some j, i gets an object and pays

b_i, obtaining the payoff of $v_i - b_i$; if $b_i < b_j$ for all j, the payoff of i is 0. Write this as a Bayesian game. Compute a symmetric Bayesian Nash equilibrium of this game in increasing differentiable strategies.

Exercise 20.14. A state government wants to construct a new road. There are n construction firms. In order to decrease the cost of delay in completion of the road, the government wants to divide the road into $k < n$ segments and construct the segments simultaneously using different firms. The cost of delay for the public is $C_p = K/k$ for some constant $K > 0$. The cost of constructing a segment for firm i is c_i/k where $(c_1, \ldots, c_n)$ are independently and uniformly distributed on $[0,1]$, where c_i is privately known by firm i. The government hires the firms through the following procurement auction.

$k+1$**st-price Procurement Auction** Simultaneously, each firm i submits a bid b_i and each of the firms with the **lowest** k bids wins one of the segments. Each winning firm is paid the lowest $k+1$st bid as the price for the construction of the segment. The ties are broken by a coin toss.

The payoff of a winning firm is the price paid minus its cost of constructing a segment, and the payoff of a losing firm is 0. For example, if $k = 2$ and the bids are $(0.1, 0.2, 0.3, 0.4)$, then firms 1 and 2 win and each is paid 0.3, resulting in payoff vector $(0.3 - c_1/2, 0.3 - c_2/2, 0, 0)$.

1. For a given fixed k, find a Bayesian Nash equilibrium of this game in which no firm bids below its cost. Verify that it is indeed a Bayesian Nash equilibrium.

2. Assume that each winning firm is to pay $\beta \in (0,1)$ share of the price to the local mafia. (In the above example it pays 0.3β to the mafia and keeps $0.3(1-\beta)$ for itself.) For a given fixed k, find a Bayesian Nash equilibrium of this game in which no firm bids below its cost. Verify that it is indeed a Bayesian Nash equilibrium.

3. Assuming that the government minimizes the sum of C_P and the total price it pays for the construction, find the condition for the optimal k for the government in Parts 1 and 2. Show that the optimal k in Part 2 is weakly lower than the optimal k in Part 1. Briefly interpret the result. [Hint: the expected value of the $k+1$st lowest cost is $(k+1)/(n+1)$.]

Exercise 20.15. There are k identical objects and n potential buyers where $n > k > 1$. Each buyer only needs one object and does not care which of the identical objects he gets. The value of the object for buyer i is v_i where $(v_1, v_2, \ldots, v_n)$ are independently and uniformly distributed on $[0,1]$. The objects are sold to k of the buyers through the following auction. Simultaneously, each buyer i submits a bid b_i, and the buyers who bid one of the k highest bids buy the object and pay their own bid. (The ties are broken by a coin toss.) That is, if $b_i > b_j$ for at least $n-k$ bidders j, then i gets an object and pays b_i, obtaining the payoff of $v_i - b_i$; if $b_i < b_j$ for at least k bidders j, the payoff of i is 0. Write this as a Bayesian game. Compute a symmetric Bayesian Nash equilibrium of this game in increasing differentiable strategies. **Hint:** Let $(x_1, \ldots, x_m)$ be independently and uniformly distributed on $[0,1]$ and let $x_{(r)}$ be rth highest x_i among $(x_1, \ldots, x_m)$. Then, the probability density function of $x_{(r)}$ is

$$f_{m,r}(x) = \frac{m!}{r!\,(m-r)!}\,(1-x)^{r-1}\,x^{m-r}.$$

Exercise 20.16. Consider n-player version of the first-price wallet auction where the common value is

$$v = t_1 + \cdots + t_n,$$

where each t_i is privately known by player i and $t_1, \ldots, t_n$ are independently and identically distributed with uniform distribution on $[0,1]$. Compute a symmetric Bayesian Nash equilibrium. Compute the expected revenue for the seller in the equilibrium you computed.

Exercise 20.17. In Exercise 20.3, imagine that the factory has two divisions, A and B, previously ran by Alice and Bob, respectively. Assume that the value of the factory for each player is

$$v = t_A + t_B$$

where t_A and t_B are the values of the divisions A and B, respectively, privately known by Alice and Bob, respectively. Assuming t_A and t_B are independently distributed with density f and cumulative distribution function F, compute the symmetric Bayesian Nash equilibrium. What is the answer when t_A and t_B are uniformly distributed on $[0,1]$?

Exercise 20.18. Compute the symmetric Bayesian Nash equilibrium in the following two cases and compare them. (You can use one solution to obtain the other.) In both cases an object is sold via first-price auctions, and there are two bidders with independent private values.

1. Values are independently and uniformly distributed on $[-r, 1-r]$ for some $r \in (0,1)$. The bids can be negative but the object is sold only if the winner's bid is non-negative.

2. Values are independently and uniformly distributed on $[0,1]$ and there is a reserve price $r \in (0,1)$. The object is sold only if the winner's bid is at least r.

Exercise* 20.19. An object is sold via first-price auction with reserve price $r \in (0, 1/2)$. There are two bidders, namely 1 and 2. The value of the object for bidder i is v_i, and it is privately known by player i. The values (v_1, v_2) are uniformly distributed on the set

$$\{(v_1, v_2) \in [0,1] \,|\, |v_1 - v_2| \leq \alpha\}$$

for some $\alpha \in (0, r)$. Compute the symmetric Bayesian Nash equilibrium (it suffices to compute the bids for values $v_i \in (r, 1-\alpha)$). What happens in the limit $\alpha \to 0$. Briefly discuss your result, comparing to your answer in the previous exercise with independent values.

Exercise 20.20. Consider the n-buyer all-pay auction in Section **??** where the values are independently and identically distributed on $[0,1]$ with uniform distribution. Compute the expected revenue for the buyer in symmetric Bayesian Nash equilibrium and compare it to the expected revenue from the symmetric Bayesian Nash equilibria of the first- and the second-price auctions.

Exercise 20.21. Compute the symmetric Bayesian Nash equilibrium in an all-pay auction with two buyers in which the value of the object is

$$v = t_1 + t_2$$

for each player where t_i is privately known by player i as in the Wallet auction. Assume that t_1 and t_2 are independent and uniformly distributed on $[0,1]$.

Exercise 20.22. Consider a competition between two athletes, namely 1 and 2. Simultaneously, each athlete i exerts effort level $x_i \geq 0$, and the one that exerts more effort wins and obtains a prize of value 1; the other does not get any prize. Each athlete i pays a cost of x_i^γ/α_i for her effort where $\gamma > 1$ is a known parameter and α_i is private information of athlete i. Assume that α_1 and α_2 are independently and uniformly distributed on $[\underline{\alpha}, \bar{\alpha}]$ for some $\bar{\alpha} > \underline{\alpha} > 0$. (The payoff of i is $1 - x_i^\gamma/\alpha_i$ if she wins and $-x_i^\gamma/\alpha_i$ if she loses; the winner is determined by a coin toss in case of a tie.) Write this as a Bayesian game. Find a symmetric Bayesian Nash equilibrium.

Exercise 20.23. Chaya is an accomplished writer. She wants to sell (the right to publish) her latest novel to a publisher, using a "royalty" auction where the bids denote the fraction of the proceeds Chaya gets from publishing the book as her royalty. There are n (profit-maximizing) publishers, $1, 2, \ldots, n$. For each publisher i, publishing the novel generates proceeds v_i where v_i is privately known by publisher i and the profits $v_1, \ldots, v_n$ are independently and identically distributed with uniform distribution on $[0, 1]$. Simultaneously, each publisher i submits a bid r_i and the highest bidder wins the auction; the winner is determined randomly if multiple publishers tie for the highest bid. The winner i^* buys the novel and pays Chaya $r_{i^*} v_{i^*}$, keeping $(1 - r_{i^*})\, v_i^*$ for itself.

1. Compute a symmetric Bayesian Nash equilibrium. [Hint: the bidding strategy may not be increasing in v_i.]

2. Compute the expected revenue for Chaya, and compare it to the expected revenue Chaya would get if she used a usual first-price auction, where the winner i^* simply pays its bid b_{i^*} to Chaya and keeps $v_{i^*} - b_i^*$ to itself. Which auction format should Chaya use if she wants to maximize her expected revenue?

3. Now imagine that the proceeds v from publishing the book is common to all publishers, and each publisher has some "local" private information about the profit as described in Exercise 20.16. Compute a symmetric Bayesian Nash equilibrium. Compute the expected revenue for Chaya and compare it to the expected revenue Chaya would get if she used the usual first-price Wallet auction (as in Exercise 20.16).Which auction format should Chaya use if she wants to maximize her expected revenue?

Exercise 20.24. In the previous exercise, assume that the proceeds are not fixed. After buying the book in the auction, the winning publisher invests $x \geq 0$ for promoting the book and generates proceeds

$$v_i = t_i\sqrt{x}$$

where $t_i \in [0,1]$ is an efficiency parameter for i, privately known by i, and $t_1, \ldots, t_n$ are independently and identically distributed with uniform distribution on $[0,1]$. The payoff of a publisher i is $v_i - x$, minus whatever it pays to Chaya, if it wins the auction, and zero otherwise.

1. For each auction format below compute a symmetric Bayesian Nash equilibrium assuming that the winner will choose x to maximize its payoff, and compute the expected revenue for Chaya:

   (a) Share auction in the previous exercise;

   (b) First-price auction;

   (c) Second-price auction.

2. Which auction format should Chaya use if she wants to maximize her expected revenue?

3. Now imagine that Chaya first fixes a royalty rate r, and the publishers compete on the amount of the advance they give after observing r. Simultaneously, each publisher i submits a bid a_i and the publisher who bids the highest amount wins. The winner i now pays Chaya $\max\{rv_i, a_i\}$, i.e., Chaya keeps rest of the advance if the total royalty payments fall below the advance. Compute a symmetric Bayesian Nash equilibrium for each $r \in [0,1]$. What r should Chaya choose if she wants to maximize her expected revenue.

Exercise 20.25. Consider the following double auction with sales taxes. A seller owns an object whose value to him is $v_S = [0,1]$. The value of the object for a buyer is $v_B = [0,1]$. The seller and the buyer simultaneously set prices p_S and p_B, respectively. If $p_S > p_B$, then there is no trade. If $p_B \geq p_S$, then the object is sold to the buyer at price $p = (p_B + p_S)/2$, and the buyer pays τp to the local government (in addition to paying p to the seller for the object), where $\tau > 0$ is a known constant. The values v_S and v_B are independent and uniformly distributed on $[0,1]$. (The prices can take any value.) Write this as a Bayesian game, and find a linear Bayesian Nash equilibrium.

Exercise 20.26. Consider the following charity auction. There are two bidders, namely 1 and 2. Each bidder i has a distinct favored charity. Simultaneously, each bidder i contributes b_i to the auction. The highest bidder wins, and the sum $b_1 + b_2$ goes to the favored charity of the winner. The winner is determined by a coin toss in case of a tie. The payoff of the bidder i is

$$u_i(b_1, b_2, \theta_i) = \begin{cases} \theta_i (b_1 + b_2) - b_i^\gamma & \text{if } i \text{ wins} \\ -b_i^\gamma & \text{otherwise,} \end{cases}$$

where $\gamma > 1$ is a known parameter, θ_i is privately known by player i, and θ_1 and θ_2 are independently and uniformly distributed on $[0, 1]$. Find a differential equation that must be satisfied by strategies in a symmetric Bayesian Nash equilibrium. (Assume that the equilibrium strategies are increasing and differentiable.)