Chapter 14

Games of Incomplete Information

We have so far focused on games in which any piece of information that is known by any player is known by all the players (and indeed common knowledge). Such games are called games of *complete information*. Informational concerns do not play any role in such games. In real life, players always have some private information that is not known by other parties. For example, one can hardly know other players' preferences and beliefs as well as they do. Informational concerns play a central role in players' decision making in such strategic environments. We now switch our focus on such informational issues. We will consider games in which a party may have some information that is not known by some other party. Such games are called games of *incomplete information* or *asymmetric information*. The informational asymmetries are modeled by Nature's moves. Some players can distinguish certain moves of nature while some others cannot.

The following example illustrates the way private information affects the strategic behavior. In this example, two players face a coordination problem under uncertainty and private information. First consider the case with no private information.

Example 14.1 (Complete Information)**.** Alice and Bob have a joint research project. The value of completing the project successfully is θ, and it can be either high, denoted by $\theta = \theta_H$, or low, denoted by $\theta = \theta_L$. The players simultaneously exert efforts, choosing between high effort level H and low effort level L. The payoffs depend on the project value as well as the effort levels as in the following table:

$\theta = \theta_H$

	H	L
H	$4,4$	$0,3$
L	$3,0$	$1,1$

$\theta = \theta_L$

	H	L
H	$0,0$	$-2,1$
L	$1,-2$	$0,0$

(14.1)

where Alice and Bob are row and column players, respectively.[1] Assume that the players know θ. If it is known that the project is of low value (i.e., $\theta = \theta_L$), L strictly dominates H, and both players put low effort. On the other hand, if it is known that the project is of high value (i.e., $\theta = \theta_H$), there are multiple equilibria, (H, H) and (L, L), so that the players can both put high effort or both put low effort, depending on the equilibrium. Clearly, they would rather both put high effort.

In the example above, the value of the project is known, and the players do not face any uncertainty. They rationally choose to put low effort when the project is low value. When the project is high value, they have a coordination problem, but they can coordinate on high effort obtaining the best outcomes at all cases. If the value of the project is not known but no player has private information, the uncertainty would prevent them putting high effort when the project is of high value and low effort when the project is of low value. This will reduce their payoffs (but this is not the worst news).

Example 14.2 (Complete Information with Uncertainty)**.** Now, imagine that the players do not know the value of the project. In particular, the value of project is high with probability q and low with the remaining probability $1 - q$ for some known q with $1/2 \leq q < 2/3$. In that case, their payoffs will be according to the left table with probability q and according to the right table with the remaining probability. Accordingly, their payoffs are obtained by taking the weighted average of the payoffs in the two tables:

	H	L
H	$4q, 4q$	$2q-2, 2q+1$
L	$2q+1, 2q-2$	q, q

[1] These payoffs can be derived from a more detailed economic model as follows. The project is completed successfully with probability $e_A e_B/4$, where e_A and e_B are the effort levels of Alice and Bob, respectively. The cost of exerting effort e is e^2. Hence, the payoffs of Alice and Bob are $u_A = \theta e_A e_B/4 - e_A^2$ and $u_B = \theta e_A e_B/4 - e_B^2$, respectively. One obtains the payoffs in the table by taking $\theta_H = 8$, $\theta_L = 4$, $H = 2$ and $L = 1$.

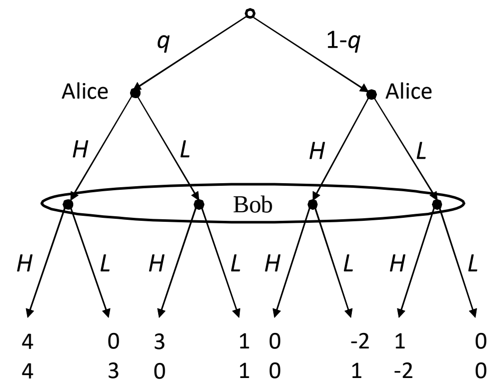

Figure 14.1: Extensive-form game for Example 14.3.

For example, the payoff from both putting high effort is $4 \times q + 0 \times (1 - q)$, which is $4q$. Since $q \geq 1/2$, there are still two equilibria in pure strategies, one in which both players exert high effort, and one in which both players exert low effort, mirroring the Nash equilibria for the case it is known that the project is of high value.

This example shows that uncertainty per se is not important for players' coordination on high effort when the expected value of such coordination is substantially high (i.e., $q \geq 1/2$). In contrast, a party having private information about the unknown value can be detrimental, as the next example shows.

Example 14.3 (Asymmetric Information)**.** Now imagine that Alice is a senior researcher with a lot of experience, while Bob is her junior partner. In particular, assume that Alice knows the value of completing the project successfully, but Bob does not know its value. He knows that Alice knows it. As in the previous case, Bob believes that the project is of high value—and Alice knows it—with probability q, and the project is of low value—and Alice knows it—with probability $1 - q$, where $1/2 \leq q < 2/3$. This game can be represented as an extensive-form game as in Figure 14.1. First, Nature picks a θ, picking $\theta = \theta_H$ with probability q and $\theta = \theta_L$ with probability $1 - q$. Alice observes θ, but Bob does not. They then simultaneously choose their effort levels. Under this specification, the game is dominance-solvable. To see this, first observe that Alice would not choose high effort level H if the project is of low value; L gives a higher payoff than H no matter what Bob does when $\theta = \theta_L$:

	Bob plays H	Bob plays L
Payoff from H	0	-2
Payoff from L	1	0

Thus, she must play L when $\theta = \theta_L$ (at her information set on the right). Knowing this, at his information set, Bob must assign probability $1 - q$ to project being of low value and Alice exerting low effort L—the rightmost node in his information set. At that node, Bob loses 2 by exerting high effort. Bob does not know what Alice plays when the project is of high value. But that is not relevant for Bob's decision. In the best case scenario for high effort, Alice plays H when θ is high—the leftmost node. At that node, Bob gets 4 from H and 3 from L, i.e., he can gain 1 by exerting high effort. That gain is not enough to offset the loss of 2 at low θ, and Bob would lose $2(1-q) - q = 2 - 3q$ in expectation by exerting high effort—because the probability of high θ is $q < 2/3$. Therefore, Bob must play L. Given that Bob plays L, Alice must play L even when the project is of high value. This leads to a unique solution: each player exerts low effort regardless of the value of the project.

Now imagine that the project is indeed of high value. Without private information, they could coordinate on high effort and gain high payoffs, and they would not lose anything if it turned out that the project is of low value. Now Alice learns privately the project is of *high* value and this information turns out to be detrimental to coordination on high effort. This disturbing result arises because Bob knows that Alice knows whether the project is of high value while Bob himself does not know it. The mechanism is as follows. Although the project is of high value, if the project were of low value, Alice would have put low effort when she learned the value of the project. Bob does not know whether Alice learned that project value is high or low, but he knows that she learned the value and will put low effort in case the value is low. Since that contingency is sufficiently likely (with probability $1 - q > 1/3$), Bob now puts low effort. Anticipating this, Alice also puts low effort despite the fact that she just learned that the project is of high value. Note that low-effort behavior in a counterfactual situation spreads to other cases due to incomplete information.

Why does not Alice tell Bob that the project is of high value? She would love to tell him that—and probably she would tell him. The problem is her saying that the project is of high value is not credible. Indeed, Alice would always want Bob to put high effort, and Bob would not put any effort if he thinks that the project is of low value. Hence, even if the project is of low value, Alice would tell Bob that the project is of high value to induce him to work. Then, Bob would not learn anything when Alice tells him that the project is of high value because Bob sees that Alice would tell that regardless of her information.

The problem described by the above example is common in real world. For example, a sales manager is typically more informed then her junior subordinates about the market conditions, and a subordinate may worry that even if he works hard he will not be able to make sales because market conditions are bad and the superiors will not give enough support—knowing the market conditions. Such a situation may arise whenever more informed managers need to collaborate with less knowledgeable subordinates. As in this example, the asymmetric information may have a detrimental effect.

More generally, asymmetric information plays an important role throughout social and economic interactions. For example, individuals may have a better idea about their health status than the insurance companies, lowering the scope of insurance. An applicant to a job may have more information about his skills and preferences than the potential employers, e.g., knowing what kind of jobs that he did in his previous employment. Likewise a potential employer may have more information about what kind of skills are necessary to advance in that company. Seller of a used car may know more about the condition of the car than a buyer can check. A defendant in a litigation may know more about his culpability, as he may know what kinds of actions that he took relevant to the case. In an auction, a bidder may want to know what other bidders think about the object auctioned, not only to be able to bid the right amount but also incorporate the other bidders' information towards evaluating the object's value. For example, he may feel buyer's remorse if he realizes that all the other bidders bid substantially lower than him.

This section presents a general formulation, namely a Bayesian game, that helps analyzing such problems, and the later chapters will explore many applications as above using this formulation. In a Bayesian game, each player's private information will be represented by her "types", allowing a suitable summary of what each player knows and what each player may think about what kind of information the other players may have.

For instance, in Example 14.3, Alice has two types: High and Low. Since Bob does not have any private information, he has only one type. As in Figure 14.1, incomplete information is modeled via imperfect-information games where Nature chooses each player's type and privately informs her. These games are called *incomplete-information games* or *Bayesian games*.

14.1 Bayesian Games

This section formally introduces Bayesian games and discusses some fine details, such as strategies and belief updating.

Definition The ingredients of a Bayesian game are

- a set $N = \{1, \ldots, n\}$ of *players* with generic members i and j;
- a set Θ of payoff parameters (aka *states*) with generic member θ;
- a set T_i of *types* for each player i with generic member t_i;
- a probability distribution p on $\Theta \times T$, where $T = T_1 \times \cdots \times T_n$ is the set of type profiles $t = (t_1, \ldots, t_n)$;
- a set A_i of *actions* for each player i with generic member a_i; and
- a utility function $u_i : A \times \Theta \times T \to \mathbb{R}$ for each player i, where $A = A_1 \times \cdots \times A_n$ is the set of action profiles $a = (a_1, \ldots, a_n)$.

At the beginning, Nature chooses some payoff parameter $\theta \in \Theta$ and type profile $t = (t_1, t_2, \ldots, t_n) \in T$, where each $(\theta, t) \in \Theta \times T$ is selected with probability $p(\theta, t)$. Then, each player i observes her own type t_i, but not the state θ or the other players' types. Finally, players simultaneously choose their actions, each player knowing her own type. The payoff of a player may depend on the state θ, all players' types and all players' actions because the utility function u_i of each player i is defined on $A \times \Theta \times T$, assigning a numerical score on each (a, θ, t). This is a Von-Neumann and Morgenstern utility function: player i chooses her action a_i towards maximizing the expected value of u_i knowing (only) her own type t_i. Once again, $u = (u_1, \ldots, u_n)$ denotes the list of utility functions. This extensive-form game is denoted by (N, Θ, T, A, p, u)—and called a *Bayesian Game*.

Here (Θ, T, p) describes the information structure in the game, and it is referred to as a *type space*. The remaining ingredients (N, A, u) describe the actions and payoffs (similarly to the normal form games, except that the utility functions depend on states and types as well as actions now).

Example 14.4. One can write the game in Example 14.3 as a Bayesian game by setting

- $N = \{\text{Alice, Bob}\}$;
- $\Theta = T_A = \{\theta_H, \theta_L\}$ and $T_B = \{t_B\}$;
- $A_A = A_B = \{H, L\}$, where actions are effort levels;
- $p(\theta_H, \theta_H, t_B) = q$ and $p(\theta_L, \theta_L, t_B) = 1 - q$;
- the payoff functions as in the table in Example 14.3.

The set of types for Alice is $T_A = \{\theta_H, \theta_L\}$ because Alice may have two different information: either the project is of high value, i.e., $t_A = \theta_H$, or the project is of low value, i.e., $t_A = \theta_L$. Since Bob does not have any private information, he has only one (dummy) type t_B, whence everything Bob knows will be known by Alice. Since it is commonly known that Alice knows the value of project θ, the equality $\theta = t_A$ holds with probability one, so that her information will always match the value of the project. In this particular example, the payoffs do not depend on the types directly. In that case, it will be customary to write the payoffs as a function of actions and the state, writing $u_i : A \times \Theta \to \mathbb{R}$. One still needs to specify the type space because the primary function of types is to encode the players' private information. Sometimes, the players' payoffs may be a function of types only, in that case, one can drop Θ from the list.

In a Bayesian game, there are three stages. The first stage is the *ex-ante* stage. This is the stage before Nature chooses θ and types t. In a card game, this corresponds to the stage before the cards are dealt. The second stage is the *interim* stage. This is when every player knows her own type, but the players have not taken their actions. In a card game, this is when each player have seen their own hand and nobody has taken any action. The last stage is the *ex-post* stage. This is after everybody has played their actions and players learned all players' types. In a card game, this is when all the cards are played and the players learned what cards each player had.

Interpretation Unlike card games, in real life, typically, there is no ex-ante stage. The players have their information when they start the game. Modelers introduce an ex-ante stage to describe the players' information and players' beliefs about what other players may know in a coherent way. For example, in the example of Alice and Bob above, one can envision the situation as the project is of high value, and Alice knows this. Unfortunately, Bob does not know the value of the project and assigns probability q to project being indeed of high value. All he knows is that Alice knows the value of the project (and this is commonly known). To describe this situation, I introduced the low value for the project and a type of Alice who knows that the project is of low value. This was necessitated by the fact that Bob thought that it was possible that the project is of low value and Alice knows that. The type θ_L is a hypothetical type that helps one to formalize what Alice would have done if she knew that the project is of low value. This is analogous to considering how a player would have played the game if she were dealt a different hand (and this may be necessary to analyze how other players would interpret possible moves by the player during the card game). Overall, Bayesian games take incomplete information situations, and represent them as card games.

When they are represented in this form, incomplete-information games are extensive-form games with chance moves as they are formulated in Chapter 2. For example, the Bayesian game in Example 14.4 has the extensive-form representation in Figure 14.1. Since every extensive-form game can also be represented as a normal-form game by listing the players' strategies and the expected utility each player gets from each strategy profile, every Bayesian game also has a normal-form representation. The latter game does not have any explicit uncertainty. In particular, it is a game of *complete* information. The players' uncertainty and private information are incorporated implicitly in the descriptions of the strategy sets and the expected payoffs. Therefore, every Bayesian game also has a complete-information representation, but the form described here makes the private information explicit, so one can focus on informational issues.

Historically, it has been known all along that incomplete-information games as in card games can be represented in this way. Indeed, in 1950, John Nash applied his theory first to a simplified version of poker. It was not clear however that genuine situations of incomplete information can be represented in this way, as there is no ex-ante stage where players obtain their information. In 1967, John Harsanyi introduced the idea that one can use a hypothetical ex-ante stage and hypothetical types to represent any form of incomplete information as a card game. Soon other researchers applied this idea to analyze strategic situations. For example, in 1968, Robert Aumann, Michael Maschler, and Richard Stearns were using this formulation to develop theories to help the US State Department in their interaction with USSR; the work remained as a state secret until it was declassified and published in 1995. This formulation played a central role in the emergence of Game Theory as the main mathematical tool for strategic analysis.

Strategies In general extensive-form games, a strategy of a player determines which action she will take at each information set of hers. A Bayesian game is an extensive-form game in which information sets are identified with types $t_i \in T_i$. Hence, a strategy of a player i is a function

$$s_i : T_i \to A_i,$$

mapping her types to her actions. A strategy of a player is a type-dependent plan, describing what action she takes depending on her type. Since she takes her action after observing her type, sometimes the types are treated as distinct players. For example, one may write "type t_i takes action a_i" to mean that player i plays action a_i when her type is t_i.

Example 14.5. In Example 14.4, Alice has four strategies: HH—meaning that she exerts high effort regardless of the project value, HL—meaning that she will exert high effort if the value of the project is high and exert low effort if the value is low, LH, and LL.[2] Bob has only two strategies: H and L. Clearly, these are also the strategies in the extensive-form representation—in Figure 14.1.

The expected utility of a player i from a strategy profile $s = (s_1, \ldots, s_n)$ is denoted by $E\left[u_i\left(s\right)\right]$, and computed as

$$E\left[u_i\left(s\right)\right] = \sum_{(\theta,t)\in\Theta\times T} u_i\left(s_1\left(t_1\right), \ldots, s_n\left(t_n\right), \theta, t\right) p\left(\theta, t\right),$$

[2] Formally, these are functions from the set $\{\theta_H, \theta_L\}$ of Alice's types to the set $\{H, L\}$ of her actions; e.g., $HH\left(\theta_H\right) = HH\left(\theta_L\right) = H$, while $HL\left(\theta_H\right) = H$ and $HL\left(\theta_L\right) = L$.

by taking a weighted average of the realized payoffs under each state and type profile where the weights are given by the probability distribution p. For example, in Example 14.4, the expected payoff of Alice from strategy profile (HL, H) is computed as

$$E[u_A(HL, L)] = qu_A(H, H, \theta_H) + (1-q)\,u_A(L, H, \theta_L) = q \times 4 + (1-q) \times 1 = 3q + 1.$$

With probability q, the state is θ_H; Alice observes θ_H and plays $HL(\theta_H) = H$, and Bob plays H, yielding a payoff of 4 for Alice. With probability $1-q$, the state is θ_L; Alice observes θ_L and plays $HL(\theta_L) = L$, and Bob plays H, yielding a payoff of 1 for Alice. One computes Alice's expected payoff by taking the weighted average of these payoffs with probabilities as weights to obtain the expected payoff of $3q + 1$.

Once again, one obtains the same expected payoff if she computes the expected payoffs from strategy profiles in extensive-form representation as in Chapter 2. In the above example, in the extensive form game, the outcome is (θ_H, H, H) with probability q and (θ_L, L, H) with the remaining probability $1-q$. Since Alice's payoffs at (θ_H, H, H) and (θ_L, L, H) are 4 and 1, respectively, her expected payoff is $4q + (1-q) = 3q + 1$.

As in any extensive-form game, one can represent a Bayesian game as a normal form game, by taking N as the set of players, the set S_i of functions $s_i : T_i \to A_i$ as the strategy set of each player i, and the expected utility $E[u_i(s)]$ as the payoff function.

Example 14.6. For $q = 1/2$, the Bayesian game in Example 14.4 can be represented in normal form as

	H	L
HH	$2, 2$	$-1, 2$
HL	$5/2, 1$	$0, 3/2$
LH	$3/2, 0$	$-1/2, 1$
LL	$2, -1$	$1/2, 1/2$

where Alice plays rows and Bob plays columns. (See above how to calculate the payoff vector from (HL, H), and the other payoff vectors are computed similarly.)

In the extensive-form game, as well as in the Bayesian game, it is explicit that Alice knows θ, and Bob does not know θ. In the normal-form game, this is implicit. Alice's strategies are functions from θ to her actions, reflecting the fact that Alice knows θ and can condition her behavior on θ. Bob has only two strategies and his actions cannot depend on θ, reflecting the fact that he does not know θ. Clearly, it is not easy to reason from such summary information without an explicit background story. The formulation in Bayesian games represent such information explicitly without resorting to extensive form.

One can use the extensive-form and normal-form representations above to apply the solution concepts developed in the previous chapters to Bayesian games. For example, one can apply rationalizability or Nash equilibrium in the normal form representation to compute the rationalizable strategies and Nash equilibria in a Bayesian game.

Example 14.7. For $q = 1/2$, one can compute the rationalizable strategies in Example 14.4 by computing the rationalizable strategies in the normal-form representation:

	H	L
HH	$2, 2$	$-1, 2$
HL	$5/2, 1$	$0, 3/2$
LH	$3/2, 0$	$-1/2, 1$
LL	$2, -1$	$1/2, 1/2$

Observe that HL strictly dominates HH and LH, and eliminating the latter strategies one obtains

	H	L
HL	$5/2, 1$	$0, 3/2$
LL	$2, -1$	$1/2, 1/2$

Now, L strictly dominates H. Thence, one eliminates H, and then HL to obtain the unique rationalizable strategy profile (LL, L), according to which both players put low effort regardless of the state. This is, of course, the only Nash equilibrium. (Example 14.8 obtains the same result for all $q \leq 2/3$ by analyzing the extensive-form game.)

Interim Beliefs Players' types may be correlated. For example, in a card game, the other players will not hold the cards held by a player. For another example, imagine that a house is to be sold in an auction, and the buyers carry out some private research about the house before bidding for it. Their findings about the house will be correlated (but not necessarily the same) because they are all about the same house. Then, as she observes her own type, a player updates her beliefs about the state and the other players' types. The new beliefs are called *interim*. She takes her actions under these interim beliefs. It is assumed throughout that the beliefs are updated according to the Bayes' Rule.[3]

Definition 14.1 (Bayes' Rule)**.** For any two events A and B, the probability that A occurs conditional on B occurring is

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)},$$

where $P(A \cap B)$ is the probability that A and B occur simultaneously and $P(B)$ is the (unconditional) probability that B occurs; $P(A \mid B)$ is called the (conditional) probability of A given B.

The above formula is used to compute the players' interim beliefs, the conditional probabilities given types. To this end, write $t_{-i} = (t_1, t_2, ..., t_{i-1}, t_{i+1}, \ldots, t_n)$ for the type profile for players other than t_i and write $T_{-i} = T_1 \times \cdots \times T_{i-1} \times T_{i+1} \times \cdots \times T_n$. Write $p(\theta, t_{-i}|t_i)$ for the conditional probability a player i assigns to state being θ and the other players' types being t_{-i} given that her type is t_i. The conditional probability distribution $p(\cdot|t_i)$ will be referred to as *interim belief* of player i at type t_i, or simply the interim belief of type t_i.

To compute interim beliefs, one first computes the (total) probability of each type t_i by adding the probabilities $p(\theta, t_i, t_{-i})$ of all cases where the type of player i is t_i:

$$p(t_i) = \sum_{(\theta, t_{-i}) \in \Theta \times T_{-i}} p(\theta, t_i, t_{-i}).$$

The resulting probability distribution on T_i is called *the marginal distribution on* T_i. The interim belief at type t_i is computed simply dividing the joint probability with the marginal probability of t_i:

$$p(\theta, t_{-i}|t_i) = \frac{p(\theta, (t_i, t_{-i}))}{p(t_i)}.$$

The next example illustrates how to compute the interim beliefs on a variation of the game between Alice and Bob.

[3]The Bayes' Rule is used when the types across players are 'correlated'. But if they are independent, then life is simpler; players do not update their beliefs.

Example 14.8 (Two-sided Incomplete Information)**.** In Example 14.3, take $q = 1/2$, and imagine that Bob also has some information about the value of the project; Alice still knows the value of the project. In particular, he observes the value of a random variable $t_B \in \{h, l\}$ that is correlated to the value of the project as in the following table of joint probabilities

	h	l
θ_H	3/8	1/8
θ_L	1/8	3/8

where (θ, t_B) takes the value (θ_H, h) with probability 3/8, (θ_H, l) with probability 1/8, (θ_L, h) with probability 1/8 and (θ_L, l) with probability 3/8. (In the type space formulation above, one writes $p(\theta_H, \theta_H, h) = p(\theta_L, \theta_L, l) = 3/8$ and $p(\theta_H, \theta_H, l) = p(\theta_L, \theta_L, h) = 1/8$, reflecting the fact that it is common knowledge that Alice knows θ.) This information structure arises if Bob can run a test t_B that tells the true value of the project with probability 3/4 and the wrong value with probability 1/4. The test result is denoted by t_B. And all these are common knowledge.

To compute the interim beliefs, first compute the marginal distributions, computing the probability of each type. The probability that $t_A = \theta_H$ is

$$p(\theta_H) = p(\theta_H, h) + p(\theta_H, l) = 3/8 + 1/8 = 1/2;$$

the remaining probability is assigned to type $t_A = \theta_L$, and $p(\theta_L) = 1/2$. Notice that the marginal distribution for Alice is computed simply by adding up the probabilities in each row. Similarly, the marginal distribution for Bob is computed by adding up the probabilities in each column:

$$p(h) = p(\theta_H, h) + p(\theta_L, h) = 3/8 + 1/8 = 1/2,$$

and $p(l) = 1/2$. For each type, one obtains the conditional probabilities by dividing the joint probability by the probability of the type, which happens to be 1/2 every time in this example. Then, in this particular example, one multiplies the probabilities in the above table with 2 to obtain the interim beliefs. For example, reading from the first row of the table, one can see that type $t_A = \theta_H$ of Alice believes that Bob's type is h with probability 3/4, i.e., $p(h|\theta_H) = 3/4$, and l with probability 1/4. On the other hand, reading from the second row, one can see that type $t_A = \theta_L$ believes that Bob's type is

h with probability $1/4$ and l with probability $3/4$. Bob's interim beliefs are computed similarly using the columns, the left column for type h, and the right column for type l.

In this example, each type occurs with equal probability, but the players' types are correlated so that the joint distribution is as in the table above. The correlation leads players to revise their beliefs at the interim stage. For example, if Bob observes a high signal $t_B = h$, then Bob updates his belief about the value of the project being high as

$$p(\theta_H|h) = \frac{p(\theta_H, h)}{p(h)} = \frac{3/8}{1/2} = 3/4,$$

increasing probability of high value project from $1/2$ to $3/4$. His view of the project improves. Similarly, he decreases his probability if he observes a low signal: $p(\theta_H|l) = 1/4$. Since Bob knows that Alice knows the value of the project, he updates his belief about Alice accordingly. For example, at the interim stage, when he observes h, he assigns probability $p(\theta_H, \theta_H|h) = 3/4$ to Alice knowing that project is of high value. Alice also updates her beliefs about Bob, in that Alice's beliefs about Bob's type depends on whether Alice knows that the project is of high value or low value. In particular,

$$p(h|\theta_H) = \frac{p(\theta_H, h)}{p(\theta_H)} = \frac{3/8}{1/2} = 3/4. \tag{14.2}$$

That is, knowing that the project is of high value, Alice thinks that Bob must have gotten a favorable signal h with probability $3/4$. Likewise, if Alice knew that the project is of low value, she would think that Bob has gotten a low signal l with probability $3/4$.

The main solution concept for Bayesian games, namely Bayesian Nash equilibrium, will use interim beliefs. Hence, it is worth introducing a couple of more pieces of notation regarding interim beliefs. The expectation under the conditional belief $p(\cdot|t_i)$ is called the *conditional expectation* given t_i and denoted by $E[\cdot|t_i]$. In particular, the conditional expectation of a player's payoff given her type is computed as

$$E[u_i(s)|t_i] = \sum_{(\theta,t)\in\Theta\times T_{-i}} u_i(s_1(t_1), \ldots, s_n(t_n), \theta, t_{-i})\, p(\theta, t_{-i}|t_i). \tag{14.3}$$

For example, if Alice plays $s_A = HL$ and Bob plays $s_B = HL$ (meaning that he plays H

if and only if his signal is h), then Alice's conditional expected utility at type $t_A = \theta_H$ is

$$\begin{aligned} E\left[u_A\left(HL, HL\right)|\theta_H\right] &= \frac{3}{4}u_A\left(s_A\left(\theta_H\right), s_B\left(h\right), \theta_H\right) + \frac{1}{4}u_A\left(s_A\left(\theta_H\right), s_B\left(l\right), \theta_H\right) \\ &= \frac{3}{4}u_A\left(H, H, \theta_H\right) + \frac{1}{4}u_A\left(H, L, \theta_H\right) \\ &= \frac{3}{4}\times 4 + \frac{1}{4}\times 0 = 3. \end{aligned}$$

The first line in this calculation reflects the fact that type θ_H of Alice believes that Bob's type is h with probability $3/4$ and l with probability $1/4$, knowing that the state is θ_H and her type is θ_H. According to the strategy profile, Alice plays H, and Bob plays H when his type is h and plays L when his type is l (with probabilities $3/4$ and $1/4$, respectively). This gives the second equality. The rest is simple algebra.

Observe that Alice's payoff does not depend on what she would have played if her type were θ_L because she knows that her type is θ_H. This is generally true, and a player's conditional expectation depends only on her action at the given type. Hence, one simply writes the expected payoff as a function of her action, as $E\left[u_i\left(a_i, s_{-i}\right)|t_i\right]$; one sometimes writes $E\left[u_i\left(a_i, s_{-i}, \theta, t\right)|t_i\right]$ to emphasize the dependence on the state and types.

14.2 Bayesian Nash Equilibrium

When the probability of each type is positive according to p, a Bayesian Nash equilibrium is defined as a Nash equilibrium of the Bayesian game. In that case, in a Nash equilibrium, for each type t_i, player i plays a best response to the others' strategies towards maximizing her expected payoff under her interim belief at t_i. If the probability of some type t_i is zero, then at the ex-ante stage player i can plan to play any action at type t_i as a best-response because the probability of that contingency is zero. This leads to spurious Nash equilibria when there are zero-probability types. The general definition of Bayesian Nash equilibrium rules out such spurious Nash equilibria, by requiring each player to play a best response to the other players' strategies under her interim belief at each of her types, regardless of whether the probability of that type is positive.

Definition 14.2. In a Bayesian game, a strategy profile $s^* = (s_1^*, ..., s_n^*)$ is said to be a *Bayesian Nash Equilibrium* if for each player i and type $t_i \in T_i$,

$$s_i^*(t_i) \in \arg\max_{a_i \in A_i} E\left[u_i\left(a_i, s_{-i}^*, \theta, t\right)|t_i\right]$$

where $E\left[u_i\left(a_i, s_{-i}^*, \theta, t\right)|t_i\right]$ is the conditional expected utility of player i from playing a_i against s_{-i}^* given her type t_i, as defined in (14.3).

That is, for each player i and for each possible type, the action chosen is a best response to the other players' strategies under the *interim beliefs* of that type. The formula (14.3) for calculating expected utility $E\left[u_i\left(a_i, s_{-i}^*, \theta, t\right)|t_i\right]$ was introduced in the previous section. Note that the utility function u_i of player i may depend on players' actions, state and players' types.[4] Notice also that a Bayesian Nash equilibrium is a Nash equilibrium of a Bayesian game with the additional property that each type plays a best reply. This property is necessarily satisfied in any Nash equilibrium if all types occur with positive probability.

The next example illustrates how to check if a strategy profile is a Bayesian Nash equilibrium.

Example 14.9. In Example 14.3, the following s^* is a Bayesian Nash equilibrium:

$$s_A^*\left(\theta_H\right) = s_A^*\left(\theta_L\right) = s_B^*\left(t_B\right) = L. \tag{14.4}$$

In order to verify that s^* is a Bayesian Nash equilibrium, one must check that no type has an incentive to deviate, checking for three types. First, start with type $t_A = \theta_H$. One can see from (14.1) that her payoff from playing L is

$$u_A\left(L, L, \theta_H\right) = 1.$$

This is because she knows that $\theta = \theta_H$ and Bob will play L according to s_B^*—because Bob has only one type. Her payoff from playing H would be

$$u_A\left(H, L, \theta_H\right) = 0.$$

Since

$$u_A\left(L, L, \theta_H\right) \geq u_A\left(H, L, \theta_H\right),$$

she does not have an incentive to deviate. Similarly, type $t_A = \theta_L$ of Alice also has no incentive to deviate, as she gets $u_A\left(L, L, \theta_L\right) = 0$ from playing $s_A^*\left(\theta_L\right) = L$ and

[4] Utility function u_i does not depend on the whole of strategies $s_1, \ldots, s_n$, but the expected value of u_i possibly does.

$u_A(H, L, \theta_L) = -2$ from playing H. Finally, consider Bob. Bob's expected payoff from L is

$$\begin{aligned} E\left[u_B\left(s_A^*\left(t_A\right), L, \theta\right)\right] &= q u_B\left(s_A^*\left(\theta_H\right), L, \theta_H\right) + (1-q)\, u_B\left(s_A^*\left(\theta_L\right), L, \theta_L\right) \\ &= q u_B\left(L, L, \theta_H\right) + (1-q)\, u_B\left(L, L, \theta_L\right) \\ &= q. \end{aligned}$$

Note how Bob calculates his expected payoff. He assigns probability q to $\theta = t_A = \theta_H$, and knowing that Alice would play $s_A^*(t_A) = L$ in that case, he figures that his payoff will be $u_B(L, L, \theta_H)$ in that case. Since Alice also plays L when $\theta = t_A = \theta_L$, his payoff will be $u_B(L, L, \theta_L)$ in that case.[5] In computing his payoff from H, one simply replaces Bob's action form L to H in the above calculation:

$$\begin{aligned} E\left[u_B\left(s_A^*\left(t_A\right), H, \theta\right)\right] &= q u_B\left(L, H, \theta_H\right) + (1-q)\, u_B\left(L, H, \theta_L\right) \\ &= -2\left(1-q\right). \end{aligned}$$

Since $q \geq -2(1-q)$, this shows that he has no incentive to deviate. Since no type has an incentive to deviate, s^* is a Bayesian Nash equilibrium.

The above example illustrates how to check whether a given strategy profile in a Bayesian Nash equilibrium. One may ask: how will I compute the set of all Bayesian Nash equilibria? Computing equilibria for a game in which strategies are functions themselves can be intimidating. Indeed, it is challenging to compute the set of Nash equilibria in Bayesian games in general, the same way it is challenging to compute the set of equilibria in extensive-form games. When there are only a few types for each player, one can compute the set of Bayesian Nash equilibria by computing the set of Nash equilibria in the normal-form representation; they coincide with Bayesian Nash equilibria when all types have positive probability and contain the Bayesian Nash equilibria otherwise. For example, Example 14.7 shows that (LL, L) is the only Nash equilibrium $q = 1/2$ using the normal form. This shows that s^* above is the only Bayesian Nash equilibrium for $q = 1/2$. More generally, Example 14.3 shows that the game is dominance solvable with the unique solution (LL, L) for all q with $1/2 \leq q < 2/3$,

[5] Note that his expected payoff is as in Example 14.1, where Alice did not have private information because Alice does not use that information under s_A^*. In general, her action would depend on her type, and the expected utility calculations are different when she has private information.

showing that this is the only Bayesian Nash equilibrium for $1/2 \leq q < 2/3$. However, it is often easier to compute the set of Bayesian Nash equilibria directly by reasoning about the types' incentives, as in the next example.

Example 14.10. In Example 14.3, one can compute the set of Bayesian Nash equilibria for all q as follows. Type θ_L of Alice must play L because L gives Alice higher payoff than H no matter what Bob plays when $t_A = \theta_L$. On the other hand, the best response for type θ_H of Alice is H if Bob plays H and L if Bob plays L. In the above example, s^* picks L for both types θ_H and t_B, yielding one equilibrium. It remains to check whether one can instead pick H for those types, i.e., whether s^{**} with

$$s_A^{**}(\theta_H) = s_B^{**}(t_B) = H \text{ and } s_A^{**}(\theta_L) = L$$

is a Bayesian Nash equilibrium. To this end, one checks that Bob has no incentive to deviate; checking Alice's incentives are as in the complete information case (Example 14.1). Bob's expected payoff from H is

$$E\left[u_B\left(s_A^{**}(t_A), H, \theta\right)\right] = qu_B(H, H, \theta_H) + (1-q)\,u_B(L, H, \theta_L) = 4q - 2(1-q) = 6q - 2;$$

with probability q the project is of high value and Alice puts high effort, giving Bob 4, and with the remaining probability, the project is of low value and Alice puts low efforts, giving Bob -2. Bob's expected payoff from L is

$$E\left[u_B\left(s_A^{**}(t_A), H, \theta\right)\right] = qu_B(H, L, \theta_H) + (1-q)\,u_B(L, L, \theta_L) = 3q + 0 \times (1-q) = 3q.$$

Bob has no incentive to deviate from H if and only if $6q - 2 \geq 3q$, i.e.,

$$q \geq 2/3.$$

When $q \geq 2/3$, there are two Bayesian Nash equilibria (in pure strategies), s^* and s^{**}; when $q < 2/3$, s^* is the only Bayesian Nash equilibrium.

Notice that although Bob knows Alice's strategy s^{**}, he does not know what action Alice plays because Alice's strategy depends on her type, which Bob does not know. This is generally the case in Bayesian Nash equilibrium, where uncertainty arises from other players' private information.

Bayesian Nash equilibrium in mixed strategies The above definition and the examples of Bayesian Nash equilibria focused on pure strategies. Bayesian Nash equilibrium can also be in mixed strategies. In a Bayesian game, a mixed strategy of a player i is a mapping σ_i that assigns a probability distribution $\sigma_i(\cdot|t_i)$ on the set A_i of actions of player i. According to σ_i, player i plays each action a_i with probability $\sigma_i(a_i|t_i)$ when her type is t_i. In a Bayesian Nash equilibrium σ^*, each type t_i plays a possibly mixed action $\sigma_i^*(\cdot|t_i)$ as a best response to the other players' mixed strategies σ_{-i}^*. As in the case of mixed-strategy Nash equilibrium, a type puts positive probability only on actions that are a best response to other players' strategies (under the interim beliefs of the type). This leads to the following characterization of a mixed-strategy Bayesian Nash equilibrium, which can also be viewed as a definition. A mixed strategy profile $\sigma^* = (\sigma_1^*, \ldots, \sigma_n^*)$ is a Bayesian Nash equilibrium if and only if for each player i, each type t_i and each action a_i with $\sigma_i^*(a_i|t_i) > 0$, a_i is a best response to σ_{-i}^* for type t_i:

$$E\left[u_i\left(a_i, \sigma_{-i}^*(t_{-i}), \theta, t\right)|t_i\right] \geq E\left[u_i\left(a_i', \sigma_{-i}^*(t_{-i}), \theta, t\right)|t_i\right] \qquad (\text{for all } a_i'),$$

where the expectation is taken both with respect to (θ, t_{-i}) using the conditional probability distribution $p(\cdot|t_i)$ and with respect to the other players' actions using the mixed strategy σ_{-i}^*. Of course, as in the mixed strategy Nash equilibrium, a type puts a positive probability on more than one action only if she is indifferent between those actions. The following example illustrates the mixed strategy Nash equilibrium.

Example 14.11. In Example 14.3, take $q > 2/3$, and consider the following mixed strategy Nash equilibrium σ^*:

$$\sigma_A^*(H|\theta_H) = \alpha; \sigma_A^*(H|\theta_L) = 0; \sigma_B^*(H|t_B) = \beta.$$

That is, Alice exerts high effort with probability α when she knows that the project is of high value, and Bob exerts high effort with probability β, where α and β are probabilities to be computed in equilibrium. Alice exerts low effort when the project is of low value (as her only best response). The pure-strategy equilibrium s^* corresponds to $\alpha = \beta = 0$, and the equilibrium s^{**} corresponds to $\alpha = \beta = 1$. There is a third equilibrium where α and β are strictly in between zero and one. Towards computing that equilibrium, compute the best response of Bob first. If he plays H, his expected payoff is

$$\begin{aligned} E\left[u_B\left(\sigma_A^*(t_A), H, t_A\right)\right] &= q\left[\alpha u_B(H, H, \theta_H) + (1-\alpha) u_B(L, H, \theta_H)\right] + (1-q) u_B(L, H, \theta_L) \\ &= 4q\alpha - 2(1-q). \end{aligned}$$

The first line is obtained as follows. When Alice's type is θ_H, she plays H with probability α, leading to payoff $u_B(H, H, \theta_H)$ for Bob, and she plays L with probability $1-\alpha$, leading to payoff $u_B(L, H, \theta_H)$ for Bob. Hence, when Alice's type is $t_A = \theta_H$, Bob's expected payoff is the expression in the square brackets. This payoff is multiplied by the probability q of type $t_A = \theta_H$. With probability $1-q$, Alice's type is θ_L, and she plays L, and this gives the second term in that line. The second line is computed by substituting the values of the payoffs above. Similarly, if Bob plays L, his expected payoff is

$$\begin{aligned} E\left[u_B\left(\sigma_A^*\left(t_A\right), L, t_A\right)\right] &= q\left[\alpha u_B\left(H, L, \theta_H\right) + (1-\alpha) u_B\left(L, L, \theta_H\right)\right] + (1-q) u_B\left(L, H, \theta_L\right) \\ &= (2\alpha + 1) q. \end{aligned}$$

If $E\left[u_B\left(\sigma_A^*\left(t_A\right), H, t_A\right)\right] > E\left[u_B\left(\sigma_A^*\left(t_A\right), L, t_A\right)\right]$, then Bob's unique best response is H, and equilibrium requires that $\beta = 1$, which leads to the equilibrium $\alpha = \beta = 1$. Likewise, if $E\left[u_B\left(\sigma_A^*\left(t_A\right), H, t_A\right)\right] < E\left[u_B\left(\sigma_A^*\left(t_A\right), L, t_A\right)\right]$, then Bob's unique best response is L, and equilibrium requires that $\beta = 0$, which leads to the equilibrium $\alpha = \beta = 0$. For β strictly in between 0 and 1, Bob must be indifferent:

$$E\left[u_B\left(\sigma_A^*\left(t_A\right), H, t_A\right)\right] = E\left[u_B\left(\sigma_A^*\left(t_A\right), L, t_A\right)\right].$$

Substituting the values above in this indifference condition, one obtains

$$4q\alpha - 2(1-q) = (2\alpha + 1) q.$$

Solving this indifference equation for α, one computes its value in the third equilibrium:

$$\alpha = \frac{2-q}{2q}.$$

Here, α takes value of 1 at $q = 2/3$, and decreases to $1/2$ as q goes up to 1. Now, for $q > 2/3$, α is strictly between 0 and 1, and the type θ_H of Alice must be playing both H and L with positive probabilities. Therefore, she must be indifferent:

$$E\left[u_A\left(H, \sigma_B^*, t_A\right) | t_A = \theta_H\right] = E\left[u_A\left(L, \sigma_B^*, t_A\right) | t_A = \theta_H\right]$$

Since she knows that $t_A = \theta_H$, the left hand-side is

$$E\left[u_A\left(H, \sigma_B^*, \theta_H\right)\right] = \beta u_A\left(H, H, \theta_H\right) + (1-\beta) u_A\left(H, L, \theta_H\right) = 4\beta,$$

and the right-hand side is

$$E\left[u_A\left(L,\sigma_B^*,\theta_H\right)\right]=\beta u_A\left(L,H,\theta_H\right)+\left(1-\beta\right)u_A\left(L,L,\theta_H\right)=2\beta+1.$$

Therefore, the indifference condition for Alice's type $t_A=\theta_H$ becomes $4\beta=2\beta+1$, yielding

$$\beta=1/2.$$

This yields the third Bayesian Nash equilibrium: Alice plays H with probability $\alpha=\frac{2-q}{2q}$ when the project is of high value and plays L for sure when it is of low value, while Bob mixes between H and L with equal probabilities. It is straightforward to verify that this is indeed a Bayesian Nash equilibrium because types θ_H and t_B are indifferent between their H and L and L is dominant for type θ_L.

In the above example, only Alice had private information, and hence interim beliefs were trivial; Alice knew everything while Bob had a single dummy type, keeping the ex-ante probability of q for $t_A=\theta_H$ in the interim stage. In general, updating plays a central role in equilibrium behavior when the types are correlated. The next example illustrates this on Example 14.8, where both Alice and Bob have private information, and their types are correlated.

Example 14.12. Consider the game in Example 14.8, in which both Alice and Bob have two types. It is straightforward to check that in a Bayesian Nash equilibrium s^* all types exert low effort:

$$s_A^*\left(\theta_H\right)=s_A^*\left(\theta_L\right)=s_B^*\left(h\right)=s_B^*\left(l\right)=L.$$

Indeed, as one can see from (14.1), when the other player plays L, the payoff from L is higher than the payoff from H at each state, showing that no type has an incentive to deviate. Can there be an equilibrium in which a type exerts high efforts? Once again, it is strictly dominated for Alice to exert high effort when the value of the project is low. Hence, type $t_A=\theta_L$ must play L in any equilibrium. It follows then that Bob must play L when he observes a low signal l—as in Example 14.3.[6] Since L is the only best

[6] Indeed, type l assigns probability $3/4$ on $\theta=t_A=\theta_L$, and Bob loses 2 by playing H in that case if Alice plays L. Under the best case scenario for H, Alice plays H when $t_A=\theta_H$. In that case, Bob gains 1 from playing H. This adds up to a loss in expectation: $1/4-2\times 3/4=-5/4$.

response when the other player always plays L, this leaves us with only one candidate for another Bayesian Nash equilibrium in pure strategies:

$$s_A^{**}(\theta_H) = s_B^{**}(h) = H \text{ and } s_A^{**}(\theta_L) = s_B^{**}(l) = L,$$

the strategy profile s^{**} in which high types exert high effort. To verify that this is indeed a Bayesian Nash equilibrium, one needs to check that high types do not have an incentive to deviate; the low types do not have an incentive to deviate to H as discussed above. To this end, first, consider type $t_A = \theta_H$ of Alice. Her expected payoff from playing H is

$$\begin{aligned} E\left[u_A\left(H, s_B^{**}(t_B), \theta\right) | t_A = \theta_H\right] &= p(h|\theta_H)\, u_A\left(H, s_B^{**}(h), \theta_H\right) + p(l|\theta_H)\, u_A\left(H, s_B^{**}(l), \theta_H\right) \\ &= \frac{3}{4} u_A(H, H, \theta_H) + \frac{1}{4} u_A(H, L, \theta_H) \\ &= \frac{3}{4} \cdot 4 + \frac{1}{4} \cdot 0 \\ &= 3. \end{aligned}$$

There are several crucial steps in this expected utility calculations. First, one uses Alice's interim belief for the given type t_A to calculate the expectation; the interim beliefs were calculated in (14.2). Bob's type is h with probability $p(h|\theta_H) = 3/4$, as this is the interim probability the type $t_A = \theta_H$ under consideration assigns to Bob's type h. If one were to calculate expected payoff for type $t_A = \theta_L$, then one would have used probability $p(h|\theta_L) = 1/4$ instead. Second, one only considers $\theta = \theta_H$ because type $t_A = \theta_H$ of Alice knows that $\theta = \theta_H$, assigning probability zero to $\theta = \theta_L$. (In general, expected utility calculation will weight the payoff at any (θ, t_B) with probability $p(\theta, t_B|t_A)$.) Finally, although Alice knows that Bob plays s_B^{**}, she does not know whether Bob exerts high effort because his effort level depends on his type, which Alice does not know. According to $t_A = \theta_H$, with probability $3/4$ Bob observes $t_B = h$ and plays $s_B^{**}(h) = H$, and with the remaining probability he observes $t_B = l$ and plays $s_B^{**}(h) = L$, as in the second line. One compares this expected payoff of 3 to her expected payoff from playing L instead. In the latter expectation, one simply replaces Alice's action from H to L everywhere in the previous calculation:

$$
\begin{aligned}
E\left[u_A\left(L, s_B^{**}\left(t_B\right), \theta\right) | t_A=\theta_H\right] &= p\left(h | \theta_H\right) u_A\left(L, s_B^{**}(h), \theta_H\right)+p\left(l | \theta_H\right) u_A\left(L, s_B^{**}(l), \theta_H\right) \\
&= \frac{3}{4} u_A\left(L, H, \theta_H\right)+\frac{1}{4} u_A\left(L, L, \theta_H\right) \\
&= \frac{3}{4} \cdot 3+\frac{1}{4} \cdot 1 \\
&= 5/2.
\end{aligned}
$$

Since

$$
E\left[u_A\left(H, s_B^{**}\left(t_B\right), \theta\right) | t_A=\theta_H\right] \geq E\left[u_A\left(L, s_B^{**}\left(t_B\right), \theta\right) | t_A=\theta_H\right],
$$

H is a best response to s_B^{**} for type $t_A=\theta_H$, and she has no incentive to deviate. One checks similarly that type $t_B=h$ does not have an incentive to deviate. His expected payoff from playing H is

$$
\begin{aligned}
E\left[u_B\left(s_A^{**}\left(t_A\right), H, \theta\right) | t_B=h\right] &= p\left(\theta_H | h\right) u_B\left(s_A^{**}\left(\theta_H\right), H, \theta_H\right)+p\left(\theta_L | h\right) u_B\left(s_A^{**}\left(\theta_L\right), H, \theta_L\right) \\
&= \frac{3}{4} u_B\left(H, H, \theta_H\right)+\frac{1}{4} u_B\left(L, H, \theta_L\right) \\
&= \frac{3}{4} \cdot 4-\frac{1}{4} \cdot 2=10/4.
\end{aligned}
$$

Here, once again, one uses the interim beliefs of type $t_B=h$ as the probability of the states θ. Since Alice's strategy depends on θ, one writes $s_A^{**}\left(\theta_H\right)$ as her action when $\theta=\theta_H$ and $s_A^{**}\left(\theta_L\right)$ as her action when $\theta=\theta_L$. Bob's action is taken as H in both states, because one is calculating the expected payoff from playing H and this action does not depend on the state. The second line is obtained by simply substituting the values of the above probabilities and actions, and the last line is computed by substituting the values of the resulting utilities. Similarly, one can calculate that his expected payoff from playing L is only

$$
E\left[u_B\left(s_A^{**}\left(t_A\right), L, \theta\right) | t_B=h\right]=\frac{3}{4} u_B\left(H, L, \theta_H\right)+\frac{1}{4} u_B\left(L, L, \theta_L\right)=9/4.
$$

Since his expected payoff from playing H, 10/4, is higher than his expected payoff from playing L, 9/4, he has no incentive to deviate either. Since no type has an incentive to deviate, s^{**} is also a Bayesian Nash equilibrium.

It is remarkable how private information affects strategic behavior. When only Alice knows the value of the project, the private information was detrimental because Alice could use that information to exert low effort when the project is of low value. This led Bob to exert low effort, which in turn led Alice to exert low effort even when the value is high. When Bob also has information about the value of the project—albeit imperfectly, this chain is broken. Now, if Bob observes a high signal $t_B = h$, he thinks that the value is unlikely to be low (with probability $1/4$) and discounts the fact that Alice would necessarily exert low effort in that case. He assigns probability $3/4$ to Alice knowing that the value is high ($t_A = \theta_H$). Moreover, and this is very important, when Alice knows that the value is high, she anticipates that Bob also received high signal, assigning probability $3/4$ to type $t_B = h$. This mutual confidence in each other's information allows them to coordinate on high or low effort without worrying much about what they would have done if they had negative information.

14.3 Cournot Duopoly with Incomplete Information

Consider the following Cournot duopoly with demand uncertainty. There are two firms. Firm 1 is has been operating in this market for a while and knows the demand, while Firm 2 is a new entrant to the market and does not know the demand. As in the usual linear Cournot duopoly, each firm i simultaneously produces q_i units of a divisible good and sells it at price

$$P = \theta - Q$$

where θ is a demand parameter and $Q = q_1 + q_2$ is the total supply by the firms. For simplicity of exposition, prices and quantities are allowed to be negative, and the marginal cost of each firm is assumed to be zero.

Unlike in the usual Cournot duopoly, the firms are asymmetrically informed about the market conditions. The demand parameter θ can take two values: it is

$$\begin{array}{ll} \theta_H & \text{with probability } p \text{, and} \\ \theta_L & \text{with probability } 1-p \end{array}$$

where $\theta_H > \theta_L > 0$. Firm 1 knows the value of the demand parameter θ, but Firm 2 does not know it, although it knows that Firm 1 knows it. Each firm maximizes its expected profit. And all of these are common knowledge.

Formally, Firm 1 has two types: θ_H and θ_L, whereas Firm 2 has only one type. Accordingly, a strategy of Firm 1 is a pair of real numbers $q_1(\theta_H)$ and $q_1(\theta_L)$, one for when the demand is high (i.e., $\theta = \theta_H$) and one for when the demand is low (i.e., $\theta = \theta_L$). Since Firm 2 has only one type, a strategy of Firm 2 is just a real number q_2. This section analyzes the Bayesian Nash equilibrium of this game and compares it to the case when the demand is known.

Bayesian Nash Equilibrium A Bayesian Nash equilibrium is a triplet $(q_1^*(\theta_H), q_1^*(\theta_L), q_2^*)$ of real numbers, where $q_1^*(\theta)$ is the production level of type θ of Firm 1, and q_2^* is the production level of Firm 2. In equilibrium, each type plays a best response. First, consider the high-demand type θ_H of Firm 1. In equilibrium, that type knows that $\theta = \theta_H$ and that Firm 2 produces q_2^*. Hence, its production level, $q_1^*(\theta_H)$, solves the maximization problem

$$\max_{q_1} [\theta_H - q_1 - q_2^*] q_1.$$

Hence,

$$q_1^*(\theta_H) = \frac{\theta_H - q_2^*}{2}. \tag{14.5}$$

Now consider the low-demand type θ_L of Firm 1. In equilibrium, that type also knows that Firm 2 produces q_2^*, but it knows that the demand is low. Hence, its production level, $q_1^*(\theta_L)$, solves the maximization problem

$$\max_{q_1} [\theta_L - q_1 - q_2^*] q_1.$$

Hence,

$$q_1^*(\theta_L) = \frac{\theta_L - q_2^*}{2}. \tag{14.6}$$

There are two important points here. First, Firm 1 knows the demand, and its production level depends on whether the demand is high or low. Second, since Firm 2 does not have private information, its production level q_2^* is known by Firm 1, and both types of Firm 1 play a best response to q_2^* (under distinct demand levels).

Now consider Firm 2. It has one type. Firm 2 knows the strategy of Firm 1, but since it does not know which type of Firm 1 it faces, it does not know the production level of Firm 1. In Firm 2's view, there are two possibilities. With probability p, the demand is high and Firm 1 produces $q_1^*(\theta_H)$ units. With probability $1 - p$, the demand is low and Firm 1 produces $q_1^*(\theta_L)$ units. Hence, the expected profit of Firm 2 from production level q_2 is

$$\begin{aligned} U_2(q_2) &= p\left[\theta_H - q_1^*(\theta_H) - q_2\right] q_2 + (1-p)\left[\theta_L - q_1^*(\theta_L) - q_2\right] q_2 \\ &= \left(E[\theta] - E[q_1^*(\theta)] - q_2\right) q_2, \end{aligned}$$

where

$$E[\theta] = p\theta_H + (1-p)\,\theta_L$$

is the expected demand and

$$E[q_1^*(\theta)] = pq_1^*(\theta_H) + (1-p)\,q_1^*(\theta_L)$$

is the expected supply by Firm 1. The equilibrium strategy, q_2^*, of Firm 2 maximizes the quadratic function $U_2(q_2)$:

$$q_2^* = \frac{E[\theta] - E[q_1^*(\theta)]}{2}. \qquad (14.7)$$

In this example, not knowing θ and the other firm's production level, Firm 1 produces the best response to the expected demand parameter and the expected production level. In general, the equilibrium action is a best response to the expected strategy of the other player when *and only when* the actions of the other players affect the payoff of the player linearly, as in this case.[7] Likewise, it is best response to the expected value of an unknown parameter if and only if the parameter enters the player's utility function linearly (as in this case). In particular, when the other players' actions have a non-linear effect on the payoff of a player, her action may **not** be a best response to expected actions of the others. It is a common mistake to take a player's action as a best response to the expected action of others; you must avoid it.

To compute the Bayesian Nash equilibrium, one simply needs to solve the three linear equations (14.5), (14.6), and (14.7) for $(q_1^*(\theta_L), q_1^*(\theta_H), q_2)$. Observe from (14.5) and (14.6) that

$$E[q_1^*(\theta)] = \frac{E[\theta] - q_2^*}{2}.$$

Substituting this in (14.7), one obtains the equilibrium production of Firm 2 as

$$q_2^* = E[\theta]/3.$$

[7] To be more precise, when $\partial U_i/\partial q_i$ is linear in q_j.

Substituting this production level for Firm 2 in equations (14.5) and (14.6), one obtains the equilibrium production of Firm 1 for each θ:

$$q_1^*(\theta) = \frac{\theta - E[\theta]/3}{2} = \frac{\theta}{2} - \frac{E[\theta]}{6}.$$

Firm 1 makes its decision under perfect information. It knows the size of the demand and knows how much the other firm produces (in equilibrium). This knowledge allows Firm 1 to calculate the residual demand, $\theta - E[\theta]/3$, left by the other firm, and chooses its production optimally, the way a monopolist would choose its production under the residual demand. On the other hand, Firm 2 makes its decision under darkness. It does not know the size of demand or the production level of the other firm. It chooses the production level of $E[\theta]/3$.

Observe that the production level of Firm 2 does not react to the variations in demand size, reflecting the fact that Firm 2 does not know the demand size. In response, knowing demand size, Firm 1 takes full advantage of this, and its production varies with respect to θ with slope 1/2:

$$q_1^*(\theta) - q_1^*(\theta') = \frac{\theta - \theta'}{2}.$$

The production level of Firm 1 reacts to the variations in θ, reflecting the fact the Firm 1 (alone) knows θ.

It is useful to compare this to a couple of other scenarios. If a firm i is a monopoly and knows the demand size, then it produces

$$q^M(\theta) = \theta/2.$$

The rate at which an informed monopolist's production reacts to the changes in demand is the same as the one for the informed firm in the duopoly above:

$$q^M(\theta) - q^M(\theta) = \frac{\theta - \theta'}{2}.$$

This is not a coincidence. In both cases, the firm optimally produces half of the residual demand. Since Firm 2 is uninformed and cannot react to a variation in demand size, the variation is fully reflected in the residual demand for the informed firm, making the changes in its production identical in two cases. In contrast, if both firms were informed, each would have produced

$$q^{NE}(\theta) = \theta/3,$$

and each firm's production would react to the changes in demand at a lower rate of $1/3$. Here, firms' reactions to the demand variations are muted because the other firm also takes advantage of the demand information, and the fluctuations in the demand is not fully reflected in the residual demand.

14.4 Bayesian Games with Dynamics

In a Bayesian game (N, A, Θ, T, u, p), after observing their types, the players simultaneously choose their "actions". Sometimes, as in Example 14.3, an action can be a simple move, such as exerting high or low effort. In general, players can be playing a dynamic game after observing their types. In that case, an action a_i corresponds to a plan of action in that dynamic game. Specifically, after observing their types, players can be playing a general extensive form game where payoffs depend on θ and t as well as the play of the game. Taking θ and t as known parameters, one can compute the normal form representation of the extensive form game, where the utility functions may vary with θ and t. This leads to a normal form game (N, A, u). The Bayesian game is then formulated as (N, A, Θ, T, u, p), by adding the type space (Θ, T, p).

Example 14.13. Consider the extensive-form game in Figure 14.2. There are two players: a Firm and a Worker. Worker can be of High ability, in which case he would like to Work when he is hired, or of Low ability, in which case he would rather Shirk. Firm would want to Hire the worker that will work but not the worker that will shirk. Worker knows his ability level. Firm does not know whether the worker is of high ability or low ability. Firm believes that the worker is of high ability with probability p and low ability with probability $1 - p$. Most importantly, the firm knows that the worker knows his own ability level. In the extensive-form game, this is modeled by Nature choosing between High and Low and revealing the choice to the worker but not to the firm.

In this game, after the worker observes his ability, they play one of the games in Figure 14.3, depending on the worker's ability. These smaller games can be represented in normal form by setting

- $N = \{F, W\}$;

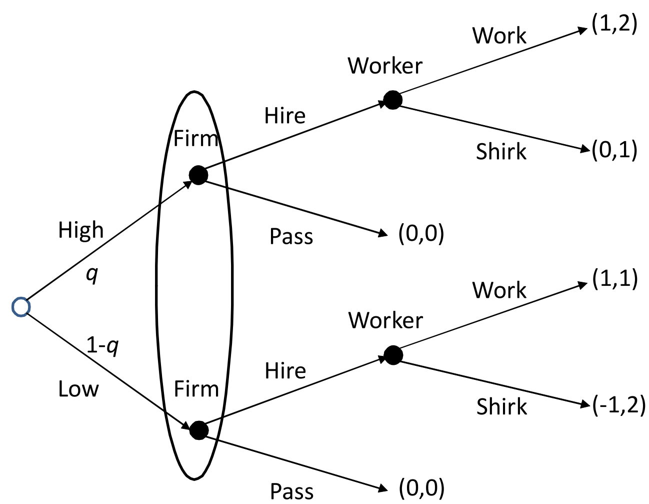

Figure 14.2: A game on employment decisions with incomplete information

- $A_F = \{\text{hire, pass}\}$, $A_W = \{\text{work, shirk}\}$;[8]
- and the utility functions u_F and u_W are defined by the following tables, where the first entry is the payoff of the firm and the tables on the left and right correspond to high- and low-ability workers, respectively,

high	work	shirk
hire	$1,2$	$0,1$
pass	$0,0$	$0,0$

low	work	shirk
hire	$1,1$	$-1,2$
pass	$0,0$	$0,0$

Then, the game in Example 14.13 can be represented as a Bayesian game (N, A, T, u, p) where N, A, and u are as above, and the type space is defined by

- $T_F = \{t_F\}, T_W = \{\text{high, low}\}$;
- $p(t_F, \text{high}) = p$, $p(t_F, \text{low}) = 1 - p$.

For example, for $p = 3/4$, consider the Nash equilibrium of the game between the firm and the worker in which the firm hires and worker works if and only if Nature

[8] Here, "work" means "work if hired", and "shirk" means "shirk if hired".

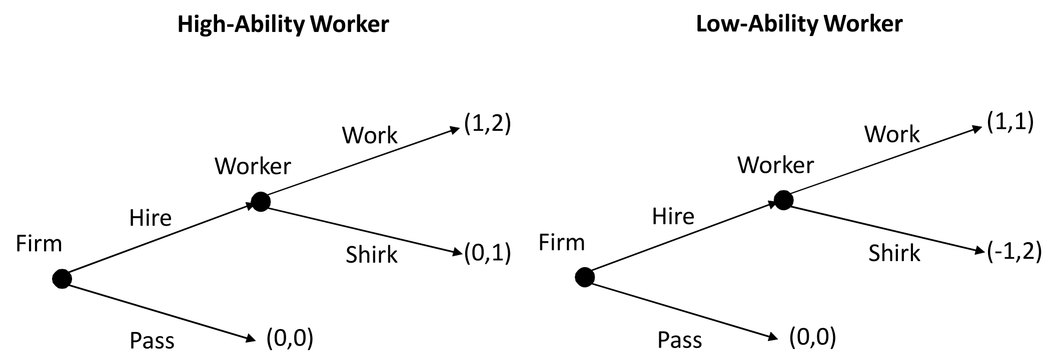

Figure 14.3: Game forms in Example 14.13, depending on workers ability

chooses high. One can formally write this strategy profile as $s^* = (s_F^*, s_W^*)$ with

$$\begin{aligned} s_F^*(t_F) &= \text{hire}, \\ s_W^*(\text{high}) &= \text{work}, \\ s_W^*(\text{low}) &= \text{shirk}. \end{aligned}$$

To see why this is a Bayesian Nash equilibrium, observe that high- and low-ability workers prefer to work and shirk, respectively, at their only information sets. Hence, work and shirk are best responses to hire for high- and low-ability workers, respectively. To check that hire is a best response for the firm, observe that, at its only type t_F, firm assigns probability 3/4 to high type and 1/4 to low type. Hence, if it hires the worker, with probability 3/4, the worker is of high type and works, yielding the payoff of $u_F(\text{hire, work, high}) = 1$, and with probability 1/4, the worker is of low type and shirks, yielding the payoff of $u_F(\text{hire, shirk, low}) = -1$. Hence, the firm's expected payoff from hire is

$$E\left[u_F\left(\text{hire}, s_W^*, t_W\right)|t_F\right] = 1 \times \frac{3}{4} + (-1) \times \frac{1}{4} = \frac{1}{2}.$$

Its expected payoff from pass is 0, and hence hire is a best response.

This game has another Bayesian Nash equilibrium s^{**}:

$$s_F^{**}(t_F) = \text{pass};\ s_W^*(\text{high}) = s_W^*(\text{low}) = \text{shirk}.$$

In this equilibrium, the firm passes on the worker, and the worker would not work if hired, regardless of his type. To see that this is a Bayesian Nash equilibrium, it is best to check the payoffs from the payoff tables above, rather than the game forms in Figure 14.3. When the firm plays pass, the worker gets 0 regardless of his type and action. Hence, shirk is a best response for each type. Likewise, pass is a best response for the firm because its expected payoff would be negative if it hired the worker:

$$E\left[u_F\left(\text{hire}, s_W^{**}, t_W\right)|t_F\right] = p \times 0 + (1-p) \times (-1) = p - 1 = -1/4.$$

This equilibrium is counterintuitive because the high-ability worker was meant to be a worker who would work if hired given the employment contract. As in Nash equilibria that are not subgame-perfect, this equilibrium relies on an irrational move at a node that is not reached in equilibrium. As Nash equilibrium, Bayesian Nash equilibrium allows players choosing irrational moves in information sets that are not supposed to arise. We will later refine Bayesian Nash equilibrium, ruling out such irrational behavior.

14.5 Bayesian Games with a Continuum of Types

The focus so far was on games with finitely many types, and a player had at most two types in the examples. In real-life applications, it is often more natural to consider a continuum of types. For example, in the collaboration example (Example 14.3), the value of completing the project successfully is a real number, and there is no reason to exclude any number between θ_H and θ_L if one thinks that both values θ_H and θ_L are possible. Allowing all real numbers within such an interval leads to a continuum of types. Such a large type space arises naturally in many real-world applications. For example, in the Cournot competition, a type can be an estimate for a demand parameter (as in Section 14.3) or a firm's marginal cost; in auctions, a player's type is often the value of the good auctioned for the player; and in coordinated investment games, a player's type is often her return from investment. In such games, computing the set of Bayesian Nash equilibria is often quite challenging and may be out of reach even for expert game theorists. One then often focuses on a class of equilibria assuming that the equilibrium strategies take a certain form. This section introduces a couple of main functional form assumptions and the basic techniques to compute such equilibria using some examples.

When both types and actions are real numbers, one often imposes some of the following functional form assumptions, assuming there are equilibria with that functional form. A Bayesian game is said to be *symmetric* if players' names do not matter: $T_i = T_j$

and $A_i = A_j$ for all i and j; the joint distribution of types is symmetric, and the payoff functions are identical and symmetric with respect to other players. In a symmetric game, a strategy profile $(s_1, \ldots, s_n)$ is said to be *symmetric* if there exists a function $\tilde{s} : T_1 \to A_1$ such that

$$s_i(t_i) = \tilde{s}(t_i) \qquad \text{for all } i \text{ and } t_i.$$

Any given two players will take the same action if they happen to have the same type. But the players will typically have different types, and they will take different actions. Usually, symmetric games have symmetric Bayesian Nash equilibria. One often focuses on symmetric Bayesian Nash equilibria in symmetric games.

Another common functional form assumption is monotonicity. A strategy s_i is said to be *monotone* if s_i is weakly increasing or weakly decreasing. A Bayesian Nash equilibrium is monotone if all players use a monotone strategy. In a monotone symmetric Bayesian Nash equilibrium, all players use the same monotone strategy $\tilde{s}$. Finally, in applications, one often imposes linearity, when there is such an equilibrium. A strategy $s_i : T_i \to A_i$ is said to be *linear* if $s_i(t_i) = \alpha_i t_i + \beta_i$ for some *constants* α_i and β_i. A linear strategy is monotone; it is increasing when $\alpha > 0$ and decreasing when $\alpha < 0$. A symmetric, linear Bayesian Nash equilibrium is a Bayesian Nash equilibrium $(s_1, \ldots, s_n)$ with $s_i(t_i) = \alpha t_i + \beta$ for some *constants* α and β, where the constants α and β are common for all players. Computing such an equilibrium reduces to computing two constants, making such equilibria attractive for researchers, but such an equilibrium may or may not exist. (Note that all of these functional forms can also be imposed when there are finitely many types, but they become especially useful when there are infinitely many types.)

The rest of this section illustrates these functional forms and techniques to compute such equilibria on simple examples. The first example considers a game where each player has a continuum of types but only two actions. In this example, higher types have higher incentive to play one action, and one naturally look for monotone equilibria, where each player uses a cutoff strategy, taking one action below the cutoff and switching to the other action above the cutoff.

14.5.1 Coordinated Investment with Incomplete Information

There are two players, 1 and 2, and a potential project. Each player may either invest in the project or not invest. If they both invest in the project, it will succeed; otherwise, it will fail and cost money to the party who invests (if there is any investment). The payoffs are as follows:

	Invest	Not Invest
Invest	x_1, x_2	$x_1 - 1, 0$
Not Invest	$0, x_2 - 1$	$0, 0$

Player 1 chooses between rows, and Player 2 chooses between columns. Each player i privately knows her return from investment, x_i, but not the other player's return from investment. Assume that the returns are independently and identically distributed with uniform distribution on a large interval $[\underline{x}, \bar{x}]$ for some $\underline{x} < 0$ and $\bar{x} > 1$.

Notice that the game is symmetric: the players have identical sets of types; they have identical sets of actions, and the type distribution and payoff functions are symmetric. In such games, one often focuses on *symmetric* equilibria. One considers an equilibrium s^*, where both players use the same strategy s:

$$s_i^*(x_i) = s(x_i) \qquad \text{(for every } x_i \text{ and every } i\text{)}$$

where s is a function from the set $[\underline{x}, \bar{x}]$ of types to the set of actions, {Invest, Not Invest}. If players had the same return from investment, they would have taken the same action, but it is likely that their returns are different, and one may invest while the other does not.

Moreover, a player's incentive to invest is increasing in her type (as in Chapter 7), so that—all else equal—if a player prefers to invest when her return is x_i, she would still prefer to invest if she had a higher return x_i'. In such games, one also focuses on *monotone* equilibria in which each player uses a cutoff strategy, investing if her return is above a cutoff, and not investing if her return is below that cutoff. Formally, a *monotone strategy* s_i for this game is a strategy with a cutoff value $\hat{x}_i$ such that player invests if and only if her return exceeds the cutoff:

$$s_i(x_i) = \begin{cases} \text{Invest} & \text{if } x_i \geq \hat{x}_i, \\ \text{Not Invest} & \text{if } x_i < \hat{x}_i. \end{cases}$$

A *symmetric Bayesian Nash equilibrium s^* in monotone strategies* has a cutoff value $\hat{x}$ such that

$$s_i^*(x_i) = \begin{cases} \text{Invest} & \text{if } x_i \geq \hat{x}, \\ \text{Not Invest} & \text{if } x_i < \hat{x}. \end{cases}$$

It is characterized by a single number.

One computes such equilibria by computing the cutoff—as follows. Consider an equilibrium with a cutoff $\hat{x}$. Take any type x_i for a player i. If player i invests, then she gets x_i for sure but loses a payoff of 1 if the other player does not invest. Moreover, in equilibrium, the other player j invests if $x_j \geq \hat{x}$ and not invest if $x_j < \hat{x}$. Hence, the expected payoff of type i from Invest is

$$U(x_i) = x_i - \Pr(x_j < \hat{x}).$$

(Here the probability does not depend on x_i because the types are independently distributed.) The payoff from Not Invest is zero. Hence, Invest is the unique best response if $x_i > \Pr(x_j < \hat{x})$, and Not Invest is the unique best response if $x_i < \Pr(x_j < \hat{x})$. Player i is indifferent if $x_i = \Pr(x_j < \hat{x})$. Since player i must be playing a best response for each type, the unique best response must coincide with the equilibrium action. In particular, she must be indifferent at the cutoff type $x_i = \hat{x}$:

$$\hat{x} = \Pr(x_j < \hat{x}).$$

The indifference condition characterizes the symmetric Bayesian Nash equilibria in monotone strategies. One computes these equilibria by solving the indifference condition. Towards this goal, observe that since the right-hand side is a probability, taking values between 0 and 1, the cutoff $\hat{x}$ must be between zero and one. Since $\underline{x} < 0$ and $\bar{x} > 1$, the cutoff must be in between $\underline{x}$ and $\bar{x}$. Hence,

$$\Pr(x_j < \hat{x}) = \frac{\hat{x} - \underline{x}}{\hat{x} - \underline{x}}.$$

Then, the indifference condition becomes

$$\hat{x} = \frac{\hat{x} - \underline{x}}{\bar{x} - \underline{x}}.$$

The unique solution to this equation is

$$\hat{x} = \frac{-\underline{x}}{\bar{x} - \underline{x} - 1}.$$

Since $-\underline{x} > 0$ and $\bar{x} > 1$, the cutoff $\hat{x}$ is in between 0 and 1.

In general, a symmetric Bayesian Nash equilibrium with cutoff strategies is computed as follows. First, one assumes that the other player j plays according to the equilibrium strategy—with cutoff $\hat{x}$. Second, one computes the best response to this strategy for each type x_i. The best response must contain a cutoff strategy as assumed; e.g., there must be a type $\tilde{x}$ such that invest is a best response for types above $\tilde{x}$ and not invest is a best response for types below $\tilde{x}$. (Otherwise, there is no symmetric equilibrium with cutoff strategies as assumed.) If the equilibrium cutoff is in the interior, then it must be that cutoff type is indifferent. The third step is to write the indifference condition. This condition is an equation for cutoff $\hat{x}$. Finally, one solves the equation to compute the cutoff. There may be multiple solutions, leading to multiple equilibria.

There is one important caveat. The indifference condition is necessary only if there are types on either side of the cutoff so that both actions are played in equilibrium. In the above example, this was ensured by the assumption that $\underline{x} < 0$ and $\bar{x} > 1$, for which not invest and invest are dominant, respectively. Otherwise, the cutoff may be at the corner, when all players can have a strict preference. For example, suppose $0 < \underline{x} < \bar{x} < 1$ above. Then, there are multiple equilibria, one in which all types play invest and one in which all type play not invest.[9] In these equilibria, all types strictly prefer equilibrium action. For example, in "all-invest" equilibrium, the payoff from investment is x_i, which is strictly positive. Imposing indifference condition would give $\hat{x} = 0$, a cutoff outside of the allowed range.

In order to illustrate how to compute symmetric equilibria with cutoff strategies, the above discussion focused on such equilibria. In this simple game, all Bayesian Nash equilibria are symmetric and in cutoff strategies. To see this, consider an arbitrary Bayesian Nash equilibrium σ^*. For each player i, let π_i be the probability that player i does not invest according to σ_i^* ($\pi_i = \frac{1}{\bar{x}-\underline{x}} \int \sigma_i^* \left(\text{not invest}|x_i\right) dx_i$). Then, the expected payoff from invest for the other player j with type x_j is $x_j - \pi_i$. Hence, for any $x_j > \pi_i$, invest gives a strictly positive payoff and it must be that type x_j plays invest (i.e., $\sigma_i^* \left(\text{Invest}|x_j\right) = 1$); likewise types $x_j < \pi_i$ must play not invest. Therefore, all equilibria are in cutoff strategies with cutoffs $\hat{x}_j = \pi_i$; the cutoff type can play either action with positive probability. Moreover, the cutoffs must be identical in equilibrium. Indeed, if

[9] There is also a third cutoff equilibrium with cutoff $\hat{x} = \frac{\underline{x}}{1-(\bar{x}-\underline{x})}$ as in the previous case.

$\hat{x}_i < \hat{x}_j$, then by definition $\pi_i < \pi_j$, showing that $\hat{x}_i = \pi_j > \pi_i = \hat{x}_j$, a contradiction.

14.5.2 Negotiation for Shares

This section illustrates the basic techniques for computing the Bayesian Nash equilibria when there are infinitely many types and actions on the following important example.

There are two players, namely 1 and 2. They want to form a partnership. If they form a partnership, they will jointly produce a surplus, which is normalized to 1. If they do not form a partnership, each player i has an *outside option* of working for some wage w_i, privately known by player i. Wages w_1 and w_2 are independently distributed with the uniform distribution on $[0,1]$. The game proceeds as follows. After privately observing her type w_i, each player i simultaneously asks a share a_i from the surplus. If $a_1 + a_2 \leq 1$, then each player i gets the share a_i she asks and half of the unclaimed surplus, $1 - (a_1 + a_2)$, getting

$$\frac{1}{2} + \frac{a_i - a_j}{2}.$$

If $a_1 + a_2 > 1$, the shares asked by the players are not feasible. In that case, they do not form a partnership, and each player i gets her outside option. The players are risk-neutral.

Formally, this can be modeled by a Bayesian game as follows. Each player's type is her outside option, and the sets of types are $T_1 = T_2 = [0,1]$, with uniform joint distribution. Each player's action is the share she asks, and the sets of actions are $A_1 = A_2 = [0,1]$. By normalizing players' payoffs in case of disagreement to zero, one can write the payoff of each player i as

$$u_i(a_1, a_2, w_1, w_2) = \begin{cases} \frac{1}{2} + \frac{a_i - a_j}{2} - w_i & \text{if } a_1 + a_2 \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

There is a continuum of Bayesian Nash equilibria in this game. I will first present the unique symmetric linear equilibrium in which the shares players ask are linearly increasing in their outside options. I will then present a continuum of equilibria in cutoff strategies.

Symmetric Linear Equilibrium One takes several basic steps in order to compute a symmetric linear equilibrium (and check whether there is such an equilibrium). These steps are explained on the present game next, focusing on the symmetric linear equilibria with strictly increasing strategies.

Step 1 *Assume a symmetric linear equilibrium with strictly increasing strategies:*

$$\begin{aligned} s_1^*(w_1) &= \alpha w_1 + \beta \\ s_2^*(w_2) &= \alpha w_2 + \beta \end{aligned}$$

for all types w_1 and w_2 for some constants $\alpha > 0$ and β, that will be determined later. The important thing here is the constants do not depend on the players or their types.

Step 2 *Compute the expected payoff of each type against the equilibrium strategy by the other player.* Fix some type w_i. Observe that the payoff from asking a_i is

$$u_i\left(a_i, s_j^*(w_j), w_i, w_j\right) = \begin{cases} \frac{1}{2} + \frac{a_i - \alpha w_j - \beta}{2} - w_i & \text{if } a_i \leq 1 - \alpha w_j - \beta, \\ 0 & \text{otherwise.} \end{cases}$$

One can conveniently write the condition $a_i \leq 1 - \alpha w_j - \beta$ as a condition on the other player's type: $w_j \leq (1 - a_i - \beta)/\alpha$. Assuming $(1 - a_i - \beta)/\alpha$ is in between 0 and 1, one then computes the expected payoff as

$$U(a_i) \equiv E\left[u_i\left(a_i, s_j^*(w_j), w_i, w_j\right) | w_i\right] = \int_0^{(1-a_i-\beta)/\alpha} \left(\frac{1}{2} + \frac{a_i - \alpha w_j - \beta}{2} - w_i\right) dw_j. \tag{14.8}$$

Here, one simply integrates the payoff within the range as w_j is uniformly distributed, and the payoff is zero outside of the range. If w_j were not uniformly distributed, one would multiply the payoff with the probability density function of w_j before integrating it.

The expected payoff is illustrated graphically in Figure 14.4. The share a_j asked by the other player is an increasing function of the other player's outside option w_j, which player i does not know. There will be a partnership until a_j hits $1 - a_i$ at $w_j = (1 - a_i - \beta)/\alpha$, and player i will obtain zero payoff after then. The share of player i from partnership depends both on her demand a_i and on the other player's demand a_j. It is given by the solid decreasing line: it starts at $(1 + a_i)/2$ at $w_j = 0$ and reduces to a_i at $w_j = (1 - a_i - \beta)/\alpha$. The payoff of player i is this share minus her outside option

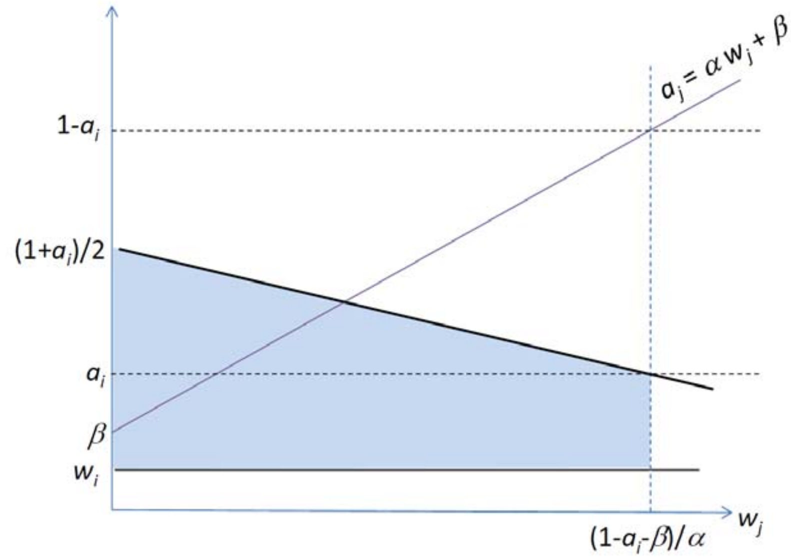

Figure 14.4: Expected payoff from asking a_i for type w_i

w_i. Since player i does not know the other player's outside option w_j, she does not know the share or the payoff she will get. Her expected payoff is the shaded area.

Remark 14.1. It is tempting to compute the integral at this step, but that computation is unnecessary because one takes the derivative of this expectation to compute the best response next, and derivative is the converse of integral. In general those unnecessary calculations may be the hardest step. Hence, it is advisable that one leaves the integral as is and use Leibniz's rule[10] to differentiate it to obtain the first-order condition.

Step 3 *Compute the best response of each type against the equilibrium strategy by the other player.* To this goal, one computes the first-order condition, by taking the derivative of U in (14.8) and setting it equal to zero. One uses Leibniz's rule to take

[10] Leibniz's Rule:

$$\frac{\partial}{\partial x}\int_{t=L(x,y)}^{U(x,y)} f(x,y,t)\,dt = \frac{\partial U}{\partial x}\cdot f(x,y,U(x,y)) - \frac{\partial L}{\partial x}\cdot f(x,y,L(x,y)) + \int_{t=L(x,y)}^{U(x,y)} \frac{\partial}{\partial x} f(x,y,t)\,dt. \quad (14.9)$$

this derivative:

$$\begin{aligned} U'(a_i) &= -\frac{1}{\alpha} \times (a_i - w_i) + \int_0^{(1-a_i-\beta)/\alpha} \frac{1}{2} dw_j \\ &= \frac{1 + 2w_i - \beta - 3a_i}{2\alpha}. \end{aligned} \tag{14.10}$$

One uses the Leibniz's rule to compute the first equality as follows. The derivative of the upper bound $(1 - a_i - \beta)/\alpha$ is $-1/\alpha$, and the payoff in the integral takes the value of $a_i - w_i$ at this bound, as shown in Figure 14.4. One multiplies the derivative with the payoff to obtain the first term. Since the lower bound of the integral is constant, its derivative is zero, and it does not contribute to the derivative. Finally, the derivative of the payoff in the integral with respect to a_i is $1/2$, and one integrates it over the range, to obtain the second term. The first-order condition is obtained by $U'(a_i) = 0$, i.e.,

$$\frac{1 + 2w_i - \beta - 3a_i}{2\alpha} = 0.$$

Solving the first-order condition for a_i, one obtains the best response for type w_i:

$$a^{BR}(w_i) = \frac{1 + 2w_i - \beta}{3}. \tag{14.11}$$

The first-order condition and its derivation is illustrated graphically in Figure 14.5. If type w_i increases the share she asks by an amount of Δ, her expected payoff changes in two ways. First, $1 - a_i$ goes down with the same amount Δ. The cutoff type $(1 - a_i - \beta)/\alpha$ moves to the left by an amount of Δ/α as a result. This results in players not forming a partnership when w_j is in between $(1 - a_i - \beta)/\alpha - \Delta/\alpha$ and $(1 - a_i - \beta)/\alpha$, costing approximately $a_i - w_i$ for player i. The expected value of this loss is approximately the area of the shaded rectangle in the figure, which is $\frac{\Delta}{\alpha} \times (a_i - w_i)$. This loss is represented by the negative term in (14.10). Second, player i gets more from partnership. This gain is $\Delta/2$ for each w_j below the above cutoff, and thus the expected gain is the area of the shaded parallelogram in the figure, which is $(\Delta/2) \times (1 - a_i - \beta)/\alpha$. This gain is represented by the second term in (14.10). As type w_i adjusts her demand, the gain and the loss above vary. At the optimum, the gain must be equal to the loss. For if the gain or loss were larger, she would improve her expected payoff by increasing or decreasing her demand slightly, respectively. This equality is the first-order condition:

$$\frac{\Delta}{\alpha} \times (a_i - w_i) = \frac{\Delta}{2\alpha} \times (1 - a_i - \beta).$$

The solution is given by (14.11).

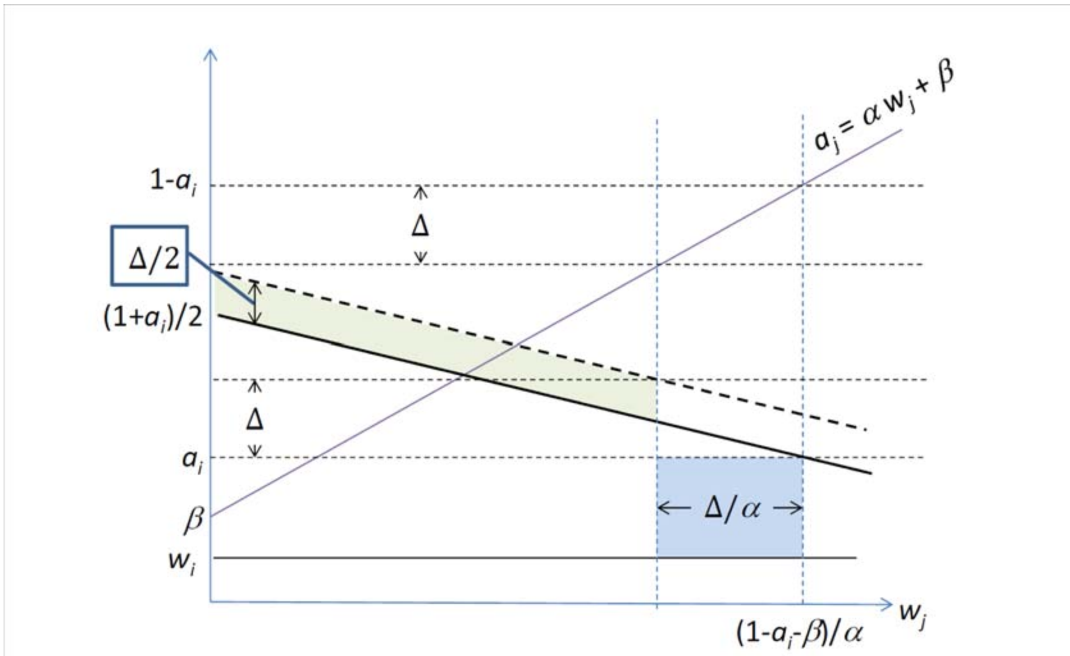

Figure 14.5: The expected payoff change from increasing the demand by Δ.

Step 4 *Verify that best-reply functions are indeed affine*, i.e., $a^{BR}(w_i)$ is of the form $a^{BR}(w_i) = \alpha w_i + \beta$. Indeed, one can rewrite (14.11) as

$$a^{BR}(w_i) = \frac{2}{3}w_i + \frac{1-\beta}{3} \tag{14.12}$$

Check that both $2/3$ and $(1-\beta)/3$ are constant, i.e., they do not depend on w_i, and they are same for both players. (If this were not the case, one would conclude that there is no symmetric, linear Bayesian Nash equilibrium with strictly increasing strategies.)

Step 5 *Compute the constants α and β.* To do this, observe that in order to have an equilibrium, the best reply $a^{BR}(w_i)$ in (14.12) must be equal to $s_i^*(w_i)$ for each w_i. That is,

$$\frac{2}{3}w_i + \frac{1-\beta}{3} = \alpha w_i + \beta$$

must be an identity, i.e., it must remain true for all values of w_i. Hence, the coefficient of w_i must be equal in both sides:

$$\alpha = \frac{2}{3}.$$

The intercept must be same in both sides, too:

$$\beta = \frac{1-\beta}{3}.$$

Thus,

$$\beta = \frac{1}{4}.$$

This yields the symmetric, linear Bayesian Nash equilibrium:

$$s_i^*(w_i) = \frac{2}{3}w_i + \frac{1}{4}.$$

In the only linear symmetric equilibrium with strictly increasing strategies, each player asks two thirds of her outside option in addition to a constant markup of 1/4. She asks more than her outside option when her outside option is low. More interestingly, when her outside option is above 3/4, she asks *less* than her outside option. This may be confusing at first glance. There are two reasons such a low demand. First, since the other player asks at least 1/4, the player knows that the partnership will not be formed—even if the other player's outside option is zero. Hence, increasing her demand does not affect her payoff. Second, in general, a player also gets half of the unclaimed share, which may be high when the other player's outside option is low, and her expected share from partnership may exceed her outside option even if she asks less than her outside option.

Remark 14.2. This illustrates a general technique for computing a symmetric linear equilibrium. One assumed that all players use the same affine function as their strategy, that maps their types to their actions. Under that assumption, one computes the best response for each player. If there is indeed such an equilibrium, the best response will be an affine function of type as assumed. In that case, one identifies the best response with the assumed equilibrium strategy, to compute the parameters for the equilibrium functions. If there is no symmetric linear equilibrium, the best response will not be a linear function of types, and observing that the best-response cannot be linear, one concludes that there is no symmetric linear equilibrium. When there is no symmetric linear equilibrium, the above exercise will be somewhat futile. Luckily, symmetric games will typically have a symmetric equilibrium. One can use the steps above without assuming linearity to compute the symmetric Bayesian Nash equilibria. The equilibrium strategy obtained in this way is often a solution to a differential equation (as in Chapter 20), making the analysis substantially more difficult.

Other Equilibria There is a continuum of other Bayesian Nash equilibria. In particular, for any $\hat{a}_1, \hat{a}_2 \in [0,1]$ with $\hat{a}_1 + \hat{a}_2 = 1$, there is a Bayesian Nash $(\hat{s}_1, \hat{s}_2)$, defined by

$$\hat{s}_i(w_i) = \begin{cases} \hat{a}_i & \text{if } w_i \leq \hat{a}_i, \\ 1 & \text{otherwise.} \end{cases}$$

That is, there is a convention for how to share the surplus, given by $\hat{a}_1$ and $\hat{a}_2$. The players ask the conventional share if that is better than their outside options, and take their outside options otherwise. There is one symmetric equilibrium in this class, according to which the players share the surplus equally.

To check that this is a Bayesian Nash equilibrium, one needs to check that each type plays a best response. First consider a type $w_i > \hat{a}_i$. She asks 1 and gets the payoff of zero, taking her outside option. Since the other player asks at least $\hat{a}_j = 1 - \hat{a}_i$, this will be the outcome as long as she asks more than $\hat{a}_j$. On the other hand, if she asks $a_i \leq \hat{a}_i$, she will receive at most $\hat{a}_i$ if the partnership is formed, and her payoff will be negative, as her share will be less than her outside option. Hence, she plays a best response by asking 1. Now consider a type $w_i \leq \hat{a}_i$. She receives a share of $\hat{a}_i$ from a partnership if $w_j \leq \hat{a}_j$, and she will receive her outside option otherwise. If she asks a higher amount, the partnership will not be formed, and she will get her outside option regardless of w_j. Since her outside option is not better than $\hat{a}_i$, she has no incentive to ask a higher share. If she asks a lower but positive share, she will get a lower share from partnership when $w_j \leq \hat{a}_j$, and she will again receive her outside option otherwise; this will lower her expected payoff. If she asks zero, the partnership will be formed for all values of w_j, but this will be because she will get zero share instead of her outside option when $w_j > \hat{a}_j$. Taken together, this shows that no type has an incentive to deviate, showing that $(\hat{s}_1, \hat{s}_2)$ is a Bayesian Nash equilibrium.

Cost of Incomplete Information The players would like to form a partnership when $w_1 + w_2 \leq 1$ and take their outside options when $w_1 + w_2 > 1$. If players' outside options were known, such a plan could be implemented by a Nash equilibrium. For example, it is a Nash equilibrium that each player i asks

$$a_i^* = w_i + \frac{1}{2} \max\{1 - w_1 - w_2, 0\}.$$

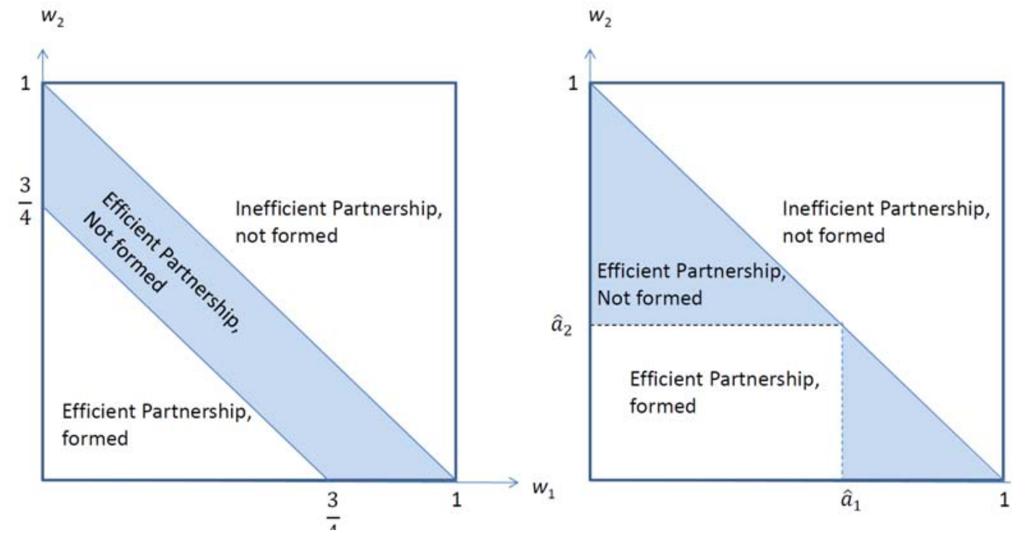

Figure 14.6: Negotiation outcome under incomplete information; symmetric linear equilibrium on the left, and cutoff equilibrium on the right.

Under this equilibrium, the players form a partnership if and only if $w_1 + w_2 \leq 1$. When the players' outside options are private information, such equilibria are not feasible, and all Bayesian Nash equilibria lead to inefficiency in the form of lost partnership opportunities. For example, in the symmetric linear equilibrium above, the players form a partnership (i.e. $s_1^*(w_1) + s_2^*(w_2) \leq 1$) if and only if

$$w_1 + w_2 \leq \frac{3}{4}.$$

Thus, when $\frac{3}{4} < w_1 + w_2 < 1$, the players could strictly improve their payoffs by forming a partnership, but in equilibrium they choose their outside options instead. This inefficiency is illustrated on the left panel of Figure 14.6. In the shaded region, players fail to form a beneficial partnership. Similarly, under a cutoff equilibrium $(\hat{s}_1, \hat{s}_2)$ above, they form a partnership if and only if $(w_1, w_2) \leq (\hat{a}_1, \hat{a}_2)$, and they lose a valuable partnership opportunity when $w_1 + w_2 < 1$ but $w_i > \hat{a}_i$ for some player i, as in the shaded area on the right panel of Figure 14.6.

14.6 Mixed Strategy and Incomplete Information†

In complete information games, in equilibrium, a player may be uncertain about another player's behavior only when the latter plays a mixed strategy. Such uncertain situations correspond to mixed-strategy Nash equilibria. In incomplete information games, a player may be uncertain about another player's behavior even if she knows his strategy. In particular, mixed strategy equilibria of a complete information game can be viewed as pure strategy equilibria of an incomplete information with negligible amount of payoff uncertainty.

For example, consider a thief and a police officer. The thief has stolen an object. He can either hide the object INSIDE his car on in the TRUNK. The police officer stops the thief. She can either check INSIDE the car or the TRUNK, but not both. (She cannot let the thief go without checking, either.) If the police officer checks the place where the thief hides the object, she catches the thief, in which case the thief gets -1 and the police officer gets 1; otherwise, she cannot catch the thief, yielding the payoffs 1 and -1 for the thief and the police officer, respectively. This is a matching-penny game. There is a unique Nash equilibrium, in which Thief hides the object INSIDE or the TRUNK with equal probabilities, and the police officer checks INSIDE or the TRUNK with equal probabilities.

Now imagine that the players know their own payoffs privately. In particular, in addition to the known payoffs above, the thief gets an extra payoff b_i form hiding the object in the TRUNK for some $b_i \in B \equiv \{b_1, b_2, \ldots, b_{100}\}$ and the police officer gets an extra payoff $d_j \in D \equiv \{d_1, d_2, \ldots, d_{100}\}$ from checking the TRUNK where

$$\begin{aligned} b_1 &< b_2 < \cdots < b_{50} < 0 < b_{51} < \cdots < b_{100}, \\ d_1 &< d_2 < \cdots < d_{50} < 0 < d_{51} < \cdots < d_{100}, \end{aligned}$$

where each pair of (b, d) is equally likely. This is a Bayesian game in which the types are b_i and d_j for the thief and the police officer, respectively. The Bayesian game has the following Bayesian Nash equilibrium: A thief of type b_i uses

$$\begin{array}{ll} \text{INSIDE} & \text{if } b_i < 0 \\ \text{TRUNK} & \text{if } b_i > 0; \end{array}$$

a police officer of type d_j checks

$$\begin{array}{ll} \text{INSIDE} & \text{if } d_j < 0 \\ \text{TRUNK} & \text{if } d_j > 0. \end{array}$$

This is a Bayesian Nash equilibrium. From the thief's point of view, the police officer is equally likely to check TRUNK or INSIDE the car. Hence, his expected payoff is b_i for TRUNK and zero for INSIDE. Therefore, it is the best response for him to hide the object in the trunk if and only if the extra benefit b_i from hiding in the trunk is positive. Similar for the police officer.

Note that from the point of view of an outside observer, the mixed-strategy equilibrium of complete information game in Part 1 and the pure strategy Bayesian Nash equilibrium of the Bayesian game in Part 2 are equivalent: in both cases, the thief hides either inside the car or in the trunk and the police officer checks inside or the trunk, where the probability of each pair is $1/4$. Moreover, in both games, the players face the same uncertainty about the action of the other player, assigning equal probabilities on both actions. The rationale for those beliefs is somewhat different, however. In the complete information game, a player thinks that the actions of the other player are equally likely because he does not know the strategy of the other player, assigning equal probabilities on those strategies. In the Bayesian game, however, each player does know what the other player's strategy is—as a function of that player's type. Yet, she does not know which action the other player takes as she does not know the other player's type. Therefore, the uncertainty about the strategies in the complete information game is replaced with uncertainty about the others' types. One can always convert a mixed strategy Nash equilibrium to a pure strategy Bayesian Nash equilibrium by introducing very small uncertainty about the players' payoffs. (This fact is known as Harsanyi's Purification Theorem.) Hence, a mixed strategy Nash equilibrium can be interpreted as coming from slight variations in players' payoffs.

14.7 Exercises with Solutions

Exercise 14.1. Consider a two-player game in which the payoffs, which depend on θ, and actions are as in the following table:

$\theta = 0$

	L	R
a	$1, -1$	$-1, 1$
b	$-1, 1$	$1, -1$

$\theta = 1$

	L	R
a	$1, 1$	$-1, -1$
b	$-1, 1$	$1, -1$

where $\Pr(\theta = 0) = \Pr(\theta = 1) = 1/2$. Only Player 2 knows whether $\theta = 0$ or $\theta = 1$. Write this as a Bayesian game, and compute a Bayesian Nash equilibrium.

Solution. Since Player 2 knows θ, one can suppress the set Θ of the unknown parameters.[11] The Bayesian game is formally as follows:

- the set of players: $N = \{1, 2\}$;
- the set of actions for each player: $A_1 = \{a, b\}$ and $A_2 = \{L, R\}$;
- the set of types for each player: $T_1 = \{t_1\}$ (it is a singleton), $T_2 = \{0, 1\}$ (possible values of θ);
- beliefs are given by $p(t_1, 0) = p(t_1, 1) = 1/2$;
- utility functions u_1 and u_2 are given by the matrices above.

Towards finding a Bayesian Nash equilibrium in pure strategies, note that a pure strategy for Player 1 is an action $s_1(t_1) \in A_1$, and a pure strategy for Player 2 is a pair $(s_2(0), s_2(1)) \in A_2 \times A_2$, assigning an action for each type of that player. To find an equilibrium, one can guess and eventually verify that there exists a BNE in which Player 1's strategy is $s_1(t_1) = a$. Player 2's best response to this strategy is $s_2(0) = R$ and $s_2(1) = L$. One needs to verify that $s_1(t_1) = a$ is a best response to the strategy of Player 2 that $s_2(0) = R$ and $s_2(1) = L$. To do that, compute that the expected payoff

[11] Alternatively, one could take $\Theta = T_2 = \{0, 1\}$.

of Player 1 from a is

$$\begin{aligned} U_1(a) &= u_1(a, s_2(0), 0)\, p(t_1, 0) + u_1(a, s_2(1), 1)\, p(t_1, 1) \\ &= u_1(a, R, 0) \cdot \frac{1}{2} + u_1(a, L, 1) \cdot \frac{1}{2} \\ &= -1 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = 0. \end{aligned}$$

Here, the first equality spells out the expected utility from a, which is the expected value of the function $u_1(a, s_2(t_2), t_2)$; the next equality substitutes the values of $s_2(t_2)$ and $p(t_1, t_2)$, and the last equality is simple algebra. Likewise, the expected utility from b is

$$U_1(b) = u_1(b, R, 0) \cdot \frac{1}{2} + u_1(b, L, 1) \cdot \frac{1}{2} = 0.$$

Hence, $U_1(a) \geq U_1(b)$, showing that a is a best response. Therefore, the strategy profile $(s_1(t_1) = a; s_2(0) = R, s_2(1) = L)$ is a Bayesian Nash equilibrium. This is the only Bayesian Nash equilibrium in pure strategies. Indeed, if Player 1 plays b, the best response of Player 2 is $s_2(0) = s_2(1) = L$, to which the unique best response of Player 1 is a, showing that Player 1 does not play b in equilibrium. There is a continuum of Bayesian Nash equilibrium in mixed strategies, in which Player 1 mixes, putting probability at most 1/2 on a, and Player 2 plays $s_2(0) = R$ and $s_2(1) = L$.

Exercise 14.2. In a two player game, suppose that the payoffs are given by the table

	L	R
X	θ, γ	$1, 2$
Y	$-1, \gamma$	$\theta, 0$

where $\theta \in \{0, 2\}$ is known by Player 1, $\gamma \in \{1, 3\}$ is known by Player 2, and all pairs of (θ, γ) have a probability of 1/4. Write this formally as a Bayesian game and compute the set of Bayesian Nash equilibria.

Solution. Since each parameter is known by some player, it is not necessary to use Θ. Take $T_1 = \{0, 2\}$ to be the set of values for θ, $T_2 = \{1, 3\}$ to be the set of values for γ. Assign probability 1/4 to each type profile

$$p(0, 1) = p(0, 3) = p(2, 1) = p(2, 3) = 1/4.$$

The set of actions for players 1 and 2 are $A_1 = \{X, Y\}$ and $A_2 = \{L, R\}$, respectively. The utility functions u_1 and u_2 are defined by the table above, e.g., $u_1(X, L, \theta, \gamma) = u_1(Y, R, \theta, \gamma) = \theta$, $u_1(X, R, \theta, \gamma) = 1$, and $u_1(Y, L, \theta, \gamma) = -1$.

To compute a Bayesian Nash equilibrium s^*, one needs to determine $s_1^*(0) \in \{X, Y\}$, $s_1^*(2) \in \{X, Y\}$, $s_2^*(1) \in \{L, R\}$, and $s_2^*(3) \in \{L, R\}$—four actions in total. First observe that when $\theta = 0$, action X strictly dominates action Y, i.e.,

$$u_1(X, a_2, \theta = 0, \gamma) > u_1(Y, a_2, \theta = 0, \gamma)$$

for all actions $a_2 \in A_2$ and types $\gamma \in \{1, 3\}$ of Player 2. Hence, it must be that

$$s_1^*(0) = X.$$

Similarly, when $\gamma = 3$, action L strictly dominates action R, and hence

$$s_2^*(3) = L.$$

Now consider the type $\theta = 2$ of Player 1. Since her payoff does not depend on γ, observe that her payoff from X is $1 + p_L$, where p_L is the probability that Player 2 plays L. Her payoff from Y is $2(1 - p_L) - p_L$, which is equal to $2 - 3p_L$. Hence, for $\theta = 2$, X is a best response if

$$1 + p_L \geq 2 - 3p_L,$$

i.e.,

$$p_L \geq 1/4.$$

When $p_L > 1/4$, X is the only best response. But type γ must play L, and the probability of that type is $1/2$. Therefore,

$$p_L \geq 1/2 > 1/4.$$

Since $s_1^*(2)$ is a best response for $\theta = 2$, it follows that

$$s_1^*(2) = X.$$

Finally, consider $\gamma = 1$. Given s_1^*, Player 2 knows that Player 1 plays X (regardless of her type). Hence, the payoff of $\gamma = 1$ is $\gamma = 1$ when he plays L and 2 when he plays R. Therefore,

$$s_2^*(1) = R.$$

To check that s^* is indeed a Bayesian Nash equilibrium, one checks that each type plays a best response (see Exercise 14.6).

Exercise 14.3. Consider the linear Cournot duopoly with zero marginal costs and with inverse-demand function

$$P = \theta - q_1 - q_2$$

where

$$\theta = 1 + \varepsilon t_1 + \varepsilon t_2,$$

$\varepsilon \in (0, 1/2)$ is a known parameter, and each $t_i \in \{-1, 1\}$ is privately known by firm i. Take $\Pr(t_i = 1) = \Pr(t_i = -1) = 1/2$ independent of t_j. (Recall that, simultaneously firms 1 and 2 choose q_1 and q_2, respectively, and the payoff of firm i is $q_i P$. The firms can produce negative amounts and the price can be negative.) Write this formally as a Bayesian game, and compute a Bayesian Nash equilibrium.

Solution. Formally, the Bayesian game is the list

- $N = \{1, 2\}$
- $T_1 = T_2 = \{-1, 1\}$
- $p(t_1, t_2) = 1/4$ everywhere
- $A_1 = A_2 = (-\infty, \infty)$
- $u_i(q_1, q_2, t_1, t_2) = q_i P.$

There is a unique Bayesian Nash equilibrium, and it is symmetric. Consider a Bayesian Nash equilibrium (q_1^*, q_2^*), where q_i^* is a function of t_i for each player i. For each type t_i, $q_i^*(t_i)$ is a best response to q_j^*:

$$q_i^*(t_i) = \frac{1}{2} E\left[\theta - q_j^*(t_j) | t_i\right],$$

where $E[\theta - q^* | t_i]$ is the conditional expectation of $\theta - q^*(t_j)$ given t_i; the expected profit $q_i\left(E\left[\theta - q_j^*(t_j) | t_i\right] - q_i\right)$ is maximized at $\frac{1}{2} E\left[\theta - q_j^*(t_j) | t_i\right]$. Compute that

$$E\left[\theta - q_j^* | t_i\right] = E[\theta | t_i] - E\left[q_j^*(t_j) | t_i\right] = 1 + \varepsilon t_i - E\left[q_j^*\right],$$

where the last equality uses the fact that $E\left[q_j^*(t_j) | t_i\right]$ does not depend on t_i, by independence. Therefore,

$$q_i^*(t_i) = \frac{1}{2} + \frac{1}{2}\varepsilon t_i - \frac{1}{2} E\left[q_j^*\right]. \tag{14.13}$$

Taking the expectations of both sides, compute

$$E\left[q_i^*\right] = \frac{1}{2} - \frac{1}{2}E\left[q_j^*\right].$$

This is true for all distinct i and j, and the unique solution to this equation system is

$$E\left[q_1^*\right] = E\left[q_2^*\right] = 1/3.$$

Substituting this in (14.13), obtain the equilibrium strategy as

$$q_i^*\left(t_i\right) = \frac{1}{3} + \frac{\varepsilon}{2}t_i.$$

When there are only two types, one can assume without loss of generality that the equilibrium strategy is linear. For example, one can assume that $q_i^*\left(t_i\right) = \bar{q}_i + \alpha_i t_i$ for constants $\bar{q}_i = E\left[q_i^*\right]$ and $\alpha_i = \left(q_i^*\left(1\right) - q_i^*\left(-1\right)\right)/2$. One can then compute the symmetric equilibrium using the technique developed for computing symmetric linear equilibrium.

Exercise 14.4. Two partners simultaneously invest in a project, where the level of investment can be any non-negative real number. If partner i invests x_i and the other partner j invests x_j, then the payoff of partners i is

$$t_i x_i x_j - x_i^3.$$

Here, t_i is privately known by partner i, and t_1 and t_2 are independently distributed with the uniform distribution on $[0,1]$. Find all symmetric Bayesian Nash equilibria.

Solution. Consider a symmetric Bayesian Nash equilibrium, where each player plays a strategy s. For any type t_i, the expected payoff from investing x_i is

$$U\left(x_i; t_i\right) = E\left[t_i x_i s\left(t_j\right) - x_i^3\right] = t_i x_i E\left[s\left(t_j\right)\right] - x_i^3.$$

The best response x_i for type t_i must satisfy the first order condition $\partial U\left(x_i; t_i\right)/\partial x_i = 0$, which can be written as

$$t_i E\left[s\left(t_j\right)\right] - 3x_i^2 = 0.$$

Hence, the best response for type t_i is

$$x_i = \sqrt{t_i E\left[s\left(t_j\right)\right]/3}.$$

Since type t_i plays a best response in equilibrium, this shows that

$$s(t_i) = \sqrt{t_i E[s(t_j)]/3}.$$

Thus, all symmetric equilibria are of the form $s(t_i) = \alpha\sqrt{t_i}$ where $\alpha = \sqrt{E[s(t_j)]/3}$. In order to compute the set equilibria, one only needs to compute the values the constant a can take. To this end, compute the expected equilibrium investment as

$$E[s(t_i)] = E\left[\alpha\sqrt{t_i}\right] = \alpha E\left[\sqrt{t_i}\right] = \alpha\int_0^1 \sqrt{t_i}dt_i = \frac{2}{3}\alpha.$$

Substituting this in $\alpha = \sqrt{E[s(t_j)]/3}$, one obtains

$$\alpha = \sqrt{\frac{2}{9}\alpha}.$$

There are two solutions to this equation. One of them is $\alpha = 0$. This corresponds to a "no-investment" equilibrium each player invests

$$s(t_i) = 0$$

regardless of her type. The other solution is $\alpha = \frac{2}{9}$. This corresponds to the symmetric Bayesian Nash equilibrium where each type t_i invests

$$s(t_i) = \frac{2}{9}\sqrt{t_i}.$$

Exercise 14.5. Consider the spacial competition game in Section 6.4.3. Assume that ice cream parlors 1 and 2 are located at 0 and 1, respectively. Assume also that for each ice cream parlor i, the marginal cost of producing ice cream is c_i where c_i is privately known by i and (c_1, c_2) are independently and identically distributed on $[0,1]$. Write this as a Bayesian game, and find a symmetric linear Bayesian Nash equilibrium.

Solution. This can be written as a Bayesian game as follows. The players are the ice cream parlors, 1 and 2; the type of player i is c_i, where the set of types is $[0,1]$, and the joint distribution of types is the uniform distribution on $[0,1]^2$. For each player, the action space is $[0,\infty)$, the set of prices. To compute the payoffs of the players, recall from (6.9) that the demand for player i is

$$Q_i(p_1, p_2) = \frac{1}{2} + \frac{p_j - p_i}{2c}.$$

Hence, the payoff of each player i is

$$u_i(p_1, p_2, c_i) = (p_i - c_i)\left(\frac{1}{2} + \frac{p_j - p_i}{2c}\right)$$

for $j \neq i$.

In order to compute a symmetric linear Bayesian Nash equilibrium, consider a Bayesian Nash equilibrium (s_1, s_2) where

$$s_i(c_i) = ac_i + b$$

for some constants a and b with $a > 0$. The expected payoff of a type c_i from charging price p_i is

$$U(p_i) \equiv E[u_i(p_i, s_j(c_j), c_i)|c_i] = (p_i - c_i)\left(\frac{1}{2} + \frac{a/2 + b - p_i}{2c}\right),$$

where one substitutes the expected value $E[s_j(c_j)] = a/2 + b$ for p_j in $u_i(p_1, p_2, c_i)$, as u_i varies linearly with p_j. The quadratic function $U(p_i)$ is maximized at

$$p_i^*(c_i) = \frac{1}{2}c_i + \frac{1}{4}a + \frac{1}{2}b + \frac{1}{2}c;$$

the easiest way to obtain the maximizer is solving $U'(p_i) = 0$. Since type c_i plays $s(c_i) = ac_i + b$ for each c_i, the best response $p_i^*(c_i)$ must be equal to $s(c_i)$ for each c_i. Since p_i^* is an affine function, this identity is possible, and obtained by setting the coefficients of c_i and the intercepts equal, obtaining

$$\begin{aligned} a &= 1/2 \\ b &= \frac{1}{4}a + \frac{1}{2}b + \frac{1}{2}c. \end{aligned}$$

The solution to this equation system is

$$a = 1/2 \text{ and } b = 1/4 + c.$$

The equilibrium strategy is given by

$$s_i(c_i) = \frac{1}{2}c_i + \frac{1}{4} + c.$$

14.8 Exercises

Exercise 14.6. In Exercise 14.2, verify that s^* is indeed a Bayesian Nash equilibrium. Following the analysis above, show that there is no other Bayesian Nash equilibrium.

Exercise 14.7. In Example 14.3, assume that Bob observes a noisy signal t_B about the value of the project; Alice knows the value of the project. The signal can take two values: high (h) or low (l). If the value of the project is high ($\theta = \theta_H$), the signal is high ($t_B = h$) with probability $r > 1/2$ and low ($t_B = h$) with probability $1 - r$. If the value of the project is low ($\theta = \theta_L$), the signal is high ($t_B = h$) with probability $1 - r$ and low ($t_B = h$) with probability r. All of these are commonly known. Compute the set of Bayesian Nash equilibria in pure strategies.[12] For each q, determine the values of r under which some types exert high effort in an equilibrium. Briefly discuss your result.

Exercise 14.8. Write the game in Exercise 4.14 as a Bayesian game and compute the set of Bayesian Nash equilibria in pure strategies for each $p \in [1/4, 1/2]$.

Exercise 14.9. Consider the two-player Bayesian game

	L	R
X	$3, \theta$	$0, 0$
Y	$2, 2\theta$	$2, \theta$
Z	$0, 0$	$3, -\theta$

where $\theta \in \{-1, 1\}$ is privately known by Player 2 and θ takes each value with probability $1/2$. Compute the Bayesian Nash equilibrium of this game. What would be the Nash equilibria in pure strategies (i) if it were common knowledge that $\theta = -1$, or (ii) if it were common knowledge that $\theta = 1$?

Exercise 14.10. In the Prisoners' Dilemma game in Chapter 1, assume that Alex is known to be vindictive but Bill may be self-interested or vindictive, each case having 50% chance. Alex does not know Bill's character but privately observes a signal that says "vindictive" with probability p and says "self-interested" with probability $1 - p$ if Bill is vindictive, and the signal says "vindictive" with probability $1 - p$ and says "self-interested" with probability p if Bill is self-interested. Bill does not see the signal

[12] In all exercises, verify that the strategy profile you identified is indeed a Bayesian Nash equilibrium.

but knows that Alex sees such a signal. And all of these are commonly known. Write this as a Bayesian game. For each $p > 1/2$, find a Bayesian Nash equilibrium, and verify your solution.

Exercise 14.11. Consider a two-player game with the following payoff matrix

	L	R
a	$\theta, \theta+\gamma$	$0, 0$
b	$0, \gamma$	θ, θ

where $\theta \in \{1, 3\}$ is privately known by Player 1 and $\gamma \in \{-2, 2\}$ is privately known by Player 2. Moreover,

$$\begin{aligned} \Pr(\theta = 3, \gamma = 2) &= \Pr(\theta = 1, \gamma = -2) = 1/3 \\ \Pr(\theta = 3, \gamma = -2) &= \Pr(\theta = 1, \gamma = 2) = 1/6. \end{aligned}$$

Write this formally as a Bayesian game, and compute a Bayesian Nash equilibrium.

Exercise 14.12. Consider the following game. There are two players; each one simultaneously chooses a real number $x_i \geq 0$, and the payoff of player i is

$$\theta_i (x_1 x_2)^{1/2} - x_i,$$

where $\theta_i \in \{0, 1, 2\}$ is privately known by player i. Each value has probability 1/3, independent of θ_j. Write this as a Bayesian game, and compute the set of Bayesian Nash equilibria in pure strategies.

Exercise 14.13. Consider a two-player game with the payoff matrix

	L	R
X	$1, \theta$	$-\theta, 0$
Y	$\theta, 0$	$1, \theta$

where $\theta \in \{-2, 2\}$ is privately known by Player 1, and $\Pr(\theta = -2) = 0.8$. Write this formally as a Bayesian game, and find a Bayesian Nash equilibrium of this game.

Exercise 14.14. Consider a two-player Bayesian game with the following payoff matrix

	R	S	P
R	$f(t_1), f(t_2)$	$f(t_1)+10, g(t_2)-10$	$f(t_1)-10, h(t_2)+10$
S	$g(t_1)-10, f(t_2)+10$	$g(t_1), g(t_2)$	$g(t_1)+10, h(t_2)-10$
P	$h(t_1)+10, f(t_2)-10$	$h(t_1)-10, g(t_2)+10$	$h(t_1), h(t_2)$

where $t_i \in \{0, 1, 2\}$ is privately known by player i; f, g, and h are known functions with $f(0) = 1$, $f(1) = f(2) = 0$, $g(1) = 1$, $g(0) = g(2) = 0$, $h(2) = 1$, and $h(0) = h(1) = 0$; and each pair (t_1, t_2) has probability $1/9$. Write this as a Bayesian game, and find a Bayesian Nash equilibrium of this game.

Exercise 14.15. Consider the two-player Bayesian game with payoff matrix

	a	b
a	$2 + \varepsilon_1, 1$	$\varepsilon_1, \varepsilon_2$
b	$0, 0$	$1, 2 + \varepsilon_2$

where ε_1 and ε_2 are privately known by players 1 and 2, respectively, and are identically and independently distributed with uniform distribution on $[-1/3, 2/3]$. Find a Bayesian Nash equilibrium of this game in which for each action (a or b) there is a realization of ε_i at which player i plays that action.

Exercise 14.16. Consider a penalty kicker and a goalkeeper. Simultaneously, penalty kicker decides whether to send the ball to the Left or to the Right, and the goalkeeper decides whether to cover the Left or the Right. The payoffs are as follows (where the penalty kicker is the row-player):

	Left	Right
Left	$x - 1, y + 1$	$x + 1, -1$
Right	$1, y - 1$	$-1, 1$

Here, x and y are independently and uniformly distributed on $[-1, 1]$; the penalty kicker knows x, and the goal keeper knows y. Find a Bayesian Nash equilibrium.

Exercise 14.17. Consider a two-player game with payoff matrix

	a	b
a	$2, 2$	$0, \theta$
b	$\theta, 0$	$1, 1$

where $\theta \in \{0, 3\}$ is privately known by Player 1, and each value is equally likely. Write this game formally as a Bayesian game, and compute two Bayesian Nash equilibria of this game.

Exercise 14.18. Consider the following price competition game with two firms. Each firm sets a price p_i and sells the quantity

$$Q_i\left(p_1, p_2, \theta\right) = \theta - p_i + p_j/2$$

where

$$\theta = 1 + \varepsilon t_1 + \varepsilon t_2,$$

$\varepsilon \in (0, 1/2)$ is a known parameter, and each $t_i \in \{-1, 1\}$ is privately known by firm i. Take $\Pr\left(t_i = 1\right) = \Pr\left(t_i = -1\right) = 1/2$ independent of t_j. The marginal costs are zero. (The firms can sell negative amounts and the prices can be negative.) Write this formally as a Bayesian game, and compute a Bayesian Nash equilibrium.

Exercise 14.19. There are two profit-maximizing firms, A and B, which produce applesauce and banana puree, respectively. (Applesauce and banana puree are imperfect substitutes.) Simultaneously, A and B sets the prices $p_A \in [0, 1]$ and $p_B \in [0, 1]$ of applesauce and banana puree, respectively. Firms A and B sell

$$q_A\left(p_A, p_B\right) = 1 - p_A - \gamma\left(1 - p_B\right) \text{ and } q_B\left(p_A, p_B\right) = 1 - p_B - \gamma\left(1 - p_A\right)$$

units of applesauce and banana puree, respectively. (The amounts can be negative.) Here, $\gamma \in \{0, 1/2\}$ is privately known by A, and each value of γ is equally likely. The marginal cost is $c = 0$ for each firm. Write this game formally as a Bayesian game, and compute the set of all Bayesian Nash equilibria.

Exercise 14.20. In Exercise 14.19, assume that

$$\gamma = t_A t_B$$

where t_A and t_B are privately known by firms A and B, respectively, and they are independently and uniformly distributed on $[0, 1]$. Compute a symmetric Bayesian Nash equilibrium. (Assume for simplicity that the prices and the quantities can take any value, including negative numbers.)

Exercise 14.21. Consider a linear Cournot oligopoly with n firms and the inverse-demand function $P = 1 - Q$ (where P is price and Q is total supply). The marginal cost c_i of each firm i is its private information, where $(c_1, \ldots, c_n)$ are independently and identically distributed; write F for the CDF and $\bar{c}$ for the expected value of c_i. Write this game formally as a Bayesian game, and find a symmetric Bayesian Nash equilibrium. (The firms can produce negative amounts and the price can be negative.)

Exercise 14.22. Consider two candidates in a primary. Each candidate i has valence v_i, independently and identically distributed with uniform distribution on $[0, 1]$. Each candidate's valence, which measures her quality as a candidate, is her private information. Simultaneously, each candidate i picks a campaign spending level $x_i \geq 0$. The candidate i with the highest $x_i + v_i$ wins the primary; the tie is broken with a coin toss. The winner then runs in the general election as the candidate for the party. Her electoral success depends on her valence and both candidates' spending levels in the primary, so that the payoff of a candidate i is $v_i + x_i - x_j - x_i^2$ if she wins the primary and $-x_i^2$ if she loses the primary. Write this game formally as a Bayesian game, and find a symmetric linear Bayesian Nash equilibrium.

Exercise 14.23. Consider an n-player game in which each player i selects a search level $s_i \in [0, 1]$ (simultaneously), receiving the payoff

$$u_i (s_1, \ldots, s_n, \theta_1, \ldots, \theta_n) = \theta_i s_1 \cdots s_n - s_i^\gamma / \gamma,$$

where $(\theta_1, \ldots, \theta_n)$ are independently and identically distributed on $[0, \infty)$. the expected value of each Here, $\gamma > 1$ is commonly known and θ_i is privately known by player i. (Denote the expected value of θ_i by $\bar{\theta}$ and the expected value of θ_i^α by $\bar{\theta}_\alpha$ for any $\alpha > 0$.)

1. For $\gamma = 2$, find the symmetric linear Bayesian Nash equilibria.

2. For $n \neq \gamma \neq 2$, find the symmetric Bayesian Nash equilibria.

Exercise 14.24. There are n players in a town. Simultaneously each player i contributes x_i to a public project, yielding a public good of amount

$$y = x_1 + \cdots + x_n,$$

where x_i is any real number. The payoff of each player i is

$$u_i = y^2 - c_i x_i^\gamma$$

where $\gamma > 2$ is a known parameter and the cost parameter $c_i \in \{1, 2\}$ of player i is his private information. The costs $(c_1, \ldots, c_n)$ are independently and identically distributed where the probability of $c_i = 1$ is $1/2$ for each player i. Write this formally as a Bayesian game, and find a Bayesian Nash equilibrium.

Exercise 14.25. There are n people, who want to produce a common public good through voluntary contributions. Simultaneously, every player i contributes x_i. The amount of public good produced is

$$y = x_1 + \cdots + x_n.$$

The payoff of each player i is

$$u_i = \theta_i y - y^2 - x_i,$$

where θ_i is a parameter privately known by player i, and $\theta_1, \theta_2, \ldots, \theta_n$ are independently and identically distributed with uniform distribution on $[1, 2]$. Assume that x_i can be positive or negative. Compute a symmetric Bayesian Nash equilibrium. [Hint: The equilibrium will be linear, in the form of $x_i(\theta_i) = a\theta_i + b$.]

Exercise 14.26. In a college, there are n students who share the college's data network. Simultaneously, each student i chooses the data size $x_i \geq 0$ to send over the network. The speed of the network is inversely proportional to the total size of the data, so that it takes $x_i \tau(x_1, \ldots, x_n)$ minutes to send the message where

$$\tau(x_1, \ldots, x_n) = x_1 + \cdots + x_n.$$

The payoff of any student i is

$$t_i x_i - x_i \tau(x_1, \ldots, x_n),$$

where $t_i \in \{1, 2\}$ is a payoff parameter, privately known by student i, and $t_1, \ldots, t_n$ are independently and identically distributed with $\Pr(t_i = 1) = \Pr(t_i = 2) = 1/2$. Write this game formally as a Bayesian game, and compute a symmetric Bayesian Nash equilibrium. (Hint: In equilibrium one of the types may choose zero.)

Exercise* 14.27. In the previous exercise, characterize the set of symmetric Bayesian Nash equilibria, assuming instead that $t_1, \ldots, t_n$ are independently and identically distributed with a density on positive real numbers where the density is strictly positive on an open set of positive reals; there is a continuum of types. Compute the equilibrium for the case of uniform distribution on some interval $[0, \bar{t}]$.

Exercise 14.28. There are n programmers. Simultaneously, each programmer i picks a specification s_i, which can be any real number. The payoff of each programmer i is

$$-\left(s_i - \theta_i\right)^2 - \alpha \sum_{j \neq i} \left(s_i - s_j\right)^2,$$

where θ_i and $\alpha \geq 0$ are real numbers. While α is a known parameter, the ideal specification θ_i of programmer i is privately known by programmer i. The parameters $\theta_1, \ldots, \theta_n$ are independently and identically distributed; write F, f, and $\bar{\theta}$ for the CDF, pdf, and expected value of θ_i, respectively. Write this as a Bayesian game, and find a symmetric Bayesian Nash equilibrium. Briefly discuss how the strategies vary with the parameters α and n.