Solutions to Problem Set 4
14.12 Fall 2023

October 13, 2023

Problem 1

One example is the following normal form game:

	L	R
U	-1	-1
D	-1	0

You can verify that playing U with probability $p = 1$ is a security strategy for Player 1 (and (U, L) is a Nash equilibrium). However, observe that D dominates U.

Problem 2

Part 1

See extensive form at end of document.

Part 2

Note that it does not make sense for either player to choose not to bet when they see Heads. So, we can limit the normal form to the following:

	BB	BN
BB	0	$(\alpha - \beta)/4$
BN	$\beta/4$	0

Part 3

Now, we will compute the security strategies. Let p denote the probability with which player 1 chooses to bet when they receive tails.

$$u_1(Bp,BB) = \frac{1}{4}(\beta+\alpha) + \frac{p}{4}(-\alpha-\beta) + \frac{1-p}{4}(-\alpha) = \frac{\beta}{4}(1-p)$$

$$u_1(Bp,BN) = \frac{1}{4}\alpha + \frac{p}{4}(-\beta-\alpha) + \frac{1-p}{4}(-\alpha) + \frac{p}{4}(\alpha) = \frac{p}{4}(\alpha-\beta)$$

We use these to construct the worst gain function. Note that this will depend on how α and β are related.

When $\alpha = \beta$, note that $u_1(Bp,BN) = 0$, while $u_1(Bp,BB)$ is always greater than or equal to zero. So, here, the worst gain is 0.

When $\alpha < \beta$, $u_1(Bp,BN) < 0$, so again this is less than $u_1(Bp,BB)$.

When $\alpha > \beta$, to find the minimum, we solve for the intersection of $u_1(Bp,BB)$ and $u_1(Bp,BN)$:

$$\frac{p}{4}(\alpha-\beta) = \frac{\beta}{4}(1-p)$$

$$\frac{\beta}{4} = \frac{p\alpha}{4} \implies p = \frac{\beta}{\alpha}$$

(As a sanity check, as a probability, this does not make sense when $\beta > \alpha$.)

$$\min\{u(Bp,BB), u(Bp,BN)\} = \begin{cases} \frac{\beta}{4}(1-\frac{\beta}{\alpha}) & \alpha > \beta \\ 0 & \alpha = \beta \\ \frac{p}{4}(\alpha-\beta) & \alpha < \beta \end{cases}$$

Now, we must find the security strategies for Player 1. To do this, we maximize the worst gain function (above):

- When $\alpha > \beta$, the security strategy is to play $BBp + (1-p)BN$ with $p = \frac{\beta}{\alpha}$ (as calculated above).

- When $\alpha = \beta$, the worst gain is zero, so any $BBp + (1-p)BN$ with $p \in [0,1]$ is a security strategy (WG is maximized for any p).
- When $\alpha < \beta$, the worst gain is $\frac{p}{4}(\alpha - \beta)$, so it is a security strategy to play $p = 0$ (i.e., BN) (WG is maximized when $p = 0$).

Problem 3

Part 1

See extensive form (end of document)

Part 2

Let $(C,C),(C,D),(D,C)$ and (D,D) denote the 4 nodes Alice is playing, where the first component is what Alice played and second is what Bob played. In these nodes, backward induction implies that Alice will choose S at C,C and D,D while she will choose T at C,D and D,C. Knowing this, Bob will make sure to choose D at both his information sets, since this guarantees him a payoff of 2, which is his strict optimum. Given these, Alice is indifferent between choosing C and D at her first decision node, since whatever she does she obtain a payoff of 1 given the continuation play we have derived. Thus the characterization of the backward induction is

- Bob: choose D in both information sets
- Alice: (C,S,T,T,S) or (D,S,T,T,S) where first component is the choice at the initial node and rest correspond to choices in $(C,C),(C,D),(D,C)$ and (D,D) respectively.

Problem 4

Part 1

At the end of this document are the extensive forms and backward induction solutions for all three cases (for concreteness, we evaluate first case at $c = 0.5$, second at $c = 1$ and third at $c = 1.5$, but the backward induction equilibrium would be the same for other c's in the given ranges.)

Case 1: $c < 1$

- Alice: (D, S, T, T, S)
- Bob: choose D in both information sets

Case 2: $1 < c < 2$

- Alice: (C, S, S, T, S)
- Bob: choose C in the information set where Alice chose C; choose D in the info set where Alice chose D.

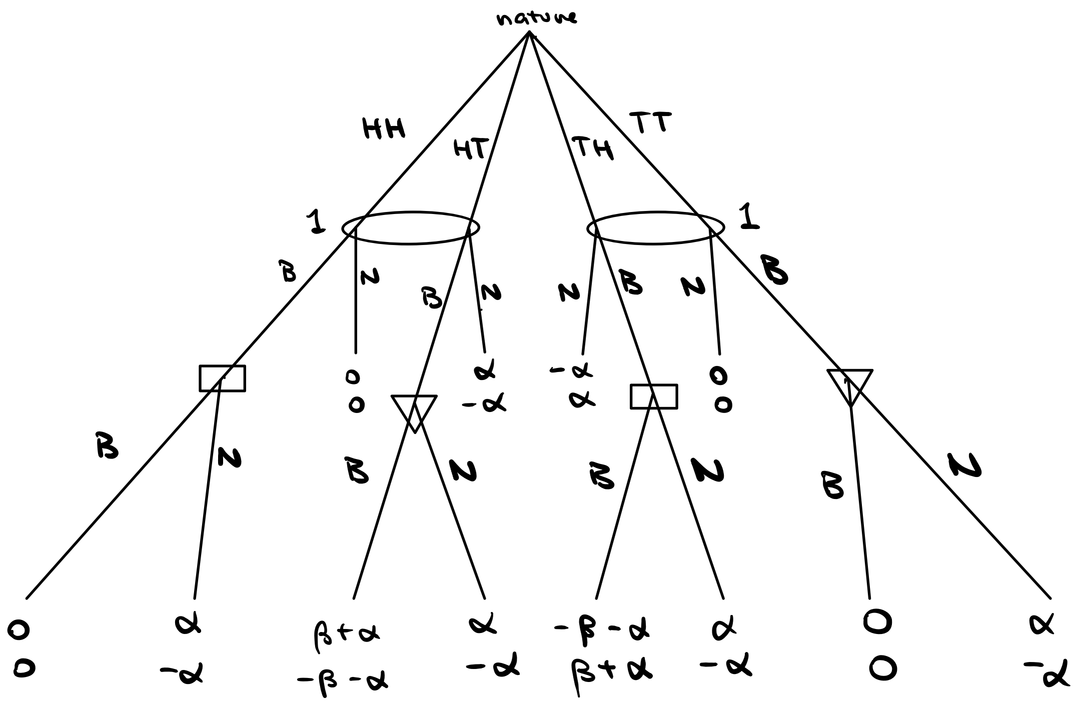
Case 3: $c > 21$

- Alice: (C, S, S, S, S)
- Bob: choose C in the information set where Alice chose C; choose D in the info set where Alice chose D.

Part 2

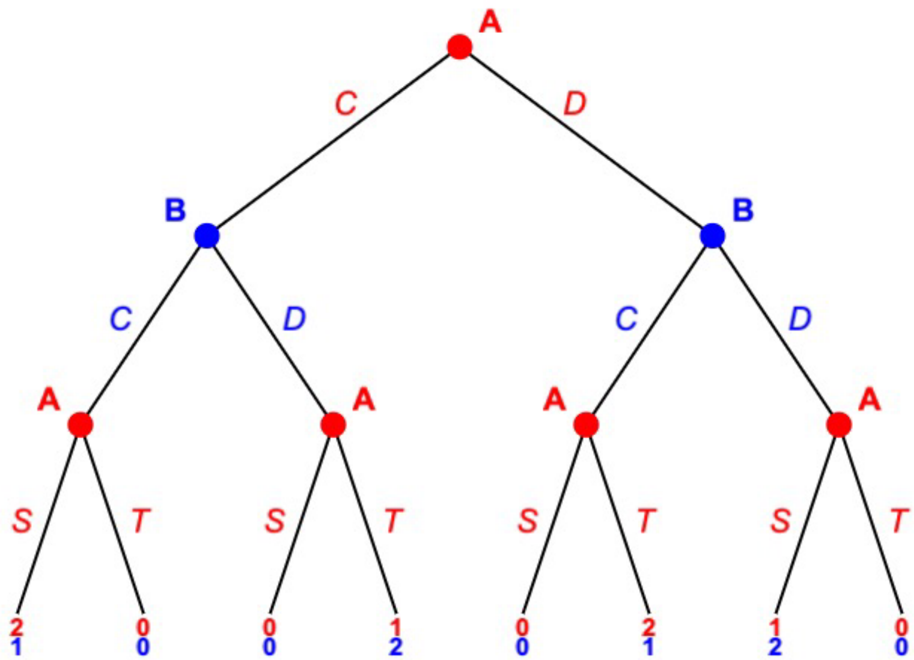
First, note that Alice never actually travels in the backward induction solution, so the case we are in, the cost does not affect Alice's payoff. However, the cost affects the backward induction solution. In the first case, Alice's cost for travelling is low, so she will travel even if she chooses C, and knowing this Bob always chooses D. However, in the other two cases, if Alice chooses C in the beginning, it is never optimal for her to travel in the last round. As a result, if she chooses C, it is optimal for Bob to choose C and this becomes the backward induction solutions. In a sense, the higher cost gives Alice power to credibly commit she will be at C when it is time to meet.

Problem 2, Extensive form



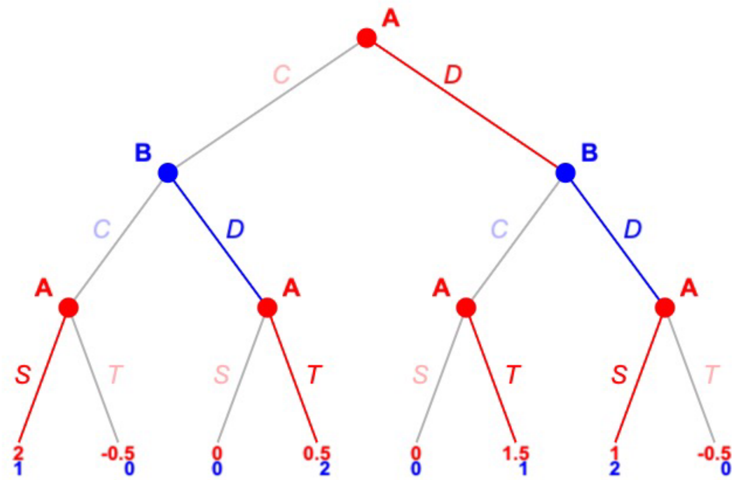
• both players have 2 info sets

Problem 3, extensive form

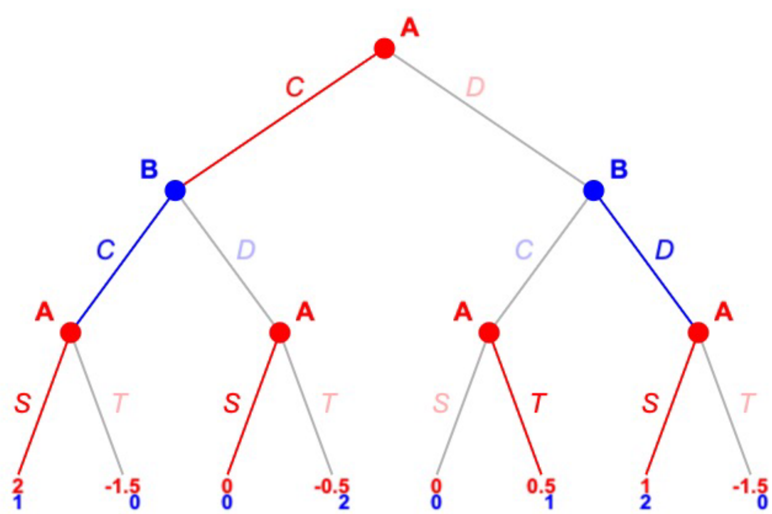


Problem 4, extensive forms:

$c < 1$



$1 < c < 2$



$c > 2$