18.650. Fundamentals of Statistics
Fall 2023. Recitation sheet 1.1

1 Standardization and the Gaussian cdf

Problem 1 Let $X_1, \dots, X_n$ be i.i.d. with mean $\mu \in \mathbb{R}$ and variance $\sigma^2 > 0$ and let $\bar{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$. Standardize $\bar{X}_n$ (standardizing a random variable Y means finding numbers a, b such that $(Y-a)/b$ has mean 0 and variance 1 (at least approximately)).

Problem 2 Let $X \sim N(2, 1.96)$. Compute the following probabilities.

1. $\mathbb{P}(X > 1)$.
2. $\mathbb{P}(X^3 - 3X^2 + 2X \geq 0)$.
3. Find $a > 1$ such that $\mathbb{P}(1 \leq \frac{|X-2|}{1.4} \leq a) = \alpha$ for some given $\alpha \in (0, 1)$.

2 Convergence of random variables

Problem 3 (AoS exercise 5.1) Let X_i be i.i.d. with mean μ and variance σ^2. Let

$$\hat{\sigma}_n^2 = \frac{1}{n}\sum_{i=1}^n (X_i - \bar{X}_n)^2$$

be the sample variance. Show that $\hat{\sigma}_n^2 \xrightarrow{\mathbb{P}} \sigma^2$. [Hint: write $\hat{\sigma}_n^2$ in terms of $n^{-1}\sum_i X_i^2$ and $\bar{X}_n$.]

Problem 4 (AoS exercise 5.9) Suppose that $\mathbb{P}(X = 1) = \mathbb{P}(X = -1) = 1/2$. Define

$$X_n = \begin{cases} X & \text{with probability } 1 - \frac{1}{n}, \\ e^n & \text{with probability } \frac{1}{n} \end{cases}.$$

Does X_n converge to X in probability? In distribution?

Problem 5 Let $X_1, \ldots, X_n$ be $n = 60$ i.i.d. Uniform random variables on the interval $[0, 1]$.

1. Find the parameters μ and σ such that $\bar{X}_n \sim N(\mu, \sigma^2)$ approximately.
2. What is the approximate distribution of $\exp(\bar{X}_n)$

Problem 6 (AoS exercise 5.13) Let $Z_1, Z_2, \ldots$ be positive i.i.d. variables with continuous probability density function f, and suppose $\lambda = \lim_{x \downarrow 0} f(x) > 0$. Let $X_n = n \times \min(Z_1, \ldots, Z_n)$. Show that $X_n \rightsquigarrow Exp(1/\lambda)$.

	Second decimal place of Z									
Z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

The table lists $P(Z \leq z)$ where $Z \sim N(0,1)$ for positive values of z.