14.12 Midterm Review

Fall 2023

About the midterm

- October 26, 10:30am-12pm, 4-237 (in class)
- 80 min, close book, close notes (unless mentioned otherwise in class next week)
- Bring writing utensils
- Study formal definitions as listed in class notes!! (they are more formal than what is in this slideshow :D)

Topics

- Extensive form, normal form, strategy
- Rationalizability
- Nash Equilibrium, beliefs, best response
- Cournot and Bertrand competition
- Zero-sum games, security strategy
- Backward Induction
- Subgame-perfect NEs
- Infinitely repeated games, One-shot deviation

Strategy & Strategy Profile

- A **<u>strategy</u>** of a player is a <u>complete contingency-plan</u> that determines which action to take at <u>each information set</u>.
- A **<u>strategy profile</u> $\mathbf{s = (s_1, s_2, \ldots, s_n)}$** is a list of strategies, one for each of the n players.

Notes:

- Remember the difference between strategy & strategy profile!
    - NEs and SPNEs are strategy profiles
- If player i has n information sets, their strategy should be n letters long
    - This means you must write down actions even when you think the player won't reasonably get there!

Extensive Form & Normal Form

Extensive form game tree has **5 things:**

1. The set of players
2. A tree
3. An allocation of non-terminal nodes of the trees to the players
4. An information partition of the nodes
5. Payoffs for each player at each terminal node

Normal form is a **list**:

$G = (N, S_1, S_2, \ldots, S_n, u_1, u_2, \ldots, u_n)$

- $N = \{1, \ldots, n\}$ is the set of players
- S_i is the set of all strategies available to i,
- $u_i : S_1 \times S_2 \times \ldots \times S_n \rightarrow \mathbb{R}$ is the utility function of i (given all the strategies for every player, what will i get).

Belief & Best Response

Belief: player i's belief of what other players are going to play (the assigned probabilities of belief for each player j should add up to 1)

Best Response: a function of β_i. "What player plays to maximize payoff given a belief about others' strategies"

Definition 2.11. For any player i, a strategy s_i is said to be a *best response* to s_{-i} if

$$u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}) \qquad \text{(for all } s_i' \in S_i\text{)}.$$

A strategy s_i is said to be a *best response* to a belief β_{-i} if playing s_i maximizes the expected payoff under β_{-i}, i.e.,

$$u_i\left(s_i, \beta_{-i}\right) \geq u_i\left(s_i', \beta_{-i}\right), \qquad \forall s_i' \in S_i.$$

Domination of Strategies

We say a strategy σ_1 (strictly,weakly) dominates another strategy σ_2 if σ_1 yields (strictly/weakly) better payoff than σ_2

- Strictly dominant strategy = all the inequalities are strict
- Weakly dominant strategy = at least one of the inequality is strict

Rationalizability

- Iterated elimination of strictly dominated strategies
  1. Initialize S_i^0 = all available strategies for player i
  2. At each round k (k >=1), assume that the actions available to each player i is S_i^{k-1}. Eliminate any strategy strictly dominated by any mixed strategy.
  3. Repeat until no strategies can be eliminated.

Emphasis on strictly dominated and mixed strategy!

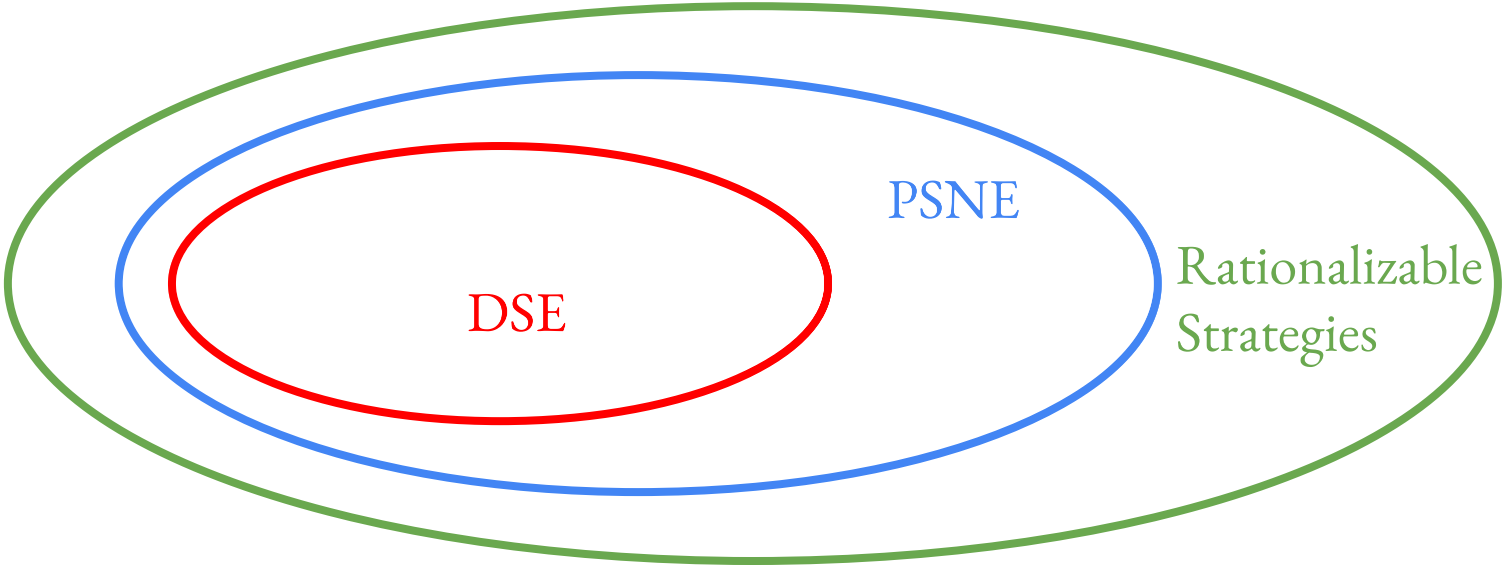
Nash Equilibrium

… is a strategy profile at which point none of the players want to deviate.

- For PSNE: highlight the best responses for each player, and select the cells where all players are playing their best responses.
- MSNE Problem-Solving Strategy:
  - First, make our lives easier by eliminating all the strictly dominated strategies
  - Assign probabilities to each strategy for a player, and then calculate the probabilities needed such that the other player is <u>indifferent</u> between each of their choices
  - You may need to consider situations where the probability of a player playing a certain move is 0.
  - Potentially, lots of casework

DSE $\subseteq$ PSNE $\subseteq$ RS

Theorem. In any game, a dominant strategy equilibrium is a pure strategy nash equilibrium, and a pure strategy nash equilibrium is rationalizable.



Cournot and Bertrand Competition

Cournot Competition

- n firms;
- each firm i produces at q_i, determined simultaneously
- everyone's price is $P = \max\{1-Q,0\}$, where Q = sum of quantity for each firm
- **NE: every player sells $(1-Q_{i}-c)/(n+1)$ in quantity**

Bertrand Competition

- n firms;
- Each firm i produce the same good and set their price $\{p_1, p_2, \ldots, p_n\}$ simultaneously
- The firm with lowest price gets all the business, rest gets none
- If every firm sets the same price p then they will each sell quantity p/n
- **NE: everyone sets their price to 0**

For both competition models:

- they can be applied to many game-theoretical concept s in this class so there might not be one question format. But in general, know how to take partial derivatives :D
- The constants (such as the 1 in 1-Q) could vary. There could also be a marginal cost c to production. As a result, NEs will change slightly.

Zero-sum game, security strategy

- Zero sum game: 2 player game, player 1's payoff = opposite of player 2's payoff
  - "Payoffs add to 0"
- Worst gain = payoff of player 1 at "worst case"
- Player 1's worst gain = player 2's worst loss
- Security strategy of player 1 maximizes worst gain for player 1
- Security strategy of player 2 minimizes worst loss for player 2
- Problem solving: start by assuming a mixed strategy of player i and compare i's payoffs for each scenario where player j plays a <u>pure</u> strategy
- Maximize the minimum of the payoffs you just calculated

Backward Induction

- For extensive form games
- Your output is one or more PSNE
- You can only do this when you have <u>perfect information</u> (in a subgame)!!
- Usually used in bargaining/negotiation questions

Subgame Perfect NE

- Know what is and what isn't a subgame!!
- A SPNE is an NE in every subgame (thus by definition, it is an NE because the entire game is also one of its subgames)
- A SPNE is an NE but an NE need not be a SPNE

Infinitely Repeated Games

General setup:

- stage game G (what actions each player have and payoff at each stage)
- horizon/timing: t = 0,1,2,...
- perfect monitoring: each player observes what everyone else does and knows what happened in the past, but not in the future
- Discounting: future values are worth less than current
    - Discounting factor δ is between 0 and 1
    - Can interpret this in several ways: e.g., interest rate, impatience, and the probability that the game could end in xxx steps

Strategies for infinitely repeated games:

- For each player i, there needs to be an action at each distinct <u>history</u>
  - History = “which time t am I at now” AND “what happened in the past”
  - Instead of “I will play A at t=0, B at t=1…”, the strategies look more like “I will play A at t=0, and at t=1, if my opponent played C, I will play A; but if my opponent played D, I will play B…”
- Yes, there are a LOT of strategies. (even more than the ~Monty Hall problem in PSET 1).
- It is pretty long to write this in normal form, so we can now describe them qualitatively. For example:
  - “Each player sells 1 unit every day, but if one of the players sells more than 1 unit on any day, every other player will sell 2 units a day for forever” (Grim and Trigger)
  - “Each player plays what the other player played last time”

One-Shot Deviation

- Start with some predetermined strategy profile (in which player i's strategy is s_i)
- "One shot" = at ONE specific time t, ONE specific player i decides to deviate from s_i
- We use one-shot deviation to find if a strategy profile is an SPNE
  - Idea: an SPNE should satisfy that FOR ALL histories h, at each step, there is no player who has a profitable one-shot deviation.

Practice Questions

(From past exams)

Midterm 2021, #1

Topic: extensive form

Problem 1 (12 points)**.** Recall the following "matching pennies" game between Alice and Bob:

		Bob L	Bob R
Alice	L	$1, -1$	$-1, 1$
Alice	R	$-1, 1$	$1, -1$

Consider a dynamic version of this game. Alice first chooses L or R. Then a *biased* coin is flipped to determine whether Bob observes Alice's move. The coin comes up Heads with probability 0.75 and Tails with probability 0.25.

- If the coin is Heads, Bob observes Alice's move and then chooses L or R.
- If the coin is Tails, Bob chooses L or R without observing Alice's move.

After Bob's move, the game ends. Payoffs are determined by the matrix above.

(a) Write out this game in extensive form.

(b) How many pure strategies does Alice have?

(c) How many pure strategies does Bob have?

(d) Express the extensive form game from part (a) in strategic form. (Be sure to explain the notation you use for Bob's strategies.) To save time, pick one of Alice's strategies and **fill in the utilities only for the cells in which Alice uses that strategy**. You can leave all other cells blank.

Midterm 2022, #2

Topics: Iterated Elimination, rationalizability, finding PSNE and MSNE

Problem 2 (20 points). Consider the following strategic-form game between two players:

Player 2

Player 1		a	b	c	d
	A	$0, 0$	$3, 1$	$-2, 0$	$4, 2$
	B	$0, 3$	$1, 1$	$4, -3$	$0, 1$
	C	$1, 2$	$2, 0$	$-1, -3$	$2, 1$
	D	$-5, 5$	$-5, 5$	$-5, 5$	$-5, 5$

1. Compute the set of rationalizable strategies for each player.

2. Find *every* (pure and mixed) Nash equilibrium.

Midterm 2022, #3

Topic: backward induction

Problem 3 (20 points). Consider the following game between two players, called the seller and the buyer. The seller has a single good to trade. The seller values the good at v_S, and the buyer values the good at v_B, where

$$0 < v_S < v_B < 1.$$

These values are measured in monetary units. Each player's utility equals her value from the good (if she gets/keeps it) net any payments she pays/receives. The buyer and seller haggle over the price as follows. There is no discounting.

- In period 0, the proposer (who will be specified below) proposes that the good is sold at a price $p_0 \in [0, 1]$. If the responder accepts this offer, the trade is carried out at the specified price and the game ends. If the responder rejects the offer, play proceeds to period 1.

- In period 1, the proposer (who will be specified below) proposes that the good is sold at a price $p_1 \in [0, 1]$. If the responder accepts this offer, the trade is carried out at the specified price and the game ends. If the responder rejects the offer, there is no trade and the game ends (i.e., the seller keeps the good).

1. Suppose the seller is the proposer in period 0 and the buyer is the proposer in period 1. Find *one* subgame perfect Nash equilibrium and describe its outcome.

2. Suppose the buyer is the proposer in period 0 and the seller is the proposer in period 1. Find *one* subgame perfect Nash equilibrium and describe its outcome.

3. Based on your answers to parts 1–2, is it better to propose in period 0 or in period 1? Explain why.

Midterm 2022, #4.1 (Tricky!)

Topic: cournot competition, Nash Equilibrium.

Problem 4 (20 points). Firm 1 and firm 2 are engaged in Cournot quantity competition. Simultaneously, each firm i chooses a production quantity $q_i \in [0, 10]$. The market price is

$$P(Q) = \max\{10 - Q, 0\},$$

where Q is the total quantity produced by the two firms.

Each firm must pay a fixed cost $c_0 > 0$ to initiate production, and a per-unit marginal cost of 2 thereafter. That is, each firm i has production cost function

$$c(q_i) = \begin{cases} 0 & \text{if } q_i = 0, \\ c_0 + 2q_i & \text{if } 0 < q_i \leq 10. \end{cases}$$

Each firm's utility equals its profits.

1. For which values of c_0 is it an equilibrium for both firms to produce 0?

Hint: start with (0,0) and think about deviating.

Midterm 2022, #4.2

Problem 4 (20 points). Firm 1 and firm 2 are engaged in Cournot quantity competition. Simultaneously, each firm i chooses a production quantity $q_i \in [0, 10]$. The market price is

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$$c(q_i) = \begin{cases} 0 & \text{if } q_i = 0, \\ c_0 + 2q_i & \text{if } 0 < q_i \leq 10. \end{cases}$$

Each firm's utility equals its profits.

2. For which values of c_0 does there exist an equilibrium in which one firm produces 0 and the other produces a strictly positive quantity?

Hint: Your answers should be inequalities involving c_0.

Q & A