Fall 2023
14.12 Game Theory
Tomasz Sadzik

Problem Set 10

You are encouraged to work together on the problem sets, but you must write up your own solutions. Consulting solutions from previous semesters (released by the instructor or written by other students) is prohibited. Problem sets must be submitted electronically through Canvas.

Late problem sets submitted within 24 hours of the deadline will be accepted with a 50% penalty. Problem sets more than 24 hours late will not be accepted. Make sure to allow yourself enough time to complete the submission process. (If you have technical difficulties, you may email your problem set to the TA by the deadline.)

Problem 1 (VCG mechanism)**.** There are two goods, α and β, and two bidders, $i = 1, 2$. The goods will be auctioned to the bidders according to a VCG mechanism. There are four possible (pure) allocations of the good: good α can be allocated to player 1 or player 2; good β can be allocated to player 1 or player 2.[1] The principal can also allocate the good randomly, e.g., by flipping a coin. Formally, a (random) *allocation* is a probability distribution over the four pure allocations of the two goods among the two players.

Each bidder i has one of three possible valuations, denoted v^A, v^B, v^C. Each player's utility depends on the set of goods allocated to him, as indicated in the table below:

[1] Here, we are prohibiting the auctioneer from keeping either good. Allowing that possibility would not change the VCG mechanism.

	v^A	v^B	v^C
$\varnothing$	0	0	0
$\{\alpha\}$	4	2	3
$\{\beta\}$	2	4	3
$\{\alpha, \beta\}$	5	5	6.5

Compute the VCG mechanism,[2] breaking ties symmetrically between players $i = 1$ and $i = 2$. That is, for all $v_1, v_2 \in \{v^A, v^B, v^C\}$ compute an efficient (random) allocation $x^*(v_1, v_2)$ and VCG transfers $t_1(v_1, v_2)$ and $t_2(v_1, v_2)$.

Problem 2 (Perfect Bayesian Equilibrium)**.** Exercise 15.12.
Hint for part b: If Nature's move is represented explicitly in the extensive form, then Bob will have one information set, which contains four nodes.

Problem 3 (Job-market signaling)**.** In class, we analyzed a simple job-market signaling game in which the sender (worker) had two possible types (abilities), ℓ and h. This problem considers a generalization of that setting.

The sender has three possible types $t \in \{\ell, m, h\}$, where $0 < \ell < m < h$. The sender's type is drawn from a prior distribution that puts probabilities π_ℓ, π_m, and π_h on types ℓ, m, and h, respectively. The sender knows her type and chooses an education level e in $[0, 1]$. The receiver (firm) sees the sender's education level, but not the sender's type, and then chooses a wage w in $\mathbf{R}_+$. The payoffs for the sender and receiver are

$$u_S(e, w; t) = w - \frac{1}{t}e^2, \qquad u_R(e, w; t) = -(w - t)^2.$$

1. For which education levels e^* does there exist a *pooling* PBE[3] in which every sender type chooses education e^*?

2. For which education levels $\underline{e}$ and $\bar{e}$ does there exists a *semi-separating* PBE in which type ℓ chooses $\underline{e}$ and types m and h choose $\bar{e}$?

3. For which education levels $\underline{e}'$ and $\bar{e}'$ does there exist a *semi-separating* PBE in which types ℓ and m choose $\underline{e}'$ and type h chooses $\bar{e}'$?

[2] Choose h_i as in the notes, so that that each player pays her externality.
[3] Here and below, PBE means *perfect Bayesian equilibrium.*

4. For which education levels e_ℓ, e_m, and e_h does there exists a *separating* equilibrium in which each type t chooses e_t, for $t = \ell, m, h$?