14.12 Pset 10 Solutions

December 2023

Problem 1

We denote an allocation by what player 1 receives. The rest must go to player 2. By symmetry, it suffices to consider the six profiles (A, A), (B, B), (C, C), (A, B), (A, C), (B, C). We specify the allocation and transfer in each case:

- (A, A): $x = (1/2)\{\alpha\} + (1/2)\{\beta\}$; $t_1 = t_2 = (1/2)(5-2) + (1/2)(5-4) = 2$
- (B, B): $x = (1/2)\{\alpha\} + (1/2)\{\beta\}$; $t_1 = t_2 = (1/2)(5-4) + (1/2)(5-2) = 2$
- (C, C): $x = (1/2)\{\alpha, \beta\} + (1/2)\emptyset$; $t_1 = t_2 = (1/2)6.5 = 3.25$
- (A, B): $x = \{\alpha\}$; $t_1 = 5 - 4 = 1$; $t_2 = 5 - 4 = 1$
- (A, C): $x = \{\alpha\}$; $t_1 = 6.5 - 3 = 3.5$; $t_2 = 5 - 4 = 1$
- (B, C): $x = \{\beta\}$; $t_1 = 6.5 - 3 = 3.5$; $t_2 = 5 - 4 = 1$

Problem 2

Part 1

First, we assume that θ is known. When the state is high, we have the following subgame:

	H	L
H	$\underline{4}$, $\underline{4}$	0, 3
L	3, 0	$\underline{1}$, $\underline{1}$

In this case, the 2 SPNE are:

$$\{(IH, H); (XL, L)\}$$

Now, when the state is low, we are left with the following subgame:

	H	L
H	0, 0	-2, $\underline{1}$
L	$\underline{1}$, -2	$\underline{0}$, $\underline{0}$

Here, the only SPNE is:

$$\{(IL, L)\}$$

Part 2

Now, we assume that θ is high with probability q and low with probability $1-q$ for some q with $\frac{1}{2} \leq q < \frac{2}{3}$, where Alice knows θ, but Bob does not.

As the hint says, Bob has one information set (with 4 nodes), and 2 possible moves, L or H; and note that we are looking for pure strategy PBE.

Suppose that we are in an equilibrium where Bob always plays H. Then, Alice would play IH in state high, and IL in state low. When Bob gets to move, he believes that it is the high state with probability q, and low with $1-q$.

His expected payoffs from playing H are: $4q - 2(1-q) = 6q - 2$

If he deviated to L, then his expected payoffs would be $3q$. But then, note that: $q < \frac{2}{3} \leftrightarrow 3q < 2 \leftrightarrow 6q - 2 < 3q$ - so, given this strategy profile and Bob's beliefs, he would prefer to deviate to L.

So, suppose instead that Bob plays L.

Here, Alice is choosing between the following potential strategies (high state decisions written before low state decisions).

$(IHIH); (ILIL); (IHIL); (ILIH); (XHXH); (XLXL); (XHXL); (XLXH)$
$(IHXH); (IHXL); (ILXH); (ILXL); (XHIH); (XHIL); (XLIH); (XLIL)$

Note that Alice will not want to play I in state high given that Bob plays L, because X will give her a payoff of 3. In the high state subgame, Alice plays L as a best response to Bob's play of L (which gives payoff 1). Then, in the low state, Alice prefers to play I and then L. This means that Bob has the belief that the true state is low (with probability 1 if he gets to move), i.e., $b(low|I) = 1$; $b(high|I) = 0$.

Now, we check whether Bob has any incentive to deviate. In this suggested PBE, if he switches to H, then his expected payoff is strictly worse (because in state low, this will give him a payoff of -2).

So, we have that:

$$s_A(high) = XL; s_A(low) = IL; s_B = L; b(low|I) = 1$$

is the unique pure strategy PBE.

(You can also think about this problem by considering whether Alice pools on I, X, or separates between I and X depending on the state, and rule out the potential equilibria).

For example, suppose that Alice pools on I. Then, we can consider Bob's best responses to the following potential strategies that Alice can play:

$ILIL \implies$ Bob has beliefs $(q, 1-q)$, and best response is L (payoff q, as opposed to $-2(1-q)$ from H).

$ILIH \implies$ Bob's best response is L (payoff of 1 as opposed to 0).

$IHIL \implies$ Bob's best response is L (expected payoff of $3q$, which is better than $6q-2$ from H).

$IHIH \implies$ Bob's best response is H (expected payoff of $4q$, which is better than $2q+1$).

Note that in the first 3 cases, Alice will prefer to switch to playing X in state H, and in the final case, Alice would prefer to play L in the low state. So, the players here are not playing best responses, and these are not PBE. Continuing with this process, all but $(XLIL, L)$ can be ruled out.

Problem 3

Define $t^* = \pi_l l + \pi_m m + \pi_h h$; $\bar{t} = \frac{1}{\pi_m + \pi h}(\pi_m m + \pi_h h)$; and $\underline{t} = \frac{1}{\pi_l + \pi m}(\pi_l l + \pi_m m)$

1. In the pooling equilibrium, we need that 1) every indifference curve is above $w = l$ line, and 2) players get positive utility in equilibrium. This requires that at the pooling education level e^*, we have the indifference curve of the lowest type being above the $w = l$ line

   $$t^* - \frac{(e^*)^2}{l} \geq l$$

   and the highest type not getting negative return from education

   $$t^* - \frac{(e^*)^2}{h} \geq 0$$

   It can be seen from the figure that the first constraint is the one that could bind. Hence the pooling level of education should satisfy

   $$(e^*)^2 \leq l(t^* - l)$$

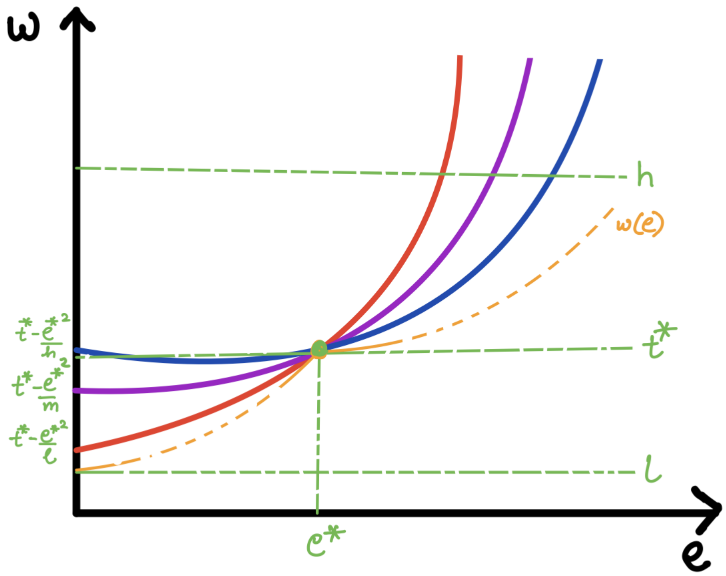

Figure 1: Everyone pools

2. We can show that $\underline{e}$ must satisfy $\underline{e} = 0$, therefore equilibrium utility for low type is $w = l$

   - Define $\tilde{e}$ as the level of education where the indifference curve of the low type intersects with $w = \bar{t}$. We must have $\bar{e} > \tilde{e}$ so that the low type doesn't have incentive to mimic the high types. $\tilde{e}$ solves $l = \bar{t} - \frac{(\tilde{e})^2}{l}$ or $\tilde{e} = \sqrt{l(\bar{t} - l)}$. The pooling education level $\bar{e}$ must satisfy
   $$\bar{e} \geq \sqrt{l(\bar{t} - l)}$$
   - Additionally medium type should not have incentive to mimic low type which is ensured when
   $$\bar{e} \leq \sqrt{m(\bar{t} - l)}$$
   - Everyone gets positive return given the above 3 constraints so IR constraints are satisfied.

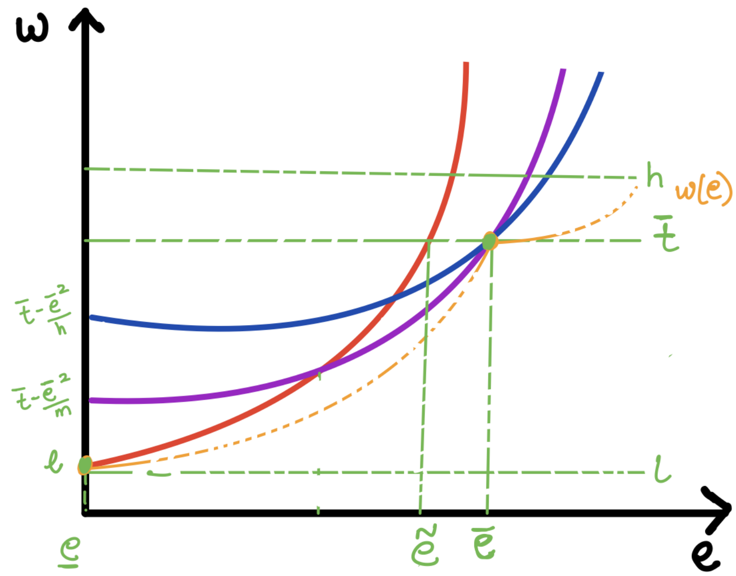

Figure 2: Only m and h type pool

3. Define $\tilde{e}'$ as the education level where the IC curve of high type intersects that of low type.

    - We must have $\underline{e}' \leq \tilde{e}'$ as this ensure that high type doesn't have an incentive to mimic low types.
    - Denote $\tilde{e}''$ as the level of education where the IC curve of the medium type intersects $w = h$ line. To keep the medium type from mimicing the high type, we need $\bar{e}' \leq \tilde{e}''$.
    - We also need the IC curve to be above $w = l$ everywhere which requires $\underline{t} - \frac{(\underline{e}')^2}{l} \geq l$

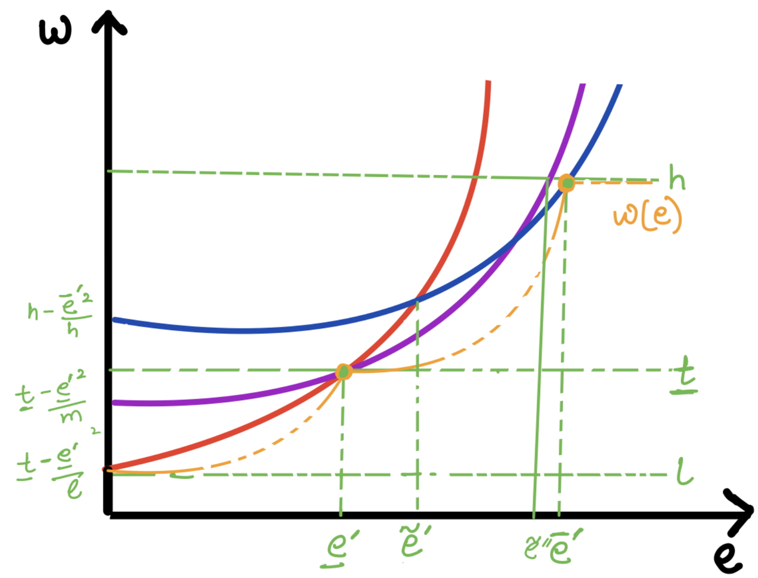

Figure 3: Only l and m pool

4. We must have $e_L = 0$. Define e_1 as the education level where the IC of the low type intersects $w = m$. Define e_2 as the education level where IC curve of the medium type intersects with $w = h$.

   - For the low type to not try and mimic the medium type we must have $e_m \geq e_1$.
   - For the medium type to not try and mimic the high type, we must have $e_H \geq e_2$.
   - We should have high type not wanting to mimic medium type (which is when the intersection of IC curve for the two types is to the right of e_m)
   - We should have medium and low type IC intersect to the right of $e_L = 0$ so that medium type doesn't want to mimic low type. That is same as saying $e_m \geq \sqrt{m(m-l)}$
   - We must also have every type getting positive payoff in the equilibrium which is ensured when $e_m \leq m$ and $e_h \leq h$. But this is satisfied given the above 4 constraints

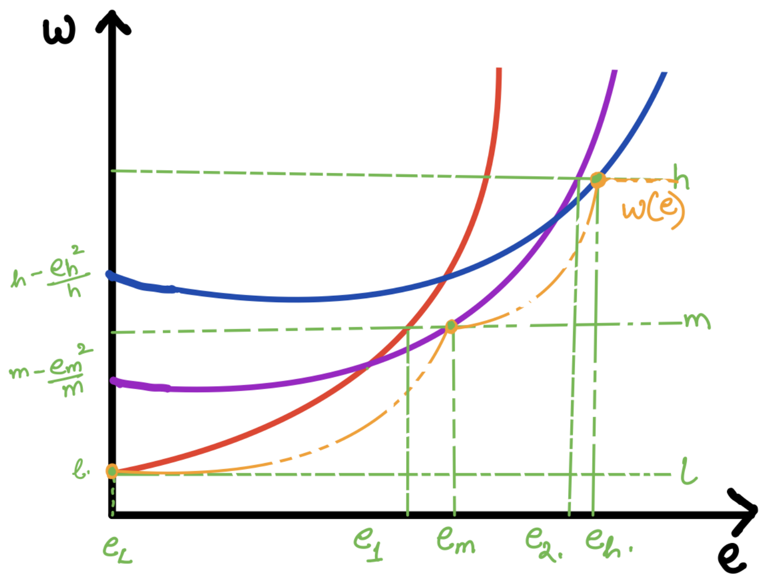

Figure 4: Fully separating equilibrium