14.12 Week 3 Recitation!!

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Sign-in

Topics

- Zero-sum games
    - Worst gain/worst loss;
    - Security strategy; Value of game
    - Minimax theorem
- Backwards Induction (Ch.8)
    - Applications: recreational games, price haggling, pre-trial negotiation

Zero-sum game

- 2 players
- Player A's gain is player B's loss! In other words:

  $\mathbf{u_1(s_1,s_2) + u_2(s_1,s_2) = 0}$ for s_1 in S_1, s_2 in S_2

- Example: Chess, Checkers, Rock Paper Scissors (RPS)

Rock paper scissors 🪨 🗒️ ✂️

Note that the payoffs aren't represented as tuples!

Whose payoffs are those?

P1\P2	R	P	S
R	0	-1	1
P	1	0	-1
S	-1	1	0

Rock paper scissors 🪨 🗒️ ✂️

Note that the payoffs aren't represented as tuples!

Whose payoffs are those?

Player 1's!

P1\P2	R	P	S
R	0	-1	1
P	1	0	-1
S	-1	1	0

Rock paper scissors 🪨 🗒️ ✂️ - worst case analysis

What is the worst case scenario for player 1 if she chooses to play **R**ock?

P1\P2	R	P	S
R	0	-1	1
P	1	0	-1
S	-1	1	0

Rock paper scissors 🪨 🗒️ ✂️ - worst case analysis

What is the worst case scenario for player 1 if she chooses to play **R**ock?

- When player 2 plays **P**aper, player 1 will lose (i.e. get payoff -1).
- This is also called <u>worst gain</u> of player 1.

What about other (pure) strategies for player 1?

P1\P2	R	P	S
R	0	-1	1
P	1	0	-1
S	-1	1	0

Rock paper scissors 🪨 📝 ✂️ - worst case analysis

- When player 1 plays R, her worst case payoff is -1
- When player 1 plays P, her worst case payoff is -1
- When player 1 plays S, her worst case payoff is -1

P1\P2	R	P	S
R	0	-1	1
P	1	0	-1
S	-1	1	0

Can we do better than -1?

Security Strategies

$$wg(\sigma_1) = \min_{\sigma_2} u(\sigma_1, \sigma_2) = \min_{s_2} u(\sigma_1, s_2) = \bar{v}$$
$$wl(\sigma_2) = \max_{\sigma_1} u(\sigma_1, \sigma_2) = \max_{s_1} u(s_1, \sigma_2) = \underline{v}$$

wg = worst gain for **player 1**

wl = worst loss for **player 2**

Note: wg is a function that takes player 1's strategy as input (vice versa for wl and p2)

First equation in English: The worst gain for player 1 given that she plays the mixed strategy σ_1 is the minimum payoff player 1 can get amongst all of player 2's possible mixed strategies.

A security strategy for player 1 is a mixed strategy σ_1 that <u>maximizes</u> worst gain.

Rock paper scissors 🪨 🗒️ ✂️ returns

- If we are confined to pure strategies, then the best we can do (i.e. the maximum worst gain) is -1
- But what about mixed strategies?
- **Proposal: player 1 plays mixed strategy $\boldsymbol{\sigma_{0.5}}$ with the following probability assignments:**
- **Play R with probability 0.5, play P and S with probability 0.25**
- When P2 plays R: $E[u_1] = 0.5*0 + 0.25*1 - 0.25*1 = 0$
- When P2 plays P: $E[u_1] = 0.5*(-1)+0.25*0+0.25*1 = -0.25$
- When P2 plays S: $E[u_1] = 0.5*1+0.25*(-1)+0.25*0 = 0.25$

P1\P2	R	P	S
R	0	-1	1
P	1	0	-1
S	-1	1	0

For player 1, the worst gain with $\boldsymbol{\sigma_{0.5}}$ is min{0,-0.25,0.25} = -0.25. **Can we do even better?**

Rock paper scissors 🪨 🗒️ ✂️ returns

- If we are confined to pure strategies, then the best we can do (i.e. the maximum worst gain) is -1
- But what about mixed strategies?
- **Proposal: player 1 plays mixed strategy $\sigma_{0.5}$ with the following probability assignments:**
- **Play R with probability 0.5, play P and S with probability 0.25**
- What is the worst gain for player 1 with this strategy?

P1\P2	R	P	S
R	0	-1	1
P	1	0	-1
S	-1	1	0

Rock paper scissors 🪨 🗒️ ✂️ returns

- If we are confined to pure strategies, then the best we can do (i.e. the maximum worst gain) is -1
- But what about mixed strategies?
- **Another proposal: player 1 plays R, P, S each with probability ⅓**
- When P2 plays R: $E[u_1] = ⅓*(0+1-1) = 0$
- When P2 plays P: $E[u_1] = ⅓*(0+1-1) = 0$
- When P2 plays S: $E[u_1] = ⅓*(0+1-1) = 0$

The worst gain is min{0,0,0} = 0!

(This strategy is the security strategy for player 1 because it maximizes player 1's worst gain.)

P1\P2	R	P	S
R	0	-1	1
P	1	0	-1
S	-1	1	0

Minimax Theorem (good to know!)

Theorem 1 (Minimax, von Neumann 1928)
In a finite two-player zero-sum strategic form game, $\underline{v} = \bar{v}$*, that is,*

$$\max_{\sigma_1 \in \Sigma_1} \min_{\sigma_2 \in \Sigma_2} u(\sigma_1, \sigma_2) = \min_{\sigma_2 \in \Sigma_2} \max_{\sigma_1 \in \Sigma_1} u(\sigma_1, \sigma_2).$$

Theorem 2 (Nash equilibria)
Consider a finite two-player zero-sum strategic form game. For any strategy profile (σ_1, σ_2)*, the following are equivalent:*

A. (σ_1, σ_2) *is a Nash equilibrium;*

B. *For* $i = 1, 2$*, strategy* σ_i *is a security strategy for player* i*.*

Adopted from Prof. Ian Ball's notes.

Security (Problem Solving) Strategy

- First of all: if the question is asking for a security strategy, it is asking for a <u>strategy</u>, not a payoff or a worst gain, worst loss, etc :D
- Finding security strategy for player 1:
    - Step 1: Assume the security strategy of player 1 is some **mixed strategy $\boldsymbol{\sigma}$**, assign probabilities for each pure strategy player 1 can play
    - Step 2: Calculate player 1's payoffs wrt σ against player 2's PURE strategies
        - Why is it unnecessary to think of player 2's mixed strategies here?
    - Step 3: Compare each of the payoffs you just calculated, find the strategy (usually a combination of probabilities on pure strategies) that gives you **WORST** payoff. The worst payoff is your worst gain.
        - Note: the worst gain is usually dependent on the probabilities you assumed at step 1
    - Now, try to maximize worst gain by toggling parameters (i.e., the probabilities you assumed in step 1).

Rock paper scissors 🪨 📄 ✂️ returns x2

General setup (I gave up typing math in ppt):

P1\P2	R	P	S
R	0	-1	1
P	1	0	-1
S	-1	1	0

1. Assign p_R, p_P, p_S to be the probabilities player 1 plays R, P, S respectively. Call this strategy σ_1.

2. let $u_1(\sigma_1, s)$ denote player 1's payoff when she plays σ_1 and player 2 plays s.
   - $u_1(\sigma_1, R) = \frac{1}{3}(p_R \times 0 + p_P \times 1 + p_S \times (-1)) = \frac{1}{3}(p_P - p_S)$
   - $u_1(\sigma_1, P) = \frac{1}{3}(p_R \times (-1) + p_P \times 0 + p_S \times 1 = \frac{1}{3}(p_S - p_R)$
   - $u_1(\sigma_1, S) = \frac{1}{3}(p_R \times 1 + p_P \times (-1) + p_S \times 0 = \frac{1}{3}(p_R - p_P)$

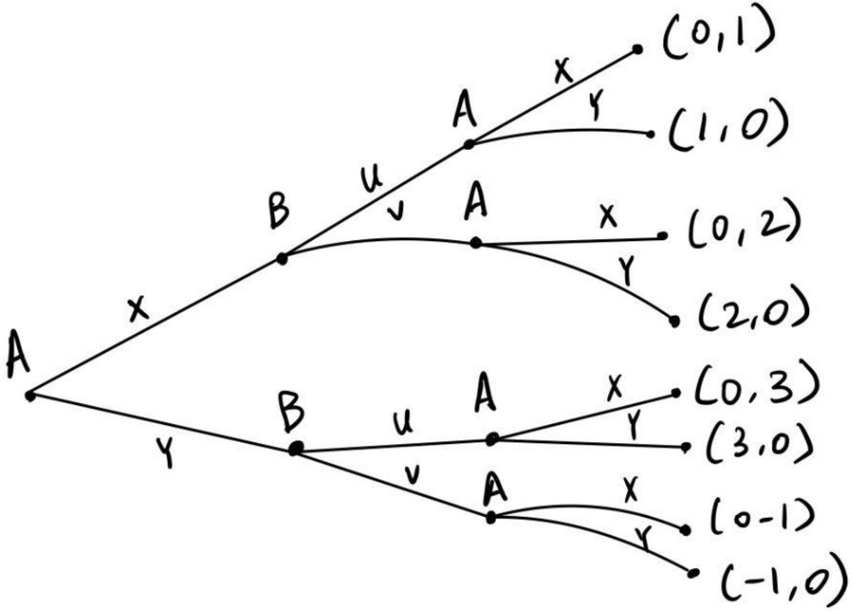
3. We want to find (p_R, p_P, p_S) that maximizes $\min\{\frac{1}{3}(p_P - p_S), \frac{1}{3}(p_S - p_R), \frac{1}{3}(p_R - p_P)\}$. By visual inspection, the best way to do this is to minimize $|p_i - p_j|$ where $i, j \in \{R, P, S\}$, and we arrive at $p_R = p_P = p_S = \frac{1}{3}$.

Backward Induction

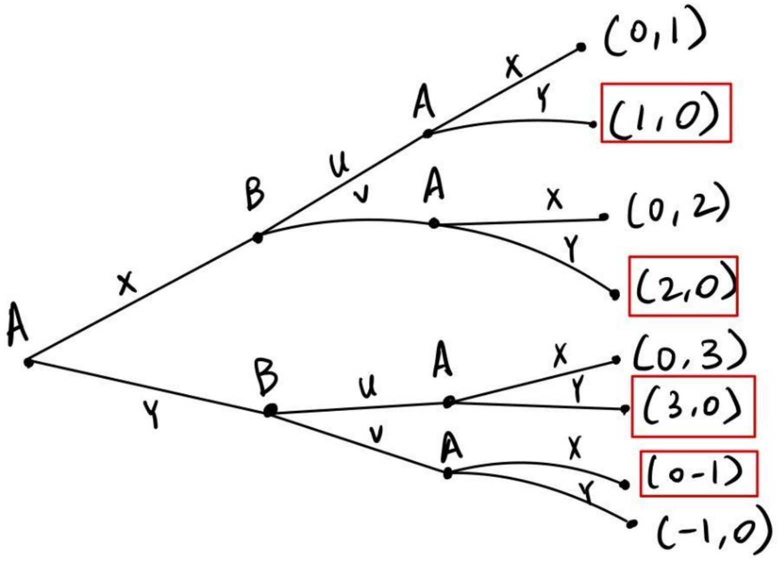
- Input: finite extensive form game with perfect information
- Output: a “special” pure strategy NE - hopefully.
    - It is not always possible to backwards induce to 1 singular solution

Backward Induction - Example

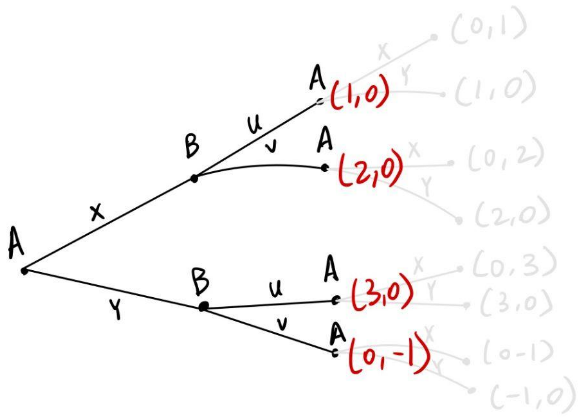
Consider the following game tree. The payoffs are (u_A, u_B). What are the PSNEs?

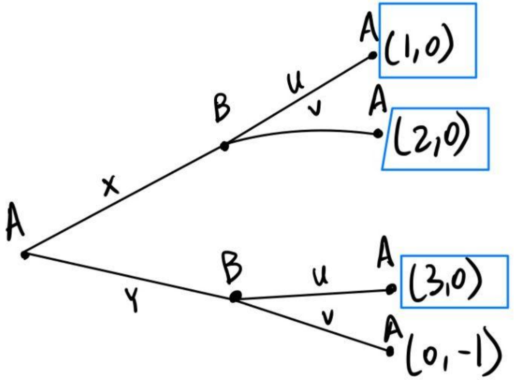


Backward Induction - Example

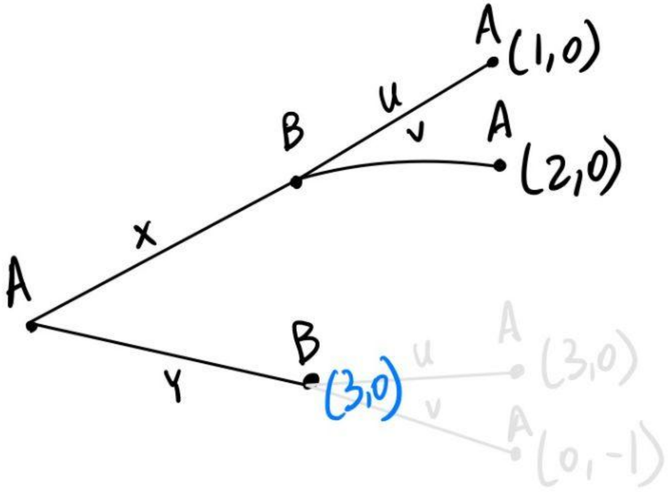


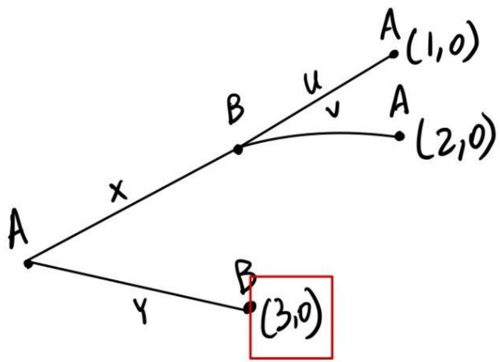
Backward Induction - Example



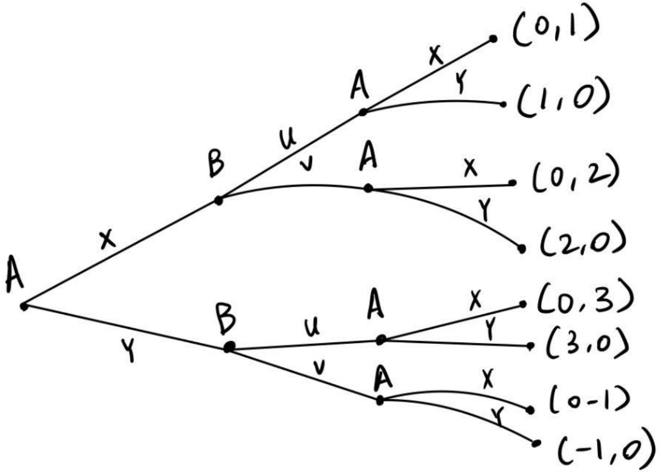


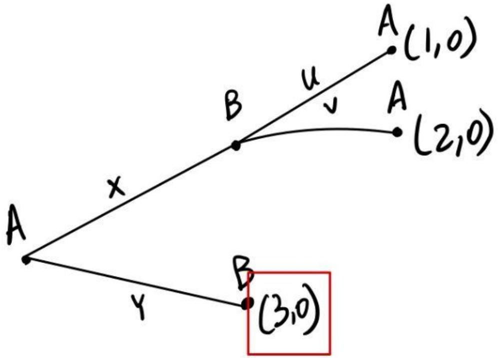
Backward Induction - Example





Backward Induction - Example





Solution: (YX,u) and (YY,u).

Explanation:

- We see that B would choose u no matter what A chooses in the first action; therefore, it does not matter what A chooses when she gets to play (i.e., in the second slot for strategy, both X and Y are acceptable for NE.)
- We also see that given the last diagram, A will choose Y for the first step.