Solutions to Problem Set 6
14.12 Fall 2023

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Problem 1

We need to check 2 things:

- player i is satisfied with its offering strategy, i.e., deviating to offering any other amount will not bring player i strictly better payoff;
- player i is satisfied with its offer, i.e., setting its accepting threshold of the offer at any other value will not bring player i strictly better payoff.

Case 1. Without loss of generality, assume player i is making the offer. Following the proposed strategy, i offers $\frac{1-\delta_j}{1-\delta_1\delta_2}$ and player j accepts this offer. Clearly, any deviation of a smaller offer is strictly dominated by the proposed strategy. If player i offers a smaller portion to j, the offer is rejected. Next period, j will offer $\frac{\delta_i(1-\delta_j)}{1-\delta_i\delta_j}$, which is (even before discounting) less than what i can get by following the equilibrium strategy.

Case 2. Without loss of generality, assume i is receiving an offer. Then it must be that accepting an offer $x_i \geq \frac{\delta_i(1-\delta_j)}{1-\delta_i\delta_j}$ is better than rejecting (case a); and rejecting an offer $x_i < \frac{\delta_i(1-\delta_j)}{1-\delta_i\delta_j}$ is better than accepting (case b). Consider case a. So the offer leaves $i \geq \frac{\delta_i(1-\delta_j)}{1-\delta_i\delta_j}$ and if i follows the strategy, i accepts it. After any deviation that results in rejection, i makes an offer of $\frac{1-\delta_j}{1-\delta_1\delta_2}$ and gets $\frac{\delta_i(1-\delta_j)}{1-\delta_1\delta_2}$ after discounting, which is equal to what i gets from following the proposed strategy, so player i will accept. For case b, we see that it is better to continue, and get a payoff of $\frac{\delta_i(1-\delta_j)}{1-\delta_1\delta_2}$ after discounting, which is better than accepting some $x_i < \frac{\delta_i(1-\delta_j)}{1-\delta_i\delta_j}$. So, there is no profitable deviation and the proposed equilibrium is a SPNE.

Problem 2

Part 1

Answer: there are $\boxed{16}$ SPNEs.

Note that a stage game Nash equilibrium S,S or H,H must be played in the final round in any SPNE. Moreover, the final round strategy of each player is a function from history h_1 to actions taken in the final period. The following strategies constitute an equilibrium

$$\sigma_0 = S \tag{1}$$

$$\sigma_1(h_1) = \begin{cases} S & \text{if } h_1 = (S,S) \\ X & \text{if } h_1 = (S,H) \\ Y & \text{if } h_1 = (H,S) \\ Z & \text{if } h_1 = (H,H) \end{cases} \tag{2}$$

for all $(X,Y,Z) \in \{S,H\}^3$. This gives us 8 different equilibria. Another 8 comes from

$$\sigma_0 = S \tag{3}$$

$$\sigma_1(h_1) = \begin{cases} H & \text{if } h_1 = (S,S) \\ X & \text{if } h_1 = (S,H) \\ Y & \text{if } h_1 = (H,S) \\ Z & \text{if } h_1 = (H,H) \end{cases} \tag{4}$$

for all $(X,Y,Z) \in \{S,H\}^3$.

Part 2

Answer: there are $\boxed{2}$ SPNEs.

To make sure S,H is played in the initial round, we must have S,S being played after it. Consider the following strategies, where superscripts denote the players.

$$\sigma_0^1 = S, \quad \sigma_0^2 = H, \tag{5}$$

$$\sigma_1^1(h_1) = \sigma_1^2(h_1) = \begin{cases} H & \text{if } h_1 = (S,S) \\ S & \text{if } h_1 = (S,H) \\ H & \text{if } h_1 = (H,S) \\ H & \text{if } h_1 = (H,H) \end{cases} \tag{6}$$

Note that both players can gain 1 by deviating in the initial period, but also lose 1 in the final period. Thus, these strategies are an equilibrium. Consider the following.

$$\sigma_0^1 = S, \quad \sigma_0^2 = H, \tag{7}$$

$$\sigma_1^1(h_1) = \sigma_1^2(h_1) = \begin{cases} H & \text{if } h_1 = (S,S) \\ S & \text{if } h_1 = (S,H) \\ S & \text{if } h_1 = (H,S) \\ H & \text{if } h_1 = (H,H) \end{cases} \tag{8}$$

Same conclusion applies here since (H,S) cannot happen under any unilateral deviation.

Problem 3

Part 1

There are two things to check, whether players would follow the strategy in state Y and state Z. In state Y, following the strategy gives

$$4+4\delta+4\delta^2+\dots \tag{9}$$

While the most profitable one shot deviation (x) gives

$$5+0\times\delta+4\delta^2+\dots \tag{10}$$

For this to be an equilibrium, it must be that $5\leq 4+4\delta \Rightarrow \delta\geq 1/4$.

In state Z, following the strategy gives

$$0+4\delta+4\delta^2+\dots \tag{11}$$

While most profitable one shot deviation (x), gives

$$2+0\times\delta+4\delta^2+\dots \tag{12}$$

For this to be an equilibrium, it must be that $2\leq 4\delta$, which is $\delta\geq 1/2$. Thus the proposed strategies is a SPNE if and only if $\delta\geq 1/2$.

Part 2

This question might seem long in the beginning, but by looking at it carefully we see that if last period's play was (x,x), following the proposed strategy gives 0 in every single period, while deviating to y gives 1 this period and at least 0 in every other period. Thus, this is a profitable deviation for all $\delta\in(0,1)$. Hence, this cannot be a SPNE.

Problem 4

Answer: the first strategy profile is a SPNE, and the second is not.

1. First, note that the stage game has a unique Nash equilibrium. Taking the FOC of the payoff, the optimal strategy of any player is

$$\frac{1}{2}y_i^{-1/2} - \frac{y_i}{n} = 0 \iff y_i^* = \frac{n^2}{4} \tag{13}$$

regardless of other's actions. This also shows that after any deviation, no one will deviate from the proposed strategies, since stage game Nash equilibrium is being played in every period. To check for a deviation in the initial phase, note that the proposed strategies give

$$1/4 + \delta 1/4 + \ldots = \frac{1}{4(1-\delta)} \tag{14}$$

We have already calculated that the best deviation is $\frac{n^2}{4}$. This results in

$$\sqrt{\frac{n^2}{4}} - \frac{n^2}{4n} - \frac{n-1}{n}\frac{1}{4} + \frac{\delta}{1-\delta}\left(\sqrt{\frac{n^2}{4}} - \frac{n^2}{4}\right) \tag{15}$$

Rearranging, we get

$$\frac{n}{4} - \frac{n-1}{n}\frac{1}{4} + \frac{\delta}{1-\delta}\left(\frac{n}{2} - \frac{n^2}{4}\right) \tag{16}$$

It is easy to see that equation 14 is higher than equation 16 for large δ, including $\delta = 0.99$.