Chapter 3

Dominance

The previous chapter focused on formal representation of strategic situations. The remainder of the book is devoted to the analysis of these games towards finding the outcomes that are deemed plausible for the strategic situations modeled. The preferred methodology in this book is to consider a set of plausible assumptions about the players' beliefs and discover their implications on what the players would play. Such analyses lead to solution concepts, picking a set of solutions for each game, where a solution yields a strategy for each player. This chapter is devoted to two solution concepts: dominance and dominant-strategy equilibrium. These solution concepts are based on the idea that a rational player does not play a strategy that is dominated by another strategy.

For an illustration, consider the Prisoners' Dilemma game:

$$\begin{array}{c|c|c|} & C & D \\ \hline C & 5,5 & 0,6 \\ \hline D & 6,0 & 1,1 \\ \hline \end{array} \tag{3.1}$$

where C and D stand for Cooperate and Defect, respectively. The payoff from strategy D is strictly higher than the payoff from strategy C for any fixed strategy of the other player:

	Other player plays C	Other player plays D
C	5	0
D	6	1

If a player knows that the other player plays C, then she would play D as it gives 6 instead of 5. If she knows that the other player plays D, then she would play D as it gives 1 instead of 0. Since she prefers D to C in each case, she would prefer D to C even if she does not know which case is true. Indeed, writing p for the probability that the other player plays C, one can see that the expected payoff from C is

$$5 \times p + 0 \times (1 - p) = 5p,$$

while the expected payoff from D is

$$6 \times p + 1 \times (1 - p) = 5p + 1.$$

Since D gives a strictly higher payoff than C, she strictly prefers D to C.

This illustrates a fundamental property of expected utility maximization, known as the Sure Thing Principle. If a strategy s_i always gives higher payoff than s_i' no matter what other players play, then player i will strictly prefer s_i to s_i' even if she does not know what the others play, and she will never play the dominated strategy s_i'. The main result of this chapter establishes that once domination by mixed strategies is allowed, the converse is also true, and a strategy will be played rationally under some belief if and only if it is not dominated in the above sense.

The Prisoners' Dilemma game has the property that Defect dominates the only other strategy, and hence a player does not need to analyze the others' motives or strategies in order to decide what to play. There is no other strategy better than Defect, and hence she can simply play Defect. More generally, a strategy is said to be dominant if it "weakly dominates" any other strategy, in that it yields a weakly higher payoff regardless of what other players play and it is strictly better in some cases. We have a dominant-strategy equilibrium if each player has a dominant strategy.

3.1 Rationality and Dominance

Consider a player $i \in N$ in a game $G = (N, S_1, \ldots, S_n, u_1, \ldots, u_n)$. It is assumed implicitly that player i forms a belief about the other players' strategies and plays a strategy that maximizes the expected value of u_i under that belief.[1] In that case, player

[1] The main concepts used in this section have been introduced in Section 2.6: a belief is a probability distribution β_{-i} on other players' strategies $s_{-i} \in S_{-i}$; a mixed strategy is a probability distribution σ_i on own strategies S_i; a pure strategy is simply a strategy $s_i \in S_i$ (as opposed to mixed), and a

i is called *rational*. This section characterizes the strategies a rational player can play under some belief. The key concept in this characterization will be dominance:

Definition 3.1. For any player $i \in N$, a strategy $s_i^* \in S_i$ is said to (*strictly*) *dominate* a strategy $s_i \in S_i$ if

$$u_i\left(s_i^*, s_{-i}\right) > u_i\left(s_i, s_{-i}\right) \qquad (\text{for all } s_{-i} \in S_{-i}).$$

That is, s_i^* gives a strictly higher payoff than s_i does, no matter what strategy s_{-i} the other players play. For example, in the Prisoners' Dilemma game, D strictly dominates C for Player 1 because $u_1(D, C) = 6 > 5 = u_1(C, C)$ and $u_1(D, D) = 6 > 5 = u_1(C, D)$.

If a strategy s_i^* strictly dominates a strategy s_i, then, under any belief about s_{-i}, the expected payoff from playing s_i^* is strictly higher than the expected payoff from playing s_i—as it was illustrated on the Prisoners' Dilemma example above. Therefore, it is not rational to play strategy s_i under any belief. The converse is not true. It may not be rational to play a strategy s_i under any belief although s_i is not strictly dominated by any pure strategy $s_i^* \in S_i$, as the next example illustrates.

Example 3.1. Consider Player 1 in the following game.

	L	R
T	$2, 0$	$-1, 1$
M	$0, 10$	$0, 0$
B	$-1, -6$	$2, 0$

(3.2)

She is contemplating about whether to play T, or M, or B. A quick inspection of her payoffs reveals that her best play depends on what she thinks Player 2 does. Write p for the probability she assigns to Player 2 playing L. Her expected payoffs from playing T, M, and B are

$$\begin{aligned} U_T &= 2p - (1-p) = 3p - 1, \\ U_M &= 0, \\ U_B &= -p + 2(1-p) = 2 - 3p, \end{aligned}$$

best response to a belief β_{-i} is a strategy s_i that maximizes the expected payoff under belief β_{-i}, i.e., $u_i\left(s_i, \beta_{-i}\right) \geq u_i\left(s_i', \beta_{-i}\right)$ for all s_i', where $u_i\left(s_i, \beta_{-i}\right)$ denotes the expected payoff from playing s_i under belief β_{-i}.

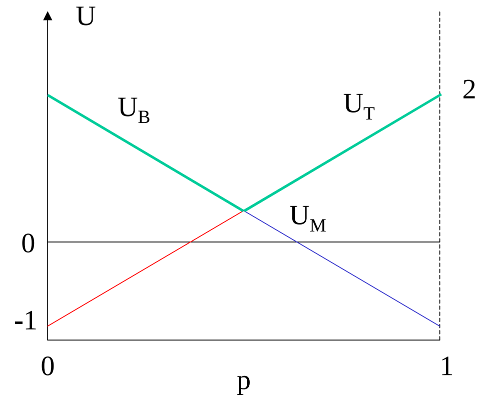

Figure 3.1: Expected payoffs in (3.2) as a function of the probability of L.

respectively. These values are plotted in Figure 3.1. As it is clear from the graph, U_T is the largest when $p > 1/2$, and U_B is the largest when $p < 1/2$. At $p = 1/2$, $U_T = U_B > 0$. Hence, if player 1 is rational, then she plays B when $p < 1/2$, T when $p > 1/2$, and B or T when $p = 1/2$. It is never rational to play M, no matter what she believes about the strategy of Player 2, although M is not dominated by T or B.[2]

As it turns out, strategy M is strictly dominated by a *mixed* strategy, and that is why it is not a best response under any belief. In general games, a strategy is not a best response to any belief if and only if it is strictly dominated by a pure or mixed strategy. Towards establishing this, dominance by a mixed strategy is defined next.

Definition 3.2. A mixed strategy σ_i is said to (*strictly*) *dominate* a strategy s_i if

$$u_i(\sigma_i, s_{-i}) > u_i(s_i, s_{-i}) \qquad (\text{for all } s_{-i} \in S_{-i}).$$

For example, in game (3.2), M is strictly dominated by the mixed strategy σ_1 that puts probability $1/2$ on each of T and B. To check this, for $s_2 = L$, one computes the expected payoff from playing σ_1 against L as

$$u_1(\sigma_1, L) = \frac{1}{2}u_1(T, L) + \frac{1}{2}u_1(B, L) = \frac{1}{2} \cdot 2 + \frac{1}{2} \cdot (-1) = \frac{1}{2}.$$

(With probability $1/2$, Player 1 ends up playing T and obtains the payoff of 2 associated with strategy profile (T, L), and with probability $1/2$, Player 1 ends up playing B and

[2] Strategy M gives higher payoff than T when R is played and higher payoff than B when L is played.

obtains the payoff of -1 associated with strategy profile (B, L).) This is higher than the payoff from M:

$$u_1(\sigma_1, L) = 1/2 > 0 = u_1(M, L).$$

The same is true for $s_2 = R$, showing that σ_1 strictly dominates M. The dominance is vividly tabulated as follows:

Strategy	Player 2 plays L	Player 2 plays R
σ_1	1/2	1/2
M	0	0

No matter what s_2 is, σ_1 yields a higher expected payoff than M does, showing that σ_1 strictly dominates M.

The next result is the main result of this section, and it mathematically links the fact that playing M is not rational under any belief to the fact that M is strictly dominated (by mixed strategy σ_1).

Theorem 3.1. *Assume that there are finitely many strategies. A strategy s_i is a best response to some belief if and only if s_i is not strictly dominated.*[3]

In other words, playing strategy s_i is never rational if and only if s_i is strictly dominated by a (mixed or pure) strategy. This important characterization can be spelled out as follows. If s_i is not strictly dominated by any strategy (mixed or pure), then there exists a belief β_{-i} against which s_i is a best response. In that case, a rational player may play s_i—under belief β_{-i} for example. Conversely, if s_i is strictly dominated (by a pure or mixed strategy), then s_i is not a best response to any belief β_{-i}. In that case, a rational player never plays s_i. Under any belief β_{-i}, there is some other pure strategy that is better than s_i. For example, in game (3.2), M is never a best response because strategy T yields a higher expected payoff when probability p of Player 2 playing L is higher than 1/3 and strategy B yields a higher expected payoff when probability p is lower than 2/3. Although the theorem deduces that M is not a best response under any belief from the dominance by the mixed strategy σ_1, irrationality of M stems from the availability of better pure strategies that do not necessarily dominate M. Even if Player

[3] The proof can be found in the appendix. If you like mathematical challenges try to prove it yourself. Finiteness of strategy set is assumed only for simplicity; this result is true for all games that will ever be considered in this book.

1 can randomize between her strategies, she would choose a mixed strategy only in the knife-edge case that $p = 1/2$.

To sum up: *if one assumes that players are rational (and that the game is as described), then one can conclude that no player plays a strategy that is strictly dominated (by some mixed or pure strategy), and this is all one can conclude.*

3.2 Dominant-strategy equilibrium

This section introduces two concepts of dominant strategy, one is stronger than the other. It uses the weaker concept to define dominant-strategy equilibrium.

Definition 3.3. A strategy s_i^* is said to be a *strictly dominant strategy* for player i if s_i^* strictly dominates all the other strategies of player i.

For example, in the prisoners' dilemma game, D strictly dominates the only other strategy of C. Hence, D is a strictly dominant strategy. If i is rational and has a strictly dominant strategy s_i^*, then she will not play any other strategy. In that case, it is reasonable to expect that she will play s_i^*. Unfortunately, there are only a few interesting strategic situations in which players have a strictly dominant strategy. Such situations can be analyzed as individual decision problems.

A slightly weaker form of dominance is more common, especially in dynamic games (which will be covered later in the book) and in situations that arise in structured environments, such as under suitably designed trading mechanisms as in auctions. This weaker form is called weak dominance:

Definition 3.4. A strategy s_i^* is said to *weakly dominate* s_i if

$$u_i(s_i^*, s_{-i}) \geq u_i(s_i, s_{-i}) \qquad \text{for all } s_{-i} \in S_{-i},$$

and at least one of these inequalities is strict.

That is, no matter what the other players play, playing s_i^* is at least as good as playing s_i, and there are some contingencies in which playing s_i^* is strictly better than s_i. In that case, if rational, player i would play s_i only if she believes that these contingencies will never occur. If she is *cautious* in the sense that she assigns some positive probability for each contingency, then she will not play s_i. This weak dominance is used in the definition of a dominant strategy:

Definition 3.5. A strategy s_i^* of a player i is said to be a *(weakly) dominant strategy* if s_i^* weakly dominates all the other strategies of player i.

When there is a weakly dominant strategy, if the player is rational and cautious, then she will play the dominant strategy.

Example 3.2. Consider the game

$$\begin{array}{c|c|c|}
 & A & B \\ \hline
A & 1,1 & 0,0 \\ \hline
B & 0,0 & 0,0 \\ \hline
\end{array} \tag{3.3}$$

In this game, neither player has a strictly dominant strategy. Indeed, when the other player plays B, a player is indifferent between her own strategies. Nevertheless, strategy A is a weakly dominant strategy: it yields a strictly higher payoff when the other player plays A and it yields at least as much payoff as B otherwise. Hence, a player plays A if she is rational and cautious.

When every player has a dominant strategy, one can make a strong prediction about the outcome. This case yields the first solution concept in this book.

Definition 3.6. A strategy profile $s^* = (s_1^*, s_2^*, ...s_n^*)$ is said to be a *dominant-strategy equilibrium* if for each player i, s_i^* is a weakly dominant strategy.

For example, in game (3.3), A is a dominant strategy for each player, and hence (A, A) is a dominant-strategy equilibrium. Likewise, (D, D) is a dominant-strategy equilibrium in the Prisoners' Dilemma game.

When it exists, the dominant-strategy equilibrium has an obvious attraction. In that case, rational cautious players will play the dominant-strategy equilibrium. Unfortunately, it does not exist in general. For example, consider an instance of the Stag-Hunt game discussed in Chapter 1:

	Stag	Hare
Stag	3, 3	0, 2
Hare	2, 0	2, 2

Clearly, no player has a dominant strategy: Stag is a strict best response to Stag, and Hare is a strict best response to Hare. Therefore, there is no dominant-strategy equilibrium.

3.3 Second-Price Auction

Under suitably designed trading mechanisms, it is possible to have a dominant-strategy equilibrium. Such mechanisms are desirable for they give the traders strong incentive to play a particular strategy (which is presumably preferred by the market designer) and eliminate the traders' uncertainty about what the other players play, as it becomes irrelevant for the trader what the other players are doing. This section presents such a mechanism: second-price auction.

From slave markets in the Roman Empire to modern electronic commerce, auctions have been used for trading goods throughout history. There are many auction formats, such as English auction, in which the buyers raise their prices until there is only one buyer left, Dutch auction (for flowers in Holland) in which price goes down until a buyer accepts it and buys the good, and first-price sealed-bid auction, in which the buyers submit their bids in sealed envelopes and the buyer who submits the highest price buys the good paying her own bid. In the above auctions, the buyers need to form a belief about what other buyers do in order to decide what is best for them. For example, in a first-price auction, bidding one's own valuation for the object is a weakly dominated strategy because one cannot capture any of the gain from trade under that strategy. A buyer instead would bid a lower number, but how low she would bid depends on what she thinks about what other players would bid. An auction designer may want to get rid of such strategic uncertainty to avoid the resulting uncertainty in buyers' behavior. In particular, he may set the price lower than the winner's bid in order to give an incentive to players to bid their true values. One can accomplish this by charging the highest bidder not her own bid but the next highest bid. This auction format is called the second-prize sealed bid auction, or Vickrey auction, named after William Vickrey, who was first to study this auction format in 1961. Variations of this auction format are used extensively from the US Treasury bill auctions to online auctions. Under this auction format, it is a dominant strategy for each buyer to submit her valuation of the good as her bid. This section establishes this for the two-buyer case with known valuations. More general cases with known valuations are covered in the exercises, and auctions with private information will be studied extensively later in the book.

Formally, there is an object to be sold through an auction. There are two buyers. The value of the object for any buyer i is v_i, which is known by the buyer i. Each buyer

i submits a bid b_i in a sealed envelope, simultaneously. Then, the envelopes are opened, and the buyer who submits the highest bid gets the object and pays the second-highest bid. That is, the buyer i^* with

$$b_{i^*} = \max\{b_1, b_2\}$$

gets the object and pays b_j with $j \neq i^*$. (If the bids are equal, a buyer is selected by a coin toss as the winner.)

Formally the game is defined by the player set $N = \{1, 2\}$, the strategies b_i, and the payoffs

$$u_i(b_1, b_2) = \begin{cases} v_i - b_j & \text{if } b_i > b_j \\ (v_i - b_j)/2 & \text{if } b_i = b_j \\ 0 & \text{if } b_i < b_j \end{cases}$$

where $i \neq j$. (When $b_i = b_j$, player i wins with probability 1/2, getting the payoff $v_i - b_j$, and she loses with probability 1/2, getting 0. The expected payoff is $(v_i - b_j)/2$.)

In this game, bidding her true valuation v_i is a dominant strategy for each player i. To see this, consider the strategy of bidding some other value $b_i' \neq v_i$. We want to show that b_i' is weakly dominated by bidding v_i. Consider the case $b_i' < v_i$. If the other player bids some $b_j < b_i'$, player i would get $v_i - b_j$ under both strategies b_i' and v_i. If the other player bids some $b_j \geq v_i$, player i would get 0 under both strategies b_i' and v_i. But if $b_j = b_i'$, bidding v_i yields $v_i - b_j > 0$, while b_i' yields only $(v_i - b_j)/2$. Likewise, if $b_i' < b_j < v_i$, bidding v_i yields $v_i - b_j > 0$, while b_i' yields only 0. Therefore, bidding v_i weakly dominates b_i'. The case $b_i' > v_i$ is similar, except for when $b_i' > b_j > v_i$, bidding v_i yields 0, while b_i' yields negative payoff $v_i - b_j < 0$. Therefore, bidding v_i is a dominant strategy. Since this is true for each player i, (v_1, v_2) is a dominant-strategy equilibrium.

3.4 Exercises with Solutions

Exercise 3.1. There are n students in a class. Simultaneously, each student i chooses an effort level x_i incurring cost cx_i^2 for some $c > 0$. The students are graded on a curve: each student i gets a grade $x_i - \alpha \sum_{j \neq i} x_j$ for some $\alpha > 0$, obtaining a payoff of

$$u_i(x_1, \ldots, x_n) = x_i - \alpha \sum_{j \neq i} x_j - cx_i^2.$$

All of the above is common knowledge.

1. Write this game in normal form.

2. Is there a dominant-strategy equilibrium? If so, compute the dominant-strategy equilibrium.

3. Compute the effort levels at which the sum of the students' payoffs is maximized.

Solution. The game is represented in normal form as follows. The set of players is $N = \{1, \ldots, n\}$. For each $i \in N$, $S_i = \mathbb{R}$, and $u_i : \mathbb{R}^n \to \mathbb{R}$ is given in the question.

Part 2: for any $x_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$, the best response can be found by

$$\frac{\partial u_i}{\partial x_i} = 1 - 2cx_i = 0.$$

The solution to this equation is the unique best response:

$$x_i^* = \frac{1}{2c}.$$

Since x_i^* is the best response to every strategy, x_i^* dominates any other strategy x_i:

$$u_i\left(x_i^*, x_{-i}\right) > u_i\left(x_i, x_{-i}\right) \qquad \left(\forall x_i \neq x_i^*, \forall x_{-i}\right).$$

Therefore, $\left(\frac{1}{2c}, \ldots, \frac{1}{2c}\right)$ is the dominant-strategy equilibrium.

Part 3: The total utility is

$$U = \sum_i \left(x_i - \alpha \sum_{j \neq i} x_j - cx_i^2\right) = \left(1 - (n-1)\alpha\right) \sum_i x_i - \sum_i cx_i^2.$$

The first-order condition for the maximization of U with respect to x_i is

$$\frac{\partial U}{\partial x_i} = 1 - (n-1)\alpha - 2cx_i = 0.$$

Therefore, U is maximized at

$$\left(\frac{1-(n-1)\alpha}{2c}, \ldots, \frac{1-(n-1)\alpha}{2c}\right).$$

Note that the dominant-strategy equilibrium corresponds to the case $\alpha = 0$, ignoring the negative impact on the other students' grades. The dominant-strategy equilibrium always yields a higher effort than the socially optimal level that maximizes U. This is a version of the commons problem, a generalization of the Prisoners' Dilemma game. In the commons problem, each player chooses a contribution level to a public good, yielding a positive impact on the other players' payoffs. In that problem, equilibrium effort is lower than the optimal one. Here, the impact is negative, and the students work harder than socially optimal.

3.5 Exercises

Exercise 3.2. Show that there cannot be a dominant strategy in mixed strategies.

Exercise 3.3. Show that a player can have at most one dominant strategy.

Exercise 3.4. In a pirate ship, $n \geq 2$ pirates are to determine the amount y of gunpowder for the ship as follows. Simultaneously, each pirate i submits a real number $s_i \geq 0$. The amount of gunpowder is determined to be

$$y = \min\{s_1, \ldots, s_n\},$$

and each pirate i pays his share y/n of the cost. The payoff of a pirate i is

$$u_i(y) = \sqrt{y} - y/n.$$

Everything above is commonly known.

1. Write this formally as a normal-form game.
2. Check whether there is a dominant-strategy equilibrium. If there is one, compute it and verify that it is indeed a dominant-strategy equilibrium. Otherwise, explain why there cannot be a dominant-strategy equilibrium.

Exercise 3.5. A group of n students go to a restaurant. Simultaneously, each student i selects a price p_i and they share the total bill $b = p_1 + \cdots + p_n$ equally, yielding payoff $\sqrt{p_i} - b/n$ for each student i. Show that there is a dominant-strategy equilibrium and compute it. Discuss the limiting cases $n = 1$ and $n \to \infty$.

Exercise 3.6. Consider the following game with two players, namely 1 and 2, and with Nature. First, Nature chooses a pair $(\theta_1, \theta_2) \in \{0, \ldots, 9\} \times \{0, \ldots, 9\}$ randomly and reveals θ_1 to player 1 and θ_2 to player 2, where the probability of each pair (θ_1, θ_2) is $1/100$. Then, simultaneously, each player i chooses a non-negative real number x_i. The payoff of each player i is

$$(\theta_1 + \theta_2)(x_1 + x_2) - x_i^2.$$

Write this game in normal form. Check if there is a dominant-strategy equilibrium and compute it if it exists.

Exercise 3.7. Alice and Beatrice are the CEOs of two firms, named A and B, respectively. The payment to each CEO depends on her firm's profit—in order to give her proper incentives—as well as the profit of the other firm's profit—in order to account for the market conditions beyond her control (the latter is called benchmarking). Simultaneously, each CEO i exerts effort $e_i > 0$, which leads to profit

$$\pi_i(e_i) = \log(e_i)$$

to her firm, and she is paid

$$w_i = \alpha\pi_i(e_i) - \beta\pi_j(e_j)$$

were $\alpha > \beta > 0$. The payoff of each CEO i is $u_i = w_i - e_i$. Compute the dominant-strategy equilibrium and briefly discuss your finding.

Exercise 3.8. Show that bidding true values is a dominant-strategy equilibrium in the n-buyer case of second-price auction of Section 3.3.

Exercise 3.9. Consider an auction in which k identical objects are sold to $n > k$ bidders. Each bidder i needs only one object and has a valuation v_i for the object. In the auction, simultaneously, every bidder i bids b_i. The highest k bidders win. Each winner gets one object and pays the $k + 1^{st}$ highest bidder (i.e., the price p is the highest bid among the bidders who do not get an object). (The ties are broken by a coin toss.) Each of the losing bidders gets a gift of value w for their participation. (The winners do not get a gift.) Show that the game has a dominant-strategy equilibrium, and compute the equilibrium.

Exercise 3.10. A government is to decide whether to construct a road between two towns, namely Arlington and Belmont. The values of the road for Arlington and Belmont are $a \geq 0$ and $b \geq 0$, respectively. The cost of constructing the road is $c > 0$. The government wants to construct the road if and only if $a + b \geq c$. The values a and b are known by the towns, but not by the government; c is known by everybody. To learn these values, the government asks each town to submit the value of the road for the town. Given the submitted valuations v_A and v_B, which need to be non-negative, the government constructs the road if and only if $v_A + v_B \geq c$ and tax Arlington and

Belmont $t_A(v_A, v_B)$ and $t_B(v_A, v_B)$, respectively, where

$$t_A(v_A, v_B) = \begin{cases} c - v_B & \text{if } v_A + v_B \geq c \text{ and } v_B < c \\ 0 & \text{otherwise} \end{cases}$$
$$t_B(v_A, v_B) = \begin{cases} c - v_A & \text{if } v_A + v_B \geq c \text{ and } v_A < c \\ 0 & \text{otherwise.} \end{cases}$$

Find the dominant-strategy equilibrium; verify that it is indeed a dominant-strategy equilibrium.

Exercise 3.11. In the previous problem, assume that the values a and b are not known. Nature selects a pair (a, b) from $\{c/3, 2c/3\}^2$ randomly (where each pair has probability $1/4$), and privately reveals a to Arlington and b to Belmont. Then, knowing their own value but not the other's value, each town submits a valuation as above (and the government makes the construction and taxation decisions as above). Find the dominant-strategy equilibrium; verify that it is indeed a dominant-strategy equilibrium.

Exercise 3.12. There are n players and an object. The game is as follows:

- First, for each player i, Nature chooses a number v_i from $\{0, 1, 2, \ldots, 99\}$, where each number is equally likely, and reveals v_i to player i and nobody else (v_i is the value of the object for player i).

- Then, each player i simultaneously bids a number b_i.

- The player who bids the highest number wins the object and pays b_j where b_j is the highest number bid by a player other than the winner. (If two or more players bid the highest bid, the winner is determined by a coin toss among the highest bidders.) The payoff of player i is $(v_i - b_j)$ if she is the winner and 0 otherwise.

1. Write this game in normal form. That is, determine the set of strategies for each player, and the payoff of each player for each strategy profile.

2. Show that there is a dominant-strategy equilibrium. State the equilibrium.

3.6 Appendix: Proof of Theorem 3.1

I now prove that a strategy is a best response to a belief if and only if it is not strictly dominated. The proof relies on a key mathematical theorem, called the Separating Hyperplane Theorem: Let A and B be convex, non-empty, and disjoint subsets of $\mathbb{R}^n$. Assume that either A is closed, or B is open. Then, there exists a non-zero $\lambda \in \mathbb{R}^n$ and $t \in \mathbb{R}$ such that $\lambda \cdot a \leq t < \lambda \cdot b$ for all $a \in A$, $b \in B$.

I will first show that, if s_i is not strictly dominated, then it is a weak best response to some belief. Denote the elements of finite set S_{-i} by $1, 2, \ldots, m$. For each mixed strategy σ_i and $k \in S_{-i}$, let $v_{\sigma_i,k}$ be the expected payoff from playing σ_i against k for player i. The vector $v_{\sigma_i} = (v_{\sigma_i,1}, \ldots, v_{\sigma_i,m})$ is in $\mathbb{R}^m$. Let A be the set of all such vectors v_{σ_i}. Clearly, A is convex. Take any s_i that is not strictly dominated, and define

$$B = \{v \in \mathbb{R}^m | v_k > v_{s_i,k} \ \forall k\}.$$

Clearly, B is also convex and open. Moreover, since s_i is not strictly dominated, $A \cap B = \varnothing$. Hence, by the Separating-Hyperplane Theorem, there exist a non-zero $\lambda \in \mathbb{R}^n$ and $t \in \mathbb{R}$ such that $\lambda \cdot a \leq t < \lambda \cdot b$ for all $a \in A$, $b \in B$. Moreover, because of the way the set B is defined $\lambda_k \geq 0$ for all k. Thus, one can normalize λ by dividing it to the sum of its elements to obtain $\lambda_1 + \cdots + \lambda_m = 1$. After such normalization, λ corresponds to a probability distribution on S_{-i} (i.e. a belief on other players' strategies). Note that, for any s_i', $v_{s_i'}$ is in A and v_{s_i} is on the boundary of B. Therefore,

$$\lambda \cdot v_{s_i'} \leq \lambda \cdot v_{s_i}.$$

Against belief λ, the expected payoffs from playing s_i and s_i' are $\lambda \cdot v_{s_i}$ and $\lambda \cdot v_{s_i'}$, respectively. Therefore, s_i is a best response to belief λ.

I next show that a strictly dominated strategy cannot be a best response. To this end, suppose s_i is a best response to a belief β_{-i} and strictly dominated by some σ_i:

$$\begin{aligned} u_i(s_i, \beta_{-i}) &\geq u_i(s_i', \beta_{-i}) \qquad (\forall s_i') \\ u_i(\sigma_i, s_{-i}) &> u_i(s_i, s_{-i}) \qquad (\forall s_{-i}). \end{aligned}$$

Then, by taking expectations with respect to σ_i and β_{-i} in the first and the second inequalities, respectively, one can obtain a clear contradiction:

$$\begin{aligned} u_i(s_i, \beta_{-i}) &\geq u_i(\sigma_i, \beta_{-i}) \\ u_i(\sigma_i, \beta_{-i}) &> u_i(s_i, \beta_{-i}). \end{aligned}$$