Chapter 7

Supermodular Games

A common exercise in economics is to understand how a particular outcome varies with a particular parameter. For example, one may want to know whether a reduction in income tax increases the investment level in equilibrium. The answer often depends on whether there is complementarity (or supermodularity) between the relevant variables, i.e., whether the payoff increase due to increasing one variable is an increasing function of the other. When there is strategic interaction, this often requires complementarity not only between a parameter and a player's strategy but also whether players' strategies are complementary to each other. For example, in the case of tax reduction above, the answer depends not only on whether tax reduction increases the return from investing one more dollar but also whether those returns get higher when other players invest more. This chapter introduces games with strategic complementarity, called *supermodular games*.

The main results establish the structure of the solution set and monotone comparative statics under complementarity. For individual decision problems, the result establishes that the set of solutions is weakly increasing in complementarity parameters. For games, the result establishes that there are extremal equilibria that bound all rationalizable strategies, and the extremal equilibria are weakly increasing in complementarity parameters.

7.1 Example

I will illustrate the main concepts and results on a simple concrete example about research and development (R&D), advertisement, and prices. Consider a firm deciding how much to invest in research and development and how much to invest in advertisement. By investing in R&D, the firm develops a product. The product quality is increasing in the expenditure level in R&D, so that higher investment in expenditure makes customers more likely to buy the product if they are aware of it. In order to make the potential customers aware of the product, the firm must advertise it. The number of customers the firm can reach is increasing with advertisement expenditure.

Formally, if the firm invests a in advertisement and b in research and development, a potential customer becomes aware of the firm's product with probability $\alpha(a)$, in which case she buys it with probability $\beta(b)$—at some fixed price p—where both functions α and β are increasing. The firm's expected profit is

$$U(a,b) = \alpha(a)\,\beta(b)\,p - a - b.$$

The expenditures in R&D and advertisement are complementary. To check this formally, one checks how R&D expenditure b affects the incentive to increase the expenditure a in advertisement. For a given $\Delta > 0$, a Δ amount increase in a results in a $U(a+\Delta,b) - U(a,b)$ increase in the profit, where the increase in profit can be negative. The expenditures a and b are complements if the increase $U(a+\Delta,b) - U(a,b)$ is increasing in b. In this example, the increase in profit is

$$U(a+\Delta,b) - U(a,b) = (\alpha(a+\Delta) - \alpha(a))\,\beta(b)\,p - \Delta.$$

Since the probability of sale, $\beta(b)$, is an increasing function of the expenditure b in research and development, $U(a+\Delta,b) - U(a,b)$ is an increasing function of b. Similarly, the profit increase due to an extra R&D investment Δ is an increasing function of advertisement investment a:

$$U(a,b+\Delta) - U(a,b) = \alpha(a)\,p\,(\beta(b+\Delta) - \beta(b)) - \Delta.$$

In that case, advertisement and R&D expenditures are called *complementary*. One can also see that there is complementarity between price p and the expenditures in advertisement and R&D.

When there is complementarity between a parameter and a choice variable, the optimal choice is increasing in the parameter. For example, the firm's optimal advertisement and R&D expenditures are increasing in price p. For an illustration, take $a, b \in [0,1]$, $\alpha(a) = a^{1/3}$ and $\beta(b) = b^{1/3}$, so that $U(a,b) = (ab)^{1/3} p - a - b$. For any fixed price p, the optimal expenditures in advertisement and R&D are[1]

$$a^*(p) = b^*(p) = (p/3)^3 .$$

The optimal expenditure in each activity is increasing in price.

When there are multiple players, the players' strategies may be complementary to each other. Such *strategic complementarity* leads to an interesting structure for sets of equilibria and rationalizable strategies. For a concrete example, now imagine that Advertisement and R&D departments of the firm are run independently by different managers, where each department gets half of the revenues and pays its own expenditure. In particular, the Advertisement department chooses the advertisement expenditure a (on this particular product), and the R&D department chooses the R&D expenditure b. The payoffs of the Advertisement and the R&D departments are

$$u_A(a,b) = \frac{1}{2}\alpha(a)\beta(b)p - a \text{ and } u_B(a,b) = \frac{1}{2}\alpha(a)\beta(b)p - b,$$

respectively. The Advertisement and R&D departments now complement each other. For example, the Advertisement department's return $u_A(a+\Delta, b) - u_A(a,b)$ from an extra expenditure Δ is increasing in the R&D department's expenditure b. Since a and b are the strategies of two players, this complementarity is called *strategic complementarity.*

Under strategic complementarity, there will be maximal and minimal equilibria, and these equilibria will be weakly increasing with complementary parameters. To illustrate this, take $\alpha(a) = a^{1/3}$ and $\beta(b) = b^{1/3}$ as before, so that

$$u_A(a,b) = \frac{p}{2}(ab)^{1/3} - a \text{ and } u_B(a,b) = \frac{p}{2}(ab)^{1/3} - b.$$

The best response functions are

$$a^*(b) = (p/6)^{3/2} b^{1/2} \text{ and } b^*(a) = (p/6)^{3/2} a^{1/2}.$$

[1] By symmetry, $a = b$ at the optimum, and $U(a,a) = a^{2/3}p - 2a$, which is maximized at $a^*(p)$.

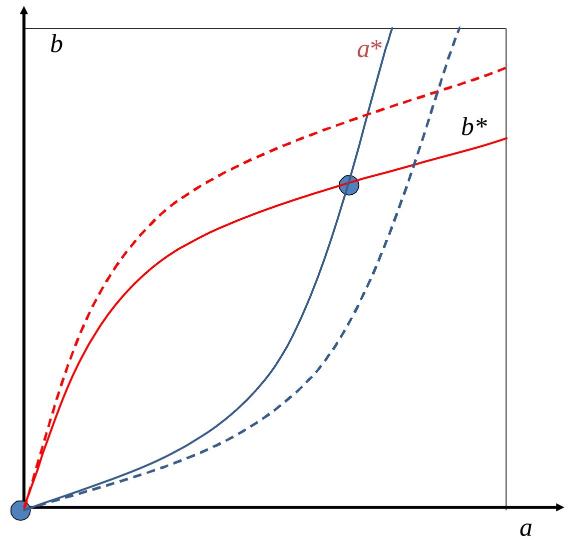

Figure 7.1: Nash equilibria in Advertisement and R&D example

As seen in Figure 7.1, the best response functions are increasing—because of strategic complementarity. The equilibria are the intersections of the graphs of the best-response functions. There are two equilibria. One is at $a = b = 0$, where neither player invests, correctly assuming that the other player will not invest, and any (unilateral) expenditure will be a total waste. The other equilibrium is at

$$\hat{a} = \hat{b} = (p/6)^3 ,$$

which is computed by substituting $b^*(a)$ for b in the formula for $a^*(b)$. Note that the equilibria are ordered: both players spend more in equilibrium $\left(\hat{a}, \hat{b}\right)$ with respect to equilibrium $(0, 0)$. Note also that both equilibria are weakly increasing in price p, which is complementary to the expenditures a and b. The smallest equilibrium $(0, 0)$ remains constant while the largest equilibrium $\left((p/6)^3, (p/6)^3\right)$ is increasing. This is vividly illustrated in Figure 7.1. When p increases, the best-response functions shift up (plotted in dashed lines). The largest equilibrium moves up as a result. As it will be shown more generally, when a and b are bounded, the set of rationalizable strategies will be $\left[0, (p/6)^3\right]$ for each player. Hence the smallest and largest rationalizable strategies are also increasing in price.

Remark 7.1. More generally, in a single-person optimization problem, the set of optimal

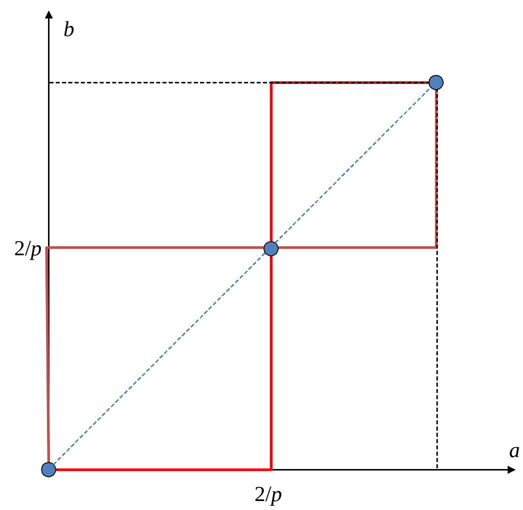

Figure 7.2: Equilibria under linear functions $\alpha(a) = a$ and $\beta(q(b)) = b$.

choices shift up when a complementary parameter increases. In particular, the largest and smallest solutions increase. Only the latter weaker conclusion holds in games with strategic complementarity. The extremal equilibria are weakly increasing in complementary parameters, but some middle equilibrium may move in the opposite direction. For example, take $a, b \in [0, 1]$, $p > 2$, and take $\alpha(a) = a$ and $\beta(b) = b$, so that

$$u_A(a, b) = \frac{p}{2}ab - a \text{ and } u_B(a, b) = \frac{p}{2}ab - b.$$

Then, there are three equilibria: $(0, 0)$, $(1, 1)$, and $(2/p, 2/p)$, as plotted in Figure 7.2. The extremal equilibria $(0, 0)$ and $(1, 1)$ remain constant while the middle equilibrium $(2/p, 2/p)$ *decreases* with price p.

7.2 Complementarity

This section defines complementarity formally and introduces the concept of supermodular functions. In general, complementarity is defined as follows.

Definition 7.1. A function $f : \mathbb{R}^2 \to \mathbb{R}$ is said to have *increasing differences* if for any $(x_1, x_2) \geq (y_1, y_2)$

$$f(x_1, x_2) - f(y_1, x_2) \geq f(x_1, y_2) - f(y_1, y_2).$$

In that case, the two inputs are *complements* (under f).

Note that the above inequality can be written as

$$f(x_1, x_2) + f(y, y_2) \geq f(x_1, y_2) + f(y_1, x_2).$$

Thus, it does not matter which marginal contribution we look at. If the marginal contribution of the first entry is increasing in the second entry, then the marginal contribution of the second entry is also increasing in the first entry. The above definition remains valid if f is restricted to any subset $X_1 \times X_2 \subset \mathbb{R}^2$. When f is defined over $\mathbb{R}^2$ and twice differentiable, the complementarity condition above can be written as

$$\frac{\partial^2 f}{\partial x_1 \partial x_2} \geq 0,$$

i.e., the cross-derivatives are non-negative. More generally, when there are complementarities between all variables, function f is called supermodular:

Definition 7.2. A function $f : \mathbb{R}^m \to \mathbb{R}$ is said to be *supermodular* if any two inputs are complements under f, i.e., for any distinct $k, l \in \{1, \ldots, m\}$ and for any $(x_k, x_l, x_{-kl}), (y_k, y_l, x_{-kl}) \in \mathbb{R}^m$ with $(x_k, x_l) \geq (y_k, y_l)$,

$$f(x_k, x_l, x_{-kl}) - f(y_k, x_l, x_{-kl}) \geq f(x_k, y_l, x_{-kl}) - f(y_k, y_l, x_{-kl}),$$

where (x_k, x_l, x_{-kl}) denotes the vector in which kth and lth entries are x_k and x_l, respectively, while the remaining entries (if any) are as in x_{-kl}.

Once again the above definition remains valid if f is restricted to any $X_1 \times \cdots \times X_m \subset \mathbb{R}^m$. When f is defined over $\mathbb{R}^m$ and twice differentiable, supermodularity simply requires that all cross-derivatives are non-negative:

$$\frac{\partial^2 f}{\partial x_k \partial x_l} \geq 0 \qquad \text{(for all distinct } k \text{ and } l\text{)}.$$

A function f is said to be *submodular* if $-f$ is supermodular. All functions $f : \mathbb{R} \to \mathbb{R}$ of one variable will be considered supermodular (and submodular) by convention.

7.3 Optimization Under Supermodular Payoffs

When the payoff function is supermodular, the set of optimal choices will shift up when the parameters move up. This section establishes this for the largest and smallest optimal choices.

Theorem 7.1 (Monotonicity Theorem)**.** *Consider any continuous supermodular utility function $u : X \times \Theta \to \mathbb{R}$ where $X = X_1 \times \cdots \times X_m \subset \mathbb{R}^m$ is a non-empty, closed and bounded set of choice variables x and $\Theta \subset \mathbb{R}^n$ is a non-empty set of payoff parameters θ. Then, the largest and smallest optimal choices,*

$$\begin{aligned} \bar{B}(\theta) &= \max\{x \in X | u(x,\theta) \geq u(y,\theta) \text{ for all } y \in X\} \text{ and} \\ \underline{B}(\theta) &= \min\{x \in X | u(x,\theta) \geq u(y,\theta) \text{ for all } y \in X\}, \end{aligned}$$

exist and are weakly increasing in θ.

Proof When $X \subset \mathbb{R}$, the existence of largest and smallest optima are immediate. I will not prove it here more generally. I will only show that $\bar{B}(\theta)$ is weakly increasing in θ. Take any $\theta, \theta' \in \Theta$ with $\theta \geq \theta'$. I will show that $\bar{B}(\theta) \geq \bar{B}(\theta')$. Define $x = (x_1, \ldots, x_m)$ and $y \in (y_1 \ldots, y_m)$ by setting

$$\begin{aligned} x_k &= \max\left\{\bar{B}_k(\theta), \bar{B}_k(\theta')\right\} \\ y_k &= \min\left\{\bar{B}_k(\theta), \bar{B}_k(\theta')\right\} \end{aligned}$$

for each dimension k. Suppose that $\bar{B}(\theta) \ngeq \bar{B}(\theta')$. Then, $x > \bar{B}(\theta)$. Hence, x is not an optimal solution under θ. Thus,

$$0 > u(x,\theta) - u\left(\bar{B}(\theta),\theta\right) \geq u(x,\theta') - u\left(\bar{B}(\theta),\theta'\right) \geq u\left(\bar{B}(\theta'),\theta'\right) - u(y,\theta') \geq 0,$$

a contradiction. Here, the strict inequality holds because $\bar{B}(\theta)$ is optimal while x is not optimal under θ. The next two inequalities follow from supermodularity of u, the first one from complementarity between θ and the strategies, and the second one from strategic complementarity. The last inequality follows from the optimality of $\bar{B}(\theta')$ under θ'. □

The Monotonicity Theorem establishes that when the utility function is supermodular—with respect to both choice variables and parameters, the largest and smallest solutions are increasing in parameters.

One can use this result to obtain comparative statics without explicitly solving the optimization problem. For example, in Section 7.1, the optimal solution is obtained only for a specific form of α and β. In that case, the optimal solution, $a = b = (p/3)^3$, was clearly increasing in p. Unfortunately, for more general functions α and β, solving the optimization problem can be quite difficult. Fortunately, using the above theorem, one can easily conclude that an increase in p increases the (extremal) optimal levels of expenditures, by checking that $U(a,b) = \alpha(a)\beta(b) - a - b$ is supermodular; this is indeed the case whenever α and β are weakly increasing as shown in Section 7.1. Sometimes one can apply the Monotonicity Theorem by transforming the payoff function—as in the next example.

Example 7.1. Consider a monopolist who chooses a price p for its product, facing a demand function $D(p,\theta)$ and marginal cost c, where θ is a demand parameter. Write

$$p^*(\theta, c) = \arg\max_{p \geq c'} (p - c) D(p, \theta)$$

for the optimal price, where $c' > c$ is a fixed lower bound for prices. Direct application of the Monotonicity Theorem to this problem may not be as useful. Observe however that optimal solution is invariant to monotone transformations of objective functions, and hence

$$p^*(\theta, c) = \arg\max_{p \geq c'} \log(p - c) + \log D(p, \theta).$$

The new objective function is supermodular with respect to p and c. Hence, the Monotonicity Theorem implies that p^* is weakly increasing in c. Moreover, the new objective function is supermodular with respect to p and θ as long as $\log D(p,\theta)$ is supermodular. Hence, the Monotonicity Theorem implies that p^* is weakly increasing in θ as long as the logarithm of the demand function is supermodular.[2]

7.4 Supermodular Games

This section formally introduces supermodular games and establishes their main properties. The set of players is $N = \{1, \ldots, n\}$. The set of strategies for each player i is a closed and bounded subset

$$S_i = S_{i1} \times \cdots \times S_{im_i} \subset \mathbb{R}^{m_i}$$

[2] The latter condition is equivalent to the price elasticity of demand $-\frac{\partial \log D(p,\theta)}{\partial \log p}$ being weakly decreasing in θ.

with the smallest and largest elements $\underline{s}_i$ and $\bar{s}_i$, respectively. Typically, $S_i = [\underline{s}_i, \bar{s}_i]$ for some real numbers $\underline{s}_i$ and $\bar{s}_i$. The general formulation allows players to have multiple choices (as in the case of R&D, advertising and pricing) and the strategy set to be discrete (as in the case of deciding whether to invest in a project). Writing

$$S = S_1 \times \cdots \times S_n$$

for the set of strategy profiles, write $\underline{s} = \min S$ and $\overline{s} = \max S$ for the smallest and largest strategy profiles, respectively. Each utility function

$$u_i : S \to \mathbb{R}$$

is supermodular[3] and continuous. In the sequel, a game is said to be *supermodular* if it satisfies the above properties.

For example, consider Advertisement and R&D departments run independently as in Section 7.1. The payoff functions are supermodular, and each player chooses a real number. In order to satisfy the conditions above, one can simply take $a \in [\underline{a}, \bar{a}]$ and $b \in [\underline{b}, \bar{b}]$ and assume that α and β are continuous.

Example 7.2 (Differentiated Bertrand Oligopoly)**.** Consider a differentiated Bertrand oligopoly model in which each player i faces constant marginal cost c_i and demand function $Q_i(p) = A - a_i p_i + \sum_{j \neq i} b_j p_j$, where A, a_i and b_j are all positive numbers (see Section 6.4.2). For each i, assume that price p_i is selected from $[c_i, \bar{p}_i]$ for some large $\bar{p}_i$. This yields a supermodular game because

$$u_i(p) = (p_i - c_i)\ Q_i(p)$$

is supermodular:

$$\frac{\partial^2 u_i}{\partial p_i \partial p_j} = b_j \geq 0.$$

[3] The results in this section hold under a weaker but more mouthful two-part assumption: (1) each u_i is a supermodular function of s_i for each fixed s_{-i} and (2) for each $s_i \geq s_i'$, the difference $u_i(s_i, s_{-i}) - u_i(s_i', s_{-i})$ is a weakly increasing function of s_{-i}. For example, when the strategy sets are one-dimensional and the utility functions are differentiable, it suffices to check that

$$\frac{\partial^2 u_i}{\partial s_i \partial s_j} \geq 0$$

for all distinct i and j.

Example 7.3 (Linear Cournot Duopoly)**.** Consider a Cournot duopoly model with inverse demand function $P = A - q_1 - q_2$ and cost functions $C_1(q_1)$ and $C_2(q_2)$ (see Section 6.2.1). Restrict the set of possible production levels to a large compact interval. This leads to a "submodular" game because the utility function of firm i is

$$u_i(q) = q_i P(q) - C_i(q_i),$$

yielding

$$\frac{\partial^2 u_i}{\partial q_i \partial q_j} = -1 < 0.$$

This is a supermodular game when q_2 is ordered in the reverse order (by denoting the strategies as negatives of the quantities for player 2):

$$\frac{\partial^2 u_i}{\partial q_1 \partial(-q_2)} = 1 > 0.$$

In general, two-player games with submodular payoff functions can be made supermodular by reversing the order on one of the strategies. Hence, they exhibit the useful properties of supermodular games. This trick does not work, however, when there are more than two players, and the "submodular" games with more than two players may exhibit dramatically different properties than the supermodular ones.

Example 7.4 (Linear Cournot Oligopoly)**.** In the above example, suppose that there are three or more players. Once again, for any $i \neq j$,

$$\frac{\partial^2 u_i}{\partial q_i \partial q_j} = -1 < 0.$$

But this game cannot be made supermodular by reversing the orders. Indeed, as seen in Chapter 6, the relation between rationalizability and Nash equilibria in Cournot oligopoly is quite different than the relation in Cournot duopoly, which inherits the special structure of Nash equilibrium and rationalizability in supermodular games.

7.4.1 Rationalizability and Equilibrium

This section establish that there exist extremal equilibria and that rationalizable strategies are bounded by the extremal equilibria. Moreover the extremal equilibria are computed by iterated application of best response to the largest and smallest strategies, a fast computation technique when one uses computers.

Fix a supermodular game (N, S, u), and for any player $i \in N$, define the largest and smallest best responses to any $s_{-i} \in S_{-i}$ as

$$\begin{aligned}\bar{B}_i(s_{-i}) &= \max\{s_i \in S_i | u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}) \text{ for all } s_i' \in S_i\} \text{ and} \\ \underline{B}_i(s_{-i}) &= \min\{s_i \in S_i | u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}) \text{ for all } s_i' \in S_i\}.\end{aligned}$$

Define also the largest and smallest best-response mappings $\bar{B} : S \to S$ and $\underline{B} : S \to S$ by setting $\bar{B}(s) = (\bar{B}_1(s_{-1}), \ldots, \bar{B}_n(s_{-n}))$ and $\underline{B} = (\underline{B}_1(s_{-1}), \ldots, \underline{B}_n(s_{-n}))$. By the Monotonicity Theorem, the mappings $\bar{B}$ and $\underline{B}$ are well-defined and weakly increasing.

In a supermodular game, one can compute the largest Nash equilibrium as follows. Take the largest strategy profile $\bar{s}$ and apply the largest best response mapping, obtaining $\bar{B}(\bar{s}) \in S$. Since $\bar{s}$ is the largest strategy profile,

$$\bar{B}(\bar{s}) \leq \bar{s}.$$

Now apply the largest best response mapping $\bar{B}$ to $\bar{B}(\bar{s})$ obtaining a new strategy profile $\bar{B}^2(\bar{s}) = \bar{B}(\bar{B}(\bar{s}))$. Since $\bar{B}(\bar{s}) \leq \bar{s}$, the Monotonicity Theorem implies that

$$\bar{B}^2(\bar{s}) \leq \bar{B}(\bar{s}).$$

Following in this fashion, one can construct a decreasing sequence $\bar{B}^k(\bar{s})$ of strategy profiles where

$$\bar{B}^k(\bar{s}) = \bar{B}(\bar{B}^{k-1}(\bar{s})) \leq \bar{B}^{k-1}(\bar{s}).$$

Since S is bounded, this decreasing sequence converges to some $s^* \in S$, which turns out to be a Nash equilibrium. Moreover, as it turns out, s^* is the largest rationalizable strategy profile. In particular, if a strategy s survives the kth round of iterated elimination of rationalizable strategies, then it must be that $s \leq \bar{B}^k(\bar{s})$. One can similarly, construct an increasing sequence of strategies obtaining the smallest Nash equilibrium, which also provides a lower bound to rationalizable strategies. This is formally stated in the next result.

Theorem 7.2. *For any supermodular game, the strategy profiles*

$$s^* = \lim_{k\to\infty} \bar{B}^k(\bar{s}) \text{ and } s^{**} = \lim_{k\to\infty} \underline{B}^k(\underline{s})$$

are Nash equilibria, and for any rationalizable strategy profile s,

$$s^* \geq s \geq s^{**}.$$

This result establishes several important facts. First of all, it establishes that there exists an equilibrium, indeed, there are extremal equilibria in pure strategies. Second, it establishes a useful procedure to compute these equilibria: one iteratively applies extremal best response functions to the largest and smallest strategy profiles. In comparison, finding a fixed point of a function is a computationally hard problem. Finally, it establishes that the rationalizable strategies are bounded by these extremal equilibria. This not only relates extreme implications of equilibrium and rationalizability to each other, but also helps in identifying rationalizable strategies. For example, when the extremal best response functions are continuous and strategy sets are convex intervals, the result implies that the rationalizable set is the convex hull of extremal equilibrium strategies. It also implies that uniqueness of Nash equilibrium in pure strategies is equivalent to dominance solvability: *a supermodular game is dominance solvable if and only if there exists a unique Nash equilibrium in pure strategies.*

In the example of advertising and R&D, the extremal equilibria were $(0,0)$ and $\left((p/6)^3, (p/6)^3\right)$. Note that if we allowed any non-negative expenditure $a \geq 0$ and $b \geq 0$, one could not eliminate any strategy because each $a \geq 0$ is a best response to $b = a^2 (6/p)^{1/3}$. Thus, every strategy is rationalizable. The theorem above establishes that if one imposes an upper bound, larger than $(p/6)^3$, then the set of rationalizable strategies reduces to $\left[0, (p/6)^3\right]$ for each player. The next example illustrates the same point in differentiated price competition.

Example 7.5. Consider the differentiated Bertrand oligopoly model above, and set $a_1 = \cdots = a_n = a$, $b_1 = \cdots = b_n = b$ and $c_1 = \cdots = c_n = c$ for some positive a, b, and c with $a > (n-1)\, b$. When the upper bound $\bar{p}$ is not binding, there is a unique Nash equilibrium, in which each firm charges

$$p^* = \frac{A + ac}{2a - (n-1)\, b}. \tag{7.1}$$

Thus, the game is dominance solvable, where p^* is the only rationalizable strategy. If $\bar{p} < p^*$, then the unique rationalizable strategy is $\bar{p}$. On the other hand, if the set of allowable prices is not restricted from above, the set of rationalizable strategies is $[p^*, \infty)$, where players can charge any price above the unique Nash equilibrium price p^* (see Section 6.4.2).

7.4.2 Comparative Statics

The next result shows that the extremal equilibria are weakly increasing in complementary parameters, extending the Monotonicity Theorem for optimization in games.

Theorem 7.3. *Consider a family of supermodular games G^{θ} with payoffs parameterized by $\theta \in \mathbb{R}^m$, where each utility function $u_i(s, \theta)$ is supermodular. Then, the largest and smallest equilibria, $s^*(\theta)$ and $s^{**}(\theta)$, are weakly increasing in θ.*

Proof. Take any θ, θ' with $\theta \geq \theta'$, and write $\bar{B}_{\theta}$ and $\bar{B}_{\theta'}$ for the largest best response functions under θ and θ', respectively. By the Monotonicity Theorem, $\bar{B}_{\theta}(s) \geq \bar{B}_{\theta'}(s)$ for every s. Since $\bar{B}_{\theta}$ and $\bar{B}_{\theta'}$ are weakly increasing, this further implies that $\bar{B}_{\theta}^k(s) \geq \bar{B}_{\theta'}^k(s)$ for every k. Therefore,

$$s^*(\theta) = \lim_{k\to\infty} \bar{B}_{\theta}^k(s) \geq \lim_{k\to\infty} \bar{B}_{\theta'}^k(s) = s^*(\theta').$$

The inequality for the smallest equilibrium is proved similarly. □

For example, in Section 7.1, the extremal equilibria were $(0, 0)$ and $\left((p/6)^3, (p/6)^3\right)$. While smallest equilibrium remains constant, the largest equilibrium increases in price p. The theorem shows that, since the payoff function is supermodular, the extremal equilibria will increase in prices so long as α and β are increasing and the expenditures are bounded from above. Similarly, in the linear Bertrand oligopoly the payoffs are supermodular with respect to the marginal costs as well, and hence the extremal equilibrium prices will be weakly increasing in marginal costs c_i, as in the symmetric case in (7.1). Remark 7.1 also shows that this result only applies to the extremal equilibria: there can be a middle equilibrium that decreases in θ.

7.5 Applications

This section presents further economic applications of supermodular games. Many of these applications are important models in economics in their own right. Some of these examples will not be supermodular, but the analysis is similar to the supermodular games.

7.5.1 Partnership

Consider an employer and a worker. The employer provides the capital $K \geq 0$ (in terms of investment in technology, etc.) and the worker provides the labor $L \geq 0$ (in terms of the investment in the human capital) to produce

$$f(K, L) = K^{\alpha} L^{\beta}$$

for some $\alpha, \beta \in (0, 1)$ with $\alpha + \beta < 1$. They share the output equally. The parties determine their investment level (the employer's capital K and the worker's labor L) simultaneously. The per-unit costs of the capital and the labor for the employer and the worker are $r > 0$ and $w > 0$, respectively. Here, the understanding is that the employer borrows the capital at interest rate r, or alternatively she could have earned interest rate r by lending her capital instead of investing in the project. Likewise, w represents the wage the worker forgoes by spending an extra unit of labor in this project. The worker cannot put more than some fixed positive $\bar{L}$. The payoffs for the employer and the worker are

$$u_E(K, L) = \frac{1}{2} f(K, L) - rK$$

and

$$u_W(K, L) = \frac{1}{2} f(K, L) - wL,$$

respectively. Everything described up to here is common knowledge.

Note that the sets of strategies for the Employer and Worker are

$$S_E = [0, \infty) \text{ and } S_W = \left[0, \bar{L}\right],$$

respectively. Moreover, the payoff functions are supermodular:

$$\frac{\partial^2 u_E}{\partial K \partial L} = \frac{\partial^2 u_W}{\partial K \partial L} = \frac{1}{2} \frac{\partial^2 f}{\partial K \partial L} = \frac{\alpha\beta}{2} K^{\alpha-1} L^{\beta-1} \geq 0.$$

Consequently, the best response functions are increasing:

$$K^*(L) = \left(\frac{1}{2} \frac{\alpha}{r} L^{\beta}\right)^{1/(1-\alpha)} \text{ and } L^*(K) = \min\left\{\left(\frac{1}{2} \frac{\beta}{w} K^{\alpha}\right)^{1/(1-\beta)}, \bar{L}\right\},$$

where K^* and L^* are the best response functions of the employer and worker, respectively. The game would have been supermodular if one also bounded the capital from

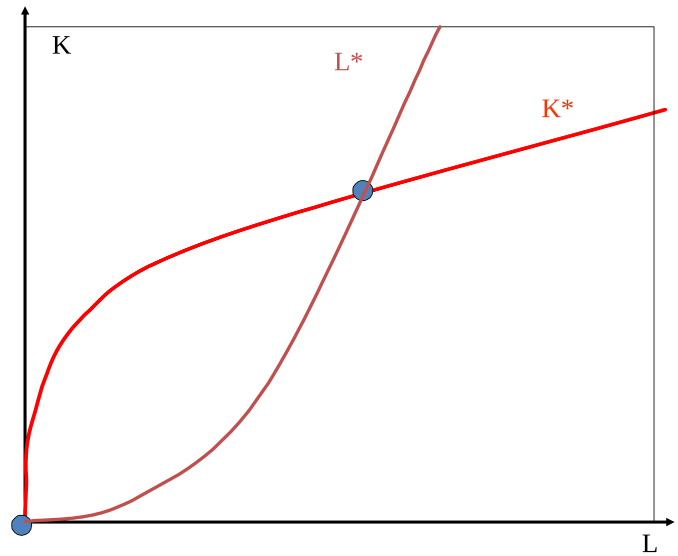

Figure 7.3: Nash equilibrium in partnership game.

above by some $\bar{K}$. Without a bound it is not supermodular. For example, if neither capital nor labor was bounded above, every strategy would have been rationalizable (why?). As it turns out all the conclusions in the previous hold with bound $\bar{L}$ on labor.

First compute the Nash equilibria. Since u_E and u_W are strictly concave in own strategies, all Nash equilibria are in pure strategies. The Nash equilibria are the intersections of the graphs of the best response functions above, i.e., $K = K^*(L)$ and $L = L^*(K)$. Clearly, $(0,0)$ is a Nash equilibrium. To find the other possible equilibria, first consider the case $L^*\left(K^*\left(\bar{L}\right)\right) < \bar{L}$. In that case, the non-zero solution to the above equation is

$$\begin{aligned}
\hat{K} &= \left(\frac{1}{2}\left(\frac{\alpha}{r}\right)^{1-\beta}\left(\frac{\beta}{w}\right)^{\beta}\right)^{1/(1-\alpha-\beta)} \\
\hat{L} &= \left(\frac{1}{2}\left(\frac{\beta}{w}\right)^{1-\alpha}\left(\frac{\alpha}{r}\right)^{\alpha}\right)^{1/(1-\alpha-\beta)}.
\end{aligned}$$

The constraint for the labor is not binding in equilibrium (i.e., $\hat{L} \leq \bar{L}$), and the only non-zero equilibrium is $\left(\hat{K}, \hat{L}\right)$. This case is plotted in Figure 7.3. When $L^*\left(K^*\left(\bar{L}\right)\right) \geq \bar{L}$, the constraint for the labor binds, and the non-zero equilibrium is $\left(K^*\left(\bar{L}\right), \bar{L}\right)$.

What about rationalizability? First observe that the strategies in $[0, \hat{K}]$ are rational-

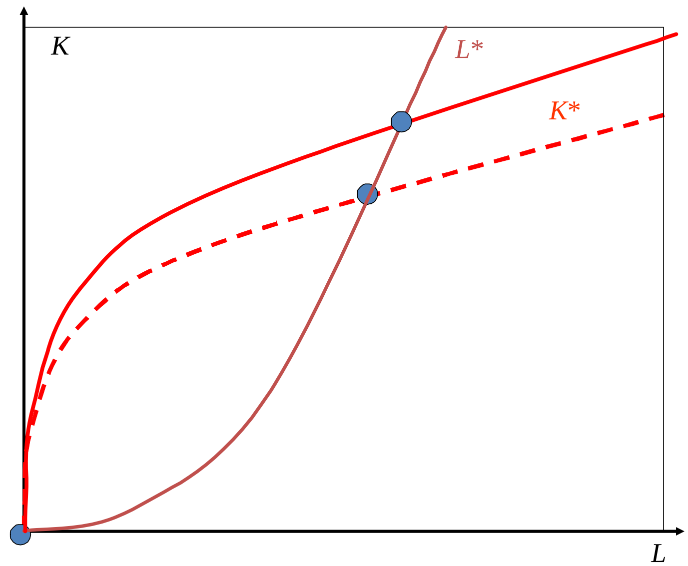

Figure 7.4: The effect of an increase in interest rate r.

izable for the employer and the strategies $[0, \hat{L}]$ are rationalizable for the worker. This is because each K in $[0, \hat{K}]$ is a best response to a strategy L in $[0, \hat{L}]$, while each L in $[0, \hat{L}]$ is a best response to a strategy K in $[0, \hat{K}]$ (see Exercise 4.6). Can there be any other rationalizable strategy? The answer would have been No (by Theorem 7.2) if the strategies were bounded. As it turns out, this is still true thanks to the upper bound on labor. Since K^* is increasing, a strategy $K > K^*\left(\bar{L}\right)$ cannot be a best response to pure strategy L, which cannot exceed $\bar{L}$. It cannot be a best response to any mixed strategy either because, as in the example of Cournot duopoly, any $K > K^*\left(\bar{L}\right)$ is strictly dominated by $K^*\left(\bar{L}\right)$. Therefore, all strategies $K > K^*\left(\bar{L}\right)$ are eliminated in the first round, and the set of strategies surviving this round is $\left[0, K^*\left(\bar{L}\right)\right]$. Since this set is closed and bounded, the game with the surviving strategies is a supermodular game, and Theorem 7.2 implies that, in this game, the sets of rationalizable strategies for the employer and the worker are $[0, \hat{K}]$ and $[0, \hat{L}]$, respectively. Since the set of rationalizable strategies in this game must coincide with the set of rationalizable strategies in the original game, this shows that the sets of rationalizable strategies are $[0, \hat{K}]$ and $[0, \hat{L}]$ in the original game.

How would an increase in the interest rate affect the investment and employment levels? In other words, how do the equilibrium levels of K and L vary with r? The effect of an increase in r is illustrated in Figure 7.4, where the best response function of the employer under a higher level of r is plotted in dashed lines. As shown in the figure, an increase in the interest rate r leads the employer to invest less against any level of anticipated labor. This does not affect the no-investment equilibrium, but decreases both capital and labor investments $\hat{K}$ and $\hat{L}$ in the largest equilibrium. It is intuitive that the employer lowers her capital investment in response to an increase in interest rate. The decrease in labor is due to strategic reasons. Anticipating a lower level of capital investment, the worker exerts lower level of labor. This further lowers the employer's incentive to invest, and so on, leading to a low level of investment by both players. This multiplier effect leads to a decrease in capital and labor investments more than what is implied by employer's behavior.

One can formally deduce this from Theorem 7.3 as follows. Write $\theta = -r$, so that the payoff function of the employer is

$$u_E(K,L) = \frac{1}{2}f(K,L) + \theta K,$$

while u_W does not depend on θ. Observe that

$$\frac{\partial^2 u_E}{\partial K \partial \theta} = 1,$$

and all the other cross-derivatives with respect to θ are zero. Hence, by Theorem 7.3, an increase in r leads to a decrease in θ, and thereby weakly decreases the extremal equilibria. Similarly, Theorem 7.3 implies that an increase in w leads to a decrease in both capital and labor investments in the partnership.

7.5.2 Searching for A Partner

Consider a set $N = \{1, \ldots, n\}$ of players for some large n. Each player i exerts effort $s_i \in [0,1]$, incurring a cost $s_i^2/2$. Let $A(s_{-i})$ be the average effort level for the players other than i. The probability that i finds a match is $s_i g(A(s_{-i}))$ for some increasing, continuous function $g : [0,1] \to [0,1]$ with $g(0) = 0$ and $g(1) < 1$. Let the payoff from match be $\theta \geq 0$. Then, the expected payoff of player i is

$$U_i(s) = \theta s_i g(A(s_{-i})) - s_i^2/2.$$

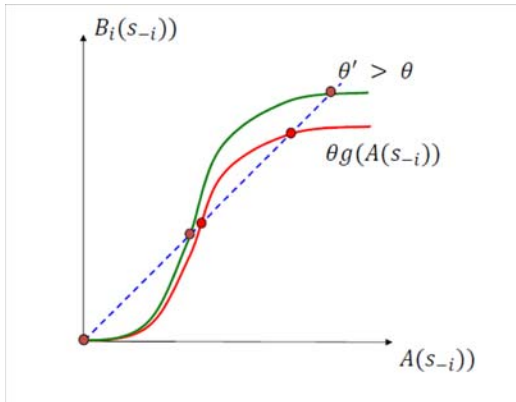

Figure 7.5: Equilibria in search model

The payoff function exhibits strategic complementarity: an increase in $A(s_{-i})$ always results in a (weakly) increase in the marginal utility $\partial U_i/\partial s_i$ of exerting more effort. This leads to an increasing best-response function:

$$B_i(s_{-i}) = \theta g(A(s_{-i})).$$

Note that the level of strategic complementarity depends on θ and the slope of g. Similarly, there is complementarity between the search level s_i and the value θ of match:

$$\partial^2 U_i/\partial s_i \partial\theta = g(A(s_{-i})) \geq 0.$$

Once again the best response is increasing in θ.

Consider the Nash equilibria of the above game. Since the game is symmetric, extremal equilibria will be symmetric. Hence, consider symmetric equilibria. These equilibria are characterized by the intersection of the graph of g with the diagonal, as in Figure 7.5. In this figure, there are three equilibria, and all of the equilibria are ordered, where the smallest equilibrium is located at the origin. While the number of equilibria depends on the shape of g, the equilibria will always be ordered (because g and the diagonal are increasing), and there will exist extremal equilibria.

How do the equilibrium search levels vary by θ? To find an answer, increase θ to a higher level θ'. Since this corresponds to scaling up the best response function (by θ'/θ), the new equilibria are formed as in the figure. The smallest equilibrium remains at zero (weakly increasing). The largest equilibrium moves up, as in Theorem 7.3.

These changes are intuitive; players search more when the match is more valuable. Note however that the middle equilibrium decreases, so that the players search less when the match is more valuable. This shows that the intuition is true only for the extreme equilibria. Finally, note that the largest equilibrium moves more than the individual best responses, i.e., $s_i^*(\theta') > B_i(s_{-i}^*(\theta), \theta')$, where $s^*(\theta)$ and $s^*(\theta')$ are the equilibria under θ and θ', respectively. That is, there is a multiplier effect.

7.5.3 Coordination in Software Development

There are n software developers, named $i = 1, 2, \ldots, n$. Each software developer i has an "ideal specification" θ_i for her product, but also would like her product to be compatible with the other software, where θ_i is a real number. Simultaneously, each i selects a specification parameter s_i, which is a real number. The payoff of a player i is

$$u_i\left(s_1, \ldots, s_n\right) = 100 - \left(s_i - \theta_i\right)^2 - \frac{1}{n-1} \sum_{j \neq i} \left(s_i - s_j\right)^2 .$$

Note that a developer pays two costs: one for being away from her ideal specification, and one for being away from the specification of other software (cost of incompatibility). Note also that in the normal-form game, the software developers are the players; each chooses a strategy s_i from real line, and the payoff function of player i is u_i.

Notice that the payoff functions are supermodular:

$$\frac{\partial^2 u_i}{\partial s_i \partial s_j} = \frac{2}{n-1} > 0,$$

while $\partial^2 u_i / \partial s_k \partial s_j = 0$ when $i \neq j \neq k$. Hence, if one can bound the strategies, the game becomes supermodular.

For each player i, the best response function is[4]

$$B_i\left(s_{-i}\right) = \frac{\theta_i}{2} + \frac{1/2}{n-1} \sum_{j \neq i} s_j .$$

[4] The first-order condition is

$$-2\left(s_i - \theta_i\right) - \frac{2}{n-1} \sum_{j \neq i} \left(s_i - s_j\right) = 0,$$

which simplifies to the displayed formula.

The Nash equilibria is computed by solving the linear equation system, $s_i = B_i(s_{-i})$, $i \in N$. Toward solving it, sum these equations up over i and obtain $\sum_i s_i = \sum_i \theta_i$. Substituting this in the above equation, one obtains the unique Nash equilibrium strategy

$$s_i^* = \frac{n-1}{2n-1}\theta_i + \frac{1}{2n-1}\sum_{j=1}^{n}\theta_j.$$

In equilibrium, a software developer chooses, roughly, the average of her own ideal specification and the average $\sum_{j=1}^{n}\theta_j/n$ of all ideal specifications, including her own.

What about the rationalizable strategies? Without any bound on rationalizable strategies, every strategy is rationalizable as one can see from the best response functions. If the strategies are bounded, then rationalizable strategies are bounded by the equilibrium as follows. For example, suppose that players cannot select a specification below some $\underline{s} \leq \min_i \theta_i$. Then, following the methodology in Section 7.4.1, one can iteratively eliminate the strategies below the equilibrium strategies s^*:

$$s^* = \lim_{k\to\infty} B^k(\underline{s}, \ldots, \underline{s}).$$

If there is also an upper bound $\bar{s} > \max_i s_i^*$, then the game is dominance solvable, with s^* as the unique rationalizable strategy profile.

7.5.4 Competition in Research and Development

Two start ups, named Firm 1 and Firm 2, are competing for leadership in a software market. The leader wins, and the other loses. Each firm can invest some $x \in [0.001, 1]$ unit for research and development by paying cost of $x/4$. If a firm invests x units and the other firm invests y units, the former wins with probability $x/(x+y)$. Therefore, the payoff of the former start up will be

$$\frac{x}{x+y} - x/4.$$

All these are common knowledge.

The revenues of the leader and the follower are 1 and 0, respectively. These numbers are multiplied with their respective probabilities, and the final payoff above is obtained after subtracting the cost of research. This is represented as a normal-form game, by taking the players as Firm 1 and Firm 2, strategy set of each player as $[0.001, 1]$, and the payoff function as above.

The payoff functions are not supermodular (the cross-derivative is $(x-y)/(x+y)^3$), but the analysis of this game mirrors the analysis of a supermodular game, thanks to increasing best response functions and concavity of utility functions with respect to own strategy.

The best response function of a firm is[5]

$$B(y) = 2\sqrt{y} - y.$$

Note that $B(y) > y$ whenever $y < 1$. Therefore, the graphs of best responses intersect each other only at $x = y = 1$. Therefore, $(1,1)$ is the only Nash equilibrium.

As it turns out, the game is dominance solvable, and $(1,1)$ is the only rationalizable strategy profile. Indeed, since $y \geq y_0 \equiv 0.001$, any strategy $x < B(y_0)$ is strictly dominated by $x_1 = B(y_0)$, and therefore eliminated. Write also $x_0 = y_0$ and $x_1 = y_1$. Now, the remaining strategy space of each player is $[x_1, 1]$. Note that $x_1 = B(.001) > 0.001 = x_0$. Now, similarly, one can eliminate any strategy $x < x_2 \equiv B(y_1)$. Applying this iteratively, after nth elimination, the remaining strategy space is $[x_n, 1]$ where

$$x_n = B(x_{n-1}) = 2\sqrt{x_{n-1}} - x_{n-1}$$

and $x_0 = .001$. The mapping B has a unique fixed point: $x^* = 1$, and $x_n \to 1$ as $n \to \infty$.[6] Hence, only strategy 1 survives iterated strict dominance.

7.5.5 Political Competition

Two candidates, Alice and Bob, are running for a political office. Simultaneously, Alice and Bob invest $x_A \geq 0$ and $x_B \geq 0$ for their campaigns, respectively. Alice wins with probability

$$p_A(x_A, x_B) = \frac{x_A}{\alpha + x_A + x_B};$$

[5] This immediately follows from the first-order condition:

$$0 = \frac{\partial}{\partial x}\left(\frac{x}{x+y} - x/4\right) = \frac{y}{(x+y)^2} - 1/4.$$

[6] A fixed point of a mapping B is defined as x such that $B(x) = x$. To see the limit, observe that $x_n = B(x_{n-1}) > x_{n-1}^{1/2}$. Hence,

$$1 > x_n > x_0^{(1/2)^{n-1}}.$$

As $n \to \infty$, since $(1/2)^{n-1} \to 0$, $x_0^{(1/2)^{n-1}} \to 1$. Therefore, $x_n \to 1$.

Bob wins with probability

$$p_B(x_A, x_B) = \frac{x_B}{\alpha + x_A + x_B},$$

and with remaining probability $\alpha/(\alpha + x_A + x_B)$, a third party candidate wins, where $\alpha > 0$ is a fixed small number. For each party, the value of winning is 1 and the cost of investment is x, so that the expected payoff of Alice and Bob are $p_A(x_A, x_B) - x_A$ and $p_B(x_A, x_B) - x_B$, respectively.

As in the case of competition in R&D, the game is not supermodular, and the best response functions are not increasing. However, the game still turns out to be dominance solvable.

Towards computing the unique Nash equilibrium, one can take the derivative of $p_A - x_A$ with respect to x_A and set it equal to zero, obtaining the first-order condition for Alice

$$\alpha + x_B = (\alpha + x_A + x_B)^2. \tag{7.2}$$

Similarly, the first-order condition for Bob is

$$\alpha + x_A = (\alpha + x_A + x_B)^2. \tag{7.3}$$

Comparing the two equations, conclude that $x_A = x_B$. Substituting this equality in (7.2), conclude further that, in equilibrium, x_A solves

$$\alpha + x_A = (\alpha + 2x_A)^2.$$

There is only one non-negative solution to this quadratic equation:

$$x^* = \frac{1 - 4\alpha + \sqrt{1 + 8\alpha}}{8}.$$

The unique Nash equilibrium is given by $x_A = x_B = x^*$.

As it turns out, this game is dominance-solvable, with x^* as the unique rationalizable strategy. From (7.2) and (7.3), the best response functions of Alice and Bob are

$$\begin{aligned} x_A^{BR}(x_B) &= \sqrt{x_B + \alpha} - (x_B + \alpha) \\ x_B^{BR}(x_A) &= \sqrt{x_A + \alpha} - (x_A + \alpha). \end{aligned}$$

The best-response functions are plotted in Figure 7.6. Since the utility functions are strictly concave, if $x_B > x_B^{BR}(x_A)$ for each x_A, then x_B is strictly dominated (see Section

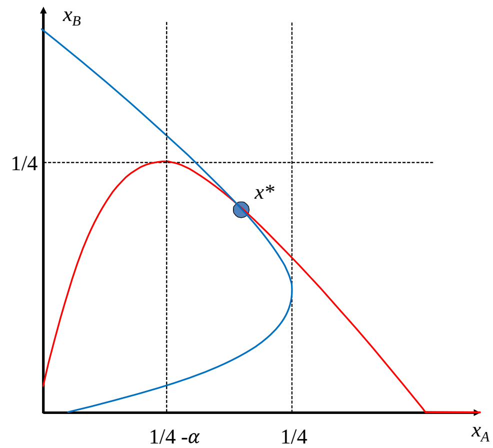

Figure 7.6: Best respponse functions in political competition

6.2.1). For example, any $x_B > 1/4$ is strictly dominated by $x_B = 1/4$. Similarly, any $x_A > 1/4$ is strictly dominated by $x_A = 1/4$. Hence, all strategies $x_A > 1/4$ and $x_B > 1/4$ are eliminated in the first round. In the next round, one eliminates the strategies $x_A < x_A^{BR}(0) = \sqrt{\alpha} - \alpha$ and $x_B < \sqrt{\alpha} - \alpha$. This is because all such strategies are now strictly dominated by $\sqrt{\alpha} - \alpha$. Elimination continues in this fashion iteratively. The elimination process stops at x^*, and $x_A = x_B = x^*$ are the only rationalizable strategies.[7]

[7] Here is a mathematical proof: Since the game is symmetric, the set of rationalizable strategies is the same for both players; call that set R. Recall from Exercise 4.6 that no rationalizable strategy is strictly dominated when we restrict the other player's strategies to be rationalizable. That is, for each $x \in R$, there exists $y \in R$ such that $x = x_A^{BR}(y) = x_B^{BR}(y)$. Suppose that $\min R < 1/4 - \alpha$. Now, $\min R = x_B^{BR}(y)$ for some $y \in R$. Since x_B^{BR} is "single-peaked", either $y = \min R$ or $y = \max R$. But, since $x^* \leq \max R \leq 1/4$, as in the figure, $x_B^{BR}(\max R) > 1/4 - \alpha$, showing that $y \neq \max R$. Hence, $y = \min R$, i.e., $\min R = x_B^{BR}(\min R)$. But this is a contradiction because it implies that $(\min R, \min R)$ is a Nash equilibrium. All in all, this proves that $\min R \geq 1/4 - \alpha$. Then, x_B^{BR} is strictly decreasing on R. That is, $\min R = x_B^{BR}(\max R)$ and $\max R = x_A^{BR}(\min R)$, i.e., $(\max R, \min R)$ is a Nash equilibrium, showing that $\max R = \min R = x^*$.

7.6 Exercises with Solutions

Exercise 7.1. Consider the investment game

	Invest	Stay Out
Invest	x_1, x_2	$x_1 - 1, 0$
Stay Out	$0, x_2 - 1$	$0, 0$

Represent this game as a supermodular game and show that the largest and the smallest equilibria are weakly increasing in x_1 and x_2.

Solution. Write 1 for Invest and 0 for Stay Out. Then,

$$u_1(1,1) - u_1(0,1) = x_1 > x_1 - 1 = u_1(1,0) - u_1(0,0),$$

showing that the payoffs are supermodular—with strategy sets $S_1 = S_2 = \{0,1\}$. Since $u_1(1,1) - u_1(0,1) = x_1$ and $u_1(1,0) - u_1(0,0) = x_1 - 1$ are increasing in x_1, the parameters x_1 and x_2 are complementary to the strategies, and hence the extremal equilibria are weakly increasing x_1 and x_2. (Compute these equilibria as a function of $x_1 = x_2$.)

The following counterexample shows that the comparative statics in Theorem 7.3 may fail when the strategy space is not bounded.

Exercise 7.2. There are three partners, namely 1, 2, and 3. Simultaneously, each partner i puts effort $e_i \geq 0$, producing output $e_1 e_2 e_3$, which is sold at price p, and costing ce_i^2 to i. The partners share the output equally; the payoff of i is

$$u_i = \frac{p}{3} e_1 e_2 e_3 - ce_i^2.$$

Compute the symmetric equilibria, and show that they are weakly *increasing in cost* c and weakly *decreasing in price* p. Compare this to Theorem 7.3.

Solution. The first-order condition is

$$e_i = \frac{p}{3c} e_j e_k.$$

Thus in any symmetric equilibrium $(\hat{e}, \hat{e}, \hat{e})$, we have

$$\hat{e} = \frac{p}{3c} \hat{e}^2.$$

Thus, there are two symmetric equilibria $(0,0,0)$ and (e^*,e^*,e^*) where

$$e^* = \frac{3c}{p}.$$

Clearly, e^* is decreasing in price p and increasing in cost c. This is counterintuitive because the payoffs are supermodular in $(e_1, e_2, e_3, p, -c)$. Thus, if there were an upper bound $\bar{e}$ on effort levels, Theorem 7.3 would imply that the extremal equilibria are weakly increasing in price p and weakly decreasing in cost c. The reason for the counterintuitive comparative statics is that there is "an equilibrium at infinity", which is not feasible, and the equilibrium (e^*,e^*,e^*) is a "middle equilibrium", exhibiting the properties of such middle equilibria in Remark 7.1. Formally, with a bound $\bar{e}$, there would be three equilibria: $(0,0,0)$, (e^*,e^*,e^*), and $(\bar{e},\bar{e},\bar{e})$. The extremal equilibria would be constant, and the middle equilibrium would move in a counterintuitive direction (as in Figure 7.2).

Exercise 7.3. There are n partners working on a joint project. Simultaneously, each partner i exerts effort level $e_i \in [0,1]$. The project succeeds with probability

$$P(e_1, \ldots, e_n) = e_1 \cdot \cdots \cdot e_n.$$

The payoff of partner e_i is $\theta - e_i^\gamma$ if the project succeeds and $-e_i^\gamma$ otherwise where θ and γ are known parameters with $\gamma > \theta > 1$.

1. Write this in normal form and show that it is a supermodular game.

2. Compute the set of Nash equilibria and briefly discuss how Nash equilibria vary with θ, γ, and n. (Hint: pay close attention to the case $n > \gamma$.)

3. Compute the set of rationalizable strategies.

Solution. (Part 1) The game is (N, S, u) where $N = \{1, \ldots, n\}$ is the set of players, $S = [0,1]^n$ is the set of strategy profiles $(e_1, \ldots, e_n)$, and

$$u_i(e_1, \ldots, e_n) = \theta e_1 \cdot \cdots \cdot e_n - e_i^\gamma$$

is the utility function of player i. Clearly, each $S_i = [0,1]$ is closed and bounded and u_i is continuous. Therefore, it suffices to check that u_i is supermodular. Indeed,

$$\frac{\partial^2 u_i}{\partial e_j \partial e_k} = \theta \geq 0$$

for all distinct j and k, showing that u_i is supermodular.

(Part 2) Since the utility functions are strictly concave in own effort level, all Nash equilibria are in pure strategies. The first-order condition for Nash equilibrium is

$$\theta \prod_{j \neq i} e_j = \gamma e_i^{\gamma-1} \qquad \text{(for each } i\text{)}.$$

One can easily show that all equilibria must be symmetric. Indeed, suppose e_i and e_k are the largest and smallest effort levels in an equilibrium. Then,

$$e_i^{\gamma-1} = \frac{\theta}{\gamma} \prod_{j \neq i} e_j \leq \frac{\theta}{\gamma} \prod_{j \neq k} e_j = e_k^{\gamma-1},$$

showing that $e_i = e_k$. Therefore, all equilibria are of the form $(\hat{e}, \dots, \hat{e})$ where

$$\theta \hat{e}^{n-1} = \gamma \hat{e}^{\gamma-1}.$$

Clearly, one equilibrium is $(0, 0, \dots, 0)$. When $\gamma = n$, this is the unique solution. For $\gamma \neq n$, there is another solution to this equation:

$$\hat{e} = \left(\frac{\theta}{\gamma}\right)^{1/(\gamma-n)}.$$

When $\gamma > n$, $\hat{e} \in (0, 1)$, and $(\hat{e}, \dots, \hat{e})$ is also an equilibrium. When $\gamma < n$, $\hat{e} > 1$, and $(\hat{e}, \dots, \hat{e})$ is not an equilibrium; the unique Nash equilibrium is $(0, \dots, 0)$. The smallest equilibrium, $(0, \dots, 0)$ is independent of the parameters θ, γ, and n. When $\gamma > n$, each player plays $(\theta/\gamma)^{1/(\gamma-n)}$ in the largest equilibrium, and this effort level is increasing in θ and decreasing in n. The effort level is increasing in γ if and only if $\gamma \leq n + \ln(\gamma/\theta)$. Note that, when $\gamma < n$, the expression $(\theta/\gamma)^{1/(\gamma-n)}$ is decreasing in θ, seemingly contradicting Theorem 7.3, but it is not an equilibrium effort for these parameter values.

(Part 3) By Theorem 7.2, when $\gamma \leq n$, the game is dominance solvable and the unique rationalizable strategy is $(0, \dots, 0)$. When $\gamma > n$, Theorem 7.2 implies that rationalizable strategies are within $\left[0, (\theta/\gamma)^{1/(\gamma-n)}\right]$. Moreover, each $e_i \in \left[0, (\theta/\gamma)^{1/(\gamma-n)}\right]$ is a best response to each player putting effort level $(\gamma/\theta)^{1/(n-1)} e_i^{(\gamma-1)/(n-1)}$, which is in $\left[0, (\theta/\gamma)^{1/(\gamma-n)}\right]$. Therefore, by Exercise 4.6, all of these strategies are rationalizable.

7.7 Exercises

Exercise 7.4. In Section 7.5.1, compute the rationalizable strategies for the case

1. $\alpha = \beta = 1/2$, $c > 1/4$, $r > 1/4$;

2. $\alpha = \beta = 1/2$, $c = r = 1/4$;

3. $\alpha = \beta = 1/2$, $c < 1/4$, $r < 1/4$.

Exercise 7.5. Alice and Bob seek each other. Simultaneously, Alice puts effort $s_A \in [0,1]$ and Bob puts effort $s_B \in [0,1]$ to search. The probability of meeting is $s_A s_B$; the value of the meeting for each of them is v, and the search costs s_A^3 to Alice and s_B^3 to Bob.

1. Find the Nash equilibria of this game.

2. How do the search efforts in equilibrium vary with v?

3. Take $v = 1$ and compute all rationalizable strategies.

Exercise 7.6. In Exercise 6.5, assume that government in Country 1 sets the tariff τ_1 in order to maximize

$$\gamma CS_1 + \kappa\pi_1 + T_1$$

foreseeing equilibrium strategies under (τ_1, τ_2) where $CS_1 = (\theta_1 - P_1)^2/2$ is the consumer surplus in Country 1, π_1 is the profit of Firm 1, and $T_1 = \tau_1 P_1 q_{21}$ is the tax revenue. How does the optimal tariff for government vary with parameters γ and κ? Briefly discuss the implication of your finding to how political base of a government affects its trade policy.

Exercise 7.7. In Exercise 7.2, assume that the strategy set is $[0, \bar{x}]$. Compute the sets of rationalizable strategies and Nash equilibria

1. for $\bar{x} > 3c/p$, and

2. for $\bar{x} < 3c/p$.

Exercise 7.8. Consider a two-player game in which each player's strategy is a real number $x \in [0,1]$. A player's payoff is

$$-(x - y/2 - 1/4)^2$$

where x is his own strategy and y is the strategy chosen by the other player. Compute the sets of all Nash equilibria and all rationalizable strategies.

Exercise 7.9. In the software development game in Section 7.5.3, compute the set of rationalizable strategies for the following cases:

1. the set of strategies is all real numbers;

2. the set of strategies is $[0,1]$ and $\theta_i \in (0,1)$ for each i.

Exercise 7.10. Redo the analysis in Section 7.5.5 for the easier case of $\alpha = 0$.

Exercise 7.11. There are $n \geq 2$ spammers who are competing for the attention of a customer, who is not a player in this game. Simultaneously, each spammer chooses an intensity $x_i \in [\underline{x}, \bar{x}]$ at which he spams the customer. The customer opens the spam from spammer i with probability

$$\frac{x_i}{x_1 + \cdots + x_n}.$$

The payoff of spammer i is $\theta - x_i$ if the customer opens the spam from i and $-x_i$ otherwise. Assume that θ, $\underline{x}$ and $\bar{x}$ are known positive numbers where $\underline{x}$ is very small and $\bar{x}$ is very large (e.g., $0 < \underline{x} < \theta/n^2$ and $\bar{x} > \theta$).

1. Compute a Nash equilibrium of this game, and briefly discuss how equilibrium varies with θ and n.

2. Compute the set of rationalizable strategies for $n = 2$ and $n = 6$. [Hint: this game is **not** supermodular.]

Exercise 7.12. There are two firms that are in a price competition but also need to advertise in order to create the market for their product. Simultaneously, each firm $i \in N = \{1,2\}$ chooses advertisement level $a_i \in [0,1]$ and price $p_i \in [0,\bar{p}]$ for some large $\bar{p}$ and then each firm i sells amount of

$$Q_i = a_1 + a_2 - p_i + bp_j$$

where $b \in (0,1)$. (All variables a_1, a_2, p_1, p_2 are set at the same time.) The payoff of firm i is $p_i Q_i - \alpha a_i^2$, where $\alpha > 0$ is a known parameter. Write this formally as a game in normal form, and compute the sets of all Nash equilibria and all rationalizable strategies.

Exercise 7.13. There are two firms that are in a quantity competition but also need to advertise in order to create the market for their product. Simultaneously, each firm $i \in N = \{1, 2\}$ chooses advertisement level $a_i \in [0, 1]$ and production level $q_i \geq 0$, and then sells it at price

$$P = a_1 + a_2 - q_1 - q_2.$$

(All variables a_1, a_2, q_1, q_2 are set at the same time.) The payoff of firm i is $Pq_i - \alpha a_i^2$, where $\alpha \in (0, 1/4)$ is a known parameter. Write this formally as a game in normal form, and compute the sets of all Nash equilibria and all rationalizable strategies.[8] How would your answer change if you had $\alpha \in (1/3, 1/2)$ instead?

Exercise 7.14. There are $n \geq 2$ students in a class. Simultaneously, each student i exerts effort $e_i \in [0, 2]$, and obtains the grade

$$g_i(e_1, \ldots, e_n) = e_i - \frac{e_1 + \cdots + e_n}{2n}.$$

The payoff of a student i is $u_i(e_1, \ldots, e_n) = 2 \log(1 + g_i(e_1, \ldots, e_n)) - e_i$. Write this formally as a game in normal form, and compute the sets of all Nash equilibria and all rationalizable strategies. (**Hint:** If $f(x) = \log(x)$, then $f'(x) = 1/x$. If $h(x) = f(g(x))$, then $h'(x) = f'(g(x))\, g'(x)$.)

[8] Hint: In computing a best response, it may be useful to compute the best q_i as a function of (a_i, a_j, q_j) and find the best a_i after substituting the best q_i in the utility function.