Chapter 15

Dynamic Bayesian Games

Players may have private information while playing a dynamic game. For example, in an English auction, the bidders typically have private information and bid dynamically, adjusting their bids or dropping out as they observe how the other bidders bid. In a price negotiation for a used good, such as a car or a house, the buyer may privately know how much she needs the good while the seller may have superior information about the quality of the good; they negotiate by making offers and counteroffers repeatedly until they reach an agreement or one of them walks out. In yet another example, employees may have private information about their abilities and preferences and may make many important observable choices such as education attainment and area of study before they go on job market.

In all these examples, as they observe the other players' actions, the players update their beliefs about what the other parties may know and react to the information revealed, and the players take this information transmission into account when they take their actions. For example, in an English auction, the players will learn about the other players' valuations from their bids, and a bidder may bid much higher than required to stay in the auction, in the hopes that the other players will take such a "jump bid" as an evidence for excessively high valuation and will drop out, allowing the player to buy the good at a lower price than the price she would end up paying otherwise. In a price negotiation, the buyer may reject a reasonably good price or stall the negotiation in the hopes that this will convince the seller that she does not need the good that much, persuading him to agree to an even lower price. Likewise, the seller may stall the negotiations to signal the buyer that the good is of high quality, and the seller is not willing to sell it for a low price. A worker who knows her own abilities may take costly actions to reveal it, e.g., by going to a graduate school or taking challenging courses. Likewise, a prosecutor may let a criminal walk away by concealing a damning evidence, as the revelation of the evidence may jeopardize a more important ongoing investigation.

This chapter introduces a class of dynamic games with incomplete information. In these games, each player observes a type, and her type remains her private information throughout the game. The actions are publicly observable. When it is a player's turn to move, she knows what players have done so far, and her only uncertainty is about what other players' types are. (Simultaneous moves as in multi-stage games are introduced in an extension.) Many dynamic applications with incomplete information are of this form, and the later chapters will present many such applications. The chapter introduces a new solution concept, called *perfect Bayesian equilibrium*, for these games; the solution concept is especially useful in understanding informational issues as it explicitly describes how players belief evolve as the game proceeds.

Bayesian Nash equilibrium can also be used to analyze the dynamic games of incomplete information. Unfortunately, Bayesian Nash equilibrium has two drawbacks in dynamic games, stemming from its inherently static formulation. First, a Bayesian Nash equilibrium does not explicitly specify how the beliefs evolve as the game proceeds, making it difficult to study information transmission and related concerns discussed above. Second, as in the case of Nash equilibrium in dynamic games with complete information, Bayesian Nash equilibrium allows players to take suboptimal actions at information sets that are not deemed to be reached.

For example, consider the game in Figure 15.1 from Chapter 14, where $p > 1/2$. As explained in Chapter 14, this game can be modeled as a Bayesian game, where the worker has two types. This game is meant to describe a situation in which a firm does not know whether a worker is of high ability or low ability; under the employment contract a high ability worker prefers to work while a low-ability worker prefers to shirk. However, there is a Bayesian Nash equilibrium in which, if hired, the worker would shirk regardless her ability, and anticipating this, the firm passes on the worker. Clearly, a high-ability worker's shirking is against her preferences. Nonetheless it is consistent with Bayesian Nash equilibrium because every strategy of the worker is a best reply to the strategy

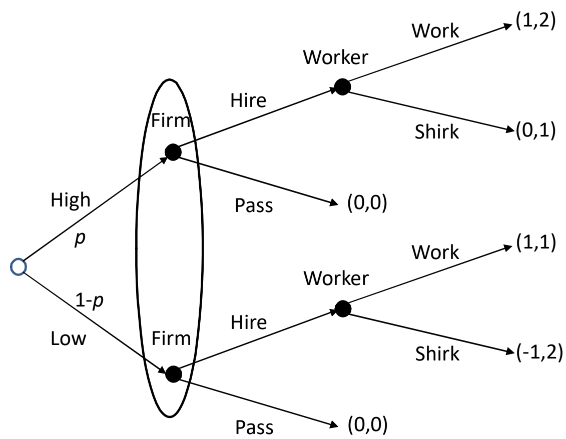


Figure 15.1: A Bayesian Nash equilibrium in which player Worker plays a suboptimal action.

Pass. (The worker gets 0 no matter what strategy she plays.) In order to solve this problem, assume that players are *sequentially rational*, i.e., they play a best response at every information set, maximizing their expected payoff knowing that they are at the information set. For example, when she is to move, the high-ability worker knows that Nature has chosen "High" and the firm has chosen "Hire". She must then play Work, as Work is the only best response under that knowledge. This would lead to the other equilibrium: firm hires, and the worker works if she is of high ability and shirks otherwise.

In this example, the latter equilibrium is the only subgame-perfect Nash equilibrium, and one could use subgame-perfect Nash equilibrium to rule out the former equilibrium. Unfortunately, in a typical dynamic game of incomplete information, there is no proper subgame, and subgame-perfect Nash equilibrium does not help; perfect Bayesian Nash equilibrium applies the idea of sequential rationality directly to solve these games.

For an illustration, in the above example, imagine that the firm is desperate to hire a worker, even a low-ability worker, but the worker needs to apply to the job in order to be hired, and application is somewhat costly. As in Figure 15.2, the firm gets v if it does not hire the worker, and the worker gets -1 if she applies to the job and gets turned down. If she opts out and does not apply, then she will get her outside option. The value of her outside option is w if she is of high type and zero otherwise. For this

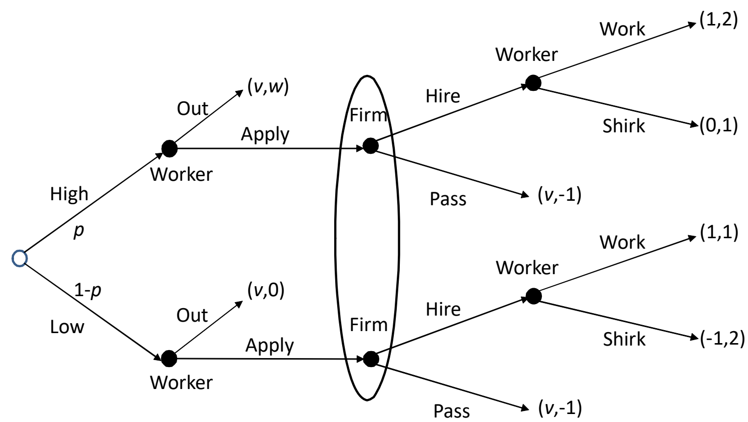


Figure 15.2: A hiring game with application process

example, take $v = -2$ and $w = 0$, so that the firm would like to hire the worker even if she is of low type. Now, as in the previous example, once hired, the worker should work if she is of high type and shirk if she is of low type. The firm does not know the worker's type, but that does not matter for its decision because its payoff from turning her down ($v = -2$) is lower than what it would get in either case. Thus, the firm must hire the worker if the worker applies to the job. Then, at the beginning, the worker should apply regardless of her type. Let us call this the *reasonable equilibrium*. There is, however, another subgame-perfect Nash equilibrium, which can be called *unreasonable*. In this equilibrium, as before, the worker would work if she is of high type and shirk if she is of low type. However, the worker does not apply to the job and the firm would have turned her down if she applied. This is a Nash equilibrium because the worker would lose 1 by applying, and the firm's strategy does not affect its payoff because its information set is not reached. This is also a subgame-perfect Nash equilibrium because the only proper subgames are the two decision nodes of the worker at which she decides whether to work or shirk after being hired.

In this example, the firm's belief about the worker's type at its information set did not matter because the option of not hiring was always worse. In general, the firm's optimal decision depends on its beliefs. This leads to the following fundamental question. What should the firm think about the worker's type if the worker applies to the job? In the reasonable equilibrium above, the firm must assign probability p to high type using the

Bayes rule, assuming that the firm knows what strategy the worker is playing. Indeed, in that case, since both types apply with probability 1, the probability that worker applies is 1:

$$\Pr(\text{Apply}) = \Pr(h)\Pr(\text{Apply}|h) + \Pr(l)\Pr(\text{Apply}|l) = p \times 1 + (1-p) \times 1 = 1.$$

Hence, the conditional probability of high type after application is

$$\Pr(h|\text{Apply}) = \frac{\Pr(h)\Pr(\text{Apply}|h)}{\Pr(\text{Apply})} = \frac{p}{1} = p.$$

On the other hand, in the unreasonable equilibrium, the information set of the firm is not reached, and the Bayes rule allows any belief, and we cannot say what firm should believe. Indeed, if one tries to compute the conditional probability of high type for the unreasonable equilibrium, she would get

$$\Pr(h|\text{Apply}) = \frac{\Pr(h)\Pr(\text{Apply}|h)}{\Pr(h)\Pr(\text{Apply}|h) + \Pr(l)\Pr(\text{Apply}|l)} = \frac{0 \cdot p}{0 \cdot p + 0 \cdot (1-p)} = \frac{0}{0}$$

because both types apply to the job with zero probability in this equilibrium. As this example illustrates, the beliefs depend on the equilibrium, and in particular they depend on the players' prescribed moves by the purported equilibrium prior to the information set. Since a player's behavior depends on her beliefs, this prevents one from using backward induction-like procedures to solve these games.

One may ask: why doesn't the firm simply assume that the worker is of high type with probability p? To answer this question, suppose that the worker's payoff from outside option is 3 for high type and 0 for the low type, and the firm gets 0 if it turns down the worker. The rest of the game is as in Figure 15.2. It is a best response for the firm to hire the worker if and only if the firm assigns at least probability $1/2$ to the high type, i.e., $\Pr(h|\text{Apply}) \geq 1/2$. Now suppose that the firm naively thought that the probability of high type is $p > 1/2$ (as in the beginning of the game) and hired her. Then, in equilibrium, anticipating this behavior, a worker of low type will definitely apply to the job and shirk once she is hired. With payoff of 2, this is clearly better than the alternative of opting out and getting zero. On the other hand, a worker of high type will not apply because her outside option is better than what she can get in this job. If the firm understands this behavior, then it must assign zero probability on high type (i.e. $\Pr(h|\text{Apply}) = 0$) and turn her down. Otherwise, sooner or later, it would realize that all the workers it hired are of low ability. (Now we figured that the firm should not hire, what should the firm believe when it sees an application? Well if it is known that the firm does not hire, no worker must apply to this job as the application is costly. Then, the firm can have any belief after seeing an application. Should it believe that the worker is of high type and hire her?)

Understanding how players update their beliefs as they play the game is fundamental to the strategic analysis of dynamic games with incomplete information. Accordingly, the perfect Bayesian equilibrium will explicitly specify a strategy profile and a belief system that describes what players believe about the other players' types at each history. It will require that the strategies are sequentially rational under the specified beliefs and beliefs are derived from the strategies using the Bayes' rule whenever it is applicable.

15.1 Formulation

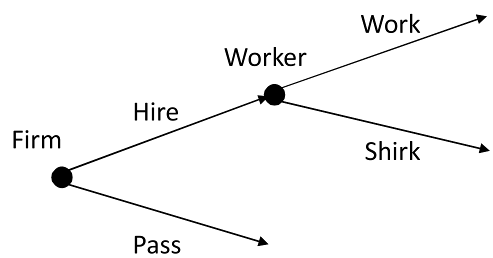
A *dynamic Bayesian game* (with publicly observed actions) consists of

- a set of players $N = \{1, \ldots, n\}$,
- a set of type profiles $T = \{T_1, \ldots, T_n\}$,
- a probability distribution p_i on T_i for each player i,
- a game tree (in which all past actions are known),
- an assignment of each non-terminal history to a player (including Nature), and
- a payoff function $u_i : T \times Z \to \mathbb{R}$ for each player where Z is the set of terminal nodes.

As in Bayesian games, Nature first picks a type profile $t = (t_1, \ldots, t_n) \in T$ and each player i observes t_i (but not the other players' types), and then they play the game described by the game tree—in which they observe the actions taken by each player and never observing the other player's types. If players did not have private information, this would be a game of perfect information. Each information set at which player i moves has as many nodes as T_{-i}, each node corresponding to a $t_{-i} \in T_{-i}$. The next example illustrates the formulation using the hiring game in the introduction.

Example 15.1. The game in Figure 15.1 can be represented as a dynamic Bayesian game as follows:

- the set of players is $N = \{\text{Firm, Worker}\}$;
- the set of types is $T_F = \{t_F\}$ for the firm and $T_W = \{H, L\}$ for the worker, where t_F is a dummy type, and H and L stand for high-ability and low-ability, respectively;
- the probability of H is $p_W(H) = p$, and the probability of L is $p_W(L) = 1 - p$;
- the "game tree" and the assignment of players are as follows



- the payoffs are as given in Figure 15.1, as a function of types.

Note that the "game tree" here describes only the order of actions, without describing the chance moves and players' uncertainty about those moves. The extensive-form game associated with the dynamic Bayesian game contains also Nature's selection of types and the information sets associated with players' uncertainty—as described above. In particular, the extensive-form game associated with the dynamic game above is as plotted in Figure 15.1.

There are a couple of specific assumptions made in the above formulation—for simplicity of exposition. First, it is assumed that the payoffs of types are independently selected. That is, probability of selecting type profile t is $p_1(t_1) \times p_2(t_2) \times \cdots \times p_n(t_n)$. In most applications in this book, there are only two players and the independence assumption will not play a significant role, but it simplifies the general analysis. Second, the formulation assumes that the "game tree" does not depend on the types, and hence the players do not have private information about the availability of some actions. (Section 15.3 below allows simultaneous moves—as in multi-stage games; Chapter 17 presents an extension that applies to arbitrary games in extensive-form.)

15.2 Perfect Bayesian equilibrium

This section formally introduces Perfect Bayesian equilibrium (henceforth PBE), the main solution concept that will be used for the analyses of the dynamic Bayesian games with observable actions. The novelty of the perfect Bayesian equilibrium is that the beliefs are explicitly stated in addition to the strategy profile as part of the solution concept. This allows one to directly impose sequential rationality requirement.

Formally, a *strategy* s_i of a player i is a type-contingent plan of action that specifies how each type plays the game. It assigns a move $s_i(t_i, h) \in A(h)$ to each history h at which player i moves; this is the action that will be taken by the type t_i at history h according to s_i. Likewise, a mixed strategy σ_i of a player i is a type-contingent mixed plan of action that assigns a probability distribution $\sigma_i(\cdot|t_i, h)$ on the available moves $A(h)$ at each history h at which player i moves. The key feature of this definition is that a player's strategy depends on her own type, so that each type has a plan of action.

A solution will also explicitly specify the players' beliefs about the other players' types, describing how the players will update their beliefs about the other players' types as they see how they play the game. Formally, a *belief system* b is a system of conditional probability distributions $b(\cdot|h)$ that assigns a probability distribution $b(\cdot|h)$ on T for each history h. Since the types are independently distributed, the conditional probabilities of types will be independently distributed, so that the conditional probability $b(t|h)$ of a type profile t at a history h will be the product of the conditional probabilities $b_i(t_i|h)$ of types t_i:

$$b(t|h) = b_1(t_1|h) \cdot \cdots \cdot b_n(t_n|h).$$

The solution will be a pair (σ, b) of a strategy profile σ and a belief system b:

Definition 15.1. An *assessment* is a pair (σ, b) of a strategy profile σ and a belief system b.

One can derive a belief system b from a strategy profile σ by computing the conditional probabilities of types at every history as follows. Imagine that the conditional probability distribution at a history h is $b(t|h)$ and a player i moves at history h, taking an action a. Now, since player i only knows about t_i, the beliefs about the other players' types should not be affected by what player i chooses at this stage. Therefore, the beliefs about the other players' types t_j remains as is:

$$b_j(t_j|h,a) = b_j(t_j|h) \qquad (\text{for all } j \neq i,\ t_j \in T_j). \tag{15.1}$$

Beliefs about types t_i will be updated, and the updating depends on what strategy player i is supposed to play. Suppose that strategy profile σ is being played. At history h, under belief $b_i(\cdot|h)$, action a will be played with probability

$$\Pr(a|\sigma_i,h) = \sum_{t_i \in T_i} \sigma_i(a|t_i,h)\, b_i(t_i|h). \tag{15.2}$$

To obtain this formula, one first computes the probability that player i is of a give type t_i and plays a. The latter probability is $\sigma_i(a|t_i,h)\, b_i(t_i|h)$ because player i is of type t_i with probability $b_i(t_i|h)$ and she plays a with probability $\sigma_i(a|t_i,h)$ in that case. One then adds up these probabilities over types of player i to obtain the total probability of playing a, as in the formula. (The total probability is computed by integration when there is a continuum of types.) Now, if the probability $\Pr(a|\sigma_i,h)$ is positive, the conditional probability is computed using the Bayes rule as follow:

$$b_i(t_i|h,a) = \frac{\sigma_i(a|t_i,h)\, b_i(t_i|h)}{\Pr(a|\sigma_i,h)} \qquad \text{if } \Pr(a|\sigma_i,h) > 0. \tag{15.3}$$

If a were not supposed to be played by i at history h regardless of her type, i.e., if $\Pr(a|\sigma_i,h) = 0$, then the above ratio is undefined, and the conditional distribution $b_i(t_i|h,a)$ can be arbitrary.

Observe that the conditional beliefs after a move a with $\Pr(a|\sigma_i,h) > 0$ are uniquely determined by the strategy profile. This also determines the beliefs at some off-path histories h. In particular, if history h is reached only because some other players deviated from σ—i.e. if history h could be reached with positive probability if i followed σ_i and others played some other strategy—then beliefs $b_i(t_i|h)$ is computed by using the Bayes rule and σ_i throughout. On the other hand, if history h is reached because player i deviated, then the beliefs about player i can be somewhat arbitrary. Hence, there can be many belief assessments derived from a strategy profile by varying the conditional distribution after those moves. Since such unexpected moves are common in games, the multiplicity can be large. In some of those equilibria, a player i may avoid an action because she is afraid that the move will lead the other players to hold some beliefs that will eventually hurt her, and those beliefs are arbitrary because she is not supposed to take that action regardless of her type.

Formally, perfect Bayesian equilibrium is defined by sequential rationality and the belief updating above as follows.

Definition 15.2. An assessment (σ, b) is said to be *sequentially rational* if at each history h, the player i who is to move at h maximizes her expected utility

1. given her type t_i and her beliefs $b(\cdot|h)$ about the other players' types at history h, and

2. given that the players will play according to σ in the continuation game.

Definition 15.3. An assessment (σ, b) is said to be a *perfect Bayesian equilibrium* (henceforth PBE) if it is sequentially rational and satisfies (15.1) and (15.3) throughout.

It is important to keep in mind that, unlike earlier solution concepts, perfect Bayesian equilibrium is an assessment, not just a strategy profile. To compute an equilibrium, one must find a strategy for each player and a belief system that assigns a conditional probability for each type at each history. The belief system is derived using the Bayes rule but at histories after a deviation by a player, the beliefs about that player is arbitrary, and hence the belief system is an important part of the solution that needs to be stated explicitly in addition to the strategies.

Example 15.2. In Example 15.1, there is a unique perfect Bayesian equilibrium: For $p > 1/2$, the firm hires the worker; the worker works if and only if her type is H, and at the initial node the firm assigns probability p on type H. Formally,

$$\begin{aligned} s_F(t_F) &= \text{Hire;} \\ s_W(H, \text{Hire}) &= \text{Work; } s_W(L, \text{Hire}) = \text{Shirk;} \\ b_F(t_F|h) &= 1 \text{ and } b_W(H|h) = p \end{aligned}$$

at every decision history h.

To verify that this assessment is indeed a PBE, one must check sequential rationality at every history and verify that b_W can be derived from strategy profile using the Bayes' rule. The derivation of beliefs using Bayes' rule is trivial in this example. One must check the sequential rationality of the worker for each of her types at history Hire, and one must check sequential rationality of the firm at history Apply. In order to check sequential rationality of type H at history Hire, observe that the worker's payoff from Work at history Hire for type H is

$$u_W\left(H, s_F, \text{ Hire } | H, \text{ Hire}\right) = u_W\left(H, \text{ Hire, Work}\right) = 2.$$

Note that one starts from history Hire in computing payoffs so the firm's strategy does not matter for this computation. In the extensive-form game, this is the history (H, Hire), where Nature has chosen H and the firm has chosen Hire. Hence, Work leads to the history (H, Hire, Work), at which the worker gets 2. She gets 2 if she follows her strategy. This payoff must be at least as high as her payoff from deviating and playing Shirk. The latter payoff is only

$$u_W\left(H, s_F, \text{ Shirk } | H, \text{ Hire}\right) = u_W\left(H, \text{ Hire, Shirk}\right) = 1,$$

showing sequential rationality for type H at history Hire. Similarly, to check sequential rationality for type L at history Hire, one checks that

$$u_W\left(L, s_F, \text{ Shirk } | L, \text{ Hire}\right) = 2 > 1 = u_W\left(L, s_F, \text{Work}|L, \text{Hire}\right).$$

One checks the sequential rationality of the firm at the initial history as follows. Since the firm does not know the worker's type, it makes its decision under uncertainty, maximizing its expected payoff under its beliefs at the initial history. Its expected payoff from its strategy $s_F\left(t_F\right) = \text{Hire}$ is

$$\begin{aligned}
E\left[u_F\left(\text{Hire}, s_W\right)\right] &= b_W(H) \cdot u_F\left(H, \text{Hire}, s_W\left(H, \text{Hire}\right)\right) + b_W(L) \cdot u_F\left(L, \text{Hire}, s_W\left(L, \text{Hire}\right)\right) \\
&= p \cdot u_F\left(H, \text{Hire}, \text{Work}\right) + (1-p) \cdot u_F\left(L, \text{Hire}, \text{Shirk}\right) \\
&= p \cdot 1 + (1-p) \cdot (-1) = 2p - 1.
\end{aligned}$$

Here the first equality is obtained as follows. At the initial history, according to the belief system, the worker is of type H with probability $b_W(H)$; in that case, she plays $s_W\left(H, \text{Hire}\right)$ according to her strategy, yielding the payoff of $u_F\left(H, \text{Hire}, s_W\left(H, \text{Hire}\right)\right)$. Similarly, with probability $b_W(L)$, the worker is of type L, and this yields the payoff of $u_F\left(L, \text{Hire}, s_W\left(L, \text{Hire}\right)\right)$. The next equality is obtained by substituting the values of the probabilities $b_W(\cdot)$ and the moves $s_W\left(\cdot, \text{Hire}\right)$; for example, if the worker is of type

H, she will play Work, resulting in $(H, \text{Hire}, \text{Work})$. Finally, the equalities in the last line are obtained by substituting the values of the payoffs, and the rest is simple algebra. One next computes the expected payoff from the only deviation Pass as

$$E\left[u_F\left(\text{Pass}, s_W\right)\right] = b_W\left(H\right) \cdot u_F\left(H, \text{Pass}, s_W\left(H, \text{Hire}\right)\right) + b_W\left(L\right) \cdot u_F\left(L, \text{Pass}, s_W\left(L, \text{Hire}\right)\right) = 0.$$

In order to check sequential rationality, one must check that

$$E\left[u_F\left(\text{Hire}, s_W\right)\right] \geq E\left[u_F\left(\text{Pass}, s_W\right)\right];$$

i.e.,

$$2p - 1 \geq 0.$$

This inequality holds if and only if $p \geq 1/2$.

To show that there is no other PBE, one observes that sequential rationality of the worker requires that $s_W\left(H, \text{Hire}\right) = \text{Work}$ and $s_W\left(L, \text{Hire}\right) = \text{Shirk}$, and the beliefs must be as above by definition. Then, sequential rationality of the firm at the initial node requires that it hires the worker.

This example fully specifies the belief about each player at every decision history. Clearly, many of these beliefs are trivial or irrelevant in that they are not held by the player who moves. It is customary to skip those beliefs in the description of a belief system. In this example, it suffices to describe the belief about the worker's type at the initial history.

Example 15.3. Consider the game in Figure 15.2. The dynamic Bayesian game here differs from the previous one in two ways: it has an additional application stage and a more general payoff function. For $v = -2$ and $w = 0$, there is a unique PBE: both types of the worker apply; the firm hires, believing that the worker is of high ability with probability p, and the high-ability worker works while the low-ability worker shirks. Formally,

$$\begin{aligned}
s_W\left(H\right) &= s_W\left(L\right) = \text{Apply}; \\
s_F\left(\text{Apply}\right) &= \text{Hire}; \\
s_W\left(H, \text{Apply,Hire}\right) &= \text{Work};\ s_W\left(L, \text{Apply,Hire}\right) = \text{Shirk}; \\
b\left(H|\text{Apply}\right) &= p.
\end{aligned}$$

Indeed, by sequential rationality of the worker, she must work if she is of type H and shirk if she is of type L. Since the firm gets either 1 or -1 from hiring depending on the worker's type, while it gets only -2 from not hiring, the firm's sequential rationality implies that it hires. Given this, the sequential rationality at the initial nodes requires that both types must apply. One then derives $b\,(H|\text{Apply}) = p$ from s_W as above.

For $v = 0$ and $w = 3$, there is a continuum of PBE, but in all of these equilibria both types of the worker opt out, not applying to the job. In one equilibrium, both types choose Out (i.e. $s_W\,(H) = s_W\,(L) = \text{Apply}$); the firm does not hire, assigning probability 0 on type H (i.e. $b\,(H|\text{Apply}) = 0$), and only the high-ability worker would have worked (i.e. $s_W\,(H, \text{Apply,Hire}) = \text{Work}$; $s_W\,(L, \text{Apply,Hire}) = \text{Shirk}$). The other equilibria differ in specifying different beliefs $b\,(H|\text{Apply})$ for worker types after an application and different strategy for the firm. Since both types play Out, any belief is consistent with the Bayes' rule after Apply. But one cannot specify $b\,(H|\text{Apply}) > 1/2$. In that case, the sequential rationality of the firm would require that it hires the worker, but then sequential rationality of the worker would require that $s_W\,(H) = \text{Out}$ and $s_W\,(L) = \text{Apply}$, and the Bayes rule would then yield $b\,(H|\text{Apply}) = 0/\,(1-p) = 0$. For each belief system with $b\,(H|\text{Apply}) \leq 1/2$, there is a sequential equilibrium. For $b\,(H|\text{Apply}) < 1/2$, the firm chooses Pass.

There is also a continuum of equilibria with $b\,(H|\text{Apply}) = 1/2$. In such equilibria, the firm is indifferent between Hire and Pass, and it chooses Hire with probability $\sigma_F\,(\text{Hire}|\text{Apply}) = q \leq 1/3$. To see the condition on q, observe that the expected payoff of type L from applying is $2q + (-1)\,(1-q) = 3q - 1$; with probability q, she will be hired and will obtain the payoff of 2 by shirking, and with probability $1-q$, she will be turned down and get -1. If she chooses Out, she will get 0. Since type H must choose Out and $b\,(H|\text{Apply}) = 1/2$, type L must also choose Out, and her expected payoff from applying $(3q-1)$ must not exceed 0. This yields $q \leq 1/3$.

15.3 Multi-Stage Games

For simplicity, the baseline model in this chapter assumed that the players can observe all previous actions, excluding possibility of simultaneous actions among other possibilities. The analysis extends to the multi-stage games immediately. Relaxing the perfect information about the moves, assume more generally that the game tree is of multi-stages. At each stage h, a set $N(h)$ of players simultaneously move, each $i \in N(h)$ choosing from a set $A_i(h)$ of actions. Note that the next stage is of the form (h, a), where a is the profile of actions a_i chosen by players $i \in N(h)$ at stage h. The rest is as in the baseline model; in particular players' types are independently distributed.

Given any strategy profile σ, a belief system b is derived as follows. Given any h and (h, a), the beliefs about the players who do not make a move remain as is, i.e.,

$$b_j(t_j|h, a) = b_j(t_j|h) \qquad (\text{for all } j \notin N(h), \, t_j \in T_j). \tag{15.4}$$

Beliefs about types of the players $i \in N(h)$ who move will be updated, and the updating depends on what strategy player i is supposed to play. Action a_i is played at h with probability

$$\Pr(a_i|\sigma_i, h) = \sum_{t_i \in T_i} \sigma_i(a_i|t_i, h) \, b_i(t_i|h), \tag{15.5}$$

as in (15.2). If the probability $\Pr(a|\sigma_i, h)$ is positive, the conditional probability is computed using the Bayes rule as follow:

$$b_i(t_i|h, a) = \frac{\sigma_i(a_i|t_i, h) \, b_i(t_i|h)}{\Pr(a_i|\sigma_i, h)} \text{ if } \Pr(a_i|\sigma_i, h) > 0. \tag{15.6}$$

If $\Pr(a_i|\sigma_i, h) = 0$, then the above ratio is undefined, and the conditional distribution $b_i(t_i|h, a)$ can be arbitrary.

The definitions of sequential rationality and PBE are extended similarly:

Definition 15.4. An assessment (σ, b) is said to be *sequentially rational* if at each h, each player $i \in N(h)$ maximizes her expected utility

1. given her type t_i and her beliefs $b(\cdot|h)$ about the other players' types at history h, and

2. given that the other players will play according to σ at h and in the continuation game.

Definition 15.5. An assessment (σ, b) is said to be a *perfect Bayesian equilibrium* (henceforth PBE) if it is sequentially rational and satisfies (15.4) and (15.6) throughout.

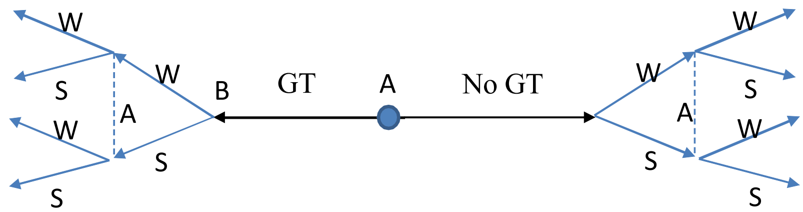


Figure 15.3: Game tree in Example 15.4.

The next example illustrates the concept of perfect Bayesian equilibrium when players can move simultaneously.

Example 15.4. Alice and Beatrice have a joint project. Once the project starts, simultaneously, each of them chooses between Work and Shirk. The payoff of a player i from Work is x_i+1 if the other player Works and x_i if the other player Shirks, while her payoff from Shirk is 0 regardless of what the other player does. It is known that the payoff parameter of Beatrice is $x_B = -1/2$. The payoff parameter $x_A \in \{-1/2, 1/2\}$ of Alice is privately known by Alice, where $\Pr(x_A = 1/2) = \Pr(x_A = -1/2) = 1/2$. Before the project starts Alice decides whether to take a difficult course on Game Theory, which does not help in the project and costs $c \in (1/2, 1)$ utils to Alice (i.e. we subtract c from Alice's payoffs above if she takes the course), and Beatrice observes whether Alice took the course before each decides whether to Work.

This can be viewed as a dynamic Bayesian game in which types of Alice and Beatrice are x_A and x_B, respectively, and the game tree is as in Figure 15.3. Although the difficult course in Game Theory is costly and has no intrinsic benefit, it may help Alice to signal her high return from investment persuading Beatrice to invest, as in the following perfect Bayesian equilibrium, plotted in Figure 15.4:

$$\begin{aligned}
s_A(1/2) &= \text{GT}; s_A(-1/2) = \text{No GT}; \\
s_A(1/2, \text{GT}) &= s_A(1/2, \text{No GT}) = s_A(-1/2, \text{GT}) = W; s_A(-1/2, \text{No GT}) = S \\
s_B(\text{GT}) &= W; s_B(\text{No GT}) = S; \\
b_A(1/2|\text{GT}) &= b_A(-1/2|\text{No GT}) = 1.
\end{aligned}$$

Alice takes the game theory course if her return is high (i.e. $x_A = 1/2$), and she does not take the course otherwise. If Beatrice observes that Alice has taken the course, she concludes that Alice's return is high (i.e. $b_A(1/2|\text{GT}) = 1$) and works; otherwise, she concludes that Alice's return is low and shirks. Alice works at all cases except for the case that $x_A = -1/2$ and she has not taken the course.

One verifies that this is indeed a perfect Bayesian equilibrium as follows. To check sequential rationality at the initial node for type $x_A = 1/2$, observe that Alice gets $3/2-c$ from GT.[1] When Alice takes the course, in the continuation game she and Beatrice both work, yielding $3/2$ for each, and Alice pays c for taking the course. Her payoff from No GT is 0; when she does not take the course, she still Works, but Beatrice does not work, so Alice gets only $1/2$. Since $c < 1$, playing GT is sequentially rational. For type $x_A = -1/2$, Alice gets $1/2 - c$ from GT because they both work after taking the course, yielding only $1/2$ to Alice this time, not counting the cost c she must incur. She gets 0 from No GT, as neither players invest in that case. Since $c > 1/2$, No GT is sequentially rational for type $x_A = -1/2$. After GT, assigning probability 1 on $x_A = 1/2$, Beatrice gets $1/2$ from Work and 0 from Shirk; her move is sequentially rational. Similarly, After No GT, assigning probability 1 on $x_A = -1/2$, Beatrice gets $-1/2$ from Work and 0 from Shirk; her move is sequentially rational. Sequential rationality of Alice's Work/Shirk decisions are straightforward. Since each type of Alice takes a different action, the Bayes' rule leads to probability one on the correct type after each action:

$$
\begin{aligned}
\Pr(x_A = 1/2|GT) &= \frac{\Pr(GT|x_A = 1/2) \times 1/2}{\Pr(GT|x_A = 1/2) \times 1/2 + \Pr(GT|x_A = -1/2) \times 1/2} \\
&= \frac{1 \times 1/2}{1 \times 1/2 + 0 \times 1/2} = 1 \\
\Pr(x_A = -1/2|NoGT) &= 1.
\end{aligned}
$$

This is the only pure-strategy perfect Bayesian equilibrium in which Alice takes the course for some type. When $x_A = -1/2$, since $c > 1/2$, the payoff of Alice is negative if she takes the course (no matter what happens after that), while she can get 0 if she does not take the course and shirks after that. Hence, she must not take the course in any equilibrium. Thus, the type that takes the course is $x_A = 1/2$. Since the different types take different actions, beliefs must be as above. Moreover, if a type of Alice takes the course in equilibrium, Beatrice must Work after the course is taken and must shirk after the course is not taken; for Alice would not take the course otherwise. Then,

[1] One-shot deviation principle applies, and one can take the players' future moves fixed and consider one-shote deviations at the given stage.

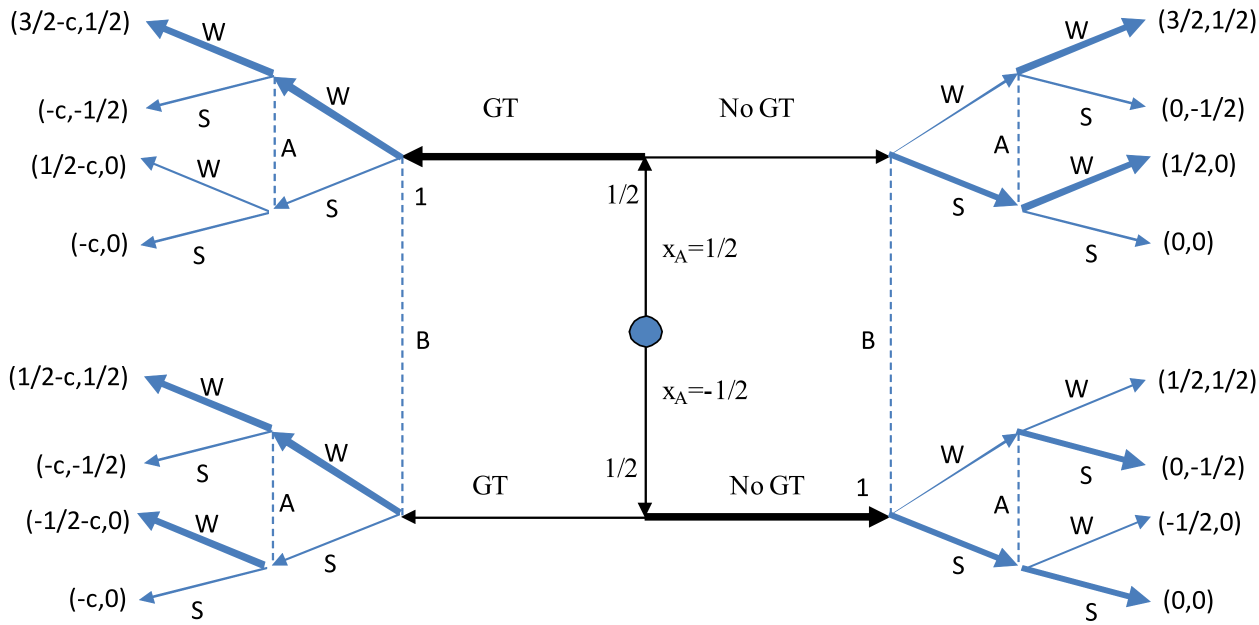


Figure 15.4:

the Work/Shirk decision of Alice must be as in the above equilibrium by sequential rationality.

15.4 Exercises with Solution

Exercise 15.1. Consider the following game between a potential applicant, named Bindu, to a Ph.D. program in Economics and the Admission Committee for the program. Bindu privately knows whether she likes Economics. She loves Economics with probability 0.1 and hates Economics with probability 0.9. She first decides whether to apply to the Ph.D. program. If she does not apply, the game ends and both Bindu and the committee get 0. If she applies, then the committee decides whether to accept or reject her. If the committee rejects, then committee gets 0, and Bindu gets -1. If the committee accepts her, the payoffs depend on whether she loves or hates Economics. If she loves Economics, she will be successful and the payoffs will be 20 for each player. If she hates Economics, the payoffs will be -10 for for each player. Find two perfect Bayesian equilibrium of this game, one in which her application is accepted, and one in which her application is rejected.

Solution. Denote the type space of Bindu by $T_B = \{L, H\}$, where L means she loves Economics and H means she hates it; the initial probabilities are $p_B(H) = 0.9$ and $p_B(L) = 0.1$. The committee has a dummy type that is omitted. Write s_B and s_C for the strategies of Bindu and the committee, respectively. The following is a perfect Bayesian equilibrium:

$$\begin{aligned} s_B(L) &= \text{Apply}; s_A(H) = \text{Do Not Apply}; \\ s_C &= \text{Accept}; \\ b_B(L|\text{Apply}) &= 1; \end{aligned}$$

Only the type who loves Economics applies to the program; upon observing the application, the committee updates its belief and becomes certain that she loves Economics and accepts her. This is despite the fact that she is not likely to love Economics ex-ante, but in the unlikely event that she applies, the committee is convinced that she loves it. Verifying that this is indeed PBE is straight forward. Now, consider the following perfect Bayesian equilibrium:

$$\begin{aligned} s_B(L) &= s_B(H) = \text{Do Not Apply}; \\ s_C &= \text{Reject}; \\ b_B(L|\text{Apply}) &= 0.1; \end{aligned}$$

In this equilibrium, even the type who loves Economics does not apply, anticipating that the committee will keep their prior belief about Bindu's preferences and reject her. In this equilibrium, the prior probability of application is zero, and hence committee can maintain any belief about Bindu's preferences. They keep their prior beliefs in this particular equilibrium. (This is an equilibrium for any $b_B(L|\text{Apply}) \leq 1/3$.)

Exercise 15.2. Consider the following version of Yankee Swap Game, played by Alice, Bob, and Caroline. There are 3 boxes, namely A, B, and C, and three prizes x, y, and z. The prizes are put in the boxes randomly, so that any combination of prizes is equally likely, and the boxes are closed without showing their contents to the players. First, Alice is to open box A, revealing its content observable. Then, in the alphabetical order, Bob and Caroline are to open the box with their own initial, making its content observable, and either keep the content as is or swap its content with the content of a box that has been opened already. Finally, Alice is given the option of swapping the content of her box with the content of any other box, ending the game when each player gets the prize in their own box.

1. Assume that it is commonly known that, for each player, the payoff from x, y, and z are 3, 2, and 0, respectively. Find a subgame-perfect Nash equilibrium.

2. Now assume that it is commonly known that the preferences of Bob and Caroline are as in Part 1, but the preferences of Alice are privately known by herself. With probability 1/2, her utility function is as above, but with probability 1/2 she gets payoffs of 2, 3, and 0 from x, y, and z, respectively. Find a perfect Bayesian equilibrium of this game.

Solution. (Part 1) In the very last stage, Alice will get the box with prize x, regardless of the box it is located. If her box contains x, she will leave it as is; she will swap her prize with x otherwise. In the previous stage, when it is Caroline's turn to swap, she then targets getting y in the end. Since Alice would not swap with y, Caroline ensures this by locating y in her own box. She keeps it as is if her box contains y and swaps the prize in her box with y otherwise. Thus, regardless of what Bob does, Alice, Bob, and Caroline get x, z, and y, respectively. There are multiple subgame-perfect Nash equilibria with the same outcome. In one equilibrium, Bob keeps the prize in his box, regardless of the contents of his and Alice's boxes. The equilibria vary according to which prizes Bob keeps and which prizes he swaps, but they all lead to the allocation above.

(Part 2) Only Alice has private information. Write $T_A = \{x, y\}$ where x is the type who likes x most and y is the type who likes y most. Moreover, Alice does not have a move until the end, as her only option at the beginning is to open her box. Hence, the beliefs remain as

$$p_A(x|h) = p_A(y|h) = 1/2$$

at every h. At the last stage, Alice with type t_A chooses the prize t_A, keeping it if t_A is in her own box, and swapping the content of her box with prize t_A otherwise. Now consider Caroline's decision. If she has x in her box at the end of her move, she will get x with probability 1/2 (when $t_A = y$), and the content of Alice's box with the remaining probability. Her expected payoff is

$$\frac{1}{2} \times 3 + \frac{1}{2} \times 2 = 5/2$$

if she has x and Alice has y; her expected payoff is

$$\frac{1}{2} \times 3 + \frac{1}{2} \times 0 = 3/2$$

if she has x and Alice has z. Similarly, one can compute her payoffs from every combination as follows, where the first, second, and the last entries are the prizes in Alice's, Bob's and Caroline's boxes, respectively, after Caroline makes her move:

(x,y,z)	(x,z,y)	(y,x,z)	(y,z,x)	(z,x,y)	(z,y,x)
0	5/2	0	5/2	1	3/2

Given the locations of the prizes, Caroline then keeps or swap the content of her box towards maximizing these payoffs. Clearly, her best option is to give z to Bob, when she she does not care where x and y are located; and she does not want z to be in her box. As a function of the locations when she makes her decision the final locations and the resulting payoffs are as follows:

Before Caroline	(x,y,z)	(x,z,y)	(y,x,z)	(y,z,x)	(z,x,y)	(z,y,x)
After Caroline	(x,z,y)	(x,z,y)	(y,z,x)	(y,z,x)	(z,y,x)	(z,y,x)
Bob's payoff	0	0	0	0	1	1
Caroline's payoff	5/2	5/2	5/2	5/2	3/2	3/2

Here, the first line describes the locations of the prizes after Bob made his move, where the prizes are ordered alphabetically according to the initial of the owner of the box. The second line describes the locations after Caroline moves, and the payoffs in the last two lines are the resulting expected payoffs of Bob and Caroline after Alice moves and makes changes according to her own preferences. For example, if the allocation is (x,y,z) after Bob's move, then Caroline swaps her z with Bob's y, yielding (x,z,y). Then, Alice keeps it as is if $t_A = x$ and swaps her prize with Caroline's if $t_A = y$. This results in payoff 0 for Bob (as he gets z in both cases) and 5/2 for Caroline, as explained above. If the allocation is (x,z,y), Caroline keeps her prize, leaving the allocation unaltered. Caroline has a unique best response in all cases, and the best response is as in the table.

Now when it is Bob's turn, he knows where each prize is; he knows the contents of his and Alice's box and infers that the remaining prize is in Caroline's box. He can only replace his prize with Alice, and he would like z to be in Alice's box when he is done; he is indifferent otherwise. Thus, if he has z in his own box, he swaps it with Alice; he keeps his box as is otherwise. (He can also swap x and y, leading to multiple equilibria.)

Exercise 15.3. Consider the following TV game, from the game show "Let's Make a Deal." There are two players, Host and Contestant. There are also three doors, L, M, and R.

- Nature puts a car behind one of these doors, and goats behind the others. The probability of having the car is same for all doors. Host knows which door, but Contestant does not.
- Then, Contestant selects a door.
- Then, Host must open one of the two doors that are not selected by Contestant and show Contestant what Nature put behind that door.
- Then, Contestant chooses any of the three doors, and receives whatever is behind that door.

Payoffs for Contestant and Host are $(1, -1)$ if Contestant receives a car, and $(0, 0)$ if he receives a goat. Compute a PBE of this game. Verify that this is indeed a PBE.

Solution. It is useful to formulate the type space formally. The set of types t_H for the Host is $T_H = \{L, M, R\}$, where the type indicates the door hiding the car. Each type t_H has probability $p(t_H) = 1/3$. The contestant has a dummy type that is ignored. Clearly, the host should not open the door with car behind; type t_H must choose a door other than t_H. Consider the following mixed strategy PBE (σ, b). If the contestant picks the door with a car, then the host opens either of the other doors with equal probability; if the contestant picks a door with a goat, then the host opens the other door with the goat.[2] The contestant picks L initially. In the last round, if there is a car behind the door opened by the host, she switches to that door; she switches to the remaining closed door if there is a goat behind the opened door. (Note that she always switches.)

[2] Formally, writing D_C for the door picked by the contestant and $T' = T_H \backslash \{t_H, D_C\}$ for the set of

For example, if the host opens M, then she chooses M if there is car behind it and R otherwise. The belief system is computed as follows. The beliefs does not change until the host opens a door. Write D_C for the door the contestant picks and D_H for the door the host open. Suppose there is a goat behind D_H. Towards computing the belief at this history, first compute the total probability that the host opens the door D_H given D_C. With probability 1/3, the host's type is $t_H = D_C$, and he opens the door D_H with probability 1/2 in that case; with probability 1/3, the host's type is $t_H = D_H$, and he opens the door D_H with probability 0 in that case; and with the remaining probability 1/3, the host's type is different from D_C and D_H, and he opens the door D_H with probability 1 in that case. Therefore, the total probability is

$$\Pr\left(D_H|D_C\right) = \frac{1}{2} \times \frac{1}{3} + 0 \times \frac{1}{3} + 1 \times \frac{1}{3} = \frac{1}{2}.$$

Hence, the conditional probability of type $t_H = D_C$ is

$$b_H\left(D_C|D_C, D_H, G\right) = \frac{\sigma_C\left(D_H|t_H = D_C, D_C\right)\Pr\left(t_H = D_C\right)}{\Pr\left(D_H|D_C\right)} = \frac{\frac{1}{2} \times \frac{1}{3}}{\frac{1}{2}} = \frac{1}{3}.$$

She assigns probability 1/3 to the door she picked originally. She assigns the remaining probability to the remaining closed door $D \in \{L, M, R\} \setminus \{D_C, D_H\}$:

$$b_H\left(D|D_C, D_H, G\right) = \frac{2}{3}.$$

What if there is a car behind the door D_H the host opens? This is a zero probability event, and the perfect Bayesian equilibrium does not restrict the beliefs in that case. But given the structure of the problem she should recognize that the the car is behind that door, assigning probability

$$b_H\left(D_H|D_C, D_H, Car\right) = 1$$

to the type associated with the door opened by the host.

To verify that this is indeed a PBE, one needs to check sequential rationality. First, at the last round, the contestant picks the door with the highest conditional probability

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doors with a goat that is not picked by the contestant, the host opens each door $D \in T'$ with probability

$$\sigma_H\left(D|t_H, D_C\right) = \frac{1}{|T'|}.$$

of the car, and hence it is sequentially rational for her to follow the strategy. To check sequential rationality for the contestant, observe that if the contestant picks the door with the car (i.e., $D_C = t_H$), then the host gets 0 whichever the remaining doors he opens, and hence mixing between them is sequentially rational. Otherwise (i.e., if $D_C \neq t_H$), the host gets -1 regardless of whether he opens the door t_H or the third door because the contestant will choose t_H in either case, and opening the third door is sequentially rational. Finally, the contestant gets 0 if he picks the door t_H initially and gets 1 otherwise. Hence, her expected payoff from each door is $2/3$, and picking L initially is sequentially rational.

Since the contestant is indifferent between the doors at the initial stage in this particular PBE, she could choose any door with any probability without violating sequential rationality. Each such randomization leads to a different PBE, yielding a continuum of equilibria. Clearly the difference between these equilibria is trivial. The labeling on the doors matter only after the contestant picks a door.

Observe also that in the PBE above, the contestant always switches the door after the host opens a door, never picking the door she picked originally. Indeed this is true in all PBE (see Exercise 15.11). This feature has puzzled many mathematicians, including Paul Erdos. They found the result paradoxical because the door was picked initially by the contestant under complete ignorance. They might have thought that, conditional on there is a goat behind the opened door, each of the remaining closed doors are equally likely to have the car behind. This would indeed be the case, if the host did not know where the car is. But the host knows where the car is. He cannot open the door picked by the contestant initially and does not open the door that has a car behind. This breaks the symmetry.

15.5 Exercises

Exercise 15.4. Consider the dynamic Bayesian game in Figure 15.5 where t can be -1, 1, or 2, each with probability $1/3$, and it is privately known by Player 1. Moreover, $u(t) = v(t) = 1$ if $t \in \{-1, 2\}$ and $u(1) = -v(1) = 2$. Compute a perfect Bayesian equilibrium.

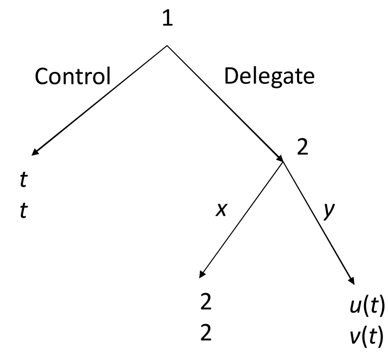

Figure 15.5:

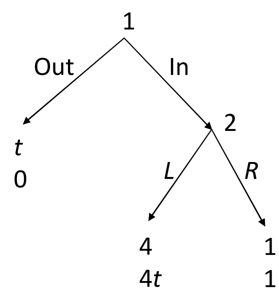

Figure 15.6:

Exercise 15.5. Consider the dynamic Bayesian game in Figure 15.6 where t can be 0 or 3, each with probability 1/2. Compute the set of all perfect Bayesian equilibria for each case below and briefly discuss the role of incomplete information by comparing your answers:

1. the payoffs are not known by either player;

2. the payoffs are privately known by Player 2;

3. the payoffs are privately known by Player 1.

Exercise 15.6. Compute the set of all perfect Bayesian equilibria of the dynamic Bayesian game in which the game tree and the payoffs are as in Figure 15.7; u and v are privately known by Player 1; and (u, v) takes values of $(4, 3)$, $(6, 3)$, and $(-1, -1)$ with probabilities 0.5, 0.4, and 0.1, respectively.

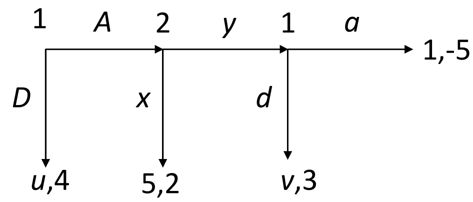

Figure 15.7:

Exercise 15.7. Consider a Stackelberg duopoly, in which Firm 1 sets its production level q_1 first, and Firm 2 sets its production level q_2 after observing q_1. Each firm sells its product at price $P = 1 - q_1 - q_2$. Assume that each firm i privately knows its own marginal cost c_i where c_1 and c_2 are independently distributed with expected values $\bar{c}_1$ and $\bar{c}_2$, respectively. Compute a perfect Bayesian equilibrium.

Exercise 15.8. In the location-choice game in Section 10.4.2, assume that each player i has marginal cost $c_i \in \{0, \gamma\}$ for some known $\gamma \in (0, 1)$. Assume that player i privately knows c_i and each pair (c_1, c_2) is equally likely. Compute a perfect Bayesian equilibrium.

Exercise 15.9. Consider a professor and a student. There are two types of students, H and L. The student knows his type, but the professor does not. The prior probability of type H is $\pi \in [0, 1]$. The events take place in the following order.

- First, the professor determines a cutoff value $\gamma \in [0, 100]$.
- Observing γ and his type, the student decides whether to take the class.
- If the student does not take the class, the game ends; the professor gets 0, and the student gets W_t, where $t \in \{H, L\}$ is his type and $0 < W_L < W_H < 100$.
- If the student takes the class, then he chooses an effort level e and takes an exam. His score in the exam is $s = e$ if $t = L$ and $s = 2e$ if $t = H$; i.e., a high type student scores higher for any effort level.
- The student gets a letter grade

$$g = \begin{cases} A & \text{if } s \geq \gamma \\ B & \text{otherwise.} \end{cases}$$

- The student's payoff is $100 - e/2$ if he gets $g = A$, and $-e/2$ if he gets B. The professor's payoff is s.

1. Consider a prestigious institution with high standards, where π is high, and W_H is not too high. In particular, $\pi > .5\,(100 - W_L) / (100 - W_H)$ and $W_H < (100 + W_L)/2$. Compute a PBE for this game.

2. Consider a prestigious institution with spoiled kids, where both π and W_H are high. In particular, $W_H > (100 + W_L)/2$ and $\pi > 1 - 2\,(100 - W_H) / (100 - W_L)$. Compute a PBE for this game.

3. Consider a lower-tier college, where both π and W_H are low; $\pi < .5\,(100 - W_L) / (100 - W_H)$ and $W_H < (100 + W_L)/2$. Compute a PBE for this game.

4. Assuming that W_L is the same at all three institutions, rank the exam scores in (1), (2) and (3).

Exercise 15.10. In the Yankee-Swap Game in Exercise 15.2, assume that z is an electronic gadget and Bob is an electronic-gadget enthusiast, having superior information about the value of z. In particular, the payoff from z is β for each player, where β is either 0 or 5, each with probability 1/2, and Bob privately knows β. The rest of the payoffs are as in Part 2 of Exercise 15.2. Compute a perfect Bayesian equilibrium.

Exercise 15.11. In Exercise 15.3, show that the contestant must switch the door (with probability 1) in every PBE.

Exercise 15.12. Alice is a senior researcher, and she has an assistant, Bob. She has a research project. She can either work on the project by herself (a move denoted by X), or ask Bob to collaborate with her (a move denoted by I). If she asks him to collaborate, they simultaneously exert efforts, which can be high or low, and the outcome depends on the effort levels as well as the value θ of the project, which can be high or low. The tree and the payoffs are depicted in Figure 15.8.

1. Assuming that θ is known, compute the set of pure-strategy subgame-perfect Nash equilibria for each θ.

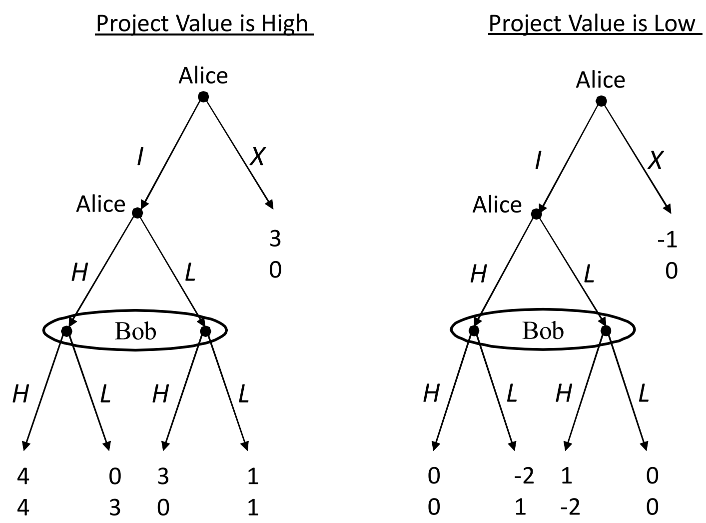

Figure 15.8:

2. Assume that θ is high with probability q and low with probability $1-q$ for some q with $1/2 \leq q < 2/3$, and Alice privately knows θ (Bob does not know θ). Compute the set of all perfect Bayesian equilibria in pure strategies.