Chapter 2

Representation of Games

This chapter formally introduces games and some fundamental concepts, such as a strategy and information. In order to analyze a strategic situation, one needs to know

- who the players are,
- which actions are available to them,
- how much each player values each outcome,
- what each player knows.

A game is just a formal representation of the above information. This is usually done in one of the following two ways:

1. The extensive-form representation, in which the above information is *explicitly* described using game trees and information sets;
2. The normal-form (or strategic-form) representation, in which the above information is *summarized* by the use of strategies.

Both forms of representation are useful in their own way, and both representations will be used extensively throughout the book.

It is important to emphasize that, when describing what a player knows, one needs to specify not only what she knows about the external parameters, such as the payoffs, but also what she knows about the other players' knowledge and beliefs about these parameters, as well as what she knows about the other players' knowledge of her own beliefs, and so on. In both representations, such information is encoded in an economical manner. The first half of this book focuses on non-informational issues and analyzes the games of complete information, in which everything that is known by a player is known by everybody. The second half will focus on informational issues, allowing players to have asymmetric information, so that one may know a piece of information that is not known by another.

The outline of this chapter is as follows. The first section is devoted to the expected utility theory that describes how a player makes her choices under uncertainty. The second section is devoted to the extensive-form representation of games. The third section is devoted to the concept of strategy. The fourth section is devoted to the normal-form representation, and the equivalence between the two representations. The final two sections contain exercises and some of their solutions.

2.1 Payoffs and Decision Making under Uncertainty

When everything is done, the game ends, resulting in an outcome. A player's behavior until that point is guided by how she feels about the possible outcomes and what she believes about the other player's behavior. In general, the outcome depends on the choices made by all players, and a player may not know what choices are made by the other players. Hence, she makes her choices under uncertainty, not knowing the outcomes of her choices. In order to analyze a strategic interaction, one then needs a theory of decision making under uncertainty. This book adopts the standard theory of decision making under uncertainty, called the *Expected Utility Theory*. This theory is described in detail in Appendix A; this section gives a brief outline of the theory needed to follow the main text.

The book assumes that when a player does not know something, she forms a belief about the things that she is uncertain about and assigns a probability on each possible case, where the probabilities are all non-negative numbers, adding up to one. Mathematically, a function that assigns a probability to each case is called a *probability distribution*. They will be called *beliefs* here when they refer to a player's likelihood assessment about things that she does not know. Every belief leads to a probability distribution on the outcomes, and such a probability distribution is called *lottery.* Hence, at any moment in the game, given the belief of a player at that moment, each of her choices can be represented as a lottery. The book assumes that each player has a numerical score for each outcome and her choice maximizes the expected value of that score—when her choices are represented as lotteries.

Formally, let Z be the set of all possible outcomes, and assume that it is finite. A *lottery* is a probability distribution p on Z, where

$$p : Z \to [0, 1]$$

is a function with

$$\sum_{z \in Z} p(z) = 1.$$

A player is said to *strictly prefer* a lottery p to a lottery q if she never chooses q when p is available, and she is said to *weakly prefer* a lottery p to a lottery q if she may choose p when q is available. It is assumed throughout the book that each player has a utility function

$$u : Z \to \mathbb{R},$$

and she weakly prefers a lottery p to a lottery q if and only if

$$U(p) \equiv \sum_{z \in Z} p(z) u(z) \geq \sum_{z \in Z} q(z) u(z) \equiv U(q).$$

Here, $U(p)$ and $U(q)$ are the expected values of the function u under the distributions p and q, respectively. It is assumed that she evaluates the lotteries according to these expected values and chooses a lottery with the highest expected value. Such a utility function u is called a Von Neumann-Morgenstern utility function. The description of a game will specify a Von Neumann-Morgenstern utility function for each player.

For a concrete example, consider the Prisoners' Dilemma game in the previous chapter, played by two prisoners, named Al and Bill. The outcome depends on both players' strategies and is given by the following table

	Cooperate	Defect
Cooperate	both walk free after a trial	Al gets a long jail sentence ... Bill gets immediate release
Defect	Al gets immediate release ... Bill gets a long jail sentence	both get a short jail sentence

where Al chooses between the rows and Bill chooses between the columns. Al is self-interested and cares only about the jail time he serves. He ranks the outcomes as follows: getting immediate release is the best outcome; walking free after a trial is the second-best; a short jail is the third, and getting a long jail sentence is the worst outcome.

Al does not know the outcome of his strategies. For example, the outcome of Cooperate can be that both players walk free after a trial or that Al gets a long jail sentence while Bill gets an immediate release depending on whether Bill plays Cooperate or Defect. Therefore, in order to understand which strategy Al will play, we need to know how he ranks these choices with unknown outcomes. It is not enough to know how he ranks the outcomes. For example, suppose Al evaluates his choices according to the worst outcome they may lead to. The worst outcome under Cooperate is that Al gets a long jail sentence while Bill gets immediate release, and the worst outcome under Defect is that both get a short jail sentence. Then, Al ranks Defect above Cooperate. Alternatively, Al could evaluate the choices according to how he feels about the most symmetric outcome they may lead to. The most symmetric outcomes are both walk free after a trial under Cooperate and both get a short jail sentence under Defect. Now Al ranks Cooperate above Defect.

Throughout the book, it will be assumed that the players are expected utility maximizer. That is, each player assigns a numerical score on each outcome, which will be called the player's payoff. She will also assign probabilities on the contingencies that she does not know, and she will evaluate each choice with unknown outcome according to the expected value of the numerical score under that choice.

In this particular example, Al assigns numerical scores v_{CC}, v_{CD}, v_{DC}, and v_{DD} to the outcomes of (Cooperate, Cooperate), (Cooperate, Defect), (Defect, Cooperate), and (Defect, Defect), respectively. He gives higher scores to the outcomes that he ranks higher: $v_{DC} > v_{CC} > v_{DD} > v_{CD}$. As in the previous chapter, one can take $v_{CC} = 5$, $v_{CD} = 0$, $v_{DC} = 6$, and $v_{DD} = 1$. If Al assigns probability p to Bill playing Cooperate and $1 - p$ to Bill playing Defect, then he assigns the numerical score $pv_{CC} + (1 - p)\, v_{CD}$ to Cooperate and the numerical score $pv_{DC} + (1 - p)\, v_{DD}$ to Defect, and he ranks his strategies according to these numerical scores. For example, if $p = 1/2$ and the payoffs are as above, he assigns numerical score $5/2 = \frac{1}{2} \times 5 + \frac{1}{2} \times 0$ to Cooperate and numerical score $7/2 = \frac{1}{2} \times 6 + \frac{1}{2} \times 1$ to Defect, and he ranks Defect above Cooperate.

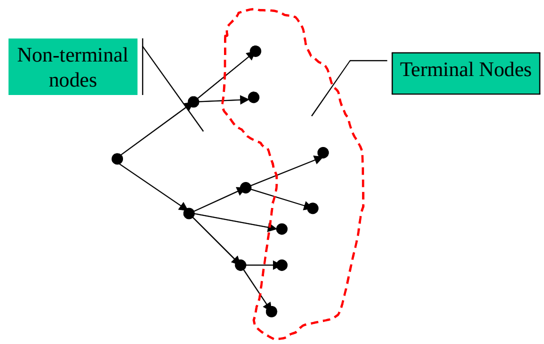

Figure 2.1: A tree—decomposed into terminal and non-terminal nodes.

2.2 Extensive-form Representation

The extensive-form representation of a game contains all the information about the game explicitly, by defining who moves when, what each player knows when she moves, what moves are available to him, and where each move leads to, etc. This is done by use of a *game tree* and *information sets*—as well as more basic information such as players and the payoffs.

2.2.1 Game Tree

Towards defining a tree, note that a *directed graph* is a set of nodes and a set of edges that connect some of these nodes; the edges are depicted by arrows. Each edge comes with a label where distinct edges may have the same label.

Definition 2.1. A *tree* is a directed graph such that

1. there is a unique initial node, for which there is no incoming edge;

2. the initial node is linked to every other node through a unique path.

For a visual aid, imagine the branches of a tree arising from its trunk. For example, the graph in Figure 2.1 is a tree. There is a unique starting node, and it branches out from there without forming a loop. It does look like a tree. On the other hand, the graphs in Figure 2.2 are not trees. In the graph on the left-hand side, there are two alternative paths to node A from the initial node, one through node B and one through

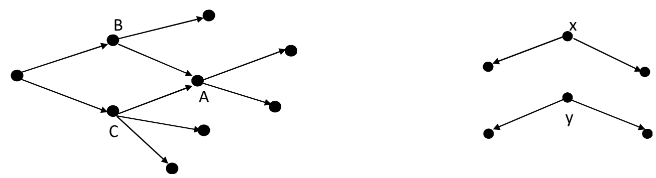

Figure 2.2: Two graphs that are not a tree

node C. This violates the second condition. On the right-hand side, there is no path that connects the nodes x and y, once again violating the last condition. Observe that the conditions in the definition of a tree imply that every non-initial node has a unique incoming edge.

In a game tree, there are two types of nodes: *terminal* nodes, at which the game ends, and *non-terminal* nodes, at which a player would need to make a further decision. This is formally stated as follows.

Definition 2.2. The nodes that are not followed by another node are called *terminal*. The other nodes are called *non-terminal* or *decision nodes* (interchangeably).

For example, the terminal and non-terminal nodes for the game tree in 2.1 are as indicated in that figure. There is no outgoing arrow in any terminal node, indicating that the game has ended. A terminal node may also be referred to as an outcome in the game. At such a node, one needs to specify the players' payoffs towards describing their preferences among the outcomes. On the other hand, there are some outgoing arrows in any non-terminal node, indicating that some further decisions are to be made. In that case, on needs to describe who makes a decision and what she knows at the time of the decision. A game is formally defined just like this, next.

2.2.2 Games in Extensive Form

An extensive-form representation of a game simply lists the information needed to describe the strategic situation, identifying who the players are, who moves and when, what she knows and what choices available to her when a player is to move, and what her preferences are on the outcomes.

Definition 2.3 (Extensive-form)**.** An *extensive-form game* consists of

- a set of players,
- a tree,
- an allocation of non-terminal nodes of the tree to the players,
- an informational partition of the non-terminal nodes (to be made precise in the next subsection), and
- payoffs for each player at each terminal node.

Players The set of players consists of the decision-makers or actors who make some decision during the course of the game. Some games may also contain a special player Nature (or Chance) that represent the uncertainty the players face, as it will be explained in Subsection 2.2.4. The set of players is denoted by

$$N = \{1, 2, \ldots, n\}$$

and $i, j \in N$ are designated as generic players.

Outcomes and Payoffs The set of terminal nodes is denoted by Z. At a terminal node, the game has ended, leading to some outcome. At that point, one specifies a payoff, which is a real number, for each player i. The mapping

$$u_i : Z \to \mathbb{R}$$

that maps each terminal node to the payoff of player i at that node is the Von-Neumann-Morgenstern utility function of player i. Recall that this means that player i tries to maximize the expected value of u_i. That is, given any two lotteries p and q on Z, she prefers p to q if and only if p leads to a higher expected value for function u_i than q does, i.e., $\sum_{z\in Z} u_i(z)\,p(z) \geq \sum_{z\in Z} u_i(z)\,q(z)$.

Decision Nodes In a non-terminal node, a new decision is to be made. Hence, in the definition of a game, a player is assigned to each non-terminal node. This is the player who will make the decision at that point. Towards describing the decision problem of the player at the time, one defines the available choices to the player at the moment. These are the outgoing arrows at the node, each of them leading to a different node.

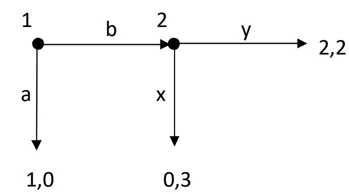

Figure 2.3: A simple game in extensive form

Each of these choices is also called a *move* or an *action* (interchangeably). Note that the moves come with their labels, and two different arrows can have the same label. In that case, they are the same move.

Formally, the set of all nodes in the tree is denoted by H, and sometimes referred to as the set of histories h. The set of all decision nodes is then $H \backslash Z$. For each decision node $h \in H \backslash Z$, the set of available moves is denoted by $A(h)$. Each move $a \in A(h)$ leads to a new node, denoted by (h, a). In this way, one can describe nodes in the tree by the action sequence that leads to the node from the initial node. The player who is to move at decision node h is denoted by $\iota(h)$.

Example 2.1. Consider the game in Figure 2.3. In this game, there are two players, namely 1 and 2. The tree consists of five nodes, two decision nodes, and three terminal nodes. At the initial node, Player 1 moves, choosing between the moves a and b. If she chooses a, the game ends, reaching to a terminal node. If she chooses b, then Player 2 moves. As in the rest of the book, the bullets for the terminal nodes are omitted; they have payoffs attached to them instead. Since there are two players, payoff vectors have two elements. The first number is the payoff of Player 1 and the second is the payoff of Player 2.[1] These payoffs are *Von Neumann-Morgenstern utilities.* That is, each player makes her moves toward maximizing the expected value of her own payoffs, given her beliefs about how the other player will play the game.

2.2.3 Information Sets

One also needs to describe what each player knows at each moment of her decision making. This is formally done by *information sets*, as follows.

[1] When the players are ordered, as in this example, the payoffs are written in the order of players.

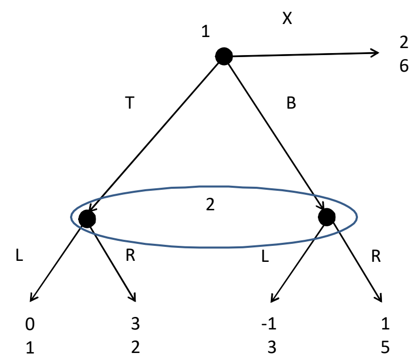

Figure 2.4: A game

Definition 2.4. An *information set* is a collection $I \subseteq H \backslash Z$ of decision nodes such that

1. the same player i is to move at each of these nodes (i.e., $\iota(h) = \iota(h')$ for all $h, h' \in I$);

2. the same moves are available at each of these nodes (i.e., $A(h) = A(h')$ for all $h, h' \in I$).

Definition 2.5. An *information partition* is an allocation of each non-terminal node of the tree to an information set; the initial node must be "alone".

The meaning of an information set is that when the player is in that information set, she knows that one of the nodes in the information set is reached, but she cannot rule out any of the nodes in the information set. Moreover, in a game, the information set belongs to the player who is to move in the given information set, representing her uncertainty. That is, the player who is to move at the information set is unable to distinguish between the points in the information set, but able to distinguish between the points outside the information set from those in it. Therefore, Definition 2.4 would be meaningless without condition 1, while condition 2 requires that the player knows her available choices. The latter condition can be taken as a simplifying assumption. When it does not lead to confusion, I also refer to information sets as *history* and write h_i for a generic history at which player i moves.

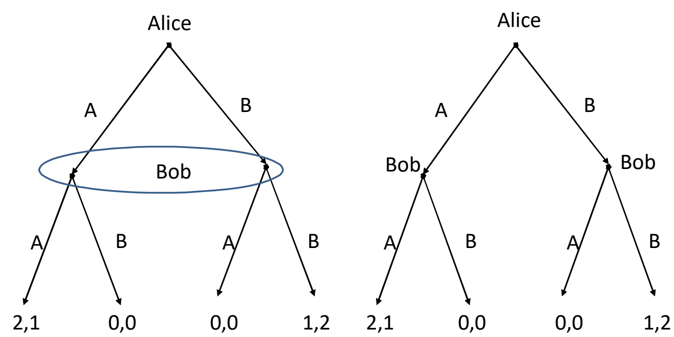

Figure 2.5: The Battle of Sexes game (Left) and a variation with sequential moves (Right)

For an example, consider the game in Figure 2.4. The information set at which Player 2 moves consists of the nodes that follow Player 1 taking actions T and B.[2] Player 2 knows that Player 1 has taken action T or B and not action X, but Player 2 cannot know for sure whether Player 1 has taken T or B. Player 1 moves at the initial node, which is in an information set that contains only that node. Player 1 does not face any uncertainty about what happened thus far; she does not know what Player 2 will play later.

Example 2.2 (Battle of the Sexes Game)**.** In Figure 2.5, consider the game on the left. This is a well-known game, known as the Battle of the Sexes. It epitomizes coordination with conflict: two players, namely Alice and Bob, would like to coordinate on (A, A) or (B, B), but Alice prefers to coordinate on (A, A), while Bob prefers to coordinate on (B, B). In this game each player has one information set. In particular, Bob does not know whether Alice played A or B. Likewise, Alice does not know whether Bob will play A or B. Neither player has information about the other player's move. Such games are called *simultaneous-action* or *simultaneous-move* games.

Example 2.3 (Sequential Battle of the Sexes Game)**.** Now, imagine that Bob knows what Alice does when he takes his action. This can be formalized via the extensive-form

[2] Throughout the book, the information sets are depicted either by the standard set notation (as in Figure 2.4), or by dashed curves connecting the nodes in the information sets, depending on convenience. Moreover, the information sets with only one node in them are not depicted in the figures. For example, in Figure 2.4, the initial node is in an information set that contains only that node.

game in Figure 2.5 on the right. In this version, Bob has two information sets, one after Alice plays A, and one after Alice plays B. Hence, Bob knows the action taken by Alice.

The informational partition in the last example is very simple. Every information set has only one element. Hence, there is no uncertainty regarding the previous play in the game. A game is said to have *perfect information* if every information set has only one element. Recall that in a tree, each node is reached through a unique path. Hence, in a perfect-information game, a player can construct the previous play perfectly. For instance, in the sequential Battle of the Sexes game, Bob knows whether Alice chose A or B. And Alice knows that when she plays A or B, Bob will know what she has played.

2.2.4 Nature as a player and representation of uncertainty

Many card and board games involve some randomization. Dice are thrown in backgammon; cards are drawn in poker, and coins are tossed in many games to determine the order in which the players move. Such explicit randomization is easily incorporated in games using a new player, called Nature, who chooses her moves according to a given probability distribution. At any point in the game such randomization occurs, Nature moves and replicates the randomization.

For a concrete example, suppose that two players want to choose between two decisions, Left and Right, and the decision making is delegated to one of the players by a coin toss, where Player 1 makes the decision if it comes up head and Player 2 makes the decision if it comes up tail. The payoffs may depend on who makes the decision. This situation can be modeled by the game in Figure 2.6. At the beginning, Nature chooses between Head and Tail, each with probability 1/2. Player 1 moves if it is Head, and Player 2 moves if it is Tail, each player choosing between Left and Right.

Representing randomizations, Nature differs from the regular players in three ways:

1. Nature does not have payoffs.

2. Nature chooses her moves probabilistically, and the probabilities of moves are given in the description of the game.

3. Nature does not have information sets.

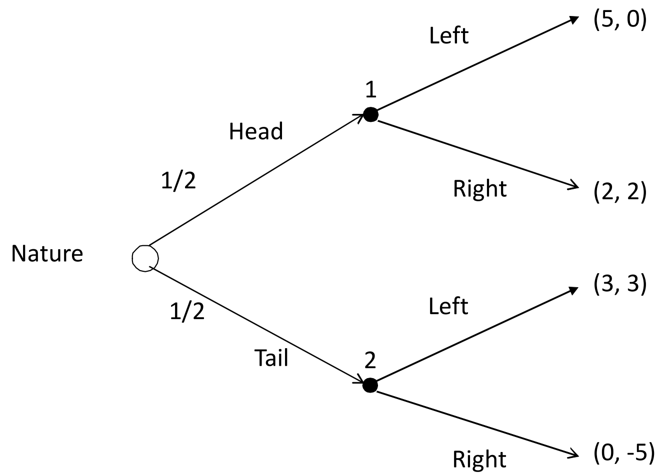

Figure 2.6: A game with chance

Nature's decision nodes are depicted by an empty circle, and each move's probability is written next to it.

In many real-world applications, players also face uncertainty about things that they take into account when they make their decision. For example, players may face uncertainty about the dividend to be paid by a particular company, the oil prices in the future, or the preferences of other players in the game. Such subjective uncertainty will also be incorporated in games using Nature, the same way explicit randomization is incorporated. First, players' beliefs are represented as coming from explicit randomization and then incorporated in the description of the game as Nature's moves.

For a concrete example, consider a player named Alice deciding whether to buy a new computer today at price \$1,000 or postpone her purchase decision to the next summer when there may be a sale. She thinks that there will be a sale and the price will drop to \$600 with probability 1/2, and there will not be a sale with the remaining probability. She will need a computer in between, and hence the value of the computer is \$1,200 if she buys it now, but it will only be \$900 if she buys it next summer. This problem can be formulated as a game with a single player as in Figure 2.7. Alice first chooses between "buy" and "postpone". If she buys, the game ends for her, and she gets the payoff of $1,200 - 1,000$. If she chooses "postpone", then Nature chooses between "sale" and "no sale", and after seeing Nature's move, Alice chooses between "buy" and "pass".

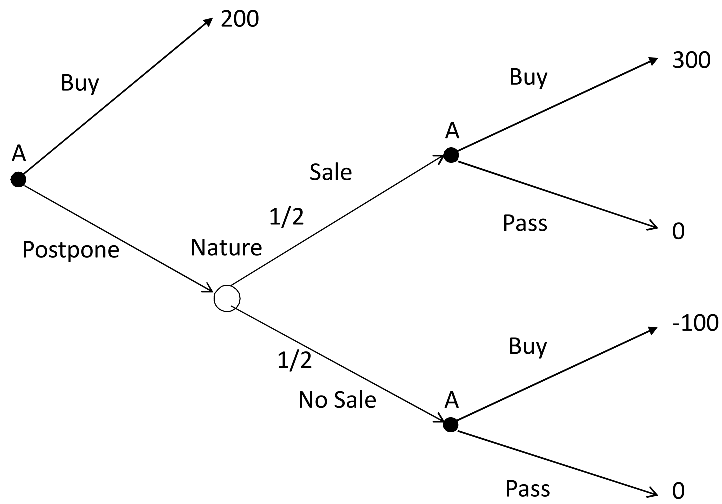

Figure 2.7: A game with uncertainty

Her payoff is 0 if she passes, and 900 minus the price if she buys.

2.2.5 Commonly Known Assumptions

The structure of a game is assumed to be known by all players, and it is assumed that all players know that all players know the structure, and so on. That is, in a more formal language, the structure of the game is *common knowledge*.[3] For example, in the game of Figure 2.4, Player 1 knows that if she chooses T or B, Player 2 will know that Player 1 has chosen one of the above actions without being able to rule out either one. Moreover, Player 2 knows that Player 1 has the above knowledge, and Player 1 knows that Player 2 knows it, and so on. Using information sets and richer game trees, one can model arbitrary information structures like this. For example, as you are asked to do in Exercise 2.14, one could also model a situation in which Player 1 does not know whether Player 2 could distinguish the actions T and B. One could do that by having three information sets for Player 2; one of them is reached only after T, one of them is reached only after B, and one of them can be reached after both T and B. Towards

[3]Formally, a proposition X is said to be common knowledge if all of the following are true: X is true; everybody knows that X is true; everybody knows that everybody knows that X is true; ...; everybody knows that ... everybody knows that X is true, *ad infinitum*.

modeling uncertainty of Player 1, one would further introduce a chance move, whose outcome either leads to the first two information sets (observable case) or to the last information case (unobservable case). In general, at any node, the following are known: which player is to move, which moves are available to the player, and which information set contains the node, summarizing the player's information at the node. Again, all these are assumed to be common knowledge.

2.3 Strategies

Definition 2.6. *A strategy* of a player *is a complete contingent-plan* that determines which action she will take at each information set she is to move (including the information sets that will not be reached according to this strategy). More mathematically, a strategy of a player i is a function s_i that maps every information set h_i of player i to an action that is available at h_i.

It is important to note the following three subtleties in the definition.

1. *Every* information set of the player must have a move assigned. (If one omits to assign a move for an information set, then she would not know what the player would have done when that information set is reached.)

2. The assigned move must be available at the information set. (If the assigned move is not available at an information set, then the plan would not be feasible as it could not be executed when that information set is reached.)

3. At all nodes in a given information set, the player plays the same move. (After all, the player cannot distinguish those nodes from each other and cannot play different moves at those nodes.)

Example 2.4 (Battle of the Sexes)**.** In Figure 2.5 on the left, each player has only one information set, choosing between moves A and B. Hence, the set of strategies is $\{A, B\}$ for each player. Although Bob has two decision nodes, they are both in the same information set. Hence, he needs to either play A at both nodes (strategy A) or B at both nodes (strategy B).

Example 2.5 (Sequential Battle of the Sexes)**.** Now consider the game on the right in Figure 2.5. Alice still has only one information set. Hence, the set of strategies for Alice is $\{A, B\}$. On the other hand, Bob has now two information sets. Hence, a strategy for Bob determines what to do at each information set, i.e., depending on what Alice does. His strategies are:

$AA = A$ if Alice plays A, and A if Alice plays B;

$AB = A$ if Alice plays A, and B if Alice plays B;

$BA = B$ if Alice plays A, and A if Alice plays B;

$BB = B$ if Alice plays A, and B if Alice plays B.

The set of strategies can be quite large, even in simple games in extensive form. For example, as it is shown in Exercise 2.4 below, the number of strategies in a 3×3 tick-tack-toe game is too large to consider all strategies for practical purposes.[4] One can then look for clever ways to reduce the number of strategies by grouping them in equivalence classes and considering only one representative strategy from each group as in Exercise 2.1 below. For such purposes, it might suffice to look at the reduced-form strategies. Formally, a reduced-form strategy is defined as an incomplete contingent plan that determines which action the player will take at each information set she is to move and that has not been precluded by this plan. Reduced-form strategies will *not* be used in the book. This is because the solution concepts for dynamic games treat strategies with identical reduced form differently when some of these strategies prescribe moves that are irrational given the information the players have at the time they make those moves.

What are the outcomes of the strategies of the players? What are the payoffs generated by those strategies? Towards answering these questions, I next introduce a couple of formalisms that will be used throughout the book.

Definition 2.7. In a game with players $N = \{1, \ldots, n\}$, a *strategy profile* is a list

$$s = (s_1, \ldots, s_n)$$

of strategies, one for each player. The set of all strategy profiles is denoted by

$$S = S_1 \times \cdots \times S_n.$$

[4] Try to compute the number of strategies yourself before you read the solution.

Definition 2.8. In a game without Nature, each strategy profile s leads to a unique terminal node $z(s)$ if all players play according to s; the node $z(s)$ is called the *outcome* of s. The payoff vector from strategy s is the payoff vector at $z(s)$.

Equivalently, the outcome can be described by the resulting history, which is called *the path of play.*

Example 2.6 (Sequential Battle of the Sexes)**.** In the game on the right in Figure 2.5, if Alice plays A and Bob plays AA, then the outcome is

`both players choose` A,

and the payoff vector is $(2, 1)$. If Alice plays A and Bob plays AB, the outcome is the same, yielding the payoff vector $(2, 1)$. If Alice plays B and Bob plays AB, then the outcome is now

`both players choose` B,

and the payoff vector is $(1, 2)$. One can compute the payoffs for the remaining strategy profiles similarly.

In games with Nature, a strategy profile leads to a *probability distribution* on the set of terminal nodes. The outcome of the strategy profile is then the resulting probability distribution. The payoff vector from the strategy profile is the expected payoff vector under the resulting probability distribution.

Example 2.7 (A game with Chance)**.** In Figure 2.6, each player has two strategies, Left and Right. The outcome of the strategy profile (Left,Left) is the lottery that

`Nature chooses Head and Player 1 plays Left`

with probability 1/2 and

`Nature chooses Tail and Player 2 plays Left`

with probability 1/2. Hence, the expected payoff vector is

$$u(Left, Left) = \frac{1}{2}(5, 0) + \frac{1}{2}(3, 3) = (4, 3/2).$$

2.4 Normal form

Sometimes, it suffices to summarize all of the information in a strategic situation by the set of strategies and the payoffs of the players from the strategy profiles, computed as above. Such a summary representation is called normal-form or strategic-form representation.

Definition 2.9 (Normal-Form)**.** A *normal-form* game is any list

$$G = (N, S_1, \ldots, S_n; u_1, \ldots, u_n),$$

where $N = \{1, \ldots, n\}$ is the set of players, and for each player $i \in N$, S_i is the set of all strategies that are available to player i, and

$$u_i : S_1 \times \cdots \times S_n \to \mathbb{R}$$

is player i's Von Neumann-Morgenstern utility function. I will often write $G = (N, S, u)$ where $S = S_1 \times \cdots \times S_n$ and $u = (u_1, \ldots, u_n)$, a mapping from S to $\mathbb{R}^n$.

The above definition makes many assumptions. First, the players choose their strategies simultaneously, and no player has any information about the other players' strategies when she chooses her own strategy. As discussed above, a strategy is a complete contingent plan that determines what moves a player takes at her information sets, and players may be able to observe other players' moves and react to them. In the normal-form representation, the players choose these contingent plans simultaneously before the game starts. Second, a player's utility may depend not only on her own strategy but also on the strategies played by other players, and each u_i is a Von-Neumann-Morgenstern utility function, meaning that player i plays a strategy that maximizes the expected value of u_i (where the expected values are computed with respect to her own beliefs). Player i is said to be *rational* in that case. Finally, all these are assumed to be common knowledge, i.e., it is common knowledge that the set of players is $N = \{1, \ldots, n\}$, that the set of strategies available to each player i is S_i, and that each i plays a strategy that maximizes the expected value of u_i given her beliefs. When there are only two players, one can represent the normal-form game by a bimatrix (i.e., by two matrices), as in the next example.

Example 2.8 (Battle of the Sexes)**.** In Figure 2.5, the extensive-form game on the left can be represented as

$$\begin{array}{c|c|c|} & A & B \\ \hline A & 2,1 & 0,0 \\ \hline B & 0,0 & 1,2 \\ \hline \end{array} \tag{2.1}$$

in normal form. Both players have strategies A and B. Alice chooses between the rows while Bob chooses between the columns. The book will conventionally refer to the player who chooses between the rows as Player 1 or "the row-player" and will refer to the other player as Player 2 or "the column-player". In each box the first number is Player 1's payoff and the second one is Player 2's payoff.

2.5 From Extensive Form to Normal Form

As it has been described in detail in the previous section, in an extensive-form game, the set of strategies is the set of all complete contingent plans, mapping information sets to the available moves. Moreover, each strategy profile s leads to an outcome $z(s)$, which is in general probability distribution on the set of terminal nodes. The payoff vector is the expected payoff vector from $z(s)$. One can always convert an extensive-form game to a normal form game in this way.

Example 2.9 (Sequential Battle of the Sexes)**.** Based on the earlier analyses, the normal-form game corresponding to the extensive-form game on the right in Figure 2.5 is

	AA	AB	BA	BB
A	2, 1	2, 1	0, 0	0, 0
B	0, 0	1, 2	0, 0	1, 2

Information sets are very important. The two extensive-form games in Figure 2.5 may appear similar, but they correspond to two distinct situations and have quite distinct normal-form representations. Under perfect information, Bob knows what Alice has done, while nobody knows about the other player's move under the version with imperfect information.

As mentioned above, when there are chance moves, one needs to compute the expected payoffs in order to obtain the normal-form representation. This is illustrated in the next example.

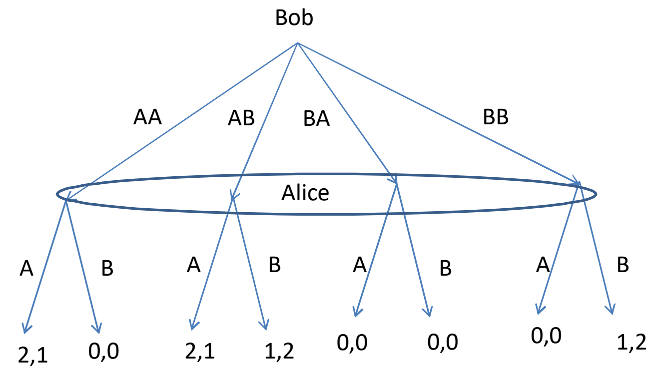

Figure 2.8: A battle of the sexes game with perfect information?

Example 2.10 (A game with Nature)**.** Consider the game in Figure 2.6, where each player has two strategies, Left and Right. Following the earlier calculations, the normal-form representation is obtained as follows:

	Left	Right
Left	$4, 3/2$	$5/2, -5/2$
Right	$5/2, 5/2$	$1, -3/2$

The payoff from (Left,Left) has been computed already. The payoff from (Left, Right) is computed as

$$\frac{1}{2}(5,0) + \frac{1}{2}(0,-5) = (5/2, -5/2).$$

While there is a unique normal-form representation for any extensive-form game (up to a relabeling of strategies), there can be many extensive-form games with the same normal-form representation. After all, any normal-form game can also be represented as a simultaneous-action game in extensive form. For example, the normal-form game of the sequential Battle of the Sexes game can also be represented as in Figure 2.8.

2.6 Fundamental Concepts

This section introduces some fundamental concepts that will be used extensively throughout the book. It formally defines concepts of a belief, a best response, and a concept called mixed strategy.

Notation 2.1. For any player $i \in N$, the notation s_{-i} denotes the list of strategies s_j played by all the players j other than i, i.e.,

$$s_{-i} = (s_1, ...s_{i-1}, s_{i+1}, ...s_n).$$

The set of other players' strategies is denoted by

$$S_{-i} = \prod_{j \neq i} S_j.$$

Finally, the notation (s_i, s_{-i}) denotes the strategy profile in which i plays s_i and other players play according to s_{-i}; i.e., $(s_i, s_{-i}) = (s_1, ...s_{i-1}, s_i, s_{i+1}, ...s_n)$.

2.6.1 Belief

Throughout the book, it will be assumed that, whenever a player does know something, she forms a belief that can be expressed as a probability distribution. In particular, they do not know other players' strategies, and they will form a belief about these strategies:

Definition 2.10. A *belief* of player i (about other players' strategies) is a probability distribution β_{-i} on S_{-i}.

Write $u_i(s_i, \beta_{-i})$ for the expected payoff from playing s_i under belief β_{-i}. When S_{-i} is finite, this is computed as

$$u_i\left(s_i, \beta_{-i}\right) = \sum_{s_{-i} \in S_{-i}} u_i\left(s_i, s_{-i}\right) \beta_{-i}\left(s_{-i}\right).$$

For example, in the Battle of the Sexes game in (2.1), suppose Alice assigns probability p to Bob playing A, and write $\beta_{-1}(A) = p$ and $\beta_{-1}(B) = 1 - p$. Then, her expected payoff from playing A is

$$u_1\left(A, \beta_{-1}\right) = pu_1(A, A) + (1-p)\, u_1(A, B) = p \times 2 + (1-p) \times 0 = 2p.$$

She thinks that with probability p Bob plays A, resulting in payoff 2 from (A, A), and with probability $1 - p$ Bob plays B, resulting in payoff 0 from (A, B). Likewise, her payoff from playing B is $u_1\left(A, \beta_{-1}\right) = 1 - p$.

2.6.2 Best Response

Throughout the book, it will further be assumed that each player plays a strategy that maximizes her expected payoff under her belief:

Definition 2.11. For any player i, a strategy s_i is said to be a *best response* to s_{-i} if

$$u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i}) \qquad \text{(for all } s'_i \in S_i\text{)}.$$

A strategy s_i is said to be a *best response* to a belief β_{-i} if playing s_i maximizes the expected payoff under β_{-i}, i.e.,

$$u_i\left(s_i, \beta_{-i}\right) \geq u_i\left(s'_i, \beta_{-i}\right), \qquad \forall s'_i \in S_i.$$

For example, in the Battle of the Sexes game, A is the unique best response to A because $u_1(A, A) = 2 > 0 = u_1(B, A)$. Likewise, B is the unique best response to B. One can compute the set of Alice's best responses to a belief β_{-1} as follows. Recall that the expected payoffs from A and B are $u_1\left(A, \beta_{-1}\right) = 2p$ and $u_1\left(B, \beta_{-1}\right) = 1 - p$, respectively, where $p = \beta_{-1}(A)$ is the probability that Bob plays A. For each p, one compares $2p$ to $1-p$. If $2p > 1-p$, then the expected payoff from A is strictly higher than the expected payoff from B, and A is the unique best response. This is the case if $p > 1/3$. Likewise, if $1 - p > 2p$, then the expected payoff from B is strictly higher, and B is the unique best response. This is the case if $p < 1/3$. Finally, for $p = 1/3$, we have $1 - p = 2p$, and the expected payoffs from strategies A and B are equal. In that case, Alice is indifferent between the strategies A and B, and both strategies A and B are a best response. As the last instance shows, in some cases, there can be multiple best responses, and a player may play any of the best responses.

When there are infinitely many strategies to choose from, finding a best response can be challenging, and there may not be any best response because there could always be a better option. For example, what would you choose as your wealth if you could choose any number? There is a best response if the utility function is continuous and the set of strategies is closed and bounded. Exercise 2.5 below shows how to compute the set of best responses in some important cases with infinitely many strategies.

2.6.3 Mixed Strategy

The set of available strategies for a player i is S_i. Sometimes it is useful to imagine that the players can randomize between their strategies. For example, a penalty kicker may toss a coin before deciding whether to kick to the left or right side of the goalkeeper. Such randomized strategies play an important role in the analyses of games, and they are formally defined as follows.

Definition 2.12. A *mixed strategy* of a player i is a probability distribution over the set S_i of her strategies; the strategies $s_i \in S_i$ are also called *pure* strategies.

When player i has finitely many strategies, a mixed strategy σ_i for player i is a function on S_i such that $0 \leq \sigma_i(s_i) \leq 1$ for each strategy s_i and the weights $\sigma_i(s_i)$ add up to 1. For example, in the Battle of Sexes game, a mixed strategy σ_1 for Alice may put equal probabilities on each strategy: $\sigma_1(A) = \sigma_1(B) = 1/2$.

For any mixed strategy σ_i and any s_{-i}, $u_i(\sigma_i, s_{-i})$ denotes the expected payoff player i gets from playing mixed strategy σ_i against s_{-i}. When S_i is finite, this is computed as

$$u_i(\sigma_i, s_{-i}) = \sum_{s_i \in S_i} u_i(\sigma_i(s_i), s_{-i}).$$

For example, in the Battle of the Sexes game above, if Alice plays the mixed strategy σ_1 with $\sigma_1(A) = \sigma_1(B) = 1/2$ against A, then her expected payoff is

$$u_1(\sigma_1, A) = \frac{1}{2} u_1(A, A) + \frac{1}{2} u_1(B, A) = 1.$$

With probability $1/2$, Alice ends up playing A and obtains the payoff of 2 associated with strategy profile (A, A), and with probability $1/2$, she ends up playing B and obtains the payoff of 0 associated with strategy profile (B, A).

The concept of a mixed strategy is somewhat less natural than the concepts of belief and best response. There are many interpretations for mixed strategies. One interpretation is deliberate randomization as in the penalty kicker example above. Another interpretation is that players in an otherwise homogeneous population play heterogeneous strategies, some playing one strategy and others playing some other. A mixed strategy is the distribution of strategies in such a population. Players are matched randomly, and they do not know the strategy played by the opponent, but they know the distribution. Yet another interpretation is that the players have some idiosyncratic personal private information; they choose different strategies depending on their information. Their actions appear random to the other players, as they are not privy to her information. For example, a penalty kicker may kick to the left or right, depending on how confident he feels at the moment. In all cases, mixed strategies serve as a device to represent the uncertainty the *other players* face regarding the strategy played by a player.

2.6.4 Basic Properties of Expected Utility Preferences

Expected utility preferences on lotteries have specific useful properties. This section briefly describes some of these properties that will be used in the book frequently.

Linearity in Probabilities The expected utility of a lottery $p : Z \to [0,1]$ is a linear function of p when p is viewed as a vector:

$$U(p) = \sum_{z \in Z} u(z) \, p(z) \, .$$

If $u(z) > u(z')$, then the player gets better off as one moves some of the probability on z' to z without affecting the other probabilities. If the player could choose any probability distribution, then she would put all the probability on the outcomes z with the highest payoffs. She would randomize only if she is indifferent between the outcomes she is putting positive probability. In particular, for a player i, the payoff from a mixed strategy σ_i under a belief β_{-i} is the weighted average of the payoffs from the pure strategies, weighted according to σ_i:

$$u_i\left(\sigma_i, \beta_{-i}\right) = \sum_{s_i \in S_i} u_i\left(s_i, \beta_{-i}\right) \sigma_i\left(s_i\right) .$$

A player chooses a mixed strategy that puts positive probabilities on multiple strategies only when she is indifferent between those strategies and each of these strategies are also best responses.

Invariance Under Affine Transformations A player's preference between any two lotteries does not change when her utility function is multiplied with a positive constant, or a constant is added to it. Specifically, consider two utility functions $u : Z \to \mathbb{R}$ and

$v : Z \to \mathbb{R}$ where $v(z) = au(z) + b$ for some constants a and b where $a > 0$. Then, the expected value of v under a lottery p is

$$V(p) = \sum_{z \in Z} v(z)\, p(z) = a \sum_{z \in Z} u(z)\, p(z) + b = aU(p) + b,$$

obtained by multiplying the expected value of u by a and adding b. Hence, p is weakly preferred to q under v if and only if p is weakly preferred to q under u:

$$V(p) \geq V(q) \iff U(p) \geq U(q).$$

Thus, the analysis of a game does not change if one adds a constant to a player's payoffs or multiplies them with a positive constant.

Conversely, nonlinear transformations do affect the preferences between the lotteries, possibly affecting the analysis of a game. For example, although a player may always like to have more wealth, her attitudes toward risky assets will be determined by the shape of her utility function. To see this, take $Z = [0, 1]$ be the set of wealth levels a player can have, and consider two increasing utility functions $u(z) = z$ and $v(z) = z^{\alpha}$ for some positive $\alpha < 1$. Consider two lotteries p and q where $p(0) = p(1) = 1/2$ and $q(\hat{z}) = 1$ for some $\hat{z}$ with $\hat{z} < 1/2 < \hat{z}^{\alpha}$. The expected utility from the risky lottery p is $1/2$ under both utility functions. On the other hand, the expected utility from the safe lottery q is $\hat{z}$ under utility function u and $\hat{z}^{\alpha}$ under the utility function v. Since $\hat{z} < 1/2 < \hat{z}^{\alpha}$, the player strictly prefers the risky lottery under u and the safe lottery under v.

Attitudes Towards Risk In applications, we will often consider problems with monetary rewards, such as the profit of a firm or the price of a good. In those problems, the utility function on monetary outcomes will reflect the player's attitudes towards risk. In particular, it is useful to know when a player is averse to taking risk, likes taking risk, or indifferent towards risks. This section relates the attitudes of an individual towards risk to the properties of her von-Neumann-Morgenstern utility function.

Consider the lotteries with monetary prizes and consider a player with utility function $u : Z \to \mathbb{R}$ where $Z = \mathbb{R}$. A player is said to be *risk-neutral* if she is indifferent between any two lotteries p and q with equal expected values, i.e., with

$$\sum_{z} zp(z) = \sum_{z} zq(z). \tag{2.2}$$

For example, she is indifferent towards a bet that gives him \$1 with 50% chance and makes him lose \$1 with 50% chance. Note that the indifference condition (2.2) corresponds to the expected utility under a specific utility function:

$$u(z) = z;$$

and indifferences are the same if the preferences are reversed: $u(z) = -z$. By the invariance property above, this implies that a player is risk-neutral if and only if her utility function is linear, i.e.,

$$u(z) = az + b$$

for some constants a and b. When a player is risk-neutral, one can simply consider the dollar amounts as payoffs, and in applications with monetary rewards, we will often assume risk neutrality.

In economic applications, players are often assumed to be averse to risk, in that they would prefer to get rid of uncertainty if they can do it without affecting the expected monetary reward. Formally, a player is said to be *risk-averse* if she prefers to get the expected value $\sum zp(z)$ of a lottery p to p itself:

$$u\left(\sum zp(z)\right) \geq \sum u(z)\, p(z)\,. \tag{2.3}$$

In particular, they would prefer $\lambda x + (1-\lambda)\, y$ to a lottery that gives x with probability λ and y with the remaining probability. That is,

$$u(\lambda x + (1-\lambda)\, y) \geq pu(x) + (1-p)u(y).$$

This is a familiar inequality from calculus: a function g is said to be *concave* if

$$g(\lambda x + (1-\lambda)y) \geq \lambda g(x) + (1-\lambda)g(y)$$

for all $\lambda \in (0,1)$. Thus, risk-aversion implies a concave utility function. The converse is also true: if the utility function u is concave, then inequality (2.3) always holds, and the player is risk-averse. Therefore, a player is risk-averse if and only if her utility function is concave. Similarly, a player is said to be *risk-seeking* if she has a *convex* utility function.

2.7 Exercises with Solutions

Exercise 2.1. What is the normal-form representation for the game in Figure 2.9?

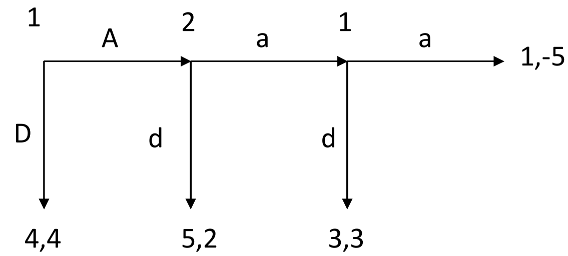

Figure 2.9: A centepede-like game

Solution. Player 1 has two information sets with two actions in each. Since the set of strategies is functions that map information sets to the available moves, she has the following four strategies: Aa, Ad, Da, Dd. The meaning here is straightforward: Ad assigns A to the first information set and d to the last information set. On the other hand, Player 2 has only two strategies: a and d. Filling in the payoffs from the tree, one obtains the following normal-form representation:

	a	d
Aa	$1,-5$	$5,2$
Ad	$3,3$	$5,2$
Da	$4,4$	$4,4$
Dd	$4,4$	$4,4$

This exercise illustrates the fact that, in computing strategies, one must also consider the decision in the information sets that are precluded by the strategy itself. For example, Player 1 plays D, the game ends, and her decision at her second information set is irrelevant. One must still assign a move at that information set, and there are two strategies, Da and Dd, that differ from each other with respect to their description at that information set. Since these two strategies lead to the same outcome, they lead to the same payoffs. Some solution concepts, such as Nash equilibrium, will treat them as equivalent, but some others, such as backward induction, will treat them differently (these solution concepts will be formally introduced later).

Exercise 2.2. Write the game in Figure 2.10 in normal form.

Solution. The important point in this exercise is that Player 2 has to play the same move in a given information set. For example, she cannot play x on the left node and y on the

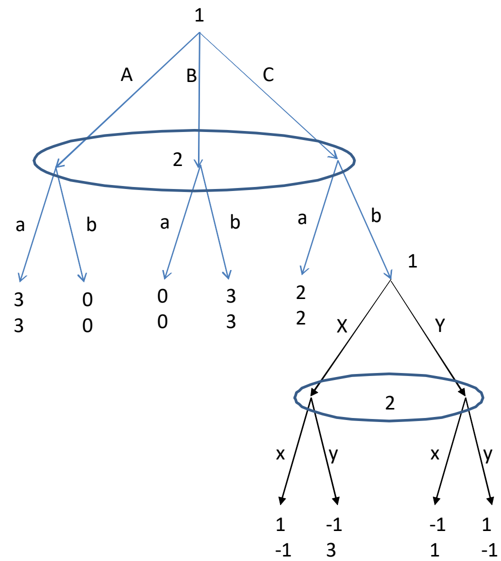

Figure 2.10:

right node of her second information set. Hence, her set of strategies is $\{ax, ay, bx, by\}$. This results in the following normal-form game:

	ax	ay	bx	by
AX	$3,3$	$3,3$	$0,0$	$0,0$
AY	$3,3$	$3,3$	$0,0$	$0,0$
BX	$0,0$	$0,0$	$3,3$	$3,3$
BY	$0,0$	$0,0$	$3,3$	$3,3$
CX	$2,2$	$2,2$	$1,-1$	$-1,3$
CY	$2,2$	$2,2$	$-1,1$	$1,-1$

Exercise 2.3. Write the game in Figure 2.11 in normal form, where the first entry is the payoff of the student and the second entry is the payoff of the professor.

Solution. In this game, the student knows whether he is sick or healthy when he decides whether to take the regular exam or ask for a make up exam. Hence, he has two information sets, and a strategy must prescribe an action when he is healthy and an action when he is sick. His strategies can be written as RR, RM, MR, and MM, where RM means Regular when Healthy and Make up when Sick, MR means Make up when

Healthy and Regular when Sick, etc. On the other hand, the professor does not know whether he was sick; she only observes that he asked for a make up. She has only one information set. Hence, she has only two strategies: "same" and "new". The normal form game is as follows:

Student\Prof	same	new
RR	$1,0$	$1,0$
RM	$3,1/2$	$3/2,(1-c)/2$
MR	$2,-1$	$1/2,-(1+c)/2$
MM	$4,-1/2$	$1,-c$

The main objective of this example is to illustrate how to compute the payoffs in games with chance moves, as in this game. One must compute the expectations of payoffs over the chance moves. In this game, the payoffs are obtained by taking expectations over whether the student is healthy or sick. For example, RR leads to $(2,1)$ and $(0,-1)$ with equal probabilities, yielding $(1,0)$, regardless of the strategy of Prof. On the other hand, (RM,new) leads to $(2,1)$ and $(1,-c)$ with equal probabilities, yielding $(3/2,(1-c)/2)$. Likewise, the strategy profile (RM,same) leads to $(2,1)$ and $(4,0)$ with equal probabilities, yielding $(3,1/2)$. The payoffs for the remaining strategy profiles are computed similarly.

Exercise 2.4. How many strategies each player has in a 3×3 tick-tack-toe game?

Solution. I will compute the number of strategies for the player who moves first. A strategy for her prescribes her first square, her second square for every information set she chooses her second square, her third square for every information set she chooses her third square, and so on. She chooses her first square out of 9. When she gets to choose her second square, she chooses one square out of 7 remaining available squares—but there are 9×8 information sets at which she chooses her second square. Likewise, she chooses her third square out of 5 remaining squares, and there are $9\times 8\times 7\times 6$ information sets at which she chooses her third square. Continuing in this fashion, one can compute that the number of strategies is

$$9\times 7^{9\times 8}\times 5^{9\times 8\times 7\times 6}\times 3^{9\times 8\times 7\times 6\times 5\times 4}\times 1^{9\times 8\times 7\times 6\times 5\times 4\times 3\times 2}.$$

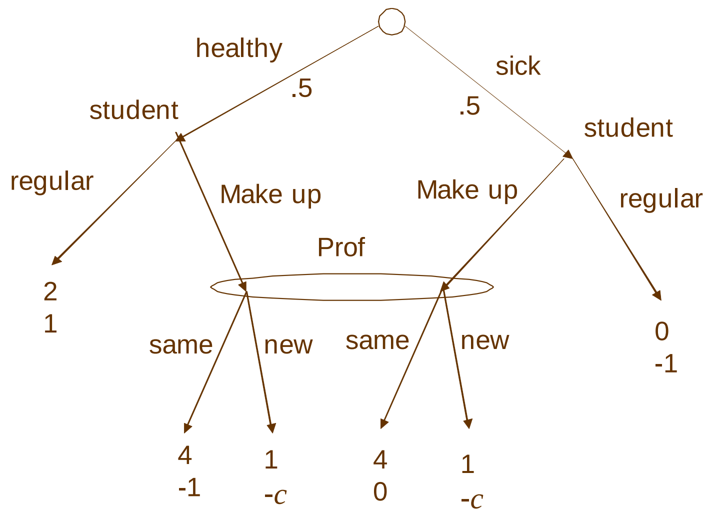

Figure 2.11:

There are approximately 6.0267×10^{31031} strategies! (In comparison, in the known observable universe, it is believed that there are less than 3×10^{23} stars and less than 10^{82} atoms.)

Exercise 2.5. For the following strategy space and utility pairs, check if best response exists for player 1, and compute it when it exists.

1. $S_1 = [0,1]$; $u_1(s_1) = s_1$ if $s_1 < 1$ and $u_1(1) = 0$.

2. $S_1 = S_2 = [0,\infty)$; $u_1(s_1) = s_1 s_2$.

3. Partnership Game: $S_1 = S_2 = [0,\infty)$; $u_1(s_1) = \theta s_1 s_2 - s_1^2$ where $\theta > 0$.

4. First-Price Auction: $S_1 = S_2 = [0,\infty)$;

$$u_1(s_1, s_2) = \begin{cases} v - s_1 & \text{if } s_1 > s_2 \\ (v - s_1)/2 & \text{if } s_1 = s_2 \\ 0 & \text{otherwise} \end{cases}$$

   where $v > 0$.

5. Price Competition: $S_1 = S_2 = [0, \infty)$;
$$u_1(s_1, s_2) = \begin{cases} (1 - s_1) s_1 & \text{if } s_1 < s_2 \\ (1 - s_1) s_1/2 & \text{if } s_1 = s_2 \\ 0 & \text{otherwise.} \end{cases}$$

6. Quantity Competition: $S_1 = S_2 = [0, \infty)$; $u_1(s_1, s_2) = (1 - s_1 - s_2) s_1 - cs_1$.

Solution. The solution to each part is as follows.

1. Clearly, there is no best response. Plot a graph for illustration. (Continuity fails here.)

2. Everything is a best response when $s_2 = 0$, and nothing is a best response when $s_2 \neq 0$. Strategy space is not bounded. This also shows that there can be more than one best response.

3. Best response exists although S_1 is not closed and bounded. Take the partial derivative with respect to s_1 and set it equal to zero in order to obtain the "first-order condition" for maximum:
$$\frac{\partial u_1}{\partial s_1} = \theta s_2 - 2s_1 = 0.$$
Solve this equation to obtain the best response:
$$s_1 = \theta s_2/2.$$
One does not need to check the second-order condition because u_1 is concave.

4. Any $s_1 \leq s_2$ is a best response when $s_2 = v$; any $s_1 < s_2$ is a best response when $s_2 > v$, and nothing is a best response when $s_2 < v$. Continuity fails.

5. Everything is a best response when $s_2 = 0$, and nothing is a best response when $0 < s_2 \leq 1/2$. The unique best response is $1/2$ whenever $s_2 > 1/2$. Continuity fails.

6. There is a unique best response. As in Part 3, the first-order condition is
$$\frac{\partial u_1}{\partial s_1} = 1 - 2s_1 - s_2 - c = 0,$$
yielding
$$s_1 = \frac{1 - s_2 - c}{2}.$$

2.8 Exercises

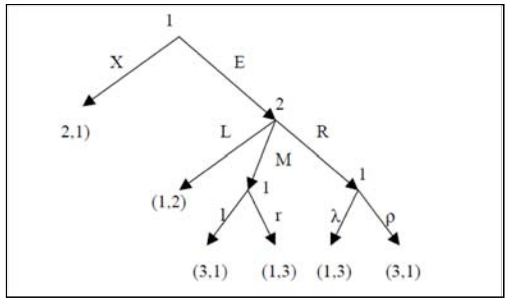

Figure 2.12:

Exercise 2.6. Write the games in Figures 2.12, 2.13, and 2.14 in normal form.

Exercise 2.7. Consider the game in which the following are commonly known. First, Ann chooses between actions a and b. Then, with probability $1/3$, Bob observes which action Ann has chosen and with probability $2/3$ he does not observe the action she has chosen. In all cases (regardless of whether he has observed Ann chose a, or he has observed Ann chose b, or he has not observed any action), Bob chooses between actions α and β. The payoff of each player is 1 after (a, α) and (b, β) and 0 otherwise.

1. Write the above game in extensive form.

2. Write the above game in normal form.

Exercise 2.8. Consider the following variation of the above game. First, Ann chooses between actions a and b. Then, Bob decides whether to observe the chosen action of

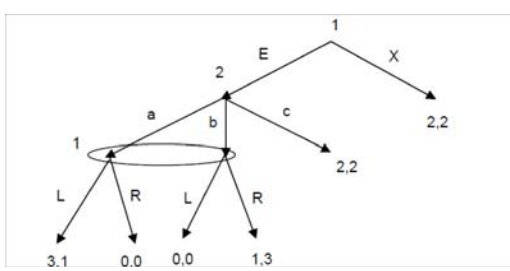

Figure 2.13:

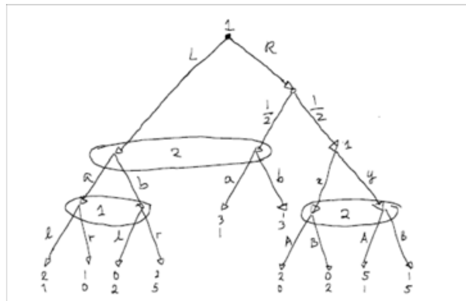

Figure 2.14:

Ann or not, by choosing between the actions Open and Shut, respectively. In all cases, Bob then chooses between actions α and β. The payoff of Ann is 1 after (a, α) and (b, β) and 0 otherwise, regardless of whether Bob chooses Open or Shut. The payoff of Bob is equal to the payoff of Ann if he has chosen Shut, and his payoff is equal to the payoff of Ann minus 1/2 if he has chosen Open.

1. Write the above game in extensive form.

2. Write the above game in normal form.

Exercise 2.9. Ann is a researcher, working on a project for Bob. First, Ann decides whether to Work or Shirk. If she works, the project succeeds with probability p and fails with probability $1 - p$ where $1 > p > 1/2$. If she shirks, the project fails for sure. Bob does not see if Ann works or shirks, but he does observe whether the project succeeds or fails. After observing whether it succeeds or fails, Bob decides whether to Renew Ann's employment contract, or Terminate it. Ann gets 3 from renewal of contract and loses 1 from working (i.e., her payoffs at the paths with work-renew, work-terminate, shirk-renew, and shirk-terminate are 2, -1, 3, and 0, respectively). Bob gets 1 if Ann works and he renews or Ann shirks and he terminates; he gets -1 otherwise (i.e., on the paths with work-terminate and shirk-renew).

1. Write this game in extensive form.

2. Write this game in normal form.

Exercise 2.10. Consider the following variation of the game in the previous problem. Ann does not know whether Bob can observe if Ann works. With probability 1/2, the situation is as in the previous problem, where Bob can only observe whether the project succeeds. With probability 1/2, Bob can observe both whether Ann works and whether the project succeeds. (Everything else is as in the previous problem).

1. Write this game in extensive form.

2. Write this game in normal form. (If you think there are too many strategies, it suffices to identify the strategy sets correctly and describe how the payoffs are computed.)

Exercise 2.11. Bob works for Ann. First, Ann chooses between Power (P) and Delegation (D). If she chooses P, then Nature either confirms (C) or reverses (R) Ann's action, where the probability of C is 1/2. If Ann chooses P and Nature confirms, then Ann gets to choose between A and B. Otherwise, Bob gets to choose between a and b without knowing what happened (i.e., without knowing whether Ann chose D or Ann chose P but her decision is reversed). The payoffs are as follows, where the first and the second entries are the payoffs of Ann and Bob, respectively:

PCA	PCB	PRa	PRb	Da	Db
$(3,0)$	$(0,0)$	$(2,-1)$	$(0,1)$	$(2,2)$	$(0,1)$

(Here, outcomes are described by the path. E.g., PCA is the terminal node after Ann chooses Power; Nature confirms, and Ann finally chooses A. Note that Nature moves only if Ann chooses P; Bob moves only if Ann Chooses D or Ann chooses P and Nature chooses R.)

1. Write this game in extensive form.

2. Write this game in normal form.

Exercise 2.12. Consider two profit-maximizing firms, namely 1 and 2. First, Firm 1 chooses a production level $q_1 \in [0,1]$. Then, observing q_1, Firm 2 chooses a production level $q_2 \in [0,1]$. Each sells its good at price $P = 1 - q_1 - q_2$. The payoff of firm i is Pq_i, which is its profit.

1. Write this game in extensive form.

2. Write this game in normal form.

Exercise 2.13. This question asks you to formalize a TV game. The players are a host and a contestant. There are three doors, named A, B, and C. Nature puts a prize behind one of the doors, leaving the other two doors empty, where each door has an equal probability of containing the prize. Host knows where the prize is but the contestant does not. First, the contestant selects a door $x \in \{A, B, C\}$. Observing x (and knowing where the prize is), the host opens one of the remaining two doors, revealing to the contestant whether the prize is behind that door. (Here, the door opened is $y \in \{A, B, C\} \setminus \{x\}$.) Finally, knowing x, y, and the content of y, the contestant picks a door $z \in \{A, B, C\}$. The host and the contestant get -1 and 1, respectively, if the prize is behind door z; they both get 0 otherwise.

1. Write this game in extensive form.

2. For each player, describe the set of strategies, and determine the number of strategies. Write down a strategy for the contestant. (Make sure that what you write is indeed a strategy; it doesn't need to be a good strategy.)

Exercise 2.14. Consider the variation of the game in Figure 2.4, in which Player 1 believes that Player 2 can distinguish actions T and B with probability 1/3 and cannot distinguish them probability 2/3, and this belief is common knowledge.

1. Write this game in extensive form.

2. Write this game in normal form.

Exercise 2.15. Describe one strategy for the player who moves first in 3×3 tick-tack-toe game. (Make sure that what you describe is indeed a strategy; it doesn't need to be a good strategy.)

Exercise 2.16. This question asks you to compute number of some reduced-form strategies in 3×3 tick-tack-toe game.

1. Compute the number of reduced-form strategies that do not condition on player's own past actions, e.g., if she has chosen the top-left square in the first-round it is not prescribed how she would have chosen her second square if she had chosen the center square in the first round.

2. Compute the set of strategies in which action at any given round depends only on which boxes have been picked by each player already (without distinguishing the order in which the boxes have been picked).