Solutions to Problem Set 7
14.12 Fall 2023

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1 Problem 1

There are n firms with no marginal cost. In the stage game, firm i produces q_i and faces the price $P = \max\{1 - \sum_i q_i, 0\}$. All the past production levels are perfectly observable. The pay-off of firm i is given by

$$u_i(\{q_{i,t}, q_{-i,t}\}_t) = (1-\delta)\sum_{t=0}^{\infty} \delta^t q_{i,t} \max\{1 - \sum_i q_i, 0\}$$

As in class, we define $f(q)$ to be the profit when everyone produces q and $g(q)$ to be the profit of the firm that best responds to everyone else producing q. That is

$$f(q) = qP(nq) = q(1-nq)\mathbf{I}(q \leq \frac{1}{n})$$

$$g(q) = \max_{q'} q'P(q' + (n-1)q) = \frac{(1-(n-1)q)^2}{4}\mathbf{I}(q \leq \frac{1}{n-1})$$

Grim-Trigger

Suppose the grim-trigger strategy is to play q^* every period if no one has deviated. If there is any deviation in the past, then the player play $\tilde{q}$.

- For the history where someone has deviated in the past, one shot deviation is not profitable only when $\tilde{q} = q^{NE} = \frac{1}{n+1}$.
- For the history where no one has deviated in the past, one shot deviation is not profitable

only when

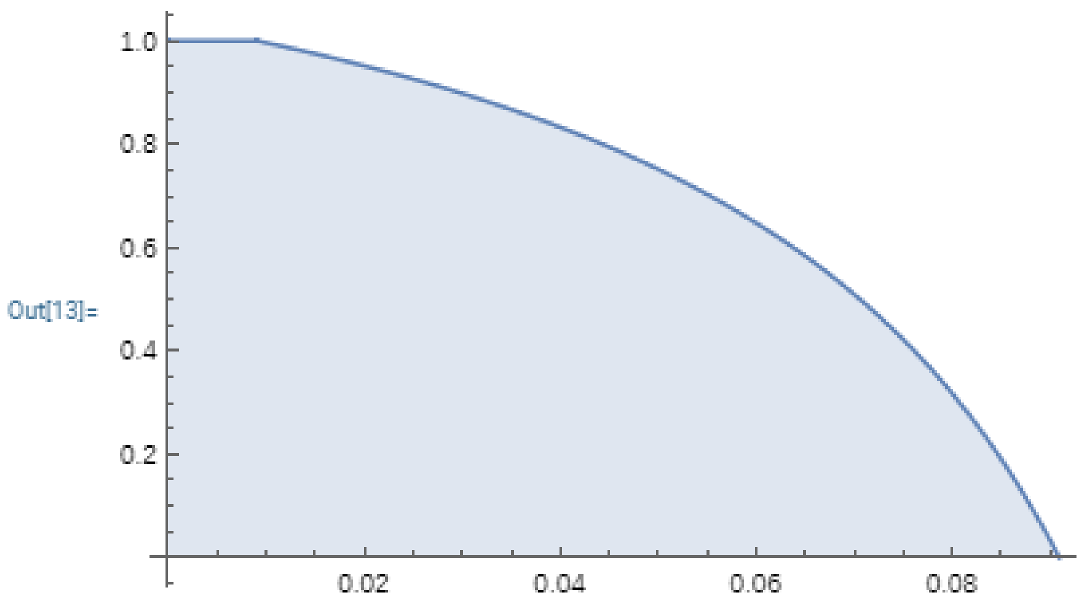
$$g(q^*)(1-\delta)+\delta f(q^{NE}) \leq f(q^*)$$

$$\iff \delta \geq \frac{g(q^*)-f(q^*)}{g(q^*)-f(q^{NE})}$$

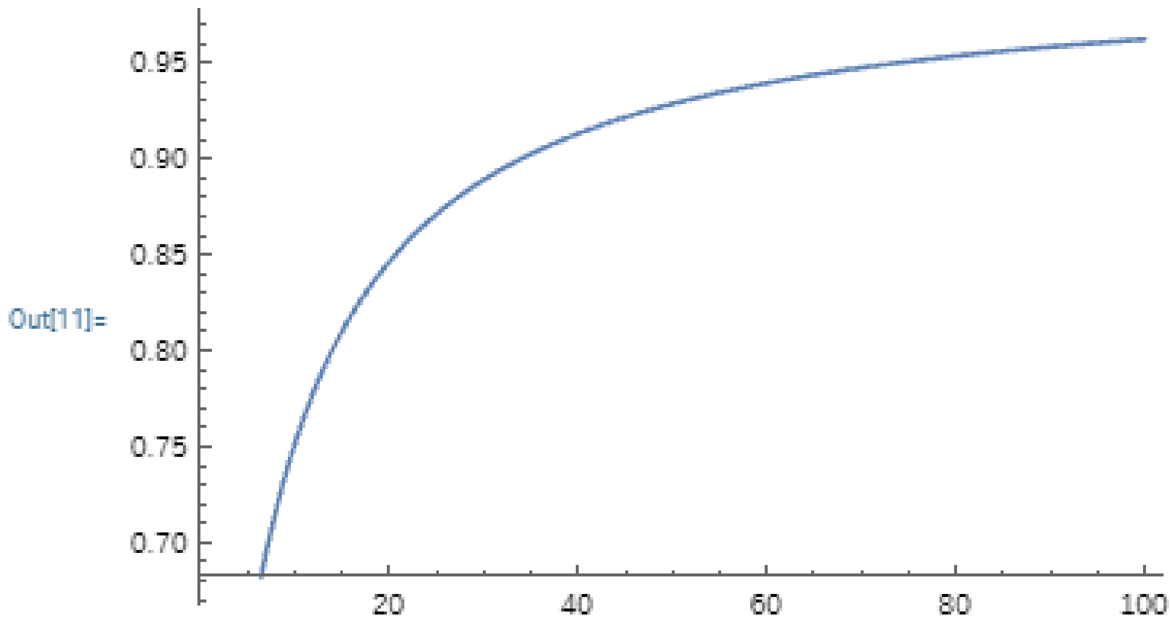
Note that for the above inequality to hold, we should always have $q^* \leq q^{NE} = \frac{1}{n+1} < \frac{1}{n} < \frac{1}{n-1}$ Therefore, we can rewrite the inequality as

$$\delta \geq \frac{\frac{(1-(n-1)q^*)^2}{4} - q^*(1-nq^*)}{\frac{(1-(n-1)q^*)^2}{4} - (\frac{1}{n+1})^2}$$

Below, I plot the region for δ against q^* for $n = 10$



Below, I plot the lower bound for δ against n for $q^* = q^M = \frac{1}{2n}$



Carrots and Sticks

Under this strategy, players produce q^C in the carrots stage and q^S in the sticks stage. If everyone produces what they are supposed to produce in period t, the game transitions to carrots stage in period $t+1$, otherwise it transitions to sticks stage.

Suppose the payoff from playing carrots stage is V_C and sticks stage is V_S. There are two relevant histories to consider.

- Suppose we are in the carrots stage. If the player continues playing as recommended she gets $V_C = f(q^C)$. If she deviates she gets $(1-\delta)g(q^C) + \delta V_S$.
- Suppose we are in sticks stage. If players continue as recommended they get $V_S = (1-\delta)f(q^S) + \delta V_C$. If they deviate, they get $(1-\delta)g(q^S) + \delta V_S$

We can derive the following two inequalities are-

$$
\begin{aligned}
f(q^C) - f(q^S) &\geq \frac{1}{\delta}(g(q^C) - f(q^C)) \\
f(q^C) - f(q^S) &\geq \frac{1}{\delta}(g(q^S) - f(q^S))
\end{aligned}
$$

which can be reduced to

$$f(q^C) - f(q^S) \geq \frac{1}{\delta} \max\{g(q^C) - f(q^C), g(q^S) - f(q^S)\}$$

Note that if $q_s \geq \frac{1}{2N}$, we must have $q_c \leq q_s$.

Suppose $q_c \leq \frac{1}{n} \leq q_s \leq \frac{1}{n-1}$. That is, we punish the firms by pushing them to zero profits, then we would have

$$f(q^C) - f(q^S) \geq \frac{1}{\delta} \max\{g(q^C) - f(q^C), g(q^S)\}$$

Suppose $q_c \leq \frac{1}{n} \leq \frac{1}{n-1} \leq q_s$. That is, we punish the firms by pushing them to zero profits, and they don't gain anything by deviating, then we would just have

$$f(q^C) - f(q^S) \geq \frac{1}{\delta}(g(q^C) - f(q^C))$$

2 Problem 2

Firms are in a repeated price competition. Each firm sets price p_i. Denote the set of firms that charge the minimum price as $M(\{p_i\}) = \{k \mid p_k = \min_i\{p_i\}\}$ The quantity sold by firm i is

$$Q_i = \frac{1 - p_i}{\#M} \mathbf{I}(i \in M)$$

Here $f(p) = p\frac{1-p}{2}$ and $g(p) = (p - \varepsilon)(1 - p + \varepsilon) \approx p(1-p)$

The game starts in the cartel state. Any deviation starts the price war which lasts K periods.

Suppose we are in the cartel state. Continuing to play the recommended strategy yields $f(p^c) = p^c \frac{1-p^c}{2}$. If the player deviates, she gets $(1-p^c)p^c$ but gets punished for K periods. This deviation will not be profitable if

$$g(p^c) - f(p^c) \leq \sum_{k=1}^{K} \delta^k (f(p^c) - f(p^w))$$

Now suppose we are in the stage where it has been m periods since a player deviated. If we continue playing recommended strategy for rest of the game we get $f(p^w)$ for $K - m$ more periods and $f(p^c)$ thereafter. If we deviate we get $g(p^w)$ now, get punished for K more periods, that is m extra periods and $f(p^c)$ thereafter. The deviation is not profitable if

$$g(p^w) - f(p^w) \leq \sum_{k=K+1}^{K+m} \delta^k (f(p^c) - f(p^w))$$

for all $m = 1, ..., K$.

Note that if the above condition holds for $m = 1$, it holds for all m. Thus we have the two conditions on parameters for the equilibrium to be SPNE.

The conditions hence become

$$p^c(1-p^c)\frac{1-\delta}{1-\delta^K} \leq (\delta(p^c(1-p^c) - p^w(1-p^w))$$
$$p^w(1-p^w)\frac{1}{\delta^K} \leq \delta(p^c(1-p^c) - p^w(1-p^w)$$

which can be rewritten as

$$\delta(p^c(1-p^c) - p^w(1-p^w)) \geq \max\{p^c(1-p^c)\frac{1-\delta}{1-\delta^K}, p^w(1-p^w)\frac{1}{\delta^K}\}$$