

ISYE 4803-CIF Final Project

Geometric Programming

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Overview and Goal

Project Goal. Introduce the basic mathematical and optimization background of geometric programming while demonstrating its utility as a problem solving framework.

Desired Outcomes.

- ▶ *Expected.* You will loosely be able to classify a problem as a Geometric Program (GP).
- ▶ *Expected.* Your barrier of entry to understanding [1] and [2] will be lowered.
- ▶ *Not Expected.* You will be able to formulate arbitrary problems as GPs.

Disclaimer

The material I present here is more thoroughly covered in [1] and [2]. Besides using the relative positioning constraint explanation discussed in [1] to *formally* write those constraints for the floor planning problem covered in [2], the examples and formulations are not my own. However, with the exception of the “Geometric program in Convex Form” slide, which I directly took from Stanford’s EE364a material, this presentation is of my own design. I specifically focused on making the graduate level material more approachable to an undergraduate audience.

Outline

- ▶ Motivation (Example 1)
- ▶ Background Mathematics
- ▶ Geometric Programming
- ▶ Example 2
- ▶ Example 3 and Computational Experiment
- ▶ Appendix

Solving Optimization Problems

General Optimization Problems

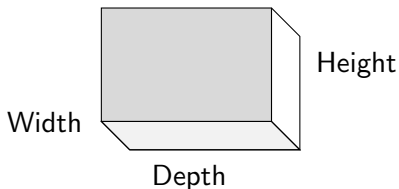
- ▶ *Very hard* to “solve.”
- ▶ Require babysitting.
- ▶ “Solutions” involve a trade-off between long computation times and not finding the global optima.

Convex Optimization Problems

- ▶ Solved *to global optimality* efficiently.
- ▶ Can be embedded into real-time systems and solved with a zero failure rate.
- ▶ Includes classes of problems such as linear programs, least-squares, and quadratic programs.

Upshot (for presentation). Can we solve? \equiv Is the problem convex?

Motivation: “Structural Engineering” Example



Suppose we want to maximize the volume of a box-shaped structure with height h , width w , and depth d subject to some physical constraints. The corresponding optimization problem, \mathcal{P}_1 , is

$$\begin{array}{ll}\text{maximize} & hwd \\ \text{subject to} & 2(hw + hd) \leq A_{\text{wall}}, \quad wd \leq A_{\text{flr}}, \\ & \alpha \leq h/w \leq \beta, \quad \gamma \leq d/w \leq \delta.\end{array}$$

Motivation

Can we solve \mathcal{P}_1 :

$$\begin{array}{ll} \text{maximize} & hwd \\ \text{subject to} & 2(hw + hd) \leq A_{\text{wall}}, \quad wd \leq A_{\text{flr}}, \\ & \alpha \leq h/w \leq \beta, \quad \gamma \leq d/w \leq \delta \end{array}$$

(is \mathcal{P}_1 convex)?

Motivation

Simply consider

$$\text{maximize } f_0(h, w, d) = hwd.$$

Motivation

The objective

$$\text{maximize } f_0(h, w, d) = hwd,$$

is clearly

- ▶ not a sum of squares (\mathcal{P}_1 not least-squares),
- ▶ nonlinear (\mathcal{P}_1 not a LP),
- ▶ **not convex** (\mathcal{P}_1 not a convex problem)!

However, \mathcal{P}_1 is a **geometric program**.

Motivation

How should we proceed handling \mathcal{P}_1 ,

$$\begin{array}{ll} \text{maximize} & hwd \\ \text{subject to} & 2(hw + hd) \leq A_{\text{wall}}, \quad wd \leq A_{\text{flr}}, \\ & \alpha \leq h/w \leq \beta, \quad \gamma \leq d/w \leq \delta? \end{array}$$

Does anyone have intuition? What do we do when we have a nonlinear LP?

Motivation

Consider the following problem, \mathcal{P}_2 :

$$\begin{array}{ll}\text{minimize} & h^{-1}w^{-1}d^{-1} \\ \text{subject to} & (2/A_{\text{wall}})hw + (2/A_{\text{wall}})hd \leq 1, \quad (1/A_{\text{flr}})wd \leq 1, \\ & \alpha h^{-1}w \leq 1, \quad (1/\beta)hw^{-1} \leq 1, \\ & \gamma wd^{-1} \leq 1, \quad (1/\delta)w^{-1}d \leq 1.\end{array}$$

We refer to \mathcal{P}_2 as a geometric program in *standard form*.

Motivation

Define the change of variable

$$y_1 = \log h, \quad y_2 = \log w, \quad y_3 = \log d$$

$$\Longleftrightarrow$$

$$h = e^{y_1}, \quad w = e^{y_2}, \quad d = e^{y_3}$$

and the associated problem \mathcal{P}_3

$$\begin{array}{ll} \text{minimize} & e^{-y_1-y_2-y_3} \\ \text{subject to} & e^{y_1+y_2+b_1} + e^{y_1+y_3+b_1} \leq 1, \quad e^{y_2+y_3+b_2} \leq 1, \\ & e^{-y_1+y_2+b_3} \leq 1, \quad e^{y_1-y_2+b_4} \leq 1, \\ & e^{y_2-y_3+b_5} \leq 1, \quad e^{-y_2+y_3+b_6} \leq 1, \end{array}$$

where $b_1 = \log(2/A_{\text{wall}})$ and the other b are defined similarly.

Motivation

Finally, take the logarithm of the objective function and constraint functions to obtain \mathcal{P}_4

$$\begin{array}{ll}\text{minimize} & -y_1 - y_2 - y_3 \\ \text{subject to} & \log(e^{y_1+y_2+b_1} + e^{y_1+y_3+b_1}) \leq 0 \\ & y_2 + y_3 + b_2 \leq 0 \\ & -y_1 + y_2 + b_3 \leq 0 \\ & y_1 - y_2 + b_4 \leq 0 \\ & y_2 - y_3 + b_5 \leq 0 \\ & -y_2 + y_3 + b_6 \leq 0\end{array}$$

which is **clearly convex!** Furthermore, we refer to \mathcal{P}_4 as a geometric program in *convex form*.

Motivation: Upshot

We obtained three equivalent problems, \mathcal{P}_2 , \mathcal{P}_3 , and \mathcal{P}_4 , such that

$$\mathcal{P}_1 \iff \mathcal{P}_2 \iff \mathcal{P}_3 \iff \mathcal{P}_4,$$

where \mathcal{P}_4 **is convex**.

This is why we like geometric programming! While a GP is not (in general) a convex problem, it is **easily transformed** to a convex program.

Monomials

A function $f : \mathbf{R}^n \rightarrow \mathbf{R}$ with $\mathbf{dom} f = \mathbf{R}_{++}^n$, defined as

$$f(x) = cx_1^{a_1}x_2^{a_2}\cdots x_n^{a_n},$$

where $c > 0$ and $a_i \in \mathbf{R}$, is called a *monomial function*, or simply, a *monomial*.

Examples (assuming x, y , and z are positive real variables)

- ▶ $2x$
- ▶ 0.23
- ▶ $3x^2y^{-.12}z$

Monomials

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Examples (assuming x, y , and z are positive real variables)

- ▶ $2x$ $(2x^1y^0z^0)$
- ▶ 0.23 $(0.23x^0y^0z^0)$
- ▶ $3x^2y^{-.12}z$ $(c = 3, a_1 = 2, a_2 = -0.12, a_3 = 1)$

Posynomials

A function $f : \mathbf{R}^n \rightarrow \mathbf{R}$ with **dom** $f = \mathbf{R}_{++}^n$, defined as (**a sum of monomials**)

$$f(x) = \sum_{k=1}^K c_k x_1^{a_{1k}} x_2^{a_{2k}} \cdots x_n^{a_{nk}},$$

where $c_k > 0$, is called a *posynomial function*, or simply, a *posynomial* (with K terms in the variables x_1, \dots, x_n).

Examples (assuming x, y , and z are positive real variables)

- ▶ $0.23 + x/y$
- ▶ $2x + 3y + 2z$
- ▶ $2(1 + xy)^3$

Posynomials

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Examples (assuming x, y , and z are positive real variables)

- ▶ $0.23 + x/y$ ($0.23x^0y^0z^0 + x^1y^{-1}z^0$)
- ▶ $2x + 3y + 2z$ ($2x^1y^0z^0 + 3x^0y^1z^0 + 2x^0y^0z^1$)
- ▶ $2(1 + xy)^3$ (closure rules)

Geometric Programming

An optimization problem of the form

$$\begin{array}{ll}\text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 1, \quad i = 1, \dots, m \\ & h_i(x) = 1, \quad i = 1, \dots, p,\end{array}$$

where f_0, \dots, f_m are posynomials and h_1, \dots, h_p are monomials is a *geometric program* (GP) in *standard form*. Note that $\mathcal{D} = \mathbf{R}_{++}^n$; the constraint $x \succ 0$ is implicit.

Geometric program in Convex Form

- ▶ Change variables $y_i = \log x_i$ and take logarithm of cost, constraints.
- ▶ Monomial $f(x) = cx_1^{a_1} \cdots x_n^{a_n}$ transforms to

$$\log f(e^{y_1}, \dots, e^{y_n}) = a^T y + b \quad (b = \log c)$$

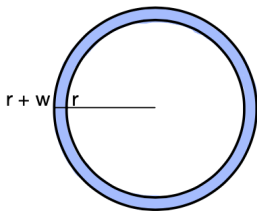
- ▶ posynomial $f(x) = \sum_{k=1}^K c_k x_1^{a_{1k}} x_2^{a_{2k}} \cdots x_n^{a_{nk}}$ transforms to

$$\log f(e^{y_1}, \dots, e^{y_n}) = \log \left(\sum_{k=1}^K e^{a_k^T y + b_k} \right) \quad (b_k = \log c_k)$$

- ▶ Geometric program transforms to convex problem

$$\begin{aligned} & \text{minimize} && \log \left(\sum_{k=1}^K \exp(a_{0k}^T y + b_{0k}) \right) \\ & \text{subject to} && \log \left(\sum_{k=1}^K \exp(a_{ik}^T y + b_{ik}) \right) \leq 0, \quad i = 1, \dots, m \\ & && Gy + d = 0 \end{aligned}$$

Example 2: Heat Flow



Problem. Maximize the total heat flow down a pipe of fixed length subject to physical and economical constraints.

Design Variables

- ▶ T is the degrees above ambient temperature a heated fluid flows through the pipe.
- ▶ r is the radius of the circular cross section.
- ▶ w , where $w \ll r$, is the thickness of the pipe's insulation.

Example 2: Heat Flow

The corresponding optimization problem is

$$\begin{aligned} &\text{maximize} && \alpha_4 Tr^2 \\ &\text{subject to} && \alpha_1 Tr/w + \alpha_2 r + \alpha_3 rw \leq C_{\max} \\ & && T_{\min} \leq T \leq T_{\max} \\ & && r_{\min} \leq r \leq r_{\max} \\ & && w_{\min} \leq w \leq w_{\max} \\ & && w \leq 0.1r. \end{aligned}$$

The velocity of the fluid is assumed fixed, so the heat flow down the pipe is proportional to Tr^2 . The heat loss is proportional to Tr/w .

Example 2: Heat Flow

The heat flow GP in standard form is expressed as

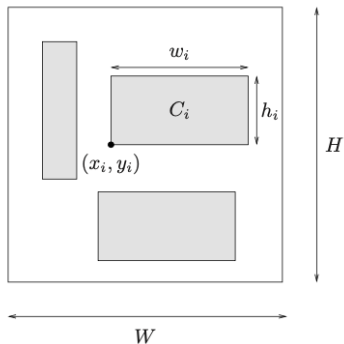
$$\begin{array}{ll}\text{minimize} & \alpha_4 T^{-1} r^{-2} \\ \text{subject to} & (\alpha_1 / C_{\max}) T r w^{-1} + (\alpha_2 / C_{\max}) r + (\alpha_3 / C_{\max}) r w \leq 1, \\ & T_{\min} T^{-1} \leq 1, \quad (1 / T_{\max}) T \leq 1, \\ & r_{\min} r^{-1} \leq 1, \quad (1 / r_{\max}) \leq 1, \\ & w_{\min} w^{-1} \leq 1, \quad (1 / w_{\max}) w \leq 1, \\ & 10 w r^{-1} \leq 1.\end{array}$$

Example 3: Floor Planning

Problem. Configure and place rectangular cells such that they don't overlap.

Objective. Minimize the area of the bounding rectangle: WH .

Design Variables. The height, h_i , width, w_i , and lower left corner, (x_i, y_i) , of the $i = 1, \dots, N$ cells subject to some area constraints.



Example 3: Floor Planning

Positioning Requirements

- ▶ All cells are required to lie in the bounding rectangle, i.e.,

$$x_i \geq 0, \quad y_i \geq 0, \quad x_i + w_i \leq W, \quad y_i + h_i \leq H, \quad i = 1, \dots, N.$$

- ▶ To remove the **combinatorial nature** of the problem, relative positioning relations \mathcal{L} and \mathcal{B} are established with the following requirement: for each (i, j) with $i \neq j$ one of the following holds

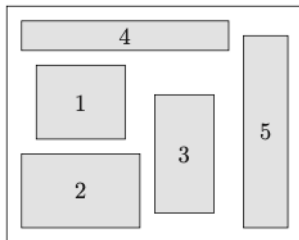
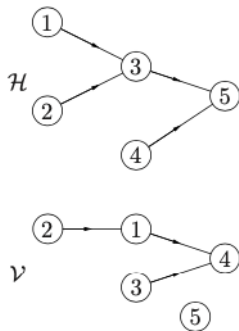
$$(i, j) \in \mathcal{L}, \quad (j, i) \in \mathcal{L}, \quad (i, j) \in \mathcal{B}, \quad (j, i) \in \mathcal{B},$$

and $(i, i) \notin \mathcal{L}, (i, i) \notin \mathcal{B}$.

Example 3: Floor Planning

From the transitivity of \mathcal{L} and \mathcal{B} , a minimal set of relative positioning constraints can be described using two **directed acyclic graphs** (DAGs) \mathcal{H} and \mathcal{V} .

Example. Consider a floor placement problem with 5 cells. The relative positioning can be described by the following DAG



Example 3: Floor Planning

The associated optimization problem

$$\begin{aligned} & \text{minimize} && WH \\ & \text{subject to} && x_i + w_i \leq x_j, \quad (i, j) \in \mathcal{H} \\ & && y_i + h_i \leq y_j, \quad (i, j) \in \mathcal{V} \\ & && x_i + w_i \leq W, \quad i \in \mathbf{sinks} \mathcal{H} \\ & && y_i + h_i \leq H, \quad i \in \mathbf{sinks} \mathcal{V} \\ & && w_1 h_1 = a_1, \quad w_2 h_2 = a_2, \quad \dots \quad w_n h_n = a_n \\ & && W = \max \left\{ \sum_{j \in P_i} w_j \mid P_i \in \mathcal{P}_{\mathcal{H}} \right\} \\ & && H = \max \left\{ \sum_{j \in P_i} h_j \mid P_i \in \mathcal{P}_{\mathcal{V}} \right\}, \end{aligned}$$

where $j \in P_i \in \mathcal{P}_{\mathcal{H}}$ is an index of a cell along the i th path from source to sink in \mathcal{H} ($j \in P_i \in \mathcal{P}_{\mathcal{V}}$ defined similarly), is **almost a generalized geometric program** (GGP)

Example 3: Floor Planning

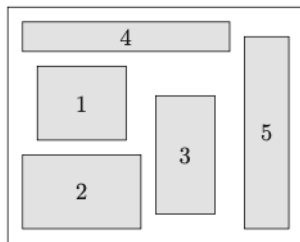
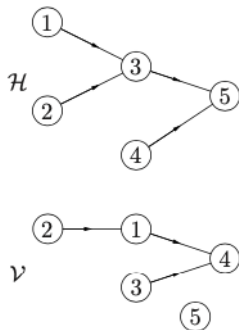
Relaxing the last two sets of equality constraints yields the **valid GGP**

$$\begin{array}{ll}\text{minimize} & WH \\ \text{subject to} & x_i + w_i \leq x_j, \quad (i, j) \in \mathcal{H} \\ & y_i + h_i \leq y_j, \quad (i, j) \in \mathcal{V} \\ & x_i + w_i \leq W, \quad i \in \mathbf{sinks} \mathcal{H} \\ & y_i + h_i \leq H, \quad i \in \mathbf{sinks} \mathcal{V} \\ & w_1 h_1 = a_1, \quad w_2 h_2 = a_2, \quad \dots \quad w_n h_n = a_n \\ & \sum_{j \in P_i} w_j \leq W, \quad P_i \in \mathcal{P}_{\mathcal{H}} \\ & \sum_{j \in P_i} h_j \leq H, \quad P_i \in \mathcal{P}_{\mathcal{V}},\end{array}$$

where the **relaxed constraints will be tight at the solution.**

Computational Experiment: Floor Planning

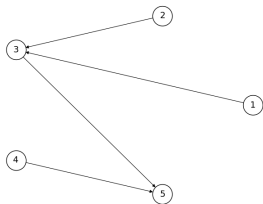
Reconsider



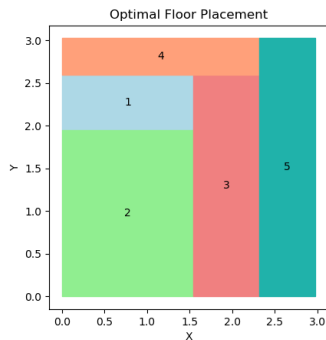
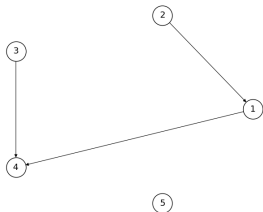
where we initialize the areas as random integers between 1 and 5.

Computational Experiment: Floor Planning

\mathcal{H}

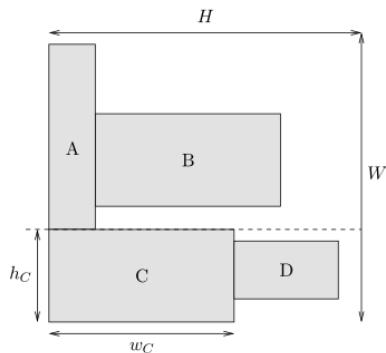


\mathcal{V}



Computational Experiment: Floor Planning

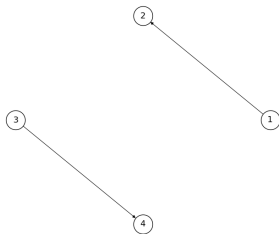
Now consider the following layout



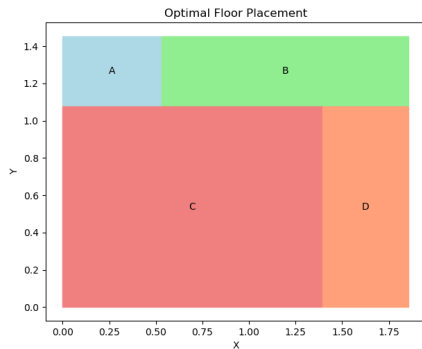
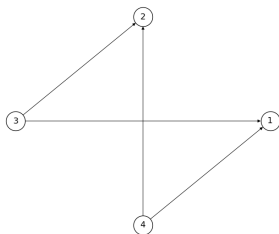
with areas $a = 0.2$, $b = 0.5$, $c = 1.5$, $d = 0.5$.

Computational Experiment: Floor Planning

\mathcal{H}



\mathcal{V}



References I



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