ISYE 4803-CIF Final Project
Geometric Programming

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April 17, 2024

Overview and Goal

Project Goal. Introduce the basic mathematical and optimization background of geometric programming while demonstrating its utility as a problem solving framework.

Desired Outcomes.

- *Expected*. You will loosely be able to classify a problem as a Geometric Program (GP).
- *Expected*. Your barrier of entry to understanding [1] and [2] will be lowered.
- *Not Expected*. You will be able to formulate arbitrary problems as GPs.

Disclaimer

The material I present here is more thoroughly covered in [1] and [2]. Besides using the relative positioning constraint explanation discussed in [1] to *formally* write those constraints for the floor planning problem covered in [2], the examples and formulations are not my own. However, with the exception of the "Geometric program in Convex Form" slide, which I directly took from Stanford's EE364a material, this presenation is of my own design. I specifically focused on making the graduate level material more approachable to an undergraduate audience.

Outline

- Motivation (Example 1)
- Background Mathematics
- Geometric Programming
- Example 2
- Example 3 and Computational Experiment
- Appendix

Solving Optimization Problems

General Optimization Problems

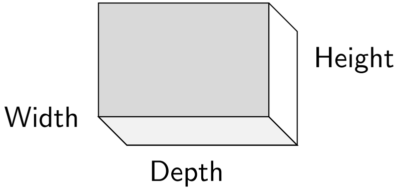
- *Very hard* to "solve."
- Require babysitting.
- "Solutions" involve a trade-off between long computation times and not finding the global optima.

Convex Optimization Problems

- Solved *to global optimality* efficiently.
- Can be embedded into real-time systems and solved with a zero failure rate.
- Includes classes of problems such as linear programs, least-squares, and quadratic programs.

Upshot (for presentation). Can we solve? $\equiv$ Is the problem convex?

Motivation: “Structural Engineering” Example



Suppose we want to maximize the volume of a box-shaped structure with height h, width w, and depth d subject to some physical constraints. The corresponding optimization problem, $\mathcal{P}_1$, is

$$\begin{array}{ll} \text{maximize} & hwd \\ \text{subject to} & 2(hw + hd) \leq A_{\text{wall}}, \quad wd \leq A_{\text{flr}}, \\ & \alpha \leq h/w \leq \beta, \qquad\quad \gamma \leq d/w \leq \delta. \end{array}$$

Motivation

Can we solve $\mathcal{P}_1$:

$$\begin{array}{lll} \text{maximize} & hwd & \\ \text{subject to} & 2(hw + hd) \leq A_{\text{wall}}, & wd \leq A_{\text{flr}}, \\ & \alpha \leq h/w \leq \beta, & \gamma \leq d/w \leq \delta \end{array}$$

(is $\mathcal{P}_1$ convex)?

Motivation

Simply consider minimize $f_0(h, w, d) = hwd$.

Motivation

The objective

$$\text{minimize } f_0(h, w, d) = hwd,$$

is clearly

- not a sum of squares ($\mathcal{P}_1$ not least-squares),
- nonlinear ($\mathcal{P}_1$ not a LP),
- **not convex** ($\mathcal{P}_1$ not a convex problem)!

However, $\mathcal{P}_1$ is a **geometric program**.

Motivation

How should we proceed handling $\mathcal{P}_1$,

$$\begin{array}{ll} \text{maximize} & hwd \\ \text{subject to} & 2(hw + hd) \le A_{\text{wall}}, \quad wd \le A_{\text{flr}}, \\ & \alpha \le h/w \le \beta, \qquad \gamma \le d/w \le \delta? \end{array}$$

Does anyone have intuition? What do we do when we have a nonlinear LP?

Motivation

Consider the following problem, $\mathcal{P}_2$:

$$
\begin{array}{ll}
\text{minimize} & h^{-1}w^{-1}d^{-1} \\
\text{subject to} & (2/A_{\text{wall}})hw + (2/A_{\text{wall}})hd \le 1, \quad (1/A_{\text{flr}})wd \le 1, \\
& \alpha h^{-1}w \le 1, \quad (1/\beta)hw^{-1} \le 1, \\
& \gamma wd^{-1} \le 1, \quad (1/\delta)w^{-1}d \le 1.
\end{array}
$$

We refer to $\mathcal{P}_2$ as a geometric program in *standard form*.

Motivation

Define the change of variable

$$y_1 = \log h, \quad y_2 = \log w, \quad y_3 = \log d$$
$$\Longleftrightarrow$$
$$h = e^{y_1}, \qquad w = e^{y_2}, \qquad d = e^{y_3}$$

and the associated problem $\mathcal{P}_3$

$$\begin{array}{lll} \text{minimize} & e^{-y_1-y_2-y_3} & \\ \text{subject to} & e^{y_1+y_2+b_1} + e^{y_1+y_3+b_1} \leq 1, & e^{y_2+y_3+b_2} \leq 1, \\ & e^{-y_1+y_2+b_3} \leq 1, & e^{y_1-y_2+b_4} \leq 1, \\ & e^{y_2-y_3+b_5} \leq 1, & e^{-y_2+y_3+b_6} \leq 1, \end{array}$$

where $b_1 = \log(2/A_{\text{wall}})$ and the other b are defined similarly.

Motivation

Finally, take the logarithm of the objective function and constraint functions to obtain $\mathcal{P}_4$

$$\begin{array}{ll} \text{minimize} & -y_1 - y_2 - y_3 \\ \text{subject to} & \log\left(e^{y_1+y_2+b_1} + e^{y_1+y_3+b_1}\right) \leq 0 \\ & y_2 + y_3 + b_2 \leq 0 \\ & -y_1 + y_2 + b_3 \leq 0 \\ & y_1 - y_2 + b_4 \leq 0 \\ & y_2 - y_3 + b_5 \leq 0 \\ & -y_2 + y_3 + b_6 \leq 0 \end{array}$$

which is **clearly convex!** Furthermore, we refer to $\mathcal{P}_4$ as a geometric program in *convex form*.

Motivation: Upshot

We obtained three equivalent problems, $\mathcal{P}_2$, $\mathcal{P}_3$, and $\mathcal{P}_4$, such that

$$\mathcal{P}_1 \Longleftrightarrow \mathcal{P}_2 \Longleftrightarrow \mathcal{P}_3 \Longleftrightarrow \mathcal{P}_4,$$

where $\mathcal{P}_4$ **is convex**.

This is why we like geometric programming! While a GP is not (in general) a convex problem, it is **easily transformed** to a convex program.

Monomials

A function $f : \mathbf{R}^n \to \mathbf{R}$ with $\mathbf{dom}\, f = \mathbf{R}^n_{++}$, defined as

$$f(x) = c x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n},$$

where $c > 0$ and $a_i \in \mathbf{R}$, is called a *monomial function*, or simply, a *monomial*.

Examples (assuming x, y, and z are positive real variables)

- $2x$
- 0.23
- $3x^2 y^{-.12} z$

Monomials

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Examples (assuming x, y, and z are positive real variables)

- $2x \qquad (2x^1 y^0 z^0)$
- $0.23 \qquad (0.23 x^0 y^0 z^0)$
- $3x^2 y^{-.12} z \qquad (c = 3, a_1 = 2, a_2 = -0.12, a_3 = 1)$

Posynomials

A function $f : \mathbf{R}^n \to \mathbf{R}$ with $\mathbf{dom}\, f = \mathbf{R}^n_{++}$, defined as (**a sum of monomials**)

$$f(x) = \sum_{k=1}^{K} c_k x_1^{a_{1k}} x_2^{a_{2k}} \cdots x_n^{a_{nk}},$$

where $c_k > 0$, is called a *posynomial function*, or simply, a *posynomial* (with K terms in the variables $x_1, \ldots, x_n$).

Examples (assuming x, y, and z are positive real variables)

- $0.23 + x/y$
- $2x + 3y + 2z$
- $2(1 + xy)^3$

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Examples (assuming x, y, and z are positive real variables)

- $0.23 + x/y \quad (0.23x^0y^0z^0 + x^1y^{-1}z^0)$
- $2x + 3y + 2z \quad (2x^1y^0z^0 + 3x^0y^1z^0 + 2x^0y^0z^1)$
- $2(1 + xy)^3 \quad$ (closure rules)

Geometric Programming

An optimization problem of the form

$$\begin{array}{lll} \text{minimize} & f_0(x) & \\ \text{subject to} & f_i(x) \leq 1, & i = 1, \ldots, m \\ & h_i(x) = 1, & i = 1, \ldots, p, \end{array}$$

where $f_0, \ldots f_m$ are posynomials and $h_1, \ldots h_p$ are monomials is a *geometric program* (GP) in *standard form*. Note that $\mathcal{D} = \mathbf{R}^n_{++}$; the constraint $x \succ 0$ is implicit.

Geometric program in Convex Form

- Change variables $y_i = \log x_i$ and take logarithm of cost, constraints.
- Monomial $f(x) = c x_1^{a_1} \cdots x_n^{a_n}$ transforms to

$$\log f(e^{y_1}, \ldots, e^{y_n}) = a^T y + b \quad (b = \log c)$$

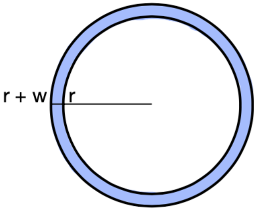
- posynomial $f(x) = \sum_{k=1}^{K} c_k x_1^{a_{1k}} x_2^{a_{2k}} \cdots x_n^{a_{nk}}$ transforms to

$$\log f(e^{y_1}, \ldots, e^{y_n}) = \log \left(\sum_{k=1}^{K} e^{a_k^T y + b_k} \right) \quad (b_k = \log c_k)$$

- Geometric program transforms to convex problem

$$\begin{array}{ll} \text{minimize} & \log \left(\sum_{k=1}^{K} \exp(a_{0k}^T y + b_{0k}) \right) \\ \text{subject to} & \log \left(\sum_{k=1}^{K} \exp(a_{ik}^T y + b_{ik}) \right) \leq 0, \quad i = 1, \ldots, m \\ & Gy + d = 0 \end{array}$$

Example 2: Heat Flow



Problem. Maximize the total heat flow down a pipe of fixed length subject to physical and economical constraints.

Design Variables

- T is the degrees above ambient temperature a heated fluid flows through the pipe.
- r is the radius of the circular cross section.
- w, where $w \ll r$, is the thickness of the pipe's insulation.

Example 2: Heat Flow

The corresponding optimization problem is

$$\begin{array}{ll} \text{maximize} & \alpha_4 Tr^2 \\ \text{subject to} & \alpha_1 Tr/w + \alpha_2 r + \alpha_3 rw \leq C_{max} \\ & T_{\min} \leq T \leq T_{\max} \\ & r_{\min} \leq r \leq r_{\max} \\ & w_{\min} \leq w \leq w_{\max} \\ & w \leq 0.1r. \end{array}$$

The velocity of the fluid is assumed fixed, so the heat flow down the pipe is proportional to Tr^2. The heat loss is proportional to Tr/w.

Example 2: Heat Flow

The heat flow GP in standard form is expressed as

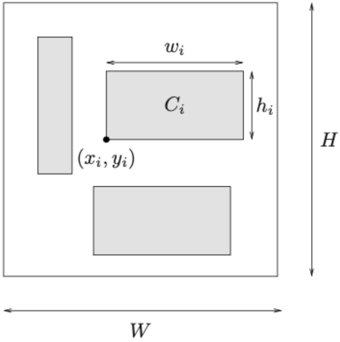
$$
\begin{array}{ll}
\text{minimize} & \alpha_4 T^{-1} r^{-2} \\
\text{subject to} & (\alpha_1/C_{max}) Trw^{-1} + (\alpha_2/C_{max}) r + (\alpha_3/C_{max}) rw \leq 1, \\
& T_{\min} T^{-1} \leq 1, \qquad (1/T_{\max}) T \leq 1, \\
& r_{\min} r^{-1} \leq 1, \qquad (1/r_{\max}) \leq 1, \\
& w_{\min} w^{-1} \leq 1, \qquad (1/w_{\max}) w \leq 1, \\
& 10 w r^{-1} \leq 1.
\end{array}
$$

Example 3: Floor Planning

Problem. Configure and place rectangular cells such that they don't overlap.

Objective. Minimize the area of the bounding rectangle: WH.

Design Variables. The height, h_i, width, w_i, and lower left corner, (x_i, y_i), of the $i = 1, \ldots, N$ cells subject to some area constraints.



Example 3: Floor Planning

Positioning Requirements

- **All cells are required to lie in the bounding rectangle**, i.e.,

  $$x_i \geq 0, \quad y_i \geq 0, \quad x_i + w_i \leq W, \quad y_i + h_i \leq H, \quad i = 1, \ldots N.$$

- To remove the **combinatorial nature** of the problem, relative positioning relations $\mathcal{L}$ and $\mathcal{B}$ are established with the following requirement: for each (i, j) with $i \neq j$ one of the following holds

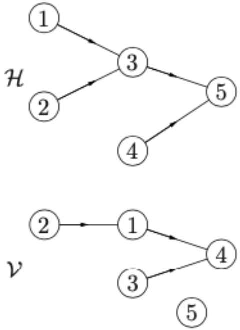
  $$(i, j) \in \mathcal{L}, \quad (j, i) \in \mathcal{L}, \quad (i, j) \in \mathcal{B}, \quad (j, i) \in \mathcal{B},$$

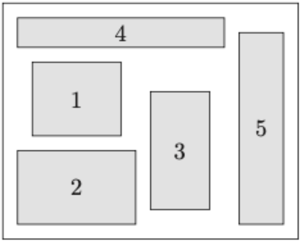
  and $(i, i) \notin \mathcal{L}, (i, i) \notin \mathcal{B}$.

Example 3: Floor Planning

From the transitivity of $\mathcal{L}$ and $\mathcal{B}$, a minimal set of relative positioning constraints can be described using two **directed aylic graphs** (DAGs) $\mathcal{H}$ and $\mathcal{V}$.

Example. Consider a floor placement problem with 5 cells. The relative positioning can be described by the following DAG





Example 3: Floor Planning

The problem can **almost** be stated as a **generalized geometric program** (GGP)

$$
\begin{array}{ll}
\text{minimize} & WH \\
\text{subject to} & x_i + w_i \leq x_j, \quad (i,j) \in \mathcal{H} \\
& y_i + h_i \leq y_j, \quad (i,j) \in \mathcal{V} \\
& x_i + w_i \leq W, \quad i \in \ \textbf{sinks}\, \mathcal{H} \\
& y_i + h_i \leq H, \quad i \in \ \textbf{sinks}\, \mathcal{V} \\
& w_1 h_1 = a_1, \quad w_2 h_2 = a_2, \quad \ldots \quad w_n h_n = a_n \\
& W = \max \left\{ \sum_{j \in P_i} w_j \mid P_i \in \mathcal{P}_{\mathcal{H}} \right\} \\
& H = \max \left\{ \sum_{j \in P_i} h_j \mid P_i \in \mathcal{P}_{\mathcal{V}} \right\},
\end{array}
$$

where $j \in P_i \in \mathcal{P}_{\mathcal{H}}$ is an index of a cell along the ith path from source to sink in $\mathcal{H}$. $j \in P_i \in \mathcal{P}_{\mathcal{V}}$ is defined similarly.

Example 3: Floor Planning

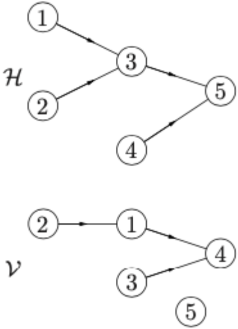
Relaxing the last two equality constraints yields a valid **generalized geometric program** (GGP)

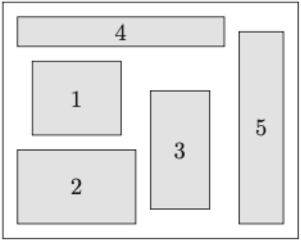
$$
\begin{array}{ll}
\text{minimize} & WH \\
\text{subject to} & x_i + w_i \leq x_j, \quad (i,j) \in \mathcal{H} \\
& y_i + h_i \leq y_j, \quad (i,j) \in \mathcal{V} \\
& x_i + w_i \leq W, \quad i \in \ \textbf{sinks}\, \mathcal{H} \\
& y_i + h_i \leq H, \quad i \in \ \textbf{sinks}\, \mathcal{V} \\
& w_1 h_1 = a_1, \quad w_2 h_2 = a_2, \quad \ldots \quad w_n h_n = a_n \\
& \sum_{j \in P_i} w_j \leq W, \quad P_i \in \mathcal{P}_{\mathcal{H}} \\
& \sum_{j \in P_i} h_i \leq H, \quad P_i \in \mathcal{P}_{\mathcal{V}},
\end{array}
$$

where the relaxed constraints will be tight at the solution.

Computational Experiment: Floor Planning

Reconsider

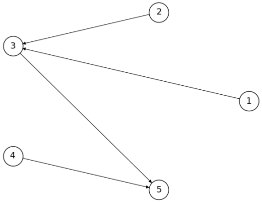




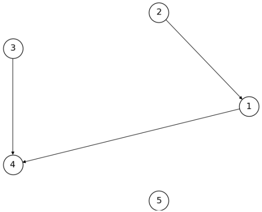
where we initialize the areas as random integers between 1 and 5.

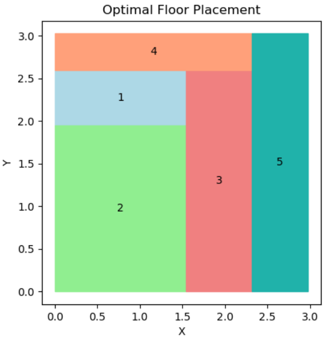
Computational Experiment: Floor Planning

$\mathcal{H}$



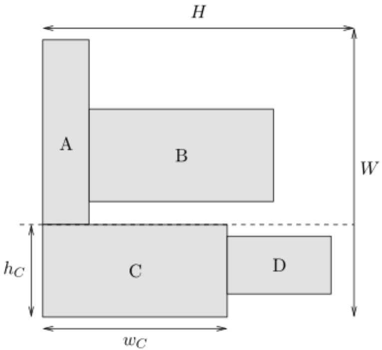
$\mathcal{V}$





Computational Experiment: Floor Planning

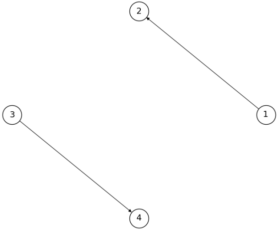
Now consider the following layout



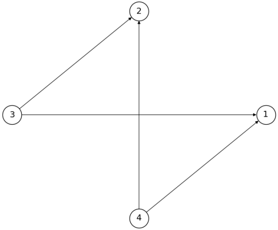
with areas $a = 0.2$, $b = 0.5$, $c = 1.5$, $d = 0.5$.

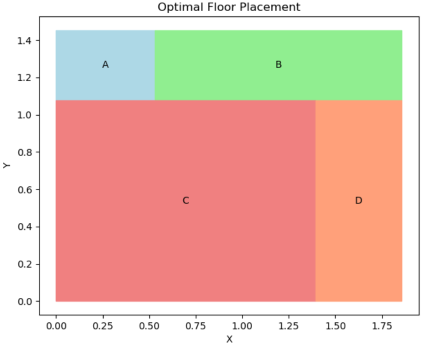
Computational Experiment: Floor Planning

$\mathcal{H}$



$\mathcal{V}$





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