CS1010S Programming Methodology

Lecture 3

Recursion, Iteration & Order of Growth

26 Aug 2020

Python Problems?

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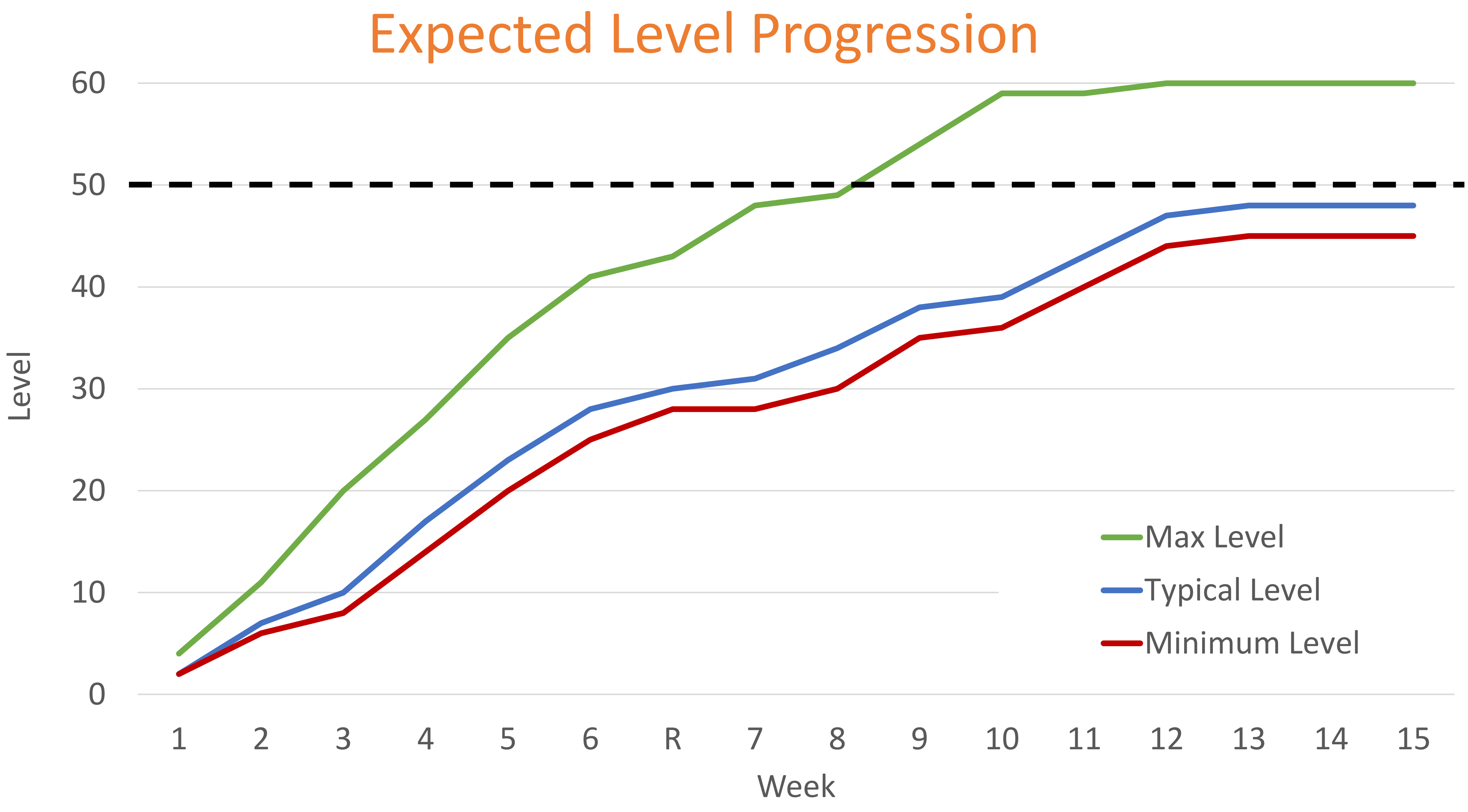
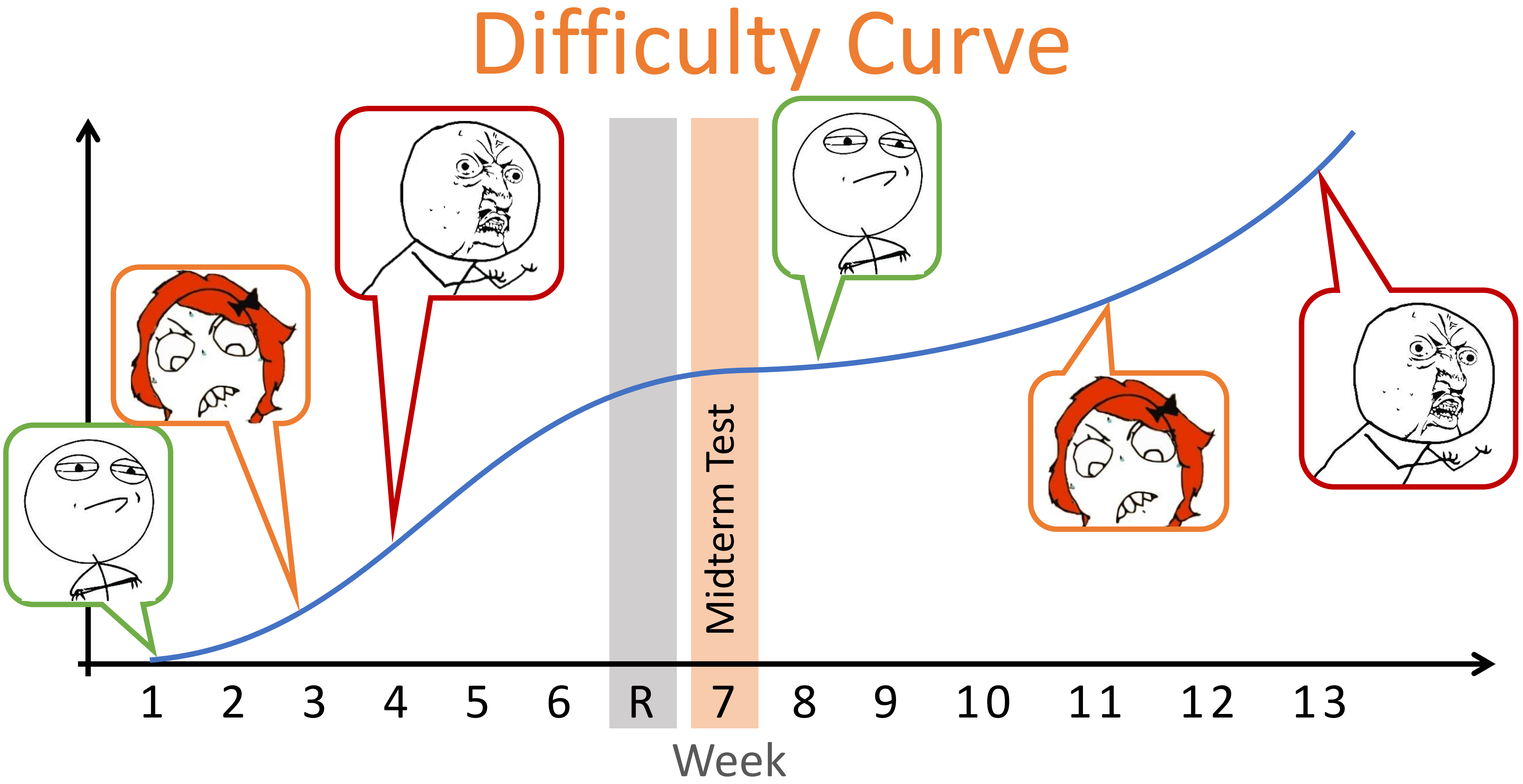

Expected Level Progression
60
50
40
30
20
10
0
Level
1
2
3
4
5
6
R
7
8
9
10
11
12
13
14
15
Week
Max Level
Typical Level
Minimum Level



Difficulty Curve
Midterm Test
1
2
3
4
5
6
R
7
8
9
10
11
12
13
Week


LEAVE NO MAN BEHIND

Reinforcements

Remedial classes

- Every week
- 6:30 – 8:30 pm
- Watch Coursemology for updates

Done with all the missions?

Got a lot of time to burn?

Optional Trainings

Contests

Due 06 Sep 2020

Winning: 400 EXP + Prize

Participation: 50 EXP

Recap

Don’t need to know how it works

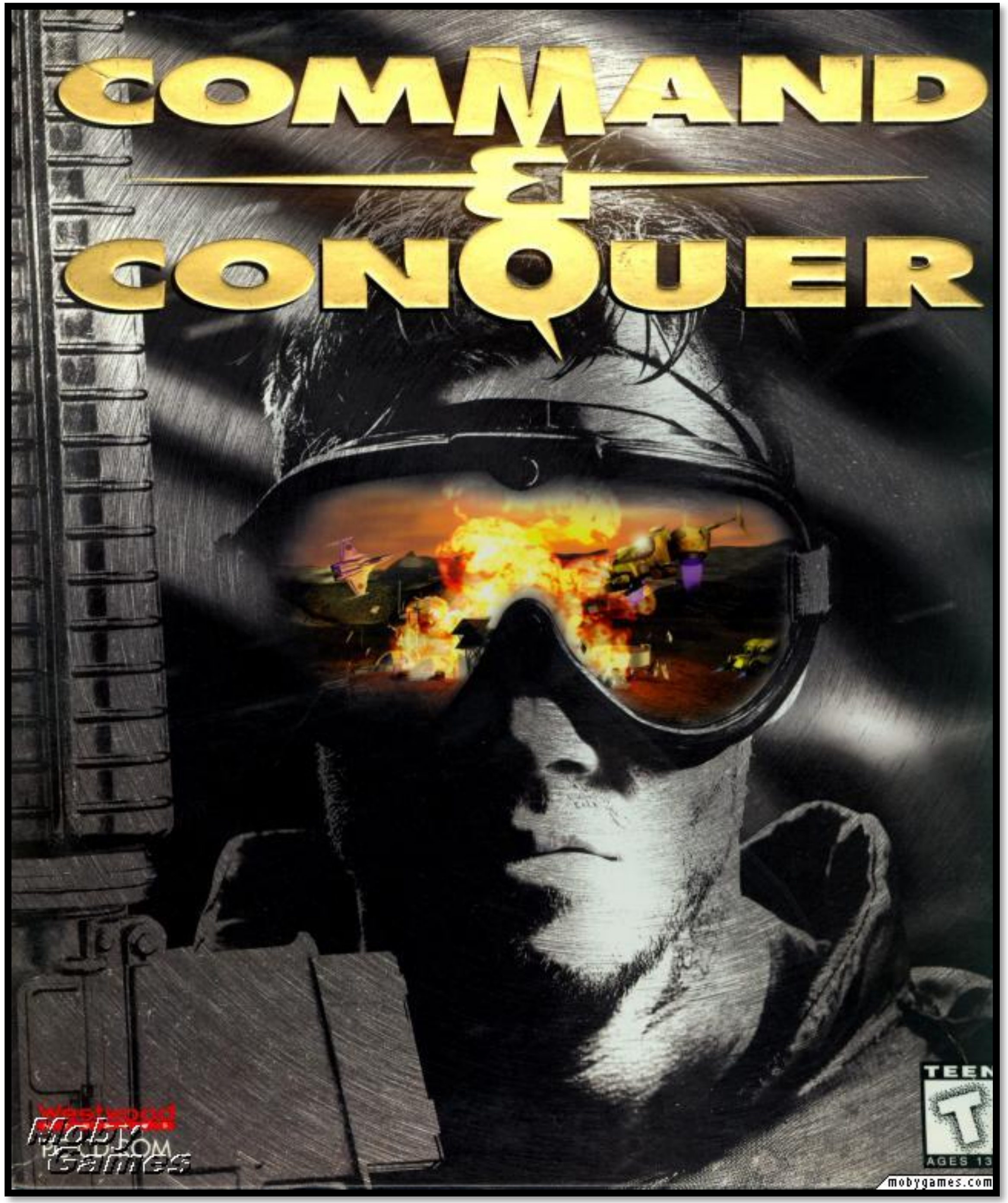
Just know what it does

(the inputs and output)

Learning Outcomes

After *this lesson, you should be able to*

- know how apply divide and conquer technique to solve a problem
- differentiate what is recursion and iteration
- state the order of growth in terms of time and space for computations


COMMAND
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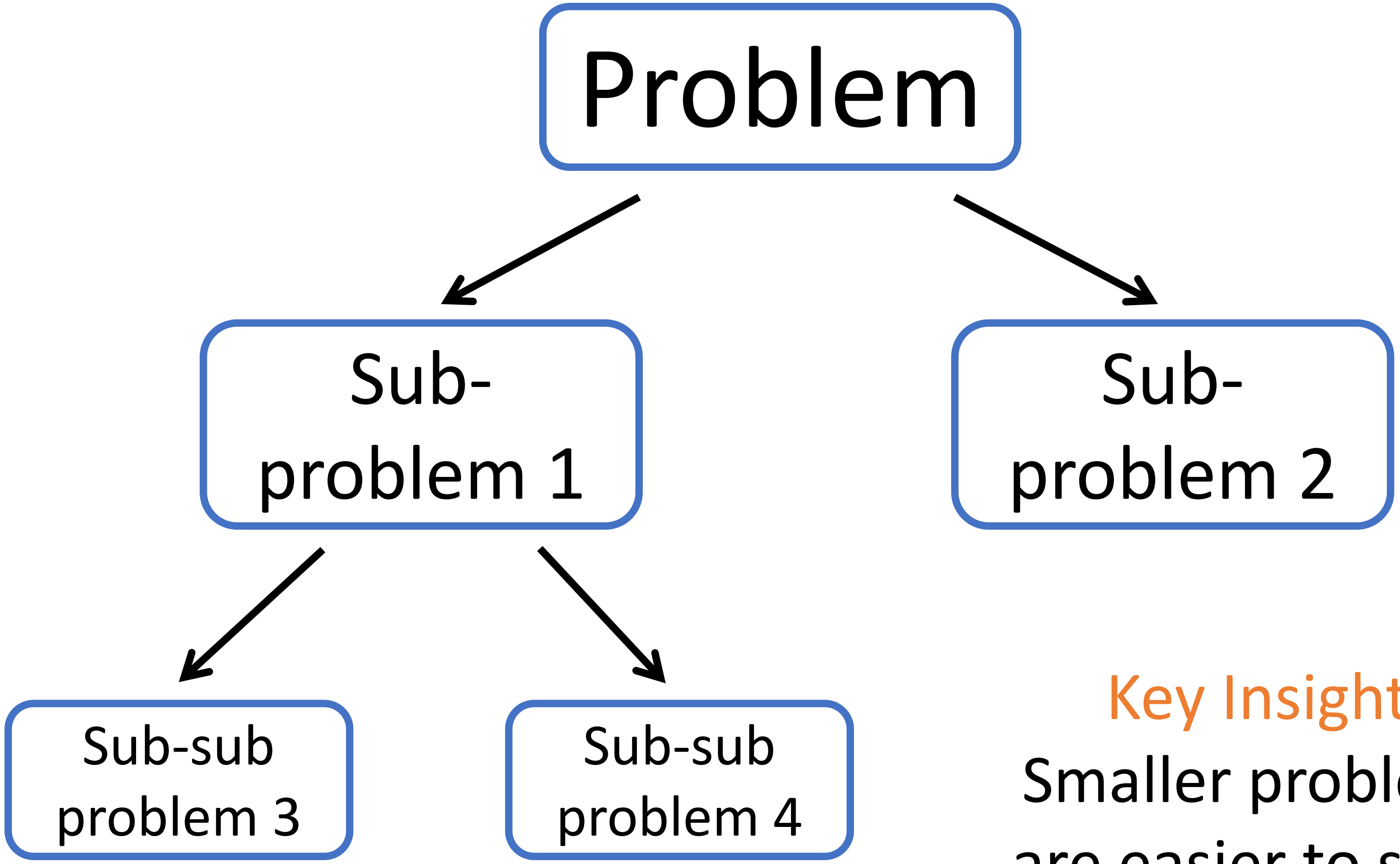
Divide
&
Conquer



Key Insight:
Smaller problems are easier to solve

What is Recursion

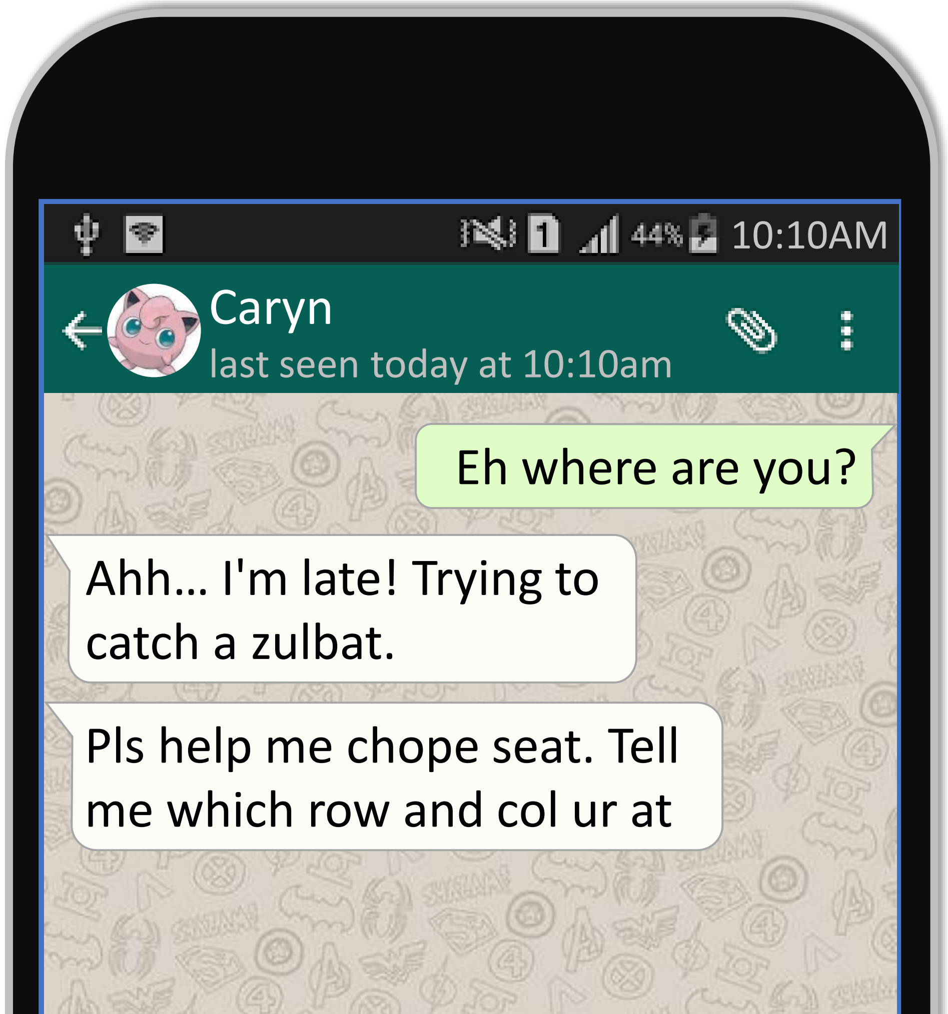
Smaller child problem(s) has same structure as the parent

A recursive function is defined as itself

e.g. $f(n) = \cdots f(m) \cdots$

Analogy

Your friend is late for lecture...



How to find your row?

The Strategy

- Your row number is 1 more than the row in front of you.
- Ask the person in front for his/her row number and add 1 to it.
- The person in front uses the same strategy.
- Eventually, person in front row simply replies 1.

This is Recursion

Example

Consider the factorial function:

$$n! = n \times (n-1) \times (n-2) \cdots \times 1$$

Rewrite:

$$n! = \begin{cases} n \times (n-1)!, & n > 1 \\ 1, & n \leq 1 \end{cases}$$

Factorial

$$n! = \begin{cases} n \times (n-1)!\,, & n > 1 \\ 1\,, & n \leq 1 \end{cases}$$

```
def factorial(n):
    if n <= 1:
        return 1
    else:
        return n * factorial(n - 1)
```

Recursion

```
def factorial(n):
    if n <= 1:
        return 1
    else:
        return n * factorial(n - 1)
```

terminating condition

recursive call

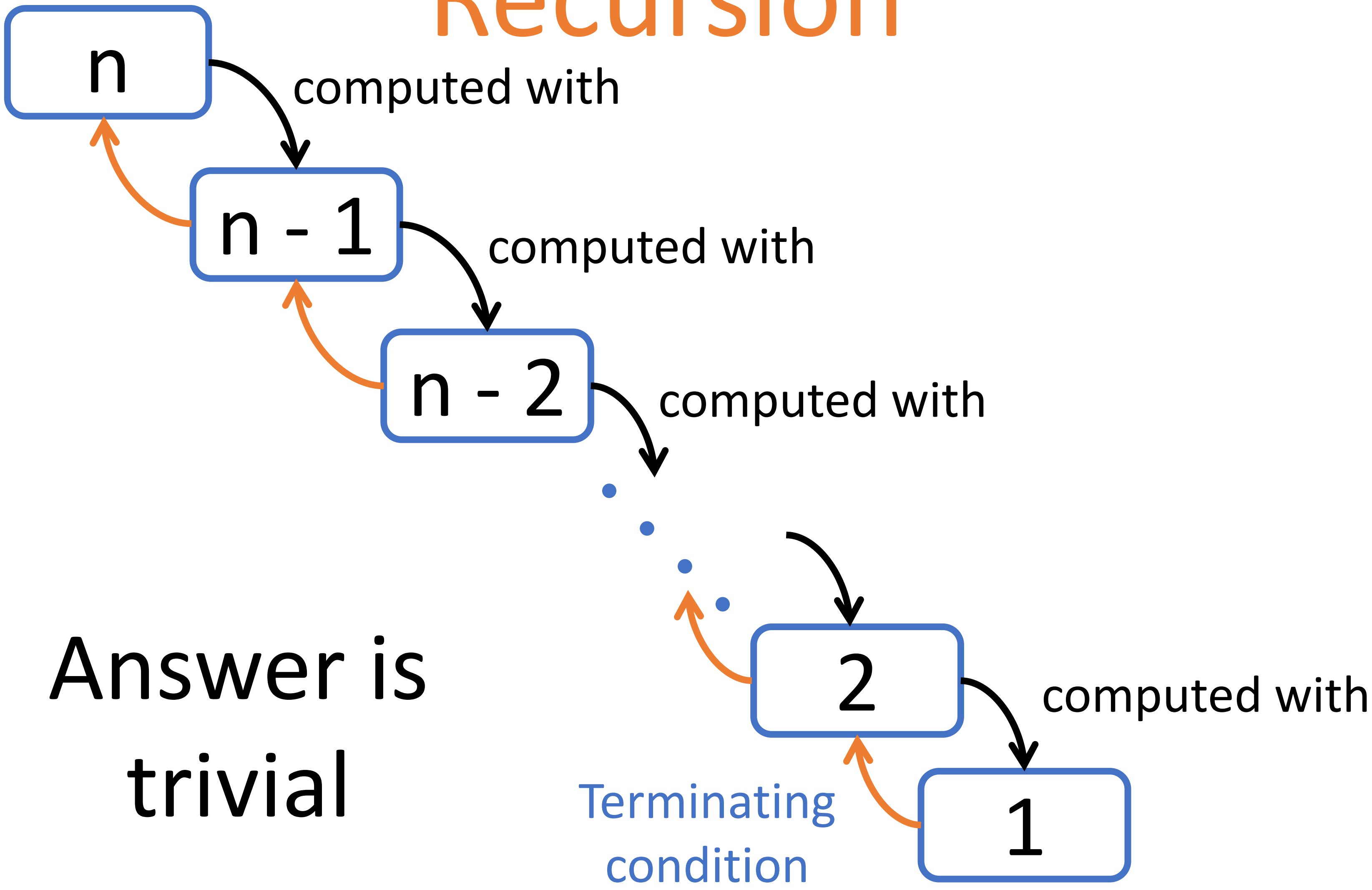
Function that calls itself is called a **recursive** function

Recursive process

```
factorial(5)
5 * factorial(4)
5 * (4 * factorial(3))
5 * (4 * (3 * factorial(2))
5 * (4 * (3 * (2 * factorial(1)))
5 * (4 * (3 * (2 * 1)))
5 * (4 * (3 * 2))
5 * (4 * 6)
5 * 24
120
```

Note the build up of pending operations.

Recursion


n
computed with
n - 1
computed with
n - 2
computed with
2
computed with
1
Terminating condition
Answer is trivial

How to write recursion

1. Figure out the base case
   - Typically n = 0 or n = 1
2. Assume you know how to solve n – 1
   - Now how to solve for n?

Factorial: Linear recursion

```
def factorial(n):
    if n <= 1:
        return 1
    else:
        return n * factorial(n – 1)
```

```
factorial(4)
     ↓
factorial(3)
     ↓
factorial(2)
     ↓
factorial(1)
```

Fibonacci Numbers

Leonardo Pisano Fibonacci (12^{th} century) is credited for the sequence:

0, 1, 1, 2, 3, 5, 8, 13, 21, …

Note: each number is the sum of the previous two.

Fibonacci in Math

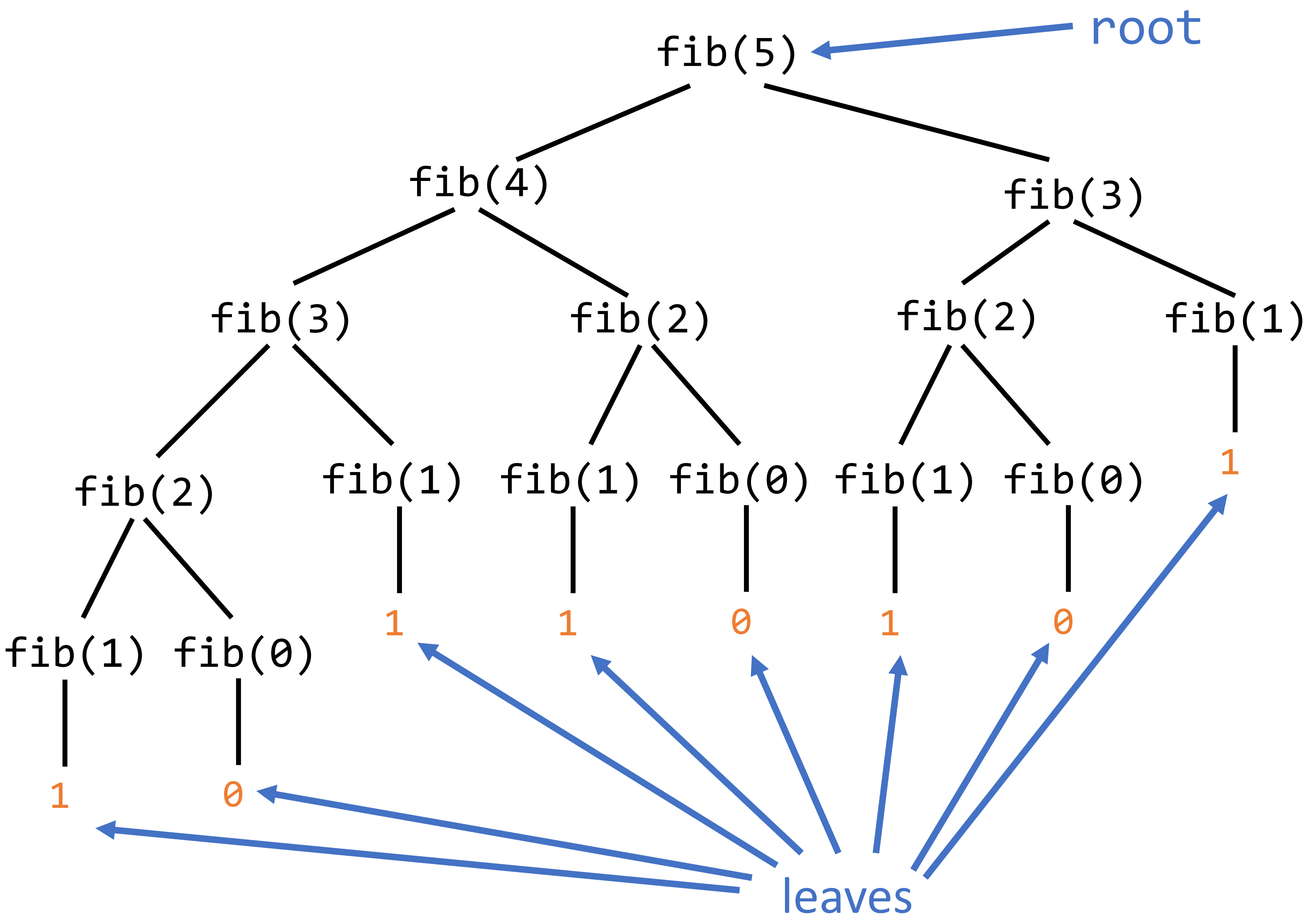
$$fib(n) = \begin{cases} 0, & n = 0 \\ 1, & n = 1 \\ fib(n-1) + fib(n-2) & n > 1 \end{cases}$$

Fibonacci in Python

$$fib(n) = \begin{cases} 0, & n = 0 \\ 1, & n = 1 \\ fib(n-1) + fib(n-2) & n > 1 \end{cases}$$

```
def fib(n):
    if (n == 0):
        return 0
    elif (n == 1):
        return 1
    else:
        return fib(n - 1) + fib(n - 2)
```

Tree recursion



Mutual recursion

```
def ping(n):
    if (n == 0):
        return n
    else:
        print("Ping!")
        pong(n - 1)

def pong(n):
    if (n == 0):
        return n
    else:
        print("Pong!")
        ping(n - 1)
```

```
ping(10)

Ping!
Pong!
Ping!
Pong!
Ping!
Pong!
Ping!
Pong!
Ping!
Pong!
```

Iteration

the act of repeating a process with the aim of approaching a desired goal, target or result.

- *Wikipedia*

Iterative Factorial

Idea

Start with 1, multiply by 2, multiply by 3, … , multiply by n.

$$n! = 1 \times 2 \times 3 \cdots \times n$$

Product		Counter
120	$\times$	[illegible]

Iterative Factorial

$$n! = 1 \times 2 \times 3 \cdots \times n$$

Computationally

Starting:

```
product = 1
counter = 1
```

Iterative (repeating) step:

```
product ← product × counter
counter ← counter + 1
```

End:

`product` contains the result

Iterative Factorial

Start with 1, multiply by 2, multiply by 3, …

$$n! = 1 \times 2 \times 3 \cdots \times n$$

Python Code

```
def factorial(n):
    product, counter = 1, 1
    while counter <= n:
        product = product * counter
        counter = counter + 1
    return product
```

while loop

```
while <expression>:
    <body>
```

expression

- Predicate (condition) to stay within the loop

body

- Statement(s) that will be evaluated if predicate is True

Yet another way

$$n! = 1 \times 2 \times 3 \cdots \times n$$

Factorial rule:

product ← product × counter

counter ← counter + 1

non-inclusive.
Up to n.

```
def factorial(n):
    product = 1
    for counter in range(2, n+1):
        product = product * counter
    return product
```

for loop

```
for <var> in <sequence>:
    <body>
```

`sequence`

- a sequence of values

`var`

- variable that take each value in the sequence

`body`

- statement(s) that will be evaluated for each value in the sequence

range function

`range([start,] stop[, step])`

creates a sequence of integers

- from start (inclusive) to stop (non-inclusive)
- incremented by step

Examples

```
for i in range(10):
    print(i)

for i in range(3, 10):
    print(i)

for i in range(3, 10, 4):
    print(i)
```

break & continue

```python
for j in range(10):
    print(j)
    if j == 3:
        break
print("done")
```

```
0
1
2
3
done
```

Break out of loop

```python
for j in range(10):
    if j % 2 == 0:
        continue
    print(j)
print("done")
```

```
1
3
5
7
9
done
```

Continue with next value

Iterative process

```
def factorial(n):
    product, counter = 1, 1
    while counter <= n:
        product = (product *
                   counter)
        counter = counter + 1
    return product

factorial(6)
```

product	counter
1	1
1	2
2	3
6	4
24	5
120	6
720	7

counter > n (7 > 6)
return product (720)

Recursion
vs
Iteration

Recursive process occurs when there are deferred operations.

Iterative process does not have deferred operations.

Recursive Process

```
factorial(5)
5 * factorial(4)
5 * (4 * factorial(3))
5 * (4 * (3 * factorial(2))
5 * (4 * (3 * (2 * factorial(1)))
5 * (4 * (3 * (2 * 1)))
5 * (4 * (3 * 2))
5 * (4 * 6)
5 * 24
120
```

deferred
operations

Orders of Growth

Like Physicists, we care about two things:

1. Space
2. Time

Rough measure of resources used by a computational process

Space: how much memory do we need to run the program

Time: how long it takes to run a program

Order of growth

Why do we care?

We want to know how much resource our algorithm needs

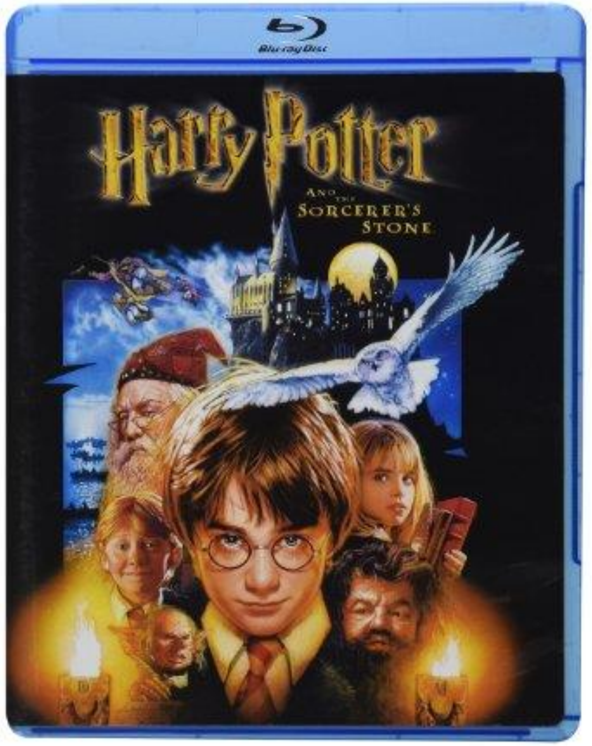
Analogy

Suppose you want to buy a Blu-ray movie from Amazon (~40GB)

Two options:

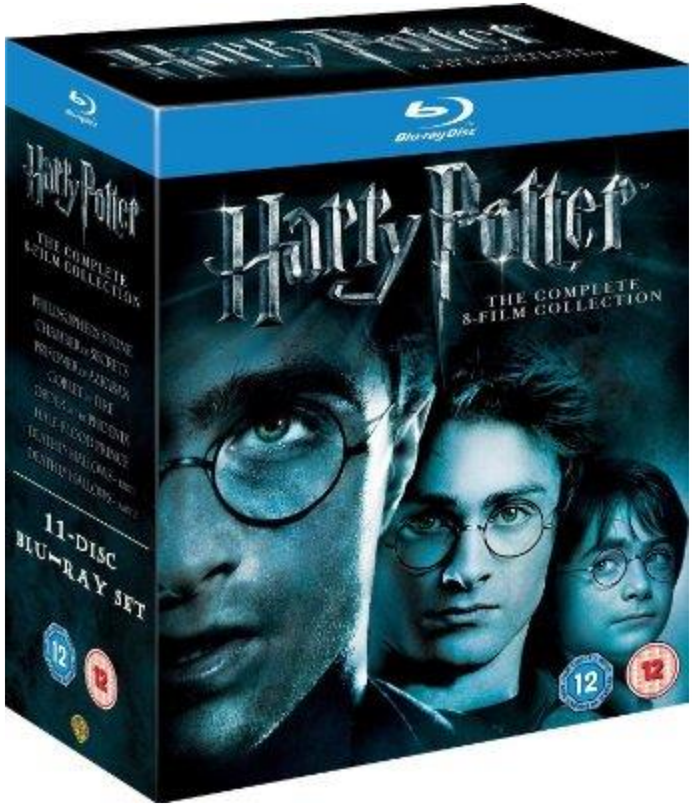
1. Download
2. 2-day Prime Shipping

Which is faster?



Buying the Entire Series

What if you want more movies?

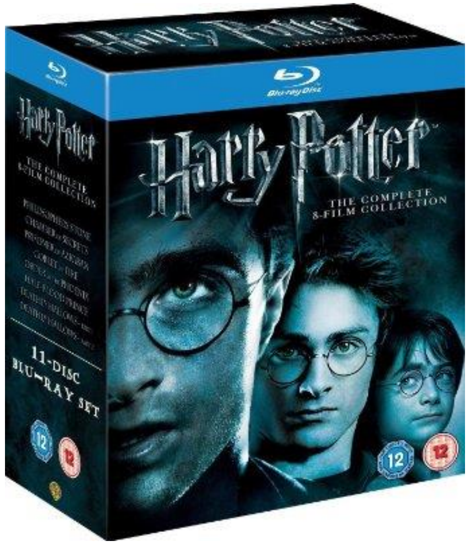


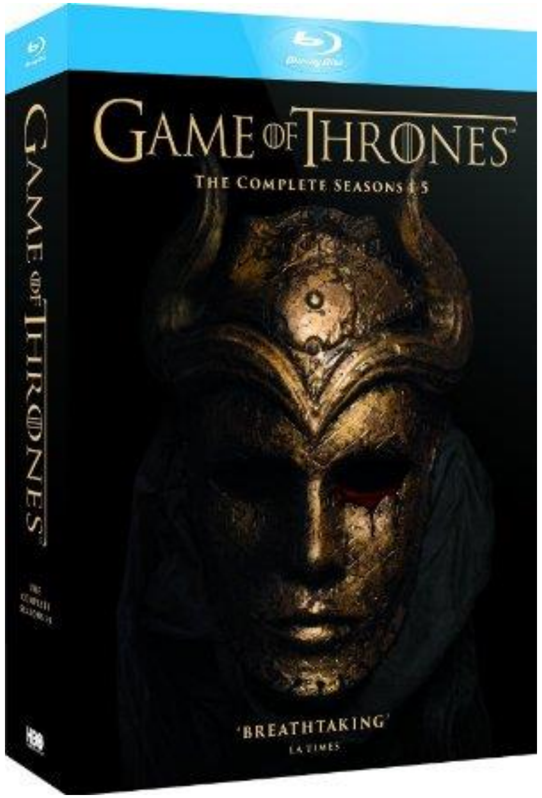
- 8 Blu-ray discs
- ~320 GB

Which is faster?

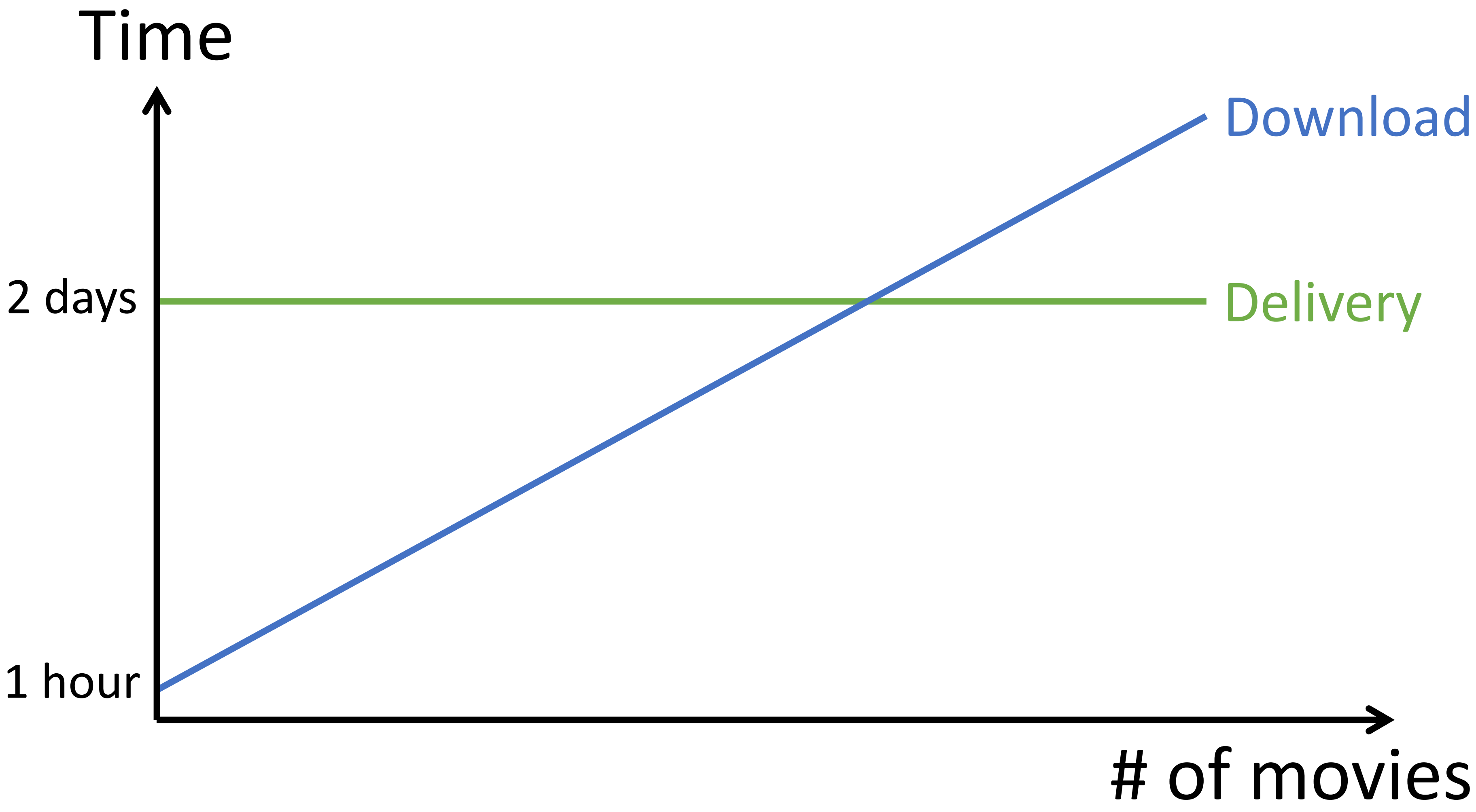
1. Download, or
2. 2-day delivery

Even more movies?





Download vs Delivery



We want to ask questions like:

factorial(5) → factorial(10) ?

fib(10) → fib(20)?

How much more time?

How much more space?

2x?

Same?

4x?

Order of Growth is NOT the absolute time or space a program takes to run

Order of Growth is the proportion of growth of the time/space of a program w.r.t. the growth of the input

Formal Definition

Let n denote size of the problem.

Let $R(n)$ denote the resources needed.

Definition:

$R(n)$ has order of growth $\Theta(f(n))$ written

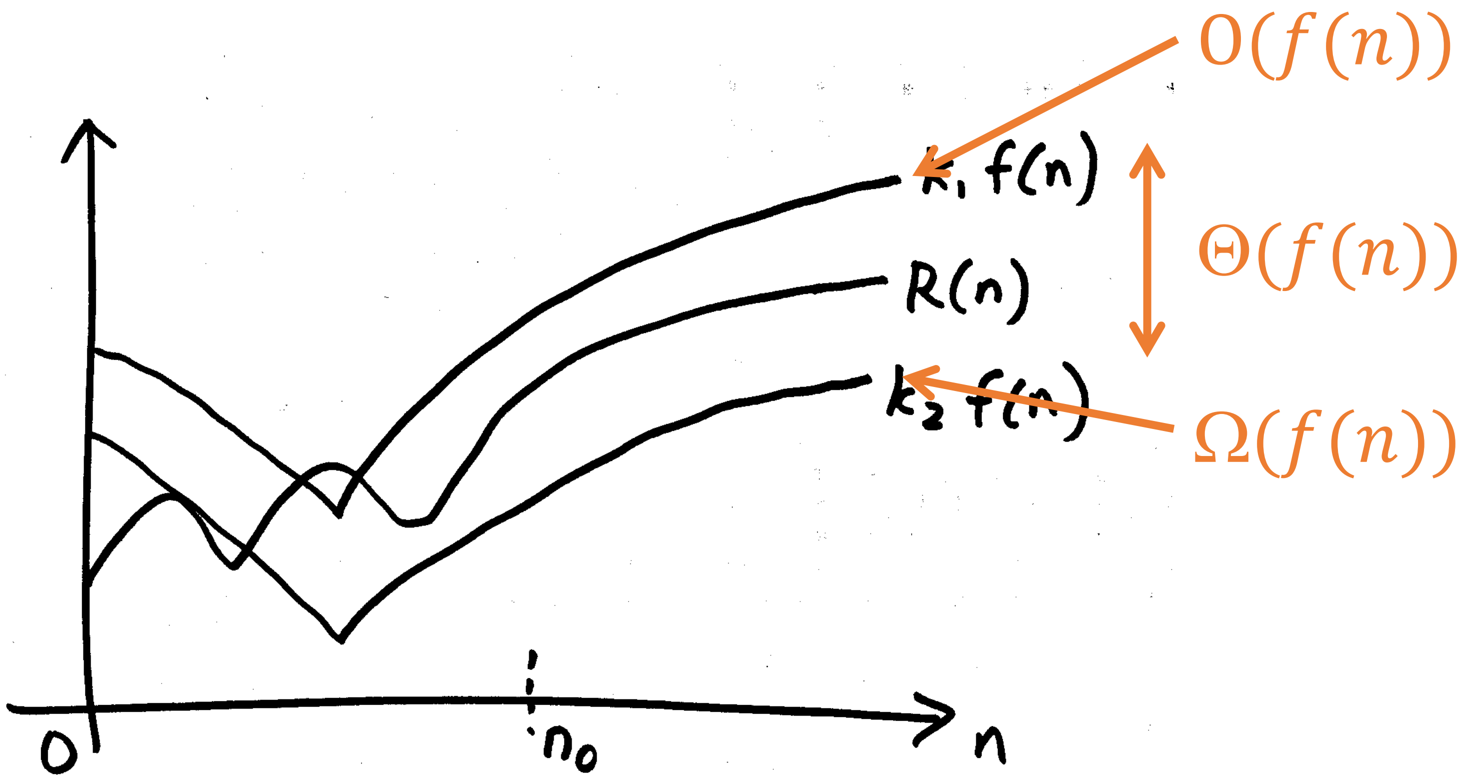
$$R(n) = \Theta(f(n))$$

If there are positive constants k_1 and k_2 such that

$$k_1 f(n) \leq R(n) \leq k_2 f(n)$$

for any sufficiently large value of n

Diagram



For $n >= n_0$, $R(n)$ is sandwiched between

Some common $f(n)$

- 1
- n
- n^2
- n^3
- $\log n$
- $n \log n$
- 2^n

Intuitively

If n is doubled

(i.e. increased to $2n$)

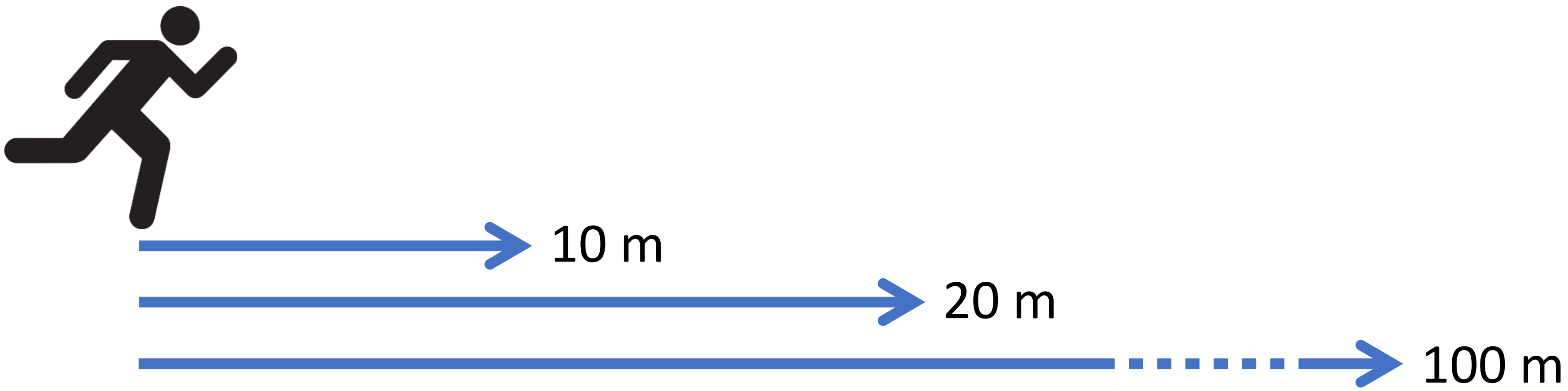
then $R(n)$

(the resource required),

is increased to $f(2n)$

Another analogy

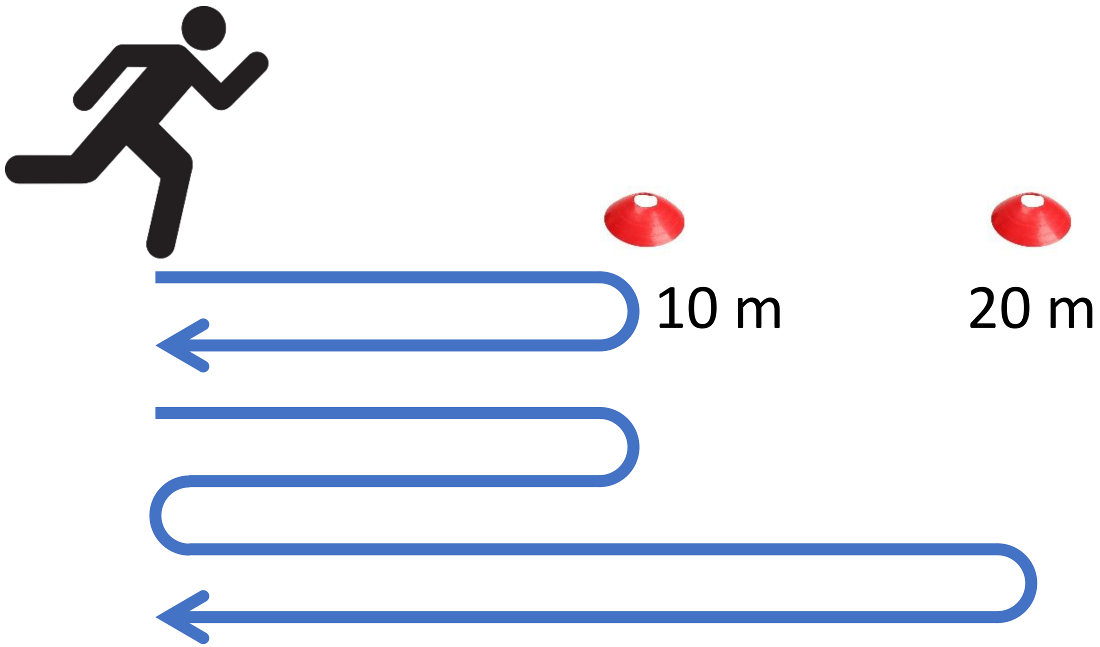
- Suppose you can run 10 m in 1.5 secs



- Time is linear to distance

Shuttle Run

- Run and return



- Time is _________ to distance

Recap: Recursive Factorial

```
def factorial(n):
    if n <= 1:
        return 1
    else:
        return n * factorial(n – 1)
```

Order of growth?

1. Time
2. Space

Recursive process

```
factorial(5)
5 * factorial(4)
5 * (4 * factorial(3))
5 * (4 * (3 * factorial(2))
5 * (4 * (3 * (2 * factorial(1)))
5 * (4 * (3 * (2 * 1)))
5 * (4 * (3 * 2))
5 * (4 * 6)
5 * 24
120
```

- Time $\propto$ #operations
- Linearly proportional to *n*

Recursive process

```
factorial(5)
5 * factorial(4)
5 * (4 * factorial(3))
5 * (4 * (3 * factorial(2))
5 * (4 * (3 * (2 * factorial(1)))
5 * (4 * (3 * (2 * 1)))
5 * (4 * (3 * 2))
5 * (4 * 6)
5 * 24
120
```

- Space $\propto$ #pending operations
- Linearly proportional to *n*

Recursive Factorial

```
factorial(5)
5 * factorial(4)
5 * (4 * factorial(3))
5 * (4 * (3 * factorial(2))
5 * (4 * (3 * (2 * factorial(1)))
5 * (4 * (3 * (2 * 1)))
5 * (4 * (3 * 2))
5 * (4 * 6)
5 * 24
120
```

Time: $O(n)$ Linear

Space: $O(n)$ Linear

Iterative Factorial

```
def factorial(n):
    product, counter = 1, 1
    while counter <= n:
        product = (product *
                   counter)
        counter = counter + 1
    return product

factorial(6)
```

product	counter
1	1
1	2
2	3
6	4
24	5
120	6
720	7

Iterative process

```
product: 720

counter: 7
```

Time (# of steps):

- linearly proportional to n

Space (memory):

- constant
- no deferred operations
- All information contained in 2 variables (old values overwritten by new)

Time: $O(n)$ Linear

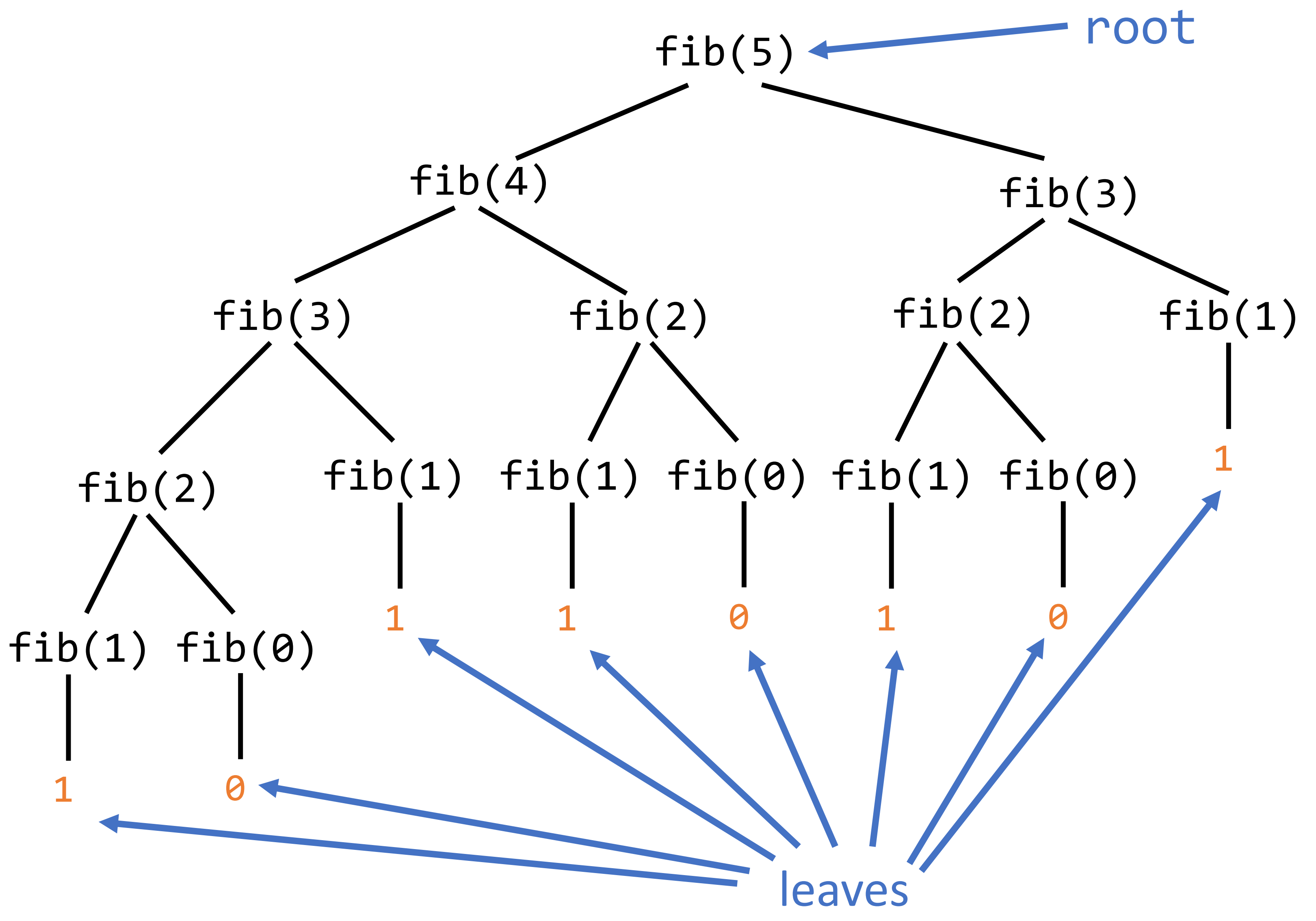
Space: $O(1)$ Constant

Recap: Fibonacci

$$fib(n) = \begin{cases} 0, & n = 0 \\ 1, & n = 1 \\ fib(n-1) + fib(n-2) & n > 1 \end{cases}$$

```
def fib(n):
    if (n == 0):
        return 0
    elif (n == 1):
        return 1
    else:
        return fib(n - 1) + fib(n - 2)
```

Tree recursion



Fibonacci

- Number of leaves in tree is $fib(n+1)$
- Can be shown that fib(n) is the closest integer to $\frac{\Phi^n}{\sqrt{5}}$
  - Where $\Phi = \frac{1+\sqrt{5}}{2} \approx 1.6180$
  - called the golden ratio
- Therefore time taken is $\approx \Phi^n$
  - (exponential in n)

Tree recursion

- Time:
  - Proportional to number of leaves, i.e., exponential in n.
- Space (memory):
  - Proportional to the depth of the tree, i.e., linear in n.

General form

Suppose a computation C takes $3n + 5$ steps to complete, what is the order of growth?

$$O(3n + 5) = O(n)$$

Take the largest term.
Drop the constants.

Another Example

How about $3^n + 4n^2 + 4$?

Order of growth

$$= O(3^n + 4n^2 + 4)$$
$$= O(3^n)$$

Tips

- Identify dominant terms, ignore smaller terms
- Ignore additive or multiplicative constants
  - $4n^2 - 1000n + 300000 = O(n^2)$
  - $\frac{n}{7} + 200n \log n = O(n \log n)$
- Note: $\log_a b = \dfrac{\log_c b}{\log_c a}$
  - So base is not important

More tricks in CS1231, CS3230

Some involve sophisticated proofs

For now…

Count the number of "basic computational steps".

- Identify the basic computation steps
- Try a few small values of *n*
- Extrapolate for really large *n*
- Look for "worst case" scenario

Numeric example

n	$\log n$	$n \log n$	n^2	n^3	2^n
1	0	0	1	1	2
2	0.69	1.38	4	8	4
3	1.098	3.29	9	27	8
10	2.3	23.0	100	1000	1024
20	2.99	59.9	400	8000	10^6
30	3.4	109	900	27000	10^9
100	4.6	461	10000	10^6	1.2×10^{30}
200	5.29	1060	40000	8×10^6	1.6×10^{60}
300	5.7	1710	90000	27×10^6	2.03×10^{90}
1000	6.9	6910	10^6	10^9	1.07×10^{301}
2000	7.6	15200	4×10^6	8×10^9	?
3000	8	24019	9×10^6	27×10^9	?
10^6	13.8	13.8×10^6	10^{12}	10^{18}	?

13.7 billion years $\approx 2^{59}$ seconds

Time: how long it takes to run a program

Space: how much memory do we need to run the program pythontutor.com

itoday.com

Moral of the story

Different ways of performing a computation (algorithms) can consume dramatically different amounts of resources.

Recursion Revisited

- Solve the problem for a simple (base) case
- Express (divide) a problem into one or more smaller similar problems
- Similar to

Mathematical Induction

Comparison

Mathematical Induction

- Start with a base case b
- Assume k works, derive a function to show $k + 1$ also works
- Therefore, it must be true for all cases $\geq b$

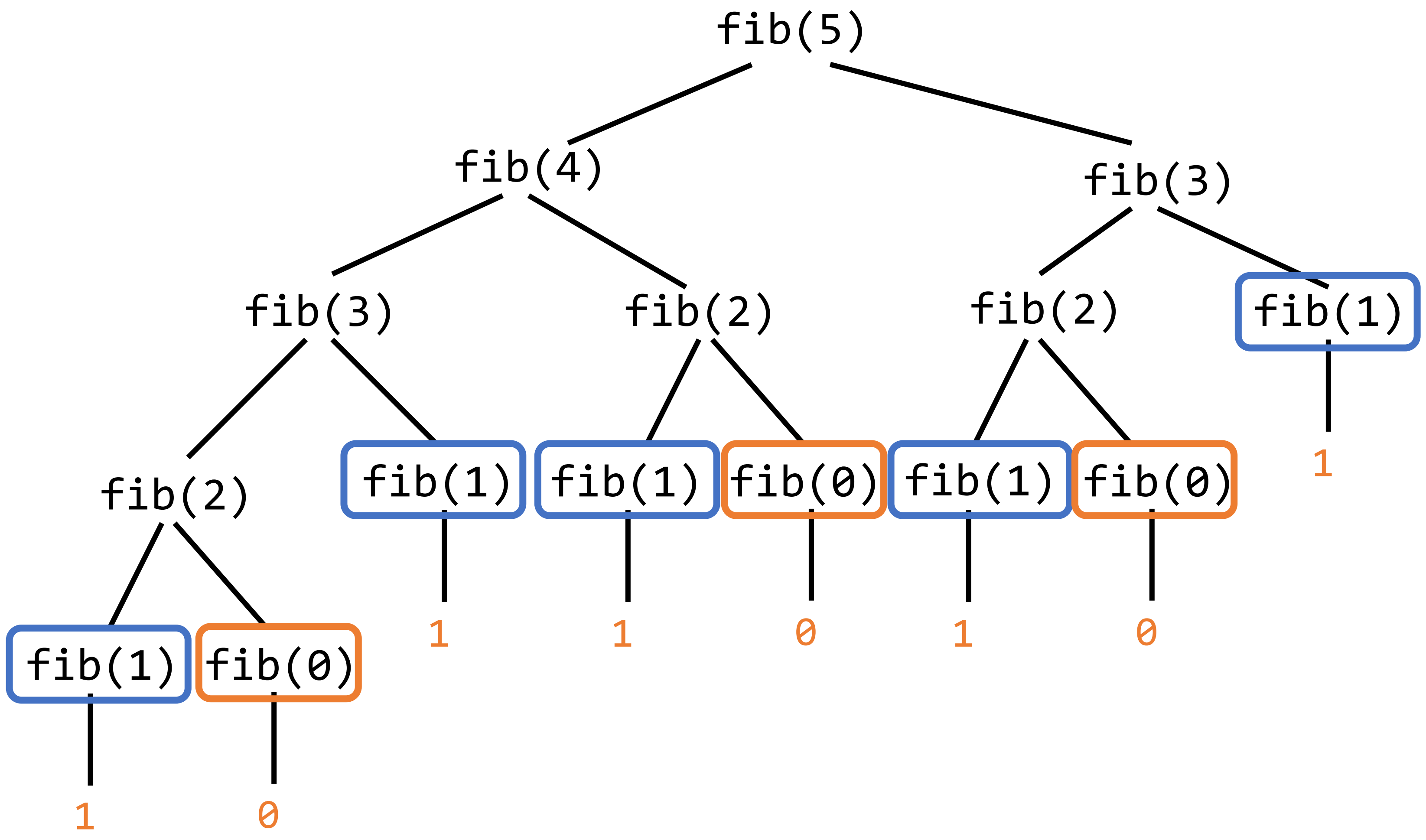
Recursion

- Find base case(s) b where we can just state the answer
- Derive a function to express the problem of size n as sub-problems of $k < n$
- The function can therefore solve all $n \geq b$

Sometimes it may be possible that you will need more than one base case?

When? Why?

Tree recursion



Other times you may have to express a problem in another form and the other form back in the present form
(mutual recursion)

- E.g. sin and cos

More Examples

Greatest Common Divisor

The GCD of two numbers a and b, is the largest positive integer that divides both a and b without remainder.

Greatest Common Divisor

Naïve Algorithm:

Given two numbers a and b

Start with 1.

Check if it divides both a and b.

Try 2, then 3, and so on… until you reach a or b.

Greatest Common Divisor

Euclid's Algorithm:

Given two numbers a and b, where $a = b \cdot Q + r$ (the remainder of the division), then we have

$$GCD(a, b) = GCD(b, r), \forall a, b > 0$$

$$GCD(a, 0) = a$$

Greatest Common Divisor

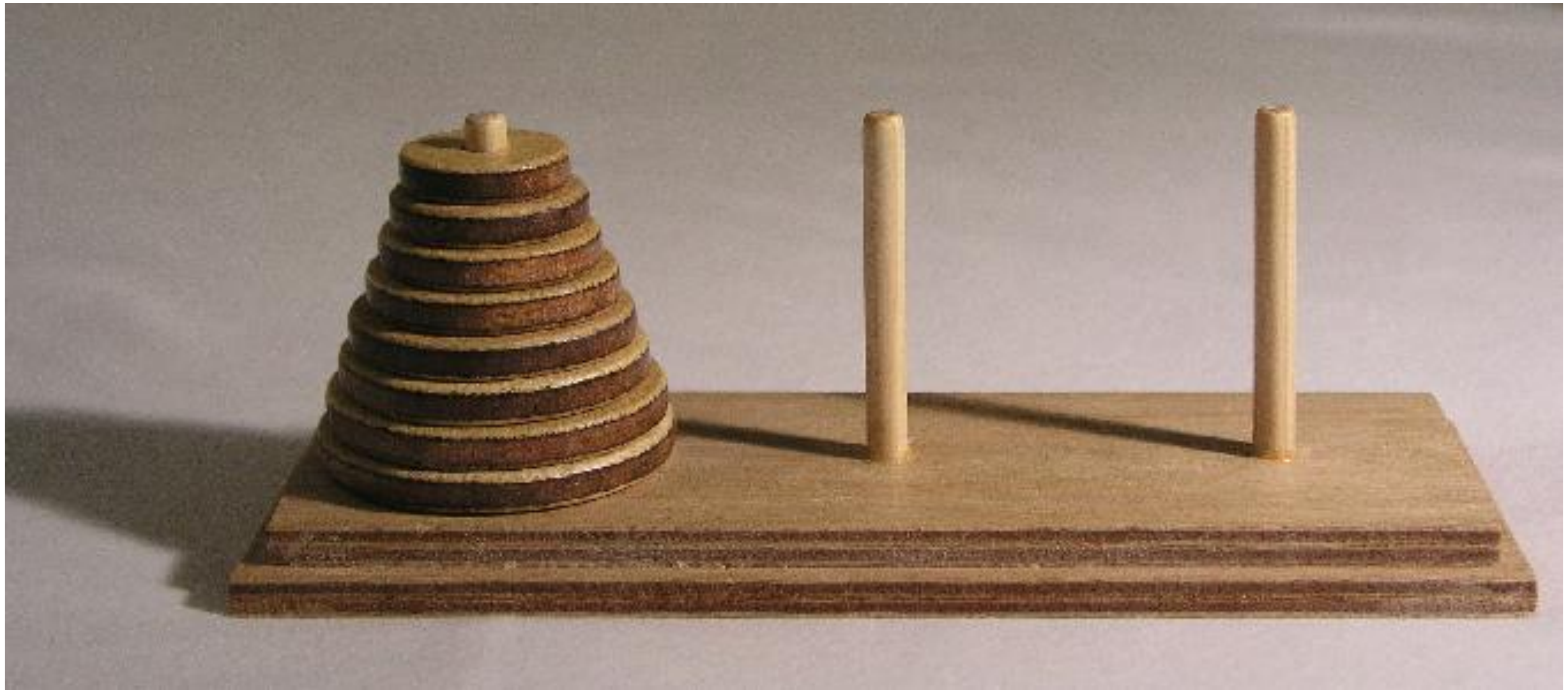
```
def gcd(a, b):
    if (b == 0):
        return a
    else:
        return gcd(b, a % b)
```

$$GCD(a,b) = GCD(b,r), \forall\, a,b > 0$$

$$GCD(a,0) = a$$

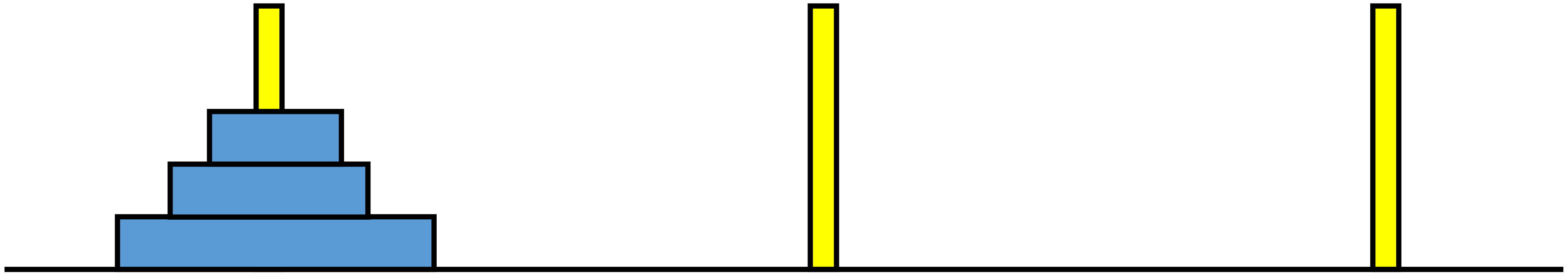
```
GCD(206,40) = GCD(40,6)
            = GCD(6,4)
            = GCD(4,2)
            = GCD(2,0)
            = 2
```

Tower of Hanoi



Towers of Hanoi

Goal: Move all discs from one stick to another

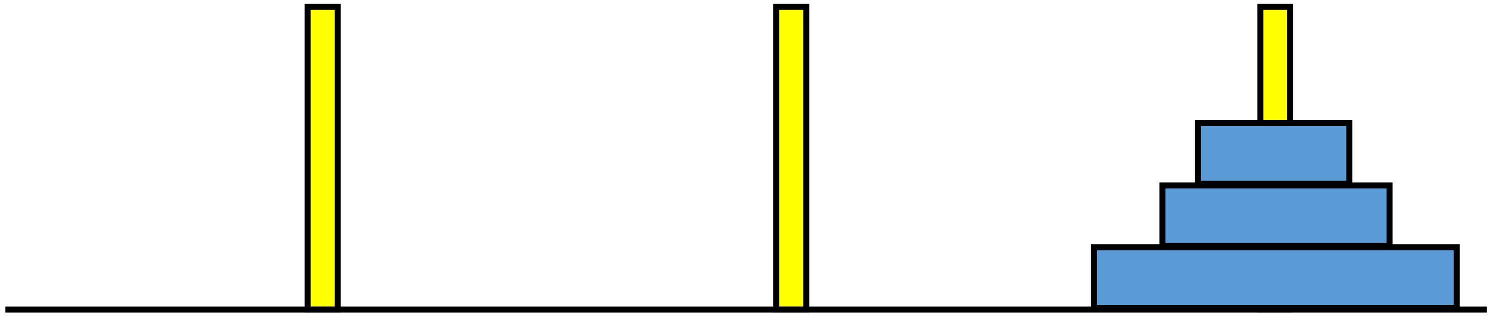


Rules:

1. Can only move one disc at a time
2. Cannot put a larger disc over a smaller disc

Towers of Hanoi

Goal: Move all discs from one stick to another



Rules:

1. Can only move one disc at a time
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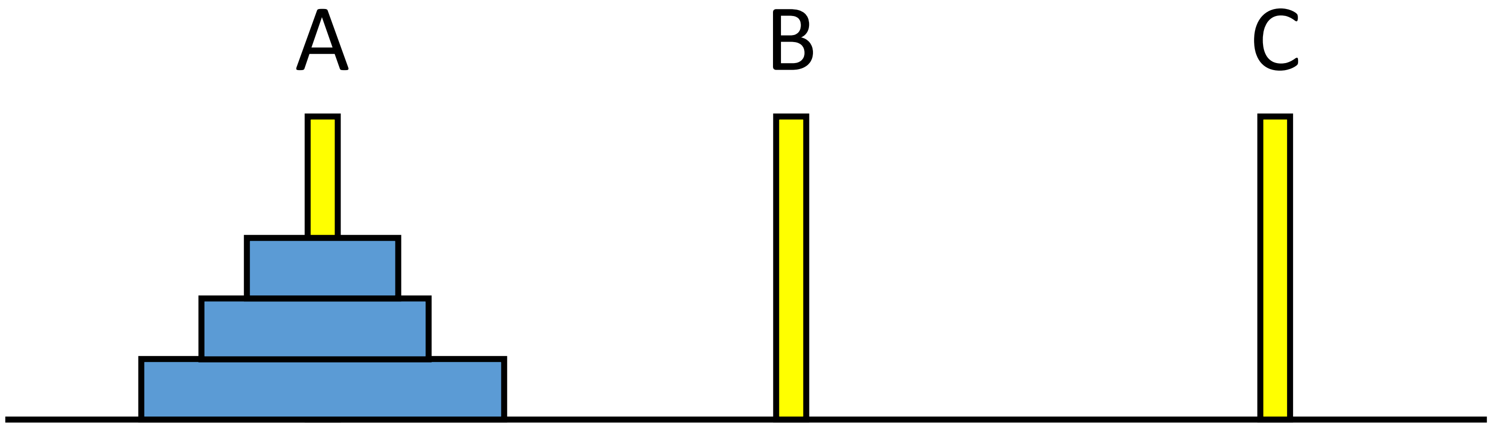
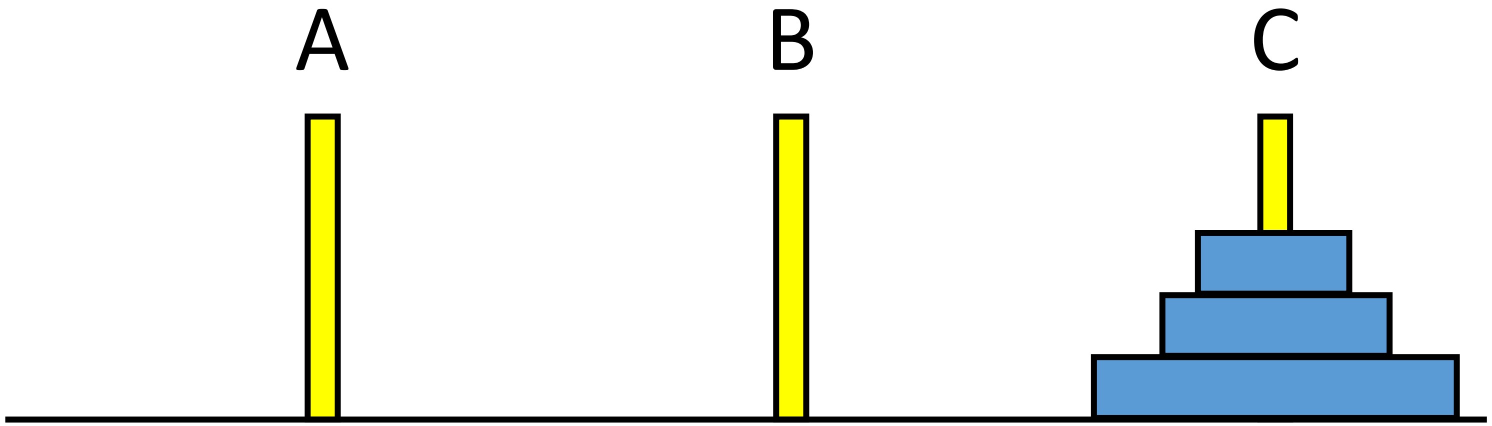
Towers of Hanoi



Suppose we know how to move 3 discs from A to C

Towers of Hanoi



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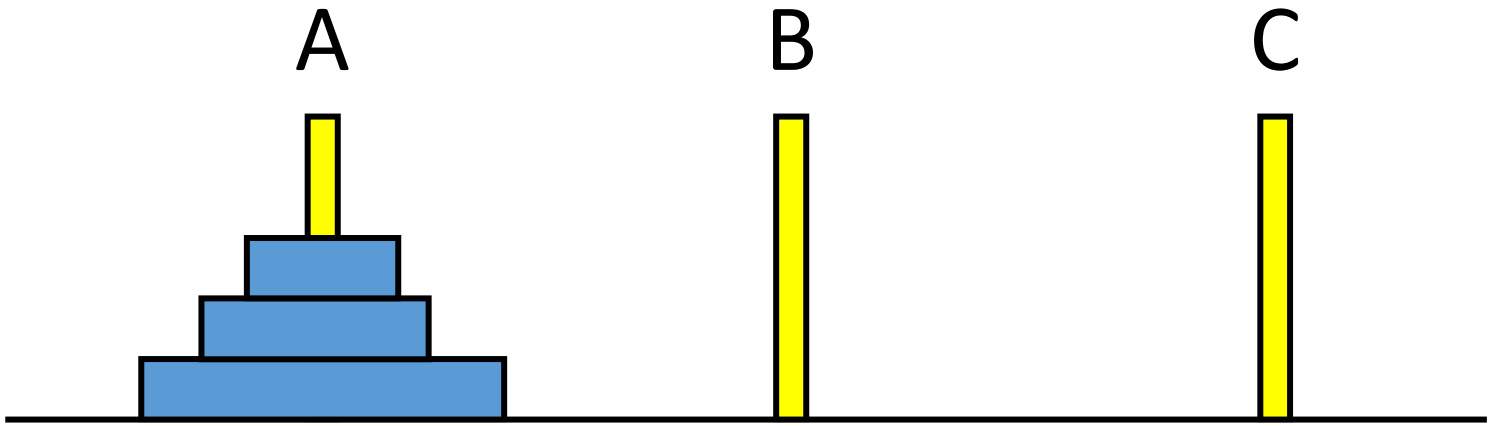
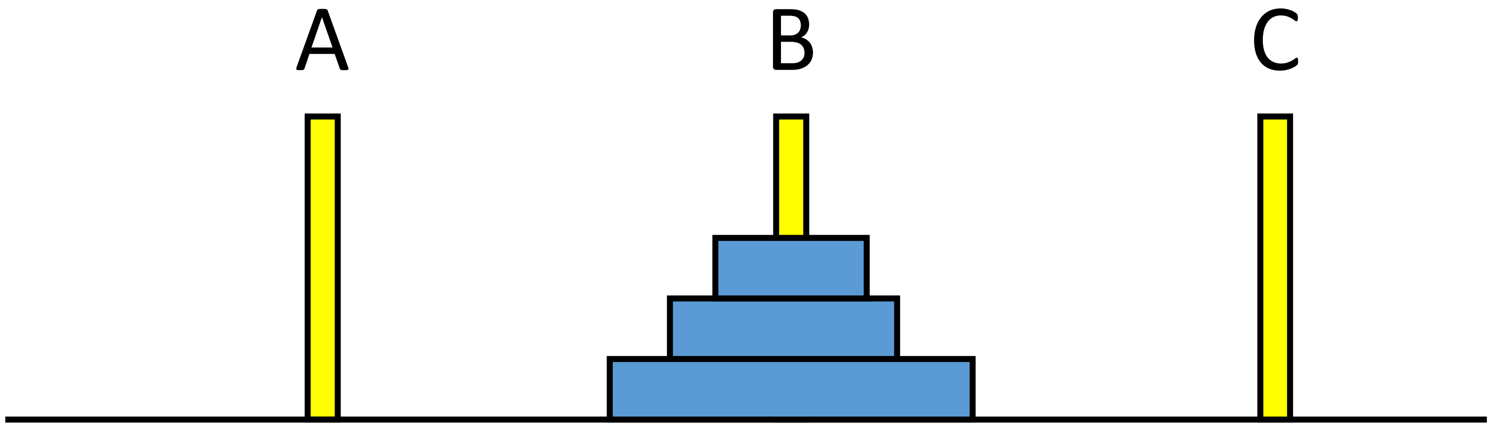
Towers of Hanoi



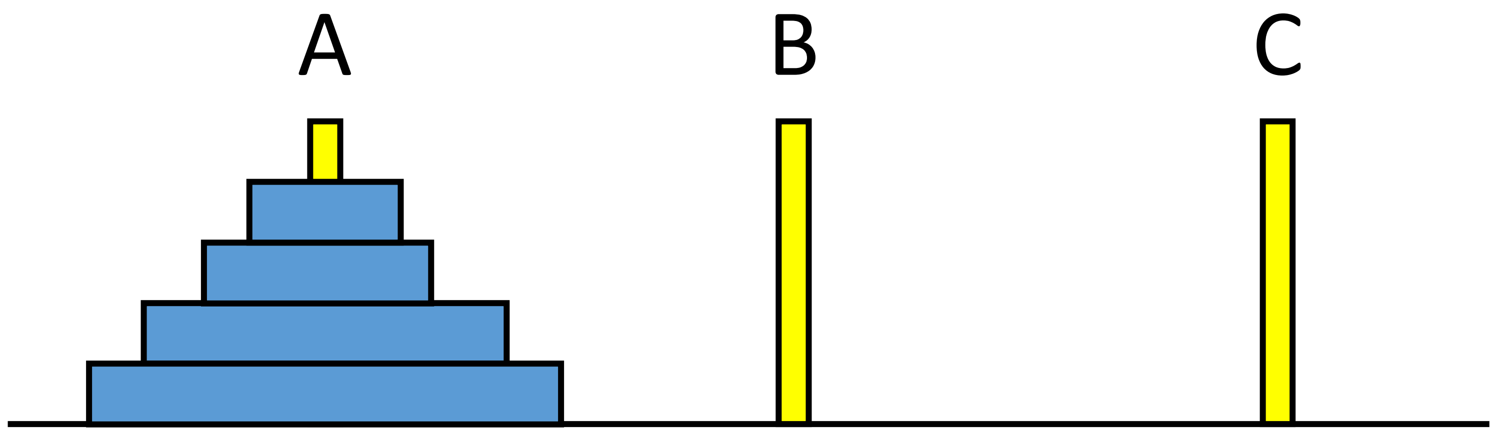
Claim: we can move 3 discs from A to B. Why?

Towers of Hanoi



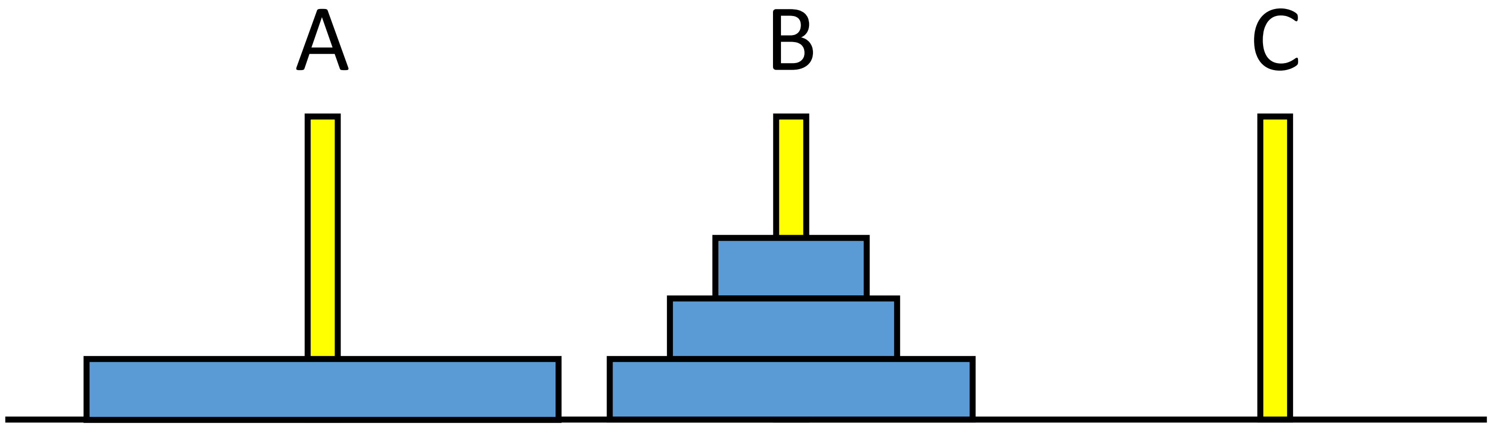
Claim: we can move 3 discs from A to B. Why?

Towers of Hanoi



How to move 4 discs from A to C?

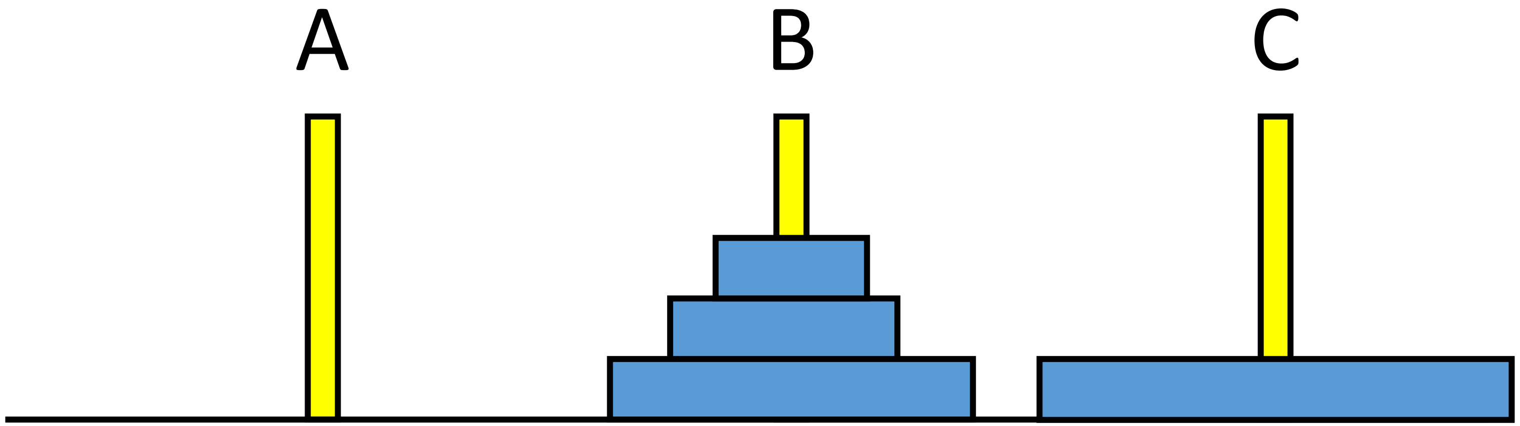
Towers of Hanoi



How to move 4 discs from A to C?

- Move 3 disc from A to B

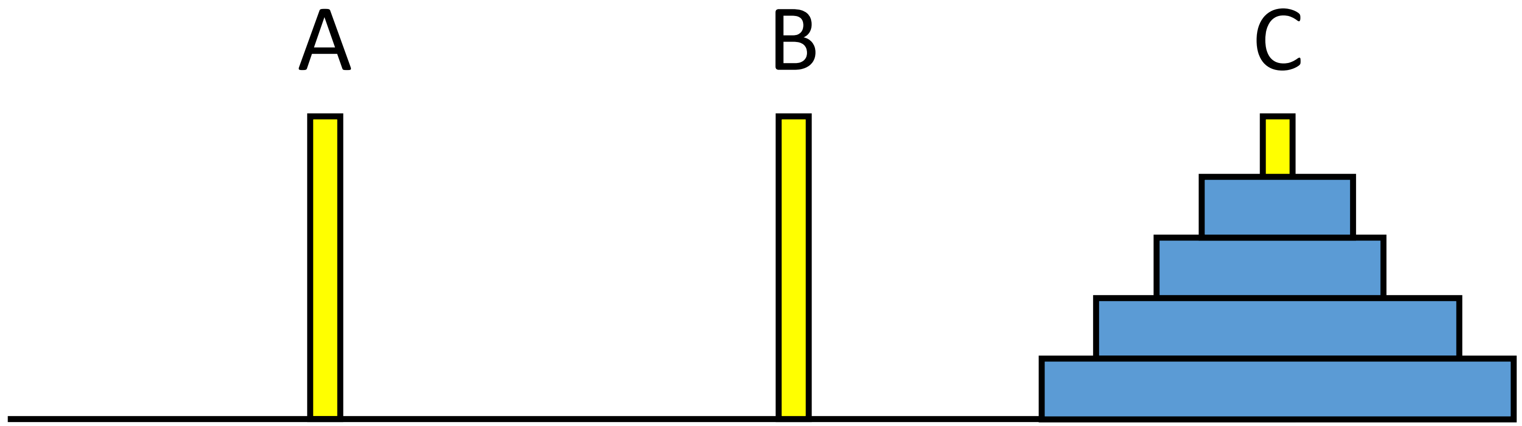
Towers of Hanoi



How to move 4 discs from A to C?

- Move 3 disc from A to B
- Move 1 disc from A to C

Towers of Hanoi



How to move 4 discs from A to C?

- Move 3 disc from A to B
- Move 1 disc from A to C
- Move 3 disc from B to C

Divided into smaller problem

- Move 4 discs → Move 3 discs
- Move 5 discs? → Move 4 discs
- Move n discs? → Move $n - 1$ discs

Recursion

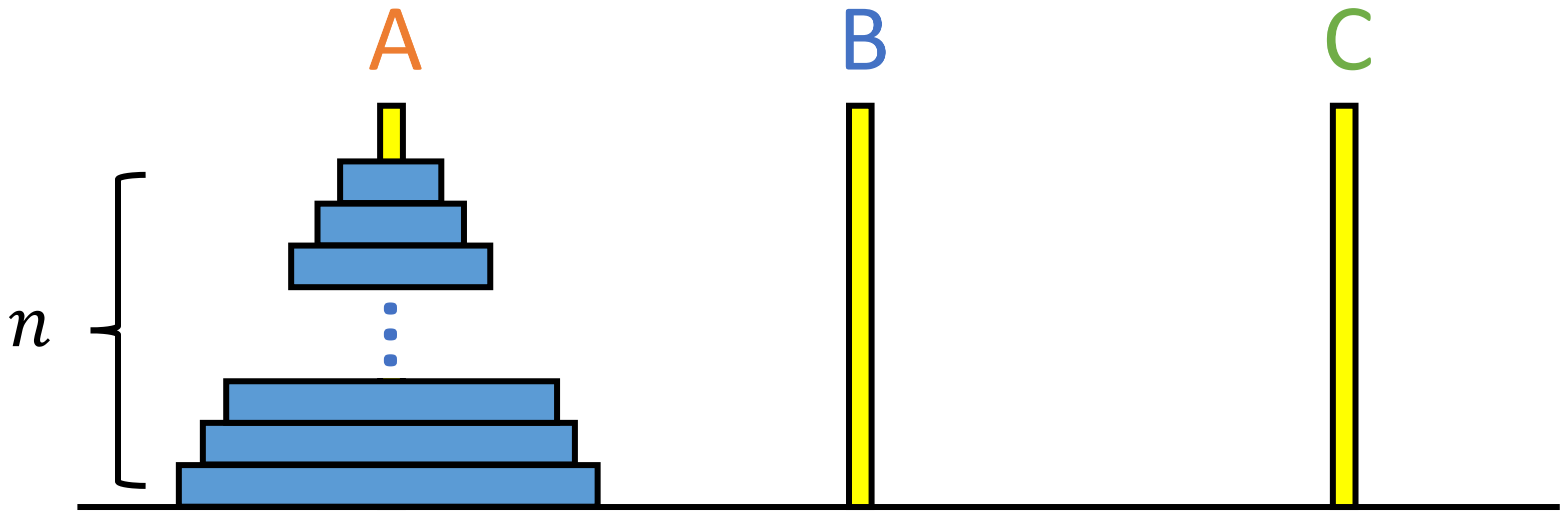
1. Expressed (divided) the problem into one or more smaller problems

$$n = f(n-1)$$

2. Solve the simple (base) case
   - 1 disc? Move directly from X to Y
   - 0 disc? Do nothing

High Level Idea

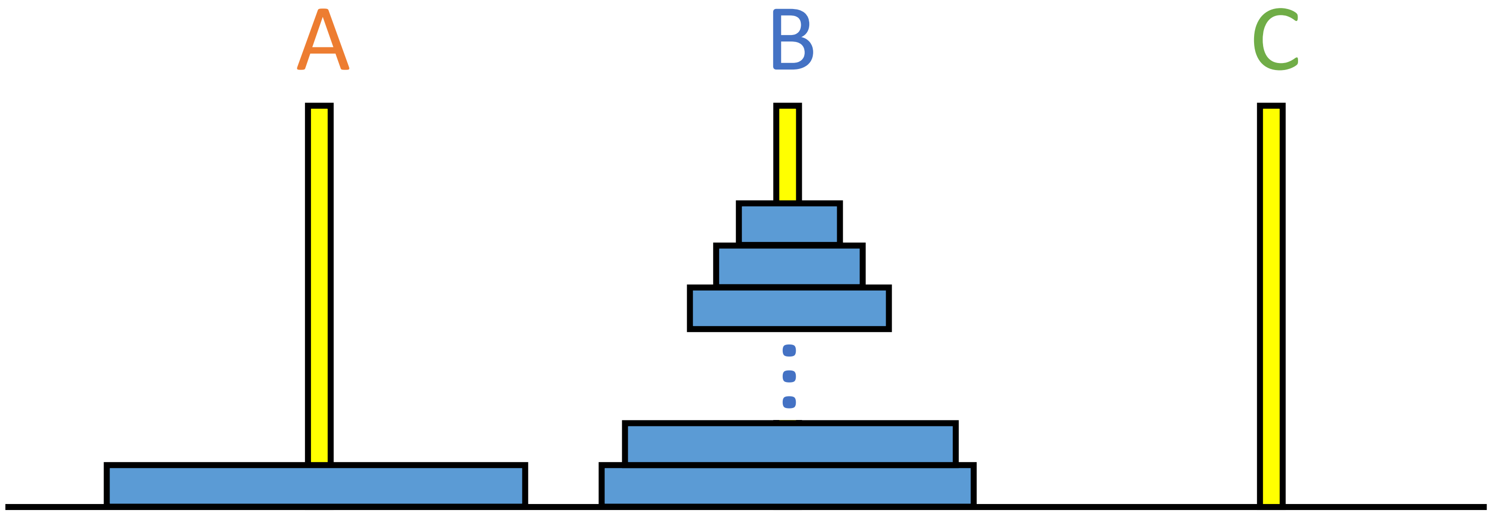
To move n discs from A to C using B



High Level Idea

To move n discs from A to C using B

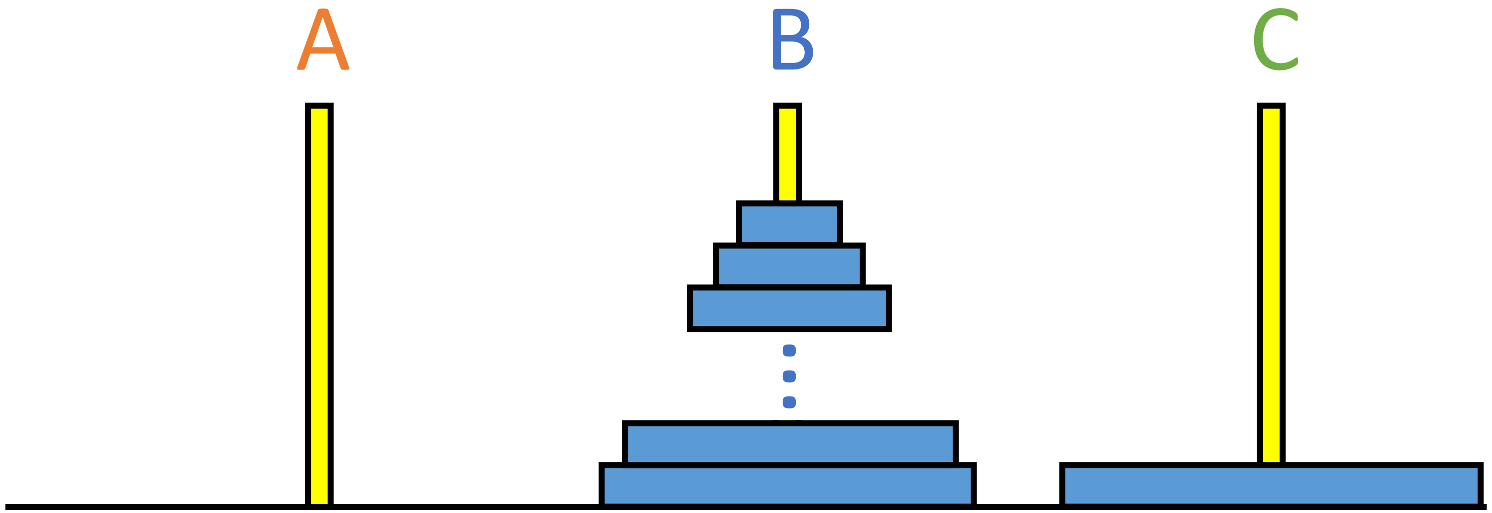
1. move $n-1$ discs from A to B using C



High Level Idea

To move n discs from A to C using B

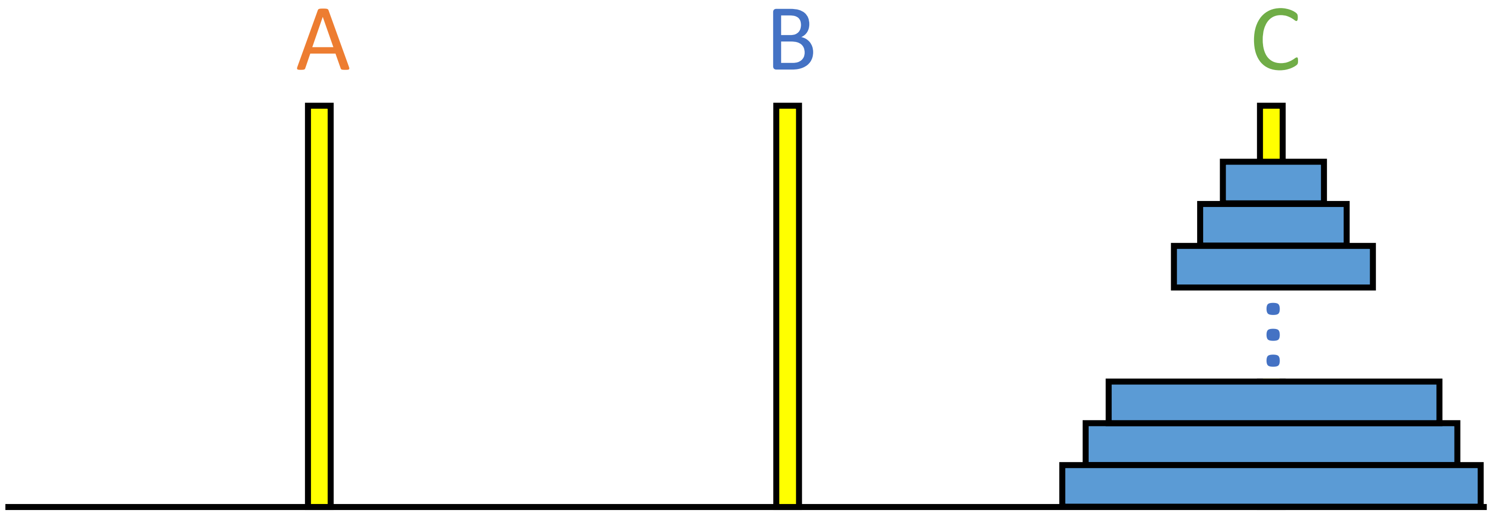
1. move $n-1$ discs from A to B using C
2. move disc from A to C



High Level Idea

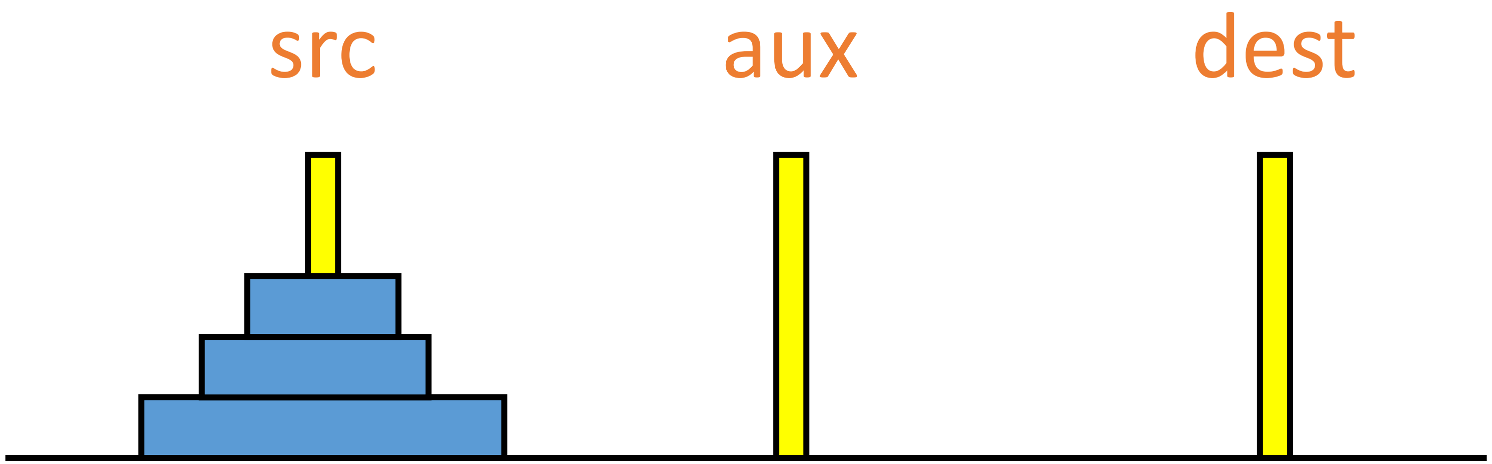
To move n discs from A to C using B

1. move $n-1$ discs from A to B using C
2. move disc from A to C
3. move $n-1$ discs from B to C using A



Towers of Hanoi

```
def move_tower(size, src, dest, aux):
    if size == 1:
        print_move(src, dest) # display the move
    else:
        move_tower(size-1, src, aux, dest)
        print_move(src, dest)
        move_tower(size-1, aux, dest, src)
```



Tower of Hanoi

```
def print_move(src, dest):
    print("move top disk from ", src" to ", dest)
```

Another example

What does this function compute?

```
def foo(x, y):
    if (y == 0):
        return 1
    else:
        return x * foo(x, y-1)
```

This?

```
def power(b, e):
    if (e == 0):
        return 1
    else:
        return b * power(b, e-1)
```

Exponentiation (b^e)

```
def power(b, e):
    if (e == 0):
        return 1
    else:
        return b * power(b, e-1)
```

- Time requirement? $O(n)$
- Space requirement? $O(n)$

Can we do better?

Another way to express b^e

$$b^e = \begin{cases} 1, & e = 0 \\ (b^2)^{\frac{e}{2}}, & e \text{ is even} \\ b^{e-1} \cdot b, & e \text{ is odd} \end{cases}$$

Fast Exponentiation

```
def fast_expt(b, e):
    if e == 0:
        return 1
    elif e % 2 == 0:
        return fast_expt(b*b, e/2)
    else:
        return b * fast_expt(b, e-1)
```

$$b^e = \begin{cases} 1, & e = 0 \\ (b^2)^{\frac{e}{2}}, & e \text{ is even} \\ b^{e-1} \cdot b, & e \text{ is odd} \end{cases}$$

- Time requirement? $O(\log n)$
- Space requirement? $O(\log n)$

Can we do this iteratively?

Summary

- Recursion
  - Solve the problem for a simple (base) case
  - Express (divide) a problem into one or more smaller similar problems

- Iteration: `while` and `for` loops

Summary

- Order of growth:
  - Time and space requirements for computations
  - Different ways of performing a computation (algorithms) can consume dramatically different amounts of resources.
  - Pay attention to efficiency!

Something to think about….

- Can you write a recursive function `sum_of_digits` that will return the sum of digits of an arbitrary positive integer?
- How about a recursive function `product_of_digits` that will return the product of the digits?

Notice a pattern?

How would you write a function that computed the sum of square roots of the digits of a number?